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An algorithmic discrete gradient field and the cohomology algebra of configuration spaces of two points on complete graphs

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We introduce and study an algorithm that constructs a discrete gradient field on any simplicial complex. With a computational complexity similar to that of existing methods, our algorithmic gradient field is always maximal and in a number of cases even optimal. We make a thorough analysis of the resulting gradient field in the case of Munkres discrete model for $\mathrm{Conf}(K_m, 2)$, the configuration space of ordered pairs of noncolliding particles moving on the complete graph K_m on m vertices. This allows us to describe in full the cohomology algebra $H^*(\mathrm{Conf}(K_m, 2); R)$ for any commutative unital ring R. As an application we prove that, although $\mathrm{Conf}(K_m, 2)$ is outside the "stable" regime, all its topological complexities are maximal when $m \geq 4$.

55R80, 57Q70; 57M15

1 Introduction

Since the development of discrete Morse theory (DMT) by R Forman [15], the concept of a discrete gradient field (DGF) has played an important role in a wide range of areas of mathematics and the sciences alike. The idea arose as a combinatorial analogue of the concept of a smooth gradient field in differential topology, and has proven to be just as important as its smooth predecessor. In particular, DGFs have become one of the main tools in the relatively recent growth of computational topology techniques. For instance, Forman's DMT has been successfully used to deal with noise-reduction problems by Bauer, Lange and Wardetzky [6], as well as in topological data analysis by Harker, Mischaikow, Mrozek and Nanda [22], and within topological visualization and mesh compression applications by Lewiner, Lopes and Tavares [26]. DMT has also seen important applications in the purely theoretical realm, for instance, in the establishment of minimal cellular structures with the homotopy type of the complement of hyperplane arrangements and, more generally, of different sorts of configuration spaces; see Farley [10], Mori and Salvetti [28], Salvetti and Settepanella [32] and Severs and White [33]. DGFs have also been used in the determination of explicit homology bases for complexes of two-connected graphs, objects that play a relevant role in Vassiliev's study of knots in the standard 3–sphere; see Shareshian [34] and Vassiliev [35; 36; 37].

We review the basics on Forman's DMT in Section 2.2. For the purposes of this introduction, the nonspecialized reader should keep in mind that a DGF encodes an organized recipe to stretch the structure of a CW complex X, without changing its homotopy type, with the aim of simplifying the original cell structure. During the stretching process, a typical (regular) cell α gets squeezed by pushing one of its faces β towards the interior of α. The DGF consists of all such pairs (α, β), the "Morse pairings". Cells that are not of the squeezable type nor the pushable type are called *critical* and carry much of the homotopy information of X. Although the roots of the idea go back to Whitehead's simple homotopy theory in the 1930s, DGF technology currently stands as an important alternative to homotopy-minded methods in algebraic topology, especially when the heart of the topological phenomenon under consideration has a combinatorial origin.

In a typical application, the goal is to construct a DGF that renders an efficient and tractable simplification of the cell structure of a given complex. Actually, in each of the applications noted above, the efficiency goal is attained by constructing a suitable though ad-hoc DGF. In contrast, our first main contribution is the description, in Section 3, of an algorithm that constructs, for *any* finite ordered abstract simplicial complex $(K, \preceq)$, a DGF W that reaches reasonable (even notable, in a number of cases) DGF-efficiency goals:

Theorem 1.1 *The discrete gradient field W on $(K, \preceq)$ constructed in Section 3 is maximal. Indeed, all faces and all cofaces of a W–critical face are involved in a Morse pairing.*

In particular, W is a steepness pairing in the sense of Lampret [25, Lemma 2.2]. More importantly, it turns out that in many cases W is either optimal (perhaps after a convenient selection of the vertex ordering $\preceq$), or close to being so. Here optimality refers to the fact that in every dimension $k \geq 0$, the resulting Morse complex, which is homotopy equivalent to $|K|$, has exactly as many k–cells as the k^{th} Betti number of the geometric realization $|K|$ of K. Indeed, the algorithm constructing W can be thought of as a generalization of the inclusion–exclusion (IE) process with respect to a chosen vertex. For instance, the IE process gives an optimal gradient field collapsing a full simplex to the chosen vertex, and our algorithm remains optimal for many other complexes. In fact, for a general ordered simplicial complex $(K, \preceq)$, the vertex ordering $\preceq$ plays a heuristic role that guides the IE process.

The flexibility and generality of our method should lead to many more applications of the sort discussed in the first paragraph of this introduction, both in the theoretical and applied realms. So, in addition to illustrating the efficiency/optimality feature of our algorithmic DGF in a number of standard examples, as our second main contribution we obtain in Section 4 a full description of the cohomology ring of configuration spaces of ordered pairs of points in complete graphs. This is attained through a thorough study of the corresponding algorithmic DGF. Our results in this direction are described in the next paragraphs, after placing our work in context.

Configuration spaces

$$\mathrm{Conf}(X, n) = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}$$

are important ubiquitous objects in mathematics and its applications. They are reasonably well understood when $X = M$, a manifold of dimension at least two. For $X = \Gamma$ a graph, $\mathrm{Conf}(\Gamma, n)$ has attracted much attention in recent years due to its role in geometric group theory, and also because graph configuration spaces provide natural models for the problem of planning collision-free motion of multiple agents performing on a system of tracks; see Farber [8], Ghrist [17] and Ghrist and Koditschek [19]. Yet, the current understanding of the topology of $\mathrm{Conf}(\Gamma, n)$ appears to be far more limited than that of the higher-dimensional case $\mathrm{Conf}(M, n)$. This is due in part to the lack of Fadell–Neuwirth fibrations relating graph configuration spaces for different values of n. Informally, unlike the higher-dimensional counterpart $\mathrm{Conf}(M, n)$, one-dimensional motion planning actually requires global knowledge of the ambient graph. Thus, while additive information about the homology of graph configuration spaces is already available in the literature (see for instance Abrams [1], An, Drummond-Cole and Knudsen [3], Chettih and Lütgehetmann [7], Farber and Hanbury [9], Ghrist [18], Ko and Park [24], Maciążek and Sawicki [27] and Ramos [31]), explicit cup product descriptions seem to be scarcer. Notable exceptions are the work of Farley and Sabalka [11; 13; 14] (see also González and Hoekstra-Mendoza [21]) and Barnett and Farber [4]. The former relates the cohomology algebra of (unordered) configurations on trees to exterior face rings, while the latter describes in full the rational cohomology algebra of ordered pairs of points on planar graphs. We close the gap by focusing on a family of graphs which is diametrically different to that considered by Barnett and Farber. Indeed, we give a full description of the cohomology algebra, with any ring coefficients, of the configuration space of ordered pairs of points on a complete graph K_m with m vertices. The complete description is slightly technical and, for the purposes of this introduction, it is more useful to offer the following detailed navigational chart for Section 4, where the cohomology ring $H^*(\mathrm{Conf}(|K_m|, 2))$ is fully determined.

We start by reviewing a standard combinatorial homotopy model for $\mathrm{Conf}(|K_m|, 2)$ in the introductory Section 2.1. The corresponding algorithmic DGF is described in Proposition 4.1, while the resulting Morse (co)differential is described in Proposition 4.3. Bases of Morse cocycles are described in Definition 4.4 and Proposition 4.6 (for dimension 1), and in Definition 4.8 (for dimension 2). Corresponding cohomological bases are derived in Corollaries 4.7 (for dimension 1) and 4.11 (for dimension 2). The Morse-theoretic cup product is fully determined at the cocycle level by (43) and Propositions 4.13 and 4.14. The cohomological cup product can then be read off from (9) using the full power of Corollary 4.11, which gives explicit formulae that allow us to recover the basis expression of the cohomology class represented by any given Morse 2–cocycle. This renders a complete and fully computer-implementable description of the ring $H^*(\mathrm{Conf}(|K_m|, 2))$.

For the reader's benefit we spell out in Example 4.15 the above navigational chart in the case of the complete graph on five vertices. Our description of the cohomology algebra of $\mathrm{Conf}(|K_5|, 2)$ reflects the well-known fact that this space is homotopy equivalent to a closed orientable surface of genus six. More interestingly, Corollary 4.16 is a simple though partial description of the cup product structure in the cohomology of any $\mathrm{Conf}(|K_m|, 2)$. In such terms, it is clear that certain cup product aspects coming from the homotopy manifold structure of $\mathrm{Conf}(|K_5|, 2)$ are kept for $\mathrm{Conf}(|K_m|, 2)$ when $m > 5$.

We close with an application to motion planning in topological robotics. Namely, after reviewing in Section 5 the basics of Farber and Rudyak's sequential topological complexity TC_s, we use Corollary 4.16

to compute Farber and Rudyak's homotopy invariant in the case of two ordered point-type robots moving without collisions on a track system in the shape of a complete graph:

Theorem 1.2 *For $m \geq 4$ and $s \geq 2$,*

$$\mathrm{TC}_s(\mathrm{Conf}(|K_m|, 2)) = s\,\mathrm{hdim}(\mathrm{Conf}(|K_m|, 2)) = \begin{cases} s & \text{if } m = 4, \\ 2s & \text{otherwise.} \end{cases}$$

Here hdim stands for homotopy dimension. The relevance of Theorem 1.2 is fully discussed in the paragraph of Section 5 containing (46)–(48).

2 Preliminaries

2.1 The Munkres model for 2–particle configuration spaces

Let D be a full subcomplex of a given abstract simplicial complex X, ie assume that every simplex of X whose vertices lie in D is itself a simplex of D. Consider the (necessarily full) subcomplex C of X consisting of the simplices σ of X whose geometric realization $|\sigma|$ is disjoint from $|D|$. The vertices of X are partitioned into those of D and those of C and, as observed in [30, Lemma 70.1], the linear homotopy

$$H\colon (|X| - |D|) \times [0,1] \to |X| - |D|, \quad H(x,s) = (1-s)x + s\sum_{i=1}^{r} \frac{t_i}{\sum_{k=1}^{r} t_k} c_i$$

exhibits $|C|$ as a strong deformation retract of $|X| - |D|$. Here $x = \sum_{i=1}^{r} t_i c_i + \sum_{j=1}^{\rho} \tau_j d_j$ is the barycentric expression of $x \in |X| - |D|$ having $t_i > 0 < \tau_j$ for all i and j, with $c_1, \dots, c_r$ vertices of C for $r \geq 1$ and $d_1, \dots, d_\rho$ vertices of D for $\rho \geq 0$.

Let K be a finite abstract *ordered* simplicial complex, meaning the vertex set V of K comes equipped with a partial ordering $\preceq$ which is linear upon restriction to any face. We will be interested in Munkres model C above when $X = K \times K$ is the ordered product, with D corresponding to the subcomplex whose geometric realization is the diagonal $\Delta_{|K|}$ in $|K \times K| = |K| \times |K|$. The vertex set of $K \times K$ is $V \times V$, with elements denoted by columns, while a k–simplex of $K \times K$ is a matrix array

$$\begin{bmatrix} v_{0,1} & v_{1,1} & \cdots & v_{k,1} \\ v_{0,2} & v_{1,2} & \cdots & v_{k,2} \end{bmatrix} \tag{1}$$

of elements in V satisfying:

- For $i = 1, 2$, $v_{0,i} \preceq v_{1,i} \preceq \cdots \preceq v_{k,i}$ with $\{v_{0,i}, v_{1,i}, \dots, v_{k,i}\}$ an l–face of K (possibly with $l \leq k$).
- For $j = 0, 1, \dots, k-1$, at least one of the inequalities $v_{j,1} \preceq v_{j+1,1}$ or $v_{j,2} \preceq v_{j+1,2}$ is strict.

Such a matrix-type simplex belongs to D provided its two rows are repeated: $v_{j,1} = v_{j,2}$ for $j = 0, 1, \dots, k$. In particular, D is a full subcomplex of $K \times K$, and we get a homotopy equivalence

$$|C| \simeq \mathrm{Conf}(|K|, 2). \tag{2}$$

Note that a simplex (1) belongs to C precisely when $v_{j,1} \neq v_{j,2}$ for $j = 0, 1, \dots, k$. In particular, the vertex set of C is $V \times V \setminus \Delta_V$ (with elements denoted by column matrices).

2.2 Discrete Morse theory

We review the notation and facts we need from Forman's discrete Morse theory. See [15; 16] for details.

As in the previous subsection, let K be a finite abstract ordered simplicial complex with ordered vertex set $(V, \preceq)$. Let $(\mathcal{F}, \subseteq)$ be the face poset of K, that is, $\mathcal{F}$ is the set of faces of K partially ordered by inclusion. For a face $\alpha \in \mathcal{F}$, we write $\alpha^{(p)}$ to indicate that α is p–dimensional, and use the notation $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_p]$, where

$$\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_p \tag{3}$$

is the ordered list of vertices of α. We choose the orientation on α determined by (3). For faces $\alpha^{(p)} \subset \beta^{(p+1)}$, consider the incidence number $\iota_{\alpha,\beta}$ of α and β, (the coefficient ± 1 of α in the expression of $\partial(\beta)$). Here ∂ stands for the standard boundary operator in the oriented simplicial chain complex $C_*(K)$,

$$\partial([v_0, v_1, \dots, v_i]) = \sum_{0 \le j \le i} (-1)^j \partial_{v_j}([v_0, v_1, \dots, v_i]) = \sum_{0 \le j \le i} (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_i],$$

where $\partial_{v_j}([v_0, v_1, \dots, v_i]) = [v_0, \dots, \hat{v}_j, \dots, v_i]$ is the face obtained by removing v_j from $[v_0, v_1, \dots, v_i]$.

We think of the Hasse diagram $H_{\mathcal{F}}$ of $\mathcal{F}$ as a directed graph: the vertex set of $H_{\mathcal{F}}$ is $\mathcal{F}$ and the directed edges are the ordered pairs $(\alpha^{(p+1)}, \beta^{(p)})$ with $\beta \subset \alpha$. Such a directed edge will be denoted by $\alpha^{(p+1)} \searrow \beta^{(p)}$.

Definition 2.1 A *partial matching* W on $H_{\mathcal{F}}$ is a directed subgraph of $H_{\mathcal{F}}$ whose vertices have degree one. The *W–modified Hasse diagram* $H_{\mathcal{F},W}$ is the directed graph obtained from $H_{\mathcal{F}}$ by reversing the orientation of all edges of W.

Note that the vertex set of W may be a proper subset of $\mathcal{F}$. In such a case, faces in $\mathcal{F}$ that are not vertices of W are called *W–critical.* On the other hand, a reversed edge is denoted by $\beta^{(p)} \nearrow \alpha^{(p+1)}$, in which case α is said to be *W–collapsible* and β is said to be *W–redundant.* The words "critical", "collapsible" and "redundant" will also be used when the partial matching W is implicit from the context.

Definition 2.2 Let W be a partial matching on $H_{\mathcal{F}}$. A *W–path* is an alternating chain of up-going and down-going directed edges of $H_{\mathcal{F},W}$ of either of the two forms

$$\alpha_0 \nearrow \beta_1 \searrow \alpha_1 \nearrow \cdots \nearrow \beta_k \searrow \alpha_k \quad \text{or} \quad \gamma_0 \searrow \delta_1 \nearrow \gamma_1 \searrow \cdots \searrow \delta_k \nearrow \gamma_k. \tag{4}$$

A W–path as the one on the left (resp. right) side of (4) is called an *upper* (resp. *lower*) W–path, and the W–path is called *elementary* (resp. *constant*) when $k = 1$ (resp. when $k = 0$). A *mixed* W–path $\tilde{\lambda}$ from a face $\beta^{(p+1)}$ to a face $\alpha^{(p)}$ is the concatenation of a directed edge $\beta \searrow \gamma$ in $H_{\mathcal{F},W}$ and an upper W–path λ from γ to α.

As above, we use the term "path" as a synonym of "W–path" when the partial matching is implicit from the context. The sets of upper and lower paths that start on a p–cell α and end on a p–cell β are denoted by $\overline{\Gamma}(\alpha, \beta)$ and $\underline{\Gamma}(\alpha, \beta)$, respectively. Note that concatenation of upper/lower paths yields product maps

$$\overline{\Gamma}(\alpha, \beta) \times \overline{\Gamma}(\beta, \gamma) \to \overline{\Gamma}(\alpha, \gamma) \quad \text{and} \quad \underline{\Gamma}(\alpha, \beta) \times \underline{\Gamma}(\beta, \gamma) \to \underline{\Gamma}(\alpha, \gamma). \tag{5}$$

For instance, any nonconstant upper/lower path is a product of corresponding elementary paths.

Definition 2.3 The *multiplicity* of a constant path γ is $\mu(\gamma) := 1$, and of elementary upper/lower paths is

$$\mu(\alpha_0 \nearrow \beta_1 \searrow \alpha_1) := -\iota_{\alpha_0,\beta_1}\iota_{\alpha_1,\beta_1} \quad \text{and} \quad \mu(\gamma_0 \searrow \delta_1 \nearrow \gamma_1) := -\iota_{\delta_1,\gamma_0}\iota_{\delta_1,\gamma_1}.$$

The multiplicity of nonelementary nonconstant paths is defined to be a multiplicative function with respect to the product maps (5). Likewise,

$$\mu(\tilde{\lambda}) := \iota_{\gamma,\beta}\,\mu(\lambda)$$

defines the multiplicity of the mixed path $\tilde{\lambda}$ given by the concatenation of the edge $\beta \searrow \gamma$ and the upper path $\lambda \in \overline{\Gamma}(\gamma, \alpha)$.

Our central tools are discrete gradient fields:

Definition 2.4 A nonconstant path as in (4) is called a cycle if $\alpha_0 = \alpha_k$ in the upper case, or $\gamma_0 = \gamma_k$ in the lower case. Note that the cycle condition can only hold with $k > 1$. A partial matching W is said to be a *gradient field* on K if no nonconstant path is a cycle. In such a case, paths are referred as *gradient paths.*

Note that W is a gradient field if and only if $H_{\mathcal{F},W}$ has no cycles (as a directed graph).

We close this preliminary section by recalling (Definition 2.5 and Proposition 2.7) the way in which the structure of critical faces and gradient paths between them can be used to assemble a (co)chain complex that recovers the (co)homology of K.

Definition 2.5 Let R be a commutative unital ring.[1] As a graded additive R–module, the *Morse chain complex* $(\mu_*(K), \partial)$ is degreewise R–free, with basis in dimension $p \geq 0$ given by the oriented critical faces $\alpha^{(p)}$ of K, and with *Morse boundary map* $\partial\colon \mu_*(K) \to \mu_{*-1}(K)$ given at a critical face $\alpha^{(p)}$ by

$$\partial(\alpha^{(p)}) = \sum_{\beta^{(p-1)}} \Big(\sum_{\tilde{\lambda}} \mu(\tilde{\lambda})\Big)\beta, \tag{6}$$

where the outer summation runs over all critical faces $\beta^{(p-1)}$, and the inner summation runs over all mixed gradient paths $\tilde{\lambda}$ from α to β. The *Morse cochain complex* $(\mu^*(K), \delta)$ is the R–dual[2] of $(\mu_*(K), \partial)$.

Thus $\mu^p(K)$ is R–free with basis given by the duals of the oriented critical faces $\alpha^{(p)}$ of K. The value of the Morse coboundary map $\delta\colon \mu^*(K) \to \mu^{*+1}(K)$ at a (dualized) critical face $\alpha^{(p)}$ is

$$\delta(\alpha^{(p)}) = \sum_{\beta^{(p+1)}} \Big(\sum_{\tilde{\lambda}} \mu(\tilde{\lambda})\Big)\beta, \tag{7}$$

where the outer summation runs over all (dualized) critical faces $\beta^{(p+1)}$, and the inner summation runs over all mixed gradient paths $\tilde{\lambda}$ from β to α.

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[1] We restrict to ring coefficients as we will ultimately be interested in cup products.

[2] For the sake of brevity, we will consistently omit writing asterisks for dualized objects; the context clarifies the intended meaning.

Remark 2.6 For critical faces $\gamma_1^{(p)}$ and $\gamma_2^{(p+1)}$, the multiplicity-counted number of mixed gradient paths λ from γ_2 to γ_1, ie the sum $[\gamma_1;\gamma_2] := \sum_\lambda \mu(\lambda)$, is called the *Morse-theoretic incidence number* of γ_1 and γ_2. In these terms, (6) and (7) take the more familiar forms

$$\partial(\alpha^{(p)}) = \sum_{\beta^{(p-1)}} [\beta;\alpha]\beta \quad \text{and} \quad \delta(\alpha^{(p)}) = \sum_{\beta^{(p+1)}} [\alpha,\beta]\beta.$$

Gradient paths yield a homotopy equivalence between the Morse cochain complex $\mu^*(K)$ and the usual simplicial cochain complex $C^*(K)$. For our purposes we need:

Proposition 2.7 *The formulae*

$$\begin{aligned} \overline{\Phi}(\alpha^{(p)}) &= \sum_{\beta^{(p)}} \Bigg(\sum_{\lambda\in\overline{\Gamma}(\beta,\alpha)} \mu(\lambda)\Bigg)\beta \quad \textit{for } \alpha \textit{ critical and } \beta \textit{ arbitrary,} \\ \underline{\Phi}(\beta^{(p)}) &= \sum_{\alpha^{(p)}} \Bigg(\sum_{\lambda\in\underline{\Gamma}(\alpha,\beta)} \mu(\lambda)\Bigg)\alpha \quad \textit{for } \beta \textit{ arbitrary and } \alpha \textit{ critical,} \end{aligned} \tag{8}$$

determine cochain maps $\overline{\Phi}\colon \mu^*(K) \to C^*(K)$ and $\underline{\Phi}\colon C^*(K) \to \mu^*(K)$ inducing cohomology isomorphisms $\overline{\Phi}^*$ and $\underline{\Phi}^*$ with $(\underline{\Phi}^*)^{-1} = \overline{\Phi}^*$.

In particular, cup products can be evaluated directly at the level of the Morse cochain complex $\mu^*(K)$. Indeed, for Morse cocycles $x, y \in \mu^*(K)$ representing respective cohomology classes $x', y' \in H^*(\mu^*(K))$, the Morse-theoretic cohomology cup product $x' \cdot y'$ is represented by the Morse cocycle

$$x \overset{\mu}{\smile} y := \underline{\Phi}(\overline{\Phi}(x) \smile \overline{\Phi}(y)) \in \mu^*(K), \tag{9}$$

where $\smile$ stands for the simplicial cup product.

3 Algorithmic gradient fields

Let K be a finite abstract ordered simplicial complex of dimension d with ordered vertex set $(V, \preceq)$. Recall that the partial order $\preceq$ is required to restrict to a linear order on simplices of K. In this section, we describe and study an algorithm $\mathcal{A}$ that constructs a discrete gradient field W (which depends on $\preceq$) on K.

By the order-extension principle, we may assume $\preceq$ is linear from the outset. Let $\mathcal{F}^i$ denote the set of i–dimensional faces of K. Recall that a face $\alpha^{(i)} \in \mathcal{F}^i$ is identified with the ordered tuple $[\alpha_0, \alpha_1, \ldots, \alpha_i]$, with $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_i$, of its vertices. In such a setting, we say that α_r appears in position r of α. The ordered tuple notation allows us to lexicographically extend $\preceq$ to a linear order (also denoted by $\preceq$) on the set $\mathcal{F}$ of faces of K. We write $\prec$ for the strict version of $\preceq$.

For a vertex $v \in V$, a face $\alpha \in \mathcal{F}^i$ and an integer $r \geq 0$, let

$$\iota_r(v,\alpha) = \begin{cases} \alpha \cup \{v\} & \text{if } \alpha \cup \{v\} \in \mathcal{F}^{i+1} \text{ with } v \text{ appearing in position } r \text{ of } \alpha \cup \{v\}, \\ \varnothing & \text{otherwise.} \end{cases}$$

3.1 Acyclicity

At the start of the algorithm we set $W := \varnothing$ and initialize auxiliary variables $F^i := \mathcal{F}^i$ for $0 \le i \le d$ which, at any moment of the algorithm, keep track of i–dimensional faces not taking part in a pairing in W. Throughout the algorithm $\mathcal{A}$, pairings $(\alpha, \beta) \in \mathcal{F}^i \times \mathcal{F}^{i+1}$ are added to W by means of a family of processes $\mathcal{P}^i$ running for $i = d-1, d-2, \dots, 1, 0$ (in that order), where $\mathcal{P}^i$ is executed provided (at the relevant moment) both F^i and F^{i+1} are not empty (so there is a chance to add new pairings to W). Process $\mathcal{P}^i$ consists of three levels of nested subprocesses:

(i) At the most external level, $\mathcal{P}^i$ consists of a family of processes $\mathcal{P}^{i,r}$ for $i+1 \ge r \ge 0$, executed in descending order with respect to r.

(ii) In turn, each $\mathcal{P}^{i,r}$ consists of a family of subprocesses $\mathcal{P}^{i,r,v}$ for $v \in V$, executed from the $\preceq$–largest vertex to the smallest.

(iii) At the innermost level, each process $\mathcal{P}^{i,r,v}$ consists of a family of instructions $\mathcal{P}^{i,r,v,\alpha}$ for $\alpha \in \mathcal{F}^i$, executed following the $\preceq$–lexicographic order.

Instruction $\mathcal{P}^{i,r,v,\alpha}$ checks whether, at the moment of its execution, $(\alpha, \iota_r(v,\alpha))$ is in $F^i \times F^{i+1}$, that is, whether $(\alpha, \iota_r(v,\alpha))$ is "available" as a new pairing. If so, the pairing $\alpha \nearrow \iota_r(v,\alpha)$ is added to W, while α and $\iota_r(v,\alpha)$ are removed from F^i and F^{i+1}, respectively. Two immediate consequences stand from the above construction. Namely, at the end of the algorithm, the resulting family of pairs W is a partial matching in $\mathcal{F}$, and all faces and cofaces of an unpaired cell are involved in a W–paring. The former fact is part of the far more important Proposition 3.1 which, together with the latter, yields Theorem 1.1.

Proposition 3.1 *W is a gradient field.*

In preparation for the proof of Proposition 3.1, we need:

Definition 3.2 Let $W_{i,r,v}$ denote the collection of pairings $\alpha \nearrow \beta$ in W constructed during $\mathcal{P}^{i,r,v}$. Consider also the collection $P_{i,r,v}$ of pairs $(\alpha, \beta) \in \mathcal{F}^i \times \mathcal{F}^{i+1}$ such that $\beta \setminus \alpha = \{v\}$ with v appearing in position r of β. Thus $W_{i,r,v} = P_{i,r,v} \cap W$.

We start by proving that, at the moment that $\mathcal{A}$ constructs a pairing $\alpha \nearrow \beta$, α is in fact the smallest (with respect to $\preceq$) of the facets of β that remain unpaired.

Lemma 3.3 *Let $\alpha = [\alpha_0, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_i] \nearrow \beta = [\alpha_0, \dots, \alpha_r, \beta_0, \alpha_{r+1}, \dots, \alpha_i]$ be a pairing in $W_{i,r+1,\beta_0}$ and let γ be a face of β with $\gamma = [\alpha_0, \dots, \alpha_r, \beta_0, \alpha_{r+1}, \dots, \widehat{\alpha}_j, \dots, \alpha_i]$ for $r+1 \le j \le i$, ie $\gamma \prec \alpha$. Then there is an integer $l \in \{j+1, j+2, \dots, i+1\}$ and a vertex δ_0 with $\alpha_j \prec \delta_0$ such that*

$$\gamma \nearrow \delta := [\alpha_0, \dots, \alpha_r, \beta_0, \alpha_{r+1}, \dots, \alpha_{j-1}, \widehat{\alpha}_j, \dots, \delta_0, \dots]$$

lies in W_{i,l,δ_0}. In particular, the pairing $\gamma \nearrow \delta$ is constructed by $\mathcal{A}$ before the pairing $\alpha \nearrow \beta$.

Proof Previous to the instruction $\mathcal{P}^{i,r+1,\beta_0,\alpha}$ that constructs $\alpha \nearrow \beta$, the algorithm $\mathcal{A}$ executes the instruction $\mathcal{P}^{i,j+1,\alpha_j,\gamma}$ that evaluates the potential pair $(\gamma,\beta) \in P_{i,j+1,\alpha_j}$. This is not an element of W, as β remains available until a later stage in $\mathcal{A}$. So γ must be paired by an instruction $\mathcal{P}^{i,l,\delta_0,\gamma}$ previous to $\mathcal{P}^{i,j+1,\alpha_j,\gamma}$, which forces the conclusion. $\square$

Proof of Proposition 3.1 Assume for a contradiction that there is a W–cycle

$$\alpha^0 \nearrow \beta^0 \searrow \alpha^1 \nearrow \beta^1 \searrow \alpha^2 \nearrow \cdots \nearrow \beta^n \searrow \alpha^{n+1} = \alpha^0 \tag{10}$$

(the condition $n \geq 1$ is forced by the definition of a gradient path). Without loss of generality, we can assume that $\alpha^0 \nearrow \beta^0$ is constructed by $\mathcal{A}$ before any other pairing $\alpha^j \nearrow \beta^j$ with $1 \leq j \leq n$. So Lemma 3.3 forces the start of the cycle to have the form

$$\begin{aligned}
\alpha^0 &= [\alpha^0_0, \ldots, \alpha^0_{j_0}, \alpha^0_{j_0+1}, \ldots, \alpha^0_k],\\
\beta^0 &= [\alpha^0_0, \ldots, \alpha^0_{j_0}, \beta^0_0, \alpha^0_{j_0+1}, \ldots, \alpha^0_k],\\
\alpha^1 &= [\alpha^0_0, \ldots, \widehat{\alpha}^0_l, \ldots, \alpha^0_{j_0}, \beta^0_0, \alpha^0_{j_0+1}, \ldots, \alpha^0_k].
\end{aligned}$$

Assume inductively $\alpha^j = [\ldots, \beta^0_0, \alpha^0_{j_0+1}, \ldots, \alpha^0_k]$ with β^0_0 appearing in position j_0 (so $\alpha^j \neq \alpha^0$). The choosing of $\alpha^0 \nearrow \beta^0$ implies that β^j is obtained from α^j by inserting a vertex v on the left of β^0_0 ($v < \beta^0_0$). A new application of Lemma 3.3 (together with the choosing of $\alpha^0 \nearrow \beta^0$) then shows that α^{j+1} must be obtained from β^j by removing a vertex other than $\beta^0_0, \alpha^0_{j_0+1}, \ldots, \alpha^0_k$. Thus $\alpha^{j+1} = [\ldots, \beta^0_0, \alpha^0_{j_0+1}, \ldots, \alpha^0_k]$, which is again different from α_0. Iterating, we get a situation incompatible with the equality in (10). $\square$

We have noted that, when K is a full simplex, $\mathcal{A}$ constructs the standard (and optimal) gradient field determined by inclusion–exclusion of a fixed vertex (the largest one in the selected order $\preceq$). As illustrated in Examples 3.4, optimality is reached in other standard situations. Example 3.6 and Corollary 4.2 deal with slightly less standard instances, while [20] deals with novel situations in which our gradient field is optimal.

Examples 3.4 Figure 1, left, gives a triangulation of the projective plane $\mathbb{R}P^2$. The gradient field shown by the heavy arrows is determined by $\mathcal{A}$ using the indicated ordering of vertices. The only critical faces are [6] (in dimension 0), [2, 5] (in dimension 1) and [1, 3, 4] (in dimension 2), so optimality of the field

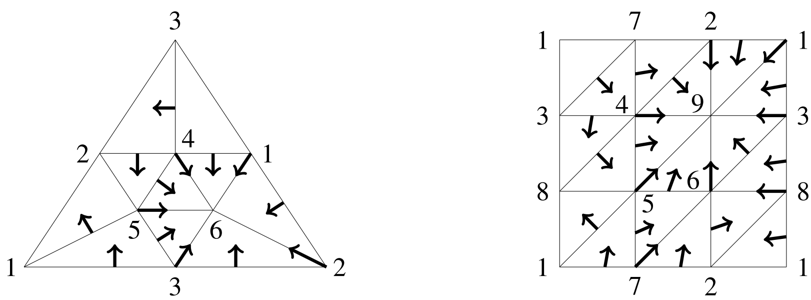

Figure 1: Algorithmic gradient fields for the projective plane (left) and the 2–torus (right).

follows from the known mod-2 homology of $\mathbb{R}P^2$. Although the gradient field depends on the ordering of vertices, we have verified with the help of a computer that, in this case, all possible 720 gradient fields (coming from the corresponding 6! possible orderings of vertices) are optimal. A corresponding optimal gradient field on the 2–torus (and the vertex order rendering it) is shown in Figure 1, right. This time the critical faces are $[9]$ (in dimension 0), $[2,8]$ and $[5,8]$ (in dimension 1) and $[1,3,7]$ (in dimension 2). The torus case is interesting in that there are vertex orderings that yield nonoptimal gradient fields. In general, a plausible strategy for choosing a convenient ordering of vertices consists of assuring the largest possible number of vertices with high $\preceq$–tag so that no two such vertices lie on a common face. For instance, in our torus example, no pair of vertices taken from 7, 8 and 9 lie on a single face.

We address the option $\alpha \prec \gamma$ ruled out by the hypotheses in Lemma 3.3:

Lemma 3.5 *Let $\alpha = [\alpha_0, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_k] \nearrow \beta = [\alpha_0, \dots, \alpha_i, \beta_0, \alpha_{i+1}, \dots, \alpha_k]$ lie in $W_{k,i+1,\beta_0}$ and let γ be a face of β with $\alpha \prec \gamma$, ie $\gamma = [\alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_i, \beta_0, \alpha_{i+1}, \dots, \alpha_k]$ for $0 \le j \le i$. Assume $\gamma \nearrow \delta$ is a pairing constructed* ***after*** *the pairing $\alpha \nearrow \beta$. Then δ is obtained from γ by inserting a vertex δ_0 which is $\prec$–smaller than β_0, ie $\delta = (\dots, \delta_0, \dots, \beta_0, \alpha_{i+1}, \dots, \alpha_k)$.*

Proof The assertion follows from the definition of the algorithm $\mathcal{A}$, noticing that α_{i+1} appears in position $i+1$ in γ. □

3.2 Gradient fields via a faster algorithm

The proof of Proposition 3.1 makes critical use of "timing" in the construction of W–pairs within the algorithm $\mathcal{A}$. We will modify this characteristic to get a more efficient and faster version of $\mathcal{A}$. While the timing of the W–pairs construction will be altered, we shall show that the new algorithm constructs the same gradient field.

The algorithm $\bar{\mathcal{A}}$ in this subsection, initialized with auxiliary variables $\overline{W}$ and $\bar{F}^i$ analogous to those for its counterpart $\mathcal{A}$, consists of a family of processes $\overline{\mathcal{P}}^i$ running for $i = d-1, d-2, \dots, 1, 0$ (in that order). Each $\overline{\mathcal{P}}^i$ is executed under the same conditions (with respect to $\bar{F}^i$ and $\bar{F}^{i+1}$) as its analogue $\mathcal{P}^i$, but consists only of two (rather than three) levels of nested subprocess. Namely, at the most external level, $\overline{\mathcal{P}}^i$ consists of a family of processes $\overline{\mathcal{P}}^{i,v}$ for $v \in V$, executed from the $\preceq$–largest vertex to the smallest. In turn, each process $\overline{\mathcal{P}}^{i,v}$ consists of a family of instructions $\overline{\mathcal{P}}^{i,v,\alpha}$ for $\alpha \in \mathcal{F}^i$, executed following the $\preceq$–lexicographic order. Instruction $\overline{\mathcal{P}}^{i,v,\alpha}$ checks whether, at that moment, $(\alpha, \{v\} \cup \alpha) \in \bar{F}^i \times \bar{F}^{i+1}$ (availability). If so, the pairing $\alpha \nearrow \{v\} \cup \alpha$ is added to $\overline{W}$, while α and $\{v\} \cup \alpha$ are removed from $\bar{F}^i$ and $\bar{F}^{i+1}$, respectively. Thus the difference with the algorithm $\mathcal{A}$ is that, in order to construct a pairing $\alpha \nearrow \{v\} \cup \alpha$ in $\overline{W}$, we do not care about the position of v in $\{v\} \cup \alpha$. As we will explain next, such a situation means that algorithm $\bar{\mathcal{A}}$ constructs some gradient pairings $\alpha \nearrow \beta$ earlier than they would be constructed by $\mathcal{A}$, thus avoiding the need to perform subsequent testing instructions related to α or β.

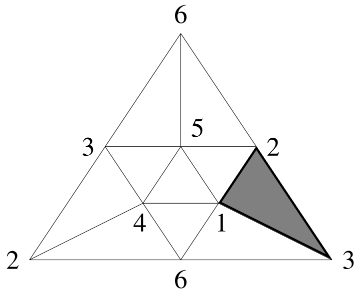

Figure 2: Vertex order in the projective plane with the facet $[1,2,3]$ removed.

Example 3.6 Consider the triangulation of the punctured projective plane shown in Figure 2. In the algorithm $\mathcal{A}$, the pairing $[2,3] \nearrow [2,3,4]$, which is constructed during the process $\mathcal{P}^{1,2,4}$, comes before the pairing $[1,5] \nearrow [1,4,5]$, which is constructed during the process $\mathcal{P}^{1,1,4}$. Instead, these two pairings arise in the opposite order in the algorithm $\bar{\mathcal{A}}$, and they both are constructed during the process $\mathcal{Q}^{1,4}$. As the reader can easily check, the (common) resulting gradient field has only two critical faces, namely $[6]$ and $[4,5]$, and is thus optimal (for the punctured projective plane has the homotopy type of the circle S^1).

The goal of this subsection is to prove Theorem 3.7, which states that $W = \overline{W}$ at the end of both algorithms. The proof is best organized by setting $\overline{W}_{i,r,v} := P_{i,r,v} \cap \overline{W}$ (cf Definition 3.2), as well as

$$W_{k,v} = \bigsqcup_r W_{k,r,v} \quad \text{and} \quad \overline{W}_{k,v} = \bigsqcup_r \overline{W}_{k,r,v}.$$

Theorem 3.7 *The pairings constructed by $\mathcal{A}$ and $\bar{\mathcal{A}}$ agree: $W_{k,r,v} = \overline{W}_{k,r,v}$ for all relevant indices k, r and v. In particular, $\overline{W}$ is acyclic.*

The proof of Theorem 3.7 uses the following elementary observations for vertices v and w with $v \preceq w$:

$$\begin{aligned} &(\alpha,\beta) \in P_{k,r,v} \text{ and } (\alpha,\gamma) \in P_{k,s,w} \implies r \le s, \text{ with equality provided } v = w, \\ &(\alpha,\beta) \in P_{k,r,v} \text{ and } (\gamma,\beta) \in P_{k,s,w} \implies r \le s, \text{ with equality provided } v = w. \end{aligned} \tag{11}$$

Remark 3.8 In the proof of Theorem 3.7, it will be convenient to keep in mind the following closer view of the central part of algorithms $\mathcal{A}$ and $\bar{\mathcal{A}}$. In the case of $\mathcal{A}$, an efficient way to execute a process $\mathcal{P}^{k,r,v}$ is by assembling the set $N_{k,r,v}$ of $(k+1)$–dimensional faces γ having v in position r and such that both γ and $\partial_v(\gamma)$ are available (neither γ nor $\partial_v(\gamma)$ have been paired previous to the start of $\mathcal{P}^{k,r,v}$). With such a preparation, $\mathcal{P}^{k,r,v}$ simply adds[3] to W all pairs $(\partial_v(\gamma),\gamma)$ with $\gamma \in N_{k,r,v}$ (construction of new pairings), and removes all faces γ and $\partial_v(\gamma)$, for $\gamma \in N_{k,r,v}$, from the corresponding lists of unpaired faces (update of available faces). Likewise, an efficient way to execute process $\overline{\mathcal{P}}^{k,v}$ in $\bar{\mathcal{A}}$ is by assembling the set $\overline{N}_{k,v}$ of $(k+1)$–dimensional faces γ containing v as a vertex (in any position) and such that both γ and $\partial_v(\gamma)$ are available at the start of $\overline{\mathcal{P}}^{k,v}$. With such a preparation, Lemma 3.9 shows that $\overline{\mathcal{P}}^{k,v}$ simply adds to $\overline{W}$ (in lexicographic order) all pairings $(\partial_v(\gamma),\gamma)$ with $\gamma \in \overline{N}_{k,v}$ (construction of new

[3]The adding of pairs is done following the $\preceq$–lexicographic order (cf. Lemma 3.9), though this much is immaterial at this point.

pairings), and removes all faces γ and $\partial_v(\gamma)$, for $\gamma \in \overline{N}_{k,v}$, from the corresponding lists of unpaired faces (update of available faces). In particular, $N_{k,r,v}$ (resp. $\overline{N}_{k,v}$) is the set of collapsible faces for the block of pairings constructed by $\mathcal{P}^{k,r,v}$ (resp. $\overline{\mathcal{P}}^{k,v}$), while the faces $\partial_v(\beta)$ for $\beta \in N_{k,r,v}$ (resp. $\beta \in \overline{N}_{k,v}$) are the corresponding redundant faces.

Lemma 3.9 *Let α and β be $(k+1)$–dimensional faces each containing v as a vertex (in any position). In terms of the $\preceq$–lexicographic order, the condition $\alpha \prec \beta$ holds if and only if $\partial_v(\alpha) \prec \partial_v(\beta)$.*

Proof We provide a detailed proof for completeness. The lexicographic order is linear (we have assumed so at the vertex level), so it suffices to show that $\partial_v(\alpha) \prec \partial_v(\beta)$ provided $\alpha \prec \beta$. Say v appears in positions i and j in α and β, respectively. The result is obvious if $i = j$ (this is why we did not need the lemma in our closer look at $\mathcal{A}$), or if the lexicographic decision for the inequality $\alpha \prec \beta$ is taken at a position smaller than $m := \min\{i, j\}$. Thus we can assume $i \neq j$ with α and β being identical up to and including position $m-1$. The inequality $\alpha \prec \beta$ then forces $i > j = m$. Thus $\partial_v(\alpha)$ and $\partial_v(\beta)$ are identical up to position $j-1$, while in position j,

- $\partial_v(\alpha)$ has the vertex α_j, which is smaller than $v = \alpha_i$, and
- $\partial_v(\beta)$ has the vertex β_{j+1}, which is larger than $v = \beta_j$.

Consequently $\partial_v(\alpha) \prec \partial_v(\beta)$. □

Proof of Theorem 3.7 Recall that d denotes the dimension of the simplicial complex under consideration. Fix $i \in \{0, 1, \dots, d-1\}$ and assume

$$\text{the equality } W_{k,r,v} = \overline{W}_{k,r,v} \text{ is valid whenever } k > i, \tag{12}$$

for all relevant values of r and v. The inductive goal is to prove

$$W_{i,r,v} = \overline{W}_{i,r,v} \quad \text{for all } v \in V \text{ and all } r \in \{0, 1, \dots, i+1\}. \tag{13}$$

(The induction is vacuously grounded by the fact that $W_{d,r,v} = \varnothing = \overline{W}_{d,r,v}$ at the start of both algorithms.) We start by arguing the case $r = i+1$ in (13), which in turn will be done by induction on the reverse ordering of vertices (starting from the largest vertex $v_{\max}$) and through a comparison of the corresponding actions of $\mathcal{A}$ and $\overline{\mathcal{A}}$ during *simultaneous* execution of these algorithms. In detail:

Case I ($r = i+1$ and $v = v_{\max}$ in (13)) Pairings in $W_{i,i+1,v_{\max}}$ are constructed during the execution of process $\mathcal{P}^{i,i+1,v_{\max}}$, while those in $\overline{W}_{i,i+1,v_{\max}}$ are constructed during the execution of process $\overline{\mathcal{P}}^{i,v_{\max}}$. In principle, the latter process would also construct pairings outside $\overline{W}_{i,i+1,v_{\max}}$. However, such a possibility is prevented by the fact that $v_{\max}$ can only appear in the last position of any face. Taking into account the inductive assumption (12), this means that processes $\mathcal{P}^{i,i+1,v_{\max}}$ in $\mathcal{A}$ and $\overline{\mathcal{P}}^{i,v_{\max}}$ in $\overline{\mathcal{A}}$ construct the same new pairings, and consequently perform the same updating of sets of available faces (this justifies the abuse of notation $\mathcal{P}^{i,i+1,v_{\max}} = \overline{\mathcal{P}}^{i,v_{\max}}$). Furthermore, after these processes conclude, no further pairings can be constructed by insertion of $v_{\max}$ (either in $\mathcal{A}$ or in $\overline{\mathcal{A}}$). Thus in fact

$$W_{i,v_{\max}} = W_{i,i+1,v_{\max}} = \overline{W}_{i,i+1,v_{\max}} = \overline{W}_{i,v_{\max}}, \tag{14}$$

which in particular grounds the inductive (on the vertices) argument for the case $r = i + 1$ in (13). As explained in the paragraph preceding Lemma 3.9, the redundant entries in (14) are the i–dimensional faces α such that $\alpha \cup \{v_{\max}\}$ is an $(i+1)$–dimensional face (so $v_{\max} \notin \alpha$) available at the start of process $\mathcal{P}^{i,i+1,v_{\max}} = \overline{\mathcal{P}}^{i,v_{\max}}$ (all i–dimensional faces are available at this point), whereas the collapsible entries in (14) are the $(i+1)$–dimensional faces available at the start of $\mathcal{P}^{i,i+1,v_{\max}} = \overline{\mathcal{P}}^{i,v_{\max}}$ that contain $v_{\max}$ as a vertex.

The above situation changes slightly in later stages of the algorithms and, in order to better appreciate subtleties, it is highly illustrative to spend a little time analyzing in detail a few of the pairings constructed right after (14).

Case II ($r = i + 1$ and $v = v_{\max -1}$ in (13)) Let $v_1, v_2, \dots, v_{\max -1}, v_{\max}$ be the elements of the vertex set V listed increasingly according to $\preceq$. Pairings in $W_{i,i+1,v_{\max -1}}$ (resp. $\overline{W}_{i,i+1,v_{\max -1}}$) are constructed during the execution of the process $\mathcal{P}^{i,i+1,v_{\max -1}}$ (resp. $\overline{\mathcal{P}}^{i,v_{\max -1}}$). In both processes, the construction is done by considering the insertion of $v_{\max -1}$ among available faces (these are common to both algorithms up to this point), either in position $i + 1$ in the case of $\mathcal{A}$, or in any position in the case of $\bar{\mathcal{A}}$. As in Case I,

$$\overline{\mathcal{P}}^{i,v_{\max -1}} \text{ might construct pairings outside } \overline{W}_{i,i+1,v_{\max -1}} \tag{15}$$

and

$$\text{any such a pairing would have to lie in } \overline{W}_{i,i,v_{\max -1}}, \tag{16}$$

as $v_{\max -1}$ cannot appear in a position smaller than i in an $(i+1)$–dimensional face. In terms of the notation introduced in the paragraph previous to Lemma 3.9, the possibility in (15) translates into a strict inclusion $N_{i,i+1,v_{\max -1}} \subset \overline{N}_{i,v_{\max -1}}$. However, an element in $\overline{N}_{i,v_{\max -1}} \setminus N_{i,i+1,v_{\max -1}}$ is forced to be an $(i+1)$–dimensional face which, in addition to being available at the start of $\mathcal{P}^{i,i+1,v_{\max -1}}$ and $\overline{\mathcal{P}}^{i,v_{\max -1}}$, has $v_{\max}$ appearing in the last position (for, as indicated in (16), $v_{\max -1}$ appears in the next-to-last position). Such a situation conflicts with the description of collapsible faces noted at the end of Case I, ruling out the possibility in (15). Thus, as above, $\mathcal{P}^{i,i+1,v_{\max -1}} = \overline{\mathcal{P}}^{i,v_{\max -1}}$ and

$$W_{i,v_{\max -1}} = W_{i,i+1,v_{\max -1}} = \overline{W}_{i,i+1,v_{\max -1}} = \overline{W}_{i,v_{\max -1}}. \tag{17}$$

While Cases I and II are essentially identical, the construction of subsequent pairings has a twist whose solution is better appreciated by taking a quick glance at the next block of pairings (those constructed by $\mathcal{P}^{i,i+1,v_{\max -2}}$ in the case of $\mathcal{A}$ and by $\overline{\mathcal{P}}^{i,v_{\max -2}}$ in the case of $\bar{\mathcal{A}}$). Namely, this time the inclusion

$$N_{i,i+1,v_{\max -2}} \subseteq \overline{N}_{i,v_{\max -2}} \tag{18}$$

may actually fail to be an equality, as illustrated in Example 3.6. As a result, the particularly strong forms of assertions (14) and (17) no longer hold true for subsequent blocks of pairings. In any case, what we do recover from (18) — and the discussion previous to Lemma 3.9 — is the fact that $W_{i,i+1,v_{\max -2}} = \overline{W}_{i,i+1,v_{\max -2}}$. We next inductively extend this conclusion to other vertices, and then explain how early pairings constructed in $\bar{\mathcal{A}}$ are eventually recovered in $\mathcal{A}$.

Case III (inductive step settling (13) for $r = i+1$) Fix a vertex $v \in V$ and assume

$$\overline{W}_{i,i+1,w} = W_{i,i+1,w} \tag{19}$$

whenever $v \prec w$, allowing the possibility that process $\overline{\mathcal{P}}^{i,w}$ in $\bar{\mathcal{A}}$ constructs more pairings than those constructed by the corresponding process $\mathcal{P}^{i,i+1,w}$ in $\mathcal{A}$. In such a setting, faces available at the start of $\overline{\mathcal{P}}^{i,v}$ are necessarily available at the start of $\mathcal{P}^{i,i+1,v}$, so

$$\overline{W}_{i,i+1,v} \subseteq W_{i,i+1,v}. \tag{20}$$

Assume for a contradiction that this inclusion is strict, and pick a pairing

$$(\alpha,\beta) \text{ in } W_{i,i+1,v} \text{ and not in } \overline{W}_{i,i+1,v}. \tag{21}$$

This means that α or β (or both) are not available at the start of $\overline{\mathcal{P}}^{i,v}$; in view of (12), this can only happen provided either

(i) $(\alpha,\beta') \in \overline{W}_{i,r,w} \subseteq P_{i,r,w}$ for some face β', some vertex w and some position r, or

(ii) $(\alpha',\beta) \in \overline{W}_{i,r,w} \subseteq P_{i,r,w}$ for some face α', some vertex w and some position r,

where in either case $v \prec w$ and $r \le i+1$. But $(\alpha,\beta) \in W_{i,i+1,v} \subseteq P_{i,i+1,v}$, so (11) yields in fact $i+1 = r$. Thus α or β is part of a pairing in $\overline{W}_{i,r,w} = \overline{W}_{i,i+1,w} = W_{i,i+1,w}$, where the latter equality comes from (19) but contradicts (21). Thus (20) is an equality. Note that the above argument does not rule out the possibility that $\overline{\mathcal{P}}^{i,v}$ constructs more pairings (by inserting v at a position smaller than $i+1$) than are constructed by $\mathcal{P}^{i,i+1,v}$.

The conclusion of the proof of Theorem 3.7 — the proof of (13) for $r \le i$ — proceeds by (inverse) induction on r, with the above discussion for $r = i+1$ grounding the induction. The new inductive argument requires an entirely different viewpoint coming from the following fact: in $\mathcal{A}$, after $\mathcal{P}^{i,i+1,v_1}$ is over, process $\mathcal{P}^i$ continues with many more subprocesses, the first of which is $\mathcal{P}^{i,i,v_{\max}}$. Yet in $\bar{\mathcal{A}}$, process $\overline{\mathcal{P}}^i$ finishes as soon $\overline{\mathcal{P}}^{i,v_1}$ is over, ie when the final inductive stage in Case III concludes. Therefore, our proof strategy from this point on requires pausing $\bar{\mathcal{A}}$ in order to analyze the rest of the actions in $\mathcal{P}^i$. In particular, we explain next how $\mathcal{P}^i$ catches up with all the "early" pairings $\bigcup_v (\overline{W}_{i,v} \setminus W_{i,i+1,v})$ constructed by $\overline{\mathcal{P}}^i$.

Case IV (double inductive step settling (13) for any r) Fix $r \in \{0,1,\dots,i\}$ and assume inductively that, as $\mathcal{P}^i$ progresses, $\mathcal{P}^{i,\rho}$ yields $\overline{W}_{i,\rho,w} = W_{i,\rho,w}$ for any vertex w and any position of insertion $\rho > r$. (The induction is grounded by Case III above.) The goal is to prove

$$\overline{W}_{i,r,w} = W_{i,r,w} \quad \text{for all vertices } w. \tag{22}$$

Since $r \le i$, we get $\overline{W}_{i,r,v_{\max}} = \varnothing = W_{i,r,v_{\max}}$. We can therefore assume in a second inductive level that, for some vertex v with $v \prec v_{\max}$, (22) holds true for all vertices w with $v \prec w$. The updated goal is to prove $\overline{W}_{i,r,v} = W_{i,r,v}$.

Inclusion $\overline{W}_{i,r,v} \subseteq W_{i,r,v}$ Suppose for a contradiction that

$$(\alpha, \beta) \in \overline{W}_{i,r,v} \tag{23}$$

is an "early" pairing (constructed during the execution of $\overline{\mathcal{P}}^{i,v}$) that cannot be constructed during the execution of $\mathcal{P}^{i,r,v}$. Then α or β (or both) must be involved as a pairing of some $W_{i,s,w}$ with $s \geq r$, and in addition with $v \prec w$ if in fact $s = r$. The double inductive equality $W_{i,s,w} = \overline{W}_{i,s,w}$, the dynamics of $\bar{\mathcal{A}}$ and (23) then force $v = w$, and consequently $s > r$. But this inequality contradicts (11) since $\overline{W}_{i,r,v} \subseteq P_{i,r,v}$ and $\overline{W}_{i,s,w} \subseteq P_{i,s,w}$.

Inclusion $W_{i,r,v} \subseteq \overline{W}_{i,r,v}$ Suppose for a contradiction that

$$(\alpha, \beta) \in W_{i,r,v} \tag{24}$$

is not one of the "early" pairings constructed during the execution of $\overline{\mathcal{P}}^{i,v}$. Then α or β (or both) must be involved in a pairing of some $\overline{W}_{i,s,w}$ with $v \prec w$. As in the previous paragraph, (11) then yields $r \leq s$. In turn, the double inductive hypothesis gives $\overline{W}_{i,s,w} = W_{i,s,w}$, which thus contains a pairing involving α or β, in contradiction to (24). □

3.3 Computational complexity and performance

Designing efficient algorithms and implementing fast software for the homological processing of large data sets is a lively technological challenge. With this in mind, we now study the computational complexity of our algorithm, and compare it with a closely related technique used within the realm of current applications.

Harker et al. [22] describe and study an efficient way of computing homology of complexes and their maps. At the core of their method there is an algorithm $\mathcal{H}$ for constructing discrete gradient fields on a well-suited class of complexes (à la Tucker). The idea is based on a Morse theory extension of the coreduction method introduced in [29]. Namely, cells α and β form a coreduction pair of a complex K provided α is a codimension-1 free face of β in K. Initializing K to be the whole initial complex, the algorithm $\mathcal{H}$ constructs Morse pairings $\alpha \nearrow \beta$ whenever α and β form a coreduction pair in K. Each time such a coreduction pair is found, its entries are removed from K before looking for the next pair of coreduction cells. If at any moment no coreduction pairs exist in K, faces of K with no boundary in K are declared to be critical (and removed from K) until creating new coreduction pairs. The algorithm repeats until K is empty, which completes the basic (iterative) building process $\mathcal{H}_0$ of $\mathcal{H}$.

Although both $\bar{\mathcal{A}}$ and $\mathcal{H}$ are based on a heuristic search of Morse pairs, the corresponding gradient fields bear no resemblance to each other. Indeed, the coreduction heuristic in $\mathcal{H}$ is replaced in $\bar{\mathcal{A}}$ by an inclusion–exclusion strategy guided by the chosen vertex ordering $\preceq$. More precisely, we highlight the main conceptual differences and apparent similarities between $\bar{\mathcal{A}}$ and $\mathcal{H}$. For starters, note that there is no reason to expect (and indeed it is usually not the case) that $\mathcal{H}_0$ yields a reasonably efficient gradient field on the original complex. Nonetheless the Tucker viewpoint of complexes allows Harker et al. to iterate $\mathcal{H}_0$ and, in doing so, the eventually stabilized gradient field turns out to be reasonably efficient — optimal in some cases. On the other hand, as illustrated in Examples 3.4 and formalized by Theorem 1.1, the corresponding efficiency property is reached by $\bar{\mathcal{A}}$ as a result of the $\preceq$–guided search of Morse pairs. But the most important issue to stress when comparing $\bar{\mathcal{A}}$ and $\mathcal{H}$ is given in terms of computational complexity. Aside from

(a) the cost of iterating $\mathcal{H}_0$ until reaching a stable gradient field, and

(b) the cost of processing the resulting Morse complex at the conclusion of each application of $\mathcal{H}_0$ in order to gather the Tucker information needed for the next application of $\mathcal{H}_0$,

the computational cost of applying $\mathcal{H}_0$ (say for the first time) is linear on the starting *complex mass*

$$m_K = \mathrm{card}\{(\alpha^{p+1}, \beta^p) : p \geq 0 \text{ and } \alpha, \beta \text{ are faces of } K \text{ with } \beta \subset \alpha\}. \tag{25}$$

See [22, Proposition 5.1]. We prove (Proposition 3.10) that, if we think of the basic $\bar{\mathcal{A}}$–instruction $\overline{\mathcal{P}}^{i,v,\alpha}$ in Section 3.2 as being performed in $O(1)$ time,[4] then $\bar{\mathcal{A}}$ also executes in $O(m_K)$ time. Consequently, for practical implementations, a profitable strategy reducing computational costs coming from (a) and (b) above can be based on a combination of algorithms $\bar{\mathcal{A}}$ and $\mathcal{H}$. In fact, the maximality condition of the gradient field W resulting from an initial application of $\bar{\mathcal{A}}$ can potentially be bypassed (and possibly turned into an optimality condition) by means of a subsequent (and then much quicker) application of $\mathcal{H}$ on the Morse–Tucker complex resulting from W.

Proposition 3.10 *For a finite abstract ordered simplicial complex K with complex mass (25), algorithm $\bar{\mathcal{A}}$ executes in $O(m_K)$ time.*

Proof An efficient implementation of $\bar{\mathcal{A}}$ requires initializing a couple of functions, $f(\sigma)$ and $g(i,v)$. The former function is binary and answers, at any moment of the algorithm, the question of whether a given face σ of the original complex belongs to the set of "available" faces $\overline{F}^{\dim(\sigma)}$ (here and below we reuse the notation set forth in Section 3.2). The latter function reports, at any moment of the algorithm, the list of available faces in a given dimension i that contain a given vertex v. The cost of initializing f (with values True) is linear on the number of faces of the original complex, and therefore can be safely neglected for the purposes of this proof. On the other hand, for a given dimension i, we start by setting $g(i,v) = \varnothing$ for all vertices v. Then, for each face $\sigma \in \mathcal{F}^i$ and for each vertex $v \in \sigma$, we append σ to $g(i,v)$. This last task takes $O\big(\sum_{\sigma \in \mathcal{F}^i} m_\sigma\big)$ time, where m_σ stands for the *boundary mass* of σ, ie the cardinality of the set of facets of σ. Thus initializing g (and f) takes $O(m_K)$ time. With this preparation, $\bar{\mathcal{A}}$ can then be executed in $O(m_K)$ time following the indications in Remark 3.8. Namely, to execute process $\overline{\mathcal{P}}^{i,v}$, select the faces $\sigma \in g(i+1,v)$ with $f(\partial_v(\sigma)) = \text{True}$, so that

$$\partial_v(\sigma) \nearrow \sigma \tag{26}$$

is a new Morse pair — in which case the corresponding values of f and g have to be updated. Naturally, some faces $\sigma \in g(i+1,v)$ will not lead to the "v–type" Morse pairing (26), and will have to be accounted for by later processes $\overline{\mathcal{P}}^{i,w}$ (with $w < v$). But, just as in the initialization of g, a fixed such σ will have to be processed at most m_σ times. Therefore the actual algorithm $\bar{\mathcal{A}}$ is executed in $O(m_K)$ time too. □

[4] Such an assumption is easily achieved through an efficient implementation of $\bar{\mathcal{A}}$; see the proof of Proposition 3.10.

Note that the estimation $O(m_K)$ for the complexity in the final part of the previous proof is rather coarse. Indeed, none of the i–dimensional redundant faces paired during the execution of $\overline{\mathcal{P}}^i$ will have to be processed during the execution of $\overline{\mathcal{P}}^{i-1}$. It is in this sense that the maximality of our algorithm (Theorem 1.1) leads, paradoxically, to a computational complexity that, in practical situations, is lower than what is estimated here.

3.4 Collapsibility conditions

This section is devoted to theoretical aspects of our gradient field. Precisely, we describe a set of "local" conditions that allow us to identify gradient pairings without having to actually run any of the two versions of our algorithm. Our local conditions determine in full the gradient field in a number of instances.[5] The main result in this section (Theorem 3.19) is presented through a series of preliminary complexity-increasing results in order to isolate the role of each of the condition ingredients.

Definition 3.11 A vertex α_i of a face $\alpha = [\alpha_0, \dots, \alpha_k] \in \mathcal{F}^k$ is said to be maximal in α if $\partial_{\alpha_i}(\alpha) \cup \{v\} \notin \mathcal{F}^k$ for all vertices v with $\alpha_i \prec v$. When α_i is nonmaximal in α, we write $\alpha(i) := \partial_{\alpha_i}(\alpha) \cup \{\alpha^i\}$, where

$$\alpha^i := \max\{v \in V : \alpha_i \prec v \text{ and } \partial_{\alpha_i}(\alpha) \cup \{v\} \in \mathcal{F}^k\}.$$

Note that α^i is maximal in $\alpha(i)$, and that α^i is not a vertex of α. Iterating the construction, for a given face $\alpha = [\alpha_0, \dots, \alpha_k] \in \mathcal{F}^k$ and a sequence of integers $0 \le i_1 < i_2 < \cdots < i_p \le k$, we say that the face $[\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_p}]$ is nonmaximal in α provided:

- α_{i_1} is nonmaximal in α, so we can form the face $\alpha(i_1)$.
- α_{i_2} is nonmaximal in $\alpha(i_1)$, so we can form the face $\alpha(i_1, i_2) := \alpha(i_1)(i_2)$.

  $\vdots$
- α_{i_p} is nonmaximal in $\alpha(i_1, \dots, i_{p-1})$, so we can form the face $\alpha(i_1, \dots, i_p) := \alpha(i_1, \dots, i_{p-1})(i_p)$.

When $p = 0$ (so there is no constructing process), $\alpha(i_1, i_2, \dots, i_p)$ is interpreted as α.

Lemma 3.12 *No vertex of a redundant k–face $\alpha \in \mathcal{F}^k$ is maximal in α.*

Proof Assume a pairing $\alpha = [\alpha_0, \dots, \alpha_k] \nearrow \beta = [\alpha_0, \dots, \alpha_{r-1}, \beta_0, \alpha_r, \dots, \alpha_k]$ and consider a vertex α_i of α. If $i < r$, the k–face $\partial_{\alpha_i}(\beta) = [\alpha_0, \dots, \widehat{\alpha}_i, \dots, \alpha_{r-1}, \beta_0, \alpha_r, \dots, \alpha_k]$ shows that the vertex α_i is nonmaximal in α. If $i \ge r$, Lemma 3.3 gives a pairing

$$\gamma := [\alpha_0, \dots, \alpha_{r-1}, \beta_0, \alpha_r, \dots, \widehat{\alpha}_i, \dots, \alpha_k] \nearrow [\alpha_0, \dots, \alpha_{r-1}, \beta_0, \alpha_r, \dots, \widehat{\alpha}_i, \dots, \delta_0, \dots] =: \delta$$

by insertion of a vertex δ_0 with $\alpha_i \prec \delta_0$, so that the k–face $\partial_{\beta_0}(\delta)$ shows that vertex α_i is nonmaximal in α. □

While maximal vertices in a face α can be thought of as giving obstructions for redundancy of α, maximality of the largest vertex in α is actually equivalent to collapsibility of α in a specific way:

[5] This holds, for instance, in the case of the projective plane and the torus in Examples 3.4, as well as in our application to spaces of ordered pairs of points on complete graphs; see Section 4.1.

Corollary 3.13 *The following conditions are equivalent for a k–face $\alpha = [\alpha_0, \dots, \alpha_k] \in \mathcal{F}^k$:*

(i) α_k *is maximal in* α.

(ii) $\partial_{\alpha_k}(\alpha) \nearrow \alpha$.

Proof Assuming (i), both α and $\partial_{\alpha_k}(\alpha)$ are available at the start of process $\mathcal{P}^{k-1,k,\alpha_k}$; the former face in view of Lemma 3.12, and the latter face by the maximality hypothesis. The W–pairing in (ii) is therefore constructed by the process $\mathcal{P}^{k-1,k,\alpha_k}$. On the other hand, if (i) fails, there is a vertex v of K which is maximal with respect to the conditions $\alpha_k \prec v$ and $\partial_{\alpha_k}(\alpha) \cup \{v\} \in \mathcal{F}^k$. As v is maximal in $\partial_{\alpha_k}(\alpha) \cup \{v\} = [\alpha_0, \dots, \alpha_{k-1}, v]$, the argument in the previous paragraph gives $[\alpha_0, \dots, \alpha_{k-1}] \nearrow [\alpha_0, \dots, \alpha_{k-1}, v]$, thus ruling out the W–pairing in (ii). □

Under additional restrictions (spelled out in (27)), maximality of other vertices also forces collapsibility in a specific way. We start with the case of the next-to-last vertex, where the additional restrictions are simple, yet the if and only if situation in Corollary 3.13 is lost; see Remark 3.15.

Proposition 3.14 *Let $\alpha = [\alpha_0, \dots, \alpha_k] \in \mathcal{F}^k$. If α_{k-1} is maximal in α but α_k is not, then $\partial_{\alpha_{k-1}}(\alpha) \nearrow \alpha$.*

Proof By Lemma 3.12, α is available at the start of process $\mathcal{P}^{k-1}$ and, in fact, at the start of process $\mathcal{P}^{k-1,k-1,\alpha_{k-1}}$, in view of Corollary 3.13 and the hypothesis on α_k. The asserted pairing follows since $\partial_{\alpha_{k-1}}(\alpha) = [\alpha_0, \dots, \widehat{\alpha}_{k-1}, \alpha_k]$ is also available at the start of process $\mathcal{P}^{k-1,k-1,\alpha_{k-1}}$. Indeed, a potential pairing $\partial_{\alpha_{k-1}}(\alpha) \nearrow \partial_{\alpha_{k-1}}(\alpha) \cup \{v\}$ constructed at a stage before $\mathcal{P}^{k-1,k-1,\alpha_{k-1}}$ would have $\alpha_{k-1} \prec v$, contradicting the maximality of α_{k-1} in α. □

Remark 3.15 Consider the gradient field on the projective plane in Examples 3.4. Neither 5 nor 2 are maximal in $[1,2,5]$ (due to the faces $[1,2,6]$ and $[1,3,5]$), yet the pairing $[1,5] \nearrow [1,2,5]$ holds.

More generally,

Proposition 3.16 *For a face $\alpha = [\alpha_0, \dots, \alpha_k] \in \mathcal{F}^k$ and an integer $r \in \{0, 1, \dots, k\}$ with α_r maximal in α, the pairing $\partial_{\alpha_r}(\alpha) \nearrow \alpha$ holds provided*

(27) *for any sequence $r+1 \le t_1 < \cdots < t_p \le k$, the face $[\alpha_{t_1}, \dots, \alpha_{t_p}]$ is nonmaximal in α.*

Proof We argue by decreasing induction on $r = k, k-1, \dots, 0$. The grounding cases $r = k$ and $r = k-1$ are covered by Corollary 3.13 and Proposition 3.14, respectively. For the inductive step, the maximality of α_r in α assures both that α is available at the start of $\mathcal{P}^{k-1}$ (Lemma 3.12), and that $\partial_{\alpha_r}(\alpha)$ is available at the start of $\mathcal{P}^{k-1,r,\alpha_r}$. It thus suffices to note that (27) implies that α is also available at the start of $\mathcal{P}^{k-1,r,\alpha_r}$. But a potential pairing $[\alpha_0, \dots, \widehat{\alpha}_{t_1}, \dots, \alpha_k] \nearrow [\alpha_0, \dots, \alpha_k]$ previous in $\mathcal{A}$ to the intended pairing $\partial_{\alpha_r}(\alpha) \nearrow \alpha$, ie with $t_1 \in \{r+1, \dots, k\}$, is inductively ruled out by the (yet previous in $\mathcal{A}$) pairing

$$[\alpha_0, \dots, \widehat{\alpha}_{t_1}, \dots, \alpha_k] \nearrow \alpha(t_1) = [\alpha_0, \dots, \widehat{\alpha}_{t_1}, \{\alpha^{t_1}, \alpha_{t_1+1}, \dots, \alpha_k\}],$$

where the use of curly braces is meant to indicate that α^{t_1} may occupy any position among the ordered vertices $\alpha_{t_1+1}, \dots, \alpha_k$. □

Not all conditions in (27) would be needed in concrete instances of Proposition 3.16. For instance, this will be (recursively) the case if, in the previous proof, some α^{t_1} turns out to be larger than some of the vertices $\alpha_{t_1+1}, \dots, \alpha_k$.

Example 3.17 The pairing $\partial_{\alpha_{k-2}}(\alpha) \nearrow \alpha = [\alpha_0, \dots, \alpha_k]$ holds provided

(i) α_{k-2} is maximal in α,

(ii) α_{k-1} is nonmaximal in α, and

(iii) α_k is nonmaximal in α as well as in $\alpha(k-1)$.

Note that (ii) is used in order to state (iii).

Theorem 3.19, a far-reaching extension of Proposition 3.16, provides sufficient conditions that allow us to identify "exceptional" pairings such as the one noted in Remark 3.15.

Definition 3.18 A vertex α_r of a face $\alpha = [\alpha_0, \dots, \alpha_k] \in \mathcal{F}^k$ is said to be collapsing in α provided

(i) the face α is not redundant,

(ii) condition (27) holds, and

(iii) for every v with $\alpha_r \prec v$ and $\partial_{\alpha_r}(\alpha) \cup \{v\} \in \mathcal{F}^k$, there is a vertex α_j of α with $v \prec \alpha_j$ such that α_j is collapsing in $\partial_{\alpha_r}(\alpha) \cup \{v\}$.

Definition 3.18(i) and (iii) hold when α_r is maximal in α. Note the recursive nature of Definition 3.18.

Theorem 3.19 *If α_r is collapsing in α, then $\partial_{\alpha_r}(\alpha) \nearrow \alpha$.*

Proof The proof is parallel to that of Proposition 3.16. This time the induction is grounded by Corollary 3.13 and the observation that, when $r = k$, Definition 3.18(iii) implies that α_k is maximal in α. The rest of the argument in the proof of Proposition 3.16 applies with two minor adjustments. First, Lemma 3.12 is not needed — neither can it be applied — in view of condition (i). Second, the fact that $\partial_{\alpha_r}(\alpha)$ is available at the start of $\mathcal{P}^{k-1,r,\alpha_r}$ comes directly from (iii) and induction. □

4 Application to configuration spaces

We use the gradient field in the previous section to describe the cohomology ring of the configuration space of ordered pairs of points on a complete graph.

4.1 Gradient fields on the Munkres homotopy simplicial model

Let K_m be the 1–dimensional skeleton of the full $(m-1)$–dimensional simplex on vertices $V_m = \{1, 2, \dots, m\}$. Thus $|K_m|$ is the complete graph on the m vertices. The homotopy type of $\mathrm{Conf}(|K_m|, 2)$ is well understood for $m \le 3$, so we assume $m \ge 4$ from now on. We think of K_m as an ordered simplicial complex with the natural order on V_m, and study $\mathrm{Conf}(|K_m|, 2)$ through its simplicial homotopy model

C_m in (2). The condition $m \geq 4$ implies that C_m is a pure 2–dimensional complex, ie all of its maximal faces have dimension 2. Furthermore, 2–dimensional faces of C_m have one of the forms

$$\begin{bmatrix} a & a & d \\ b & c & c \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a' & c' & c' \\ b' & b' & d' \end{bmatrix} \tag{28}$$

where

$$d > a \notin \{b, c\}, \quad b < c \neq d, \quad d' > b' \notin \{a', c'\} \quad \text{and} \quad a' < c' \neq d'. \tag{29}$$

Note that the matrix-type notation in (28) is compatible with the notation $\alpha = [\alpha_0, \dots, \alpha_k]$ in previous sections; each α_i now stands for a column-type vertex $\genfrac{}{}{0pt}{}{a}{b}$ (with $a \neq b$). In what follows, the conditions in (29) on the integers $a, b, c, d, a', b', c', d' \in V_m$ will generally be implicit and omitted when writing a 2–simplex or one of its faces. For instance, the forced relations $a \neq b < d \neq a$ are omitted in item (a) of:

Proposition 4.1 *Let W_m be the gradient field on C_m constructed in Section 3 with respect to the lexicographic order on the vertices $\genfrac{}{}{0pt}{}{a}{b} = (a, b) \in V_m \times V_m \setminus \Delta_{V_m}$ of C_m. The full list of W_m–pairings is:*

(a) $\left[\begin{smallmatrix} a & a \\ b & d \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & a & m \\ b & d & d \end{smallmatrix}\right]$ *for* $a < m > d$.

(b) $\left[\begin{smallmatrix} a & a \\ b & m \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & a & m-1 \\ b & m & m \end{smallmatrix}\right]$ *for* $a < m - 1$.

(c) $\left[\begin{smallmatrix} a & c \\ b & b \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & c & c \\ b & b & m \end{smallmatrix}\right]$ *for* $b < m > c$.

(d) $\left[\begin{smallmatrix} a & m \\ b & b \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & m & m \\ b & b & m-1 \end{smallmatrix}\right]$ *for* $b < m - 1$.

(e) $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & c & c \\ b & b & d \end{smallmatrix}\right]$ *for* $a < c$, $b < d$, $b \neq c$ *and either* $c < m > d$ *or* $c = m > d + 1$.

(f) $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ *for* $a < c$, $b < d$, $a \neq d$ *and either* $b = c < m > d$ *or* $c + 1 < m = d$.

(g) $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & m \\ b & m-1 \end{smallmatrix}\right]$ *for either* $b < m - 1$ *or* $a < m - 1 = b$.

(h) $\left[\begin{smallmatrix} a \\ m \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} a & m-1 \\ m & m \end{smallmatrix}\right]$ *for* $a < m - 1$.

(i) $\left[\begin{smallmatrix} m-1 \\ m \end{smallmatrix}\right] \nearrow \left[\begin{smallmatrix} m-1 & m-1 \\ m-2 & m \end{smallmatrix}\right]$.

In particular:

(j) *In dimension* 0, *the critical face is the vertex* $\left[\begin{smallmatrix} m \\ m-1 \end{smallmatrix}\right]$.

(k) *In dimension* 1, *the critical faces are the simplices*

- (k.1) $\left[\begin{smallmatrix} a & m-1 \\ b & m \end{smallmatrix}\right]$ *with either* $a = m - 1 > b + 1$ *or* $a < m - 1 \geq b$,
- (k.2) $\left[\begin{smallmatrix} m & m \\ b & d \end{smallmatrix}\right]$ *with* $d < m - 1$,
- (k.3) $\left[\begin{smallmatrix} a & c \\ m & m \end{smallmatrix}\right]$ *with* $c < m - 1$.

(l) *In dimension* 2, *the critical faces are the simplices* $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ *with* $b \neq c < m > d$.

Note that the condition $a \neq d$ in (f) is forced to hold in the stronger form $a < d$.

Proof All pairings, except for the one in (f) when $b \neq c$ (so that $c + 1 < m = d$), are given by Corollary 3.13 and Proposition 3.14. The exceptional case requires the stronger Theorem 3.19. On the other hand, direct inspection shows that the faces listed as critical are precisely those not taking part in the list of W_m–pairings. The proof is then complete by observing that the criticality of any d–dimensional face α in (j)–(l) is forced by the fact that all possible $(d-1)$–faces and all possible $(d+1)$–cofaces of α are involved in one of the pairings (a)–(i). For instance, a face $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ in (l) is not collapsible since the three potential pairings

$$\begin{bmatrix} a & a \\ b & d \end{bmatrix} \nearrow \begin{bmatrix} a & a & c \\ b & d & d \end{bmatrix}, \quad \begin{bmatrix} a & c \\ b & d \end{bmatrix} \nearrow \begin{bmatrix} a & a & c \\ b & d & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & c \\ d & d \end{bmatrix} \nearrow \begin{bmatrix} a & a & c \\ b & d & d \end{bmatrix}$$

are ruled out by (a), (e) and (c), respectively. □

The next-to-last sentence in the proof above reflects the maximality of W_m in Theorem 1.1. On the other hand, a straightforward counting shows that the number c_d of critical faces in dimension $d \in \{0,1,2\}$ is given by

$$c_0 = 1, \quad c_1 = 2(m-2)^2 - 1 \quad \text{and} \quad c_2 = \tfrac{1}{4}(m-1)(m-2)(m-3)(m-4). \tag{30}$$

In particular, the Euler characteristic of $\mathrm{Conf}(|K_m|,2)$ is given by

$$\tfrac{1}{4}m(m^3 - 10m^2 + 27m - 18),$$

which yields an explicit expression for the conclusion of [4, Corollary 1.2] in the case of complete graphs. Note in particular that the gradient field W_4 is optimal:

Corollary 4.2 *There is a homotopy equivalence* $\mathrm{Conf}(|K_4|,2) \simeq \bigvee_7 S^1$.

Corollary 4.2 should be compared to the fact that the configuration space of *unordered* pairs of points in $|K_4|$ has the homotopy type of $\bigvee_4 S^1$; see [12, Example 4.5].

4.2 The Morse cochain complex

The Morse coboundary map $\delta\colon \mu^i(C_m) \to \mu^{i+1}(C_m)$ is forced to vanish for $i = 0$ since $c_0 = 1$. It is more interesting to describe the situation for $i = 1$:

Proposition 4.3 *The coboundary* $\delta\colon \mu^1(C_m) \to \mu^2(C_m)$ *vanishes on the duals of the critical faces of types* $(k.2)$ *and* $(k.3)$ *in Proposition 4.1. For the duals of the critical faces of type* $(k.1)$ *we have*

$$\delta\left(\begin{bmatrix} a & m-1 \\ b & m \end{bmatrix}\right) = \sum \begin{bmatrix} a & a & x \\ y & b & b \end{bmatrix} - \sum \begin{bmatrix} a & a & x \\ b & y & y \end{bmatrix} + \sum \begin{bmatrix} x & x & a \\ b & y & y \end{bmatrix} - \sum \begin{bmatrix} x & x & a \\ y & b & b \end{bmatrix}, \tag{31}$$

where all four summands on the right hand-side of (31) run over all integers x and y that render critical 2–faces. Explicitly, $a < x < m$ in the first and second summations, $x < a$ in the third and fourth summations, $b < y < m$ in the second and third summations, $y < b$ in the first and fourth summations, and $b \neq x \neq y \neq a$ in all four summations.

Note that the first two summations in (31) are empty when $a = m-1$ (so $b < m-2$).

Proof The complete trees of mixed paths $\beta \searrow\nearrow \cdots \nearrow\searrow \alpha$ from critical 2–dimensional faces β to either critical or collapsible 1–dimensional faces α are given in Figures 3–11, where we indicate a positive (resp. negative) face with a bold (resp. regular) arrow. Types of pairings involved are indicated using the item

$$\begin{cases} \searrow \left[\begin{smallmatrix} a & c \\ d & m \end{smallmatrix}\right] \overset{\text{(f)}}{\nearrow} \left[\begin{smallmatrix} a & a & c \\ d & m & m \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} a & a \\ d & m \end{smallmatrix}\right] \overset{\text{(b)}}{\nearrow} \left[\begin{smallmatrix} a & a & m-1 \\ d & m & m \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} a & m-1 \\ d & m \end{smallmatrix}\right] & (+) \\ \searrow \left[\begin{smallmatrix} a & m-1 \\ m & m \end{smallmatrix}\right] & \text{(h)} \end{cases} \\ \searrow \left[\begin{smallmatrix} a & c \\ m & m \end{smallmatrix}\right] \quad (+) \end{cases} \\ \searrow \left[\begin{smallmatrix} c & c \\ d & m \end{smallmatrix}\right] \overset{\text{(b)}}{\nearrow} \left[\begin{smallmatrix} c & c & m-1 \\ d & m & m \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} c & m-1 \\ d & m \end{smallmatrix}\right] & (-) \\ \searrow \left[\begin{smallmatrix} c & m-1 \\ m & m \end{smallmatrix}\right] & \text{(h)} \end{cases} \end{cases}$$

Figure 3: Gradient paths evolving from $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right] \searrow \left[\begin{smallmatrix} a & c \\ d & d \end{smallmatrix}\right] \overset{\text{(c)}}{\nearrow} \left[\begin{smallmatrix} a & c & c \\ d & d & m \end{smallmatrix}\right]$ for $b \neq c \leq m-2 \geq d$.

names (a)–(i) in Proposition 4.1. At the end of each branch we indicate either the type of paring that shows α is collapsible or, if α is critical, the multiplicity with which the path must be accounted for in (6) and (7).

The first assertion follows by observing that, in Figures 3–11, there are two mixed paths departing from a fixed critical 2–dimensional face and arriving to a fix critical 1–dimensional face of the form (k.2) or (k.3). These two mixed paths have opposite multiplicities, so they cancel each other out in (7). For instance, each mixed path from $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} a & c \\ m & m \end{smallmatrix}\right]$ in Figure 3 cancels out the corresponding path in Figure 4.

To get at (31), start by noticing from Figures 3–11 that there are only four types of mixed paths departing from a given critical 2–dimensional face $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ that arrive to some critical 1–dimensional faces of type (k.1). Namely:

- There is a mixed path $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right] \searrow\nearrow \cdots \nearrow\searrow \left[\begin{smallmatrix} a & m-1 \\ d & m \end{smallmatrix}\right]$ with multiplicity $+1$; see Figures 3, 6 and 9,
- There is a mixed path $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right] \searrow\nearrow \cdots \nearrow\searrow \left[\begin{smallmatrix} a & m-1 \\ b & m \end{smallmatrix}\right]$ with multiplicity -1; see Figures 4, 7 and 10,
- There is a mixed path $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right] \searrow\nearrow \cdots \nearrow\searrow \left[\begin{smallmatrix} c & m-1 \\ b & m \end{smallmatrix}\right]$ with multiplicity $+1$; see Figures 4, 7 and 10.
- There is a mixed path $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right] \searrow\nearrow \cdots \nearrow\searrow \left[\begin{smallmatrix} c & m-1 \\ d & m \end{smallmatrix}\right]$ with multiplicity -1 provided $(c,d) \neq (m-1, m-2)$; see Figures 3, 6 and 9.

$$\begin{cases} \searrow \left[\begin{smallmatrix} a & c \\ b & b \end{smallmatrix}\right] \overset{\text{(c)}}{\nearrow} \left[\begin{smallmatrix} a & c & c \\ b & b & m \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} a & c \\ b & m \end{smallmatrix}\right] \overset{\text{(f)}}{\nearrow} \left[\begin{smallmatrix} a & a & c \\ b & m & m \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} a & a \\ b & m \end{smallmatrix}\right] \overset{\text{(b)}}{\nearrow} \left[\begin{smallmatrix} a & a & m-1 \\ b & m & m \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} a & m-1 \\ b & m \end{smallmatrix}\right] & (-) \\ \searrow \left[\begin{smallmatrix} a & m-1 \\ m & m \end{smallmatrix}\right] & \text{(h)} \end{cases} \\ \searrow \left[\begin{smallmatrix} a & c \\ m & m \end{smallmatrix}\right] \quad (-) \end{cases} \\ \searrow \left[\begin{smallmatrix} c & c \\ b & m \end{smallmatrix}\right] \overset{\text{(b)}}{\nearrow} \left[\begin{smallmatrix} c & c & m-1 \\ b & m & m \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} c & m-1 \\ b & m \end{smallmatrix}\right] & (+) \\ \searrow \left[\begin{smallmatrix} c & m-1 \\ m & m \end{smallmatrix}\right] & \text{(h)} \end{cases} \end{cases} \\ \searrow \left[\begin{smallmatrix} c & c \\ b & d \end{smallmatrix}\right] \overset{\text{(a)}}{\nearrow} \left[\begin{smallmatrix} c & c & m \\ b & d & d \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} c & m \\ b & d \end{smallmatrix}\right] \overset{\text{(e)}}{\nearrow} \left[\begin{smallmatrix} c & m & m \\ b & b & d \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} c & m \\ b & b \end{smallmatrix}\right] \overset{\text{(d)}}{\nearrow} \left[\begin{smallmatrix} c & m & m \\ b & b & m-1 \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} c & m \\ b & m-1 \end{smallmatrix}\right] & \text{(g)} \\ \searrow \left[\begin{smallmatrix} m & m \\ b & m-1 \end{smallmatrix}\right] & \text{(g)} \end{cases} \\ \searrow \left[\begin{smallmatrix} m & m \\ b & d \end{smallmatrix}\right] \quad (-) \end{cases} \\ \searrow \left[\begin{smallmatrix} c & m \\ d & d \end{smallmatrix}\right] \overset{\text{(d)}}{\nearrow} \left[\begin{smallmatrix} c & m & m \\ d & d & m-1 \end{smallmatrix}\right] \begin{cases} \searrow \left[\begin{smallmatrix} c & m \\ d & m-1 \end{smallmatrix}\right] & \text{(g)} \\ \searrow \left[\begin{smallmatrix} m & m \\ d & m-1 \end{smallmatrix}\right] & \text{(g)} \end{cases} \end{cases} \end{cases}$$

Figure 4: Gradient paths evolving from $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right] \searrow \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] \overset{\text{(e)}}{\nearrow} \left[\begin{smallmatrix} a & c & c \\ b & b & d \end{smallmatrix}\right]$ for $b \neq c \leq m-2 \geq d$.

$$
\begin{cases}
\searrow \left[\begin{smallmatrix} a & m \\ b & d \end{smallmatrix}\right] \overset{\text{(e)}}{\nearrow} \left[\begin{smallmatrix} a & m & m \\ b & b & d \end{smallmatrix}\right]
\begin{cases}
\searrow \left[\begin{smallmatrix} a & m \\ b & b \end{smallmatrix}\right] \overset{\text{(d)}}{\nearrow} \left[\begin{smallmatrix} a & m & m \\ b & b & m-1 \end{smallmatrix}\right]
\begin{cases}
\searrow \left[\begin{smallmatrix} a & m \\ b & m-1 \end{smallmatrix}\right] & \text{(g)} \\
\searrow \left[\begin{smallmatrix} m & m \\ b & m-1 \end{smallmatrix}\right] & \text{(g)}
\end{cases} \\
\searrow \left[\begin{smallmatrix} m & m \\ b & d \end{smallmatrix}\right] \quad (+)
\end{cases} \\
\searrow \left[\begin{smallmatrix} a & m \\ d & d \end{smallmatrix}\right] \overset{\text{(d)}}{\nearrow} \left[\begin{smallmatrix} a & m & m \\ d & d & m-1 \end{smallmatrix}\right]
\begin{cases}
\searrow \left[\begin{smallmatrix} a & m \\ d & m-1 \end{smallmatrix}\right] & \text{(g)} \\
\searrow \left[\begin{smallmatrix} m & m \\ d & m-1 \end{smallmatrix}\right] & \text{(g)}
\end{cases}
\end{cases}
$$

Figure 5: Gradient paths evolving from $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right] \searrow \left[\begin{smallmatrix} a & a \\ b & d \end{smallmatrix}\right] \overset{\text{(a)}}{\nearrow} \left[\begin{smallmatrix} a & a & m \\ b & d & d \end{smallmatrix}\right]$ for $b \neq c \leq m-2 \geq d$.

Therefore the value of the boundary map $\partial\colon \mu_2(C_m) \to \mu_1(C_m)$ at a critical face $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ for $b \neq c < m > d$ with $(c,d) \neq (m-1, m-2)$ is

$$
\partial\left(\begin{bmatrix} a & a & c \\ b & d & d \end{bmatrix}\right) = \begin{bmatrix} a & m-1 \\ d & m \end{bmatrix} - \begin{bmatrix} a & m-1 \\ b & m \end{bmatrix} + \begin{bmatrix} c & m-1 \\ b & m \end{bmatrix} - \begin{bmatrix} c & m-1 \\ d & m \end{bmatrix}, \tag{32}
$$

whereas, for $(c,d) = (m-1, m-2)$,

$$
\partial\left(\begin{bmatrix} a & a & m-1 \\ b & m-2 & m-2 \end{bmatrix}\right) = \begin{bmatrix} a & m-1 \\ m-2 & m \end{bmatrix} - \begin{bmatrix} a & m-1 \\ b & m \end{bmatrix} + \begin{bmatrix} m-1 & m-1 \\ b & m \end{bmatrix}. \tag{33}
$$

(Note that (32) is valid when $(c,d) = (m-1, m-2)$ provided the fourth noncritical term

$$
\begin{bmatrix} m-1 & m-1 \\ m-2 & m \end{bmatrix}
$$

is omitted.) That (31) follows by dualizing (32) and (33) is a straightforward exercise left to the reader. $\square$

4.3 Cohomology bases

By Corollary 4.2, we can assume $m \geq 5$ throughout the rest of the paper. We start by identifying (in Corollary 4.7) an explicit basis for $H^1(\mathrm{Conf}(|K_m|, 2))$, ie for the kernel of the Morse coboundary $\delta\colon \mu^1(C_m) \to \mu^2(C_m)$. By Proposition 4.3, it is enough to focus on the submodule $\mu_0^1(C_m)$ of $\mu^1(C_m)$ generated by the duals of the basis elements of type (k.1). Thus $\mu_0^1(C_m)$ is free on elements $\left\{\begin{smallmatrix} a \\ b \end{smallmatrix}\right\}$ satisfying $a < m > b \neq a$ and $(a,b) \neq (m-1, m-2)$, where $\left\{\begin{smallmatrix} a \\ b \end{smallmatrix}\right\}$ stands for the dual of $\left[\begin{smallmatrix} a & m-1 \\ b & m \end{smallmatrix}\right]$.

Definition 4.4 Consider the elements $\left\langle\begin{smallmatrix} a \\ b \end{smallmatrix}\right\rangle \in \mu_0^1(C_m)$ defined for $a < m > b \neq a$ and $(a,b) \neq (m-1, m-2)$ according to the following cases (see Figure 12):

(R_1) For $1 \leq a \leq 2$, or for $1 \leq a \leq m-3$ with $b = m-1$, or for $(a,b) = (3,1)$,

$$
\left\langle\begin{matrix} a \\ b \end{matrix}\right\rangle := \sum_{a \neq j \leq b} \left\{\begin{matrix} a \\ j \end{matrix}\right\}.
$$

$$
\begin{cases}
\searrow \left[\begin{smallmatrix} a & m-1 \\ d & m \end{smallmatrix}\right] & (+) \\
\searrow \left[\begin{smallmatrix} m-1 & m-1 \\ d & m \end{smallmatrix}\right] & \begin{matrix} (-) & \text{if } d < m-2 \\ \text{(i)} & \text{if } d = m-2 \end{matrix}
\end{cases}
$$

Figure 6: Gradient paths evolving from $\left[\begin{smallmatrix} a & a & m-1 \\ b & d & d \end{smallmatrix}\right] \searrow \left[\begin{smallmatrix} a & m-1 \\ d & d \end{smallmatrix}\right] \overset{\text{(c)}}{\nearrow} \left[\begin{smallmatrix} a & m-1 & m-1 \\ d & d & m \end{smallmatrix}\right]$ for $b \neq m-1 > d$.

$$\left\{\begin{array}{l}
\searrow \begin{bmatrix} a & m-1 \\ b & b \end{bmatrix} \overset{(c)}{\nearrow} \begin{bmatrix} a & m-1 & m-1 \\ b & b & m \end{bmatrix} \left\{\begin{array}{l} \searrow \begin{bmatrix} a & m-1 \\ b & m \end{bmatrix} \quad (-) \\ \searrow \begin{bmatrix} m-1 & m-1 \\ b & m \end{bmatrix} \quad (+) \end{array}\right. \\[2ex]
\searrow \begin{bmatrix} m-1 & m-1 \\ b & d \end{bmatrix} \overset{(a)}{\nearrow} \begin{bmatrix} m-1 & m-1 & m \\ b & d & d \end{bmatrix} \left\{\begin{array}{l}
\searrow \begin{bmatrix} m-1 & m \\ b & d \end{bmatrix} \overset{(e)}{\nearrow} \begin{bmatrix} m-1 & m & m \\ b & b & d \end{bmatrix} \left\{\begin{array}{l} \searrow \begin{bmatrix} m-1 & m \\ b & b \end{bmatrix} \overset{(d)}{\nearrow} \begin{bmatrix} m-1 & m & m \\ b & b & m-1 \end{bmatrix} \left\{\begin{array}{l} \searrow \begin{bmatrix} m-1 & m \\ b & m-1 \end{bmatrix} \quad (g) \\ \searrow \begin{bmatrix} m & m \\ b & m-1 \end{bmatrix} \quad (g) \end{array}\right. \\ \searrow \begin{bmatrix} m & m \\ b & d \end{bmatrix} \quad (-) \end{array}\right. \\[2ex]
\searrow \begin{bmatrix} m-1 & m \\ d & d \end{bmatrix} \overset{(d)}{\nearrow} \begin{bmatrix} m-1 & m & m \\ d & d & m-1 \end{bmatrix} \left\{\begin{array}{l} \searrow \begin{bmatrix} m-1 & m \\ d & m-1 \end{bmatrix} \quad (g) \\ \searrow \begin{bmatrix} m & m \\ d & m-1 \end{bmatrix} \quad (g) \end{array}\right.
\end{array}\right.
\end{array}\right.$$

Figure 7: Gradient paths evolving from $\begin{bmatrix} a & a & m-1 \\ b & d & d \end{bmatrix} \searrow \begin{bmatrix} a & m-1 \\ b & d \end{bmatrix} \overset{(e)}{\nearrow} \begin{bmatrix} a & m-1 & m-1 \\ b & b & d \end{bmatrix}$ for $b \neq m-1 > d$.

(R$_2$) For $4 \le a \le m-1$ with $b=1$, or for $a = m-1$ with $1 \le b \le m-4$,

$$\left\langle {a \atop b} \right\rangle := \sum_{b \neq i \le a} \left\{ {i \atop b} \right\}.$$

(R$_3$) $\left\langle {3 \atop 2} \right\rangle := \left\{ {3 \atop 2} \right\} + \left\{ {3 \atop 1} \right\} + \left\{ {2 \atop 3} \right\} + \left\{ {2 \atop 1} \right\} + \left\{ {1 \atop 3} \right\} + \left\{ {1 \atop 2} \right\}$.

(R$_4$) For $4 \le a \le m-2$,

$$\left\langle {a \atop 2} \right\rangle := \sum_{2 \neq i \le a} \left\{ {i \atop 2} \right\} + \sum_{i \le a-1} \left\{ {i \atop a} \right\}.$$

(R$_5$) For $a, b \in \{3, 4, \dots, m-2\}$,

$$\left\langle {a \atop b} \right\rangle := \sum_{j \neq i \le a \neq j \le b} \left\{ {i \atop j} \right\} + \sum_{i \le a-1} \left\{ {i \atop a} \right\}.$$

(R$_6$) $\left\langle {m-1 \atop m-3} \right\rangle := \left\{ {m-1 \atop m-3} \right\} - \sum_{i,j} \left\{ {i \atop j} \right\}$, where the sum runs over i and j with $m-3 \neq j \neq i \le m-2 \ge j$.

(R$_7$) $\left\langle {m-2 \atop m-1} \right\rangle := \left\{ {m-2 \atop m-1} \right\} - \sum_{i,j} \left\{ {i \atop j} \right\}$, where the sum runs over i and j with $m-2 \ge j \neq i \le m-3$.

Direct inspection of the defining formulae yields:

Lemma 4.5 *The following relations hold under the indicated conditions*:

(i) $\left\langle {a \atop b} \right\rangle - \left\langle {a \atop b-1} \right\rangle = \sum_{b \neq i \le a} \left\{ {i \atop b} \right\}$, *provided* $3 \le a \le m-2$ *and* $4 \le b \le m-2$ *with* $a \neq b \neq a+1$.

(ii) $\left\langle {a \atop 3} \right\rangle - \left\langle {a \atop 2} \right\rangle - \left\langle {a \atop 1} \right\rangle = \sum_{3 \neq i \le a} \left\{ {i \atop 3} \right\}$, *provided* $4 \le a \le m-2$.

(iii) $\left\langle {b-1 \atop b} \right\rangle - \left\langle {b-1 \atop b-2} \right\rangle = \sum_{i < b} \left\{ {i \atop b} \right\}$, *provided* $5 \le b \le m-2$.

(iv) $\left\langle {3 \atop 4} \right\rangle - \left\langle {3 \atop 2} \right\rangle = \sum_{i \le 3} \left\{ {i \atop 4} \right\}$, *provided* $m \ge 6$.

Proposition 4.6 *The elements* $\left\langle {a \atop b} \right\rangle$ *for* $a < m > b \neq a$ *and* $(a,b) \neq (m-1, m-2)$ *yield a basis of* $\mu_0^1(C_m)$.

$$
\left\{
\begin{array}{l}
\searrow \begin{bmatrix} a & m \\ b & d \end{bmatrix} \overset{\text{(e)}}{\nearrow} \begin{bmatrix} a & m & m \\ b & b & d \end{bmatrix}
\left\{
\begin{array}{l}
\searrow \begin{bmatrix} a & m \\ b & b \end{bmatrix} \overset{\text{(d)}}{\nearrow} \begin{bmatrix} a & m & m \\ b & b & m-1 \end{bmatrix}
\left\{
\begin{array}{ll}
\searrow \begin{bmatrix} a & m \\ b & m-1 \end{bmatrix} & \text{(g)} \\
\searrow \begin{bmatrix} m & m \\ b & m-1 \end{bmatrix} & \text{(g)}
\end{array}
\right. \\
\searrow \begin{bmatrix} m & m \\ b & d \end{bmatrix} \quad (+)
\end{array}
\right. \\
\searrow \begin{bmatrix} a & m \\ d & d \end{bmatrix} \overset{\text{(d)}}{\nearrow} \begin{bmatrix} a & m & m \\ d & d & m-1 \end{bmatrix}
\left\{
\begin{array}{ll}
\searrow \begin{bmatrix} a & m \\ d & m-1 \end{bmatrix} & \text{(g)} \\
\searrow \begin{bmatrix} m & m \\ d & m-1 \end{bmatrix} & \text{(g)}
\end{array}
\right.
\end{array}
\right.
$$

Figure 8: Gradient paths evolving from $\begin{bmatrix} a & a & m-1 \\ b & d & d \end{bmatrix} \searrow \begin{bmatrix} a & a \\ b & d \end{bmatrix} \overset{\text{(a)}}{\nearrow} \begin{bmatrix} a & a & m \\ b & d & d \end{bmatrix}$ for $b \neq m-1 > d$.

Proof In view of the one-to-one correspondence $\left\{\begin{smallmatrix} a \\ b \end{smallmatrix}\right\} \leftrightarrow \left\langle\begin{smallmatrix} a \\ b \end{smallmatrix}\right\rangle$, it suffices to check that

(34) each $\left\{\begin{smallmatrix} a \\ b \end{smallmatrix}\right\}$ is a $\mathbb{Z}$–linear combination of the elements $\left\langle\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right\rangle$.

In most cases (34) follows by a simple recursive argument based on the observation that, in all cases,

$$\left\{\begin{matrix} a \\ b \end{matrix}\right\} = \left\langle\begin{matrix} a \\ b \end{matrix}\right\rangle + \sum_{(a',b') \neq (a,b)} \pm 1 \left\{\begin{matrix} a' \\ b' \end{matrix}\right\}. \tag{35}$$

Namely, the recursive argument applies for (R_1) when $a \leq 2$ or $b = 1$, for (R_2) when $b = 1$, and for (R_3). The recursive argument also applies in the cases (R_6) and (R_7), as well as in the remaining instances of (R_1) and (R_2) provided

(36) (34) holds true when a and b fall in cases (R_4) and (R_5).

Since (R_4) and (R_5) are empty for $m = 5$, we only need to verify (36) assuming $m \geq 6$.

Direct computation gives $\left\{\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right\} = \left\langle\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right\rangle$, while Lemma 4.5(iii) and (iv) yield

$$\left\{\begin{matrix} a \\ 2 \end{matrix}\right\} = \left(\left\langle\begin{matrix} a \\ 2 \end{matrix}\right\rangle - \left\langle\begin{matrix} a-1 \\ 2 \end{matrix}\right\rangle\right) + \left(\left\langle\begin{matrix} a-2 \\ a-1 \end{matrix}\right\rangle - \left\langle\begin{matrix} a-2 \\ a-3 \end{matrix}\right\rangle\right) - \left(\left\langle\begin{matrix} a-1 \\ a \end{matrix}\right\rangle - \left\langle\begin{matrix} a-1 \\ a-2 \end{matrix}\right\rangle\right)$$

for $5 \leq a \leq m-2$, which establishes (36) in the case of (R_4). The validity of (36) in the case $a = 3$ of (R_5) is established in a similar way. Note first that the idea in the recursive argument based on (35) works to give (36) for $(a, b) = (3, 4)$; then use Lemma 4.5(i) to get

$$\left\{\begin{matrix} 3 \\ b \end{matrix}\right\} = \left(\left\langle\begin{matrix} 3 \\ b \end{matrix}\right\rangle - \left\langle\begin{matrix} 3 \\ b-1 \end{matrix}\right\rangle\right) - \left\langle\begin{matrix} 2 \\ b \end{matrix}\right\rangle + \left\langle\begin{matrix} 2 \\ b-1 \end{matrix}\right\rangle - \left\langle\begin{matrix} 1 \\ b \end{matrix}\right\rangle + \left\langle\begin{matrix} 1 \\ b-1 \end{matrix}\right\rangle$$

$$
\left\{
\begin{array}{l}
\searrow \begin{bmatrix} a & c \\ m-1 & m \end{bmatrix} \overset{\text{(f)}}{\nearrow} \begin{bmatrix} a & a & c \\ m-1 & m & m \end{bmatrix}
\left\{
\begin{array}{l}
\searrow \begin{bmatrix} a & a \\ m-1 & m \end{bmatrix} \overset{\text{(b)}}{\nearrow} \begin{bmatrix} a & a & m-1 \\ m-1 & m & m \end{bmatrix}
\left\{
\begin{array}{ll}
\searrow \begin{bmatrix} a & m-1 \\ m-1 & m \end{bmatrix} & (+) \\
\searrow \begin{bmatrix} a & m-1 \\ m & m \end{bmatrix} & \text{(h)}
\end{array}
\right. \\
\searrow \begin{bmatrix} a & c \\ m & m \end{bmatrix} \quad (+)
\end{array}
\right. \\
\searrow \begin{bmatrix} c & c \\ m-1 & m \end{bmatrix} \overset{\text{(b)}}{\nearrow} \begin{bmatrix} c & c & m-1 \\ m-1 & m & m \end{bmatrix}
\left\{
\begin{array}{ll}
\searrow \begin{bmatrix} c & m-1 \\ m-1 & m \end{bmatrix} & (-) \\
\searrow \begin{bmatrix} c & m-1 \\ m & m \end{bmatrix} & \text{(h)}
\end{array}
\right.
\end{array}
\right.
$$

Figure 9: Gradient paths evolving from $\begin{bmatrix} a & a & c \\ b & m-1 & m-1 \end{bmatrix} \searrow \begin{bmatrix} a & c \\ m-1 & m-1 \end{bmatrix} \overset{\text{(c)}}{\nearrow} \begin{bmatrix} a & c & c \\ m-1 & m-1 & m \end{bmatrix}$ for $b \neq c < m-1$.

$$
\left\{
\begin{array}{l}
\searrow \begin{bmatrix} a & c \\ b & b \end{bmatrix} \overset{(c)}{\nearrow} \begin{bmatrix} a & c & c \\ b & b & m \end{bmatrix}
\left\{
\begin{array}{l}
\searrow \begin{bmatrix} a & c \\ b & m \end{bmatrix} \overset{(f)}{\nearrow} \begin{bmatrix} a & a & c \\ b & m & m \end{bmatrix}
\left\{
\begin{array}{l}
\searrow \begin{bmatrix} a & a \\ b & m \end{bmatrix} \overset{(b)}{\nearrow} \begin{bmatrix} a & a & m-1 \\ b & m & m \end{bmatrix}
\left\{
\begin{array}{ll}
\searrow \begin{bmatrix} a & m-1 \\ b & m \end{bmatrix} & (-) \\
\searrow \begin{bmatrix} a & m-1 \\ m & m \end{bmatrix} & (h)
\end{array}
\right. \\
\searrow \begin{bmatrix} a & c \\ m & m \end{bmatrix} \quad (-)
\end{array}
\right. \\
\searrow \begin{bmatrix} c & c \\ b & m \end{bmatrix} \overset{(b)}{\nearrow} \begin{bmatrix} c & c & m-1 \\ b & m & m \end{bmatrix}
\left\{
\begin{array}{ll}
\searrow \begin{bmatrix} c & m-1 \\ b & m \end{bmatrix} & (+) \\
\searrow \begin{bmatrix} c & m-1 \\ m & m \end{bmatrix} & (h)
\end{array}
\right.
\end{array}
\right. \\
\searrow \begin{bmatrix} c & c \\ b & m-1 \end{bmatrix} \overset{(a)}{\nearrow} \begin{bmatrix} c & c & m \\ b & m-1 & m-1 \end{bmatrix}
\left\{
\begin{array}{ll}
\searrow \begin{bmatrix} c & m \\ b & m-1 \end{bmatrix} & (g) \\
\searrow \begin{bmatrix} c & m \\ m-1 & m-1 \end{bmatrix} & (g)
\end{array}
\right.
\end{array}
\right.
$$

Figure 10: Gradient paths evolving from $\begin{bmatrix} a & a & c \\ b & m-1 & m-1 \end{bmatrix} \searrow \begin{bmatrix} a & c \\ b & m-1 \end{bmatrix} \overset{(e)}{\nearrow} \begin{bmatrix} a & c & c \\ b & b & m-1 \end{bmatrix}$ for $b \neq c < m-1$.

for $5 \leq b \leq m-2$. The validity of (36) in the case $b = 3$ of (R_5) is established using the idea in the recursive argument at the beginning of the proof, except that (35) is replaced by the expression

$$\genfrac{\{}{\}}{0pt}{}{a}{3} = \genfrac{\langle}{\rangle}{0pt}{}{a}{3} - \genfrac{\langle}{\rangle}{0pt}{}{a}{2} - \genfrac{\langle}{\rangle}{0pt}{}{a}{1} - \sum_{3 \neq i < a} \genfrac{\{}{\}}{0pt}{}{i}{3}$$

coming from Lemma 4.5(ii). Lastly, the validity of (36) in the remaining case $a, b \in \{4, \dots, m-2\}$ of (R_5) is established by the formulae

- $\genfrac{\{}{\}}{0pt}{}{a}{b} = \bigl(\genfrac{\langle}{\rangle}{0pt}{}{a}{b} - \genfrac{\langle}{\rangle}{0pt}{}{a}{b-1}\bigr) - \bigl(\genfrac{\langle}{\rangle}{0pt}{}{a-1}{b} - \genfrac{\langle}{\rangle}{0pt}{}{a-1}{b-1}\bigr)$, when $a+1 \neq b \neq a-1$,
- $\genfrac{\{}{\}}{0pt}{}{a}{b} = \bigl(\genfrac{\langle}{\rangle}{0pt}{}{a}{a-1} - \genfrac{\langle}{\rangle}{0pt}{}{a}{a-2}\bigr) - \bigl(\genfrac{\langle}{\rangle}{0pt}{}{a-2}{a-1} - \genfrac{\langle}{\rangle}{0pt}{}{a-2}{a-3}\bigr)$, when $b = a-1$ (so $a \geq 5$),
- $\genfrac{\{}{\}}{0pt}{}{a}{b} = \bigl(\genfrac{\langle}{\rangle}{0pt}{}{a}{a+1} - \genfrac{\langle}{\rangle}{0pt}{}{a}{a-1}\bigr) - \bigl(\genfrac{\langle}{\rangle}{0pt}{}{a-1}{a+1} - \genfrac{\langle}{\rangle}{0pt}{}{a-1}{a}\bigr)$, when $b = a+1$ (so $a \leq m-3$),

which use Lemma 4.5(i), (iii) and (iv). □

Recall that the Morse coboundary $\delta\colon \mu^0(C_m) \to \mu^1(C_m)$ is trivial, so that the 1–dimensional cohomology of $\mathrm{Conf}(|K_m|, 2)$ is given by the kernel of $\delta\colon \mu^1(C_m) \to \mu^2(C_m)$.

Corollary 4.7 *A basis for* $H^1(\mathrm{Conf}(|K_m|, 2))$ *is given by*

(i) *the duals of the critical* 1*–dimensional faces of type* (k.2) *and* (k.3) *in* Proposition 4.1, *and*

(ii) *the* (*already dualized*) *elements* $\genfrac{\langle}{\rangle}{0pt}{}{a}{b}$ *satisfying* $a = m-1$, $b = m-1$ or $(a, b) = (m-2, m-3)$.

Proof Recall $m \geq 5$. A straightforward counting shows that the number of elements in (i) and (ii) is $(m-1)(m-2)$, which is also the first Betti number of $\mathrm{Conf}(|K_m|, 2)$; see [9, Corollary 23]. Since the

$$
\left\{
\begin{array}{ll}
\searrow \begin{bmatrix} a & m \\ b & m-1 \end{bmatrix} & (g) \\
\searrow \begin{bmatrix} a & m \\ m-1 & m-1 \end{bmatrix} & (g)
\end{array}
\right.
$$

Figure 11: Gradient paths evolving from $\begin{bmatrix} a & a & c \\ b & m-1 & m-1 \end{bmatrix} \searrow \begin{bmatrix} a & a \\ b & m-1 \end{bmatrix} \overset{(a)}{\nearrow} \begin{bmatrix} a & a & m \\ b & m-1 & m-1 \end{bmatrix}$ for $b \neq c < m-1$.

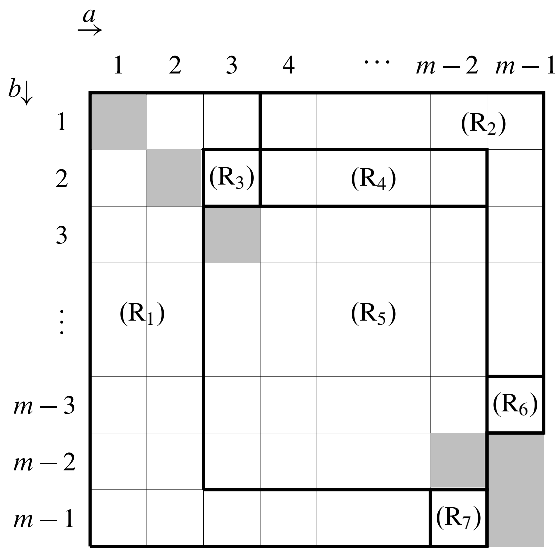

Figure 12: Defining regions for the basis elements $\left\langle \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rangle$.

homology of $\mathrm{Conf}(|K_m|, 2)$ is torsion-free [9, Proposition 2], Proposition 4.6 implies that the proof will be complete once it is checked that $\delta\colon \mu^1(C_m) \to \mu^2(C_m)$ vanishes on each of the elements in (i) and (ii). Indeed, this actually implies that δ is injective on the submodule generated by the basis elements $\left\langle \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rangle$ not included in (ii).

The vanishing of δ on the elements in (i) comes directly from Proposition 4.3, whereas the vanishing of δ on the elements in (ii) is verified by direct calculation using the expression of δ in Proposition 4.3. The arithmetic manipulations needed are illustrated next in a representative case, namely, that of $\left\langle \begin{smallmatrix} m-1 \\ m-3 \end{smallmatrix} \right\rangle$.

Use Proposition 4.3 and the defining formula (R_6) to get

$$\begin{aligned}\delta\left\langle \begin{matrix} m-1 \\ m-3 \end{matrix} \right\rangle = \sum_{x,y} \begin{bmatrix} x & x & m-1 \\ m-3 & y & y \end{bmatrix} - \sum_{x,y} \begin{bmatrix} x & x & m-1 \\ y & m-3 & m-3 \end{bmatrix} \\ - \sum_{m-3\neq j\neq i\leq m-2\geq j} \left(\sum_{x,y} \begin{bmatrix} i & i & x \\ y & j & j \end{bmatrix} - \sum_{x,y} \begin{bmatrix} i & i & x \\ j & y & y \end{bmatrix} + \sum_{x,y} \begin{bmatrix} x & x & i \\ j & y & y \end{bmatrix} - \sum_{x,y} \begin{bmatrix} x & x & i \\ y & j & j \end{bmatrix} \right).\end{aligned}$$

The summands with $y = m-1$ in the second inner summation cancel out the corresponding ones in the third inner summation. (The corresponding fact for $y < m-1$, dealt with below, is more subtle since $i \leq m-2$ in the third inner summation, while $x \leq m-1$ in the second inner summation.) Noticing in addition that $y = m-2$ is forced in the first outer summation, we then get

$$\begin{aligned}\delta\left\langle \begin{matrix} m-1 \\ m-3 \end{matrix} \right\rangle = \sum_{x} \begin{bmatrix} x & x & m-1 \\ m-3 & m-2 & m-2 \end{bmatrix} - \sum_{x,y} \begin{bmatrix} x & x & m-1 \\ y & m-3 & m-3 \end{bmatrix} \\ - \sum_{m-3\neq j\neq i\leq m-2\geq j} \left(\sum_{x,y} \begin{bmatrix} i & i & x \\ y & j & j \end{bmatrix} - \sum_{\substack{x \\ y\leq m-2}} \begin{bmatrix} i & i & x \\ j & y & y \end{bmatrix} + \sum_{\substack{x \\ y\leq m-2}} \begin{bmatrix} x & x & i \\ j & y & y \end{bmatrix} - \sum_{x,y} \begin{bmatrix} x & x & i \\ y & j & j \end{bmatrix} \right).\end{aligned}$$

In the last expression, the summands with $x \le m-2$ in the second inner summation cancel out the third inner summation, so

$$\delta\left\langle\begin{matrix}m-1\\m-3\end{matrix}\right\rangle = \sum_x \begin{bmatrix}x & x & m-1\\ m-3 & m-2 & m-2\end{bmatrix} - \sum_{x,y}\begin{bmatrix}x & x & m-1\\ y & m-3 & m-3\end{bmatrix}$$
$$- \sum_{m-3\neq j\neq i\le m-2\ge j}\left(\sum_{x,y}\begin{bmatrix}i & i & x\\ y & j & j\end{bmatrix} - \sum_{y\le m-2}\begin{bmatrix}i & i & m-1\\ j & y & y\end{bmatrix} - \sum_{x,y}\begin{bmatrix}x & x & i\\ y & j & j\end{bmatrix}\right).$$

Likewise, in the last expression, summands with $x \le m-2$ in the first inner summation cancel out the third inner summation, so

$$\delta\left\langle\begin{matrix}m-1\\m-3\end{matrix}\right\rangle = \sum_x \begin{bmatrix}x & x & m-1\\ m-3 & m-2 & m-2\end{bmatrix} - \sum_{x,y}\begin{bmatrix}x & x & m-1\\ y & m-3 & m-3\end{bmatrix}$$
$$- \sum_{m-3\neq j\neq i\le m-2\ge j}\left(\sum_{y}\begin{bmatrix}i & i & m-1\\ y & j & j\end{bmatrix} - \sum_{y\le m-2}\begin{bmatrix}i & i & m-1\\ j & y & y\end{bmatrix}\right).$$

Lastly, merge the first (resp. second) outer summation and the second (resp. first) inner summation in the last expression to get

$$\delta\left\langle\begin{matrix}m-1\\m-3\end{matrix}\right\rangle = \sum_{j\neq i\le m-2\ge y}\begin{bmatrix}i & i & m-1\\ j & y & y\end{bmatrix} - \sum_{j\neq i\le m-2\ge j}\begin{bmatrix}i & i & m-1\\ y & j & j\end{bmatrix} = 0,$$

as asserted. □

We next identify (in Corollary 4.11) an explicit basis for $H^2(\mathrm{Conf}(|K_m|,2))$, ie for the cokernel of the Morse coboundary $\delta\colon \mu^1(C_m)\to\mu^2(C_m)$. In what follows, the conditions $1\le a<c<m$, $1\le b<d<m$ and $c\neq d\neq a\neq b\neq c<m>d$ for critical faces $\left[\begin{smallmatrix}a & a & c\\ b & d & d\end{smallmatrix}\right]$ identified in Proposition 4.1(l) will be implicit (and generally omitted).

Definition 4.8 Let $\mathcal{C}$ be the collection of the critical faces $\left[\begin{smallmatrix}a & a & c\\ b & d & d\end{smallmatrix}\right]$ of one of the four types

$$\left[\begin{smallmatrix}1 & 1 & c\\ 2 & d & d\end{smallmatrix}\right] \text{ with } c,d\in\{3,4,\dots,m-1\},\tag{37}$$

$$\left[\begin{smallmatrix}1 & 1 & 2\\ 3 & d & d\end{smallmatrix}\right] \text{ with } d\in\{4,5,\dots,m-1\},\tag{38}$$

$$\left[\begin{smallmatrix}2 & 2 & c\\ 1 & 3 & 3\end{smallmatrix}\right] \text{ with } c\in\{4,5,\dots,m-1\},\tag{39}$$

$$\left[\begin{smallmatrix}2 & 2 & 3\\ 1 & 4 & 4\end{smallmatrix}\right],\tag{40}$$

and let $\mathcal{B}$ stand for the collection of all other critical faces $\left[\begin{smallmatrix}a & a & c\\ b & d & d\end{smallmatrix}\right]$.

The following change of basis is used to show that the duals of critical faces in $\mathcal{B}$ form a basis of $H^2(\mathrm{Conf}(|K_m|,2))$:

Definition 4.9 For each $\left[\begin{smallmatrix}a & a & c\\ b & d & d\end{smallmatrix}\right]\in\mathcal{B}$, consider the element $\left\langle\begin{smallmatrix}a & a & c\\ b & d & d\end{smallmatrix}\right\rangle\in\mu_2(C_m)$ defined through:

- (i) Case $a=1$ and $c\ge 3$ with $\left[\begin{smallmatrix}a & a & c\\ b & d & d\end{smallmatrix}\right]$ not fitting in (37):
  - (a) $\left\langle\begin{smallmatrix}1 & 1 & c\\ b & d & d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix}1 & 1 & c\\ b & d & d\end{smallmatrix}\right] - \left[\begin{smallmatrix}1 & 1 & c\\ 2 & d & d\end{smallmatrix}\right] + \left[\begin{smallmatrix}1 & 1 & c\\ 2 & b & b\end{smallmatrix}\right]$, for $b\ge 3$.

(ii) Case $a=1$ and $c=2$ with $\left[\begin{smallmatrix} a&a&c\\ b&d&d\end{smallmatrix}\right]$ not fitting in (38):

(b) $\left\langle\begin{smallmatrix} 1&1&2\\ b&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 1&1&2\\ b&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&2\\ 3&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&2\\ 3&b&b\end{smallmatrix}\right]$, for $b \geq 4$.

(iii) Case $a=2$, $b=1$ and $c \geq 4$ with $\left[\begin{smallmatrix} a&a&c\\ b&d&d\end{smallmatrix}\right]$ not fitting in (39):

(c) $\left\langle\begin{smallmatrix} 2&2&c\\ 1&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 2&2&c\\ 1&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 2&2&c\\ 1&3&3\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&2\\ 3&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&3&3\end{smallmatrix}\right]$, for $d \geq 4$.

(iv) Case $a=2$, $b=1$ and $c=3$ with $\left[\begin{smallmatrix} a&a&c\\ b&d&d\end{smallmatrix}\right]$ not fitting in (40):

(d) $\left\langle\begin{smallmatrix} 2&2&3\\ 1&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 2&2&3\\ 1&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 2&2&3\\ 1&4&4\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&2\\ 3&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&2\\ 3&4&4\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&3\\ 2&4&4\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&3\\ 2&d&d\end{smallmatrix}\right]$, for $d \geq 5$.

(v) Case $a=2$ and $b \geq 3$:

(e) $\left\langle\begin{smallmatrix} 2&2&c\\ 3&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 2&2&c\\ 3&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&2\\ 3&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&3&3\end{smallmatrix}\right]$,

(f) $\left\langle\begin{smallmatrix} 2&2&c\\ b&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 2&2&c\\ b&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&2\\ 3&b&b\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&2\\ 3&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&b&b\end{smallmatrix}\right]$, for $b \geq 4$.

(vi) Case $a=3$ and $b=1$:

(g) $\left\langle\begin{smallmatrix} 3&3&c\\ 1&2&2\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 3&3&c\\ 1&2&2\end{smallmatrix}\right] + \left[\begin{smallmatrix} 2&2&3\\ 1&4&4\end{smallmatrix}\right] - \left[\begin{smallmatrix} 2&2&c\\ 1&3&3\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&2\\ 3&4&4\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&3\\ 2&4&4\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&3&3\end{smallmatrix}\right]$,

(h) $\left\langle\begin{smallmatrix} 3&3&c\\ 1&4&4\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 3&3&c\\ 1&4&4\end{smallmatrix}\right] + \left[\begin{smallmatrix} 2&2&3\\ 1&4&4\end{smallmatrix}\right] - \left[\begin{smallmatrix} 2&2&c\\ 1&3&3\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&2\\ 3&4&4\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&4&4\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&3&3\end{smallmatrix}\right]$,

(i) $\left\langle\begin{smallmatrix} 3&3&c\\ 1&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 3&3&c\\ 1&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 2&2&3\\ 1&4&4\end{smallmatrix}\right] - \left[\begin{smallmatrix} 2&2&c\\ 1&3&3\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&2\\ 3&4&4\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&3&3\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&3\\ 2&4&4\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&3\\ 2&d&d\end{smallmatrix}\right]$, for $d \geq 5$.

(vii) Case $a=3$ and $b \geq 2$:

(j) $\left\langle\begin{smallmatrix} 3&3&c\\ 2&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 3&3&c\\ 2&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&3\\ 2&d&d\end{smallmatrix}\right]$,

(k) $\left\langle\begin{smallmatrix} 3&3&c\\ b&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} 3&3&c\\ b&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&b&b\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&3\\ 2&b&b\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&3\\ 2&d&d\end{smallmatrix}\right]$, for $b \geq 4$.

(viii) Case $a \geq 4$:

(l) $\left\langle\begin{smallmatrix} a&a&c\\ 1&2&2\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} a&a&c\\ 1&2&2\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&a\\ 2&3&3\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&3&3\end{smallmatrix}\right] - \left[\begin{smallmatrix} 2&2&c\\ 1&3&3\end{smallmatrix}\right] + \left[\begin{smallmatrix} 2&2&a\\ 1&3&3\end{smallmatrix}\right]$,

(m) $\left\langle\begin{smallmatrix} a&a&c\\ 1&3&3\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} a&a&c\\ 1&3&3\end{smallmatrix}\right] - \left[\begin{smallmatrix} 2&2&c\\ 1&3&3\end{smallmatrix}\right] + \left[\begin{smallmatrix} 2&2&a\\ 1&3&3\end{smallmatrix}\right]$,

(n) $\left\langle\begin{smallmatrix} a&a&c\\ 1&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} a&a&c\\ 1&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 2&2&c\\ 1&3&3\end{smallmatrix}\right] + \left[\begin{smallmatrix} 2&2&a\\ 1&3&3\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&3&3\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&a\\ 2&3&3\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&a\\ 2&d&d\end{smallmatrix}\right]$, for $d \geq 4$,

(o) $\left\langle\begin{smallmatrix} a&a&c\\ 2&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} a&a&c\\ 2&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&a\\ 2&d&d\end{smallmatrix}\right]$,

(p) $\left\langle\begin{smallmatrix} a&a&c\\ b&d&d\end{smallmatrix}\right\rangle := \left[\begin{smallmatrix} a&a&c\\ b&d&d\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&c\\ 2&d&d\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&c\\ 2&b&b\end{smallmatrix}\right] - \left[\begin{smallmatrix} 1&1&a\\ 2&b&b\end{smallmatrix}\right] + \left[\begin{smallmatrix} 1&1&a\\ 2&d&d\end{smallmatrix}\right]$, for $b \geq 3$.

Direct inspection shows that

(41) each $\left\langle\begin{smallmatrix} a&a&c\\ b&d&d\end{smallmatrix}\right\rangle - \left[\begin{smallmatrix} a&a&c\\ b&d&d\end{smallmatrix}\right]$ is a linear combination of basis elements in $\mathcal{C}$.

Therefore $\mathcal{B}' \cup \mathcal{C}$ is a new basis of $\mu_2(C_m)$, where $\mathcal{B}'$ stands for the collection of elements $\left\langle\begin{smallmatrix} a&a&c\\ b&d&d\end{smallmatrix}\right\rangle$. Further, routine verifications using (32) and (33) show that $\mathcal{B}'$ lies in the kernel of $\partial\colon \mu_2(C_m) \to \mu_1(C_m)$. In fact:

Lemma 4.10 *$\mathcal{B}'$ is a basis of the kernel of the Morse boundary map $\partial\colon \mu_2(C_m) \to \mu_1(C_m)$.*

Proof The argument is parallel to that in the proof of Corollary 4.7. Namely, by direct counting, the cardinality of $\mathcal{C}$ is $|\mathcal{C}| = m^2 - 5m + 5$. In view of (30), this leads to

$$|\mathcal{B}'| = |\mathcal{B}| = \tfrac{1}{4} m(m-2)(m-3)(m-5) + 1,$$

which is the second Betti number of $\mathrm{Conf}(|K_m|,2)$; see [9, Corollary 23]. Hence $\partial\colon \mu_2(C_m) \to \mu_1(C_m)$ is forced to be injective on the submodule spanned by $\mathcal{C}$ and, in particular, $\mathcal{B}'$ spans (and is thus a basis of) $\ker(\partial\colon \mu_2(C_m) \to \mu_1(C_m))$. $\square$

The two sequences

$$\begin{aligned} 0 \to H_2(\mathrm{Conf}(|K_m|,2)) &\overset{\iota}{\hookrightarrow} \mu_2(C_m) \xrightarrow{\partial} \mu_1(C_m), \\ 0 \leftarrow H^2(\mathrm{Conf}(|K_m|,2)) &\overset{\iota^*}{\twoheadleftarrow} \mu^2(C_m) \xleftarrow{\delta} \mu^1(C_m), \end{aligned} \tag{42}$$

are exact; the first one by definition, and the second one, which is the dual of the first one, because $H_1(\mathrm{Conf}(|K_m|,2))$ is torsion-free. Thus the cohomology class represented by an element $e \in \mu^2(C_m)$ is given by the ι^*–image of e. Such an interpretation of cohomology classes is used in the proof of:

Corollary 4.11 *A basis of $H^2(\mathrm{Conf}(|K_m|,2))$ is given by the classes represented by the duals of the critical faces in $\mathcal{B}$. Furthermore, the expression (as a linear combination of basis elements) of the cohomology class represented by the dual of a critical face in $\mathcal{C}$ is obtained from the following equations, which are congruences modulo the image of $\delta\colon \mu^1(C_m) \to \mu^2(C_m)$:*

(E₁) $\sum_x \left[\begin{smallmatrix} x & x & c \\ 2 & 3 & 3 \end{smallmatrix}\right] - \sum_x \left[\begin{smallmatrix} c & c & x \\ 2 & 3 & 3 \end{smallmatrix}\right] \equiv \sum_{\substack{y \neq 3 \\ z \in \{1,3\}}} \left(\sum_x \left[\begin{smallmatrix} x & x & c \\ z & y & y \end{smallmatrix}\right] - \sum_x \left[\begin{smallmatrix} c & c & x \\ z & y & y \end{smallmatrix}\right]\right)$ *for* $c > 3$.

(E₂) $\sum_x \left[\begin{smallmatrix} x & x & 3 \\ 2 & 4 & 4 \end{smallmatrix}\right] - \sum_x \left[\begin{smallmatrix} 3 & 3 & x \\ 2 & 4 & 4 \end{smallmatrix}\right] \equiv \sum_{\substack{y \neq 4 \\ z \in \{1,4\}}} \left(\sum_x \left[\begin{smallmatrix} x & x & 3 \\ z & y & y \end{smallmatrix}\right] - \sum_x \left[\begin{smallmatrix} 3 & 3 & x \\ z & y & y \end{smallmatrix}\right]\right)$.

(E₃) $\sum_{x,y} \left[\begin{smallmatrix} x & x & c \\ y & d & d \end{smallmatrix}\right] - \sum_{x,y} \left[\begin{smallmatrix} c & c & x \\ y & d & d \end{smallmatrix}\right] \equiv \sum_{x,y} \left[\begin{smallmatrix} x & x & c \\ d & y & y \end{smallmatrix}\right] - \sum_{x,y} \left[\begin{smallmatrix} c & c & x \\ d & y & y \end{smallmatrix}\right]$ *for* $3 \le c \neq d \ge 4$ *with* $(c,d) \neq (3,4)$.

(E₄) $\sum_{x,y} \left[\begin{smallmatrix} x & x & 2 \\ y & 4 & 4 \end{smallmatrix}\right] - \sum_{(x,y)\neq(3,1)} \left[\begin{smallmatrix} 2 & 2 & x \\ y & 4 & 4 \end{smallmatrix}\right] \equiv \left(\sum_{x,y} \left[\begin{smallmatrix} x & x & 2 \\ 4 & y & y \end{smallmatrix}\right] - \sum_{x,y} \left[\begin{smallmatrix} 2 & 2 & x \\ 4 & y & y \end{smallmatrix}\right]\right) - \left(\sum_{\substack{y \ge 4 \\ x}} \left[\begin{smallmatrix} x & x & 3 \\ 1 & y & y \end{smallmatrix}\right] - \sum_{x,y} \left[\begin{smallmatrix} 3 & 3 & x \\ 1 & y & y \end{smallmatrix}\right]\right)$.

(E₅) $\sum_{x,y} \left[\begin{smallmatrix} x & x & 2 \\ y & d & d \end{smallmatrix}\right] - \sum_{x,y} \left[\begin{smallmatrix} 2 & 2 & x \\ y & d & d \end{smallmatrix}\right] \equiv \sum_{x,y} \left[\begin{smallmatrix} x & x & 2 \\ d & y & y \end{smallmatrix}\right] - \sum_{x,y} \left[\begin{smallmatrix} 2 & 2 & x \\ d & y & y \end{smallmatrix}\right]$ *for* $d > 4$.

(E₆) $\sum_{x,y} \left[\begin{smallmatrix} x & x & c \\ 1 & y & y \end{smallmatrix}\right] \equiv \sum_{x,y} \left[\begin{smallmatrix} c & c & x \\ 1 & y & y \end{smallmatrix}\right]$ *for* $c > 2$.

Note that (E₅) says that the congruence in (E₃) also holds for $c = 2$ provided $d > 4$. Likewise, (E₁) and (E₂) can be stated simultaneously as

$$\sum_x \begin{bmatrix} x & x & c \\ 2 & q & q \end{bmatrix} - \sum_x \begin{bmatrix} c & c & x \\ 2 & q & q \end{bmatrix} \equiv \sum_{\substack{y \neq q \\ z \in \{1,q\}}} \left(\sum_x \begin{bmatrix} x & x & c \\ z & y & y \end{bmatrix} - \sum_x \begin{bmatrix} c & c & x \\ z & y & y \end{bmatrix} \right),$$

with $c > 3 = q$ or $c = 3 = q - 1$. We have chosen the structure stated in (E₁)–(E₆) for proof-organization purposes; see Figure 13.

Proof The first assertion follows from (41) and (42). For the second assertion, start by noting that the listed congruences are obtained by dualizing the 16 formulae (in Definition 4.9) that describe the inclusion ι. Indeed, the validity of the congruences is obtained by a straightforward verification (left as an exercise for the reader) of the fact that both sides of each congruence evaluate the same at each basis element $\left\langle\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right\rangle$. Furthermore, direct inspection shows that, in each equation ($\mathrm{E_i}$), there is a single

i	1	2	3	4	5	6	6
s_i	$\left[\begin{smallmatrix}1&1&c\\2&3&3\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1&3\\2&4&4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1&c\\2&d&d\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1&2\\3&4&4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1&2\\3&d&d\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2&2&c\\1&3&3\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2&2&3\\1&4&4\end{smallmatrix}\right]$
type	(37)	(37)	(37)	(38)	(38)	(39)	(40)
restrictions	$c>3$		$3\le c\ne d\ge 4$ $(c,d)\ne(3,4)$		$d>4$	$c>3$	

Figure 13: Elements coming from $\mathcal{C}$ in the congruences ($\mathrm{E_i}$) of Corollary 4.11.

summand s_i (spelled out in Figure 13) that fails to come from $\mathcal{B}$. Therefore ($\mathrm{E_i}$) can be thought of as expressing the cohomology class represented by s_i as a $\mathbb{Z}$–linear combination of basis elements. The second assertion of the corollary then follows by observing, from Figure 13, that each element in $\mathcal{C}$ arises as one, and only one, of the special summands s_i. $\square$

4.4 Cohomology ring

In previous sections we have described explicit cocycles in $\mu^*(C_m)$ representing basis elements in cohomology. We now make use of (9) to assess the corresponding cup products at both the critical cochain and homology levels. Since cup products in $C^*(C_m)$ are elementary (see Remark 4.12), the bulk of the work amounts to giving a (suitable) description of the cochain maps $\overline{\Phi}\colon \mu^*(C_m)\to C^*(C_m)$ and $\underline{\Phi}\colon C^*(C_m)\to\mu^*(C_m)$.

Remark 4.12 Recall that basis elements in $C^1(C_m)$ are given by the dualized 1–dimensional faces $\left[\begin{smallmatrix}a&c\\b&d\end{smallmatrix}\right]$. (As in earlier parts of the paper, upper stars for dualized elements are omitted, and arithmetic restrictions among the numbers assembling critical faces are usually not written down.) From the usual formula for cup products in the simplicial setting, we see that the only nontrivial products in $C^*(C_m)$ have the form

$$\begin{bmatrix}a&a\\b&d\end{bmatrix}\smile\begin{bmatrix}a&c\\d&d\end{bmatrix}=\begin{bmatrix}a&a&c\\b&d&d\end{bmatrix}\quad\text{or}\quad\begin{bmatrix}a&c\\b&b\end{bmatrix}\smile\begin{bmatrix}c&c\\b&d\end{bmatrix}=\begin{bmatrix}a&c&c\\b&b&d\end{bmatrix}. \tag{43}$$

(So every 2–face is uniquely a product of two 1–faces.) In particular, for the purposes of applying (9), all basis elements $\left[\begin{smallmatrix}a&c\\b&d\end{smallmatrix}\right]$ with $a<c$ and $b<d$ can be ignored in the expression for $\overline{\Phi}$.

Proposition 4.13 *The values of the cochain map $\overline{\Phi}\colon\mu^1(C_m)\to C^1(C_m)$ on the basis elements* (k.1)–(k.3) *of Proposition 4.1 satisfy the following family of congruences taken modulo basis elements $\left[\begin{smallmatrix}a&c\\b&d\end{smallmatrix}\right]$ with $a<c$ and $b<d$:*

(i) $\overline{\Phi}\left(\left[\begin{smallmatrix}a&m-1\\b&m\end{smallmatrix}\right]\right)\equiv\left[\begin{smallmatrix}a&a\\b&m\end{smallmatrix}\right]+\sum\left[\begin{smallmatrix}a&x\\b&b\end{smallmatrix}\right]-\sum\left[\begin{smallmatrix}y&a\\b&b\end{smallmatrix}\right]$, *where the first summation runs over $x\in\{1,\dots,m-1\}$ with $a<x\neq b$, and the second summation runs over $y\in\{1,\dots,m-1\}$ with $b\neq y<a$.*

(ii) $\overline{\Phi}\left(\left[\begin{smallmatrix}m&m\\b&d\end{smallmatrix}\right]\right)\equiv\sum\left[\begin{smallmatrix}x&x\\b&d\end{smallmatrix}\right]$, *where the summation runs over $x\in\{1,\dots,m\}$ with $b\neq x\neq d$.*

(iii) $\overline{\Phi}\left(\left[\begin{smallmatrix}a&c\\m&m\end{smallmatrix}\right]\right)\equiv\sum\left[\begin{smallmatrix}a&c\\y&y\end{smallmatrix}\right]$, *where the summation runs over $y\in\{1,\dots,m\}$ with $a\neq y\neq c$.*

$$
\overset{(b<m-2)}{\left[\begin{smallmatrix} m-1 & m-1 \\ b & m \end{smallmatrix}\right]} \swarrow \overset{(b\neq y<m-1)}{\left[\begin{smallmatrix} y & m-1 & m-1 \\ b & b & m \end{smallmatrix}\right]} \overset{(c)}{\nwarrow} \left[\begin{smallmatrix} y & m-1 \\ b & b \end{smallmatrix}\right] \left\{ \begin{array}{l} \swarrow \overset{(b<f<m-1)}{\left[\begin{smallmatrix} y & m-1 & m-1 \\ b & b & f \end{smallmatrix}\right]} \overset{(e)}{\nwarrow} \left[\begin{smallmatrix} y & m-1 \\ b & f \end{smallmatrix}\right] \swarrow \overset{(y\neq f)}{\left[\begin{smallmatrix} y & y & m-1 \\ b & f & f \end{smallmatrix}\right]} \quad \text{(critical)} \\ \swarrow \overset{(y\neq z<b)}{\left[\begin{smallmatrix} y & y & m-1 \\ z & b & b \end{smallmatrix}\right]} \quad \text{(critical)} \end{array} \right.
$$

Figure 14: Gradient paths landing on a critical cell $\left[\begin{smallmatrix} m-1 & m-1 \\ b & m \end{smallmatrix}\right]$ of type (k.1) with $b<m-2$.

Proof The congruences follow from (8) and from direct inspection of Figures 14–17, where we spell out the complete trees of gradient paths landing on critical 1–dimensional faces. Here we follow the notational conventions used in Figures 3–11, except that we now keep track of relevant numerical restrictions and, at the start of each path, we indicate the reason that prevents the path from pulling back one further step. $\square$

Proposition 4.14 *For critical* 1*–faces* x *and* y, *the product* $\overline{\Phi}(x)\smile\overline{\Phi}(y)$ *appearing in* (9) *is a linear combination* $\sum \pm z$ *of dualized* 2*–faces* z, *each of which has one of the following forms*:

(i) $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ *for* $a<c<m>d>b$ *and* $b\neq a\neq d\neq c$, *with trivial* $\underline{\Phi}$*–image unless* $b\neq c$, *in which case*

$$
\underline{\Phi}\left(\begin{bmatrix} a & a & c \\ b & d & d \end{bmatrix}\right) = \begin{bmatrix} a & a & c \\ b & d & d \end{bmatrix},
$$

(ii) $\left[\begin{smallmatrix} a & c & c \\ b & b & d \end{smallmatrix}\right]$ *for* $a<c\leq m-1>d>b$ *and* $a\neq b\neq c\neq d$, *with trivial* $\underline{\Phi}$*–image unless* $a\neq d$, *in which case*

$$
\underline{\Phi}\left(\begin{bmatrix} a & c & c \\ b & b & d \end{bmatrix}\right) = -\begin{bmatrix} a & a & c \\ b & d & d \end{bmatrix},
$$

$$
\left\{ \begin{array}{l}
\swarrow \left[\begin{smallmatrix} a & a & m-1 \\ b & m & m \end{smallmatrix}\right] \overset{(b)}{\nwarrow} \left[\begin{smallmatrix} a & a \\ b & m \end{smallmatrix}\right] \left\{ \begin{array}{l}
\swarrow \overset{(b\neq y<a)}{\left[\begin{smallmatrix} y & a & a \\ b & b & m \end{smallmatrix}\right]} \overset{(c)}{\nwarrow} \left[\begin{smallmatrix} y & a \\ b & b \end{smallmatrix}\right] \left\{ \begin{array}{l}
\swarrow \overset{(y\neq w<b,\, w\neq a)}{\left[\begin{smallmatrix} y & y & a \\ w & b & b \end{smallmatrix}\right]} \quad \text{(critical)} \\
\swarrow \overset{(a<b)}{\left[\begin{smallmatrix} y & y & a \\ a & b & b \end{smallmatrix}\right]} \overset{(f)}{\nwarrow} \left[\begin{smallmatrix} y & a \\ a & b \end{smallmatrix}\right] \quad \text{(free)} \\
\swarrow \overset{(b<z<m,\, z\neq a)}{\left[\begin{smallmatrix} y & a & a \\ b & b & z \end{smallmatrix}\right]} \overset{(e)}{\nwarrow} \left[\begin{smallmatrix} y & a \\ b & z \end{smallmatrix}\right] \swarrow \overset{(y\neq z)}{\left[\begin{smallmatrix} y & y & a \\ b & z & z \end{smallmatrix}\right]} \quad \text{(critical)}
\end{array} \right. \\
\swarrow \overset{(a<x<m-1)}{\left[\begin{smallmatrix} a & a & x \\ b & m & m \end{smallmatrix}\right]} \overset{(f)}{\nwarrow} \left[\begin{smallmatrix} a & x \\ b & m \end{smallmatrix}\right] \swarrow \overset{(x\neq b)}{\left[\begin{smallmatrix} a & x & x \\ b & b & m \end{smallmatrix}\right]} \overset{(c)}{\nwarrow} \left[\begin{smallmatrix} a & x \\ b & b \end{smallmatrix}\right] \left\{ \begin{array}{l}
\swarrow \overset{(b<r<m,\, r\neq x)}{\left[\begin{smallmatrix} a & x & x \\ b & b & r \end{smallmatrix}\right]} \overset{(e)}{\nwarrow} \left[\begin{smallmatrix} a & x \\ b & r \end{smallmatrix}\right] \swarrow \overset{(a\neq r)}{\left[\begin{smallmatrix} a & a & x \\ b & r & r \end{smallmatrix}\right]} \quad \text{(critical)} \\
\swarrow \overset{(a\neq s<b,\, s\neq x)}{\left[\begin{smallmatrix} a & a & x \\ s & b & b \end{smallmatrix}\right]} \quad \text{(critical)} \\
\swarrow \overset{(x<b)}{\left[\begin{smallmatrix} a & a & x \\ x & b & b \end{smallmatrix}\right]} \overset{(f)}{\nwarrow} \left[\begin{smallmatrix} a & x \\ x & b \end{smallmatrix}\right] \quad \text{(free)}
\end{array} \right.
\end{array} \right. \\
\swarrow \overset{(b<m-1)}{\left[\begin{smallmatrix} a & m-1 & m-1 \\ b & b & m \end{smallmatrix}\right]} \overset{(c)}{\nwarrow} \left[\begin{smallmatrix} a & m-1 \\ b & b \end{smallmatrix}\right] \left\{ \begin{array}{l}
\swarrow \overset{(b<x<m-1)}{\left[\begin{smallmatrix} a & m-1 & m-1 \\ b & b & x \end{smallmatrix}\right]} \overset{(e)}{\nwarrow} \left[\begin{smallmatrix} a & m-1 \\ b & x \end{smallmatrix}\right] \swarrow \overset{(x\neq a)}{\left[\begin{smallmatrix} a & a & m-1 \\ b & x & x \end{smallmatrix}\right]} \quad \text{(critical)} \\
\swarrow \overset{(a\neq y<b)}{\left[\begin{smallmatrix} a & a & m-1 \\ y & b & b \end{smallmatrix}\right]} \quad \text{(critical)}
\end{array} \right.
\end{array} \right.
$$

Figure 15: Gradient paths landing on a critical cell $\left[\begin{smallmatrix} a & m-1 \\ b & m \end{smallmatrix}\right]$ of type (k.1) with $a<m-1\geq b$.

$$\overset{(d<m-1)}{\begin{bmatrix} m & m \\ b & d \end{bmatrix}} \swarrow \overset{(b\neq x<m)}{\begin{bmatrix} x & m & m \\ b & b & d \end{bmatrix}} \overset{\text{(e)}}{\nwarrow} \begin{bmatrix} x & m \\ b & d \end{bmatrix} \swarrow \overset{(x\neq d)}{\begin{bmatrix} x & x & m \\ b & d & d \end{bmatrix}} \overset{\text{(a)}}{\nwarrow} \begin{bmatrix} x & x \\ b & d \end{bmatrix} \begin{cases} \swarrow \overset{(x<y<m,\, b\neq y\neq d)}{\begin{bmatrix} x & x & y \\ b & d & d \end{bmatrix}} & \text{(critical)} \\ \swarrow \overset{(x<b)}{\begin{bmatrix} x & x & b \\ b & d & d \end{bmatrix}} \overset{\text{(f)}}{\nwarrow} \begin{bmatrix} x & b \\ b & d \end{bmatrix} & \text{(free)} \\ \swarrow \overset{(b\neq z<x)}{\begin{bmatrix} z & x & x \\ b & b & d \end{bmatrix}} \overset{\text{(e)}}{\nwarrow} \begin{bmatrix} z & x \\ b & d \end{bmatrix} \swarrow \overset{(d\neq z)}{\begin{bmatrix} z & z & x \\ b & d & d \end{bmatrix}} & \text{(critical)} \end{cases}$$

Figure 16: Gradient paths landing on a critical cell $\begin{bmatrix} m & m \\ b & d \end{bmatrix}$ of type (k.2).

(iii) $\begin{bmatrix} a & a & c \\ b & m & m \end{bmatrix}$ *for* $a<c<m-1\geq b$ *and* $b\neq a\neq m\neq c$, *with trivial* $\underline{\Phi}$*–image unless* $b\neq c$, *in which case*

$$\underline{\Phi}\left(\begin{bmatrix} a & a & c \\ b & m & m \end{bmatrix}\right) = \sum_{\substack{y<b \\ a\neq y\neq c}} \begin{bmatrix} a & a & c \\ y & b & b \end{bmatrix} - \sum_{\substack{b<x<m \\ a\neq x\neq c}} \begin{bmatrix} a & a & c \\ b & x & x \end{bmatrix},$$

(iv) $\begin{bmatrix} a & c & c \\ b & b & m \end{bmatrix}$ *for* $a<c\leq m-1\geq b$, $a\neq b\neq c$ *and either* $c<m-1$ *or* $c=m-1>b+1$, *with* $\underline{\Phi}$*–image*

$$\underline{\Phi}\left(\begin{bmatrix} a & c & c \\ b & b & m \end{bmatrix}\right) = \sum_{\substack{b<x<m \\ a\neq x\neq c}} \begin{bmatrix} a & a & c \\ b & x & x \end{bmatrix} - \sum_{\substack{y<b \\ a\neq y\neq c}} \begin{bmatrix} a & a & c \\ y & b & b \end{bmatrix}.$$

Proof By (43), the only 1–faces in the expression of $\overline{\Phi}(\delta)$ that can lead to a summand $\pm\begin{bmatrix} r & r & t \\ s & u & u \end{bmatrix}$ in the product $\overline{\Phi}(\gamma)\smile\overline{\Phi}(\delta)$ have the form $\pm\begin{bmatrix} r & t \\ u & u \end{bmatrix}$. From the expressions of $\overline{\Phi}$ in Proposition 4.13, this can hold only with $t<m$ and in fact $t<m-1$ whenever $u=m$, in view of the form of the basis elements of type (k.3). So $\begin{bmatrix} r & r & t \\ s & u & u \end{bmatrix}$ fits either (i) or (iii). Likewise, the only 1–faces in the expression of $\overline{\Phi}(\delta)$ that can lead to a summand $\pm\begin{bmatrix} r & t & t \\ s & s & u \end{bmatrix}$ in $\overline{\Phi}(\gamma)\smile\overline{\Phi}(\delta)$ have the form $\pm\begin{bmatrix} t & t \\ s & u \end{bmatrix}$, which can hold only under one of the following conditions:

- $u=m$ and $t\leq m-1$, as well as $s<m-2$ if $t=m-1$ (recall the form of basis elements of type (k.1)).
- $u<m-1$ (recall the form of basis elements of type (k.2)).

In the former possibility, $\begin{bmatrix} r & t & t \\ s & s & u \end{bmatrix}$ fits (iv). In the latter, $\begin{bmatrix} r & t & t \\ s & s & u \end{bmatrix}$ fits (ii) unless $t=m$, in which case

(44) the expression of $\overline{\Phi}(\gamma)$ should include a summand of the form $\pm\begin{bmatrix} r & m \\ s & s \end{bmatrix}$.

$$\overset{(c<m-1)}{\begin{bmatrix} a & c \\ m & m \end{bmatrix}} \swarrow \overset{(a\neq y<m)}{\begin{bmatrix} a & a & c \\ y & m & m \end{bmatrix}} \overset{\text{(f)}}{\nwarrow} \begin{bmatrix} a & c \\ y & m \end{bmatrix} \swarrow \overset{(c\neq y)}{\begin{bmatrix} a & c & c \\ y & y & m \end{bmatrix}} \overset{\text{(c)}}{\nwarrow} \begin{bmatrix} a & c \\ y & y \end{bmatrix} \begin{cases} \swarrow \overset{(y<z<m,\, z\neq c)}{\begin{bmatrix} a & c & c \\ y & y & z \end{bmatrix}} \overset{\text{(e)}}{\nwarrow} \begin{bmatrix} a & c \\ y & z \end{bmatrix} \swarrow \overset{(a\neq z)}{\begin{bmatrix} a & a & c \\ y & z & z \end{bmatrix}} & \text{(critical)} \\ \swarrow \overset{(a\neq z<y,\, z\neq c)}{\begin{bmatrix} a & a & c \\ z & y & y \end{bmatrix}} & \text{(critical)} \\ \swarrow \overset{(c<y)}{\begin{bmatrix} a & a & c \\ c & y & y \end{bmatrix}} \overset{\text{(f)}}{\nwarrow} \begin{bmatrix} a & c \\ c & y \end{bmatrix} & \text{(free)} \end{cases}$$

Figure 17: Gradient paths landing on a critical cell $\begin{bmatrix} a & c \\ m & m \end{bmatrix}$ of type (k.3).

	λ_{41}	λ_{42}	λ_{14}	λ_{24}	λ_{34}	λ_{32}	δ_{12}	δ_{13}	δ_{23}	υ_{12}	υ_{13}	υ_{23}
λ_{41}												$-g$
λ_{42}							g	$-g$	g	g		g
λ_{14}									$-g$			
λ_{24}								g				
λ_{34}								$-g$	g	g	$-g$	g
λ_{32}							$-g$	g	$-g$	$-g$	g	$-g$
δ_{12}		$-g$				g						
δ_{13}		g		$-g$	g	$-g$						
δ_{23}		$-g$	g		$-g$	g						
υ_{12}		$-g$			$-g$	g						
υ_{13}					g	$-g$						
υ_{23}	g	$-g$			$-g$	g						

Table 1: Here g stands for the generator of $H^2(\mathrm{Conf}(|K_m|,2))$, zeros are not shown, and brackets for cohomology classes are omitted.

But inspection of the expressions of $\overline{\Phi}$ in Proposition 4.13 rules out (44). Lastly, the four asserted expressions for the cochain map $\underline{\Phi}$ follow from (8) and the analysis of gradient paths in Figures 3–11. □

Example 4.15 For $m=5$, Corollaries 4.7 and 4.11 render the following list of cocycles representing a graded basis for $H^*(\mathrm{Conf}(|K_m|,2))$. In dimension 2 there is the single cocycle $\left[\begin{smallmatrix}3&3&4\\1&2&2\end{smallmatrix}\right]$, while in dimension 1 there are the twelve cocycles:

$$\delta_{12} := \left[\begin{smallmatrix}5&5\\1&2\end{smallmatrix}\right],\quad \upsilon_{12} := \left[\begin{smallmatrix}1&2\\5&5\end{smallmatrix}\right],\quad \delta_{13} := \left[\begin{smallmatrix}5&5\\1&3\end{smallmatrix}\right],\quad \upsilon_{13} := \left[\begin{smallmatrix}1&3\\5&5\end{smallmatrix}\right],\quad \delta_{23} := \left[\begin{smallmatrix}5&5\\2&3\end{smallmatrix}\right],\quad \upsilon_{23} := \left[\begin{smallmatrix}2&3\\5&5\end{smallmatrix}\right],$$

$$\lambda_{14} := \left[\begin{smallmatrix}1&4\\4&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}1&4\\3&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}1&4\\2&5\end{smallmatrix}\right],\qquad \lambda_{24} := \left[\begin{smallmatrix}2&4\\4&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}2&4\\1&5\end{smallmatrix}\right],$$

$$\lambda_{41} := \left[\begin{smallmatrix}4&4\\1&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}3&4\\1&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}2&4\\1&5\end{smallmatrix}\right],\qquad \lambda_{42} := \left[\begin{smallmatrix}4&4\\2&5\end{smallmatrix}\right]-\left[\begin{smallmatrix}3&4\\1&5\end{smallmatrix}\right]-\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]-\left[\begin{smallmatrix}2&4\\1&5\end{smallmatrix}\right]-\left[\begin{smallmatrix}1&4\\3&5\end{smallmatrix}\right],$$

$$\lambda_{32} := \left[\begin{smallmatrix}3&4\\2&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}3&4\\1&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}2&4\\1&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}1&4\\3&5\end{smallmatrix}\right]+\left[\begin{smallmatrix}1&4\\2&5\end{smallmatrix}\right],\quad \lambda_{34} := \left[\begin{smallmatrix}3&4\\4&5\end{smallmatrix}\right]-\left[\begin{smallmatrix}2&4\\3&5\end{smallmatrix}\right]-\left[\begin{smallmatrix}2&4\\1&5\end{smallmatrix}\right]-\left[\begin{smallmatrix}1&4\\3&5\end{smallmatrix}\right]-\left[\begin{smallmatrix}1&4\\2&5\end{smallmatrix}\right].$$

Then the complete algebra structure of $H^*(\mathrm{Conf}(|K_m|,2))$ is spelled out by the matrix of cup products in Table 1. In particular, replacing λ_{42} by $\lambda'_{42} := \lambda_{42}+\lambda_{41}+\lambda_{34}+\lambda_{32}+\lambda_{24}+\lambda_{14}$, λ_{34} by $\lambda'_{34} := \lambda_{34}+\lambda_{32}$ and λ_{32} by $\lambda'_{32} := \lambda_{32}+\lambda_{42}$, we get a cohomology basis whose only (up to anticommutativity) nontrivial products are

$$\lambda'_{42} \smile \upsilon_{12} = \lambda_{24} \smile \delta_{13} = \lambda'_{32} \smile \upsilon_{13} = g \quad \text{and} \quad \lambda_{14} \smile \delta_{23} = \lambda_{41} \smile \upsilon_{23} = \lambda'_{34} \smile \delta_{12} = -g. \tag{45}$$

The cohomology ring $H^*(\mathrm{Conf}(|K_m,2|))$ becomes richer as m increases (with $\mathrm{Conf}(|K_m|,2)$ no longer being a homotopy closed surface). Yet, some aspects of the particularly simple structure in (45) are kept for all $m>5$. Explicitly, let $\upsilon_{a,c}$, $\delta_{b,d}$ and $\lambda_{e,f}$ stand for the basis elements of $H^1(\mathrm{Conf}(|K_m,2|))$ represented, respectively, by the 1–cocycles $\left[\begin{smallmatrix}a&c\\m&m\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}m&m\\b&d\end{smallmatrix}\right]$ and $\left\langle\begin{smallmatrix}e\\f\end{smallmatrix}\right\rangle$ described in Corollary 4.7. Then:

$\delta_{b,d}\cdot\upsilon_{a,c}$, $a\in\{b,d\}$	$\upsilon_{a,c}\cdot\delta_{b,d}$, $c\in\{b,d\}$
$\lambda_{m-1,f_1}\cdot\lambda_{m-1,f_2}$, $1\le f_i\le m-4$ $\lambda_{m-1,f}\cdot\lambda_{m-1,m-3}$, $1\le f\le m-4$ $\lambda_{m-1,f}\cdot\lambda_{m-2,m-3}$, $1\le f\le m-4$ $\lambda_{m-1,f}\cdot\lambda_{m-2,m-1}$, $1\le f\le m-4$ $\lambda_{m-1,m-3}\cdot\lambda_{m-2,m-3}$ $\lambda_{m-1,m-3}\cdot\lambda_{m-2,m-1}$ $\lambda_{m-2,m-3}\cdot\lambda_{m-2,m-1}$	$\lambda_{m-1,f}\cdot\lambda_{e,m-1}$, $1\le f\le m-4$ and $1\le e\le m-3$ $\lambda_{m-1,m-3}\cdot\lambda_{e,m-1}$, $1\le e\le m-3$ $\lambda_{e_1,m-1}\cdot\lambda_{e_2,m-1}$, $1\le e_i\le m-3$ with $e_1>e_2$ $\lambda_{m-2,m-3}\cdot\lambda_{e,m-1}$, $1\le e\le m-3$ $\lambda_{m-2,m-1}\cdot\lambda_{e,m-1}$, $1\le e\le m-3$

Table 2

Corollary 4.16 *Any cup product of the form $\delta_{b_1,d_1}\cdot\delta_{b_2,d_2}$, $\upsilon_{a_1,c_1}\cdot\upsilon_{a_2,c_2}$ or $\lambda_{e_1,f_1}\cdot\lambda_{e_2,f_2}$ vanishes. On the other hand, a cup product $\delta_{b,d}\cdot\upsilon_{a,c}$ is nonzero if and only if $\{a,b\}\cap\{c,d\}=\varnothing$, in which case $\delta_{b,d}\cdot\upsilon_{a,c}$ is represented by $\left[\begin{smallmatrix} a & a & c\\ b & d & d\end{smallmatrix}\right]$.*

Proof This is a straightforward calculation using Proposition 4.13. We only indicate the two main checking steps for the reader's benefit. In what follows we assume $m\ge 6$. First, Proposition 4.13(i) is used to check that, modulo 1–faces not taking part on nonzero products (43), $\overline{\Phi}\left\langle\begin{smallmatrix} e\\ f\end{smallmatrix}\right\rangle$ is congruent to

- $\sum_{i<m}\left[\begin{smallmatrix} i & i\\ f & m\end{smallmatrix}\right]$ for $e=m-1$ and $1\le f\le m-4$,
- $\sum_{j<m}\left(\left[\begin{smallmatrix} e & e\\ j & m\end{smallmatrix}\right]+\sum_{x<m}\left[\begin{smallmatrix} e & x\\ j & j\end{smallmatrix}\right]-\sum_{y<m}\left[\begin{smallmatrix} y & e\\ j & j\end{smallmatrix}\right]\right)$ for $1\le e\le m-3$ and $f=m-1$,
- $\left[\begin{smallmatrix} m-1 & m-1\\ m-3 & m\end{smallmatrix}\right]-\sum_{\substack{i<m-1>j\\ j\ne m-3}}\left[\begin{smallmatrix} i & i\\ j & m\end{smallmatrix}\right]-\sum_{i<m-1>j}\left[\begin{smallmatrix} i & m-1\\ j & j\end{smallmatrix}\right]$ for $(e,f)=(m-1,m-3)$,
- $\left[\begin{smallmatrix} m-2 & m-2\\ m-1 & m\end{smallmatrix}\right]+\left[\begin{smallmatrix} m-2 & m-1\\ m-1 & m-1\end{smallmatrix}\right]-\sum_y\left[\begin{smallmatrix} y & m-2\\ m-1 & m-1\end{smallmatrix}\right]-\sum_{\substack{i<m-1>j\\ i\ne m-2}}\left(\left[\begin{smallmatrix} i & i\\ j & m\end{smallmatrix}\right]+\sum_{\substack{x=m-\varepsilon\\ \varepsilon\in\{1,2\}}}\left[\begin{smallmatrix} i & x\\ j & j\end{smallmatrix}\right]\right)$ for $(e,f)=(m-2,m-1)$,
- $\sum_{i<m-1>j}\left(\left[\begin{smallmatrix} i & i\\ j & m\end{smallmatrix}\right]+\left[\begin{smallmatrix} i & m-1\\ j & j\end{smallmatrix}\right]\right)$ for $(e,f)=(m-2,m-3)$.

The above congruences together with those in Proposition 4.13(ii) and (iii) are then used to check that each of the products asserted to vanish do so because there is no room for nonzero products (43) in the corresponding portion $\overline{\Phi}(x)\smile\overline{\Phi}(y)$ of (9). Such an assertion is easily seen for products $\delta_{b_1,d_1}\cdot\delta_{b_2,d_2}$ and $\upsilon_{a_1,c_1}\cdot\upsilon_{a_2,c_2}$, but the explicit details are not so direct for $\delta_{b,d}\cdot\upsilon_{a,c}$ and $\lambda_{e_1,f_1}\cdot\lambda_{e_2,f_2}$. In fact, in the latter two cases, a convenient order of factors needs to be chosen in order to ensure the vanishing of the corresponding $\overline{\Phi}(x)\smile\overline{\Phi}(y)$; see Table 2. The order chosen is immaterial for the trivial-product conclusion, as cohomology cup products are anticommutative. Keep in mind that $H^*(\mathrm{Conf}(|K_m|,2))$ is torsion-free, so cup squares of 1–dimensional classes are trivial for free.

Lastly, the fact that $\delta_{b,d}\cdot\upsilon_{a,c}$ is represented by $\left[\begin{smallmatrix} a & a & c\\ b & d & d\end{smallmatrix}\right]$ when $\{a,b\}\cap\{c,d\}=\varnothing$ follows by noticing that $\underline{\Phi}\left(\overline{\Phi}\left[\begin{smallmatrix} m & m\\ b & d\end{smallmatrix}\right]\smile\overline{\Phi}\left[\begin{smallmatrix} a & c\\ m & m\end{smallmatrix}\right]\right)=\left[\begin{smallmatrix} a & a & c\\ b & d & d\end{smallmatrix}\right]$. Here $\left[\begin{smallmatrix} a & a & c\\ b & d & d\end{smallmatrix}\right]$ fails to represent one of our basis elements when

$(a,b)=(1,2)$, $(a,b,c)=(1,3,2)$, $(a,b,d)=(2,1,3)$ or $(a,b,c,d)=(2,1,3,4)$; recall (37)–(40). In each such case, one of Corollary 4.11(E_1)–(E_6) applies to write (the cohomology class of) $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ in terms of basis elements. Either way, inspection of (E_1)–(E_6) shows that $\left[\begin{smallmatrix} a & a & c \\ b & d & d \end{smallmatrix}\right]$ represents a nonzero cohomology class. □

5 Topological complexity

Fix a positive integer $s \geq 2$ and a path-connected space X. The s^{th} topological complexity $\mathrm{TC}_s(X)$ of X is the sectional category of the evaluation map $e_s\colon PX \to X^s$ which sends a (free) path on X, $\gamma \in PX$, to

$$e_s(\gamma)=\left(\gamma\left(\frac{0}{s-1}\right),\gamma\left(\frac{1}{s-1}\right),\ldots,\gamma\left(\frac{s-1}{s-1}\right)\right).$$

The term "sectional category" is used in the reduced sense, so $\mathrm{TC}_s(X)+1$ stands for the smallest number of open sets covering X^s on each of which e_s admits a section. For instance, the (reduced) Lusternik–Schnirelmann category $\mathrm{cat}(X)$ of X is the sectional category of the evaluation map $e_1\colon P_0X \to X$ sending a based path $\gamma \in P_0X$ (ie $\gamma(0)=\star$ for a fixed base point $\star \in X$) to $e_1(\gamma)=\gamma(1)$.

Proposition 5.1 [5, Theorem 3.9] *For a c–connected space X having the homotopy type of a CW complex,*

$$\mathrm{cl}(X) \leq \mathrm{cat}(X) \leq \mathrm{hdim}(X)/(c+1) \quad \textit{and} \quad \mathrm{zcl}_s(X) \leq \mathrm{TC}_s(X) \leq s\,\mathrm{cat}(X).$$

Here $\mathrm{hdim}(X)$ denotes the minimal dimension of cell complexes homotopy equivalent to X, while $\mathrm{cl}(X)$ and $\mathrm{zcl}_s(X)$ stand, respectively, for the cup length of X and the s^{th} zero-divisor cup length of X. Explicitly, $\mathrm{cl}(X)$ is the largest integer $l \geq 0$ such that there are classes[6] $c_j \in \widetilde{H}^*(X)$ for $1 \leq j \leq l$ with nonzero cup product. Likewise, $\mathrm{zcl}_s(X)$ is the largest integer $l \geq 0$ such that there are classes $z_j \in H^*(X^s)$ for $1 \leq j \leq l$ ("zero divisors") with nonzero cup product and such that each factor restricts trivially under the diagonal $X \hookrightarrow X^s$.

Let Γ be a 1–dimensional cell complex — a graph. While the fundamental group of $\mathrm{Conf}(\Gamma,n)$ is a central character in geometric group theory, the topological complexity of $\mathrm{Conf}(\Gamma,n)$ becomes relevant for the task of planning collision-free motion of n autonomous distinguishable agents moving on a Γ–shaped system of tracks. It is known that $\mathrm{hdim}(\mathrm{Conf}(\Gamma,n))$ is bounded from above by $m=m(\Gamma)$, the number of essential vertices of Γ; see for instance [12, Theorem 4.4]. Thus Proposition 5.1 yields

$$\mathrm{TC}_s(\mathrm{Conf}(\Gamma,n)) \leq sm. \tag{46}$$

For $s=2$, Farber proved in [8] that (46) is an equality when Γ is a tree and $n \geq 2m$, with the single (and well known) exception of $(n,m)=(2,1)$ where the (unique) essential vertex of Γ has valency 3 — which we call the "Y_2–exception". Farber also conjectured that the tree restriction is superfluous in obtaining equality in (46). The conjecture has recently been confirmed in [23] by Knudsen, who proved

[6] For the purposes of this section, cohomology will be taken with mod 2 coefficients.

equality in (46) for any $s \geq 2$ and any graph Γ, as long as the "stable" restriction $n \geq 2m$ is kept (and the Y_2–exception is avoided). Note that the stable condition forces $\operatorname{hdim}(\operatorname{Conf}(\Gamma, n)) = m$. More generally, it would be interesting to characterize the triples (s, Γ, n) for which the (in principle) improved bound

$$\mathrm{TC}_s(\operatorname{Conf}(\Gamma, n)) \leq s \operatorname{hdim}(\operatorname{Conf}(\Gamma, n)) \tag{47}$$

holds as an equality, preferably determining the value of $\operatorname{hdim}(\operatorname{Conf}(\Gamma, n))$. For instance, it is known from [2, Section 5] that, for any s and n (possibly with $n < 2m$),

$$\operatorname{hdim}(\operatorname{Conf}(\Gamma, n)) = \operatorname{cat}(\operatorname{Conf}(\Gamma, n)) = \min\{\lfloor n/2 \rfloor, m\} \tag{48}$$

when Γ is a tree, in which case (47) is an equality — the Y_2–exception still applies. The goal of this section is to prove Theorem 1.2, which adds a new and completely different family of instances where equality holds in (47) outside the stable regime $n \geq 2m$.

Note that $\operatorname{Conf}(|K_m|, 2)$ is empty for $m = 1$, and disconnected for $m = 2$, while $\operatorname{Conf}(|K_3|, 2)$ leads to the Y_2–exception. On the other hand, the cases $m = 4$ and $m = 5$ in Theorem 1.2 are well known in view of Corollary 4.2 and the last assertion in Example 4.15. We prove Theorem 1.2 for $m \geq 6$ by constructing $2s$ zero divisors in $\operatorname{Conf}(|K_m|, 2)$ with a nonzero cup product, and using Proposition 5.1 together with the obvious fact that $\operatorname{hdim}(\operatorname{Conf}(|K_m|, 2)) \leq 2$. It is natural to think that the expected richness of cup products in general graph configuration spaces might lead to many more instances where (47) would hold as an equality — even if $n < 2m$.

For integers $1 \leq i \leq s \geq 2$ and a cohomology class x in a space X, consider the exterior tensor product $x_{(i)} := 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \in H^*(X)^{\otimes s} = H^*(X^s)$, where the tensor factor x appears in the i^{th} position. The following result is straightforward to check:

Lemma 5.2 *Let x, y, z and w be four elements in the mod 2 cohomology of a space X satisfying the relations $x^2 = y^2 = xz = yz = yw = 0$. Then*

$$\Bigl(\prod_{i=2}^{s}(x_{(1)} + x_{(i)})\Bigr)\Bigl(\prod_{i=2}^{s}(y_{(1)} + y_{(i)})\Bigr)(z_{(1)} + z_{(s)})(w_{(1)} + w_{(s)})$$
$$= zw \otimes xy \otimes xy \otimes \cdots \otimes xy + xy \otimes xy \otimes \cdots \otimes xy \otimes zw.$$

Proof of Theorem 1.2 for $m \geq 6$ In view of Corollary 4.16 and Lemma 5.2, the 1–dimensional basis elements $x := \delta_{1,2}$, $y := \upsilon_{3,4}$, $z := \upsilon_{1,3}$, $w := \delta_{2,4} \in H^* \operatorname{Conf}(|K_m|, 2)$ yield a product of $2s$ zero divisors, with product-representative

$$\begin{bmatrix}1&1&3\\2&4&4\end{bmatrix} \otimes \begin{bmatrix}3&3&4\\1&2&2\end{bmatrix} \otimes \begin{bmatrix}3&3&4\\1&2&2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}3&3&4\\1&2&2\end{bmatrix} + \begin{bmatrix}3&3&4\\1&2&2\end{bmatrix} \otimes \begin{bmatrix}3&3&4\\1&2&2\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}3&3&4\\1&2&2\end{bmatrix} \otimes \begin{bmatrix}1&1&3\\2&4&4\end{bmatrix}. \tag{49}$$

The tensor factor $\left[\begin{smallmatrix}3&3&4\\1&2&2\end{smallmatrix}\right]$ represents one of the basis elements in the previous section. However, as indicated in Figure 13, we need to apply relation (E_2) in Corollary 4.11 in order to write the (cohomology class of the) tensor factor $\left[\begin{smallmatrix}1&1&3\\2&2&4\end{smallmatrix}\right]$ as a sum $\sum b_i$ of basis elements b_i (recall, we work mod 2). If $m \geq 6$, the basis element $\left[\begin{smallmatrix}3&3&5\\2&4&4\end{smallmatrix}\right]$ appears as a summand b_i, from which the nontriviality of the cohomology class represented by (49) follows. □

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