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Spectral diameter of Liouville domains

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The group of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold is endowed with a natural bi-invariant distance, due to Viterbo, Schwarz, Oh, Frauenfelder and Schlenk, coming from spectral invariants in Hamiltonian Floer homology. This distance has found numerous applications in symplectic topology. However, its diameter is still unknown in general. In fact, for closed symplectic manifolds there is no unifying criterion for the diameter to be infinite. We prove that for any Liouville domain this diameter is infinite if and only if its symplectic cohomology does not vanish. This generalizes a result of Monzner, Vichery and Zapolsky and has applications in the setting of closed symplectic manifolds.

57R17, 51F99, 53D05, 57R58

1 Introduction and results

Liouville domains are a special kind of compact symplectic manifold with boundary. They are characterized by their exact symplectic form $\omega = \mathrm{d}\lambda$ and the fact that their boundary is of contact type. Liouville domains allow us to study under a common theoretical framework many important classes of symplectic manifolds. Examples of such manifolds include cotangent disk bundles over closed manifolds, complements of Donaldson divisors [Giroux 2017], preimages of some intervals under exhausting functions of Stein manifolds [Cieliebak and Eliashberg 2012], positive regions of convex hypersurfaces in contact manifolds [Giroux 1991] and total spaces of Lefschetz fibrations.

A key invariant of a Liouville domain D is its symplectic cohomology $\mathrm{SH}^*(D)$. It was first defined in [Floer and Hofer 1994; Cieliebak et al. 1995] and later developed in [Viterbo 1999]. Symplectic cohomology allows one to study the behavior of periodic Reeb orbits on the boundary of D. It is defined in terms of the Floer cohomology groups of a specific class of Hamiltonian functions on the completion $\widehat{D}$ of D which results from the gluing of the cylinder $[1,\infty)\times\partial D$ to ∂D.

The primary goal of this paper is to relate symplectic cohomology and spectral invariants, an important tool in Hamiltonian dynamics. When defined on a symplectic manifold (M,ω), spectral invariants associate to any pair $(\alpha, H)\in \mathrm{H}^*(M)\times C_c^\infty(S^1\times M)$ a real number $c(\alpha,H)$ that belongs to the spectrum of the action functional associated to H.[1] Spectral invariants were first defined in $\mathbb{R}^{2n}$ from the point of view of generating functions in [Viterbo 1992]. They were then constructed on closed symplectically aspherical manifolds in [Schwarz 2000] and general closed symplectic manifolds in [Oh 2005] (see also [Usher 2013]).

Frauenfelder and Schlenk [2007] constructed spectral invariants on Liouville domains. These spectral invariants are homotopy invariant in the Hamiltonian term in the following sense. If two compactly supported Hamiltonians H and F generate the same time-1 map, $\varphi_H=\varphi_F$, then $c(\alpha,H)=c(\alpha,F)$. Thus $c(\alpha,\cdot)$ descends to the group of compactly supported Hamiltonian diffeomorphisms $\mathrm{Ham}_c(D)$. This allows one to define a bi-invariant norm γ on $\mathrm{Ham}_c(D)$, called the spectral norm, by

$$\gamma(\varphi)=c(1,\varphi)+c(1,\varphi^{-1}).$$

One key feature of the spectral norm γ is the fact that it acts as a lower bound to the celebrated Hofer norm [1990] (see [Lalonde and McDuff 1995] and the book [Polterovich 2001] for further developments in the subject). It is thus natural to ask whether the spectral diameter

$$\mathrm{diam}_\gamma(M)=\sup\{\gamma(\varphi)\mid \varphi\in\mathrm{Ham}_c(M)\}$$

is finite or not. In particular, if $\mathrm{diam}_\gamma(M)=+\infty$ then the Hofer norm is assured to be unbounded. Further links between the spectral norm and Hofer geometry are discussed in Section 1.4.

1.1 Main results

We find a characterization of the finiteness of $\mathrm{diam}_\gamma(D)$ in the case of a Liouville domain $(D,\mathrm{d}\lambda)$ in terms of its symplectic cohomology.

Our main technical result shows that if $\mathrm{SH}^*(D)\neq 0$ then $c(1,H)$ can be made arbitrarily large. This, combined with the converse implication which was proved in [Benedetti and Kang 2022], implies:

Theorem A1 *Let (D,λ) be a Liouville domain. Then $\mathrm{diam}_\gamma(D)=+\infty$ if and only if $\mathrm{SH}^*(D)\neq 0$.*

As an intermediate step to proving Theorem A1, we show the following auxiliary result.

Lemma B *Let H be a compactly supported Hamiltonian on a Liouville domain (D,λ). Then,*

$$c(1,H)\geq 0.$$

[1] At least if the Hamiltonian satisfies certain technical conditions.

Lemma B is a cohomological adaptation for Liouville domains of a result of [Ganor and Tanny 2023, Lemma 4.1]. They show that $c([\mathrm{pt}], F) \le 0$ for Hamiltonians F supported in certain incompressible domains of closed aspherical manifolds. It follows from [loc. cit., Section 5.1], that this inequality extends to Liouville domains. We remark that the inequality $c([M], F) \ge 0$ for the same class of Hamiltonians F was already shown by Humilière, Le Roux and Seyfaddini [Humilière et al. 2016]. It follows directly from [loc. cit., Theorem 45]. The main difference in Lemma B here is that we consider spectral invariants on Floer cohomology (instead of Floer homology) with respect to the unit (instead of the point class). Furthermore it applies to general Liouville domains without any ambient symplectic manifold. The proof uses an adaptation to Floer cohomology on Liouville domains of the barricade construction introduced in [Ganor and Tanny 2023].

In fact, when the symplectic cohomology of a Liouville domain is nonvanishing, the implication of Theorem A1 follows from a sharper result. Denote by $d_\gamma(\varphi, \psi) = \gamma(\varphi \circ \psi^{-1})$ the spectral distance on $\mathrm{Ham}_c(D)$ and by d_{st} the standard Euclidean distance on $\mathbb{R}$.

Theorem A2 *Let (D, λ) be a Liouville domain such that $\mathrm{SH}^*(D) \neq 0$. Then there exists an isometric group embedding $(\mathbb{R}, d_{\mathrm{st}}) \to (\mathrm{Ham}_c(D), d_\gamma)$.*

The proof of Theorem A2 uses an explicit construction of an isometric group embedding. This construction is a generalization of the procedure used by Monzner, Vichery and Zapolsky to prove Theorem 3 below. The construction of the aforementioned embedding relies primarily on the computation of spectral invariants of Hamiltonians which are constant on the skeleton of D, a special subset of Liouville domains which we define in Section 2.1.

Lemma C *Suppose (D, λ) is a Liouville domain such that $\mathrm{SH}^*(D) \neq 0$. Let H be a compactly supported autonomous Hamiltonian on D such that*

$$H|_{\mathrm{Sk}(D)} = -A \quad \text{and} \quad -A \le H|_D \le 0$$

for a constant $A > 0$. Then

$$c(1, H) = A.$$

1.2 What is already known for Liouville domains

Following the [Benedetti and Kang 2022], it is known that the spectral diameter of a Liouville domain D is bounded if its symplectic cohomology vanishes. This result was achieved using a special capacity derived from the filtered symplectic cohomology of D. To better understand how this is done, let us give an overview of the construction of $\mathrm{SH}^*(D)$ following [Viterbo 1999].

Consider the class of admissible Hamiltonians $H : \widehat{D} \to \mathbb{R}$ which are affine in the radial coordinate on the cylindrical part of $\widehat{D}$.[2] We can define the filtered Floer cohomology groups $\mathrm{HF}^*_{(a,b)}(H)$ of such

[2] See Definition 7 for the precise conditions.

Hamiltonians by considering only the 1–periodic orbits with action inside the interval (a,b).[3] Taking an increasing sequence of admissible Hamiltonians $\{H_i\}_i$ with corresponding slopes $\{\tau_i\}_i$ satisfying $\tau_i \to +\infty$, one can define the filtered symplectic cohomology $\mathrm{SH}^*_{(a,b)}(D)$ of D as

$$\mathrm{SH}^*_{(a,b)}(D) = \varinjlim_{H_i} \mathrm{HF}^*_{(a,b)}(H_i).$$

It follows from the above definition that for $a \le a'$ and $b \le b'$ there is a natural map

$$\iota^{b,b'}_{a,a'} \colon \mathrm{SH}^*_{(a,b)}(D) \longrightarrow \mathrm{SH}^*_{(a',b')}(D).$$

Moreover, the full symplectic cohomology $\mathrm{SH}^*(D) = \mathrm{SH}^*_{(-\infty,\infty)}(D)$ comes with a natural map

$$v^* \colon \mathrm{H}^*(D) \longrightarrow \mathrm{SH}^*(D)$$

called the Viterbo map. The failure of v^* to be an isomorphism signals the presence of Reeb orbits on the boundary of D. Thus, $\mathrm{SH}^*(D)$ is a useful tool to study the Weinstein conjecture [1979], which claims that on any closed contact manifold the Reeb vector field should admit at least one periodic orbit. For instance, Viterbo [1999] proved the Weinstein conjecture for the boundary of subcritical Stein manifolds.

We can extend any compactly supported Hamiltonian $H \in C^\infty_c(S^1 \times D)$ to an admissible Hamiltonian with small slope H^ε and define its Floer cohomology as $\mathrm{HF}^*(H) = \mathrm{HF}^*(H^\varepsilon)$. A key property of Floer cohomology on Liouville domains is that if an admissible Hamiltonian F has a slope close enough to zero, then we have an isomorphism $\Phi_F \colon \mathrm{H}^*(D) \to \mathrm{HF}^*(F)$. Thus, the Floer cohomology of compactly supported Hamiltonians on D is well-defined.

Let H be a compactly supported Hamiltonian. Following [Frauenfelder and Schlenk 2007], the spectral invariant associated to $(\alpha, H) \in \mathrm{H}^*(D) \times C^\infty_c(S^1 \times D)$ corresponds to the real number

$$c(\alpha, H) = \inf\{c \in \mathbb{R} \mid \Phi_H(\alpha) \in \operatorname{im} \iota^{<c}\},$$

where

$$\iota^{<c} = \iota^{c,+\infty}_{-\infty,-\infty} \colon \mathrm{HF}^*_{(-\infty,c)}(H) \longrightarrow \mathrm{HF}^*(H)$$

is the map induced by natural inclusion of subcomplexes.

Now, define the SH-capacity of D as

$$c_{\mathrm{SH}}(D) = \inf\{c > 0 \mid \iota^{\varepsilon,c}_{-\infty,-\infty} = 0\} \in (0,\infty],$$

where, for $\epsilon > 0$ sufficiently small,

$$\iota^{\varepsilon,c}_{-\infty,-\infty} \colon \mathrm{SH}^*_{(-\infty,\varepsilon)}(D) \longrightarrow \mathrm{SH}^*_{(-\infty,c)}(D).$$

It is known that $c_{\mathrm{SH}}(D)$ is finite if and only if $\mathrm{SH}^*(D)$ vanishes. Using this, Benedetti and Kang proved the following upper bound on spectral invariants of compactly supported Hamiltonians with respect to the unit.

[3] See Section 2.2.1 for details on the action convention we use in this paper.

Theorem 1 [Benedetti and Kang 2022] *Let $(D, \mathrm{d}\lambda)$ be a Liouville domain with $\mathrm{SH}^*(D) = 0$. Then,*

$$\sup\{c(1, H)\} \leq c_{\mathrm{SH}}(D) < +\infty,$$

where the supremum is taken over all compactly supported Hamiltonians in D.

In particular, by the definition of the spectral norm, if $\mathrm{SH}^*(D) = 0$, then for any compactly supported Hamiltonian H generating $\varphi_H \in \mathrm{Ham}_c(D)$, we have

$$\gamma(\varphi) = c(1, \varphi) + c(1, \varphi^{-1}) \leq 2c_{\mathrm{SH}}(D) < +\infty.$$

Therefore, Theorem 1 provides the *only if* part of Theorem A1.

On the other hand, symplectic cohomology is known to be nonzero in many cases [Seidel 2008, Section 5]. Since we will be using $\mathbb{Z}_2$ coefficients throughout this article, one case of particular interest to us is the following.

Proposition 2 [Viterbo 1999] *Suppose D contains a closed exact Lagrangian submanifold L. Then,* $\mathrm{SH}^*(D) \neq 0$.

This result of Viterbo can be used, in conjunction with Theorem A1, to prove that the spectral diameter is infinite for quite general classes of Liouville domains.

1.2.1 Cotangent bundles Monzner, Vichery and Zapolsky [Monzner et al. 2012] showed the following.

Theorem 3 *Let N be a closed manifold. There exists an isometric group embedding of $(\mathbb{R}, d_{\mathrm{st}})$ in $(\mathrm{Ham}_c(T^*N), d_\gamma)$.*

Note that Theorem 3 follows directly from Theorem A2 and Proposition 2. Indeed, since the zero section $N \subset DT^*N$ is an exact closed Lagrangian submanifold, Proposition 2 assures us that $\mathrm{SH}^*(DT^*N) \neq 0$. Therefore, Theorem A2 guarantees the existence of an isometric group embedding

$$(\mathbb{R}, d_{\mathrm{st}}) \longrightarrow (\mathrm{Ham}_c(T^*N), d_\gamma).$$

Theorem 3 immediately implies:

Corollary 4 *Let N be a closed manifold. Then* $\mathrm{diam}_\gamma(DT^*N) = +\infty$.

Similarly to Theorem 3, Corollary 4 follows directly from Proposition 2 and Theorem A1.

1.3 The spectral diameter of other symplectic manifolds

It has been known for a long time [Entov and Polterovich 2003] that for $(\mathbb{C}P^n, \omega\mathrm{FS})$,

$$\mathrm{diam}_\gamma(\mathbb{C}P^n) \leq \int_{\mathbb{C}P^1} \omega\mathrm{FS}.$$

The above upper bound was latter optimized in [Kislev and Shelukhin 2021, Theorem G] to

$$\operatorname{diam}_\gamma(\mathbb{C}P^n) = \frac{n}{n+1}\int_{\mathbb{C}P^1} \omega \mathrm{FS}.$$

However, for a surface Σ_g of genus $g \geq 1$, the spectral diameter is infinite. This case is covered by the following theorem [Kislev and Shelukhin 2021, Theorem D], which is a sharpening of [Usher 2013, Theorem 1.1].

Theorem 5 *Let (M,ω) be a closed symplectic manifold that admits an autonomous Hamiltonian $H \in C^\infty(M,\mathbb{R})$ such that*

(U1) *all the contractible periodic orbits of X_H are constant.*

Then $\operatorname{diam}_\gamma(M) = +\infty$.

Theorem 5 allows one to prove that the spectral diameter is infinite in many cases. A list of examples in which condition (U1) holds can be found in [Usher 2013, Section 1]. As mentioned above, surfaces of positive genus satisfy (U1). Also, if (N,ω_N) satisfies (U1) then so does $(M\times N, \omega_M \oplus \omega_N)$ for any other closed symplectic manifold (M,ω_M).

Kawamoto [2022b] proved that the spectral diameters of the quadrics Q^2 and Q^4 (of real dimensions 4 and 8 respectively), and certain stabilizations of them, are infinite.

1.3.1 Symplectically aspherical manifolds Recall that a symplectic manifold (M,ω_M) is symplectically aspherical if both ω_M and the first Chern class $c_1(M)$ of M vanish on $\pi_2(M)$; namely, for every continuous map $f\colon S^2 \to M$,

$$\langle [\omega_M], f_*[S^2]\rangle = 0 = \langle c_1(M), f_*[S^2]\rangle.$$

An open subset $U \subset M$ is said to be incompressible if the map $\pi_1(U) \to \pi_1(M)$ induced by the inclusion is injective.

As pointed out in [Buhovsky et al. 2021], it has been conjectured that $\operatorname{diam}_\gamma(M) = +\infty$ on all closed symplectically aspherical manifolds. Here, we prove that conjecture in the case of the twisted product $(M\times M, \omega\oplus -\omega)$ of a closed symplectically aspherical manifold (M,ω) with itself. But first, a more general result.

Proposition D *Let (M,ω) be a closed symplectically aspherical manifold of dimension $2n$. Suppose there exists an incompressible Liouville domain D of codimension 0 embedded inside M with $\mathrm{SH}^*(D) \neq 0$. Then*, $\operatorname{diam}_\gamma(M) = +\infty$.

Proof Let H be a compactly supported Hamiltonian in D and denote by $\iota\colon D \to M$ the embedding. By a cohomological analogue of [Ganor and Tanny 2023, Claim 5.2], we have that

$$c_D(\beta, H) = \max_{\substack{\alpha \in \mathrm{H}^*(M) \\ \iota^*(\alpha)=\beta}} c_M(\alpha, H)$$

for all $\beta \in \mathrm{H}^*(D)$, where c_D and c_M are the spectral invariants on D and M respectively. In particular, we know that the unit $1_M \in \mathrm{H}^*(M)$ is sent to the unit $1_D \in \mathrm{H}^*(D)$ under the map $\iota^*\colon \mathrm{H}^*(M) \to \mathrm{H}^*(D)$. Moreover, it is well known that the spectral invariant with respect to the unit can be implicitly written as

$$c_M(1_M, H) = \max_{\alpha \in \mathrm{H}^*(M)} c_M(\alpha, H)$$

(see Lemma 26). Therefore, fixing $\beta = 1_D$, we have

$$c_D(1_D, H) = c_M(1_M, H).$$

Using Theorem A1, the above equation thus yields the desired result. □

Corollary E *Let (M, ω) be a closed symplectically aspherical manifold. Then,*

$$\mathrm{diam}_\gamma(\mathrm{Ham}(M \times M, \omega \oplus -\omega)) = +\infty.$$

Proof Consider the closed Lagrangian given by the diagonal $L = \Delta$ inside $(M \times M, \omega_M \oplus -\omega_M)$. By virtue of the Weinstein neighborhood theorem, there exists an open neighborhood U of L and a symplectomorphism $\psi\colon U \to D_\varepsilon T^*L$ such that $\varphi(L)$ coincides with the zero section of an ϵ–radius codisk bundle $D_\varepsilon T^*L$ over L. The Liouville structure on $D_\varepsilon T^*L$ pulls back to a Liouville structure on U. Note that, inside $M \times M$, L is incompressible; ie the map $\pi_1(L) \to \pi_1(M \times M)$ of first homotopy groups induced by the inclusion $L \to M \times M$ is injective. Therefore, by homotopy equivalence, U and $D_\varepsilon T^*L$ are also incompressible. The desired result follows directly from Proposition D. □

1.4 Hofer geometry

As hinted at above, the finiteness of the spectral diameter plays a role in Hofer geometry. In particular, it can be used to study the following question posed in [Le Roux 2010]:

Question For any $A > 0$, let

$$E_A(M, \omega) := \{\varphi \in \mathrm{Ham}(M, \omega) \mid d_H(\mathrm{Id}, \varphi) > A\}$$

be the complement of the closed ball of radius A in Hofer's metric. For all $A > 0$, does $E_A(M, \omega)$ have nonempty C^0–interior?

Indeed, in the case of closed symplectically aspherical manifolds with infinite spectral diameter, a positive answer to the Question above was given by Buhovsky, Humilière and Seyfaddini (see also [Kawamoto 2022a; 2022b] for the positive and negative monotone cases).

Theorem 6 [Buhovsky et al. 2021] *Let (M,ω) be a closed, connected and symplectically aspherical manifold. If $\mathrm{diam}_\gamma(M)=+\infty$, then $E_A(M,\omega)$ has nonempty C^0–interior for all $A>0$.*

Using Theorem 6 in conjunction with Corollary E, we directly obtain the following answer to the Question above in the specific setting of Corollary E.

Corollary F *Let (M,ω) be a closed symplectically aspherical manifold. Then, $E_A(M\times M,\omega\oplus-\omega)$ has a nonempty C^0–interior for all $A>0$.*

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2 Liouville domains and admissible Hamiltonians

In this subsection we recall the definition of Liouville domains, specify the class of Hamiltonians we will restrict our attention to and describe how their Floer trajectories behave at infinity.

2.1 Completion of Liouville domains

A Liouville domain $(D,\mathrm{d}\lambda,Y)$ is an exact symplectic manifold with boundary on which the vector field Y, defined by $Y\lrcorner\,\mathrm{d}\lambda=\lambda$ and called the Liouville vector field, points outwards along ∂D. Denote by $\widehat{D}=D\cup[1,\infty)\times\partial D$ the completion of D and (r,x) the coordinates on $[1,\infty)\times\partial D$. Here, we glue ∂D and $\{1\}\times\partial D$ with respect to the reparametrization $\psi_Y^{\ln r}$ of the Liouville flow generated by Y. Given $\delta>0$, let

$$D^\delta=\psi_Y^{\ln\delta}(D)=\widehat{D}\setminus(\delta,\infty)\times\partial D.$$

We extend the Liouville form λ to $\widehat{D}$ by defining $\hat{\lambda}\colon T\widehat{D}\to\mathbb{R}$ as

$$\hat{\lambda}\,|_D=\lambda\quad\text{and}\quad\hat{\lambda}\,|_{\widehat{D}\setminus D}=r\alpha,$$

where $\alpha=\lambda|_{\partial D}$. The cylindrical portion $[1,\infty)\times\partial D$ of $\widehat{D}$ is thus equipped with the symplectic form $\omega=\mathrm{d}(r\alpha)$. See Figure 1.

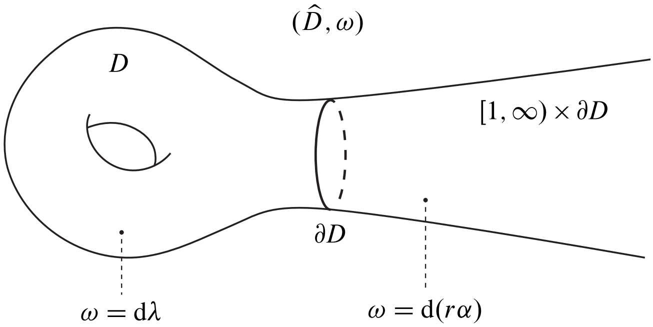


Figure 1: A Liouville domain with its completion.

The skeleton $\mathrm{Sk}(D)$ of $(D, \mathrm{d}\lambda, Y)$ is defined by

$$\mathrm{Sk}(D) = \bigcap_{0<r<1} \psi_Y^{\ln r}(D).$$

Denote by R_α the Reeb vector field on ∂D associated to α, meaning

$$R_\alpha \lrcorner \, \mathrm{d}\alpha = 0, \quad \alpha(R) = 1.$$

We define $\mathrm{Spec}(\partial D, \alpha)$ to be the set of periods of closed characteristics, the periodic orbits generated by R_α, on ∂D and put

$$T_0 = \min \mathrm{Spec}(\partial D, \lambda).$$

As a subset of $\mathbb{R}$, $\mathrm{Spec}(\partial D, \alpha)$ is known to be closed and nowhere dense. For any $A \in \mathbb{R}$, let η_A denote the distance between A and $\mathrm{Spec}(\partial D, \lambda)$.

2.2 Admissible Hamiltonians and almost-complex structures

2.2.1 Periodic orbits and action functional Given a Hamiltonian $H \colon S^1 \times \widehat{D} \to \mathbb{R}$, one defines its time-dependent Hamiltonian vector field $X_H^t \colon \widehat{D} \to T\widehat{D}$ by

$$X_H^t \lrcorner \, \omega = -dH_t,$$

where $H_t(p) = H(t, p)$. We denote by $\varphi_H^t \colon \widehat{D} \to \widehat{D}$ the flow generated by X_H^t. The set of all contractible 1–periodic orbits of φ_H^t is denoted by $\mathcal{P}(H)$. An orbit $x \in \mathcal{P}(H)$ is said to be *nondegenerate* if

$$\det(\mathrm{id} - d_{x(0)}\varphi_H^1) \neq 0$$

and *transversally nondegenerate* if the eigenspace associated to the eigenvalue 1 of the map $d_{x(0)}\varphi^1$ is of dimension 1.

Let $\mathcal{L}\widehat{D}$ be the space of contractible loops in $\widehat{D}$. For a Hamiltonian $H \colon S^1 \times \widehat{D} \to \mathbb{R}$, the *Hamiltonian action functional* $\mathcal{A}_H \colon \mathcal{L}\widehat{D} \to \mathbb{R}$ associated to H is defined as

$$\mathcal{A}_H(x) = \int_0^1 x^*\hat{\lambda} - \int_0^1 H_t(x(t))\, \mathrm{d}t.$$

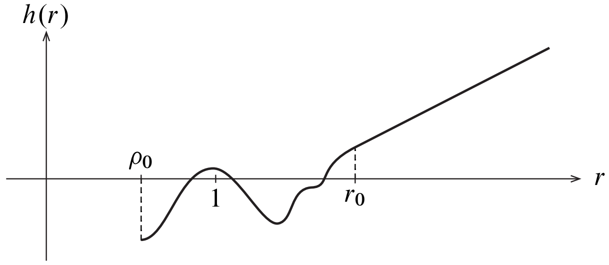


Figure 2: An r_0–admissible Hamiltonian.

It is well known that the elements of $\mathcal{P}(H)$ correspond to the critical points of $\mathcal{A}_H$; see [Audin and Damian 2014, Section 6]. The image of $\mathcal{P}(H)$ under the Hamiltonian action functional is called the *action spectrum of* H and is denoted by $\operatorname{Spec}(H)$. For an open set $U \subset \widehat{D}$ we define

$$\mathcal{P}_U(H) = \{x \in \mathcal{P}(H) \mid \operatorname{im} x \subset U\}.$$

2.2.2 Admissible Hamiltonians The completion of a Liouville domain is obviously noncompact. We thus need to control the behavior at infinity of Hamiltonians we use in order for them to have finitely many 1–periodic contractible orbits.

Definition 7 Let $r_0 > 1$. A Hamiltonian H is r_0–admissible if there exists $\rho_0 \in (0, r_0)$ such that

- $H(t, x, r) = h(r)$ on $\widehat{D} \setminus D^{\rho_0}$,
- $h(r) = \tau_H r + \eta_H$ on $(r_0, +\infty)$ for $\tau_H \in (0, \infty) \setminus \operatorname{Spec}(\partial D, \alpha)$,
- H is regular: every element of $\mathcal{P}_{D^\rho}(H)$ is nondegenerate and every element of $\mathcal{P}_{\widehat{D}\setminus D^\rho}(H)$ is transversally nondegenerate.

We denote the set of such Hamiltonians by $\mathcal{H}_{r_0}$. See Figure 2.

We will also consider the set $\mathcal{H}^0_{r_0} \subset \mathcal{H}_{r_0}$ of r_0–admissible Hamiltonians which are negative on D. In some cases, it is not necessary to specify r_0 as long as it is greater than 1. For that purpose, we define

$$\mathcal{H} = \bigcup_{r_0>1} \mathcal{H}_{r_0}, \quad \mathcal{H}^0 = \bigcup_{r_0>1} \mathcal{H}^0_{r_0}.$$

Remark 8 Suppose $H \in \mathcal{H}$. If $x \in \mathcal{P}_{\widehat{D}\setminus D^{\rho_0}}(H)$ is nonconstant, then it is necessarily transversally nondegenerate. Indeed, since H is time-independent there by definition, for any $c \in \mathbb{R}$, we know $x(t-c)$ is also a 1–periodic orbit of H.

Lemma 9 *If $H \in \mathcal{H}$, then $|\mathcal{P}_{D^{\rho_0}}(H)|$ is finite and $\mathcal{P}_{\widehat{D}\setminus D^{\rho_0}}(H)$ consists of a finite number of periodic orbits and S^1 families of periodic orbits.*

Proof Since $\overline{D^{\rho_0}}$ is compact and elements of $\mathcal{P}_{D^{\rho_0}}(H)$ are nondegenerate, there is a finite number of 1–periodic orbits of H inside it.

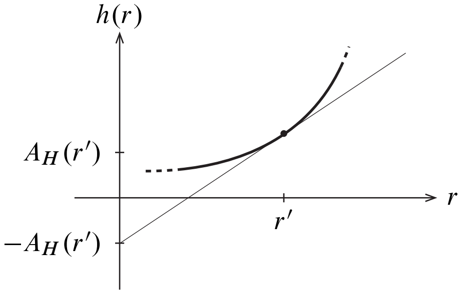


Figure 3: Action value of a periodic orbit contained in $\{r'\} \times \partial D$.

Next, we look at the elements of $\mathcal{P}_{\widehat{D} \setminus D^{\rho_0}}(H)$. On $\widehat{D} \setminus D^{\rho_0}$, we know that $H = h(r)$ and $\omega = \mathrm{d}\hat{\lambda}$. Therefore, on $\widehat{D} \setminus D^{\rho_0}$

$$\begin{aligned} X_H \lrcorner\, \omega &= X_H \lrcorner\, (\mathrm{d}r \wedge \alpha + r\mathrm{d}\alpha) \\ &= \mathrm{d}r(X_H)\alpha - \alpha(X_H)\mathrm{d}r + rX_H \lrcorner\, \mathrm{d}\alpha \end{aligned}$$

and $\mathrm{d}H = h'(r)\mathrm{d}r$. Hamilton's equation thus yields

$$\mathrm{d}r(X_H) = 0 = X_H \lrcorner\, \mathrm{d}\alpha, \quad \alpha(X_H) = h'(r).$$

The three equations above imply the following two facts:

- On $\widehat{D} \setminus D^{\rho_0}$, $X_H = h'(r)R_\alpha$.
- If $x \in \mathcal{P}(H)$ is such that $x \cap \widehat{D} \setminus D^{\rho_0} \neq \varnothing$, then $\operatorname{im} x \subset \{r\} \times \partial D$ for some $r > \rho_0$.

We conclude that a 1–periodic orbit x of H which lies inside $\{r\} \times \partial D$ corresponds to a Reeb orbit of period $h'(r)$. Notice that since $\tau_H \notin (0, +\infty) \cap \operatorname{Spec}(\partial D, \alpha)$, we have $\mathcal{P}_{\widehat{D} \setminus D^{\rho_0}}(H) = \mathcal{P}_{D^{r_0} \setminus D^{\rho_0}}(H)$. Therefore, since $\overline{D^{r_0} \setminus D^{\rho_0}}$ is compact and every element of $\mathcal{P}_{D^{r_0} \setminus D^{\rho_0}}(H)$ is transversally nondegenerate by definition, $\mathcal{P}_{D^{r_0} \setminus D^{\rho_0}}(H)$ is finite. □

Remark 10 The fact that admissible Hamiltonians are radial on the cylindrical part of $\widehat{D}$ allows us to express the action of the 1–periodic orbits inside $\widehat{D} \setminus D$ in terms of that radial function. To see this, we fix $H \in \mathcal{H}$ and compute the action of a nonconstant orbit $x \in \mathcal{P}(H) \cap (\widehat{D} \setminus D)$ which we suppose lies inside $\{r\} \times \partial D$ for $r > 1$:

$$\mathcal{A}_H(x) = \int_0^1 x^*\hat{\lambda} - \int_0^1 H \circ x \,\mathrm{d}t = \int_0^1 r\alpha(X_H)\,\mathrm{d}t - \int_0^1 h(r)\,\mathrm{d}t = rh'(r) - h(r).$$

The function $A_H(r) = rh'(r) - h(r)$ on the right-hand side of the above equation has a nice geometric interpretation. Looking at the graph of h, we notice that $A_H(r')$ corresponds to minus the y–coordinate of the intersection of the tangent at the point $(r', h(r'))$ and the y–axis. See Figure 3.

2.2.3 Monotone homotopies We will need to also restrict the types of Hamiltonian homotopies we consider to the following class.

Definition 11 Let $H_s = \{H_s\}_{s\in\mathbb{R}}$ be a smooth homotopy from $H_+ \in \mathcal{H}_{r_0}$ to $H_- \in \mathcal{H}_{r_0'}$ We say that H_s is a *monotone homotopy* if the following conditions hold:

- There exists $S > 0$ such that $H_{s'} = H_-$ for $s' < -S$ and $H_{s'} = H_+$ for $s' > S$.
- $H_s = h_s(r)$ on $\widehat{D} \setminus D^\rho$ for $\rho = \max\{\rho_0, \rho_0'\}$.
- For $R = \max\{r_0, r_0'\}$, we have $h_s(r) = \tau_s r + \eta_s$ on $(R, +\infty)$ for smooth functions τ_s, η_s of s.
- $\partial_s H_s(t, p) \le 0$ for $(t, p, s) \in S^1 \times \widehat{D} \times \mathbb{R}$.

For $H_+ \in \mathcal{H}_{r_0}$ and $H_- \in \mathcal{H}_{r_0'}$ such that $H_+ \le H_-$ pointwise everywhere on $\widehat{D}$, we can explicitly construct a monotone homotopy in the following way. Fix a positive constant $S > 0$. Let $\beta : \mathbb{R} \to [0, 1]$ be a smooth function such that $\beta(s) = 0$ for $s \le -S$, $\beta(s) = 1$ for $s \ge S$ and $\beta'(s) > 0$ for all $s \in (-S, S)$. Define

$$H_s = H_- + \beta(s)(H_+ - H_-).$$

Notice that, since $\beta'(s) \ge 0$ and $H_+ \le H_-$, we have

$$\partial_s H_s = \beta'(s)(H_+ - H_-) \le 0.$$

For $R = \max\{r_0, r_0'\}$ we have, on $\widehat{D} \setminus D^R$,

$$H_s(t, r, p) = (\beta(s)(\tau_+ - \tau_-) + \tau_-)r + \beta(s)(\eta_+ - \eta_-) + \eta_- = h_s(r)$$

as desired and

$$\partial_s \partial_r h_s(r) = \beta(s)'(\tau_+ - \tau_-) \le 0. \tag{1}$$

This inequality will be needed for the maximum principle of Section 2.3.2. Equation (1) also holds for general monotone homotopies. Indeed, by definition, $H_s(r)$ decreases in s and $H_s(r)$ is affine for $r \ge R$.

2.2.4 Admissible almost-complex structures

Let J be an almost-complex structure on $\widehat{D}$. Recall that J is ω–compatible if the map $g_J : TM \otimes TM \to \mathbb{R}$ defined by

$$g_J(v, w) = \omega(v, Jw)$$

is a Riemannian metric. To control the behavior of ω–compatible almost-complex structures at infinity, we make the following definition.

Definition 12 Let J be an ω–compatible almost-complex structure on $\widehat{D}$. We say that J is *admissible* if $J_1 = J|_{\widehat{D}\setminus D}$ is of *contact type*. Namely, we ask that

$$J_1^* \hat{\lambda} = \mathrm{d}r.$$

We denote the set of such almost-complex structures by $\mathcal{J}$. A pair (H, J), where $H \in \mathcal{H}_{r_0}$ and $J \in \mathcal{J}$, is called an r_0*–admissible pair*.

2.3 Floer trajectories and maximum principle

In this subsection, we recall some analytical aspects of Floer theory on Liouville domains. Issues regarding transversality will be dealt with in the next section.

2.3.1 Floer trajectories Consider a Hamiltonian $H: S^1 \times \widehat{D} \to \mathbb{R}$ and two 1–periodic orbits $x_\pm \in \mathcal{P}(H)$. Let J be an ω–compatible almost-complex structure on $\widehat{D}$. A *Floer trajectory* between x_- and x_+ is a solution $u: \mathbb{R} \times S^1 \to \widehat{D}$ to the *Floer equation*

$$\partial_s u + J(\partial_t u - X_H) = 0$$

that converges uniformly in t to x_- and x_+ as $s \to \pm\infty$:

$$\lim_{s \to \pm\infty} u(s,t) = x_\pm(t).$$

We denote the moduli space of such trajectories by $\mathcal{M}'(x_-, x_+; H)$. We may reparametrize a solution $u \in \mathcal{M}'(x_-, x_+; H)$ in the $\mathbb{R}$–coordinate by adding a constant. Thus, Floer trajectories occur in $\mathbb{R}$–families. The space of unparametrized solutions is denoted by $\mathcal{M}(x_-, x_+; H) = \mathcal{M}'(x_-, x_+; H)/\mathbb{R}$. When the context is clear, we will drop H from the notation and simply write $\mathcal{M}(x_-, x_+)$.

If we replace H with a monotone homotopy $H_\bullet = \{H_s\}_{s \in \mathbb{R}}$, then we can instead consider solutions $u: \mathbb{R} \times S^1 \to \widehat{D}$ to the *s-dependent Floer equation*

$$\partial_s u + J(\partial_t u - X_{H_s}) = 0$$

that converge uniformly in t to $x_\pm \in \mathcal{P}(H_\pm)$ as $s \to \pm\infty$. The moduli space of such trajectories is denoted by $\mathcal{M}(x_-, x_+; H_\bullet)$. Unlike the s–independent case, $\mathcal{M}(x_-, x_+; H_\bullet)$ does not admit a free $\mathbb{R}$–action by which we can quotient.

2.3.2 Maximum principle To define Floer cohomology of $\widehat{D}$, we need to control the behavior of the Floer trajectories. In particular, we have to make sure they do not escape to infinity. Admissible Hamiltonians and admissible complex structures allow us to achieve that requirement. The first result in that direction is the maximum principle for Floer trajectories. In what follows we say that v is a local Floer solution of (H, J) in $\widehat{D} \setminus D$ if

$$v = u|_{u^{-1}(\operatorname{im} u \cap \widehat{D} \setminus D)} : u^{-1}(\operatorname{im} u \cap \widehat{D} \setminus D) \longrightarrow \widehat{D} \setminus D$$

for some $u \in \mathcal{M}(x_-, x_+; H)$.

Lemma 13 (generalized maximum principle [Viterbo 1999]) *Let (H, J) be an r_0–admissible pair on $\widehat{D}$. Suppose v is a local Floer solution of (H, J) in $\widehat{D} \setminus D^{r_0}$. Then, the r–coordinate $r \circ v$ of v does not admit an interior maximum unless $r \circ v$ is constant.*

Remark 14 The generalized maximum principle still holds if we replace $H \in \mathcal{H}$ by a monotone homotopy H_s between $H_+ \in \mathcal{H}_{r_0}$ and $H_- \in \mathcal{H}_{r_0'}$ and if v is a local solution of the s–dependent Floer equation

$$\partial_s v + J(\partial_t v - X_{H_s}) = 0$$

inside $\widehat{D} \setminus D^R$, where $r = \max\{r_0, r_0'\}$. Here it is crucial that $\partial_s \partial_r h_s(r) \leq 0$ for large enough r. From the maximum principle above, we immediately obtain the following corollary which guarantees that Floer trajectories do not escape to infinity.

Corollary 15 *Let (H, J) be an r_0–admissible pair on $\widehat{D}$ and let $x_\pm \in \mathcal{P}(H)$. If $u \in \mathcal{M}(x_-, x_+)$, then*

$$\operatorname{im} u \subset D^R \quad \text{for } R = \max\{r \circ x_-, r \circ x_+, r_0\}.$$

If H_s is a monotone homotopy between $H_- \in \mathcal{H}_{r_0}$ and $H_+ \in \mathcal{H}_{r_0'}$ and u is a solution to the s–dependent Floer equation between $x_- \in \mathcal{P}(H_-)$ and $x_+ \in \mathcal{P}(H_+)$, then

$$\operatorname{im} u \subset D^R, \quad \text{for } R = \max\{r \circ x_-, r \circ x_+, r_0, r_0'\}.$$

2.3.3 Energy An important quantity which is associated to a Floer trajectory is its *energy*. It is defined as

$$E(u) = \frac{1}{2}\int_{\mathbb{R}\times S^1} (|\partial_s u|_J^2 + |\partial_t u - X_H|_J^2)\, \mathrm{d}s \wedge \mathrm{d}t,$$

where $|\cdot|_J$ is the norm corresponding to g_J. Using the Floer equation, we can write

$$|\partial_t u - X_H|_J^2 = \omega(J\partial_s u, -\partial_s u) = \omega(\partial_s u, J\partial_s u) = |\partial_s u|_J^2.$$

Thus, the energy can be written more compactly as

$$E(u) = \int_{\mathbb{R}\times S^1} |\partial_s u|_J^2\, \mathrm{d}s \wedge \mathrm{d}t.$$

It is often useful to estimate the difference in Hamiltonian action of the ends of a Floer trajectory in terms of the energy of that trajectory. This can be achieved using the maximum principle and Stokes' theorem.

Lemma 16 *Let (H, J) be an r_0–admissible pair and let $u \in \mathcal{M}'(x_-, x_+; H)$ for $x_\pm \in \mathcal{P}(H)$. Then,*

$$0 \leq E(u) = \mathcal{A}_H(x_+) - \mathcal{A}_H(x_-).$$

If H_s is a monotone homotopy between $H_+ \in \mathcal{H}_{r_0}$ and $H_- \in \mathcal{H}_{r_0'}$ that is constant in the s–coordinate for $s > |S|$ then

$$0 \leq E(u) \leq \mathcal{A}_{H_+}(x_+) - \mathcal{A}_{H_-}(x_-) + \sup_{\substack{s\in[-S,S],\\ t\in S^1, p\in D^R}} \partial_s H_s(t, p),$$

where $R = \max\{r \circ x_-, r \circ x_+, r_0, r_0'\}$.

3 Filtered Floer and symplectic cohomology

We present in this section a brief overview of Floer cohomology for completions of Liouville domains and their symplectic cohomology. For more details we refer the reader to [Cieliebak et al. 1995; 1996; 2010; Viterbo 1999; Weber 2006; Ritter 2013].

3.1 Filtered Floer cohomology

3.1.1 The Floer cochain complex Let (H, J) be an admissible pair. As mentioned in Remark 8, the 1–periodic orbits of H on $\widehat{D} \setminus D^{\rho_0}$ come in a finite number of S^1–families, which we denote by $\hat{x}_i$. To break each $\hat{x}_i$ in a finite number of isolated periodic orbits, we first choose an open neighborhood U_i of each $\hat{x}_i$ such that $U_i \cap U_j = \varnothing$ for $i \neq j$. Then, we define on each $\hat{x}_i$ a Morse function f_i having exactly two critical points: one of index 0 and another of index 1. We extend each f_i to its corresponding U_i. When added to H, these perturbations, which can be chosen as small as we want, break each of the S^1–families into two critical points. By virtue of the action formula derived in Remark 10, the actions of the new critical points are as close as we want to the action of their original S^1–family. We denote by H_1 the Hamiltonian resulting from this procedure. By abuse of notation we will write $\mathcal{P}(H)$ for the set of 1–periodic orbits of H_1.

We define the *Floer cochain group of* H as the $\mathbb{Z}_2$–vector space[4]

$$\mathrm{CF}^*(H) = \bigoplus_{x \in \mathcal{P}(H)} \mathbb{Z}_2\langle x \rangle.$$

As the notation above suggests, $\mathrm{CF}^*(H)$ is in fact a *graded* $\mathbb{Z}_2$–vector space. Assuming that the first Chern class $c_1(\omega) \in \mathrm{H}^2(\widehat{D}; \mathbb{Z})$ of $(T\widehat{D}, J)$ vanishes on $\pi_2(\widehat{D})$, the Conley-Zehnder index $\mathrm{CZ}(x) \in \mathbb{Z}$ of a 1–periodic orbit $x \in \mathcal{P}(H)$ is well-defined [Salamon and Zehnder 1992]. We can therefore equip $\mathrm{CF}^*(H)$ with the degree

$$|x| = \tfrac{1}{2} \dim \widehat{D} - \mathrm{CZ}(x)$$

and define

$$\mathrm{CF}^k(H) = \bigoplus_{\substack{x \in \mathcal{P}(H) \\ |x| = k}} \mathbb{Z}_2\langle x \rangle.$$

Here, CZ is normalized such that for a C^2–small time-independent admissible Hamiltonian F,

$$\mathrm{CZ}(x) = \tfrac{1}{2} \dim \widehat{D} - \mathrm{ind}(x),$$

where $\mathrm{ind}(x)$ corresponds to the Morse index of $x \in \mathrm{Crit}(F) = \mathcal{P}(F)$. In particular, if x is a local minimum of F, then $|x| = 0$. This convention therefore ensures that the cohomological unit has degree 0.

[4] We use $\mathbb{Z}_2$ coefficients here for simplicity but the cohomological construction that follows can be carried out with any coefficient ring.

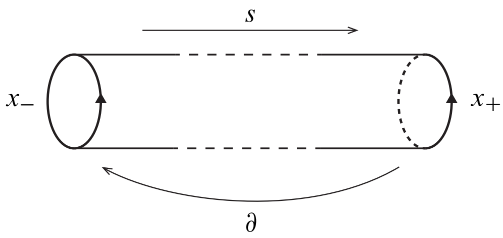


Figure 4: The differential in Floer cohomology goes from right to left.

For a generic perturbation of J, the space $\mathcal{M}(x_-, x_+; H)$ is a smooth manifold of dimension

$$\dim \mathcal{M}(x_-, x_+; H) = \mathrm{CZ}(x_+) - \mathrm{CZ}(x_-) - 1.$$

In the case where $|x_-| = |x_+| + 1$, Corollary 15 and Lemma 16 allow us to use the standard compactness arguments, as in [Audin and Damian 2014, Chapter 8], to show that $\mathcal{M}(x_-, x_+; H)$ is a compact manifold of dimension 0. Knowing that, we define the coboundary operator $\partial\colon \mathrm{CF}^k(H) \to \mathrm{CF}^{k+1}(H)$ by

$$\partial x_+ = \sum_{|x_-|=k+1} \#_2\mathcal{M}(x_-, x_+; H)x_-,$$

where $\#_2\mathcal{M}(x_-, x_+; H)$ is the count modulo 2 of components in $\mathcal{M}(x_-, x_+, H)$. See Figure 4.

Using once again Corollary 15, $\partial \circ \partial = 0$ holds by standard arguments which appear in [Audin and Damian 2014, Chapter 9]. The pair $(\mathrm{CF}^*(H), \partial)$ is thus a graded cochain complex that we call the *Floer cochain complex of H*.

3.1.2 Filtered Floer cochain complex The Hamiltonian action functional induces a filtration on the Floer cochain complex. For $a \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \mathrm{Spec}(H)$, we define

$$\mathrm{CF}^k_{<a}(H) = \bigoplus_{\substack{x\in\mathcal{P}(H)\\ |x|=k,\ \mathcal{A}_H(x)<a}} \mathbb{Z}_2\langle x\rangle.$$

By definition, we have $\mathrm{CF}^*(H) = \mathrm{CF}^*_{<+\infty}(H)$. Lemma 16 assures that ∂ decreases the action. Thus, the restriction $\partial_{<a}\colon \mathrm{CF}^k_{<a}(H) \to \mathrm{CF}^{k+1}_{<a}(H)$ of the coboundary operator is well-defined and $(\mathrm{CF}^*_{<a}(H), \partial_{<a})$ is a subcomplex of $(\mathrm{CF}^*(H), \partial)$. Now, for $a, b \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \mathrm{Spec}(H)$ such that $a < b$, we can define the Floer cochain complex in the action window (a, b) as the quotient

$$\mathrm{CF}^*_{(a,b)}(H) = \frac{\mathrm{CF}^*_{<b}(H)}{\mathrm{CF}^*_{<a}(H)},$$

on which we denote the projection of the coboundary operator by

$$\partial_{(a,b)}\colon \mathrm{CF}^k_{(a,b)}(H) \longrightarrow \mathrm{CF}^{k+1}_{(a,b)}(H).$$

Therefore, for $a, b, c \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \operatorname{Spec}(H)$ such that $a < b < c$, we have an inclusion and a projection

$$\iota_{a,a}^{b,c} : \mathrm{CF}^*_{(a,b)}(H) \longrightarrow \mathrm{CF}^*_{(a,c)}(H), \quad \pi_{a,b}^{c,c} : \mathrm{CF}^*_{(a,c)}(H) \longrightarrow \mathrm{CF}^*_{(b,c)}(H)$$

that produce the short exact sequence

$$0 \longrightarrow \mathrm{CF}^*_{(a,b)}(H) \xrightarrow{\iota_{a,a}^{b,c}} \mathrm{CF}^*_{(a,c)}(H) \xrightarrow{\pi_{a,b}^{c,c}} \mathrm{CF}^*_{(b,c)}(H) \longrightarrow 0.$$

For simplicity, we define $\iota^{<c} = \iota_{-\infty,-\infty}^{+\infty,c}$ and $\pi_{>b} = \pi_{-\infty,b}^{+\infty,+\infty}$.

3.1.3 Filtered Floer cohomology Let $a, b \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \operatorname{Spec}(H)$ such that $a < b$. The above filtered cochain complexes allow us to define the *Floer cohomology group of H* in the action window (a, b) *as*

$$\mathrm{HF}^*_{(a,b)}(H) = \frac{\ker \partial_{(a,b)}}{\operatorname{im} \partial_{(a,b)}}.$$

The full Floer cohomology group of H is defined as $\mathrm{HF}^*(H) = \mathrm{HF}^*_{(-\infty,+\infty)}(H)$. For $a, b, c \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \operatorname{Spec}(H)$ such that $a < b < c$, the short exact sequence on the cochain level induces a long exact sequence in cohomology:

(2)
$$\begin{array}{ccc} \mathrm{HF}^*_{(a,b)}(H) & & \\ \uparrow{\scriptstyle [+1]} & \searrow{\scriptstyle [\iota_{a,a}^{b,c}]} & \\ & & \mathrm{HF}^*_{(a,c)}(H) \\ & \swarrow{\scriptstyle [\pi_{a,b}^{c,c}]} & \\ \mathrm{HF}^*_{(b,c)}(H) & & \end{array}$$

For C^2–small admissible Hamiltonians with small slope at infinity, the Floer cohomology recovers the standard cohomology of D.

Lemma 17 [Ritter 2013, Section 15.2] *Let $H \in \mathcal{H}$ be a C^2–small Hamiltonian with $\tau_H < T_0$ for $T_0 = \min \operatorname{Spec}(\partial D, \lambda)$. Then, we have an isomorphism*

$$\Phi_H : \mathrm{H}^*(D) \longrightarrow \mathrm{HF}^*(H).$$

Remark 18 We can endow $\mathrm{HF}^*(H)$ with a ring structure [Ritter 2013] where the product is given by the pair of pants product. The unit in $\mathrm{HF}^*(H)$, which we denote by 1_H, coincides with $\Phi_H(e_D)$, where e_D is the unit in $\mathrm{H}^*(D)$.

3.1.4 Compactly supported Hamiltonians We can define the Floer cohomology of compactly supported Hamiltonians on Liouville domains by first extending to affine functions on the cylindrical portion of $\widehat{D}$.

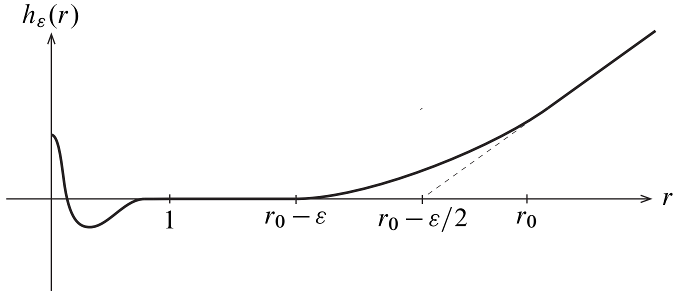


Figure 5: The τ–extension of a compactly supported Hamiltonian.

Definition 19 Denote by $\mathcal{C}(D)$ the set of Hamiltonians with support in $S^1\times(D\setminus\partial D)$. Let $H\in\mathcal{C}(D)$. For $\tau\in(0,\infty)\setminus\mathrm{Spec}(\partial D,\lambda)$, we define the τ–extension $H^\tau\in\mathcal{H}_1$ of H as follows. Fix $0<\varepsilon<1$ and $r_0>1$ so that $1<r_0-\varepsilon$,

- $H^\tau=H$ on D and $H^\tau=0$ on $D^{r_0-\varepsilon}\setminus D$,
- $H^\tau=h_\varepsilon(r)$ on $\widehat{D}\setminus D^{r_0-\varepsilon}$,
- $h_\varepsilon(r)$ is convex for $r\in[r_0-\varepsilon,r_0]$, with $h_\varepsilon^{(k)}(1)=0$ for all $k\geq 0$, $h'_\varepsilon(1+\varepsilon)=\tau$ and $h_\varepsilon^{(\ell)}(1+\varepsilon)=0$ for all $\ell>1$,
- $h_\varepsilon(r)=\tau(r-(r_0-\varepsilon/2))$ for $r\in[r_0,+\infty)$.

We perturb H^τ so that it is r_0–admissible. The Floer cohomology of H is defined as

$$\mathrm{HF}^*_{(a,b)}(H)=\mathrm{HF}^*_{(a,b)}(H^\tau),$$

where $0<\tau<T_0$. See Figure 5.

Since we take a slope τ smaller than the minimum Reeb period to define $\mathrm{HF}^*_{(a,b)}(H)$, the above definition doesn't depend on the choice of τ, ε and r_0, as we will see in Lemma 20 below.

3.1.5 Continuation maps Let $K\in\mathcal{H}_{r_0}$ and $F\in\mathcal{H}_{r'_0}$ such that $F\leq K$. Consider a monotone homotopy $H_\bullet$ from F to K. Then from Corollary 15 and Lemma 16 in the case of homotopies, we can apply the techniques shown in [Audin and Damian 2014, Chapter 11] to show that, for $x_-\in\mathcal{P}(K)$ and $x_+\in\mathcal{P}(F)$ with $|x_-|=|x_+|$, $\mathcal{M}(x_-,x_+;H_\bullet)$ is a smooth compact manifold of dimension 0. The *continuation map* $\Phi^{H_\bullet}\colon\mathrm{CF}^k(F)\to\mathrm{CF}^k(K)$ *induced by* H_s on the cochain level is defined as

$$\Phi^{H_\bullet}(x_+)=\sum_{|x_-|=k}\#_2\mathcal{M}(x_-,x_+;H_\bullet)x_-,$$

where $\#_2\mathcal{M}(x_-,x_+;H_\bullet)$ is the count modulo 2 of components in $\mathcal{M}(x_-,x_+;H_\bullet)$. The map

$$[\Phi^{H_\bullet}]\colon\mathrm{HF}^*(F)\longrightarrow\mathrm{HF}^*(K)$$

is independent of the chosen monotone homotopy and we can denote it by $[\Phi^{K,F}]$. Consider the monotone homotopy

$$H_s = K + \beta(s)(F - K)$$

described in Section 2.2. We note that $\partial_s H_s \leq 0$ since $F \leq K$ and $\beta' \geq 0$. Thus the action estimate given by Lemma 16 for homotopies yields

$$\mathcal{A}_K(x_-) \leq \mathcal{A}_H(x_+) + \sup_{\substack{s\in[-S,S],\\ t\in S^1, p\in D^R}} \partial_s H_s(t,p) \leq \mathcal{A}_H(x_+)$$

for $x_- \in \mathcal{P}(K)$ and $x_+ \in \mathcal{P}(F)$. Therefore, the continuation map decreases the action and hence induces maps

$$[\Phi^{K,F}_{(a,b)}]\colon \mathrm{HF}^*_{(a,b)}(F) \longrightarrow \mathrm{HF}^*_{(a,b)}(K)$$

that commute with the inclusion and restriction maps as follows [Ritter 2013, Section 8]:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathrm{HF}^*_{(a,b)}(F) & \xrightarrow{[\iota^{b,c}_{a,a}]} & \mathrm{HF}^*_{(a,c)}(F) & \xrightarrow{[\pi^{c,c}_{a,b}]} & \mathrm{HF}^*_{(b,c)}(F) & \longrightarrow \cdots \\
& \Big\downarrow {\scriptstyle [\Phi^{K,F}_{(a,b)}]} & & \Big\downarrow {\scriptstyle [\Phi^{K,F}_{(a,c)}]} & & \Big\downarrow {\scriptstyle [\Phi^{K,F}_{(b,c)}]} & \\
\cdots \longrightarrow & \mathrm{HF}^*_{(a,b)}(K) & \xrightarrow[{[\iota^{b,c}_{a,a}]}]{} & \mathrm{HF}^*_{(a,c)}(K) & \xrightarrow[{[\pi^{c,c}_{a,b}]}]{} & \mathrm{HF}^*_{(b,c)}(K) & \longrightarrow \cdots
\end{array} \tag{3}$$

Suppose we are given another Hamiltonian $H \geq K$. Then we have the commutative diagram

$$\mathrm{HF}^*_{(a,b)}(F) \xrightarrow{[\Phi^{K,F}_{(a,b)}]} \mathrm{HF}^*_{(a,b)}(K) \xrightarrow{[\Phi^{H,K}_{(a,b)}]} \mathrm{HF}^*_{(a,b)}(H), \qquad [\Phi^{H,F}_{(a,b)}] = [\Phi^{H,K}_{(a,b)}] \circ [\Phi^{K,F}_{(a,b)}].$$

As opposed to the closed case, for completion of Liouville domains, continuation maps do not necessarily yield isomorphisms. One case in which they do is when both Hamiltonians have the same slope.

Lemma 20 [Ritter 2009, Section 2.12] *Let $F, K \in \mathcal{H}$ and suppose τ_F and τ_K are both contained in an open interval that does not intersect* $\mathrm{Spec}(\partial D, \alpha)$. *Then, if $\tau_F \leq \tau_K$,*

$$[\Phi^{K,F}]\colon \mathrm{HF}^*(F) \longrightarrow \mathrm{HF}^*(K)$$

is an isomorphism. Under $[\Phi^{K,F}]$, 1_F and 1_K are identified.

In action windows, we have the following isomorphisms.

Lemma 21 [Viterbo 1999, Proposition 1.1] *Let $H_\bullet$ be a monotone homotopy between $H_\pm \in \mathcal{H}$ that is constant in the s–coordinate for $|s| > S > 0$. Suppose $a_s, b_s\colon \mathbb{R} \to \mathbb{R}$ are functions which are constant* *outside* $[-S, S]$ *and* $a_s, b_s \notin \mathrm{Spec}(H_s)$ *for all* s. *Then*,

$$[\Phi^{H_-,H_+}]\colon \mathrm{HF}^*_{(a_+,b_+)}(H_+) \xrightarrow{\cong} \mathrm{HF}^*_{(a_-,b_-)}(H_-)$$

for $a_\pm = \lim_{s\to\pm\infty} a_s$ *and* $b_\pm = \lim_{s\to\pm\infty} b_s$.

3.2 Filtered symplectic cohomology

Equip the set of admissible Hamiltonians $\mathcal{H}^0$ negative on D with the partial order

$$H \preceq K \iff H(t,p) \le K(t,p) \quad \text{for all } (t,p) \in S^1 \times \widehat{D}.$$

Let $\{H_i\}_{i\in I} \subset \mathcal{H}^0$ be a cofinal sequence with respect to $\preceq$. We define the *symplectic cohomology of* D as the direct limit

$$\mathrm{SH}^*_{(a,b)}(D) = \varinjlim_{H_i} \mathrm{HF}^*_{(a,b)}(H_i)$$

taken with respect to the continuation maps

$$[\Phi^{H_j,H_i}_{(a,b)}]\colon \mathrm{HF}^*_{(a,b)}(H_i) \longrightarrow \mathrm{HF}^*_{(a,b)}(H_j)$$

for $i < j$. We let $\mathrm{SH}^*(D) = \mathrm{SH}^*_{(-\infty,+\infty)}(D)$. The long exact sequence on Floer cohomology carries through the direct limit and we also have a long exact sequence on symplectic cohomology:

$$\begin{array}{ccc} \mathrm{SH}^*_{(a,b)}(D) & \xrightarrow{[\iota^{b,c}_{a,a}]} & \mathrm{SH}^*_{(a,c)}(D) \\ {\scriptstyle [+1]}\uparrow & & \downarrow{\scriptstyle [\pi^{c,c}_{a,b}]} \\ & \mathrm{SH}^*_{(b,c)}(D) & \end{array}$$

The Viterbo map Let $F \in \mathcal{H}$ and consider $H \in \mathcal{H}^0$ with $\tau_H = \tau_F$. Then, by Lemma 20, we have $\mathrm{HF}^*(F) \cong \mathrm{HF}^*(H)$ and there exist, by the definition of symplectic cohomology, a map

$$j_F\colon \mathrm{HF}^*(F) \cong \mathrm{HF}^*(H) \longrightarrow \mathrm{SH}^*(D) \tag{4}$$

sending each element of $\mathrm{HF}^*(H)$ to its equivalence class. Now, for $H \in \mathcal{H}^0$ with slope $\tau_H < T_0$ we can define, by Lemma 17, the map $v^*\colon \mathrm{H}^*(D) \to \mathrm{SH}^*(D)$ first introduced in [Viterbo 1999] by

$$\mathrm{H}^*(D) \xrightarrow{\Phi_H} \mathrm{HF}^*(H) \xrightarrow{j_H} \mathrm{SH}^*(D), \qquad v^* = j_H \circ \Phi_H.$$

This map induces a unit on symplectic cohomology. Recall that 1_H denotes the unit in $\mathrm{HF}^*(H)$ (see Remark 18).

Theorem 22 [Ritter 2013] *The ring structure on* $\mathrm{HF}^*(H)$ *induces a ring structure on* $\mathrm{SH}^*(D)$. *The unit on* $\mathrm{SH}^*(D)$ *is given by the image of the unit* $e_D \in \mathrm{H}^*(D)$ *under the map* v^*. *Moreover,*

$$v^*(e_D) \in \mathrm{im}\big([\iota^{\varepsilon,\infty}_{-\infty,\infty}] \colon \mathrm{SH}^*_{(-\infty,\varepsilon)}(D) \longrightarrow \mathrm{SH}^*_{(-\infty,\infty)}(D)\big).$$

4 Spectral invariants and spectral norm

4.1 Spectral invariants

Denote by $\mathrm{Ham}_c(D, \mathrm{d}\lambda)$ the group of compactly supported Hamiltonian diffeomorphisms of $(D, \mathrm{d}\lambda)$ and by $\mathrm{Symp}_c(D, \mathrm{d}\lambda)$ the group of compactly supported symplectomorphisms of $(D, \mathrm{d}\lambda)$. The Hofer norm of a compactly supported Hamiltonian $H \in \mathcal{C}(D)$ is defined as

$$\|H\| = \int_0^1 \Big(\sup_{p\in D} H(t,p) - \inf_{p\in D} H(t,p)\Big)\,\mathrm{d}t.$$

Using the Hofer norm, we can define a bi-invariant metric [Hofer 1990; Lalonde and McDuff 1995] on $\mathrm{Ham}_c(D, \mathrm{d}\lambda)$ by

$$d_H(\varphi,\psi) = d_H(\varphi\psi^{-1}, \mathrm{id}), \quad d_H(\varphi, \mathrm{id}) = \inf\{\|H\| \mid \varphi = \varphi_H\}.$$

Recall that $\mathcal{C}(D)$ forms a group under the multiplication

$$H \# K(t,p) = H(t,p) + K(t, (\varphi^t_H)^{-1}(p)),$$

with the inverse of some $H \in \mathcal{C}(D)$ given by $\overline{H}(t,p) = -H(t, \varphi^t_H(p))$.

From Lemma 17 and by the definition of $\mathrm{HF}^*(H)$ for $H \in \mathcal{C}(D)$, we know that $\mathrm{HF}^*(H) \cong \mathrm{H}^*(D)$. For $\beta \in \mathrm{H}^*(D)$, we define, following [Schwarz 2000], the *spectral invariant of* H *relative to* β *as*

$$c(\beta, H) = \inf\big\{\ell \in \mathbb{R} \mid \Phi_H(\beta) \in \mathrm{im}\big([\iota^{\ell,\infty}_{-\infty,-\infty}] \colon \mathrm{HF}^*_{(-\infty,\ell)}(H) \to \mathrm{HF}^*(H)\big)\big\},$$

which is, by exactness of the long exact sequence (2), equivalent to

$$c(\beta, H) = \inf\{\ell \in \mathbb{R} \mid [\pi^{\infty,\infty}_{-\infty,\ell}] \circ \Phi_H(\beta) = 0\}.$$

The following proposition gathers all the properties of spectral invariants we need for the rest of the text. Proofs of these properties can be found[5] in [Frauenfelder and Schlenk 2007, Section 5].

Proposition 23 *Let* $\beta, \eta \in H^*(D)$ *and let* $H, K \in \mathcal{C}(D)$. *Then:*

- ***Continuity*** $\int_0^1 \min_{x\in D}(K-H)\,\mathrm{d}t \le c(\beta,H) - c(\beta,K) \le \int_0^1 \max_{x\in D}(K-H)\,\mathrm{d}t$.
- ***Spectrality*** $c(\beta, H) \in \mathrm{Spec}(H)$.
- ***Triangle inequality*** $c(\beta \smile \eta, H \# K) \le c(\beta, H) + c(\eta, K)$.
- ***Monotonicity*** *If* $H(t,x) \le K(t,x)$ *for all* $(t,x) \in [0,1] \times D$, *then* $c(\beta, H) \ge c(\beta, K)$.

[5] Note that the signs for continuity and monotonicity differ from [Frauenfelder and Schlenk 2007, Section 5] because of differences in sign conventions.

Remark 24 The continuity property of Proposition 23 allows us to define spectral invariants of compactly supported continuous Hamiltonians $H \in C_c^0([0,1] \times D)$. They satisfy continuity, the triangle inequality and monotonicity.

4.1.1 Additional properties of *c* The following lemma assures us that spectral invariants are well-defined on $\mathrm{Ham}_c(D, \mathrm{d}\lambda)$. The proof relies on the spectrality and the triangle inequality.

Lemma 25 *Let $H, K \in \mathcal{C}(D)$ such that $\varphi_H = \varphi_K$ and let $\beta \in \mathrm{H}^*(D)$. Then,*

$$c(\beta, H) = c(\beta, K).$$

Proof We have $\varphi_{H \# \overline{K}} = \varphi_0 = \mathrm{id}$ and in that case $\mathrm{Spec}(H \# \overline{K}) = \{0\}$. Now, by spectrality of spectral invariants, $c(\beta, H \# \overline{K}) = 0$. Thus, the triangle inequality yields

$$c(\beta, H) = c(\beta, H \# \overline{K} \# K) \le c(\beta, H \# \overline{K}) + c(\beta, K) = c(\beta, K).$$

Repeating the same argument with $K \# \overline{H}$ instead of $H \# \overline{K}$, we obtain $c(\beta, K) \le c(\beta, H)$, which concludes the proof. □

The spectral invariant with respect to the cohomological unit admits an implicit definition which depends on the spectral invariants with respect to all other cohomology classes in $\mathrm{H}^*(D)$. This follows directly from the triangle inequality.

Lemma 26 *Let $H \in \mathcal{C}(D)$. Then,*

$$c(1, H) = \max_{\beta \in \mathrm{H}^*(D)} c(\beta, H).$$

Proof Let $\beta \in H^*(D)$. By the definition of the unit and the concatenation of Hamiltonians, we have

$$c(\beta, H) = c(\beta \smile 1, H) = c(\beta \smile 1, 0 \# H).$$

Then, since $c(\beta, 0) = 0$, the triangle inequality guarantees that

$$c(\beta, H) = c(\beta \smile 1, 0 \# H) \le c(\beta, 0) + c(1, H) = c(1, H).$$

The choice of β being arbitrary, this concludes the proof. □

4.1.2 The symplectic contraction principle We conclude this section by recalling the symplectic contraction technique introduced in [Polterovich 2014, Section 5.4]. This principle allows one to describe the effect of the Liouville flow $\{\psi_Y^{\log r}\}_{0<r<1}$ on spectral invariants.

First, we need to describe how the Liouville flow acts on the symplectic form ω of D and on compactly supported Hamiltonians on D. Since $L_Y \omega = \omega$, we have that the Liouville flow contracts the symplectic form:

$$(\psi_Y^{\log r})^* \omega = r\omega.$$

Now, consider a Hamiltonian $H \in \mathcal{C}(D)$ supported in $U \subset D$. For fixed $0 < r < 1$ define the Hamiltonian

$$H_r(t,x) = \begin{cases} rH(t,(\psi_Y^{\log r})^{-1}(x)) & \text{if } x \in \psi_Y^{\log r}(U), \\ 0 & \text{if } x \notin \psi_Y^{\log r}(U). \end{cases} \tag{5}$$

It then follows from the two previous equations that $\operatorname{Spec}(H_r) = r\operatorname{Spec}(H)$. This allows one to prove:

Lemma 27 [Polterovich 2014] *Suppose $H \in \mathcal{C}(D)$ and let $H_r \in \mathcal{C}(D)$ be as in (5). Then,*

$$c(1,H_r) = rc(1,H).$$

4.2 Spectral norm

We define the *spectral norm $\gamma(H)$ of $H \in \mathcal{C}(D)$* as

$$\gamma(H) = c(1,H) + c(1,\overline{H}).$$

For $\varphi \in \operatorname{Ham}_c(D, \mathrm{d}\lambda)$ such that $\varphi = \varphi_H$, define

$$\gamma(\varphi) = \gamma(H).$$

By virtue of Lemma 25, this is well-defined.

From [Frauenfelder and Schlenk 2007, Section 7], we have the following theorem which justifies calling γ a norm.

Theorem 28 *Let $\varphi, \psi \in \operatorname{Ham}_c(D, \mathrm{d}\lambda)$ and let $\chi \in \operatorname{Symp}_c(D, \mathrm{d}\lambda)$. Then:*

- ***Nondegeneracy*** $\gamma(\mathrm{id}) = 0$ *and* $\gamma(\varphi) > 0$ *if* $\gamma \neq \mathrm{id}$.
- ***Triangle inequality*** $\gamma(\varphi\psi) \leq \gamma(\varphi) + \gamma(\psi)$.
- ***Symplectic invariance*** $\gamma(\chi \circ \varphi \circ \chi^{-1}) = \gamma(\varphi)$.
- ***Symmetry*** $\gamma(\varphi) = \gamma(\varphi^{-1})$.
- ***Hofer bound*** $\gamma(\varphi) \leq d_H(\varphi, \mathrm{id})$.

5 Cohomological barricades on Liouville domains

Ganor and Tanny [2023] introduced a particular perturbation of Hamiltonians compactly supported inside *contact incompressible boundary* domains (CIB) of closed aspherical symplectic manifolds. For instance, if $U \subset M$ is an incompressible open set which is a Liouville domain, then U is a CIB. In Floer homology, the aforementioned Hamiltonian perturbation, which is called a barricade, prohibits the existence of Floer trajectories exiting and entering the CIB. We consider barricades in the particular case of Liouville domains and adapt them to Floer cohomology.

In the present setting, we define barricades for a special class of admissible Hamiltonians.

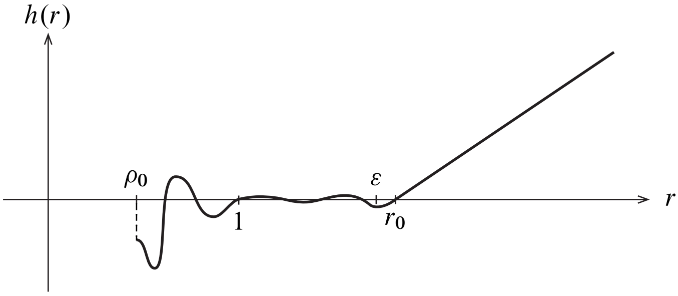


Figure 6: An r_0–barricade-admissible Hamiltonian.

Definition 29 A Hamiltonian H is said to be r_0-*barricade-admissible* if $H \in \mathcal{H}_{r_0}$ and the following conditions hold:

- $H(t,x,r) = h(r)$ on $\widehat{D} \setminus D^{\rho_0}$ for some $\rho_0 \in (0,1)$.
- $h(r)$ is C^2–small on $(1, r_0 - \varepsilon)$.
- $h(r)$ is strictly convex on $(r_0 - \varepsilon, r_0)$.

(See Figure 6.) Here $\varepsilon > 0$ is small enough so that $1 < r_0 - \varepsilon$. We denote the set of r_0–barricade-admissible Hamiltonians by $\overline{\mathcal{H}}_{r_0}$.

We say that $(F_\bullet, J)$ is an r_0-*barricade-admissible pair* if $F_\bullet$ is a monotone homotopy such that $F_s \in \overline{\mathcal{H}}_{r_0}$ for all s and J is an admissible almost-complex structure.

Remark 30 By Definition 19, the extension H^τ of any Hamiltonian H compactly supported in D can be chosen so that it is r_0–barricade-admissible.

Definition 31 Let $r_0 > 1$ and $0 < \varepsilon < r_0 - 1$. Define $B_{r_0,\varepsilon} = D^{r_0-\varepsilon} \setminus D$, where, for $\rho > 0$, $D^\rho = \Psi_Y^{\log\rho}(D)$. Suppose $(F_\bullet, J)$ is an r_0–barricade-admissible pair from F_+ to F_-. We say that $(F_\bullet, J)$ admits a barricade on $B_{r_0,\varepsilon}$ if for every $x_\pm \in \mathcal{P}(F_\pm)$ and every Floer trajectory $u\colon \mathbb{R} \times S^1 \to \widehat{D}$ connecting $x_\pm$, we have, for $D_{\mathrm{b}} := D^{r_0-\varepsilon} = D \cup B_{r_0,\varepsilon}$:

(1) If $x_- \in D$, then $\mathrm{im}(u) \subset D$.

(2) If $x_+ \in D_b$, then $\mathrm{im}(u) \subset D_{\mathrm{b}}$.

Remark 32 In the language of [Ganor and Tanny 2023], a barricade on $B_{r_0,\varepsilon}$ as described above would be called a barricade in $D^{r_0-\varepsilon}$ around D.

5.1 How to construct barricades

To construct barricades, we need to consider special classes of pairs of Hamiltonians and almost-complex structures. These are defined using a refinement of Definition 3.5 in [Ganor and Tanny 2023].

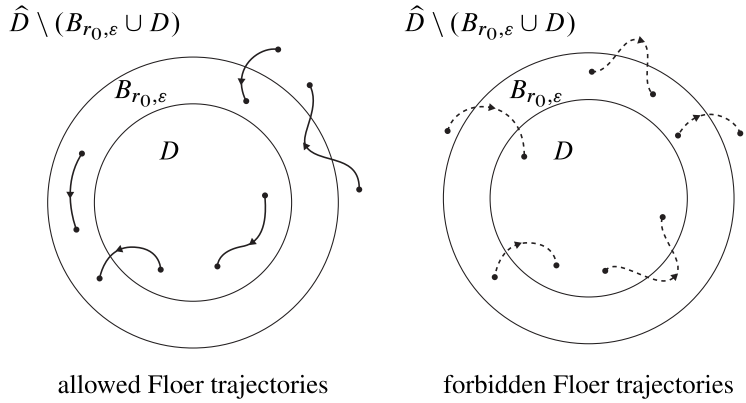


Figure 7: Floer cylinders in a barricade. The arrows follow the direction of the Floer differential and the continuation map: from x_+ to x_-.

Definition 33 Let $r_0 > 1$, $\sigma \in (0, +\infty) \setminus \mathrm{Spec}(\partial D, \lambda)$ and $0 < \varepsilon < r_0 - 1$. An r_0–barricade-admissible pair $(F_\bullet, J)$ admits a cylindrical bump of slope σ on $B_{r_0,\varepsilon}$ if:

(1) $F = 0$ on $\partial B_{r_0,\varepsilon} \times S^1 \times \mathbb{R}$.

(2) $JY = R_\alpha$ for Y the Liouville vector field on D, on a neighborhood of $\partial B_{r_0,\varepsilon}$; ie J is cylindrical near $\partial B_{r_0,\varepsilon} = \partial D \sqcup (\{r_0 - \varepsilon\} \times \partial D)$.

(3) $\nabla_J F = \sigma Y$ near $(\{1\} \times \partial D) \times S^1 \times \mathbb{R}$ and $\nabla_J F = -\sigma Y$ near $(\{r_0 - \varepsilon\} \times \partial D) \times S^1 \times \mathbb{R}$. Here, ∇_J denotes the gradient induced by the metric g_J.

(4) All 1–periodic orbits of $F_\pm$ contained in $B_{r_0,\varepsilon}$ are critical points with values in the interval $(-\sigma, \sigma)$. (In particular, $\sigma < T_0$.)

A cohomological adaptation of Lemma 3.3 in [Ganor and Tanny 2023] yields the following action estimates for pairs with cylindrical bumps.

Lemma 34 *Suppose that the r_0–barricade-admissible pair (F, J) admits a cylindrical bump of slope σ on $B_{r_0,\varepsilon}$. For every finite-energy solution u connecting $x_\pm \in \mathcal{P}(F_\pm)$:*

(1) $\operatorname{im} x_- \subset D$ *and* $\operatorname{im} x_+ \subset \widehat{D} \setminus D \Longrightarrow \mathcal{A}_{F_+}(x_+) > \sigma$.

(2) $\operatorname{im} x_+ \subset D_{\mathrm{b}}$ *and* $\operatorname{im} x_- \subset \widehat{D} \setminus D_{\mathrm{b}} \Longrightarrow \mathcal{A}_{F_-}(x_-) < -\sigma$.

See Figure 7.

Lemma 34 and the maximum principle are all we need to prove that every pair with a cylindrical bump admits a barricade. More precisely, we have:

Proposition 35 *Let (F, J) be an r_0–barricade-admissible pair with a cylindrical bump of slope σ on $B_{r_0,\varepsilon}$. Then, (F, J) admits a barricade on $B_{r_0,\varepsilon}$.*

Proof Suppose $u \colon \mathbb{R} \times S^1 \to \widehat{D}$ is a Floer trajectory between $x_\pm \in \mathcal{P}(F_\pm)$. We only need to study the case where $\operatorname{im} x_- \subset D$ and the case where $\operatorname{im} x_+ \subset D_{\mathrm{b}}$.

Suppose that $\operatorname{im} x_- \subset D$. We first establish that x_+ must lie inside D. Indeed, if $\operatorname{im} x_+ \subset \widehat{D} \setminus D$, part (1) of Lemma 34 assures us that $\mathcal{A}_{F_+}(x_+) > \sigma$, which contradicts the fact that orbits on $\widehat{D} \setminus D$ must have action in the interval $(-\sigma, \sigma)$ by the construction of the cylindrical bump. Therefore, $\operatorname{im} x_+ \subset D$ as desired. Now, since $\operatorname{im} x_\pm \subset D$, the maximum principle guarantees that $\operatorname{im} u \subset D$.

To finish the proof, we look at the case where $\operatorname{im} x_+ \subset D_{\mathrm{b}}$. Similarly to the previous case, we prove that x_- also lies inside D_{b}. If $\operatorname{im} x_- \subset \widehat{D} \setminus D_{\mathrm{b}}$, part (2) of Lemma 34 imposes $\mathcal{A}_{F_-}(x_-) < -\sigma$, which is again impossible by construction of the cylindrical bump. Therefore, $\operatorname{im} x_- \subset D_{\mathrm{b}}$ and the maximum principle implies $\operatorname{im} u \subset D_{\mathrm{b}}$. □

Given a pair (F, J) and $\sigma > 0$ small, we can add to F a $\mathcal{C}^\infty$–small radial bump function χ with support inside $B_{r_0,\varepsilon}$ such that $(F + \chi, J)$ has a cylindrical bump of slope σ on $B_{r_0,\varepsilon}$. By Proposition 35, the perturbed pair will also admit a barricade on $B_{r_0,\varepsilon}$. A second perturbation of the Hamiltonian term at its ends, under which the barricade survives, allows us to achieve Floer regularity for the pair. This procedure is carried out carefully in [Ganor and Tanny 2023, Section 9] and yields the following.

Theorem 36 [Ganor and Tanny 2023] *Let $F_\bullet$ be a monotone homotopy. Then, there exists a $\mathcal{C}^\infty$–small perturbation $f_\bullet$ of $F_\bullet$ and an almost-complex structure J such that the pairs $(f_\bullet, J)$ and $(f_\pm, J)$ are Floer-regular and have a barricade on $B_{r_0,\varepsilon}$.*

5.2 Decomposition of the Floer cochain complex

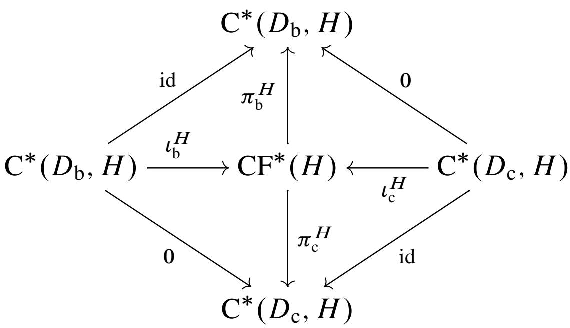
Let us investigate what structure barricades impose on the Floer cochain complex. Let $H \in \overline{\mathcal{H}}_{r_0}$ and suppose the pair (H, J) admits a barricade on $B_{r_0,\varepsilon}$. For an open subset $U \subset \widehat{D}$, denote by $\mathrm{C}^*(U, H)$ the set of 1–periodic orbits of H in U. By the definition of the differential ∂ on Floer cohomology, $\mathrm{C}^*(D_{\mathrm{b}}, H)$ is closed under ∂ and it therefore forms a subcomplex of $\mathrm{CF}^*(H)$. Moreover, for $D_{\mathrm{c}} = \widehat{D} \setminus D_{\mathrm{b}}$, we also have that

$$\mathrm{C}^*(D_{\mathrm{c}}, H) = \frac{\mathrm{CF}^*(H)}{\mathrm{C}^*(D_{\mathrm{b}}, H)}$$

is a well-defined cochain complex. In terms of vector spaces, we have the decomposition

$$\mathrm{CF}^*(H) \cong \mathrm{C}^*(D_{\mathrm{b}}, H) \oplus \mathrm{C}^*(D_{\mathrm{c}}, H).$$

The direct product gives us injections ι_{b}^H, ι_{c}^H and projections π_{b}^H, π_{c}^H for which the diagram



commutes and the equation

$$\iota_{\mathrm{b}}^{H} \circ \pi_{\mathrm{b}}^{H}(q) + \iota_{\mathrm{c}}^{H} \circ \pi_{\mathrm{c}}^{H}(q) = q$$

holds for any $q \in \mathrm{CF}^*(H)$. Here, the projection π_{c}^{H} coincides with the canonical projection

$$\mathrm{CF}^*(H) \twoheadrightarrow \frac{\mathrm{CF}^*(H)}{\mathrm{C}^*(D_{\mathrm{b}}, H)}.$$

The differential ∂_{b} on $\mathrm{C}^*(D_{\mathrm{b}})$ is simply the restriction of the differential ∂ of $\mathrm{CF}^*(H)$ on $\mathrm{C}^*(D_{\mathrm{b}})$. The differential ∂_{c} on $\mathrm{C}^*(D_{\mathrm{c}})$ is the quotient complex differential defined by

$$\partial_{\mathrm{c}} \pi_{\mathrm{c}}^{H}(p) = \pi_{\mathrm{c}}^{H}(\partial p).$$

5.2.1 Continuation maps Let $(F_\bullet, J)$ be an r_0–barricade-admissible pair that admits a barricade on $B_{r_0,\varepsilon}$. Then, since the continuation map $\Phi_{F_\bullet} : \mathrm{CF}^*(F_+) \to \mathrm{CF}^*(F_-)$ counts Floer trajectories of F connecting 1–periodic orbits of F_+ to 1–periodic orbits of F_-, it restricts, due to the barricade, to a chain map

$$\Phi_F^{\mathrm{b}} : \mathrm{C}^*(D_{\mathrm{b}}, F_+) \longrightarrow \mathrm{C}^*(D_{\mathrm{b}}, F_-).$$

Moreover, in virtue of Lemma 38 below, Φ_F projects to a chain map

$$\Phi_F^{\mathrm{c}} : \mathrm{C}^*(D_{\mathrm{c}}, F_+) \longrightarrow \mathrm{C}^*(D_{\mathrm{c}}, F_-)$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{CF}^*(F_+) & \xrightarrow{\ \Phi_{F_\bullet}\ } & \mathrm{CF}^*(F_-) \\ {\scriptstyle \pi_{\mathrm{b}}^{+}} \big\downarrow & & \big\downarrow {\scriptstyle \pi_{\mathrm{b}}^{-}} \\ \mathrm{C}^*(D_{\mathrm{c}}, F_+) & \xrightarrow[\ \Phi_{F_\bullet}^{\mathrm{c}}\ ]{} & \mathrm{C}^*(D_{\mathrm{c}}, F_-) \end{array}$$

where we write $\pi_{\mathrm{b}}^{+} = \pi_{\mathrm{b}}^{F_+}$ and $\pi_{\mathrm{b}}^{-} = \pi_{\mathrm{b}}^{F_-}$.

5.2.2 Chain homotopies Let $(F_\pm, J)$ be r_0–barricade-admissible pairs that admit cylindrical bumps of slope σ on $B_{r_0,\varepsilon}$ such that F_+ and F_- have the same slope $\tau_+ = \tau_-$ at infinity. Consider the linear homotopy

$$F_s = F_- + \beta(s)(F_+ - F_-),$$

where $\beta : \mathbb{R} \to [0,1]$ is a smooth function such that $\beta(s) = 0$ for $s \le -1$, $\beta(s) = 1$ for $s \ge 1$ and $\beta'(s) > 0$ for all $s \in (-1,1)$. Denote by $\overline{F}_\bullet$ the inverse homotopy defined by $\overline{F}_s = F_{-s}$. For $\rho > 1$ large, we define the concatenation $F \,\#\, \overline{F}_\bullet$ as

$$(F \,\#\, \overline{F})_s = \begin{cases} F_{s+\rho} & \text{for } s \le 0, \\ \overline{F}_{s-\rho} & \text{for } s \ge 0. \end{cases}$$

Using the definition of $F_\bullet$ and $\overline{F}_\bullet$, we can simply write

$$(F \,\#\, \overline{F})_s = F_- + \beta_\rho(s)(F_+ - F_-)$$

for $\beta_\rho(s)=\beta(-|s|+\rho)$. The homotopy $F\#\overline{F}_\bullet$ generates the composition of continuation homomorphisms $\Phi_F\circ\Phi_{\overline{F}}\colon \mathrm{CF}^*(F_-)\to\mathrm{CF}^*(F_-)$, which is chain homotopic to the identity on $\mathrm{CF}^*(F_-)$,

$$\Phi_{F_\bullet}\circ\Phi_{\overline{F}_\bullet}-\mathrm{id}_-=\partial_-\circ\Psi_--\Psi_-\circ\partial_-$$

for $\Psi_-\colon\mathrm{CF}^*(F_-)\to\mathrm{CF}^{*-1}(F_-)$ and ∂_- the differential on $\mathrm{CF}^*(F_-)$. The chain homotopy Ψ_- is built by counting Floer solutions of the homotopy $\{\Gamma^\kappa\}_{\kappa\in[0,1]}$ between $F\#\overline{F}_\bullet$ and the constant homotopy F_-, which is defined by

$$\Gamma^\kappa_s=F_-+\kappa\beta_\rho(s)(F_+-F_-).$$

For $x\in\mathcal{P}(F_-)$ and $y\in\mathcal{P}(F_+)$, define

$$\mathcal{M}^\Gamma(x,y)=\{(\kappa,u)\mid\kappa\in[0,1],\ u\in\mathcal{M}(x,y;\Gamma^k_\bullet)\}.$$

We can perturb Γ with a C^∞–small function in order to make it regular [Audin and Damian 2014, Chapter 11]. Now, since the pairs $(F_\pm,J)$ admit cylindrical bumps of slope σ on $B_{r_0,\varepsilon}$, and thus have barricades on $B_{r_0,\varepsilon}$, solutions to the parametric Floer equation for Γ^κ also admit cylindrical bumps of slope σ on $B_{r_0,\varepsilon}$ and have barricades on $B_{r_0,\varepsilon}$. To see this, first fix $\kappa\in[0,1]$, We need to show that Γ^κ satisfies conditions (1) through (4) of Definition 33. For (1), we have, on $\partial B_{r_0,\varepsilon}\times S^1\times\mathbb{R}$,

$$\Gamma^\kappa=F_-+\kappa\beta_\rho(s)(F_+-F_-)=0+\kappa\beta_\rho(s)(0-0)=0.$$

Condition (2) is automatically satisfied since J is fixed. For condition (3), we have on $(\{1\}\times\partial D)\times S^1\times\mathbb{R}$,

$$\nabla_J\Gamma^\kappa=\nabla_JF_-+\kappa\beta_\rho(s)(\nabla_JF_+-\nabla_JF_-)=\sigma Y+\kappa\beta_\rho(s)(\sigma Y-\sigma Y)=\sigma Y$$

and, by the same computation, $\nabla_J\Gamma^\kappa=-\sigma Y$ on $(\{r_0-\varepsilon\}\times\partial D)\times S^1\times\mathbb{R}$. Condition (4) is also satisfied since $\Gamma^\kappa_{\pm\infty}=F_-$. All of this still holds with regular perturbations of Γ.

Lemma 37 *Let $F_-,F_+\in\overline{\mathcal{H}}_{r_0}$ with same slope at infinity and suppose they both admit barricades on $B_{r_0,\varepsilon}$. Furthermore, suppose that solutions to the parametric Floer equation for Γ^κ also admit barricades on $B_{r_0,\varepsilon}$. Then, for any C^∞–small perturbation Γ' of Γ which satisfies $\mathcal{P}(F'_\pm)=\mathcal{P}(F_\pm)$, Floer trajectories in $\mathcal{M}^{\Gamma'}$ follow the rules of the barricade on $B_{r_0,\varepsilon}$.*

Proof The proof follows the same ideas as the proof of Proposition 9.21 in [Ganor and Tanny 2023]. By Gromov compactness, any sequence $(\kappa_n,u_n)\in\mathcal{M}^\Gamma(x_-,y_+)$ of solutions to the parametric Floer equation converges, up to taking a subsequence, to a broken trajectory $(\kappa,\bar{v})$, where $\bar{v}=(v_1,\dots,v_k,w,v'_1,\dots,v'_\ell)$ connects two orbits $x_\pm\in\mathcal{P}(F_\pm)$. The fact that $F_\pm$ both admit a barricade on $B_{r_0,\varepsilon}$ assures us that

- $x_-\in D\implies\bar{v}\subset D$,
- $x_+\in D\implies\bar{v}\subset D_\flat$.

Now, consider a sequence of regular homotopies $\{\Gamma_n\}_n$ with ends $\lim_{s\to\pm\infty}\Gamma_{s,n}=F_{n\pm}$ converging to Γ such that $\mathcal{P}(F_{n\pm})=\mathcal{P}(F_\pm)$ for all n. Then, the above two implications regarding broken trajectories imply that every trajectory $(\kappa_n,u'_n)\in\mathcal{M}^{\Gamma'}(x_-,x_+)$, for $x_\pm\in\mathcal{P}(F_\pm)$, obey the rules of the barricade. □

Thus, Ψ_- restricts to a map $\Psi_-^b\colon \mathrm{C}^*(D_\mathrm{b}, F_-) \to \mathrm{C}^{*-1}(D_\mathrm{b}, F_-)$ and by Lemma 39 below, we can define its projection $\Psi_-^c\colon \mathrm{C}^*(D_\mathrm{c}, F_-) \to \mathrm{C}^{*-1}(D_\mathrm{c}, F_-)$.

Technical lemmas When adapting computations from homology to cohomology, we often have to rely on quotient complexes instead of subcomplexes. Here are a few simple results from homological algebra which will be useful in that regard. Let (A, d_A) and (C, d_C) be cochain complexes and let $B \subset A$ and $D \subset C$ be subcomplexes.

Lemma 38 *Suppose $f\colon (A, B) \to (C, D)$ is a chain map. Then, there exists a unique chain map $\bar{f}\colon A/B \to C/D$ such that the following diagram commutes*

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & C \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_D} \\ A/B & \xrightarrow[\;\bar{f}\;]{} & C/D \end{array}$$

for π_B and π_D the canonical projections. It follows that, on cohomology, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^*(A) & \xrightarrow{\;[f]\;} & \mathrm{H}^*(C) \\ \downarrow{\scriptstyle [\pi_B]} & & \downarrow{\scriptstyle [\pi_D]} \\ \mathrm{H}^*(A/B) & \xrightarrow[\;[\bar{f}]\;]{} & \mathrm{H}^*(C/D) \end{array}$$

Proof Define, for all $x \in A$,

$$\bar{f}(\pi_B(x)) = \pi_D(f(x)).$$

We first need to show that $\bar{f}$ is well-defined. Suppose $x' = x + b$ for $x \in A$ and $b \in B$. Then, since f restricts to a map from B to D, there exists $d \in D$ such that $f(b) = d$ and we have

$$\bar{f}(\pi_B(x')) = \pi_D(f(x + b)) = \pi_D(f(x) + d) = \pi_D(f(x)).$$

Thus, $\bar{f}$ is well-defined.

To prove uniqueness, we simply use the definition of $\bar{f}$. Suppose we have another map $\bar{g}\colon A/B \to C/D$ which makes the above diagram commute as well. Then, for all $x \in A$,

$$\bar{f}(\pi_B(x)) - \bar{g}(\pi_B(x)) = \pi_D(f(x)) - \pi_D(f(x)) = 0. \qquad \square$$

Lemma 39 *Suppose $f\colon (A, B) \to (C, D)$ and $g\colon (C, D) \to (A, B)$ are chain maps such that $f \circ g$ is chain homotopic to the identity*

$$f \circ g - \mathrm{id}_C = d_C \circ \psi - \psi \circ d_C,$$

where the chain homotopy is a map $\psi\colon (C, D) \to (C, D)$. Then, $\bar{f} \circ \bar{g}\colon C/D \to C/D$ is also chain homotopic to the identity.

Proof Since the chain homotopy $\psi\colon (C,D)\to(C,D)$ is a chain map of pairs, Lemma 38 allows us to define its projection $\bar\psi\colon C/D\to C/D$. Thus, for all $y\in C$,

$$\begin{aligned}\bar f\circ\bar g(\pi_D(y))-\mathrm{id}_{C/D}(\pi_D(y))&=\bar f\circ\pi_B(g(y))-\pi_D(\mathrm{id}_C(y))\\&=\pi_D(f\circ g(y))-\pi_D(\mathrm{id}_C(y))\\&=\pi_D((d_C\circ\psi-\psi\circ d_C)(y))\\&=(d_{C/D}\circ\pi_D\circ\psi-\pi_D\circ\psi\circ d_C)(y)\\&=d_{C/D}\circ\bar\psi(\pi_D(y))-\bar\psi\circ d_{C/D}(\pi_D(y)),\end{aligned}$$

which proves that $\bar f\circ\bar g$ is chain homotopic to the identity on C/D since any $z\in C/D$ is of the form $z=\pi_D(y)$. □

6 Proofs of main results

6.1 Proof of Theorem A1

Fix $A\in(0,\infty)\setminus\mathrm{Spec}(\partial D,\lambda)$. The idea of the proof is to construct a special admissible Hamiltonian for which $c(1,\cdot)$ is bounded from below by $A-\varepsilon$ for ε a small constant which depends on A. This construction is inspired by [Cieliebak et al. 2010, Proposition 2.5]. Then, we use the fact that $c(1,\cdot)\ge 0$ to conclude.

6.1.1 Construction of the Hamiltonian Fix some $r_0>1$. For any $\delta\in(0,1)$ and $\sigma\in(0,T_0)$, we define the Hamiltonian $H_{\delta,A}$ as follows:

- $H_{\delta,A}$ is the constant function $A(\delta-1)$ on D^δ.
- $H_{\delta,A}(r,x)=A(r-1)$ on $D\setminus D^\delta$.
- $H_{\delta,A}(r,x)=0$ on $D^{r_0}\setminus D$.
- $H_{\delta,A}(r,x)=\sigma(r-r_0)$ on $\widehat D\setminus D^{r_0}$.

See Figure 8. We add a small perturbation to $H_{\delta,A}$ so that it lies in $\overline{\mathcal H}_{r_0}$. Denote by $h_{\delta,A}$ the function of one variable for which $H_{\delta,A}=h_{\delta,A}\circ r$ on D^c. If γ is a 1–periodic orbit of $h_{\delta,A}$ inside the level set $\{r\}\times\partial D$, its action can be written as

$$\mathcal A_{H_{\delta,A}}(\gamma)=\mathcal A_{H_{\delta,A}}(r)=rh'_{\delta,A}(r)-h_{\delta,A}(r).$$

The 1–periodic orbits of $H_{\delta,A}$ can be classified into three different categories. Recall that η_A denotes the distance between A and $\mathrm{Spec}(\partial D,\alpha)$.

(I) Critical points in D^δ with action close to $r_{\mathrm I}:=(1-\delta)A$.

(II) Nonconstant 1–periodic orbits near $\{\delta\}\times\partial D$ with action in a small neighborhood of the interval

$$I_{\mathrm{II}}=[\delta T_0+(1-\delta)A,\,A-\delta\eta_A].$$

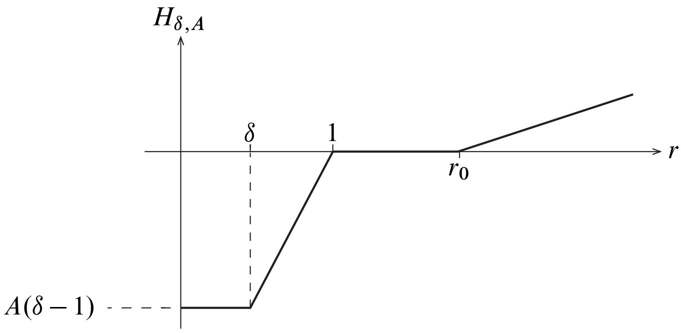


Figure 8: Radial portion of the Hamiltonian $H_{\delta,A}$.

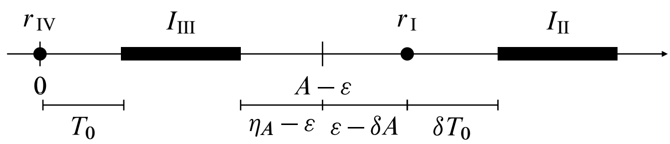


Figure 9: Distances that separate the action windows under consideration.

(III) Nonconstant 1–periodic orbits near $\{1\}\times\partial D$ with action in a small neighborhood of the interval

$$I_{\mathrm{III}}=[T_0, A-\eta_A].$$

(IV) Critical points in $D^{r_0}\setminus D$ with action close to $r_{\mathrm{IV}}:=0$.

Note that there are no nonconstant 1–periodic orbits near $\{r_0\}\times\partial D$, since the slope of the Hamiltonian there ranges from 0 to σ, which is less than T_0 by assumption.

We now want to construct a Floer complex $\mathrm{C}^*_{\mathrm{I,II}}$ which will contain the orbits of type (I) and (II) and another complex $\mathrm{C}^*_{\mathrm{III,IV}}$ containing orbits of type (III) and (IV). To that end, pick $0<\delta<1$ small enough so that $\delta A<\eta_A$. Now choose $\varepsilon>0$ such that

$$\delta A<\varepsilon<\eta_A.$$

Then, we have the inequalities

$$r_{\mathrm{IV}}<I_{\mathrm{III}}<A-\varepsilon<r_{\mathrm{I}}<I_{\mathrm{II}}.$$

As shown in Figure 9, r_{I}, I_{II}, I_{III} and r_{IV} are all separated by distances which depend only on T_0, A, η_A, δ and ε. Thus, we can choose the perturbation we add to $H_{\delta,A}$ to be small enough so that, in terms of action, we have

$$(\mathrm{IV})<(\mathrm{III})<A-\varepsilon<(\mathrm{I})<(\mathrm{II}).$$

Therefore, since the Floer differential decreases the action, we can define the Floer cochain complexes as

$$\mathrm{C}^*_{\mathrm{III,IV}}=\mathrm{CF}^*_{(-\infty,A-\varepsilon)}(H_{\delta,A}),\quad \mathrm{C}^*_{\mathrm{I,II}}=\frac{\mathrm{CF}^*(H_{\delta,A})}{\mathrm{C}^*_{\mathrm{III,IV}}}=\mathrm{CF}^*_{(A-\varepsilon,\infty)}(H_{\delta,A})$$

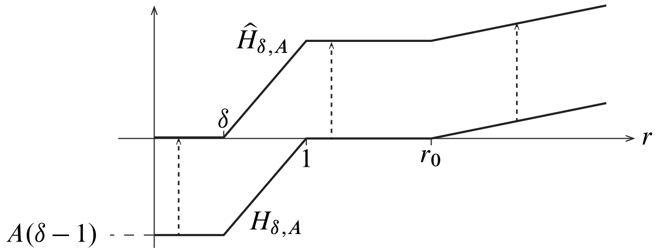


Figure 10: Homotopy from $H_{\delta,A}$ to $\widehat{H}_{\delta,A}$.

and they yield the Floer cohomology groups

$$\mathrm{H}^*(\mathrm{C}^*_{\mathrm{III,IV}}) = \mathrm{HF}^*_{(-\infty,A-\varepsilon)}(H_{\delta,A}), \quad \mathrm{H}^*(\mathrm{C}^*_{\mathrm{I,II}}) = \mathrm{HF}^*_{(A-\varepsilon,\infty)}(H_{\delta,A}).$$

A quick look at the action windows under consideration informs us that the above complexes fit into the short exact sequence

$$0 \longrightarrow \mathrm{C}^*_{\mathrm{III,IV}} \xrightarrow{\iota^{A-\varepsilon,+\infty}_{-\infty,-\infty}} \mathrm{CF}^*(H_{\delta,A}) \xrightarrow{\pi^{+\infty,+\infty}_{-\infty,A-\varepsilon}} \mathrm{C}^*_{\mathrm{I,\,II}} \longrightarrow 0,$$

which in turn yields an exact triangle in cohomology:

$$\begin{array}{ccc} \mathrm{H}^*(\mathrm{C}^*_{\mathrm{III,IV}}) & \xrightarrow{[\iota^{A-\varepsilon,+\infty}_{-\infty,-\infty}]} & \mathrm{HF}^*(H_{\delta,A}) \\ & {\scriptstyle [+1]}\nwarrow \qquad \swarrow{\scriptstyle [\pi^{+\infty,+\infty}_{-\infty,A-\varepsilon}]} & \\ & \mathrm{H}^*(\mathrm{C}^*_{\mathrm{I,II}}) & \end{array}$$

6.1.2 Factoring a map to SH$^*(D)$ We now build maps Ψ and $\Psi_{\mathrm{I,II}}$ such that the diagram

$$\begin{array}{ccc} \mathrm{HF}^*(H_{\delta,A}) & \xrightarrow{[\pi^{\infty,\infty}_{-\infty,A-\varepsilon}]} & \mathrm{H}^*(\mathrm{C}^*_{\mathrm{I,II}}) \\ & {\scriptstyle \Psi}\searrow & \downarrow{\scriptstyle \Psi_{\mathrm{I,II}}} \\ & & \mathrm{SH}^*(D) \end{array} \tag{6}$$

commutes. We need to construct Ψ so that it coincides with the map $j_{H_{\delta,A}}\colon \mathrm{HF}^*(H_{\delta,A}) \to \mathrm{SH}^*(D)$ (see (4)). By virtue of Theorem 22, this assures us that Ψ is a map of unital algebras.

First, we construct $\Psi_{\mathrm{I,II}}$ in three steps.

Step 1 $[\Phi_1]\colon \mathrm{H}^*(\mathrm{C}^*_{\mathrm{I,II}}) \cong \mathrm{HF}^*_{(\delta A-\varepsilon,\infty)}(H_{\delta,A} + A(1-\delta))$. This isomorphism follows from a simple shift of $A(1-\delta)$ in the Hamiltonian term, which translates to a shift of $A(\delta-1)$ in action (see Figure 10). In what follows, we let $\widehat{H}_{\delta,A} := H_{\delta,A} + A(1-\delta)$.

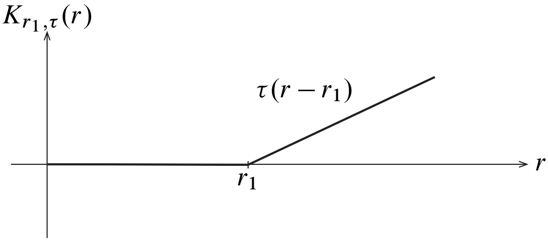


Figure 11: Radial portion of the Hamiltonian $K_{r_1,\tau}$.

For the next steps, we need to define another special family of Hamiltonians. Given $r_1 \in (0, +\infty)$ and $\tau \in (0,\infty) \setminus \mathrm{Spec}(\partial D, \lambda)$, define the Hamiltonian $K_{r_1,\tau}$ as follows (see Figure 11):

- $K_{r_1,\tau}$ is the constant zero function on D^{r_1}.
- $K_{r_1,\tau}(x,r) = \tau(r - r_1)$ on $\widehat{D} \setminus D^{r_1}$.

We add a small perturbation to $K_{r_1,\tau}$ so that it r_1–admissible. The 1–periodic orbits of $K_{r_1,\tau}$ fall in two categories:

(I′) Critical points in D^{r_1} with action near zero.

(II′) Nonconstant 1–periodic orbits near $\{r_1\} \times \partial D$ with action in a small neighborhood of the interval

$$[r_1 T_0, r_1\tau - r_1\eta_\tau].$$

By the same argument used for $H_{\delta,A}$, the action windows (I′) and (II′) are separated if we choose a small enough perturbation.

Step 2 $[\Phi_2]\colon \mathrm{HF}^*_{(\delta A-\varepsilon,\infty)}(\widehat{H}_{\delta,A}) \cong \mathrm{HF}^*(K_{\delta,A})$. Consider the homotopy

$$F_s = (1-\beta(s))K_{\delta,A} + \beta(s)\widehat{H}_{\delta,A},$$

where $\beta\colon \mathbb{R} \to [0,1]$ is a smooth function such that $\beta(s) = 0$ for $s \le -1$, $\beta(s) = 1$ for $s \ge 1$ and $\beta'(s) > 0$ for all $s \in (-1,1)$ (see Figure 12). Denote by

$$\Phi_{F_\bullet}\colon \mathrm{CF}^*(\widehat{H}_{\delta,A}) \longrightarrow \mathrm{CF}^*(K_{\delta,A})$$

the continuation map generated by $F_\bullet$.

Notice that since $H_{\delta,A} \preceq K_{\delta,A}$ we can restrict the continuation map on the action window $(\delta A - \varepsilon, \infty)$. Thus,

$$[\Phi_{F_\bullet}]\colon \mathrm{HF}^*_{(\delta A-\varepsilon,\infty)}(\widehat{H}_{\delta,A}) \longrightarrow \mathrm{HF}^*_{(\delta A-\varepsilon,\infty)}(K_{\delta,A})$$

is well-defined. Moreover, since $\delta A - \varepsilon < 0$, $K_{\delta,A}$ has no orbits outside the action window $(\delta A - \varepsilon, \infty)$ and thus

$$[\iota^{-\infty,\infty}_{\delta A-\varepsilon,\infty}]\colon \mathrm{HF}^*_{(\delta A-\varepsilon,\infty)}(K_{\delta,A}) \longrightarrow \mathrm{HF}^*(K_{\delta,A})$$

is an isomorphism. We define $[\Phi_2]$ to be the composition $[\iota^{-\infty,\infty}_{\delta A-\varepsilon,\infty}] \circ [\Phi_{F_\bullet}]$.

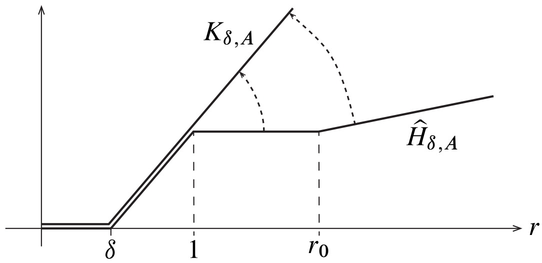


Figure 12: Homotopy from $\widehat{H}_{\delta,A}$ to $K_{\delta,A}$.

Step 3 Recall from (4), that we have a natural map

$$j_{K_{\delta,A}}\colon \mathrm{HF}^*(K_{\delta,A}) \longrightarrow \mathrm{SH}^*(D).$$

We define $\Psi_{\mathrm{I,II}}\colon \mathrm{H}^*(\mathrm{C}^*_{\mathrm{I,II}}) \to \mathrm{SH}^*(D)$ to be the composition

$$\Psi_{\mathrm{I,II}} = j_{K_{\delta,A}} \circ [\Phi_2] \circ [\Phi_1].$$

The morphism Ψ is built in a similar fashion. We define it as the following composition of maps:

$$\begin{array}{ccccc} \mathrm{HF}^*(H_{\delta,A}) & \xrightarrow[{[\Phi'_1]}]{\cong} & \mathrm{HF}^*(\widehat{H}_{\delta,A}) & & \\ & & \big\downarrow {\scriptstyle [\Phi'_2]} & & \\ & & \mathrm{HF}^*(K_{\delta,A}) & \xrightarrow{j_{K_{\delta,A}}} & \mathrm{SH}^*(D) \end{array}$$

Here, the isomorphism $[\Phi'_1]$ follows from the fact that both $H_{\delta,A}$ and $\widehat{H}_{\delta,A}$ have the same slope at infinity. We defined $[\Phi'_2]$ to be the continuation map $[\Phi^{K_{\delta,A}\widehat{H}_{\delta,A}}]$. The last map is given, just as in Step 3, by $j_{K_{\delta,A}}\colon \mathrm{HF}^*(K_{\delta,A}) \to \mathrm{SH}^*(D)$. By construction, we therefore have

$$\Psi = j_{K_{\delta,A}} \circ [\Phi'_2] \circ [\Phi'_1] = j_{K_{\delta,A}} \circ [\Phi_{K_{\delta,A}\widehat{H}_{\delta,A}}] \circ [\Phi'_1] = j_{H_{\delta,A}}$$

as desired.

Now, we need to prove that diagram (6) commutes. Writing the maps Ψ and $\Psi_{\mathrm{I,II}}$ explicitly, we have the following diagram:

(7)
$$\begin{array}{ccc} \mathrm{HF}^*(H_{\delta,A}) & \xrightarrow{[\pi^{+\infty,+\infty}_{-\infty,A-\varepsilon}]} & \mathrm{H}^*(\mathrm{C}^*_{\mathrm{I,II}}) \\ \big\downarrow {\scriptstyle [\Phi'_1]} & & {\scriptstyle [\Phi_1]} \big\downarrow \\ \mathrm{HF}^*(\widehat{H}_{\delta,A}) & \xrightarrow{[\pi^{+\infty,+\infty}_{-\infty,\delta A-\varepsilon}]} & \mathrm{HF}^*_{(\delta A-\varepsilon,+\infty)}(\widehat{H}_{\delta,A}) \\ \big\downarrow {\scriptstyle [\Phi_{K_{\delta,A}\widehat{H}_{\delta,A}}]} & & {\scriptstyle [\Phi_2]} \big\downarrow \\ \mathrm{HF}^*(K_{\delta,A}) & \xrightarrow{\mathrm{id}} & \mathrm{HF}^*(K_{\delta,A}) \\ & & {\scriptstyle j_{K_{\delta,A}}} \big\downarrow \\ & & \mathrm{SH}^*(D) \end{array}$$

The top square in diagram (7) commutes because, since $\widehat{H}_{\delta,A} \geq H_{\delta,A}$, there exists a continuation map from $\mathrm{HF}^*(H_{\delta,A}) \cong \mathrm{HF}^*_{(\delta A-\varepsilon,\infty)}(H_{\delta,A})$ to $\mathrm{HF}^*_{(\delta A-\varepsilon,+\infty)}(\widehat{H}_{\delta,A})$, where the isomorphism follows from the fact that $H_{\delta,A}$ has no orbits outside the action window $(\delta A-\varepsilon,\infty)$. Now, since the projection $[\pi^{+\infty,+\infty}_{-\infty,\delta A-\varepsilon}]$ commutes with continuation maps (see diagram (3)), the bottom square in diagram (7) also commutes. Therefore, we can conclude that diagram (6) commutes.

6.1.3 Spectral invariant and spectral norm of $H_{\delta,A}$

Recall that, by definition,

$$c(1, H_{\delta,A}) = \inf\{\ell \in \mathbb{R} \mid [\pi^{+\infty,+\infty}_{-\infty,\ell}] \circ [\iota^{\ell,+\infty}_{-\infty,-\infty}](1) = 0\}.$$

Since Ψ is a morphism of unital algebras, the commutative diagram (6) assures us that

$$[\pi^{+\infty,+\infty}_{-\infty,A-\varepsilon}](1_{H_{\delta,A}}) \neq 0$$

since we assume that $\mathrm{SH}^*(D) \neq 0$. Thus, from the exact triangle in cohomology induced by $[\iota^{A-\varepsilon,+\infty}_{-\infty,-\infty}]$ and $[\pi^{+\infty,+\infty}_{-\infty,A-\varepsilon}]$, we have $1 \notin \mathrm{im}[\iota^{A-\varepsilon,+\infty}_{-\infty,-\infty}]$ and therefore

$$c(1, H_{\delta,A}) \geq A-\varepsilon.$$

Now, we turn our attention to the spectral norm $\gamma(H_{\delta,A})$. We know from Lemma B that

$$c(1, H_{\delta,A}), c(1, \overline{H}_{\delta,A}) \geq 0.$$

It thus follows from the previous inequality that

$$\gamma(H_{\delta,A}) = c(1, H_{\delta,A}) + c(1, \overline{H}_{\delta,A}) \geq A-\varepsilon$$

as desired. This completes the proof.

6.2 Proof of Lemma B

We give a proof of Lemma B which relies on the decomposition of the Floer complex induced by the barricade. We expect that Lemma B could also be proven using Poincaré duality between filtered Floer cohomology and filtered Floer homology (as in [Cieliebak and Oancea 2018, Section 3]) and Lemma 4.1 of [Ganor and Tanny 2023].

Let $H \in \mathcal{H}_{r_0}$ with slope $0 < \tau_H < T_0$. Consider a linear homotopy $F_\bullet$ from $F_+ = K_{r_0,\tau_H}$ (see Figure 11) to $F_- = H$. There exists a small perturbation $f_\bullet$ of $F_\bullet$ and an almost-complex structure J such that the pairs $(f_\bullet, J)$ and $(f_\pm, J)$ admit a barricade on $B_{r_0,\varepsilon}$ for $\varepsilon > 0$ small enough. Fix $\delta > 0$. The construction of Theorem 36 allows us to choose J time independent [Ganor and Tanny 2023, Remark 3.7] and f such that

$$-\delta \leq \int_0^1 \min_{x\in\widehat{D}\setminus(r_0,+\infty)\times\partial D} (f_- - H)\,\mathrm{d}t \leq \delta.$$

We may assume further that f_+ has a local minimum point $p \in D_\mathrm{c} = \widehat{D} \setminus D_\mathrm{b}$, since f_+ is C^2–small there. It follows from Lemma 17 that $1_{f_+} = [p] \in \mathrm{HF}^*(f_+)$ is the image of the unit $e_D \in \mathrm{H}^*(D)$ under the isomorphism $\Phi_{f_+}\colon \mathrm{H}^*(D) \to \mathrm{HF}^*(f_+)$. Moreover, since f_+ and f_- have the same slope at infinity, Lemma 20 assures us that the isomorphism $[\Phi_{f_\bullet}]\colon \mathrm{HF}^*(f_+) \to \mathrm{HF}^*(f_-)$ induced by the continuation morphism $\Phi_{f_\bullet}\colon \mathrm{CF}^*(f_+) \to \mathrm{CF}^*(f_-)$ preserves the unit. To summarize, we have

$$\Phi_{f_+}(e_D) = [p] = 1_{f_+} \quad \text{and} \quad [\Phi_{f_\bullet}(p)] = [\Phi_{f_\bullet}](1_{f_+}) = 1_{f_-}.$$

By the continuity of spectral invariants, we know that

$$c(1,H) - c(1,f_-) \geq \int_0^1 \min_{x\in D^{r_0}} (f_- - H)\,\mathrm{d}t.$$

Therefore, by our choice of f_-, we have $c(1,H) \geq -\delta + c(1,f_-)$. To complete the proof, it suffices to show that $c(1,f_-) \geq -k\delta$ for $k \geq 0$ independent of f_-. However, the definition of spectral invariants guarantees the existence of $q \in \mathrm{CF}^*(f_-)$ cohomologous to 1 for which $c(1,f_-) \geq \mathcal{A}_{f_-}(q) - \delta$. We thus only need to prove that $\mathcal{A}_{f_-}(q) \geq -\delta$. In the case where q is a combination $q_1 + \cdots + q_k$ of orbits, the action of q is defined as

$$\mathcal{A}_{f_-}(q) = \max_i \mathcal{A}_{f_-}(q_i).$$

Recall from Section 5.2 that the barricade construction assures that we have, in terms of vector spaces, the decomposition

$$\mathrm{CF}^*(f_\pm) \cong \mathrm{C}^*(D_\mathrm{b}, f_\pm) \oplus \mathrm{C}^*(D_\mathrm{c}, f_\pm)$$

with inclusions and projections respectively given by

$$\iota^\pm_\heartsuit\colon \mathrm{C}^*(D_\heartsuit, f_\pm) \to \mathrm{CF}^*(f_\pm) \quad \text{and} \quad \pi^\pm_\heartsuit\colon \mathrm{CF}^*(f_\pm) \to \mathrm{C}^*(D_\heartsuit, f_\pm)$$

for $\heartsuit \in \{\mathrm{b},\mathrm{c}\}$. Moreover, Floer trajectories starting in D_b must have ends in D_b and Floer trajectories starting in D_c can have ends in D_b and D_c. Thus,

$$\Phi_{f_\bullet}(p) = p_\mathrm{b} + p_\mathrm{c} \quad \text{and} \quad q = p_\mathrm{b} + p_\mathrm{c} + \partial(r_\mathrm{b} + r_\mathrm{c})$$

for $p_\mathrm{b}, r_\mathrm{b} \in \operatorname{im} i_\mathrm{b}^-$ and $p_\mathrm{c}, r_\mathrm{c} \in \operatorname{im} \iota_\mathrm{c}^-$. Furthermore,

$$\partial(r_\mathrm{b}) = r_\mathrm{bb} \quad \text{and} \quad \partial(r_\mathrm{c}) = r_\mathrm{cb} + r_\mathrm{cc},$$

where $r_\mathrm{bb}, r_\mathrm{cb} \in \operatorname{im} \iota_\mathrm{b}^-$ and $r_\mathrm{cc} \in \operatorname{im} \iota_\mathrm{c}^-$. See Figure 13 for an illustration of the Floer trajectories under consideration here.

Notice that since f_- is C^2–small on D_c, we have $\mathcal{A}_{f_-}(p_\mathrm{c} + r_\mathrm{cc}) \geq -\delta$. Thus, if $p_\mathrm{c} + r_\mathrm{cc} \neq 0$, we have

$$\mathcal{A}_{f_-}(q) = \mathcal{A}_{f_-}(p_\mathrm{b} + p_\mathrm{c} + r_\mathrm{bb} + r_\mathrm{cb} + r_\mathrm{cc}) \geq \mathcal{A}_{f_-}(p_\mathrm{c} + r_\mathrm{cc}) \geq -\delta.$$

We now prove that $p_\mathrm{c} + r_\mathrm{cc} \neq 0$. This is equivalent to showing that the class $[\pi_\mathrm{c}^-(p_\mathrm{c})]$ in $\mathrm{H}^*(D_\mathrm{c}, f_-)$ is nonzero. Indeed, if $p_\mathrm{c} + r_\mathrm{cc} = 0$, we have, by the definition of r_cc, $p_\mathrm{c} = -\partial r_\mathrm{c}$ and thus

$$[\pi_\mathrm{c}^-(p_\mathrm{c})] = [\pi_\mathrm{c}^-(-\partial r_\mathrm{c})] = [-\partial_\mathrm{c}\pi_\mathrm{c}^-(r_\mathrm{c})] = 0.$$

Denote by $\Phi_{\bar{f}_\bullet}\colon \mathrm{CF}^*(f_-) \to \mathrm{CF}^*(f_+)$ the continuation map generated by the inverse homotopy $\bar{f}_s = f_{-s}$. We know that both $\Phi_{\bar{f}_\bullet} \circ \Phi_{f_\bullet}$ and $\Phi_{f_\bullet} \circ \Phi_{\bar{f}_\bullet}$ are chain homotopic to the identity:

$$\Phi_{\bar{f}_\bullet} \circ \Phi_{f_\bullet} - \mathrm{id}_+ = \partial_+ \circ \Psi_+ - \Psi_+ \circ \partial_+,$$
$$\Phi_{f_\bullet} \circ \Phi_{\bar{f}_\bullet} - \mathrm{id}_- = \partial_- \circ \Psi_- - \Psi_- \circ \partial_-$$

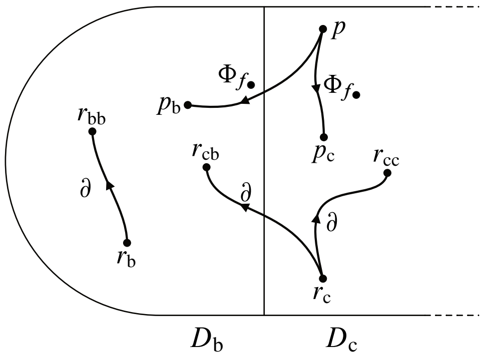


Figure 13: The possible trajectories for the differential of r_b, r_c and the continuation map applied to p according to the rules of the barricade.

for the differentials $\partial_\pm\colon \mathrm{CF}^*(f_\pm) \to \mathrm{CF}^{*+1}(f_\pm)$ and chain homotopies $\Psi_\pm\colon \mathrm{CF}^*(f_\pm) \to \mathrm{CF}^{*-1}(f_\pm)$. (In fact, for our purpose here, we only need the first homotopy relation.) Since $\Psi_\pm$ also obey the rules of the barricade by Lemma 37, the composition of the projections $\Phi^c_{f_\bullet}: \mathrm{C}^*(D_c, f_+) \to \mathrm{C}^*(D_c, f_-)$ and $\Phi^c_{\bar f_\bullet}: \mathrm{C}^*(D_c, f_-) \to \mathrm{C}^*(D_c, f_+)$ is chain homotopic to the identity on $\mathrm{C}^*(D_c, f_+)$ by Lemma 39. Therefore, on cohomology, the morphism

$$[\Phi^c_{\bar f_\bullet} \circ \Phi^c_{f_\bullet}]\colon \mathrm{H}^*(D_c, f_+) \to \mathrm{H}^*(D_c, f_+)$$

is given by the identity. Moreover, recall that by definition, $p \in D_c$, which guarantees that, as a cycle, $p \in \operatorname{im} \iota_c^+$ and since $[p] = 1_{f_+}$, we have $[\pi_c^+(p)] \neq 0$. Therefore,

$$[\pi_c^-(p_c)] = [\Phi^c_{f_\bullet} \circ \pi_c^+(p)] = [\Phi^c_{f_\bullet}]([\pi_c^+(p)]) \neq 0.$$

This concludes the proof.

6.3 Proof of Lemma C

Let $0 < \delta < 1$ be small enough so that

$$\delta A < \delta A + \delta\eta_A < \eta_A.$$

Then, following the proof of Theorem A1 with $\varepsilon = \delta(A + \eta_A)$, we have that

$$c(1, H_{\delta,A}) \geq A - \delta(A + \eta_A).$$

Notice that $H_{\delta,A}$ converges uniformly as $\delta \to 0$ to the continuous function $H_{0,A}$ (see Figure 14). Then, by continuity of spectral invariants and the previous equation, we have

$$c(1, H_{0,A}) = \lim_{\delta\to 0} c(1, H_{\delta,A}) \geq \lim_{\delta\to 0} (A - \delta(A + \eta_A)) = A.$$

Moreover, since $H_{0,A} \geq -A$, continuity of spectral invariants yields

$$c(1, H_{0,A}) \leq \max_{x\in D} -H_{0,A} = A,$$

which allows us to conclude that $c(1, H_{0,A}) = A$.

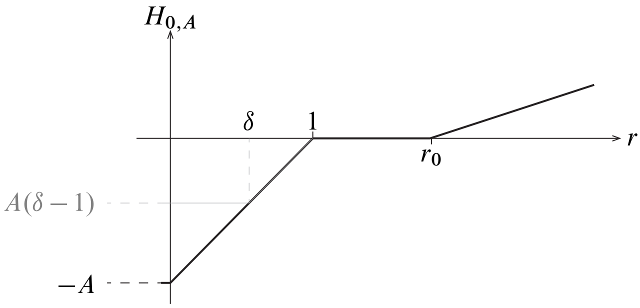


Figure 14: The continuous Hamiltonian $H_{0,A}$.

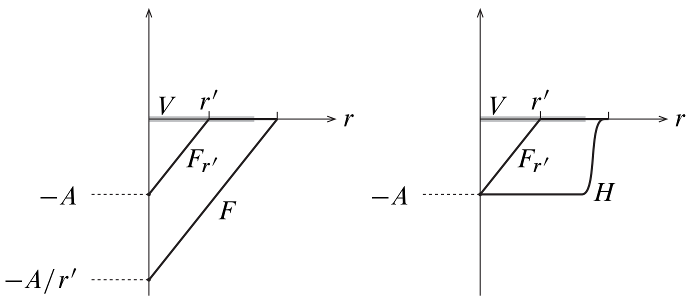


Figure 15: The Hamiltonians F, $F_{r'}$ and H.

First, we prove the lemma for Hamiltonians which are constant on an open neighborhood of the skeleton of D. Consider an autonomous Hamiltonian $H \in \mathcal{C}(D)$ such that $H|_V = -A$ and $-A \le H \le 0$ for an open neighborhood V of $\mathrm{Sk}(D)$ and a constant $A > 0$. The last condition on H allows us to use continuity of spectral invariance to conclude that

$$c(1, H) \le A. \tag{8}$$

All we need to do now is prove that A bounds $c(1, H)$ from below.

Define $F \in \mathcal{C}(D)$ to be the continuous autonomous Hamiltonian that agrees with $H_{0,A/r'}$ on D for some $0 < r' < 1$. Since $H|_V = -A$, we can choose r' so that the r'–contraction $F_{r'}$ of F under the Liouville flow (see (5) and Figure 15) has support in V and $-A \le F_{r'} \le 0$. Therefore,

$$F_{r'}(x) \ge H(x) \quad \text{for all } x \in D. \tag{9}$$

From the contraction principle stated in Lemma 27 and the computation of $c(1, H_{0,A})$ above, we have

$$c(1, F_{r'}) = r' c(1, F) = r' c(1, H_{0,A/r'}) = A.$$

This computation and (9) yield, by virtue of the monotonicity of spectral invariants, the lower bound $A = c(1, F_{r'}) \le c(1, H)$ as desired. In conjunction with (8), we conclude that $c(1, H) = A$.

Now, we prove the lemma in general. Suppose $H|_{\mathrm{Sk}(D)} = -A$ and $-A \le H \le 0$. For any $\varepsilon \in (0,1)$, there exists a compactly supported Hamiltonian H_ε such that $H_\varepsilon|_{V_\varepsilon} = -A$ for an open neighborhood V_ε of $\mathrm{Sk}(D)$ and $H_\varepsilon \le H$ everywhere. Indeed, define H_ε as follows: $H_\varepsilon|_{\mathrm{Sk}(D)} = -A$,

$$H_\varepsilon|_{D^\varepsilon \setminus \mathrm{Sk}(D)} = \beta_\varepsilon(r)H + (1-\beta_\varepsilon(r))(-A),$$

where $\beta_\varepsilon \colon (0,1) \to \mathbb{R}$ is such that

- $\beta_\varepsilon|_{(0,\varepsilon]} \equiv 0$,
- $\beta'_\varepsilon|_{(\varepsilon,2\varepsilon/3)} > 0$,
- $\beta_\varepsilon|_{(2\varepsilon/3,1)} \equiv 1$.

Then, H_ε satisfies the required conditions and converges uniformly to H as $\varepsilon \to 0$. We have $c(1,H_\varepsilon) = A$ by the previous computation, and by continuity of spectral invariants, we can conclude that

$$c(1,H) = c(1,H_\varepsilon) = A.$$

This completes the proof.

6.4 Proof of Theorem A2

Let $H \in \mathcal{C}(D)$ be an autonomous Hamiltonian such that $H|_V = -1$ and $-1 \le H \le 0$ everywhere for an open neighborhood V of $\mathrm{Sk}(D)$.

Define $\iota \colon \mathbb{R} \to \mathrm{Ham}_c(D)$ as

$$\iota(s) = \varphi_{sH},$$

where $\varphi_{sH} \in \mathrm{Ham}_c(D)$ is the time-1 map associated to sH. We claim that ι is the desired embedding.

We first bound $d_\gamma(\iota(s),\iota(s'))$ from above. If $F \in \mathcal{C}(D)$, then $\gamma(\varphi_F) \le \|F\|$. Moreover, since H is autonomous, $sH \# \overline{s'H} = (s-s')H$. Therefore,

$$d_\gamma(\iota(s),\iota(s')) = \gamma(\iota(s)\iota(s')^{-1}) \le \|(s-s')H\| = |s-s'|.$$

Now, we bound $d_\gamma(\iota(s),\iota(s'))$ from below. Since d_γ is symmetric, we can assume that $s \ge s'$. Then, by Lemmas B and C, we have

$$d_\gamma(\iota(s),\iota(s')) \ge c(1,(s-s')H) = s-s',$$

which completes the proof.

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