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CINDY TAN





# Smallest nonabelian quotients of surface braid groups

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We give a sharp lower bound on the size of nonabelian quotients of the surface braid group  $B_n(\Sigma_g)$  and classify all quotients that attain the lower bound: depending on  $n$  and  $g$ , a quotient of minimum order is either a symmetric group or a 2-step nilpotent  $p$ -group.

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## 1 Introduction

The Artin braid group  $B_n$  arises as the fundamental group of  $\text{UConf}_n(\mathbb{D})$ , the configuration space of  $n$  distinct unordered points on the open disk  $\mathbb{D}$ . One can generalize this construction to define, for an oriented closed genus- $g$  surface  $\Sigma_g$ , the *surface braid groups*

$$B_n(\Sigma_g) = \pi_1(\text{UConf}_n(\Sigma_g)).$$

It was shown by Kolay [4] that for  $n = 3$  or  $n \geq 5$ , the smallest noncyclic finite quotient of  $B_n$  is the symmetric group  $S_n$ , in the sense that  $S_n$  has minimum order amongst noncyclic quotients of  $B_n$  and  $S_n$  is the unique noncyclic quotient of  $B_n$  of minimum order.

We consider the analogous question for surface braid groups. With our main result we show that whilst  $S_n$  is a quotient of  $B_n(\Sigma_g)$ , it is not in general the smallest nonabelian quotient.

For  $g \geq 1$  the inclusion of a disk into  $\Sigma_g$  induces an embedding  $B_n \hookrightarrow B_n(\Sigma_g)$  (see Birman [2]); any two such embeddings are conjugate in  $B_n(\Sigma_g)$ . By a *braid-free* quotient of  $B_n(\Sigma_g)$  we mean a finite quotient with (any such embedding of)  $B_n$  having cyclic image. Our main result is the following theorem:

**Theorem 1** (Smallest nonabelian quotients of  $B_n(\Sigma_g)$ ) *Let  $n \geq 5$  and  $g \geq 1$ . Suppose that  $G$  is a finite nonabelian quotient of  $B_n(\Sigma_g)$ .*

- (a) *If  $G$  is not braid-free then  $|G| \geq n!$  with equality if and only if  $G \cong S_n$ .*
- (b) *If  $G$  is braid-free then  $G$  is 2-step nilpotent and  $|G| \geq p^{2g+j}$ , where  $p$  is the smallest prime dividing  $g+n-1$  and  $j = 1$  or  $2$  according to whether  $p$  is odd or 2, respectively. Equality occurs if and only if either  $G \cong \text{I}(p^j, g)$  or  $G \cong \text{II}(p^j, g)$  (these two groups are nonisomorphic 2-step nilpotent  $p$ -groups defined in Construction 10).*

*In particular, the smallest nonnilpotent quotient of  $B_n(\Sigma_g)$  is  $S_n$ .*

Note that Theorem 1 implies the following qualitative result:

- Corollary 2** (a) Fix  $g \geq 1$ . For all sufficiently large  $n$ , the smallest nonabelian quotients of  $B_n(\Sigma_g)$  are 2-step nilpotent  $p$ -groups (in particular, the smallest nonabelian quotient is not  $S_n$ ).
- (b) Fix  $n \geq 5$ . For all sufficiently large  $g$ , the smallest nonabelian quotient of  $B_n(\Sigma_g)$  is  $S_n$ . Also, there exists a (small)  $g$  for which this not true.

**Remarks 3** (Smaller cases) (a) If  $n = 1, 2, 3, 4$  and  $g \geq 1$  (with the exception of  $(n, g) = (1, 1)$  where  $B_n(\Sigma_g) = \pi_1(T^2) = \mathbb{Z}^2$  is abelian) then the symmetric group  $S_3$  is the smallest nonabelian quotient of  $B_n(\Sigma_g)$ .

(b) If  $g = 0$  then  $B_n(\Sigma_g)$  is the spherical braid group  $B_n(S^2)$ , which is an intermediate quotient of the map  $B_n \rightarrow S_n$ ; see Fadell and Van Buskirk [3]. It follows from the result of Kolay [4] that the smallest quotient of  $B_n(S^2)$  is  $S_n$  for  $n \geq 5$  and  $S_3$  for  $n = 3, 4$ . For  $n = 1, 2$  we note that  $B_n(S^2)$  is abelian.

From Theorem 1 we obtain partial confirmation of a conjecture of Chen [2, Conjecture 1.3]:

**Corollary 4** Let  $n \geq 5$  and  $m \geq 3$ , and let  $g, h \geq 0$ . If  $n > m$  then there are no surjective homomorphisms

$$B_n(\Sigma_g) \rightarrow B_m(\Sigma_h).$$

**Proof method** Theorem 1(a) follows from Kolay: By mapping a braid to its permutation on points,  $S_n$  is a finite quotient of  $B_n(\Sigma_g)$ . If  $B_n \rightarrow B_n(\Sigma_g) \rightarrow G$  has noncyclic image then  $|G| \geq n!$  with the bound attained only by  $G \cong S_n$ .

Our primary contribution here is Theorem 1(b), which considers the braid-free quotients. We utilize a presentation of  $B_n(\Sigma_g)$  (Theorem 13) due to Bellingeri [1] and assume that  $B_n$  has cyclic image to reduce the relations, and conclude that a braid-free quotient  $G$  must be nilpotent. If we further assume that  $G$  is a nonabelian braid-free quotient of minimum order, then  $G$  belongs to a class of nilpotent groups called JN2 groups (Definition 5) which were classified by Newman in 1960 [5]. It then suffices to find the smallest JN2 groups which can be realized as a quotient of  $B_n(\Sigma_g)$ , a straightforward task given the concrete nature of Newman's classification.

Section 2 provides a self-contained exposition of the classification of JN2 groups. In Section 3 we prove Theorem 1(b), as well as Corollary 4.

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## 2 Just 2–step nilpotent groups

In this section we introduce and classify JN2 groups, a class of nilpotent groups which includes all minimal nonabelian braid-free quotients of  $B_n(\Sigma_g)$ .

**Definition 5** A group  $G$  is *just 2–step nilpotent* (JN2) if  $G$  is 2–step nilpotent (in particular, nonabelian) and every proper quotient of  $G$  is abelian.

Finite JN2 groups admit a complete and explicit classification due to Newman [5]: any finite JN2 group can be assigned a unique class  $(p^j, m)$  where  $p$  is a prime and  $j$  and  $m$  are positive integers; up to isomorphism, there are precisely two JN2 groups of a given class  $(p^j, m)$ . We will state and prove this classification theorem in Theorem 11, following the general ideas of [5].

All JN2 groups will hereafter be assumed to be finite. The following proposition will allow us to define the class  $(p^j, m)$  of a JN2 group:

**Proposition 6** (Characterization of JN2 groups [5, Theorem 1]) *A finite group  $G$  is JN2 if and only if there exists a prime  $p$  such that*

- (a)  $G' := [G, G]$  is cyclic of order  $p$ ,
- (b) the center  $ZG$  is cyclic of order a power of  $p$ , and
- (c)  $G/ZG$  is elementary abelian of exponent  $p$ .

*In particular, a JN2 group is a  $p$ –group.*

**Proof** ( $\Rightarrow$ ) Let  $G$  be a finite JN2 group. For every nontrivial normal subgroup  $N \trianglelefteq G$ , we have that  $G' \leq N$  since any proper quotient of  $G$  is abelian. Since  $G$  is 2–step nilpotent,  $G' \leq ZG$ . Consequently:

- (a)  $G'$  is abelian and admits no proper nontrivial subgroups so  $G' \cong \mathbb{Z}/p\mathbb{Z}$  for some prime  $p$ .
- (b)  $ZG$  cannot be properly decomposed as a direct sum: any nontrivial subgroup of  $ZG$  contains  $G'$  so no two nontrivial subgroups intersect trivially. Since  $ZG$  is finite abelian, it must be cyclic of prime power order. The prime must be  $p$  because  $G' \leq ZG$ .
- (c)  $G/ZG$  is abelian because  $G' \leq ZG$ . For  $x, y \in G$ , we have that  $[x^p, y] = [x, y]^p$  by using the identity

$$[xz, y] = z[x, y]z^{-1}[z, y]$$

and noting that  $[x, y]$  is central because  $G' \leq ZG$ . But  $G'$  has order  $p$ , so in fact  $[x^p, y] = 1$ . Thus  $x^p \in ZG$  for all  $x \in G$ , which is to say that  $G/ZG$  has exponent  $p$ .

( $\Leftarrow$ ) Suppose  $G$  is a finite group satisfying (a), (b), and (c). Then  $G' \neq \{1\}$  by (a) and  $G' \leq ZG$  by (c), so  $G$  is 2–step nilpotent.

If  $N \trianglelefteq G$  is a normal subgroup with  $G' \not\leq N$  then  $N \cap G' = \{1\}$  by (a). Since  $N$  is normal,  $[N, G] \leq N \cap G' = \{1\}$  so  $N \leq ZG$ . But  $G' \leq ZG$ , and (a) and (b) imply that any nontrivial subgroup of  $ZG$  intersects  $G'$  nontrivially. Thus  $N = \{1\}$ . We conclude that every proper quotient of  $G$  is abelian.  $\square$

An immediate corollary of (c) is that  $V := G/ZG$  has the structure of an  $\mathbb{F}_p$ -vector space. Note that vector addition in  $V$  is written multiplicatively and scalar multiplication of an element  $x \bmod ZG \in V$  by a scalar  $r \in \mathbb{F}_p$  is written as

$$r(x \bmod ZG) = x^r \bmod ZG.$$

Fix a generator  $z$  of  $ZG$ . This fixes a generator  $z^{p^j-1}$  of  $G'$ , and hence an identification of  $G'$  with  $\mathbb{F}_p$ . Define a pairing

$$V \times V \rightarrow G' = \mathbb{F}_p \quad \text{given by } (x \bmod ZG, y \bmod ZG) \mapsto [x, y].$$

This pairing is a well-defined bilinear nondegenerate alternating form, which makes  $V$  into a symplectic vector space. In particular,  $\dim V$  is even.

Thus associated to each JN2 group  $G$  is a class  $(p^j, m)$  where  $|ZG| = p^j$  and  $\dim V = 2m$ , so  $G$  fits into the short exact sequence

$$1 \rightarrow \mathbb{Z}/p^j\mathbb{Z} \rightarrow G \rightarrow (\mathbb{Z}/p\mathbb{Z})^{2m} \rightarrow 0.$$

The symplectic structure on central factor groups  $V = G/ZG$  is key to the classification theorem because symplectic automorphisms on central factor groups can be used to construct isomorphisms between certain JN2 groups of the same class. The following lemma extracts from a JN2 group a normalized symplectic basis on its associated vector space  $V$ :

**Lemma 7** *Let  $G$  be JN2 of class  $(p^j, m)$  where  $p^j \neq 2$ , with a fixed generator  $z$  of  $ZG$ . Then there exists a symplectic basis  $\mathcal{B} = \{a_i \bmod ZG, b_i \bmod ZG\}_{i=1}^m$  of  $V = G/ZG$  such that the representatives  $a_i, b_i \in G$  satisfy either*

- (I)  $a_i^p = b_i^p = 1$  for all  $i$ , or
- (II)  $a_1^p = b_1^p = z$  and  $a_i^p = b_i^p = 1$  for  $2 \leq i \leq m$ .

We will say that  $\mathcal{B}$  is type I or II accordingly.

**Remark 8** (Nomenclature) For the reader familiar with existing terminology from [5], a “type I (respectively II) basis” as named in our Lemma 7 corresponds to a “canonic normal basis with zero (respectively one) pairs of type II” in the vocabulary of Newman.

**Proof** Note that  $x^p \in ZG$  for all  $x \in G$  because  $G/ZG$  has exponent  $p$ . Let  $(ZG)^p = \{u^p : u \in ZG\}$  and identify  $ZG/(ZG)^p$  with  $\mathbb{F}_p$  by the mapping  $z \bmod (ZG)^p \mapsto 1$ . Define a map

$$\nu: V \rightarrow ZG/(ZG)^p = \mathbb{F}_p \quad \text{given by } x \bmod ZG \mapsto x^p \bmod (ZG)^p.$$

Viewing  $V = G/ZG$  as a vector space written multiplicatively,  $\nu$  commutes with scalar multiplication and

$$\nu((x \bmod ZG)(y \bmod ZG)) = (xy)^p \bmod (ZG)^p = [y, x]^{p(p-1)/2} x^p y^p \bmod (ZG)^p$$

for  $x, y \in G$ , so  $\nu$  is a linear functional as long as  $[y, x]^{p(p-1)/2} = 1 \bmod (ZG)^p$ . This holds if  $p^j \neq 2$ : If  $p$  is odd then  $p \mid \frac{1}{2}p(p-1)$ , so  $[y, x]^{p(p-1)/2} = 1$  because  $G'$  has order  $p$ . If  $j \geq 2$  then  $G' \not\leq ZG$  so  $G' \leq (ZG)^p$ .

If  $\nu$  is the trivial linear functional on  $V$ , take  $\mathcal{B}$  to be any symplectic basis of  $V$ . Otherwise there exists a symplectic basis  $\mathcal{B}$  of  $V$  such that  $\nu$  written with respect to  $\mathcal{B}$  is the row vector

$$\nu = [1 \ 1 \ 0 \ \cdots \ 0],$$

because symplectic automorphisms act transitively on nontrivial vectors.

For each basis vector  $x_j \bmod ZG \in \mathcal{B}$ ,

$$x_j^p \bmod (ZG)^p = \nu(x_j \bmod ZG) = z^{\nu_j} \bmod (ZG)^p,$$

so there exists  $u_j \in ZG$  such that  $x_j^p = z^{\nu_j} u_j^p$ . Then  $x_j u_j^{-1} \equiv x_j \bmod ZG$  and  $(x_j u_j^{-1})^p = z^{\nu_j}$ . Thus  $x_j u_j^{-1} \in G$  are representatives of the basis  $\mathcal{B}$  satisfying (I) if  $\nu$  is trivial and (II) otherwise.  $\square$

We will now construct two standard nonisomorphic JN2 groups for each given class  $(p^j, m)$ . The proof of the classification theorem will exhibit an isomorphism from any arbitrary JN2 group to a standard one. The primary method of constructing larger JN2 groups from smaller ones is taking a central product.

**Definition 9** (Central product) Let  $G$  and  $H$  be groups for which  $ZG \cong ZH$ . Define the *central product* of  $G$  and  $H$  (with respect to an isomorphism  $\varphi : ZG \rightarrow ZH$ ) to be

$$G \odot H = (G \times H)/N \quad \text{where } N = \langle (g, \varphi(g)^{-1}) : g \in ZG \rangle,$$

i.e. identifying  $ZG \times 1$  with  $1 \times ZH$  by the isomorphism  $\varphi$ . By  $G^{\odot n}$  we mean the central product of  $n$  copies of  $G$  with the identity isomorphism on  $ZG$ .

Note that if  $G$  and  $H$  are JN2 of class  $(p^j, m_1)$  and  $(p^j, m_2)$ , then  $G \odot H$  is JN2 of class  $(p^j, m_1 + m_2)$  by Proposition 6 since

- (a)  $(G \odot H)' = G' \times H' / N \cong G' \cong H' \cong \mathbb{Z}/p\mathbb{Z}$ ,
- (b)  $Z(G \odot H) \cong ZG \cong ZH$ , and
- (c)  $(G \odot H)/Z(G \odot H) \cong (G/ZG) \times (H/ZG)$ .

**Construction 10** (Standard JN2 groups) Define the groups

$$\begin{aligned} M(p^j) &= \langle z, a, b : [z, a] = [z, b] = 1; [a, b] = z^{p^{j-1}}; z^{p^j} = a^p = b^p = 1 \rangle, \\ N(p^j) &= \langle z, a, b : [z, a] = [z, b] = 1; [a, b] = z^{p^{j-1}}; z^{p^j} = 1; a^p = b^p = z \rangle, \\ I(p^j, m) &= M(p^j)^{\odot m}, \\ II(p^j, m) &= N(p^j) \odot M(p^j)^{\odot(m-1)}. \end{aligned}$$

Observe the following:

- (1)  $M(p^j)$  and  $N(p^j)$  are JN2 (by Proposition 6) of class  $(p^j, 1)$ , with each center generated by  $z$  and  $\{a, b\}$  as a symplectic basis of  $V$ .
- (2)  $I(p^j, m)$  and  $II(p^j, m)$  are JN2 of class  $(p^j, m)$  by the remarks following Definition 9.

(3)  $I(p^j, m)$  and  $II(p^j, m)$  are not isomorphic when  $p^j \neq 2$ : the group  $N(p^j)$  has an element of order  $p^{j+1}$  (for example,  $a$  or  $b$ ) and therefore so does  $II(p^j, m)$ . On the contrary, the group  $M(p^j)$ , and consequently also  $I(p^j, m) = M(p^j)^{\odot m}$ , has exponent at most  $p^j$ : the linear functional  $\nu$  (as in the proof of Lemma 7) is trivial on the symplectic basis  $\{a, b\}$  so  $M(p^j)^p \leq (ZG)^p$ , and hence  $M(p^j)^{p^j} \leq (ZG)^{p^j} = 1$ .

**Note** If  $p^j = 2$ , then  $I(p^j, m)$  and  $II(p^j, m)$  are still nonisomorphic:  $M(2)$  is the dihedral group  $D_8$  and  $N(2)$  is the quaternion group  $Q_8$ , which contain two and six elements of order 4, respectively, and both have centers of order 2. In particular no elements of order 4 are central. The larger groups  $I(p^j, m)$  and  $II(p^j, m)$  can then be distinguished by counting the number of elements of order 4 because only central elements are identified in the central product. We will not require this case.

**Theorem 11** (Classification of finite JN2 groups [5, Theorems 5 and 7(c) and Lemma 8(i)]) *Let  $G$  be JN2 of class  $(p^j, m)$ . Suppose that  $p^j \neq 2$ . Then  $G$  is isomorphic to either  $I(p^j, m)$  or  $II(p^j, m)$ .*

**Proof** Let  $z$  be a generator of  $ZG$  and let  $\mathcal{B}$  be the symplectic basis given by Lemma 7. In the notation of Lemma 7, let  $H_i = \langle z, a_i, b_i \rangle$ . If  $\mathcal{B}$  is type I then  $H_i = M(p^j)$  for all  $i$ . If  $\mathcal{B}$  is type II then  $H_1 = N(p^j)$  and  $H_i = M(p^j)$  for  $i \geq 2$ .

The subgroups  $H_i$  commute pairwise, together generate  $G$ , and intersect precisely in their centers  $\langle z \rangle$ , so  $G \cong \odot_{i=1}^m H_i$ . Hence  $G$  is isomorphic to  $I(p^j, m)$  or  $II(p^j, m)$ , according to the type of the basis  $\mathcal{B}$ .  $\square$

**Remarks 12** (a) **Generalizations** For brevity, we have excluded the case of  $p^j = 2$  and specialized to finite groups. With additional work, the  $p^j = 2$  case and some infinite JN2 groups (those with a countable symplectic basis) also admit a classification as central products of elementary JN2 groups, see [5].

(b) **Special cases** Note that  $M(p)$  and  $N(p)$  are the only two groups of order  $p^3$ . The group  $M(p) = I(p, 1)$  is isomorphic to the Heisenberg group over  $\mathbb{F}_p$ . A generalization of the finite Heisenberg groups are the *extraspecial groups*, which are defined to be  $p$ -groups  $G$  with  $ZG$  order  $p$  and  $G/ZG$  nontrivial elementary abelian. In particular, extraspecial groups are JN2, and it follows from Theorem 11 that there are precisely two distinct extraspecial groups of order  $p^{1+2m}$  for each choice of a prime  $p$  and positive integer  $m$  and that this exhausts all extraspecial groups.

### 3 Minimal nonabelian quotients of $B_n(\Sigma_g)$

In this section we provide the proof of Theorem 1(b). The strategy of the proof will be to utilize an explicit presentation of the surface braid groups (Theorem 13) to characterize braid-free quotients by the relations that they must satisfy (Lemma 15). We will then show that many JN2 groups are realized as nonabelian braid-free quotients of  $B_n(\Sigma_g)$  (Lemma 16) and finally prove that all nonabelian braid-free quotients of minimum order belong to the list of JN2 groups in Lemma 16.

**Theorem 13** (Presentation of  $B_n(\Sigma_g)$ , Bellingeri [1, Theorem 1.2]) *For  $g \geq 1$  and  $n \geq 2$ , the surface braid group  $B_n(\Sigma_g)$  admits the presentation given by*

- generators  $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g, b_1, \dots, b_g$ ,
- braid relations

$$[\sigma_i, \sigma_j] = 1 \quad \text{for } 1 \leq i, j \leq n-1 \text{ and } |i-j| \geq 2,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n-2,$$

and mixed relations

- (R1)  $[a_r, \sigma_i] = [b_r, \sigma_i] = 1$  for  $1 \leq r \leq g$  and  $i \neq 1$ ,
- (R2)  $[a_r, \sigma_1^{-1} a_r \sigma_1^{-1}] = [b_r, \sigma_1^{-1} b_r \sigma_1^{-1}] = 1$  for  $1 \leq r \leq g$ ,
- (R3)  $[a_s, \sigma_1 a_r \sigma_1^{-1}] = [b_s, \sigma_1 b_r \sigma_1^{-1}] = 1$  for  $1 \leq s < r \leq g$ ,
- $[b_s, \sigma_1 a_r \sigma_1^{-1}] = [a_s, \sigma_1 b_r \sigma_1^{-1}] = 1$  for  $1 \leq s < r \leq g$ ,
- (R4)  $[a_r, \sigma_1^{-1} b_r \sigma_1^{-1}] = \sigma_1^2$  for  $1 \leq r \leq g$ ,
- (TR)  $[a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] = \sigma_1 \sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2 \sigma_1$ .

**Remark 14** (Geometric interpretation of the presentation) The Bellingeri generators  $\sigma_i$  can be identified as the images of the Artin braid generators under a choice of embedding  $B_n \hookrightarrow B_n(\Sigma_g)$ . The remaining generators  $a_r$  and  $b_r$  can be understood loosely to be the standard generators of  $\pi_1(\Sigma_g)$ .

More precisely, let  $\{p_1, \dots, p_n\} \in \text{UConf}_n(\Sigma_g)$  denote the basepoint of  $B_n(\Sigma_g)$  and let  $D \subset \Sigma_g$  be an open disk with  $p_1 \in \partial D$ , with  $p_2, \dots, p_n$  in the interior of  $D$ . There is an inclusion

$$\pi_1(\Sigma_g - D, p_1) \hookrightarrow B_n(\Sigma_g)$$

which takes a loop  $\gamma$  in  $\Sigma_g - D$  to the braid on  $\Sigma_g$  with first strand  $\gamma$  and all other strands trivial. The group  $\pi_1(\Sigma_g - D, p_1)$  is free on  $2g$  generators and surjects onto  $\pi_1(\Sigma_g, p_1)$ , which has a standard presentation. The surface braid group generators  $a_r, b_r \in B_n(\Sigma_g)$  can then be understood as a choice of a free generating set of  $\pi_1(\Sigma_g - D, p_1)$  which lifts the standard generating set of  $\pi_1(\Sigma_g, p_1)$ . It should be emphasized that the lifts are not canonical and that the presentation depends on the choices; the curious reader may refer to [1] for illustrations of the loops which produce this particular presentation.

**Lemma 15** (Characterization of braid-free quotients) *Let  $n \geq 3$  and  $g \geq 1$ . A finite group  $G$  is a braid-free quotient of  $B_n(\Sigma_g)$  if and only if  $G$  admits a generating set  $\{\sigma, a_1, b_1, \dots, a_g, b_g\}$  satisfying the relations*

- (R1')  $[a_r, \sigma] = [b_r, \sigma] = 1$  for  $1 \leq r \leq g$ ,
- (R3')  $[a_s, a_r] = [b_s, b_r] = [b_s, a_r] = [a_s, b_r] = 1$  for  $1 \leq s < r \leq g$ ,
- (R4')  $[a_r, b_r] = \sigma^2$  for  $1 \leq r \leq g$ ,
- (TR')  $\sigma^{2(g+n-1)} = 1$ .

**Proof** A finite quotient of  $B_n(\Sigma_g)$  is presented by Theorem 13 with additional relations. The condition that  $B_n$  has cyclic image in a quotient is equivalent to adding the relations

$$\sigma_i = \sigma_1, \quad 1 \leq i \leq n.$$

If we add these relations and write  $\sigma = \sigma_1$ , the relation (R2) is made redundant and (R1), (R3), and (R4) respectively reduce to the relations (R1'), (R3'), and (R4'), as in the statement of the lemma. The final relation (TR) reduces to

$$[a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] = \sigma^{2(n-1)},$$

which is equivalent to (TR') because from (R4') we can write  $a_r = b_r^{-1} \sigma^{-2} a_r b_r$  so that

$$[a_r, b_r^{-1}] = a_r b_r^{-1} a_r^{-1} b_r \stackrel{(R4')}{=} (b_r^{-1} \sigma^{-2} a_r b_r) b_r^{-1} a_r^{-1} b_r = b_r^{-1} \sigma^{-2} b_r \stackrel{(R1')}{=} \sigma^{-2}. \quad \square$$

The following lemma proves that many JN2 groups are braid-free quotients:

**Lemma 16** *Let  $n \geq 3$  and  $g \geq 1$ . Let  $p$  be a prime dividing  $g + n - 1$ .*

- (a) *If  $p = 2$  then  $I(2^2, g)$  and  $II(2^j, g)$  for all  $j \geq 2$  are nonabelian braid-free quotients of  $B_n(\Sigma_g)$ .*
- (b) *If  $p$  is odd then  $I(p, g)$  and  $II(p^j, g)$  for all  $j \geq 1$  are nonabelian braid-free quotients of  $B_n(\Sigma_g)$ .*

**Proof** Let  $p$  be a prime dividing  $g + n - 1$ . By Lemma 15 we need to exhibit a generating set  $\{\sigma, a_r, b_r\}$  of each group satisfying relations (R1'), (R3'), (R4'), and (TR').

In any of the JN2 groups in the statement of the theorem, fix a generator  $z$  of the center and choose  $a_1, b_1, \dots, a_g, b_g$  to be the representatives of a symplectic basis of  $V$  given by Lemma 7. By Theorem 11 this basis will be type I for  $I(2^2, g)$  and  $I(p, g)$ , and type II for  $II(2^j, g)$  and  $II(p^j, g)$ . Note that with the given symplectic form, the condition that a basis is symplectic is simply that all basis elements commute except symplectic pairs  $[a_r, b_r] = z^{p^{j-1}}$ . In particular, (R3') is satisfied.

We will now choose  $\sigma$  for each group and verify that  $\{\sigma, a_r, b_r\}$  generate the group and satisfy (R4').

- (1)  $I(2^2, g)$  is generated by  $\sigma = z$  and  $\{a_r, b_r\}$ . These satisfy (R4') because  $[a_r, b_r] = z^2 = \sigma^2$ .
- (2)  $II(2^j, g)$ , for a given  $j \geq 2$ , is generated by  $\{a_r, b_r\}$  alone because  $a_1^p = z$ . If we choose  $\sigma = z^{2^{j-2}}$  then (R4') is satisfied because  $[a_r, b_r] = z^{2^{j-1}} = \sigma^2$ .
- (3)  $I(p, g)$  for odd prime  $p$  is generated by  $\sigma = z^{(p^j+1)/2}$  and the  $a_r$  and  $b_r$ . Then (R4') is satisfied because  $[a_r, b_r] = z = \sigma^2$ .
- (4)  $II(p^j, g)$ , for given odd prime  $p$  and  $j \geq 1$ , is generated by  $a_r$  and  $b_r$  alone because  $a_1^p = z$ . Then set  $\sigma = z^{(p^j+p^{j-1})/2}$  so that (R4') is satisfied because  $[a_r, b_r] = z^{p^{j-1}} = \sigma^2$ .

In all cases  $\sigma$  was chosen to be central, and hence (R1') is satisfied.

It remains to check that (TR') holds, namely that  $|\sigma|$  divides  $2(g + n - 1)$ . Recall that we are assuming that  $p \mid (g + n - 1)$ . In cases (1) and (2), we have  $p = 2$  and  $|\sigma| = 4 = 2p \mid 2(g + n - 1)$ . In cases (3) and (4), we have  $|\sigma| = p \mid (g + n - 1)$ . □

**Proof of Theorem 1(b)** Let  $G$  be a nonabelian braid-free quotient and let  $\{\sigma, a_1, b_1, \dots, a_g, b_g\}$  denote the generating set of  $G$  as given by Lemma 15. By (R1') and (R3'), all pairs of these generators commute except for pairs  $a_r$  and  $b_r$ , so  $G' = \langle \sigma^2 \rangle$  by (R4'). Then  $G'$  is central and nontrivial, which is to say that  $G$  is 2-step nilpotent.

Assume now that  $G$  is of minimum order amongst nonabelian braid-free quotients of  $B_n(\Sigma_g)$ . Then  $G$  has no proper nonabelian quotients and thus is JN2 of some class  $(p^j, m)$ .

We make three claims:

- (1)  $p^j \neq 2$ ,
- (2)  $m = g$ , and
- (3)  $p \mid (g + n - 1)$ .

These claims will complete the proof: Since  $|G| = p^{2m+j}$ , it follows from claims (1) and (2) that  $|G| \geq p^{2g+1}$  if  $p$  is odd and  $|G| \geq 2^{2g+2}$  if  $p = 2$ . Claims (1) and (3) along with the minimality of  $G$  together imply that  $G$  is one of (in particular, the smallest of) the quotients constructed in Lemma 16. Explicitly, if  $g + n - 1$  is even then  $p^j = 2^2$ . Otherwise  $p^j = p$ , where  $p$  is the smallest prime dividing  $g + n - 1$ . Finally,  $G$  must be isomorphic to either  $I(p^j, g)$  or  $II(p^j, g)$  by Theorem 11.

**Proof of claims** Let  $d = |\sigma|$ . By (R1'),  $\sigma$  is central so  $d \mid p^j$ . But  $p = |G'| = |\sigma^2|$  so  $d \mid 2p$ . Thus either  $p$  is odd and  $p = d$ , or  $p = 2$  and  $d = 4$ .

- (1) If  $p$  is odd then  $p^j \neq 2$ . If  $p = 2$  then  $p^j \geq d = 4$  so  $p^j \neq 2$ .
- (2) We will show that  $\dim V = 2g$  by proving that

$$\mathcal{B} = \{a_r \text{ mod } ZG, b_r \text{ mod } ZG\}_{r=1}^g$$

is a basis of  $V$ . Every element  $x \in G$  can be written uniquely in the form

$$x = \sigma^k a_1^{i_1} \dots a_g^{i_g} b_1^{j_1} \dots b_g^{j_g}$$

using commuting relations (R1'), (R3'), and (R4') so  $\mathcal{B}$  is a generating set. To prove that  $\mathcal{B}$  is linearly independent, let

$$y = a_1^{i_1} \dots a_g^{i_g} b_1^{j_1} \dots b_g^{j_g} \in G$$

and suppose that  $y = 0 \text{ mod } ZG$ , which is to suppose that an arbitrary linear combination of elements of  $\mathcal{B}$  is trivial in  $V$ . Then  $y$  is central, so

$$[y, b_1] = [a_1^{i_1}, b_1] = \sigma^{-2i_1} = 1,$$

which implies that  $d \mid 2i_1$  and thus  $i_1 = 0 \text{ mod } p$ : If  $p$  is odd then  $d = p$ , so  $p \mid i_1$ . If  $p = 2$  then  $d = 4 \mid 2i_1$  so,  $p = 2 \mid i_1$ .

Similarly  $i_r = j_r = 0 \text{ mod } p$  for all  $r$ , which is to say that all coefficients of the linear combination are trivial over the base field  $\mathbb{F}_p$ . This proves the linear independence of  $\mathcal{B}$ .

(3) The relation (TR') imposes the relation  $d \mid 2(g + n - 1)$ . Either  $d = p$  is odd or  $d = 4$  and  $p = 2$ ; in both cases (TR') implies that  $p \mid (g + n - 1)$ .  $\square$

**Proof of Corollary 4** Let  $n \geq 5$  and  $m \geq 3$ , and let  $g, h \geq 0$ . If there is a surjection  $B_n(\Sigma_g) \rightarrow B_m(\Sigma_h)$  then the composition  $B_n(\Sigma_g) \rightarrow B_m(\Sigma_h) \rightarrow S_m$  is also surjective. Since  $S_m$  is not nilpotent when  $m \geq 3$ , we must have  $m \geq n$ .  $\square$

**Remark 17** (Punctured surfaces, surfaces with boundary) Bellingeri [1] also gives a presentation of the braid group of a genus- $g$  surface with  $m$  punctures (equivalently for the purposes of braid groups,  $m$  boundary components). The above methods can be used nearly verbatim to prove that the smallest nonabelian quotient of  $B_n(\Sigma_{g,m})$  is the smaller of  $S_n$  or  $I(2^2, g)$  and  $\Pi(2^2, g)$ .

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*Department of Mathematics, University of Chicago  
Chicago, IL, United States*

cindy@math.uchicago.edu

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
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