A closed ball compactification of a maximal component via cores of trees

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We show that, in the character variety of surface group representations into the Lie group $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$, the compactification of the maximal component introduced by the second author is a closed ball upon which the mapping class group acts. We study the dynamics of this action. Finally, we describe the boundary points geometrically as $(\overline{A_1 \times A_1}, 2)$–valued mixed structures.

1 Introduction

A recurring theme in higher Teichmüller theory is to relate surface group representations into higher-rank Lie groups with geometric objects. Taking its cue from classical Teichmüller theory, one is often interested in studying the degeneration of these associated geometric objects when the representation leaves all compact sets in the character variety. The celebrated Thurston compactification of Teichmüller space regards Fuchsian representations as marked hyperbolic metrics, where degenerating families of hyperbolic metrics subconverge to projectivized measured laminations. One key aspect of this compactification is that it is a closed ball upon which the mapping class group acts. In years following, there have been numerous different perspectives of the Thurston compactification, using a variety of methods, topological, geometric, analytic and algebraic (see [Bonahon 1988; Bestvina 1988; Paulin 1988; Wolf 1989; Morgan and Shalen 1984; Brumfiel 1988]).

When the Lie group $\mathrm{PSL}(2, \mathbb{R})$ is replaced with a higher-rank one, the relevant geometric object is not always immediately clear. In rank 2 however, combined work of Schoen [1993], Labourie [2017], Loftin [2001], Collier [2016], Alessandrini and Collier [2019], and Collier, Tholozan and Toulisse [Collier et al. 2019] provides a geometric interpretation to representations in the various distinguished components of the relevant character variety. These components are usually maximal components or Hitchin components, which maximize a topological quantity, the Toledo invariant, or contain a deformation of the classical Teichmüller space. Parreau [2012] compactifies them by attaching at infinity surface group actions on a Euclidean building.

This paper will primarily be concerned with the rank-2 semisimple split Lie group $G = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$. The product structure of G makes our study more amenable towards techniques from classical Teichmüller theory. For S a closed, orientable, smooth surface of genus $g > 1$, work of Goldman [1988] shows the connected components of the character variety $\chi(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$ are determined by the Euler number. In particular, the distinguished component with maximal Euler number of $2g-2$ is the Teichmüller space $\mathrm{Teich}(S)$. If we denote the character variety for $G = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ by $\chi(\pi_1(S), G)$, then the connected components are merely products of the connected components of $\chi(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$. The maximal component $\mathrm{Max}(S, G)$ of $\chi(\pi_1(S), G)$ is the collection of conjugacy classes of pairs of representations, each of which is a Fuchsian representation. Hence $\mathrm{Max}(S) := \mathrm{Max}(S, \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}))$ is the product of two copies of Teichmüller space.

Elements in the component $\mathrm{Max}(S)$ have a number of related geometric interpretations. Schoen [1993] has shown these representations correspond to equivariant minimal Lagrangians in $\mathbb{H}^2 \times \mathbb{H}^2$. At the same time, the group $G = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ is the isometry group of AdS^3, and Mess [2007] has shown the holonomy representations of GHMC-AdS^3 manifolds are precisely the ones in $\mathrm{Max}(S)$. Krasnov and Schlenker [2007] have shown to each GHMC-AdS^3 manifold there is a unique equivariant space-like maximal surface, whose image under the Gauss map is the aforementioned minimal Lagrangian.

In seeking a compactification of $\mathrm{Max}(S)$ via degeneration of geometric objects, the second author in his thesis [Ouyang 2023] showed the natural limits to the minimal Lagrangians were given by cores of $\mathbb{R}$–trees dual to measured laminations. These are topologically and group-theoretically defined distinguished subcomplexes of the product of two trees, where some parts are two-dimensional and the remaining parts are one-dimensional. Denote by $\mathrm{Core}(\mathcal{T}, \mathcal{T})$, the space of cores in the product of trees dual to measured laminations. Observe that there is a natural $\mathbb{R}^+$–action on $\mathrm{Core}(\mathcal{T}, \mathcal{T})$ and denote by $\mathbb{P}\,\mathrm{Core}(\mathcal{T}, \mathcal{T})$ the resulting projectivization. We equip $\mathrm{Max}(S)$ and $\mathbb{P}\,\mathrm{Core}(\mathcal{T}, \mathcal{T})$ with the equivariant Gromov–Hausdorff topology. One natural question one might ask is what exactly is the topology of the resulting compactification. Our first main result is the following.

Theorem A *The disjoint union*

$$\mathfrak{B} = \mathrm{Max}(S) \sqcup \mathbb{P}\,\mathrm{Core}(\mathcal{T}, \mathcal{T})$$

is homeomorphic to a closed ball of dimension $12g-12$.

More precisely, we will show that the interior of $\mathfrak{B}$ can be identified with $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$ and its boundary with $\mathbb{P}(\mathcal{MF}(S) \times \mathcal{MF}(S))$. A point in $\mathfrak{B}$ will thus be represented by a pair (x_1, x_2), where x_1 and x_2 are either both marked hyperbolic structures or both measured foliations up to simultaneous projective equivalence.

The key new contribution of Theorem A is the description of the topology of a compactification of a higher Teichmüller space. Even in the case of Teichmüller space, Thurston's original proof requires the construction of charts in order to show that the compactified space has the structure of a manifold with boundary and then uses the Schönflies theorem (see [Fathi et al. 2012, pages 162–164]). We overcome these difficulties in proving Theorem A by considering a more analytic approach inspired by the compactification of Teichmüller space using harmonic maps in [Wolf 1989]: we naturally identify $\mathrm{Max}(S)$ with a unit ball in a vector space of pairs of holomorphic quadratic differentials and $\mathbb{P}\,\mathrm{Core}(\mathcal{T}, \mathcal{T})$ with its boundary. To the best of our knowledge, this is the first example of a higher Teichmüller component of a closed surface that is compactified to a closed ball.

It is not too difficult to see from the construction of this compactification that the action of the mapping class group extends continuously to the boundary. Following Thurston, we study the action of the mapping class group $\mathrm{MCG}(S)$ on our compactification $\mathfrak{B}$.

Proposition 1.1 *Suppose $\phi \in \mathrm{MCG}(S)$ and $\phi(x) = x$ for some $x = (x_1, x_2) \in \mathfrak{B}$, where $\mathfrak{B}$ is as defined in Theorem A.*

   (1) *If ϕ is periodic, then x_1 and x_2 are any two points fixed by ϕ in the Thurston compactification of Teichmüller space such that $(x_1, x_2) \in \mathfrak{B}$.*

   (2) *If ϕ is pseudo-Anosov, then $(x_1, x_2) \in \partial\mathfrak{B}$ and $x_1 = 0$, or $x_2 = 0$ or $x_1 = x_2$.*

The action of the mapping class group appears to be more interesting if we consider its action on a natural quotient of $\mathfrak{B}$. In fact, given a maximal representation ρ, there is a unique equivariant minimal Lagrangian $\widetilde{\Sigma}_\rho$ in $\mathbb{H}^2 \times \mathbb{H}^2$. The induced metric on $\widetilde{\Sigma}_\rho$ descends to a negatively curved Riemannian metric on S. We denote by $\mathrm{Ind}(S)$ the space of such metrics. It turns out that $\mathrm{Ind}(S) = \mathrm{Max}(S)/S^1$, since there is an S^1–family of maximal representations with intrinsically isometric equivariant minimal Lagrangians. (However, these minimal Lagrangians are not extrinsically isometric in $\mathbb{H}^2 \times \mathbb{H}^2$: their second fundamental form, which is completely determined by a holomorphic quadratic differential on S, differs under rotation; see [Ouyang 2023, Proposition 4.3]). Similarly, the distance on the core of the product of two trees dual to a pair of measured laminations can be recovered from a mixed structure, that is, a hybrid geometric object on S that is in part a measured lamination and in part a finite-area flat metric induced by a meromorphic quadratic differential on subsurfaces glued along annuli. The space of projectivized mixed structures can then be identified with the boundary of $\mathrm{Ind}(S)$ in the length spectrum topology [Ouyang 2023]. The mapping class group acts on $\overline{\mathrm{Ind}(S)}$ and we prove the following:

Theorem B   *Assume $\phi \in \mathrm{MCG}(S)$ fixes $\mu \in \partial \overline{\mathrm{Ind}(S)}$.*

(1)   *If μ is **purely flat**, ie μ is a mixed structure without laminar pieces, then ϕ is periodic.*

(2)   *If μ is **properly mixed**, ie μ is a mixed structure with at least one flat subsurface and one laminar part, then ϕ is not pseudo-Anosov.*

Note that the remaining case of μ a *purely laminar* mixed structure, in other words a genuine measured lamination on S, is handled by the Nielson–Thurston classification theorem. Theorem 5.12 will give a more detailed description of item (2) in Theorem B when ϕ is reducible. In particular, we will show that the subdivision of S induced by μ is a refinement of the one induced by ϕ if μ has no trivial parts.

The absence of a product structure for the other simple split Lie groups of rank 2 makes the study of the topology of any compactification considerably more difficult. Furthermore, for $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$, quadratic differentials are intimately related to pairs of measured laminations, and for higher-order differentials, which appear for the other rank-2 cases, there are no obvious analogous topological objects. However, it is possible to describe our compactification without explicit references to $\mathbb{R}$–trees, and we conjecture this perspective can be extended to the other rank-2 Lie groups. In particular, given any Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{a}$ and positive Weyl chamber $\mathfrak{a}^+$, we define $\overline{\mathfrak{a}^+}$–valued measured laminations and $(\overline{\mathfrak{a}^+}, k)$–mixed structures obtained by gluing these vector-valued laminations together with $1/k$–translation surfaces of finite area along annuli (see Section 6 for details). We will consider this notion for the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$: in this case its Cartan subalgebra is of type $A_1 \times A_1$ and we denote by $\overline{A_1^+ \times A_1^+}$ the closure of a fixed positive Weyl chamber. Concretely, in this case the Cartan subalgebra can be chosen to be the space of pairs of 2×2 traceless diagonal matrices, so it is homeomorphic to $\mathbb{R}^2$ and $\overline{A_1^+ \times A_1^+}$ is homeomorphic to a quadrant. We can rephrase our main result as follows:

Theorem C   *The boundary of $\mathrm{Max}(S)$ can be identified with the space of $(\overline{A_1^+ \times A_1^+}, 2)$–mixed structures on S, which is thus topologically a sphere of dimension $12g - 13$.*

Moreover, we prove in Lemma 6.8 that $(\overline{A_1^+ \times A_1^+}, 2)$–mixed structures are dual to the subcomplexes of a Euclidean building introduced and studied in [Parreau 2022]. Theorem C has the advantage of being easily adaptable to other higher Teichmüller components (see Conjecture 6.7 for the precise statements in rank 2).

Historical remarks

In analogy with the classical case, compactifications of higher Teichmüller spaces are fruitfully studied using different techniques and perspectives. Parreau [2012] compactifies the character variety of surface group representations into noncompact semisimple connected real Lie groups with finite center using Euclidean buildings. For Hitchin and maximal connected components, one can obtain additional information on the boundary points by using the (Θ-)positivity properties of the representations as in [Alessandrini 2008; Burger and Pozzetti 2017; Fock and Goncharov 2006; Le 2016; Martone 2019a; 2019b;

Parreau 2022]. For rank-2 Lie groups, the second and third authors used analytic methods to study degenerations of geometric objects associated to these representations in [Ouyang 2023; Ouyang and Tamburelli 2021; 2023]. In a series of papers, Burger, Iozzi, Parreau and Pozzetti [Burger et al. 2017; 2021a; 2024] use geodesic currents and real algebrogeometric methods to study the Weyl chamber length spectrum compactification of general character varieties introduced in [Parreau 2012]. Their results apply in particular to Hitchin and maximal components, which are fundamental examples of higher Teichmüller spaces, and establish several structural properties of the boundary points. While we refer to their announcement [Burger et al. 2021b] for an account of their general framework and results, here we describe in greater detail their independent work [Burger et al. 2021c] on the compactification of n–copies of Teich(S). Burger, Iozzi, Parreau and Pozzetti identify the boundary of the Weyl chamber length spectrum compactification of Teich(S)^n with the projectivization of $\mathcal{MF}(S)^n$, which is a sphere of dimension $n(6g-6)-1$. In addition, they show that MCG(S) acts properly discontinuously on the space of *positive joint systole n–tuples* of measured foliations [Burger et al. 2021c, Theorem 1.1]. This result provides a new geometric description of the domain of discontinuity introduced in [Burger et al. 2021a] for the MCG(S) action on the boundary of the Weyl chamber length spectrum compactification in the case of the Lie group PSL$(2, \mathbb{R})^n$. Finally, when $n = 2$, they describe the boundary points as vector-valued mixed structures (in their language, $\mathbb{R}^2$–mixed structures) and associate to these objects a dual tree-graded $\mathbb{R}^2$–space in the sense of [Druțu and Sapir 2005] (see Theorems 1.2 and 1.3 in [Burger et al. 2021c]). Their results lead to an (a priori different) compactification of Max(S).

2 Background

2.1 Foliations, laminations and $\mathbb{R}$–trees

We recall some classical facts about measured foliations and laminations. This material can be found in [Fathi et al. 2012]. Let S be a closed, orientable, smooth surface of genus $g > 1$. A *measured foliation* is a singular foliation (with k–pronged singularities) equipped with a measure on transverse arcs, invariant under transverse homotopy.

If S is given a hyperbolic metric σ, then a *measured lamination* is a closed set of disjoint simple geodesics on (S, σ) together with a transverse measure. There is a natural homeomorphism between the space $\mathcal{MF}(S)$ of measured foliations on S and the space $\mathcal{ML}(S)$ of measured laminations on (S, σ), so that the role of σ is an auxiliary one. Thurston showed $\mathcal{MF}(S)$ is topologically trivial, being a ball of dimension $6g-6$. The space $\mathbb{PMF}(S)$ is the boundary of Teichmüller space under the Thurston compactification.

If S is given a complex structure J, then to any holomorphic quadratic differential $q = q(z)\,dz^2$, one may consider the foliation obtained by integrating the line field $q(v, v) > 0$. When further given the transverse measure defined by $\int_\alpha |\text{Im}(\sqrt{q})|$, the resulting measured foliation is called the *horizontal foliation* of q. Likewise integrating the line field $q(v, v) < 0$ and taking the measure $\int_\alpha |\text{Re}(\sqrt{q})|$ gives the *vertical foliation* of q. The theorem of [Hubbard and Masur 1979] states that for a fixed Riemann surface (S, J)

and any measured foliation $\mathcal{F}$ on S, there is a unique holomorphic quadratic differential q, whose horizontal foliation is Whitehead equivalent (ie it differs at most by isotopies or expanding or collapsing pronged singularities along straight arcs) to $\mathcal{F}$. Any measured foliation $\mathcal{F}$ on S lifts to a measured foliation $\widetilde{\mathcal{F}}$ on the universal cover $\widetilde{S}$. Taking the leaf space of $\widetilde{\mathcal{F}}$ together with a distance induced by the pushforward of the transverse measure gives an $\mathbb{R}$–tree. When an $\mathbb{R}$–tree is constructed from a measured foliation in this way, the $\mathbb{R}$–tree comes equipped with a $\pi_1(S)$–action from $\widetilde{\mathcal{F}}$. This action is *small*, that is, the stabilizer of an arc never contains a free group of rank 2, and *minimal*, that is, the action does not preserve any proper subtree. A result of [Skora 1996] says that any $\mathbb{R}$–tree with a $\pi_1(S)$–action which is both small and minimal is constructed from a measured foliation on S. Such $\mathbb{R}$–trees are said to be dual to a measured foliation, and for our purposes, all $\mathbb{R}$–trees we consider will be dual to a measured foliation.

2.2 Half-translation surfaces, flat metrics and mixed structures

A Riemann surface equipped with a holomorphic quadratic differential q is called a half-translation surface. This terminology comes from the fact these can be realized by gluing polygons in $\mathbb{C}$ via translations or rotations of angle π.

A half-translation surface is naturally endowed with a singular flat metric $|q|$, where the singularities are at the zeros of q. Duchin, Leininger and Rafi [Duchin et al. 2010] have studied the degeneration of unit-area quadratic differential metrics, and have shown the limits are precisely projectivized (quadratic) mixed structures. A *mixed structure* is a collection of integrable meromorphic quadratic differential metrics on subsurfaces and measured laminations on other subsurfaces, glued along flat annuli to recover the surface S. Trivial examples of mixed structures include singular flat metrics on S and measured laminations on S. We say that a mixed structure is *properly mixed* if it has a flat piece but it is not a singular flat metric. Mixed structures, when the meromorphic differential is cubic or quartic, appear in the compactification of Hitchin components for $SL(3, \mathbb{R})$ and $Sp(4, \mathbb{R})$ (see [Ouyang and Tamburelli 2021; 2023]).

A measured lamination λ on S is said to *fill* if the complement $S \setminus \lambda$ is a disjoint union of topological disks. A pair $\mathcal{F}_1, \mathcal{F}_2$ of measured foliations on S is said to *fill* or is *transverse* if, for any third foliation $\mathcal{G}$, one has $i(\mathcal{F}_1, \mathcal{G}) + i(\mathcal{F}_2, \mathcal{G}) > 0$. Here $i(\cdot, \cdot)$ denotes the Bonahon intersection pairing, which generalizes the topological intersection number between curves. We remark that the intersection number for the corresponding measured laminations is the same; therefore we can define filling for a pair of measured laminations analogously. Notice that given a holomorphic quadratic differential q, the vertical and horizontal foliations of q fill. Conversely, the result of [Gardiner and Masur 1991] says that, given any pair of filling measured foliations, there exists a unique Riemann surface structure and a unique holomorphic quadratic differential which realizes the original pair as its vertical and horizontal foliation (up to Whitehead equivalence). In particular, a pair of filling measured foliations will determine a unique half-translation surface structure and consequently a unique singular flat quadratic differential metric.

2.3 Minimal Lagrangians in $\mathbb{H}^2 \times \mathbb{H}^2$

A *minimal Lagrangian* $\widetilde{\Sigma}$ in $\mathbb{H}^2 \times \mathbb{H}^2$ is a minimal surface which is Lagrangian with respect to the symplectic form $\omega \oplus -\omega$, where ω is the standard Kähler form on $\mathbb{H}^2$. Any $\rho \in \mathrm{Max}(S)$ acts on $\mathbb{H}^2 \times \mathbb{H}^2$, and Schoen [1993] has shown to each such ρ there is a unique ρ–equivariant minimal Lagrangian $\widetilde{\Sigma}_\rho$ in $\mathbb{H}^2 \times \mathbb{H}^2$, thereby providing a geometric interpretation to representations in $\mathrm{Max}(S)$. The second author [Ouyang 2023] has studied the degeneration of these minimal Lagrangians and has shown that one may interpret the space $\mathbb{P}\,\mathrm{Core}(\mathcal{T}, \mathcal{T})$ as the boundary of the maximal component $\mathrm{Max}(S)$.

2.4 Induced metrics and projectivized mixed structures

The induced metric on the unique ρ–equivariant minimal Lagrangian descends to a metric on *S*. It is not too difficult (see [Ouyang 2023, Proposition 4.2]) to see this metric is in fact negatively curved. Hence, by the result of [Otal 1990], its marked length spectrum determines the metric. The marked length spectrum is the data of both the curve class and the length of its geodesic representative in the given homotopy class. Let $\mathrm{Ind}(S)$ denote the space of induced metrics coming from the ρ–equivariant minimal Lagrangians. Then in fact one may embed $\mathrm{Ind}(S)$ into the space of projectivized marked length spectra. Its closure is then determined to be precisely the space $\mathrm{Ind}(S)$ together with the projectivized mixed structures [Duchin et al. 2010, Theorem 5; Ouyang 2023, Theorem 5.5].

3 Core of a product of trees

In this section we recall the notion of core of a product of trees and describe its geometry in the case of trees dual to measured laminations. The core of a product of two $\mathbb{R}$–trees can actually be defined for any pair of $\mathbb{R}$–trees each admitting a $\pi_1(S)$–action. It is not necessary that the $\mathbb{R}$–trees be dual to measured foliations. However, we will specifically mention when particular properties of cores are germane only to $\mathbb{R}$–trees dual to measured foliations. The main reference for the material covered here is [Guirardel 2005].

Given an $\mathbb{R}$–tree T, a *direction* δ based at a point $p \in T$ is a connected component of $T \setminus \{p\}$. For a product $T_1 \times T_2$ of $\mathbb{R}$–trees, a *quadrant* Q based at $(p_1, p_2) \in T_1 \times T_2$ is a product $\delta_1 \times \delta_2$ of directions. If the $\mathbb{R}$–trees T_1, T_2 are equipped with a $\pi_1(S)$–action by isometries, then we say a quadrant Q is *heavy* if there exists a sequence $\{\gamma_n\} \subset \pi_1(S)$ for which, for $i = 1, 2$,

  (i) $\gamma_n \cdot p_i \in \delta_i$, and

  (ii) $d_i(\gamma_n \cdot p_i, p_i) \to \infty$ as $n \to \infty$.

Otherwise the quadrant is said to be *light*. Following [Guirardel 2005], the *core* $\mathcal{C}(T_1, T_2)$ of $T_1 \times T_2$ is

$$ T_1 \times T_2 \setminus \bigsqcup_{Q \text{ light}} Q. $$

When T_1 and T_2 are dual to measured laminations, the core $\mathcal{C}(T_1, T_2)$ is always nonempty since the $\pi_1(S)$–actions are irreducible [Guirardel 2005, Proposition 3.1].

However, even when T_1 and T_2 are dual to measured foliations, one pathology may still occur: $\mathcal{C}(T_1, T_2)$ may be disconnected. This happens, for instance, when $T_1 = T_2$ and T_1 is dual to a multicurve. However, in such cases, Guirardel introduced a canonical way of extending the core to a connected subset of $T_1 \times T_2$ with convex fibers. (Here, a subset $E \subset T_1 \times T_2$ has convex fibers if for every $x \in T_i$ the set $E \cap p_i^{-1}(x)$ is convex, where $p_i : T_1 \times T_2 \to T_i$ denotes the canonical projection.) With abuse of terminology, we will still refer to this canonical extension as the core of $T_1 \times T_2$. The following result completely characterizes when this extension needs to be considered.

Definition 3.1 Given two real trees T and T' endowed with an action of $\pi_1(S)$, we say that T is a *refinement* of T' if there is an equivariant map $f : T \to T'$ such that for all $x, y, z \in T$ if z lies in the geodesic $[x, y]$ connecting x and y, then $f(z)$ belongs to $[f(x), f(y)]$.

Proposition 3.2 [Guirardel 2005, Proposition 4.14] *Let T_1 and T_2 be trees dual to measured laminations. Then the core $\mathcal{C}(T_1, T_2)$ is disconnected if and only if T_1 and T_2 are refinements of a common nontrivial simplicial tree T.*

For example the assumptions of Proposition 3.2 are satisfied if T_1 and T_2 are dual to measured laminations λ_1 and λ_2 with common isolated leaves.

When T_1 and T_2 are both dual to measured laminations λ_1 and λ_2, we can actually realize the core $\mathcal{C}(T_1, T_2)$ more concretely. Before describing this construction, we need the following result, which can be seen as a special case of the decomposition theorem for general geodesic currents in [Burger et al. 2017] (see also [Burger et al. 2021a]) about how two measured laminations interact on subsurfaces. Here, when we refer to measured laminations on open surfaces S', usually arising as subsurfaces of S, we will always assume them to be compactly supported in S'.

Lemma 3.3 *Let λ_1 and λ_2 be measured laminations on S. Then there is a system of nontrivial, pairwise nonhomotopic, disjoint, simple closed curves $\gamma_1, \ldots, \gamma_n$ such that on each connected component S' of $S \setminus \bigcup_j \gamma_j$ either*

(i) *$\lambda_1 + \lambda_2$ is a (possibly zero) measured lamination on S', or*

(ii) *λ_1 and λ_2 are transverse and fill S'; ie for all measured laminations ν on S' we have*

$$i(\lambda_1, \nu) + i(\lambda_2, \nu) \neq 0.$$

Proof Consider a maximal collection of nontrivial, pairwise nonhomotopic, disjoint, simple closed curves γ_j such that

$$i(\lambda_1, \gamma_j) + i(\lambda_2, \gamma_j) = 0.$$

We claim that this collection of curves satisfies the requirement of the lemma. Indeed, let S' be a connected component of $S \setminus \bigcup_j \gamma_j$. We need to show that if the pair (λ_1, λ_2) does not fill the subsurface S', then

$\lambda_1 + \lambda_2$ is a lamination on S', or, equivalently, λ_1 and λ_2 are nowhere transverse on S'. The claim is clearly true if the support of either λ_1 or λ_2 does not intersect S', so we can assume that both have support on S'. Because the pair (λ_1, λ_2) does not fill S' by assumption, there is a measured lamination ν on S' such that $i(\lambda_1, \nu) + i(\lambda_2, \nu) = 0$. On the other hand, by hypothesis, $i(\lambda_1, \gamma) + i(\lambda_2, \gamma) \neq 0$ for all nonperipheral simple closed curves γ on S'. Therefore, the measured lamination ν does not contain isolated closed leaves. Let us first consider the case in which ν fills the subsurface S', in the sense that the complement of ν (in S') only consists of disks and annuli. We note that then necessarily the support of ν must contain the support of λ_1 and λ_2 because otherwise λ_1 and λ_2 would intersect ν transversely somewhere. But this implies that λ_1 and λ_2 are nowhere transverse, being both contained in the support of a measured lamination. We now reduce the general case to this, by showing that ν must fill S'. Assume the opposite, and let $S'' \subset S'$ be a subsurface filled by ν. Note that at least one between λ_1 and λ_2 intersects the boundaries of S'' transversely. Without loss of generality we assume it is λ_1. Since ν fills S'', the support of λ_1 intersects ν transversely, but this contradicts the fact that $i(\nu, \lambda_1) = 0$. $\square$

The last ingredient we need is an explicit realization of a tree T_λ dual to a measured lamination λ. The construction goes as follows (see [Morgan and Otal 1993] for more details). Fix an auxiliary hyperbolic metric on S and identify $\widetilde{S}$ with $\mathbb{H}^2$. Let $\tilde{\lambda}$ be the lift of λ under the covering map $\pi : \mathbb{H}^2 \to S$. We define the metric space $\mathrm{pre}(T_\lambda)$, where points of $\mathrm{pre}(T_\lambda)$ are the connected components of $\mathbb{H}^2 \setminus \tilde{\lambda}$ and the distance is computed as follows: if $x, y \in \mathrm{pre}(T_\lambda)$ correspond to connected components C_x, C_y of $\mathbb{H}^2 \setminus \tilde{\lambda}$ then

$$d_\lambda(x, y) = \inf\left\{ \int_\gamma d\tilde{\lambda} \mid \gamma : [0,1] \to \mathbb{H}^2,\ \gamma(0) \in C_x,\ \gamma(1) \in C_y \right\}.$$

The tree T_λ is then the unique $\mathbb{R}$–tree that contains $\mathrm{pre}(T_\lambda)$ such that any point of T_λ lies in a segment with vertices in $\mathrm{pre}(T_\lambda)$. Note that we have a natural projection map $p_\lambda : \mathbb{H}^2 \setminus \tilde{\lambda} \to T_\lambda$. If λ has no isolated leaves, this map extends continuously to a map, still denoted by p_λ, defined on the entire $\mathbb{H}^2$. Otherwise, the continuous extension is obtained by first replacing each isolated leaf ℓ in $\tilde{\lambda}$ with a strip $\ell \times [-\epsilon, \epsilon]$ endowed with a uniform measure with total mass equal to $\tilde{\lambda}_{|\ell}$.

There is also another way of realizing the tree dual to a measured lamination using the language of measured foliations. Let $\mathcal{F}$ denote the measured foliation corresponding to the measured lamination λ under the homeomorphism between $\mathcal{MF}(S)$ and $\mathcal{ML}(S)$. Let $\widetilde{\mathcal{F}}$ be its lift to $\mathbb{H}^2$. Then the tree T_λ can be defined as the quotient $\mathbb{H}^2 / \sim$, where $\sim$ denotes the equivalence relation

$$x \sim y \iff d_{\mathcal{F}}(x, y) = 0$$

and

$$d_{\mathcal{F}}(x, y) = \inf\{ i(\widetilde{\mathcal{F}}, \gamma) \mid \gamma : [0,1] \to \mathbb{H}^2,\ \gamma(0) = x,\ \gamma(1) = y \}.$$

More concretely, T_λ identifies with the leaf space of $\widetilde{\mathcal{F}}$ with distance given by integrating the measure of $\widetilde{\mathcal{F}}$ along arcs transverse to the leaves. We denote by π_λ the natural projection $\pi_\lambda : \mathbb{H}^2 \to T_\lambda$.

We are now ready to describe the core of a product of two trees T_1 and T_2 dual to measured laminations λ_1 and λ_2 on S. Lemma 3.3 furnishes a decomposition of S into subsurfaces that we lift to a decomposition

of $\mathbb{H}^2$. The regions of this decomposition come in two flavors according to whether they project to subsurfaces where $\lambda_1 + \lambda_2$ is a lamination or to subsurfaces where the pair (λ_1, λ_2) fills. Following the statement of Lemma 3.3, we call these regions of type i and type ii, respectively. On the regions $\Omega \subset \mathbb{H}^2$ of type i, the union of the lifts $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ can be regarded as the lift of the measured lamination $\lambda_0 = \lambda_1 + \lambda_2$. We denote by T_0 the tree dual to λ_0. Note that, for each region $\Omega \subset \mathbb{H}^2$ of type i, we have a map $p_0 := p_{\lambda_0}$, as defined before, and two natural collapsing maps $c_j : T_0 \to T_j$ for $j = 1, 2$. On the regions of type ii, we replace the measured laminations $\tilde{\lambda}_i$ with the corresponding measured foliations $\tilde{\mathcal{F}}_i$ and consider the projections $\pi_i := \pi_{\lambda_i}$ as described previously. Following [Guirardel 2005, Example 4; 2005, Proposition 6.1], the core $\mathcal{C}(T_1, T_2)$ is the image of the map $F : \tilde{S} \to T_1 \times T_2$ defined as follows:

$$(1) \qquad F(x) = \begin{cases} (\pi_1 \times \pi_2)(x) & \text{if } x \text{ belongs to a region of type } ii, \\ (c_1 \times c_2)(p_0(x)) & \text{if } x \text{ belongs to a region of type } i. \end{cases}$$

Note that F is well-defined and continuous on the boundary $\tilde{\gamma}$ between two different regions of $\tilde{S}$ because $\tilde{\gamma}$ is the lift of a curve γ_j given by Lemma 3.3 which, by definition, has vanishing intersection number with λ_0, $\mathcal{F}_1$, and $\mathcal{F}_2$; hence $(\pi_1 \times \pi_2)(\tilde{\gamma})$ and $(c_1 \times c_2)(p_0(\tilde{\gamma}))$ is a single point.

It follows from this explicit description of $\mathcal{C}(T_1, T_2)$ that the core is, in general, a 2–dimensional subcomplex of $T_1 \times T_2$ that is invariant under the diagonal action of $\pi_1(S)$. Moreover, the 2–dimensional pieces of $\mathcal{C}(T_1, T_2)$ are exactly the images of regions of type ii and are foliated by two families of transverse foliations. Their quotients under the group action are the union of the subsurfaces of S in which λ_1 and λ_2 fill, endowed with the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ [Guirardel 2005, Example 4]. In particular, the 2–dimensional pieces of $\mathcal{C}(T_1, T_2)$ are the universal covers of half-translation surfaces. On the other hand, the images under F of regions Ω of type i are 1–dimensional subcomplexes of $T_1 \times T_2$. Each such Ω can be seen as the universal cover of a subsurface S' of S where the restriction of $\lambda_1 + \lambda_2$ is a measured lamination. Let $T_1' \subset T_1$ and $T_2' \subset T_2$ be the corresponding subtrees. It turns out [Guirardel 2005, Section 6] that $F(\Omega)$ is an $\mathbb{R}$–tree that is a common refinement of T_1' and T_2' if endowed with the distance

$$d_0(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2), \quad x = (x_1, x_2), \ y = (y_1, y_2) \in T_1' \times T_2',$$

where d_j denotes the distance on T_j.

Lemma 3.4 *The $\mathbb{R}$–tree $(F(\Omega), d_0)$ is isometric to the tree dual to the measured lamination $\lambda_0 = \lambda_1 + \lambda_2$ restricted to S'.*

Proof The tree $F(\Omega)$ inherits from $T_1' \times T_2'$ an isometric action of $\pi_1(S')$. We can define a length function

$$\ell : \pi_1(S') \to \mathbb{R}^+, \quad \gamma \mapsto \lim_{n \to +\infty} \frac{1}{n} d_0(x, \gamma^n \cdot x),$$

where x is any point in $F(\Omega)$ (the definition is independent of the choice of x). The limit in the formula above is well-defined and coincides, indeed, with the minimal translation distance of $\gamma \in \pi_1(S')$ [Guirardel and Levitt 2017, Section A.3]. Since the action of $\pi_1(S')$ on $F(\Omega)$ is minimal and irreducible, by [Guirardel and Levitt 2017, Theorem A.5], the isometry class of $(F(\Omega), d_0)$ is completely determined

by its length function. However, it is clear from the definition of ℓ and d_0 that $\ell = \ell_0 := \ell_1 + \ell_2$, where ℓ_j denotes the analogously defined length functions on T_1' and T_2'. On the other hand, ℓ_0 is exactly the length function of the tree dual to the measured lamination λ_0, and the claim follows. $\qquad\square$

The ambient space $T_1 \times T_2$ has, however, another natural distance defined by

$$d(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}, \quad x = (x_1, x_2),\ y = (y_1, y_2) \in T_1 \times T_2.$$

This induces a path metric $d_{\mathcal{C}}$ on the core $\mathcal{C}$ of $T_1 \times T_2$, where the $d_{\mathcal{C}}$–distance between two points in the core is the infimum of the length of all paths connecting the points and entirely contained in the core, where the length is computed using the distance d. Guirardel [2005, Proposition 4.9] showed that the core is a CAT(0) space if endowed with this path distance $d_{\mathcal{C}}$. In particular, since $F(\Omega)$ does not contain topological circles by Lemma 3.4, we can conclude that $F(\Omega)$ endowed with the restriction of $d_{\mathcal{C}}$ is still an $\mathbb{R}$–tree.

We will denote by $\mathrm{Core}(\mathcal{T}, \mathcal{T})$ the space of cores of the product of two trees dual to measured laminations on S endowed with this path distance.

Proposition 3.5 $\mathrm{Core}(\mathcal{T}, \mathcal{T})$ *is homeomorphic to* $\mathcal{ML}(S) \times \mathcal{ML}(S)$.

Proof Since the core of a product of trees is uniquely determined by the two factors, the result follows immediately from the homeomorphism between the space of trees dual to measured laminations and $\mathcal{ML}(S)$. $\qquad\square$

We note that there is a natural $\mathbb{R}^+$–action on $\mathrm{Core}(\mathcal{T}, \mathcal{T})$ given by rescaling the induced metric on the core, which, under the homeomorphism above, corresponds to the diagonal action of $\mathbb{R}^+$ by scalar multiplication on the measures. We denote by $\mathbb{P}\,\mathrm{Core}(\mathcal{T}, \mathcal{T})$ the quotient $\mathrm{Core}(\mathcal{T}, \mathcal{T})/\mathbb{R}^+$. It follows that $\mathbb{P}\,\mathrm{Core}(\mathcal{T}, \mathcal{T})$ is homeomorphic to $\mathbb{P}(\mathcal{ML}(S) \times \mathcal{ML}(S))$. In particular, it is topologically a sphere of dimension $12g-13$.

4 Thurston's compactification

Recall that we denote by $\mathrm{Max}(S)$ the space of conjugacy classes of representations $\rho = (\rho_1, \rho_2)$ of the fundamental group of a closed connected oriented surface S of negative Euler characteristic into the Lie group $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ such that $e(\rho_1) + e(\rho_2) = 4g - 4$. Here, e denotes the Euler number of the representation. It follows from [Goldman 1988] that ρ_1 and ρ_2 are both Fuchsian representations. Therefore, as $\mathrm{Max}(S)$ may be thought of as the product of two copies of Teichmüller space, it is homeomorphic to an open cell of dimension $12g-12$.

The main goal of this section is to prove Theorem A from the Introduction, which we restate below for the convenience of the reader.

Theorem 4.1 *The disjoint union*

$$\mathcal{B} = \mathrm{Max}(S) \sqcup \mathbb{P}\,\mathrm{Core}(\mathcal{T}, \mathcal{T})$$

is homeomorphic to a closed ball of dimension $12g-12$.

We begin by recalling the topology placed on $\mathfrak{B}$. The maximal component $\mathrm{Max}(S)$ is naturally homeomorphic to the product of two copies of Teichmüller space. This in turn, by the result of Schoen, is homeomorphic to the space of equivariant minimal Lagrangians in $\mathbb{H}^2 \times \mathbb{H}^2$. Under the Gromov–Hausdorff topology, diverging sequences of minimal Lagrangians subconverge to the (projective) core of a product of two trees [Ouyang 2023, Theorem 8.1]. The two trees are dual to a pair of measured laminations, and the topology on $\mathfrak{B}$ is compatible with the Thurston compactification on $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$ in the following way: if $(\rho_{1,n}, \rho_{2,n}) \to [\lambda_1, \lambda_2]$, then the associated minimal Lagrangians converge to the core of $T_1 \times T_2$, where T_i is dual to λ_i.

Fix a complex structure J on S and denote by X the Riemann surface (S, J). Then for any hyperbolic metric $h \in \mathrm{Teich}(S)$ there is a unique harmonic map $w_h \colon X \to (S, h)$ in the homotopy class of the identity [Eells and Sampson 1964; Hartman 1967]. Harmonicity of w_h ensures that the Hopf differential $q_h = (w_h^* h)^{(2,0)}$ is a holomorphic quadratic differential on X. The vector space $\mathrm{QD}(X)$ of holomorphic quadratic differentials on X has a natural norm given by the L^2–norm with respect to the uniformizing hyperbolic metric σ of X. With an abuse of notation, we will still denote by X the hyperbolic surface (S, σ). The map which assigns to a point in Teichmüller space its corresponding Hopf differential is a homeomorphism [Wolf 1989].

Proof of Theorem 4.1 By Theorem 6.13 of [Ouyang 2023], the space $\mathrm{Max}(S) \sqcup \mathbb{P}\,\mathrm{Core}(\mathcal{T}, \mathcal{T})$ is naturally homeomorphic to $\mathrm{Teich}(S) \times \mathrm{Teich}(S) \sqcup \mathbb{P}(\mathcal{MF}(S) \times \mathcal{MF}(S))$, so it suffices to prove the latter is homeomorphic to a closed ball of dimension $12g-12$.

As $\mathbb{P}(\mathcal{MF}(S) \times \mathcal{MF}(S))$ is homeomorphic to a sphere of dimension $12g-13$, the remainder of the proof consists of describing how to attach this topological space to the open cell $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$ to obtain a closed ball.

We start by fixing a complex structure J on S. Let $X = (S, J)$ be the resulting Riemann surface. By the Wolf parametrization [1989]

$$\mathrm{Teich}(S) \times \mathrm{Teich}(S) \cong \mathrm{QD}(X) \oplus \mathrm{QD}(X)$$

via the map $\Phi(\rho_1, \rho_2) = (q_{\rho_1}, q_{\rho_2})$. We equip $\mathrm{QD}(X) \oplus \mathrm{QD}(X)$ with the norm

$$\|q\| = \max(\|q_1\|, \|q_2\|),$$

and consider

$$\mathrm{BPQD}(X) = \{q = (q_1, q_2) : \|q\| < 1\},$$

which is, topologically, a ball of dimension $12g-12$. We will need the following lemma.

Lemma 4.2 *The map*

$$\beta \colon \mathrm{QD}(X) \oplus \mathrm{QD}(X) \to \mathrm{BPQD}(X), \quad q = (q_1, q_2) \mapsto \frac{4q}{1 + 4\|q\|},$$

is continuous, injective, and proper. Hence β is a homeomorphism.

Proof Suppose $\beta(q_1, q_2) = \beta(\phi_1, \phi_2)$. It follows then that $q_1 = k\phi_1$ and $q_2 = k\phi_2$ for some $k \in \mathbb{R}$. Writing out $\beta(q_1, q_2) = \beta(q_1/k, q_2/k)$, basic algebra shows $k = 1$. Continuity and properness follow by inspection. $\qquad\square$

We will now describe the attaching map. Consider the map

$$\psi : \mathrm{Teich}(S) \times \mathrm{Teich}(S) \sqcup \mathbb{P}(\mathcal{MF}(S) \times \mathcal{MF}(S)) \to \overline{\mathrm{BPQD}(X)}$$

defined by

$$\psi(x) = \begin{cases} \beta(\Phi(x)) & \text{if } x \in \mathrm{Teich}(S) \times \mathrm{Teich}(S), \\ \lim_{n \to +\infty} \beta(\Phi(x_n)) & \text{if } x \in \mathbb{P}(\mathcal{MF}(S) \times \mathcal{MF}(S)) \text{ and } x_n \to x. \end{cases}$$

We show first that the map ψ is well-defined. Suppose $x_n = (X_{1,n}, X_{2,n}) \to x$ and $x_n' = (X_{1,n}', X_{2,n}') \to x$, where $x = [\lambda_1, \lambda_2] \in \mathbb{P}(\mathcal{MF}(S) \times \mathcal{MF}(S))$. That is to say, there exist sequences of real numbers c_n, d_n for which the rescaled hyperbolic surfaces $\widetilde{X}_{i,n}/c_n$ and $\widetilde{X}_{i,n}'/d_n$ converge to $\mathbb{R}$–trees T_i, T_i' dual to laminations λ_i and λ_i' such that $[(\lambda_1, \lambda_2)] = [(\lambda_1', \lambda_2')]$. By [Wolf 1989], the sequences c_n and d_n can be taken to be $\|\Phi(x_n)\|$ and $\|\Phi(x_n')\|$. Note that, a priori, $\lambda_i' = k\lambda_i$ for some $k > 0$. With such rescaling, the harmonic maps $h_{i,n} \colon \widetilde{X} \to \widetilde{X}_{i,n}/c_n$ converge to the harmonic map $h_i \colon \widetilde{X} \to T_i$ given by projection onto the leaf space of the measured foliation $\widetilde{\mathcal{F}}_i$ corresponding to $\tilde{\lambda}_i$ [Wolf 1995, Corollary 5.2]. Moreover the sequence of Hopf differentials $q_{i,n}$ of $h_{i,n}$ converges to the Hopf differential q_i of h_i (here take the quotient so that q_i is a holomorphic quadratic differential on X and not $\widetilde{X}$). Finally, the differential q_i is the unique holomorphic quadratic differential on X whose horizontal foliation is Whitehead equivalent to $\mathcal{F}_i$. Likewise the sequence of harmonic maps $h_{i,n}' \colon \widetilde{X} \to \widetilde{X}_{i,n}'/d_n$ converges to the harmonic map $h' \colon \widetilde{X} \to T_i'$, whose Hopf differential q_i' is the limit of the Hopf differentials $q_{i,n}'$ of $h_{i,n}'$ and has horizontal foliation $\mathcal{F}_i'$ corresponding to the lamination λ_i'. Notice, in addition, that $(q_{1,n}, q_{2,n}) = \Phi(x_n)/\|\Phi(x_n)\|$ and similarly $(q_{1,n}', q_{2,n}') = \Phi(x_n')/\|\Phi(x_n')\|$. It follows that the limits of $\beta(\Phi(x_n))$ and $\beta(\Phi(x_n'))$ as $n \to +\infty$ exist and coincide with (q_1, q_2) and (q_1', q_2'). As the distance functions d_i and d_i' on T_i and T_i' satisfy $d_i = k \cdot d_i'$, by homogeneity of the Hopf differential, one has $q_i = k \cdot q_i'$. Since the pairs (q_1, q_2) and (q_1', q_2') both have unit norm, we conclude that $k = 1$ and the limits of $\beta(\Phi(x_n))$ and $\beta(\Phi(x_n'))$ as $n \to +\infty$ are equal.

Continuity follows almost immediately: the map $\beta \circ \Phi$ is continuous on the interior and extends continuously to the boundary by a diagonal argument. Indeed, we can approximate a sequence along the boundary by sequences in the interior.

Bijectivity of ψ on the interior also follows by [Wolf 1989] and Lemma 4.2. On the boundary, given $q = (q_1, q_2)$ with $\|q\| = 1$, if $X_{i,t}$ is the hyperbolic surface corresponding to the rays tq_i in Wolf's parameterization of $\mathrm{Teich}(S)$, we have that $\beta(\Phi(X_{1,t}, X_{2,t})) \to q$ as $t \to \infty$; thus ψ is surjective on the boundary. Since the limit of $\beta(\Phi(x_n))$ along diverging sequences in $x_n \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ only depends on the projective class of the limit of x_n and not on the particular sequence, we deduce that ψ is injective on the boundary, because every point in $\mathbb{P}(\mathcal{ML}(S) \times \mathcal{ML}(S))$ can be obtained as a limit along a ray defined above and the limit of $\beta \circ \Phi$ along distinct rays is different.

It remains to prove ψ^{-1} is continuous. We can actually write the inverse explicitly:

$$\psi^{-1}(q_1, q_2) = \begin{cases} \Phi^{-1}(\beta^{-1}(q_1, q_2)) & \text{if } \|(q_1, q_2)\| < 1, \\ [\lambda_1, \lambda_2] & \text{if } \|(q_1, q_2)\| = 1, \end{cases}$$

where λ_i is the measured lamination corresponding to the horizontal foliation of q_i. Continuity of ψ^{-1} on $\mathrm{BPQD}(X)$ is then a consequence of Lemma 4.2 and Wolf's parameterization. Continuity on the boundary follows from the Hubbard–Masur theorem [1979]. In general, if $q_n = (q_{1,n}, q_{2,n}) \in \mathrm{BPQD}(X)$ converges to $(q_1, q_2) \in \partial\overline{\mathrm{BPQD}(X)}$, then there is a sequence of scaling factors c_n such that the pair of hyperbolic surfaces $x_n = \psi^{-1}(q_{1,n}, q_{2,n})$ rescaled by c_n converges to real trees T_1, T_2 dual to measured laminations λ_1, λ_2. We need to show that $\psi^{-1}(q_1, q_2)$ is equal to $[\lambda_1, \lambda_2]$. Assume not; then we would have, by injectivity and continuity of ψ,

$$(q_1, q_2) = \psi(\psi^{-1}(q_1, q_2)) \neq \psi([\lambda_1, \lambda_2]) = \lim_{n \to +\infty} \psi(x_n) = \lim_{n \to +\infty} (q_{1,n}, q_{2,n}),$$

which contradicts the assumption on $(q_{1,n}, q_{2,n})$.

Finally, we remark the compactification in [Ouyang 2023] is independent of the choice of a base point, so that the role of the base point (S, J) is merely an auxiliary one. This completes the proof of the theorem. $\square$

5 Fixed point for the mapping class group action

In this section, we study the action of the mapping class group $\mathrm{MCG}(S)$ on the compactification $\mathfrak{B} = \overline{\mathrm{Max}(S)}$ constructed in Theorem 4.1. We wish to study the fixed points of this action. We will need the following observations.

Lemma 5.1 *The action of the mapping class group on* $\mathrm{Max}(S)$ *extends continuously to the closure* $\mathfrak{B} = \overline{\mathrm{Max}(S)}$.

Corollary 5.2 *For every* $\phi \in \mathrm{MCG}(S)$*, there exists* $x \in \mathfrak{B}$ *such that* $\phi(x) = x$.

The first main goal of this section is to analyze these fixed points via the celebrated Nielsen–Thurston classification, which we recall for future reference.

Theorem 5.3 (Nielsen–Thurston classification; see [Farb and Margalit 2012, Chapter 13]) *Any diffeomorphism* ϕ *on* S *is isotopic to a map* ϕ' *satisfying one of the following mutually exclusive conditions*:

(1) **Periodic** ϕ' *is of finite order.*

(2) **Reducible** ϕ' *is not periodic, and there is a nonempty set* $\{c_1, \ldots, c_r\}$ *of isotopy classes of essential pairwise disjoint simple closed curves in* S *such that* $\{\phi'(c_i)\}_{i=1}^r = \{c_i\}_{i=1}^r$.

(3) **Pseudo-Anosov/pA** *There exist* $\lambda > 1$ *and two transverse measured foliations* $\mathcal{F}$ *and* $\mathcal{F}'$ *such that*

$$\phi'(\mathcal{F}) = \lambda\mathcal{F} \quad \text{and} \quad \phi'(\mathcal{F}') = \frac{1}{\lambda}\mathcal{F}'.$$

Remark 5.4 Note that our definition of reducible mapping class is nonstandard as we assume that if ϕ is reducible, then it is not periodic. We do so to improve our exposition. The set $\{c_1, \ldots, c_r\}$ in item (2) is a *reduction system* of ϕ. The *canonical reduction system* $\{\bar{c}_1, \ldots, \bar{c}_k\}$ of ϕ reducible is the intersection of all the maximal (with respect to inclusion) reduction systems. Equivalently, each $\bar{c}_j$ is part of a reduction system and if $i(\bar{c}_j, c) \neq 0$ and $n \neq 0$, then $\phi^n(c) \neq c$.

Remark 5.5 The Nielsen–Thurston classification theorem also applies to surfaces S' with boundary [Fathi et al. 2012, Theorem 11.6]. In this case a diffeomorphism of S' is considered up to isotopies that do not necessarily fix pointwise the boundary components. We can thus still talk about pseudo-Anosov diffeomorphisms of S', which are exactly the mapping classes that are neither reducible nor periodic and preserve two transverse measured foliations on S'.

We are ready to characterize the fixed points of a mapping class acting on $\mathfrak{B}$ and establish Proposition 1.1 from the Introduction.

Proposition 5.6 *Suppose $\phi \in \mathrm{MCG}(S)$ and $\phi(x) = x$ for some $x = (x_1, x_2) \in \mathfrak{B}$.*

(1) *If ϕ is periodic, then x_1 and x_2 are any two points fixed by ϕ in the Thurston compactification of Teichmüller space such that $(x_1, x_2) \in \mathfrak{B}$.*

(2) *If ϕ is pA, then $(x_1, x_2) \in \partial\mathfrak{B}$ and $x_1 = 0$, or $x_2 = 0$ or $x_1 = x_2$.*

Proof (1) If ϕ fixes x projectively, there exists $\alpha > 0$ such that $\phi(x_1, x_2) = (\alpha x_1, \alpha x_2)$. Since ϕ is periodic, we can check that $\alpha = 1$.

(2) Since ϕ fixes the projective class of (x_1, x_2), there exists $\alpha > 0$ such that $\phi(x_1, x_2) = (\alpha x_1, \alpha x_2)$. On the other hand, since ϕ is pseudo-Anosov, there exist two measured laminations y_1 and y_2 and $\lambda > 1$ such that $\phi(y_1) = \lambda y_1$ and $\phi(y_2) = (1/\lambda) y_2$. Since ϕ does not fix any other projective class of measured laminations [Fathi et al. 2012, Corollary 12.4], it follows that $x_i = 0$, y_1 or y_2 for $i = 1, 2$. We claim that $x \neq (y_1, y_2)$ (and, symmetrically, $x \neq (y_2, y_1)$). Otherwise, because $i(y_1, y_2) \neq 0$,

$$\lambda \cdot i(y_1, y_2) = i(\phi(y_1), y_2) = \alpha \cdot i(y_1, y_2) = i(y_1, \phi(y_2)) = \frac{1}{\lambda} \cdot i(y_1, y_2),$$

which is a contradiction. $\square$

There is a natural continuous projection map $\pi : \mathfrak{B} \to \overline{\mathrm{Ind}(S)}$ defined as follows. For $x \in \mathrm{Max}(S)$, consider the corresponding equivariant minimal Lagrangian $\widetilde{\Sigma}_x$. Then, $\pi(x)$ is the induced metric on $\widetilde{\Sigma}_x$. Otherwise, if $x \in \partial\mathfrak{B}$, consider the core of the tree corresponding to $x \in \mathbb{P}(\mathcal{ML}(S) \times \mathcal{ML}(S))$: its length spectrum coincides with that of a mixed structure μ on S. Set $\pi(x) = \mu$. This projection π is continuous then by [Ouyang 2023, Theorem 6.13]. We consider the corresponding action of $\mathrm{MCG}(S)$ on $\overline{\mathrm{Ind}(S)}$ given by push-forward.

Lemma 5.7 *The actions of $\mathrm{MCG}(S)$ on $\mathfrak{B}$ and $\overline{\mathrm{Ind}(S)}$ commute. In other words, for every $\phi \in \mathrm{MCG}(S)$*

$$\pi \circ \phi = \phi \circ \pi.$$

Proof If $x = (x_1, x_2)$ is in the interior of $\mathfrak{B}$, then $\pi(\phi(x)) = \phi(\pi(x))$ because $\phi(\pi(x))$ has the same length spectrum as the induced metric on the minimal Lagrangian associated to $\phi(x_1)$ and $\phi(x_2)$. Suppose $x \in \partial\mathfrak{B}$ and consider a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathrm{Max}(S)$ such that $x_n \to x$. Since, $\pi(\phi(x_n)) = \phi(\pi(x_n))$ for all $n \in \mathbb{N}$, the result follows by continuity of ϕ and π. $\qquad\square$

We are now ready to establish the main theorems of this section. In particular, Theorem 5.8 below is Theorem B from the Introduction.

Theorem 5.8 *Assume* $\phi \in \mathrm{MCG}(S)$ *fixes* $\mu \in \partial\overline{\mathrm{Ind}(S)}$.

  (1) *If* μ *is **purely flat**, then* ϕ *is periodic.*

  (2) *If* μ *is **properly mixed**, then* ϕ *is not pA.*

Proof For item (1), if ϕ fixes projectively a geodesic current coming from a flat metric, then ϕ rescales the flat metric by some positive constant. Therefore, it is an automorphism of the underlying conformal structure, and hence is of finite order by the Hurwitz automorphism theorem.

We establish item (2). Suppose μ is properly mixed, ie μ is not flat but it has at least one flat piece. We can decompose S as

$$(\{S_\alpha\}_{\alpha\in\mathcal{A}}, \{d_\beta\}_{\beta\in\mathcal{B}}, \{\mu_\alpha\}_{\alpha\in\mathcal{A}}),$$

where μ_α is a flat structure or a (possibly zero) laminar structure on S_α and d_β is a maximal collection of closed geodesics so that

$$i(d_\beta, d_{\beta'}) = 0 \quad \text{and} \quad i(d_\beta, \mu) = 0$$

for all $\beta, \beta' \in \mathcal{B}$ and for every c that intersects some d_β transversely, $i(c, \mu) > 0$. Note that there exists a unique set $\{d_\beta\}_{\beta\in\mathcal{B}}$ with these properties (see [Burger et al. 2017, Theorem 1.1]).

Claim 5.9 *The map* ϕ *fixes the set* $\{d_\beta\}_{\beta\in\mathcal{B}}$.

Proof Observe that

$$i(\phi(d_\beta), \phi(d_{\beta'})) = i(d_\beta, d_{\beta'}) = 0 \quad \text{and} \quad i(\mu, \phi(d_\beta)) = i(\phi^{-1}(\mu), d_\beta) = 0.$$

If c is a curve that intersects $\phi(d_\beta)$ transversely, then

$$i(\phi^{-1}(c), d_\beta) = i(c, \phi(d_\beta)) > 0 \quad \text{and} \quad i(c, \mu) = i(\phi^{-1}(c), \phi^{-1}(\mu)) > 0.$$

Thus, by uniqueness, $\{\phi(d_\beta)\}_{\beta\in\mathcal{B}} = \{d_\beta\}_{\beta\in\mathcal{B}}$. $\qquad\square$

Item (2) now follows immediately from the claim above as ϕ must fix the set of closed curves $\{d_\beta\}_{\beta\in\mathcal{B}}$, but pseudo-Anosov diffeomorphisms do not preserve any closed curve. $\qquad\square$

Remark 5.10 For an explicit example of μ purely flat and ϕ periodic such that $\phi(\mu) = \mu$, consider a singular flat metric on a surface of genus 2 obtained by doubling a singular flat metric on a torus with boundary.

Remark 5.11 [Theorem 5.8](1) holds more generally, and with the same proof, in the case in which S has punctures and μ gives a conformal class of metrics with a finite group of conformal automorphisms. In particular, conformal structures on a surface (with or without punctures) with negative Euler characteristic will have finite conformal automorphism group; see [Oikawa 1956]. From this, we deduce that if $\phi \in \mathrm{MCG}(S)$ fixes a purely flat structure μ on a surface S (possibly with punctures), then ϕ is necessarily periodic.

Theorem 5.12 *Suppose $\phi \in \mathrm{MCG}(S)$ is reducible and fixes $\mu \in \partial\overline{\mathrm{Ind}(S)}$, which is properly mixed. Let $S = (S_\alpha, \{d_\beta\}_{\beta\in\mathcal{B}}, \mu_\alpha)$ be the subdivision of S induced by μ.*

(1) *If, for some $N > 0$, we have $\psi_\alpha = (\phi^N)_{|S_\alpha} : S_\alpha \to S_\alpha$ is pA, then $\mu_\alpha = 0$.*

(2) *If $\mu_\alpha \neq 0$ for all $\alpha \in \mathcal{A}$, then the canonical reduction system of ϕ is contained in $\{d_\beta\}_{\beta\in\mathcal{B}}$.*

Proof By the hypotheses, we can decompose S as $(\{S_\alpha\}_{\alpha\in\mathcal{A}}, \{d_\beta\}_{\beta\in\mathcal{B}}, \{\mu_\alpha\}_{\alpha\in\mathcal{A}})$. By [Claim 5.9](), there exists $N > 0$ such that ϕ^N fixes d_β for all $\beta \in \mathcal{B}$ and $\phi^N(S_\alpha) = S_\alpha$. Set $\psi_\alpha = (\phi^N)_{|S_\alpha} : S_\alpha \to S_\alpha$.

In order to prove item (1), we need to consider three cases.

(a) If $\mu_\alpha = 0$, then ψ_α can be any element in $\mathrm{MCG}(S_\alpha)$.

(b) If (S_α, μ_α) is purely flat (there exists at least one α for which this happens), then ψ_α can only be periodic by [Theorem 5.8]() and [Remark 5.11](), as incompressibility of the subsurfaces rules out the case of the once-punctured sphere and annuli.

(c) Suppose (S_α, μ_α) is purely laminar and nonzero. Since μ has a flat piece μ_β, we know that ψ_β is periodic and hence it fixes μ_β (not just projectively). We deduce that $\phi^N(\mu) = \mu$; otherwise we could find $z \neq 1$ such that $\psi_\alpha(\mu_\alpha) = z\mu_\alpha$, but then ϕ^N would not fix μ projectively. We can now conclude that ψ_α cannot be pA. This is because if c is a curve such that $i(\mu_\alpha, c) > 0$, then

$$i(\mu_\alpha, c) = i(\psi_\alpha^{-1}(\mu_\alpha), c) = i(\mu_\alpha, \psi_\alpha(c)),$$

but $i(\mu_\alpha, \psi_\alpha(c)) \neq i(\mu_\alpha, c)$ because ψ_α would change the length of curves transverse to μ_α.

This completes the proof of item (1).

For item (2), we wish to prove that the canonical reduction system $\{\bar{c}_1, \ldots, \bar{c}_k\}$ of ϕ is a subset of $\{d_\beta\}_{\beta\in\mathcal{B}}$ under the additional assumption that $\mu_\alpha \neq 0$ for all $\alpha \in \mathcal{A}$. First, observe that by [Claim 5.9]() $\{d_\beta\}_{\beta\in\mathcal{B}}$ is contained in a maximal reduction system for ϕ. In particular $i(\bar{c}_j, d_\beta) = 0$ for all j and β. Moreover, since μ is properly mixed, there exists β such that μ_β is flat; hence ψ fixes μ, not just its projective class, as observed before.

Assume, by contradiction, $\bar{c}_j \notin \{d_\beta\}_{\beta\in\mathcal{B}}$. Suppose $\bar{c}_j$ is contained in a purely flat piece (S_α, μ_α). Then, by [Theorem 5.8]() and [Remark 5.11](), ψ_α is necessarily periodic. But this contradicts the property that if $i(\bar{c}_j, c) \neq 0$ and $n \neq 0$, then $\phi^n(c) \neq c$ since there exists m such that ψ_α^m is the identity. Therefore $\bar{c}_j$ is contained in a purely laminar piece μ_α.

By the definition of $\{d_\beta\}_{\beta \in \mathcal{B}}$, ψ_α fixes a measured lamination $\mathcal{F}$ which is filling in S_α. Hence, by the Nielsen–Thurston classification theorem, ψ_α is necessarily pseudo-Anosov or periodic. If ψ_α is pA, then this contradicts item (1). Assume that ψ_α is periodic, so that there exists $m > 0$ such that ψ_α^m is the identity. Then, we achieve again a contradiction because there would exist c such that $i(\bar{c}_j, c) \neq 0$ but $\phi^m(c) = c$. Therefore, $\bar{c}_j$ cannot be contained in a purely laminar part either. By the definition of $\{d_\beta\}_{\beta \in \mathcal{B}}$, this forces the curve $\bar{c}_j$ to be one of the d_β's. $\qquad \square$

6 $\overline{\mathfrak{a}^+}$–valued measured laminations and mixed structures

In this final section we introduce Weyl-chamber-valued measured laminations and use them to refine the notion of mixed structures on a closed surface defined in [Duchin et al. 2010], and generalized to higher-order differentials in [Ouyang and Tamburelli 2021; 2023]. We show that the core of the product of two trees dual to measured laminations is dual to such a mixed structure, thus giving a new interpretation of the boundary objects in our compactification of $\mathrm{Max}(S)$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra. The choice of a maximal compact subalgebra $\mathfrak{k}$ induces an orthogonal decomposition of $\mathfrak{g}$ for the Killing form:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

A Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is a maximal abelian subspace of $\mathfrak{m}$. This induces a decomposition of $\mathfrak{g}$ in $\mathrm{ad}(\mathfrak{a})$–eigenspaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

Elements of $\Sigma \subset \mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$ are called restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$. Here we can extract a subset Δ of *simple* roots with the property that any $\alpha \in \Sigma$ can be expressed as a linear combination of simple roots with coefficients all of the same sign. This distinguishes, thus, a subset of positive roots that we denote by $\Sigma^+ \subset \Sigma$. The closed positive Weyl chamber of $\mathfrak{a}$ associated to Σ^+ is then the cone

$$\overline{\mathfrak{a}^+} = \{X \in \mathfrak{a} \mid \alpha(X) \geq 0 \text{ for all } \alpha \in \Sigma^+\}.$$

We also denote by W the Weyl group of $\mathfrak{g}$, ie $W = N(\mathfrak{a})/\mathfrak{a}$, and by r the opposition involution. Moreover, recall that $\mathfrak{a}$ has a partial order: if $x, y \in \mathfrak{a}$, then $x \leq y$ if $x - y \in \mathfrak{a}^+$. The following definition is due to [Parreau 2012, Section 2.2.3]:

Definition 6.1 A function $d_{\mathfrak{a}^+} : Y \times Y \to \overline{\mathfrak{a}^+}$ on a topological space Y is an $\overline{\mathfrak{a}^+}$–valued distance if

(i) $d_{\mathfrak{a}^+}(x, y) = 0$ if and only if $x = y$,

(ii) $d_{\mathfrak{a}^+}(x, y) = r(d_{\mathfrak{a}^+}(y, x))$ for all $x, y \in Y$,

(iii) $d_{\mathfrak{a}^+}(x, y) \leq d_{\mathfrak{a}^+}(x, z) + d_{\mathfrak{a}^+}(y, z)$ for all $x, y, z \in Y$.

We introduce the notion of Weyl-chamber-valued measured lamination.

Definition 6.2 An $\overline{\mathfrak{a}^+}$–valued measured lamination on a (not necessarily closed) surface S is a geodesic lamination λ on S that supports a measure μ on transverse arcs that takes values in $\overline{\mathfrak{a}^+}$ and satisfies the following properties:

(a) $\mu(\gamma) \neq 0$ if γ intersects λ transversely.

(b) If γ and γ' are homotopic arcs transverse to λ and there is a homotopy between them that preserves transversality at every time, then $\mu(\gamma) = \mu(\gamma')$.

(c) μ is additive on concatenation of paths, ie $\mu(\gamma\gamma') = \mu(\gamma) + \mu(\gamma')$ for all γ and γ' transverse to λ such that concatenation is defined.

Remark 6.3 If $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, then we can identify the closed positive Weyl chamber with $\mathbb{R}_{\geq 0}$. Thus, in this case, Definition 6.2 recovers the standard notion of measured laminations. Similarly, if $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, then $\overline{\mathfrak{a}^+}$–valued laminations can be identified with ordered pairs (λ_1, λ_2) such that λ_1, λ_2 and $\lambda_1 + \lambda_2$ are measured laminations (ie λ_1 and λ_2 are nowhere transverse).

We can also extend the classical notion of trees dual to a measured lamination to this context.

Definition 6.4 Let (T, d) be an $\mathbb{R}$–tree acted upon by the fundamental group of S. We say that the action of $\pi_1(S)$ is dual to an $\overline{\mathfrak{a}^+}$–valued measured lamination μ if there is an equivariant map $p \colon \widetilde{S} \to T$ and an $\overline{\mathfrak{a}^+}$–valued distance $d_{\mathfrak{a}+} \colon T \times T \to \overline{\mathfrak{a}^+}$ such that:

(a) For all $x, y \in \widetilde{S}$, we have $d_{\mathfrak{a}+}(p(x), p(y)) = \mu(\gamma)$ for some (hence any) arc $\gamma \colon [0, 1] \to \widetilde{S}$ transverse to the support of μ with $\gamma(0) = x$ and $\gamma(1) = y$.

(b) Given a geodesic path $\gamma \colon [0, 1] \to T$, we have $d(\gamma(0), \gamma(1)) \geq \|d_{\mathfrak{a}+}(\gamma(0), \gamma(1))\|$. Here $\|\cdot\|$ denotes the standard Euclidean norm of a vector in $\overline{\mathfrak{a}^+}$.

We now combine $\overline{\mathfrak{a}^+}$–valued measured laminations with the classical notion of $1/k$–translation surfaces in order to define a hybrid structure on S.

Definition 6.5 Let $\overline{\mathfrak{a}^+}$ be a closed Weyl chamber and $k \geq 1$ an integer. An $(\overline{\mathfrak{a}^+}, k)$–mixed structure on a closed surface S is the datum of

(a) a collection of nonhomotopically trivial, pairwise nonhomotopic, disjoint simple closed curves $\gamma_1, \ldots, \gamma_n$ on S;

(b) for each connected component S' of $S \setminus \bigcup_j \gamma_j$ either
   - an $\overline{\mathfrak{a}^+}$–valued measured lamination λ, where we allow each γ_j to be in the support; or
   - a meromorphic k–differential of finite area that endows S' with a $1/k$–translation surface structure.

These $(\overline{\mathfrak{a}^+}, k)$–mixed structures can be interpreted as dual to the $(\mathfrak{a}, W)$–complexes studied by Anne Parreau [2022] in the context of $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$. Let us recall briefly how these complexes are defined and explain in which sense these notions can be considered dual to each other.

Following [Parreau 2022], an $(\mathfrak{a}, W)$–complex K is the union of (possibly degenerate) polygons in $\mathfrak{a}$ glued together along boundary segments via elements of $W_{\text{aff}} = W \ltimes \mathbb{R}$. More precisely, there is a family of affine simplices $P_\mu \subset \mathfrak{a}$ and injective maps $\phi_\mu \colon P_\mu \to K$ such that if $K_\mu = \phi_\mu(P_\mu)$ and $K_{\mu'} = \phi_{\mu'}(P_{\mu'})$ have nonempty intersection then there is $w_{\mu,\mu'} \in W_{\text{aff}}$ such that $\phi_\mu(x) = \phi_{\mu'}(x')$ if and only if $x' = w_{\mu,\mu'}(x)$ and $P_\mu \cap w_{\mu,\mu'}^{-1}(P_{\mu'})$ is a face in P_μ. We only consider connected and simply connected $(\mathfrak{a}, W)$–complexes acted upon by $\pi_1(S)$. Note that, since the gluing maps between simplices are Euclidean isometries, the Euclidean distance on $\mathfrak{a}$ induces a distance on K. We will only work with $(\mathfrak{a}, W)$–complexes whose induced distance is CAT(0). Similarly, K is also endowed with an $\overline{\mathfrak{a}^+}$–valued distance inherited from $\mathfrak{a}$.

Examples of $(\mathfrak{a}, W)$–complexes are subcomplexes of an Euclidean building modeled on W_{aff}. We will see that cores of products of two trees dual to measured laminations are indeed $(\mathfrak{a}, W)$–complexes, where $\mathfrak{a}$ is the Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ and $W = \{\pm \text{Id}\}$.

Definition 6.6 We say that an $(\mathfrak{a}, W)$–complex K acted upon by $\pi_1(S)$ is dual to an $(\overline{\mathfrak{a}^+}, k)$–mixed structure μ on S if we can decompose K into a 1–dimensional part K_1 and a 2–dimensional part K_2 such that

- K_1 is the union of $\mathbb{R}$–trees dual to the laminar part of μ,
- K_2 is endowed with a $1/k$–translation surface structure isomorphic to the universal cover of the flat parts of μ.

Note that the 2–dimensional part of an $(\mathfrak{a}, W)$–complex can be endowed with a $1/k$–translation surface structure only if W contains the subgroup generated by rotations of angle $2\pi/k$.

We believe that these mixed structures naturally appear in a harmonic map compactification of the Hitchin and maximal components of the character variety for real Lie groups G of rank 2. In this context, Labourie [2017], Collier [2016] and Collier, Tholozan and Toulisse [Collier et al. 2019] proved that given a Hitchin or maximal representation $\rho \colon \pi_1(S) \to G$ there exists a unique ρ–equivariant minimal surface $\widetilde{\Sigma}_\rho$ in G/K, where K is a maximal compact subgroup of G. One could then find a compactification of these components by studying the limiting behavior of $\widetilde{\Sigma}_{\rho_n}$ when ρ_n leaves all compact sets in the character variety. Up to subsequences, and after rescaling the metric on G/K appropriately, $\widetilde{\Sigma}_{\rho_n}$ should converge to a subcomplex $\widetilde{\Sigma}_\infty \subset B$, where B is a nondiscrete Euclidean building modeled on the affine Weyl group of G. We conjecture that Σ_∞ is dual to a mixed structure as in Definition 6.6, where $\overline{\mathfrak{a}^+}$ is a Cartan subalgebra of the Lie algebra of G and k depends on the particular group. More precisely, we conjecture the following:

Conjecture 6.7 (a) *Let G be a real split semisimple Lie group of rank 2. Then the boundary of $\text{Hit}(S, G)$ can be identified with the space of projective classes of $(\overline{\mathfrak{a}^+}, k)$–mixed structures where:*

- *If $G = \text{SL}(3, \mathbb{R})$, then $\mathfrak{a} = A_2$ and $k = 3$.*
- *If $G = \text{Sp}(4, \mathbb{R})$, then $\mathfrak{a} = B_2$ and $k = 4$.*
- *If $G = G_2^{\mathbb{R}}$, then $\mathfrak{a} = G_2$ and $k = 6$.*

(b) *Let G be a real semisimple Lie group of Hermitian type and rank 2. Then the boundary of $\mathrm{Max}(S, G)$ can be identified with the space of projective classes of $(\overline{\mathfrak{a}^+}, k)$–mixed structures where:*

- *If $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$, then $\mathfrak{a} = A_1 \times A_1$ and $k = 2$.*
- *If $G = \mathrm{SO}(2, n)$ with $n \geq 3$, then $\mathfrak{a} = B_2$ and $k = 4$.*

In support of this conjecture, we show that the core of the product of two trees dual to measured laminations is dual to an $(\overline{A_1^+ \times A_1^+}, 2)$–mixed structure and that we can identify $\mathrm{Core}(\mathcal{T}, \mathcal{T})$ with the space of such structures, thus proving the conjecture for $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. Moreover, in [Loftin et al. 2022], Loftin, Wolf, and the third author give further evidence towards Conjecture 6.7 by describing the geometry of the harmonic maps to buildings arising from some diverging sequences of $\mathrm{SL}(3, \mathbb{R})$–Hitchin representations. It would be interesting to introduce a higher-rank version of our vector-valued mixed structures, at least for the case of $\mathrm{SL}(d, \mathbb{R})$–Hitchin components, and relate it to the subspaces of the Euclidean building studied in [Le 2016; Martone 2019a].

Lemma 6.8 *Let T_1 and T_2 be real trees dual to measured laminations λ_1 and λ_2 and let C be the core of $T_1 \times T_2$. Then C is an $(A_1 \times A_1, \{\pm \mathrm{Id}\})$–complex dual to an $(\overline{A_1^+ \times A_1^+}, 2)$–mixed structure on S.*

Proof We already saw in Section 3 that C is the union of a 1–dimensional subcomplex C_1 and a 2–dimensional subcomplex C_2 of $T_1 \times T_2$. Moreover, we showed that each connected component of C_2 is the universal cover of a half-translation surface structure on a subsurface S' of S, on which the laminations λ_1 and λ_2 fill. Thus, it only remains to show that each connected component C_1' of C_1 is a tree dual to an $\overline{A_1^+ \times A_1^+}$–valued measured lamination.

Recall from Section 3 that C_1' is the image under the map F defined in (1) of a domain $\Omega \subset \mathbb{H}^2$ that can be identified with the universal cover of a subsurface S' of S on which λ_1 and λ_2 are nowhere transverse. Moreover, we observe that C_1' has a natural distance d induced by the ambient space

$$d((x_0, y_0), (x_1, y_1)) = \sqrt{d_1(x_0, y_0)^2 + d_2(x_1, y_1)^2}$$

and a natural $\overline{A_1^+ \times A_1^+}$–valued distance $\vec{d}$ defined by

$$\vec{d}((x_0, y_0), (x_1, y_1)) = (d_1(x_0, y_0), d_2(x_1, y_1)).$$

We claim that (C_1', d) is an $\mathbb{R}$–tree dual to the $\overline{A_1^+ \times A_1^+}$–valued measured lamination $\vec{\lambda} = (\lambda_1, \lambda_2)$ (see Remark 6.3). By Lemma 3.4, C_1' can be identified with the $\mathbb{R}$–tree dual to the measured lamination $\lambda_0 = \lambda_1 + \lambda_2$ if endowed with the distance d_0 introduced in Section 3. In particular, there is a continuous $\pi_1(S')$–equivariant map $p := p_{\lambda_0} \colon \Omega \to C_1'$. It follows immediately from the definitions and the fact that T_1 and T_2 are dual to the laminations λ_1 and λ_2 that for all $x, y \in \Omega'$ we have

$$\vec{d}(p(x), p(y)) = \vec{\lambda}(\gamma)$$

for all $\gamma \colon [0, 1] \to \Omega$ transverse to the support of λ_0 with $\gamma(0) = x$ and $\gamma(1) = y$.

Property (b) in Definition 6.4 also holds. Indeed, a geodesic path $\gamma = (\gamma_1, \gamma_2) \colon [0, 1] \to C_1' \subset T_1 \times T_2$, seen in the quadrant $\gamma_1 \times \gamma_2$, consists of a concatenation of horizontal, vertical or diagonal paths in which the projections onto the two factors are always nondecreasing. Hence,

$$d(\gamma(0), \gamma(1)) \geq \|\vec{d}(\gamma(0), \gamma(1))\|,$$

and the proof is complete.   □

Theorem 6.9   *The space of $(\overline{A_1^+ \times A_1^+}, 2)$–mixed structures on S is homeomorphic to $\mathrm{Core}(\mathcal{T}, \mathcal{T})$.*

Proof   Let Y denote the set of $(\overline{A_1^+ \times A_1^+}, 2)$–mixed structures on S. We still need to define a topology on Y. We will construct a bijection

$$\varphi \colon Y \to \mathcal{ML}(S) \times \mathcal{ML}(S)$$

with the property that for all $y \in Y$ the core of the product of trees corresponding to $\varphi(y)$ is dual to the $(\overline{A_1^+ \times A_1^+}, 2)$–mixed structure y. We then give Y the topology that makes φ a homeomorphism, thus proving the result.

Given $y \in Y$, let $\gamma_1, \ldots, \gamma_n$ be the simple closed curves subdividing S into its laminar and flat parts, as in Definition 6.5. Let S_i for $i = 1, \ldots, m$ denote the connected components of $S \setminus \bigcup_j \gamma_j$. If S_i is endowed with a half-translation surface structure induced by a meromorphic quadratic differential q_i of finite area, then the horizontal and vertical foliations of q_i determine a pair of measured laminations $(\lambda_1^i, \lambda_2^i)$. Here we are implicitly using the well-known homeomorphism between the space of measured foliations arising this way and the space of measured laminations; see for instance [Levitt 1983; Lindenstrauss and Mirzakhani 2008]. On the other hand, by Remark 6.3, if S_i carries an $\overline{\mathfrak{a}^+}$–valued measured lamination, then this is equivalent to a pair of measured laminations $(\lambda_1^i, \lambda_2^i)$ possibly containing some boundary curves γ_j in their support. We can then associate to $y \in Y$ the pair of measured laminations $(\lambda_1, \lambda_2) \in \mathcal{ML}(S) \times \mathcal{ML}(S)$ defined as $\lambda_j = \sum_i^m \lambda_j^i$ for $j = 1, 2$. Since the horizontal and vertical measured foliations uniquely determine a meromorphic quadratic differential of finite area [Gardiner and Masur 1991], using Remark 6.3 and Lemma 3.3, it is clear that φ is a bijection.

Moreover, comparing the definition of the map φ with Lemma 6.8, it is easy to verify that the core of the product of trees dual to the pair $\varphi(y)$ is dual to the $(\overline{A_1^+ \times A_1^+}, 2)$–mixed structure y we started with.   □

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