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Profinite isomorphisms and fixed-point properties

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We describe a flexible construction that produces triples of finitely generated, residually finite groups $M \hookrightarrow P \hookrightarrow \Gamma$, where the maps induce isomorphisms of profinite completions $\widehat{M} \cong \widehat{P} \cong \widehat{\Gamma}$, but M and Γ have Serre's property FA while P does not. In this construction, P is finitely presented and Γ is of type F_∞. More generally, given any positive integer d, one can demand that M and Γ have a fixed point whenever they act by semisimple isometries on a complete CAT(0) space of dimension at most d, while P acts without a fixed point on a tree.

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1 Introduction

In the quest for profinite invariants of discrete groups, fixed-point properties have been a source of disappointment. For example, Aka [1] proved that the profinite completion of a finitely generated, residually finite group does not determine whether the group has property (T), ie whether the group can act without a global fixed point as a group of affine isometries of a Hilbert space. Cheetham-West, Lubotzky, Reid and Spitler [13] proved a similar theorem for actions on trees: they construct pairs of finitely presented, residually finite groups G_1 and G_2 such that $\widehat{G}_1 \cong \widehat{G}_2$ but G_1 has Serre's property FA whereas G_2 does not. (Here, $\widehat{G}_i$ denotes the profinite completion of G_i.)

In the present paper, we will improve upon this last result in two ways. First, we construct groups with these properties for which (G_2, G_1) is a Grothendieck pair, ie the isomorphism $\widehat{G}_1 \cong \widehat{G}_2$ is induced by a monomorphism of discrete groups $G_1 \hookrightarrow G_2$ (cf. [13, Question 4.1]). Secondly, we extend this result from actions on trees (the 1–dimensional case) to actions on d–dimensional CAT(0) spaces, with $d \geq 1$ arbitrary.

We say that a group G has *property* Fix_d if G fixes a point whenever it acts by semisimple isometries on a complete CAT(0) space of covering dimension at most d. Every isometry of a simplicial tree is semisimple, so Fix_1 implies Serre's property FA (and extends it to cover actions on complete $\mathbb{R}$–trees).

Theorem A *For every integer $d \geq 1$, there exist triples of residually finite groups $M \xrightarrow{i} P \xrightarrow{j} \Gamma$ such that*

(1) *i and j induce isomorphisms $\widehat{M} \cong \widehat{P} \cong \widehat{\Gamma}$;*

(2) *M is finitely generated, P is finitely presented, and Γ is of type F_∞;*

(3) *M and Γ have property* Fix_d, *but*

(4) *P is a nontrivial amalgamated free-product and therefore acts on a simplicial tree without a global fixed point.*

An artefact of our proof is that although M and Γ have Fix_d, they each contain a subgroup of finite index that can act on a tree without fixing a point (Remark 6.1).

The fixed-point properties required in the above theorem will be established using the following criterion, which is drawn from the circle of ideas developed in [6; 4].

Theorem B (Corollary 5.5) *If A is a finitely generated group with finite abelianisation and B is a finite group, then $A \wr B$ has Fix_d for $d = |B| - 1$.*

The first steps in our construction of the triples $M \hookrightarrow P \hookrightarrow \Gamma$ in Theorem A follow the template for constructing finitely presented Grothendieck pairs that originates in [10] and is explicit in Section 8 of [7]. We craft finitely presented groups Q that enjoy an array of properties relevant to our aims (Section 3.1); we use a suitably adapted form of the Rips construction (Proposition 3.1) to produce short exact sequences $1 \to N \to G \to Q \to 1$ with G finitely presented and residually finite, N finitely generated, and both N and G perfect; and we take a fibre product of several copies of $G \to Q$ to produce $N^d \hookrightarrow P_d \hookrightarrow G^d$ with P_d finitely presented. (A novel feature here is that we take the fibre product of several copies of $G \to Q$, not just two.) The triples $M \overset{i}{\hookrightarrow} P \overset{j}{\hookrightarrow} \Gamma$ we seek are obtained by taking finite extensions of N^d, P_d and G^d in a way that allows us to apply Theorem B.

There is a great deal of flexibility in this construction — see Section 7.

2 Preliminaries

In this section we gather the basic definitions and facts we need concerning profinite completions of groups and isometries of CAT(0) spaces.

2.1 Profinite completions

If $M_1 < M_2$ are normal subgroups of finite index in a group G, then there is a natural map $G/M_1 \to G/M_2$. Thus the finite quotients of G form a directed system. The *profinite completion* of G is the inverse limit of this system:

$$\widehat{G} := \varprojlim G/M.$$

The natural map $i : G \to \widehat{G}$ is injective if and only if G is residually finite. If G is finitely generated then, for every finite group Q, composition with i defines a bijection $\mathrm{Hom}(\widehat{G}, Q) \to \mathrm{Hom}(G, Q)$ that restricts to a bijection on the set of epimorphisms. In particular, G and $\widehat{G}$ have the same set of finite images, which we denote by $\mathfrak{F}(G)$. Thus $\widehat{G}_1 \cong \widehat{G}_2$ implies $\mathfrak{F}(G_1) = \mathfrak{F}(G_2)$. Less obviously, for finitely generated groups, $\mathfrak{F}(G_1) = \mathfrak{F}(G_2)$ implies $\widehat{G}_1 \cong \widehat{G}_2$ — see [17, pages 88–89]. (Note that $\widehat{G}_1 \cong \widehat{G}_2$ does not imply that there are any nontrivial homomorphisms $G_1 \to G_2$.)

A property $\mathfrak{P}$ of finitely generated, residually finite groups is said to be a *profinite invariant* if $\widehat{G}_1 \cong \widehat{G}_2$ implies that G_2 has $\mathfrak{P}$ whenever G_1 has $\mathfrak{P}$. Theorem A shows that Fix_d is not a profinite invariant.

A pair of finitely generated, residually finite groups $G_1 \overset{\iota}{\hookrightarrow} G_2$ is called a *Grothendieck pair* [15] if the induced map $\hat{\iota}\colon \widehat{G}_1 \to \widehat{G}_2$ is an isomorphism. For fixed G_2, there can be infinitely many nonisomorphic subgroups G_1 such that $G_1 \hookrightarrow G_2$ is a Grothendieck pair, even if one requires both G_1 and G_2 to be finitely presented [8].

2.2 Isometries of CAT(0) spaces

We refer the reader to [11] for basic facts about CAT(0) spaces. We write $\mathrm{Isom}(X)$ for the group of isometries of a CAT(0) space X and $\mathrm{Fix}(H)$ for the set of points in X fixed by each element of a subset $H \subset \mathrm{Isom}(X)$. Note that $\mathrm{Fix}(H)$ is closed and convex.

If X is complete, each closed, nonempty bounded subset is contained in a unique smallest ball; see [11, page 178]. If the bounded subset is an orbit of a subgroup $H < \mathrm{Isom}(X)$, then the centre of the ball will be fixed by H. This proves the following standard proposition.

Proposition 2.1 *If X is a complete* CAT(0) *space, then every finite subgroup of* $\mathrm{Isom}(X)$ *fixes a point in X.*

By combining the preceding bounded-orbit observation with the fact that $\mathrm{Fix}(H)$ is itself a CAT(0) space, one can prove the following standard fact — see [6, Corollary 2.5], for example.

Proposition 2.2 *Let X be a complete* CAT(0) *space. If the subgroups $H_1, \dots, H_n < \mathrm{Isom}(X)$ commute and $\mathrm{Fix}(H_i)$ is nonempty for $i = 1, \dots, n$, then $\bigcap_i \mathrm{Fix}(H_i)$ is nonempty.*

For an isometry $\gamma \in \mathrm{Isom}(X)$,

$$\mathrm{Min}(\gamma) := \{p \in X \mid d(p, \gamma.p) = |\gamma|\},$$

where $|\gamma| := \inf\{d(x, \gamma.x) \mid x \in X\}$. By definition, γ is *semisimple* if $\mathrm{Min}(\gamma)$ is nonempty. Every isometry of a complete $\mathbb{R}$–tree is semisimple. Semisimple isometries are divided into *hyperbolics* (also called loxodromics), for which $|\gamma| > 0$, and *elliptics*, which are the isometries with $\mathrm{Fix}(\gamma) \neq \varnothing$. If γ is hyperbolic then there exist γ–invariant isometrically embedded lines $\mathbb{R} \hookrightarrow X$ on which γ acts as a translation by $|\gamma|$; each such line is called an axis for γ. The union of these axes is $\mathrm{Min}(\gamma)$. The following extract from pages 229–231 of [11] summarises the properties of $\mathrm{Min}(\gamma)$ that we require.

Proposition 2.3 *Let X be a complete* CAT(0) *space and let $\gamma \in \mathrm{Isom}(X)$ be a hyperbolic isometry. Then*:

1. $\mathrm{Min}(\gamma)$ *splits isometrically* $\mathrm{Min}(\gamma) = Y \times \mathbb{R}$, *where $Y \times \{0\}$ is a closed, convex subspace of X.*
2. *γ acts trivially on Y and acts as translation by $|\gamma|$ on each of the lines $\{y\} \times \mathbb{R}$.*

(3) *The centraliser* $C(\gamma) < \mathrm{Isom}(X)$ *leaves* $\mathrm{Min}(\gamma)$ *and its splitting invariant, acting by translations of the second factor.*

(4) *If* $\delta \in C(\gamma)$ *is hyperbolic, then* $\mathrm{Min}(\gamma)$ *contains an axis for* δ.

3 A Rips construction and input groups

The purpose of this section is to produce the short exact sequences $1 \to N \to G \to Q \to 1$ described in the introduction.

3.1 The input groups Q

Our constructions require as input a group Q with the following properties:

- Q is of type F_3 (ie has a classifying space $K(Q,1)$ with finite 3–skeleton).
- $H_2(Q,\mathbb{Z}) = 0$.
- $\hat{Q} = 1$.
- Q is a nontrivial amalgamated free product (and therefore does not have FA).

There are many ways to concoct groups Q with these properties. Indeed, *every* finitely presented group can be embedded (explicitly, with controlled geometry) into a finitely presented group that has no nontrivial finite quotients [3]; by replacing this enveloping group with its universal central extension one can force it to have trivial second homology; and by taking a free product of two copies of the resulting group one obtains a group Q satisfying all but the first of the above properties. If the group that one starts with is of type F_3, then so is Q. Likewise for type F (having a finite classifying space) and type F_∞ (having a classifying space with finite skeleta).

One can also find explicit groups of the desired form in the literature. For example, from [10] one could take

$$Q = \langle a, b, \alpha, \beta \mid ba^{-2}b^{-1}a^3,\ \beta\alpha^{-2}\beta^{-1}\alpha^3,\ [bab^{-1}, a]\beta^{-1},\ [\beta\alpha\beta^{-1}, \alpha]b^{-1} \rangle.$$

3.2 A convenient version of the Rips construction

There are many refinements of the Rips construction in the literature, with various properties imposed on the groups constructed. The following version suits our needs.

Proposition 3.1 *There exists an algorithm that, given a finite presentation* $\langle X \mid R\rangle$ *of a group* Q, *will construct a finite aspherical presentation* $\langle X \cup \{a_1, a_2\} \mid \widetilde{R} \cup V\rangle$ *for a group* G *so that*

(1) G *is hyperbolic, residually finite, of type* F, *and virtually special;*

(2) $N := \langle a_1, a_2\rangle$ *is normal in* G;

(3) G/N *is isomorphic to* Q;

(4) *G is perfect if Q is perfect*;

(5) *if $\widehat{Q} = 1$ and $H_2(Q,\mathbb{Z}) = 0$, then N and G are both perfect.*

Proof With the exception of item (5), the proof is covered by Proposition 2.10 of [9]. The crucial property of residual finiteness is due to Wise [18; 19].

For item (5), we consider the 5–term exact sequence extracted from the corner of the LHS spectral sequence for $1 \to N \to G \to Q \to 1$:

$$H_2(Q,\mathbb{Z}) \to H_0(Q,\ H_1(N,\mathbb{Z})) \to H_1(G,\mathbb{Z}) \to H_1(Q,\mathbb{Z}) \to 0.$$

As all the other terms are zero, $H_0(Q,\ H_1(N,\mathbb{Z})) = 0$. By definition, $H_0(Q,\ H_1(N,\mathbb{Z}))$ is the group of coinvariants for the action of Q on $H_1(N,\mathbb{Z})$ that is induced by conjugation in G. As the abelian group $H_1(N,\mathbb{Z})$ is finitely generated, its automorphism group is residually finite. As Q has no nontrivial finite quotients, its action on $H_1(N,\mathbb{Z})$ must be trivial. Therefore $H_1(N,\mathbb{Z}) = H_0(Q,\ H_1(N,\mathbb{Z})) = 0$. □

4 Fibre products

Our proof of Theorem A relies on the various properties of fibre products that we establish in this section. These properties cover three topics: the finiteness properties of fibre products, their behaviour with respect to profinite completions, and their interaction with wreath products.

4.1 Fibre products and finiteness properties

For $i = 1,\dots,d$, let $p_i\colon G_i \to Q$ be an epimorphism of groups. The *fibre product* of this family of maps is

$$P_d = \{(g_1,\dots,g_d) \mid p_i(g_i) = p_j(g_j),\ i,j = 1,\dots,d\} < G_1 \times \cdots \times G_d.$$

The case $p_1 = \cdots = p_d$ will be of particular interest in this article.

P_d is the preimage of the diagonal subgroup

$$Q_d^{\Delta} := \{(q,\dots,q) \mid q \in Q\} < Q \times \cdots \times Q$$

and there is a short exact sequence

$$1 \to N^{(d)} \to P_d \to Q_d^{\Delta} \to 1, \tag{4-1}$$

where $N_i = \ker p_i$ and $N^{(d)} = N_1 \times \cdots \times N_d$.

We need a criterion to ensure that P_d is finitely presented; we will deduce this from the following *Asymmetric* 1-2-3 *Theorem* [12].

Theorem 4.1 [12] *For $i = 1,2$, let $1 \to N_i \to G_i \xrightarrow{p_i} Q \to 1$ be a short exact sequence of groups. If G_1 and G_2 are finitely presented, Q is of type F_3, and at least one of the groups N_1, N_2 is finitely generated, then the fibre product $P < G_1 \times G_2$ is finitely presented.*

Corollary 4.2 *Suppose $d \geq 2$ and let $1 \to N_i \to G_i \xrightarrow{p_i} Q \to 1$ be a short exact sequence of groups, for $i = 1, \dots, d$. If the groups G_i are all finitely presented, the groups N_i are finitely generated, and Q is of type F_3, then the associated fibre product $P_d < G_1 \times \cdots \times G_d$ is finitely presented.*

Proof We proceed by induction on d; the case $d = 2$ is covered by the theorem. Let $P_d < G_1 \times \cdots \times G_d$ be the fibre product of $p_1, \dots, p_d$. For the inductive step, first note that

$$P_d \ < \ P_{d-1} \times G_d \ < \ G_1 \times \cdots \times G_d$$

is the fibre product of the map $p_d \colon G_d \to Q$ and the composition $P_{d-1} \to Q^{\Delta}_{d-1} \to Q$, where $P_{d-1} \to Q^{\Delta}_{d-1}$ is the map from (4-1) and $Q^{\Delta}_{d-1} \to Q$ is the isomorphism $(q, \dots, q) \mapsto q$. To complete the proof, we apply the theorem, noting that P_{d-1} is finitely presented, by induction. □

We shall also need the following more elementary result.

Lemma 4.3 *For $i = 1, \dots, d$, let $p_i \colon G_i \twoheadrightarrow Q$ be an epimorphism of groups. If the groups G_i are finitely generated and Q is finitely presented, then the fibre product $P_d < G_1 \times \cdots \times G_d$ is finitely generated.*

Proof As in the preceding proof, induction reduces us to the case $d = 2$. We fix a finite presentation $Q = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$ and for $i = 1, 2$ choose $a_{ij} \in G_i$ such that $p_i(a_{ij}) = a_j$. We then add a finite set of elements $B_i \subset \ker p_i$ to obtain a finite generating set for G_i, and denote by ρ_{ik} the word obtained from r_k by replacing each a_j with a_{ij}. It is easy to check that the fibre product $P < G_1 \times G_2$ is generated by

$$\big\{(b_{1s}, 1),\ (1, b_{2s}),\ (a_{1j}, a_{2j}),\ (\rho_{1k}, 1) \mid j = 1, \dots, n;\ k = 1, \dots, m;\ b_{is} \in B_i\big\}. \qquad \square$$

4.2 Fibre products and Grothendieck pairs

The idea of constructing Grothendieck pairs using fibre products originates in the work of Platonov and Tavgen [16] and was extended in [2; 8; 10].

Lemma 4.4 [10, Lemma 2.2] *Let $1 \to N \to G \to Q \to 1$ be an exact sequence of finitely generated groups. If $\widehat{Q} = 1$ and $H_2(Q, \mathbb{Z}) = 0$, then $N \hookrightarrow G$ induces an isomorphism of profinite completions.*

The following variant of the Platonov–Tavgen argument will be useful.

Proposition 4.5 [8, Theorem 2.2] *Let $p_1 \colon G_1 \to Q$ and $p_2 : G_2 \to Q$ be epimorphisms with G_1 and G_2 finitely generated and Q finitely presented. Let $P < G_1 \times G_2$ be the associated fibre product. If $\widehat{Q} = 1$ and $H_2(Q, \mathbb{Z}) = 0$, then $P \hookrightarrow G_1 \times G_2$ induces an isomorphism of profinite completions.*

We need an extension to the case of $d \geq 2$ factors.

Theorem 4.6 *For $i = 1, \dots, d$, let $p_i \colon G_i \to Q$ be an epimorphism of finitely generated groups, and let $P_d < \Gamma := G_1 \times \cdots \times G_d$ be the associated fibre product. If Q is finitely presented, $\widehat{Q} = 1$ and $H_2(Q, \mathbb{Z}) = 0$, then $P_d \hookrightarrow \Gamma$ induces an isomorphism of profinite completions.*

Proof We argue by induction on d, as in the proof of Corollary 4.2. In the inductive step, we appeal to Lemma 4.3 to ensure that P_{d-1} is finitely generated. We can then apply Proposition 4.5 to $p_d\colon G_d \to Q$ and $P_{d-1} \to Q^{\Delta}_{d-1} \cong Q$, noting that P_d is the fibre product of these maps. □

4.3 Fibre products and wreath products

Given groups A and B, with B finite, the *wreath product* $A \wr B$ is the semidirect product $A^B \rtimes B$, or more precisely $\left(\bigoplus_{b\in B} A_b\right) \rtimes B$, with fixed isomorphisms $\mu_b\colon A \to A_b$ so that $b \in B$ acts on $A_{b'}$ as $\mu_{bb'} \circ \mu_{b'}^{-1}$. We identify $A^{\Delta} < A \times \cdots \times A$ with its image under $(\mu_b)_{b\in B}$. The following trivial observation will play an important role in what follows.

Lemma 4.7 *$\langle A^{\Delta}, B\rangle < A \wr B$ is the direct product $A^{\Delta} \times B \cong A \times B$.*

Given B and a short exact sequence of groups $1 \to N \to G \to Q \to 1$, we take the direct product of $|B|$ copies of the sequence, indexed by the elements of B, and let B permute these copies by its left action on the indices. The resulting semidirect products give us a (nonexact) sequence of groups

$$N \wr B \hookrightarrow G \wr B \twoheadrightarrow Q \wr B.$$

The action of B preserves the fibre product $P_B < G^B = \bigoplus_{b\in B} G_b$ of the maps $G_b \to Q_b$, giving a semidirect product

$$P_B \rtimes B = \langle P_B, B\rangle < G \wr B$$

and a (nonexact) sequence of groups

$$N \wr B \hookrightarrow P_B \rtimes B \twoheadrightarrow \langle Q^{\Delta}, B\rangle < Q \wr B.$$

From Lemma 4.7 we deduce:

Lemma 4.8 *With the notation established above, there is surjection*

$$P_B \rtimes B \twoheadrightarrow Q^{\Delta} \cong Q.$$

5 Fixed point criteria

In this section we present criteria that guarantee fixed points for group actions on complete $\mathrm{CAT}(0)$ spaces of finite dimension. These criteria are extracted from the more general criteria explained in [4; 6].

The following result is a special case of [6, Corollary 3.6].

Proposition 5.1 *Let d be a positive integer and let X be a complete $\mathrm{CAT}(0)$ space of dimension less than d. Let $S_1, \dots, S_d \subset \mathrm{Isom}(X)$ be conjugates of a subset $S \subset \mathrm{Isom}(X)$ with $[s_i, s_j] = 1$ for all $s_i \in S_i$ and $s_j \in S_j$ with $i \neq j$. If every element of S (hence S_i) has a fixed point in X, then so does every finite subset of S (hence S_i).*

Corollary 5.2 *Let d be a positive integer, let X be a complete* CAT(0) *space of dimension less than d, and let $H_1, \dots, H_d < \mathrm{Isom}(X)$ be conjugate subgroups that pairwise commute. If each H_i is generated by a finite set of elliptic elements, then $D = \langle H_1, \dots, H_d \rangle$ has a fixed point in X.*

Proof Let $S = S_1$ be a finite set of elliptics generating H_1. We conjugate S to obtain a generating set S_i for each H_i. The proposition says that $\mathrm{Fix}(S_i) = \mathrm{Fix}(H_i)$ is nonempty, whence $\mathrm{Fix}(D)$ is nonempty, by Proposition 2.2. □

For $n \in \mathbb{N}$, an *n–flat* in a metric space X is an isometrically embedded copy of Euclidean space $\mathbb{E}^n \hookrightarrow X$.

Lemma 5.3 *If $K_0, \dots, K_d$ are groups with $\mathrm{Hom}(K_i, \mathbb{R}) = 0$ and X is a complete* CAT(0) *space that does not contain any $(d+1)$–flats, then there does not exist an action $\rho\colon K_0 \times \dots \times K_d \to \mathrm{Isom}(X)$ such that each $\rho(K_i)$ contains a hyperbolic isometry.*

Proof We shall prove the lemma by induction, the case $d = 0$ being trivial. Assume that the lemma is true for $d' \le d - 1$. The induction will be complete if we can derive a contradiction from the assumption that there are hyperbolic isometries $\gamma_i \in \rho(K_i)$ for $i = 0, \dots, d$. If this were the case, then, according to Proposition 2.3, the subspace $\mathrm{Min}(\gamma_0)$ would split isometrically as $Y \times \mathbb{R}$ and the centraliser $C(\gamma_0)$ of γ_0 in $\mathrm{Isom}(X)$ would preserve $\mathrm{Min}(\gamma_0)$ and its splitting, acting by translations on the second factor of $Y \times \mathbb{R}$. The group of translations is $\mathbb{R}$ and $\mathrm{Hom}(K_i, \mathbb{R}) = 0$, so $K_1 \times \dots \times K_d$ must act trivially on the second factor. Thus we obtain an action of $K_1 \times \dots \times K_d$ on $Y_0 = Y \times \{0\}$. Part (1) of Proposition 2.3 assures us that $Y_0 \subset X$ is closed and convex, hence a CAT(0) space, and part (4) tells us that $\gamma_i \in K_i$ acts as a hyperbolic isometry of Y_0, for $i = 1, \dots, d$. But $Y \times \mathbb{R} = \mathrm{Min}(\gamma_0)$ embeds isometrically in X, so Y_0 does not contain a d–flat. This contradicts our inductive hypothesis. □

Theorem 5.4 *Let G be a group and suppose that there is a subgroup $D = H_0 \times \dots \times H_d < G$ with H_i conjugate to H_0 in G for $i = 1, \dots, d$. If H_0 is finitely generated and has finite abelianisation, then D has a fixed point whenever G acts by semisimple isometries on a complete* CAT(0) *space of dimension at most d.*

Proof The hypothesis $\dim(X) \le d$ is stronger than requiring that X contains no $(d+1)$–flat, so the preceding lemma tells us that there are no hyperbolic elements in the subgroups H_i. Because H_0 is finitely generated, Corollary 5.2 completes the proof. □

The following result was stated as Theorem B in the introduction.

Corollary 5.5 *If A is a finitely generated group with finite abelianisation and B is a finite group, then $A \wr B$ has Fix_d, where $d = |B| - 1$.*

Proof Let $G = A \wr B = \left(\bigoplus_{b \in B} A_b\right) \rtimes B$ and $D = \bigoplus_{b \in B} A_b$. Theorem 5.4 tells us that D has a fixed point whenever $A \wr B$ acts by semisimple isometries on a complete CAT(0) space X with $\dim(X) \le |B| - 1$. Because $B < A \wr B$ normalises D, it leaves its set of fixed points $\mathrm{Fix}(D) \subset X$ invariant. $\mathrm{Fix}(D)$ is closed and convex, hence a complete CAT(0) space. Proposition 2.1 provides a point in $\mathrm{Fix}(D)$ that is fixed by B and hence by $A \wr B = \langle D, B \rangle$. $\square$

6 Proof of Theorem A

Let Q be a group satisfying the conditions listed in Section 3.1. By Proposition 3.1, there is a short exact sequence

$$1 \to N \to G \to Q \to 1$$

with N finitely generated and perfect, and G hyperbolic (hence type F_∞), residually finite and perfect. Given $d \geq 2$, we fix a finite group B with $|B| = d+1$. Proceeding as in Section 4.3, we take the direct product of $|B|$ copies of this sequence, indexed by the elements of B, and take the fibre product of the maps $G_b \to Q$ to obtain

$$N^B \hookrightarrow P_B \hookrightarrow G^B.$$

The action of B permuting the direct factors of G^B leaves N^B and P_B invariant, so the above inclusions extend to

$$N \wr B \overset{i}{\hookrightarrow} P_B \rtimes B \overset{j}{\hookrightarrow} G \wr B.$$

We claim that this triple of groups has the properties required in Theorem A.

Towards showing that i induces an isomorphism of profinite completions, note first that Lemma 4.4 applies to $N^B \hookrightarrow P_B$, because N^B is normal in P_B with quotient Q. Likewise, Theorem 4.6 assures us that $P_B \hookrightarrow G^B$, the restriction of j, induces an isomorphism of profinite completions. Thus the maps $N^B \hookrightarrow P_B \hookrightarrow G^B$ induce isomorphisms $\widehat{N^B} \cong \widehat{P_B} \cong \widehat{G^B}$. The action of B permuting the factors of G^B extends to $\widehat{G^B}$, where it preserves the dense subgroups P_B and N^B. Since the operations of profinite completion and semidirect product with a finite group commute, we conclude that $\hat{i}$ and $\hat{j}$ give isomorphisms $\widehat{N^B} \rtimes B \cong \widehat{P_B} \rtimes B \cong \widehat{G^B} \rtimes B$. This establishes Theorem A(1).

$N \wr B$ is finitely generated, since N is. Corollary 4.2 assures us that P_B is finitely presented, whence the finite extension $P_B \rtimes B$ is too. And since G is of type F_∞, so is $G \wr B$. (Indeed, Proposition 3.1 produces a group G that is of type F, ie has a finite classifying space, so G^B is of type F and $G \wr B$ is virtually of type F.) This establishes Theorem A(2).

N and G are finitely generated and perfect, so Corollary 5.5 tells us that $N \wr B$ and $G \wr B$ have Fix_d, since $|B| = d+1$. In contrast, $P_B \rtimes B$ maps onto Q, as in Lemma 4.8, and therefore it is a nontrivial amalgamated free product — in particular it does not have property FA or Fix_d. $\square$

Remark 6.1 We remarked in the introduction that although $M = N \wr B$ and $\Gamma = G \wr B$ have Fix_d, where $d = |B| - 1$, they each have a subgroup of finite index that can act without a fixed point on a tree. For Γ this is obvious, since $G^B < \Gamma$ maps onto Q via projection to G. More indirectly, from Proposition 3.1

we know that G is virtually special and hence large, ie there is a finite-index subgroup $G_0 < G$ that maps onto a nonabelian free group, say $\mu\colon G_0 \twoheadrightarrow L$. Thus we can map G_0^B, which has finite index in Γ, onto L to produce fixed-point-free actions on trees.

For M, we consider $N_0 = N \cap G_0$, noting that as $\mu(N_0) < L$ is finitely generated and normal, $\mu(N_0)$ is either trivial or of finite index, since L is free. In fact, $\mu(N_0)$ must be the whole of L: as Q has no finite quotients, the restriction of $G \twoheadrightarrow Q$ to G_0 is onto and $G_0/N_0 \cong Q$, which means that G_0/N_0 cannot map onto $L/\mu(N_0)$ if the latter is a nontrivial free or finite group. It follows that N_0^B, which has finite index in M, maps onto L.

7 Flexibility and a decision problem

It is clear from the discussion in Section 3.1 that there is a great deal of flexibility in how one chooses the input groups Q. Consequently, one is free to impose various extra conditions on the Grothendieck pairs $P \rtimes B \hookrightarrow G \wr B$ that we have constructed. In particular, the range of pairs that one obtains is sufficient to accommodate many of the undecidability phenomena described in [5] and elsewhere. For example, by following the proof of [5, Theorem B] we obtain the following theorem. Similar results hold with condition Fix_d in place of FA.

Theorem 7.1 *There does not exist an algorithm that, given a finitely presented, residually finite group Γ that has property FA and a finitely presentable subgroup $u\colon P \hookrightarrow \Gamma$ with $\hat{u}\colon \hat{P} \to \hat{\Gamma}$ an isomorphism, can determine whether or not P has property FA.*

Proof As in [5], one can enhance the groups constructed in [14] to obtain a recursive sequence of finite presentations $\mathcal{Q}^{(m)} \equiv \langle S \mid R^{(m)} \rangle$ for groups $Q^{(m)}$, with S and $|R^{(m)}|$ fixed, so that (i) there is no algorithm to determine which of the groups are trivial, but (ii) if $Q^{(m)} \neq 1$ then it satisfies the properties listed in Section 3.1. We apply the algorithm of Proposition 3.1 to the presentations $\mathcal{Q}^{(m)}$ to obtain $G^{(m)} \twoheadrightarrow Q^{(m)}$, with an explicit presentation for $G^{(m)}$ and hence $G^{(m)} \times G^{(m)}$. The fibre product $P^{(m)} \hookrightarrow G^{(m)} \times G^{(m)}$ is given by the finite generating set described in Lemma 4.3, with B_i the given relators of $G^{(m)}$. Theorem 4.1 assures us that $P^{(m)}$ is finitely presentable. We pass from $P^{(m)} \hookrightarrow G^{(m)} \times G^{(m)}$ to $u_m\colon P^{(m)} \rtimes (\mathbb{Z}/2) \hookrightarrow G^{(m)} \wr (\mathbb{Z}/2)$ and then argue as in the proof of Theorem A to see that u_m induces an isomorphism of profinite completions, that $G^{(m)} \wr (\mathbb{Z}/2)$ has property FA (in fact Fix_1), and that $P^{(m)} \rtimes (\mathbb{Z}/2)$ maps onto $Q^{(m)}$.

Note that the groups $G^{(m)} \wr (\mathbb{Z}/2)$ are given by a recursive sequence of presentations and the maps u_m are given by a recursive sequence of generating sets for the subgroups $P^{(m)} \rtimes (\mathbb{Z}/2)$.

If $Q^{(m)} \neq 1$ then $P^{(m)} \rtimes (\mathbb{Z}/2)$ does not have FA, since it maps onto $Q^{(m)}$. But if $Q^{(m)} = 1$ then u_m is an isomorphism, so $P^{(m)}$ does have FA. And by construction, there is no algorithm to decide which of these alternatives holds. □

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