The real-oriented cohomology of infinite stunted projective spaces

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Let $E\mathbb{R}$ be an even-periodic real Landweber exact C_2–spectrum, and ER be its spectrum of fixed points. We compute the ER–cohomology of the infinite stunted real projective spectra P_j. These cohomology groups combine to form the $RO(C_2)$–graded coefficient ring of the C_2–spectrum

$$b(ER) = F(EC_{2+}, i_*ER),$$

which we show is related to $E\mathbb{R}$ by a cofiber sequence $\Sigma^\sigma b(ER) \to b(ER) \to E\mathbb{R}$. We illustrate our description of $\pi_\star b(ER)$ with the computation of some ER–based Mahowald invariants.

1 Introduction

The spectrum MU of complex cobordism plays a central role in both our conceptual and computational understanding of stable homotopy theory. Landweber [1968] introduced what is now known as the C_2–equivariant spectrum $M\mathbb{R}$ of real bordism, with underlying spectrum MU and fixed points $MR = MU^{hC_2}$ the homotopy fixed points for the action of C_2 on MU by complex conjugation. Work of Araki [1979], Hu and Kriz [2001], and others, has shown that essentially all of the theory of complex-oriented homotopy theory may be carried out in the C_2–equivariant setting with $M\mathbb{R}$ in place of MU, leading to the rich subject of real-oriented homotopy theory. This subject has seen extensive study over the past two decades, with a notable increase in interest following the use of $M\mathbb{R}$ by Hill, Hopkins and Ravenel [Hill et al. 2016] to resolve the Kervaire invariant one problem.

There are real analogues of most familiar complex-oriented cohomology theories. An important family of examples is given by the real Johnson–Wilson theories $E\mathbb{R}(n)$, refining the usual Johnson–Wilson theories $E(n)$. These theories are Landweber flat over $M\mathbb{R}$, in the sense that they are $M\mathbb{R}$–modules and satisfy

$$E\mathbb{R}(n)_\star X \cong E\mathbb{R}(n)_\star \otimes_{M\mathbb{R}_\star} M\mathbb{R}_\star X$$

for any C_2–spectrum X. The fixed points $ER(n) = E\mathbb{R}(n)^{C_2} = E(n)^{hC_2}$ are nonequivariant cohomology theories that are interesting in their own right; for example, $ER(1) \simeq KO_{(2)}$, and $ER(2)$ is a variant of $\mathrm{TMF}_0(3)_{(2)}$. One may regard the descent from $E(n)$ to $ER(n)$ as encoding a portion of the $E(n)$–based Adams–Novikov spectral sequence, and accordingly each $ER(n)$ detects infinite families in $\pi_* S$.

There is in general a tradeoff between the richness of a homology theory and the ease with which it may be computed. Kitchloo, Lorman, and Wilson have carried out extensive computations with $ER(n)$–theory [Kitchloo and Wilson 2007b; 2015; Lorman 2016; Kitchloo et al. 2017; 2018a], and their program has shown that these theories strike a very pleasant balance between richness and computability. Computations of $ER(2)^*\mathbb{R}P^n$ in particular have been applied to the nonimmersion problem for real projective spaces, with computations for $n = 2k$ in [Kitchloo and Wilson 2008a], $n = 16k + 1$ in [Kitchloo and Wilson 2008b], and $n = 16k + 9$ by Banerjee [2010].

This paper contributes to the above story. Let $E\mathbb{R}$ be a real Landweber exact C_2–spectrum in the sense of Hill and Meier [2017, Section 3.2]; we take this to include the assumption that $E\mathbb{R}$ is strongly even. Write E for the underlying spectrum of $E\mathbb{R}$ and $ER = E\mathbb{R}^{C_2} = E^{hC_2}$ for its fixed points. Suppose moreover that $E\mathbb{R}$ is even-periodic, in the sense that $\pi_{1+\sigma} E\mathbb{R}$ contains a unit. This is equivalent to asking that the $M\mathbb{R}$–orientation of $E\mathbb{R}$ extends to an $MP\mathbb{R}$–orientation, where

$$MP\mathbb{R} \simeq \bigoplus_{n \in \mathbb{Z}} \Sigma^{n(1+\sigma)} M\mathbb{R}$$

is the real analogue of 2–periodic complex cobordism.

The primary goal of this paper is to compute the ER–cohomology of the infinite stunted projective spectra P_j. When $j > 0$, these are the spaces

$$P_j = \mathbb{R}P^\infty / \mathbb{R}P^{j-1};$$

in general, P_j is the Thom spectrum of $j\sigma$, where σ is the sign representation of C_2 regarded as a vector bundle over $BC_2 = \mathbb{R}P^\infty$. The cohomology $ER^* P_*$ is of interest for at least a few reasons: first, it is one long exact sequence away from the groups $ER^*\mathbb{R}P^j$, which have so far only been studied at heights ≤ 2; second, there are C_2–equivariant Hurewicz maps $\pi_{c+w\sigma} S_{C_2} \to ER^{-c} P_w$, which are at least as nontrivial as the nonequivariant Hurewicz maps for ER; third, there is an interesting interplay between the C_2 appearing in $ER \simeq E^{hC_2}$ and the C_2 appearing in $ER^*(P_w) \simeq ER^*(S_{hC_2}^{w\sigma})$ which sheds some light on the nature of the C_2–spectrum $E\mathbb{R}$.

We record the basic properties of ER in Section 3. In particular, $\pi_0 ER \cong \pi_0 E$, the torsion in $\pi_* ER$ is supported on a single class $x \in \pi_1 ER$, there is a cofiber sequence $\Sigma ER \xrightarrow{x} ER \to E$, and the x–Bockstein spectral sequence for ER–cohomology agrees with the homotopy fixed point spectral sequence (HFPSS) from the E_2–page on.

Write $b(ER) = F(EC_{2+}, i_* ER)$ for the Borel C_2–spectrum on ER with trivial C_2–action. This satisfies $\pi_{c+w\sigma} b(ER) = ER^{-c} P_w$, and we shall compute $ER^* P_*$ using the x–Bockstein spectral sequence

$$\pi_\star b(E)[x] \Rightarrow \pi_\star b(ER).$$

This concludes an investigation we began in [Balderrama 2021]. There, we computed the HFPSS $H^*(C_2; \pi_\star b(KU_2^\wedge)) \Rightarrow \pi_\star b(KO_2^\wedge)$ as a step in our description of the C_2–equivariant $K(1)$–local sphere. At the time, we were able to put the E_2–page into a more general context by computing $H^*(C_2; \pi_\star b(E))$

for more general even-periodic Landweber exact spectra E, but had no information about possible higher differentials. In this paper, we carry out the rest of the computation for real-oriented E. The results are summarized in Section 2 below.

1.1 Remark The reader may observe that by restricting to even-periodic spectra, we have ruled out the real Johnson–Wilson theories $ER(n)$ for $n \geq 2$. However, any real Landweber exact C_2–spectrum $E\mathbb{R}$ is a summand of the even-periodic theory $\bigoplus_{n \in \mathbb{Z}} \Sigma^{n(1+\sigma)} E\mathbb{R}$, so no real information has been lost. A more subtle point is that implicit in the definition of real Landweber exactness is the assumption that $E\mathbb{R}$ is a ring up to homotopy, and it is not known whether $E\mathbb{R}(n)$ always satisfies this. However, the partial multiplicative structure given in [Kitchloo et al. 2018b] is sufficient for our computation to apply to 2–periodic $ER(n)$–theory.

2 Summary

We now describe our results. We start with the following, which serves as the linchpin for our computation of $\pi_\star b(ER)$. Write $\rho \in \pi_{-\sigma} S_{C_2}$ for the Euler class of the sign representation and $\tau^{-2} \in \pi_{2\sigma-2} b(E)$ for the Thom class of $2\sigma = \mathbb{C} \otimes \sigma$. These classes are sometimes denoted by a_σ and u_σ^{-2}, but we will reserve those symbols for $E\mathbb{R}$ and $C_{2+} \otimes i_* ER$. Write $u \in \pi_2 E$ for the chosen unit, and set

$$\xi = \rho\tau^{-2}u \in \pi_\sigma b(E).$$

2.1 Theorem (Section 4) *The class ξ is a permanent cycle in the x–Bockstein spectral sequence, detecting a lift of x. Moreover, there is a cofiber sequence*

$$(1) \qquad\qquad \Sigma^\sigma b(ER) \xrightarrow{\xi} b(ER) \to E\mathbb{R}$$

of C_2–spectra.

This cofiber sequence is a twisted form of the standard cofiber sequence

$$(2) \qquad\qquad \Sigma^{-\sigma} b(ER) \xrightarrow{\rho} b(ER) \to C_{2+} \otimes i_* ER.$$

2.2 Example When $E = KU$, one can identify $b(ER) = F(EC_{2+}, KO_{C_2})$ and $E\mathbb{R} = K\mathbb{R}$, and $\xi = \pm\eta_{C_2}$ is the C_2–equivariant Hopf map. In this case, Theorem 2.1 recovers the real Wood cofibering $KO_{C_2}/(\eta_{C_2}) \simeq K\mathbb{R}$ (cf [Guillou et al. 2020, Proposition 10.13]).

To show that ξ is a permanent cycle detecting a lift of x, we first reduce to the universal case $E = MP$, then show that this is the only possibility compatible with norms on $b(MPR)$. Given this, the cofiber sequence of (1) is a mostly formal consequence of (2) and the fact that ξ differs from ρ by a unit in $\pi_\star b(E)$.

We now describe $\pi_\star b(ER)$. We start by fixing some notation for $\pi_\star b(E)$. Write $[2](z) \in E_0[\![z]\!]$ for the 2–series of the formal group law of E, and write $u_n \in E_0$ for the elements corresponding to the usual $v_n \in \pi_{2(2^n-1)}E$ by $u_n = u^{-(2^n-1)}v_n$. We may find series $h_n(z) \in E_0[\![z]\!]$ for $n \geq 0$, of the form $h_n(z) = u_n + O(z)$ and satisfying

$$[2](z) = zh_0(z), \quad h_n(z) \equiv u_n + z^{2^n}h_{n+1}(z) \pmod{u_0, \ldots, u_{n-1}}.$$

Note in particular

$$[2](z) \equiv z^{2^n}h_n(z) \pmod{u_0, \ldots, u_{n-1}}.$$

We now specialize to $\pi_\star b(E)$. Set

$$z = \rho\xi = \rho^2\tau^{-2}u, \quad h_n = h_n(z), \quad w_n = \rho^{2^{n+1}}h_{n+1} \equiv \tau^{2^{n+1}}u^{-2^n}(h_n - u_n),$$

the last congruence being modulo $(u_0, \ldots, u_{n-1})$. We abbreviate $h = h_0$. This is the transfer element in $\pi_0 b(E) = E^0 BC_2$, and we have

$$\pi_0 b(E) = \frac{E_0[\![z]\!]}{([2](z))}, \quad \pi_\star b(E) = \frac{E_0[\rho, \tau^{\pm 2}, u^{\pm 1}]^\wedge_\rho}{(\rho \cdot h)};$$

see for instance [Balderrama 2021, Section 2.1].

2.3 Theorem (Section 5) *Define the subring $Z \subset \pi_\star b(E)$ by*

$$Z = E_0(\rho, \xi, \tau^{2^{n+2}l}u^{2^{n+1}k}u_n, \tau^{2^{n+1}(2l+1)}u^{2^{n+1}k}h_n : n \geq 0; k, l \in \mathbb{Z})^\wedge_{(\rho, \xi)} \subset \pi_\star b(E),$$

and let $B \subset Z[x]$ be the ideal generated by the elements

$$\tau^{2^{n+2}l}u^{2^{n+1}k}u_n \cdot x^{2^{n+1}-1}, \quad \tau^{2^{n+1}(2l+1)}u^{2^{n+1}k}h_n \cdot x^{2^{n+1}-1}, \quad \tau^{2^{n+2}l}u^{2^{n+1}k}w_n \cdot x^{2^{n+1}-1}$$

for $n \geq 0$ and $k, l \in \mathbb{Z}$. Then $Z[x]/B$ is the x–adic associated graded of $\pi_\star b(ER)$.

2.4 Remark In integer degrees, $\pi_* b(ER)$ is very simply described:

$$\pi_* b(ER) \cong ER_*[\![z]\!]/([2](z));$$

see Corollary 4.3. This does not require the full computation of $\pi_\star b(ER)$, and follows as soon as one knows that ξ is a permanent cycle. In particular, $\pi_0 b(ER) \cong E^0 BC_2$. To get a feeling for $\pi_\star b(ER)$ outside integer degrees, the reader may wish to peruse Tables 1 and 2, described in Remark 5.5, which list $\pi_0 b(ER)$–module generators for the groups $\pi_{c+w\sigma}b(ER)$ in a range.

2.5 Remark Implicit in Theorem 2.3 is the fact that $\tau^{2^{n+2}l}u^{2^{n+1}k}w_n \in Z$ for $n \geq 0$ and $k, l \in \mathbb{Z}$. In particular,

$$\tau^{2^{n+2}(2l+1)}u^{2^{n+2}k}w_n = \rho^{2^{n+1}} \cdot \tau^{2^{n+2}(2l+1)}u^{2^{n+2}k}h_{n+1},$$
$$\tau^{2^{n+3}l}u^{2^{n+1}(2k+1)}w_n = \xi^{2^{n+1}} \cdot \tau^{2^{n+2}(2l+1)}u^{2^{n+2}k}h_{n+1},$$
$$\tau^{2^{n+3}l}u^{2^{n+2}k}w_n \equiv \rho^{2^{n+1}} \cdot \tau^{2^{n+3}l}u^{2^{n+2}k}u_{n+1} + \xi^{2^{n+1}} \cdot \tau^{2^{n+3}l}u^{2^{n+2}k}w_{n+1},$$
$$\tau^{2^{n+2}(2l-1)}u^{2^{n+1}(2k+1)}w_n \equiv \xi^{2^{n+1}} \cdot \tau^{2^{n+3}l}u^{2^{n+2}k}u_{n+1} + \rho^{2^{n+1}} \cdot \tau^{2^{n+3}(l-1)}u^{2^{n+2}(k+1)}w_{n+1},$$

where the last two formulas hold mod $u_0, \ldots, u_n$.

The ring $Z[x]/B$ is the E_∞–page of the x–Bockstein spectral sequence for $\pi_\star b(ER)$, obtained after running differentials which are generated by

$$d_{2^{n+1}-1}(u^{2^n}) = u_n x^{2^{n+1}-1}, \quad d_{2^{n+1}-1}(\tau^{2^{n+1}}) = -w_n x^{2^{n+1}-1}.$$

The differentials on u^{2^n} appear in the x–Bockstein spectral sequence for $\pi_* ER$, and are consequences of the computation of $\pi_\star M\mathbb{R}$ by Hu and Kriz [2001], as we review in Section 3. The differentials on $\tau^{2^{n+1}}$ are the core of our computation. These differentials turn out to be forced by the permanent cycle $\xi = \rho\tau^{-2}u$, by a Leibniz rule argument based on $d_{2^{n+1}-1}(\xi^{2^n}) = 0$. This argument would not be possible if one tried to compute each $ER^* P_j$ individually, and illustrates the strength of using the C_2–spectrum $b(ER)$ as a tool for packaging information about the cohomology of all stunted projective spectra into one object.

One might also try to understand $\pi_\star b(ER)$ through the ρ–Bockstein or the ξ–Bockstein spectral sequences. Using the cofiber sequences (2) and (1), these are of signature

$$\pi_\star(C_{2+} \otimes i_* ER)[\rho] \Rightarrow \pi_\star b(ER), \quad \pi_\star E\mathbb{R}[\xi] \Rightarrow \pi_\star b(ER).$$

Here, $\pi_\star(C_{2+} \otimes i_* ER) \cong \pi_* ER[u_\sigma^{\pm 1}]$ with $|u_\sigma| = 1 - \sigma$, and in degrees $* + w\sigma$ the ρ–Bockstein spectral sequence is exactly the Atiyah–Hirzebruch spectral sequence for $ER^* P_w$ based on the standard cell structure of P_w. By construction, the differentials in these spectral sequences are controlled by the boundary maps

$$\mathrm{tr}(u_\sigma^{-1} \cdot -): \pi_{\star+1-\sigma}(C_{2+} \otimes i_* ER) \to \pi_\star b(ER), \quad \partial: \pi_{\star+1+\sigma} E\mathbb{R} \to \pi_\star b(ER)$$

for the cofiber sequences (2) and (1). This first boundary map is exactly the transfer for the C_2–spectrum $b(ER)$. Although we do not know whether it is feasible to compute either the ρ–Bockstein or ξ–Bockstein spectral sequence directly, we can use our computation of $\pi_\star b(ER)$ to deduce the following.

Write $\bar{u} \in \pi_{1+\sigma} E\mathbb{R}$ for the invertible element guaranteed by the $MP\mathbb{R}$–orientation of $E\mathbb{R}$.

2.6 Theorem (Section 6) *The above transfer and boundary maps satisfy*

$$\mathrm{tr}(u_\sigma^{-1} \cdot u_\sigma^{2^n(2k+1)}) = \rho^{2^n-1}\tau^{2^{n+1}k} h_n x^{2^n-1} + O(x^{2^n}),$$
$$\partial(\bar{u}^{2^n(2k+1)}) = \xi^{2^n-1}\tau^{-2^{n+1}k} u^{2^{n+1}k} h_n x^{2^n-1} + O(x^{2^n})$$

for $n \geq 0$ and $k \in \mathbb{Z}$.

The error terms here are necessary as the classes $\tau^{2^{n+1}k} h_n$ and $\tau^{-2^{n+1}k} u^{2^{n+1}k} h_n$ have only been defined mod x. It is amusing to observe that Theorem 2.6 produces elements of arbitrarily high x–adic filtration in the C_2–equivariant Hurewicz image of $b(MPR)$; as far as we know, such families have not yet been constructed in the nonequivariant Hurewicz image of MPR.

Theorem 2.3 does not quite describe the ring $\pi_\star b(ER)$, but only its x–adic associated graded $Z[x]/B$. The latter is a good approximation to the former, particularly when compared to the ρ–adic and ξ–adic associated graded rings, where the classes ρ and ξ appear as simple 2–torsion classes. Still, taking the

x–adic associated graded does kill some information, and it seems to be a subtle problem to completely reconstruct the ring $\pi_\star b(ER)$. Although we shall not completely resolve this, we do discuss where to find hidden ρ and ξ–extensions. The importance of ρ–extensions is clear: as

$$\pi_{c+w\sigma}(b(ER)/(\rho^m)) = ER^{-c}(P_{w-m}^{w-1}),$$

one must understand the action of ρ if one wishes to extract information about the ER–cohomology of finite projective spaces. The importance of ξ–extensions is clear from the perspective of C_2–equivariant homotopy theory: just as important classes in the Hurewicz image of ER are supported on x, important classes in the C_2–equivariant Hurewicz image of $b(ER)$ are supported on ξ, such as the equivariant Hopf fibrations η_{C_2}, ν_{C_2}, and σ_{C_2} detected in $\pi_\star b(ER)$ by $h_1\xi$, $h_2\xi^2 x$, and $h_3\xi^4 x^3$ respectively, and so the action of ξ gives information about the behavior of these elements. The cofiber sequences (1) and (2) give information about ρ and ξ–extensions, leading to the following.

2.7 Theorem (Section 7) *There are extensions*

$$\rho \cdot \tau^{2(2^{n+1}k-r)} u^{2^{n+1}(2l+1)} h = \left(\tau^{2^{n+2}k} u^{2^{n+2}l} h_{n+1}\xi^{2r-1} + O(\rho)\right)x^{2^{n+2}-2r} + O(x^{2^{n+2}-2r+1}),$$

$$\xi \cdot \tau^{2(2^{n+1}k+r)} u^{2(2^n(2l+1)-r)} h = \left(\tau^{2^{n+2}k} u^{2^{n+2}l} h_{n+1}\rho^{2r-1} + O(\xi)\right)x^{2^{n+2}-2r} + O(x^{2^{n+2}-2r+1})$$

for $k, l \in \mathbb{Z}$, $n \geq 0$, and $1 \leq r \leq 2^{n+1} - 1$.

As with Theorem 2.3, implicit in this theorem is the fact that the terms on the left and right do in fact live in $\pi_\star b(ER)$, for example $\tau^4 h = 2\tau^4 + \rho\xi \cdot \tau^4 h_1$. The error terms are present to remind the reader that these are extensions and not products: to resolve them would require describing how to lift classes from Z to $\pi_\star b(ER)$, and we shall not pursue this. In particular, if k is even then the h_{n+1} terms on the right may be replaced with u_{n+1} without affecting the theorem statement.

This concludes our description of $\pi_\star b(ER)$. Although $\pi_\star b(ER)$ is complicated, it is not impossible to work with. We illustrate this in Section 8 by computing some MPR–based Mahowald invariants. Li, Shi, Wang and Xu [Li et al. 2019] have shown that real bordism detects the Hopf elements, Kervaire classes, and $\bar{\kappa}$ family. These are the elements in $\pi_* S$ detected in the classical Adams spectral sequence by the Sq^0–families generated by h_0, h_0^2, and g_1. We compute the iterated MPR–based Mahowald invariants of 2, 4, and $\bar{\kappa}$, showing that they line up with these Sq^0–families exactly.

3 Even-periodic real Landweber exact spectra

We begin by recording some properties of $E\mathbb{R}$ and ER. The material of this section is essentially a translation to the even-periodic setting of familiar facts about the real Johnson–Wilson theories. We would like to avoid confusion between elements of $\pi_\star b(ER)$ and $\pi_\star E\mathbb{R}$, so in this section we write a_σ instead of ρ for the Euler class of the sign representation, and use the symbol u_σ for what would previously have been written τ. In particular, these symbols have degrees

$$|a_\sigma| = -\sigma, \quad |u_\sigma| = 1 - \sigma.$$

Before considering $E\mathbb{R}$, we consider the C_2–HFPSS in general. There is a cofiber sequence

(3) $$S^{-\sigma} \xrightarrow{a_\sigma} S^0 \to C_{2+}.$$

Let X be a C_2–spectrum. Then we may identify

$$\pi_\star(C_{2+} \otimes X) \cong \pi_*^e X[u_\sigma^{\pm 1}],$$

and so the a_σ–Bockstein spectral sequence for X is of signature

$$E_1 = \pi_*^e X[u_\sigma^{\pm 1}, a_\sigma] \Rightarrow \pi_\star X,$$

where $\pi_*^e X$ are the homotopy groups of the underlying spectrum of X. This spectral sequence converges conditionally to the homotopy groups of the a_σ–completion of X, which may be identified as its Borel completion $F(EC_{2+}, X)$. Moreover we have the following fact; see for instance [Hill and Meier 2017, Lemma 4.8].

3.1 Lemma *For any C_2–spectrum X, the a_σ–Bockstein spectral sequence for X agrees with the HFPSS for X from the E_2–page on.* □

The proof amounts to identifying the a_σ–Bockstein spectral sequence with the Borel cohomology spectral sequence induced by the standard cellular filtration of EC_{2+}. This identification leads to the following.

3.2 Lemma *Let X be a C_2–spectrum, and write ψ^{-1} for the involution on $\pi_*^e X$. Then the d_1–differential in the a_σ–Bockstein spectral sequence for X is given by*

$$d_1(\alpha u_\sigma^n a_\sigma^m) = (\alpha - (-1)^n \psi^{-1}(\alpha)) u_\sigma^{n-1} a_\sigma^{m+1}$$

for $\alpha \in \pi_*^e X$. In particular, if X carries a product, then the differentials satisfy the Leibniz rule

$$d_r(\alpha\beta) = d_r(\alpha)\beta + \psi^{-1}(\alpha)d_r(\beta)$$

for $r \geq 1$, where the ψ^{-1} may be omitted for $r \geq 2$. □

Now let $E\mathbb{R}$ be as in the introduction: a strongly even and even-periodic and real Landweber exact C_2–spectrum in the sense of [Hill and Meier 2017, Section 3.2], with underlying spectrum E. This set of assumptions means three things. First, $E\mathbb{R}$ is a homotopy commutative C_2–ring spectrum equipped with a multiplicative orientation $MP\mathbb{R} \to E\mathbb{R}$. In particular, there is an invertible element $\bar{u} \in \pi_{1+\sigma} E\mathbb{R}$ coming from the generator of the $n = 1$ summand of $MP\mathbb{R} \simeq \bigoplus_{n \in \mathbb{Z}} \Sigma^{n(1+\sigma)} M\mathbb{R}$. Second, $\pi_0 E\mathbb{R} \cong \pi_0 E$, and in general

$$E\mathbb{R}_\star X \cong E_0 \otimes_{MP_0} MP\mathbb{R}_\star X$$

for any C_2–spectrum X. Third, $\pi_{-1} E\mathbb{R} = 0$. Implicit in these is the fact that $MP\mathbb{R}$ itself satisfies these conditions. This is nontrivial, and follows from work of Hu and Kriz [2001] on real cobordism. We recall the key calculation.

3.3 Lemma *The C_2–spectrum $E\mathbb{R}$ is Borel complete, with a_σ–Bockstein spectral sequence*

$$E_1 = E_0[\bar{u}^{\pm 1}, u_\sigma^{\pm 1}, a_\sigma] \Rightarrow \pi_\star E\mathbb{R},$$

where

$$|\bar{u}| = 1 + \sigma, \quad |u_\sigma| = 1 - \sigma, \quad |a_\sigma| = -\sigma.$$

The differentials are $E_0[\bar{u}^{\pm 1}, a_\sigma]$–linear, and are generated by

$$d_{2^{n+1}-1}(u_\sigma^{2^n}) = u_n \bar{u}^{2^n - 1} a_\sigma^{2^{n+1}-1},$$

where $u_n = u^{-(2^n - 1)} v_n \in E_0$. In particular, $\pi_0 E\mathbb{R} = \pi_0 E$.

Proof We first verify the given description of the a_σ–Bockstein spectral sequence. The E_1–page of the a_σ–Bockstein spectral sequence is given by $E_*[u_\sigma^{\pm 1}, a_\sigma]$. To put this in the desired form, we set $\bar{u} = u_\sigma^{-1} u$ with $u \in \pi_2 E$ the unit; when $E = MP$, this generates the $n = 1$ summand of $MP\mathbb{R} \simeq \bigoplus_{n \in \mathbb{Z}} \Sigma^{n(1+\sigma)} M\mathbb{R}$. As $E\mathbb{R}$ is Landweber exact over $MP\mathbb{R}$, the a_σ–Bockstein spectral sequence for $E\mathbb{R}$ is tensored down from the a_σ–Bockstein spectral sequence for $MP\mathbb{R}$, and here the computation is known by work of Hu and Kriz [2001].

The C_2–spectrum $M\mathbb{R}$ is shown to be Borel complete in [Hu and Kriz 2001, Theorem 4.1], and Landweber exactness extends the proof to $E\mathbb{R}$. By the Tate fracture square, $E\mathbb{R}$ is Borel complete if and only if the map $\Phi^{C_2} E\mathbb{R} \to E\mathbb{R}^{tC_2}$ is an equivalence, where Φ^{C_2} denotes the functor of geometric fixed points. Landweber exactness implies

$$\pi_* \Phi^{C_2} E\mathbb{R} \cong E_0 \otimes_{MP_0} \pi_* \Phi^{C_2} MP\mathbb{R} \cong E_0/(u_0, u_1, \ldots)[x^{\pm 1}],$$

where $x = a_\sigma \bar{u} \in \pi_1 E\mathbb{R}$, the last identification coming from the equivalence $\Phi^{C_2} M\mathbb{R} \simeq MO$. This is exactly what one obtains computing $\pi_* E\mathbb{R}^{tC_2}$ by the Tate spectral sequence, which may itself be obtained from the above description of the a_σ–Bockstein spectral sequence by inverting a_σ. Thus $E\mathbb{R}$ is Borel complete as claimed. $\qquad \square$

We now pass to the nonequivariant spectrum $ER = E\mathbb{R}^{C_2} \simeq E^{hC_2}$. Note that $\pi_* ER$ is the portion of $\pi_\star E\mathbb{R}$ located in integer degrees, and write $x = a_\sigma \bar{u} \in \pi_1 ER$. We then have the following analogue of [Kitchloo and Wilson 2007a] and [Kitchloo and Wilson 2008a, Theorem 4.2].

3.4 Proposition *There is a cofiber sequence*

$$\Sigma ER \xrightarrow{x} ER \to E,$$

and thus for any spectrum X an x–Bockstein spectral sequence

$$E_1 = (E^* X)[x] \Rightarrow ER^* X,$$

and this agrees with the HFPSS

$$E_2 = H^*(C_2; E^* X) \Rightarrow ER^* X$$

from the E_2–page on. Write ψ^{-1} for the involution on E^*X. Then the d_1–differential in the x–Bockstein spectral sequence is given by

$$d_1(\alpha) = (\alpha - \psi^{-1}(\alpha))u^{-1}x$$

for $\alpha \in E^*X$, and the differentials satisfy the Leibniz rule

$$d_r(\alpha\beta) = d_r(\alpha)\beta + \psi^{-1}(\alpha)d_r(\beta)$$

for $r \geq 1$, where the ψ^{-1} may be omitted for $r \geq 2$. All x–Bockstein differentials are E_0–linear, and when $X = S^0$ they are generated by

$$d_{2^{n+1}-1}(u^{2^n}) = u_n x^{2^{n+1}-1}.$$

Proof As $x = a_\sigma \bar{u}$ and $\bar{u}$ is invertible in $E\mathbb{R}$, (3) implies that there is a cofiber sequence

$$\Sigma E\mathbb{R} \xrightarrow{x} E\mathbb{R} \to C_{2+} \otimes E\mathbb{R}$$

of C_2–spectra. Passing to fixed points yields the corresponding cofiber sequence for ER. The remaining facts follow from the previous lemmas. $\square$

3.5 Remark Figure 1 depicts the E_∞ page of the x–Bockstein spectral sequence for ER.

The lines of slope 1 depict x–towers. Everything in filtration $\geq 2^n - 1$ is a module over $E_0/(u_0, \ldots, u_{n-1})$, and these regions are separated by dashed lines. The terms on the bottom describe the 0–line $(\pi_* ER)/(x)$. For example, $(\pi_{16} ER)/(x) \subset \pi_{16} E \cong E_0\{u^8\}$ is the E_0–submodule generated by $(2u^8, u_1u^8, u_2u^8)$; as $2u^8 \cdot x = 0$ and $u_1 u^8 \cdot x^3 = 0$, the x–tower out of this is supported on $u_1 u^8$ and $u_2 u^8$, and on just
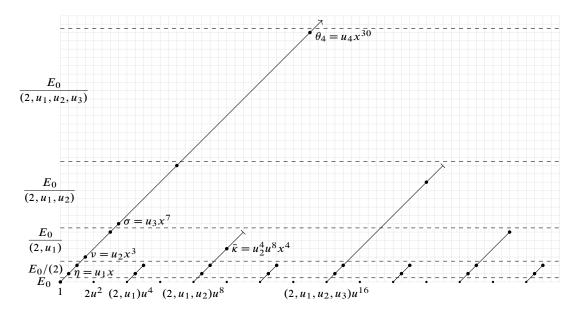

Figure 1

$u_2 u^8$ starting in filtration 3. The solid circles in positive filtration indicate degrees where $\pi_* ER$ has nontrivial Hurewicz image, with some notable elements labeled (see Section 8).

Similar charts appear in [Hahn and Shi 2020, Section 6].

4 Comparing $b(ER)$ and $E\mathbb{R}$

We are now in a position to consider Theorem 2.1. The first order of business is to identify $\xi = \rho \tau^{-2} u$ in $\pi_\sigma b(E)$ as a permanent cycle in the x–Bockstein spectral sequence for $\pi_\star b(ER)$. As

$$\rho \colon \pi_\sigma b(ER) \to \pi_0 b(ER) = ER^0 \mathbb{R}P^\infty$$

is the inclusion of a summand, the fact that ξ is a permanent cycle in the x–Bockstein spectral sequence for $\pi_\star b(ER)$ is predicted by the computation of $ER(n)^* \mathbb{R}P^\infty$ by Kitchloo and Wilson [2008a, Theorem 1.2]; see also [Kitchloo et al. 2017]. However, we take a different approach that sheds light on additional aspects of ξ.

Because the x–Bockstein spectral sequence for $\pi_\star b(ER)$ agrees with the HFPSS

$$E_2 = H^*(C_2; \pi_\star b(E)) \Rightarrow \pi_\star b(ER)$$

from the E_2–page on, we can just as well work with the HFPSS in this section.

4.1 Lemma *We have*

$$\pi_\star b(E) = \frac{E_0[\rho, \tau^{\pm 2}, u^{\pm 1}]_\rho^\wedge}{(\rho \cdot h)},$$

and C_2 acts on $\pi_\star b(E)$ by the E_0–linear multiplicative involution ψ^{-1} satisfying

$$\psi^{-1}(\rho) = \rho, \quad \psi^{-1}(u) = -u, \quad \psi^{-1}(\tau^2) = \tau^2(h-1).$$

In particular, ξ is fixed under the action of ψ^{-1}.

Proof The structure of $\pi_\star b(E)$ is as described in [Balderrama 2021, Section 2.1]. That ψ^{-1} fixes ξ follows immediately. □

4.2 Proposition *The class ξ is a permanent cycle in the HFPSS for $\pi_\star b(ER)$, detecting a lift of x.*

Proof By assumption, $E\mathbb{R}$ is $MP\mathbb{R}$–oriented. As $\xi = \rho \tau^{-2} u$ lifts to $\pi_\sigma b(MP)$ and x lifts to $\pi_1 MPR$, it suffices to prove the proposition with ER replaced by MPR.

As MP is an $\mathbb{E}_\infty$ ring, and complex conjugation acts on MP by $\mathbb{E}_\infty$ automorphisms, there is a C_2–equivariant external squaring operation

$$\mathrm{Sq} \colon \pi_n MP \to \pi_{n(1+\sigma)} b(MP).$$

As Sq is additive modulo transfers and ρ annihilates the transfer ideal, the composite $\rho \cdot \mathrm{Sq}$ is additive and so induces a map

$$Q \colon H^1(C_2; \pi_{n+1} MP) \to H^1(C_2; \pi_{n(1+\sigma)+1} b(MP))$$

in group cohomology. By [Balderrama 2024, Theorem 1.0.1], if $a \in H^1(C_2; \pi_{n+1} MP)$ is a permanent cycle detecting $\alpha \in \pi_n MPR$, then $Q(a)$ is a permanent cycle weakly detecting $\mathrm{Sq}(\alpha) \in \pi_{n(1+\sigma)} b(MPR)$. Now recall that x represents the generator of $H^1(C_2; \mathbb{Z}\{u\}) \cong \mathbb{Z}/(2) \subset H^1(C_2; \pi_2 MP)$. The $\mathbb{E}_\infty$ structure on periodic cobordism is such that $\mathrm{Sq}(u) = \tau^{-2} u^2$; see for instance the paragraph after [Ando et al. 2004, Lemma 4.3], noting that u and $\tau^{-2} u^2$ are the periodic Thom classes for $\mathbb{C}$ and $\mathbb{C}[C_2]$ respectively. Thus $Q(x) = \xi x$, and it follows that ξx detects $\mathrm{Sq}(x)$. As $\mathrm{Sq}(x)$ lifts x^2, this is only possible if ξ is a permanent cycle detecting a lift of x as claimed. $\square$

The following corollary is not needed for Theorem 2.1, but will be useful later on in understanding the structure of $\pi_\star b(ER)$. It is a direct analogue of [Kitchloo and Wilson 2008a, Theorem 1.2].

4.3 Corollary *In integer degrees, we have*

$$\pi_* b(ER) \cong ER_*[\![z]\!]/([2](z)),$$

where $z = \rho \xi$. In particular, $\pi_\star b(ER)$ is a module over $\pi_0 b(ER) \cong E^0 BC_2 \cong E_0[\![z]\!]/([2](z))$.

Proof The x–Bockstein spectral sequence for $\pi_* b(ER)$ takes the form

$$E_1 = \pi_* b(E)[x] \Rightarrow \pi_* b(ER).$$

Recall that

$$\pi_* b(E) \cong E_*[\![z]\!]/([2](z)), \quad z = \rho \xi.$$

As ρ and ξ are permanent cycles, so is z. Thus the differentials in the x–Bockstein spectral sequence for $\pi_* b(ER)$ are induced by those for $\pi_* ER$, leading to the given description of $\pi_* b(ER)$. $\square$

We now relate $b(ER)$ and $E\mathbb{R}$. These live in the full subcategory $\mathrm{Sp}^{BC_2} \subset \mathrm{Sp}^{C_2}$ of Borel complete C_2–spectra, equivalent to the category of spectra with C_2–action. The functor

$$b: \mathrm{Sp} \to \mathrm{Sp}^{BC_2}, \quad b(X) = F(EC_{2+}, i_* X)$$

is the diagonal, endowing a spectrum with the trivial C_2–action. In particular, it is left adjoint to the functor of homotopy fixed points, and if $X \in \mathrm{Sp}^{BC_2}$ then the counit of this adjunction gives a canonical map

$$b(X^{\mathrm{h}C_2}) \to X.$$

Specializing to $X = E\mathbb{R}$, we have the following.

4.4 Theorem *The canonical map $b(ER) \to E\mathbb{R}$ fits into a cofiber sequence*

(4) $$\Sigma^\sigma b(ER) \xrightarrow{\xi} b(ER) \to E\mathbb{R}$$

of C_2–spectra.

Proof As $E\mathbb{R}$ is strongly even, we have $\pi_\sigma E\mathbb{R} = 0$. As the maps in (4) are $b(ER)$–linear, their composite must be null. As $b(ER)$ and $E\mathbb{R}$ are x–complete, it then suffices to show that (4) is a cofiber

sequence after coning off x. As $b(ER)/(x) \simeq b(E)$ and $E\mathbb{R}/(x) \simeq C_{2+} \otimes i_* E$, (4) with x coned off takes the form

$$\Sigma^\sigma b(E) \xrightarrow{\xi} b(E) \to C_{2+} \otimes i_* E,$$

which is a cofiber sequence as now $\xi = \rho \cdot \tau^{-2} u$ differs from ρ by a unit. □

Theorem 2.1 follows by combining Proposition 4.2 and Theorem 4.4.

5 The Bockstein spectral sequence

We now compute the x–Bockstein spectral sequence

$$\pi_\star b(E)[x] \Rightarrow \pi_\star b(ER).$$

We maintain notation from the introduction. In particular, recall that h_n and w_n are defined in terms of the 2–series of E by specializing

$$[2](z) = zh_0(z), \quad h_n(z) \equiv u_n + z^{2^n} h_{n+1}(z) \pmod{u_0, \ldots, u_{n-1}},$$

$$w_n = \rho^{2^{n+1}} h_{n+1} \equiv \tau^{2^{n+1}} u^{-2^n} (h_n - u_n) \pmod{u_0, \ldots, u_{n-1}}$$

to $z = \rho\xi = \rho^2 \tau^{-2} u$. As with u_n, these classes are well defined modulo $(u_0, \ldots, u_{n-1})$. We begin by describing what will be the cycles and boundaries of the x–Bockstein spectral sequence for $\pi_\star b(ER)$. Let $Z_{2^{n+1}-1} \subset \pi_\star b(E)$ be the subring

$$E_0\big(\rho, \xi, u^{\pm 2^{n+1}}, \tau^{2^{\pm n+2}}, \tau^{2^{i+2}l} u^{2^{i+1}k} u_i, \tau^{2^{i+1}(2l+1)} u^{2^{i+1}k} h_i : 0 \le i \le n; k, l \in \mathbb{Z}\big)^{\wedge}_{(\rho, \xi)},$$

and let $B_{2^{n+1}-1} \subset Z_{2^{n+1}-1}[x]$ be the ideal generated by

$$\tau^{2^{i+2}l} u^{2^{i+1}k} u_i \cdot x^{2^{i+1}-1}, \quad \tau^{2^{i+1}(2l+1)} u^{2^{i+1}k} h_i \cdot x^{2^{i+1}-1}, \quad \tau^{2^{i+2}l} u^{2^{i+1}k} w_i \cdot x^{2^{i+1}-1}$$

for $0 \le i \le n$ and $k, l \in \mathbb{Z}$. We also declare $Z_0 = \pi_\star b(E)$ and $B_0 = (0)$, and for $2^{n+1} - 1 \le r < 2^{n+2} - 1$ we write $Z_r = Z_{2^{n+1}-1}$ and $B_r = Z_{2^{n+1}-1}$. Thus there are inclusions

$$0 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset Z_2[x] \subset Z_1[x] \subset Z_0[x] = \pi_\star b(E)[x].$$

5.1 Theorem *The x–Bockstein spectral sequence for $\pi_\star b(ER)$ supports differentials*

$$d_{2^{n+1}-1}(u^{2^n}) = u_n x^{2^{n+1}-1}, \quad d_{2^{n+1}-1}(\tau^{2^{n+1}}) = -w_n x^{2^{n+1}-1},$$

and we may identify $Z_r[x]$ and B_r as its r–cycles and r–boundaries.

Proof We proceed by induction, treating the inductive step first.

Let $n \ge 1$, and suppose we have verified $E_{2^n} \cong Z_{2^n-1}[x]/B_{2^n-1}$. In particular, E_{2^n} is generated by the permanent cycles ρ and ξ, the classes $\tau^{2^{i+2}l} u^{2^{i+1}k} u_i$ and $\tau^{2^{i+1}(2l+1)} u^{2^{i+1}k} h_i$ for $i < n$, and the classes $u^{\pm 2^n}$ and $\tau^{\pm 2^{n+1}}$. As the classes $\tau^{2^{i+2}l} u^{2^{i+1}k} u_i$ and $\tau^{2^{i+1}(2l+1)} u^{2^{i+1}k} h_i$ are x^{2^n-1}–torsion for $i < n$,

having survived to the E_{2^n}–page they must be permanent cycles. It follows that the next differentials are determined by their effect on u^{2^n} and $\tau^{2^{n+1}}$.

The differential $d_{2^{n+1}-1}(u^{2^n}) = u_n x^{2^{n+1}-1}$ follows from Proposition 3.4. Now write

$$d_{2^{n+1}-1}(\tau^{2^{n+1}}) = \alpha \cdot x^{2^{n+1}-1}.$$

As ξ is a permanent cycle, the Leibniz rule implies

$$0 = d_{2^{n+1}-1}(\xi^{2^n}) = d_{2^{n+1}-1}(\rho^{2^n} \tau^{-2^{n+1}} u^{2^n}) = \rho^{2^n}(\tau^{-2^{n+2}} \alpha u^{2^n} + \tau^{-2^{n+1}} u_n) x^{2^{n+1}-1}.$$

This is on the $E_{2^{n+1}-1} = E_{2^n}$–page, and so combines with our inductive hypothesis to imply

$$z^{2^{n-1}}(u_n + \tau^{-2^{n+1}} u^{2^n} \alpha) \equiv 0 \pmod{u_0, \dots, u_{n-1}, z^{2^{n-1}} h_n}.$$

As $\alpha \equiv 0 \pmod{\rho}$, this is only possible if

$$u_n + \tau^{-2^{n+1}} u^{2^n} \alpha \equiv h_n \pmod{u_0, \dots, u_{n-1}},$$

and thus

$$d_{2^{n+1}-1}(\tau^{2^{n+1}}) = \alpha x^{2^{n+1}-1} = \tau^{2^{n+1}} u^{-2^n}(h_n - u_n) x^{2^{n+1}-1} = w_n x^{2^{n+1}-1}$$

as claimed.

To identify boundaries and cycles, observe that as a general property of the x–Bockstein spectral sequence, if we write $Z'_r[x]$ for its r–cycles then

$$Z'_r = \mathrm{Ker}(d_r \colon Z_{r-1} \to E_r) = \mathrm{Ker}(d_r \colon Z_{r-1} \to E_r[x^{-1}]),$$

ie to compute cycles it suffices to work in the x–inverted x–Bockstein spectral sequence, or equivalently the x–Bockstein spectral sequence with x set to 1. Our inductive hypothesis implies

$$\frac{E_{2^n}}{(x-1)} \cong \frac{(E_0/(u_0, \dots, u_{n-1}))[\rho, \xi, u^{\pm 2^n}, \tau^{\pm 2^{n+1}}]^{\wedge}_{(\rho,\xi)}}{(\xi^{2^n} - \rho^{2^n} \tau^{-2^{n+1}} u^{2^n}, \rho^{2^n}(u_n \tau^{2^{n+1}} + \rho^{2^{n+1}} h_{n+1} u^{2^n}))},$$

and we have just produced the differentials

$$d(u^{2^n}) = u_n, \quad d(\tau^{2^{n+1}}) = \rho^{2^{n+1}} h_{n+1}.$$

Thus $\mathrm{Ker}(d)$ is generated over $E_0(\rho, \xi, u^{\pm 2^{n+1}}, \tau^{\pm 2^{n+2}})^{\wedge}_{(\rho,\xi)}$ by $u_n \tau^{2^{n+1}} + \rho^{2^{n+1}} h_{n+1} u^{2^n} = \tau^{2^{n+1}} h_n$, and this leads to $Z'_{2^{n+1}-1} = Z_{2^{n+1}-1}$ as claimed. The identification of boundaries follows immediately.

The base case, concerning the d_1–differential and identification of the E_2–page, can be handled by considering $0 = d_1(\xi)$ just like the above, only taking into account the twist in the Leibniz rule for d_1 given in Proposition 3.4. Alternately, one may just use the formula $d_1(a) = (a - \psi^{-1}(a))x$ given there, where the action of ψ^{-1} is given in Lemma 4.1.   □

The ring Z and ideal $B \subset Z[x]$ of the introduction may be identified as $Z = \bigcap_r Z_r$ and $B = \bigcup_r B_r$. Thus Theorem 2.3 follows from Theorem 5.1 by letting $r \to \infty$.

5.2 Remark   Although we have relied on the known computation of $\pi_* MPR$ in computing $\pi_\star b(MPR)$, this was not actually necessary: the proof of Theorem 5.1 gives an independent computation, as we now explain.

Note that no computation was needed to produce $x \in \pi_1 MP\mathbb{R}$ or prove $MP\mathbb{R}/(x) \simeq C_{2+} \otimes i_* MP$, as $x = a_\sigma \bar{u}$ where $\bar{u}$ generates the $n = 1$ summand of $MP\mathbb{R} \simeq \bigoplus_{n \in \mathbb{Z}} \Sigma^{n(1+\sigma)} M\mathbb{R}$. Thus it suffices to describe the x–Bockstein spectral sequence $MP_0[u^{\pm 1}, x] \Rightarrow MPR_*$. This is MP_0–linear by [Hu and Kriz 2001, Proposition 2.27], which uses the theory of real orientations but not the computation of $\pi_\star M\mathbb{R}$. Thus it suffices to produce the differentials $d_{2^{n+1}-1}(u^{2^n}) = u_n x^{2^{n+1}-1}$. The differential $d_1(u) = 2x$ follows from the involution $\psi^{-1}(u) = -u$, so suppose inductively that we have computed $d_{2^{n+1}-1}(u^{2^n}) = u_n x^{2^{n+1}-1}$.

Next note that no computation was needed in Proposition 4.2 to prove ξ is a permanent cycle. The argument in Theorem 5.1 now applies to show $d_{2^{n+1}-1}(\tau^{2^{n+1}}) = -\rho^{2^{n+1}} h_{n+1} x^{2^{n+1}-1}$.

As in Section 4, there is a canonical map $q \colon b(MP^{hC_2}) \to F(EC_{2+}, MP\mathbb{R})$. Here, we write MP^{hC_2} and $F(EC_{2+}, MP\mathbb{R})$ instead of MPR and $MP\mathbb{R}$ as the proof that $MP\mathbb{R}$ is Borel complete relies on knowledge of its x–Bockstein spectral sequence. The map q fits into a diagram of cofiber sequences:

$$
\begin{array}{ccccccc}
\Sigma b(MP^{hC_2}) & \xrightarrow{\ x\ } & b(MP^{hC_2}) & \longrightarrow & b(MP) & \xrightarrow{\ \partial\ } & \Sigma^2 b(MP^{hC_2}) \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\Sigma F(EC_{2+}, MP\mathbb{R}) & \xrightarrow{\ x\ } & F(EC_{2+}, MP\mathbb{R}) & \longrightarrow & C_{2+} \otimes i_* MP & \xrightarrow{\ \partial'\ } & \Sigma^2 F(EC_{2+}, MP\mathbb{R})
\end{array}
$$

The x–Bockstein differential $d_{2^{n+1}-1}(\tau^{2^{n+1}}) = -\rho^{2^{n+1}} h_{n+1} x^{2^{n+1}-1}$ implies

$$\partial(\tau^{2^{n+1}}) = -\rho^{2^{n+1}} h_{n+1} x^{2^{n+1}-2}$$

mod higher filtration, and as $p(\tau^2) = u_\sigma^2$ and $q(\rho) = a_\sigma$ it follows that

$$
\begin{aligned}
\partial'(u^{2^{n+1}}) = \partial'(\bar{u}^{2^{n+1}} u_\sigma^{2^{n+1}}) &= \bar{u}^{2^{n+1}} q(\partial(\tau^{2^{n+1}})) \\
&= \bar{u}^{2^{n+1}} q(-\rho^{2^{n+1}} h_{n+1} x^{2^{n+1}-2}) \\
&= \bar{u}^{2^{n+1}} a_\sigma^{2^{n+1}} u_{n+1} x^{2^{n+1}-2} = u_{n+1} x^{2^{n+2}-2}
\end{aligned}
$$

mod higher filtration. This gives the next x–Bockstein differential

$$d_{2^{n+2}-1}(u^{2^{n+1}}) = u_{n+1} x^{2^{n+2}-1},$$

completing the induction.

We end this subsection with some observations about the structure of $\pi_\star b(ER)$.

5.3 Proposition   *The C_2–spectrum $b(ER)$ has the gap*

$$\pi_{*\sigma - 1} b(ER) = 0.$$

u_0,u_1 $\rho^8 x^8, h_2$	$\xi u_1, \xi h_2$ $\rho^7 x^8$	$\xi^2 u_1, \xi^2 h_2$ $\rho^6 x^8$	$\xi^3 u_1, \xi^3 h_2$ $\rho^5 x^8$	$u_0, \xi^4 h_2$ $\rho^4 x^8$	$\rho^3 u_1, \xi^5 h_2$ $\rho^3 x^8$	$\rho^2 u_1, \xi^6 h_2$ $\rho^2 x^8$	$\rho u_1, \xi^7 h_2$ ρx^8	u_0,u_1 x^8
$\rho^7 x^7$	$\rho^6 x^7$	$\rho^5 x^7$	$\rho^4 x^7$	$\rho^3 x^7$	$\rho^2 x^7$	ρx^7	x^7	ξx^7
h $\rho^6 x^6$	$\rho^5 x^6$	$h_1 x^2$ $\rho^4 x^6$	$\xi h_1 x^2$ $\rho^3 x^6$	h $\rho^2 x^6$	ρx^6	x^6	ξx^6	h $\xi^2 x^6$
$\rho^5 x^5$	$h_1 x$ $\rho^4 x^5$	$\xi h_1 x$ $\rho^3 x^5$	$\rho^2 x^5$	ρx^5	x^5	ξx^5	$\xi^2 x^5$	$\rho h_1 x$ $\xi^3 x^5$
u_0, h_1 $\rho^4 x^4$	ξh_1 $\rho^3 x^4$	$\xi^2 h_1$ $\rho^2 x^4$	$\xi^3 h_1$ ρx^4	u_0 x^4	$\rho^3 h_1$ ξx^4	$\rho^2 h_1$ $\xi^2 x^4$	ρh_1 $\xi^3 x^4$	u_0, h_1 $\xi^4 x^4$
$\rho^3 x^3$	$\rho^2 x^3$	ρx^3	x^3	ξx^3	$\xi^2 x^3$	$\xi^3 x^3$	$\xi^4 x^3$	$\xi^5 x^3$
h $\rho^2 x^2$	ρx^2	x^2	ξx^2	h $\xi^2 x^2$	$\xi^3 x^2$	$\xi^4 x^2$	$\xi^5 x^2$	h $\xi^6 x^2$
ρx	x	ξx	$\xi^2 x$	$\xi^3 x$	$\xi^4 x$	$\xi^5 x$	$\xi^6 x$	$\rho u_1 x$ $\xi^7 x$
1	ξ	ξ^2	ξ^3	u_0 ξ^4	$\rho^3 u_1$ ξ^5	$\rho^2 u_1$ ξ^6	ρu_1 ξ^7	u_0,u_1 ξ^8

Table 1

Proof Declare the *coweight* of a degree $c + w\sigma$ to be the quantity c, so that we are claiming $\pi_\star b(ER)$ vanishes in coweight -1. By Theorem 5.1, $\pi_\star b(ER)$ is generated over the coweight 0 classes $E_0(\rho, \xi)^\wedge_{(\rho,\xi)}$ by the class x in coweight 1 and the classes

$$\tau^{2^{n+2}l} u^{2^{n+1}k} u_n, \quad \tau^{2^{n+1}(2l+1)} u^{2^{n+1}k} h_n$$

in coweights of the form $2^{n+1}t$. These classes are killed by $x^{2^{n+1}-1}$, and therefore cannot support long enough x–towers to reach coweight -1. $\qquad\square$

5.4 Proposition *If E is L_d–local, then $b(ER)$ is $u^{\pm 2^{d+1}}$ and $\tau^{\pm 2^{d+1}}$–periodic. Moreover,*

$$x^{2^{d+1}-1} = 0, \quad \rho^{2^d} x^{2^d-1} = 0, \quad \xi^{2^d} x^{2^d-1} = 0$$

in $Z[x]/B$.

Proof Recall that E is L_d–local provided u_d is invertible in $E_0/(u_0, \ldots, u_{d-1})$, or equivalently if the ideal $(u_0, \ldots, u_d) \subset E_0$ generates the entire ring. Thus as $u_i u^{\pm 2^{d+1}}$ is a permanent cycle for $i \leq d$, it follows that $u^{\pm 2^{d+1}}$ is also a permanent cycle. Likewise, as $u_i x^{2^{d+1}-1} = 0$ for $i \leq d$, it follows that $x^{2^{d+1}-1} = 0$.

$\xi u_1, \rho^7 h_2$ ξx^8	$\xi^2 u_1, \rho^6 h_2$ $\xi^2 x^7$	$\xi^3 u_1, \rho^5 h_2$ $\xi^3 x^7$	$u_0, \rho^4 h_2$ $\xi^4 x^8$	$\rho^3 u_1, \rho^3 h_2$ $\xi^5 x^7$	$\rho^2 u_1, \rho^2 h_2$ $\xi^6 x^8$	$\rho u_1, \rho h_2$ $\xi^7 x^8$	u_0, u_1 $\xi^8 x^8, h_2$
$\xi^2 x^7$	$\xi^3 x^7$	$\xi^4 x^7$	$\xi^5 x^7$	$\xi^6 x^7$	$\xi^7 x^7$	$\xi^8 x^7$	$\xi^9 x^7$
$\rho h_1 x^2$ $\xi^3 x^6$	$x^2 h_1$ $\xi^4 x^6$	$x^2 \xi h_1$ $\xi^5 x^6$	h $\xi^6 x^6$	$\xi^7 x^6$	$\xi^8 x^6$	$\xi^9 x^6$	h $\xi^{10} x^6$
$h_1 x$ $\xi^4 x^5$	$x \xi h_1$ $\xi^5 x^5$	$\xi^6 x^5$	$\xi^7 x^5$	$\xi^8 x^5$	$\xi^9 x^5$	$\xi^{10} x^5$	$x \rho h_1$ $\xi^{11} x^5$
ξh_1 $\xi^5 x^4$	$\xi^2 h_1$ $\xi^6 x^4$	$\xi^3 h_1$ $\xi^7 x^4$	u_0 $\xi^8 x^4$	$\rho^3 h_1$ $\xi^9 x^4$	$\rho^2 h_1$ $\xi^{10} x^4$	ρh_1 $\xi^{11} x^4$	u_0, h_1 $\xi^{12} x^4$
$\xi^6 x^3$	$\xi^7 x^3$	$\xi^8 x^3$	$\xi^9 x^3$	$\xi^{10} x^3$	$\xi^{11} x^3$	$\xi^{12} x^3$	$x^3 \rho^3 u_2$ $\xi^{13} x^3 k$
$\rho u_1 x^2$ $\xi^7 x^2$	$x^2 u_1$ $\xi^8 x^2$	$x^2 \xi u_1$ $\xi^9 x^2$	$x^2 \xi^2 u_1 / x^2 \rho^6 u_2$ $h, \xi^{10} x^2$	$x^2 \rho^5 u_2$ $\xi^{11} x^2$	$x^2 \rho^4 u_2$ $\xi^{12} x^2$	$x^2 \rho^3 u_2$ $\xi^{13} x^2$	$h, x^2 \rho^2 u_2$ $\xi^{14} x^2$
$u_1 x$ $\xi^8 x$	$x \xi u_1$ $\xi^9 x$	$x \xi^2 u_1 / x \rho^6 u_2$ $\xi^{10} x$	$x \rho^5 u_2$ $\xi^{11} x$	$x \rho^4 u_2$ $\xi^{12} x$	$x \rho^3 u_2$ $\xi^{13} x$	$x \rho^2 u_2$ $\xi^{14} x$	$x \rho u_1, x \rho u_2$ $\xi^{15} x$
$\xi u_1, \rho^7 u_2$ ξ^9	$\xi^2 u_1, \rho^6 u_2$ ξ^{10}	$\xi^3 u_1, \rho^5 u_2$ ξ^{11}	$u_0, \rho^4 u_2$ ξ^{12}	$\rho^3 u_1, \rho^3 u_2$ ξ^{13}	$\rho^2 u_1, \rho^2 u_2$ ξ^{14}	$\rho u_1, \rho u_2$ ξ^{15}	u_0, u_1 ξ^{16}, u_2

Table 2

Next, as $h_d(z) = u_d + O(z)$, it follows by Weierstrass preparation that

$$(u_0, \ldots, u_{d-1}, h_d) \subset \pi_0 b(ER) \cong E^0 BC_2$$

(see Corollary 4.3) generates the entire ring. As $u_i \tau^{\pm 2^{d+1}}$ for $i < d$ and $h_d \tau^{\pm 2^{d+1}}$ are permanent cycles, it follows that $\tau^{\pm 2^{d+1}}$ is a permanent cycle. Next, note that

$$w_{d-1} = \rho^{2^d} h_d, \quad \tau^{2^{d+1}} u^{2^d} w_{d-1} = \xi^{2^d} h_d.$$

As $u_i x^{2^d - 1} = 0$ for $i < d$, the identities $w_{d-1} x^{2^d - 1} = 0$ and $\tau^{2^{d+1}} u^{2^d} w_{d-1} x^{2^d - 1} = 0$ then imply $\rho^{2^d} x^{2^d - 1} = 0$ and $\xi^{2^d} x^{2^d - 1} = 0$. □

5.5 Remark Tables 1 and 2 may be helpful in getting acquainted with the general shape of $\pi_\star b(ER)$, and especially for visualizing the arguments in Sections 6 and 7.

These describe generators of $\pi_{c+w\sigma} b(ER)$ as a module over $\pi_0 b(ER) \cong E^0 BC_2$ in coweight $0 \le c \le 8$ and stem $0 \le c + w \le 16$, the first table containing stems $0 \le c + w \le 8$ and second $9 \le c + w \le 16$. It is arranged by stem and coweight: the box at coordinate (s, c) contains a list of generators for $\pi_{c+(s-c)\sigma} b(ER)$. For space reasons, we have omitted any $\tau^{2i} u^j$ terms. These may recovered by

comparing degrees: for example, the box in coordinate $(8,5)$ has entries $\rho h_1 x$ and $\xi^3 x^5$, and this means that $\pi_{5+3\sigma} b(ER)$ is generated over $\pi_0 b(ER)$ by $\rho \cdot \tau^{-4} u^4 h_1 \cdot x$ and $\xi^3 x^5$.

The entry $x\xi^2 u_1 / x\rho^6 u_2$ indicates that either $x\xi^2 u_1$ or $x\rho^6 u_2$ may be chosen as a generator, and likewise for $x^2 \xi^2 u_1 / x^2 \rho^6 u_2$. This sort of choice also appears on the 0–line: for example, in box $(5,0)$ one could replace $\rho^3 u_1$ with ξu_0.

These tables assume that E has sufficiently large height, say $E = MP$.

6 Transfers

Recall that there are cofiber sequences

$$(5) \qquad \Sigma^{-\sigma} b(ER) \xrightarrow{\rho} b(ER) \to C_{2+} \otimes i_* ER, \quad \Sigma^\sigma b(ER) \xrightarrow{\xi} b(ER) \to E\mathbb{R}$$

of C_2–spectra. The first is a general cofiber sequence that exists for any C_2–spectrum, given that $C_{2+} \otimes b(ER) \simeq C_{2+} \otimes i_* ER$, and the second was shown in Theorem 4.4. Here,

$$\pi_\star (C_{2+} \otimes i_* ER) \cong \pi_* ER[u_\sigma^{\pm 1}], \quad |u_\sigma| = 1 - \sigma,$$

and $\pi_\star E\mathbb{R}$ was described in Section 3.

Associated to the cofiber sequences of (5) are boundary maps

$$\mathrm{tr}(u_\sigma^{-1} \cdot -) \colon \pi_{\star+1-\sigma}(C_{2+} \otimes i_* ER) \to \pi_\star b(ER), \quad \partial \colon \pi_{\star+1+\sigma} E\mathbb{R} \to \pi_\star b(ER).$$

The first of these is the transfer for the C_2–spectrum $b(ER)$. Both are $\pi_\star b(ER)$–linear.

6.1 Proposition *The above transfer and boundary maps satisfy*

$$\mathrm{tr}(u_\sigma^{-1} \cdot u_\sigma^{2^n(2k+1)}) = \rho^{2^n-1} \tau^{2^{n+1}k} h_n x^{2^n-1} + O(x^{2^n}),$$
$$\partial(\bar{u}^{2^n(2k+1)}) = \xi^{2^n-1} \tau^{-2^{n+1}k} u^{2^{n+1}k} h_n x^{2^n-1} + O(x^{2^n})$$

for $n \geq 0$ and $k \in \mathbb{Z}$.

Proof The error terms are present just because $\tau^{2^{n+1}k} h_n$ and $\tau^{-2^{n+1}k} u^{2^{n+1}k} h_n$ have only been defined mod x, so we omit them in the proof.

First consider the case $n = 0$. These claimed values are not hidden in the x–Bockstein spectral sequence, so it suffices to show that they hold after coning off x. After coning off x, the cofiber sequences (5) take the form

$$(6) \qquad \Sigma^{-\sigma} b(E) \xrightarrow{\rho} b(E) \to C_{2+} \otimes i_* E, \quad \Sigma^\sigma b(E) \xrightarrow{\rho\tau^{-2}u} b(E) \to C_{2+} \otimes i_* E.$$

In particular, $\partial(\bar{u}\alpha) = \mathrm{tr}(\alpha)$. By [Hopkins et al. 2000, Remark 6.15], the transfer

$$\mathrm{tr} \colon \pi_0 E \to E^0 BC_2 \cong \pi_0 b(E)$$

satisfies $\mathrm{tr}(1) = h$. The proof is to observe that h is the unique class which satisfies

$$\rho \cdot h = 0, \quad h \equiv 2 \pmod{\rho}.$$

As tr and ∂ are $\pi_\star b(E)$–linear, we deduce

$$\mathrm{tr}(u_\sigma^{-1} \cdot u_\sigma^{2n+1}) = \tau^{2n} \mathrm{tr}(1) = \tau^{2n} h, \quad \partial(\bar{u}^{2n+1}) = \tau^{-2n} u^{2n} \partial(\bar{u}) = \tau^{-2n} u^{2n} h$$

as claimed. The argument is essentially the same for $n \geq 1$. Observe that

$$\rho^{2^n} \tau^{2^{n+1} k} h_n = \tau^{2^{n+1} k} w_{n-1}, \quad \xi^{2^n} \tau^{-2^{n+1} k} h_n = \tau^{-2^{n+1}(k+1)} u^{2^n} w_{n-1}.$$

In particular $\rho^{2^n - 1} \tau^{2^{n+1} k} h_n x^{2^n - 1}$ and $\xi^{2^n - 1} \tau^{-2^{n+1} k} u^{2^{n+1} k} h_n x^{2^n - 1}$ generate the kernels of ρ and ξ in their respective degrees. As the kernels of ρ and ξ are generated by the images of tr and ∂, this gives the claimed values of tr and ∂ up to multiplication by a unit, which may then be ruled out by working in the universal case $E = MP$. $\qquad\square$

7 Hidden extensions

We now turn our attention to hidden extensions. We begin with a general discussion. Write $Z[x]/B$ for the x–adic associated graded of $\pi_\star b(ER)$, as computed in Section 5. In general, hidden extensions in the x–Bockstein spectral sequence arise from the failure of $\pi_\star b(ER)$ to be isomorphic to $Z[x]/B$, and especially for relations to fail to lift through the map

$$(7) \qquad\qquad \pi_\star b(ER) \to (\pi_\star b(ER))/(x) \cong Z \subset \pi_\star b(E).$$

Recall that

$$\pi_\star b(E) = \frac{E_0[\rho, \tau^{\pm 2}, u^{\pm 1}]^\wedge_\rho}{(\rho \cdot h)}.$$

This indicates that the simple indecomposable hidden extensions will be those ρ and ξ–extensions lifting relations of the form

$$(8) \qquad\qquad \rho \cdot \tau^{2i} u^j h = 0, \quad \xi \cdot \tau^{2i} u^j h = 0,$$

where i and j are such that $\tau^{2i} u^j h \in Z$.

If a relation of this sort lifts to $\pi_\star b(ER)$, then necessarily the corresponding $\tau^{2i} u^j h$ is in the image of the transfer or boundary studied in the previous section. These classes are generally not in the image of the transfer or boundary, and so one knows from the start that the relations in (8) generally lift to nontrivial hidden extensions in $\pi_\star b(ER)$.

One can use Proposition 6.1 to compute some of these directly:

$$\xi \cdot \tau^{2^{n+1}(2k+1)} h = \xi \cdot \mathrm{tr}(u_\sigma^{2^{n+1}(2k+1)}) = \mathrm{tr}(x u_\sigma^{-1} \cdot u_\sigma^{2^{n+1}(2k+1)}) = \rho^{2^{n+1} - 1} \tau^{2^{n+1} k} h_{n+1} x^{2^{n+1}}$$

by Frobenius reciprocity, and likewise

$$\rho \cdot \tau^{-2^{n+1}(2k+1)} u^{2^{n+1}(2k+1)} h = \rho \cdot \partial(\bar{u}^{2^{n+1}(2k+1)+1}) = \xi^{2^{n+1} - 1} \tau^{2^{n+2} k} u^{2^{n+2} k} h_{n+1} x^{2^{n+1}}.$$

In general however, a more indirect approach is necessary. Consider the cofiber sequences

$$(9) \qquad \Sigma^{-\sigma} b(ER) \xrightarrow{\rho} b(ER) \xrightarrow{p} C_{2+} \otimes i_* ER, \quad \Sigma^{\sigma} b(ER) \xrightarrow{\xi} b(ER) \xrightarrow{q} E\mathbb{R}.$$

The long exact sequences associated to these imply that the image of ρ is equal to the kernel of the forgetful map $p \colon \pi_\star b(ER) \to \pi_{|\star|} ER$, and that the image of ξ is equal to the kernel of the canonical map $q \colon \pi_\star b(ER) \to \pi_\star E\mathbb{R}$. To find elements of these kernels, one looks for elements in $\pi_\star b(ER)$ that lift the relations $u_n x^{2^{n+1}-1} = 0$. This relation already holds in $\pi_\star b(ER)$, so we need only consider lifts involving the filtration-shifting identities $p(\xi) = x$ and $q(\rho) = a_\sigma$. In this way we focus our attention on those classes of the form

$$(10) \qquad \tau^{2^{n+2}k} u^{2^{n+2}l} h_{n+1} \xi^r x^s, \quad \tau^{2^{n+2}k} u^{2^{n+2}l} h_{n+1} \rho^r x^s,$$

where $r + s \geq 2^{n+1} - 1$ and $r \geq 1$ and $s < 2^{n+2} - 1$. By the preceding discussion, the former must be in the image of ρ and the latter in the image of ξ, and when this is not the case in $Z[x]/B$ there must be a hidden extension making it so. If $r + s > 2^{n+2} - 1$, then the witness to the classes in (10) being in the image of ρ or ξ may be obtained by multiplying a smaller witness with some suitable power of ρ or ξ and x. Thus we are led to focus on the case where $r + s = 2^{n+2} - 1$. We will show that when s is even, the necessary hidden extensions are exactly those lifting the relations in (8). First, a couple observations.

7.1 Lemma *Fix positive integers $r + s = 2^{n+2} - 1$ with s even. Then the classes*

$$\tau^{2^{n+2}k} u^{2^{n+2}l} h_{n+1} \xi^r x^s, \quad \tau^{2^{n+2}k} u^{2^{n+2}l} h_{n+1} \rho^r x^s$$

are not in the image of ρ or ξ respectively in $Z[x]/B$, at least when $E = MP$.

Proof Consider the first case. Suppose towards contradiction that

$$\tau^{2^{n+2}k} u^{2^{n+2}l} h_{n+1} \xi^r x^s = \rho \alpha x^s$$

for some $\alpha \in Z$. As the x–Bockstein spectral sequence has only odd differentials and s is even, necessarily we can divide out by x to obtain

$$(11) \qquad (\tau^{2^{n+2}k} u^{2^{n+2}l} h_{n+1} \xi^r - \rho \alpha) x^{s-1} = 0.$$

This means that $\tau^{2^{n+2}k} u^{2^{n+2}l} h_{n+1} \xi^r - \rho \alpha$ detects some class $\theta \in \pi_\star b(MPR)$ satisfying $\theta \cdot x^{s-1} = 0$. Write $p \colon \pi_\star b(MPR) \to \pi_{|\star|} MPR$ for the restriction. As $p(\xi) = x$, necessarily $p(\theta)$ is detected by $u^{2^{n+2}l} u_{n+1} x^r$. Thus

$$0 = p(\theta \cdot x^{s-1}) \equiv u^{2^{n+2}l} u_{n+1} x^{r+s-1} \pmod{x^{r+s}}$$

in $\pi_* MPR$. As $r + s - 1 < 2^{n+2} - 1$, this is incompatible with the structure of the x–Bockstein spectral sequence for $\pi_* MPR$, a contradiction. The second case is identical, only instead using the map $b(MPR) \to MP\mathbb{R}$ in place of the restriction. $\qquad \square$

7.2 Lemma *Suppose that i and j are such that $\tau^{2i} u^j h \in Z$. Then $\tau^{2i} u^j h$ generates the kernels of ρ and ξ in its degree of $Z[x]/B$ as a module over $\pi_0 b(ER)$.*

Proof The class $\tau^{2i}u^j h$ generates the kernels of ρ and ξ in Z, as this is the case in $\pi_\star b(E)$. Thus the lemma follows from the following observation: $Z[x]/B$ contains no x–divisible elements in the kernel of ρ or ξ in even degrees, that is in degrees of the form $c + w\sigma$ with both c and w even. Indeed, any x–divisible element in even degree and in a given filtration must be of the form $\alpha x^{2r} = 0$ with $\alpha \in Z$ in even degree. As α is in even degree and B is generated by classes of the form $w \cdot x^?$ with w in even degree, relations $\rho\alpha x^{2r} = 0$ or $\xi\alpha x^{2r} = 0$ are only possible if $\alpha x^{2r} = 0$ already, proving the lemma. $\square$

We may now give the main theorem of this subsection.

7.3 Theorem *There are extensions*

$$\rho \cdot \tau^{2(2^{n+1}k-r)}u^{2^{n+1}(2l+1)}h = \left(\tau^{2^{n+2}k}u^{2^{n+2}l}h_{n+1}\xi^{2r-1} + O(\rho)\right)x^{2^{n+2}-2r} + O(x^{2^{n+2}-2r+1}),$$

$$\xi \cdot \tau^{2(2^{n+1}k+r)}u^{2(2^n(2l+1)-r)}h = \left(\tau^{2^{n+2}k}u^{2^{n+2}l}h_{n+1}\rho^{2r-1} + O(\xi)\right)x^{2^{n+2}-2r} + O(x^{2^{n+2}-2r+1})$$

for $k, l \in \mathbb{Z}$, $n \geq 0$, and $1 \leq r \leq 2^{n+1} - 1$.

Proof It suffices to produce these extensions in the universal case $E = MP$. This ensures that the terms on the right are nonzero, so that these are nontrivial extensions. As discussed above, the cofiber sequences of (9) show that the terms

$$\tau^{2^{n+2}k}u^{2^{n+2}l}h_{n+1}\xi^{2r-1}x^{2^{n+2}-2r}, \quad \tau^{2^{n+2}k}u^{2^{n+2}l}h_{n+1}\rho^{2r-1}x^{2^{n+2}-2r}$$

must be in the image of ρ and ξ respectively. By [Lemma 7.1](#), this is not the case in $Z[x]/B$, so there must be hidden extensions making it so. In other words, there must be hidden extensions of the form

$$\rho \cdot \alpha = \left(\tau^{2^{n+2}k}u^{2^{n+2}l}h_{n+1}\xi^{2r-1} + O(\rho)\right)x^{2^{n+2}-2r} + O(x^{2^{n+2}-2r+1}),$$

$$\xi \cdot \beta = \left(\tau^{2^{n+2}k}u^{2^{n+2}l}h_{n+1}\rho^{2r-1} + O(\xi)\right)x^{2^{n+2}-2r} + O(x^{2^{n+2}-2r+1}),$$

where α and β are detected by classes in $Z[x]/B$ killed by ρ and ξ respectively. The error terms ensure that we do not need to pin down α and β precisely, but only the $\pi_0 b(MPR)$–submodule of $Z[x]/B$ that they generate. By [Lemma 7.2](#), the extensions given in the theorem statement are the only possibilities in these degrees. $\square$

7.4 Remark This leaves open the problem of finding witnesses to the classes of (10) being in the image of ρ or ξ in the case where $r + s = 2^{n+2} - 1$ and r is even. In some cases no hidden extension is necessary, for example

$$\rho^{2^{n+1}}h_{n+1}x^{2^{n+1}-1} = w_n x^{2^{n+1}-1} = 0,$$

$$\xi^{2^{n+1}}h_{n+1}x^{2^{n+1}-1} = \tau^{-2^{n+2}}u^{2^{n+1}}w_n x^{2^{n+1}-1} = 0.$$

However, the general situation seems to be rather subtle. For example, for $h_2\xi^2 x^5$ to be in the image of ρ, the only possibility is that $\rho\tau^{-4}u^4 h_1 x$ detects a class satisfying

$$\rho \cdot \rho\tau^{-4}u^4 h_1 x = h_2\xi^2 x^5 + O(\rho).$$

On the other hand,

$$\rho^2 \tau^{-4} u^4 h_1 = \tau^{-4} u^4 w_0, \quad \tau^{-4} u^4 w_0 \cdot x = 0$$

in $Z[x]/B$. This indicates the existence of a mixed extension along the lines of

$$\rho^2 \tau^{-4} u^4 h_1 = \tau^{-4} u^4 w_0 + h_1 \xi^2 x^4 + O(\rho).$$

Note that if $\theta \in \pi_\star b(ER)$ is detected by $\tau^{-4} u^4 w_0$, then so is $\theta + h_1 \xi^2 x^4$. Thus for such an extension to even be defined, one must specify some information about how one lifts elements from Z to $\pi_\star b(ER)$, and these considerations are outside the scope of our investigation.

8 Some Mahowald invariants

We end by giving some examples of computations within the ring $\pi_\star b(ER)$. Our examples will center around the following definition.

8.1 Definition Given a spectrum A, the *A–based Mahowald invariant* is a multivalued function

$$R_A : \pi_* A \rightharpoonup \pi_* A,$$

ie a relation on $\pi_* A$, defined as follows: given $y \in \pi_n A$ and $z \in \pi_{n+k} A$, we say $z \in R_A(y)$ if z lifts to a class $\zeta \in \pi_\star b(A)$ such that $\rho^N y = \rho^{N+k} \zeta$ for $N \gg 0$, and moreover k is as large as possible.

8.2 Remark There are natural maps $\pi_n A \to \pi_n A^{tC_2}$ and $\pi_{c+w\sigma} b(A) \to \pi_c A^{tC_2}$, and the condition $\rho^N y = \rho^{N+k} \zeta$ for $N \gg 0$ amounts to asking that $y = \zeta$ in $\pi_* A^{tC_2}$. When $A = S$, this construction recovers the classical Mahowald invariant, commonly called the root invariant. See [Mahowald and Ravenel 1993] for additional background, [Bruner and Greenlees 1995] for the relation to C_2–equivariant homotopy theory, which connects Definition 8.1 to other definitions, [Behrens 2007] for the state of the art in S–based Mahowald invariants at the prime 2, [Quigley 2022] for further discussion of A–based Mahowald invariants with $A \neq S$, and [Li et al. 2022] for more information about spectra related to ER^{tC_2}.

Li, Shi, Wang and Xu [Li et al. 2019] prove that the Hurewicz image of real bordism detects the Hopf elements, Kervaire classes, and $\bar{\kappa}$ family. These are the elements in $\pi_* S$ detected on the E_2–page of the Adams spectral sequence by the classes h_i, h_j^2, and g_{k+1} respectively; note there is no claimed relation between h_i here and the elements h_i in $\pi_\star b(E)$. These classes arrange into Sq^0 families, ie

(12) $$\mathrm{Sq}^0(h_i) = h_{i+1}, \quad \mathrm{Sq}^0(h_j^2) = h_{j+1}^2, \quad \mathrm{Sq}^0(g_{k+1}) = g_{k+2}.$$

Informally, this means that they arise as iterated Mahowald invariants at the level of Ext. Of course this cannot lift to the level of homotopy, as not all of these classes are permanent cycles; still, it is known that $\eta \in R_S(2)$, $\nu \in R_S(\eta)$, and $\sigma \in R_S(\nu)$, and it is conjectured that $\theta_{j+1} \in R_S(\theta_j)$ for $j \geq 3$ provided θ_{j+1} exists, see [Mahowald and Ravenel 1993, Proposition 2.4].

We can compute the iterated MPR–based Mahowald invariants of the classes 2, $\theta_0 = 4$, and $\bar{\kappa}$, yielding an analogue of (12). Our computation works just as well for ER in a range depending on the height of E. First we need to know how $\bar{\kappa}$ sits inside $\pi_\star MPR$.

8.3 Lemma *The class $\bar{\kappa}$ is detected by MPR, with Hurewicz image $u_2^4 u^8 x^4$.*

Proof If $\bar{\kappa}$ is detected by MPR, then it is detected by MR. As $\pi_{20} MR = \mathbb{Z}/(2)\{u_2^4 u^8 x^4\} \subset \pi_{20} MPR$, it suffices just to show that $\bar{\kappa}$ is detected by MR, which was shown in [Li et al. 2019]. Alternately, as there is a ring map $MR \to \mathrm{TMF}_0(3)$ [Hill and Meier 2017], it suffices to show that $\bar{\kappa}$ is detected in the latter, and here one may appeal to [Mahowald and Rezk 2009]. $\qquad\square$

We now abbreviate $R = R_{MPR}$.

8.4 Theorem *Define elements*

$$a_n = u_n x^{2^n - 1} \in \pi_{2^n - 1} MPR, \quad b_m = u_{m+1}^4 u^{2^{m+2}} x^{2^{m+2} - 4} \in \pi_{4(3 \cdot 2^m - 1)} MPR$$

for $n \geq 0$ and $m \geq 1$, so that for example $a_0 = 2$ and $b_1 = \bar{\kappa}$. Then there are MPR–based Mahowald invariants

$$a_{n+1} \in R(a_n), \quad a_{n+1}^2 \in R(a_n^2), \quad b_{m+1} \in R(b_m).$$

Proof First consider a_n. As

$$h_n \equiv u_n + \rho^{2^n} \xi^{2^n} h_{n+1} \pmod{u_0, \ldots, u_{n-1}},$$

the relation $\rho^{2^n} h_n \cdot x^{2^n - 1} = 0$ implies

$$(13) \qquad\qquad \rho^{2^n} \cdot u_n x^{2^n - 1} = -\rho^{2^{n+1}} \cdot h_{n+1} \xi^{2^n} x^{2^n - 1}.$$

There are no further relations and $h_{n+1} \xi^{2^n} x^{2^n - 1}$ lifts $u_{n+1} x^{2^{n+1} - 1} = a_{n+1}$, yielding $a_{n+1} \in R(a_n)$. The case of a_n^2 is identical, only we must apply (13) twice:

$$\rho^{2^n} \cdot u_n^2 x^{2(2^n - 1)} = -\rho^{2^{n+1}} \cdot h_{n+1} \xi^{2^n} u_n x^{2(2^n - 1)} = \rho^{3 \cdot 2^n} \cdot h_{n+1}^2 \xi^{2^{n+1}} x^{2(2^n - 1)}.$$

Now consider b_m. As $2^{m+2} - 4 \geq 2^{m+1} - 1$ for $m \geq 1$, we may apply (13) thrice to obtain

$$(14) \qquad \rho^{2^{m+1}} \cdot u_{m+1}^4 u^{2^{m+2}} x^{2^{m+2} - 4} = \rho^{2^{m+3}} \cdot u_{m+1} u^{2^{m+2}} \cdot \xi^{3 \cdot 2^{m+1}} h_{m+2}^3 x^{2^{m+2} - 4}.$$

At this point additional care is needed: we cannot apply (13) again, as despite appearances $u_{m+1} u^{2^{m+2}}$ is indecomposable. Instead, the relation $\rho \cdot h = 0$ gives

$$0 \equiv u_{m+1} \rho^{2^{m+2} - 1} \tau^{-2^{m+2} + 2} u^{2^{m+1} - 1} + h_{m+2} \rho^{2^{m+3} - 1} \tau^{-2^{m+3} + 2} u^{2^{m+2} - 1} \pmod{u_0, \ldots, u_m}$$

in $\pi_\star b(MP)$, and thus

$$u_{m+1} u^{2^{m+2}} \cdot \xi^{2^{m+2} - 1} = u_{m+1} u^{2^{m+2}} \cdot \rho^{2^{m+2} - 1} \tau^{-2^{m+3} - 2} u^{2^{m+2} - 1}$$

$$\equiv \tau^{-2^{m+3}} u^{2^{m+3}} h_{m+3} \cdot \rho^{3 \cdot 2^{m+1}} \xi^{2^{m+1} - 1} \pmod{u_0, \ldots, u_m}.$$

Substituting this into (14) yields

$$\rho^{2^{m+1}} \cdot u_{m+1}^4 u^{2^{m+2}} x^{2^{m+2}-4} = \rho^{7 \cdot 2^{m+1}} \cdot \tau^{-2^{m+3}} u^{2^{m+3}} h_{m+2}^4 \cdot \xi^{2^{m+2}} x^{2^{m+2}-4}.$$

We cannot pull this class back any further. Thus, as $\tau^{-2^{m+3}} u^{2^{m+3}} h_{m+2}^4 \cdot \xi^{2^{m+2}} x^{2^{m+2}-4}$ lifts

$$u_{m+2}^4 u^{2^{m+3}} x^{2^{m+3}-4} = b_{m+1},$$

we obtain $b_{m+1} \in R(b_m)$. $\qquad\square$

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Department of Mathematics, University of Virginia
Charlottesville, VA, United States

williamb@math.uni-bonn.de

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