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On products of beta and gamma elements in the homotopy of the first Smith–Toda spectrum

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We determine the first cohomology of the monochromatic comodule M_2^1 at an odd prime, and apply the results to show nontrivialities of some products of beta and gamma elements in the homotopy groups of the Smith–Toda spectrum $V(1)$. The cohomology for a prime greater than three was previously determined by the first author. Here we verify them and determine the cohomology at the prime 3 by elementary calculation. The cohomology will be a stepping stone for computing the cohomology of the monochromatic comodule M_0^3, which we hope to determine for a long time.

55Q45; 55Q51, 55T15

1 Introduction

Let p be an odd prime number and $\mathcal{S}_{(p)}$ denote the stable homotopy category of p–local spectra. Let $S \in \mathcal{S}_{(p)}$ denote the sphere spectrum. Then the mod p Moore spectrum M and the first Smith–Toda spectrum $V(1)$ are given by the cofiber sequences

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S \quad \text{and} \quad \Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} M. \tag{1.1}$$

Here $p \in \pi_0(S) \cong \mathbb{Z}_{(p)}$, and $\alpha \in [M, M]_q$ denotes the Adams map. Hereafter we put

$$q = 2p - 2 \in \mathbb{Z}.$$

To study the homotopy groups $\pi_*(X)$ of a spectrum X, we adopt the Adams–Novikov spectral sequence

$$E_2^{s,t}(X) = H^{s,t}\,\mathrm{BP}_*(X) \Rightarrow \pi_{t-s}(X). \tag{1.2}$$

Hereafter we abbreviate as

$$H^{s,t} M = \mathrm{Ext}^{s,t}_{\mathrm{BP}_*(\mathrm{BP})}(\mathrm{BP}_*, M)$$

for a $\mathrm{BP}_*(\mathrm{BP})$–comodule M over the Hopf algebroid

$$(\mathrm{BP}_*, \mathrm{BP}_*(\mathrm{BP})) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], \mathrm{BP}_*[t_1, t_2, \dots]) \tag{1.3}$$

based on the Brown–Peterson spectrum $\mathrm{BP} \in \mathcal{S}_{(p)}$. We note that the v_i are Hazewinkel's generators and the degrees of v_i and t_i are $|v_i| = 2p^i - 2 = |t_i|$; see Miller, Ravenel and Wilson [2, (1.1)].

Let

$$I_n = (p, v_1, \dots, v_{n-1}) \quad \text{and} \quad J_j = (p, v_1, v_2^j) \tag{1.4}$$

for $v_0 = p$ denote the invariant ideals of BP_*. Since $\mathrm{BP}_*(\alpha) = v_1$, the cofiber sequences (1.1) induce the short exact sequences

$$0 \to \mathrm{BP}_* \xrightarrow{p} \mathrm{BP}_* \xrightarrow{i_*} \mathrm{BP}_*/I_1 \to 0 \quad \text{and} \quad 0 \to \mathrm{BP}_*/I_1 \xrightarrow{v_1} \mathrm{BP}_*/I_1 \xrightarrow{(i_1)_*} \mathrm{BP}_*/I_2 \to 0 \tag{1.5}$$

along with the isomorphisms

$$\mathrm{BP}_*(S) = \mathrm{BP}_*, \quad \mathrm{BP}_*(M) = \mathrm{BP}_*/I_1 \quad \text{and} \quad \mathrm{BP}_*(V(1)) = \mathrm{BP}_*/I_2.$$

Furthermore, we have a short exact sequence

$$0 \to \mathrm{BP}_*/I_2 \xrightarrow{v_2^j} \mathrm{BP}_*/I_2 \xrightarrow{\bar{i}_j} \mathrm{BP}_*/J_j \to 0 \tag{1.6}$$

for $j \geq 1$. We denote by

$$\delta_0 \colon H^s\mathrm{BP}_*/I_1 \to H^{s+1}\mathrm{BP}_*, \quad \delta_1 \colon H^s\mathrm{BP}_*/I_2 \to H^{s+1}\mathrm{BP}_*/I_1, \quad \bar{\delta}_j \colon H^s\mathrm{BP}_*/J_j \to H^{s+1}\mathrm{BP}_*/I_2$$

the connecting homomorphisms associated to the short exact sequences (1.5) and (1.6). We define the Greek letter elements by

$$\begin{aligned} \bar{\beta}'_s &= \delta_1(v_2^s) \in E_2^1(M) = H^1\,\mathrm{BP}_*/I_1 && \text{for } v_2^s \in H^0\,\mathrm{BP}_*/I_2, \\ \bar{\beta}_s &= \delta_0\delta_1(v_2^s) \in E_2^2(S) = H^2\,\mathrm{BP}_* && \text{for } v_2^s \in H^0\,\mathrm{BP}_*/I_2, \\ \bar{\gamma}''_{s/j} &= \bar{\delta}_j(v_3^s) \in E_2^1(V(1)) = H^1\,\mathrm{BP}_*/I_2 && \text{for } v_3^s \in H^0\,\mathrm{BP}_*/J_j, \end{aligned}$$

and $\bar{\gamma}''_s = \bar{\gamma}''_{s/1} \in E_2^1(V(1))$. We notice that $1 \leq j \leq p^n$ if $p^n \mid s$, so that $v_3^s \in H^0\,\mathrm{BP}_*/J_j$.

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all integers and its subset consisting of all nonnegative integers, respectively. We denote by $\mathbb{Z}^{(p)}$ $(= \mathbb{Z} \setminus p\mathbb{Z})$ and $\mathbb{N}^{(p)}$ $(= \mathbb{N} \setminus p\mathbb{N})$ the sets of the integers prime to p, and decompose $\mathbb{Z}^{(p)}$ into the three summands

$$\mathbb{Z}^{(p)} = \mathbb{Z}_0 \coprod \mathbb{Z}_1 \coprod \mathbb{Z}_2,$$

for

$$\begin{gathered} \mathbb{Z}_0 = \{s \in \mathbb{Z}^{(p)} : p \nmid (s+1)\}, \quad \mathbb{Z}_1 = \{s \in \mathbb{Z}^{(p)} : p^2 \mid (s+1)\}, \\ \mathbb{Z}_2 = \{s \in \mathbb{Z}^{(p)} : p \mid (s+1) \text{ and } p^2 \nmid (s+1)\}. \end{gathered} \tag{1.7}$$

We consider subsets of $\mathbb{N}$:

$$\begin{aligned} 2\mathbb{N}_{>0} &= \{s \in \mathbb{N} : s \text{ is even} \geq 2\}, & \mathbb{N}_1 &= \{s \in \mathbb{N}^{(p)} : p^2 \nmid (s+p+1) \text{ or } p^3 \mid (s+p+1)\}, \\ \overline{2\mathbb{N}} &= \{s \in \mathbb{N} : s \text{ is odd}\}, & \mathbb{N}_2 &= \{s \in \mathbb{N}^{(p)} : p \nmid (s+2) \text{ or } p^3 \mid (s+2)(s+2+p)\}. \end{aligned}$$

Furthermore, we put $\mathbb{Z}_i^+ = \mathbb{Z}_i \cap \mathbb{N}$ for $i = 0, 1, 2$. We introduce the subsets U_1, U'_1, U_2 and U'_2 of $\mathbb{N}^{(p)} \times \mathbb{N}$ given by

$$\begin{aligned} U_1 &= (\mathbb{N}^{(p)} \times 2\mathbb{N}) \cup (\mathbb{Z}_0^+ \times \mathbb{N}), \\ U'_1 &= (\mathbb{N}^{(3)} \times \{0\}) \cup (\mathbb{N}_1 \times 2\mathbb{N}_{>0}) \cup ((\mathbb{Z}_0^+ \cap \mathbb{N}_2) \times \mathbb{N}) \cup (\mathbb{Z}_0^+ \times \{1\}), \\ U_2 &= (\mathbb{N}_1 \times 2\mathbb{N}) \cup (((\mathbb{Z}_0^+ \cap \mathbb{N}_2) \cup \mathbb{Z}_1^+) \times \mathbb{N}) \cup (\mathbb{N}^{(p)} \times \{1\}), \\ U'_2 &= (\mathbb{N}_1 \times \{0\}) \cup (\mathbb{N}^{(3)} \times (\{1\} \cup 2\mathbb{N}_{>0})) \cup ((\mathbb{Z}_0^+ \cup \mathbb{Z}_1^+) \times \mathbb{N}). \end{aligned}$$

Our main result is the following:

Theorem 1.8 *Let p be an odd prime. In the Adams–Novikov E_2–term for computing $\pi_*(V(1))$, $\bar\beta_1$ and $\bar\beta_2$ act on the gamma elements $\bar\gamma''_{sp^r/j}$ (for $(s,r)\in\mathbb{N}^{(p)}\times\mathbb{N}$ and $1\le j\le p^r$) by*

$$\bar\gamma''_{sp^r/j}\bar\beta_1\ne 0 \quad \text{for } (s,r)\in U_1 \text{ if } p\ge 5, \text{ and for } (s,r)\in U'_1 \text{ if } p=3,$$
$$\bar\gamma''_{sp^r/j}\bar\beta_2\ne 0 \quad \text{for } (s,r)\in U_2 \text{ if } p\ge 5, \text{ and for } (s,r)\in U'_2 \text{ if } p=3,$$

in $E_2^3(V(1))$.

There is a way to define $\gamma''_{sp^r/j}$ for $j\le a_r$ (a_r is defined in (2.7)) so that $v_2^{j-1}\gamma''_{sp^r/j}=\gamma''_{sp^r}$, and the theorem holds for such extended gamma elements. Also $\bar\beta_s\equiv\binom{s}{2}v_2^{s-2}\bar\beta_2+s(2-s)v_2^{s-1}\bar\beta_1 \bmod I_2$ (see Oka and Shimomura [5, Lemma 4.4]), and so

$$\bar\gamma''_{sp^r/j}\bar\beta_t=\binom{t}{2}\bar\gamma''_{sp^r/j-t+2}\bar\beta_2+t(2-t)\bar\gamma''_{sp^r/j-t+1}\bar\beta_1.$$

Thus Theorem 1.8 implies nontriviality of the products of $\bar\gamma''_{sp^r/j}$ and $\bar\beta_t$.

The Adams–Novikov differential d_r is 0 if $q\nmid(r-1)$ by the sparseness of the spectral sequence (1.2). This shows that the products in the theorem are not in the image of any differentials d_r. It is well known that the elements $\bar\beta_1$ and $\bar\beta_2$ converge to the homotopy elements $\beta_1,\beta_2\in\pi_*(S)$, respectively, in the spectral sequence (1.2) for $X=S$.

Corollary 1.9 *Let p be an odd prime. If $\bar\gamma''_{sp^r/j}\in E_2^1(V(1))$ is a permanent cycle detecting $\gamma''_{sp^r/j}\in\pi_*(V(1))$, then $\gamma''_{sp^r/j}\beta_i\ne 0$ for $i=1,2$ in the homotopy groups $\pi_*(V(1))$ for (s,r) given in Theorem 1.8.*

Toda [12, Theorem 1] and Oka [4, Theorem 4.2] showed that γ''_s and $\gamma''_{sp/2}$ are permanent cycles for $p\ge 7$.

Corollary 1.10 *Let $p\ge 7$, and r and s be integers with $(s,r)\in\mathbb{N}^{(p)}\times\mathbb{N}$. Then, in $\pi_*(V(1))$,*

$$\gamma''_{sp^r/j}\beta_1\ne 0 \quad \text{if } r \text{ is even or } p\nmid(s+1),$$
$$\gamma''_{sp^{2r}/j}\beta_2\ne 0 \quad \text{if } p^2\nmid(s+p+1) \text{ or } p^3\mid(s+p+1),$$
$$\gamma''_{sp^{2r+1}/j}\beta_2\ne 0 \quad \text{for } r\ge 1 \text{ if } p\nmid(s+1)(s+2),\ p^2\mid(s+1) \text{ or } p^3\mid(s+2)(s+2+p),$$

and $\gamma''_{sp/j}\beta_2\ne 0$, where $j=1,2$.

Theorem 1.8 follows from Theorem 2.9, which states the structure of the first cohomology of the monochromatic comodule M_2^1. The cohomology $H^1M_2^1$ was determined by the first author [10] based on the computation in [9] at a prime ≥ 5. Here we determine the cohomology based on elementary calculation at an odd prime. The generators are explicitly given so that we can use the result easily in further computation. This result will be a stepping stone for determining the long-desired cohomology $H^*M_0^3$.

This paper is organized as follows: In Section 2, we state the main result, Theorem 2.9, which gives the structure of $H^1M_2^1$. In Section 3, we prove Theorems 2.9 and 1.8 assuming Lemma 3.4, whose proof will be given in the Section 6. Section 4 is devoted to introducing some formulas, cochains and relations for the following sections. We refine the elements $x_{3,i}$ given by Miller, Ravenel and Wilson [2, (5.11)] to define x_i, which induce the cochains $y_{s,i}, y'_{s,i}\in\Omega^1E(3)_*$ in Section 5.

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2 The structure of $H^1M_2^1$

In this section, we state the structure of $H^1M_2^1$ for an odd prime p. The structure was given in [10] for primes $p \geq 5$.

We begin with defining the monochromatic $\mathrm{BP}_*(\mathrm{BP})$–comodules N_n^s and M_n^s inductively by

$$N_n^0 = \mathrm{BP}_*/I_n, \quad M_n^s = v_{s+n}^{-1}N_n^s,$$

for the ideal I_n in (1.4) and the short exact sequence

$$0 \to N_n^s \xrightarrow{\iota_n^s} M_n^s \xrightarrow{\kappa_n^s} N_n^{s+1} \to 0 \tag{2.1}$$

see [2, Section 3.A]. Since BP_* is a $\mathrm{BP}_*(\mathrm{BP})$–comodule with structure map η_R, the right unit map of the Hopf algebroid $\mathrm{BP}_*(\mathrm{BP})$, these monochromatic comodules have the structure maps induced from η_R.

Let $E(3)$ denote the third Johnson–Wilson spectrum, which yields a Hopf algebroid

$$\big(E(3)_*, E(3)_*(E(3))\big) = (\mathbb{Z}_{(p)}[v_1, v_2, v_3, v_3^{-1}], E(3)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*(\mathrm{BP}) \otimes_{\mathrm{BP}_*} E(3)_*).$$

Its structure maps are induced from the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*(\mathrm{BP}))$ in (1.3). Since we have the Miller–Ravenel change of rings theorem

$$H^*M = \mathrm{Ext}^*_{\mathrm{BP}_*(\mathrm{BP})}(\mathrm{BP}_*, M) \cong \mathrm{Ext}^*_{E(3)_*(E(3))}(E(3)_*, E(3)_* \otimes_{\mathrm{BP}_*} M)$$

for a v_3–local $\mathrm{BP}_*(\mathrm{BP})$–comodule M [1, Theorem 3.10], we denote the cohomology of an $E(3)_*(E(3))$–comodule M also by

$$H^sM = \mathrm{Ext}^s_{E(3)_*(E(3))}(E(3)_*, M).$$

By virtue of the change of rings theorem, we denote simply by M_n^s the $E(3)_*(E(3))$–comodule $E(3)_* \otimes_{\mathrm{BP}_*} M_n^s$. We consider the Ext group as the cohomology group of the cobar complex

$$\Omega^sM = M \otimes_{E(3)_*} E(3)_*(E(3)) \otimes_{E(3)_*} \cdots \otimes_{E(3)_*} E(3)_*(E(3)) \tag{2.2}$$

for s factors of $E(3)_*(E(3))$, with well-known differentials $d_r : \Omega^rM \to \Omega^{r+1}M$; see (4.1).

The cohomology $H^tM_n^s$ of the monochromatic comodules with $s+n=3$ are determined in the following cases (see [8, Theorems 6.3.12 and 6.3.14; 2, Theorem 5.10]):

$$\begin{aligned} H^0M_3^0 &= K(3)_*, \\ H^1M_3^0 &= K(3)_*\{h_0, h_1, h_2, \zeta_3\}, \\ H^2M_3^0 &= K(3)_*\{g_i, k_i, b_i, h_i\zeta_3 : i \in \mathbb{Z}/3\}, \\ H^0M_2^1 &= K(2)_*/k(2)_* \oplus \bigoplus_{i\geq 0, s\in\mathbb{Z}^{(p)}} k(2)_*/(v_2^{a_i})\{x_i^s/v_2^{a_i}\}. \end{aligned} \tag{2.3}$$

Indeed, we read off $H^s M_3^0 = K(3)_* \otimes H^s S(3)$ from [8, Proposition 6.2.1], where $S(3)$ is the Hopf algebra defined in [8, Section 6.2]. The cohomology groups $H^* M_3^0$ and $H^0 M_1^2$ for $p \geq 5$ are also determined by Ravenel [8, Theorem 6.3.34] and Nakai [3], respectively. Here

$$k(2)_* = \mathbb{Z}/p[v_2], \quad K(2)_* = \mathbb{Z}/p[v_2, v_2^{-1}] \quad \text{and} \quad K(3)_* = \mathbb{Z}/p[v_3, v_3^{-1}],$$

where $K(3)_* = E(3)_*/I_3 = M_3^0$. The elements x_i $(= x_{3,i})$ are introduced in [2, (5.11)] and are such that $x_i \equiv v_3^{p^i}$ mod I_3 (see Lemma 5.1), and the generators h_i, ζ_3, g_i, k_i and b_i are defined by cocycles in the cobar complex $\Omega^* E(3)_*/I_3$ as follows:

(2.4) $$h_i = [t_1^{p^i}], \quad \zeta_3 = [Z], \quad g_i = [G_i], \quad k_i = [K_i] \quad \text{and} \quad b_i = [b_{1,i}].$$

Hereafter $[x]$ denotes the cohomology class represented by a cocycle x, and the representatives in (2.4) are defined by

(2.5) $$\begin{aligned} Z &= -v_3^{-1} ct_3 + v_3^{-p} t_3^p + v_3^{-p^2} t_3^{p^2} - v_3^{-p} t_1^p t_2^{p^2}, &\quad G_i &= t_1^{p^i} \otimes t_2^{p^i} + \tfrac{1}{2} t_1^{2p^i} \otimes t_1^{p^{i+1}}, \\ K_i &= t_2^{p^i} \otimes t_1^{p^{i+1}} + \tfrac{1}{2} t_1^{p^i} \otimes t_1^{2p^{i+1}}, &\quad b_{1,i} &= \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} t_1^{kp^i} \otimes t_1^{(p-k)p^i}. \end{aligned}$$

Here ct_3 is the Hopf conjugation of t_3 (see Lemma 4.3). We notice that G_i, K_i and $b_{1,i}$ are also cocycles of $\Omega^* E(3)_*/I_2$, and of $\Omega^* \mathrm{BP}_*/I_2$ in [2, (1.9)].

Remark 2.6 The generators g_i and k_i in (2.3) are given by the Massey products $\langle h_i, h_{i+1}, h_i \rangle$ and $\langle h_{i+1}, h_{i+1}, h_i \rangle$, respectively, in [8, Theorem 6.3.4]. These are represented by cocycles

$$G_i'' = t_2^{p^i} \otimes t_1^{p^{i+1}} + t_1^{p^i} \otimes ct_2^{p^i}$$

and K_i' in (4.20) in the cobar complex $\Omega^* E(3)_*/I_2$, since these Massey products have no indeterminacy. By (4.21), K_i' is homologous to K_i. Also $d_1(t_1^{p^i} t_2^{p^i}) = -2G_i - G_i''$, and G_i'' is homologous to $-2G_i$. Since p is odd, we may replace generators g_i and k_i by $[G_i]$ and $[K_i]$, and set as in (2.4).

We introduce integers $e(n)$, a_n, $j_{s,n}$ and $j'_{s,n}$ for integers $n(\geq 0)$ and s by

(2.7) $$\begin{aligned} e(n) &= \frac{p^n - 1}{p - 1} \quad \text{for } n \geq 0, \\ a_n &= \begin{cases} 1 & \text{for } n = 0, \\ p^n + (p^{n-1} - 1)/(p+1) & \text{for odd } n \geq 1, \\ p^n + p(p^{n-2} - 1)/(p+1) & \text{for even } n \geq 2, \end{cases} \\ j_{s,n} &= \begin{cases} 2 & \text{for } s \in \mathbb{Z}_0 \text{ and } n = 0, \\ 2p^2 - p + 1 & \text{for } s \in \mathbb{Z}_0 \text{ and } n = 2, \\ 2a_n + \bar{1} & \text{for } s \in \mathbb{Z}_0, \text{ even } n \geq 4, \\ a_{n+2} - a_{n+1} & \text{for } s \in \mathbb{Z}_1 \text{ and even } n \geq 0, \\ p + 1 & \text{for } s \in \mathbb{Z}^{(p)} \text{ and } n = 1, \\ e(3)p^{n-2} - p + 1 & \text{for } s \in \mathbb{Z}^{(p)} \text{ and odd } n \geq 3, \end{cases} \end{aligned}$$

$$j'_{s,0} = \begin{cases} 2 & \text{for } p \nmid s(s-1), \\ 2p & \text{for } s = tp+1 \text{ and } p \nmid t(t-1), \\ p^2+1 & \text{for } s = tp^2+1 \text{ and } p \nmid t, \\ a_n + p & \text{for } s = tp^n+1 \text{ with } n > 2 \text{ and } p \nmid t, \\ a_n + 1 & \text{for } s = tp^n + e(n) \text{ with even } n \geq 2 \text{ and } p \nmid (t-1), \\ a_n + 2 & \text{for } s = tp^n + e(n) \text{ with odd } n > 2 \text{ and } p \nmid (t-1), \end{cases}$$

$$j'_{s,n} = \begin{cases} 2p & \text{for } s \in \mathbb{Z}_0 \text{ and } n = 1, \\ 2pa_{n-1} + p & \text{for } s \in \mathbb{Z}_0 \text{ and odd } n \geq 3, \\ pa_{n+1} - pa_n & \text{for } s \in \mathbb{Z}_1 \text{ and odd } n \geq 1, \\ p^2 + p & \text{for } s \in \mathbb{Z}^{(p)} \text{ and } n = 2, \\ e(3)p^{n-2} - 1 + \bar{1} & \text{for } s \in \mathbb{Z}^{(p)} \text{ and even } n \geq 4. \end{cases}$$

Here $\bar{1} = 0$ if $p \geq 5$ and 1 if $p = 3$, the $\mathbb{Z}_i$ are the subsets of the integers $\mathbb{Z}$ defined in (1.7), and the integers a_n are $a_{3,n}$ in [2, (5.13)]. We note that

(2.8) $$a_n + a_{n-1} = e(3)p^{n-2} - 1 \quad \text{for } n \geq 2 \qquad \text{and} \qquad p^n + a_{n-2} - p^{n-3} = a_n \quad \text{for } n \geq 3.$$

Theorem 2.9 *Let p be an odd prime. Then $H^1M_2^1$ is the direct sum of the $k(2)_*$–module $B_\infty = K(2)_*/k(2)_*\{h_0, h_1, \tilde{\zeta}_2, \zeta_3\}$ and the $k(2)_*$–cyclic modules generated by*

$$\begin{aligned} &(\zeta_3)_{sp^n/a_n} && \text{for } (s,n) \in \mathbb{Z}^{(p)} \times \mathbb{N}, \\ &(h_0)_{sp^n/j_{s,n}} && \text{for } (s,n) \in ((\mathbb{Z}_0 \cup \mathbb{Z}_1) \times 2\mathbb{N}) \cup (\mathbb{Z}^{(p)} \times \overline{2\mathbb{N}}), \\ &(h_1)_{sp^n/j'_{s,n}} && \text{for } (s,n) \in ((\mathbb{Z}_0 \cup \mathbb{Z}_1) \times \overline{2\mathbb{N}}) \cup ((\mathbb{Z}^{(p)} \times 2\mathbb{N}) \setminus \{(1,0)\}), \\ &(h_2)_{tp-1/p-1} && \text{for } t \in \mathbb{Z}. \end{aligned}$$

There is a little difference between the cases for $p \geq 5$ and $p = 3$. In the theorem, $\tilde{\zeta}_2$ $(= (h_1)_1)$ denotes the homology class of z in (4.18) (see also (3.8)), and the generators $(\xi)_{s/j}$ for $\xi = [X]$ in $H^1M_3^0$ denote

$$(\xi)_{s/j} = [v_3^s X / v_2^j + *]$$

for a cocycle $v_3^s X / v_2^j + *$ of the cobar complex $\Omega^1 M_2^1$ with an element $*$ killed by v_2^{j-1}. The element v_2 acts on $(\xi)_{s/j}$ by

(2.10) $$v_2(\xi)_{s/j} = (\xi)_{s/j-1} \quad \text{and} \quad v_2(\xi)_{s/1} = 0,$$

and so $(\xi)_{s/j}$ generates a cyclic $k(2)_*$–module isomorphic to $k(2)_*/(v_2^j)$:

$$k(2)_*\{(\xi)_{s/j}\} \cong k(2)_*/(v_2^j).$$

3 Proofs of Theorems 2.9 and 1.8

In this section, we assume Lemma 3.4, which will be verified by a routine calculation in Section 6.

3.1 Proof of Theorem 2.9

For the monochromatic comodules defined in Section 2, we have a short exact sequence

(3.1)
$$0 \to M_3^0 \xrightarrow{\eta} M_2^1 \xrightarrow{v_2} M_2^1 \to 0,$$

where $\eta(x) = x/v_2$ (see [2, (3.10)]), which induces the long exact sequence

(3.2)
$$\cdots \to H^0 M_2^1 \xrightarrow{\delta_0} H^1 M_3^0 \xrightarrow{\eta_*} H^1 M_2^1 \xrightarrow{v_2} H^1 M_2^1 \xrightarrow{\delta_1} H^2 M_3^0 \to \cdots.$$

From [2, (5.18)], we read off the following:

Proposition 3.3 *The cokernel of $\delta_0\colon H^0 M_2^1 \to H^1 M_3^0$ is a $\mathbb{Z}/p$–module generated by $(h_0)_0$, $(h_1)_0$,*

$$(h_0)_{sp^{2k}} \text{ for } s \in \mathbb{Z}_0 \cup \mathbb{Z}_1, \quad (h_0)_{tp^{2k+1}} \text{ for } t \in \mathbb{Z}^{(p)}, \quad (h_1)_{tp^{2k}} \text{ for } t \in \mathbb{Z}^{(p)},$$
$$(h_1)_{sp^{2k+1}} \text{ for } s \in \mathbb{Z}_0 \cup \mathbb{Z}_1, \quad (h_2)_{tp-1} \text{ for } t \in \mathbb{Z}, \quad (\zeta_3)_t \text{ for } t \in \mathbb{Z},$$

for $k \geq 0$. Here $\mathbb{Z}_i$ is a subset of $\mathbb{Z}$ given in (1.7), and $(\xi)_s = v_3^s \xi$ for $\xi \in \{h_i, \zeta_3 : i \in \mathbb{Z}/3\}$.

Let $(x)_s \in \Omega^1 E(3)_*$ denote a cochain satisfying

$$(x)_s \equiv v_3^s x \mod I_3.$$

Lemma 3.4 *The following cochains exist in $\Omega^1 E(3)_*/I_2$:*

(1) $(t_1)_{sp^{2k}}$ *and* $(t_1^p)_{sp^{2k+1}}$ *for $s \in \mathbb{Z}_0$ such that*

$$d_1((t_1)_{sp^{2k}}) \equiv \begin{cases} s(s+1)v_2^2 v_3^{s-1-p} G_2 & k = 0, \\ s(s+1)v_2^{2p^2-p+1} v_3^{sp^2-2p} G_1 & k = 1, \\ -3s(s+1)v_2^{2a_{2k}} v_3^{(sp-2)p^{2k-1}} K_0 & k \geq 2,\ p \geq 5, \\ -2s(s+1)v_2^{2a_{2k}+1} v_3^{3^{2k-1}(3s-2)}(b_{1,0} + t_1^p \otimes Z') & k \geq 2,\ p = 3, \end{cases}$$

$$d_1((t_1^p)_{sp^{2k+1}}) \equiv \begin{cases} s(s+1)v_2^{2p} v_3^{sp-2} G_0 & k = 0, \\ s(s+1)v_2^{2pa_{2k}+p} v_3^{(sp-2)p^{2k}} b_{1,1} & k \geq 1, \end{cases}$$

(2) $(t_1)_{sp^{2k}}$ *and* $(t_1^p)_{sp^{2k+1}}$ *for $s = tp^2 - 1 \in \mathbb{Z}_1$ such that*

$$d_1((t_1)_{sp^{2k}}) \equiv v_2^{a_{2k+2}-a_{2k+1}} v_3^{(tp-1)p^{2k+1}} b_{1,0}, \quad d_1((t_1^p)_{sp^{2k+1}}) \equiv v_2^{pa_{2k+2}-pa_{2k+1}} v_3^{(tp-1)p^{2k+2}} b_{1,1},$$

for $k \geq 0$,

(3) $(t_1)_{sp^{2k+1}}$ *and* $(t_1^p)_{sp^{2k}}$ *for $s \in \mathbb{Z}^{(p)}$ such that*

$$d_1((t_1^p)_{tp^k+1}) \equiv \begin{cases} t(t-1)v_2^{2p} v_3^{tp-1} G_0 & k = 1, \\ -tv_2^{p^2+1} v_3^{(tp-1)p} G_1 & k = 2, \\ -2tv_2^{a_k+p} v_3^{(tp-1)p^{k-1}} G_0 & \text{odd } k \geq 3, \\ 2tv_2^{a_k+p} v_3^{(tp-1)p^{k-1}} K_0 & \text{even } k \geq 4, \end{cases}$$

$$d_1((t_1^p)_{tp^k+e(k)}) \equiv \begin{cases} -(t-1)v_2^{a_k+1} v_3^{tp^k+pe(k-2)} G_1 & \text{even } k \geq 2, \\ -(t-1)v_2^{a_k+2} v_3^{tp^k+pe(k-2)} b_{1,1} & \text{odd } k \geq 3, \end{cases}$$

$$d_1((t_1^p)_{sp^{2k}}) \equiv \begin{cases} s(s-1)v_2^2 v_3^{s-2} K_1 & k=0, \\ -sv_2^{p^2+p} v_3^{sp^2-p-1} K_0 & k=1, \\ -3sv_2^{e(3)p^{2k-2}-1} v_3^{(sp^2-p-1)p^{2k-2}} K_0 & p \geq 5, k \geq 2, \\ -sv_2^{3^{2k-2}e(3)} v_3^{(9s-4)3^{2k-2}} (b_{1,0} + Z' \otimes t_1^p) & p=3, k \geq 2, \end{cases}$$

$$d_1((t_1)_{sp^{2k+1}}) \equiv \begin{cases} -sv_2^{p+1} v_3^{(s-2)p} K_2 & k=0, \\ sv_2^{e(3)p^{2k-1}-p+1} v_3^{(sp^2-p-1)p^{2k-1}} b_{1,1} & k \geq 1, \end{cases}$$

(4) $(t_1^{p^2})_{tp-1}$ *such that* $d_1((t_1^{p^2})_{tp-1}) \equiv v_2^{p-1} v_3^{tp-p} b_{1,2}$.

Here G_i, K_i *and* $b_{1,i}$ *are the cocycles of* $\Omega^2 E(3)_*/I_2$ *in* (2.5), Z' *is an element in Lemma 5.1, and* $x \equiv v_2^a y$ *denotes the congruence modulo* J_{a+1}.

Let $d_1((x)_t) \equiv v_2^j y \mod J_{j+1}$ be a congruence in Lemma 3.4. Then $\delta_1(([x])_{t/j}) = [y]$ for the connecting homomorphism δ_1 in (3.2). Here $([x])_{t/j}$ $(= [(x)_t/v_2^j]) \in H^1 M_2^1$ denotes the cohomology class of the cocycle $(x)_t/v_2^j$ of $\Omega^1 M_2^1$. Thus the cochains in Lemma 3.4 give rise to elements $(h_0)_{sp^r/j_{s,r}}$ and $(h_1)_{sp^r/j'_{s,r}}$ of $H^1 M_2^1$ as well as their δ_1–images. Furthermore, we have elements

$$(\zeta_3)_{tp^n/a_n} = x_n^t \zeta_3 / v_2^{a_n} \in H^1 M_2^1$$

for the elements x_n $(= x_{3,n})$ introduced in [2, (5.11)] (see Lemma 5.1) with

(3.5)
$$\delta_1((\zeta_3)_{tp^n/a_n}) = \begin{cases} (h_2\zeta_3)_{t-1} & n=0, \\ (h_0\zeta_3)_{(tp-1)p^{n-1}} & n \text{ is odd}, \\ (h_1\zeta_3)_{(tp-1)p^{n-1}} & n \text{ is even } \geq 2, \end{cases}$$

by [2, (5.18)] (or Lemma 5.1). As a $k(2)_*$–module, $K(2)_*/k(2)_*\{\xi\} = \mathbb{Z}/p\{(\xi)_{0/j} : j \geq 1\}$ with $v_2(\xi)_{0/j} = (\xi)_{0/j-1}$ and $v_2(\xi)_{0/1} = 0$; see (2.10).

Let B be the $k(2)_*$–module of the theorem. Each direct summand of B is a submodule of $H^1 M_2^1$, which defines a $k(2)_*$–module map $f : B \to H^1 M_2^1$. Furthermore, assigning $(\xi)_{s/1} \in B$ to the generator $(\xi)_s$ of the cokernel of δ_0, we have a homomorphism $\bar{\eta}_* : H^1 M_3^0 \to B$ by Proposition 3.3. These homomorphisms fit in the commutative diagram

$$\begin{array}{ccccccccc} H^0 M_2^1 & \xrightarrow{\delta_0} & H^1 M_3^0 & \xrightarrow{\bar{\eta}_*} & B & \xrightarrow{v_2} & B & \xrightarrow{\delta'_1} & H^2 M_3^0 \\ \| & & \| & & \downarrow f & & \downarrow f & & \| \\ H^0 M_2^1 & \xrightarrow{\delta_0} & H^1 M_3^0 & \xrightarrow{\eta_*} & H^1 M_2^1 & \xrightarrow{v_2} & H^1 M_2^1 & \xrightarrow{\delta_1} & H^2 M_3^0 \end{array}$$

where we define δ'_1 by $\delta_1 f$. It suffices to show that the upper sequence is exact by [2, Remark 3.11]. By the definition of B, the subsequence $H^0 M_2^1 \xrightarrow{\delta_0} H^1 M_3^0 \xrightarrow{\bar{\eta}_*} B \xrightarrow{v_2} B$ is exact and the composite $B \xrightarrow{v_2} B \xrightarrow{\delta'_1} H^2 M_3^0$ is zero.

Suppose that the δ_1'–images of the generators are linearly independent, and take $\xi \in \operatorname{Ker}\delta_1'$ to be a homogeneous element. Then

$$\xi = \sum_k c_k \xi_k \quad \text{for generators } \xi_k \text{ of } B \text{ and scalars } c_k \in k(2)_* \qquad \text{and} \qquad 0 = \delta_1'(\xi) = \sum_k \bar{c}_k \delta_1'(\xi_k)$$

for the image $\bar{c}_k$ of c_k under the projection $k(2)_* \to \mathbb{Z}/p$ sending v_2 to zero. Since the $\delta_1'(\xi_k)$ are linearly independent we see $\bar{c}_k = 0$, and so we have $c_k' \in k(2)_*$ such that $c_k = v_2 c_k'$. Therefore

$$\xi = \sum_k v_2 c_k' \xi_k \in \operatorname{Im} v_2,$$

and we see that the upper sequence of the above diagram is exact if the δ_1'–images of the generators are linearly independent.

The δ_1'–image is a $\mathbb{Z}/p$–submodule of $H^2M_3^0$ in (2.3) with generators of the form $(\rho)_s$ for $\rho \in \{g_i, k_i, b_i, h_i\zeta_3 : i \in \mathbb{Z}/3\}$ by Lemma 3.4 and (3.5). Moreover, Lemma 3.4 and (3.5) show that the δ_1'–image of each generator ξ_k has only one summand of form $(\rho)_s$,

$$(h_0\zeta_3)_{(tp-1)p^{2n}}, \quad (h_1\zeta_3)_{(tp-1)p^{2n-1}}, \quad (h_2\zeta_3)_{t-1}, \quad (g_2)_{s-1-p},$$
$$(k_1)_{s-2}, \quad (k_2)_{(s-2)p}, \quad (b_0)_{(tp-1)p^{2n+1}} \text{ for } p \ge 5, \quad (b_2)_{tp-p},$$

except for

g_0	$(g_0)_{sp-2}$	$(g_0)_{(tp-1)p^{2n}}$		
g_1	$(g_1)_{(sp-2)p}$	$(g_1)_{(tp-1)p}$	$(g_1)_{tp^{2n}+pe(2n-2)}$	
k_0	$(k_0)_{(sp-2)p^{2n-1}}$	$(k_0)_{(tp-1)p^{2n-1}}$	$(k_0)_{(sp^2-p-1)p^{2n}}$	$(p \ge 5)$
k_0	$(k_0)_{3^{2n-1}(3t-1)}$	$(k_0)_{9s-4}$		$(p=3)$
b_0	$(b_0)_{3^{2n-1}(3s-2)}$	$(b_0)_{3^{2n+1}(3t-1)}$	$(b_0)_{3^{2n-2}(9s-4)}$	$(p=3)$
b_1	$(b_1)_{(sp-2)p^{2n}}$	$(b_1)_{(tp-1)p^{2n+2}}$	$(b_1)_{tp^{2n+1}+pe(2n-1)}$	$(b_1)_{(sp^2-p-1)p^{2n-1}}$

These show that the δ_1'–images $\delta_1'(\xi_k)$ for the generators ξ_k of B are different from each other, and so they are linearly independent. □

3.2 Proof of Theorem 1.8

Let $\delta_2^0\colon H^*N_2^1 \to H^{*+1}N_2^0$ be the connecting homomorphism associated to the short exact sequence (2.1), and consider the diagram

$$\begin{array}{ccccc} & & H^2M_2^0 \xrightarrow{(\kappa_2^0)_*} & H^2N_2^1 & \xrightarrow{\delta_2^0} H^3N_2^0 = E_2^3(V(1)) \\ & & & \downarrow{\scriptstyle \iota_2^1} & \\ H^1M_2^1 \xrightarrow{\delta_1} & H^2M_3^0 & \xrightarrow{\eta_*} & H^2M_2^1 & \end{array}$$

of exact sequences for δ_1 in (3.2). The connecting homomorphism $\bar{\delta}_j$ associated to (1.6) is factorized into the composite $\bar{\delta}_j : H^s\mathrm{BP}_*/J_j \xrightarrow{\hat{\iota}_j} H^sN_2^1 \xrightarrow{\delta_2^0} H^{s+1}N_2^0$ for the homomorphism $\hat{\iota}_j$ given by $\hat{\iota}_j(x) = x/v_2^j$. It follows that

$$\textbf{(3.6)} \qquad \bar{\gamma}''_{sp^r/j} = \delta_2^0(v_3^{sp^r}/v_2^j) \in H^1N_2^0 = E_2^1(V(1)) \quad \text{for } v_3^{sp^r}/v_2^j \in H^0N_2^1.$$

Since δ_2^0 is a $k(2)_*$–module map, we have

$$v_2^{j-1}\overline{\gamma}''_{sp^r/j} = v_2^{j-1}\delta_2^0(v_3^{sp^r}/v_2^j) = \delta_2^0(v_2^{j-1}v_3^{sp^r}/v_2^j) = \delta_2^0(v_3^{sp^r}/v_2) = \overline{\gamma}''_{sp^r}. \tag{3.7}$$

It is well known that

$$\bar{\beta}_1 = -b_0 = [-b_{1,0}] \quad \text{and} \quad \bar{\beta}_2 = 2k_0 = [2K_0] \in H^2N_3^0$$

for the cocycles $b_{1,0}$ and K_0 in (2.5); see [5, Lemma 4.4]. This defines elements $v_3^{sp^r}\bar{\beta}_i/v_2 \in H^2N_2^1$ for $i = 1, 2$, and

$$\delta_2^0(v_3^{sp^r}\bar{\beta}_i/v_2) = \gamma''_{sp^r}\bar{\beta}_i \in E_2^3(V(1)) \quad \text{(by (3.6))}.$$

We also see that for $v_3^{sp^r}\bar{\beta}_i \in H^2M_3^0$,

$$\eta_*(v_3^{sp^r}\bar{\beta}_i) = \iota_2^1(v_3^{sp^r}\bar{\beta}_i/v_2) \in H^2M_2^1.$$

From Lemma 3.4, the elements $v_3^{sp^r}\bar{\beta}_1 = -(b_0)_{sp^r}$ and $v_3^{sp^r}\bar{\beta}_2 = 2(k_0)_{sp^r} \in H^2M_3^0$ may be in the image of δ_1 if

$$p \geq 5 \quad \text{and} \quad (s,r) \in (\mathbb{Z}_1^+ \cup \mathbb{Z}_2^+) \times \overline{2\mathbb{N}}$$

or

$$p = 3 \quad \text{and} \quad (s,r) \in (\overline{\mathbb{N}}_1 \times 2\mathbb{N}_{>0}) \cup ((\mathbb{Z}_1^+ \cup \mathbb{Z}_2^+) \times \overline{2\mathbb{N}}) \cup (\overline{\mathbb{N}}_2 \times \overline{2\mathbb{N}}_{>1}),$$

and if

$$p \geq 5 \quad \text{and} \quad (s,r) \in (\overline{\mathbb{N}}_1 \times 2\mathbb{N}) \cup (\mathbb{Z}_2^+ \times \overline{2\mathbb{N}}_{>1}) \cup (\overline{\mathbb{N}}_2 \times \overline{2\mathbb{N}}_{>1})$$

or

$$p = 3 \quad \text{and} \quad (s,r) \in (\overline{\mathbb{N}}_1 \times \{0\}) \cup (\mathbb{Z}_2^+ \times \overline{2\mathbb{N}}_{>1}),$$

respectively. Here $\overline{\mathbb{N}}_i = \mathbb{N}^{(p)} \setminus \mathbb{N}_i$ for $i = 1, 2$. Therefore, if a pair (s,r) satisfies the condition of the theorem, then the element $v_3^{sp^r}\bar{\beta}_i$ is not in the image of δ_1, and survives to $\iota_2^1(v_3^{sp^r}\bar{\beta}_i/v_2)$ under the homomorphism η_*. Thus $v_3^{sp^r}\bar{\beta}_i/v_2 \neq 0 \in H^2N_2^1$ under the conditions.

Ravenel determined in [8, Theorem 6.3.24; 7, Theorem 3.2] that

$$H^2M_2^0 = \begin{cases} K(2)_*\{h_0\tilde{\zeta}_2, h_1\tilde{\zeta}_2, b_0, b_1, \xi\} & p = 3, \\ K(2)_*\{h_0\tilde{\zeta}_2, h_1\tilde{\zeta}_2, g_0, g_1\} & p \geq 5, \end{cases} \tag{3.8}$$

where $\tilde{\zeta}_2 = v_2^{p+1}\zeta_2 = [-z]$ for ζ_2 in [2, Proposition 3.18)] and z in (4.18). This shows that the elements $v_3^{sp^r}\bar{\beta}_i/v_2$ for $i = 1, 2$ are not in the image of $(\kappa_2^0)_*$, and hence survive to $\gamma''_{sp^r}\bar{\beta}_i \in E_2^3(V(1))$. Moreover, $\gamma''_{sp^r/j}\bar{\beta}_i \neq 0 \in E_2^3(V(1))$ if $v_2^{j-1}\gamma''_{sp^r/j}\bar{\beta}_i = \gamma''_{sp^r}\bar{\beta}_i$ is not zero, where the equality follows by (3.7). □

4 Some cochains in the cobar complex $\Omega^*E(3)_*$

In the rest of this paper, we consider $E(3)_*(E(3))$–comodules whose structure maps are induced from the right unit map $\eta_R\colon E(3)_* \to E(3)_*(E(3))$. We consider the cobar complex Ω^*M of a comodule M in (2.2), whose differentials are given by

$$d_0(v) = \eta_R(v) - v \in \Omega^1E(3)_*, \quad \text{and} \quad d_1(x) = 1 \otimes x - \Delta(x) + x \otimes 1 \in \Omega^2E(3)_* \tag{4.1}$$

for $v \in \Omega^0 E(3)_* = E(3)_*$ and $x \in \Omega^1 E(3)_* = E(3)_*(E(3))$. For the differentials d_0 and d_1, we have relations (see [11, (2.3.2)])

$$\begin{aligned} d_0(vv') &= vd_0(v') + d_0(v)\eta_R(v'), \\ d_1(vx) &= d_0(v)\otimes x + vd_1(x), \\ d_1(xy) &= -x\otimes y - y\otimes x + d_1(x)\Delta y + (x\otimes 1 + 1\otimes x)d_1(y), \\ d_1(x\eta_R(v)) &= d_1(x)(1\otimes \eta_R(v)) - x\otimes d_0(v), \end{aligned} \tag{4.2}$$

for $v, v' \in E(3)_*$ and $x, y \in E(3)_*(E(3))$. A formula for the Hopf conjugation $c\colon \mathrm{BP}_*(\mathrm{BP}) \to \mathrm{BP}_*(\mathrm{BP})$ is given in [6, (3)], and immediately implies the following:

Lemma 4.3 *The Hopf conjugation $c\colon E(3)_*(E(3)) \to E(3)_*(E(3))$ acts as*

$$ct_1 = -t_1, \quad ct_2 = t_1^{p+1} - t_2 \quad \text{and} \quad ct_3 \equiv t_2 t_1^{p^2} - t_1 ct_2^p - t_3 \mod I_2.$$

For the right unit $\eta_R\colon \mathrm{BP}_* \to \mathrm{BP}_*(\mathrm{BP})$, we have a well-known formula (see [6, (11)])

$$\eta_R(v_n) \equiv v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1 \mod I_{n-1}. \tag{4.4}$$

A routine calculation using (4.1) and (4.4) shows the following:

Lemma 4.5 *Put $\sigma_n = \sum_{k=0}^{n-1} v_2^{p^{2k}a_{2n-2k-1}-p^{2k+1}} v_3^{p^{2k}} \in E(3)_*$. Then*

$$d_0(\sigma_n) \equiv v_2^{p^{2n-2}} t_1^{p^{2n}} - v_2^{a_{2n-1}} t_1 \mod I_2.$$

In $E(3)_*(E(3))$ we have $\eta_R(v_4) = 0 = \eta_R(v_5)$, which give rise to relations

$$v_3 t_1^{p^3} \equiv t_1\eta_R(v_3)^p - v_2 t_2^{p^2} + v_2^{p^2} t_2 \quad \text{and} \quad v_3 t_2^{p^3} \equiv t_2\eta_R(v_3)^{p^2} - v_2 t_3^{p^2} - v_2 w^p + v_2^{p^3} t_3 \mod I_2 \tag{4.6}$$

(see [6, (12) and (16); 8, Corollary 4.3.21]), where $w \in E(3)_*(E(3))$ $(= w_1(v_3, v_2 t_1^{p^2}, -v_2^p t_1)$ in [8, Corollary 4.3.21]) is an element defined by

$$pw = v_3^p + v_2^p t_1^{p^3} - v_2^{p^2} t_1^p + y^p - \eta_R(v_3)^p \tag{4.7}$$

for $y \in (p, v_1)$ in $\eta_R(v_3) = v_3 + v_2 t_1^{p^2} - v_2^p t_1 + y$; see (4.4).

The diagonal $\Delta\colon E(3)_*(E(3)) \to E(3)_*(E(3)) \otimes_{E(3)_*} E(3)_*(E(3))$ of the Hopf algebroid $E(3)_*(E(3))$ acts on the elements t_i and ct_i by

$$\begin{aligned} \Delta(t_1) &= t_1\otimes 1 + 1\otimes t_1, \\ \Delta(t_2) &\equiv t_2\otimes 1 + t_1\otimes t_1^p + 1\otimes t_2 - v_1 b_{1,0} \mod (p, v_1^2), \\ \Delta(t_3) &\equiv t_3\otimes 1 + t_2\otimes t_1^{p^2} + t_1\otimes t_2^p + 1\otimes t_3 - v_2 b_{1,1} \mod I_2, \\ \Delta(t_4) &\equiv t_4\otimes 1 + t_3\otimes t_1^{p^3} + t_2\otimes t_2^{p^2} + t_1\otimes t_3^p + 1\otimes t_4 - v_3 b_{1,2} \mod I_3, \end{aligned} \tag{4.8}$$

(see [6, Theorem 8; 8, Corollary 4.3.15]) and so

(4.9)
$$\begin{aligned} d_1(ct_2) &\equiv -t_1^p \otimes t_1, \\ d_1(ct_3) &\equiv ct_2^p \otimes t_1 + t_1^{p^2} \otimes ct_2 - v_2 b_{1,1} \mod I_2, \\ d_1(ct_4) &\equiv t_1^{p^3} \otimes ct_3 - ct_2^{p^2} \otimes ct_2 + ct_3^p \otimes t_1 - v_3 b_{1,2} \mod I_3, \end{aligned}$$

since $\Delta(cx) = (c \otimes c)T\Delta(x)$ for the switching map T given by $T(x \otimes y) = y \otimes x$, where $b_{1,k}$ is the cocycle in (2.5).

The fact $d_1(t_1^{p^{k+1}}) \equiv -pb_{1,k} \mod (p^2)$ implies not only that the cochain $b_{1,k} \in \Omega^2 E(3)_*/(p)$ is a cocycle, but also the following lemma:

Lemma 4.10 *The cochain w in (4.7) satisfies*
$$w \equiv -v_2 v_3^{p-1} t_1^{p^2} \mod J_2 \quad \textit{and} \quad d_1(w) \equiv -v_2^p b_{1,2} + v_2^{p^2} b_{1,0} \mod I_2.$$

Corollary 4.11 *Put $W_n = \sum_{i=0}^{n-1} v_2^{p^{2i} a_{2n-2i} - p^{2i+2}} w^{p^{2i}}$. Then*
$$d_1(W_n) \equiv -v_2^{p^{2n-1}} b_{1,2n} + v_2^{a_{2n}} b_{1,0} \mod I_2.$$

We generalize the relations (4.6) and obtain the following proposition from [8, (4.3.1) and Lemma 4.3.11]; 6, Theorem 1] (see [9, Proposition 2.1]):

Proposition 4.12 *There exist elements T_n for $n \geq 0$ satisfying $T_n \equiv t_n^p \mod I_3$ and*
$$v_2^{p^{k+1}} t_{k+1} + t_k \eta_R(v_3)^{p^k} \equiv v_1 T_{k+2} + v_2 T_{k+1}^p + v_3 T_k^{p^2} \mod (p, v_1^2)$$
for $k \geq 0$. In particular, $T_0 = 1$, $T_1 \equiv t_1^p$, $T_2 \equiv t_2^p$ and $T_3 \equiv t_3^p + w \mod I_2$.

Proof We begin by recalling some notation from [8, Section 4.3]. For a sequence $J = (j_1, j_2, \dots, j_m)$ of positive integers we set $|J| = m$ and $\|J\| = \sum_{i=1}^m j_i$, and an element $v_J \in E(3)_*$ is defined recursively by $v_{(j,J)} = v_j v_J^{p^j}$. Let $w_k(S)$ for a set S be symmetric polynomials of degree p^n such that $w_0(S) = \sum_{x \in S} x$ and $\sum_{x \in S} x^{p^n} = \sum_{k=0}^n p^k w_k(S)^{p^{n-k}}$. We then define sets S_n out of a set $S = \{a_{i,j}\}$ recursively by
$$S_n = \{a_{i,j} : i + j = n\} \cup \bigcup_{|J|>0} \{v_J w_{|J|}(S_{n-\|J\|})^{p^{\|J\|-|J|}}\}.$$

By [8, (4.3.1) and Lemma 4.3.11],

(4.13)
$$w_0(C_n) \equiv \sum_{i+j=n}^F t_i \eta_R(v_j)^{p^i} \equiv \sum_{i+j=n}^F v_i t_j^{p^i} \equiv w_0(D_n) \mod (p)$$

for the sets
$$C = \{t_i \eta_R(v_j)^{p^i}\} \quad \text{and} \quad D = \{v_i t_j^{p^i}\}.$$

In $E(3)_*(E(3))$, put
$$w(S_n) = \sum_J v_J^p w_{|J|+1}(S_{n-\|J\|})^{p^{\|J\|-|J|}} \quad \text{and} \quad T_n = t_n^p - w(C_n) + w(D_n).$$

Then the proposition follows from (4.13) and the congruences

$$w_0(C_n) \equiv v_2^{p^{n-2}} t_{n-2} + t_{n-3}\eta_R(v_3)^{p^{n-3}} + v_1 w(C_{n-1}) + v_2 w(C_{n-2})^p + v_3 w(C_{n-3})^{p^2},$$
$$w_0(D_n) \equiv v_1 t_{n-1}^p + v_2 t_{n-2}^{p^2} + v_3 t_{n-3}^{p^3} + v_1 w(D_{n-1}) + v_2 w(D_{n-2})^p + v_3 w(D_{n-3})^{p^2},$$

seen by the relation

$$v_{(k,J)} w_{|(k,J)|}(S_{n-\|(k,J)\|})^{p^{\|(k,J)\|-|(k,J)|}} = v_k v_J^{p^k} w_{|J|+1}(S_{n-k-\|J\|})^{p^{\|J\|-|J|+k-1}}. \qquad \square$$

Lemma 4.14 *For $n \geq 0$,*

$$\eta_R(v_2^{p-1} v_3^{e(n)}) \equiv \sum_{i=0}^{n} (-1)^{n-i} v_2^{p^{i+1}e(n-i)+p-1} v_3^{e(i)} t_{n-i}^{p^i} - v_2^p w_n^p + v_1 v_2^{p-2} w_{n+1} \quad \bmod (p, v_1^2).$$

Here

$$w_n = \sum_{i=1}^{n} (-1)^i v_2^{e(i-1)} T_i \eta_R(v_3^{p^{i-1}e(n-i)}). \tag{4.15}$$

Proof In this proof, every congruence is considered modulo (p, v_1^2). By Proposition 4.12, $t_k \eta_R(v_3^{p^k}) \equiv \widetilde{T}_k - v_2^{p^{k+1}} t_{k+1}$ for $\widetilde{T}_k = v_1 T_{k+2} + v_2 T_{k+1}^p + v_3 T_k^{p^2}$, which implies inductively

$$t_1 \eta_R(v_3^{pe(n)}) \equiv -\sum_{i=1}^{n} (-1)^i v_2^{p^2 e(i-1)} \widetilde{T}_i \eta_R(v_3^{p^{i+1}e(n-i)}) + (-1)^n v_2^{p^2 e(n)} t_{n+1},$$

and hence

$$\begin{aligned} t_1 \eta_R(v_3^{pe(n)}) \equiv{}& -v_1 v_2^{-p-1} w_{n+2} + v_2^{1-p} w_{n+1}^p - v_3 w_n^{p^2} + (-1)^n v_2^{p^2 e(n)} t_{n+1} \\ & - v_1 v_2^{-p-1}(t_1^p \eta_R(v_3) - v_2 t_2^p)\eta_R(v_3^{pe(n)}) + v_2^{1-p} t_1^{p^2} \eta_R(v_3^{pe(n)}). \end{aligned} \tag{4.16}$$

Now we prove the lemma by induction. For $n = 0$, it follows from the facts $\eta_R(v_2) \equiv v_2 + v_1 t_1^p$ by (4.4) and $w_1 = -t_1^p$.

Assuming the case for n, we obtain the case for $n+1$ from (4.16) and

$$\begin{aligned} \eta_R(v_2^{p-1} v_3^{e(n+1)}) \equiv{}& v_2^{-p^2+2p-1} v_3 \eta_R(v_2^{p-1} v_3^{e(n)})^p + v_2^{p-1}(v_2 t_1^{p^2} + v_1 t_2^p)\eta_R(v_3^{pe(n)}) \\ & - v_2^{2p-1} t_1 \eta_R(v_3^{pe(n)}) - v_1 v_2^{p-2} t_1^p \eta_R(v_3^{e(n+1)}), \end{aligned}$$

given by $\eta_R(v_2^{p-1} v_3) \equiv v_2^{p-1}(v_3 + v_2 t_1^{p^2} - v_2^p t_1 + v_1 t_2^p) - v_1 v_2^{p-2} t_1^p \eta_R(v_3)$. Here $\eta_R(v_3)$ is given in [2, (5.7)]. $\square$

Evaluate the congruence in Lemma 4.14 under d_1, and compare the v_1–multiples. Then we deduce the following corollary; see [9, Proposition 2.3]. Indeed, if $v_1 v_2^{p-2} d_1(w_{n+1}) \equiv A + v_1 B \bmod (p, v_1^2)$ for some A and B involving no v_1, then $A \equiv 0 \bmod (p, v_1^2)$ and $v_2^{p-2} d_1(w_{n+1}) \equiv B \bmod I_2$.

Corollary 4.17 *For the elements w_n in (4.15),*

$$d_1(w_{n+1}) \equiv -\sum_{i=0}^{n-1}(-1)^{n-i} v_2^{p^{i+1}e(n-i)} w_{i+1}\otimes t_{n-i}^{p^i} - (-1)^n v_2^{e(n+1)}\mathfrak{b}_n \mod I_2.$$

Here $\mathfrak{b}_n$ is an element in $d_1(t_n)\equiv \mathfrak{a}_n + v_1\mathfrak{b}_n \mod (p, v_1^2)$ for $\mathfrak{a}_n$ and $\mathfrak{b}_n$ involving no v_1. In particular, $\mathfrak{b}_2 = b_{1,0}$ by (4.8).

We have the cocycle z in $\Omega^1 E(3)_*/I_2$ given by

(4.18) $$z = v_3 t_1^p + v_2 c t_2^p - v_2^p t_2 = t_1^p \eta_R(v_3) - v_2 t_2^p + v_2^p c t_2 = -w_2 + v_2^p c t_2,$$

which represents the element $-v_2^{p+1}\zeta_2 \in H^1 M_2^0$; see [2, Proposition 3.18(c)] and (3.8). In particular,

(4.19) $$t_1^p \eta_R(v_3) \equiv z + v_2 t_2^p - v_2^p c t_2 \mod I_2.$$

We further have cocycles G_i' and $K_i' \in \Omega^2 E(3)_*/I_2$ for $i \in \{0,1,2\}$ defined by

(4.20) $$G_i' = c t_2^{p^i}\otimes t_1^{p^i} + \tfrac{1}{2} t_1^{p^{i+1}}\otimes t_1^{2p^i} \quad\text{and}\quad K_i' = t_1^{p^{i+1}}\otimes c t_2^{p^i} + \tfrac{1}{2} t_1^{2p^{i+1}}\otimes t_1^{p^i},$$

which are homologous to G_i and K_i in (2.5), respectively. Indeed,

(4.21) $$d_1(\mathfrak{g}_i) \equiv G_i' - G_i \quad\text{and}\quad d_1(\mathfrak{k}_i) \equiv K_i' - K_i \mod I_2,$$

for $i \in \{0,1,2\}$, and for $\mathfrak{g}_i, \mathfrak{k}_i \in \Omega^1 E(3)_*$ given by

(4.22) $$\mathfrak{g}_i = t_1^{p^i} t_2^{p^i} - \tfrac{1}{2} t_1^{p^{i+1}+2p^i} \quad\text{and}\quad \mathfrak{k}_i = t_1^{p^{i+1}} t_2^{p^i} - \tfrac{1}{2} t_1^{2p^{i+1}+p^i}.$$

We also have a similar relation

(4.23) $$d_1(t_1^p t_2) \equiv -(t_1^p\otimes t_2 + c t_2 \otimes t_1^p) - 2K_0 \mod I_2.$$

Lemma 4.24 *In $\Omega^1 E(3)_*$, put*

$$\omega_1 = \eta_R(v_3) t_2 - v_2 t_3 + v_2^p t_1 t_2,\quad \omega_2 = \tfrac{1}{2}\eta_R(v_3) t_1^{2p} - v_2^p \mathfrak{k}_0 \quad\text{and}\quad \widetilde{\omega}_2 = -w_3 - v_2^{pe(2)} t_1^p t_2.$$

Then, modulo I_2,

$$d_1(\omega_1) \equiv -t_1\otimes z - v_2^2 b_{1,1} - 2v_2^p G_0,\quad d_1(\omega_2) \equiv -t_1^p\otimes z - v_2 G_1 + v_2^p K_0,$$
$$d_1(\widetilde{\omega}_2) \equiv v_2^{p^2} z\otimes t_1^p + 2v_2^{p^2+p} K_0 + v_2^{e(3)} b_{1,0}.$$

Proof In this proof we consider congruences modulo I_2. A routine calculation shows the congruence for $d_1(\omega_1)$:

$$d_1(\eta_R(v_3)t_2) \equiv -t_1\otimes(z + \underbrace{v_2 t_2^p}_{a} + \underset{\sim\sim}{v_2^p t_2} - \underbrace{v_2^p t_1^{p+1}}_{c}) - t_2\otimes(\underbrace{v_2 t_1^{p^2}}_{b} - \underbrace{v_2^p t_1}_{d}), \qquad \text{(by (4.2) and (4.19))}$$

$$d_1(-v_2 t_3) \equiv v_2(\underbrace{t_1\otimes t_2^p}_{a} + \underbrace{t_2\otimes t_1^{p^2}}_{b} - v_2 b_{1,1}), \qquad \text{(by (4.8))}$$

$$d_1(v_2^p t_1 t_2) \equiv -v_2^p(\underset{\sim\sim}{t_1\otimes t_2} + \underbrace{t_2\otimes t_1}_{d} + \underset{\sim\sim}{t_1^2\otimes t_1^p} + \underbrace{t_1\otimes t_1^{p+1}}_{c}). \qquad \text{(by (4.8) and (4.2))}$$

Here the underlined terms with the same label cancel each other and the wavy underlined terms make $-2v_2^p G_0$.

For $d_1(\omega_2)$, we calculate

$$d_1\big(\tfrac{1}{2}\eta_R(v_3)t_1^{2p}\big) \equiv -t_1^p \otimes (z + \underbrace{v_2 t_2^p}_{G} - \underbrace{v_2^p c t_2}_{K'}) - \underbrace{\tfrac{1}{2}v_2 t_1^{2p} \otimes t_1^{p^2}}_{G} + \underbrace{\tfrac{1}{2}v_2^p t_1^{2p} \otimes t_1}_{K'}. \quad \text{(by (4.2) and (4.19))}$$

Add $d_1(-v_2^p \mathfrak{t}_0)$, and we obtain the desired congruence by (4.21).

We verify $d_1(\widetilde{\omega}_2)$ by

$$\begin{aligned} d_1(w_3) &\equiv -v_2^{pe(2)} w_1 \otimes t_2 + v_2^{p^2} w_2 \otimes t_1^p - v_2^{e(3)} b_{1,0} && \text{(by Corollary 4.17)} \\ &\equiv -\underbrace{v_2^{pe(2)}(-t_1^p) \otimes t_2}_{a} + v_2^{p^2}(-z + \underbrace{v_2^p c t_2}_{b}) \otimes t_1^p - v_2^{e(3)} b_{1,0}, && \text{(by (4.18) and (4.15))} \\ d_1(v_2^{pe(2)} t_1^p t_2) &\equiv -v_2^{pe(2)}((\underbrace{t_1^p \otimes t_2}_{a} + \underbrace{c t_2 \otimes t_1^p}_{b}) + 2K_0). && \text{(by (4.23))} \end{aligned}$$

□

5 The elements x_i and deriving elements y_i and y_i'

In [2, (5.11)], Miller, Ravenel and Wilson introduced elements $x_{3,i} \in v_3^{-1}\mathrm{BP}_*$. We refine them, and define the elements $x_i \in E(3)_*$ by

$$\begin{aligned} x_i &= v_3^{p^i} \quad \text{for } i = 0, 1, 2, \\ x_3 &= x_2^p - v_2^{p^3-1} v_3^{(p-1)p^2+1}, \\ x_4 &= x_3^p - v_2^{e(2)p^3-p-1} v_3^{(p^2-e(2))p^2+p+1}, \\ x_{2k+1} &= x_{2k}^p - v_2^{pa_{2k}-1} x_{2k-1}^{(p-1)p} v_3 - v_2^{e(3)p^{2k-1}-e(3)} v_3^{(p^2-e(2))p^{2k-1}+p+1}, \\ x_{2k+2} &= x_{2k+1}^p - 2v_2^{e(3)p^{2k}-e(3)} v_3^{(p^2-e(2))p^{2k}+p+1}, \end{aligned}$$

for $k \geq 2$.

Lemma 5.1 (see [9, Proposition 3.1]) *In $\Omega^1 E(3)_*$, we have*

$$\begin{aligned} d_0(x_0) &\equiv v_2 t_1^{p^2} - v_2^p t_1 \mod I_2, \\ d_0(x_1) &\equiv v_2^p v_3^{p-1} t_1 - v_2^{p+1} v_3^{-1} t_2^{p^2} \mod J_{2p}, \\ d_0(x_i) &\equiv v_2^{a_i}(x_{i-1}^{p-1} t_1^{p^{\varepsilon_i}} + B_i) \mod J_{e(3)p^{i-2}} \quad \text{for } i \geq 2. \end{aligned}$$

Here $\varepsilon_i = \frac{1}{2}(1+(-1)^i)$, and the B_i are given by

i	2	3	$2k$	$2k+1$
B_i	$-v_2^p v_3^{c(2)} t_2$	$v_2^{p^2-p} v_3^{c(3)}(z - v_2^p t_1^{p+1})$	$v_2^{a_{2k-1}-p} v_3^{c(2k)}(z - v_2^p t_2)$	$v_2^{a_{2k}-p} v_3^{c(2k+1)}(2z - v_2^p c t_2)$

for $c(k) = (p^2-p-1)p^{k-2}$. For $i \geq 4$, add $v_2^{a_{i-1}+1} v_3^{c(i)} Z'$ to B_i if we consider the congruence modulo $J_{e(3)p^{i-2}+1}$. Here Z' is a cocycle homologous to aZ for some $a \in \mathbb{Z}/p$.

Proof This follows from a routine calculation: For $i \leq 2$, it follows from (4.4) and from (4.6).

We obtain $d_0(x_3)$ from (4.19) and

$$d_0(v_3^{(p-1)p^2+1}) \equiv v_3^{(p-1)p^2}(v_2t_1^{p^2} - v_2^p t_1) - v_2^{a_2}v_3^{(p-1)p^2-p}(t_1^p\eta_R(v_3) - v_2^p t_2) \bmod J_{e(3)}$$

by (4.2), (4.4) and the congruence on $d_0(x_2)$. We note that $\eta_R(v_3^{p+1}) = v_3^{p+1} + v_2z^p - v_2^{p^2}z$ by [2, (3.20)], and obtain

$$\begin{aligned} &d_0(v_3^{(p^2-e(2))p^2+p+1}) \\ &\quad\equiv v_3^{(p^2-e(2))p^2}(v_2z^p - v_2^{p^2}z) - v_2^{a_2}v_3^{(p^2-e(2))p^2-p}t_1^p(v_3^{p+1}+v_2z^p) + v_2^{p^2+p}v_3^{(p^2-e(2))p^2}t_2 \bmod J_{e(3)}. \end{aligned}$$

The congruence on $d_0(x_4)$ follows from this and the congruence on $d_0(x_3)$, together with the definition of the element x_3.

Inductively suppose that

$$d_0(x_{2k}) \equiv v_2^{a_{2k}}x_{2k-1}^{p-1}t_1^p + v_2^{e(3)p^{2k-2}-e(2)}v_3^{(p^2-e(2))p^{2k-2}}(z - v_2^p t_2) \quad \bmod J_{e(3)p^{2k-2}}.$$

Then we calculate

$$\begin{aligned} d_0(x_{2k}^p) &\equiv \underset{a}{\underline{v_2^{pa_{2k}}x_{2k-1}^{(p-1)p}t_1^{p^2}}} + v_2^{e(3)p^{2k-1}-e(2)p}v_3^{(p^2-e(2))p^{2k-1}}(\underset{b}{\underline{z^p}} - \underset{c}{\underline{v_2^{p^2}t_2^p}}) \\ d_0(-v_2^{pa_{2k}-1}&x_{2k-1}^{(p-1)p}v_3) \\ &\equiv -v_2^{pa_{2k}-1}x_{2k-1}^{(p-1)p}(\underset{a}{\underline{v_2t_1^{p^2}}} - v_2^p t_1) + v_2^{e(3)p^{2k-1}-p-1}x_{2k-1}^{p^2-p-1}(z + \underset{c}{\underline{v_2t_2^p}} - v_2^p c t_2), \end{aligned}$$

where the second congruence follows by (4.2) and (4.19), and

$$d_0(-v_2^{e(3)p^{2k-1}-e(3)}v_3^{(p^2-e(2))p^{2k-1}+p+1}) \equiv -v_2^{e(3)p^{2k-1}-e(3)}v_3^{(p^2-e(2))p^{2k-1}}(\underset{b}{\underline{v_2z^p}} - v_2^{p^2}z).$$

Therefore

$$d_0(x_{2k+1}) \equiv v_2^{pa_{2k}+p-1}x_{2k-1}^{(p-1)p}t_1 + v_2^{e(3)p^{2k-1}-e(2)}v_3^{(p^2-e(2))p^{2k-1}}(2z - v_2^p c t_2).$$

Also

$$\begin{aligned} d_0(x_{2k+1}^p) &\equiv v_2^{pa_{2k+1}}x_{2k-1}^{(p-1)p^2}t_1^p + v_2^{e(3)p^{2k}-e(2)p}v_3^{(p^2-e(2))p^{2k}}(2z^p - v_2^{p^2}ct_2^p) \\ &\equiv v_2^{pa_{2k+1}}(x_{2k+1}^{p-1} - v_2^{pa_{2k}-1}x_{2k-1}^{(p^2-p-1)p}v_3)t_1^p + v_2^{e(3)p^{2k}-e(2)p}v_3^{(p^2-e(2))p^{2k}}(2z^p - v_2^{p^2}ct_2^p) \\ &\equiv v_2^{pa_{2k+1}}x_{2k+1}^{p-1}t_1^p + v_2^{e(3)p^{2k}-e(2)p}v_3^{(p^2-e(2))p^{2k}}(2z^p - v_2^{p^2-1}z - v_2^{p^2+p-1}t_2) \end{aligned}$$

and

$$d_0(-2v_2^{e(3)p^{2k}-e(3)}v_3^{(p^2-e(2))p^{2k}+p+1}) \equiv -2v_2^{e(3)p^{2k}-e(3)}v_3^{(p^2-e(2))p^{2k}}(v_2z^p - v_2^{p^2}z).$$

Therefore

$$d_0(x_{2k+2}) \equiv v_2^{pa_{2k+1}}x_{2k+1}^{p-1}t_1^p + v_2^{e(3)p^{2k}-e(2)}v_3^{(p^2-e(2))p^{2k}}(z - v_2^p t_2).$$

These complete the induction.

Put $d_0(x_i) \equiv v_2^{a_i}(x_{i-1}^{p-1}t_1^{p^{\varepsilon_i}} + B_i + v_2^{a_{i-1}+1}C) \bmod J_{e(3)p^{i-1}+1}$ for a cochain C. It is easy to see

$$d_1(v_2^{a_i}(x_{i-1}^{p-1}t_1^{p^{\varepsilon_i}} + B_i)) \equiv 0 \quad \bmod J_{e(3)p^{i-1}+1}.$$

It follows that C is a cocycle of $\Omega^1 M_3^0$, and so C represents a cohomology class $av_3^{c(i)}\zeta_3 \in H^1 M_3^0$ for some $a \in \mathbb{Z}/p$ by (2.3). □

Put

$$d_0(x_i) \equiv v_2^{a_i} A_i + v_2^{a_i} B_i \quad \text{for } A_i = x_{i-1}^{p-1} t_1^{p^{\varepsilon_i}},$$

where $\varepsilon_i = \frac{1}{2}(1+(-1)^i)$. We introduce elements $y_{s,i}$ and $y'_{s,i} \in \Omega^1 E(3)_*$ by

$$y_{s,i} = x_i^s t_1^{p^{\varepsilon_i+1}} - s x_i^{s-p+1} B_{i+1} \quad \text{and} \quad y'_{s,i} = x_i^s t_1^{p^{\varepsilon_i}} + \tfrac{1}{2} s v_2^{a_i} x_i^{s-1} A_i t_1^{p^{\varepsilon_i}}.$$

Lemma 5.2 *For the elements $y_{s,i}$ and $y'_{s,i}$,*

$$d_1(y_{s,0}) \equiv s(s+1) v_2^2 v_3^{s-p-1} G_2, \quad d_1(y_{s,1}) \equiv s(s+1) v_2^{2p} v_3^{sp-2} G_0,$$
$$d_1(y_{s,2}) \equiv -s(s+1) v_2^{2p^2-p} v_3^{sp^2-2p} (t_1^p \otimes z - v_2^p x),$$
$$d_1(y_{s,i}) \equiv \begin{cases} -s(s+1) v_2^{2a_{2k+1}-p} x_{2k}^{sp-2} (t_1 \otimes z - v_2^p G_0) & i = 2k+1, \\ -s(s+1) v_2^{2a_{2k+2}-p} x_{2k+1}^{sp-2} (2t_1^p \otimes z - v_2^p K'_0) & i = 2k+2, \end{cases}$$
$$d_1(y'_{s,1}) \equiv -s v_2^{p+1} v_3^{sp-2p} K_2, \quad d_1(y'_{s,2}) \equiv -s v_2^{p^2+p} v_3^{sp^2-p-1} K_0,$$
$$d_1(y'_{s,3}) \equiv s v_2^{a_3+p^2-p} v_3^{sp^3-p^2-p} (z \otimes t_1 - v_2^p x'),$$
$$d_1(y'_{s,i}) \equiv \begin{cases} s v_2^{e(3)p^{i-2}-p-1} v_3^{(sp^2-p-1)p^{2k-2}} (z \otimes t_1^p - v_2^p K_0) & i = 2k, \\ s v_2^{e(3)p^{i-2}-p-1} v_3^{(sp^2-p-1)p^{2k-1}} (2z \otimes t_1 - v_2^p G'_0) & i = 2k+1. \end{cases}$$

Here $x = (t_2 + t_1^{p+1}) \otimes t_1^p + t_1^p \otimes t_1^{p+1} + \frac{1}{2} t_1^{2p} \otimes t_1$ and $x' = t_1^{p+1} \otimes t_1 + \frac{1}{2} t_1^p \otimes t_1^2$, and these congruences are considered modulo J_{a+1}, where a is the largest power of v_2 in each congruence. Furthermore, replace K'_0 and K_0 in the congruences on $d_1(y_{s,2k+2})$ and $d_1(y'_{s,2k})$ by $K'_0 + v_2 t_1^p \otimes Z'$ and $K_0 + v_2 Z' \otimes t_1^p$, respectively, if we consider the congruences modulo J_{a+2}.

Proof We note that

$$d_1(B_{i+1}) \equiv -d_1(A_{i+1}) \equiv -d_0(x_i^{p-1}) \otimes t_1^{p^{\varepsilon_i+1}} \mod I_2,$$
$$d_0(x_i^s) + s x_i^{s+1-p} d_0(x_i^{p-1}) \equiv \binom{s+1}{2} x_i^{s-2} d_0(x_i)^2 \mod J_{3a_i}.$$

Indeed, $d_0(x_i^s) \equiv s x_i^{s-1} d_0(x_i) + \binom{s}{2} x_i^{s-2} d_0(x_i)^2 \mod J_{3a_i}$. Also,

$$d_1(A_i t_1^{p^{\varepsilon_i}}) \equiv d_0(x_{i-1}^{p-1}) \otimes t_1^{2p^{\varepsilon_i}} - 2x_{i-1}^{p-1} t_1^{p^{\varepsilon_i}} \otimes t_1^{p^{\varepsilon_i}} \equiv d_0(x_{i-1}^{p-1}) \otimes t_1^{2p^{\varepsilon_i}} - 2A_i \otimes t_1^{p^{\varepsilon_i}} \mod J_{a_{i-1}+2}.$$

Then we calculate

$$\begin{aligned} d_1(y_{s,i}) &\equiv d_0(x_i^s) \otimes t_1^{p^{\varepsilon_i+1}} - s d_0(x_i^{s+1-p}) \otimes B_{i+1} + s x_i^{s+1-p} d_0(x_i^{p-1}) \otimes t_1^{p^{\varepsilon_i+1}} \quad \text{(by (4.2))} \\ &\equiv \binom{s+1}{2} x_i^{s-2} d_0(x_i)^2 \otimes t_1^{p^{\varepsilon_i+1}} - s(s+1) x_i^{s-p} d_0(x_i) \otimes B_{i+1} \mod J_{2a_i+p}, \\ d_1(y'_{s,i}) &\equiv s x_i^{s-1} d_0(x_i) \otimes t_1^{p^{\varepsilon_i}} + \tfrac{1}{2} s v_2^{a_i} x_i^{s-1} d_0(x_{i-1}^{p-1}) \otimes t_1^{2p^{\varepsilon_i}} - s v_2^{a_i} x_i^{s-1} A_i \otimes t_1^{p^{\varepsilon_i}} \quad \text{(by (4.2))} \\ &\equiv s v_2^{a_i} x_i^{s-1} (B_i \otimes t_1^{p^{\varepsilon_i}} + \tfrac{1}{2} d_0(x_{i-1}^{p-1}) \otimes t_1^{2p^{\varepsilon_i}}) \mod J_{e(3)p^{i-2}+1}. \end{aligned}$$

Now we obtain the lemma from Lemma 5.1. □

6 Proof of Lemma 3.4

In this section, we define the cochains $(t_1^{p^i})_s$ and verify their d_1–differential.

6.1 The cochains $(t_1)_{sp^{2k}}$ and $(t_1^p)_{sp^{2k+1}}$ for $s \in \mathbb{Z}_0$

We define the cochains by

$$\begin{aligned}(t_1)_s = y_{s,0}, (t_1^p)_{sp} = y_{s,1}, \quad (t_1)_{sp^2} &= y_{s,2} - s(s+1)v_2^{2p^2-p}v_3^{sp^2-2p}\omega_2,\\ (t_1^p)_{sp^{2k+1}} &= y_{s,2k+1} - s(s+1)v_2^{2a_{2k+1}-p}x_{2k}^{sp-2}\omega_1,\\ (t_1)_{sp^{2k+2}} &= y_{s,2k+2} - s(s+1)v_2^{2a_{2k+2}-p^2-p}x_{2k+1}^{sp-2}(2\tilde{\omega}_2 + v_2^{p^2}(2zt_1^p + v_2^p\mathfrak{k}_0)),\end{aligned}$$

for $k \geq 1$. Then the lemma for this case follows immediately from Lemmas 5.2, 5.1 and 4.24 together with (4.21). Note also $2a_{2k+1} - p + 2 = 2pa_{2k} + p$. For example, for the case $p = 3$ and $k \geq 2$, we compute modulo $J_{2a_{2k}+2}$

$$\begin{aligned}d_1((t_1)_{3^{2k}s}) &\equiv d_1(y_{s,2k}) - s(s+1)v_2^{2a_{2k}-12}x_{2k-1}^{3s-2}d_1(2\tilde{\omega}_2 + v_2^9(2zt_1^3 + v_2^3\mathfrak{k}_0))\\ &\equiv -s(s+1)v_2^{2a_{2k}-3}x_{2k-1}^{3s-2}(\underbrace{2t_1^3\otimes z}_{a} - v_2^3(\underbrace{K_0'}_{b} + v_2t_1^3\otimes Z')) - s(s+1)v_2^{2a_{2k}-12}x_{2k-1}^{3s-2}\\ &\qquad \cdot\big(2(\underbrace{v_2^9 z\otimes t_1^3}_{c} + \underbrace{2v_2^{12}K_0}_{d} + v_2^{13}b_{1,0}) + v_2^9(-2(\underbrace{z\otimes t_1^3}_{c} + \underbrace{t_1^3\otimes z}_{a}) + v_2^3(\underbrace{K_0'}_{b} - \underbrace{K_0}_{d}))\big),\end{aligned}$$

where the second equivalence follows by Lemmas 5.2 and 4.24, and Equation (4.21)

6.2 The cochains $(t_1)_{sp^{2k}}$ and $(t_1^p)_{sp^{2k+1}}$ for $s \in \mathbb{Z}_1$

We put $s = tp^2 - 1$, and define the cochains $(t_1)_{(tp^2-1)p^{2k}}$ and $(t_1^p)_{(tp^2-1)p^{2k+1}}$ by

$$\begin{aligned}v_2^{a_{2k+1}}(t_1)_{(tp^2-1)p^{2k}} &= -v_3^{(t-1)p^{2k+2}}w^{p^{2k+1}} - d_0(v_2^{p^{2k+1}-p^{2k-2}}v_3^{(tp^2-1)p^{2k}}\sigma_k)\\ &\qquad + v_2^{p^{2k+2}-p^{2k-1}}v_3^{(tp-1)p^{2k+1}}W_k,\\ (t_1^p)_{(tp^2-1)p^{2k+1}} &= (t_1)^p_{(tp^2-1)p^{2k}},\end{aligned}$$

for the elements σ_k in Lemma 4.5, w in (4.7) and W_k in Corollary 4.11. Then this case follows from Lemmas 4.5 and 4.10, Corollary 4.11 and (2.8). We also use relations $w^{p^{2k+1}} \equiv -v_2^{p^{2k+1}}v_3^{p^{2k+2}-p^{2k}}t_1^{p^{2k}}$ mod $J_{a_{2k+1}+1}$ by Lemma 4.10 and (4.6), and $b_{1,2}^{p^{2k+1}} \equiv b_{1,2k+3} \equiv v_3^{(p-1)p^{2k+1}}b_{1,2k}$ mod I_3 by (4.6). For example,

$$\begin{aligned}v_2^{a_{2k+1}}(t_1)_{(tp^2-1)p^{2k}} &\equiv v_3^{(t-1)p^{2k+2}}(v_2^{p^{2k+1}}v_3^{p^{2k+2}-p^{2k}}t_1^{p^{2k}}) && \text{(by Lemma 4.5)}\\ &\qquad - v_2^{p^{2k+1}-p^{2k-2}}v_3^{(tp^2-1)p^{2k}}(v_2^{p^{2k-2}}t_1^{p^{2k}} - v_2^{a_{2k-1}}t_1)\\ &\equiv v_2^{a_{2k+1}}v_3^{(tp^2-1)p^{2k}}t_1 \quad \text{mod } J_{a_{2k+1}+1},\end{aligned}$$

since $p^{2k+1}-p^{2k-2}+a_{2k-1}=a_{2k+1}$ in (2.8), and

$$v_2^{a_{2k+1}}d_1((t_1)_{(tp^2-1)p^{2k}}) \equiv \underbrace{v_2^{p^{2k+2}}v_3^{(t-1)p^{2k+2}}b_{1,2}^{p^{2k+1}}}_{a}+v_2^{p^{2k+2}-p^{2k-1}}v_3^{(tp-1)p^{2k+1}}(-\underbrace{v_2^{p^{2k-1}}b_{1,2k}}_{a}+v_2^{a_{2k}}b_{1,0}) \mod J_{a_{2k+2}+1}$$

by Lemma 4.10 and Corollary 4.11. Since $p^{2k+2}-p^{2k-1}+a_{2k}=a_{2k+2}$ in (2.8), we obtain the case for $(t_1)_{sp^{2k}}$.

6.3 The cochains $(t_1)_{sp^{2k+1}}$ and $(t_1^p)_{sp^{2k}}$ for $s\in\mathbb{Z}^{(p)}$

We begin by defining

$$(t_1^p)_s=v_3^st_1^p+sv_2v_3^{s-1}ct_2^p-s(s-1)v_2^2v_3^{s-2}\mathfrak{k}_1.$$

Then we calculate by (4.2), (4.4), (4.8) and (4.22), and obtain

$$d_1((t_1^p)_s)\equiv s(s-1)v_2^2v_3^{s-2}K_1 \mod J_3.$$

Now we consider the cases for $p\mid s(s-1)$.

6.3.1 The cochains $(t_1^p)_{tp^k+1}$ for $k\geq 1$ We define the cochains by

$$\begin{aligned}(t_1^p)_{tp+1}&=v_3^{tp}z+tv_2^pv_3^{tp}t_2-tv_2^{p+1}v_3^{tp-p}ct_3^p,\\(t_1^p)_{tp^2+1}&=x_2^tz+tv_2^{a_2}v_3^{(tp-1)p}\omega_2,\\(t_1^p)_{tp^{2k+1}+1}&=x_{2k+1}^tz+tv_2^{a_{2k+1}}v_3^{(tp-1)p^{2k}}\omega_1+tv_2^{a_{2k}+p+1}(t_1^p)_{(tp^2-1)p^{2k-1}},\\(t_1^p)_{tp^{2k+2}+1}&=x_{2k+2}^tz+tv_2^{a_{2k+2}-p^2}v_3^{(tp-1)p^{2k+1}}(\widetilde{\omega}_2+v_2^{p^2}zt_1^p),\end{aligned}$$

in $\Omega^1E(3)_*$ for $k\geq 1$, $t\in\mathbb{Z}^{(p)}$, x_n in 5.1, z in (4.18) and ω_i in Lemma 4.24. We verify this case by a routine calculation using (4.2), (4.4), (4.18), (4.8) and (4.9). We see that

$$t_1^{p^3}\otimes z\equiv\eta_R(v_3)t_1^{p^3}\otimes t_1^p+v_2t_1^{p^3}\otimes ct_2^p-v_2^pv_3^{p-1}t_1\otimes t_2 \quad\text{and}\quad \eta_R(v_3)t_1^{p^3}\equiv v_3^pt_1+v_2ct_2^{p^2} \mod J_{p+1}$$

by (4.18), (4.4) and (4.6). It follows that $t_1^{p^3}\otimes z\equiv -d_1(v_3^pt_2)+v_2d_1(ct_3^p) \mod J_{p+1}$, and then $d_1(v_3^{tp}z)\equiv tv_2^pv_3^{tp-p}(-d_1(v_3^pt_2)+v_2d_1(ct_3^p))+\binom{t}{2}v_2^{2p}v_3^{tp-1}t_1^2\otimes t_1^p \mod J_{2p+1}$. Thus we obtain $d_1((t_1^p)_{tp+1})$.

The congruences on $d_1((t_1^p)_{tp^k+1})$ for $k\geq 2$ follow directly from Lemmas 5.1 and 4.24 and the results on $d_1((t_1^p)_{(tp^2-1)p^{2k-1}})$ shown in the previous subsection. For example,

$$\begin{aligned}d_1((t_1^p)_{tp^{2k+1}+1})&\equiv d_1(x_{2k+1}^t)\otimes z+tv_2^{a_{2k+1}}v_3^{(tp-1)p^{2k}}d_1(\omega_1)+tv_2^{a_{2k}+p+1}d_1((t_1^p)_{(tp^2-1)p^{2k-1}})\\&\equiv \underbrace{tv_2^{a_{2k+1}}v_3^{(tp-1)p^{2k}}t_1\otimes z}_{a}+tv_2^{a_{2k+1}}v_3^{(tp-1)p^{2k}}(-\underbrace{t_1\otimes z}_{a}-\underbrace{v_2^2b_{1,1}}_{b}-2v_2^pG_0)\\&\qquad+\underbrace{tv_2^{a_{2k}+p+1+pa_{2k}-pa_{2k-1}}v_3^{(tp-1)p^{2k}}b_{1,1}}_{b} \mod J_{a_{2k+1}+p+1},\end{aligned}$$

where the second equivalence follows by Lemmas 5.1, 4.24 and 3.4(2).

6.3.2 The cochains $(t_1^p)_{tp^k+e(k)}$ for $k \geq 2$ We put $r = 2n-1+\varepsilon$ ($\varepsilon \in \{0,1\}$), and

$$(t_1^p)'_{tp^r+e(r)} = x_r^t(w_{r+1} + v_2^{p^r-p^{r-3}} w_r \eta_R(\sigma_{n-1}^{p^\varepsilon}) + v_2^{a_r} w_r t_1^{p^\varepsilon})$$

for w_r in (4.15). Note that $w_r \equiv v_3^{pe(r-2)} w_2 \equiv -v_3^{pe(r-2)} z \bmod J_p$ by (4.15) and (4.18). Then $(t_1^p)'_{tp^r+e(r)} \equiv x_r^t w_{r+1} \equiv -v_3^{tp^r+e(r)} t_1^p \bmod I_3$. Furthermore, we calculate

$$\begin{aligned} d_1((t_1^p)'_{tp^r+e(r)}) &\equiv \underset{\sim\sim\sim}{t v_2^{a_r} v_3^{(tp-1)p^{r-1}} t_1^{p^\varepsilon} \otimes w_{r+1}} \qquad \text{(by Lemmas 5.1 and 4.5 and Corollary 4.17)} \\ &\quad + x_r^t\big(\underbrace{v_2^{p^r} w_r \otimes t_1^{p^{r-1}}}_{a} - v_2^{p^r-p^{r-3}} w_r \otimes (\underbrace{v_2^{p^{2n-4+\varepsilon}} t_1^{p^{2n-2+\varepsilon}}}_{a} - \underbrace{v_2^{a_{2n-3+\varepsilon}} t_1^{p^\varepsilon}}_{b}) \\ &\qquad - v_2^{a_r}(\underbrace{w_r \otimes t_1^{p^\varepsilon}}_{b} + \underset{\sim\sim\sim}{t_1^{p^\varepsilon} \otimes w_r})\big) \\ &\equiv -(t-1) v_2^{a_r} v_3^{tp^r+pe(r-2)} t_1^{p^\varepsilon} \otimes z \mod J_{a_r+p} \end{aligned}$$

together with (4.2) and (2.8). This case now follows from Lemma 4.24 by setting $(t_1^p)_{tp^r+e(r)} = -(t_1^p)'_{tp^r+e(r)} + (t-1)\, v_2^{a_r} v_3^{tp^r+pe(r-2)} \omega_{1+\varepsilon}$.

6.3.3 The cochains $(t_1^p)_{sp^{2k}}$ for $k \geq 1$ and $(t_1)_{sp^{2k+1}}$ for $k \geq 0$ We define $(t_1^{\varepsilon_i})_{sp^i}$ by

$$\begin{aligned} &(t_1)_{sp} = y'_{s,1}, \quad (t_1^p)_{sp^2} = y'_{s,2}, \quad (t_1)_{sp^3} = y'_{s,3} + s v_2^{e(3)p-p-1} v_3^{(sp^2-p-1)p}(zt_1 - \omega_1), \\ &(t_1^p)_{sp^4} = y'_{s,4} - \tfrac{1}{2} s v_2^{e(3)p^2-p^2-p-1} v_3^{(sp^2-p-1)p^2}(\widetilde{\omega}'_2 - v_2^{p^2} z t_1^p), \\ &(t_1)_{sp^{2k+1}} = y'_{s,2k+1} + 2s v_2^{e(3)p^{2k-1}-p-1} v_3^{(sp^2-p-1)p^{2k-1}}(zt_1 - \omega_1), \\ &(t_1^p)_{sp^{2k+2}} = y'_{s,2k+2} - s v_2^{e(3)p^{2k}-p^2-p-1} v_3^{(sp^2-p-1)p^{2k}} \widetilde{\omega}_2, \end{aligned}$$

where $\widetilde{\omega}'_2 = \widetilde{\omega}_2 - v_2^{p^2+p} t_1^p t_2 - v_2^{e(3)} v_3^{-p^2} c t_4^p$. Except for $d_1((t_1^p)_{sp^4})$, the lemma for this case follows from Lemmas 5.2 and 4.24 with (4.2).

For $d_1((t_1^p)_{sp^4})$, we make a calculation:

$$\begin{aligned} \widetilde{\omega}_2 &\equiv -w_3 && \text{(by Lemma 4.24)} \\ &\equiv t_1^p \eta_R(v_3^{p+1}) - v_2 t_2^p \eta_R(v_3^p) + v_2^{p+1} t_3^p && \text{(by (4.15) and Proposition 4.12)} \\ &\equiv (z + \underbrace{v_2 t_2^p}_{a} - v_2^p c t_2)\eta_R(v_3^p) - \underbrace{v_2 t_2^p \eta_R(v_3^p)}_{a} + v_2^{p+1} t_3^p && \text{(by (4.19))} \\ &\equiv v_3^p(z + v_2^p t_2) - v_2^{p+1} c t_3^p \mod J_{p+2}. && \text{(by (4.4))} \end{aligned}$$

Applying the Hopf conjugation c to the congruences of (4.6) shows the relations

(6.1) $$t_1^{p^3} \eta_R(v_3) \equiv v_3^p t_1 + v_2 c t_2^{p^2} \quad \text{and} \quad c t_2^{p^3} \eta_R(v_3) \equiv v_3^{p^2} c t_2 - v_2 c t_3^{p^2} \mod J_{p+1}.$$

Then, modulo J_{p+2},

$$\begin{aligned}
t_1^{p^4}\otimes v_3^p z &\equiv t_1^{p^4}\eta_R(v_3)^p\otimes z \\
&\equiv (v_3^{p^2}t_1^p+v_2^p ct_2^{p^3})\otimes z && \text{(by (6.1))}\\
&\equiv v_3^{p^2}t_1^p\otimes z+v_2^p ct_2^{p^3}\eta_R(v_3)\otimes t_1^p+v_2^{p+1}ct_2^{p^3}\otimes ct_2^p && \text{(by (4.18))}\\
&\equiv v_3^{p^2}t_1^p\otimes z+v_2^p(v_3^{p^2}ct_2-v_2ct_3^{p^2})\otimes t_1^p+v_2^{p+1}ct_2^{p^3}\otimes ct_2^p, && \text{(by (6.1))}\\
t_1^{p^4}\otimes v_2^p v_3^p t_2 &\equiv v_2^p t_1^{p^4}\eta_R(v_3)^p\otimes t_2 && \text{(by (6.1))}\\
&\equiv v_2^p v_3^{p^2}t_1^p\otimes t_2.
\end{aligned}$$

Therefore

$$\begin{aligned}
d_1(v_2^p v_3^{p^2}t_1^p t_2+v_2^{p+1}ct_4^p) &\equiv -v_2^p v_3^{p^2}(\underbrace{t_1^p\otimes t_2}_{a}+\underline{t_2\otimes t_1^p+t_1^{p+1}\otimes t_1^p+t_1\otimes t_1^{2p}})\\
&\qquad +v_2^{p+1}(\underbrace{t_1^{p^4}\otimes ct_3^p}_{b}-\underbrace{ct_2^{p^3}\otimes ct_2^p}_{c}+\underbrace{ct_3^{p^2}\otimes t_1^p}_{d}-v_3^p b_{1,2}^p),\\
t_1^{p^4}\otimes\widetilde{\omega}_2 &\equiv v_3^{p^2}t_1^p\otimes z+v_2^p(\underline{v_3^{p^2}ct_2}-\underbrace{v_2ct_3^{p^2}}_{d})\otimes t_1^p+\underbrace{v_2^{p+1}ct_2^{p^3}\otimes ct_2^p}_{c}\\
&\qquad +\underbrace{v_2^p v_3^{p^2}t_1^p\otimes t_2}_{a}-\underbrace{v_2^{p+1}t_1^{p^4}\otimes ct_3^p}_{b},
\end{aligned}$$

where the first equation follows from (4.2), (4.8) and (4.9). The sum of the wavy underlined terms is $-v_2^p v_3^{p^2}(2t_2\otimes t_1^p+t_1\otimes t_1^{2p})=-v_2^p v_3^{p^2}K_0$, and $b_{1,2}^p\equiv v_3^{p^2-p}b_{1,0}$ mod I_3 by (4.6). Then, modulo J_{p+2},

(6.2) $$t_1^{p^4}\otimes\widetilde{\omega}_2+d_1(v_2^p v_3^{p^2}t_1^p t_2+v_2^{p+1}ct_4^p)\equiv v_3^{p^2}t_1^p\otimes z-2v_2^p v_3^{p^2}K_0-v_2^{p+1}v_3^{p^2}b_{1,0}.$$

Now we calculate $d_1((t_1^p)_{sp^4})$ mod $J_{e(3)p^2+1}$ for odd prime p as

$$d_1(y'_{s,4})\equiv sv_2^{e(3)p^2-p-1}v_3^{(sp^2-p-1)p^2}(\underbrace{z\otimes t_1^p}_{a}-v_2^p(K_0+v_2Z'\otimes t_1^p))\qquad\text{(by 5.2)}$$

and

$$\begin{aligned}
&d_1\big(-\tfrac12 sv_2^{e(3)p^2-p^2-p-1}v_3^{(sp^2-p-1)p^2}(\widetilde{\omega}'_2-v_2^{p^2}zt_1^p)\big)\\
&\equiv \tfrac12 sv_2^{e(3)p^2-p-1}v_3^{(sp^2-p-2)p^2}\underline{t_1^{p^4}\otimes\widetilde{\omega}_2}-\tfrac12 sv_2^{e(3)p^2-p^2-p-1}v_3^{(sp^2-p-1)p^2} && \text{(by Lemma 4.24)}\\
&\qquad\cdot(v_2^{p^2}z\otimes t_1^p+2v_2^{p^2+p}K_0+v_2^{p^2+p+1}b_{1,0}-\underline{d_1(v_2^{p^2+p}t_1^p t_2+v_2^{e(3)}v_3^{-p^2}ct_4^p)}+v_2^{p^2}(z\otimes t_1^p+t_1^p\otimes z))\\
&\equiv \tfrac12 sv_2^{e(3)p^2-p-1}v_3^{(sp^2-p-2)p^2}(\underbrace{v_3^{p^2}t_1^p\otimes z}_{b}-2v_2^p v_3^{p^2}K_0-v_2^{p+1}v_3^{p^2}b_{1,0}) && \text{(by (6.2))}\\
&\qquad -\tfrac12 sv_2^{e(3)p^2-p^2-p-1}v_3^{(sp^2-p-1)p^2}(\underbrace{v_2^{p^2}z\otimes t_1^p}_{a}+2v_2^{p^2+p}K_0+v_2^{p^2+p+1}b_{1,0}+v_2^{p^2}(\underbrace{z\otimes t_1^p}_{a}+\underbrace{t_1^p\otimes z}_{b})).
\end{aligned}$$

6.4 The cochains $(t_1^{p^2})_{tp-1}$ for $t \in \mathbb{Z}$

Put

$$(t_1^{p^2})_{tp-1} = -v_2^{-1} v_3^{(t-1)p} w.$$

Then the lemma for this case follows from Lemma 4.10.

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