Algebraic & Geometric Topology

Volume 24 (2024)

Phase transition for the existence of van Kampen 2–complexes in random groups

Tsung-Hsuan Tsai

msp

Algebraic & Geometric Topology 24:7 (2024) 3897–3917
DOI: 10.2140/agt.2024.24.3897
Published: 9 December 2024

Phase transition for the existence of van Kampen 2–complexes in random groups

TSUNG-HSUAN TSAI

Gromov (1993) showed that every reduced van Kampen diagram D of a random group at density d satisfies the isoperimetric inequality $|\partial D| \geq (1-2d-\varepsilon)|D|\ell$. Adapting Gruber and Mackay's (2021) method for random triangular groups, we obtain a nonreduced van Kampen 2–complex version of this inequality.

The main result of this article is a phase transition: given a geometric form Y of 2–complexes, we find a critical density $d_c(Y)$ such that, in a random group at density d, if $d < d_c$, then there is no reduced van Kampen 2–complex of the form Y; while if $d > d_c$, then there exists reduced van Kampen 2–complexes of the form Y.

As an application, we exhibit phase transitions for small-cancellation conditions in random groups, giving explicitly the critical densities for the conditions $C'(\lambda)$, $C(p)$, $B(p)$ and $T(q)$.

20F06; 20F05, 20P05

1 Introduction

Random groups The first occurrence of random group presentations is the density model by M Gromov [1993, 9.B]. Formally, a random group is a random variable with values in a given set of groups, often constructed by group presentations with a fixed set of generators and a random set of relators. The goal is to study the asymptotic behaviors of a sequence of random groups (G_ℓ) when the maximal relator lengths ℓ goes to infinity. We say that G_ℓ satisfies some property Q_ℓ *asymptotically almost surely* (a.a.s.) if the probability that G_ℓ satisfies Q_ℓ converges to 1 as ℓ goes to infinity.

Let us consider the *permutation invariant density model* of random groups introduced by Gromov [1993, page 272] and developed in [Tsai 2022]. Fix the set of generators $X_m = \{x_1, \dots, x_m\}$ with $m \geq 2$ for group presentations. Let B_ℓ be the set of cyclically reduced words of $X_m^\pm$ of length at most ℓ. We shall construct random groups by *densable* and *permutation invariant* random subsets of B_ℓ.

Definition 1.1 [Gromov 1993, page 272; Tsai 2022, Definitions 1.5 and 2.5] A sequence of random subsets (R_ℓ) of the sequence of sets (B_ℓ) is called *densable with density* $d \in \{-\infty\} \cup [0,1]$ if the sequence of random variables $\operatorname{dens}_{B_\ell}(R_\ell) := \log_{|B_\ell|}(|R_\ell|)$ converges in probability to the constant d.

The sequence (R_ℓ) is called *permutation invariant* if R_ℓ is a permutation measure-invariant random subset of B_ℓ.

Many natural models of random subsets are densable and permutation invariant. For example, the uniform distribution on all subsets of cardinality $\lfloor B_\ell^d \rfloor$ considered in [Ollivier 2004; 2005; 2007], or the Bernoulli sampling of parameter $|B_\ell|^{d-1}$ considered in [Antoniuk et al. 2015] for random triangular groups.

Some other natural models are densable but not permutation invariant. For instance, consider the Bernoulli sampling of parameter $(2m)^{(d-1)\ell}$ on the set of nonreduced words of length ℓ, and reduce these words to form a random subset of B_ℓ. This is also the case for Gromov's expander graph model [2003], in which the random relators are the words read on the simple cycles of a randomly labeled expander graph.

Definition 1.2 [Gromov 1993, page 273; Tsai 2022, Definition 4.1] A sequence of random groups $(G_\ell(m,d))$ with m generators at density d is defined by

$$G_\ell(m,d) = \langle X_m \mid R_\ell \rangle,$$

where (R_ℓ) is a densable sequence of permutation invariant random subsets of (B_ℓ) with density d.

For detailed surveys on random groups, we refer the reader to work by E Ghys [2004], Y Ollivier [2005], I Kapovich and P Schupp [2008], and F Bassino, C Nicaud and P Weil [Bassino et al. 2020].

Van Kampen 2–complexes We consider oriented combinatorial 2–complexes and van Kampen diagrams as in [Lyndon and Schupp 1977, III.2 and III.9], with an additional precision that each face has an orientation given by its boundary path.

A 2–*complex* is hence a triplet $Y=(V,E,F)$ where V is the set of vertices, E is the set of oriented edges and F is the set of oriented faces. Its underlying graph is denote by $Y^{(1)}=(V,E)$. Every edge $e\in E$ has a starting point $\alpha(e)\in V$, an ending point $\omega(e)\in V$ and an inverse edge $e^{-1}\in E$, satisfying $\alpha(e^{-1})=\omega(e)$, $\omega(e^{-1})=\alpha(e)$ and $(e^{-1})^{-1}=e$. Every face $f\in F$ has a boundary path ∂f that is a cyclically reduced loop on the underlying graph $Y^{(1)}$, and an inverse face $f^{-1}\in F$ whose boundary path is the inverse. That is to say, it satisfies $(f^{-1})^{-1}=f$ and $\partial(f^{-1})=(\partial f)^{-1}$. The *starting point* of a face f is the starting point of its boundary path. Note that f^{-1} has the same starting point as f.

A *geometric edge* is a pair of inverse edges $\{e,e^{-1}\}$, denoted by $\bar{e}$. Similarly, a *geometric face* is a pair of inverse faces $\{f,f^{-1}\}$, denoted by $\bar{f}$. Throughout this article, we will carefully distinguish oriented edges (faces) and geometric edges (faces). We denote by $|Y^{(1)}|$ the number of geometric edges and $|Y|$ the number of geometric faces.

Definition 1.3 A *van Kampen* 2–*complex* with respect to a group presentation $G=\langle X \mid R\rangle$ is a 2–complex $Y=(V,E,F)$ with labels on edges by generators $\varphi_1\colon E\to X^{\pm}$ and labels on faces by relators $\varphi_2\colon F\to R^{\pm}$ such that $\varphi_1(e^{-1})=\varphi_1(e)^{-1}$, $\varphi_2(f^{-1})=\varphi_2(f)^{-1}$ and $\varphi_1(\partial f)=\varphi_2(f)$.

We denote briefly $Y=(V,E,F,\varphi_1,\varphi_2)$.

The data of the labels φ_1,φ_2 on Y is equivalently given by a combinatorial map $Y\to K(X,R)$, where $K(X,R)$ is the standard 2–complex with respect to the group presentation $G=\langle X\mid R\rangle$ (with one vertex, an edge for each generator and its inverse, and a face for each relator and its inverse).

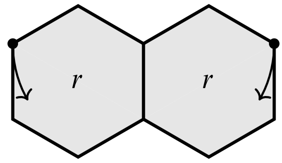

Figure 1: A reducible pair of faces.

A *van Kampen diagram* D is a finite, planar (embedded in a Euclidean plane) and simply connected van Kampen 2–complex. Its boundary length $|\partial D|$ is the length of a boundary path, passing once by every edge adjacent to one face and twice by every edge adjacent to zero faces.

A pair of faces in a van Kampen 2–complex is called *reducible* if they have the same relator label and their boundaries share a common edge at the same respective position (see Figure 1). A van Kampen 2–complex is called *reduced* if there is no reducible pair of faces.

Isoperimetric inequalities In order to prove the hyperbolicity of a random group at density $d < \frac{1}{2}$, Gromov [1993, 9.B] showed that a.a.s. *reduced* local van Kampen diagrams of $G_\ell(m,d)$ satisfy an isoperimetric inequality depending on the density d.

Theorem 1.4 [Gromov 1993, page 274; Ollivier 2004, Chapter 2] *Let $(G_\ell(m,d))$ be a sequence of random groups with $m \geq 2$ generators at density d. For any $\varepsilon > 0$ and $K > 0$, a.a.s. every* ***reduced*** *van Kampen diagram D of $G_\ell(m,d)$ with $|D| \leq K$ satisfies the isoperimetric inequality*

$$|\partial D| \geq (1 - 2d - \varepsilon)|D|\ell.$$

Ollivier's proof [2004] can achieve a slightly stronger[1] inequality,

$$|D^{(1)}| \geq \left(1 - d - \tfrac{1}{2}\varepsilon\right)|D|\ell.$$

One may expect such an inequality to hold for *every* reduced van Kampen 2–complex Y with $|Y| \leq K$. D Gruber and J Mackay [2021, Section 2] showed that in the triangular model of random groups,[2] the above inequality holds for every *nonreduced van Kampen 2–complex Y* with $|Y| \leq K$ if the *reduction degree* (Definition 2.1) $\mathrm{Red}(Y)$ is added in the left-hand side of the inequality.

However, the result fails in the regular Gromov density model: the condition $|Y| \leq K$ is not enough (see Remark 2.4). In Section 2 of this paper, we introduce the notion of *complexity* (Definition 2.2) to adapt Gruber and Mackay's inequality in the Gromov density model, establishing a *nonreduced van Kampen 2–complex* version of Theorem 1.4. A similar approach was given in the preprint [Odrzygóźdź 2021].

[1] Note that every van Kampen diagram composed of relators of lengths at most ℓ satisfies $2|D^{(1)}| - |\partial D| \leq |D|\ell$, so the given inequality implies the isoperimetric inequality.

[2] A model where the relator length $\ell = 3$ is fixed, and we are interested in asymptotic behaviors when the number of generators m goes to infinity.

Theorem 1.5 *Let $(G_\ell(m,d))$ be a sequence of random groups with $m \geq 2$ generators at density d. Let $\varepsilon > 0$, $K > 0$. For any $d < \frac{1}{2}$, a.a.s. every van Kampen 2–complex Y of complexity K of $G_\ell(m,d)$ satisfies*

$$|Y^{(1)}| + \mathrm{Red}(Y) \geq (1-d-\varepsilon)|Y|\ell.$$

Phase transition for the existence of van Kampen 2–complexes We are now interested in the converse of Theorem 1.5: Given a 2–complex Y satisfying the inequality of Theorem 1.5, is it true that a.a.s. there exists a *reduced* van Kampen 2–complex of $G_\ell(m,d)$ whose underlying 2–complex is Y?

A 2–complex Y is said to be *fillable* by a group presentation $G = \langle X \mid R\rangle$ (or by the set of relators R) if there exists a *reduced* van Kampen 2–complex of G whose underlying 2–complex is Y. An edge of a 2–complex is called *isolated* if it is not adjacent to any face. Since isolated edges do not affect fillability, we will only consider finite 2–complexes without isolated edges in the following.

To better formulate the problem, we consider a sequence of 2–complexes (Y_ℓ) and introduce the notion of *geometric form* of 2–complexes (Y,λ) (Definition 3.1), together with its density $\mathrm{dens}\, Y$ and its *critical density* $\mathrm{dens}_c\, Y$ (Definition 3.2). The main result of this article is the phase transition at density $1-\mathrm{dens}_c(Y)$, for the fillability of the 2–complex Y_ℓ.

Theorem 1.6 *Let $(G_\ell(m,d))$ be a sequence of random groups with $m \geq 2$ generators at density d. Let (Y_ℓ) be a sequence of 2–complexes with some geometric form (Y,λ).*

(i) *If $d < 1-\mathrm{dens}_c\, Y$, then a.a.s. Y_ℓ is not fillable by $G_\ell(m,d)$.*

(ii) *If $d > 1-\mathrm{dens}_c\, Y$ and Y_ℓ is fillable by B_ℓ, then a.a.s. Y_ℓ is fillable by $G_\ell(m,d)$.*

In Section 3, we prove Theorem 1.6 using the multidimensional intersection formula for random subsets (Theorem 3.6, [Tsai 2022, Theorem 3.7]), which generalizes the proof for the $C'(\lambda)$ phase transition in [Tsai 2022, Theorem 1.4]. We will see in Remark 3.3 that the second assertion of the theorem is equivalent to the following corollary.

Corollary 1.7 *Let $(G_\ell(m,d))$ be a sequence of random groups with $m \geq 2$ generators at density d. Let $s > 0$ and $K > 0$. Let (Y_ℓ) be a sequence of 2–complexes of the same geometric form such that Y_ℓ is fillable by B_ℓ. If every sub-2–complex Z_ℓ of Y_ℓ satisfies*

$$|Z_\ell^{(1)}| \geq (1-d+s)|Z_\ell|\ell,$$

then a.a.s. Y_ℓ is fillable by $G_\ell(m,d)$.

Note that we need Y_ℓ to have at least one filling by the set of all possible relators B_ℓ. It is automatically satisfied for planar and simply connected 2–complexes. In addition, if every face boundary length of Y_ℓ is exactly ℓ, then the given inequality is equivalent to an isoperimetric inequality similar the inequality of Theorem 1.4. Hence the following corollary.

Corollary 1.8 *Let $(G_\ell(m,d))$ be a sequence of random groups with $m \geq 2$ generators at density d. Let $s > 0$ and $K > 0$. Let (D_ℓ) be a sequence of finite planar 2–complexes of the same geometric form such that every face boundary length of D_ℓ is exactly ℓ. If every sub-2–complex D'_ℓ of D_ℓ satisfies*

$$|\partial D'_\ell| \geq (1 - 2d + s)|D'_\ell|\ell,$$

then a.a.s. D_ℓ is fillable by $G_\ell(m,d)$.

It is mentioned in [Ollivier and Wise 2011, Proposition 1.8] that when $d < 1/p$, a.a.s., a random group at density d has the $C(p)$ small cancellation condition. As an application of Theorem 1.6, we show that there is a phase transition: if $d > 1/p$, then a.a.s. a random group at density d *does not* have $C(p)$ (see Proposition 4.2).

Acknowledgements The content of this article was completed during my PhD thesis at the University of Strasbourg. I would like to thank my thesis advisor, Thomas Delzant, for his guidance and interesting discussions on the subject.

2 Isoperimetric inequality for van Kampen 2–complexes

We shall prove Theorem 1.5 in this section.

2.1 Reduction degree and complexity

Given a (nonreduced) van Kampen diagram $Y = (V, E, F, \varphi_1, \varphi_2)$ with respect to a group presentation $\langle X \mid R \rangle$, its *reduction degree* is the total number of geometric edges causing reducible pair of faces, counted with *multiplicity*: for any edge $e \in E$, any relator $r \in R$ and any integer j, we count the number of faces $f \in F$ labeled by r and having e as the j^{th} boundary edge. If this number is k, we add $(k-1)^+$ to the reduction degree where $(\cdot)^+$ is the positive part function. Here is the formal definition given by Gruber and Mackay [2021].

Definition 2.1 (reduction degree [Gruber and Mackay 2021, Definition 2.5]) Let $Y = (V, E, F, \varphi_1, \varphi_2)$ be a van Kampen 2–complex of a group presentation $G = \langle X \mid R \rangle$. Let ℓ be the maximal boundary length of faces of Y. The reduction degree of Y is

$$\operatorname{Red}(Y) = \sum_{e \in E} \sum_{r \in R} \sum_{1 \leq j \leq \ell} \left(\left| \{ f \in F \mid \varphi_2(f) = r,\ e \text{ is the } j^{\text{th}} \text{ edge of } \partial f \} \right| - 1 \right)^+.$$

It is not hard to see that a van Kampen 2–complex Y is reduced if and only if $\operatorname{Red}(Y) = 0$. Since isolated edges (edges that are not attached by any face) do not affect the reduction degree, we will only consider 2–complexes without isolated edges in the following.

A *maximal arc* of a 2–complex is a reduced combinatorial path passing only by vertices of degree 2 whose endpoints are not of degree 2. The *complexity* of a 2–complex encodes the number of maximal arcs with the number of faces.

Definition 2.2 (complexity of a 2–complex) Let Y be a 2–complex. Let $K > 0$. We say that Y is of complexity K if $|Y| \leq K$ and if for any face f of Y, the boundary path ∂f is divided into at most K maximal arcs.

If D is a planar and simply connected 2–complex with $|D| \leq K$, then the complexity of D is $6K$. Indeed, as the rank of its underlying graph is K, the number of its maximal arcs is at most $3K$, and every boundary path is divided into at most $6K$ maximal arcs (an arc may be used twice).

Lemma 2.3 *Let $K > 0$. There exists a number $C(K)$, depending only on K, such that the number of 2–complexes of complexity K with face boundary lengths at most ℓ is bounded by $C(K)\ell^{K^2}$.*

Proof Recall that we only consider 2–complexes without isolated edges, so the number of maximal arcs in a 2–complex of complexity K is at most K^2 (each of the K faces has at most K arcs). Since the face boundary lengths are at most ℓ, these K^2 maximal arcs have lengths at most ℓ. So there are at most ℓ^{K^2} choices for their lengths. Now let $C(K)$ be the number of choices to attach these K^2 maximal arcs to form a 2–complex. The number of ways to construct a 2–complex of complexity K with boundary lengths at most ℓ is hence bounded by $C(K)\ell^{K^2}$. □

Remark 2.4 While the number of 2–complexes with a *bounded complexity* grows polynomially with the maximal face boundary length ℓ, it is not the case for 2–complexes with a *bounded number of faces*, not even for 2–complexes with a *bounded number of maximal arcs*.

For example, consider the set of 2–complexes with one single face of boundary length ℓ whose underlying graph is 8–shaped with one vertex and two edges. There are only two maximal arcs, while the number of such 2–complexes equals to the number of words on two letters and their inverses of length ℓ, which grows exponentially with ℓ. Our polynomial bound will be useful in the proof of Theorem 1.5.

Remark 2.5 Actually, there are van Kampen 2–complexes that contradict the inequality of Theorem 1.5. For instance, D Calegari and A Walker [2015] proved that at any density $d < \frac{1}{2}$, there exists a number K depending only on d such that, in $G_\ell(m,d)$ there is a.a.s. a reduced van Kampen 2–complex Y homeomorphic to a surface of genus $O(\ell)$ (hence with complexity $O(\ell)$) with at most K faces.

Since every edge is adjacent to two faces in a surface, we have $|Y^{(1)}| \leq \frac{1}{2}|Y|\ell$, while according to Theorem 1.5 we expect that

$$|Y^{(1)}| \geq (1-d-\varepsilon)|Y|\ell > \tfrac{1}{2}|Y|\ell.$$

2.2 Abstract van Kampen 2–complexes

Let $(G_\ell(m,d))$ be a sequence of random groups at density d, defined by $G_\ell(m,d) = \langle x_1,\dots,x_m \mid R_\ell\rangle$. Recall that B_ℓ is the set of all cyclically reduced words of length at most ℓ and $|B_\ell| = (2m-1)^{\ell+O(1)}$. Let $0 < \varepsilon < 1-d$. Since $\log_{|B_\ell|}|R_\ell|$ converges in probability to the constant d, the probability event

$$Q_\ell := \left\{(2m-1)^{(d-(\varepsilon/4)\ell)} \leq |R_\ell| \leq (2m-1)^{(d+(\varepsilon/4)\ell)}\right\}$$

is a.a.s. true (see [Tsai 2022, Proposition 1.8]).

If we consider the Bernoulli density model where the events $\{r \in R_\ell\}$ through $r \in B_\ell$ are independent of the same probability $(2m-1)^{(d-1)\ell}$, it is obvious that we have $\mathbf{Pr}(r_1,\dots,r_k \in R_\ell) = (2m-1)^{k(d-1)\ell}$ for distinct $r_1,\dots,r_k$ in B_ℓ. In the permutation invariant density model, we have the following corresponding proposition, which is a variant of [Tsai 2022, Lemma 3.10].

Proposition 2.6 *Let $r_1,\dots,r_k$ be pairwise different relators in B_ℓ. We have*

$$\mathbf{Pr}(r_1,\dots,r_k \in R_\ell \mid Q_\ell) \le (2m-1)^{k(d-1+(\varepsilon/2))\ell}. \qquad \square$$

Abstract van Kampen 2–complexes, as abstract van Kampen diagrams introduced by Ollivier [2004], is a structure between 2–complexes and van Kampen 2–complexes that helps us solve 2–complex problems in random groups. Recall that since isolated edges do not affect fillability, we will only consider finite 2–complexes without isolated edges.

Definition 2.7 (abstract van Kampen 2–complex) An abstract van Kampen 2–complex $\widetilde{Y}$ is a 2–complex (V, E, F) with a labeling function on faces by integer numbers and their inverses

$$\widetilde{\varphi}_2\colon F \to \{1, 1^-, 2, 2^-, \dots, k, k^-\}$$

such that $\widetilde{\varphi}_2(f^{-1}) = \widetilde{\varphi}_2(f)^-$. We denote it simply by $\widetilde{Y} = (V, E, F, \widetilde{\varphi}_2)$.

By convention $(i^-)^- = i$. The integers $\{1,\dots,k\}$ are called abstract relators. Similar to a van Kampen diagram, a pair of faces $f, f' \in F$ is *reducible* if they are labeled by the same abstract relator, and they share an edge at the same position of their boundaries. An abstract diagram is called *reduced* if there is no reducible pair of faces. Let ℓ be the maximal boundary length of faces. The *reduction degree* of the 2–complex $\widetilde{Y}$ can be similarly defined as

$$\operatorname{Red}(\widetilde{Y}) = \sum_{e\in E}\sum_{1\le i\le k}\sum_{1\le j\le \ell}\bigl(\bigl|\{f \in F \mid \widetilde{\varphi}_2(f) = i,\ e \text{ is the } j^{\text{th}} \text{ edge of } \partial f\}\bigr| - 1\bigr)^+.$$

We say that an abstract van Kampen 2–complex with k abstract relators $\widetilde{Y} = (V, E, F, \widetilde{\varphi}_2)$ is *fillable* by a group presentation $G = \langle X \mid R\rangle$ (or by a set of relators R) if there exists k *different* relators $r_1,\dots,r_k \in R$ such that the construction $\varphi_2(f) := r_{\widetilde{\varphi}_2(f)}$ gives a van Kampen 2–complex $Y = (V, E, F, \varphi_1, \varphi_2)$[3] of G. The k–tuple of relators $(r_1,\dots,r_k)$, or the van Kampen 2–complex Y, is called a *filling* of $\widetilde{Y}$; see Figure 2, left. As we picked different relators for different abstract relators, if Y is a filling of $\widetilde{Y}$, then $\operatorname{Red}(Y) = \operatorname{Red}(\widetilde{Y})$, and $\widetilde{Y}$ is reduced if and only if Y is reduced.

Denote ℓ_i the length of the abstract relator i for $1 \le i \le k$. Let $\ell = \max\{\ell_1,\dots,\ell_k\}$ be the maximal boundary length of faces. The pairs of integers $(i,1),\dots,(i,\ell_i)$ are called *abstract letters* of i. The set of abstract letters of $\widetilde{Y}$ is then a subset of the product set $\{1,\dots,k\}\times\{1,\dots,\ell\}$. The geometric edges of $\widetilde{Y}$ are decorated by abstract letters and directions: Let $f \in F$ be labeled by i and let $e \in E$ be at the j^{th}

[3] Note that the edge labeling φ_1 is determined by the face labeling φ_2 as there are no isolated edges.

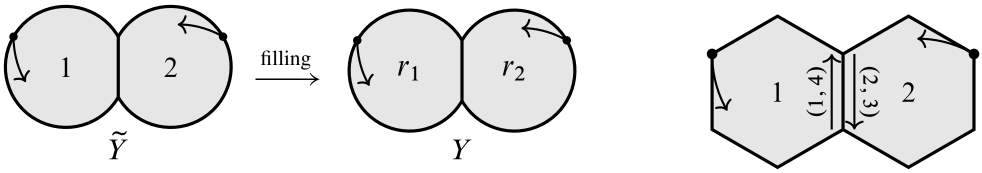

Figure 2: Left: filling an abstract van Kampen 2–complex. Right: a geometric edge decorated by two abstract letters.

position of ∂f. The geometric edge $\bar{e}$ is decorated, on the side of $\bar{f}$, by an arrow indicating the direction of e and the abstract letter (i, j). The number of decorations on a geometric edge is the number of its adjacent faces with multiplicity (an edge may be attached twice by the same face); see Figure 2, right.

Definition 2.8 (free-to-fill) An abstract letter (i, j) of $\widetilde{D}$ is *free-to-fill* if, for any edge $\bar{e}$ decorated by (i, j), it is the minimal decoration on $\bar{e}$.

Denote α_i the number of faces labeled by the abstract relator i and η_i the number of free-to-fill edges of i. We have the following estimation.

Lemma 2.9 *Let $\widetilde{Y} = (V, E, F, \widetilde{\varphi}_2)$ be an abstract van Kampen 2–complex with k abstract relators. Then*

$$\sum_{i=1}^{k} \alpha_i \eta_i \leq |\widetilde{Y}^{(1)}| + \mathrm{Red}(\widetilde{Y}).$$

Proof Denote by $\bar{E}$ the set of geometric edges and $\bar{F}$ the set of geometric faces. For any geometric edge $\bar{e}$, an adjacent face $\bar{f}$ from which the decoration is minimal is called a *preferred face* of $\bar{e}$. For any face $\bar{f}$, let $\bar{E}_{\bar{f}}$ be the set of geometric edges $\bar{e}$ on its boundary such that $\bar{f}$ is a preferred face of $\bar{e}$. Note that an edge will never be counted twice as the decorations given by one face are all different. According to Definition 2.8, for any face f with $\widetilde{\varphi}_2(f) = i$, we have $\eta_i \leq |\bar{E}_{\bar{f}}|$. Hence,

$$\sum_{i=1}^{k} \alpha_i \eta_i \leq \sum_{\bar{f} \in \bar{F}} |\bar{E}_{\bar{f}}|.$$

Denote by $\mathrm{Red}(\bar{e})$ the reduction degree caused by the edge $\bar{e}$. That is,

$$\mathrm{Red}(\bar{e}) := \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq \ell} \big(\big|\{ f \in F \mid \widetilde{\varphi}_2(f) = i,\ e \text{ or } e^{-1} \text{ is the } j^{\text{th}} \text{ edge of } \partial f \}\big| - 1 \big)^+,$$

so that the number of preferred faces of $\bar{e}$ is bounded by $1 + \mathrm{Red}(\bar{e})$. Hence,

$$\sum_{\bar{f} \in \bar{F}} |\bar{E}_{\bar{f}}| \leq \sum_{\bar{e} \in \bar{E}} (1 + \mathrm{Red}(\bar{e})) = |\widetilde{Y}^{(1)}| + \mathrm{Red}(\widetilde{Y}). \qquad \square$$

Probability of filling We shall estimate the probability that an abstract van Kampen 2–complex $\widetilde{Y}$ is fillable by a random group $G_\ell(m, d)$. This step is the key to prove Theorem 1.5. Recall that

$$Q_\ell := \big\{ (2m-1)^{(d-(\varepsilon/4)\ell)} \leq |R_\ell| \leq (2m-1)^{(d+(\varepsilon/4)\ell)} \big\}$$

is an a.a.s. true probability event.

Lemma 2.10 *Let $\widetilde{Y}$ be an abstract van Kampen 2–complex with k abstract relators. We have*

$$\mathbf{Pr}(\widetilde{Y} \text{ is fillable by } G_\ell(m,d) \mid Q_\ell) \le \left(\frac{2m}{2m-1}\right)^k (2m-1)^{\sum_{i=1}^k (\eta_i + (d-1+\varepsilon/2)\ell)}.$$

Proof Let us estimate the number of fillings of $\widetilde{Y}$. For every free-to-fill abstract letter (i,j), there are at most $2m$ ways to fill a generator if $j=1$, at most $(2m-1)$ ways to fill if $j \neq 1$ for avoiding reducible word. As there are η_i free-to-fill abstract letters on the i^{th} abstract relator, there are at most $2m(2m-1)^{\eta_i - 1}$ ways to fill it. So there are at most $\prod_{i=1}^k (2m(2m-1)^{\eta_i-1})$ ways to fill $\widetilde{Y}$.

Let Y be a van Kampen 2–complex, which is a filling of $\widetilde{Y}$. The 2–complex Y is labeled by k different relators in B_ℓ, denoted $r_1, \dots, r_k$. By Proposition 2.6,

$$\mathbf{Pr}(Y \text{ is a 2–complex of } G_\ell(m,d) \mid Q_\ell) = \mathbf{Pr}(r_1, \dots, r_k \in R_\ell \mid Q_\ell) \le (2m-1)^{k(d-1+\varepsilon/2)\ell}.$$

Hence

$$\begin{aligned}\mathbf{Pr}(\widetilde{Y} \text{ is fillable by } G_\ell(m,d) \mid Q_\ell) &\le \sum_{Y \text{ fills } \widetilde{Y}} \mathbf{Pr}(Y \text{ is a 2–complex of } G_\ell(m,d) \mid Q_\ell) \\ &\le \prod_{i=1}^k (2m(2m-1)^{\eta_i-1})(2m-1)^{k(d-1+\varepsilon/2)\ell} \\ &\le \left(\frac{2m}{2m-1}\right)^k (2m-1)^{\sum_{i=1}^k (\eta_i + (d-1+\varepsilon/2)\ell)}. \end{aligned}$$ □

Lemma 2.11 *Let $\widetilde{Y}$ be an abstract van Kampen 2–complex with k abstract relators. Suppose that $\widetilde{Y}$ does not satisfy the inequality given in Theorem 1.5, ie*

$$|\widetilde{Y}^{(1)}| + \mathrm{Red}(\widetilde{Y}) < (1-d-\varepsilon)|\widetilde{Y}|\ell,$$

then

$$\mathbf{Pr}(\widetilde{Y} \text{ is fillable by } G_\ell(m,d) \mid Q_\ell) \le \left(\frac{2m}{2m-1}\right)(2m-1)^{-(\varepsilon/2)\ell}.$$

Proof Let $\widetilde{Y}_i$ be the sub-2–complex of $\widetilde{Y}$ consisting of faces labeled by the i first abstract relators. Let $P_i = \mathbf{Pr}(\widetilde{Y}_i$ is fillable by $G_\ell(m,d) \mid Q_\ell)$. Apply Lemma 2.10 on $\widetilde{Y}_i$; we have

$$P_i \le \left(\frac{2m}{2m-1}\right)^i (2m-1)^{\sum_{j=1}^i (\eta_j + (d-1+\varepsilon/2)\ell)}.$$

Note that if $\widetilde{Y}$ is fillable by $G_\ell(m,d)$ then its sub-2–complex $\widetilde{Y}_i$ is fillable by the same group. So for any $1 \le i \le k$,

$$\log_{2m-1}(P_k) \le \log_{2m-1}(P_i) \le \sum_{j=1}^i \left(\eta_j + \left(d-1+\tfrac{1}{2}\varepsilon\right)\ell + \log_{2m-1}\left(\frac{2m}{2m-1}\right)\right).$$

Without loss of generality, suppose that $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_k$. Note that $\log_{2m-1}(P_k)$ is negative and

$\alpha_1 \le |\widetilde{Y}|$, so $|\widetilde{Y}|\log_{2m-1}(P_k) \le \alpha_1 \log_{2m-1}(P_k)$. By Abel's summation formula, with convention $\alpha_{k+1} = 0$,

$$\begin{aligned}|\widetilde{Y}|\log_{2m-1}(P_k) &\le \alpha_1 \log_{2m-1}(P_k) = \sum_{i=1}^{k}(\alpha_i - \alpha_{i+1})\log_{2m-1}(P_k)\\ &\le \sum_{i=1}^{k}(\alpha_i-\alpha_{i+1})\sum_{j=1}^{i}\Big[\eta_i + \big(d-1+\tfrac{1}{2}\varepsilon\big)\ell + \log_{2m-1}\Big(\frac{2m}{2m-1}\Big)\Big]\\ &= \sum_{i=1}^{k}\alpha_i\Big[\eta_i + \big(d-1+\tfrac{1}{2}\varepsilon\big)\ell + \log_{2m-1}\Big(\frac{2m}{2m-1}\Big)\Big]\\ &= \sum_{i=1}^{k}\alpha_i\eta_i + \Big(\sum_{i=1}^{k}\alpha_i\Big)\Big[\big(d-1+\tfrac{1}{2}\varepsilon\big)\ell + \log_{2m-1}\Big(\frac{2m}{2m-1}\Big)\Big].\end{aligned}$$

Note that $\sum_{i=1}^{k}\alpha_i = |\widetilde{Y}|$. By Lemma 2.9 and the hypothesis of the current lemma,

$$\sum_{i=1}^{k}\alpha_i\eta_i \le |\widetilde{Y}^{(1)}| + \mathrm{Red}(\widetilde{Y}) < (1-d-\varepsilon)|\widetilde{Y}|\ell.$$

Hence,

$$\begin{aligned}|\widetilde{Y}|\log_{2m-1}(P_k) &\le (1-d-\varepsilon)|\widetilde{Y}|\ell + |\widetilde{Y}|\Big[\big(d-1+\tfrac{1}{2}\varepsilon\big)\ell + \log_{2m-1}\Big(\frac{2m}{2m-1}\Big)\Big]\\ &\le |\widetilde{Y}|\Big[-\tfrac{1}{2}\varepsilon\ell + \log_{2m-1}\Big(\frac{2m}{2m-1}\Big)\Big].\end{aligned}$$

□

2.3 Proof of Theorem 1.5

Under the condition $Q_\ell := \{(2m-1)^{(d-(\varepsilon/4)\ell)} \le |R_\ell| \le (2m-1)^{(d+(\varepsilon/4)\ell)}\}$, the probability that there exists a van Kampen 2–complex of complexity K of $G_\ell(m,d)$ satisfying the inverse inequality

$$(*)\qquad |Y^{(1)}| + \mathrm{Red}(Y) < (1-d-\varepsilon)|Y|\ell$$

is bounded by

$$\sum_{\widetilde{Y}\text{ of complexity }K,\text{ satisfying }(*)} \mathbf{Pr}(\widetilde{Y}\text{ is fillable by }G_\ell(m,d) \mid Q_\ell).$$

By Lemma 2.3 and the face that there at most K^{2K} ways to label a 2–complex with K faces by abstract relators $\{1^{\pm},\dots,K^{\pm}\}$, there are at most $\ell^{3K}\times K^{2K}$ terms in the sum. By Lemma 2.11, every term is bounded by

$$\Big(\frac{2m}{2m-1}\Big)(2m-1)^{-(\varepsilon/2)\ell}.$$

So the sum is smaller than

$$\ell^{3K}K^{2K}\Big(\frac{2m}{2m-1}\Big)(2m-1)^{-(\varepsilon/2)\ell},$$

which converges to 0 as $\ell\to\infty$.

By definition $\mathbf{Pr}(Q_\ell) \xrightarrow{\ell\to\infty} 1$, so the probability that there exists a van Kampen 2–complex of $G_\ell(m,d)$ of complexity K satisfying $(*)$ converges to 0 as ℓ goes to infinity. That is to say, a.a.s. every van Kampen diagram of $G_\ell(m,d)$ of complexity K satisfies the inequality

$$|Y^{(1)}| + \mathrm{Red}(Y) \geq (1-d-\varepsilon)|Y|\ell. \qquad \square$$

Collapsible 2–complexes and closed surfaces Recall that an *elementary collapse* of a 2–complex, in the sense of Whitehead [1939], is the removal of a face together with one of its edges that is not adjacent to other faces. A 2–complex is called *collapsible*[4] to a graph if it can be collapsed to a graph by a sequence of elementary collapses.

Let Y be a 2–complex of complexity K. If Y is *not* collapsible, then after all possible elementary collapses, we obtain a sub-2–complex Y' having only edges that are adjacent to at least 2 faces, which gives $|Y'^{(1)}| \leq \frac{1}{2}|Y'|\ell$, where ℓ is the maximal boundary length of faces. Since it contradicts the inequality of Theorem 1.5 for *any* density $d < \frac{1}{2}$, the 2–complex Y cannot be fillable by any random group. Hence the following proposition.

Proposition 2.12 *Let $(G_\ell(m,d))$ be a sequence of random groups with $m \geq 2$ generators at density d. For any $d < \frac{1}{2}$ and $K > 0$, a.a.s. every reduced van Kampen 2–complex of complexity K of $G_\ell(m,d)$ is collapsible to a graph.*

Consequently, a 2–complex with K faces that is homeomorphic to a *closed surface* of a fixed genus[5] g is not fillable by any random group, since a surface is not collapsible and the complexity is bounded by a number depending only on K and g.

3 Phase transition for the existence of van Kampen 2–complexes

In this section, we work on the proof of Theorem 1.6.

Motivation and a counterexample Let $(G_\ell(m,d))$ be a sequence of random groups at density d. We are interested in the converse of Theorem 1.5 without the reduction part: if a 2–complex Y_ℓ with bounded complexity satisfies the inequality

$$|Y_\ell^{(1)}| \geq (1-d+s)|Y_\ell|\ell$$

with some $s > 0$, does there exist a face labeling by relators and an edge labeling by generators, so that Y_ℓ becomes a reduced van Kampen 2–complex of $G_\ell(m,d)$?

The motivation for this question comes from the well-known phase transition at density $d = \frac{1}{2}\lambda$, mentioned in [Gromov 1993, page 274]: if $d < \frac{1}{2}\lambda$ then a.a.s. $G_\ell(m,d)$ has the $C'(\lambda)$ small cancellation condition;

[4] In the original context [Whitehead 1939], the removal of an isolated edge is also an elementary collapse, and a 2–complex is *collapsible* if it can be collapsed to a point.

[5] Note that the genus g need to be fixed, otherwise by Calegari and Walker's result [2015] there exists a closed surface (see Remark 2.5).

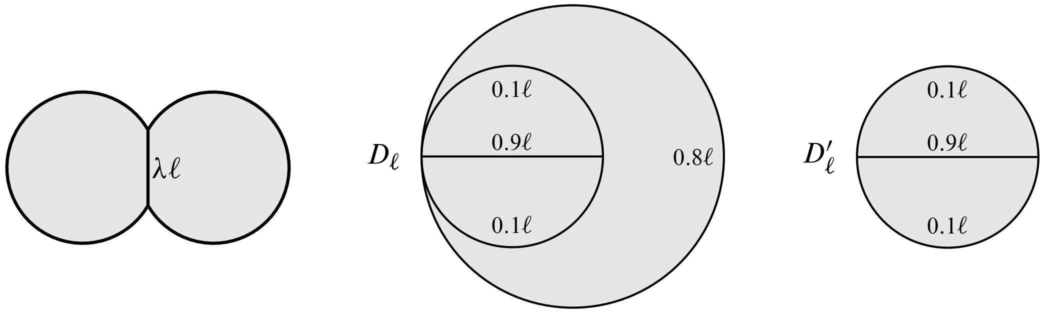

Figure 3: Left: a van Kampen diagram denying the $C'(\lambda)$ condition. Middle and right: A 2–complex that satisfies the isoperimetric inequality with a sub-2–complex that does not.

while if $d > \frac{1}{2}\lambda$ then a.a.s. $G_\ell(m,d)$ does not have $C'(\lambda)$. The first assertion is a simple application of Theorem 1.4. For the second assertion, we need to show that a.a.s. there exists a van Kampen 2–complex D of $G_\ell(m,d)$ with exactly 2 faces of boundary length ℓ, sharing a common path of length at least $\lambda\ell$ (Figure 3, left).

The first detailed proof of such an existence is given in [Bassino et al. 2020, Theorem 2.1], using an analog of the probabilistic pigeonhole principle. Another proof is given in [Tsai 2022, Theorem 1.4]. An intuitive explanation using the "dimension reasoning" is given in [Ollivier 2005, page 30]: The dimension of the set of couples $R_\ell \times R_\ell$ is $2d\ell$. Sharing a common subword of length L imposes L equations, so the "dimension" of the set of couples of relators sharing a common subword of length $\lambda\ell$ is $2d\ell - \lambda\ell$. If $d > \lambda/2$, then there will exist such a couple because the dimension will be positive. However, this argument is not true for *any* 2–complex in general. Here is a counterexample:

At density $d = 0.4$, let (D_ℓ) be a sequence of 2–complexes where D_ℓ is given in Figure 3, middle. The given inequality is satisfied because $|D_\ell^{(1)}| = 1.9\ell > 1.8\ell = (1-d)|D_\ell|\ell$. However, the subdiagram D'_ℓ (Figure 3, right) gives $|D_\ell'^{(1)}| = 1.1\ell < 1.2\ell = (1-d)|D'_\ell|\ell)$, which contradicts the isoperimetric inequality of Theorem 1.5 and cannot be a van Kampen diagram of $G_\ell(m,d)$.

3.1 Geometric form and critical density

Let us define the *geometric form* of 2–complexes and the *critical density* of a geometric form. To simplify the notation, for a 2–complex $Y = (V, E, F)$, we denote by $\mathrm{Edge}(Y)$ the set of geometric edges of Y and e instead of $\bar{e}$ for geometric edges.

Definition 3.1 A *geometric form* of 2–complexes is a couple (Y, λ) where $Y = (V, E, F)$ is a finite connected 2–complex without isolated edges, and λ is a length labeled on edges defined by

$$\lambda\colon \mathrm{Edge}(Y) \to \,]0,1], \quad e \mapsto \lambda_e,$$

such that for every face f of Y, the boundary length $|\partial f|$ is bounded by 1.

A sequence of 2–complexes (Y_ℓ) is said to be *of the geometric form* (Y,λ) if Y_ℓ is obtained from Y by dividing every edge e of Y into $\lfloor \lambda_e \ell \rfloor$ edges[6] of length 1.

A sequence of 2–complexes (Y_ℓ) is briefly said to be *of the same geometric form* if the geometric form (Y,λ) is not specified. Note that the boundary length of every face f of Y_ℓ is at most ℓ. If Z is a sub-2–complex of Y, we denote $Z \leq Y$. By convention, if (Z_ℓ) is a sequence of 2–complexes of the geometric form $(Z, \lambda|_Z)$, we have $Z_\ell \leq Y_\ell$ for any integer ℓ.

Definition 3.2 Let (Y,λ) be a geometric form of 2–complexes. The *density* of Y is

$$\operatorname{dens}(Y) := \frac{\sum_{e \in \operatorname{Edge}(Y)} \lambda_e}{|Y|}.$$

The *critical density* of Y is

$$\operatorname{dens}_c(Y) := \min_{Z \leq Y} \{\operatorname{dens}(Z)\}.$$

The intuition of this definition can be found in Lemma 3.8: the density of Y is actually the density of all possible van Kampen 2–complexes that fill Y_ℓ.

Remark 3.3 Taking Definitions 3.2 and 3.1 together, we have

$$\operatorname{dens}(Y) = \frac{\sum_{e \in \operatorname{Edge}(Y)} \lambda_e}{|Y|} = \lim_{\ell \to \infty} \frac{\sum_{e \in \operatorname{Edge}(Y)} \lfloor \lambda_e \ell \rfloor}{|Y_\ell| \ell} = \lim_{\ell \to \infty} \frac{|Y_\ell^{(1)}|}{|Y_\ell| \ell}.$$

Hence, the condition "$\operatorname{dens}_c(Y) + d > 1$" is equivalent to the following statement: Given $s > 0$, for ℓ large enough, every sub-2–complex Z_ℓ of Y_ℓ satisfies

$$|Z_\ell^{(1)}| \geq (1 - d + s)|Z_\ell|\ell.$$

This argument shows that the second assertion of Theorem 1.6 is equivalent to Corollary 1.7.

Proof of Theorem 1.6(i) We will use Theorem 1.5 without the reduction part. Let $(G_\ell(m,d))$ be a sequence of random groups with m generators at density d. Recall that a 2–complex Y_ℓ is said to be *fillable* by $G_\ell(m,d)$ if there exists a *reduced* van Kampen 2–complex of $G_\ell(m,d)$ whose underlying 2–complex is Y_ℓ.

Let (Y,λ) be a geometric form of 2–complexes with $\operatorname{dens}_c Y + d < 1$. Let (Y_ℓ) be a sequence of 2–complexes of the geometric form (Y,λ). We shall prove that a.a.s. the 2–complex Y_ℓ is *not* fillable by the random group $G_\ell(m,d)$. By the definition of critical density, there exists a sub-2–complex $Z \leq Y$ satisfying $\operatorname{dens} Z + d < 1$. Let (Z_ℓ) be the sequence of 2–complexes of the geometric form $(Z, \lambda|_Z)$. We shall prove that a.a.s. Z_ℓ is not fillable by $G_\ell(m,d)$.

[6] We can replace $\lfloor \lambda_e \ell \rfloor$ by any function with $\lambda\ell + o(\ell)$ and slightly smaller than $\lambda\ell$. Note that the sum of edge lengths on every face boundary of Y_ℓ is at most ℓ

Let $\varepsilon > 0$ such that $\operatorname{dens} Z = 1 - d - 3\varepsilon$. By definition,

$$\lim_{\ell\to\infty} \frac{|Z_\ell^{(1)}|}{|Z_\ell|\ell} = 1 - d - 3\varepsilon,$$

so for ℓ large enough,

$$|Z_\ell^{(1)}| \le (1 - d - 2\varepsilon)|Z_\ell|\ell < (1 - d - \varepsilon)|Z_\ell|\ell.$$

The complexity of Z_ℓ is

$$K = \max\Bigl\{|Z|, |Z^{(1)}|, \max\Bigl\{\frac{1}{\lambda_e} \Bigm| e \in \operatorname{Edge}(Z)\Bigr\}\Bigr\},$$

independent of ℓ. By Theorem 1.5 with ε and K given above, a.a.s. every van Kampen 2–complex Z_ℓ of $G_\ell(m, d)$ of complexity K should satisfy

$$|Z_\ell^{(1)}| \ge (1 - d - \varepsilon)|Z_\ell|\ell.$$

Hence, a.a.s. the given 2–complex Z_ℓ is not fillable by $G_\ell(m, d)$, which implies that a.a.s. Y_ℓ is not fillable by $G_\ell(m, d)$. □

3.2 The multidimensional intersection formula for random subsets

To prove the second assertion of Theorem 1.6, we need the *multidimensional intersection formula* for random subsets with density, introduced in [Tsai 2022, Section 3].

Recall that B_ℓ is the set of cyclically reduced words of $X_m^\pm = \{x_1^\pm, \dots, x_m^\pm\}$ of length at most ℓ, and that $|B_\ell| = (2m-1)^{\ell+o(\ell)}$. Let $k \ge 1$ be an integer. Denote by $B_\ell^{(k)}$ the set of k–tuples of pairwise distinct relators $(r_1, \dots, r_k)$ in B_ℓ. Such notation can be used for any set or any random set.

Note that $|B_\ell^{(k)}| = (2m-1)^{k\ell+o(\ell)}$. Recall that a sequence of fixed subsets $(\mathcal{Y}_\ell)$ of the sequence $(B_\ell^{(k)})$ is called *densable with density* $\alpha \in \{-\infty\} \cup [0, 1]$ if the sequence of real numbers $(\log_{|B_\ell^{(k)}|} |\mathcal{Y}_\ell|)$ converges to α (see [Gromov 1993, page 272; Tsai 2022, Definition 1.5]). That is to say, $|\mathcal{Y}_\ell| = (2m-1)^{\alpha k\ell+o(\ell)}$.

Definition 3.4 (self-intersection partition [Tsai 2022, Definition 3.4]) Let $(\mathcal{Y}_\ell)$ be a sequence of fixed subsets of the sequence $(B_\ell^{(k)})$. Let $0 \le i \le k$ be an integer. The i^{th} self-intersection of $\mathcal{Y}_\ell$ is

$$S_{i,\ell} := \{(x, y) \in \mathcal{Y}_\ell^2 \mid |x \cap y| = i\},$$

where $|x \cap y|$ is the number of common elements between the sets $x = (r_1, \dots, r_k)$ and $y = (r_1', \dots, r_k')$.

The family of subsets $\{S_{i,\ell} \mid 0 \le i \le k\}$ is a partition of $\mathcal{Y}_\ell^2$, called the *self-intersection partition* of $\mathcal{Y}_\ell$. Note that $(S_{i,\ell})_{\ell\in\mathbb{N}}$ is a sequence of subsets of the sequence $((B_\ell^{(k)})^2)_{\ell\in\mathbb{N}}$, with density smaller than $\operatorname{dens}_{((B_\ell^{(k)})^2)}(\mathcal{Y}_\ell^2) = \operatorname{dens}_{(B_\ell^{(k)})}(\mathcal{Y}_\ell)$.

Definition 3.5 (d–small self-intersection condition [Tsai 2022, Definition 3.5]) Let $(\mathcal{Y}_\ell)$ be a sequence of fixed subsets of $(B_\ell^{(k)})$ with density α. Let $S_{i,\ell}$ with $0 \le i \le k$ be its self-intersection partition. Let $d > 1-\alpha$. We say that $(\mathcal{Y}_\ell)$ satisfies the d–small self-intersection condition if, for every $1 \le i \le k-1$,

$$\operatorname{dens}_{((B_\ell^{(k)})^2)}(S_{i,\ell}) < \alpha - (1-d) \times \frac{i}{2k}.$$

Theorem 3.6 (multidimensional intersection formula [Tsai 2022, Theorem 3.6]) *Let (R_ℓ) be a sequence of permutation invariant random subsets of (B_ℓ) of density d. Let $(\mathcal{Y}_\ell)$ be a sequence of fixed subsets of $(B_\ell^{(k)})$ of density $\alpha > 1-d$. If $(\mathcal{Y}_\ell)$ satisfies the d–small self-intersection condition, then the sequence of random subsets $(\mathcal{Y}_\ell \cap R_\ell^{(k)})$ is densable with density $\alpha + d - 1$.*

In particular, a.a.s. the random subset $\mathcal{Y}_\ell \cap R_\ell^{(k)}$ of $B_\ell^{(k)}$ is not empty.

3.3 Proof of Theorem 1.6(ii)

Let (Y_ℓ) be a sequence of 2–complexes of the same geometric form (Y, λ) with k faces. In the following, we denote by $\mathcal{Y}_\ell$ the set of pairwise distinct relators in B_ℓ that fill Y_ℓ, which is a subset of $B_\ell^{(k)}$.

Let $(G_\ell(m,d))$ be a sequence of random groups at density d, defined by $G_\ell(m,d) = \langle X_m \mid R_\ell \rangle$, where (R_ℓ) is a sequence of random subsets with density d. The intersection $\mathcal{Y}_\ell \cap R_\ell^{(k)}$ is hence the set of k–tuples of pairwise distinct relators in R_ℓ that fill Y_ℓ. We want to prove that this intersection is not empty, so that Y_ℓ is fillable by $G_\ell(m,d)$. According to Theorem 3.6, it remains to prove that if $\operatorname{dens}_c Y > 1-d$, then the sequence $(\mathcal{Y}_\ell)$ is densable and satisfies the d–small self-intersection condition.

We will prove in Lemma 3.8 that $(\mathcal{Y}_\ell)$ is densable with density exactly $\operatorname{dens}(Y)$, and in Lemma 3.9 that it satisfies the d–small self-intersection condition.

Lemma 3.7 *Let $\overline{\mathcal{Y}_\ell}$ be the set of k–tuples of relators in B_ℓ that fill Y_ℓ, not necessarily pairwise distinct. If Y_ℓ is fillable by B_ℓ, then*

$$\operatorname{dens}_{(B_\ell^k)}(\overline{\mathcal{Y}_\ell}) = \operatorname{dens} Y.$$

Proof We shall estimate the number $|\overline{\mathcal{Y}_\ell}|$ by counting the number of labelings on edges of Y_ℓ that produce van Kampen 2–complexes with respect to all possible relators B_ℓ.

We start by filling edges in the neighborhoods of vertices that are originally vertices of the geometric form Y (before dividing). Consider the set of oriented edges of Y_ℓ starting at some vertex that is originally a vertex of Y before dividing. A *vertex labeling* is a labeling on these edges by $X_m^\pm$ that does not produce any reducible pair of edges on face boundaries: for every pair of different edges e_1, e_2 starting at the same vertex, if they are labeled by the same generator $x \in X_m^\pm$, then the path $e_1^{-1}e_2$ is not cyclically part of any face boundary loop. Since the 2–complex Y_ℓ is fillable, the set of vertex labelings is not empty. Denote by $C \ge 1$ the number of vertex labelings of Y_ℓ.

As $m \ge 2$ and $\lfloor \lambda_e \ell \rfloor \ge 3$ for ℓ large enough, if there exists a vertex labeling, then the other edges of Y_ℓ can be completed as a van Kampen 2–complex of B_ℓ, and the number C depends only on the geometric form Y.

To label the remaining $\lfloor \lambda \ell \rfloor - 2$ edges on the arc divided from the edge $e \in \mathrm{Edge}(Y)$, there are $2m-1$ choices for the first $\lfloor \lambda \ell \rfloor - 3$ edges, and $2m-2$ or $2m-1$ choices for the last edge. So

$$C \prod_{e \in \mathrm{Edge}(Y)} (2m-1)^{\lfloor \lambda_e \ell \rfloor - 3}(2m-2) \le |\overline{\mathcal{Y}_\ell}| \le C \prod_{e \in \mathrm{Edge}(Y)} (2m-1)^{\lfloor \lambda_e \ell \rfloor - 2}.$$

Recall that $k = |Y| = |Y_\ell|$ and that $|B_\ell^k| = (2m-1)^{k\ell + o(\ell)}$. We have

$$\mathrm{dens}_{(B_\ell^k)}(\overline{\mathcal{Y}_\ell}) = \frac{\sum_{e \in \mathrm{Edge}(Y)} \lambda_e}{|Y|} = \mathrm{dens}\, Y. \qquad \square$$

Lemma 3.8 *If* $\mathrm{dens}_c\, Y > \frac{1}{2}$ *and* Y_ℓ *is fillable by* B_ℓ, *then* $(\mathcal{Y}_\ell)$ *is densable in* $(B_\ell^{(k)})$ *and*

$$\mathrm{dens}_{(B_\ell^{(k)})}(\mathcal{Y}_\ell) = \mathrm{dens}\, Y.$$

Proof Suppose that $|Y| \ge 2$. The case $|Y| = 1$ is trivial. Let Z be a sub-2–complex of Y with exactly two faces f_1, f_2. As $\mathrm{dens}\, Z \ge \mathrm{dens}_c\, Y > \frac{1}{2}$, by Definition 3.2, we have

$$\sum_{e \in \mathrm{Edge}(Z)} \lambda_e > \tfrac{1}{2}|Z| = 1 \ge |\partial f_1|.$$

Let $\overline{\mathcal{Y}_\ell^Z}$ be the set of fillings of Y_ℓ by B_ℓ such that the two faces of Z are filled by the same relator. By the same arguments of the previous lemma,

$$|\overline{\mathcal{Y}_\ell^Z}| \le C(2m-1)^{|\partial f_1|} \prod_{e \in \mathrm{Edge}(Y) \setminus \mathrm{Edge}(Z)} (2m-1)^{\lfloor \lambda_e \ell \rfloor - 2},$$

so

$$\begin{aligned}\mathrm{dens}_{(B_\ell^k)}(\overline{\mathcal{Y}_\ell^Z}) &\le \frac{1}{|Y|}\Bigg[\sum_{e \in \mathrm{Edge}(Y)} \lambda_e + \Bigg(|\partial f_1| - \sum_{e \in \mathrm{Edge}(Z)} \lambda_e\Bigg)\Bigg] \\ &< \frac{\sum_{e \in \mathrm{Edge}(Y)} \lambda_e}{|Y|} = \mathrm{dens}\, Y = \mathrm{dens}_{(B_\ell^k)}(\overline{\mathcal{Y}_\ell}).\end{aligned}$$

Knowing that

$$\mathcal{Y}_\ell = \overline{\mathcal{Y}_\ell} \Big\backslash \bigcup_{Z < Y,\ |Z|=2} \overline{\mathcal{Y}_\ell^Z},$$

we have

$$|\overline{\mathcal{Y}_\ell}| - \sum_{Z < Y,\ |Z|=2} |\overline{\mathcal{Y}_\ell^Z}| \le |\mathcal{Y}_\ell| \le |\overline{\mathcal{Y}_\ell}|.$$

There are $\binom{|Y|}{2}$ terms in the sum, in every term we have $\mathrm{dens}_{(B_\ell^k)}(\overline{\mathcal{Y}_\ell^Z}) < \mathrm{dens}_{(B_\ell^k)}(\overline{\mathcal{Y}_\ell})$, so (see [Tsai 2022, Propositions 2.7 and 2.8])

$$\mathrm{dens}_{(B_\ell^k)}(\mathcal{Y}_\ell) = \mathrm{dens}_{(B_\ell^k)}(\overline{\mathcal{Y}_\ell}).$$

Together with Lemma 3.7, we have $\mathrm{dens}_{(B_\ell^k)}(\mathcal{Y}_\ell) = \mathrm{dens}\, Y$. As $\mathrm{dens}_{(B_\ell^k)}(B_\ell^{(k)}) = 1$, we get

$$\mathrm{dens}_{(B_\ell^{(k)})}(\mathcal{Y}_\ell) = \mathrm{dens}\, Y. \qquad \square$$

Lemma 3.9 *Suppose that* $\mathrm{dens}_c\, Y > 1-d$. *Let* $S_{i,\ell}$ *be the* i^{th} *self-intersection of the set* $\mathcal{Y}_\ell$. *We have*

$$\mathrm{dens}_{((B_\ell^{(k)})^2)}(S_{i,\ell}) < \mathrm{dens}\, Y - (1-d) \times \frac{i}{2k}.$$

Proof Let Z, W be two sub-2–complexes of Y with $|Z| = |W| = i < k = |Y|$. Let $(Z_\ell), (W_\ell)$ be the corresponding sequences of 2–complexes of the geometric forms Z and W, respectively. Denote by $S_\ell(Z, W)$ the set of pairs of pairwise distinct fillings $((r_1, \dots, r_k), (r'_1, \dots, r'_k))$ of Y_ℓ by all possible relators B_ℓ such that, the i relators in the first filling $(r_1, \dots, r_k)$ corresponding to Z_ℓ are identical to the i relators in the second filling $(r'_1, \dots, r'_k)$ corresponding to W_ℓ, and that the remaining $2k-2i$ relators are pairwise different, not repeating the relators in Z_ℓ and W_ℓ.

Let us estimate the cardinality $|S_\ell(Z, W)|$. First, fill the k–tuple $(r_1, \dots, r_k)$ so the i relators in the next k–tuple $(r'_1, \dots, r'_k)$ corresponding to the sub-2–complex W_ℓ is determined. There are at most $i!$ choices for ordering these i relators. To fill the remaining $k-i$ relators in $(r'_1, \dots, r'_k)$, by the same arguments of Lemma 3.7, we get

$$|S_\ell(Z, W)| \le |\mathcal{Y}_\ell| \times i! \times C \prod_{e \in \mathrm{Edge}(Y) \setminus \mathrm{Edge}(W)} (2m-1)^{\lfloor \lambda_e \ell \rfloor - 2}.$$

Recall that the density of Y is defined by $(1/|Y|)\big(\sum_{e \in \mathrm{Edge}(Y)} \lambda_e\big)$, and that $\mathrm{dens}\, W \ge \mathrm{dens}_c\, Y > 1-d$ by Definition 3.2. Together with the hypothesis $\mathrm{dens}_c\, Y > 1-d$, we have

$$\begin{aligned}
\mathrm{dens}_{((B_\ell^{(k)})^2)}(S_\ell(Z, W)) &\le \frac{1}{2k}\Bigg(\sum_{e \in \mathrm{Edge}(Y)} \lambda_e + \sum_{e \in \mathrm{Edge}(Y) \setminus \mathrm{Edge}(W)} \lambda_e\Bigg) \\
&= \frac{1}{2k}\Bigg(2 \sum_{e \in \mathrm{Edge}(Y)} \lambda_e - \sum_{e \in \mathrm{Edge}(W)} \lambda_e\Bigg) \\
&= \mathrm{dens}\, Y - \frac{i}{2k}\, \mathrm{dens}\, W \\
&< \mathrm{dens}\, Y - \frac{i}{2k}(1-d).
\end{aligned}$$

Note that

$$S_{i,\ell} = \bigcup_{\substack{Z < Y,\ W < Y \\ |Z| = |W| = i}} S_\ell(Z, W).$$

It is a union of $\binom{k}{i}^2$ subsets of densities strictly smaller than $\mathrm{dens}\, Y - \frac{i}{2k}(1-d)$. According to [Tsai 2022, Proposition 2.7], we have

$$\mathrm{dens}_{((B_\ell^{(k)})^2)}(S_{i,\ell}) < \mathrm{dens}\, Y - \frac{i}{2k}(1-d). \qquad \square$$

This completes the proof of Theorem 1.6.

4 Phase transitions for small cancellation conditions

Let us recall small cancellation notions in [Lyndon and Schupp 1977, page 240]. A *piece* with respect to a set of relators is a cyclic subword that appears at least twice. A group presentation satisfies the $C'(\lambda)$ small cancellation condition for some $0 < \lambda < 1$ if the length of a piece is at most λ times the length of any relator in which it appears. It satisfies the $C(p)$ small cancellation condition for some integer $p \geq 2$ if no relator is a product of fewer than p pieces.

The $C'(\lambda)$ condition Let $(G_\ell(m,d))$ be a sequence of random groups at density d. It is known that there is a phase transition at density $d = \lambda/2$ for the $C'(\lambda)$ condition (see [Gromov 1993, page 274; Bassino et al. 2020, Theorem 2.1; Tsai 2022, Theorem 1.4]). We give here a much simpler proof using Theorem 1.6.

Proposition 4.1 *Let $0 < \lambda < 1$. Let $(G_\ell(m,d))$ be a sequence of random groups at density d. There is a phase transition at density $d = \lambda/2$:*

(i) *If $d < \lambda/2$, then a.a.s. $G_\ell(m,d)$ satisfies $C'(\lambda)$.*

(ii) *If $d > \lambda/2$, then a.a.s. $G_\ell(m,d)$ does not satisfy $C'(\lambda)$.*

Proof (i) Let us prove by contradiction. Suppose that a.a.s. $G_\ell(m,d)$ does not satisfy $C'(\lambda)$. That is to say, a.a.s. there exists a piece w that appears in relators r_1, r_2 with $|w| > \lambda |r_1|$. It is possible that $r_1 = r_2$, but the piece should be at different positions.

Construct a van Kampen diagram D by gluing two combinatorial disks with one face, labeled respectively by r_1 and r_2, along with the paths where the piece w appears (Figure 4, left). As $r_1 \neq r_2$ or $r_1 = r_2$ but the piece appears at different positions, we obtain a reduced van Kampen diagram. The diagram satisfies $|D^{(1)}| = |r_1| + |r_2| + |w| < \ell + \ell + \lambda\ell < (1 - \lambda/2)|D|\ell$, which contradicts Theorem 1.5.

(ii) Consider a geometric form Y with two faces sharing a common edge of length λ, the other two edges are of length $1-\lambda$ (Figure 4, right). We have $\operatorname{dens} Y = \frac{1}{2}(2(1-\lambda)+\lambda) > 1-d$, and every sub-2–complex with one face is with density $1 > 1-d$. So $\operatorname{dens}_c Y > 1-d$.

Let (Y_ℓ) be a sequence of 2–complexes of the geometric form Y. By Theorem 1.6, a.a.s. Y_ℓ is fillable by $G_\ell(m,d)$, hence a.a.s. $G_\ell(m,d)$ does not satisfy $C'(\lambda)$. □

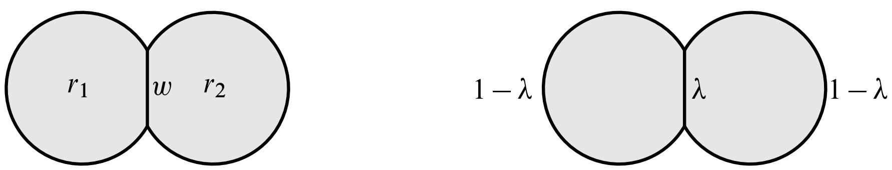

Figure 4: Left: a van Kampen 2–complex constructed from a $C'(\lambda)$ group. Right: the geometric form for the $C'(\lambda)$ condition.

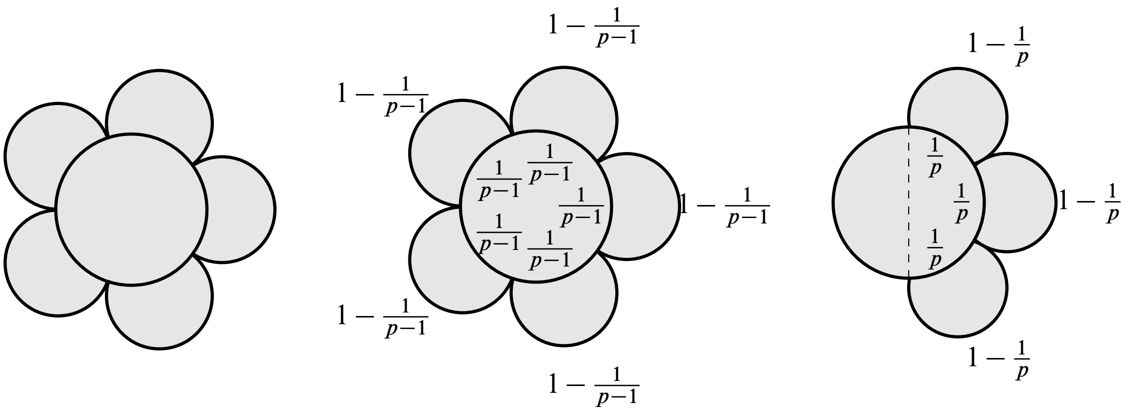

Figure 5: Left: a van Kampen 2–complex constructed from a non-$C(p)$ group. Middle: the geometric form for the $C(p)$ condition. Right: the geometric form for the $B(2p)$ condition.

The $C(p)$ condition We shall prove by Theorem 1.6 that for random groups with density, there is a phase transition at density $1/p$ for the $C(p)$ condition.

Proposition 4.2 *Let $p \geq 2$ be an integer. Let $(G_\ell(m,d))$ be a sequence of random groups at density d. There is a phase transition at density $1/p$:*

(i) *If $d < 1/p$, then a.a.s. $G_\ell(m,d)$ satisfies $C(p)$.*

(ii) *If $d > 1/p$, then a.a.s. $G_\ell(m,d)$ does not satisfy $C(p)$.*

Proof (i) Let us prove by contradiction. Suppose that a.a.s. $G_\ell(m,d)$ does not satisfy $C(p)$. That is to say, a.a.s. there is a relator that is a product of q pieces with $q \leq p-1$. By this relator we construct a reduced van Kampen diagram D with $q+1$ faces, one face is placed in the center, attached by the other q faces on the whole boundary, and there is no other attachments (Figure 5, left).

Observer that $|D| = q+1$ and $|D^{(1)}| \leq q\ell$ (sum of the boundary lengths of the outer q faces). Let $\varepsilon = (1/(q+1)-d)/2$, which is positive since $d < 1/p \leq 1/(q+1)$. We have

$$1-d-\varepsilon = \frac{q}{q+1} + \varepsilon > \frac{q}{q+1}.$$

Hence $|D^{(1)}| < (1-d-\varepsilon)|D|\ell$, which contradicts Theorem 1.5.

(ii) Consider a geometric form Y with p faces, one of the faces is placed in the center, having $p-1$ edges of length $1/(p-1)$, such that every edge is attached by another face with two edges of lengths $1/(p-1)$ and $1-1/(p-1)$. There are no other attachments (Figure 5, middle).

The density of Y is $(p-1)/p > 1-d$. If Z is a sub-2–complex of Y not containing the center face, then $\operatorname{dens} Z = 1 > 1-d$. If Z contains the center face and $i \leq p$ other faces, then

$$\operatorname{dens} Z = \frac{1+i(1-1/(p-1))}{i+1} > 1-d.$$

So $\operatorname{dens}_c Y > 1-d$.

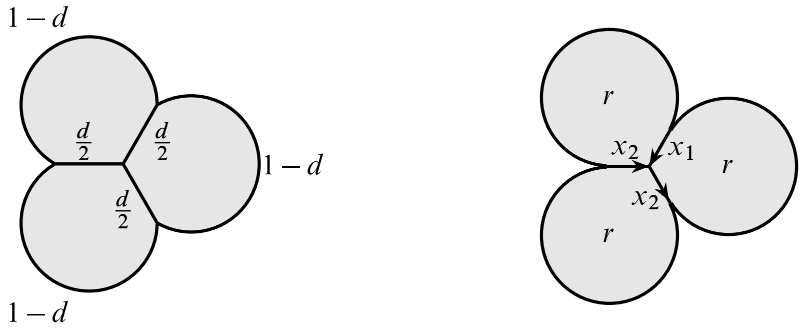

Figure 6: Left: the geometric form for the $T(q)$ condition. Right: a random relator r denying the $T(q)$ condition.

Let (Y_ℓ) be a sequence of 2–complexes of the geometric form Y. By Theorem 1.6, a.a.s. Y_ℓ is fillable by $G_\ell(m,d)$, hence a.a.s. $G_\ell(m,d)$ does not satisfy $C(p)$. □

The $B(2p)$ condition The same argument holds for the $B(2p)$ condition, introduced in [Ollivier and Wise 2011, Definition 1.7]: half of a relator cannot be the product of fewer than p pieces. One can construct a geometric form with $p+1$ faces, one of the faces is in the center, with half of its boundary attached by the other p faces, each with length $1/p$ (Figure 5, right). Its critical density is $\left(p+\frac{1}{2}\right)/(p+1)$, so a phase transition occurs at density $d = 1/(2p+2)$.

Proposition 4.3 *Let $p \geq 1$ be an integer. Let $(G_\ell(m,d))$ be a sequence of random groups at density d. There is a phase transition at density $d = 1/(2p+2)$:*

(i) *If $d < 1/(2p+2)$, then a.a.s. $G_\ell(m,d)$ satisfies $B(2p)$.*

(ii) *If $d > 1/(2p+2)$, then a.a.s. $G_\ell(m,d)$ does not satisfy $B(2p)$.* □

The $T(q)$ condition Recall that [Lyndon and Schupp 1977, page 241] a group presentation satisfies the $T(q)$ small cancellation condition for some $q \geq 4$ if, in every of its reduced van Kampen diagram, every vertex of valency at least 3 is actually of valency at least q.

Proposition 4.4 *For any density $0 < d \leq 1$, a.a.s. $G_\ell(m,d)$ does not satisfy $T(4)$.*

Proof We shall construct a reduced van Kampen diagram with a vertex of valency exactly 3. Consider the geometric form Y with 3 faces sharing one common vertex, attaching to each other with common segments of length $d/2$ (Figure 6, left). The critical density of Y is $1-d/2 > 1-d$, so by Theorem 1.6, a.a.s. the random group $G_\ell(m,d)$ has a van Kampen diagram of the form Y. □

Remark 4.5 Proposition 4.4 holds for the few relator model. For example, for a one relator random group $\langle x_1,\dots,x_m \mid r\rangle$ with $m \geq 2$, a.a.s. (when $|r| \to \infty$) the three subwords x_1x_2, x_2^{-2} and $x_2x_1^{-1}$ appear in the random relator r at different places. By these subwords, we can construct a reduced van Kampen diagram with 3 faces that has a vertex of valency exactly 3 (Figure 6, right), denying the $T(4)$ small cancellation condition.

References

[Antoniuk et al. 2015] **S Antoniuk**, **T Łuczak**, **J Świątkowski**, *Random triangular groups at density $\frac{1}{3}$*, Compos. Math. 151 (2015) 167–178 MR Zbl

[Bassino et al. 2020] **F Bassino**, **C Nicaud**, **P Weil**, *Random presentations and random subgroups: a survey*, from "Complexity and randomness in group theory", de Gruyter, Berlin (2020) 45–76

[Calegari and Walker 2015] **D Calegari**, **A Walker**, *Random groups contain surface subgroups*, J. Amer. Math. Soc. 28 (2015) 383–419 MR Zbl

[Ghys 2004] **É Ghys**, *Groupes aléatoires (d'après Misha Gromov,…)*, from "Séminaire Bourbaki, 2002/2003", Astérisque 294, Soc. Math. France, Paris (2004) exposé 916, 173–204 MR Zbl

[Gromov 1993] **M Gromov**, *Finitely presented groups*, from "Geometric group theory, II: Asymptotic invariants of infinite groups", Lond. Math. Soc. Lect. Note Ser. 182, Cambridge Univ. Press (1993) 269–282 Zbl

[Gromov 2003] **M Gromov**, *Random walk in random groups*, Geom. Funct. Anal. 13 (2003) 73–146 MR Zbl

[Gruber and Mackay 2021] **D Gruber**, **J M Mackay**, *Random triangular Burnside groups*, Israel J. Math. 244 (2021) 75–94 MR Zbl

[Kapovich and Schupp 2008] **I Kapovich**, **P Schupp**, *On group-theoretic models of randomness and genericity*, Groups Geom. Dyn. 2 (2008) 383–404 MR Zbl

[Lyndon and Schupp 1977] **R C Lyndon**, **P E Schupp**, *Combinatorial group theory*, Ergebnisse der Math. 89, Springer (1977) MR Zbl

[Odrzygóźdź 2021] **T Odrzygóźdź**, *Nonplanar isoperimetric inequality for random groups*, preprint (2021) arXiv 2104.13903

[Ollivier 2004] **Y Ollivier**, *Sharp phase transition theorems for hyperbolicity of random groups*, Geom. Funct. Anal. 14 (2004) 595–679 MR Zbl

[Ollivier 2005] **Y Ollivier**, *A January* 2005 *invitation to random groups*, Ensaios Mat. 10, Soc. Brasil. Mat., Rio de Janeiro (2005) MR Zbl

[Ollivier 2007] **Y Ollivier**, *Some small cancellation properties of random groups*, Int. J. Algebra Comput. 17 (2007) 37–51 MR Zbl

[Ollivier and Wise 2011] **Y Ollivier**, **D T Wise**, *Cubulating random groups at density less than $1/6$*, Trans. Amer. Math. Soc. 363 (2011) 4701–4733 MR Zbl

[Tsai 2022] **T-H Tsai**, *Density of random subsets and applications to group theory*, J. Comb. Algebra 6 (2022) 223–263 MR Zbl

[Whitehead 1939] **J H C Whitehead**, *Simplicial spaces, nuclei and m–groups*, Proc. Lond. Math. Soc. 45 (1939) 243–327 MR Zbl

Institut de Recherche Mathématique Avancée, Université de Strasbourg
Strasbourg, France

Current address: *Institut Camille Jordan, Université Claude Bernard Lyon 1*
Villeurbanne, France

tsai71517@gmail.com

Received: 23 November 2022 Revised: 4 September 2023

Algebraic & Geometric Topology

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 7 (pages 3571–4137) 2024