AGT

Algebraic & Geometric Topology

Volume 24 (2024)

Smallest nonabelian quotients of surface braid groups

CINDY TAN

msp

msp

Algebraic & Geometric Topology **24**:7 (2024) 3997–4006
DOI: 10.2140/agt.2024.24.3997
Published: 9 December 2024

Smallest nonabelian quotients of surface braid groups

CINDY TAN

We give a sharp lower bound on the size of nonabelian quotients of the surface braid group $B_n(\Sigma_g)$ and classify all quotients that attain the lower bound: depending on n and g, a quotient of minimum order is either a symmetric group or a 2–step nilpotent p–group.

20F65; 57K20

1 Introduction

The Artin braid group B_n arises as the fundamental group of $\mathrm{UConf}_n(\mathbb{D})$, the configuration space of n distinct unordered points on the open disk $\mathbb{D}$. One can generalize this construction to define, for an oriented closed genus-g surface Σ_g, the *surface braid groups*

$$B_n(\Sigma_g) = \pi_1(\mathrm{UConf}_n(\Sigma_g)).$$

It was shown by Kolay [4] that for $n = 3$ or $n \geq 5$, the smallest noncyclic finite quotient of B_n is the symmetric group S_n, in the sense that S_n has minimum order amongst noncyclic quotients of B_n and S_n is the unique noncyclic quotient of B_n of minimum order.

We consider the analogous question for surface braid groups. With our main result we show that whilst S_n is a quotient of $B_n(\Sigma_g)$, it is not in general the smallest nonabelian quotient.

For $g \geq 1$ the inclusion of a disk into Σ_g induces an embedding $B_n \hookrightarrow B_n(\Sigma_g)$ (see Birman [2]); any two such embeddings are conjugate in $B_n(\Sigma_g)$. By a *braid-free* quotient of $B_n(\Sigma_g)$ we mean a finite quotient with (any such embedding of) B_n having cyclic image. Our main result is the following theorem:

Theorem 1 (Smallest nonabelian quotients of $B_n(\Sigma_g)$) *Let $n \geq 5$ and $g \geq 1$. Suppose that G is a finite nonabelian quotient of $B_n(\Sigma_g)$.*

(a) *If G is not braid-free then $|G| \geq n!$ with equality if and only if $G \cong S_n$.*

(b) *If G is braid-free then G is 2–step nilpotent and $|G| \geq p^{2g+j}$, where p is the smallest prime dividing $g+n-1$ and $j = 1$ or 2 according to whether p is odd or 2, respectively. Equality occurs if and only if either $G \cong \mathrm{I}(p^j, g)$ or $G \cong \mathrm{II}(p^j, g)$* (these two groups are nonisomorphic 2–step nilpotent p–groups defined in Construction 10).

In particular, the smallest nonnilpotent quotient of $B_n(\Sigma_g)$ is S_n.

© 2024 The Author, under license to MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

Note that Theorem 1 implies the following qualitative result:

Corollary 2 (a) *Fix $g \geq 1$. For all sufficiently large n, the smallest nonabelian quotients of $B_n(\Sigma_g)$ are 2–step nilpotent p–groups (in particular, the smallest nonabelian quotient is not S_n).*

(b) *Fix $n \geq 5$. For all sufficiently large g, the smallest nonabelian quotient of $B_n(\Sigma_g)$ is S_n. Also, there exists a (small) g for which this not true.*

Remarks 3 (Smaller cases) (a) If $n = 1, 2, 3, 4$ and $g \geq 1$ (with the exception of $(n, g) = (1, 1)$ where $B_n(\Sigma_g) = \pi_1(T^2) = \mathbb{Z}^2$ is abelian) then the symmetric group S_3 is the smallest nonabelian quotient of $B_n(\Sigma_g)$.

(b) If $g = 0$ then $B_n(\Sigma_g)$ is the spherical braid group $B_n(S^2)$, which is an intermediate quotient of the map $B_n \to S_n$; see Fadell and Van Buskirk [3]. It follows from the result of Kolay [4] that the smallest quotient of $B_n(S^2)$ is S_n for $n \geq 5$ and S_3 for $n = 3, 4$. For $n = 1, 2$ we note that $B_n(S^2)$ is abelian.

From Theorem 1 we obtain partial confirmation of a conjecture of Chen [2, Conjecture 1.3]:

Corollary 4 *Let $n \geq 5$ and $m \geq 3$, and let $g, h \geq 0$. If $n > m$ then there are no surjective homomorphisms*

$$B_n(\Sigma_g) \to B_m(\Sigma_h).$$

Proof method Theorem 1(a) follows from Kolay: By mapping a braid to its permutation on points, S_n is a finite quotient of $B_n(\Sigma_g)$. If $B_n \to B_n(\Sigma_g) \to G$ has noncyclic image then $|G| \geq n!$ with the bound attained only by $G \cong S_n$.

Our primary contribution here is Theorem 1(b), which considers the braid-free quotients. We utilize a presentation of $B_n(\Sigma_g)$ (Theorem 13) due to Bellingeri [1] and assume that B_n has cyclic image to reduce the relations, and conclude that a braid-free quotient G must be nilpotent. If we further assume that G is a nonabelian braid-free quotient of minimum order, then G belongs to a class of nilpotent groups called JN2 groups (Definition 5) which were classified by Newman in 1960 [5]. It then suffices to find the smallest JN2 groups which can be realized as a quotient of $B_n(\Sigma_g)$, a straightforward task given the concrete nature of Newman's classification.

Section 2 provides a self-contained exposition of the classification of JN2 groups. In Section 3 we prove Theorem 1(b), as well as Corollary 4.

Acknowledgements I am grateful to my advisor Benson Farb for continued support throughout this project and for detailed comments on many revisions of this paper, as well as for suggesting this problem in the first place. I thank Peter Huxford for useful discussions about braid groups and small p–groups, and for many helpful suggestions during the editing process. I also thank Dan Margalit for taking the time to read and comment on an earlier draft.

2 Just 2–step nilpotent groups

In this section we introduce and classify JN2 groups, a class of nilpotent groups which includes all minimal nonabelian braid-free quotients of $B_n(\Sigma_g)$.

Definition 5 A group G is *just 2–step nilpotent* (JN2) if G is 2–step nilpotent (in particular, nonabelian) and every proper quotient of G is abelian.

Finite JN2 groups admit a complete and explicit classification due to Newman [5]: any finite JN2 group can be assigned a unique class (p^j, m) where p is a prime and j and m are positive integers; up to isomorphism, there are precisely two JN2 groups of a given class (p^j, m). We will state and prove this classification theorem in Theorem 11, following the general ideas of [5].

All JN2 groups will hereafter be assumed to be finite. The following proposition will allow us to define the class (p^j, m) of a JN2 group:

Proposition 6 (Characterization of JN2 groups [5, Theorem 1]) *A finite group G is JN2 if and only if there exists a prime p such that*

(a) *$G' := [G, G]$ is cyclic of order p,*

(b) *the center ZG is cyclic of order a power of p, and*

(c) *G/ZG is elementary abelian of exponent p.*

In particular, a JN2 group is a p–group.

Proof ($\Rightarrow$) Let G be a finite JN2 group. For every nontrivial normal subgroup $N \trianglelefteq G$, we have that $G' \leq N$ since any proper quotient of G is abelian. Since G is 2–step nilpotent, $G' \leq ZG$. Consequently:

(a) G' is abelian and admits no proper nontrivial subgroups so $G' \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p.

(b) ZG cannot be properly decomposed as a direct sum: any nontrivial subgroup of ZG contains G' so no two nontrivial subgroups intersect trivially. Since ZG is finite abelian, it must be cyclic of prime power order. The prime must be p because $G' \leq ZG$.

(c) G/ZG is abelian because $G' \leq ZG$. For $x, y \in G$, we have that $[x^p, y] = [x, y]^p$ by using the identity

$$[xz, y] = z[x, y]z^{-1}[z, y]$$

and noting that $[x, y]$ is central because $G' \leq ZG$. But G' has order p, so in fact $[x^p, y] = 1$. Thus $x^p \in ZG$ for all $x \in G$, which is to say that G/ZG has exponent p.

($\Leftarrow$) Suppose G is a finite group satisfying (a), (b), and (c). Then $G' \neq \{1\}$ by (a) and $G' \leq ZG$ by (c), so G is 2–step nilpotent.

If $N \trianglelefteq G$ is a normal subgroup with $G' \not\leq N$ then $N \cap G' = \{1\}$ by (a). Since N is normal, $[N, G] \leq N \cap G' = \{1\}$ so $N \leq ZG$. But $G' \leq ZG$, and (a) and (b) imply that any nontrivial subgroup of ZG intersects G' nontrivially. Thus $N = \{1\}$. We conclude that every proper quotient of G is abelian. □

An immediate corollary of (c) is that $V := G/ZG$ has the structure of an $\mathbb{F}_p$–vector space. Note that vector addition in V is written multiplicatively and scalar multiplication of an element $x \bmod ZG \in V$ by a scalar $r \in \mathbb{F}_p$ is written as

$$r(x \bmod ZG) = x^r \bmod ZG.$$

Fix a generator z of ZG. This fixes a generator $z^{p^{j-1}}$ of G', and hence an identification of G' with $\mathbb{F}_p$. Define a pairing

$$V \times V \to G' = \mathbb{F}_p \quad \text{given by } (x \bmod ZG, y \bmod ZG) \mapsto [x, y].$$

This pairing is a well-defined bilinear nondegenerate alternating form, which makes V into a symplectic vector space. In particular, $\dim V$ is even.

Thus associated to each JN2 group G is a *class* (p^j, m) where $|ZG| = p^j$ and $\dim V = 2m$, so G fits into the short exact sequence

$$1 \to \mathbb{Z}/p^j\mathbb{Z} \to G \to (\mathbb{Z}/p\mathbb{Z})^{2m} \to 0.$$

The symplectic structure on central factor groups $V = G/ZG$ is key to the classification theorem because symplectic automorphisms on central factor groups can be used to construct isomorphisms between certain JN2 groups of the same class. The following lemma extracts from a JN2 group a normalized symplectic basis on its associated vector space V:

Lemma 7 *Let G be JN2 of class (p^j, m) where $p^j \neq 2$, with a fixed generator z of ZG. Then there exists a symplectic basis $\mathscr{B} = \{a_i \bmod ZG, b_i \bmod ZG\}_{i=1}^m$ of $V = G/ZG$ such that the representatives $a_i, b_i \in G$ satisfy either*

(I) $a_i^p = b_i^p = 1$ *for all i, or*

(II) $a_1^p = b_1^p = z$ *and* $a_i^p = b_i^p = 1$ *for $2 \le i \le m$.*

We will say that $\mathscr{B}$ is type I or II accordingly.

Remark 8 (Nomenclature) For the reader familiar with existing terminology from [5], a "type I (respectively II) basis" as named in our Lemma 7 corresponds to a "canonic normal basis with zero (respectively one) pairs of type II" in the vocabulary of Newman.

Proof Note that $x^p \in ZG$ for all $x \in G$ because G/ZG has exponent p. Let $(ZG)^p = \{u^p : u \in ZG\}$ and identify $ZG/(ZG)^p$ with $\mathbb{F}_p$ by the mapping $z \bmod (ZG)^p \mapsto 1$. Define a map

$$\nu\colon V \to ZG/(ZG)^p = \mathbb{F}_p \quad \text{given by } x \bmod ZG \mapsto x^p \bmod (ZG)^p.$$

Viewing $V = G/ZG$ as a vector space written multiplicatively, ν commutes with scalar multiplication and

$$\nu((x \bmod ZG)(y \bmod ZG)) = (xy)^p \bmod (ZG)^p = [y, x]^{p(p-1)/2} x^p y^p \bmod (ZG)^p$$

for $x, y \in G$, so ν is a linear functional as long as $[y, x]^{p(p-1)/2} = 1 \bmod (ZG)^p$. This holds if $p^j \neq 2$: If p is odd then $p \mid \frac{1}{2}p(p-1)$, so $[y, x]^{p(p-1)/2} = 1$ because G' has order p. If $j \ge 2$ then $G' \lneqq ZG$ so $G' \le (ZG)^p$.

If ν is the trivial linear functional on V, take $\mathscr{B}$ to be any symplectic basis of V. Otherwise there exists a symplectic basis $\mathscr{B}$ of V such that ν written with respect to $\mathscr{B}$ is the row vector

$$\nu = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \end{bmatrix},$$

because symplectic automorphisms act transitively on nontrivial vectors.

For each basis vector $x_j \bmod ZG \in \mathscr{B}$,

$$x_j^p \bmod (ZG)^p = \nu(x_j \bmod ZG) = z^{\nu_j} \bmod (ZG)^p,$$

so there exists $u_j \in ZG$ such that $x_j^p = z^{\nu_j} u_j^p$. Then $x_j u_j^{-1} \equiv x_j \bmod ZG$ and $(x_j u_j^{-1})^p = z^{\nu_j}$. Thus $x_j u_j^{-1} \in G$ are representatives of the basis $\mathscr{B}$ satisfying (I) if ν is trivial and (II) otherwise. □

We will now construct two standard nonisomorphic JN2 groups for each given class (p^j, m). The proof of the classification theorem will exhibit an isomorphism from any arbitrary JN2 group to a standard one. The primary method of constructing larger JN2 groups from smaller ones is taking a central product.

Definition 9 (Central product) Let G and H be groups for which $ZG \cong ZH$. Define the *central product* of G and H (with respect to an isomorphism $\varphi : ZG \to ZH$) to be

$$G \odot H = (G \times H)/N \quad \text{where } N = \langle (g, \varphi(g)^{-1}) : g \in ZG \rangle,$$

i.e. identifying $ZG \times 1$ with $1 \times ZH$ by the isomorphism φ. By $G^{\odot n}$ we mean the central product of n copies of G with the identity isomorphism on ZG.

Note that if G and H are JN2 of class (p^j, m_1) and (p^j, m_2), then $G \odot H$ is JN2 of class $(p^j, m_1 + m_2)$ by Proposition 6 since

(a) $(G \odot H)' = G' \times H'/N \cong G' \cong H' \cong \mathbb{Z}/p\mathbb{Z}$,

(b) $Z(G \odot H) \cong ZG \cong ZH$, and

(c) $(G \odot H)/Z(G \odot H) \cong (G/ZG) \times (H/ZG)$.

Construction 10 (Standard JN2 groups) Define the groups

$$\begin{aligned} M(p^j) &= \langle z, a, b : [z,a] = [z,b] = 1; [a,b] = z^{p^{j-1}}; z^{p^j} = a^p = b^p = 1 \rangle, \\ N(p^j) &= \langle z, a, b : [z,a] = [z,b] = 1; [a,b] = z^{p^{j-1}}; z^{p^j} = 1; a^p = b^p = z \rangle, \\ \mathrm{I}(p^j, m) &= M(p^j)^{\odot m}, \\ \mathrm{II}(p^j, m) &= N(p^j) \odot M(p^j)^{\odot (m-1)}. \end{aligned}$$

Observe the following:

(1) $M(p^j)$ and $N(p^j)$ are JN2 (by Proposition 6) of class $(p^j, 1)$, with each center generated by z and $\{a, b\}$ as a symplectic basis of V.

(2) $\mathrm{I}(p^j, m)$ and $\mathrm{II}(p^j, m)$ are JN2 of class (p^j, m) by the remarks following Definition 9.

(3) $\mathrm{I}(p^j,m)$ and $\mathrm{II}(p^j,m)$ are not isomorphic when $p^j \neq 2$: the group $N(p^j)$ has an element of order p^{j+1} (for example, a or b) and therefore so does $\mathrm{II}(p^j,m)$. On the contrary, the group $M(p^j)$, and consequently also $\mathrm{I}(p^j,m) = M(p^j)^{\odot m}$, has exponent at most p^j: the linear functional ν (as in the proof of Lemma 7) is trivial on the symplectic basis $\{a,b\}$ so $M(p^j)^p \leq (ZG)^p$, and hence $M(p^j)^{p^j} \leq (ZG)^{p^j} = 1$.

Note If $p^j = 2$, then $\mathrm{I}(p^j,m)$ and $\mathrm{II}(p^j,m)$ are still nonisomorphic: $M(2)$ is the dihedral group D_8 and $N(2)$ is the quaternion group Q_8, which contain two and six elements of order 4, respectively, and both have centers of order 2. In particular no elements of order 4 are central. The larger groups $\mathrm{I}(p^j,m)$ and $\mathrm{II}(p^j,m)$ can then be distinguished by counting the number of elements of order 4 because only central elements are identified in the central product. We will not require this case.

Theorem 11 (Classification of finite JN2 groups [5, Theorems 5 and 7(c) and Lemma 8(i)]) *Let G be JN2 of class (p^j,m). Suppose that $p^j \neq 2$. Then G is isomorphic to either $\mathrm{I}(p^j,m)$ or $\mathrm{II}(p^j,m)$.*

Proof Let z be a generator of ZG and let $\mathscr{B}$ be the symplectic basis given by Lemma 7. In the notation of Lemma 7, let $H_i = \langle z, a_i, b_i\rangle$. If $\mathscr{B}$ is type I then $H_i = M(p^j)$ for all i. If $\mathscr{B}$ is type II then $H_1 = N(p^j)$ and $H_i = M(p^j)$ for $i \geq 2$.

The subgroups H_i commute pairwise, together generate G, and intersect precisely in their centers $\langle z\rangle$, so $G \cong \bigodot_{i=1}^m H_i$. Hence G is isomorphic to $\mathrm{I}(p^j,m)$ or $\mathrm{II}(p^j,m)$, according to the type of the basis $\mathscr{B}$. □

Remarks 12 (a) **Generalizations** For brevity, we have excluded the case of $p^j = 2$ and specialized to finite groups. With additional work, the $p^j = 2$ case and some infinite JN2 groups (those with a countable symplectic basis) also admit a classification as central products of elementary JN2 groups, see [5].

(b) **Special cases** Note that $M(p)$ and $N(p)$ are the only two groups of order p^3. The group $M(p) = \mathrm{I}(p,1)$ is isomorphic to the Heisenberg group over $\mathbb{F}_p$. A generalization of the finite Heisenberg groups are the *extraspecial groups*, which are defined to be p–groups G with ZG order p and G/ZG nontrivial elementary abelian. In particular, extraspecial groups are JN2, and it follows from Theorem 11 that there are precisely two distinct extraspecial groups of order p^{1+2m} for each choice of a prime p and positive integer m and that this exhausts all extraspecial groups.

3 Minimal nonabelian quotients of $B_n(\Sigma_g)$

In this section we provide the proof of Theorem 1(b). The strategy of the proof will be to utilize an explicit presentation of the surface braid groups (Theorem 13) to characterize braid-free quotients by the relations that they must satisfy (Lemma 15). We will then show that many JN2 groups are realized as nonabelian braid-free quotients of $B_n(\Sigma_g)$ (Lemma 16) and finally prove that all nonabelian braid-free quotients of minimum order belong to the list of JN2 groups in Lemma 16.

Theorem 13 (Presentation of $B_n(\Sigma_g)$, Bellingeri [1, Theorem 1.2]) *For $g \geq 1$ and $n \geq 2$, the surface braid group $B_n(\Sigma_g)$ admits the presentation given by*

- *generators $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g, b_1, \dots, b_g$,*
- *braid relations*

$$[\sigma_i, \sigma_j] = 1 \qquad \text{for } 1 \leq i, j \leq n-1 \text{ and } |i-j| \geq 2,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \text{for } 1 \leq i \leq n-2,$$

and mixed relations

$$\text{(R1)} \quad [a_r, \sigma_i] = [b_r, \sigma_i] = 1 \qquad \text{for } 1 \leq r \leq g \text{ and } i \neq 1,$$
$$\text{(R2)} \quad [a_r, \sigma_1^{-1} a_r \sigma_1^{-1}] = [b_r, \sigma_1^{-1} b_r \sigma_1^{-1}] = 1 \qquad \text{for } 1 \leq r \leq g,$$
$$\text{(R3)} \quad [a_s, \sigma_1 a_r \sigma_1^{-1}] = [b_s, \sigma_1 b_r \sigma_1^{-1}] = 1 \qquad \text{for } 1 \leq s < r \leq g,$$
$$[b_s, \sigma_1 a_r \sigma_1^{-1}] = [a_s, \sigma_1 b_r \sigma_1^{-1}] = 1 \qquad \text{for } 1 \leq s < r \leq g,$$
$$\text{(R4)} \quad [a_r, \sigma_1^{-1} b_r \sigma_1^{-1}] = \sigma_1^2 \qquad \text{for } 1 \leq r \leq g,$$
$$\text{(TR)} \quad [a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] = \sigma_1 \sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2 \sigma_1.$$

Remark 14 (Geometric interpretation of the presentation) The Bellingeri generators σ_i can be identified as the images of the Artin braid generators under a choice of embedding $B_n \hookrightarrow B_n(\Sigma_g)$. The remaining generators a_r and b_r can be understood loosely to be the standard generators of $\pi_1(\Sigma_g)$.

More precisely, let $\{p_1, \dots, p_n\} \in \mathrm{UConf}_n(\Sigma_g)$ denote the basepoint of $B_n(\Sigma_g)$ and let $D \subset \Sigma_g$ be an open disk with $p_1 \in \partial D$, with $p_2, \dots, p_n$ in the interior of D. There is an inclusion

$$\pi_1(\Sigma_g - D, p_1) \hookrightarrow B_n(\Sigma_g)$$

which takes a loop γ in $\Sigma_g - D$ to the braid on Σ_g with first strand γ and all other strands trivial. The group $\pi_1(\Sigma_g - D, p_1)$ is free on $2g$ generators and surjects onto $\pi_1(\Sigma_g, p_1)$, which has a standard presentation. The surface braid group generators $a_r, b_r \in B_n(\Sigma_g)$ can then be understood as a choice of a free generating set of $\pi_1(\Sigma_g - D, p_1)$ which lifts the standard generating set of $\pi_1(\Sigma_g, p_1)$. It should be emphasized that the lifts are not canonical and that the presentation depends on the choices; the curious reader may refer to [1] for illustrations of the loops which produce this particular presentation.

Lemma 15 (Characterization of braid-free quotients) *Let $n \geq 3$ and $g \geq 1$. A finite group G is a braid-free quotient of $B_n(\Sigma_g)$ if and only if G admits a generating set $\{\sigma, a_1, b_1, \dots, a_g, b_g\}$ satisfying the relations*

$$\text{(R1}') \quad [a_r, \sigma] = [b_r, \sigma] = 1 \qquad \text{for } 1 \leq r \leq g,$$
$$\text{(R3}') \quad [a_s, a_r] = [b_s, b_r] = [b_s, a_r] = [a_s, b_r] = 1 \qquad \text{for } 1 \leq s < r \leq g,$$
$$\text{(R4}') \quad [a_r, b_r] = \sigma^2 \qquad \text{for } 1 \leq r \leq g,$$
$$\text{(TR}') \quad \sigma^{2(g+n-1)} = 1.$$

Proof A finite quotient of $B_n(\Sigma_g)$ is presented by Theorem 13 with additional relations. The condition that B_n has cyclic image in a quotient is equivalent to adding the relations

$$\sigma_i = \sigma_1, \qquad 1 \le i \le n.$$

If we add these relations and write $\sigma = \sigma_1$, the relation (R2) is made redundant and (R1), (R3), and (R4) respectively reduce to the relations (R1′), (R3′), and (R4′), as in the statement of the lemma. The final relation (TR) reduces to

$$[a_1, b_1^{-1}]\cdots[a_g, b_g^{-1}] = \sigma^{2(n-1)},$$

which is equivalent to (TR′) because from (R4′) we can write $a_r = b_r^{-1}\sigma^{-2}a_r b_r$ so that

$$[a_r, b_r^{-1}] = a_r b_r^{-1} a_r^{-1} b_r \overset{(R4')}{=} (b_r^{-1}\sigma^{-2}a_r b_r) b_r^{-1} a_r^{-1} b_r = b_r^{-1}\sigma^{-2} b_r \overset{(R1')}{=} \sigma^{-2}. \qquad \square$$

The following lemma proves that many JN2 groups are braid-free quotients:

Lemma 16 *Let $n \ge 3$ and $g \ge 1$. Let p be a prime dividing $g + n - 1$.*

(a) *If $p = 2$ then $\mathrm{I}(2^2, g)$ and $\mathrm{II}(2^j, g)$ for all $j \ge 2$ are nonabelian braid-free quotients of $B_n(\Sigma_g)$.*

(b) *If p is odd then $\mathrm{I}(p, g)$ and $\mathrm{II}(p^j, g)$ for all $j \ge 1$ are nonabelian braid-free quotients of $B_n(\Sigma_g)$.*

Proof Let p be a prime dividing $g+n-1$. By Lemma 15 we need to exhibit a generating set $\{\sigma, a_r, b_r\}$ of each group satisfying relations (R1′), (R3′), (R4′), and (TR′).

In any of the JN2 groups in the statement of the theorem, fix a generator z of the center and choose $a_1, b_1, \ldots, a_g, b_g$ to be the representatives of a symplectic basis of V given by Lemma 7. By Theorem 11 this basis will be type I for $\mathrm{I}(2^2, g)$ and $\mathrm{I}(p, g)$, and type II for $\mathrm{II}(2^j, g)$ and $\mathrm{II}(p^j, g)$. Note that with the given symplectic form, the condition that a basis is symplectic is simply that all basis elements commute except symplectic pairs $[a_r, b_r] = z^{p^{j-1}}$. In particular, (R3′) is satisfied.

We will now choose σ for each group and verify that $\{\sigma, a_r, b_r\}$ generate the group and satisfy (R4′).

(1) $\mathrm{I}(2^2, g)$ is generated by $\sigma = z$ and $\{a_r, b_r\}$. These satisfy (R4′) because $[a_r, b_r] = z^2 = \sigma^2$.

(2) $\mathrm{II}(2^j, g)$, for a given $j \ge 2$, is generated by $\{a_r, b_r\}$ alone because $a_1^p = z$. If we choose $\sigma = z^{2^{j-2}}$ then (R4′) is satisfied because $[a_r, b_r] = z^{2^{j-1}} = \sigma^2$.

(3) $\mathrm{I}(p, g)$ for odd prime p is generated by $\sigma = z^{(p^j+1)/2}$ and the a_r and b_r. Then (R4′) is satisfied because $[a_r, b_r] = z = \sigma^2$.

(4) $\mathrm{II}(p^j, g)$, for given odd prime p and $j \ge 1$, is generated by a_r and b_r alone because $a_1^p = z$. Then set $\sigma = z^{(p^j + p^{j-1})/2}$ so that (R4′) is satisfied because $[a_r, b_r] = z^{p^{j-1}} = \sigma^2$.

In all cases σ was chosen to be central, and hence (R1′) is satisfied.

It remains to check that (TR′) holds, namely that $|\sigma|$ divides $2(g+n-1)$. Recall that we are assuming that $p \mid (g+n-1)$. In cases (1) and (2), we have $p = 2$ and $|\sigma| = 4 = 2p \mid 2(g+n-1)$. In cases (3) and (4), we have $|\sigma| = p \mid (g+n-1)$. $\square$

Proof of Theorem 1(b) Let G be a nonabelian braid-free quotient and let $\{\sigma, a_1, b_1, \dots, a_g, b_g\}$ denote the generating set of G as given by Lemma 15. By (R1′) and (R3′), all pairs of these generators commute except for pairs a_r and b_r, so $G' = \langle \sigma^2 \rangle$ by (R4′). Then G' is central and nontrivial, which is to say that G is 2–step nilpotent.

Assume now that G is of minimum order amongst nonabelian braid-free quotients of $B_n(\Sigma_g)$. Then G has no proper nonabelian quotients and thus is JN2 of some class (p^j, m).

We make three claims:

(1) $p^j \neq 2$,

(2) $m = g$, and

(3) $p \mid (g + n - 1)$.

These claims will complete the proof: Since $|G| = p^{2m+j}$, it follows from claims (1) and (2) that $|G| \geq p^{2g+1}$ if p is odd and $|G| \geq 2^{2g+2}$ if $p = 2$. Claims (1) and (3) along with the minimality of G together imply that G is one of (in particular, the smallest of) the quotients constructed in Lemma 16. Explicitly, if $g + n - 1$ is even then $p^j = 2^2$. Otherwise $p^j = p$, where p is the smallest prime dividing $g + n - 1$. Finally, G must be isomorphic to either $\mathrm{I}(p^j, g)$ or $\mathrm{II}(p^j, g)$ by Theorem 11.

Proof of claims Let $d = |\sigma|$. By (R1′), σ is central so $d \mid p^j$. But $p = |G'| = |\sigma^2|$ so $d \mid 2p$. Thus either p is odd and $p = d$, or $p = 2$ and $d = 4$.

(1) If p is odd then $p^j \neq 2$. If $p = 2$ then $p^j \geq d = 4$ so $p^j \neq 2$.

(2) We will show that $\dim V = 2g$ by proving that

$$\mathscr{B} = \{a_r \bmod ZG, b_r \bmod ZG\}_{r=1}^{g}$$

is a basis of V. Every element $x \in G$ can be written uniquely in the form

$$x = \sigma^k a_1^{i_1} \cdots a_g^{i_g} b_1^{j_1} \cdots b_g^{j_g}$$

using commuting relations (R1′), (R3′), and (R4′) so $\mathscr{B}$ is a generating set. To prove that $\mathscr{B}$ is linearly independent, let

$$y = a_1^{i_1} \cdots a_g^{i_g} b_1^{j_1} \cdots b_g^{j_g} \in G$$

and suppose that $y = 0 \bmod ZG$, which is to suppose that an arbitrary linear combination of elements of $\mathscr{B}$ is trivial in V. Then y is central, so

$$[y, b_1] = [a_1^{i_1}, b_1] = \sigma^{-2i_1} = 1,$$

which implies that $d \mid 2i_1$ and thus $i_1 = 0 \bmod p$: If p is odd then $d = p$, so $p \mid i_1$. If $p = 2$ then $d = 4 \mid 2i_1$ so, $p = 2 \mid i_1$.

Similarly $i_r = j_r = 0 \bmod p$ for all r, which is to say that all coefficients of the linear combination are trivial over the base field $\mathbb{F}_p$. This proves the linear independence of $\mathscr{B}$.

(3) The relation (TR′) imposes the relation $d \mid 2(g+n-1)$. Either $d = p$ is odd or $d = 4$ and $p = 2$; in both cases (TR′) implies that $p \mid (g+n-1)$. □

Proof of Corollary 4 Let $n \geq 5$ and $m \geq 3$, and let $g, h \geq 0$. If there is a surjection $B_n(\Sigma_g) \to B_m(\Sigma_h)$ then the composition $B_n(\Sigma_g) \to B_m(\Sigma_h) \to S_m$ is also surjective. Since S_m is not nilpotent when $m \geq 3$, we must have $m \geq n$. □

Remark 17 (Punctured surfaces, surfaces with boundary) Bellingeri [1] also gives a presentation of the braid group of a genus-g surface with m punctures (equivalently for the purposes of braid groups, m boundary components). The above methods can be used nearly verbatim to prove that the smallest nonabelian quotient of $B_n(\Sigma_{g,m})$ is the smaller of S_n or $\mathrm{I}(2^2, g)$ and $\mathrm{II}(2^2, g)$.

References

[1] **P Bellingeri**, *On presentations of surface braid groups*, J. Algebra 274 (2004) 543–563 MR Zbl

[2] **L Chen**, *Surjective homomorphisms between surface braid groups*, Israel J. Math. 232 (2019) 483–500 MR Zbl

[3] **E Fadell**, **J Van Buskirk**, *On the braid groups of E^2 and S^2*, Bull. Amer. Math. Soc. 67 (1961) 211–213 MR Zbl

[4] **S Kolay**, *Smallest noncyclic quotients of braid and mapping class groups*, Geom. Topol. 27 (2023) 2479–2496 MR Zbl

[5] **M F Newman**, *On a class of nilpotent groups*, Proc. London Math. Soc. 10 (1960) 365–375 MR Zbl

Department of Mathematics, University of Chicago
Chicago, IL, United States

cindy@math.uchicago.edu

Received: 16 January 2023 Revised: 7 June 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Robert Lipshitz	University of Oregon lipshitz@uoregon.edu
Tara E Brendle	University of Glasgow tara.brendle@glasgow.ac.uk	Norihiko Minami	Yamato University minami.norihiko@yamato-u.ac.jp
Indira Chatterji	CNRS & Univ. Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Robert Oliver	Université Paris 13 bobol@math.univ-paris13.fr
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Jessica S Purcell	Monash University jessica.purcell@monash.edu
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
David Futer	Temple University dfuter@temple.edu	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Vesna Stojanoska	Univ. of Illinois at Urbana-Champaign vesna@illinois.edu
Ian Hambleton	McMaster University ian@math.mcmaster.ca	Zoltán Szabó	Princeton University szabo@math.princeton.edu
Matthew Hedden	Michigan State University mhedden@math.msu.edu	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Chris Wendl	Humboldt-Universität zu Berlin wendl@math.hu-berlin.de
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu	Daniel T Wise	McGill University, Canada daniel.wise@mcgill.ca
Thomas Koberda	University of Virginia thomas.koberda@virginia.edu	Lior Yanovski	Hebrew University of Jerusalem lior.yanovski@gmail.com
Markus Land	LMU München markus.land@math.lmu.de		

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

© 2024 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 7 (pages 3571–4137) 2024

Geography of bilinearized Legendrian contact homology 3571

FRÉDÉRIC BOURGEOIS and DAMIEN GALANT

The deformation spaces of geodesic triangulations of flat tori 3605

YANWEN LUO, TIANQI WU and XIAOPING ZHU

Finite presentations of the mapping class groups of once-stabilized Heegaard splittings 3621

DAIKI IGUCHI

On the structure of the top homology group of the Johnson kernel 3641

IGOR A SPIRIDONOV

The Heisenberg double of involutory Hopf algebras and invariants of closed 3–manifolds 3669

SERBAN MATEI MIHALACHE, SAKIE SUZUKI and YUJI TERASHIMA

A closed ball compactification of a maximal component via cores of trees 3693

GIUSEPPE MARTONE, CHARLES OUYANG and ANDREA TAMBURELLI

An algorithmic discrete gradient field and the cohomology algebra of configuration spaces of two points on complete graphs 3719

EMILIO J GONZÁLEZ and JESÚS GONZÁLEZ

Spectral diameter of Liouville domains 3759

PIERRE-ALEXANDRE MAILHOT

Classifying rational G–spectra for profinite G 3801

DAVID BARNES and DANNY SUGRUE

An explicit comparison between 2–complicial sets and Θ_2–spaces 3827

JULIA E BERGNER, VIKTORIYA OZORNOVA and MARTINA ROVELLI

On products of beta and gamma elements in the homotopy of the first Smith–Toda spectrum 3875

KATSUMI SHIMOMURA and MAO-NO-SUKE SHIMOMURA

Phase transition for the existence of van Kampen 2–complexes in random groups 3897

TSUNG-HSUAN TSAI

A qualitative description of the horoboundary of the Teichmüller metric 3919

AITOR AZEMAR

Vector fields on noncompact manifolds 3985

TSUYOSHI KATO, DAISUKE KISHIMOTO and MITSUNOBU TSUTAYA

Smallest nonabelian quotients of surface braid groups 3997

CINDY TAN

Lattices, injective metrics and the $K(\pi, 1)$ conjecture 4007

THOMAS HAETTEL

The real-oriented cohomology of infinite stunted projective spaces 4061

WILLIAM BALDERRAMA

Fourier transforms and integer homology cobordism 4085

MIKE MILLER EISMEIER

Profinite isomorphisms and fixed-point properties 4103

MARTIN R BRIDSON

Slice genus bound in DTS^2 from s–invariant 4115

QIUYU REN

Relatively geometric actions of Kähler groups on CAT(0) cube complexes 4127

COREY BREGMAN, DANIEL GROVES and KEJIA ZHU