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Classifying rational G–spectra for profinite G

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For G an arbitrary profinite group, we construct an algebraic model for rational G–spectra in terms of G–equivariant sheaves over the space of subgroups of G. This generalises the known case of finite groups to a much wider class of topological groups. It improves upon earlier work of the first author on the case where G is the p–adic integers.

As the purpose of an algebraic model is to allow one to use homological algebra to study questions of homotopy theory, we prove that the homological dimension (injective dimension) of the algebraic model is determined by the Cantor–Bendixson rank of the space of closed subgroups of the profinite group G. This also provides a calculation of the homological dimension of the category of rational Mackey functors.

55P91; 54B40, 55P42, 55Q91

1 Introduction

The usefulness of equivariant cohomology theories in equivariant (stable) homotopy theory has long been proven. Examples of equivariant cohomology theories include the equivariant K–theory of Segal [1968] and the equivariant cobordism spectra used in Hill, Hopkins and Ravenel, [Hill et al. 2016]. To effectively study equivariant cohomology theories, one studies the category of their representing objects: equivariant spectra. That is, Brown representability holds equivariantly.

The study of equivariant spectra up to homotopy is even more demanding than the nonequivariant case, so one often works with rational equivariant spectra. Under Brown representability, rational equivariant spectra correspond to equivariant cohomology theories that take values in rational vector spaces. Rationalising preserves most of the interesting behaviour coming from the group, while removing much of the topological complexity.

A major goal in the study of rational equivariant stable homotopy theory is to find a more tractable model category that has the same homotopy theory as rational equivariant spectra. That is, one chooses a group G of interest, constructs an abelian category $\mathcal{A}(G)$ and a Quillen equivalence between the *algebraic model* $\mathrm{Ch}(\mathcal{A}(G))$ and the model category of rational G–spectra. The Quillen equivalence induces an equivalence of categories between the homotopy category of rational G–spectra and the homotopy category of the algebraic model. The primary advantage of having an algebraic model is that one can use the simplicity of the abelian category $\mathcal{A}(G)$ and the tools of homological algebra to construct objects and calculate sets of maps in the rational G–equivariant stable homotopy category. For an introduction to algebraic models and summary of the known cases see [Barnes and Kędziorek 2022].

In this paper, the authors generalise the known case of algebraic models for finite groups (see [Schwede and Shipley 2003, Example 5.1.2]) to profinite groups. Profinite groups are a commonly encountered class of compact topological groups, appearing most often as Galois groups or when one has a diagram of finite groups. They are defined as the compact Hausdorff totally disconnected topological groups. It can be shown that a group is profinite if and only if it is the limit of a filtered system of finite groups. For example, the Morava stabiliser group $\mathbb{S}_n$ from chromatic homotopy theory is profinite. Profinite groups occur in many other mathematical fields: number theory makes substantial use of profinite groups, as seen in [Bley and Boltje 2004], and the étale fundamental groups of algebraic geometry are profinite. This ubiquity drives our interest in rational G–spectra for profinite G and hence our interest in finding algebraic models in the profinite case.

1.1 Main results

Let G be a profinite group and let $\mathcal{S}G$ be the space of closed subgroups of G, topologised as the inverse limit of the finite discrete spaces $\mathcal{S}G/N$, for N open and normal in G. There is an abelian category of rational G–equivariant sheaves over $\mathcal{S}G$. Consider the full subcategory of those equivariant sheaves E such that the stalk E_K over the closed subgroup K is K–fixed. These are called rational Weyl-G–sheaves and are introduced in earlier work of the authors, [Barnes and Sugrue 2023; 2022]. In this paper, the category of rational Weyl-G–sheaves occur as the abelian category $\mathcal{A}(G)$ used to make the algebraic model $\mathrm{Ch}(\mathcal{A}(G))$ for rational G–spectra.

Theorem A (Corollary 4.10) *For G a profinite group, there is a zigzag of Quillen equivalences between the model category of rational G–spectra and the model category of chain complexes of rational Weyl-G–sheaves.*

This result generalises work of the first author [Barnes 2011] on the case where $G = \mathbb{Z}_p^\wedge$ is the p–adic integers. As with the current work, that paper uses the tilting theory (Morita theory) of Schwede and Shipley [2003, Theorem 5.1] to obtain a Quillen equivalence between rational G–spectra and a category of chain complexes in an abelian category. The p–adic case uses a hand-crafted abelian category designed for that specific group. Contrastingly, the current work uses rational G–Mackey functors in the tilting theory

step, then applies the equivalence between rational G–Mackey functors and rational Weyl-G–sheaves of [Barnes and Sugrue 2023, Theorem A] to obtain the final result.

There are two reasons why the sheaf description of the algebraic model is important. The first is that Greenlees' conjecture [2006] on algebraic models (for compact Lie groups) is described in terms of sheaves over $\mathcal{S}G$, the space of closed subgroups of G. This result is the first realisation of that conjecture, albeit for profinite groups.

The second is that using the sheaf description we can calculate the homological dimension (also called the injective dimension) of the abelian category. This dimension is a measure of the complexity of the abelian model. In this case, the result is phrased in terms of the Cantor–Bendixson rank of the space of subgroups $\mathcal{S}G$; see Section 5 for details. When G is profinite, $\mathcal{S}G$ is a profinite space (one that is compact, Hausdorff and totally disconnected). The Cantor–Bendixson rank of a profinite space can be thought of as a measure of how far the space is from being discrete. To illustrate, a discrete space has rank 1 and $\mathcal{S}\mathbb{Z}_p^\wedge$, consisting of countably many points with one accumulation point, has rank 2.

Theorem B (Corollary 5.16) *Let G be a profinite group whose space of subgroups $\mathcal{S}G$ is scattered of Cantor–Bendixson rank n. The homological dimension of the category of rational Weyl-G–sheaves is $n-1$.*

If $\mathcal{S}G$ has infinite Cantor–Bendixson rank, then the homological dimension of the category of rational Weyl-G–sheaves is infinite.

Using the equivalence between rational G–Mackey functors and rational Weyl-G–sheaves, [Barnes and Sugrue 2023, Theorem A], this result gives the homological dimension of categories of rational G–Mackey functors.

Corollary C *Let G be a profinite group whose space of subgroups $\mathcal{S}G$ is scattered of Cantor–Bendixson rank n. The homological dimension of the category of rational G–Mackey functors is $n-1$.*

If $\mathcal{S}G$ has infinite Cantor–Bendixson rank, then the homological dimension of the category of rational G–Mackey functors is infinite.

As well as the two cases in the previous results there is a third possibility, that $\mathcal{S}G$ is of finite rank, but not scattered. We make the following conjecture for this case, which occurs as Conjecture 5.17 in the main body.

Conjecture D *Let G be a profinite group. If the G–space X has finite Cantor–Bendixson rank and nonempty perfect hull, then the homological dimension of rational G–sheaves over X is infinite.*

If $X = \mathcal{S}G$, then the homological dimension of the category of rational Weyl-G–sheaves is infinite.

1.2 Future questions

The question of how the change of groups functors on spectra compare with functors relating algebraic models for varying G is surprisingly involved. The currently known cases are for (co)free equivariant

spectra; see [Williamson 2022]. For the case of (pro)finite groups, the authors expect that having the sheaf description and the Mackey functor description will be vital.

The Quillen equivalences as given are not monoidal. There are two sources of difficulty, firstly that the Quillen equivalences of the tilting theory of Schwede and Shipley are not monoidal. Resolving this would require a fundamentally different approach to the classification. Secondly, it is not known how the equivalence of [Barnes and Sugrue 2023] interacts with the tensor product of sheaves and the two known monoidal structures on G–Mackey functors: the box product and the equivariant tensor product of Hill and Mazur [2019].

A further question is whether one can construct an Adams spectral sequence which takes values in the abelian category. The difficulty here is expected to be around constructing a suitable set of geometric fixed point functors for the closed subgroups of G, which should be the topological equivalent of taking the stalk of an equivariant sheaf and hence detect equivalences in the homotopy category of rational G–spectra. The geometric fixed point functors and the Adams spectral sequence are needed to give a good set of examples of rational G–spectra and their image in the algebraic model.

1.3 Strategy of the classification

The following diagram gives the major steps in the classification of rational G–spectra, for profinite G. The tilting theorem (Morita theory) of Schwede and Shipley [2003, Theorem 5.1] is used in Section 4.2 to create a Quillen equivalence between rational G–spectra and a category of (chain complexes of) "topological Mackey functors". This gives the upper horizontal functor. The key input to apply the tilting theorem is Theorem 2.14, which proves that the homotopical information of rational G–spectra is concentrated in degree zero.

In Section 3 we study spans and the stable orbit category. The aim is to prove Theorem 3.11, which gives an equivalence between $\pi_0(\mathfrak{O}_G)$, the G–equivariant stable orbit category and $\mathrm{Span}(G_{df}\text{–sets})$, a category of spans of finite discrete G–sets. That equivalence provides the upper vertical functor of the diagram. The lower vertical functor is an equivalence of categories describing Mackey functors in terms of spans, as detailed in Section 4.1. Theorem 4.4 is where these results are combined to give the zigzag of Quillen equivalences between rational G–spectra and chain complexes of rational G–Mackey functors.

The lower horizontal functor is an equivalence by earlier work of the authors, [Barnes and Sugrue 2023, Theorem A], which proves that the category of rational G–Mackey functors is equivalent to the category of rational Weyl-G–sheaves over the space of closed subgroups of G.

$$\begin{array}{ccc}
\text{rational } G\text{–spectra} & \xleftarrow{\ \simeq\ } & \mathrm{Ch}\big(\mathrm{Func}_{\mathrm{Ab}}(\pi_0(\mathfrak{O}_G^{\mathbb{Q}}), \mathbb{Q}\text{–mod})\big) \\
 & & \big\uparrow\big\downarrow \cong \\
 & & \mathrm{Ch}\big(\mathrm{Func}_{\mathrm{Ab}}(\mathrm{Span}(G_{df}\text{–sets}), \mathbb{Q}\text{–mod})\big) \\
 & & \big\uparrow\big\downarrow \cong \\
\mathrm{Ch}(\text{Weyl-}G\text{–sheaf}_{\mathbb{Q}}(\mathcal{S}G)) & \xleftarrow{\ \cong\ } & \mathrm{Ch}(\mathrm{Mackey}_{\mathbb{Q}}(G))
\end{array}$$

In Section 5 we look at the homological dimension of the algebraic model and relate it to the Cantor–Bendixson rank of $\mathcal{S}G$.

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2 Basics on equivariant spectra for profinite G

We recap the construction of the model category of rational orthogonal G–spectra, for G a profinite group. We then give various properties of this homotopy theory that will be used in the classification. Along the way we will need some facts about profinite groups, profinite sets, and topological spaces with a profinite group action. The constructions will all be generalisations of the finite group case. The expert reader may like to skip to Section 3.

2.1 Profinite groups

We give a few reminders of useful facts on profinite groups. More details can be found in [Wilson 1998] or [Ribes and Zalesskii 2000].

A *profinite group* is a compact, Hausdorff, totally disconnected topological group. A profinite group G is homeomorphic to the inverse limit of its finite quotients:

$$G \cong \lim_{N \trianglelefteq_{\text{open}} G} G/N \subseteq \prod_{N \trianglelefteq_{\text{open}} G} G/N.$$

The limit has the canonical topology which can either be described as the subspace topology on the product or as the topology generated by the preimages of the open sets in G/N under the projection map $G \to G/N$, as N runs over the open normal subgroups of G.

Closed subgroups and quotients by closed subgroups of profinite groups are also profinite. A subgroup of a profinite group is open if and only if it is finite index and closed. The trivial subgroup $\{e\}$ is open if and only if the group is finite. The intersection of all open normal subgroups is $\{e\}$. Any open subgroup H contains an open normal subgroup, the *core* of H in G, which is defined as the finite intersection

$$\mathrm{Core}_G(H) = \bigcap_{g \in G} gHg^{-1}.$$

2.2 Equivariant orthogonal spectra

Recall that an orthogonal spectrum is a sequence of based spaces indexed by finite dimensional inner product spaces, related by suspension maps that are suitably compatible with linear isometries of those vector spaces. For details, see the work of Mandell, May Schwede and Shipley [Mandell et al. 2001] or work of Barnes and Roitzheim [2020].

Equivariantly, the picture is similar, starting from based spaces with G–action indexed by finite dimensional G–inner product spaces, with suspension maps that are compatible with both the linear isometries and G–actions. This construction was first given in work of Mandell and May [2002] and adapted to the profinite setting by Fausk [2008].

The starting point is to describe the model category of based topological G–spaces that we will use to create our G–spectra. We focus on those model structures built using the *open* subgroups H of G as then G/H is a finite set. This ensures that the forgetful functor from G–spaces to spaces is a left Quillen functor. Throughout this section G will be a profinite group.

Proposition 2.1 *There is a cofibrantly generated proper model structure on the category of based topological G–spaces with weak equivalences those maps f such that f^H is a weak equivalence of spaces, for all open subgroups H of G. Similarly, fibrations are those maps f such that f^H is a fibration of based spaces for all open subgroups H of G. This model structure is denoted $G\mathrm{Top}_*$.*

The generating cofibrations are the standard inclusions

$$G/H_+ \wedge S^{n-1}_+ \to G/H_+ \wedge D^n_+, \quad G/H_+ \wedge D^n_+ \to G/H_+ \wedge (D^n \times [0,1])_+$$

for H an open subgroup of G and $n \geqslant 0$.

Just as CW–complexes are built from iteratively attaching cells by taking the pushout over the inclusion $S^{n-1} \to D^n$, one can define G–CW–complexes using the inclusions

$$G/H \times S^{n-1} \to G/H \times D^n.$$

That is, take X_0 to be a disjoint union of copies of G/H_+, then attach cells of the above form with $n = 1$ and H varying to obtain X_1. Continuing inductively gives X_n and X is defined as the union of the X_n. We see that if X is built using finitely many cells, the stabiliser of X (the intersection of all open subgroups H used in the cells) is also open. The evident pointed analogue gives the definition of pointed G–CW–complexes. Since the spaces G/H are finite, we see that every G–CW–complex is indeed a CW–complex, after forgetting the group action. Note that our choices mean that G itself is not a G–CW–complex. Indeed, as it has no fixed points, the space G is weakly equivalent to the empty set.

Our category of G–spectra will be indexed by the finite dimensional sub-G–inner product spaces of a complete G–universe $\mathcal{U}$, as defined below.

Definition 2.2 A *G–universe* $\mathcal{U}$ is a countably infinite direct sum $\mathcal{U} = \bigoplus_{i=1}^{\infty} U$ of a real G–inner product space U, such that

(1) there is a canonical choice of trivial representation $\mathbb{R} \subset U$,

(2) $\mathcal{U}$ is topologised as the union of all finite dimensional G–subspaces of U (each equipped with the norm topology).

A G–universe is said to be *complete* if every finite dimensional irreducible representation is contained (up to isomorphism) within $\mathcal{U}$.

A complete G–universe always exists, one can be obtained by setting U to be direct sum of a representative of each isomorphism class of irreducible representations of G, then defining $\mathcal{U}$ to be the direct sum of countably many copies of U.

Remark 2.3 Since a profinite group G is compact and Hausdorff, the action of G on a finite dimensional G–inner product space factors through a Lie group quotient by [Fausk 2008, Lemma A.1]. The only such quotients of a profinite group are finite, hence if V is a finite dimensional G–inner product space, there is an open normal subgroup N of G such that $V^N = V$.

In particular, the one-point compactification of V, denoted S^V is fixed by some open normal subgroup N. Thus, S^V can be given the structure of a finite (pointed) G/N–CW–complex, by [Illman 1983], and hence is a finite (pointed) G–CW–complex.

For brevity we define equivariant orthogonal spectra in terms of enriched functors from a particular enriched category made using Thom spaces. Recall that $G\mathrm{Top}_*$ is enriched over itself, via the space of (not-necessarily equivariant) maps, where G acts by conjugation

$$(f : X \to Y) \mapsto (g_Y \circ f \circ g_X^{-1} : X \to Y).$$

Definition 2.4 Define an *indexing space* to be a finite dimensional sub-G–inner product space of $\mathcal{U}$. We define $\mathcal{L}$ to be the category of all real G–inner product spaces that are isomorphic to indexing G-spaces in $\mathcal{U}$ and morphisms the (not-necessarily equivariant) linear isometries.

Definition 2.5 For each $V \subseteq W$ there is a vector bundle (a subset of the product bundle)

$$\gamma(V, W) = \{(f, x) \subseteq \mathcal{L}(V, W) \times W \mid x \in W - f(V)\}$$

over $\mathcal{L}(V, W)$, where $W - f(V)$ is the orthogonal complement of $f(V)$ in W.

Let $\mathcal{J}(V, W)$ be the Thom space of $\gamma(V, W)$, with G–action given by $g(f, x) = (gfg^{-1}, gx)$.

Lemma 2.6 *For inclusions of indexing spaces $U \subseteq V \subseteq W$ the G–equivariant map*

$$\gamma(V, W) \times \gamma(U, V) \to \gamma(U, W), \quad ((f, x), (k, y)) \mapsto (f \circ k, x + f(y))$$

induces a composition for the $G\mathrm{Top}_*$–enriched category $\mathcal{J}$ whose objects are the objects of $\mathcal{L}$ and morphism G–spaces are given by $\mathcal{J}(V, W)$.

Definition 2.7 An *orthogonal G–spectrum* X on a universe $\mathcal{U}$ is an $G\mathrm{Top}_*$–enriched functor from $\mathcal{J}$ to $G\mathrm{Top}_*$. A map of orthogonal G–spectra is a $G\mathrm{Top}_*$–enriched natural transformation. The category of orthogonal G–spectra is denoted $G\mathrm{Sp}^O$.

In particular, an orthogonal G–spectrum X defines based G–spaces $X(V)$ for each indexing space $V \subset \mathcal{U}$ and based G–maps

$$\sigma_{V,W} : S^{W-V} \wedge X(V) \to X(W)$$

for each $V \subseteq W$. A map $f : X \to Y$ of orthogonal G–spectra defines a set of G–maps $f : X(V) \to Y(V)$, which commute with the maps $\sigma_{V,W}$ for each $V \subseteq W$.

2.3 Model categories of spectra

With the category of G–spectra defined, the next step is to give model structures. This follows the usual path, a levelwise model structure, then a stable model structure and finally a rational model structure (as a Bousfield localisation of the stable model structure). In each case, the weak equivalences are defined using the *open* subgroups of G. For brevity, we state the existence of the rational model structure as a theorem, giving the essential properties afterwards. These results are standard, details can be found in [Barnes 2008, Section 2.2].

Definition 2.8 Let X be a G–spectrum and n a nonnegative integer. We define the *H-homotopy groups* of X as

$$\pi_n^H(X) = \operatorname*{colim}_V \pi_n^H(\Omega^V X(V)), \quad \pi_{-n}^H(X) = \operatorname*{colim}_{V \supset \mathbb{R}^n} \pi_0^H(\Omega^{V-\mathbb{R}^n} X(V)).$$

In the first case the colimit runs over the indexing spaces of $\mathcal{U}$, in the second case over the indexing spaces of $\mathcal{U}$ that contain $\mathbb{R}^n$.

A map $f : X \to Y$ of orthogonal G–spectra is called a *π_*–isomorphism* if $\pi_k^H(f)$ is an isomorphism for all open subgroups H of G and all integers k. We call f a *rational π_*–isomorphism* if $\pi_k^H(f) \otimes \mathbb{Q}$ is an isomorphism for all open subgroups H of G and all integers k.

The rational model structure on G–spectra is made using the rational sphere spectrum $S^0\mathbb{Q}$; see [Barnes 2008, Definition 1.5.2] for a construction as an equivariant Moore spectrum for $\mathbb{Q}$.

Theorem 2.9 *There is a cofibrantly generated proper stable model structure on the category of orthogonal G–spectra whose weak equivalences are the class of rational π_*–isomorphisms and whose cofibrations are the class of q–cofibrations. This model structure is called the* ***rational model structure*** *and we write $G\mathrm{Sp}^O_{\mathbb{Q}}$ for the model category of orthogonal G–spectra equipped with the rational model structure.*

Proof The stable model structure on orthogonal G–spectra is cofibrantly generated, proper and stable. Thus [Barnes and Roitzheim 2020, Theorem 7.2.17] implies that any left Bousfield localisation of the stable model structure at a set of maps which are closed under desuspension is also cofibrantly generated, proper and stable.

We construct the rational model structure by localising the stable model structure at the set of all suspensions and desuspensions of the map from the sphere spectrum to the rational sphere spectrum

$$S^0 \to S^0\mathbb{Q}.$$

That this gives the correct weak equivalences is a consequence of Proposition 2.10. □

Proposition 2.10 *There is a natural isomorphism*

$$\pi_*^H(X \wedge S^0\mathbb{Q}) \cong \pi_*^H(X) \otimes \mathbb{Q}.$$

A G–spectrum X is fibrant in $G\mathrm{Sp}^O_{\mathbb{Q}}$ if and only if X is an Ω–spectrum and has rational homotopy groups.

The last statement gives the following zigzag of π_*–isomorphisms for any G–spectrum X, where $\hat{f}_{\mathbb{Q}}X$ is the fibrant replacement of X in $G\mathrm{Sp}^O_{\mathbb{Q}}$:

$$\hat{f}_{\mathbb{Q}}X \to \hat{f}_{\mathbb{Q}}X \wedge S^0\mathbb{Q} \leftarrow X \wedge S^0\mathbb{Q}.$$

This shows that our localisation is a smashing localisation.

Corollary 2.11 *If the G–spectrum A is compact in the homotopy category of $G\mathrm{Sp}^O$, then there is a natural isomorphism*

$$[A, X]_{\mathbb{Q}} \cong [A, X] \otimes \mathbb{Q}.$$

In particular, for $\hat{f}_{\mathbb{Q}}X$ the fibrant replacement of X in $G\mathrm{Sp}^O_{\mathbb{Q}}$, there is a natural isomorphism

$$\pi_*^H(\hat{f}_{\mathbb{Q}}X) \cong \pi_*^H(X) \otimes \mathbb{Q}.$$

Since the weak equivalences of the stable model structure are defined in terms of π_*–isomorphisms, the triangulated category $\mathrm{Ho}(G\mathrm{Sp}^O)$ has a set of compact generators: the suspension spectra $\Sigma^\infty G/H_+$, for H an open subgroup of G. Similarly, this set is also a set of compact generators for $\mathrm{Ho}(G\mathrm{Sp}^O_{\mathbb{Q}})$, as the weak equivalences of $G\mathrm{Sp}^O_{\mathbb{Q}}$ are defined in terms of rational π_*–isomorphisms.

Corollary 2.12 *For G a profinite group, the homotopy category of $G\mathrm{Sp}^O_{\mathbb{Q}}$ is generated by the set of compact objects $\Sigma^\infty G/H_+$, for H an open subgroup of G.*

This completes the construction of the model category we wish to model with algebra. Our next task is to study maps between objects like $\Sigma^\infty G/H_+$. We use the profinite version of tom Dieck splitting to show these maps are concentrated in degree zero.

Proposition 2.13 [Fausk 2008, Proposition 7.10] *For G a profinite group and X a based G–space, there is an isomorphism of abelian groups*

$$\bigoplus_{(H)\leqslant_{\text{open}} G} \pi_*(\Sigma^\infty EW_G H_+ \wedge_{W_G H} X^H) \to \pi_*^G(\Sigma^\infty X).$$

The sum runs over the conjugacy classes of open subgroups of G.

Theorem 2.14 [Barnes 2011, Theorem 2.9] *For G a profinite group, the graded $\mathbb{Q}$–module*

$$[\Sigma^\infty G/H_+, \Sigma^\infty G/K_+]^G_* \otimes \mathbb{Q}$$

is concentrated in degree zero.

3 Spans and the stable orbit category

The aim for this section is Theorem 3.11, which provides a combinatorial description of the stable orbit category for G. That is, the full triangulated subcategory of $\mathrm{Ho}(G\mathrm{Sp}^O)$ (the homotopy category of G–spectra) defined by the suspension spectra of finite pointed G–sets is shown to be equivalent to the category of spans of finite G–sets. This result is well-known in the case of finite groups but is seemingly new in the case of profinite groups. The method of proof is to relate the profinite case to the finite case by describing maps in $\mathrm{Ho}(G\mathrm{Sp}^O)$ as a colimit of maps in $\mathrm{Ho}((G/N)\mathrm{Sp}^O)$, as N varies over the open normal subgroups of G, see Lemma 3.9.

3.1 The Burnside category

In the case of a finite group G, the Burnside category is the category of G–sets with (equivalence classes) of spans as morphisms. In this subsection we generalise this construction to profinite groups and show how it relates to the finite group case.

Definition 3.1 A set X with an action of G is said to be *discrete* if the canonical map

$$\operatorname*{colim}_{H \leqslant_{\text{open}} G} X^H \to X$$

is an isomorphism. The category of finite discrete G–sets and equivariant maps is denoted G_{df}–sets.

Equally, one can define a discrete G–set as a G–set such that the stabiliser of each point is open. We then see that a G–set X is discrete if and only if the action on G is continuous, when X is equipped with the discrete topology. If X is a finite discrete G–set, then the stabiliser of each point is open, as is the intersection of all the stabilisers. Thus a finite G–set A is discrete if and only if there is an open subgroup H of G such that A is H–fixed.

The class of discrete G–sets is closed under arbitrary coproducts and finite products. The set of finite coproducts of G–sets of the form G/H, for H an open subgroup of G, is a skeleton for the class of finite discrete G–sets. We also note that the empty set is a finite discrete G–set.

Definition 3.2 The *Burnside ring* of G, written as $\boldsymbol{A}(G)$ is the Grothendieck ring of finite discrete G–sets. We further define the *rational Burnside ring* of G as $\boldsymbol{A}_{\mathbb{Q}}(G) = \boldsymbol{A}(G) \otimes \mathbb{Q}$.

Lemma 3.3 *For G a profinite group there is a natural isomorphism*

$$\varepsilon^* \colon \operatorname*{colim}_{N \trianglelefteqslant_{\text{open}} G} \boldsymbol{A}(G/N) \to \boldsymbol{A}(G).$$

Proof A finite G/N–set A can be considered as a G–set via inflation, $\varepsilon^* A$. Since each point of $\varepsilon^* A$ is fixed by N, this is a finite discrete G–set. Inflation gives an injective ring map

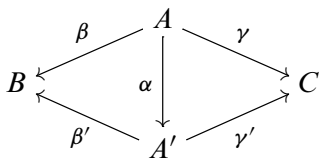
$$\varepsilon_N^* \colon \boldsymbol{A}(G/N) \to \boldsymbol{A}(G).$$

The maps ε_N^* are compatible with the maps forming the colimit (as these are also inflation maps), giving the map ε^*. As the colimit is filtered, ε^* is also injective.

Any finite discrete G–set A is fixed by some open subgroup H, which must contain an open normal subgroup N. Thus A is in the image of ε_N^* and so ε^* is surjective. □

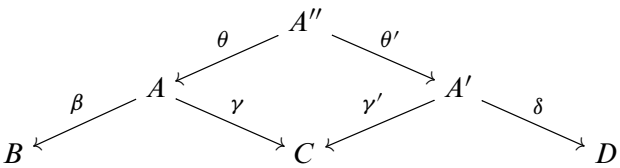
We can generalise the construction of the Burnside ring to make a category.

Definition 3.4 Let G be a profinite group. A *span* of finite discrete G–sets is a pair of equivariant maps $B \xleftarrow{\beta} A \xrightarrow{\gamma} C$, sometimes shortened to (β, γ). Two spans $B \xleftarrow{\beta} A \xrightarrow{\gamma} C$ and $B \xleftarrow{\beta'} A' \xrightarrow{\gamma'} C$ are *equivalent* if there is an equivariant isomorphism $A \to A'$ such that the following diagram commutes.



We write $[\beta, \gamma]$ for the equivalence class of (β, γ).

We recall the notion of composition of spans. Take two spans $B \xleftarrow{\beta} A \xrightarrow{\gamma} C$ and $C \xleftarrow{\gamma'} A' \xrightarrow{\delta} D$, then construct A'' as the pullback of γ and γ'



The composite of (β, γ) and (γ', δ) is the span $(\beta \circ \theta, \delta \circ \theta')$. This composition is well-defined under equivalence of spans.

Further, there is an addition on (equivalence classes of) spans with the same codomains. Consider two spans $B \xleftarrow{\beta} A \xrightarrow{\gamma} C$ and $B \xleftarrow{\beta'} A' \xrightarrow{\gamma'} C$. Their sum is the span

$$B \xleftarrow{\beta,\beta'} A \coprod A' \xrightarrow{\gamma,\gamma'} C.$$

This addition rule is associative, commutative, compatible with equivalence of spans and the unit is the span $B \leftarrow \varnothing \rightarrow C$.

Definition 3.5 The *Burnside category* of G is the category with objects the finite discrete G–sets and morphisms given by Grothendieck construction on sets of equivalence classes of spans of finite discrete G–sets, denoted $\mathrm{Span}(G_{df}\text{–sets})$.

Thus, a map $A \to B$ in the Burnside category is a formal difference of equivalence classes of spans. We also see that the Burnside ring of G is the ring $\mathrm{Span}(G_{df}\text{–sets})(*, *)$.

Just as for the Burnside ring, we can relate the Burnside category for G to the Burnside categories for the groups G/N, where N is an open normal subgroup of G.

Lemma 3.6 *If A and B are finite discrete G–sets, for G a profinite group, then there is an isomorphism*

$$\operatorname*{colim}_{\substack{N \trianglelefteqslant G \\ \text{open}}} \mathrm{Span}(G/N_{df}\text{–sets})(A, B) \cong \mathrm{Span}(G_{df}\text{–sets})(A, B).$$

Proof The maps of the colimit are given by inflation functors and inflation from G/N to G induces the map to the codomain. For the inverse, take a span

$$B \leftarrow A \rightarrow C,$$

and choose an open normal subgroup N that fixes each of element of A, B and C. Then this span appears in term N of the colimit. □

3.2 The stable orbit category

In this subsection we study maps in the G–equivariant stable homotopy category between spectra of the form $\Sigma^\infty A_+$, where A is a finite discrete G–set. Our aim is to relate this to unstable homotopy classes of maps of G/N–spaces, where N runs over the open normal subgroups of G. We start by comparing maps in the G–equivariant stable homotopy category to the G–equivariant unstable homotopy category.

We need two categories, one defined via the G–equivariant stable category and one via the rational analogue. It is important to note that these categories are not graded, we use π_0 in the notation of the categories as a reminder of this fact. The last sentence of the definition holds due to Corollary 2.11.

Definition 3.7 We define a category $\pi_0(\mathfrak{O}_G)$, called the *G–equivariant stable orbit category*. The objects are the class of G–spectra of the form $\Sigma^\infty A_+$, for A a finite discrete G–set. The morphisms are given by

$$\pi_0(\mathfrak{O}_G)(\Sigma^\infty A_+, \Sigma^\infty B_+) = [\Sigma^\infty A_+, \Sigma^\infty B_+]^G,$$

the set of maps in the homotopy category of $G\mathrm{Sp}^O$.

Similarly, we define a category $\pi_0(\mathfrak{O}_G^{\mathbb{Q}})$, called the *rational G–equivariant stable orbit category*. The objects are the same as for $\pi_0(\mathfrak{O}_G)$, but the morphisms are given by

$$\pi_0(\mathfrak{O}_G^{\mathbb{Q}})(\Sigma^\infty A_+, \Sigma^\infty B_+) = [\Sigma^\infty A_+, \Sigma^\infty B_+]^G \otimes \mathbb{Q},$$

the set of maps in the homotopy category of $G\mathrm{Sp}^O_{\mathbb{Q}}$.

Recall that a finite discrete G–set is a disjoint union of homogeneous spaces G/H for H open, hence every finite discrete G–set A is a finite G–CW–complex and A_+ is finite pointed G–CW–complex.

Lemma 3.8 [Fausk 2008, Corollary 7.2] *For G a profinite group, there is an isomorphism of abelian groups*

$$[\Sigma^\infty A, \Sigma^\infty B]^G \cong \operatorname*{colim}_{W \in \mathcal{U}} [A \wedge S^W, B \wedge S^W]^G,$$

where the right-hand terms indicate maps in the homotopy category of pointed G–spaces and A and B are finite pointed G–CW–complexes.

Now we show how the equivariant stable homotopy category for G relates to the equivariant stable homotopy category for finite quotients G/N. Whenever we talk about G/N–spectra, we will use the G/N–universe $\mathcal{U}^N$. This universe is a complete G/N–universe as any finite-dimensional G/N–inner product space V can be written as $(\varepsilon^* V)^N$ and $\varepsilon^* V$ is isomorphic to an indexing space of $\mathcal{U}$, as $\mathcal{U}$ is complete.

Let A and B be finite pointed G–CW–complexes and let $N_1 \leqslant N_2$ be open normal subgroups of G, which fix all of A and B. Then A can be considered as either a G/N_2–CW–complex or a G/N_1–CW–complex and the inflation functor ε^* from G/N_2–spaces to G/N_1–spaces sends the G/N_2–version of A to the G/N_1–version. Hence, the inflation functor ε^* induces a natural map

$$[\Sigma^\infty A, \Sigma^\infty B]^{G/N_2} \to [\Sigma^\infty A, \Sigma^\infty B]^{G/N_1}.$$

We can find a more direct description of this map using Lemma 3.8. An element of the domain can be represented as a map of pointed G/N_2–spaces

$$f\colon A \wedge S^V \to B \wedge S^V$$

for some indexing space $V \subset \mathcal{U}^{N_2}$. Applying the inflation functor gives a map of pointed G/N_1–spaces $\varepsilon^* f$ and since V is N_2–fixed, this defines a element of

$$\operatorname*{colim}_{W \in \mathcal{U}^{N_1}} [A \wedge S^W, B \wedge S^W]^{G/N_1}.$$

This assignment is compatible with taking colimits over $V^{N_2} \subset \mathcal{U}^{N_2}$ and taking G/N_2–equivariant homotopy classes. Hence, we may construct the colimit term of the following result.

Lemma 3.9 *If A and B are finite pointed G–CW–complexes, for G a profinite group, then there is an isomorphism of abelian groups*

$$\operatorname*{colim}_{\substack{N \trianglelefteqslant G \\ \text{open}}} [\Sigma^\infty A, \Sigma^\infty B]^{G/N} \cong [\Sigma^\infty A, \Sigma^\infty B]^G.$$

Proof We have seen that there are isomorphisms

$$\operatorname*{colim}_{\substack{N \trianglelefteqslant G \\ \text{open}}} [\Sigma^\infty A, \Sigma^\infty B]^{G/N} \cong \operatorname*{colim}_{\substack{N \trianglelefteqslant G \\ \text{open}}} \left(\operatorname*{colim}_{W \in \mathcal{U}^N} [A \wedge S^W, B \wedge S^W]^{G/N} \right)$$

The right-hand term maps to

$$\operatorname*{colim}_{W \in \mathcal{U}} [A \wedge S^W, B \wedge S^W]^G \cong [\Sigma^\infty A, \Sigma^\infty B]^G$$

via the inflation functors.

We construct an inverse. Take a representative $f : A \wedge S^V \to B \wedge S^V$. The finite pointed G–CW–complexes A and B are fixed by some open normal subgroup. By Remark 2.3, the G–inner product space V must also be fixed by some open normal subgroup of G. By taking intersections there is an open normal subgroup N of G, which also fixes all of A, B and V. It follows that f defines an element of $[A \wedge S^V, B \wedge S^V]^{G/N}$. One can verify that this is an inverse to the map of the statement. □

Recall the grading convention $[\Sigma^n X, \Sigma^m Y]^G = [X, Y]^G_{n-m}$. Since suspension by S^1 preserves finite (pointed) G–CW–complexes, we have the following extension.

Corollary 3.10 *For G a profinite group, there is an isomorphism of graded abelian groups*

$$\operatorname*{colim}_{\substack{N \trianglelefteq G \\ \text{open}}} [\Sigma^\infty A, \Sigma^\infty B]^{G/N}_* \cong [\Sigma^\infty A, \Sigma^\infty B]^G_*$$

for A and B finite G–CW–complexes.

Theorem 3.11 *For G a profinite group, there is an equivalence of categories*

$$\begin{aligned} \psi_G : \operatorname{Span}(G_{df}\text{–sets}) &\to \pi_0(\mathfrak{O}_G), \\ A &\mapsto \Sigma^\infty A_+ \qquad \text{on objects}, \\ [B \xleftarrow{\beta} A \xrightarrow{\alpha} C] &\mapsto \Sigma^\infty \beta \circ \tau(\alpha) \qquad \text{on morphisms}, \end{aligned}$$

where $\tau(\alpha)$ is the transfer map construction associated to α; see [Lewis et al. 1986, Construction II.5.1] *or work of the second author* [Sugrue 2019b, Construction 3.1.11].

Proof That ψ_G is an equivalence for finite groups G is well-known; see [Lewis et al. 1986, Proposition V.9.6]. We will use that result to extend from the finite case to the profinite case.

For an inclusion $N \to N'$ of open normal subgroups, the inflation functor from $\operatorname{Span}(G/N_{df}\text{–sets})$ to $\operatorname{Span}(G/N'_{df}\text{–sets})$ commutes with $\psi_{G/N}$ and $\psi_{G/N'}$, hence the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Span}(G_{df}\text{–sets})(A, B) & \xrightarrow{\psi_G} & [\Sigma^\infty A_+, \Sigma^\infty B_+]^G \\ \cong \uparrow & & \cong \uparrow \\ \operatorname*{colim}_{\substack{N \trianglelefteq G \\ \text{open}}} \operatorname{Span}(G/N_{df}\text{–sets})(A, B) & \xrightarrow[\cong]{\psi_{G/N}} & \operatorname*{colim}_{\substack{N \trianglelefteq G \\ \text{open}}} [\Sigma^\infty A_+, \Sigma^\infty B_+]^{G/N} \end{array}$$

This proves that ψ_G is full and faithful; essential surjectivity is immediate. □

4 The classification

We give the main result, the classification (in terms of Quillen equivalences) of rational G–equivariant spectra, for profinite G, in terms of a simple algebraic model. In fact, by previous work of the authors [Barnes and Sugrue 2023] we give two equivalent algebraic models. The first is the category of (chain

complexes of) rational G–Mackey functors, the second is the category of Weyl G–sheaves over the space of closed subgroups of G. The relative advantages of the two descriptions are explained in that reference, though we will need the sheaf description in Section 5.

4.1 Mackey functors

There are several equivalent definitions of Mackey functors, we briefly describe the three most common variations, leaving the axioms of the first two versions for references such as [Lindner 1976] or [Thiel 2011]. These three definitions are shown to be equivalent in [Sugrue 2019b, Section 2.1], which follows the work of Lindner [1976].

(1) A set of abelian groups $M(G/H)$, for H open in G, with induction, restriction and conjugation maps relating these groups that satisfy a list of axioms (unital, transitivity, associativity, equivariance and the Mackey axiom).

(2) A pair of functors from the category of finite discrete G–sets to abelian groups, one covariant, one contravariant that agree on objects and satisfy a pullback axiom and a coproduct axiom. These are sometimes known as categorical Mackey functors.

(3) An additive functor from the Burnside category $\mathrm{Span}(G_{df}\text{–sets})$ to abelian groups.

The choice of focussing on the open subgroups (or equally the discrete finite G–sets) matches with the "finite natural Mackey system" of [Bley and Boltje 2004, Definition 2.1 and Examples 2.2; Thiel 2011, Definition 2.2.12]. In the case of a finite group, this choice restricts to the usual definitions. For our purposes we use the last definition.

Definition 4.1 A *Mackey functor* for a profinite group G is an additive functor from the Burnside category $\mathrm{Span}(G_{df}\text{–sets})$ to abelian groups. We will write $\mathrm{Mackey}(G)$ for the category of Mackey functors and additive natural transformations between them.

A *rational Mackey functor* is an additive functor from the Burnside category $\mathrm{Span}(G_{df}\text{–sets})$ to $\mathbb{Q}$–modules. We will write $\mathrm{Mackey}_{\mathbb{Q}}(G)$ for the category of rational Mackey functors and additive natural transformations between them.

General examples of Mackey functors can be found in the references given at the start of the section. However, there is one class of rational Mackey functors of particular relevance to this paper.

Example 4.2 By Theorem 3.11 and Corollary 2.11, if X is a G–spectrum then we have a rational G–Mackey functor,

$$\pi_0(\mathfrak{O}_G^{\mathbb{Q}}) \to \mathbb{Q}\text{–mod}, \quad G/H_+ \mapsto \pi_0^H(X) \otimes \mathbb{Q} \cong [G/H_+, X]_{\mathbb{Q}}^G,$$

called the *homotopy group Mackey functor* of X.

We write out the definition of Mackey functors in terms of our notation and apply Theorem 3.11. We use $\mathrm{Func}_{\mathrm{Ab}}(-,-)$ to denote the category of enriched functors and natural transformations over abelian

groups. This is the same category as taking additive functors, as an additive functor between additive categories is precisely the data of a functor enriched over abelian groups:

$$\mathrm{Mackey}(G) = \mathrm{Func}_{\mathrm{Ab}}(\mathrm{Span}(G_{df}\text{–sets}), \mathrm{Ab}) \cong \mathrm{Func}_{\mathrm{Ab}}(\pi_0(\mathfrak{O}_G), \mathrm{Ab}),$$
$$\mathrm{Mackey}_{\mathbb{Q}}(G) = \mathrm{Func}_{\mathrm{Ab}}(\mathrm{Span}(G_{df}\text{–sets}), \mathbb{Q}\text{–mod}) \cong \mathrm{Func}_{\mathrm{Ab}}(\pi_0(\mathfrak{O}_G), \mathbb{Q}\text{–mod}).$$

It remains to relate these categories of G–Mackey functors to the model category of rational G–spectra. For this we will need a model structure on chain complexes of G–Mackey functors.

Lemma 4.3 *There is a cofibrantly generated model structure on the category of chain complexes of (rational) G–Mackey functors where a map is a fibration if and only if it is a surjection and the class of weak equivalences is the class of homology isomorphisms.*

Proof Since $\mathrm{Ch}(\mathrm{Mackey}(G)) = \mathrm{Func}_{\mathrm{Ab}}(\mathrm{Span}(G_{df}\text{–sets}), \mathrm{Ch}(\mathbb{Z}))$, we use the cofibrant generation of $\mathrm{Ch}(\mathbb{Z})$ to obtain generating sets for $\mathrm{Ch}(\mathrm{Mackey}(G))$ in terms of the representable functors. □

4.2 Tilting theory

We give our classification theorem for rational G–spectra, where G is a profinite group.

Theorem 4.4 *For G a profinite group, there is a zigzag of Quillen equivalences between the model category of rational G–spectra and the model category of chain complexes of rational G–Mackey functors.*

Proof Choose a skeleton $\mathcal{G}$ of $\pi_0(\mathfrak{O}_G^{\mathbb{Q}})$ (such as the set of finite coproducts of the G–sets G/H for H an open subgroup of G). Define $\pi_0(\mathcal{G})$ to be the category whose objects are the elements of $\mathcal{G}$ and whose morphisms are given by the abelian groups

$$[\Sigma^\infty A_+, \Sigma^\infty B_+]^G \otimes \mathbb{Q}.$$

The objects of $\mathcal{G}$ define a set of compact generators for $G\mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}$ by Corollary 2.12 and the set of graded maps between them in $\mathrm{Ho}(G\mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}})$ is concentrated in degree zero by Theorem 2.14. Thus, we can use [Schwede and Shipley 2003, Theorem 5.1.1 and Proposition B.2.1] to obtain a zigzag of Quillen equivalences,

$$G\mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}} \simeq \mathrm{Ch}\big(\mathrm{Func}_{\mathrm{Ab}}(\pi_0(\mathcal{G}), \mathrm{Ab})\big).$$

As $\pi_0(\mathcal{G})$ is a skeleton of $\pi_0(\mathfrak{O}_G^{\mathbb{Q}})$, we have the first equivalence of categories below. The second is Lemma 4.5, which applies as Corollary 2.11 shows that $\pi_0(\mathfrak{O}_G^{\mathbb{Q}}) = \pi_0(\mathfrak{O}_G) \otimes \mathbb{Q}$:

$$\mathrm{Ch}\big(\mathrm{Func}_{\mathrm{Ab}}(\pi_0(\mathcal{G}), \mathrm{Ab})\big) \cong \mathrm{Ch}\big(\mathrm{Func}_{\mathrm{Ab}}(\pi_0(\mathfrak{O}_G^{\mathbb{Q}}), \mathrm{Ab})\big) \cong \mathrm{Ch}\big(\mathrm{Func}_{\mathrm{Ab}}(\pi_0(\mathfrak{O}_G), \mathbb{Q}\text{–mod})\big).$$

Applying Theorem 3.11 gives the final step,

$$\mathrm{Ch}(\mathrm{Mackey}_{\mathbb{Q}}(G)) = \mathrm{Ch}\big(\mathrm{Func}_{\mathrm{Ab}}(\mathrm{Span}(G_{df}\text{–sets}), \mathbb{Q}\text{–mod})\big) \cong \mathrm{Ch}\big(\mathrm{Func}_{\mathrm{Ab}}(\pi_0(\mathfrak{O}_G), \mathbb{Q}\text{–mod})\big).$$ □

Lemma 4.5 *Let $\mathcal{C}$ be a small additive category. The rationalisation functor $i:\mathcal{C}\to\mathcal{C}\otimes\mathbb{Q}$ (defined in the proof) and the forgetful functor $U:\mathbb{Q}$–mod $\to$ Ab induce equivalences*

$$\mathrm{Func}_{\mathrm{Ab}}(\mathcal{C},\mathbb{Q}\text{–mod}) \xleftarrow{i_*} \mathrm{Func}_{\mathrm{Ab}}(\mathcal{C}\otimes\mathbb{Q},\mathbb{Q}\text{–mod}) \xrightarrow{U^*} \mathrm{Func}_{\mathrm{Ab}}(\mathcal{C}\otimes\mathbb{Q},\mathrm{Ab}).$$

Proof A small additive category $\mathcal{C}$ has a rationalisation $\mathcal{C}\otimes\mathbb{Q}$, this category has the same objects and morphisms given by

$$(\mathcal{C}\otimes\mathbb{Q})(c,c') = \mathcal{C}(c,c')\otimes\mathbb{Q}.$$

Composition is induced from that of $\mathcal{C}$ and $\mathcal{C}\otimes\mathbb{Q}$ is an additive category. Moreover, there is an additive functor from $\mathcal{C}$ to its rationalisation $i:\mathcal{C}\to\mathcal{C}\otimes\mathbb{Q}$.

The functors are equivalences as in each case the functor must take values in $\mathbb{Q}$–modules. □

Example 4.6 Let A be a finite discrete G–set. Let M_A be the homotopy group Mackey functor of $\Sigma^\infty A_+$ from Example 4.2. Then M_A is the representable functor given by

$$\mathrm{Span}(G_{df}\text{–sets})(-,A)\otimes\mathbb{Q}.$$

We recall the notion of Weyl-G–sheaves over the space of closed subgroups of G from [Barnes and Sugrue 2023, Section 2; 2022, Section 10]. For G a profinite group, let $\mathcal{S}G$ denote the set of closed subgroups of G. We topologise this set as the limit of finite discrete spaces

$$\mathcal{S}G := \lim_{N\trianglelefteq_{\text{open}} G} \mathcal{S}(G/N)$$

using the maps which send $K\in\mathcal{S}G$ to $KN/N\in\mathcal{S}(G/N)$.

Definition 4.7 A *G–equivariant sheaf* of $\mathbb{Q}$–modules over $\mathcal{S}G$ is a map $p:E\to\mathcal{S}G$ such that

(1) p is a G–equivariant map $p:E\to\mathcal{S}G$ of spaces with continuous G–actions,

(2) (E,p) is a sheaf space of $\mathbb{Q}$–modules,

(3) each map $g:p^{-1}(x)\to p^{-1}(gx)$ is a map of $\mathbb{Q}$–modules for every $x\in\mathcal{S}G,\ g\in G$.

We will write this as either the pair (E,p) or simply as E. A map $f:(E,p)\to(E',p')$ of G–sheaves of $\mathbb{Q}$–modules over $\mathcal{S}G$ is a G–equivariant map $f:E\to E'$ such that $p'f=p$ and $f_x:E_x\to E'_x$ is a map of $\mathbb{Q}$–modules for each $x\in\mathcal{S}G$.

Definition 4.8 A *rational Weyl-G–sheaf* E is a G–sheaf of $\mathbb{Q}$–modules over $\mathcal{S}G$ such that E_K is K–fixed and hence is a discrete $\mathbb{Q}[W_GK]$–module. A map of Weyl-G–sheaves is a map of G–sheaves of $\mathbb{Q}$ modules over $\mathcal{S}G$. We write this category as Weyl-G–sheaf$_{\mathbb{Q}}(\mathcal{S}G)$

Theorem 4.9 [Barnes and Sugrue 2023, Theorem A] *If G is a profinite group then the category of rational G–Mackey functors is equivalent to the category of rational Weyl-G–sheaves over $\mathcal{S}G$. Furthermore, this is an exact equivalence.*

Corollary 4.10 *For G a profinite group, there is a zigzag of Quillen equivalences between the model category of rational G–spectra and the model category of chain complexes of rational Weyl-G–sheaves.*

5 Homological dimension of equivariant sheaves

Using the Weyl-G–sheaf description of rational G–spectra we can calculate the homological dimension (also known as the injective dimension) of the algebraic model. This gives an indication of the homological complexity of the algebraic model. It is already known that the algebraic model in the case of finite groups has homological dimension zero (we will recover this result). The algebraic model in the case of an r–torus $(S^1)^{\times r}$ is r, as shown in [Greenlees 2012, Theorem 8.1].

We prove that the Cantor–Bendixson rank (Definition 5.4) of the space $\mathcal{S}G$ will determine the homological dimension of rational (Weyl) G–sheaves on $\mathcal{S}G$. This is an equivariant generalisation of the results of the second author [Sugrue 2019a].

5.1 Cantor–Bendixson rank

We start with the basic definitions; see [Gartside and Smith 2010a; 2010b].

Definition 5.1 For a topological space X we define the *Cantor–Bendixson process* on X. Denote by X' the set of all isolated points of X.

(1) Let $X^{(0)} = X$ and $X^{(1)} = X \setminus X'$ have the subspace topology with respect to X.

(2) For successor ordinals suppose we have $X^{(\alpha)}$ for an ordinal α, we define
$$X^{(\alpha+1)} = X^{(\alpha)} \setminus X^{(\alpha)'}.$$

(3) If λ is a limit ordinal we define
$$X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}.$$

Every Hausdorff topological space X has a minimal ordinal α such that $X^{(\alpha)} = X^{(\lambda)}$ for all $\lambda \geq \alpha$; see [Gartside and Smith 2010a, Lemma 2.7].

Definition 5.2 For X a Hausdorff topological space, the *Cantor–Bendixson rank* of X, written $\mathrm{rank}_{CB}(X)$, is the minimal ordinal α such that $X^{(\alpha)} = X^{(\lambda)}$ for all $\lambda \geq \alpha$.

A topological space X is called *perfect* if it has no isolated points, whereupon $\mathrm{rank}_{CB}(X) = 0$.

Remark 5.3 The definition given above agrees with that of [Gartside and Smith 2010b]. The convention of Dickmann, Schwartz and Tressl [Dickmann et al. 2019, Definition 4.3.1] is to take one less than the rank as defined above.

There are two ways that the Cantor–Bendixson process can stabilise, by reaching the empty set or a perfect subspace.

Definition 5.4 If X is a Hausdorff space with Cantor–Bendixson rank λ, then we define the *perfect hull* of X to be the subspace $X^{(\lambda)}$, written X_H.

We write X_S for the complement $X \setminus X_H$ and call it the *scattered part* of X. The space X is said to be *scattered* if $X_H = \varnothing$.

Example 5.5 The Cantor–Bendixson rank of the empty set is zero and the Cantor–Bendixson rank of a nonempty discrete space is 1 as every point is isolated.

The space $\mathcal{S}\mathbb{Z}_p^\wedge$ of closed subgroups of the p–adics is the subspace of $\mathbb{R}$ consisting of the points

$$\{1/n \mid n \in \mathbb{N}\} \cup \{0\}.$$

The isolated points are those of the form $1/n$, which are removed in the first stage of the Cantor–Bendixson process, thus $(\mathcal{S}\mathbb{Z}_p^\wedge)^{(1)} = \{0\}$. For $k > 1$, $(\mathcal{S}\mathbb{Z}_p^\wedge)^{(k)} = \varnothing$, so $\operatorname{rank}_{CB}(\mathcal{S}\mathbb{Z}_p^\wedge) = 2$.

Definition 5.6 If X is a space and $x \in X_S$, we define the *height* of x denoted $\operatorname{ht}(X, x)$, to be the ordinal κ such that $x \in X^{(\kappa)}$ but $x \notin X^{(\kappa+1)}$. We denote this by $\operatorname{ht}(x)$ when the background space X is understood.

We may rephrase the definitions to see that a point x of height $k > 0$ is a limit of points of height $k-1$. Consequently, an open neighbourhood of x contains infinitely many points of height j for each $j < k$.

Lemma 5.7 *Let X be a topological space with an action of a topological group G. The height of points of X is invariant under the action of G.*

If $x \in X$ has height $k > 0$, then every neighbourhood of x contains infinitely many points from orbits other than Gx.

Proof As G acts through homeomorphisms, the first statement holds. The second statement follows from our preceding discussion and the first statement. □

5.2 Equivariant Godement resolutions

We recap equivariant Godement resolutions from [Barnes and Sugrue 2022, Section 9]. The key change from the nonequivariant case is the use orbits in place of points.

Definition 5.8 Let $p\colon E \to X$ be a G–equivariant sheaf of $\mathbb{Q}$–modules over a G–space X, for G a profinite group. Define a G–sheaf

$$I^0(E) = \prod_A i_A^* E|_A$$

with the product taken over the G–orbits of X and i_A^* is the extension by zero functor induced by the map $A \to X$.

The restriction–extension adjunction induces morphisms $E \to i_A^* E|_A$, which combine to a monomorphism

$$\delta_E\colon E \to I^0(E).$$

Lemma 5.9 *Let $p\colon E \to X$ be a G–equivariant sheaf of $\mathbb{Q}$–modules over a G–space X, for G a profinite group. The G–sheaf $I^0(E)$ is injective.*

Proof We prove that $I^0(E)$ is the product of injective sheaves. For each orbit A of X, pick an element $x_A \in A$. By [Barnes and Sugrue 2022, Lemma 8.3] there is an isomorphism of sheaves over A

$$G \underset{\mathrm{stab}_G(x_A)}{\times} E_{x_A} \cong E|_A.$$

The left-hand sheaf is known as the equivariant skyscraper sheaf of E_{x_0} at x_0. It can be viewed as part of an adjunction with the left adjoint being taking the stalk at x_0. As this left adjoint preserves monomorphisms, the equivariant skyscraper sheaf construction preserves injective objects.

The result is completed by [Castellano and Weigel 2016, Proposition 3.1], which states that every object of the category of discrete $\mathbb{Q}[\mathrm{stab}_G(x_0)]$–modules is injective. □

Remark 5.10 The result [Castellano and Weigel 2016, Proposition 3.1] also fixes an omission in work of the first author [Barnes 2011, Lemma 6.2] which implicitly assumes that all discrete $\mathbb{Q}[\mathbb{Z}_p^\wedge]$–modules are injective.

Iterating this construction I^0 gives an injective resolution.

Definition 5.11 Let G be a profinite group. If E is a G–sheaf of R–modules over a profinite G–space we define the *equivariant Godement resolution* as follows:

$$0 \longrightarrow E \xrightarrow{\delta_E} I^0(E) \longrightarrow I^0(\mathrm{Coker}\,\delta_E) = I^1(E) \longrightarrow \cdots$$
$$\downarrow p \qquad \nearrow \delta_{\mathrm{Coker}\,\delta_E}$$
$$\mathrm{Coker}\,\delta_E$$

We now connect the Cantor–Bendixson rank of a G–space X to the length of the equivariant Godement resolution of rational G–sheaves over X.

Theorem 5.12 *Let E be a rational G–sheaf over a profinite G–space X, for G a profinite group. For $n \in \mathbb{N}$ and $x \in X$, the stalk $I^n(E)_x$ is zero unless the height of x is at least n.*

Proof The proof is by induction. The base case follows from the fact that $(\delta_E)_x$ is an isomorphism for any isolated point of X. Similarly, if x has a open neighbourhood where the only nontrivial stalk is at x, then $(\delta_E)_x$ is an isomorphism. The equivariant Godement resolution with the Cantor–Bendixson process gives the inductive step using Lemma 5.7. Further details are given in the nonequivariant case of [Sugrue 2019a, Lemma 3.7]. □

If we restrict ourselves to the case of scattered spaces of finite rank (the Cantor–Bendixson process ends in the empty set after finitely many steps) this theorem gives an upper bound for the length of the equivariant Godement resolutions.

Corollary 5.13 *Let G be a profinite group. Let X be a scattered profinite G–space of Cantor–Bendixson rank $n \in \mathbb{N}$. The category of rational G–sheaves over X has homological dimension at most $n-1$.*

Proof If the rank is n, then $X^{(n)} = \varnothing$ and every point has height at most $n-1$. Hence, every stalk of $I^n(X)$ is zero, so the sheaf is itself zero. □

5.3 The case of the constant sheaf

It remains to prove that the upper bound on homological dimension is in fact an equality. To that end, we take the equivariant Godement resolution of the constant sheaf at $\mathbb{Q}$, $\operatorname{Const}\mathbb{Q}$,

$$0 \longrightarrow \operatorname{Const}\mathbb{Q} \xrightarrow{\delta_0} I^0 \xrightarrow{\delta_1} I^1 \xrightarrow{\delta_2} \cdots \longrightarrow I^{n-1}.$$

Adding a small assumption on X we can prove this resolution has length exactly $n-1$.

Proposition 5.14 *Let G be a profinite group, let X be a profinite scattered G–space of Cantor–Bendixson rank n and assume that each $x \in X$ has a neighbourhood basis $\mathcal{B}_x$ of $\operatorname{stab}_G(x)$–invariant sets. In the equivariant Godement resolution of the constant sheaf, the cokernel of $\delta_i(U)$ has a nonzero $\operatorname{stab}_G(x)$–equivariant section for each $U \in \mathcal{B}_x$ whenever i is smaller than the height of x.*

Proof We start with the case of δ_0 with x of height at least 1. Take a $\operatorname{stab}_G(x)$–invariant open neighbourhood of x. By Lemma 5.7, U contains infinitely many points of other orbits which are of lower height than x.

Choose a nonzero element of $\operatorname{Const}\mathbb{Q}(U)_x = \mathbb{Q}$ represented by a section $s \in \operatorname{Const}\mathbb{Q}(U)$. We define a section t of $I^0 = \prod_A i_A^* E|_A$ by the sequence

$$A \mapsto \begin{cases} 0 & \operatorname{ht}(A) \text{ has the same parity as } \operatorname{ht}(x), \\ s|_A & \text{otherwise.} \end{cases}$$

Since s is nonzero at infinitely many points near x, t_x is nonzero. If $t = \delta_0(s')$ for some $s' \in \operatorname{Const}\mathbb{Q}(U)$ then $s'|_{Gx} = 0$. This implies that there is an open neighbourhood of x where s' restricts to zero and hence $s'_y = 0$ for all y in that open neighbourhood, which implies that $t_x = 0$, a contradiction.

The rest follows inductively, as with the nonequivariant case of [Sugrue 2019a, Lemma 4.3], with two changes. The first is that when we need to construct a new nonzero section we use the alternating process described previously. The second is that since our sets U are $\operatorname{stab}_G(x)$–invariant and we begin with a $\operatorname{stab}_G(x)$–equivariant section, all sections constructed in the proof are $\operatorname{stab}_G(x)$–equivariant.

Note that the assumption on i and the height of x is required to ensure that the new section we construct is nonzero at infinitely many orbits. □

We can now give the equivariant analogue of [Sugrue 2019a, Theorem 4.4].

Theorem 5.15 *Let X be profinite G–space such that each $x \in X$ has a neighbourhood basis $\mathcal{B}_x$ of $\operatorname{stab}_G(x)$–invariant sets, for G a profinite group.*

If X is a scattered G–space of Cantor–Bendixson rank n and x has height $n-1$, then

$$\mathrm{Ext}^{n-1}\big(G \underset{\mathrm{stab}_G(x)}{\times} \mathbb{Q}, \mathrm{Const}\,\mathbb{Q}\big) \neq 0.$$

Hence, the homological dimension of the category of rational G–sheaves over X is $n-1$.

If X has infinite Cantor–Bendixson rank then the homological dimension of the category of rational G–sheaves over X is infinite.

Proof We begin with a general calculation of maps out of an equivariant skyscraper sheaf into $I^0(E)$ for E some rational G–sheaf over X,

$$\begin{aligned}\hom\big(G \underset{\mathrm{stab}_G(x)}{\times} \mathbb{Q}, I^0(E)\big) &\cong \prod_A \hom\big(G \underset{\mathrm{stab}_G(x)}{\times} \mathbb{Q}, i_A^* E|_A\big)\\ &\cong \hom\big(G \underset{\mathrm{stab}_G(x)}{\times} \mathbb{Q}, E|_{Gx}\big)\\ &\cong \hom\big(G \underset{\mathrm{stab}_G(x)}{\times} \mathbb{Q}, G \underset{\mathrm{stab}_G(x)}{\times} E_x\big)\\ &\cong E_x^{\mathrm{stab}_G x}.\end{aligned}$$

The final term has fixed points as $\mathbb{Q}$ has the trivial $\mathrm{stab}_G\, x$–action.

Assume that X is a scattered G–space of Cantor–Bendixson rank n and x has height $n-1$. Applying our calculation to our resolution of $\mathrm{Const}\,\mathbb{Q}$ we see that our Ext groups are the homology of the chain complex

$$\mathbb{Q} \xrightarrow{\alpha_0} (\mathrm{Coker}\,\delta_0)_x^{\mathrm{stab}_G(x)} \xrightarrow{\alpha_1} (\mathrm{Coker}\,\delta_1)_x^{\mathrm{stab}_G(x)} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} (\mathrm{Coker}\,\delta_{n-2})_x^{\mathrm{stab}_G(x)}.$$

By Proposition 5.14 we have a nonzero $\mathrm{stab}_G(x)$–equivariant section which implies that

$$(\delta_i)_x^{\mathrm{stab}_G(x)} \neq 0$$

whenever i is smaller than $n-1$. By a similar argument to Proposition 5.14 we see that α_{n-1} is not surjective, hence the n^{th} Ext group is nonzero. This calculation and Corollary 5.13 show that the homological dimension is $n-1$.

In the case of infinite Cantor–Bendixson rank we see that for each n there is a point of height n, hence our earlier work shows that the homological dimension is infinite. □

The space of closed subgroups $\mathcal{S}G$ of a profinite group G always satisfies the condition on the invariant neighbourhood basis; see [Barnes and Sugrue 2023, Section 2].

Corollary 5.16 *Let G be a profinite group whose space of subgroups $\mathcal{S}G$ is scattered of Cantor–Bendixson rank n. The homological dimension of the category of rational Weyl-G–sheaves is $n-1$.*

If $\mathcal{S}G$ has infinite Cantor–Bendixson rank, then the homological dimension of the category of rational Weyl-G–sheaves is infinite.

Proof All the sheaves used in the theorem proof were stalkwise fixed and hence were Weyl-G–sheaves. □

There is a remaining case of spaces which have finite Cantor–Bendixson rank and nonempty perfect hull. We make an equivariant version of [Sugrue 2019a, Conjecture 4.6].

Conjecture 5.17 *Let G be a profinite group. If the G–space X has finite Cantor–Bendixson rank and nonempty perfect hull, then the homological dimension of rational G–sheaves over X is infinite.*

Using the equivalence between Weyl-G–sheaves and rational G–Mackey functors [Barnes and Sugrue 2023, Theorem A] we obtain the following calculation of homological dimensions for categories of rational G–Mackey functors.

Corollary 5.18 *Let G be a profinite group whose space of subgroups $\mathcal{S}G$ is scattered of Cantor–Bendixson rank n. The homological dimension of the category of rational G–Mackey functors is $n-1$.*

If $\mathcal{S}G$ has infinite Cantor–Bendixson rank, then the homological dimension of the category of rational G–Mackey functors is infinite.

Example 5.19 Using our calculations from Example 5.5 we can give the homological dimension of some algebraic models.

For G a finite group, the homological dimension of rational Weyl-G–sheaves is zero. In terms of Mackey functors, this says that every rational G–Mackey functor is injective, as was proven independently by two sources [Greenlees and May 1995, Appendix A; Thévenaz and Webb 1995, Theorems 8.3 and 9.1]. See also [Barnes and Kędziorek 2022, Theorem 4.28].

For $G = \mathbb{Z}_p^\wedge$, the p–adic integers, the homological dimension of rational Weyl-G–sheaves is one, which agrees with [Barnes 2011, Lemma 6.2].

References

[Barnes 2008] **D Barnes**, *Rational equivariant spectra*, PhD thesis, University of Sheffield (2008) arXiv 0802.0954

[Barnes 2011] **D Barnes**, *Rational $\mathbb{Z}_p$–equivariant spectra*, Algebr. Geom. Topol. 11 (2011) 2107–2135 MR Zbl

[Barnes and Kędziorek 2022] **D Barnes**, **M Kędziorek**, *An introduction to algebraic models for rational G–spectra*, from "Equivariant topology and derived algebra", Lond. Math. Soc. Lect. Note Ser. 474, Cambridge Univ. Press (2022) 119–179 MR Zbl

[Barnes and Roitzheim 2020] **D Barnes**, **C Roitzheim**, *Foundations of stable homotopy theory*, Cambridge Stud. Adv. Math. 185, Cambridge Univ. Press (2020) MR Zbl

[Barnes and Sugrue 2022] **D Barnes**, **D Sugrue**, *Equivariant sheaves for profinite groups*, Topology Appl. 319 (2022) art. id. 108215 MR Zbl

[Barnes and Sugrue 2023] **D Barnes**, **D Sugrue**, *The equivalence between rational G–sheaves and rational G–Mackey functors for profinite G*, Math. Proc. Cambridge Philos. Soc. 174 (2023) 345–368 MR Zbl

[Bley and Boltje 2004] **W Bley**, **R Boltje**, *Cohomological Mackey functors in number theory*, J. Number Theory 105 (2004) 1–37 MR Zbl

[Castellano and Weigel 2016] **I Castellano**, **T Weigel**, *Rational discrete cohomology for totally disconnected locally compact groups*, J. Algebra 453 (2016) 101–159 MR Zbl

[Dickmann et al. 2019] **M Dickmann**, **N Schwartz**, **M Tressl**, *Spectral spaces*, New Math. Monogr. 35, Cambridge Univ. Press (2019) MR Zbl

[Fausk 2008] **H Fausk**, *Equivariant homotopy theory for pro-spectra*, Geom. Topol. 12 (2008) 103–176 MR Zbl

[Gartside and Smith 2010a] **P Gartside**, **M Smith**, *Classifying spaces of subgroups of profinite groups*, J. Group Theory 13 (2010) 315–336 MR Zbl

[Gartside and Smith 2010b] **P Gartside**, **M Smith**, *Counting the closed subgroups of profinite groups*, J. Group Theory 13 (2010) 41–61 MR Zbl

[Greenlees 2006] **J P C Greenlees**, *Triangulated categories of rational equivariant cohomology theories*, Oberwolfach Reports 3 (2006) 480–489 Part of the conference report "Thick subcategories: classifications and applications"

[Greenlees 2012] **J P C Greenlees**, *Rational torus-equivariant stable homotopy, II: Algebra of the standard model*, J. Pure Appl. Algebra 216 (2012) 2141–2158 MR Zbl

[Greenlees and May 1995] **J P C Greenlees**, **J P May**, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 543, Amer. Math. Soc., Providence, RI (1995) MR Zbl

[Hill and Mazur 2019] **M A Hill**, **K Mazur**, *An equivariant tensor product on Mackey functors*, J. Pure Appl. Algebra 223 (2019) 5310–5345 MR Zbl

[Hill et al. 2016] **M A Hill**, **M J Hopkins**, **D C Ravenel**, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. 184 (2016) 1–262 MR

[Illman 1983] **S Illman**, *The equivariant triangulation theorem for actions of compact Lie groups*, Math. Ann. 262 (1983) 487–501 MR Zbl

[Lewis et al. 1986] **L G Lewis, Jr**, **J P May**, **M Steinberger**, **J E McClure**, *Equivariant stable homotopy theory*, Lecture Notes in Math. 1213, Springer (1986) MR Zbl

[Lindner 1976] **H Lindner**, *A remark on Mackey-functors*, Manuscripta Math. 18 (1976) 273–278 MR Zbl

[Mandell and May 2002] **M A Mandell**, **J P May**, *Equivariant orthogonal spectra and S–modules*, Mem. Amer. Math. Soc. 755, Amer. Math. Soc., Providence, RI (2002) MR Zbl

[Mandell et al. 2001] **M A Mandell**, **J P May**, **S Schwede**, **B Shipley**, *Model categories of diagram spectra*, Proc. Lond. Math. Soc. 82 (2001) 441–512 MR Zbl

[Ribes and Zalesskii 2000] **L Ribes**, **P Zalesskii**, *Profinite groups*, Ergebnisse der Math. 40, Springer (2000) MR Zbl

[Schwede and Shipley 2003] **S Schwede**, **B Shipley**, *Stable model categories are categories of modules*, Topology 42 (2003) 103–153 MR Zbl

[Segal 1968] **G Segal**, *Equivariant K–theory*, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 129–151 MR Zbl

[Sugrue 2019a] **D Sugrue**, *Injective dimension of sheaves of rational vector spaces*, J. Pure Appl. Algebra 223 (2019) 3112–3128 MR Zbl

[Sugrue 2019b] **D Sugrue**, *Rational G–spectra for profinite G*, PhD thesis, Queen's University Belfast (2019) arXiv 1910.12951

[Thévenaz and Webb 1995] **J Thévenaz**, **P Webb**, *The structure of Mackey functors*, Trans. Amer. Math. Soc. 347 (1995) 1865–1961 MR Zbl

[Thiel 2011] **U Thiel**, *Mackey functors and abelian class field theories*, preprint (2011) arXiv 1111.0942

[Williamson 2022] **J Williamson**, *Algebraic models of change of groups functors in (co)free rational equivariant spectra*, J. Pure Appl. Algebra 226 (2022) art. id. 107108 MR

[Wilson 1998] **J S Wilson**, *Profinite groups*, Lond. Math. Soc. Monogr. 19, Oxford Univ. Press (1998) MR Zbl

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