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An explicit comparison between 2–complicial sets and Θ_2–spaces

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We produce a direct Quillen equivalence between two models of $(\infty, 2)$–categories: the complete Segal Θ_2–spaces due to Rezk and the 2–complicial sets due to Verity.

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Introduction

The language of higher categories provides a way to describe many phenomena in areas of mathematics as diverse as topology, algebra, geometry, and mathematical physics. In a higher categorical structure, we not only have functions between objects, but functions between those functions and possibly further iterations of this idea, encoded by the notion of a k–morphism between $(k-1)$–morphisms. One might initially assume that these higher morphisms should satisfy conditions like associativity in the usual way, but for many natural examples they only hold up to isomorphism or, in topological settings, up to homotopy. In the latter situation, it is convenient to work in the setting of (∞, n)–categories, in which we have k–morphisms for arbitrarily large k, but they are all weakly invertible for $k > n$. These higher invertible morphisms provide a means for conveniently encoding the "up to isomorphism" data in the lower morphisms.

There have been many different approaches to realizing (∞, n)–categories as concrete mathematical objects; such realizations are often called *models* for (∞, n)–categories. A natural question, then, is whether these different models really do encode the same information, namely, whether we can establish an appropriate equivalence between them. Much work has been done in this direction, but there are still proposed models for which we do not have such comparisons. In some other cases, we know by general results that models must be equivalent, but do not have an explicit equivalence.

The motivation for this paper is the desire for an explicit comparison between two of these models, the complete Segal Θ_n–spaces as defined by [Rezk 2010] and the n–complicial sets as defined by [Verity 2008b] (see also [Ozornova and Rovelli 2020; Riehl 2018]); we give such a comparison when $n = 2$, for which more tools are available. Let us give a brief description of these two models.

A complete Segal Θ_2–space is described by a diagram of spaces indexed by 2–categories freely generated by pasting diagrams such as

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=A, below}] \arrow[r, ""{name=B, above}, ""{name=C, below}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet \arrow[r] & \bullet \arrow[r, bend left=60, ""{name=E, below}] \arrow[r, ""{name=F, above}] & \bullet
\arrow[Rightarrow, from=A, to=B] \arrow[Rightarrow, from=C, to=D] \arrow[Rightarrow, from=E, to=F]
\end{tikzcd}$$

which the expert reader may recognize as the generic element of Joyal's *cell category* Θ_2. In contrast, a 2–complicial set is given by a simplicial set with a suitable marking in which a k–simplex represents a diagram indexed by a truncated *oriental*, which is a free 2–category generated by a standard simplex, such as

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] & & \bullet & \bullet \arrow[l] \\
\bullet \arrow[u] \arrow[r] \arrow[ur] & \bullet \arrow[u] & = & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[ul]
\arrow[from=1-1, to=2-2, phantom, "\Nearrow" pos=0.3, "\Searrow" pos=0.7]
\arrow[from=1-5, to=2-4, phantom, "\Nearrow" pos=0.3, "\Searrow" pos=0.7]
\end{tikzcd}$$

A common way to show that two models are equivalent is to show that appropriate model categories for each are Quillen equivalent to each other. In this paper, we seek to establish such a Quillen equivalence between the model structure $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$ for complete Segal Θ_2–spaces and the model structure $ms\mathcal{S}et_{(\infty,2)}$ for 2–complicial sets.

Combining several prior results by different groups of authors, we already know that the two model categories are Quillen equivalent via a rather lengthy zigzag of Quillen equivalences between different models. Although we do not expect the reader to be familiar with all these models of $(\infty, 2)$–categories, to give an idea of the complexity of the comparison Figure 1 shows a diagram of an essentially optimal zigzag of Quillen equivalences, extracted from [Gagna et al. 2022].

To simplify the comparison, the goal of this paper is to produce the following direct Quillen equivalence.

Theorem *There is a Quillen equivalence between complete Segal Θ_2–spaces, presented by the model category $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$, and 2–complicial sets, presented by the model category $ms\mathcal{S}et_{(\infty,2)}$.*

$$\begin{tikzcd}
s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \arrow[d, equal, "\wr"', "\text{[Rezk 2001]}"] & & ms\mathcal{S}et_{(\infty,2)} \arrow[d, equal, "\text{[Gagna et al. 2022]}"', "\wr"] \\
s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)} \arrow[d, equal, "\wr"', "\text{[Bergner and Rezk 2020]}"] & & s\mathcal{S}et^{sc}_{(\infty,2)} \arrow[d, equal, "\text{[Lurie 2009b]}"', "\wr"] \\
s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)} \arrow[d, equal, "\wr"', "\text{[Bergner and Rezk 2020]}"] & & \mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}} \arrow[d, equal, "\text{[Lurie 2009a]}"', "\wr"] \\
P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)} \arrow[r, "\simeq", "\text{[Bergner and Rezk 2013]}"'] & \mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}} \arrow[r, "\simeq", "\text{[Joyal and Tierney 2007]}"'] & \mathcal{C}at_{\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}
\end{tikzcd}$$

Figure 1

In addition to providing a more transparent comparison between the two models, this direct comparison facilitates the transport of constructions between these model structures. We now briefly illustrate the advantages of each model structure via the examples of duals and joins, and we refer the reader to Section 4 for a more detailed treatment of these cases, as well as other applications.

The structure of the category Θ_2 makes the description of duals straightforward in Θ_2–spaces, thanks to the globular shape of the objects. We can think of 1–dimensional duals as given by reversing the direction of the arrows, and 2–dimensional duals as given by similarly reversing the direction of 2–cells. Describing such 2–dimensional duals in the simplicial framework is more complicated, due to the triangular shape of the cells.

On the other hand, the join construction has been described for 2–complicial sets by [Verity 2008b] and is similar to familiar join constructions for simplicial sets. One can adjoin 1–simplices connecting the vertices of the two simplicial sets being joined, and higher-dimensional simplices analogously. In this case, working in a simplicial framework is much more straightforward than that of Θ_2.

While the existence of such a direct Quillen equivalence follows formally, for example using methods of [Dugger 2001], we find it valuable to have an explicit description.

Let us now describe the main ingredients of the proof of our main theorem.

(i) We use the compatibility of the 2–categorical nerve valued in marked simplicial sets, established by [Ozornova and Rovelli 2022], to construct a left Quillen functor

$$L\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)} \to ms\mathcal{S}et_{(\infty,2)}.$$

(ii) To show that this left Quillen functor is in fact a Quillen equivalence, we use a result of [Barwick and Schommer-Pries 2021] to reduce the problem to showing that it preserves cells in dimensions 0, 1, and 2. In Section 3 we use the intermediate comparisons of models from the diagram above to identify these cells in each model and thereby show that L does indeed preserve cells.

The outline of the paper is as follows. In Section 1 we recall some necessary results about model structures for 2–categories, Θ_2–spaces, and simplicial sets with marking, as well as functors between them, such as suspension and nerve functors. In Section 2 we construct the adjunction between Θ_2–spaces and simplicial sets with marking and we show that it is a Quillen pair. We then describe how it follows from [Barwick and Schommer-Pries 2021] that this adjunction is indeed a Quillen equivalence, modulo an explicit identification of the cells in the two models. In Section 3 we then provide the desired identification of the cells in the two models. In Section 4 we discuss some applications of the main theorem.

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1 Models of $(\infty, 2)$–categories

We assume the reader is familiar with the basics of strict 2–category theory (see eg [Borceux 1994]) and with the language of model categories (see eg [Hirschhorn 2003; Hovey 1999]), and we now recall some further preliminary material that we need in this paper.

1.1 Strict 2–categories

The category $2\mathscr{C}at$ of 2–categories is defined as the category whose objects are (small) categories enriched over the category $\mathscr{C}at$ of 1–categories. In particular, a *2–category* $\mathscr{D}$ consists of a set of objects and, for any objects x and x', a 1–category $\mathrm{Hom}_{\mathscr{D}}(x, x')$ together with a horizontal composition that defines a functor of hom-categories $\circ\colon \mathrm{Hom}_{\mathscr{D}}(x, x') \times \mathrm{Hom}_{\mathscr{D}}(x', x'') \to \mathrm{Hom}_{\mathscr{D}}(x, x'')$.

We consider the following model structure on $2\mathscr{C}at$ that was constructed by [Lack 2002, Theorem 3.3] (with a correction in [Lack 2004, Theorem 4]).

Theorem 1.1 *The category $2\mathscr{C}at$ of 2–categories supports a model structure in which*

- *all 2–categories are fibrant, and*
- *the weak equivalences are precisely the biequivalences of 2–categories.*

An important source of examples of 2–categories is given by suspending 1–categories, as follows.

Definition 1.2 Let $\mathscr{D}$ be a 1–category. The *suspension* of $\mathscr{D}$ is the 2–category $\Sigma\mathscr{D}$ in which

(a) there are two objects $x_\perp$ and $x_\top$;

(b) the hom-1–categories are given by

$$\operatorname{Hom}_{\Sigma\mathscr{D}}(a,b) := \begin{cases} \mathscr{D} & \text{if } a = x_\perp \text{ and } b = x_\top, \\ [0] & \text{if } a = b, \\ \varnothing & \text{if } a = x_\top \text{ and } b = x_\perp; \end{cases}$$

(c) there is no nontrivial horizontal composition.

This construction extends to a functor $\Sigma\colon \mathscr{C}at \to 2\mathscr{C}at_{*,*}$ valued in the category of bipointed categories, namely categories endowed with a pair of (possibly equal) specified objects, and basepoint-preserving functors.

The 2–categorical suspension $\Sigma\mathscr{D}$ appears in [Barwick and Schommer-Pries 2021], where it is denoted by $\sigma(\mathscr{D})$. It is also often described in the literature as a special case of a simplicial suspension. For instance, applying the nerve to hom-categories of the suspension $\Sigma\mathscr{D}$ gives a simplicial category $N_*(\Sigma\mathscr{D})$ that agrees with what was denoted by $U(N\mathscr{D})$ in [Bergner 2007b], as $S(N\mathscr{D})$ in [Joyal 2007], as $[1]_{N\mathscr{D}}$ in [Lurie 2009a], and as $\mathbb{2}[N\mathscr{D}]$ in [Riehl and Verity 2020].

Notation 1.3 We record some notation for the following (nondisjoint) families of 2–categories.

- For $m \geq -1$, we denote by $[m]$ the finite ordinal with $m+1$ elements.
- For $j = 0, 1, 2$, we denote by C_j the free j–cell. These 2–categories can be pictured as

$$C_0 = \bullet \qquad C_1 = \begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd} \qquad C_2 = \begin{tikzcd} \bullet \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd}.$$

- For $m \geq 0$ and $k_1, \ldots, k_m \geq 0$, we denote by $[m|k_1, \ldots, k_m]$ a generic object of Joyal's cell category Θ_2, namely the full subcategory Θ_2 of $2\mathscr{C}at$ from [Joyal 1997].
- We denote by $\mathbb{I}$ the free-living isomorphism category. This category can be pictured as

$$\mathbb{I} = \begin{tikzcd} \bullet \arrow[r, bend left=50] \arrow[r, phantom, "\cong"] & \bullet \arrow[l, bend left=50] \end{tikzcd}.$$

1.2 Complete Segal spaces as a model for $(\infty, 1)$–categories

We briefly recall the theory of complete Segal spaces, as first defined by [Rezk 2001], of which the next model we discuss for $(\infty, 2)$–categories is a generalization.

First, consider functors $X\colon \Delta^{\mathrm{op}} \to s\mathcal{S}et$. For any $n \geq 1$, consider the *Segal map*

$$X_n \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}$$

induced by the inclusion

$$\underbrace{\Delta[1] \amalg_{\Delta[0]} \Delta[1] \amalg_{\Delta[0]} \cdots \amalg_{\Delta[0]} \Delta[1]}_{n} \to \Delta[n]$$

of the spine of the n–simplex into the n–simplex $\Delta[n]$.

Definition 1.4 A *Segal space* is a functor $X\colon \Delta^{\mathrm{op}} \to s\mathcal{S}et$ such that the Segal maps are weak equivalences of simplicial sets for all $n \geq 1$.

The idea is that a Segal space behaves something like a category, with simplicial sets of objects and morphisms, but with composition defined only up to homotopy.

However, to have a model for $(\infty, 1)$–categories, we do not want a simplicial set of objects, as in an internal category, but instead a discrete set of objects. The most straightforward way to get such a model is to ask for the simplicial set X_0 to be discrete.

Definition 1.5 A *Segal precategory* is a functor $X\colon \Delta^{\mathrm{op}} \to s\mathcal{S}et$ such that X_0 is a discrete simplicial set. We denote by $P\mathcal{C}at$ the full subcategory of $s\mathcal{S}et^{\Delta^{\mathrm{op}}}$ spanned by all Segal precategories. A *Segal category* is a Segal precategory that is also a Segal space.

There are two model structures for Segal precategories, the first of which has all objects cofibrant and is originally due to [Pellissier 2002, Theorem 6.4.4]; another proof is given in [Bergner 2007a, Theorem 5.1]. However, in this paper we make use of the following model structure that has cofibrations defined similarly to those in the projective model structure.

Theorem 1.6 [Bergner 2007a, Theorem 7.1; Bergner 2007c, Theorem 4.2] *The category $P\mathcal{C}at$ of Segal precategories admits a model structure in which*

- *the fibrant objects are the projectively fibrant Segal categories, and*
- *the cofibrations are projective cofibrations.*

We denote this model structure by $P\mathcal{C}at_{(\infty,1)}$.

However, from the point of view of homotopy theory, asking for discreteness is awkward. The completeness condition that we now describe can be more convenient from this perspective.

Let $N\mathbb{I}$ denote the nerve of the groupoid $\mathbb{I}$, and denote by X_{heq} the simplicial set $\mathrm{Hom}(N\mathbb{I}, X)$, which is sometimes called the space of *homotopy equivalences* of X. The unique map $N\mathbb{I} \to \Delta[0]$ induces a map

$$X_0 \to X_{\mathrm{heq}}.$$

Definition 1.7 A Segal space is *complete* if this map $X_0 \to X_{\mathrm{heq}}$ is a weak equivalence of simplicial sets.

Rezk builds a supporting model structure for the homotopy theory of complete Segal spaces.

Theorem 1.8 [Rezk 2001, Theorem 7.2] *The category $s\mathcal{S}et^{\Delta^{\mathrm{op}}}$ of simplicial spaces admits a model structure in which*

- *the fibrant objects are the injectively fibrant complete Segal spaces*, *and*
- *the cofibrations are the monomorphisms.*

We denote this model structure by $s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}$.

This model structure can be obtained by taking the left Bousfield localization of the injective model structure on $s\mathcal{S}et^{\Delta^{\mathrm{op}}}$ with respect to the following set of maps:

(1) the *Segal acyclic cofibrations*

$$\underbrace{\Delta[1] \amalg_{\Delta[0]} \Delta[1] \amalg_{\Delta[0]} \cdots \amalg_{\Delta[0]} \Delta[1]}_{n} \to \Delta[n]$$

for $n \geq 1$, and

(2) the *completeness cofibration*, given by either inclusion of the form

$$\Delta[0] \to N\mathbb{I}.$$

Complete Segal spaces, the fibrant objects in $s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}$, are then precisely the injectively fibrant simplicial spaces that are local with respect to the maps of type (1) and (2).

Remark 1.9 As briefly addressed in [Rezk 2010, Section 10], in presence of the maps of type (1), for the purpose of the localization one could replace the map of type (2) as completeness acyclic cofibration with

(2′) either inclusion of the form

$$\Delta[0] \to \Delta[0] \amalg_{\Delta[1]} \Delta[3] \amalg_{\Delta[1]} \Delta[0],$$

where the right-hand side is the colimit of the diagram

$$\Delta[0] \leftarrow \Delta[1] \xrightarrow{02} \Delta[3] \xleftarrow{13} \Delta[1] \to \Delta[0].$$

The following theorem establishes that the homotopy theories of Segal categories and complete Segal spaces are equivalent.

Theorem 1.10 [Bergner 2007a, Theorems 6.3 and 7.5] *The inclusion functor from the category of Segal precategories to the category of simplicial spaces induces a left Quillen equivalence*

$$I\colon P\mathcal{C}at_{(\infty,1)} \to s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}.$$

1.3 Complete Segal Θ_2–spaces as a model of $(\infty, 2)$–categories

We now recall the notion of complete Segal Θ_2–spaces, which give a model for $(\infty, 2)$–categories.

Let Θ_2 be Joyal's cell category. For a precise account on how Θ_2 is defined, we refer the reader to the original source [Joyal 1997], or to [Berger 2007, Definition 3.3] or [Rezk 2010, Section 1.1] for an inductive approach; we give a brief review here.

Recall that Θ_2 is a full subcategory of $2\mathcal{C}at$ and that a generic object of Θ_2 is a 2–category $[m|k_1, \dots, k_m]$ generated by gluing horizontally the suspensions of $[k_i]$ for $i = 1, \dots, m$. An example is the 2–category $[3|2, 0, 1]$, which is generated by the following data:

$$\begin{tikzcd}
x \arrow[r, bend left=60, "f"] \arrow[r, "g" near end] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow\beta"] & y \arrow[r, "l"] & z \arrow[r, bend left=60, "m"] \arrow[r, "k" near end] \arrow[r, phantom, shift left=3.5ex, "\Downarrow\gamma"] & w
\end{tikzcd}$$

Definition 1.11 A Θ_2–*set* is a presheaf $A\colon \Theta_2^{\mathrm{op}} \to \mathcal{S}et$, and we denote the category of Θ_2–sets and natural transformations by $\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$. Similarly, a Θ_2–*space* is a simplicial presheaf $A\colon \Theta_2^{\mathrm{op}} \to s\mathcal{S}et$, and we denote the category of Θ_2–spaces by $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$.

Remark 1.12 The reader familiar with [Rezk 2010] might observe that we are using the term "Θ_2–space" in a more general sense than he does. His Θ_2–spaces satisfy additional Segal and completeness conditions that we discuss below; we further specify such objects by calling them "complete Segal Θ_2–spaces".

Remark 1.13 The canonical inclusion $\mathcal{S}et \hookrightarrow s\mathcal{S}et$ of sets as discrete simplicial sets induces a canonical inclusion $\mathcal{S}et^{\Theta_2^{\mathrm{op}}} \hookrightarrow s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$, which is both a left and right adjoint. In particular, we often regard Θ_2–sets as discrete Θ_2–spaces without further specification.

Notation 1.14 For any object θ of Θ_2, we denote by $\Theta_2[\theta]$ the Θ_2–set represented by θ.

Remark 1.15 As a special case of [Ara 2014, Section 3.1], given any Θ_2–set A and any space B one can consider the Θ_2–space $A \boxtimes B$, which is defined levelwise as the simplicial set

$$(A \boxtimes B)_\theta := A_\theta \times B.$$

The construction extends to a bifunctor

$$\boxtimes\colon \mathcal{S}et^{\Theta_2^{\mathrm{op}}} \times s\mathcal{S}et \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$$

that preserves colimits in each variable.

In preparation for a localization on the category $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$, we introduce the following class of maps. The reader may notice the analogy with the maps treated in Section 1.2.

Definition 1.16 An *elementary acyclic cofibration* is a map of discrete Θ_2–spaces of the following kinds.

1. A *vertical Segal acyclic cofibration* is given by, for some $k \geq 1$, the canonical map
$$\Theta_2[1|1] \underset{\Theta_2[1|0]}{\amalg} \cdots \underset{\Theta_2[1|0]}{\amalg} \Theta_2[1|1] \hookrightarrow \Theta_2[1|k].$$
2. A *horizontal Segal acyclic cofibration* is given by, for some $m \geq 1$ and $k_i \geq 0$, where $0 \leq i \leq m$, the canonical map
$$\Theta_2[1|k_1] \underset{\Theta_2[0]}{\amalg} \cdots \underset{\Theta_2[0]}{\amalg} \Theta_2[1|k_m] \hookrightarrow \Theta_2[m|k_1, \dots, k_m].$$
3. The *horizontal completeness acyclic cofibration* is either of the inclusions of the form
$$\Theta_2[0] \to \Theta_2[0] \underset{\Theta_2[1|0]}{\amalg} \Theta_2[3|0,0,0] \underset{\Theta_2[1|0]}{\amalg} \Theta_2[0],$$
where the right-hand side is the colimit of the diagram
$$\Theta_2[0] \leftarrow \Theta_2[1|0] \xrightarrow{02} \Theta_2[3|0,0,0] \xleftarrow{13} \Theta_2[1|0] \to \Theta_2[0].$$
4. The *vertical completeness acyclic cofibration* is the canonical map
$$\Theta_2[1|0] \to \Theta_2[1|0] \underset{\Theta_2[1|1]}{\amalg} \Theta_2[1|3] \underset{\Theta_2[1|1]}{\amalg} \Theta_2[1|0],$$
induced by suspending the previous one.

We now describe two model structures on the category $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$, both established by [Rezk 2010, Section 2.13, Proposition 11.5]. Our description, in terms of the elementary acyclic cofibrations defined above, differs slightly from his, but is designed to facilitate some of our proofs in the next section. We explain in Remark 1.21 why the two approaches give the same model structures.

Theorem 1.17 *The category $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$ of Θ_2–spaces supports the following two cofibrantly generated model structures*:

- *the model structure*
$$s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$$
obtained by taking the left Bousfield localization of the injective model structure $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{\mathrm{inj}}$ with respect to the set of elementary acyclic cofibrations from Definition 1.16, *and*
- *the model structure*
$$s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$$
obtained by taking the left Bousfield localization of the projective model structure $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{\mathrm{proj}}$ with respect to the set of elementary acyclic cofibrations from Definition 1.16.

Although the model structure $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$ is more common in the literature, for technical reasons that we discuss in Remark 2.3, in this paper we focus more on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$. In this model structure

- the projectively fibrant objects, which we call *complete Segal Θ_2–spaces*, are precisely the projectively fibrant Θ_2–spaces that are local with respect to the elementary acyclic cofibrations from Definition 1.16, and
- the cofibrations are precisely the projective cofibrations.

Remark 1.18 Combining [Hirschhorn 2003, Theorem 11.6.1 and Definitions 11.5.33 and 11.5.25], we can obtain an explicit description of the generating cofibrations and generating acyclic cofibrations of $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{\mathrm{proj}}$. In particular,

(1) a set of generating cofibrations for the projective model structure on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$ is given by all maps of the form

$$\Theta_2[\theta]\boxtimes\partial\Delta[\ell]\to\Theta_2[\theta]\boxtimes\Delta[\ell]\quad\text{for }\ \theta\in\mathrm{Ob}(\Theta_2)\text{ and }\ell\geq 0;$$

(2) a set of generating acyclic cofibrations for the projective model structure on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$ is given by all maps of the form

$$\Theta_2[\theta]\boxtimes\Lambda^k[\ell]\to\Theta_2[\theta]\boxtimes\Delta[\ell]\quad\text{for }\ \theta\in\mathrm{Ob}(\Theta_2)\text{ and }0\leq k\leq\ell.$$

The following equivalence between the two model structures can alternatively also be seen as a direct application of [Hirschhorn 2003, Theorem 3.3.20].

Theorem 1.19 [Rezk 2010, Sections 2.5–2.13] *The identity functor defines a Quillen equivalence*

$$s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}\rightleftarrows s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}.$$

We want to consider the suspension of a simplicial space to a Θ_2–space. In [Rezk 2010, Section 4.4], the notation $V[1](X)$ is used for what we denote here by ΣX to emphasize the analogy with similar constructions we have discussed.

Definition 1.20 The *suspension* ΣX of a simplicial space X is the Θ_2–space obtained by applying the cocontinuous functor $\Sigma\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}}\to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{*,*}$ defined on representable simplicial spaces as

$$\Sigma(\Delta[k]\boxtimes\Delta[\ell]):=\Theta_2[1|k]\boxtimes\Delta[\ell].$$

This construction extends to a functor $\Sigma\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}}\to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{*,*}$ valued in bipointed Θ_2–spaces.

Remark 1.21 In the original construction from [Rezk 2010, Section 2.13, Proposition 11.5], two model structures on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$ are obtained by localizing the injective and projective model structure with respect to the set of maps of the following kinds:

(1′) a family of maps that can be recognized to be precisely the family of vertical Segal acyclic cofibrations, using [Rezk 2010, Proposition 11.7];

(2′) a family of maps that can be recognized to be precisely the family of horizontal Segal acyclic cofibrations, using [Rezk 2010, Proposition 11.7];

(3′) the unique map
$$N^{\Theta_2}\mathbb{I} \to \Theta_2[0];$$

(4′) the map
$$\Sigma N^{\Theta_2}\mathbb{I} \to \Theta_2[1|0]$$
obtained by suspending the map from (3′).

However, in presence of the maps of type (1) and (2), it is shown in [Rezk 2010, Section 10] and also in [Barwick and Schommer-Pries 2021, Section 13] that for the purpose of the localization the maps of type (3) and (4) are equivalent to the maps of type (3′) and (4′), respectively. It follows that, although presented differently, these two model structures in fact agree with the model structures $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$ and $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$ from Theorem 1.17.

1.4 Complicial sets as a model of $(\infty, 2)$–categories

The next model of $(\infty, 2)$–categories that we consider is based on the following structure, originally referred to as a *simplicial set with hollowness* in [Street 1987] and later as a *stratified simplicial set* in [Verity 2008a].

Definition 1.22 A *simplicial set with marking* is a simplicial set endowed with a subset of simplices of strictly positive dimensions that contains all degenerate simplices, called *thin* or *marked*. We denote by $ms\mathcal{S}et$ the category of simplicial sets with marking and marking-preserving simplicial maps.

We want to consider a model structure on the category of simplicial sets with marking, in which the fibrant objects, called 2*–complicial sets*, provide a model for $(\infty, 2)$–categories. The idea is that, in a 2–complicial set, the marked k–simplices are precisely the k–equivalences. We refer the reader to [Riehl 2018] for further elaboration on this viewpoint.

Remark 1.23 As discussed in [Verity 2008a, Observation 97], the underlying simplicial set functor $ms\mathcal{S}et \to s\mathcal{S}et$ fits into an adjoint triple

$$\begin{tikzcd} ms\mathcal{S}et \arrow[r] & s\mathcal{S}et. \arrow[l, bend right=40, "(-)^{\flat}"', ""{name=U, below}] \arrow[l, bend left=40, "(-)^{\sharp}", ""{name=D, above}] \end{tikzcd}$$

For any simplicial set X, the left adjoint $X^{\flat}$ (sometimes also denoted simply by X) is obtained by marking only the degenerate simplices of X, and the right adjoint $X^{\sharp}$ is obtained by marking all simplices in positive dimensions.

Remark 1.24 As described in detail in [Verity 2008a, Observation 109], the category $ms\mathcal{S}et$ of simplicial sets with marking is complete and cocomplete, with limits and colimits constructed as follows.

- The underlying simplicial set of a limit $\lim_{i\in I} X_i$ of simplicial sets with marking is the limit of the corresponding underlying simplicial sets of X_i, and a simplex is marked in a limit of simplicial sets with marking $\lim_{i\in I} X_i$ if and only if it is marked in each component X_i for $i \in I$.
- The underlying simplicial set of a colimit $\operatorname{colim}_{i\in I} X_i$ of simplicial sets with marking is the colimit of the corresponding underlying simplicial sets of X_i, and a simplex is marked in a colimit of simplicial sets with marking $\operatorname{colim}_{i\in I} X_i$ if and only if it admits a marked representative in X_i for some $i \in I$.

The following model structure is one instance of the family of model structures constructed by [Verity 2008b, Theorem 100], and is described in more detail in [Riehl 2018, Section 3.3].

Theorem 1.25 [Ozornova and Rovelli 2020, Theorem 1.25] *The category $ms\mathcal{S}et$ of simplicial sets with marking supports a cofibrantly generated cartesian closed model structure in which*

- *the fibrant objects are the 2–complicial sets, as recalled in* [Ozornova and Rovelli 2020, Definition 1.21], *and*
- *the cofibrations are precisely the monomorphisms on underlying simplicial sets.*

We denote this model structure by $ms\mathcal{S}et_{(\infty,2)}$.

We warn the reader that the fibrant objects in this model structure have been given different names in the literature, and could perhaps more accurately be called "2–trivial saturated weak complicial sets". We have chosen to call them "2–complicial sets" for the sake of brevity; in what follows we do not make explicit use of their definition. We recall the key results we need, in particular about the weak equivalences in this model structure, in the remainder of this section.

Remark 1.26 Because of the way the model structure $ms\mathcal{S}et_{(\infty,2)}$ is constructed, if $\Delta[3]_{\mathrm{eq}}$ denotes the 3–simplex $\Delta[3]$ in which the nondegenerate marked 1–simplices are precisely the one between the vertices 0 and 2 and the one between the vertices 1 and 3, and all simplices in dimension 2 or higher are marked, the canonical map $\Delta[3]_{\mathrm{eq}} \to \Delta[3]^\sharp$ is a weak equivalence. Indeed, the model structure $ms\mathcal{S}et_{(\infty,2)}$ is a Cisinski–Olschok model structure (in the sense of [Olschok 2011]) for which the map $\Delta[3]_{\mathrm{eq}} \to \Delta[3]^\sharp$ is an anodyne extension.

Lemma 1.27 *The functor*

$$(-)^\sharp\colon s\mathcal{S}et_{(\infty,0)} \to ms\mathcal{S}et_{(\infty,2)}$$

is a left Quillen functor, where $s\mathcal{S}et_{(\infty,0)}$ denotes the Kan–Quillen model structure on the category $s\mathcal{S}et$.

Proof The fact that the functor admits a right adjoint, often called the *core functor*, is discussed in [Riehl and Verity 2022, Definition D.1.2]. It is straightforward from its description that the functor $(-)^\sharp$

preserves cofibrations, and it is shown in [Ozornova and Rovelli 2020, Lemma 2.16] that it also sends acyclic cofibrations of $s\mathcal{S}et_{(\infty,0)}$ to weak equivalences of $ms\mathcal{S}et_{(\infty,2)}$. It follows that $(-)^\sharp$ defines indeed a left Quillen functor between the desired model categories. □

For $n = 2$, the Street nerve was studied in detail by [Duskin 2001], and can be described explicitly as follows.

Definition 1.28 The *nerve* $N\mathfrak{D}$ of a 2–category $\mathfrak{D}$ is the 3–coskeletal simplicial set in which

(0) a 0–simplex consists of an object of $\mathfrak{D}$:

$$x;$$

(1) a 1–simplex consists of a 1–morphism of $\mathfrak{D}$:

$$x \xrightarrow[a]{} y;$$

(2) a 2–simplex consists of a 2–cell of $\mathfrak{D}$ of the form $c \Rightarrow b \circ a$:

$$\begin{tikzcd}
& y \arrow[dr, "b"] & \\
x \arrow[ur, "a"] \arrow[rr, "c"'] & {} \arrow[u, Rightarrow, "\varphi"] & z
\end{tikzcd}$$

(3) a 3–simplex consists of four 2–cells of $\mathfrak{D}$ that satisfy the relation

$$\begin{tikzcd}
w & & z \arrow[ll, "e"'] & & w & & z \arrow[ll, "e"'] \\
{} \arrow[r, Rightarrow, shift left] & {} \arrow[r, Rightarrow] & {} & = & {} \arrow[r, Rightarrow] & {} \arrow[r, Rightarrow] & {} \\
x \arrow[uu, "a"] \arrow[rr, "d"'] \arrow[uurr, "b" description] & & y \arrow[uu, "c"'] & & x \arrow[uu, "a"] \arrow[rr, "d"'] & & y \arrow[uu, "c"'] \arrow[uull, "f" description]
\end{tikzcd}$$

and in which the simplicial structure is as indicated in the pictures.

Definition 1.29 [Verity 2008a, Chapter 10] The *Roberts–Street nerve* of a 2–category $\mathfrak{D}$ is the simplicial set with marking $N^{\mathrm{RS}}\mathfrak{D}$, defined by the following properties.

(0) The underlying simplicial set is the Duskin nerve $N\mathfrak{D}$.

(1) Only degenerate 1–simplices are marked.

(2) A 2–simplex of $N\mathfrak{D}$ is marked in $N^{\mathrm{RS}}\mathfrak{D}$ if and only if corresponding 2–morphism $\varphi\colon c \Rightarrow b \circ a$ is an identity.

(3) Any m–simplex of $N\mathfrak{D}$ for $m \geq 3$ is marked in $N^{\mathrm{RS}}\mathfrak{D}$.

This construction extends to a functor $N^{\mathrm{RS}}\colon 2\mathcal{C}at \to ms\mathcal{S}et$.

The Roberts–Street nerve is a right adjoint functor, but, as proved by the second- and third-named authors, does not preserve fibrant objects on the model structures we want to consider. However, it is a *homotopical* functor between model categories, in the sense that it preserves weak equivalences.

Proposition 1.30 [Ozornova and Rovelli 2022, Proposition 1.18] *The Roberts–Street nerve defines a homotopical functor of model categories*

$$N^{\mathrm{RS}}\colon 2\mathscr{C}at \to ms\mathscr{S}et_{(\infty,2)}.$$

The following two technical results essentially tell us that horizontal and vertical composition of 2–cells can be encoded via Segal-type maps that are acyclic cofibrations in the model structure for 2–complicial sets.

Theorem 1.31 [Ozornova and Rovelli 2022, Corollary 2.10] *For any $m \geq 0$ and $k_i \geq 0$ for $i = 1, \dots, m$ there is a canonical map of simplicial sets with marking*

$$N^{\mathrm{RS}}[1|k_1] \underset{N^{\mathrm{RS}}[0]}{\amalg} \cdots \underset{N^{\mathrm{RS}}[0]}{\amalg} N^{\mathrm{RS}}[1|k_m] \hookrightarrow N^{\mathrm{RS}}[m|k_1, \dots, k_m]$$

that is an acyclic cofibration, and in particular a weak equivalence, in $ms\mathscr{S}et_{(\infty,2)}$.

Theorem 1.32 [Ozornova and Rovelli 2022, Corollary 2.11] *For any $k \geq 0$ there is a canonical map of simplicial sets with marking*

$$N^{\mathrm{RS}}[1|1] \underset{N^{\mathrm{RS}}[1]}{\amalg} \cdots \underset{N^{\mathrm{RS}}[1]}{\amalg} N^{\mathrm{RS}}[1|1] \hookrightarrow N^{\mathrm{RS}}[1|k]$$

that is an acyclic cofibration, and in particular a weak equivalence, in $ms\mathscr{S}et_{(\infty,2)}$.

Note that, when taking the nerve we simply write $N^{\mathrm{RS}}[1]$ rather than $N^{\mathrm{RS}}[1|0]$, since the 2–category $[1|0]$ is just the category $[1]$ thought of as a 2–category.

An important construction in this paper is the suspension of a simplicial set with marking. We conclude this section with the definition and some key results about it.

Definition 1.33 [Ozornova and Rovelli 2022, Definition 2.6] The *suspension* ΣX of a simplicial set with marking X is the simplicial set with marking defined as follows.

- It has precisely two 0–simplices that we denote by $x_\perp$ and $x_\top$.
- The set of m–simplices for $m > 0$ is given by all k–simplices of X for $0 \leq k \leq m-1$, as well as the m–fold degeneracies of the two 0–simplices $x_\perp$ and $x_\top$, namely
$$(\Sigma X)_m \cong \{s_0^m x_\perp\} \amalg X_{m-1} \amalg \cdots \amalg X_0 \amalg \{s_0^m x_\top\}.$$
- The simplicial structure can be read off from [Ozornova and Rovelli 2022, Definition 2.6].
- The set of nondegenerate m–simplices for $m > 0$ is given by the nondegenerate $(m-1)$–simplices of X.
- A nondegenerate m–simplex σ is marked in ΣX if and only if it is marked as an $(m-1)$–simplex of X.

This construction extends to a functor $\Sigma\colon ms\mathscr{S}et \to ms\mathscr{S}et_{*,*}$ valued in bipointed marked simplicial sets.

We now recall that this functor can be upgraded to a left Quillen functor of model categories. Recall from [Hirschhorn 2021] that, given any cofibrantly model category $\mathcal{M}$, there is a model structure on the category $\mathcal{M}_{*,*}$ of bipointed objects in $\mathcal{M}$, in which cofibrations, fibrations, and weak equivalences are created in $\mathcal{M}$.

Lemma 1.34 [Ozornova and Rovelli 2022, Lemma 2.7] *Regarding ΣX as a simplicial set with marking bipointed on $x_\perp$ and $x_\top$, the marked suspension defines a left Quillen functor*

$$\Sigma\colon ms\mathcal{S}et_{(\infty,2)} \to (ms\mathcal{S}et_{(\infty,2)})_{*,*}.$$

In particular, it is homotopical and it respects connected colimits as a functor $\Sigma\colon ms\mathcal{S}et \to ms\mathcal{S}et$.

Finally, we recall that the suspension of a marked simplicial set is homotopically compatible with the Roberts–Street nerve, as one would expect.

Theorem 1.35 [Ozornova and Rovelli 2022, Theorem 2.9] *For any 1–category $\mathcal{D}$ there is a canonical map*

$$\Sigma N^{\mathrm{RS}}\mathcal{D} \to N^{\mathrm{RS}}\Sigma\mathcal{D}$$

that is a weak equivalence in $ms\mathcal{S}et_{(\infty,2)}$.

2 The comparison of models of $(\infty, 2)$–categories

In this section, we set up our explicit comparison between the two models for $(\infty, 2)$–categories that we are considering. We first establish the desired Quillen pair of functors between the unlocalized model structure on the category of Θ_2–spaces and the model structure on simplicial sets with marking, then show that it is still a Quillen pair after localization of the former model category. We then show that it is a Quillen equivalence, deferring some steps in the proof to later sections.

2.1 The Quillen pair before localizing

Let us begin by defining the functor that we use to make our comparison.

Construction 2.1 The functor $\Theta_2 \times \Delta \subseteq s\mathcal{S}et^{\Theta_2^{\mathrm{op}}} \to ms\mathcal{S}et$ given by

$$(\theta, [\ell]) \mapsto (\Theta_2 \times \Delta)[\theta, \ell] = \Theta_2[\theta] \boxtimes \Delta[\ell] \mapsto N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp$$

induces an adjunction

$$L\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}} \rightleftarrows ms\mathcal{S}et : R.$$

Roughly speaking, for any Θ_2–space W, the simplicial set with marking LW is obtained by gluing together a copy of the Roberts–Street nerve of θ, for any θ in Θ_2 that maps to W. While describing this gluing explicitly is complicated, it is essentially specified by the definition of left Kan extension.

We now show that these adjoint functors define Quillen pair on unlocalized model categories.

Proposition 2.2 *The adjunction*

$$L\colon s\mathcal{S}et_{\mathrm{proj}}^{\Theta_2^{\mathrm{op}}} \rightleftarrows ms\mathcal{S}et_{(\infty,2)} : R$$

is a Quillen pair.

Proof We want to show that the functor L preserves cofibrations and acyclic cofibrations. From Remark 1.18 we know that

(1) a set of generating cofibrations for the projective model structure on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$ is given by all maps of the form

$$\Theta_2[\theta] \boxtimes \partial\Delta[\ell] \to \Theta_2[\theta] \boxtimes \Delta[\ell] \quad \text{for } \theta \in \Theta_2 \text{ and } \ell \geq 0;$$

(2) a set of generating acyclic cofibrations for the projective model structure on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$ is given by all maps of the form

$$\Theta_2[\theta] \boxtimes \Lambda^k[\ell] \to \Theta_2[\theta] \boxtimes \Delta[\ell] \quad \text{for } \theta \in \Theta_2 \text{ and } 0 \leq k \leq \ell.$$

Using the facts that $(-)^\sharp$ commutes with colimits, which is a consequence of Lemma 1.27, and that the box product $\boxtimes$ preserves colimits in each variable, which was recalled in Remark 1.15, we see that

(1) the image of the generating cofibration via L is the map

$$N^{\mathrm{RS}}\theta \times \partial\Delta[\ell]^\sharp \to N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp \quad \text{for } \theta \in \Theta_2 \text{ and } \ell \geq 0;$$

(2) the image of the generating acyclic cofibration via L is the map

$$N^{\mathrm{RS}}\theta \times \Lambda^k[\ell]^\sharp \to N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp \quad \text{for } \theta \in \Theta_2 \text{ and } 0 \leq k \leq \ell.$$

Since the model structure $ms\mathcal{S}et_{(\infty,2)}$ is cartesian closed by Theorem 1.25 and $(-)^\sharp$ is a left Quillen functor by Lemma 1.27, we conclude that

(1) the map $N^{\mathrm{RS}}\theta \times \partial\Delta[\ell]^\sharp \to N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp$ is a cofibration, and

(2) the map $N^{\mathrm{RS}}\theta \times \Lambda^k[\ell]^\sharp \to N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp$ is an acyclic cofibration.

It follows that L preserves cofibrations and acyclic cofibrations, so it is a left Quillen functor, as desired. □

Remark 2.3 One might wonder, in contrast with much of the literature on the subject, why we have chosen to use the projective, rather than the injective, model structure on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$. However, it is not clear whether the functor

$$L\colon s\mathcal{S}et_{\mathrm{inj}}^{\Theta_2^{\mathrm{op}}} \to ms\mathcal{S}et_{(\infty,2)}$$

is a left Quillen functor, since we do not know whether it preserves cofibrations. More precisely, it is unclear whether L sends the injective cofibration

$$\partial\Theta_2[3|1,0,1] \to \Theta_2[3|1,0,1]$$

to a cofibration of $ms\mathcal{S}et_{(\infty,2)}$.

2.2 The Quillen pair after localizing

We now show that we still have a Quillen pair after localizing the projective model structure on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$.

Theorem 2.4 *The adjunction*

$$L\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \rightleftarrows ms\mathcal{S}et_{(\infty,2)} : R$$

is a Quillen pair.

Since cofibrations are unchanged by localization, it suffices to prove that L preserves acyclic cofibrations. We do so by proving that L preserves all elementary acyclic cofibrations, in the following sequence of propositions.

Proposition 2.5 *The functor L sends the vertical Segal acyclic cofibrations*

$$\Theta_2[1|1] \amalg_{\Theta_2[1|0]} \cdots \amalg_{\Theta_2[1|0]} \Theta_2[1|1] \hookrightarrow \Theta_2[1|k] \text{ for } k \geq 0$$

from Definition 1.16 to weak equivalences in $ms\mathcal{S}et_{(\infty,2)}$.

Proof The functor L sends the elementary acyclic cofibration

$$\Theta_2[1|1] \amalg_{\Theta_2[1|0]} \cdots \amalg_{\Theta_2[1|0]} \Theta_2[1|1] \hookrightarrow \Theta_2[1|k]$$

to the canonical inclusion

$$N^{\mathrm{RS}}[1|1] \amalg_{N^{\mathrm{RS}}[1]} \cdots \amalg_{N^{\mathrm{RS}}[1]} N^{\mathrm{RS}}[1|1] \hookrightarrow N^{\mathrm{RS}}[1|k],$$

which is an acyclic cofibration by Theorem 1.32. □

Proposition 2.6 *The functor L sends the horizontal Segal acyclic cofibrations*

$$\Theta_2[1|k_1] \amalg_{\Theta_2[0]} \cdots \amalg_{\Theta_2[0]} \Theta_2[1|k_m] \hookrightarrow \Theta_2[m|k_1,\dots,k_m] \text{ for } m \geq 0 \text{ and } k_i \geq 0$$

from Definition 1.16 to weak equivalences in $ms\mathcal{S}et_{(\infty,2)}$.

Proof The functor L sends the elementary acyclic cofibration

$$\Theta_2[1|k_1] \amalg_{\Theta_2[0]} \cdots \amalg_{\Theta_2[0]} \Theta_2[1|k_m] \hookrightarrow \Theta_2[m|k_1,\dots,k_m]$$

to the canonical inclusion

$$N^{\mathrm{RS}}[1|k_1] \amalg_{N^{\mathrm{RS}}[0]} \cdots \amalg_{N^{\mathrm{RS}}[0]} N^{\mathrm{RS}}[1|k_m] \hookrightarrow N^{\mathrm{RS}}[m|k_1,\dots,k_m],$$

which is an acyclic cofibration by Theorem 1.31. □

Proposition 2.7 *The functor L sends the horizontal completeness acyclic cofibration*

$$\Theta_2[0] \to \Theta_2[0] \amalg_{\Theta_2[1|0]} \Theta_2[3|0,0,0] \amalg_{\Theta_2[1|0]} \Theta_2[0]$$

from Definition 1.16 to a weak equivalence in $ms\mathcal{S}et_{(\infty,2)}$.

To prove this proposition, we need the following preliminary lemma.

Lemma 2.8 *The unique map*

$$\Delta[0] \coprod_{\Delta[1]} N^{\mathrm{RS}}[3] \coprod_{\Delta[1]} \Delta[0] \to \Delta[0]$$

is a weak equivalence in $ms\mathcal{S}et_{(\infty,2)}$.

Proof We observe that this map fits into a commutative diagram of simplicial sets with marking

$$\begin{tikzcd}
\Delta[0] \coprod_{\Delta[1]} N^{\mathrm{RS}}[3] \coprod_{\Delta[1]} \Delta[0] \arrow[r] & \Delta[0] \\
\Delta[1]^{\sharp} \coprod_{\Delta[1]} N^{\mathrm{RS}}[3] \coprod_{\Delta[1]} \Delta[1]^{\sharp} \cong \Delta[3]_{\mathrm{eq}} \arrow[r] \arrow[u] & \Delta[3]^{\sharp} \arrow[u]
\end{tikzcd}$$

where $\Delta[1]^{\sharp}$ denotes the standard 1–simplex with the maximal marking. In this diagram, we observe that

- the bottom horizontal map is an acyclic cofibration by Remark 1.26;
- the left vertical map is a map between (homotopy) pushouts induced by the identity of $N^{\mathrm{RS}}[3]$ and two copies of the weak equivalence $\Delta[1]^{\sharp} \to \Delta[0]$; and
- the right vertical map is a weak equivalence since $(-)^{\sharp}$ preserves weak equivalences by Lemma 1.27.

It follows by two-out-of-three that the top horizontal map is a weak equivalence, as desired. □

We can now use this lemma to prove Proposition 2.7.

Proof of Proposition 2.7 The functor L sends the map

$$\Theta_2[0] \to \Theta_2[0] \coprod_{\Theta_2[1|0]} \Theta_2[3|0,0,0] \coprod_{\Theta_2[1|0]} \Theta_2[0]$$

to a map

$$\Delta[0] \to \Delta[0] \coprod_{\Delta[1]} N^{\mathrm{RS}}[3] \coprod_{\Delta[1]} \Delta[0]$$

that we want to show is a weak equivalence. However, we can conclude this fact by the two-out-of-three property, since we know from Lemma 2.8 that the unique map

$$\Delta[0] \coprod_{\Delta[1]} N^{\mathrm{RS}}[3] \coprod_{\Delta[1]} \Delta[0] \to \Delta[0]$$

is a weak equivalence in $ms\mathcal{S}et_{(\infty,2)}$. □

To complete the proof of Theorem 2.4, it remains to show that L preserves one more acyclic cofibration.

Proposition 2.9 *The functor L sends the vertical completeness acyclic cofibration*

$$\Theta_2[1|0] \to \Theta_2[1|0] \coprod_{\Theta_2[1|1]} \Theta_2[1|3] \coprod_{\Theta_2[1|1]} \Theta_2[1|0]$$

from Definition 1.16 to a weak equivalence of $ms\mathcal{S}et_{(\infty,2)}$.

Proof The functor L sends the map

$$\Theta_2[1|0] \to \Theta_2[1|0] \coprod_{\Theta_2[1|1]} \Theta_2[1|3] \coprod_{\Theta_2[1|1]} \Theta_2[1|0]$$

to a map

$$N^{\mathrm{RS}}[1|0] \to N^{\mathrm{RS}}[1|0] \coprod_{N^{\mathrm{RS}}[1|1]} N^{\mathrm{RS}}[1|3] \coprod_{N^{\mathrm{RS}}[1|1]} N^{\mathrm{RS}}[1|0],$$

which we want to show is a weak equivalence. By the two-out-of-three property, it suffices to show that the map

$$N^{\mathrm{RS}}[1|0] \coprod_{N^{\mathrm{RS}}[1|1]} N^{\mathrm{RS}}[1|3] \coprod_{N^{\mathrm{RS}}[1|1]} N^{\mathrm{RS}}[1|0] \to N^{\mathrm{RS}}[1|0],$$

induced by the unique map $[1|3] \to [1|0]$ in Θ_2 that is bijective on objects, is a weak equivalence in $ms\mathcal{S}et_{(\infty,2)}$. This map can be rewritten in terms of suspensions of 1–categories, as in Definition 1.2, as

$$N^{\mathrm{RS}}\Sigma[0] \coprod_{N^{\mathrm{RS}}\Sigma[1]} N^{\mathrm{RS}}\Sigma[3] \coprod_{N^{\mathrm{RS}}\Sigma[1]} N^{\mathrm{RS}}\Sigma[0] \to N^{\mathrm{RS}}\Sigma[0].$$

By Theorem 1.35, this map fits into a commutative diagram of simplicial sets with marking

$$\begin{tikzcd}
N^{\mathrm{RS}}\Sigma[0] \coprod_{N^{\mathrm{RS}}\Sigma[1]} N^{\mathrm{RS}}\Sigma[3] \coprod_{N^{\mathrm{RS}}\Sigma[1]} N^{\mathrm{RS}}\Sigma[0] \arrow[r] & N^{\mathrm{RS}}\Sigma[0] \\
\Sigma N^{\mathrm{RS}}[0] \coprod_{\Sigma N^{\mathrm{RS}}[1]} \Sigma N^{\mathrm{RS}}[3] \coprod_{\Sigma N^{\mathrm{RS}}[1]} \Sigma N^{\mathrm{RS}}[0] \arrow[r] \arrow[u, "\simeq"] & \Sigma N^{\mathrm{RS}}[0] \arrow[u, "\cong"']
\end{tikzcd}$$

in which the two vertical maps are weak equivalences. Note that for the left-hand map we are using the fact that these pushouts are actually homotopy pushouts. In particular, by the two-out-of-three property, to prove the theorem it is enough to prove that the bottom map is a weak equivalence. Using the fact that suspension commutes with pushouts by Lemma 1.34, this map can be rewritten as

$$\Sigma\left(\Delta[0] \coprod_{\Delta[1]} N^{\mathrm{RS}}[3] \coprod_{\Delta[1]} \Delta[0]\right) \to \Sigma\Delta[0],$$

namely the suspension of the map

$$\Delta[0] \coprod_{\Delta[1]} N^{\mathrm{RS}}[3] \coprod_{\Delta[1]} \Delta[0] \to \Delta[0],$$

which was shown in Lemma 2.8 to be a weak equivalence. Since suspension is a left Quillen functor by Lemma 1.34, we are done. ☐

2.3 The Quillen equivalence

It remains to show that this Quillen pair is in fact a Quillen equivalence. Our proof, however, is not done directly via the definition, but instead uses some machinery due to [Barwick and Schommer-Pries 2021] that we now briefly recall.

The first thing we need to consider is their criterion for when a model category is a "model of $(\infty, 2)$–categories". We begin with some notation.

Notation 2.10 Given a model category $\mathcal{M}$, we denote by $\mathcal{M}_\infty$ the underlying $(\infty,1)$–category of $\mathcal{M}$. While we do not need one here, for explicit (different but equivalent) constructions of $\mathcal{M}_\infty$ in the model of quasicategories, we refer the reader to [Hinich 2016] or [Lurie 2009a, Section A.2].

Notation 2.11 Given a Quillen pair $F\colon \mathcal{M} \rightleftarrows \mathcal{M}' : G$ between model categories $\mathcal{M}$ and $\mathcal{M}'$, we denote by

$$F_\infty\colon \mathcal{M}_\infty \rightleftarrows \mathcal{M}'_\infty : G_\infty$$

the adjunction that (F,G) induces at the level of underlying $(\infty,1)$–categories.

On objects, the value of F_∞ on any object of $\mathcal{M}$ can be computed up to equivalence in $\mathcal{M}'_\infty$ by applying F to any cofibrant replacement of the given object. Similarly the value of G_∞ on any object of $\mathcal{M}'$ can be computed up to equivalence in $\mathcal{M}_\infty$ by applying G to any fibrant replacement of the given object. Moreover, if (F,G) is a Quillen equivalence, the induced adjunction (F_∞, G_∞) is an equivalence of $(\infty,1)$–categories. For more details on how to obtain this adjunction of $(\infty,1)$–categories in the model of quasicategories we refer the reader to [Hinich 2016, Proposition 1.5.1].

Definition 2.12 (Barwick–Schommer-Pries) A model category $\mathcal{M}$ is a *model of* $(\infty,2)$*–categories* if the underlying $(\infty,1)$–category is equivalent to the colossal model $\mathcal{K}$ from [Barwick and Schommer-Pries 2021, Section 8], namely if there exists an equivalence of $(\infty,1)$–categories

$$\mathcal{M}_\infty \simeq \mathcal{K}.$$

The colossal model is constructed as an $(\infty,1)$–category in [Barwick and Schommer-Pries 2021, Section 8]. As we discuss in the appendix, with standard techniques one can also present the colossal model as the underlying $(\infty,1)$–category of a model category. More precisely, we show as Theorem A.3 that it is the underlying $(\infty,1)$–category $\bigl(s\mathcal{S}et^{\Upsilon_2^{\mathrm{op}}}_{(\infty,2)}\bigr)_\infty$ of a model category $s\mathcal{S}et^{\Upsilon_2^{\mathrm{op}}}_{(\infty,2)}$.

In any case, for the main purpose of this paper the arguments are packaged in a way that no explicit construction for the colossal model is needed.

Theorem 2.13 *The model categories $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$ and $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$ are models of $(\infty,2)$–categories.*

Proof The fact that $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$ is a model of $(\infty,2)$–categories is shown in [Barwick and Schommer-Pries 2021, Section 13] and there is an equivalence

$$\bigl(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}\bigr)_\infty \simeq \bigl(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}\bigr)_\infty$$

induced by the Quillen equivalence from Theorem 1.19. □

Theorem 2.14 *The model category $ms\mathcal{S}et_{(\infty,2)}$ is a model of $(\infty,2)$–categories.*

Proof An equivalence of $(\infty,1)$–categories

$$(ms\mathcal{S}et_{(\infty,2)})_\infty \simeq \bigl(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}\bigr)_\infty$$

can be obtained combining several equivalences of $(\infty, 1)$–categories induced by Quillen equivalences due to [Bergner and Rezk 2013; 2020; Gagna et al. 2022; Joyal and Tierney 2007; Lurie 2009b], as we recalled in Figure 1. □

Next, we recall the definition of a j–cell in a model of $(\infty, 2)$–categories.

Definition 2.15 Let $\mathcal{M}$ be a model category that is a model for $(\infty, 2)$–categories. An object of $\mathcal{M}$ is a representative of the *j–cell* for $j = 0, 1, 2$ if it corresponds to the j–cell of the colossal model through any equivalence of $(\infty, 1)$–categories $\mathcal{M}_\infty \simeq \big(s\mathcal{S}et^{\Upsilon_2^{\mathrm{op}}}_{(\infty,2)}\big)_\infty$.

For completeness, the definitions of the 0–, 1– and 2–cells in the colossal model are recalled in the appendix, but will not be needed explicitly.

Remark 2.16 The object in $\mathcal{M}$ that represents the j–cell is unique up to equivalence in $\mathcal{M}_\infty$ and also up to isomorphism in $\operatorname{Ho}\mathcal{M}$, the homotopy category of $\mathcal{M}$. The definition makes sense in particular because any auto-equivalence of $\big(s\mathcal{S}et^{\Upsilon_2^{\mathrm{op}}}_{(\infty,2)}\big)_\infty$ preserves j–cells for $j = 0, 1, 2$, as shown in [Barwick and Schommer-Pries 2021, Theorem 7.3].

The following statements describe j–cells in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$ and $ms\mathcal{S}et_{(\infty,2)}$.

Proposition 2.17 *In $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$ the object $\Theta_2[C_j]$ is a representative of the j–cell for $j = 0, 1, 2$.*

Proposition 2.18 *In $ms\mathcal{S}et_{(\infty,2)}$ the object $N^{\mathrm{RS}}C_j$ is a representative of the j–cell for $j = 0, 1, 2$.*

Although the two statements are not surprising, the argument to identify cells in $ms\mathcal{S}et_{(\infty,2)}$ requires significant work and makes use of many external results. We therefore postpone both proofs to Section 3.

Finally, the following theorem is the key ingredient to prove that the functor L is a Quillen equivalence.

Theorem 2.19 [Barwick and Schommer-Pries 2021, Proposition 15.10] *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories that are models for $(\infty, 2)$–categories, and $L\colon \mathcal{M} \rightleftarrows \mathcal{N} : R$ a Quillen pair between them. Then the Quillen pair (L, R) is a Quillen equivalence if and only if the derived functor of L sends j–cells to j–cells for $j = 0, 1, 2$.*

Once the proofs of Propositions 2.17 and 2.18 are provided in Section 3, we can then apply Theorem 2.19 to the Quillen pair from Theorem 2.4 to conclude the desired Quillen equivalence.

Theorem 2.20 *The adjunction*

$$L\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \rightleftarrows ms\mathcal{S}et_{(\infty,2)} : R$$

is a Quillen equivalence, and in particular induces an equivalence of $(\infty, 1)$–categories

$$L_\infty\colon \big(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}\big)_\infty \rightleftarrows (ms\mathcal{S}et_{(\infty,2)})_\infty : R_\infty.$$

3 Recognizing cells in models of $(\infty, 2)$–categories

The goal of this section is to identify the j–cells in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$, and most importantly the j–cells in $ms\mathcal{S}et_{(\infty,2)}$, as defined in Definition 2.15. The structure of the argument involves the identification of the j–cells in several established model categories that are models of $(\infty, 2)$–categories.

In Figure 2, we display the equivalences used to identify the cells in the marked simplicial sets, and the propositions displayed show how the cells behave under the corresponding equivalence.

While it is impractical to make this section completely self-contained, we have included precise references for all relevant constructions and definitions.

Lemma 3.1 *Suppose that a functor $F\colon \mathcal{M} \to \mathcal{M}'$ is a left (resp. right) Quillen equivalence between models of $(\infty, 2)$–categories, and an object X is cofibrant (resp. fibrant) in $\mathcal{M}$. Then X is a j–cell in $\mathcal{M}$ for some $0 \le j \le 2$ if and only if $F(X)$ is a j–cell in $\mathcal{M}'$.*

Proof Consider the induced functor $F_\infty\colon \mathcal{M}_\infty \to \mathcal{M}'_\infty$, which is an equivalence of $(\infty, 1)$–categories. It follows that, for any $j = 0, 1, 2$ and j–cell X_j of $\mathcal{M}$, the object $F_\infty(X_j)$ is a cell in $\mathcal{M}'_\infty$, either by direct verification, or by appealing to Theorem 2.19. Now, an object X is a j–cell in $\mathcal{M}$ if and only if there is an isomorphism $X \cong X_j$ in $\operatorname{Ho}\mathcal{M}$. Again using the fact that F_∞ is an equivalence, this statement is equivalent to saying that there is an isomorphism $F_\infty(X) \cong F_\infty(X_j)$ in $\operatorname{Ho}(\mathcal{M}')$. But the existence of such an isomorphism is equivalent to having $F_\infty(X)$ being a j–cell of $\mathcal{M}'_\infty$ because $F_\infty(X_j)$ is one. Since $F(X)$ computes $F_\infty(X)$, the result follows. □

3.1 Recognizing cells in Θ_2–models of $(\infty, 2)$–categories

We now begin the work of identifying j–cells in different models for $(\infty, 2)$–categories. We begin with the j–cells in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$, which have been identified by Barwick and Schommer-Pries.

$$\begin{tikzcd}
s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \arrow[d, "\wr"', "\text{Proposition 2.17}"] & & ms\mathcal{S}et_{(\infty,2)} \arrow[d, "\text{Proposition 2.18}"', "\wr"] \\
s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)} \arrow[d, "\wr"', "\text{Proposition 3.5}"] & & s\mathcal{S}et^{sc}_{(\infty,2)} \arrow[d, "\text{Proposition 3.22}"', "\wr"] \\
s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)} \arrow[d, "\wr"', "\text{Proposition 3.8}"] & & \mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}} \arrow[d, "\text{Proposition 3.18}"', "\wr"] \\
P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)} \arrow[r, "\simeq", "\text{Proposition 3.12}"'] & \mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}} \arrow[r, "\simeq", "\text{Proposition 3.16}"'] & \mathcal{C}at_{\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}
\end{tikzcd}$$

Figure 2

Proposition 3.2 [Barwick and Schommer-Pries 2021, Section 13] *In $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$ the object $\Theta_2[C_j]$ is a representative of the j–cell for $j = 0, 1, 2$.*

We can now prove Proposition 2.17, which identifies the cells in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$.

Proof of Proposition 2.17 Consider the identity functor on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$, which by Theorem 1.19 defines a left Quillen equivalence

$$\mathrm{id}\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}.$$

By Proposition 3.2 we know that $\Theta_2[C_j]$ is a j–cell in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$ for $j = 0, 1, 2$. Moreover, the object $\Theta_2[C_j]$ is projectively cofibrant by [Hirschhorn 2003, Proposition 11.6.2], since it is representable. It then follows from Lemma 3.1 that $\Theta_2[C_j]$ is a j–cell in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$. □

3.2 Recognizing cells in multisimplicial models of $(\infty, 2)$–categories

We now turn to identifying j–cells in multisimplicial models of $(\infty, 2)$–categories. Because we have not yet considered these models in this paper, we describe them briefly as we go.

Theorem 3.3 [Barwick 2005, Chapter 2] *The category $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}$ of bisimplicial spaces admits a model structure in which*

- *the fibrant objects are the injectively fibrant complete Segal objects in complete Segal spaces; and*
- *the cofibrations are the monomorphisms, and in particular every object is cofibrant.*

We denote this model structure by $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)}$.

The idea behind complete Segal objects in complete Segal spaces is that we apply similar Segal and completeness conditions to functors $\Delta^{\mathrm{op}} \to s\mathcal{S}et^{\Delta^{\mathrm{op}}}$, where the target category is equipped with the complete Segal space model structure. Thus, the Segal and completeness maps are now equivalences in this model structure, rather than equivalences of simplicial sets. For more details on the definition of complete Segal objects in complete Segal spaces, see [Barwick 2005, Chapter 2; Bergner and Rezk 2020, Definition 5.3; Lurie 2009b].

There is an explicit equivalence of model categories between this model structure and the one for complete Segal Θ_2–spaces. See also [Barwick and Schommer-Pries 2021] for a different proof of this equivalence.

Theorem 3.4 [Bergner and Rezk 2020, Corollary 7.1] *The functor $d\colon \Delta\times\Delta \to \Theta_2$ given by*

$$[m, k] \mapsto [m|k, \dots, k]$$

induces a left Quillen equivalence

$$d^*\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)} \to s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)}.$$

In particular, $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)}$ is a model for $(\infty, 2)$–categories. We now characterize the j–cells in this model.

Proposition 3.5 *In* $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)}$ *the object* $d^*\Theta_2[C_j]$ *is a representative of the* j*–cell for* $j=0,1,2$.

Proof We consider the functor d^*, which defines a left Quillen equivalence

$$d^*\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}\to s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)}.$$

By Proposition 3.2, for $j=0,1,2$, the object $\Theta_2[C_j]$ is a j–cell in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$. Moreover, every object is cofibrant in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$. It follows from Lemma 3.1 that $d^*\Theta_2[C_j]$ is a j–cell in $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)}$. □

We can now generalize the notion of Segal precategory to this context; in analogy with the notion of complete Segal objects described above, we can define *Segal precategory objects* in complete Segal spaces, given by functors $X\colon\Delta^{\mathrm{op}}\to s\mathcal{S}et^{\Delta^{\mathrm{op}}}$ such that X_0 is a discrete object and the Segal maps are weak equivalences in the complete Segal space model structure. See [Bergner and Rezk 2013, Section 6] for more details.

Let us briefly describe the comparison with complete Segal objects, which is analogous to Theorem 1.10. We denote by $P\mathcal{C}at(\mathcal{S}et^{\Delta^{\mathrm{op}}})$ the full subcategory of $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}$ given by Segal precategory objects in simplicial spaces, namely those bisimplicial spaces $X\colon\Delta^{\mathrm{op}}\to s\mathcal{S}et^{\Delta^{\mathrm{op}}}$ for which X_0 is discrete, and we denote by $I\colon P\mathcal{C}at(\mathcal{S}et^{\Delta^{\mathrm{op}}})\to s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}$ the inclusion functor.

Theorem 3.6 [Bergner and Rezk 2013, 6.12] *The category* $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$ *of precategories in simplicial spaces admits a model structure in which*

- *the fibrant objects are the projectively fibrant Segal category objects, and*
- *the cofibrations are the projective cofibrations.*

We denote this model structure by $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$.

This model was compared to the previous ones by [Bergner and Rezk 2020].

Theorem 3.7 [Bergner and Rezk 2020, Theorem 9.6 and Propositions 7.1 and 9.5] *The natural inclusion functor from* [Bergner and Rezk 2020, Section 9] *induces a left Quillen equivalence*

$$I\colon P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}\to s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)}.$$

In particular, $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$ is a model for $(\infty,2)$–categories.

For each $j=0,1,2$, the bisimplicial space $d^*\Theta_2[C_j]$, a priori an object of $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}$, is actually a precategory, so it can be regarded as an object of $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$.

Proposition 3.8 *In* $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$ *the object* $d^*\Theta_2[C_j]$ *is a representative of the* j*–cell for* $j=0,1,2$.

To prove this result, we make use of the following lemma.

Lemma 3.9 *In $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$ the object $d^*\Theta_2[C_j]$ is a cofibrant for $j=0,1,2$.*

Proof If $\Delta[\varnothing]$ denotes the initial bisimplicial space, we show that the canonical map $\Delta[\varnothing]\to d^*\Theta_2[C_j]$ is a cofibration in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$ for $j=0,1,2$.

For $j=0$ and $j=1$, the object $d^*\Theta_2[C_j]$ is representable as an object of $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}$, so by [Hirschhorn 2003, Proposition 11.6.2] the map $\Delta[\varnothing]\to d^*\Theta_2[C_j]$ is a projective cofibration and hence $d^*\Theta_2[C_j]$ is cofibrant in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$.

For $j=2$, we recall from [Bergner and Rezk 2013, Section 6.2] that any map of the form $A_{[p]}\to B_{[p]}$, where $p\geq 0$ and $A\to B$ is a cofibration of $s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}$, is a cofibration in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$. Recall that $A_{[p]}$ is defined as the pushout in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$

$$\begin{tikzcd}
A\boxtimes(\Delta[p])_0 \arrow[r, hook] \arrow[d] & A\boxtimes\Delta[p] \arrow[d] \\
\Delta[0]\boxtimes(\Delta[p])_0 \arrow[r, hook] & A_{[p]}
\end{tikzcd}$$

for any $p\geq 0$ and any simplicial space A. We can now write the map $\Delta[\varnothing]\to d^*\Theta_2[C_2]$ as the following composite of three cofibrations:

$$\Delta[\varnothing]\to d^*\Theta_2[C_0]\cong\Delta[0]_{[0]}\to(\Delta[0]\amalg\Delta[0])_{[0]}\cong\Delta[\varnothing]_{[1]}\to\Delta[1]_{[1]}\cong d^*\Theta_2[C_2],$$

concluding the proof. □

We can now prove the proposition.

Proof of Proposition 3.8 We consider the inclusion functor, which defines a left Quillen equivalence

$$I\colon P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}\to s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)}.$$

For $j=0,1,2$, by Proposition 3.5 we know that $I(d^*\Theta_2[C_j])$ is a j–cell in $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{i,(\infty,2)}$. Moreover, the object is cofibrant in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$ by Lemma 3.9. It follows from Lemma 3.1 that $d^*\Theta_2[C_i]$ is a j–cell in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$. □

3.3 Recognizing cells in enriched models of $(\infty,2)$–categories

We now turn to recognizing cells in models that are given by enriched categories. Many model structures on enriched categories can be obtained by the following general result of Lurie.

Theorem 3.10 [Lurie 2009a, Theorem A.3.2.24] *Let $\mathcal{V}$ be an excellent monoidal model category, in the sense of* [Lurie 2009a, Definition A.3.2.16]. *The category of small categories enriched over $\mathcal{V}$ admits a model structure in which*

- *the fibrant objects are the* ***locally fibrant categories****, ie the enriched categories C such that for any pair of objects c,c' in C, the mapping object $\mathrm{Hom}_C(c,c')$ is fibrant in $\mathcal{V}$;*

- *the weak equivalences, which are described in* [Lurie 2009a, Definition A.3.2.1] *and* [Lawson 2017], *are enriched functors* $F: C \to D$ *such that*
  1. *for every pair of objects* c, c' *of* C, *the map induced by* F *of mapping objects*
  $$F_{c,d}: \mathrm{Hom}_C(c, c') \to \mathrm{Hom}_D(Fc, Fc'),$$
  is a weak equivalence in $\mathscr{V}$, *and*
  2. *the functor induced by* F *on (underlying categories) of* $\mathrm{Ho}\,\mathscr{V}$*–categories is essentially surjective*;
- *the cofibrations are those described in* [Lurie 2009a, Proposition A.3.2.4].

We denote this model structure by $\mathscr{C}at_{\mathscr{V}}$.

To give an idea, the technical condition for a combinatorial monoidal model category to be *excellent* requests a closure property for cofibrations and weak equivalences, in addition to compatibility of the model structure with the monoidal structure. Lurie's original definition also requires a further condition, known as "invertibility hypothesis", which was shown to follow from the other conditions by [Lawson 2017, Theorem 0.1].

We specialize this construction to the following situations.

- Let $\mathscr{V} = \mathscr{C}at$ be the canonical model structure on the category $\mathscr{C}at$ of small categories from [Rezk 1996], which is seen to be excellent using the fact that the nerve functor creates weak equivalences and commutes with filtered colimits. We then obtain precisely the model category $\mathscr{C}at_{\mathscr{C}at} = 2\mathscr{C}at$ as discussed in [Berger and Moerdijk 2013, Example 1.8].
- Let $\mathscr{V} = s\mathscr{S}et_{(\infty,1)}$ be the Joyal model structure on the category $s\mathscr{S}et$ of simplicial sets from [Joyal 2008, Theorem 6.12], which is excellent by [Lurie 2009a, Example A.3.2.23]. We then obtain the model category $\mathscr{C}at_{s\mathscr{S}et_{(\infty,1)}}$.
- Let $\mathscr{V} = s\mathscr{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}$ be Rezk's model structure from Theorem 1.8 on the category $s\mathscr{S}et^{\Delta^{\mathrm{op}}}$ of simplicial spaces, which is discussed to be excellent [Bergner and Rezk 2013, Theorem 3.11]. We then obtain the model category $\mathscr{C}at_{s\mathscr{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$.
- Let $\mathscr{V} = s\mathscr{S}et^{+}_{(\infty,1)}$ be Lurie's model structure on the category $s\mathscr{S}et^{+}$ of marked simplicial sets from [Lurie 2009a, Proposition 3.1.3.7], which is excellent by [Lurie 2009a, Example A.3.2.22]. We then obtain the model category $\mathscr{C}at_{s\mathscr{S}et^{+}_{(\infty,1)}}$.

We now turn to an explicit Quillen equivalence between one of these enriched models and one of the models we have already discussed.

Theorem 3.11 [Bergner and Rezk 2013, 7.1–7.6] *The enriched nerve functor from* [Bergner and Rezk 2013, Definition 7.3], *obtained by regarding a bisimplicial category as a simplicial object in simplicial spaces, defines a right Quillen equivalence*

$$R: \mathscr{C}at_{s\mathscr{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}} \to P\mathscr{C}at(s\mathscr{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}.$$

In particular, $\mathscr{C}at_{s\mathscr{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$ is a model for $(\infty, 2)$–categories.

Now, we would like to identify the j–cells in the model structure $\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$, for which we make use of the discrete nerve functor $N^{\mathrm{disc}}\colon \mathcal{C}at \to s\mathcal{S}et^{\Delta^{\mathrm{op}}}$ considered in [Rezk 2001]. Since it preserves products, being a right adjoint functor, it induces a functor $N^{\mathrm{disc}}_*\colon \mathcal{C}at_{\mathcal{C}at} \to \mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}}$, given by applying N^{disc} to each mapping category.

Proposition 3.12 *In $\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$ the object $N^{\mathrm{disc}}_* C_j$ is a representative of the j–cell for $j = 0, 1, 2$.*

Before proving this proposition, we establish two lemmas that tell us more about the structure of these discrete nerves.

Lemma 3.13 *For any $j = 0, 1, 2$, the $s\mathcal{S}et^{\Delta^{\mathrm{op}}}$–enriched category $N^{\mathrm{disc}}_*\Theta_2[C_j]$ is fibrant in $\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$.*

Proof For $j = 0, 1, 2$, all hom-categories of C_j are of the form $\varnothing = [-1]$, $[0]$, or $[1]$. So all hom-bisimplicial sets of $N^{\mathrm{disc}}_*\Theta_2[C_j]$ are of the form $N^{\mathrm{disc}}[-1]$, $N^{\mathrm{disc}}[0]$ or $N^{\mathrm{disc}}[1]$, which are all complete Segal spaces, namely fibrant in $s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}$, since the categories $[-1]$, $[0]$, and $[1]$ do not have any nontrivial isomorphisms. □

Lemma 3.14 *For any θ in Θ_2, there is an isomorphism of precategories*

$$R(N^{\mathrm{disc}}_*\theta) \cong d^*\Theta_2[\theta].$$

Proof For $i, j, k \geq 0$, we first compute the set $(RN^{\mathrm{disc}}_*\theta)_{[i],[j],[k]}$. If $\mathcal{D}$ is a bisimplicial category with object set $\mathcal{D}_0$, and $\mathcal{D}_1$ denotes the bisimplicial space

$$\mathcal{D}_1 = \coprod_{a,b\in\mathcal{D}_0} \mathrm{Hom}_{\mathcal{D}}(a,b),$$

by definition of R (as given in [Bergner and Rezk 2013, Definition 7.3]) for any $i \geq 0$ there is an isomorphism of bisimplicial sets

$$(R\mathcal{D})_{[i]} \cong \underbrace{\mathcal{D}_1 \underset{\mathcal{D}_0}{\times} \mathcal{D}_1 \underset{\mathcal{D}_0}{\times} \cdots \underset{\mathcal{D}_0}{\times} \mathcal{D}_1}_{i}$$

that is natural in i. When specializing to the case $\mathcal{D} = N^{\mathrm{disc}}_*\theta$, we obtain a natural isomorphism

$$(RN^{\mathrm{disc}}_*\theta)_{[i]} \cong \underbrace{(N^{\mathrm{disc}}_*\theta)_1 \underset{(N^{\mathrm{disc}}_*\theta)_0}{\times} (N^{\mathrm{disc}}_*\theta)_1 \underset{(N^{\mathrm{disc}}_*\theta)_0}{\times} \cdots \underset{(N^{\mathrm{disc}}_*\theta)_0}{\times} (N^{\mathrm{disc}}_*\theta)_1}_{i}.$$

In particular, if θ_0 denotes the set of objects of θ and θ_1 denotes the category

$$\theta_1 := \coprod_{a,b\in\theta_0} \mathrm{Hom}_{\theta}(a,b),$$

for any $j, k \geq 0$ we have a bijection

$$(RN^{\mathrm{disc}}_*\theta)_{[i],[j],[k]} \cong \underbrace{N_j\theta_1 \underset{\theta_0}{\times} N_j\theta_1 \underset{\theta_0}{\times} \cdots \underset{\theta_0}{\times} N_j\theta_1}_{i},$$

that is natural in i, j, k.

Next, for $i, j, k \geq 0$, we compute the set $(d^*\Theta_2[\theta])_{[i],[j],[k]}$. By definition of d^*, and using the fact that Θ_2 is a full subcategory of $2\mathscr{C}at$, we have bijections

$$\begin{aligned}(d^*\Theta_2[\theta])_{[i],[j],[k]} &\cong \mathrm{Hom}_{\Theta_2}([i \,|\, \underbrace{j, j, \ldots, j}_{i}], \theta)\\ &\cong \mathrm{Hom}_{2\mathscr{C}at}([i \,|\, \underbrace{j, j, \ldots, j}_{i}], \theta)\\ &\cong \mathrm{Hom}_{2\mathscr{C}at}(\underbrace{[1|j]\amalg_{[0]}[1|j]\amalg_{[0]}\cdots\amalg_{[0]}[1|j]}_{i}, \theta)\\ &\cong \underbrace{\mathrm{Hom}_{2\mathscr{C}at}([1|j], \theta) \underset{\mathrm{Hom}_{2\mathscr{C}at}([0],\theta)}{\times} \cdots \underset{\mathrm{Hom}_{2\mathscr{C}at}([0],\theta)}{\times} \mathrm{Hom}_{2\mathscr{C}at}([1|j], \theta)}_{i}\\ &\cong \underbrace{\mathrm{Hom}_{2\mathscr{C}at}([1|j], \theta) \underset{\theta_0}{\times} \cdots \underset{\theta_0}{\times} \mathrm{Hom}_{2\mathscr{C}at}([1|j], \theta)}_{i}\end{aligned}$$

that are natural in i, j, k.

Finally, we show that there is a bijection

$$\mathrm{Hom}_{2\mathscr{C}at}([1|j], \theta) \cong N_j\theta_1$$

that is natural in j, from which the lemma follows. To do so, we observe that there are natural bijections

$$\begin{aligned}\mathrm{Hom}_{2\mathscr{C}at}([1|j], \theta) &\cong \coprod_{a,b\in\theta_0} \mathrm{Hom}_{2\mathscr{C}at_{*,*}}([1|j], (\theta, a, b))\\ &\cong \coprod_{a,b\in\theta_0} \mathrm{Hom}_{\mathscr{C}at}([j], \mathrm{Hom}_\theta(a, b))\\ &\cong \mathrm{Hom}_{\mathscr{C}at}\Big([j], \coprod_{a,b\in\theta_0} \mathrm{Hom}_\theta(a, b)\Big)\\ &\cong \mathrm{Hom}_{\mathscr{C}at}([j], \theta_1) \cong N_j\theta_1,\end{aligned}$$

as desired. □

Proof of Proposition 3.12 Consider the right Quillen equivalence

$$R\colon \mathscr{C}at_{s\mathscr{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}} \to P\mathscr{C}at(s\mathscr{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}.$$

By Proposition 3.5 and Lemma 3.14, we know that for any $j = 0, 1, 2$, the object $d^*\Theta_2[C_j] \cong R(N^{\mathrm{disc}}_* C_j)$ is a j–cell in $P\mathscr{C}at(s\mathscr{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,2)}$. Moreover, by Lemma 3.13 the object $N^{\mathrm{disc}}_* C_j$ is fibrant in $\mathscr{C}at_{s\mathscr{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$. It follows from Lemma 3.1 that $N^{\mathrm{disc}}_* C_j$ is a j–cell in $\mathscr{C}at_{s\mathscr{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$, as desired. □

We now compare the model structure for categories enriched in complete Segal spaces to the model structure for categories enriched in quasicategories.

Theorem 3.15 *The functor induced by taking $(-)_0$ on each hom-simplicial space defines a right Quillen equivalence*

$$\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}} \to \mathcal{C}at_{\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}.$$

In particular, $\mathcal{C}at_{\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$ is a model for $(\infty,2)$–categories.

Proof The functor $p\colon \Delta\times\Delta\to\Delta$, defined by $[m,n]\mapsto[m]$, induces an adjoint triple

$$\begin{tikzcd} \mathcal{S}et^{\Delta^{\mathrm{op}}} \arrow[r, "p^*" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}} = s\mathcal{S}et^{\Delta^{\mathrm{op}}}, \arrow[l, bend right=40, "p_!"'] \arrow[l, bend left=40, "p_*"] \end{tikzcd}$$

where p^* is given by precomposition with p, while $p_!$ and p_* are the left and right Kan extensions along p, respectively. In particular, the functor p^* is (strong) monoidal with respect to cartesian product because it is a right adjoint. Moreover, it is shown as [Joyal and Tierney 2007, Theorem 4.11] that the adjunction

$$p^*\colon \mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)} \rightleftarrows s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)} : p_*$$

is a Quillen equivalence. One can then apply [Lurie 2009a, Remark A.3.2.6] to obtain the desired Quillen equivalence, observing that p_* is the functor $(-)_0$. □

We can now use this equivalence to identify the j–cells in $\mathcal{C}at_{\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$.

Proposition 3.16 *In $\mathcal{C}at_{\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$ the object N_*C_j is a representative of the j–cell for $j=0,1,2$.*

Proof Consider the right Quillen equivalence from Theorem 3.15

$$\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}} \to \mathcal{C}at_{\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}.$$

By Proposition 3.12 and Lemma 3.13 we know that for each $j=0,1,2$, the object $N_*^{\mathrm{disc}}C_j$ is a j–cell in $\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$ and is fibrant. It follows from Lemma 3.1 that N_*C_j, the image of $N_*^{\mathrm{disc}}C_j$ under the above Quillen equivalence, is a j–cell in $\mathcal{C}at_{\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$. □

We now make a similar comparison between categories enriched in quasicategories and categories enriched in marked simplicial sets.

Theorem 3.17 *The functor induced by taking the underlying simplicial set U on each mapping object defines a right Quillen equivalence*

$$U_*\colon \mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}} \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}.$$

In particular, $\mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}}$ is a model for $(\infty,2)$–categories.

Proof The desired right Quillen equivalence is an instance of [Lurie 2009a, Remark A.3.2.6] applied to the right Quillen equivalence

$$U\colon s\mathcal{S}et^{+}_{(\infty,1)} \to \mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}$$

from [Lurie 2009a, Theorem 3.1.5.1]. □

Once again, our goal is to identify the j–cells in this model structure. To do so, consider the flat nerve functor $N^{\flat}\colon \mathcal{C}at \to s\mathcal{S}et^{+}$, obtained by regarding the nerve of a category in which the marked 1–simplices are precisely those corresponding to identity morphisms in the category. One can check that the functor $N^{\flat}$ preserves finite cartesian products, from which we obtain an induced functor $N^{\flat}_{*}\colon \mathcal{C}at_{\mathcal{C}at} \to \mathcal{C}at_{s\mathcal{S}et^{+}}$, given by applying $N^{\flat}$ to each mapping category.

Proposition 3.18 *In $\mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}}$ the object $N^{\flat}_{*}C_j$ is a representative of the j–cell for $j = 0, 1, 2$.*

We begin with a lemma establishing that these objects are fibrant.

Lemma 3.19 *For $j = 0, 1, 2$, the object $N^{\flat}_{*}C_j$ is fibrant in $\mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}}$.*

Proof For $j = 0, 1, 2$, all hom-marked simplicial sets of $N^{\flat}_{*}C_j$ are of the form $N^{\flat}[-1]$, $N^{\flat}[0]$ or $N^{\flat}[1]$, which are naturally marked quasicategories, and therefore fibrant in $\mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}}$, since the categories $[-1]$, $[0]$ and $[1]$ have no nontrivial isomorphisms. □

Proof of Proposition 3.18 We consider the right Quillen equivalence

$$U_{*}\colon \mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}} \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}.$$

By Proposition 3.16 we know that for each $j = 0, 1, 2$, the object $U_{*}N^{\flat}_{*}C_j \cong N_{*}C_j$ is a j–cell in $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$, and $N^{\flat}_{*}C_j$ is fibrant in $\mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}}$ by Lemma 3.19. It follows from Lemma 3.1 that $N^{\flat}_{*}C_j$ is a j–cell in $\mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}}$. □

3.4 Recognizing cells in simplicial models of $(\infty, 2)$–categories

Finally, we want to identify the j–cells in the model of marked simplicial sets. To aid in doing so, we look first at the related model of scaled simplicial sets. A *scaled simplicial set* is a simplicial set with a collection of marked 2–simplices including degenerate 2–simplices.

Theorem 3.20 [Lurie 2009b, Theorem 4.2.7] *The category $s\mathcal{S}et^{sc}$ of scaled simplicial sets admits a model structure in which*

- *the fibrant objects are the ∞–bicategories from* [Lurie 2009b, Definition 4.2.8], *and*
- *the cofibrations are the monomorphisms (and in particular every object is cofibrant).*

We denote this model structure by $s\mathcal{S}et^{sc}_{(\infty,2)}$.

Lurie enhances the classical homotopy coherent nerve functor $\mathfrak{N}\colon \mathcal{C}at_{s\mathcal{S}et} \to s\mathcal{S}et$ to the context of scaled simplicial sets by taking into account the marking, obtaining a scaled homotopy coherent nerve functor $\mathfrak{N}\colon \mathcal{C}at_{s\mathcal{S}et^+} \to s\mathcal{S}et^{sc}$.

Theorem 3.21 [Lurie 2009b, Theorem 0.0.3] *The scaled homotopy coherent nerve functor from* [Lurie 2009b, Definition 3.1.10] *defines a right Quillen equivalence*

$$\mathfrak{N}^{sc}\colon \mathcal{C}at_{s\mathcal{S}et^+ (\infty,1)} \to s\mathcal{S}et^{sc}_{(\infty,2)}.$$

In particular, $s\mathcal{S}et^{sc}_{(\infty,2)}$ is a model for $(\infty,2)$–categories. We now describe the j–cells in this model structure.

The description of the j–cells in this model structure makes use of a similar scaled nerve construction, in the form of a functor $N^{sc}\colon 2\mathcal{C}at \to s\mathcal{S}et^{sc}$, as described in [Gagna et al. 2022, Definition 8.1]. Given any 2–category $\mathfrak{D}$, the scaled nerve $N^{sc}\mathfrak{D}$ is given by the Duskin nerve of $\mathfrak{D}$ together with the marking of all 2–simplices arising from 2–isomorphisms.

Proposition 3.22 *In $s\mathcal{S}et^{sc}_{(\infty,2)}$ the object $N^{sc}C_j$ is a representative of the j–cell for $j = 0, 1, 2$.*

Proof As a preliminary observation, we mention that there is an isomorphism of scaled simplicial sets

$$N^{sc}C_j \cong \mathfrak{N}^{sc}N^{\flat}_{*}C_j$$

for $j = 0, 1, 2$. This fact can be deduced combining [Gagna et al. 2022, Definition 8.1] together with [Gagna et al. 2022, Proposition 8.2].

Consider now the right Quillen equivalence

$$\mathfrak{N}^{sc}\colon \mathcal{C}at_{s\mathcal{S}et^+ (\infty,1)} \to s\mathcal{S}et^{sc}_{(\infty,2)}.$$

By Proposition 3.18 we know that for each $j = 0, 1, 2$, the object $N^{\flat}_{*}C_j$ is a j–cell in $\mathcal{C}at_{s\mathcal{S}et^+_{(\infty,1)}}$. We have also proved that $N^{\flat}_{*}C_j$ is fibrant in $\mathcal{C}at_{s\mathcal{S}et^+_{(\infty,1)}}$ in Lemma 3.19. It follows from Lemma 3.1 that $N^{sc}C_j \cong \mathfrak{N}^{sc}N^{\flat}_{*}C_j$ is a j–cell in $s\mathcal{S}et^{sc}_{(\infty,2)}$. □

Finally, we can compare the models of scaled simplicial sets and marked simplicial sets.

Theorem 3.23 [Gagna et al. 2022, Theorem 7.7] *The forgetful functor defines a right Quillen equivalence*

$$U\colon ms\mathcal{S}et_{(\infty,2)} \to s\mathcal{S}et^{sc}_{(\infty,2)}.$$

We can now prove Proposition 2.18, which characterizes the cells in $ms\mathcal{S}et_{(\infty,2)}$.

Proof of Proposition 2.18 We consider the right Quillen equivalence

$$U\colon ms\mathcal{S}et_{(\infty,2)} \to s\mathcal{S}et^{sc}_{(\infty,2)}.$$

By [Gagna et al. 2022, Definition 8.1] we know that $UN^{\mathrm{RS}}C_j \cong N^{sc}C_j$ for each $j = 0, 1, 2$. Moreover, we know that $N^{sc}C_j$ is a j–cell in $s\mathcal{S}et^{sc}_{(\infty,2)}$ by Proposition 3.18 and that it is fibrant in $ms\mathcal{S}et_{(\infty,2)}$ by [Ozornova and Rovelli 2020, Theorem 5.1(1)]. It follows from Lemma 3.1 that $N^{\mathrm{RS}}C_j$ is a j–cell in $ms\mathcal{S}et_{(\infty,2)}$. □

4 Applications

Here we discuss four situations in which one can exploit the explicit Quillen equivalence

$$L\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \rightleftarrows ms\mathcal{S}et_{(\infty,2)} : R \tag{4-1}$$

from Theorem 2.20 to produce new theorems, new proofs or export constructions given some existing ones. Precisely, we show the following.

(1) The nerve construction for 2–categories is compatible with the suspension construction and the wedge constructions in an appropriate sense in the globular setting, using the analogous statement proven in the complicial setting in [Ozornova and Rovelli 2022].

(2) The nerve construction for 2–categories is compatible with the cone construction in an appropriate sense in the globular setting, using the analogous statement proven in the complicial setting in [Gagna et al. 2023].

(3) The nerve construction for 2–categories is compatible with the co-dual construction in an appropriate sense in the complicial setting, using the analogous statement that is formal in the globular setting.

(4) Weak equivalences can be tested on homotopy categories and homs in the complicial setting, using the analogous statement for the globular setting from [Bergner and Rezk 2020].

We expect that similar techniques can be applied to translate any new results from the setting of complicial sets to that of Θ_2–spaces, and vice versa.

As a preliminary preparation that is common to many of the applications, we define the *complicial nerve* to be the homotopical functor

$$N^{\mathrm{cmp}}\colon 2\mathcal{C}at \to \mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)} \to ms\mathcal{S}et_{(\infty,2)}$$

obtained as a composite of the right Quillen functor $N^{\natural}\colon 2\mathcal{C}at \to \mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}$ from [Ozornova and Rovelli 2021, Theorem 4.12] with the left Quillen functor $\mathrm{Refl}\colon \mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)} \to ms\mathcal{S}et_{(\infty,2)}$. Similarly, we consider the *globular nerve* construction to be the right Quillen functor

$$N^{\mathrm{gl}}\colon 2\mathcal{C}at \to \mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)} \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)} \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$$

obtained by composing the right Quillen nerve $N\colon 2\mathcal{C}at \to \mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}$ from [Campbell 2020, Theorem 5.10] with the right Quillen equivalence $\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)} \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}$ from [Ara 2014, Corollary 8.8] and the identity viewed as a right Quillen equivalence $\mathrm{id}\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)} \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$. The two nerve constructions N^{gl} and N^{cmp} induce functors at the level of $(\infty,1)$–categories that have the "correct" universal property, namely they realize the $(\infty,1)$–category of strict 2–categories as a localization of the $(\infty,1)$–category of $(\infty,2)$–categories, as in [Moser et al. 2022, Remark 6.37].

The globular and complicial nerves are compatible in the following sense.

Proposition 4.1 *For every 2–category $\mathcal{D}$ there is a natural weak equivalence*

$$L((N^{\mathrm{gl}}\mathcal{D})^{\mathrm{cof}}) \xrightarrow{\simeq} N^{\mathrm{cmp}}\mathcal{D}$$

in $ms\mathcal{S}et_{(\infty,2)}$, where $(N^{\mathrm{gl}}\mathcal{D})^{\mathrm{cof}}$ denotes a functorial cofibrant replacement of $N^{\mathrm{gl}}\mathcal{D}$ in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$.

Proof As a preliminary fact, we observe that with techniques analogous to the ones employed to construct the Quillen equivalence (4-1), one could also show that there is a Quillen equivalence

$$L'\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \rightleftarrows \mathcal{S}et^{t\,\Delta^{\mathrm{op}}}_{(\infty,2)} :R'$$

by setting, for all θ in Θ_2 and $k \geq 0$,

$$L'(\Theta_2[\theta] \boxtimes \Delta[k]) := N^{\natural}(\theta) \times \Delta[k]^{\sharp}.$$

Here, $\mathcal{S}et^{t\,\Delta^{\mathrm{op}}}_{(\infty,2)}$ denotes the model structure from [Ozornova and Rovelli 2020, Theorem 1.28], and it is useful to recall that, by [Ozornova and Rovelli 2020, Proposition 1.35], there is a Quillen equivalence

$$\mathrm{Refl}\colon \mathcal{S}et^{t\,\Delta^{\mathrm{op}}}_{(\infty,2)} \rightleftarrows ms\mathcal{S}et_{(\infty,2)}.$$

By construction, one then has $L = \mathrm{Refl}\, L'$. Now, for all θ in Θ_2 and $k \geq 0$, we have a commutative diagram in $2\mathcal{C}at$

$$\begin{tikzcd}
c^{\mathrm{gl}}N^{\mathrm{gl}}(\theta) \arrow[dr, "\epsilon", "\simeq"'] & & c^{\natural}N^{\natural}(\theta) \arrow[ll] \arrow[dl, "\epsilon"', "\simeq"] \\
& \theta &
\end{tikzcd}$$

Here, $c^{\mathrm{gl}}\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}} \to 2\mathcal{C}at$ and $c^{\natural}\colon \mathcal{S}et^{t\,\Delta^{\mathrm{op}}} \to 2\mathcal{C}at$ denote the left adjoint functors to N^{gl} and $N^{\natural}$, respectively, and the top map is adjoint to $N^{\natural}(\epsilon_\theta\colon c^{\mathrm{gl}}N^{\mathrm{gl}}(\theta) \to \theta)$ with respect to the adjunction $c^{\natural} \dashv N^{\natural}$, and the vertical maps can be seen to be weak equivalences in $2\mathcal{C}at$ combining [Ara 2014, Corollary 8.8; Campbell 2020, Section 5.1; Ozornova and Rovelli 2021, Theorem 4.10]. So there is a weak equivalence

$$c^{\mathrm{gl}}\Theta[\theta] \cong c^{\mathrm{gl}}N^{\mathrm{gl}}(\theta) \xleftarrow{\simeq} c^{\natural}N^{\natural}(\theta) = c^{\natural}L'\Theta[\theta]$$

in $2\mathcal{C}at$. Applying [Dugger 2001, Lemma 9.7] on the left Quillen functors

$$c^{\mathrm{gl}}, c^{\natural}L'\colon s\mathcal{S}et_p^{\Theta_2^{\mathrm{op}}} \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \to 2\mathcal{C}at,$$

it follows that, for all W cofibrant in $(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}})_{p,(\infty,2)}$, there is a natural weak equivalence

$$c^{\mathrm{gl}}W \xleftarrow{\simeq} c^{\natural}L'W$$

in $2\mathcal{C}at$. By [Hovey 1999, Corollary 1.4.4], it follows that for every (necessarily fibrant) 2–category $\mathfrak{D}$ we obtain a natural equivalence

$$N^{\mathrm{gl}}\mathfrak{D} \xrightarrow{\simeq} R'N^{\natural}\mathfrak{D}$$

in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$. Hence, using that the left Quillen functor L' preserves weak equivalences between cofibrant objects in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$ and that the derived counit of $L' \dashv R'$ at $N^{\mathrm{gl}}\mathfrak{D}$ is a weak equivalence in $\mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}$, we obtain that there are weak equivalences

$$L'((N^{\mathrm{gl}}\mathfrak{D})^{\mathrm{cof}}) \xrightarrow{\simeq} L'((R'N^{\natural}\mathfrak{D})^{\mathrm{cof}}) \xrightarrow{\simeq} N^{\natural}\mathfrak{D}$$

in $\mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}$. Finally, using that the functor Refl is homotopical we obtain a weak equivalence

$$L((N^{\mathrm{gl}}\mathfrak{D})^{\mathrm{cof}}) = \mathrm{Refl}\, L'((N^{\mathrm{gl}}\mathfrak{D})^{\mathrm{cof}}) \xrightarrow{\simeq} \mathrm{Refl}\, N^{\natural}\mathfrak{D} = N^{\mathrm{cmp}}\mathfrak{D}$$

in $ms\mathcal{S}et_{(\infty,2)}$, as desired. ☐

4.1 Compatibility of the suspension and wedge constructions with the nerve

We consider the following two constructions in the globular setting:

(1) the *globular suspension* construction from Definition 1.20

$$\Sigma^{\mathrm{gl}}\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{p,(\infty,1)} \to (s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)})_{*,*},$$

which is a left Quillen functor; and

(2) the *globular wedge* construction

$$\vee^{\mathrm{gl}}\colon (s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)})_* \times (s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)})_* \to (s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)})_*,$$

defined by

$$W \vee^{\mathrm{gl}} Z := W \underset{\Theta_2[0]}{\amalg} Z,$$

which is a left Quillen bifunctor.

These constructions induce functors at the level of underlying $(\infty,1)$–categories that have the following recognized universal properties.

(1) The suspension construction induces precisely the construction studied in [Gepner and Haugseng 2015, Definition 4.3.21].

(2) The wedge construction induces the $(\infty,1)$–categorical coproduct in the $(\infty,1)$–category of pointed $(\infty,2)$–categories.

The analogous constructions can also be implemented in the complicial context as well, as

(1) the *complicial suspension* construction from [Ozornova and Rovelli 2022, Definition 2.6]

$$\Sigma^{\mathrm{cmp}}\colon ms\mathcal{S}et_{(\infty,1)} \to (ms\mathcal{S}et_{(\infty,2)})_{*,*};$$

(2) the *complicial wedge* construction from [Ozornova and Rovelli 2022, Definition 4.7]

$$\vee^{\mathrm{cmp}}\colon (ms\mathcal{S}et_{(\infty,2)})_* \times (ms\mathcal{S}et_{(\infty,2)})_* \to (ms\mathcal{S}et_{(\infty,2)})_*.$$

We record a precise relation between the globular and complicial suspension and wedge.

As a preliminary fact, we observe that with techniques analogous to the ones employed to construct the Quillen equivalence (4-1), we could also show that there is a Quillen equivalence

$$L_1\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{p,(\infty,1)} \rightleftarrows \mathcal{S}et^{t\,\Delta^{\mathrm{op}}}_{(\infty,1)} : R_1$$

by setting, for all $m,k\geq 0$,

$$L_1(\Delta[m]\boxtimes\Delta[k]) := N^{\mathrm{RS}}([m])\times\Delta[k]^\sharp = \mathrm{th}_1(\Delta[m])\times\Delta[k]^\sharp.$$

Here, $s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{p,(\infty,1)}$ denotes the model structure for complete Segal spaces obtained by localizing the projective model structure, $ms\mathcal{S}et_{(\infty,1)}$ denotes the model structure for saturated 1–complicial sets, and $\mathrm{th}_1\colon ms\mathcal{S}et_{(\infty,1)}\to ms\mathcal{S}et_{(\infty,2)}$ is the left Quillen functor from [Verity 2008b, Notation 13].

Proposition 4.2 (a) *For all W cofibrant in $(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,1)}$ there is a weak equivalence*

$$\Sigma^{\mathrm{cmp}}L_1W \xrightarrow{\simeq} L\Sigma^{\mathrm{gl}}W$$

in $(ms\mathcal{S}et_{(\infty,2)})_{*,*}$.

(b) *Given any W and Z in $s\mathcal{S}et_*^{\Theta_2^{\mathrm{op}}}$, there is an isomorphism in $ms\mathcal{S}et$*

$$L(W\vee^{\mathrm{gl}}Z)\cong LW\vee^{\mathrm{cmp}}LZ.$$

Proof (a) By [Ozornova and Rovelli 2022, Theorem 2.9], there is a natural weak equivalence in $(ms\mathcal{S}et_{(\infty,2)})_{*,*}$ given by the composite

$$L\Sigma^{\mathrm{gl}}\Theta_2[m]\cong L\Theta_2[1|m]=N^{\mathrm{cmp}}[1|m]=N^{\mathrm{cmp}}\Sigma[m]\xleftarrow{\simeq}\Sigma^{\mathrm{cmp}}N^{\mathrm{cmp}}[m]=\Sigma^{\mathrm{cmp}}L_1\Delta[m].$$

The first three isomorphisms are given by the definitions of Σ^{gl}, L, and Σ, respectively, and the last is given by the definition L_1. The weak equivalence was established in [Ozornova and Rovelli 2022, Theorem 2.9]. Now, using [Dugger 2001, Lemma 9.7] on the functors

$$L\Sigma^{\mathrm{gl}},\Sigma^{\mathrm{cmp}}L_1\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}}_p\to s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{p,(\infty,1)}\to(ms\mathcal{S}et_{(\infty,2)})_{*,*}$$

it follows that for all W cofibrant in $(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,1)}$ there is a weak equivalence

$$\Sigma^{\mathrm{cmp}}L_1W \xrightarrow{\simeq} L\Sigma^{\mathrm{gl}}W$$

in $(ms\mathcal{S}et_{(\infty,2)})_{*,*}$.

(b) Given any W and Z in $s\mathcal{S}et_*^{\Theta_2^{\mathrm{op}}}$, there is an isomorphism in $ms\mathcal{S}et$

$$L(W\vee^{\mathrm{gl}}Z)\cong L\Big(W\coprod_{\Theta_2[0]}Z\Big)\cong LW\coprod_{\Delta[0]}LZ\cong LW\vee^{\mathrm{cmp}}LZ,$$

concluding the proof. □

Remark 4.3 By [Hovey 1999, Corollary 1.4.4], for all (Y,c,d) fibrant in $(ms\mathcal{S}et_{(\infty,2)})_{*,*}$ there is a weak equivalence

$$\mathrm{Hom}^{\mathrm{gl}}_{RY}(c,d) \xrightarrow{\simeq} R_1\,\mathrm{Hom}^{\mathrm{cmp}}_{Y}(c,d) \tag{4-2}$$

in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,1)}$ (and hence also in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,1)}$), where

$$\mathrm{Hom}^{\mathrm{gl}}\colon (s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)})_{*,*} \to s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{p,(\infty,1)}$$

denotes the right Quillen adjoint functor to Σ^{gl}, and similarly

$$\mathrm{Hom}^{\mathrm{cmp}}\colon (ms\mathcal{S}et_{(\infty,2)})_{*,*} \to ms\mathcal{S}et_{(\infty,1)}$$

denotes the right Quillen adjoint to Σ^{cmp}.

Recall that there are functors implementing the strict suspension construction

$$\Sigma\colon \mathcal{C}at \to 2\mathcal{C}at_{*,*}$$

and the strict wedge construction

$$\vee\colon 2\mathcal{C}at_{*} \times 2\mathcal{C}at_{*} \to 2\mathcal{C}at_{*}.$$

As an application of Theorem 2.20, combined with results from [Ozornova and Rovelli 2022], we can prove the following corollary, asserting that the suspension and wedge construction along a sieve/cosieve object are both compatible with the globular nerve of 2–categories. Recall from [Ozornova and Rovelli 2022, Definition 4.3] the definition of sieve and cosieve object in a 2–category, which are used to determine which 2–categories can be wedged together.

Corollary 4.4 (a) *Given any 1–category $\mathcal{D}$, there is a weak equivalence*

$$\Sigma^{\mathrm{gl}} N^{\mathrm{gl}}\mathcal{D} \xrightarrow{\simeq} N^{\mathrm{gl}}\Sigma\mathcal{D}$$

in $(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)})_{*,*}$.

(b) *Given 2–categories $\mathcal{A}$ and $\mathcal{B}$ endowed with a sieve and a cosieve object, respectively, there is a weak equivalence in $(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)})_{*}$,*

$$N^{\mathrm{gl}}\mathcal{A} \vee^{\mathrm{gl}} N^{\mathrm{gl}}\mathcal{B} \xrightarrow{\simeq} N^{\mathrm{gl}}(\mathcal{A}\vee\mathcal{B}).$$

Proof We prove (b) and leave (a) to the interested reader.

First, in the commutative square

$$\begin{tikzcd}
N^{\mathrm{RS}}\mathcal{A} \vee^{\mathrm{cmp}} N^{\mathrm{RS}}\mathcal{B} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & N^{\mathrm{cmp}}\mathcal{A} \vee^{\mathrm{cmp}} N^{\mathrm{cmp}}\mathcal{B} \arrow[d] \\
N^{\mathrm{RS}}(\mathcal{A}\vee\mathcal{B}) \arrow[r, "\simeq"'] & N^{\mathrm{cmp}}(\mathcal{A}\vee\mathcal{B})
\end{tikzcd}$$

the left vertical map is a weak equivalence in $ms\mathcal{S}et_{(\infty,2)}$ by [Ozornova and Rovelli 2022, Theorem 4.9]. The two horizontal maps are weak equivalences in $ms\mathcal{S}et_{(\infty,2)}$, which can be seen by combining [Ozornova and Rovelli 2021, Theorem 5.2] and [Ozornova and Rovelli 2020, Proposition 1.31]. By the two-out-of-three property, the right-hand map is then a weak equivalence in $ms\mathcal{S}et_{(\infty,2)}$.

Next, in the commutative square

$$\begin{tikzcd}
L((N^{\mathrm{gl}}\mathscr{A})^{\mathrm{cof}}) \vee^{\mathrm{cmp}} L((N^{\mathrm{gl}}\mathscr{B})^{\mathrm{cof}}) \arrow[r, "\simeq"] \arrow[d] & N^{\mathrm{cmp}}\mathscr{A} \vee^{\mathrm{cmp}} N^{\mathrm{cmp}}\mathscr{B} \arrow[d, "\simeq"] \\
L((N^{\mathrm{gl}}(\mathscr{A} \vee \mathscr{B}))^{\mathrm{cof}}) \arrow[r, "\simeq"'] & N^{\mathrm{cmp}}(\mathscr{A} \vee \mathscr{B})
\end{tikzcd}$$

the two horizontal maps are weak equivalences in $ms\mathcal{S}et_{(\infty,2)}$ by Proposition 4.1. By the two-out-of-three property, the left vertical map is a weak equivalence.

We have the commutative triangle

$$\begin{tikzcd}
L((N^{\mathrm{gl}}\mathscr{A})^{\mathrm{cof}} \vee^{\mathrm{gl}} (N^{\mathrm{gl}}\mathscr{B})^{\mathrm{cof}}) \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & L((N^{\mathrm{gl}}\mathscr{A} \vee^{\mathrm{gl}} N^{\mathrm{gl}}\mathscr{B})^{\mathrm{cof}}) \arrow[d] \\
& L((N^{\mathrm{gl}}(\mathscr{A} \vee \mathscr{B}))^{\mathrm{cof}})
\end{tikzcd}$$

By [Hovey 1999, Corollary 1.3.16], the left Quillen equivalence L creates weak equivalences between cofibrant objects in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$, so we obtain a commutative square

$$\begin{tikzcd}
(N^{\mathrm{gl}}\mathscr{A} \vee^{\mathrm{gl}} N^{\mathrm{gl}}\mathscr{B})^{\mathrm{cof}} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (N^{\mathrm{gl}}(\mathscr{A} \vee \mathscr{B}))^{\mathrm{cof}} \arrow[d, "\simeq"] \\
N^{\mathrm{gl}}\mathscr{A} \vee^{\mathrm{gl}} N^{\mathrm{gl}}\mathscr{B} \arrow[r] & N^{\mathrm{gl}}(\mathscr{A} \vee \mathscr{B})
\end{tikzcd}$$

as desired. By the two-out-of-three property, the bottom map in the square is a weak equivalence in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$, as desired. □

While (a) can be essentially read off from [Rezk 2010], tackling directly (b) within the globular setting would require significant combinatorial work.

4.2 The cone construction and compatibility with the nerve

In the globular setting there does not seem to be a straightforward way to define join constructions, or even cone constructions, which play an important role in the development of the theory of limits and colimits. By contrast, the complicial setting is well-suited to implementing formal join constructions in general, and cones in particular. A *cone construction*

$$\mathrm{Cone}^{\mathrm{cmp}} := \Delta[0] \star -\colon ms\mathcal{S}et_{(\infty,1)} \to (ms\mathcal{S}et_{(\infty,2)})_*$$

is defined in [Gagna et al. 2023] in the form of a left Quillen functor.

Given any W in $(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}})_{p,(\infty,1)}$, taking advantage of the explicit Quillen equivalence (4-1), it is possible to define the cone construction for W in terms of the one for L_1W, by setting

$$\mathrm{Cone}^{\mathrm{gl}}W := R((\mathrm{Cone}^{\mathrm{cmp}}(L_1(W^{\mathrm{cof}})))^{\mathrm{fib}}).$$

While the formula is fairly complicated, there is currently no competing way of treating cones in Θ_2–spaces.

Remark 4.5 For every W cofibrant in $(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}})_{p,(\infty,2)}$ there is a zigzag of weak equivalences

$$\begin{aligned} L((\mathrm{Cone}^{\mathrm{gl}}W)^{\mathrm{cof}}) = L((R((\mathrm{Cone}^{\mathrm{cmp}}(L_1(W^{\mathrm{cof}})))^{\mathrm{fib}}))^{\mathrm{cof}}) &\xrightarrow{\simeq} (\mathrm{Cone}^{\mathrm{cmp}}(L_1(W^{\mathrm{cof}})))^{\mathrm{fib}} \\ &\xleftarrow{\simeq} \mathrm{Cone}^{\mathrm{cmp}}(L_1(W^{\mathrm{cof}})) \\ &\xrightarrow{\simeq} \mathrm{Cone}^{\mathrm{cmp}}(L_1W) \end{aligned}$$

in $ms\mathcal{S}et_{(\infty,2)}$.

Recall that there is a functor implementing the strict cone construction $\mathrm{Cone}\colon \mathcal{C}at \to 2\mathcal{C}at_*$. As an application of Theorem 2.20, combined with results from [Gagna et al. 2023], we can prove the following corollary, asserting that the cone construction is compatible with the nerve construction in a suitable sense in the globular setting for 1–categories that are freely generated by a loop-free graph. Such 1–categories are called *strong Steiner* in [Ara and Maltsiniotis 2020, Section 2.15].

Corollary 4.6 *Given any 1–category $\mathcal{D}$ that is freely generated by a loop-free graph, there is a zigzag of weak equivalences in $(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}})_{p,(\infty,2)}$,*

$$N^{\mathrm{gl}}\mathrm{Cone}\mathcal{D} \simeq \mathrm{Cone}^{\mathrm{gl}}N^{\mathrm{gl}}\mathcal{D}.$$

Proof There is a zigzag of weak equivalences

$$\begin{aligned} L((N^{\mathrm{gl}}\mathrm{Cone}\mathcal{D})^{\mathrm{cof}}) &\xrightarrow{\simeq} N^{\mathrm{cmp}}\mathrm{Cone}\mathcal{D} \\ &\xleftarrow{\simeq} \mathrm{Cone}^{\mathrm{cmp}}N^{\mathrm{cmp}}\mathcal{D} \\ &\xleftarrow{\simeq} \mathrm{Cone}^{\mathrm{cmp}}L_1((N^{\mathrm{gl}}\mathcal{D})^{\mathrm{cof}}) \simeq L(\mathrm{Cone}^{\mathrm{gl}}(N^{\mathrm{gl}}\mathcal{D}^{\mathrm{cof}})^{\mathrm{cof}}) \end{aligned}$$

in $ms\mathcal{S}et_{(\infty,2)}$ given by Proposition 4.1, [Gagna et al. 2023, Theorem 5.5], Proposition 4.1 for L_1, and Remark 4.5, respectively. Given that the left Quillen equivalence L creates weak equivalences between cofibrant objects, we obtain a zigzag of weak equivalences in $(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}})_{p,(\infty,2)}$

$$N^{\mathrm{gl}}\mathrm{Cone}\mathcal{D} \simeq (\mathrm{Cone}^{\mathrm{gl}}N^{\mathrm{gl}}\mathcal{D})^{\mathrm{cof}} \simeq (\mathrm{Cone}^{\mathrm{gl}}((N^{\mathrm{gl}}\mathcal{D})^{\mathrm{cof}}))^{\mathrm{cof}} \xrightarrow{\simeq} \mathrm{Cone}^{\mathrm{gl}}((N^{\mathrm{gl}}\mathcal{D})^{\mathrm{cof}}) \xrightarrow{\simeq} \mathrm{Cone}^{\mathrm{gl}}N^{\mathrm{gl}}\mathcal{D},$$

as desired. □

4.3 Dual constructions and compatibility with the nerve

It is determined in [Barwick and Schommer-Pries 2021, Theorem 7.3] that there are four types of dualities for $(\infty, 2)$–categories: identity; op-dual, which reverses the direction of the 1–morphisms; co-dual, which reverses the direction of the 2–morphisms; and co-op-dual, which reverses both.

In the complicial setting, one can implement the op-dual construction $(-)^{\mathrm{op}}\colon ms\mathcal{S}et_{(\infty,2)} \to ms\mathcal{S}et_{(\infty,2)}$ in a straightforward way. However, there is no formal way to define the co-dual construction in $ms\mathcal{S}et_{(\infty,2)}$, and a co-dual construction $(-)^{\mathrm{co}}$ was only proposed recently in [Loubaton 2022, Proposition 4.2.7] in the form of a (highly nontrivial) left Quillen functor $(-)^{\mathrm{co}}\colon ms\mathcal{S}et_{(\infty,2)} \to ms\mathcal{S}et_{(\infty,2)}$.

By contrast, the globular setting is well-suited to implementing all four dualities; see [Haugseng 2021]. In particular, the co-dual construction can be realized as an isomorphism that is both a left and right Quillen equivalence for both the projectively-based and injectively-based model structures,

$$(-)^{\mathrm{co}}\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \quad\text{and}\quad (-)^{\mathrm{co}}\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)} \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{i,(\infty,2)}.$$

Given the Quillen equivalence (4-1), for all Y fibrant in $ms\mathcal{S}et_{(\infty,2)}$, there is a zigzag of weak equivalences

$$Y^{\mathrm{co}} \simeq L(((RY)^{\mathrm{co}})^{\mathrm{cof}}), \tag{4-3}$$

in $ms\mathcal{S}et_{(\infty,2)}$, allowing one to express the co-dual construction of Y in terms of the one for RY.

Remark 4.7 For every fibrant object Y in $ms\mathcal{S}et_{(\infty,2)}$, by (4-3) there is zigzag of weak equivalences

$$L(((RY)^{\mathrm{co}})^{\mathrm{cof}}) \simeq Y^{\mathrm{co}}$$

in $ms\mathcal{S}et_{(\infty,2)}$. By taking a functorial fibrant replacement in $ms\mathcal{S}et_{(\infty,2)}$, we obtain a zigzag of weak equivalences between fibrant objects in $ms\mathcal{S}et_{(\infty,2)}$

$$(L(((RY)^{\mathrm{co}})^{\mathrm{cof}}))^{\mathrm{fib}} \simeq (Y^{\mathrm{co}})^{\mathrm{fib}}.$$

Applying R then gives a zigzag of weak equivalences

$$R(L(((RY)^{\mathrm{co}})^{\mathrm{cof}}))^{\mathrm{fib}}) \simeq R((Y^{\mathrm{co}})^{\mathrm{fib}})$$

in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$. By composing with the component of the derived unit of $L \dashv R$ on $((RY)^{\mathrm{co}})^{\mathrm{cof}}$, we obtain a zigzag of weak equivalences

$$(RY)^{\mathrm{co}} \simeq ((RY)^{\mathrm{co}})^{\mathrm{cof}} \simeq R((Y^{\mathrm{co}})^{\mathrm{fib}})$$

in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$.

As an application of Theorem 2.20, we can prove the following corollary, asserting that the co-dual construction is compatible with the nerve of 2–categories in the complicial setting.

Corollary 4.8 *Given any 2–category $\mathcal{D}$, there is a zigzag of weak equivalences in $ms\mathcal{S}et_{(\infty,2)}$*

$$N^{\mathrm{cmp}}\mathcal{D}^{\mathrm{co}} \simeq (N^{\mathrm{cmp}}\mathcal{D})^{\mathrm{co}}.$$

Proof There a zigzag of weak equivalences

$$RN^{\mathrm{cmp}}(\mathcal{D}^{\mathrm{co}}) \simeq N^{\mathrm{gl}}(\mathcal{D}^{\mathrm{co}}) \simeq (N^{\mathrm{gl}}\mathcal{D})^{\mathrm{co}} \simeq (RN^{\mathrm{cmp}}\mathcal{D})^{\mathrm{co}} \simeq R((N^{\mathrm{cmp}}\mathcal{D})^{\mathrm{co}})^{\mathrm{fib}})$$

in $(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}})_{p,(\infty,2)}$, where the weak equivalences are given by Proposition 4.1, inspection, Proposition 4.1, and Remark 4.7, respectively. By [Hovey 1999, Corollary 1.3.16], the right Quillen equivalence R creates weak equivalences between fibrant objects in $(s\mathcal{S}et^{\Theta_2^{\mathrm{op}}})_{p,(\infty,2)}$, so we obtain a zigzag of weak equivalences

$$N^{\mathrm{cmp}}\mathscr{D}^{\mathrm{co}} \simeq ((N^{\mathrm{cmp}}\mathscr{D})^{\mathrm{co}})^{\mathrm{fib}} \simeq (N^{\mathrm{cmp}}\mathscr{D})^{\mathrm{co}}$$

in $ms\mathcal{S}et_{(\infty,2)}$, as desired. □

4.4 A fundamental theorem for 2–complicial sets

We define the *globular hom* construction

$$\mathrm{Hom}^{\mathrm{gl}}\colon (s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)})_{*,*} \to s\mathcal{S}et^{\Theta_1^{\mathrm{op}}}_{p,(\infty,1)}$$

as the right (Quillen) adjoint functor of the suspension Σ^{gl}. Similarly, we define the *globular homotopy category* construction

$$\mathrm{Ho}^{\mathrm{gl}}\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \to \mathscr{C}at$$

to be given by

$$\mathrm{Ho}^{\mathrm{gl}}\, X = h\tau_{\Theta}^{*} X,$$

where $h\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}} \to \mathscr{C}at$ is the homotopy category functor from [Rezk 2010, Section 7.3] and

$$\tau_{\Theta}\colon \Delta = \Theta_1 \to \Theta_2$$

is the functor defined by $\tau_{\Theta}[m] = [m]([0], \dots, [0])$ from [Bergner and Rezk 2020, Section 3.2]; it is also defined in [Rezk 2010, Section 4.1] in more generality.

The following statement, which is essentially in [Bergner and Rezk 2020], can be thought of as a *fundamental theorem for* $(\infty, 2)$*–categories*, referring to the terminology from [Rezk 2022], where the $(\infty, 1)$–categorical case is treated in the model of quasicategories.

Proposition 4.9 *Let W and Z be fibrant in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$. A map $f\colon W \to Z$ in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$ is a weak equivalence if and only if*

(1) *the map f is* ***essentially surjective****, meaning that it induces an essentially surjective functor of homotopy categories*

$$\mathrm{Ho}^{\mathrm{gl}}\, f\colon \mathrm{Ho}^{\mathrm{gl}}\, W \to \mathrm{Ho}^{\mathrm{gl}}\, Z;$$

(2) *the map f is* ***homotopically fully faithful****, namely that it induces a weak equivalence*

$$f_{c,d}\colon \mathrm{Hom}^{\mathrm{gl}}_{W}(c, d) \simeq \mathrm{Hom}^{\mathrm{gl}}_{Z}(fc, fd)$$

in $s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{p,(\infty,1)}$.

Proof As a consequence of [Bergner and Rezk 2020, Theorem 6.4], the functor

$$d^{*}\colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)} \to s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{p,(\infty,2)}$$

creates weak equivalences, so saying that $f\colon W\to Z$ is a weak equivalence in $s\mathcal{S}et_{p,(\infty,2)}^{\Theta_2^{\mathrm{op}}}$ is equivalent to saying that

$$d^*f\colon d^*W\to d^*Z$$

is a weak equivalence in $s\mathcal{S}et_{p,(\infty,2)}^{(\Delta\times\Delta)^{\mathrm{op}}}$. Let us take a functorial fibrant replacement of d^*f in $s\mathcal{S}et_p^{\Theta_2^{\mathrm{op}}}$, and denote it by $(d^*f)^{\mathrm{pf}}\colon(d^*W)^{\mathrm{pf}}\to(d^*Z)^{\mathrm{pf}}$. Using the fact that d^* preserves fibrant objects [Bergner and Rezk 2020, Proposition 6.3], we can deduce that $d^*f\colon d^*W\to d^*Z$ is a weak equivalence in $s\mathcal{S}et_{p,(\infty,2)}^{\Theta_2^{\mathrm{op}}}\to s\mathcal{S}et_{p,(\infty,2)}^{(\Delta\times\Delta)^{\mathrm{op}}}$ if and only if

$$(d^*f)^{\mathrm{pf}}\colon(d^*W)^{\mathrm{pf}}\to(d^*Z)^{\mathrm{pf}}$$

is a weak equivalence in $s\mathcal{S}et_{p,(\infty,2)}^{(\Delta\times\Delta)^{\mathrm{op}}}$. Since $(d^*W)^{\mathrm{pf}}$ and $(d^*Z)^{\mathrm{pf}}$ are fibrant in $s\mathcal{S}et_{p,(\infty,2)}^{(\Delta\times\Delta)^{\mathrm{op}}}$, we know that $(d^*f)^{\mathrm{pf}}$ is a weak equivalence in $s\mathcal{S}et_{p,(\infty,2)}^{(\Delta\times\Delta)^{\mathrm{op}}}$ if and only if the same map is a weak equivalence in $s\mathcal{S}et_p^{(\Delta\times\Delta)^{\mathrm{op}}}$. Using the fact that Dwyer–Kan equivalences between complete Segal objects are precisely levelwise weak equivalences [Bergner and Rezk 2020, Proposition 8.17 and Definition 8.2], the map $(d^*f)^{\mathrm{pf}}$ is a weak equivalence in $s\mathcal{S}et_{p,(\infty,2)}^{(\Delta\times\Delta)^{\mathrm{op}}}$ if and only if

(1′) the map

$$\mathrm{Ho}((d^*f)^{\mathrm{pf}})\colon\mathrm{Ho}((d^*W)^{\mathrm{pf}})\to\mathrm{Ho}((d^*Z)^{\mathrm{pf}})$$

is an essentially surjective functor on homotopy categories, where Ho denotes the homotopy category from [Bergner and Rezk 2020, Section 8.1]; and

(2′) the map

$$((d^*f)^{\mathrm{pf}})_{a,b}\colon M^{\Delta}_{(d^*W)^{\mathrm{pf}}}((d^*f)^{\mathrm{pf}}(a),(d^*f)^{\mathrm{pf}}(b))\to M^{\Delta}_{(d^*Z)^{\mathrm{pf}}}((d^*f)^{\mathrm{pf}}(a),(d^*f)^{\mathrm{pf}}(b))$$

is a weak equivalence in $s\mathcal{S}et_{p,(\infty,1)}^{\Delta^{\mathrm{op}}}$, where M^{Δ} denotes the mapping object from [Bergner and Rezk 2020, Section 8.1].

Given the natural equivalence of categories

$$\mathrm{Ho}((d^*W)^{\mathrm{pf}})\simeq h\tau_{\Theta}^*W=\mathrm{Ho}^{\mathrm{gl}}\,W$$

and the natural weak equivalence

$$M^{\Delta}_{(d^*W)^{\mathrm{pf}}}(a,b)\simeq M^{\Delta}_{d^*W}(a,b)\cong\mathrm{Hom}^{\mathrm{gl}}_W(a,b)$$

from [Bergner and Rezk 2020, Proposition 3.10] in $s\mathcal{S}et_{p,(\infty,1)}^{\Delta^{\mathrm{op}}}$, we then obtain an equivalence to the conditions (1) and (2) from the statement, as desired. □

Aiming at providing a proof of the fundamental theorem for $(\infty,2)$–categories in the complicial context, recall the *complicial hom* construction

$$\mathrm{Hom}^{\mathrm{cmp}}\colon(ms\mathcal{S}et_{(\infty,2)})_{*,*}\to ms\mathcal{S}et_{(\infty,1)},$$

as the right (Quillen) adjoint functor of the suspension Σ^{cmp}, and the *complicial homotopy category* construction

$$\mathrm{Ho}^{\mathrm{cmp}}\colon m\mathcal{S}et_{(\infty,2)} \to \mathcal{C}at$$

given by

$$\mathrm{Ho}^{\mathrm{cmp}}\, X = c_1^{\natural}\mathrm{sp}_1 X,$$

where $\mathrm{sp}_1\colon m\mathcal{S}et \to m\mathcal{S}et$ is the right Quillen functor of [Verity 2008b, Notation 13] and $c_1^{\natural}\colon m\mathcal{S}et \to \mathcal{C}at$ is the left adjoint functor to the 1–dimensional natural nerve functor.

The homotopy category and hom constructions for the globular and complicial setting are compatible as follows:

Lemma 4.10 *Given a fibrant object X in $m\mathcal{S}et_{(\infty,2)}$, there is an isomorphism of categories*

$$\mathrm{Ho}^{\mathrm{gl}}\, RX \cong \mathrm{Ho}^{\mathrm{cmp}}\, X.$$

Proof Given a fibrant object X in $m\mathcal{S}et_{(\infty,2)}$, there is an isomorphism of categories

$$\mathrm{Ho}^{\mathrm{gl}}\, RX = h\tau_{\Theta}^{*} RX \cong hR_1\mathrm{sp}_1 X \cong c_1^{\natural}\mathrm{sp}_1 X = \mathrm{Ho}^{\mathrm{cmp}}\, X,$$

as desired. □

Using the Theorem 2.20, combined with Proposition 4.9, we can prove the following fundamental theorem for $(\infty, 2)$–categories in the complicial setting.

Theorem 4.11 *Let X and Y be fibrant in $m\mathcal{S}et_{(\infty,2)}$. A map $f\colon X \to Y$ in $m\mathcal{S}et_{(\infty,2)}$ is a weak equivalence if and only if*

(1) *the map f is **essentially surjective**, meaning f induces an essentially surjective functor of homotopy categories*

$$\mathrm{Ho}^{\mathrm{cmp}}\, f\colon \mathrm{Ho}^{\mathrm{cmp}}\, X \simeq \mathrm{Ho}^{\mathrm{cmp}}\, Y;$$

(2) *the map f is **homotopically fully faithful**, namely f induces a weak equivalence in $m\mathcal{S}et_{(\infty,1)}$,*

$$f_{c,d}\colon \mathrm{Hom}_X^{\mathrm{cmp}}(c,d) \simeq \mathrm{Hom}_Y^{\mathrm{cmp}}(fc, fd).$$

Proof A map $f\colon X \to Y$ is a weak equivalence in $m\mathcal{S}et_{(\infty,2)}$ if and only if, by (4-1), the map $Rf\colon RX \to RY$ is a weak equivalence in $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{p,(\infty,2)}$. By Proposition 4.9, we can equivalently say that Rf is a Dwyer–Kan equivalence, in that the map

$$(Rf)^{\mathrm{gl}}_{c,d}\colon \mathrm{Hom}^{\mathrm{gl}}_{RX}(c,d) \to \mathrm{Hom}^{\mathrm{gl}}_{RY}(fc, fd)$$

is a weak equivalence in $(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{p,(\infty,1)}$ for all $c, d \in (RX)_{0,[0]}$ and that

$$\mathrm{Ho}^{\mathrm{gl}}\, Rf\colon \mathrm{Ho}^{\mathrm{gl}}\, RX \simeq \mathrm{Ho}^{\mathrm{gl}}\, RY$$

is an essentially surjective functor of homotopy categories. By Remark 4.3, these conditions are equivalent to having the analogous ones for $R_1(f^{\mathrm{cmp}})$, namely, that the map

$$R_1(f^{\mathrm{cmp}}_{c,d})\colon R_1 \operatorname{Hom}^{\mathrm{cmp}}_X(c,d) \to R_1 \operatorname{Hom}^{\mathrm{cmp}}_Y(fc,fd)$$

is a weak equivalence in $ms\mathcal{S}et_{(\infty,1)}$ for all $c,d \in X_0$ and that the map

$$\mathrm{Ho}^{\mathrm{gl}} Rf\colon \mathrm{Ho}^{\mathrm{gl}} RX \to \mathrm{Ho}^{\mathrm{gl}} RY$$

is essentially surjective. Then applying Lemma 4.10, we can equivalently say that f^{cmp} is a Dwyer–Kan equivalence, in that the map

$$f^{\mathrm{cmp}}_{c,d}\colon \operatorname{Hom}^{\mathrm{cmp}}_X(c,d) \to \operatorname{Hom}^{\mathrm{cmp}}_Y(fc,fd)$$

is a weak equivalence in $ms\mathcal{S}et_{(\infty,1)}$ for all $c,d \in X_0$ and that

$$\mathrm{Ho}^{\mathrm{cmp}} f\colon \mathrm{Ho}^{\mathrm{cmp}} X \to \mathrm{Ho}^{\mathrm{cmp}} Y$$

is an essentially surjective functor of homotopy categories, as desired. □

A proof of this fact internal to the complicial setting was outlined in [Campbell 2019], and provided recently in [Loubaton 2022, Corollary 3.2.11], but it relies on highly nontrivial combinatorics. Using our comparison with the Θ_2–model gives a much less technical proof.

Appendix The colossal model of $(\infty, 2)$–categories

In this section, we give a model categorical variant of the colossal model by Barwick–Schommer-Pries.

In order to recall the original definition of the colossal model, we fix the following notations. We denote by Υ_2 the indexing category for the colossal model, namely the full subcategory of $2\mathcal{C}at$ as described by [Barwick and Schommer-Pries 2021, Definition 6.2]. In particular, $(\Upsilon_2^{\mathrm{op}})_\infty$ is the $(\infty,1)$–category obtained by regarding the category Υ_2^{op} as an $(\infty,1)$–category. We denote by $\mathcal{S}_\infty$ the $(\infty,1)$–category of spaces, namely $\mathcal{S}_\infty = (s\mathcal{S}et_{(\infty,0)})_\infty$.

Definition A.1 [Barwick and Schommer-Pries 2021] The *colossal model* is the $(\infty,1)$–category

$$\mathcal{L}_\infty(\mathcal{S}_\infty^{(\Upsilon_2^{\mathrm{op}})_\infty}),$$

obtained by localizing the presheaf $(\infty,1)$–category $\mathcal{S}_\infty^{(\Upsilon_2^{\mathrm{op}})_\infty}$ at the set of maps from [Barwick and Schommer-Pries 2021, Notation 8.3].

From the definition, we see that the colossal model is obtained by considering the $(\infty,1)$–category of spaces, taking a presheaf $(\infty,1)$–category valued in it, and then localizing. By contrast, one could instead present the same $(\infty,1)$–category by considering the Quillen model structure, which presents the $(\infty,1)$–category of spaces, taking the injective model structure on a presheaf category of functors valued in the Quillen model structure, and then a left Bousfield localization of it. More precisely, one can consider the following model structure.

Proposition A.2 *The category $s\mathcal{S}et^{\Upsilon_2^{\mathrm{op}}}$ of Υ_2–spaces supports a cofibrantly generated model structure obtained by taking the left Bousfield localization of the injective model structure $s\mathcal{S}et_{\mathrm{inj}}^{\Upsilon_2^{\mathrm{op}}}$ with respect to the set of elementary acyclic cofibrations from* [Barwick and Schommer-Pries 2021, Notation 6.5]. *We denote this model structure by $s\mathcal{S}et_{(\infty,2)}^{\Upsilon_2^{\mathrm{op}}}$.*

We want to prove that this model structure does present the colossal model, in the sense of the following theorem.

Theorem A.3 *There is an equivalence of $(\infty,1)$–categories*

$$\mathcal{L}_\infty\big((s\mathcal{S}et_{(\infty,0)})_\infty^{(\Upsilon_2^{\mathrm{op}})_\infty}\big)\simeq\big(s\mathcal{S}et_{(\infty,2)}^{\Upsilon_2^{\mathrm{op}}}\big)_\infty.$$

The proof is an application of the following result, which guarantees that one can build localizations of presheaf categories either as model categories or directly as $(\infty,1)$–categories.

Proposition A.4 *Let $\mathcal{A}$ be a category, $\mathcal{M}$ a left proper combinatorial simplicial model category, and Λ a set of maps in $\mathcal{M}^{\mathcal{A}}$. There is an equivalence of $(\infty,1)$–categories*

$$\mathcal{L}_\infty(\mathcal{M}_\infty^{\mathcal{A}_\infty})\simeq(\mathcal{L}(\mathcal{M}_{\mathrm{inj}}^{\mathcal{A}}))_\infty,$$

where $\mathcal{L}(\mathcal{M}_{\mathrm{inj}}^{\mathcal{A}})$ denotes the Bousfield localization of the injective model structure $\mathcal{M}_{\mathrm{inj}}^{\mathcal{A}}$ at Λ, and $\mathcal{L}_\infty(\mathcal{M}_\infty^{\mathcal{A}_\infty})$ denotes the localization of the $(\infty,1)$–category $\mathcal{M}_\infty^{\mathcal{A}_\infty}$ at Λ.

The proof of the proposition requires the following two ingredients.

Theorem A.5 [Lurie 2009a, Proof of Proposition A.3.7.8] *Let $\mathcal{N}$ be a left proper combinatorial simplicial model category, and Λ be a set of maps in $\mathcal{N}$. There is an equivalence of $(\infty,1)$–categories*

$$\mathcal{L}_\infty\mathcal{N}_\infty\simeq(\mathcal{L}\mathcal{N})_\infty,$$

where $\mathcal{L}\mathcal{N}$ denotes the Bousfield localization of the model structure $\mathcal{N}$ at Λ, and $\mathcal{L}_\infty\mathcal{N}_\infty$ denotes the localization of the $(\infty,1)$–category $\mathcal{N}_\infty$ at Λ_∞.

Theorem A.6 [Lurie 2009a, Proposition 4.2.4.4] *Let $\mathcal{A}$ be a category and $\mathcal{M}$ a combinatorial simplicial model category. There is an equivalence of $(\infty,1)$–categories*

$$\mathcal{M}_\infty^{\mathcal{A}_\infty}\simeq(\mathcal{M}_{\mathrm{inj}}^{\mathcal{A}})_\infty,$$

where $\mathcal{M}_{\mathrm{inj}}^{\mathcal{A}}$ denotes the injective model structure on $\mathcal{M}^{\mathcal{A}}$.

We can now prove the proposition.

Proof of Proposition A.4 Combining Theorems A.5 and A.6, we obtain an equivalence of $(\infty,1)$–categories

$$\mathscr{L}_\infty(\mathscr{M}_\infty^{\mathscr{A}_\infty}) \simeq \mathscr{L}_\infty(\mathscr{M}^{\mathscr{A}})_\infty \simeq (\mathscr{L}(\mathscr{M}^{\mathscr{A}}))_\infty,$$

as desired. □

We can now prove the theorem.

Proof of Theorem A.3 Applying Proposition A.4 with $\mathscr{M} = \mathscr{S} = s\mathscr{S}et_{(\infty,0)}$, $\mathscr{A} = \Upsilon_2^{\mathrm{op}}$ and $\Lambda = S$, the set of maps in $s\mathscr{S}et^{\Upsilon_2^{\mathrm{op}}}$ from [Barwick and Schommer-Pries 2021, Notation 8.3], we obtain the equivalence of $(\infty,1)$–categories

$$\mathscr{L}_\infty\big(\mathscr{S}_\infty^{(\Upsilon_2^{\mathrm{op}})_\infty}\big) = \mathscr{L}_\infty\big((s\mathscr{S}et_{(\infty,0)})_\infty^{(\Upsilon_2^{\mathrm{op}})_\infty}\big) \simeq \big(\mathscr{L}\big(s\mathscr{S}et_{(\infty,0)}^{\Upsilon_2^{\mathrm{op}}}\big)\big)_\infty = \big(s\mathscr{S}et_{(\infty,2)}^{\Upsilon_2^{\mathrm{op}}}\big)_\infty,$$

as desired. □

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