Algebraic & Geometric Topology

Volume 24 (2024)

Vector fields on noncompact manifolds

TSUYOSHI KATO

DAISUKE KISHIMOTO

MITSUNOBU TSUTAYA

msp

Algebraic & Geometric Topology **24**:7 (2024) 3985–3996
DOI: 10.2140/agt.2024.24.3985
Published: 9 December 2024

Vector fields on noncompact manifolds

TSUYOSHI KATO
DAISUKE KISHIMOTO
MITSUNOBU TSUTAYA

Let M be a noncompact connected manifold with a cocompact and properly discontinuous action of a discrete group G. We establish a Poincaré–Hopf theorem for a bounded vector field on M satisfying a mild condition on zeros. As an application, we show that such a vector field must have infinitely many zeros whenever G is amenable and the Euler characteristic of M/G is nonzero.

57R25, 58K45

1 Introduction

Let M be a noncompact connected manifold. Then M admits a nonvanishing vector field as M admits a vector field with isolated zeros which can be swept out to infinity. However, the resulting nonvanishing vector field is not satisfactory because it may not be bounded in the following sense. A vector field v on a Riemannian manifold is *bounded* if both $|v|$ and $|dv|$ are bounded, where dv denotes the derivative of v. Note that our boundedness condition is different from the one in [Cima and Llibre 1990] and its related works. Bounded vector fields appear in the study of manifolds of bounded geometry. Now we ask whether or not a nonvanishing bounded vector field exists on M. Weinberger [2009, Theorem 1] proved that a manifold M of bounded geometry has a nonvanishing vector field v with $|v|$ constant and $|dv|$ bounded if and only if the Euler class of M in the bounded de Rham cohomology $\widehat{H}^*(M)$ is trivial. As one may think of the Poincaré–Hopf theorem as a refinement of the Euler class criterion for the existence of a nonvanishing vector field on a compact orientable manifold, it is natural to ask whether or not one can establish the Poincaré–Hopf theorem for bounded vector fields on noncompact manifolds of bounded geometry.

A typical manifold of bounded geometry is a covering space of a compact manifold, which we equip with a lift of a metric on the base compact manifold. We say that an action of a group G on a space X is properly discontinuous if every point $x \in X$ has a neighborhood U such that $gU \cap U \neq \varnothing$ implies $g = 1$. Equivalently, the quotient map $X \to X/G$ is a covering. So we consider a connected noncompact manifold M on which a cocompact and properly discontinuous action of a group G is given, and will establish the Poincaré–Hopf theorem for a bounded vector field on M. To state it, we set notation. Let $\ell^\infty(G)$ denote the vector space of bounded functions $G \to \mathbb{R}$, and let G act on $\ell^\infty(G)$ from the left by

$$(g\phi)(h) = \phi(hg)$$

for $g, h \in G$ and $\phi \in \ell^\infty(G)$. We define the module of coinvariants of $\ell^\infty(G)$ by

$$\ell^\infty(G)_G = \ell^\infty(G)/\langle \phi - g\phi \mid \phi \in \ell^\infty(G),\ g \in G\rangle,$$

where $\langle S\rangle$ denotes the vector subspace of $\ell^\infty(G)$ generated by a subset $S \subset \ell^\infty(G)$. Let $\mathbb{1} \in \ell^\infty(G)$ denote the constant function with value 1. Let $D \subset M$ be a fundamental domain (its definition is given in Section 3). We will define the index $\operatorname{ind}(v)$ of a bounded vector field v on M as an element of $\ell^\infty(G)_G$, and will prove

$$\operatorname{ind}(v)(g) = \sum_{x \in \operatorname{Zero}(v) \cap gD} \operatorname{ind}_x(v)$$

whenever v is strongly tame, which is a mild condition on the zeros of v defined in Definition 5.5, where $\operatorname{ind}_x(v)$ denotes the local index of a vector field v at the zero $x \in \operatorname{Zero}(v)$. Here, the above equality means that there is a representative $\phi \in \ell^\infty(G)$ of $\operatorname{ind}(v) \in \ell^\infty(G)_G$ such that $\phi(g)$ is the right-hand side of the equality. Now we are ready to state the Poincaré–Hopf theorem for a bounded vector field on M.

Theorem 1.1 *Let M be a noncompact connected manifold equipped with a cocompact and properly discontinuous action of a group G such that M/G is orientable. If a vector field v on M is strongly tame and bounded, then*

$$\operatorname{ind}(v) = \chi(M/G)\mathbb{1} \quad \textit{in } \ell^\infty(G)_G.$$

As an application of Theorem 1.1, we will prove:

Theorem 1.2 *Let M and G be as in Theorem 1.1. If G is amenable and $\chi(M/G) \neq 0$, then every tame bounded vector field on M must have infinitely many zeros.*

Let M and G be as in Theorem 1.1. Then by the abovementioned result of Weinberger [2009, Theorem 1] together with [Attie et al. 1992], one can deduce that a vector bundle v on M with $|v|$ constant and $|dv|$ bounded must have a zero. But one cannot deduce further information on zeros, such as their numbers, from these results. As an application of Theorem 1.2 we will get the following result, where tameness of a diffeomorphism is defined in Definition 5.6 in an analogous way to tameness of a vector field:

Corollary 1.3 (cf. [Weinberger 2009, Corollary to Theorem 1]) *Let M and G be as in Theorem 1.1. If G is amenable and $\chi(M/G) \neq 0$, then every tame diffeomorphism of M which is C^1 close to the identity map must have infinitely many fixed points.*

Example 1.4 Let L be the noncompact surface called Jacob's ladder, a surface with infinite genus and two ends, which admits an infinite cyclic covering map onto the closed oriented surface of genus 2. Then we can apply Corollary 1.3 to L, and conclude that any tame diffeomorphism of L which is C^1 close to the identity map must have infinitely many zeros. This can be easily generalized to the infinite connected sum $M = \#^\infty N$ of a closed connected oriented even-dimensional manifold N with $\chi(N) \neq 2$.

We briefly describe the strategy of our proof, as well as some of the tools we exploit. Let M and G be as in Theorem 1.1. Recall that the Poincaré–Hopf theorem for a compact manifold can be proved by using a suitable integral in top-dimensional de Rham cohomology. Motivated by the compact case, we will define the integral

$$\int_M : \hat{H}^n(M) \to \ell^\infty(G)_G, \tag{1}$$

where $\dim M = n$, and will prove Theorem 1.1 by using it similarly to the compact case. So our approach is an extension of the classical case by means of $\ell^\infty(G)_G$. However, unlike in the compact case, the target module $\ell^\infty(G)_G$ of the integral has some interesting algebraic properties we will use to deduce Theorem 1.2.

Let us observe possible connections of our results to other contexts. Our results could be connected to the index theory on open manifolds by Roe [1988]. More specifically, our index could be related to the index of the Dirac operator

$$d + d^* : \hat{\Omega}^{\mathrm{even}}(M) \to \hat{\Omega}^{\mathrm{odd}}(M)$$

on the bounded de Rham complex $\hat{\Omega}^*(M)$, which lives in the operator K–theory $K_*(C^*_u(|G|))$ of the uniform Roe algebra $C^*_u(|G|)$. There is another possible connection. In [Kato et al. 2024], the pushforward of a vector bundle on M to M/G is defined, and its structure group is the group of unitary operators with finite propagation on the Hilbert space of square integrable functions $G \to \mathbb{C}$. On the other hand, as in [Kato et al. 2023a; 2023b], the module of coinvariants of bounded functions $\mathbb{Z} \to \mathbb{Z}$ appears in the homotopy groups of such a group of unitary operators of finite propagation for $G = \mathbb{Z}$. Then our results could be connected to the obstruction theory for the pushforward of TM onto M/G.

As mentioned in [Block and Weinberger 1992], there is an isomorphism $\hat{H}^n(M) \cong H_0^{\mathrm{uf}}(M)$ (see [Attie and Block 1998] for the proof), where $\dim M = n$ and $H_*^{\mathrm{uf}}(M)$ denotes the uniformly finite homology of M as in [Block and Weinberger 1992]. Since uniformly finite homology is a quasi-isometry invariant, there is an isomorphism $H_0^{\mathrm{uf}}(M) \cong H_0^{\mathrm{uf}}(G)$. On the other hand, as in [Brodzki et al. 2012], there is an isomorphism $H_0^{\mathrm{uf}}(G) \cong \ell^\infty(G)_G$. Then we get an isomorphism

$$\hat{H}^n(M) \cong \ell^\infty(G)_G.$$

However, this isomorphism is not explicit as it is given by a zigzag of several isomorphisms. We believe that the integral (1) gives a direct and explicit description of this isomorphism. Our intuition relies on the case $M = \mathbb{R}$ and $G = \mathbb{Z}$, which we treat in Proposition 4.5, and we propose the following:

Conjecture 1.5 The integral (1) is an isomorphism.

Throughout this paper manifolds will be smooth and without boundary, unless otherwise specified, and group actions on manifolds will be smooth too.

Acknowledgments The authors were partially supported by JSPS KAKENHI grants JP22H01123 (Kato), JP22K03284 and JP19K03473 (Kishimoto), and JP22K03317 (Tsutaya). The authors deeply appreciate the referees' useful advice and comments, which especially improved Section 2.

2 Module of coinvariants

In this section, we collect properties of the module of coinvariants $\ell^\infty(G)_G$ that we are going to use. Block and Weinberger [1992] introduced the uniformly finite homology $H_*^{\mathrm{uf}}(X)$ of a metric space X, and showed its basic properties. Later, Brodzki, Niblo, and Wright [Brodzki et al. 2012] studied amenability of discrete groups by using the uniformly finite homology, where every discrete group will be equipped with a word metric. They observed that if G is finitely generated, then the uniformly finite chain complex $C_*^{\mathrm{uf}}(G)$ is naturally isomorphic to the chain complex $C_*(G;\ell^\infty(G))$. Then since $H_0(G,\ell^\infty(G)) = \ell^\infty(G)_G$, there is a natural isomorphism

$$H_0^{\mathrm{uf}}(G) \cong \ell^\infty(G)_G \tag{2}$$

whenever G is finitely generated.

Proposition 2.1 *Let G and H be finitely generated groups. Then a quasi-isometric homomorphism $G \to H$ induces an isomorphism*

$$\ell^\infty(H)_H \xrightarrow{\cong} \ell^\infty(G)_G.$$

Proof By [Block and Weinberger 1992, Corollary 2.2], a quasi-isometric homomorphism $G \to H$ induces an isomorphism $H_*^{\mathrm{uf}}(G) \xrightarrow{\cong} H_*^{\mathrm{uf}}(H)$. Then the statement follows from (2). $\square$

Corollary 2.2 *If G is a finite group, then*

$$\ell^\infty(G)_G \cong \mathbb{R}.$$

Proof Let 1 denote the trivial group. Since G is finite, the inclusion $1 \to G$ is a quasi-isometry. Then since $\ell^\infty(1)_1 \cong \mathbb{R}$, the statement is proved by Proposition 2.1. $\square$

Proposition 2.3 *Let G be a finitely generated infinite group, and let $\phi \in \ell^\infty(G)$. If $\phi(g) = 0$ for all but finitely many $g \in G$, then ϕ is zero in $\ell^\infty(G)_G$.*

Proof Whyte [1999, Theorem 7.6] gave a necessary and sufficient condition for an element of $C_0^{\mathrm{uf}}(G)$ to be trivial in $H_0^{\mathrm{uf}}(G)$. Through the natural isomorphism $C_0^{\mathrm{uf}}(G) \cong C_0(G;\ell^\infty(G)) = \ell^\infty(G)$, this condition is stated as follows: an element $\phi \in \ell^\infty(G)$ is zero in $\ell^\infty(G)_G$ if and only if there are $C > 0$ and $r > 0$ such that for any finite subset $S \subset G$,

$$\left|\sum_{g\in S} \phi(g)\right| \le C \cdot \#\{g \in G \mid 0 < d(g,S) \le r\},$$

where d denotes a word metric of G. If G is infinite, then for any nonempty finite subset $S \subset G$, we have $\#\{g \in G \mid 0 < d(g,S) \le 1\} \ge 1$. Suppose $\phi \in \ell^\infty(G)$ satisfies $\phi(g) = 0$ for all but finitely many $g \in G$. Then if we set $C = \sum_{g\in G} |\phi(g)|$ and $r = 1$, the above inequality holds for any finite subset $S \subset G$, and so ϕ is zero in $\ell^\infty(G)_G$. $\square$

Recall that a mean on a group G is a linear map

$$\mu\colon \ell^\infty(G) \to \mathbb{R}$$

such that $\mu(\mathbb{1}) = 1$ and $\mu(\phi) \geq 0$ whenever $\phi(g) \geq 0$ for all $g \in G$, where $\mathbb{1} \in \ell^\infty(G)$ denotes the constant function with value 1 as in Section 1. A group G is *amenable* if it admits a G–invariant mean. The proof of [Block and Weinberger 1992, Theorem 3.1] together with (2) implies the following:

Proposition 2.4 *For a finitely generated group G, the following statements are equivalent*:

(i) *G is amenable.*

(ii) $\ell^\infty(G)_G \neq 0$.

(iii) *$\mathbb{1} \in \ell^\infty(G)$ is nonzero in $\ell^\infty(G)_G$.*

3 Basic properties of fundamental domains

In this section, we define a fundamental domain of a manifold with a free group action, and show its basic properties. Throughout this section, let M be a connected manifold of dimension n, possibly with boundary, on which a cocompact and properly discontinuous action of a group G is given. Since G is a quotient of the fundamental group of a compact manifold M/G, which is finitely generated, G is finitely generated.

We define a *fundamental domain* D of M as the closure of an open set of M such that

$$M = \bigcup_{g\in G} gD \quad \text{and} \quad \mathrm{Int}(D) \cap \mathrm{Int}(gD) = \varnothing$$

for all $1 \neq g \in G$. Remark that D need not be connected. A manifold M admits a fundamental domain. Indeed, given a triangulation of M/G, we can lift it to get a triangulation of M such that the G–action is free and simplicial. We choose one lift of the interior of each maximal simplex of M/G to M, so the closure of the union of these open simplices of M is a fundamental domain of M. We choose such a fundamental domain, so that D is a simplicial complex such that each $D \cap gD$ is a subcomplex of D and

$$\partial D = \Bigl(\bigcup_{1\neq g\in G} D \cap gD\Bigr) \cup (D \cap \partial M). \tag{3}$$

If $gD \cap hD$ is $(n-1)$–dimensional, then we call it a *facet* of gD (and hD). We also call $gD \cap \partial M$ a facet of gD when $\partial M \neq \varnothing$. Then the boundary of D is the union of its facets. Clearly the G–action on M restricts to ∂M, and $D \cap \partial M$ is a fundamental domain of ∂M.

We construct a generating set of G by using a fundamental domain D. Let S be a subset of G consisting of elements $g \in G$ such that $D \cap gD$ is a facet of D.

Proposition 3.1 *The set S is a symmetric generating set of G.*

Proof Let $g \in G$ and $x \in \mathrm{Int}(D)$. Then gx belongs to $\mathrm{Int}(gD)$, and so since M is connected there is a path ℓ from x to gx which passes $g_0D, g_1D, \dots, g_kD$ in order for $1 = g_0, g_1, \dots, g_{k-1}, g_k = g \in G$ such that $g_iD \cap g_{i+1}D$ is a facet and $\ell \cap g_iD \cap g_{i+1}D$ is a single point sitting in the interior of a facet $g_iD \cap g_{i+1}D$ of g_iD for $i = 0, 1, \dots, k-1$. Since $g_iD \cap g_{i+1}D = g_i(D \cap g_i^{-1}g_{i+1}D)$ is a facet of g_iD, $D \cap g_i^{-1}g_{i+1}D$ is a facet of D, implying $g_i^{-1}g_{i+1} \in S$. Thus since

$$g = g_k = (g_0^{-1}g_1)(g_1^{-1}g_2)\cdots(g_{k-1}^{-1}g_k),$$

we obtain that S is a generating set of G. If $g \in S$, then $g(D \cap g^{-1}D) = gD \cap D$ is a facet of D, and so $D \cap g^{-1}D$ is a facet of D too. Hence $g^{-1} \in S$, that is, S is symmetric, completing the proof. □

Corollary 3.2 *There is a partition $S = S^+ \sqcup S^- \sqcup S^0$ such that $(S^+)^{-1} = S^-$ and $(S^0)^2 = \{1\}$.*

Proof Let S^0 be the subset of S consisting of elements of order 2. Then the statement follows because S is symmetric. □

Let $S^+ = \{s_1, \dots, s_k\}$ and $S^0 = \{t_1, \dots, t_l\}$, where S^+ and S^0 are finite because G is finitely generated as mentioned above. We put

$$E = D \cap \partial M, \quad F_i^+ = D \cap s_iD, \quad F_i^- = D \cap s_i^{-1}D, \quad \text{and} \quad F_j^0 = D \cap t_jD$$

for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, l$.

Lemma 3.3 *The facets of D are $E, F_1^+, \dots, F_k^+, F_1^-, \dots, F_k^-, F_1^0, \dots, F_l^0$.*

Proof The statement follows from Corollary 3.2. □

We consider an orientation of a facet of gD:

Lemma 3.4 *Suppose that M is oriented. If $F = gD \cap hD$ is a facet for $g, h \in G$, then the orientations of F induced from gD and hD are opposite.*

Proof An outward vector of gD rooted at F is an inward vector of hD. Then the statement follows. □

4 The integral in bounded cohomology

In this section, we define the integral in bounded cohomology. Let M be a connected Riemannian manifold of dimension n, possibly with boundary. As in [Roe 1988], we say that a differential form ω on M is *bounded* if both $|\omega|$ and $|d\omega|$ are bounded. Let $\widehat{\Omega}^p(M)$ denote the set of bounded p–forms on M. Then by definition, $\widehat{\Omega}^*(M)$ is closed under differential, and so it is a differential graded algebra. We define the *bounded de Rham cohomology* of M as the cohomology of $\widehat{\Omega}^*(M)$, which we denote by $\widehat{H}^*(M)$. We record the following obvious fact:

Lemma 4.1 *If a map $f\colon M \to N$ between manifolds has bounded derivative, then it induces a map $f^*\colon \widehat{\Omega}^*(N) \to \widehat{\Omega}^*(M)$.*

Now we consider a cocompact and properly discontinuous action of a discrete group G on a manifold M, and choose a fundamental domain $D \subset M$. A Riemannian metric of M will be chosen to be the lift of a Riemannian metric of M/G. We assume that M/G is oriented. Then in particular, the fundamental domain D is oriented. We define the *integral* of a bounded differential form on M by

$$\int_M : \widehat{\Omega}^n(M) \to \ell^\infty(G), \quad \Big(\int_M \omega\Big)(g) = \int_{gD} \omega. \tag{4}$$

We may think of the above integral as the external transfer of the covering $M \to M/G$. Note that we can similarly define the integral for ∂M by using a fundamental domain $D \cap \partial M$ of ∂M. We prove Stokes' theorem:

Proposition 4.2 *For $\omega \in \widehat{\Omega}^{n-1}(M)$, we have*

$$\int_M d\omega = \int_{\partial M} \omega \quad \text{in } \ell^\infty(G)_G.$$

Proof We consider the facets of D described in Lemma 3.3. Define $\phi_i^\pm, \phi_j^0 \in \ell^\infty(G)$ by

$$\phi_i^\pm(g) = \int_{gF_i^\pm} \omega \quad \text{and} \quad \phi_j^0(g) = \int_{gF_i^0} \omega$$

for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, l$, where the orientations of $gF_i^\pm$ and gF_i^0 are induced from gD. Then by Lemma 3.3 and the usual Stokes' theorem, we have

$$\int_{gD} d\omega = \int_{gE} \omega + \sum_{i=1}^{k} (\phi_i^+(g) + \phi_i^-(g)) + \sum_{j=1}^{l} \phi_j^0(g),$$

where the orientation of gE is induced from gD. Since $gF_i^- = gs_i^{-1} F_i^+$, it follows from Lemma 3.4 that $\phi_i^-(g) = -\phi_i^+(gs_i^{-1})$. Then

$$\phi_i^+ + \phi_i^- = \phi_i^+ - s_i^{-1}\phi_i^+.$$

Quite similarly,

$$\phi_j^0 = \tfrac{1}{2}(\phi_j^0 - t_j^{-1}\phi_j^0).$$

Thus since $E = D \cap \partial M$ is a fundamental domain of ∂M, we obtain

$$\int_M d\omega = \int_{\partial M} \omega + \sum_{i=1}^{k} (\phi_i^+ - s_i^{-1}\phi_i^+) + \sum_{j=1}^{l} \tfrac{1}{2}(\phi_j^0 - t_j^{-1}\phi_j^0). \qquad \square$$

The following is immediate from Proposition 4.2:

Corollary 4.3 *If M is without boundary, then the integral (4) induces a map*

$$\int_M : \widehat{H}^n(M) \to \ell^\infty(G)_G.$$

By considering n–forms with support in gD, we can easily see that the integral in bounded cohomology is always surjective. We give two supporting examples for Conjecture 1.5:

Proposition 4.4 *Conjecture 1.5 is true for G finite.*

Proof If G is finite, then M is compact, and so $\widehat{H}^n(M)$ coincides with the usual n^{th} de Rham cohomology of M, which is isomorphic with $\mathbb{R}$. On the other hand, by Corollary 2.2, $\ell^\infty(G)_G \cong \mathbb{R}$. So since the integral in bounded cohomology is surjective, as mentioned above, it is actually an isomorphism, as stated. □

Proposition 4.5 *Conjecture 1.5 is true for $M = \mathbb{R}$ and $G = \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation.*

Proof We choose the interval $[0, 1] \subset \mathbb{R}$ as a fundamental domain. Let $g = 1 \in \mathbb{Z}$. Suppose that

$$\int_{\mathbb{R}} f(x)\, dx = \phi - g\phi \tag{5}$$

for a bounded function $f(x)$ on $\mathbb{R}$ and $\phi \in \ell^\infty(\mathbb{Z})$, where the integral is taken in the sense of (4). Equation (5) is equivalent to the fact that the 1–form $f(x)\, dx$ belongs to the kernel of the integral in bounded cohomology because

$$\phi - g^n\phi = (\phi + g\phi + \cdots + g^{n-1}\phi) - g(\phi + g\phi + \cdots + g^{n-1}\phi).$$

Note that $(\phi - g\phi)(i) = \phi(i) - \phi(i+1)$. Now we define

$$h(x) = \int_0^x f(t)\, dt.$$

To see that the integral in bounded cohomology is injective, it is sufficient to show that $h(x) \in \widehat{\Omega}^0(\mathbb{R})$. Since $dh(x) = f(x)\, dx$, $dh(x)$ is bounded. For $0 \le n \le x < n+1$, we have

$$h(x) = \sum_{i=0}^{n-1} \int_i^{i+1} f(t)\, dt + \int_n^x f(t)\, dt = \phi(0) - \phi(n) + \int_n^x f(t)\, dt.$$

Since $f(x)$ is bounded, $\int_n^x f(t)\, dt$ is bounded too as x and n vary. Then $h(x)$ is bounded for $x \ge 0$. Quite similarly, we can show that $h(x)$ is bounded for $x < 0$ too, and so $h(x) \in \widehat{\Omega}^0(\mathbb{R})$. Thus we obtain the injectivity. Since the integral in bounded cohomology is surjective as mentioned above, it is an isomorphism. □

5 Poincaré–Hopf theorem

In this section, we prove Theorems 1.1 and 1.2. Throughout this section, let M be a connected manifold of dimension n equipped with a cocompact and properly discontinuous action of a discrete group G such that M/G is oriented. The metric of M will be the lift of a metric of M/G.

Let Φ denote a representative of the Thom class of M/G. Then as in [Bott and Tu 1982], the support of Φ is compactly supported, and so Φ is a bounded n–form on $T(M/G)$. Let $\pi\colon M \to M/G$ denote the projection. Then the derivative of π is bounded, and so by Lemma 4.1 we get the induced map $\pi^*\colon \widehat{\Omega}^*(T(M/G)) \to \widehat{\Omega}^*(TM)$. In particular, $\pi^*(\Phi)$ is a bounded n–form on TM. Note that $\pi^*(\Phi)$

represents the Thom class of M in bounded cohomology. Let v be a vector field on M with $|dv|$ bounded. Then by Lemma 4.1, $v^*(\pi^*(\Phi))$ is a bounded n–form on M, and so we can define the index of v by

$$\operatorname{ind}(v) = \int_M v^*(\pi^*(\Phi)) \in \ell^\infty(G)_G.$$

We remark that the index $\operatorname{ind}(v)$ is independent of the choice of a representative Φ of the Thom class of M/G. Indeed, if Ψ is another representative of the Thom class of M/G, then $\Phi - \Psi = d\alpha$ for some compactly supported $(n-1)$–form α on $T(M/G)$, where Ψ is compactly supported. Hence we get $\pi^*(\Phi) - \pi^*(\Psi) = d\pi^*(\alpha)$, where all differential forms are bounded, and so by Corollary 4.3 the indices of v defined by Φ and Ψ are equal. We also remark that by Proposition 2.4, the index of a bounded vector field on M is always zero whenever G is not amenable; see [Weinberger 2009, Theorem 2]

We now show some properties of the index. Let v_0 denote the zero vector field, that is, the zero section $M \to TM$. Then $v_0^*(\pi^*(\Phi))$ is a representative of the Euler class $e(M)$ in bounded cohomology, which was considered by Weinberger [2009].

Proposition 5.1 *There is an equality*

$$\int_M e(M) = \chi(M/G)\mathbb{1}.$$

Proof Let $\bar{v}_0$ denote the zero vector field on M/G, so v_0 is the lift of $\bar{v}_0$. Since the projection $\pi : \operatorname{Int}(gD) \to M/G - \pi(\partial(gD))$ is a diffeomorphism and both $\partial(gD)$ and $\pi(\partial(gD))$ have measure zero,

$$\int_{gD} v_0^*(\pi^*(\Phi)) = \int_{M/G} \bar{v}_0^*(\Phi) = \int_{M/G} e(M/G) = \chi(M/G). \qquad \square$$

Lemma 5.2 *If vector fields v and w on M with $|dv|$ and $|dw|$ bounded are homotopic by a homotopy with bounded derivative, then*

$$\operatorname{ind}(v) = \operatorname{ind}(w).$$

Proof Let $v_t : M \times [0,1] \to TM$ be a homotopy with bounded derivative such that $v_0 = v$ and $v_1 = w$. Since the induced maps $v_t^* : \widehat{\Omega}^*(TM) \to \widehat{\Omega}^*(M \times [0,1])$ and $\pi^* : \widehat{\Omega}^*(T(M/G)) \to \widehat{\Omega}^*(TM)$ commute with the differential,

$$\int_{M\times[0,1]} dv_t^*(\pi^*(\Phi)) = \int_{M\times[0,1]} v_t^*(\pi^*(d\Phi)) = 0,$$

where Φ is a closed n–form representing the Thom class of $T(M/G)$. On the other hand, by Proposition 4.2

$$\int_{M\times[0,1]} dv_t^*(\pi^*(\Phi)) = \int_{M\times 1} w^*(\pi^*(\Phi)) - \int_{M\times 0} v^*(\pi^*(\Phi)). \qquad \square$$

Proposition 5.3 *Let v be a bounded vector field on M. Then we have*

$$\operatorname{ind}(v) = \operatorname{ind}(v_0).$$

Proof Clearly tv is a homotopy from v_0 to v with bounded derivative. So by Lemma 5.2, we are done. $\square$

We consider a mild condition on zeros of a vector field. Let $B_\delta(x)$ denote the open δ–neighborhood of $x \in M$, and let $B_\epsilon(M)$ denote the open ϵ–neighborhood of M in TM.

Definition 5.4 A vector field v on a manifold M is *tame* if there are $\delta > 0$ and $\epsilon > 0$ such that

(i) $B_\delta(x) \cap B_\delta(y) = \varnothing$ for $x \neq y \in \mathrm{Zero}(v)$, and

(ii) $v^{-1}(B_\epsilon(M)) \subset \bigcup_{x \in \mathrm{Zero}(v)} B_\delta(x)$.

Definition 5.5 A vector field v on M is *strongly tame* if it is tame and there is $\delta > 0$ such that for each $x \in \mathrm{Zero}(v)$, we have $B_\delta(x) \subset gD$ for some $g \in G$.

We also define a tame diffeomorphism, in analogy with tame vector fields:

Definition 5.6 A diffeomorphism $f: M \to M$ is *tame* if there are $\delta > 0$ and $\epsilon > 0$ such that

(i) $B_\delta(x) \cap B_\delta(y) = \varnothing$ for $x \neq y \in \mathrm{Fix}(f)$, and

(ii) $d(x, f(x)) > \epsilon$ for $x \in M - \bigcup_{y \in \mathrm{Fix}(y)} B_\delta(y)$,

where d stands for the metric of M.

We prove a technical lemma:

Lemma 5.7 *Let $f: \mathbb{R}^n \to T\mathbb{R}^n$ be a section of the tangent bundle $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for some $\delta, \epsilon > 0$, we have $f^{-1}(B_\epsilon(0)) \subset B_\delta(0)$. Let ω be a representative of the Thom class of $T\mathbb{R}^n$ such that* $\mathrm{supp}(\omega) \subset \mathbb{R}^n \times B_{\epsilon/2}(0)$. *Then*

$$\mathrm{ind}_0(f) = \int_{B_\delta(0)} f^*(\omega).$$

Proof Let B denote the closure of $B_\delta(0)$. Then by definition, $\mathrm{ind}_0(f)$ is the mapping degree of the composite

$$\partial B \xrightarrow{f} \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \xrightarrow{p} \mathbb{R}^n \setminus \{0\} \xrightarrow{q} S^{n-1},$$

where $p: T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the second projection and $q: \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ denotes the natural projection onto the unit sphere. There is a function $\rho: [0, \infty) \to \mathbb{R}$ such that $\rho(x) = 0$ for x sufficiently close to 0 and $\rho(x) = 1$ for $x \geq \frac{1}{2}\epsilon$. Let α be an $(n-1)$–form on the unit sphere S^{n-1} of $\mathbb{R}^n$ such that $\int_{S^{n-1}} \alpha = 1$. Now we define

$$\eta = p^*(d\rho \wedge q^*(\alpha)).$$

By definition $\mathrm{supp}(\eta) \subset \mathbb{R}^n \times B_{\epsilon/2}(0)$, and since $\int_{\mathbb{R}^n} d\rho \wedge q^*(\alpha) = 1$, η represents the Thom class of $T\mathbb{R}^n$. Then by the uniqueness of the Thom class, there is an $(n-1)$–form τ on $T\mathbb{R}^n$ with vertically compact support such that

$$\omega - \eta = d\tau.$$

We have $\mathrm{supp}(\tau) \subset \mathbb{R}^n \times B_{\epsilon/2}(0)$, implying $f^*(\tau)|_{\partial B} = 0$. So by Stokes' theorem, we get

$$\int_B f^*(\omega) - \int_B f^*(\eta) = \int_B f^*(d\tau) = \int_B df^*(\tau) = \int_{\partial B} f^*(\tau) = 0.$$

Note that $\eta = dp^*(\rho \cdot q^*(\alpha))$ and $\rho(f(\partial B)) = 1$. Then by Stokes' theorem, we have

$$\int_B f^*(\eta) = \int_B df^* \circ p^*(\rho \cdot q^*(\alpha)) = \int_{\partial B} f^* \circ p^*(\rho \cdot q^*(\alpha)) = \int_{\partial B} f^* \circ p^* \circ q^*(\alpha) = \mathrm{ind}_0(f).$$

Thus since $\int_{B_\epsilon(0)} f^*(\omega) = \int_B f^*(\omega)$, the proof is finished. □

We compute the index of a strongly tame bounded vector field:

Proposition 5.8 *Let v be a strongly tame bounded vector field on M. Then*

$$\mathrm{ind}(v)(g) = \sum_{x \in \mathrm{Zero}(v) \cap gD} \mathrm{ind}_x(v).$$

Proof Let δ and ϵ be as in the definition of a tame vector field. As in [Bott and Tu 1982], we may assume that the support of $\pi^*(\Phi)$ is contained in $B_{\epsilon/2}(M)$. Then

$$\mathrm{ind}(v)(g) = \sum_{x \in \mathrm{Zero}(v) \cap gD} \int_{B_\delta(x)} v^*(\pi^*(\Phi))$$

for each $g \in G$. On the other hand, by tameness of v, $\pi^*(\Phi)|_{B_\delta(x)}$ is compactly supported for $x \in \mathrm{Zero}(v)$, and so by Lemma 5.7, we get

$$\int_{B_\delta(x)} v^*(\pi^*(\Phi)) = \mathrm{ind}_x(v)$$

for each $x \in \mathrm{Zero}(v)$. □

Proof of Theorem 1.1 Combine Propositions 5.1, 5.3, and 5.8. □

Proof of Theorem 1.2 As mentioned at the beginning of Section 3, G is finitely generated. Then we can apply the results in Section 2. Let v be a tame bounded vector field on M, and suppose that v has finitely many zeros. We can easily see that v is homotopic to a strongly tame vector field by a homotopy with bounded derivative. Then by Lemma 5.2, we may assume that v itself is strongly tame, and so by Theorem 1.1, we have

$$\mathrm{ind}(v) = \chi(M/G)\mathbb{1}.$$

Since $\chi(M/G) \neq 0$, it follows from Proposition 2.4 that $\chi(M/G)\mathbb{1}$ is nonzero in $\ell^\infty(G)_G$. Then by Proposition 2.3, we obtain that v must have infinitely many zeros, a contradiction. Thus v must have infinitely many zeros. □

Proof of Corollary 1.3 Observe that a tame diffeomorphism is the composite of a tame vector field and the exponential map if it is C^1 close to the identity map. Then the statement follows from Theorem 1.2. □

References

[Attie and Block 1998] **O Attie**, **J Block**, *Poincaré duality for L^p cohomology and characteristic classes*, preprint 98-48, DIMACS (1998) Available at http://dimacs.rutgers.edu/archive/TechnicalReports/TechReports/1998/98-48.ps.gz

[Attie et al. 1992] **O Attie**, **J Block**, **S Weinberger**, *Characteristic classes and distortion of diffeomorphisms*, J. Amer. Math. Soc. 5 (1992) 919–921 MR Zbl

[Block and Weinberger 1992] **J Block**, **S Weinberger**, *Aperiodic tilings, positive scalar curvature and amenability of spaces*, J. Amer. Math. Soc. 5 (1992) 907–918 MR Zbl

[Bott and Tu 1982] **R Bott**, **L W Tu**, *Differential forms in algebraic topology*, Graduate Texts in Math. 82, Springer (1982) MR Zbl

[Brodzki et al. 2012] **J Brodzki**, **G A Niblo**, **N Wright**, *Pairings, duality, amenability and bounded cohomology*, J. Eur. Math. Soc. 14 (2012) 1513–1518 MR Zbl

[Cima and Llibre 1990] **A Cima**, **J Llibre**, *Bounded polynomial vector fields*, Trans. Amer. Math. Soc. 318 (1990) 557–579 MR Zbl

[Kato et al. 2023a] **T Kato**, **D Kishimoto**, **M Tsutaya**, *Homotopy type of the space of finite propagation unitary operators on $\mathbb{Z}$*, Homology Homotopy Appl. 25 (2023) 375–400 MR Zbl

[Kato et al. 2023b] **T Kato**, **D Kishimoto**, **M Tsutaya**, *Homotopy type of the unitary group of the uniform Roe algebra on $\mathbb{Z}^n$*, J. Topol. Anal. 15 (2023) 495–512 MR Zbl

[Kato et al. 2024] **T Kato**, **D Kishimoto**, **M Tsutaya**, *Hilbert bundles with ends*, J. Topol. Anal. 16 (2024) 291–322 MR Zbl

[Roe 1988] **J Roe**, *An index theorem on open manifolds, I*, J. Differential Geom. 27 (1988) 87–113 MR Zbl

[Weinberger 2009] **S Weinberger**, *Fixed-point theories on noncompact manifolds*, J. Fixed Point Theory Appl. 6 (2009) 15–25 MR Zbl

[Whyte 1999] **K Whyte**, *Amenability, bilipschitz equivalence, and the von Neumann conjecture*, Duke Math. J. 99 (1999) 93–112 MR Zbl

Department of Mathematics, Kyoto University
Kyoto, Japan

Faculty of Mathematics, Kyushu University
Fukuoka, Japan

Faculty of Mathematics, Kyushu University
Fukuoka, Japan

tkato@math.kyoto-u.ac.jp, kishimoto@math.kyushu-u.ac.jp, tsutaya@math.kyushu-u.ac.jp

Received: 21 December 2022 Revised: 6 September 2023

Algebraic & Geometric Topology

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 7 (pages 3571–4137) 2024