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Projective twists and the Hopf correspondence

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Given Lagrangian (real, complex) projective spaces $K_1, \dots, K_m$ in a Liouville manifold (X, ω) satisfying a certain cohomological condition, we show there is a Lagrangian correspondence (in the sense of Wehrheim and Woodward (2012)) that assigns a Lagrangian sphere $L_i \subset K$ of another Liouville manifold (Y, Ω) to any given projective Lagrangian $K_i \subset X$ for $i = 1, \dots, m$.

We use the Hopf correspondence to study *projective twists*, a class of symplectomorphisms akin to Dehn twists, but defined starting from Lagrangian projective spaces. When this correspondence can be established, we show that it intertwines the autoequivalences of the compact Fukaya category $\mathcal{F}\mathrm{uk}(X)$ induced by the projective twists $\tau_{K_i} \in \pi_0(\mathrm{Symp}_{\mathrm{ct}}(X))$ with the autoequivalences of $\mathcal{F}\mathrm{uk}(Y)$ induced by the Dehn twists $\tau_{L_i} \in \pi_0(\mathrm{Symp}_{\mathrm{ct}}(Y))$ for $i = 1, \dots, m$.

Using the Hopf correspondence, we obtain a free generation result for projective twists in a *clean plumbing* of projective spaces and various results about products of positive powers of Dehn/projective twists in Liouville manifolds.

The same techniques are also used to show that the Hamiltonian isotopy class of the projective twist (along the zero section in $T^*\mathbb{CP}^n$) in $\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$ does depend on a choice of framing for $n \geqslant 19$. Another application of the Hopf correspondence delivers smooth homotopy complex projective spaces $K \simeq \mathbb{CP}^n$ that do not admit Lagrangian embeddings into $(T^*\mathbb{CP}^n, d\lambda_{\mathbb{CP}^n})$ for $n = 4, 7$.

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1 Introduction

1.1 Questions

Given a symplectic manifold (M,ω) with contact boundary, an interesting object of study is the group $\mathrm{Symp}_{\mathrm{ct}}(M)$ of compactly supported symplectomorphisms that are the identity in a neighbourhood of the boundary. Its quotient $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(M))$ by the relation of symplectic isotopy is the *symplectic mapping class group*, and is already a highly nontrivial object. When $H^1(M;\mathbb{R})=0$, a symplectic isotopy is automatically Hamiltonian, and $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(M))$ coincides with the quotient $\mathrm{Symp}_{\mathrm{ct}}(M)/\mathrm{Ham}_{\mathrm{ct}}(M)$ by the subgroup $\mathrm{Ham}_{\mathrm{ct}}(M)\subset\mathrm{Symp}_{\mathrm{ct}}(M)$ of (compactly supported) Hamiltonian symplectomorphisms (namely time-1 maps of compactly supported Hamiltonian flows).

The symplectic mapping class group carries a (forgetful) comparison map

$$c\colon \pi_0(\mathrm{Symp}_{\mathrm{ct}}(M))\to\pi_0(\mathrm{Diff}^+_{\mathrm{ct}}(M)) \tag{1}$$

to the (compactly supported and orientation-preserving) smooth mapping class group of M. In general, the map is neither injective nor surjective. Its kernel is of particular interest as it captures phenomena which are exclusively symplectic and not visible in the smooth mapping class group. The question of whether a symplectomorphism $\varphi\in\mathrm{Symp}_{\mathrm{ct}}(M)$ is a nontrivial element of the kernel of c (ie is smoothly isotopic to the identity but not symplectically so) is called the *symplectic isotopy problem*.

In dimension two, the kernel of c is always trivial, and the symplectic mapping class group is isomorphic to the smooth mapping class group $\pi_0(\mathrm{Diff}^+_{\mathrm{ct}}(M))$; this is a consequence of *Moser's argument* [1965].

Dehn twists often provide examples of nontrivial symplectomorphisms that lie in the kernel of (1). Given a sphere L (and a choice of parametrisation, called a *framing*; see Definition 2.7), the periodicity of the (co)geodesic flow can be used to construct a compactly supported symplectomorphism of the cotangent bundle $\tau_L\in\mathrm{Symp}_{\mathrm{ct}}(T^*L)$ (see Definition 2.8), called a standard Dehn twist.

The standard Dehn twist has infinite symplectic order, ie infinite order in $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(T^*S^n))$ [Seidel 2000] and, for $n=2$, it generates the entire mapping class group $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(T^*S^2))$ [Seidel 1998].

Given a general symplectic manifold (M,ω) and an embedded Lagrangian sphere $L\subset M$, the local construction of the standard Dehn twist can be implanted in a neighbourhood of L via Weinstein's neighbourhood theorem, to yield a compactly supported symplectomorphism $\tau_L\in\mathrm{Symp}_{\mathrm{ct}}(M)$. When $\dim(L)$ is even, the Dehn twist has finite order in $\mathrm{Diff}^+_{\mathrm{ct}}(M)$ but often has infinite order in $\mathrm{Symp}_{\mathrm{ct}}(M)$. Seidel's early investigations provided the first global examples of (symplectically) nontrivial Dehn twists, in particular nontrivial elements of the kernel of the comparison map (1). For example, for a $K3$–surface (M,ω) containing two disjoint Lagrangian spheres $L_1,L_2\subset M$, the class of τ_{L_1} has infinite order in $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(M))$, and hence in that case c has infinite kernel [Seidel 2000]. Other important examples in which the kernel of c is large include Dehn twists in Milnor fibres of any isolated hypersurface singularity

[Keating 2014] and Dehn twists in projective hypersurfaces of degree $d > 2$ (and more general divisors [Tonkonog 2015]).

One of the widely employed methodologies used in these investigations is symplectic Picard–Lefschetz theory. In this context, Dehn twists are regarded as the class of symplectomorphisms that encode symplectic monodromy maps associated to nodal degenerations, ie monodromies of *Lefschetz fibrations* (see Section 2).

For an exact symplectic manifold (M,ω), any Dehn twist τ_L along a Lagrangian sphere $L \subset (M,\omega)$ can be realised as the local monodromy of an exact Lefschetz fibration (with exact smooth fibre (M,ω) and exact base). One important result that has been proved recently in [Barth et al. 2019] (an alternative proof of which can be found in this paper) is that the global monodromy of such Lefschetz fibrations can never be isotopic to the identity in the symplectic mapping class group.

Theorem A [Barth et al. 2019, Theorem 1.4] *Let (M,ω) be a Liouville manifold, and let $L_1,\dots,L_m \subset M$ be Lagrangian spheres. Let $\phi = \prod_{i=1}^k \tau_{L_{j_i}} \in \mathrm{Symp}_{\mathrm{ct}}(M)$, $j_i \in \{1,\dots,m\}$ be a positive word of Dehn twists. Then ϕ is not compactly supported isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(M)$.*

As a result, Dehn twists represent an extremely important source of (symplectically) nontrivial symplectic automorphisms of exact symplectic manifolds.

In a more general setting, we can consider both positive as well as negative powers of Dehn twists. In this case, the intersection pattern of the Lagrangians generating the twists determines the behaviour of a product of such twists. For example, if L and L' are two Lagrangian spheres of a Liouville manifold (M,ω) which intersect in a single point, then the corresponding twists $\tau_L, \tau_{L'} \in \mathrm{Symp}_{\mathrm{ct}}(M)$ satisfy the *braid relation*, an isotopy $\tau_L\tau_{L'}\tau_L \simeq \tau_{L'}\tau_L\tau_{L'}$ in $\mathrm{Symp}_{\mathrm{ct}}(M)$ [Seidel 1999; Seidel and Thomas 2001]. In a general situation, Keating showed that the suitable quantifier that obstructs the possibility of a nontrivial relation between the twists τ_L and $\tau_{L'}$ is the rank of the Floer cohomology group $\mathrm{HF}(L,L')$, as follows:

Theorem 1.1 [Keating 2014, Theorems 1.1 and 1.2] *Let (M,ω) be a Liouville manifold of dimension greater than 2, and $L, L' \subset M$ be two Lagrangian spheres satisfying $\operatorname{rank}\mathrm{HF}(L,L') \geqslant 2$ and that are such that L and L' are not quasi-isomorphic in the Fukaya category. Then the Dehn twists τ_L and $\tau_{L'}$ generate a free subgroup of $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(M))$, and the associated functors T_L and T_L' generate a free subgroup of $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(M))$.*

Note that the two-dimensional case holds via a result due to Ishida [1996].

Seidel [2000] introduced a class of symplectomorphisms defined from Lagrangian submanifolds with periodic geodesic flow. This type of Lagrangian includes spheres — in which case the symplectomorphisms are squared Dehn twists — and projective spaces. This paper focuses on the latter class of symplectomorphisms, which we call *projective twists* (the appellation *Dehn* will be associated exclusively to Dehn twists along spheres). The complex projective analogues of Dehn twists are *always* contained in the kernel of the comparison map (1) [Seidel 2000, Proposition 4.6], which means that they are a class of symplectomorphisms which are never detectable by the smooth structure. Similarly to Dehn twists, projective twists arise as local monodromies of fibration-like structures [Perutz 2007]; these fibrations are called *Morse–Bott–Lefschetz* fibrations and their singularities are Morse–Bott degenerations.

Unlike their spherical counterparts, projective twists have not been in the spotlight of research in symplectic topology, and this is for a number of reasons. The definition of projective twist requires the existence of a Lagrangian embedding of a projective space in the ambient symplectic manifold, which can result in strong topological restrictions. Moreover, the symplectic Picard–Lefschetz theory of [Seidel 2008a] does not have such immediate applications as for Dehn twists.

Nevertheless, a series of recent results indicates that projective twists do have interesting properties of the calibre of Dehn twists [Evans 2011; Harris 2011; Mak and Wu 2018].

The results of the present research are driven by the following questions, which in the existing literature have been considered for Dehn twists exclusively:

Questions 1 Let (M,ω) be a Liouville manifold.

(a) Can a reduced word of projective twists be symplectically isotopic to the identity (ie are there twists satisfying any nontrivial relations) in $\mathrm{Symp}_{\mathrm{ct}}(M)$?

(b) Can a reduced *positive* word (ie a product of positive powers) of projective twists be symplectically isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(M)$?

1.2 Methods: the Hopf correspondence

How can we study projective twists? Because much of the scholarship that emerged from the study of Dehn twists is the result of successful applications of symplectic Picard–Lefschetz theory, a first intuitive move is to approach the study of projective twists by means of their presentation as monodromies of Morse–Bott–Lefschetz fibrations. One strategy could be to adapt some of the arguments originally tailored for Dehn twists to a more general Picard–Morse–Bott–Lefschetz theory, as developed in [Wehrheim and Woodward 2016]. However, this setting presents serious complications related to a potential loss of compactness of the moduli spaces of pseudoholomorphic curves of these fibrations (the total space of Morse–Bott–Lefschetz fibrations is in general not exact, and the singular locus — a smooth manifold of the singular fibre — often admits rational curves).

To examine the properties of these symplectomorphisms, in this paper we adopt a strategy that allows to reduce the study of projective twists to that of Dehn twists in an auxiliary Liouville manifold; this is made possible via the theoretical device of *Lagrangian correspondences*.

A Lagrangian correspondence between two symplectic manifolds (W,ω) and (Y,Ω) is a Lagrangian submanifold of the product $W^- \times Y := (W \times Y, -\omega \oplus \Omega)$. By [Wehrheim and Woodward 2012; 2010a; 2010b; Gao 2017a; 2017b], under suitable conditions, a Lagrangian correspondence induces a functor which associates a Lagrangian in Y to a Lagrangian in W.

In a first stage, we define an appropriate Lagrangian correspondence that relates a set of Lagrangian projective spaces in a given Liouville manifold (W,ω) to a set of Lagrangian spheres in an auxiliary manifold (Y,Ω) expressly built under some cohomological conditions. Fix a tuple $(\mathbb{A},k,*,R) \in \{(\mathbb{R},0,1,\mathbb{Z}/2\mathbb{Z}),(\mathbb{C},1,2,\mathbb{Z})\}$. Assume there are Lagrangian projective spaces $\mathbb{AP}^n \cong K_1,\dots,K_m \subset W$ and a nontrivial class $\alpha \in H^*(W;R)$ such that $\alpha|_{K_i}$ generates $H^*(\mathbb{AP}^n;R)$. Then there is a Liouville manifold (Y,Ω), realised as a T^*S^k–bundle $q\colon Y \to W$, which contains an S^k–fibred coisotropic submanifold $V \to W$, defining a Lagrangian correspondence $\Gamma := \{(q(y),y) \mid y \in V\} \subset W^- \times Y$ in the sense of [Perutz 2008]. Over each projective Lagrangian $K_i \subset W$, the correspondence yields a Lagrangian sphere $L_i \subset Y$ for $i=1,\dots,m$ (Sections 3.1 and 3.2). We name Γ the *Hopf correspondence*.

Once the Hopf correspondence is constructed, we use Ma'u–Wehrheim–Woodward theory and Gao's extension for nonclosed correspondences to show that there is an induced functor $\Theta_\Gamma\colon \mathcal{F}\mathrm{uk}(W) \to \mathcal{F}\mathrm{uk}(Y)$ between the compact Fukaya categories (see Section 4.2). We then prove the existence of a commuting diagram (Section 4.4)

$$\begin{array}{ccc} \mathcal{F}\mathrm{uk}(Y) & \xrightarrow{\;T_{L_i}\;} & \mathcal{F}\mathrm{uk}(Y) \\ {\scriptstyle\Theta_\Gamma}\uparrow & & \uparrow{\scriptstyle\Theta_\Gamma} \\ \mathcal{F}\mathrm{uk}(W) & \xrightarrow{\;T_{K_i}\;} & \mathcal{F}\mathrm{uk}(W) \end{array} \tag{2}$$

where $T_{K_i} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ and $T_{L_i} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ are the twist functors induced by the graphs of the respective twists $\tau_{K_i} \in \mathrm{Symp}_{\mathrm{ct}}(W)$ and $\tau_{L_i} \in \mathrm{Symp}_{\mathrm{ct}}(Y)$.

1.3 Results

1.3.1 Free groups generated by projective twists In Section 5 we consider Question 1(b) and give a first answer to it. We consider a *clean plumbing* (see Definition 5.1) of Lagrangian projective spaces: a symplectic construction in which two copies of cotangent bundles $T^*\mathbb{AP}^n$ are glued along a common submanifold of the zero sections, and prove the following:

Theorem B *Let $W := T^*\mathbb{AP}^n \#_{\mathbb{AP}^l} T^*\mathbb{AP}^n$ be a clean plumbing of (real, complex) projective spaces along a linearly embedded subprojective space $\mathbb{AP}^l \subset W$, $\mathbb{A} \in \{\mathbb{R},\mathbb{C}\}$. Let $K_1, K_2 \cong \mathbb{AP}^n \subset W$ denote the Lagrangian core components of the plumbing. Then the projective twists τ_{K_1} and τ_{K_2} generate a free group inside $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(W))$, and the associated functors T_{K_1} and T_{K_2} generate a free subgroup of* $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$.

In the complex case, this theorem yields a new criterion for projective twists to generate a free subgroup of the kernel of the comparison map (1).

We prove Theorem B using the Hopf correspondence to relate the functors $T_{K_1}, T_{K_2} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ to functors $T_{L_1}, T_{L_2} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ induced by Dehn twists in a Liouville manifold (Y,Ω) constructed as a T^*S^k–bundle over W for $k \in \{0,1\}$. This is made possible via the commuting diagram (2).

We can then apply Keating's result (Theorem 1.1) to our setting to obtain a free generation result for T_{L_1} and T_{L_2}, which we translate into a free generation result for T_{K_1} and T_{K_2} via the Hopf correspondence.

Remark 1.2 The case $W := T^*\mathbb{CP}^1_1 \#_{\mathrm{pt}} T^*\mathbb{CP}^1_2$ can be obtained with the current literature [Seidel 1999; Seidel and Thomas 2001; Khovanov and Seidel 2002], by considering W as an A_2–configuration and using the isotopies $\tau_{\mathbb{CP}^1_i} \simeq \tau^2_{S^2_i}$ (see Remark 5.2). ◁

1.3.2 Positive products of twists in Liouville manifolds In Section 6.3, we restrict our attention to products of positive powers of twists, ie Question 1(b). In a first instance, we analyse this question for Dehn twists, and we present an alternative proof of Theorem A, which was originally proved (by Barth, Geiges and Zehmisch [Barth et al. 2019]) via techniques involving open book decompositions. Our proof is implemented using Picard–Lefschetz theory. The idea is to build a Lefschetz fibration $\pi\colon E \to \mathbb{C}$ with smooth fibre the Liouville manifold (M, ω) and vanishing cycles the given Lagrangian spheres involved in the product $\phi \in \mathrm{Symp}_{\mathrm{ct}}(M)$. In that way, the monodromy of π is given by ϕ. Assuming that there exists an isotopy $\phi \simeq \mathrm{Id}$ as in the statement, we extend π to a fibration over $\mathbb{CP}^1$, and, by analysing the moduli space of pseudoholomorphic sections (following [Seidel 2003]), obtain a contradictory statement.

It can happen that a product of Dehn twists, despite being necessarily not (compactly supported symplectically) isotopic to the identity, preserves some Lagrangian submanifolds of M. The question arises whether one can find a Lagrangian $T \subset M$ such that there can be no compactly supported symplectic isotopy $\phi(T) \simeq T$. The existence of such a Lagrangian would result in a stronger version of Theorem A. In Section 6.2, we address this question. We find one possible candidate Lagrangian $T \subset M$ with the above properties, but unfortunately cannot prove that such a Lagrangian always exist.

A Lagrangian $T \subset M$ is called conical if it is an exact, properly embedded Lagrangian that is preserved by the Liouville flow over the cylindrical ends of M.

Theorem C *Let (M^{2n}, ω) be a Liouville manifold containing embedded Lagrangian spheres $L_1, \dots, L_m$ and a conical Lagrangian disc T intersecting one of the spheres L_j transversely in a point. Let $\phi := \prod_{i=1}^k \tau_{L_{j_i}} \in \mathrm{Symp}_{\mathrm{ct}}(M)$ with $j_i \in \{1, \dots, m\}$ be a positive word of Dehn twists involving τ_{L_j}. Then the Lagrangians T and $\phi(T)$ are not isotopic via a compactly supported Lagrangian isotopy.*

Example 1.3 For $m > 0$, consider an iterated transverse plumbing

$$M := T^*S^m \#_{\mathrm{pt}} T^*S^m \#_{\mathrm{pt}} T^*S^m \#_{\mathrm{pt}} \cdots \#_{\mathrm{pt}} T^*S^m$$

(see Section 5.1 for the definition of plumbing). Let $\phi \in \mathrm{Symp}_{\mathrm{ct}}(M)$ be a product of Dehn twists along the Lagrangian spheres of M such that ϕ contains the Dehn twist along the j^{th} sphere. In this case, there is at least one conical Lagrangian disc $T \subset M$ as in Theorem C; any cotangent fibre of the j^{th} summand will do. ◁

The arguments we use in the proof of Theorem C are centred around the same principles as the method used for Theorem A, with some necessary adjustments due to the noncompactness of the Lagrangian $T \subset M$.

At last, in Section 6.3, we turn to applications related to projective twists. Using the Hopf correspondence, we prove a result that can be considered the (real) projective counterpart to Theorem A.

Theorem D *Let (W^{2n}, ω) be a Liouville manifold containing Lagrangian real projective spaces $K_1, \dots, K_m$ with $K_i \cong \mathbb{RP}^n$. Suppose that there is a class $\alpha \in H^1(W; \mathbb{Z}/2\mathbb{Z})$ such that, for every $i = 1, \dots, m$, $\alpha|_{K_i}$ generates $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$. Let $\varphi \in \mathrm{Symp}_{\mathrm{ct}}(W)$ be a positive word in the subset of projective twists $\{\tau_{K_i}\}_{i \in \{1,\dots,m\}}$. Then φ is not isotopic to the identity in $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(W))$.*

Using the cohomological assumption of the theorem, we establish the Hopf correspondence and prove the theorem by contradiction. The idea is that, in these circumstances, there exists a product of Dehn twists $\phi \in \mathrm{Symp}_{\mathrm{ct}}(\widetilde{W})$ in the symplectic double cover $q\colon (\widetilde{W}, \widetilde{\omega}) \to (W, \omega)$ such that $q \circ \phi = \varphi \circ q$. Then an isotopy $\varphi \simeq \mathrm{Id}$ in $\mathrm{Symp}_{\mathrm{ct}}(W)$ can be lifted to an isotopy $\phi \simeq \mathrm{Id}$ in $\mathrm{Symp}_{\mathrm{ct}}(\widetilde{W})$, contradicting Theorem A.

Unfortunately, the same techniques do not yield a result for complex projective twists; in that case, the auxiliary manifold (Y, Ω) defines a $\mathbb{C}^*$–bundle $q\colon Y \to W$ and a compactly supported isotopy on W does not lift to a compactly supported isotopy on Y, so that the arguments used for Theorem D do not apply here.

1.3.3 Framing of projective twists and Lagrangian embeddings of homotopy projective spaces

The last section uses the Hopf correspondence to examine the ways in which the symplectic structure interferes with the underlying topological structures, such as diffeomorphism and homeomorphism class, of Lagrangian homotopy projective spaces. In this paper, a manifold that is homeomorphic but not diffeomorphic to a standard (real or complex) projective space is called an AD projective space. A manifold that is homotopy equivalent but not homeomorphic to a standard (real or complex) projective space is called an AT projective space. Similarly, an AD sphere is a sphere that is homeomorphic but not diffeomorphic to the standard sphere (we decide to drop the usual epithet *exotic*; see Definition 7.6).

A notorious conjecture, the *nearby Lagrangian conjecture*, states that, given a closed smooth manifold Q, any closed exact Lagrangian submanifold of $(T^*Q, d\lambda_Q)$ is Hamiltonian isotopic to the zero section. This conjecture has generated a lot of interest in the symplectic community, but its statement is currently confirmed only up to simple homotopy equivalence [Abouzaid 2012b; Kragh 2013; Abouzaid and Kragh 2018]; in Section 7.1 we summarise the state of the art of this conjecture. For a homotopy sphere L, it is known that the choice of smooth structure can be an obstruction to the existence of a Lagrangian embedding $L \hookrightarrow T^*S^n$. Namely, for $n > 4$ odd, AD spheres which do not bound parallelisable manifolds admit no Lagrangian embedding into T^*S^n [Abouzaid 2012a; Ekholm et al. 2016].

Using the existing literature about S^1–actions on AD spheres [Bredon 1967; James 1980; Kasilingam 2016], we find, in Section 7.1, examples of nonstandard homotopy complex projective spaces which do not admit Lagrangian embeddings into $T^*\mathbb{CP}^n$. These results are compatible with the predictions derived from the nearby Lagrangian conjecture.

Theorem E *There is a manifold P homotopy equivalent to $\mathbb{CP}^4$ and with the same first Pontryagin class such that neither P nor $P \# \Sigma^8$ admits an exact Lagrangian embedding into $T^*\mathbb{CP}^4$.*

Theorem F *There is an element Σ^{14} in the group of homotopy 14–spheres Θ_{14} such that $\mathbb{CP}^7 \# \Sigma^{14}$ does not admit an exact Lagrangian embedding into $T^*\mathbb{CP}^7$.*

On the other hand, in Section 7.2, we present new results which prove that, in general, the Hamiltonian isotopy class of projective twists does depend on a choice of framing, ie a choice of smooth parametrisation $f : \mathbb{CP}^n \to L$ (see Definition 2.7). It was proved by Dimitroglou Rizell and Evans [2015] that a nonstandard parametrisation $S^n \to L$ of a Lagrangian sphere can give rise to a Dehn twist that is not isotopic to the standard Dehn twist τ_{S^n}.

We use classical homotopy theory and the Hopf correspondence to transpose the existence of nonstandard parametrisations of Dehn twists of [Dimitroglou Rizell and Evans 2015] into instances of projective twists depending on their framing.

Theorem G *The $\mathbb{CP}^n$–twist depends on the framing when $n = 19, 23, 25, 29$.*

This shows that, in general, $\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$ is not generated by the standard projective twist along the zero section $\tau_{\mathbb{CP}^n}$ (see Corollary 7.26). Moreover, we also note that the use of advanced topological technology (*topological modular forms*) can prove the existence of infinitely many nonstandardly framed (complex) projective twists (Proposition 7.24).

Organisation of the paper

The rest of the paper is organised as follows.

Sections 3 and 4 are the two theoretical cores that support the arguments throughout the paper. After recalling the principal properties of twists in Section 2, in Section 3 we prove commutative diagrams involving Dehn twists, the Hopf map and projective twists in the geometric setting. In Section 4, we define the Hopf correspondence and its applications for diagrams of functors of the Fukaya category induced by Dehn/projective twists.

The central body of the paper is divided in three parts, in which we apply the methods developed. We prove a free group generation criterion for projective twists in plumbings in Section 5, we study positive products of twists in general Liouville manifolds in Section 6, and we study framings of projective twists as well as Lagrangian embeddings of homotopy projective spaces in Section 7.

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This is my final contribution to a mathematical practice held captive by the institution — as a monolithic body of knowledge and a machine producing universal truths. My journey has led me to honour relational, living, and life-affirming mathematical praxes that exist beyond Cartesian binaries and the necropolitics of technoscience.

May we remember that mathematics is a verb, conjugated plurally.

2 Twists

This section provides the contextualisation necessary for studying Dehn twists and projective twists in symplectic topology; it can be skipped by the expert reader. We summarise the constructions of twists from a geodesic flow perspective (Section 2.1), and as local monodromies of fibrations (Section 2.2).

2.1 Twists from geodesic flow

In this section we recall the definitions of Dehn and projective twists that employ the periodicity of the geodesic flow of spheres and projective spaces (the main references are [Seidel 2003, Section 1.2; 2000, Section 4.b; Mak and Wu 2018, Section 2.1]). Let (K, g) be a closed connected Riemannian manifold admitting a periodic cogeodesic flow $\Phi^t_K : T^*K \to T^*K$ on its cotangent bundle $(T^*K, d\lambda_{T^*K})$ such that each geodesic of length 2π is closed (so that the shortest period of a unit-speed geodesic is 2π).

Let $\|\cdot\|_K$ be the norm associated to the given Riemannian metric g. The normalised cogeodesic flow satisfies $\Phi^{2\pi}_K = \mathrm{Id}$ and can be extended to a Hamiltonian S^1–action σ^H_t on $T^*K \setminus K$, with moment map $H : T^*K \setminus K \to \mathbb{R}$, $H(v) = \|v\|_K$.

Definition 2.1 Let K be diffeomorphic to S^n. For $\varepsilon > 0$, define an auxiliary smooth cut-off function $r_\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ such that $0 < r_\varepsilon(t) < \pi$ for all $t < \varepsilon$ and

$$r_\varepsilon(t) = \begin{cases} \pi - t & \text{if } t \ll \varepsilon, \\ 0 & \text{if } t \geqslant \varepsilon. \end{cases} \tag{3}$$

The model Dehn twist $\tau^{\mathrm{loc}}_K : T^*K \to T^*K$ is defined as

$$\tau^{\mathrm{loc}}_K(\xi) = \begin{cases} \sigma^H_{r_\varepsilon(\|\xi\|_K)}(\xi) & \text{if } \xi \notin K, \\ -\xi & \text{if } \xi \in K. \end{cases} \tag{4}$$

◁

Definition 2.2 Let K be diffeomorphic to $\mathbb{AP}^n$ for $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. For $\varepsilon > 0$, let $r_\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ be a smooth cut-off function such that $0 < r_\varepsilon(t) < 2\pi$ for all $t < \varepsilon$ and

$$r_\varepsilon(t) = \begin{cases} 2\pi - t & \text{if } t \ll \varepsilon, \\ 0 & \text{if } t \geqslant \varepsilon. \end{cases} \tag{5}$$

The model projective twist $\tau^{\mathrm{loc}}_K : T^*K \to T^*K$ is defined as

$$\tau^{\mathrm{loc}}_K(\xi) = \begin{cases} \sigma^H_{r_\varepsilon(\|\xi\|_K)}(\xi) & \text{if } \xi \notin K, \\ \xi & \text{if } \xi \in K. \end{cases} \tag{6}$$

◁

Remark 2.3 Our choice of cut-off functions r_ε follows [Mak and Wu 2018, Section 2.1], but the construction is independent of such choices, up to suitable isotopy [Seidel 2000]. ◁

Theorem 2.4 [Seidel 2000, Corollary 4.5] *Let (K, g) be a Riemannian manifold admitting a periodic (co)geodesic flow and satisfying $H^1(K; \mathbb{R}) = 0$. Then the symplectomorphisms τ^{loc}_K have infinite order in $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(T^*K))$.*

Theorem 2.5 [Seidel 2000, Proposition 4.6] *The symplectomorphism $\tau^{\mathrm{loc}}_{\mathbb{CP}^n}$ of Definition 2.1 is isotopic to the identity in $\mathrm{Diff}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$.*

Remark 2.6 We will often denote the standard twists by $\tau_{S^n} := \tau^{\mathrm{loc}}_{S^n}$ or, for $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $\tau_{\mathbb{AP}^n} := \tau^{\mathrm{loc}}_{\mathbb{AP}^n}$. With the conventions (3) and (5), the isomorphisms $S^1 \cong \mathbb{RP}^1$, $S^2 \cong \mathbb{CP}^1$ and $S^4 \cong \mathbb{HP}^1$ induce isotopies $\tau^2_{S^1} \simeq \tau_{\mathbb{RP}^1}$, $\tau^2_{S^2} \simeq \tau_{\mathbb{CP}^1}$ and $\tau^2_{S^4} \simeq \tau_{\mathbb{HP}^1}$, respectively (see [Seidel 2000; Harris 2011]). ◁

Now suppose (L, g) is a Riemannian manifold admitting a Lagrangian embedding $L \subset M$ into a general symplectic manifold (M, ω).

Definition 2.7 Let $K \in \{S^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n\}$. A *framed Lagrangian* sphere/projective space is a Lagrangian submanifold $L \subset M$ together with an equivalence class $[f]$ of diffeomorphisms $f : K \to L$, where $f_1 \sim f_2$ if and only if $f_2^{-1} f_1$ is isotopic, in $\mathrm{Diff}(K)$, to an element of the isometry group $\mathrm{Iso}(K, g)$. An equivalence class $[f]$ as above is called a *framing*. ◁

Definition 2.8 Let $(L, [f])$ be a framed Lagrangian sphere/projective space in (M, ω). Using Weinstein's neighbourhood theorem, extend a framing representative $f : K \to L$ to a symplectic embedding $\iota\colon D_s T^*K \to M$, where $D_s T^*K := \{v \in T^*K \mid \|v\|_K < s\}$ for $s > 0$. There is a model twist τ_K^{loc}, supported in the interior of $D_s T^*K$, and we define

$$\tau_L \cong \begin{cases} \iota \circ \tau_K^{\mathrm{loc}} \circ \iota^{-1} & \text{on } \mathrm{Im}(\iota), \\ \mathrm{Id} & \text{elsewhere.} \end{cases}$$

In the case where L is a sphere, the map τ_L is the well-known *Dehn* twist. When L is a projective space, the resulting map is called a *projective twist*. In this paper, the term *Dehn* twist is exclusively reserved for twists that are constructed from a Lagrangian sphere. ◁

Remark 2.9 (1) A Dehn twist along an exact Lagrangian sphere, or a projective twist along an exact projective Lagrangian in an exact symplectic manifold, is an exact symplectomorphism in the sense of Definition 2.11. The same holds for products of such twists. This follows by construction (for direct computations, see for example [Barth et al. 2019, Lemma 4.4; Chiang et al. 2016, Lemma 2.1]).

(2) Theorem 2.5 implies that, given a symplectic manifold (M, ω), any Lagrangian $L \cong \mathbb{CP}^n \subset M$ will define an element $\tau_L \in \mathrm{Symp}_{\mathrm{ct}}(M)$ that is isotopic to the identity in $\mathrm{Diff}_{\mathrm{ct}}(M)$. ◁

As shown by Dimitroglou Rizell and Evans [2015], the choice of framing does play a role in determining the symplectic isotopy class of a spherical Dehn twist. In Section 7, we prove that this is also the case for projective twists. Before then, any given Lagrangian submanifold involved in the construction of a twist is assumed to be endowed with a choice of framing and we omit mentioning this datum, as the results of this paper, up to the last section, are independent of such choices. This is because the autoequivalence of the Fukaya category induced by a Dehn twist (see Section 2.3) is independent of the choice of framing (as a consequence of the shape of the functor; see [Seidel 2008a, Corollary 17.17]). The same is true for the functor induced by the projective twist [Mak and Wu 2018, Theorem 6.10].

2.2 Twists as monodromies

This section approaches twists from a different perspective, one that presents these symplectomorphisms as monodromy maps of fibration-like structures. Dehn twists occur as (local) monodromies of Lefschetz fibrations, and this is one of the features that has made the study of Dehn twists particularly productive. On the other hand (it is a lesser known fact that) projective twists can be modelled as local monodromies of *Morse–Bott–Lefschetz* fibrations, another class of fibrations admitting more degenerate singularities, which we will not discuss here.

Below we give a brief review of Lefschetz fibrations (mainly following [Seidel 2008a; Maydanskiy and Seidel 2010]) on Liouville manifolds, aimed at setting the notation for future sections, and recall the well-known *Picard–Lefschetz theorem*.

Definition 2.10 A Liouville manifold of finite type is an exact symplectic manifold $(W, \omega = d\lambda_W)$, where $\lambda_W \in \Omega^1(W)$ is called the Liouville form, such that there exists a proper function $h_W : W \to [0,\infty)$ and $c_0 > 0$ with the following property: for all $c \in (c_0, \infty)$ and $x \in h_W^{-1}(c)$, the vector field Z_W dual to λ_W, called the Liouville vector field, satisfies $dh_W(Z_W)(x) > 0$.

For a regular value c of h_W, a closed sublevel set $M := h_W^{-1}([0,c])$ of a Liouville manifold $(W, d\lambda_W)$ is a compact symplectic manifold with contact type boundary $(\Sigma := h_W^{-1}(c), \lambda_W|_\Sigma)$, and it is called a Liouville domain. ◁

Definition 2.11 An exact symplectomorphism between two Liouville manifolds $(W_1, d\lambda_1)$ and $(W_2, d\lambda_2)$ is a diffeomorphism $\psi\colon W_1 \to W_2$ satisfying $\psi^*\lambda_2 - \lambda_1 = df$ for a compactly supported function $f : W_1 \to \mathbb{R}$. ◁

Definition 2.12 Now let $(M, d\lambda)$ be a Liouville domain with contact boundary $(\Sigma = \partial M, \alpha = \lambda|_\Sigma)$. The negative Liouville flow identifies a collar neighbourhood $C(\Sigma)$ of the boundary with $(-\varepsilon, 0] \times \partial M$, so that $\lambda|_{C(\Sigma)} = e^t\alpha$. An almost complex structure J of *contact type near the boundary* is one that satisfies $de^t \circ J = -\lambda$. ◁

Definition 2.13 Given a Liouville domain $(M, d\lambda)$ as above, we can use the identification of the collar neighbourhood $C(\Sigma)$ to glue an infinite cone and define the symplectic completion of M,

$$(W, \omega_W) := \big(M \cup [0,\infty) \times \partial M, d(e^t\alpha)\big), \tag{7}$$

where t is the coordinate on $(0,\infty)$, such that the Liouville flow extends to Z_W with $Z_W|_{[0,\infty)\times\partial M} = \partial_t$.

An almost complex structure J of contact type extends to an almost complex structure J_W on the completion satisfying:

- $J_W(\partial/\partial t) = R_\alpha$, where R_α is the Reeb vector field associated to α.
- J_W is invariant under translations in the t–direction.
- $J_W|_M = J$.

This kind of almost complex structure will be called cylindrical. ◁

We will only consider Liouville manifolds that are complete (ie with complete Liouville vector field) and of finite type, which we can identify as the union of a Liouville domain with a cylindrical noncompact end, equipped with an almost complex structure cylindrical at infinity.

Let $(E^{2n+2}, \Omega_E, \lambda_E)$ be a Liouville manifold, with a compatible almost complex structure J_E, and consider the complex plane with its standard symplectic form and complex structure $j_{\mathbb{C}}$. Let $\pi\colon E \to \mathbb{C}$ be a map with finitely many critical points, which are all nondegenerate, and contained in a compact set of E. Denote by $\mathrm{Crit}(\pi) := \{x \in E \mid D_x\pi = 0\}$ the set of critical points, and by $\mathrm{Crit}\,v(\pi) := \pi(\mathrm{Crit}(\pi))$ the set of critical values.

Definition 2.14 A Lefschetz fibration on (the Liouville manifold) E is a $(J_E, j_{\mathbb{C}})$–holomorphic map π, ie $D\pi \circ J_E = j_{\mathbb{C}} \circ D\pi$, with the above properties and the following additional features:

(1) For all $x \in E \setminus \mathrm{Crit}(\pi)$, $\ker(D_x\pi) \subset T_xE$ is symplectic.

(2) Every smooth fibre is symplectomorphic to the completion of a Liouville domain $(M, d\lambda_M)$.

(3) There is an open neighbourhood $U^h \subset E$ such that $\pi\colon E \setminus U^h \to \mathbb{C}$ is proper and $\pi|_{U^h}$ can be trivialised via an isomorphism $f\colon U^h \cong \mathbb{C} \times ([0,\infty) \times \partial M)$ such that

$$f^*(\lambda_E) = \lambda_{\mathbb{C}} + e^t \lambda_M. \tag{8}$$ ◁

For more details about how this fibration is modelled outside of a neighbourhood of the critical points, see [Maydanskiy and Seidel 2010, (2.1)].

By the first point above, there is a symplectic splitting

$$T_xE = \ker(D_x\pi) \oplus T_xE^h, \tag{9}$$

where T_xE^h is the symplectic complement of $\ker(D_x\pi)$ with respect to Ω_E. The decomposition in (9) defines a canonical connection over $\mathbb{C} \setminus \mathrm{Crit}\,v(\pi)$. By the triviality condition (3), for every path $\gamma\colon [0,1] \to D \setminus \mathrm{Crit}(\pi)$, there are well-defined parallel transport maps $h_\gamma\colon E_{\gamma(0)} \to E_{\gamma(1)}$ which yield symplectomorphisms between smooth fibres.

Definition 2.15 A pair $(J_E, j_{\mathbb{C}})$ is said to be *compatible* with π if the following holds:

- $D\pi \circ J_E = j_{\mathbb{C}} \circ D\pi$.
- There is a local Kähler structure J_0 such that $J_E = J_0$ in a neighbourhood of $\mathrm{Crit}(\pi)$.
- On the neighbourhood U^h, J_E is a product, $f^*(J_E) = (j_{\mathbb{C}}, J^{vv})$, where J^{vv} is a cylindrical almost complex structure compatible with $d(e^t\lambda_M)$.
- $\Omega_E(\cdot, J_E\cdot)$ is symmetric and positive definite. ◁

Remark 2.16 This choice of almost complex structure is not generic. However, the space of compatible almost complex structures on the total space of an exact Lefschetz fibration is contractible [Seidel 2003, Section 2.1], and the moduli spaces we will consider still meet the usual regularity requirements [Seidel 2003, Section 2.2]. ◁

For a Lefschetz fibration on a Liouville manifold (E, Ω_E), the proper fibration obtained as $E \setminus U^h \to \mathbb{C}$ for an open neighbourhood $U^h \subset E$ as above carries the same symplectic information as π with the difference that its fibres are Liouville domains, and as a result the total space admits a nontrivial *horizontal boundary*, given by the union of the boundaries of all fibres.

In most of the paper we will employ this latter type of Lefschetz fibration (for notational simplicity), and, unless specified, an *exact* Lefschetz fibration will denote a fibration obtained in this way.

Now let $\pi\colon E \to \mathbb{C}$ be an exact Lefschetz fibration, with smooth fibre given by the Liouville domain $(M, d\lambda)$. By the triviality assumption of Definition 2.14, there is a neighbourhood of $U^\partial \subset E$ of the horizontal boundary $\partial^h E$ that is isomorphic to an open neighbourhood of the trivial bundle $\mathbb{C} \times \partial M$,

$$U^\partial \cong \mathbb{C} \times M^{\text{out}} \subset \mathbb{C} \times M, \tag{10}$$

where $M^{\text{out}} \subset M$ is a collar neighbourhood of ∂M. The isomorphism is compatible with the Liouville forms and the almost complex structures.

Let $\pi\colon E \to \mathbb{C}$ be a Lefschetz fibration with exact compact fibre (M, ω) and distinct critical values $\operatorname{Crit} v(\pi) = \{w_0, \dots, w_m\} \subset D_R$, where $D_R \subset \mathbb{C}$ is a disc of radius R. Fix a basepoint $z_* \in \mathbb{R}$ such that $z_* \gg R$, and an identification $\pi^{-1}(z_*) \cong M$. In what follows we will frequently use the fact that, via parallel transport, any fibre $\pi^{-1}(z)$ for $z \in \mathbb{C}$ with $\operatorname{Re}(z) > R$ can be symplectically identified with the smooth fixed fibre M via parallel transport.

Definition 2.17 (1) A vanishing path associated to a critical value $w_i \in \operatorname{Crit} v(\pi)$ is a properly embedded path $\gamma_i\colon \mathbb{R}^+ \to \mathbb{C}$ with $\gamma_i^{-1}(\operatorname{Crit} v(\pi)) = \{0\}$, $\gamma_i(0) = w_i$ and $\lim_{t\to\infty} \operatorname{Re}(\gamma(t)) = \infty$ such that, outside of a compact set containing the critical values, the image of γ_i is a horizontal half ray at height $a_i \in \mathbb{R}$:

$$\exists T > 0 \;\; \forall t > T \quad \operatorname{Re}(\gamma_i(t)) > R \;\text{ and }\; \operatorname{Im}(\gamma_i(t)) = a_i. \tag{11}$$

(2) A distinguished basis of vanishing paths for π is a collection of $m+1$ disjoint paths $(\gamma_0, \dots, \gamma_m) \subset \mathbb{C}$ defined as above, with pairwise distinct heights satisfying $a_0 < a_1 < \dots < a_m$.

(3) The corresponding basis of Lefschetz thimbles is the unique set of Lagrangian discs $(\Delta_{\gamma_0}, \dots, \Delta_{\gamma_m})$ in E, where Δ_{γ_i} is defined as the set of points which under the limit $t \to 0$ of the parallel transport maps over γ_i are mapped to the critical point in $\pi^1(w_i)$ (the proof of uniqueness can be found in [Seidel 2008a, (16b)]). Given a general Lefschetz thimble $\mathcal{L}$, define its height $a(\mathcal{L})$ as the value defined in (11). For a pair of thimbles $(\mathcal{L}_0, \mathcal{L}_1)$, set $\mathcal{L}_0 > \mathcal{L}_1$ if $a(\mathcal{L}_0) > a(\mathcal{L}_1)$.

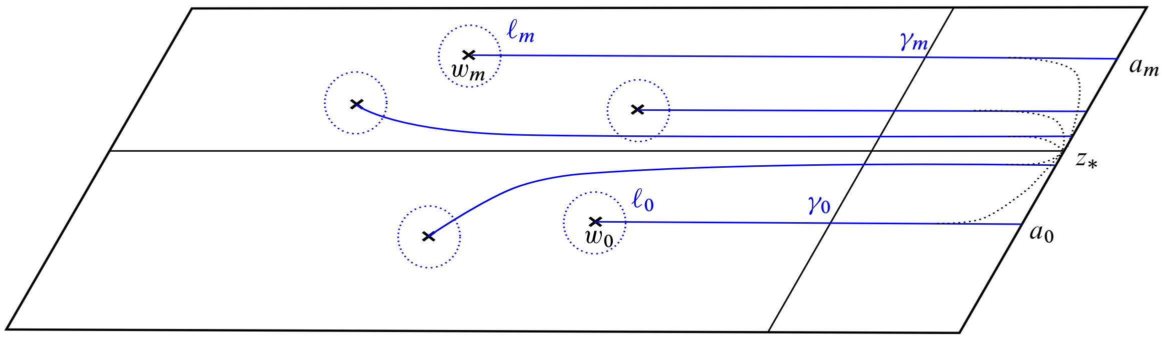

Figure 1: A distinguished basis of vanishing paths $(\gamma_0, \dots, \gamma_m)$.

(4) There is an associated basis of vanishing cycles $(V_0, \dots, V_m)$ where, for all $i = 0, \dots, m$,

$$V_i = \partial \Delta_{\gamma_i} = \Delta_{\gamma_i} \cap M \subset M$$

(using the above identification for smooth fibres). Every vanishing cycle $V_i \subset M$ is an exact Lagrangian sphere which comes with an equivalence class in of diffeomorphisms $S^n \to V_i$ defined up to the action of $\mathrm{O}(n+1)$ (called a *framing*). This is induced by the restriction of a diffeomorphism $D^{n+1} \to \Delta_i$ (which is canonical; see [Seidel 2003, Lemma 1.14]). ◁

Definition 2.18 The global monodromy is the symplectomorphism $\phi \in \mathrm{Symp}_{\mathrm{ct}}(M)$ whose Hamiltonian isotopy class is defined by the anticlockwise parallel transport map around a loop through the basepoint z_* encircling all the critical values of the fibration. (Typically, this loop is defined as the smoothing of the concatenation of the loops centred at z_* going around a single critical value as in Figure 1.) ◁

The symplectic Picard–Lefschetz theorem [Arnold 1995] states that the global monodromy ϕ is isotopic to the product of the Dehn twists along the vanishing cycles $(V_0, \dots, V_m)$,

$$\phi \simeq \tau_{V_0} \cdots \tau_{V_m} \in \mathrm{Symp}_{\mathrm{ct}}(M), \tag{12}$$

and the Hamiltonian isotopy class is independent of the choice of basis of vanishing paths.

On the other hand, given the data $\{(M, \omega), (V_0, \dots, V_m)\}$, there is an exact Lefschetz fibration $\pi : E \to \mathbb{C}$ with fibre (M, ω), and vanishing cycles $(V_0, \dots, V_m) \subset M$, unique up to exact symplectomorphism [Seidel 2008a, (16e)].

Remark 2.19 Lefschetz fibrations can be viewed as a special case of *Morse–Bott–Lefschetz* (abbreviated MBL) fibrations, a class of fibrations which allows nonisolated singularities. The monodromies of such fibrations are symplectomorphisms called *fibred twists* [Perutz 2007], which naturally generalise Dehn twists. Projective twists are a special type of fibred twists, and therefore also admit a presentation as local monodromies. However, in this paper, we won't study projective twists from this perspective. ◁

2.3 Functor twists

This section only contains the notation (and the general notions involved) that we will use for the functors of the Fukaya category that are induced by twists.

Let (M, ω) be a Liouville manifold and let k be a field of characteristic 2. Given two closed exact Lagrangian submanifolds $L_0, L_1 \subset M$ the Floer complex is freely generated as a vector space by the intersection points of the (perturbed) Lagrangians $\mathrm{CF}(L_0, L_1; k) := \bigoplus_{x \in L_0 \cap L_1} k\langle x\rangle$. The boundary operator $\partial\colon \mathrm{CF}(L_0, L_1; k) \to \mathrm{CF}(L_0, L_1; k)$ counts J_M–holomorphic strips with boundary conditions on (L_0, L_1) and asymptotic conditions on intersection points. For a compatible cylindrical almost complex structure J_M, the moduli spaces of such curves are compact oriented manifolds [Seidel 2008a, Sections 8–9] and the operator ∂ squares to zero [Seidel 2008a, (9e)], so that $(\mathrm{CF}(L_0, L_1; k), \partial)$ is a well-defined cochain complex whose cohomology is the Floer cohomology ring $\mathrm{HF}(L_0, L_1; k)$. Floer cohomology is designed to be invariant under Hamiltonian isotopies; if ϕ is the flow of a Hamiltonian vector field, then $\mathrm{HF}(L_0, \phi(L_1)) \cong \mathrm{HF}(L_0, L_1)$.

Very simply put, the compact Fukaya category, $\mathcal{F}\mathrm{uk}(M)$, is an A_∞–category whose objects are closed exact Lagrangian *branes*, which are Lagrangian submanifolds with additional algebraic data, and morphisms the Floer cochain groups between transversely intersecting Lagrangians [Seidel 2008a, (9j) and (12g)]. This category encodes intersection data associated to all its objects, including the Floer differential $\partial = \mu^1$, the Floer cup product μ^2 and higher-order products μ^k (see eg [Seidel 2008a, (9j) and (12g)]). It is well defined for any Liouville manifold (see [Seidel 2008a]).

Two Lagrangians that are Hamiltonian isotopic are quasi-isomorphic objects in the Fukaya category, which means they are isomorphic objects of the associated cohomological category, which we denote by $H(\mathcal{F}\mathrm{uk}(M))$. We denote the automorphisms of $H(\mathcal{F}\mathrm{uk}(M))$ (ie the automorphisms of the Fukaya category up to quasi-isomorphism) by $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(M))$.

Let $\mathrm{Tw}(\mathcal{F}\mathrm{uk}(M))$ be the category of twisted complexes in $\mathcal{F}\mathrm{uk}(M)$ (see [Seidel 2008a, (3k)]), and $D^\flat\,\mathcal{F}\mathrm{uk}(M) := H(\mathrm{Tw}\,\mathcal{F}\mathrm{uk}(M))$ the cohomology category of $\mathrm{Tw}(\mathcal{F}\mathrm{uk}(M))$.

There is a map

$$\Phi\colon \mathrm{Symp}_{\mathrm{ct}}(M) \to \mathrm{Auteq}(D^\flat\,\mathcal{F}\mathrm{uk}(M)) \tag{13}$$

to the group of auto-equivalences of the Fukaya category (modulo quasi-isomorphism) such that, given $\phi \in \mathrm{Symp}_{\mathrm{ct}}(M)$, $\Phi(\phi)$ sends a Lagrangian $L \subset M$ to another Lagrangian $\phi(L) \subset M$ (we avoid discussing a graded situation in this context). The map factors through the quotient by the subgroup $\mathrm{Ham}_{\mathrm{ct}}(M) \subset \mathrm{Symp}_{\mathrm{ct}}(M)$ of compactly supported Hamiltonian diffeomorphisms, so, given an exact Lagrangian sphere/projective space L and its associated twist τ_L, $\Phi(\tau_L)$ defines a well-defined element of $\mathrm{Aut}(D^\flat\,\mathcal{F}\mathrm{uk}(M))$, which we denote by T_L.

Seidel [2003] showed that, for a Dehn twist τ_L, the induced functor $T_L \in \mathrm{Aut}(D^\flat \mathcal{F}\mathrm{uk}(M))$ fits into an exact triangle (see [Seidel 2008a, (17j)]).

Recently, there have been generalisations of Seidel's triangle for a wider class of symplectomorphisms, achieved through a range of different techniques. Wehrheim and Woodward [2016] proved the existence of an exact triangle for fibred twists using quilt theory adapted to Morse–Bott Lefschetz fibrations.

Mak and Wu [2018] treated the case of projective twists, using Lagrangian cobordism theory as developed in [Biran and Cornea 2013; 2014]. They proved that the autoequivalence induced by a (real, complex, quaternionic) projective twist is isomorphic to a double cone of functors in $\mathrm{Aut}(\mathrm{Tw}\,\mathcal{F}\mathrm{uk}(M))$ [Mak and Wu 2018, Theorem 6.10].

Under the appropriate circumstances, the mirror symmetry conjecture gives conjectural descriptions of such functors. If a symplectic manifold (M, ω) has a mirror complex manifold (X, J), there are autoequivalences of the Fukaya category of M that are induced by autoequivalences of the derived category of coherent sheaves of X (we call such autoequivalences *algebraic twist functors*, and will only refer to them in Remark 4.15).

3 Commuting diagrams of twists

In this section we introduce the geometric ideas underpinning the philosophy of the Hopf correspondence. We prove a criterion for relating projective twists in a Liouville manifold (W, ω) to Dehn twists in another Liouville manifold (Y, Ω).

3.1 Complex projective Lagrangians

We begin by considering Lagrangian complex projective spaces.

Fix the round metric on S^{2n+1}, with norm $\|\cdot\|_S$, and consider the free S^1–action on S^{2n+1} by complex multiplication. The orbits of the action are great circles ("Hopf circles"), hence geodesics, and the action is isometric.

Consider the quotient map $h\colon S^{2n+1} \to \mathbb{CP}^n$, which is the (generalised) Hopf fibration. It is a Riemannian submersion that uniquely defines the Fubini–Study metric g_P on $\mathbb{CP}^n$. Identify the tangent bundles with their corresponding cotangent bundles $TS^{2n+1} \cong T^*S^{2n+1}$ and $T\mathbb{CP}^n \cong T^*\mathbb{CP}^n$ via the canonical isomorphism induced by the metrics.

The Hopf action on S^{2n+1} lifts to a Hamiltonian S^1–action on the cotangent bundle $(T^*S^{2n+1}, \omega_{T^*S^n})$ [Guillemin and Sternberg 1990]. Let $\mu\colon T^*S^{2n+1} \to \mathbb{R}$ be the moment map of this action. Assume 0 is a regular value of μ and consider the level set $V := \mu^{-1}(0) \subset T^*S^{2n+1}$, which has the structure of a principal S^1–bundle $p\colon V \to T^*\mathbb{CP}^n$ over the symplectic quotient $T^*S^{2n+1} /\!/ S^1 := V/S^1 \cong T^*\mathbb{CP}^n$.

Lemma 3.1 *Let $\tau_{S^{2n+1}} \in \mathrm{Symp}_{\mathrm{ct}}(T^*S^{2n+1})$ and $\tau_{\mathbb{CP}^n} \in \mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$ be the model Dehn and projective twists, respectively. Let $p\colon V := \mu^{-1}(0) \to T^*\mathbb{CP}^n$ be the symplectic quotient map as above. There is a commuting diagram*

$$\begin{array}{ccc} V & \xrightarrow{\tau_{S^{2n+1}}|_V} & V \\ \downarrow p & & \downarrow p \\ T^*\mathbb{CP}^n & \xrightarrow{\tau_{\mathbb{CP}^n}} & T^*\mathbb{CP}^n \end{array} \tag{14}$$

Proof The Hopf action is isometric, ie for any $g \in S^1$, the induced map $\psi_g \in \mathrm{Diff}(S^{2n+1})$ is an isometry. This implies that the differential maps on the tangent bundles $D_x\psi_g : T_xS^{2n+1} \to T_{\psi_g(x)}S^{2n+1}$ commute (for any $x \in S^{2n+1}$) with the geodesic flow.

The cogeodesic flow Φ^t_H on T^*S^{2n+1} is induced by the Hamiltonian function

$$\widetilde{H} : T^*S^{2n+1} \to \mathbb{R}, \quad (x, \xi) \mapsto \|\xi\|_S. \tag{15}$$

This is S^1–invariant, so there is a Hamiltonian function $H : T^*\mathbb{CP}^n \to \mathbb{R}$ defined on the quotient with respect to the submersion metric g_P, which induces the (co)geodesic flow on $T^*\mathbb{CP}^n$. Since p is induced by a Riemannian submersion, we have the relation $p \circ \Phi^t_{\widetilde{H}}|_V = \Phi^t_H \circ p|_V$, and, for any choice of cut-off function r_ε as in Section 2.1,

$$p \circ \sigma^{\widetilde{H}}_{r_\varepsilon(\|\xi\|_S)}(\xi) = \sigma^H_{r_\varepsilon(\|p(\xi)\|_P)} \circ p(\xi), \quad \xi \in V \subset T^*S^{2n+1}, \tag{16}$$

where σ^H_t and $\sigma^{\widetilde{H}}_t$ are the Hamiltonian S^1–actions induced by H and $\widetilde{H}$, respectively, as in Section 2.1.

Any geodesic connecting a point on S^{2n+1} to its antipode projects, under p, to a closed geodesic of minimal period on $\mathbb{CP}^n$ (it cannot collapse to a point since the Hopf action is isometric), so the definitions of the twists in Section 2.1 imply that $p \circ \tau_{S^{2n+1}}|_V = \tau_{\mathbb{CP}^n} \circ p|_V$. □

We now extend the above discussion to a more global situation; in order to do that it is necessary to set the following assumption:

Assumption (CX) Let (W, ω) be a $4n$–dimensional Liouville manifold with a homology class $\alpha \in H^2(W; \mathbb{Z})$ and Lagrangian complex projective spaces $K_1, \dots, K_m \subset W$ such that

$$\alpha|_{K_i} = x \in H^2(\mathbb{CP}^n; \mathbb{Z}) \quad \text{for all } i,$$

where $x = c_1(\mathcal{O}_{\mathbb{CP}^n}(-1))$ is the generator of the cohomology ring $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}$.

Proposition 3.2 *Let (W, ω) be a $4n$–dimensional Liouville manifold containing embedded Lagrangian complex projective spaces $K_1, \dots, K_m \subset W$. Assume there exists a class $\alpha \in H^2(W; \mathbb{Z})$ satisfying Assumption (CX). Then there is a $(4n+2)$–dimensional Liouville manifold (Y, Ω) with Lagrangian* *spheres $L_1, \dots, L_m \subset Y$, a coisotropic submanifold $V \subset Y$ with the structure of an S^1–fibre bundle $p\colon V \to W$ such that, for each $i \in \{1, \dots, m\}$, $L_i \subset V$, and there is a commuting diagram*

$$\begin{array}{ccc} V & \xrightarrow{\ \tau_{L_i}|_V\ } & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ W & \xrightarrow{\ \tau_{K_i}\ } & W \end{array} \tag{17}$$

The class $\alpha \in H^2(W;\mathbb{Z})$ restricts to a generator $x \in H^2(\mathbb{CP}^n;\mathbb{Z})$ on each Lagrangian K_i, so there is a complex line bundle $\mathcal{L} \to W$ satisfying $c_1(\mathcal{L}) = \alpha$ which is modelled on the tautological line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$ over K_i for $i = 1, \dots, m$. Fix a metric $\|\cdot\|_{\mathcal{L}}$ on $\mathcal{L}$, and for $u \in \mathcal{L}$ define a function $r(u) := \|u\|_{\mathcal{L}}$. Set $V := \{u \in \mathcal{L},\ r(u) = 1\}$. Over K_i, V defines a sphere $L_i := V|_{K_i}$.

Lemma 3.3 *The $\mathbb{C}^*$–bundle associated to $\mathcal{L}$ is a Liouville domain, where the spheres L_i are embedded as Lagrangian submanifolds.*

Proof enote this bundle by $q\colon Y \to W$. Following [Ritter 2014, Section 7.2], we build a symplectic form Ω on Y, making the spheres L_i Lagrangian, and find the appropriate vector field which will be Liouville with respect to Ω.

The metric induces a connection one-form γ^∇ on $\mathcal{L} \setminus 0$ satisfying

$$\gamma^\nabla|_{H_u^\nabla} = 0, \quad \gamma^\nabla|_{T_u^v\mathcal{L}} = \gamma \quad \text{for all } u \in \mathcal{L} \setminus 0, \qquad [d\gamma^\nabla] = -q^*(c_1(\mathcal{L})) = -q^*(\alpha), \tag{18}$$

where $H_u^\nabla\mathcal{L}$ is the horizontal distribution associated to the connection ∇ at u, $T_u^v\mathcal{L}$ the vertical distribution, and γ the fibrewise angular form defined by the metric. Let $\Omega := q^*\omega + d(f(r)\gamma^\nabla)$ for a function $f \in C^\infty(\mathbb{R})$ with

$$f(1) = 0, \qquad f'(r) > 0 \quad \text{for all } r \in \mathbb{R}.$$

Then Ω defines a symplectic form in a neighbourhood of $\{r = 1\}$, and L_i is Lagrangian with respect to Ω. Let λ be the Liouville one-form on W with $d\lambda = \omega$. Define $\lambda_Y := q^*\lambda + f(r)\gamma^\nabla$, so that $d(\lambda_Y) = \Omega$. Then (λ_Y, Ω) defines a Liouville structure near $\{r = 1\}$ (the symplectic dual to λ_Y points outwards along a small neighbourhood of $\{r = 1\}$). Therefore, a symplectic completion along this neighbourhood yields a Liouville manifold that is diffeomorphic to Y, containing the Lagrangian spheres $L_1, \dots, L_m$. □

Proof of Proposition 3.2 Let $\mathcal{L} \to W$ be the complex line bundle we have constructed above with $c_1(\mathcal{L}) = \alpha$. For each Lagrangian projective space $K_i \subset W$, the restriction of the bundle $\mathcal{L}|_{K_i}$ is modelled on the tautological line bundle, which implies that $L_i \to K_i$ is modelled on the Hopf quotient map $h\colon S^{2n+1} \to \mathbb{CP}^n$. The commutativity of (17) follows by the local commuting diagram of cotangent bundles (14). □

Example 3.4 Without Assumption (CX), Proposition 3.2 is in general not true, as the following example illustrates. Consider the manifold W obtained by attaching a 3–handle to the contact boundary of $D_sT^*\mathbb{CP}^2$ ($s > 0$) such that $H^2(W;\mathbb{Z}) = 0$. On one hand, W contains a nontrivial Lagrangian $K = \mathbb{CP}^2 \subset W$ coming from the zero section (which is preserved by the handle attachment, since it is disjoint from the boundary; note that the handle attachment is subcritical, so in fact the whole wrapped Fukaya category is preserved; see [Ganatra et al. 2024]). However, as there is no nontrivial 2–cohomology class on W, there is no nontrivial S^1–bundle over W that can be used to build a sphere over K. ◁

3.2 Real projective Lagrangians

A similar procedure can be applied to a Liouville manifold containing real projective Lagrangians with an appropriate cohomology criterion. First recall the following.

Let $S^0 \cong \mathbb{Z}/2\mathbb{Z}$ act on the sphere S^n by the antipodal map. The quotient map $h\colon S^n \to \mathbb{RP}^n$ is in this case a covering map, and induces a symplectic double cover $q\colon T^*S^n \to T^*\mathbb{RP}^n$ with $q^*\omega_{T^*\mathbb{RP}^n} = \omega_{T^*S^n}$.

Lemma 3.5 [Mak and Wu 2018, Lemma 2.4] *Let $\tau_{\mathbb{RP}^n} \in \mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{RP}^n)$ be the $\mathbb{RP}^n$–twist defined as in Section 2.1. Then the diagram*

$$\begin{array}{ccc} T^*S^n & \xrightarrow{\tau_{S^n}} & T^*S^n \\ \downarrow q & & \downarrow q \\ T^*\mathbb{RP}^n & \xrightarrow{\tau_{\mathbb{RP}^n}} & T^*\mathbb{RP}^n \end{array} \tag{19}$$

commutes.

Assumption (RE) Let (W,ω) be a $2n$–dimensional Liouville manifold with a homology class $\alpha \in H^1(W;\mathbb{Z}/2\mathbb{Z})$ and Lagrangian real projective spaces $K_1,\dots,K_m \subset W$ such that

$$\alpha|_{K_i} = x \in H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \quad \text{for all } i,$$

where $x = e(\gamma_{\mathbb{R}}^{1,n+1})$ is the Euler class of the real tautological bundle $\gamma_{\mathbb{R}}^{1,n+1} \to \mathbb{RP}^n$, and generator of the cohomology ring $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x]/x^{n+1}$.

Proposition 3.6 *Let (W,ω) be a $2n$–dimensional Liouville manifold containing embedded Lagrangian real projective spaces $K_1,\dots,K_m \subset W$. Assume there is a class $\alpha \in H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ satisfying Assumption (RE). Then there is a $2n$–dimensional Liouville manifold $(\widetilde{W},\widetilde{\omega})$ containing Lagrangian spheres $L_1,\dots,L_m \subset \widetilde{W}$ and a commuting diagram*

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\tau_{L_i}} & \widetilde{W} \\ \downarrow q & & \downarrow q \\ W & \xrightarrow{\tau_{K_i}} & W \end{array} \tag{20}$$

Proof In this case, the class $\alpha \in H^1(W;\mathbb{Z}/2\mathbb{Z})$ defines a symplectic double cover $q\colon (\widetilde{W},\widetilde{\omega}) \to (W,\omega)$. Each Lagrangian $K_i \cong \mathbb{RP}^n$ then lifts to its double cover L_i, which is a sphere $S^n \subset \widetilde{W}$. Let λ be the Liouville form on W. As q is symplectic, $\widetilde{\omega} = q^*(\omega) = q^*(d\lambda) = d(q^*(\lambda))$, and $\tilde{\lambda} := q^*(\lambda)$ defines a Liouville form on $\widetilde{W}$, which gives $\widetilde{W}$ the structure of a Liouville manifold. Then the result follows by the local case, illustrated by Lemma 3.5. □

Remark 3.7 It is possible to obtain an analogous diagram for the quaternionic twist as follows. Consider the free $S^3 \simeq \mathrm{Sp}(1)$–action on S^{4n+3} inducing the quotient map $h\colon S^{4n+3} \to \mathbb{HP}^n$. This is a submersion as in the complex case, and the same arguments (with the natural metrics) yield the local commuting diagram

(21)
$$\begin{array}{ccc} \mu^{-1}(0) & \xrightarrow{\tau_{S^{4n+3}}|_{\mu^{-1}(0)}} & \mu^{-1}(0) \\ \downarrow p & & \downarrow p \\ T^*\mathbb{HP}^n & \xrightarrow{\tau_{\mathbb{HP}^n}} & T^*\mathbb{HP}^n \end{array}$$

where $p\colon \mu^{-1}(0) \to T^*\mathbb{HP}^n$ is the S^3–fibre bundle induced given by the symplectic quotient map of the Hamiltonian action induced on T^*S^{4n+3}.

Given an $8n$–dimensional symplectic manifold (W,ω) containing quaternionic projective Lagrangians, one would hope to find a cohomological condition to ensure the existence of a symplectic $(8n+6)$–dimensional manifold (Y,Ω) with corresponding Lagrangian spheres, as we did for the real and complex cases. However, homotopy classes of maps $W \to \mathbb{HP}^\infty \cong \mathrm{Sp}(1)$ do not classify quaternionic line bundles over W, so there is no analogue of Assumptions (CX) and (RE) to ensure the existence of such a manifold and a commuting diagram of the form of (17). ◁

4 The Hopf correspondence

In this section we discuss the main theoretical device in action in this paper; Lagrangian correspondences. We begin by reviewing the main concepts from Wehrheim–Woodward Lagrangian correspondence theory (Section 4.1). The rest of the section is then focussed on the correspondence that will be used in our applications, the *Hopf correspondence*. Given a real/complex projective Lagrangian $K \subset W$ in a Liouville manifold (W,ω) satisfying (RE)/(CX), the Hopf correspondence associates to it a Lagrangian sphere $L \subset Y$ in an auxiliary Liouville manifold (Y,Ω). The key use of the Hopf correspondence in this section is aimed at achieving a categorical version of the commuting diagrams of the previous section. To do this, we first show that the Hopf correspondence $\Gamma \subset W^- \times Y$ induces a well-defined functor $\Theta_\Gamma\colon \mathcal{F}\mathrm{uk}(W) \to \mathcal{F}\mathrm{uk}(Y)$ (Sections 4.2 and 4.3). We then show that the functors of $\mathcal{F}\mathrm{uk}(W)$ induced by projective twists are entwined, via the correspondence, with the functors of $\mathcal{F}\mathrm{uk}(Y)$ induced by the Dehn twists (Section 4.4). In Section 4.5, we show that the Hopf correspondence can be used to build a symplectic Gysin sequence as established in [Perutz 2008].

4.1 Lagrangian correspondences

We summarise the basic definitions and results associated to Lagrangian correspondences in the setting of [Wehrheim and Woodward 2012; 2010a; 2010b; Ma'u et al. 2018]. For the entire section we let k be a coefficient field of characteristic two.

Definition 4.1 [Wehrheim and Woodward 2010b] A *Lagrangian correspondence* between two symplectic manifolds (M_k,ω_k) and (M_{k+1},ω_{k+1}) ("from M_k to M_{k+1}") is a Lagrangian submanifold $L_{k,k+1}\subset(M_k^-\times M_{k+1}):=(M_k\times M_{k+1},-\omega_k\oplus\omega_{k+1})$. A *cycle of Lagrangian correspondences* of length $r\geqslant 1$ is a sequence of symplectic manifolds $(M_0,\dots,M_{r+1}=M_0)$ together with a sequence of Lagrangian correspondences $\underline{L}:=(L_{01},L_{12},\dots,L_{(r-1)r},L_{r0})$ such that $L_{k(k+1)}\subset M_k^-\times M_{k+1}$ for $k=0,\dots,r$. ◁

For example, a Lagrangian submanifold L of a symplectic manifold (M,ω) is a trivial example of Lagrangian correspondence, seen as $L\subset\{\mathrm{pt}\}^-\times M=M$ (see other examples below).

Definition 4.2 [Wehrheim and Woodward 2010a, Definition 2.0.4] Let (M_i,ω_i) for $i=0,1,2$ be symplectic manifolds and $L_{01}\subset M_0^-\times M_1$ and $L_{12}\subset M_1^-\times M_2$ be Lagrangian correspondences.

(1) The correspondence transpose to L_{01} is defined as $L_{01}^t:=\{(m_1,m_0)\mid(m_0,m_1)\in L_{01}\}\subset M_1^-\times M_0$. Note that, for a simple Lagrangian $L\subset M$ of a single symplectic manifold M, we won't distinguish L from its conjugate.

(2) The composition of L_{01} and L_{12} is defined as

$$L_{01}\circ L_{12}:=\{(m_0,m_2)\in M_0^-\times M_2\mid(m_0,m_1)\in L_{01},\,(m_1,m_2)\in L_{12}\text{ for some }m_1\in M_1\}\subset M_0^-\times M_2 \tag{22}$$

and it is called embedded if it defines an embedded Lagrangian submanifold of $M_0^-\times M_2$. ◁

Example 4.3 [Perutz 2008, 1.1] Let (M^{2n},ω_M) be a symplectic manifold with a coisotropic embedding $\iota\colon V\hookrightarrow M$. If the foliation defined by the integrable distribution TV^ω is a fibration $p\colon V\to B$, then the leaf space is a symplectic manifold (B,ω_B) satisfying $p^*\omega_B=\iota^*(\omega_M)$. The (transpose) graph of p,

$$\Gamma:=\{(p(v),v)\mid v\in V\}\subset(B\times M,-\omega_B\oplus\omega_M),$$

is a Lagrangian correspondence. ◁

A special case of Example 4.3 is when the coisotropic submanifold is obtained as a level set of a moment map induced by a Hamiltonian action:

Example 4.4 [Wehrheim and Woodward 2010b, Example 2.0.2(e)] Let (M,ω_M) be a symplectic manifold. Let G be a compact Lie group acting on M Hamiltonianly with moment map $\mu\colon M\to\mathfrak{g}^*$. If G acts freely on $\mu^{-1}(0)$, the latter is a smooth G–fibred coisotropic over the symplectic quotient $W:=M/\!\!/G=\mu^{-1}(0)/G$. W is a symplectic manifold with symplectic structure $\omega_{M/\!\!/G}$ given by the

Marsden–Weinstein theorem (see for example [McDuff and Salamon 2017, Section 5.4]). The graph of the quotient map $p\colon \mu^{-1}(0) \to W$ is a Lagrangian submanifold of $(M \times W, -\omega_M \oplus \omega_W)$ and defines a Lagrangian correspondence, relating Lagrangians of M with Lagrangians of its symplectic quotient. $\triangleleft$

4.2 Induced functors

Wehrheim and Woodward [2010a; 2010b] introduced a Floer cohomology theory adapted to cycles of closed Lagrangian correspondences $\underline{L} := (L_{01}, \dots, L_{r0})$, called *quilted Floer cohomology* and denoted by $\mathrm{HF}(\underline{L}; k)$. Pseudoholomorphic quilts are a generalisation of the usual pseudoholomorphic strips used in standard Lagrangian Floer theory, and the quilted invariant is defined by counting pseudoholomorphic quilts with boundary constraints defined by the Lagrangian correspondences [Wehrheim and Woodward 2010b, Section 5]. It can be viewed as a Floer theory in product symplectic manifolds (we refer to [Wehrheim and Woodward 2010b, Section 4.3] for definitions).

One of the main features is that, given a cycle $\underline{L}$ of Lagrangian correspondences, quilted Floer cohomology is invariant under embedded composition (as in Definition 4.2) of subsequent Lagrangians in $\underline{L}$.

Theorem 4.5 [Wehrheim and Woodward 2010b, Theorem 5.4.1] *Let $\underline{L} = (L_{01}, \dots, L_{r0})$ be a cyclic sequence of closed, exact embedded and oriented Lagrangian correspondences between Liouville manifolds $(M_0, \dots, M_{r+1} = M_0)$ such that $L_{(i-1)i} \circ L_{i(i+1)}$ is embedded for each i. Then, for $\underline{L}' := (L_{01}, \dots, L_{(j-1)j} \circ L_{j(j+1)}, \dots, L_{r0})$, there is an isomorphism $\mathrm{HF}(\underline{L}; k) \cong \mathrm{HF}(\underline{L}'; k)$.*

Ma'u et al. [2018] proved that, under certain assumptions, a Lagrangian correspondence Γ_{01} between given symplectic manifolds (M_0, ω_0) and (M_1, ω_1) defines an A_∞–functor $\Theta_{\Gamma_{01}}$ between $\mathcal{F}\mathrm{uk}(M_0)$ and the dg-category of A_∞–modules over $\mathcal{F}\mathrm{uk}(M_1)$. The functor is realised as the geometric composition $(\cdot) \circ \Gamma_{01}$ of Lagrangians submanifolds of M_0 with the correspondence, and this important result relies on the invariance of Theorem 4.5. If for every Lagrangian in M_0 the composition outputs an embedded Lagrangian of M_1, the induced functor is between Fukaya categories.

Theorem 4.6 [Ma'u et al. 2018, Theorem 1.1] *Assume M_0 and M_1 are Liouville manifolds, and let $\Gamma_{01} \subset M_0^- \times M_1$ be a closed, exact and embedded correspondence such that, for any closed embedded Lagrangian $K_0 \subset M_0$, the geometric composition*

$$L_1 := K_0 \circ \Gamma_{01} = \{m_1 \in M_1 \mid (m_0, m_1) \in \Gamma_{01} \text{ for some } m_0 \in K_0\} \subset M_1 \tag{23}$$

is a closed embedded Lagrangian in M_1. This assignment defines an A_∞–functor

$$\Theta_{\Gamma_{01}}\colon \mathcal{F}\mathrm{uk}(M_0) \to \mathcal{F}\mathrm{uk}(M_1), \quad \Theta_{\Gamma_{01}}(K_0) = L_1. \tag{24}$$

In the above theorem, the correspondences are required to be closed, exact (or satisfy suitable monotonicity conditions) and embedded. Gao [2017a; 2017b] developed noncompact generalisations of Theorem 4.6, including noncompact Lagrangian correspondences, in the setting of wrapped Fukaya categories.

In both cases, the main theoretical device at work behind a result such as Theorem 4.6 (or Gao's equivalent) is quilted Floer theory, which, in [Gao 2017b], was adapted to a version suitable for noncompact correspondences. In this work we focus on a Lagrangian correspondence in a setting that features some properties of both theories. Before introducing our setting (see below), we review the types of Lagrangians that are admitted in a Gao's setting.

Let (M_0, ω_0) and (M_1, ω_1) be Liouville manifolds with cylindrical almost complex structures J_0 and J_1 and Liouville flows Z_0 and Z_1, respectively. The product manifold $(M_0 \times M_1, -\omega_0 \times \omega_1)$ is a Liouville manifold with respect to the product almost complex structure $J_{01} := -J_0 \times J_1$ and Liouville flow $Z_{01} := \pi_0^*(Z_0) + \pi_1^*(Z_1)$ for the projections $\pi_i \colon M_0 \times M_1 \to M_i$ for $i = 1, 2$.

Let $\Sigma \subset M_0 \times M_1$ be the contact hypersurface given in [Gao 2017b, Section 2.2], so that we can fix a choice of cylindrical end that is compatible with the choices above. In other words, there is a compact set $U \subset M_0 \times M_1$ bounded by Σ such that there is a diffeomorphism $M_0 \times M_1 \setminus U \cong [0, \infty) \times \Sigma$ [Gao 2017b, (2.5)].

Definition 4.7 A Lagrangian submanifold is said to be *conical* if it is an exact, properly embedded Lagrangian which is preserved by the Liouville vector field over the cylindrical end. ◁

Definition 4.8 [Gao 2017b, Definition 3.9] A Lagrangian submanifold $\Gamma_{01} \subset M_0^- \times M_1$ is called admissible if it is

(1) either a product of conical Lagrangian submanifolds of M_0^- and M_1,

(2) or a Lagrangian that is conical with respect to the cylindrical end $\Sigma \times [0, +\infty)$. ◁

Gao [2017b, Theorem 1.5] defines geometric composition for this type of Lagrangian correspondences and proves the analogue of Theorem 4.5. Moreover, he shows the open version of Theorem 4.6, namely that such a Lagrangian correspondence induces a functor of wrapped Fukaya categories [Gao 2017a, Theorem 1.2].

Below we focus on the type of correspondences we consider in this paper, which arises as a special case of Example 4.3 for a noncompact coisotropic. It is a class of exact, embedded, but not closed correspondences between Liouville manifolds.

Setting Let (M_0, ω_0) and (M_1, ω_1) be Liouville manifolds such that there is a fibration $q \colon M_1 \to M_0$ with Liouville fibres.

Let $\Gamma_{01} \subset M_0^- \times M_1$ be a Lagrangian correspondence obtained as the (transpose) graph of a proper fibration $p \colon V \to M_0$, where $V \subset M_1$ is a fibred coisotropic as in Example 4.3, and $q|_V = p$.

On $M_0^- \times M_1$ set the product almost complex structure $J_{01} = -J_0 \times J_1$ for cylindrical almost complex structures on M_0 and M_1, so that the fibration $(\mathrm{id}, q) \colon M_0^- \times M_1 \to M_0^- \times M_0$ is (J_{01}, J_{00})–holomorphic for $J_{00} = -J_0 \times J_0$.

Then $\Gamma_{01} = \{(p(v), v) \mid v \in V\}$ is properly fibred over the diagonal $\Delta_{M_0} := \{(p(v), p(v)) \mid v \in V\} \subset M_0^- \times M_0$, which is a conical Lagrangian correspondence in $M_0^- \times M_0$. However, the original correspondence Γ_{01} is not conical, or more generally admissible in the sense of Definition 4.8.

Consequently, the above setting doesn't exactly fit either the compact or the open quilted theories, but it is a combination of the two: it depicts a class of noncompact correspondences which nevertheless induces a functor of compact Fukaya categories.

Axiom 1 The type of Lagrangian correspondence $\Gamma_{01} \subset M_0^- \times M_1$ defined in the above setting induces a functor

(25) $$\Theta_{\Gamma_{01}} \colon \mathcal{F}\mathrm{uk}(M_0) \to \mathcal{F}\mathrm{uk}(M_1), \quad \Theta_{\Gamma_{01}}(K_0) = L_1.$$

Experts will recognise the validity of the above statement that we have set as an axiom. Proving it as a theorem would require a lengthy digression necessary to fill in all details covered in [Wehrheim and Woodward 2010a; Gao 2017b; 2017a]. In Lemma 4.9, we restrict to proving the invariance of quilted Floer cohomology under Lagrangian correspondences. Given invariance, the results of [Wehrheim and Woodward 2010a] yield a functor on the cohomological category. The extension to an A_∞–functor, which would turn Axiom 1 into a theorem, can then be obtained by considering higher A_∞–products, which we omit here.

Lemma 4.9 *Let $K \subset M_0$ and $L' \subset M_1$ be closed exact Lagrangians and consider the cycle of correspondences $(K, \Gamma_{01}, L') \subset (\mathrm{pt}, M_0, M_1)$. Then the quilted Floer cohomology group $\mathrm{HF}(K, \Gamma_{01}, L')$ is well defined and satisfies the invariance property*

(26) $$\mathrm{HF}(K, \Gamma_{01}, L') \cong \mathrm{HF}(K \circ \Gamma_{01}, L') = \mathrm{HF}(L, L').$$

Proof By definition (see [Wehrheim and Woodward 2010b, Section 4.3]), the generators of the cochain complex $\mathrm{CF}(K, \Gamma_{01}, L')$ are given by the generators of $\mathrm{CF}(K \times L, \Gamma_{01})$. These intersection points must be contained in a compact region, since $K \subset M_0$ and $L' \subset M_1$ are closed Lagrangians. By [Wehrheim and Woodward 2012, Proposition 2.2.1], the cochain groups $\mathrm{CF}(K \circ \Gamma_{01}, L') = \mathrm{CF}(L, L')$ and $\mathrm{CF}(K \times L, \Gamma_{01})$ are isomorphic.

We now analyse the Floer trajectories involved in the computation of $\mathrm{HF}(K, \Gamma_{01}, L')$.

By the maximum principle, the only noncompactness phenomenon that could occur would be a J_{01}–holomorphic curve escaping a compact set on the noncompact boundary condition Γ_{01}. However, all such curves, and any Floer trajectory of interest, are contained in a compact set, as we now explain.

By assumption, J_{01}–holomorphic curves with boundary conditions on (K, Γ_{01}, L') project under (id, q) to J_{00}–holomorphic curves involved in the complex for the tuple (K, Δ_{M_0}, K'), where $K' \circ \Gamma_{01} = L'$ and $(\mathrm{id}, q)(\Gamma_{01}) = \Delta_{M_0}$.

The (quilted) Floer cohomology group for the cycle of Lagrangian correspondences $(K, \Delta_{M_0}, K') \subset (\mathrm{pt}, M_0, M_0)$ can be defined as the Floer cohomology group $\mathrm{HF}(K, \Delta_{M_0}, K') := \mathrm{HF}^*(K \times K', \Delta_{M_0})$ [Gao 2017b, Lemma 4.8]. Moreover, by [Gao 2017a, Theorem 1.2], Δ_{M_0} induces the identity functor, so clearly all the J_{00}–holomorphic strips involved in the complex $\mathrm{CF}(K, \Delta_{M_0}, K')$ are well behaved, and moreover we have $\mathrm{HF}(K, \Delta_{M_0}, K') \cong \mathrm{HF}(K, K')$.

Because of properness of $(q, \mathrm{id})|_{\Gamma_{01}} : \Gamma_{01} \to M_0 \times M_0$, if there were any J_{01}–holomorphic curve escaping to infinity at the boundary condition Γ_{01}, then it would project to a J_{00}–holomorphic curve escaping to infinity at the boundary condition on Δ_{M_0}, which cannot happen. □

Remark 4.10 Let $K, K' \subset (M_0, \omega_0)$ be closed exact Lagrangians. For any conical correspondence (not just the diagonal) $\Gamma_{00} \subset M_0^- \times M_0$, compactness of moduli spaces of curves involved in the quilted complex $\mathrm{CF}(K, \Gamma_{00}, K')$ (for compact Lagrangians $K, K' \subset M_0$) is preserved. Namely, all intersection points lie in a compact region, so, by exactness, both energy and symplectic area are bounded. We can apply a reverse isoperimetric inequality, according to which the length of the boundary of such a curve is bounded by a quantity proportional to its area [Groman and Solomon 2014, Theorem 1.4].

This ensures that the boundary of all pseudoholomorphic curves is contained in a compact set, which can then be determined by using a monotonicity lemma in the likes of [Seidel and Smith 2005, Lemma 13]. Again, by exactness there is no bubbling, so the moduli spaces of such curves are compact. ◁

4.3 The Hopf correspondence

We can finally introduce the correspondence of interest, the *Hopf correspondence*. This is a Lagrangian correspondence obtained as the graph of a spherically fibred coisotropic submanifold as in Example 4.3.

We use the discussions of Sections 3.1 and 3.2 to explain how, for each type of Lagrangian projective space $K \cong \mathbb{AP}^n \subset W$ with $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$ in a Liouville manifold (W, ω) satisfying the appropriate cohomology assumption (RE) or (CX), there is a Lagrangian correspondence relating K to a Lagrangian sphere L in an auxiliary Liouville manifold (Y, Ω).

4.3.1 Lagrangian $\mathbb{CP}^n$ Let (W^{4n}, ω) be a Liouville manifold admitting Lagrangian submanifolds $K_i \cong \mathbb{CP}^n \hookrightarrow W$ for $i = 1, \dots, m$. Assume there is a class $\alpha \in H^2(W; \mathbb{Z})$ satisfying Assumption (CX). The discussion of Section 3.1 delivers a $\mathbb{C}^*$–bundle $q : Y \to W$ (associated to the complex line bundle $\mathcal{L} \to W$ with $c_1(\mathcal{L}) = \alpha$), whose total space is a Liouville manifold (Y, Ω) (proof of Proposition 3.2). Set $V := Y|_{\{r=1\}}$, the unit length bundle (determined by the metric on Y induced by a choice of hermitian metric on $\mathcal{L}$). If $V \hookrightarrow Y$ is the inclusion, then, by construction, $\iota^*\Omega = q^*\omega|_V$, so the symplectic reduction of V by S^1 is given by (W, ω), and V is a fibred coisotropic submanifold of (Y, Ω) with S^1–fibre bundle structure $p = q|_V : V \to W$.

For any Lagrangian projective space $K_i \subset W$, the restriction $V|_{K_i} \to K_i$ is a Lagrangian sphere $L_i \cong S^{2n+1} \subset Y$.

Definition 4.11 The (transpose) graph

$$\Gamma := \{(p(y), y), y \in V\} \subset W^- \times Y \tag{27}$$

defines a Lagrangian correspondence [Perutz 2008, Proposition 1.1], which we call the *Hopf correspondence*. By construction, for $K_i \cong \mathbb{CP}^n \subset \{\mathrm{pt}\} \times W$, the correspondence maps K_i to the embedded Lagrangian sphere $L_i := K_i \circ \Gamma \cong S^{2n+1} \subset \{\mathrm{pt}\} \times Y \cong Y$ for $i = 1, \dots, m$ via geometric composition. ◁

Remark 4.12 This Lagrangian correspondence can equivalently be thought of as a correspondence of the type of Example 4.4, where the coisotropic V is a regular level set of a Hamiltonian S^1 "Hopf" action, and (W, ω) its symplectic quotient (note that the local models (14), (19) and (21) are obtained from this perspective). This explains the choice of name for the correspondence. ◁

4.3.2 Lagrangian $\mathbb{RP}^n$ Let (W^{2n}, ω) be a Liouville manifold admitting Lagrangian embeddings $K_i \cong \mathbb{RP}^n \hookrightarrow W$ for $i = 1, \dots, m$. Assume there is a cohomology class $\alpha \in H^1(W; \mathbb{Z}/2\mathbb{Z})$ satisfying Assumption (RE).

Then there is a Liouville manifold $(Y, \Omega) = (\widetilde{W}^{2n}, \widetilde{\omega})$ obtained as the symplectic double cover of W and containing Lagrangian spheres $L_1, \dots, L_m \subset \widetilde{W}$. The double cover $q \colon \widetilde{W} \to W$ defines an S^0–fibration over W, and in this case the "coisotropic submanifold" is the total space itself. As above, we define the Hopf correspondence as $\Gamma := \{(q(y), y) \mid y \in \widetilde{W}\} \subset W^- \times \widetilde{W}$.

4.4 Commuting diagrams of functors

Let (W, ω) and (Y, Ω) be Liouville manifolds and $K_1, \dots, K_m \subset W$ be real/complex projective Lagrangians satisfying (RE)/(CX). Let $q \colon Y \to W$ be the fibration we constructed in the previous subsections, and $\Gamma \subset W^- \times Y$ be the Hopf correspondence obtained as the graph $\Gamma = \{(p(v), v) \mid v \in V\}$ of the spherically fibred coisotropic $p = q|_V \colon V \to W$. This correspondence is properly fibred over the diagonal $\Delta_W = \{(p(v), p(v)) \mid v \in V\} \subset W^- \times W$, via $(\mathrm{id}, q) \colon W \times Y \to W \times W$, and satisfies the conditions of Axiom 1. Therefore, there is a well-defined functor

$$\Theta_\Gamma \colon \mathcal{F}\mathrm{uk}(W) \to \mathcal{F}\mathrm{uk}(Y), \quad \Theta_\Gamma(K) = K \circ \Gamma =: L. \tag{28}$$

Let $L_1, \dots, L_m \subset Y$ be the Lagrangian spheres associated to $K_1, \dots, K_m$ through the correspondence. For each $i = 1, \dots, m$, let $T_{K_i} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ and $T_{L_i} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ be the (geometric) twist functors induced by the graphs of the respective twists $\tau_{K_i} \in \mathrm{Symp}_{\mathrm{ct}}(W)$ and $\tau_{L_i} \in \mathrm{Symp}_{\mathrm{ct}}(Y)$.

Corollary 4.13 *There is a commuting diagram at the level of compact Fukaya categories*

$$\begin{array}{ccc} \mathcal{F}\mathrm{uk}(Y) & \xrightarrow{T_{L_i}} & \mathcal{F}\mathrm{uk}(Y) \\ {\scriptstyle\Theta_\Gamma}\uparrow & & \uparrow{\scriptstyle\Theta_\Gamma} \\ \mathcal{F}\mathrm{uk}(W) & \xrightarrow{T_{K_i}} & \mathcal{F}\mathrm{uk}(W) \end{array} \tag{29}$$

In particular, iterative applications of this diagram yield

$$\Theta_\Gamma \circ \prod T_{K_i}^{k_i} = \prod T_{L_i}^{k_i} \circ \Theta_\Gamma. \tag{30}$$

Proof Consider the functors T_{K_i} and T_{L_i} as correspondences induced by the graphs of the respective twists $\tau_{K_i} \in \mathrm{Symp}_{\mathrm{ct}}(W)$ and $\tau_{L_i} \in \mathrm{Symp}_{\mathrm{ct}}(Y)$. Then we have to check that the compositions of correspondences $\Theta_\Gamma \circ T_{K_i} = T_{L_i} \circ \Theta_\Gamma$, as Lagrangians in $W^- \times Y$, coincide. By construction, this equality amounts to the commutativity of the diagram (17) or (20), respectively. $\square$

Remark 4.14 For a coefficient field of characteristic zero, the functor associated to the real projective twist has a different shape which produces a different diagram [Mak and Wu 2019, Corollary 1.3]. $\triangleleft$

Remark 4.15 Given a hypothetical mirror pair (X, M) for a symplectic manifold (M, ω) with $c_1(M) = 0$ and complex manifold (X, J), we can make the following observation.

Huybrechts and Thomas [2006] conjectured that the functors induced by projective twists on the derived Fukaya category $D^\flat(\mathcal{F}\mathrm{uk}(M))$ should be mirror to a class of autoequivalences of $D^\flat(X)$, induced by so-called $\mathbb{P}$–objects (see [Huybrechts and Thomas 2006, Definition 1.1]). This is the analogue of the statement proved by Seidel that autoequivalences of $D^\flat(\mathcal{F}\mathrm{uk}(M))$ induced by Dehn twists should be mirror to autoequivalences of $D^\flat(X)$ induced by "spherical objects" (see [Seidel and Thomas 2001, Definition 1.1]).

From this perspective, we can view the diagram 4.13 as a conjectural mirror to the following situation.

By [Huybrechts and Thomas 2006, Proposition 1.4], a $\mathbb{P}$–object $\mathcal{P} \in D^\flat(X)$ in the central fibre of an algebraic deformation $j : X \hookrightarrow \mathcal{X}$ and satisfying $0 \neq A(\mathcal{P}) \cdot \kappa(\mathcal{X}) \in \mathrm{Ext}^2(\mathcal{P}, \mathcal{P})$ has an associated spherical object given by $j_*(\mathcal{P}) \in D^\flat(\mathcal{X})$. Here, $A(\mathcal{P}) \in \mathrm{Ext}^1(\mathcal{P}, \mathcal{P} \otimes \Omega^1_X)$ is the Atiyah class of $\mathcal{P}$ and $\kappa(\mathcal{X}) \in H^1(X, \mathcal{T}_X)$ the Kodaira–Spencer class of the family $\mathcal{X}$. Furthermore, the autoequivalences associated to each object (also called "twists"), $T_\mathcal{P}$ and $T_{j_*\mathcal{P}}$, are related by a commutative diagram [Huybrechts and Thomas 2006, Proposition 2.7]

$$\begin{array}{ccc} D^\flat(X) & \xrightarrow{\ j_*\ } & D^\flat(\mathcal{X}) \\ \downarrow{\scriptstyle T_\mathcal{P}} & & \downarrow{\scriptstyle T_{j_*\mathcal{P}}} \\ D^\flat(X) & \xrightarrow{\ j_*\ } & D^\flat(\mathcal{X}) \end{array} \tag{31}$$

$\triangleleft$

4.5 Lagrangian Gysin sequence

Let $\Gamma \subset W^- \times Y$ be the Hopf correspondence. Given real/complex projective Lagrangian submanifolds $K, K' \subset W$ and their corresponding spherical lifts $L, L' \subset Y$ through the functor Θ_Γ, a version of Perutz's Gysin sequence [2008] can be used to establish a relationship between the ranks of the Floer cohomology groups $\mathrm{HF}(K, K')$ and $\mathrm{HF}(L, L')$. We will need this relation in the next section for the proof of Theorem B.

Let $V \to W$ be the S^k–fibred coisotropic defining the correspondence, $k \in \{0, 1\}$, with Euler class $\alpha \in H^{k+1}(W; R)$, $R \in \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}\}$ and Lagrangian projective spaces $K, K' \subset W$ satisfying (RE)/(CX), respectively.

Let $L = \Theta_\Gamma(K) = K \circ \Gamma \subset Y$ and $L' = \Theta_\Gamma(K') = K' \circ \Gamma \subset Y$ be the associated Lagrangian spheres given by the correspondence.

Lemma 4.16 *There is an exact triangle of the shape*

(32)
$$\begin{array}{ccc} \mathrm{HF}^*(K,K') & \xrightarrow{\alpha\cup\cdot} & \mathrm{HF}^{*+k+1}(K,K') \\ & \nwarrow \quad \swarrow_{\Gamma_*} & \\ & \mathrm{HF}^{*+k+1}(L,L') & \end{array}$$

Proof This exact sequence follows from the Gysin triangle proved by Perutz [2008, Theorem 1], which has the more general form

(33)
$$\cdots \to \mathrm{HF}^*(K,K') \xrightarrow{e(V)\cup\cdot} \mathrm{HF}^{*+k+1}(K,K') \xrightarrow{\Gamma_*} \mathrm{HF}^{*+k+1}(K,\Gamma^t,\Gamma,K') \to \cdots,$$

where the last group is the quilted Floer cohomology group of the cycle of Lagrangian correspondences $\underline{L} := (K,\Gamma,\Gamma^t,K') \subset (\mathrm{pt},W,Y,W)$, satisfying $\mathrm{HF}(K,\Gamma,\Gamma^t,K') \cong \mathrm{HF}^*(K\circ\Gamma, K'\circ\Gamma) \cong \mathrm{HF}(L,L')$.

The isomorphism follows from Axiom 1 (in particular Lemma 4.9 applied to a sequence of four Lagrangian correspondences). The compositions $L = K \circ \Gamma$ and $L' = \Gamma^t \circ K' = K' \circ \Gamma$ are embedded, and coincide with the spheres (which are Lagrangian in W) in the sphere bundle V over K and K', respectively.

The first map in the original exact sequence (32) is the quantum cup product with the Euler class $e(V) \in QH^*(W)$. In this case the exactness assumptions on the ambient symplectic manifold W ensure the well-definedness of the operation and, as $QH^*(W) \cong H^*(W)$ is a ring isomorphism, there is no quantum deformation involved and we obviously have $e(V) = \alpha \in H^*(W)$. The second map, Γ_*, is induced by the Lagrangian correspondence, and needs to be understood in the context of quilted Floer theory. We refer the reader to [Perutz 2008, Section 4.1] for a more refined description of the maps (in the setting of Hamiltonian Floer theory). □

Corollary 4.17 *The Gysin sequence produces the rank inequality*

(34)
$$\mathrm{hf}(L,L') := \operatorname{rank}\mathrm{HF}(L,L') \leqslant 2\operatorname{rank}\mathrm{HF}(K,K').$$

In Section 5, we will need to compare functors induced by (projective and Dehn) twists to the identity functor. In particular, it will be necessary to distinguish objects of $\mathcal{F}\mathrm{uk}(W)$ with their image under the twist functors. The following lemma gives a helpful criterion:

Lemma 4.18 *Let K' and $\overline{K}'$ be quasi-isomorphic objects in $\mathcal{F}\mathrm{uk}(W)$. Then the maps*

$$f_1 : \mathrm{CF}^*(K,K') \xrightarrow{\alpha\cup\cdot} \mathrm{CF}^{*+k+1}(K,K') \quad \text{and} \quad f_2 : \mathrm{CF}^*(K,\overline{K}') \xrightarrow{\alpha\cup\cdot} \mathrm{CF}^{*+k+1}(K,\overline{K}')$$

have quasi-isomorphic mapping cones.

Proof Consider the long exact sequences associated to the mapping cones of the cup product maps $f_1 : \mathrm{CF}^*(K,K') \to \mathrm{CF}^{*+k+1}(K,K')$ and $f_2 : \mathrm{CF}^*(K,\overline{K}') \to \mathrm{CF}^{*+k+1}(K,\overline{K}')$.

These sequences fit in a diagram of the shape

$$\begin{array}{ccccccc} CF^*(K,K') & \xrightarrow{f_1=\alpha\cup\cdot} & CF^{*+k+1}(K,K') & \longrightarrow & \mathrm{Cone}(f_1) & \longrightarrow & CF^{*+k+1}(K,K') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ CF^*(K,\overline{K}') & \xrightarrow{f_2=\alpha\cup\cdot} & CF^{*+k+1}(K,\overline{K}') & \longrightarrow & \mathrm{Cone}(f_2) & \longrightarrow & CF^{*+k+1}(K,\overline{K}') \end{array}$$

Since K' and $\overline{K}'$ are quasi-isomorphic objects in $\mathcal{F}\mathrm{uk}(W)$, there is a characteristic element $\eta \in \mathrm{HF}(K', \overline{K}')$ which induces an isomorphism $\mathrm{HF}(K,K') \to \mathrm{HF}(K,\overline{K}')$ (the Floer product with η; see [Seidel 2008a, (8k)]). Therefore, the vertical maps $CF^*(K,K') \to CF^*(K,\overline{K}')$ are well defined, and they are quasi-isomorphisms. By the five lemma, the mapping cones $\mathrm{Cone}(f_1)$ and $\mathrm{Cone}(f_2)$ are also quasi-isomorphic. □

5 Free groups generated by projective twists

In this section we apply the Hopf correspondence to prove our first result about products of projective twists.

Consider a transverse plumbing $W := T^*\mathbb{A}\mathbb{P}^n \#_{\mathrm{pt}} T^*\mathbb{A}\mathbb{P}^n$ of cotangent bundles of projective spaces for $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$. Then the main result of this section (Theorem B) shows that the Lagrangian cores of the plumbing define two projective twists which generate a free subgroup of $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(W))$. In fact, Theorem B is a stronger statement, which holds not only for transverse plumbings but also, more generally, for *clean* plumbings along subprojective spaces (see Definition 5.1).

For the proof, we use the Hopf correspondence to reduce the statement of Theorem B to a statement about Dehn twists, and apply *Keating's free generation result* [2014] (Theorem 5.3) for Dehn twists.

As a corollary, we show that there are infinitely many Lagrangian isotopy classes of embeddings $\mathbb{C}\mathbb{P}^n \hookrightarrow W$ which are smoothly isotopic, but pairwise not Lagrangian isotopic.

5.1 Clean Lagrangian plumbing

We first recall a construction from [Abouzaid 2011, Appendix A] of clean *Lagrangian plumbing* of two Riemannian manifolds Q_1 and Q_2 along a submanifold $B \subset Q_i$ for $i = 1, 2$. Fix three closed smooth manifolds B, Q_1 and Q_2, for each $i = 1, 2$ an embedding $B \hookrightarrow Q_i$, and an isomorphism $\varrho\colon \nu_{B/Q_1} \to \nu^*_{B/Q_2}$ from the normal bundle ν_{B/Q_1} to the conormal bundle ν^*_{B/Q_2}.

Pick a Riemannian metric on B, an inner product and a connection on $\nu_{B/Q_1} \cong \nu^*_{B/Q_2}$ (which induces an inner product and connection on $\nu_{B/Q_2} \cong \nu^*_{B/Q_1}$). This data induces a metric on the total spaces ν_{B/Q_i}, and a neighbourhood U_i of $B \subset Q_i$ can be identified with a disc subbundle $D_\varepsilon \nu_{B/Q_i}$ of radius $\varepsilon > 0$. With this identification we write $x \in U_i$ as $x = (a, b)$ for $b \in B$ and $a \in D_\varepsilon(\nu_{B/Q_i})_b$ (the fibre over b).

For each $x = (a, b) \in U_i$, the connection gives a decomposition of the fibres $T_x^* Q_i \cong T_b^* B \oplus (\nu_{B/Q_i}^*)_b$. We get an identification of a neighbourhood of $B \subset T^* Q_i$ as

$$D_\varepsilon \nu_{B/Q_i} \oplus D_\varepsilon T^* B \oplus D_\varepsilon \nu_{B/Q_i}^*. \tag{35}$$

Let V_i be a neighbourhood of $Q_i \subset T^* Q_i$ which in (35) coincides with $D_\varepsilon T^* B \oplus D_\varepsilon \nu_{B/Q_i}^*$ over $U_i \cong D_\varepsilon \nu_{B/Q_i}$.

Definition 5.1 (1) As a smooth manifold, the clean plumbing of Q_1 and Q_2 along B, denoted by $M := D_\varepsilon(T^* Q_1 \#_B T^* Q_2)$, is defined by gluing V_1 to V_2 along $D_\varepsilon \nu_{B/Q_1} \oplus D_\varepsilon T^* B \oplus D_\varepsilon \nu_{B/Q_1}^* \subset V_1$ identified with $D_\varepsilon \nu_{B/Q_2}^* \oplus D_\varepsilon T^* B \oplus D_\varepsilon \nu_{B/Q_2}$ via $(\varrho, \mathrm{id}_{T^*B}, -\varrho^*)$. Its Liouville completion will be denoted by $T^* Q_1 \#_B T^* Q_2$.

(2) The plumbing construction inherits an exact symplectic structure, since the identification maps of (1) preserve the canonical structures on $D^* Q_i$. Let Z_i be the standard radial Liouville vector field on V_i. We define a Liouville vector field Z on the plumbing by letting $Z = \rho_1 Z_1 + \rho_2 Z_2$ for smooth functions $\rho_i : M \to [0, 1]$ supported on V_i such that $\rho_1 + \rho_2 = 1$. This endows M with the structure of an exact symplectic manifold. ◁

In the next sections, we will apply this plumbing construction to cotangent bundles of projective spaces and spheres. We will work with (ungraded) Floer cohomology groups $\mathrm{HF}(Q_1, Q_2, k)$, where k is a coefficient field of characteristic two. Note that, by exactness of the Lagrangians and the manifold W, the Floer differentials in $\mathrm{CF}(Q_i, Q_i)$ vanish, and, as Q_1 and Q_2 intersect cleanly along B, there is an isomorphism $\mathrm{HF}(Q_1, Q_2) \cong H^*(B)$ [Poźniak 1994].

5.2 Proof of Theorem B

We now prove the main theorem of this section.

Theorem B *Let $W := T^* \mathbb{AP}^n \#_{\mathbb{AP}^l} T^* \mathbb{AP}^n$ be a clean plumbing of (real, complex) projective spaces along a linearly embedded subprojective space $\mathbb{AP}^l \subset W$, $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$. Let $K_1, K_2 \cong \mathbb{AP}^n \subset W$ denote the Lagrangian core components of the plumbing. Then the projective twists τ_{K_1} and τ_{K_2} generate a free group inside $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(W))$, and the associated functors T_{K_1} and T_{K_2} generate a free subgroup of* $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$.

Remark 5.2 The case $W := T^* \mathbb{CP}_1^1 \#_{\mathrm{pt}} T^* \mathbb{CP}_2^1$ can be deduced from the existing literature by considering X as an A_2–configuration and the isotopies $\tau_{\mathbb{CP}_i^1} \simeq \tau_{S_i^2}^2$ of Remark 2.6. There is a homomorphism [Seidel 1999, Proposition 8.4] $\rho \colon \mathrm{Br}_3 \to \pi_0(\mathrm{Symp}_{\mathrm{ct}}(W))$ sending the generators of the braid group σ_i to $\rho(\sigma_i) = \tau_{S_i^2}$ for $i = 1, 2$. The associated homomorphism $\hat{\rho} \colon \mathrm{Br}_3 \to \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ fits in the diagram

$$\begin{array}{ccc} \mathrm{Br}_3 & \xrightarrow{\ \rho\ } & \pi_0(\mathrm{Symp}_{\mathrm{ct}}(W)) \\ & \searrow^{\hat{\rho}} & \downarrow \\ & & \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W)) \end{array} \tag{36}$$

and its injectivity [Seidel and Thomas 2001] implies the injectivity of ρ. Then, as $\langle \sigma_1^2, \sigma_2^2 \rangle \cong \mathrm{Free}_2$, it follows that $\langle \tau_{K_1}, \tau_{K_2} \rangle \cong \mathrm{Free}_2$.

Also note that ρ is in fact an isomorphism [Wu 2014], so $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(T^*S^2 \#_{\mathrm{pt}} T^*S^2)) = \mathrm{Br}_3$. ◁

Theorem B takes inspiration from Keating's free generation result for Dehn twists in Liouville manifolds.

Theorem 5.3 [Keating 2014, Theorem 1.1 and 1.2] *Let (Y, Ω) be a Liouville manifold of dimension greater than 2, and $L, L' \subset Y$ be two Lagrangian spheres satisfying* $\operatorname{rank} \mathrm{HF}(L, L') \geqslant 2$ *and such that L and L' are not quasi-isomorphic in $\mathcal{F}\mathrm{uk}(Y)$. The Dehn twists $\tau_L, \tau_{L'}$ generate a free subgroup of* $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(Y))$, *and the associated functors $T_L, T_{L'} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ generate a free subgroup of* $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$.

Keating proves the geometric part of Theorem B by making a categorical detour, first proving that the associated functors $T_L, T_{L'} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ induced by the Dehn twists generate a free subgroup of $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$, so that the composition

$$\mathrm{Free}_2 \to \pi_0 \mathrm{Symp}_{\mathrm{ct}}(Y) \to \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y)) \tag{37}$$

is injective.

By identifying a Dehn twist with its associated functor, Keating exploits the algebraic properties of the latter to arrive at the following rank inequalities (which are central in her final proof):

Lemma 5.4 [Keating 2014, Lemma 8.1] *Let $\widetilde{L}, L, L' \subset Y$ be Lagrangians such that $\widetilde{L}$ is a sphere, $\widetilde{L} \not\cong L$ in the Fukaya category, and* $\mathrm{hf}(\widetilde{L}, L) := \mathrm{rank}(\mathrm{HF}(\widetilde{L}, L')) \geqslant 2$. *Then, for all $n \neq 0$,*

$$\mathrm{hf}(\widetilde{L}, L') > \mathrm{hf}(L, L') \implies \mathrm{hf}(\widetilde{L}, \tau^n_{\widetilde{L}}(L')) < \mathrm{hf}(L, \tau^n_{\widetilde{L}}(L')). \tag{38}$$

Lemma 5.5 [Keating 2014, Claim 8.2] *Let $L, L' \subset Y$ be two Lagrangian spheres in an exact symplectic manifold as in Theorem 5.3 satisfying* $\mathrm{hf}(L, L') = 2$. *Then, for all $m \neq 0$,*

$$\mathrm{hf}(L', L) = \mathrm{hf}(L', \tau^m_{L'} L) < \mathrm{hf}(L, \tau^m_{L'} L). \tag{39}$$

We will apply these inequalities to Lagrangian spheres obtained from the Hopf correspondence, to produce similar results for projective twists and prove Theorem B.

5.2.1 Strategy The plumbing (W, ω) and its real/complex projective Lagrangian cores $K_1, K_2 \subset W$ satisfy the cohomological conditions (RE)/ (CX).

In the case in which W is a transverse plumbing (which retracts to the wedge sum of the two spheres), there is a ring isomorphism (with $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$ and $R \in \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}\}$)

$$\widetilde{H}^*(W; R) \cong \widetilde{H}^*(K_1; R) \oplus \widetilde{H}^*(K_2; R) \cong \widetilde{H}^*(\mathbb{AP}^n; R) \oplus \widetilde{H}^*(\mathbb{AP}^n; R),$$

so it is immediate to see the existence of a class $\alpha = (\alpha_1, \alpha_2) \in H^k(W; R)$ restricting in K_1 and K_2 to the generator of $H^*(\mathbb{AP}^n; R)$ for $k \in \{1, 2\}$. For a clean plumbing along a linearly embedded subprojective space $\mathbb{AP}^l$, this restriction property still holds because the "difference" map of the Mayer–Vietoris sequence is always zero.

By Propositions 3.2 and 3.6, the cohomological condition ensures the existence of a Liouville manifold $(Y, \Omega) \to (W, \omega)$ and a Hopf correspondence $\Gamma \subset W^- \times Y$ that gives rise to associated Lagrangian spheres $S^m \cong L_i = K_i \circ \Gamma \subset Y$ for $i = 1, 2$ and commuting diagrams of twist functors (29). Then, given a product (a word in τ_{K_1} and τ_{K_2}) $\varphi \in \mathrm{Symp}_{\mathrm{ct}}(W)$ of projective twists, the Hopf correspondence yields a corresponding product of Dehn twists (a word in τ_{L_1} and τ_{L_2}) $\phi \in \mathrm{Symp}_{\mathrm{ct}}(Y)$.

In the real projective case, the geometric statement of Theorem B can be obtained by an isotopy-lifting argument using the geometric diagrams of Section 3 (the strategy adopted in Section 6.3). Assuming the projective twists do satisfy a relation, this procedure lifts the isotopy to $\mathrm{Symp}_{\mathrm{ct}}(Y)$, producing a relation between Dehn twists, which cannot hold, by Keating's theorem. However, this geometric argument does not give a statement at the level of Fukaya categories, for which the use of the Hopf correspondence at the level of functors in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ is necessary (Sections 4.4 and 4.5).

The spheres $L_1, L_2 \cong S^m$ intersect cleanly along a subsphere S^r for a tuple $(\mathbb{A}, m, r)$ that is one of $(\mathbb{R}, n, l)$ or $(\mathbb{C}, 2n+1, 2l+1)\}$ $(n, l \in \mathbb{N}^*)$, and, as noted before, $\mathrm{HF}(L_1, L_2; k) \cong H^*(S^r; k)$. Since $L_i \subset Y$ are exact spheres, $\mathrm{HF}(L_i, L_i; k) \cong H^*(S^m; k)$ [Floer 1988] and therefore

$$\mathrm{rank}\,\mathrm{HF}(L_1, L_2) = \mathrm{rank}\,\mathrm{HF}(L_i, L_i) = 2 \quad \text{for } i = 1, 2. \tag{40}$$

In the following sections we will study the ranks of the Floer cohomology groups $\mathrm{HF}(\cdot, \varphi(\cdot))$ and show that there is always a Lagrangian $\widehat{K} \subset W$ such that

$$\mathrm{HF}(\widehat{K}, \widehat{K}) \ncong \mathrm{HF}(\widehat{K}, \varphi(\widehat{K})). \tag{41}$$

As a result, $\widehat{K}$ and $\varphi(\widehat{K})$ are not quasi-isomorphic objects in $\mathcal{F}\mathrm{uk}(W)$, and therefore the functor induced by φ cannot be isomorphic to the identity in $\mathrm{Auteq}\,\mathcal{F}\mathrm{uk}(W)$. This will also rule out the possibility of φ being isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(W)$.

We prove (41) by applying the rank inequalities (38) and (39) to Lagrangian spheres in (Y, Ω) obtained via the correspondence Γ, in combination with the symplectic Gysin sequence associated to Γ (Corollary 4.17).

In the following section, we rederive a part of Keating's proof of Theorem 5.3 using the rank inequalities (38) and (39), which hold for the word of Dehn twists $\phi \in \langle \tau_{L_1}, \tau_{L_2} \rangle$ associated to φ via the Hopf correspondence. This will clarify the methods used in proving the analogous statement for projective twists (namely Theorem B) in Section 5.2.3.

5.2.2 Associated word of Dehn twists Let $\varphi \in \langle \tau_{L_1}, \tau_{L_2} \rangle \subset \mathrm{Symp}_{\mathrm{ct}}(W)$ be a word of projective twists as in the statement of Theorem B. Consider the Hopf correspondence $\Gamma \subset W^- \times Y$ and the associated word of Dehn twists $\phi \in \mathrm{Symp}_{\mathrm{ct}}(Y)$ as above. In this section we replicate the last steps in Keating's proof of the injectivity of the homomorphism

$$\mathrm{Free}_2 \to \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W)).$$

We first make the following observation about a word of twists and its conjugates:

Lemma 5.6 *Let $\phi \in \mathrm{Symp}_{\mathrm{ct}}(Y)$ be a symplectomorphism which has the shape of a global conjugate, ie $\phi = \psi^{-1}\phi'\psi$ for $\psi, \phi' \in \mathrm{Symp}_{\mathrm{ct}}(Y)$ not isotopic to the identity. Then there is a closed Lagrangian $\widetilde{L} \subset Y$ such that $\mathrm{HF}(\widetilde{L}, \phi(\widetilde{L})) \ncong \mathrm{HF}(\widetilde{L}, \widetilde{L})$ if and only if ϕ' satisfies $\mathrm{HF}(\widehat{L}, \phi'(\widehat{L})) \ncong \mathrm{HF}(\widehat{L}, \widehat{L})$ for some closed Lagrangian $\widehat{L} \subset Y$.*

Proof Assume there is a Lagrangian $\widehat{L} \subset Y$ such that $\mathrm{HF}(\widehat{L}, \phi'(\widehat{L})) \ncong \mathrm{HF}(\widehat{L}, \widehat{L})$. Then, by invariance of Floer cohomology under symplectomorphisms, $\mathrm{HF}(\psi^{-1}\widehat{L}, \psi^{-1}\phi'\psi(\psi^{-1}\widehat{L})) \cong \mathrm{HF}(\psi^{-1}\widehat{L}, \psi^{-1}(\phi'\widehat{L}) \cong \mathrm{HF}(\widehat{L}, \phi'\widehat{L}) \ncong \mathrm{HF}(\widehat{L}, \widehat{L})$, so, for $\widetilde{L} := \psi^{-1}(\widehat{L})$, we have $\mathrm{HF}(\widetilde{L}, \phi(\widetilde{L})) \ncong \mathrm{HF}(\widetilde{L}, \widetilde{L})$. The other direction is similar. □

We will apply the above lemma to a word of Dehn twists $\phi \in \mathrm{Symp}_{\mathrm{ct}}(Y)$ which in its reduced has the shape of a global conjugate, ie a word $\phi = \psi^{-1}\phi'\psi$ for two reduced words $\psi, \phi' \in \langle \tau_{L_1}, \tau_{L_2} \rangle \subset \mathrm{Symp}_{\mathrm{ct}}(Y)$ not isotopic to the identity. Then the lemma shows that it is always possible to switch between ϕ and its conjugate, as the correct choice of Lagrangian keeps track of the Floer cohomological action of the original word.

Without loss of generality, we can therefore use this conjugation argument to restrict the focus on reduced words $\phi \in \langle \tau_{L_1}, \tau_{L_2} \rangle$ which are either (for $i, j \in \{1, 2\}$)

(1) a power of a single Dehn twist, ie $\phi = \tau_{L_i}^s$, $s \in \mathbb{Z}^*$ (Lemma 5.7); or

(2) a word starting with a power of τ_{L_i} and ending in a power of τ_{L_j} with $i \neq j$ (Lemma 5.9).

Lemma 5.7 *Let $\phi = \tau_{L_i}^s \in \mathrm{Symp}_{\mathrm{ct}}(Y)$ be a reduced word of Dehn twists which is a power of a single Dehn twist, with $i \in \{1, 2\}$ and $s \in \mathbb{Z}^*$. The associated functor is not isomorphic to the identity in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$, so, in particular, ϕ cannot be isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(Y)$.*

Proof We show that there exists a closed Lagrangian $\widehat{L} \subset Y$ such that

$$\mathrm{HF}(\widehat{L}, \phi(\widehat{L})) \ncong \mathrm{HF}(\widehat{L}, \widehat{L}).$$

For $\phi = \tau_{L_i}^s$, a possible candidate is given by $\widehat{L} = L_j$ for $i, j \in \{1, 2\}$ and $i \neq j$.

Namely, the rank inequality stated by Lemma 5.5, gives

$$2 = \mathrm{hf}(L_j, L_j) = \mathrm{hf}(L_i, L_j) = \mathrm{hf}(L_i, \tau_{L_i}^s L_j) < \mathrm{hf}(L_j, \tau_{L_i}^s L_j).$$ □

Remark 5.8 The geometric result of the above lemma can also be proven independently from Keating's results, as a corollary to Theorem A (see Section 6.1). ◁

Lemma 5.9 *Let $\phi \in \langle \tau_{L_1}, \tau_{L_2} \rangle \subset \mathrm{Symp}_{\mathrm{ct}}(Y)$ be a reduced word of Dehn twists around the Lagrangian (spherical) cores which is a product where the first and last factors are powers of distinct Dehn twists. Then the functor associated to ϕ is not isomorphic to the identity in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$, so, in particular, ϕ cannot be isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(Y)$.*

Proof We show that there is a closed Lagrangian $\hat{L} \subset Y$ such that $\mathrm{HF}(\hat{L}, \phi\hat{L}) \ncong \mathrm{HF}(\hat{L}, \hat{L})$.

We can assume without loss of generality that the first factor of ϕ is a power of τ_{L_2} and the last is a power of τ_{L_1} (otherwise consider ϕ^{-1}), so that we have a word of shape

$$\phi = \tau_{L_2}^{b_k} \tau_{L_1}^{a_k} \cdots \tau_{L_2}^{b_1} \tau_{L_1}^{a_1}, \quad a_i, b_i \in \mathbb{Z}^* \text{ for } 1 \leqslant i \leqslant k. \tag{42}$$

In the case we are considering, we have $\mathrm{hf}(L_1, L_2) = 2$. Apply Lemma 5.5 to get

$$2 = \mathrm{hf}(L_1, L_1) = \mathrm{hf}(L_2, L_1) = \mathrm{hf}(L_2, \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1) < \mathrm{hf}(L_1, \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1).$$

Now apply Lemma 5.4 (with $n = a_2$, $\tilde{L} = L_1$, $L = L_2$ and $L' = \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1$) and get

$$\mathrm{hf}(L_1, \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1) = \mathrm{hf}(L_1, \tau_{L_1}^{a_2} \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1) < \mathrm{hf}(L_2, \tau_{L_1}^{a_2} \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1).$$

Apply Lemma 5.4 again (with $n = b_2$, $\tilde{L} = L_2$, $L = L_1$ and $L' = \tau_{L_1}^{a_2} \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1$)

$$\mathrm{hf}(L_2, \tau_{L_1}^{a_2} \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1) = \mathrm{hf}(L_2, \tau_{L_2}^{b_2} \tau_{L_1}^{a_2} \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1) < \mathrm{hf}(L_1, \tau_{L_2}^{b_2} \tau_{L_1}^{a_2} \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1).$$

Continue to apply Lemma 5.4 iteratively until the final step

$$\mathrm{hf}(L_2, \tau_{L_2}^{b_k} \tau_{L_1}^{a_k} \cdots \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1) < \mathrm{hf}(L_1, \tau_{L_2}^{b_k} \tau_{L_1}^{a_k} \cdots \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1).$$

Then

$$\mathrm{hf}(L_1, \tau_{L_2}^{b_k} \tau_{L_1}^{a_k} \cdots \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1) > 2 + 2k - 1 = 2k + 1.$$

So, setting $\hat{L} = L_1$, we have $\mathrm{HF}(\hat{L}, \phi(\hat{L})) \ncong \mathrm{HF}(\hat{L}, \hat{L})$. □

Corollary 5.10 *Let $\phi \in \langle \tau_{L_1}, \tau_{L_2} \rangle \subset \mathrm{Symp}_{\mathrm{ct}}(Y)$ be a word of Dehn twists that is a product of the shape (42). Then there is a Lagrangian $\hat{L} \subset Y$ such that*

$$\lim_{s \to \infty} \operatorname{rank} \mathrm{HF}^*(\hat{L}, \phi^s(\hat{L})) = \infty.$$

Proof Let ϕ be of the shape (42). Then

$$\phi^s = (\tau_{L_2}^{b_k} \tau_{L_1}^{a_k} \cdots \tau_{L_2}^{b_1} \tau_{L_1}^{a_1})(\cdots)(\cdots)(\tau_{L_2}^{b_k} \tau_{L_1}^{a_k} \cdots \tau_{L_2}^{b_1} \tau_{L_1}^{a_1})$$

has "factor length" $k \cdot s$ (in the sense of (42)). By the proof of Lemma 5.9, the rank of $\mathrm{HF}(L_1, \phi(L_1))$ depends on the number $k \in \mathbb{N}$ appearing in the factor decomposition of ϕ. Therefore,

$$\mathrm{hf}(L_1, \phi^s(L_1)) > 2ks + 1,$$

so we can set $\hat{L} := L_1$. □

5.2.3 Proof We now go back to the original word $\varphi \in \langle \tau_{K_1}, \tau_{K_2} \rangle \subset \mathrm{Symp}_{\mathrm{ct}}(W)$ of projective twists in the statement of Theorem B, and we show that it cannot induce the identity functor in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$. Lemma 5.6 holds for any symplectomorphism, so, by the same conjugation argument explained before, we can focus the attention on words that are either (for $i, j \in \{1,2\}$)

(1) a power of a single twist $\varphi = \tau_{K_i}^s$, $s \in \mathbb{Z}^*$, or

(2) a mixed product of the shape $\varphi := \tau_{K_i}^{b_k} \tau_{K_j}^{a_k} \cdots \tau_{K_i}^{b_1} \tau_{K_j}^{a_1} \in \mathrm{Symp}_{\mathrm{ct}}(W)$ with $i \neq j$ and $a_m, b_m \in \mathbb{Z}^*$ for $1 \leqslant m \leqslant k$.

Proposition 5.11 *Let $\varphi = \tau_{K_i}^s \in \langle \tau_{K_1}, \tau_{K_2} \rangle \subset \mathrm{Symp}_{\mathrm{ct}}(W)$ be a reduced word of projective twists which is a power of a single twist with $i \in \{1,2\}$ and $s \in \mathbb{Z}^*$. Then the functor induced by φ is not isomorphic to the identity in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$, and in particular φ cannot be isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(W)$.*

Proof Let $\varphi = \tau_{K_i}^s \in \mathrm{Symp}_{\mathrm{ct}}(W)$ with $s \in \mathbb{Z}^*$. Assume by contradiction that the functor induced by φ (still denoted by φ) is isomorphic to the identity, so that any Lagrangian $\widehat{K} \subset W$ is quasi-isomorphic, as an object of $\mathcal{F}\mathrm{uk}(W)$, to $\varphi(\widehat{K})$.

By Lemma 4.18, there is a quasi-isomorphism of the mapping cones of the cup product maps

$$f_1 \colon \mathrm{CF}^*(\widehat{K}, \widehat{K}) \to \mathrm{CF}^{*+k+1}(\widehat{K}, \widehat{K}) \quad \text{and} \quad f_2 \colon \mathrm{CF}^*(\widehat{K}, \varphi(\widehat{K})) \to \mathrm{CF}^{*+k+1}(\widehat{K}, \varphi(\widehat{K}))$$

(we are considering ungraded Floer cohomology groups, so technically the degrees are irrelevant here). Therefore, by the exact triangle of Lemma 4.16, if $\widehat{L} \subset Y$ is the Lagrangian lift of $\widehat{K}$ through the correspondence Γ and $\phi \in \mathrm{Symp}_{\mathrm{ct}}(Y)$ the symplectomorphism associated to φ, then $\mathrm{HF}(\widehat{L}, \widehat{L}) \cong \mathrm{HF}(\widehat{L}, \phi(\widehat{L}))$.

So, if we set $\widehat{K} := K_j$ with $j \neq i$, by assumption we have $\mathrm{HF}(K_j, \varphi(K_j)) \cong \mathrm{HF}(K_j, K_j)$ and the above argument yields $\mathrm{HF}(L_j, \phi(L_j)) = \mathrm{HF}(L_j, \tau_i^s(L_j)) \cong \mathrm{HF}(L_j, L_j)$, which is clearly in contradiction to (the proof of) Lemma 5.7 (according to which these two groups have distinct ranks). Hence, φ cannot be isomorphic to the identity functor in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$. □

Proposition 5.12 *Let $\varphi \in \langle \tau_{K_1}, \tau_{K_2} \rangle \subset \mathrm{Symp}_{\mathrm{ct}}(W)$ be a reduced word of projective twists around the Lagrangian cores which is a product where the first and last factors are powers of distinct projective twists. Then the functor induced by φ is not isomorphic to the identity in $\mathrm{Auteq}\,\mathcal{F}\mathrm{uk}(W)$; so, in particular, φ is not isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(W)$.*

Proof By the analogous discussion in the proof of Lemma 5.9, it is enough to prove the statement for a word whose reduced form is of the shape

$$\varphi := \tau_{K_2}^{b_k} \tau_{K_1}^{a_k} \cdots \tau_{K_2}^{b_1} \tau_{K_1}^{a_1} \in \mathrm{Symp}_{\mathrm{ct}}(W), \quad a_m, b_m \in \mathbb{Z}^*,\ 1 \leqslant m \leqslant k. \tag{43}$$

Denote the product of twist functors induced by (43) also by $\varphi \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$. By iteratively using commutativity of the functors in diagram (29), one can define the corresponding composition of (Dehn) twist functors $\phi \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$, which, by Theorem 5.3, cannot be isomorphic to the identity functor.

Moreover, Corollary 5.10 shows that, not only is $\mathrm{HF}(L_1, \phi(L_1))$ nonisomorphic to $\mathrm{HF}(L_1, L_1)$, but also $\lim_{s\to\infty} \mathrm{hf}(L_1, \phi^s(L_1)) \to \infty$.

The Lagrangian $L_1 = \Gamma \circ K_1 \subset Y$ is the Lagrangian associated to K_1 via the Hopf correspondence, and the symplectic Gysin exact sequence (Corollary 4.17) applied to the Hopf correspondence gives the inequality

$$\mathrm{hf}(L_1, \phi(L_1)) \leqslant 2\mathrm{hf}(K_1, \varphi(K_1)), \tag{44}$$

which implies that rank $\mathrm{hf}(K_1, \varphi^s(K_1))$ also grows at least linearly with s. □

Corollary 5.13 *Let $\varphi \in \langle \tau_{K_1}, \tau_{K_2}\rangle \subset \mathrm{Symp}_{\mathrm{ct}}(W)$ be a word of projective twists of the shape (43). Then there is a Lagrangian $\hat{K} \subset W$ such that*

$$\lim_{s\to\infty} \operatorname{rank} \mathrm{HF}^*(\hat{K}, \varphi^s(\hat{K})) = \infty.$$ □

Finally, we can summarise the proof of Theorem B.

Proof of Theorem B Let $\varphi \in \langle \tau_{K_1}, \tau_{K_2}\rangle \subset \mathrm{Symp}_{\mathrm{ct}}(W)$ be a word in the projective twists along the Lagrangian cores of W.

(1) If the word has the shape $\varphi = \tau_{K_i}^s \in \mathrm{Symp}_{\mathrm{ct}}(W)$ with $i \in \{1,2\}$ and $s \in \mathbb{Z}^*$, then its induced functor is not isomorphic to the identity in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ by Proposition 5.11.

(2) If the word has the shape $\varphi := \tau_{K_i}^{b_k}\tau_{K_j}^{a_k}\cdots\tau_{K_i}^{b_1}\tau_{K_j}^{a_1} \in \mathrm{Symp}_{\mathrm{ct}}(W)$ with $i, j \in \{1,2\}$ and $i \neq j$, and $a_m, b_m \in \mathbb{Z}^*$ for $1 \leqslant m \leqslant k$, then its induced functor is not isomorphic to the identity in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ by Proposition 5.12.

(3) If φ has any other form, then it must be a conjugate of a word of shape (1) or (2) and hence the induced functor is not isomorphic to the identity by Lemma 5.6. □

5.3 Knotted Lagrangian projective spaces

The phenomenon that a single (smooth) isotopy class of submanifolds contains infinitely many Lagrangian isotopy classes is called Lagrangian "knottedness" [Seidel 1999; Evans 2010; Hind 2012; Li and Wu 2012; Wu 2014]. Often, the quest for knottedness is intimately related to the study of isotopy classes of Dehn twists.

In the plumbing of spheres $L_i \cong S^2$ and $Y := T^*L_1 \#_{\mathrm{pt}} T^*L_2$, we know that, for any $r \in \mathbb{Z}$, $\tau_{L_2}^{2r}(L_1)$ is smoothly isotopic to the identity, but not symplectically; as first shown by Seidel [1999, Theorem 1.1], none of the powers $\tau_{L_2}^{2r}(L_1)$ are Hamiltonian isotopic. Our results yield the analogue for plumbing of complex projective spaces (of any dimension):

Corollary 5.14 *Let $W := T^*\mathbb{CP}^n \#_{\mathbb{CP}^l} T^*\mathbb{CP}^n$ be a clean plumbing along a projective subspace $\mathbb{CP}^l \subset \mathbb{CP}^n$. Each Lagrangian core $K_i \cong \mathbb{CP}^n$ of W defines a smooth isotopy class which contains infinitely many symplectic isotopy classes of Lagrangian projective spaces.*

Proof Let $K_1, K_2 \cong \mathbb{CP}^n \subset W$ be the two Lagrangian cores of the plumbing. For $i, j \in \{1,2\}$ with $i \neq j$, define the element $\varphi := \tau_{K_i} \tau_{K_j}$. Then, by Proposition 5.12,

$$\lim_{s\to\infty} \operatorname{rank} \mathrm{HF}(K_j, \varphi^s(K_j)) = \infty, \tag{45}$$

which, in particular, means that $\varphi^{s_a}(K_j)$ is not Lagrangian isotopic to $\varphi^{s_b}(K_j)$ for any $s_a \neq s_b$, despite being smoothly isotopic (by Theorem 2.5). □

Remark 5.15 The low-dimensional case $n = 1$ corresponds to a transverse plumbing of spheres $W := T^*L_1 \#_{\mathrm{pt}} T^*L_2$ and $L_i \cong S^2$. In that case, the symplectic mapping class group $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(W))$ is generated by the Dehn twists τ_{L_1} and τ_{L_2} (see Remark 5.2). Moreover, Hind [2012] proved that, for any Lagrangian sphere $L \subset W$, there is a word $\tau \in \langle \tau_{L_1}, \tau_{L_2} \rangle$ such that $\tau(L)$ is isotopic to one of the cores L_1 or L_2. ◁

6 Positive products of twists in Liouville manifolds

The present section covers our results about products of positive powers of Dehn and projective twists.

In the first part, Section 6.1, we analyse products of (positive powers of) Dehn twists. We reprove a theorem by Barth, Geiges and Zehmisch (Theorem A) asserting that, in a Liouville manifold (M, ω), no product $\phi \in \mathrm{Symp}_{\mathrm{ct}}(M)$ of positive powers of Dehn twists can be symplectically isotopic to the identity. We provide an alternative proof that was suggested by Paul Seidel. Based on symplectic Picard–Lefschetz theory, the argument for the proof relies on a count of pseudoholomorphic sections of a Lefschetz fibration constructed from the data given by ϕ and the Lagrangian spheres associated to the Dehn twists.

Using similar tools, we then prove Theorem C (Section 6.2), which can be interpreted as a relative version of Theorem A. This states that a Liouville manifold (M, ω) containing Lagrangian spheres and a conical Lagrangian disc T (Definition 4.7) intersecting one of the spheres transversely at a point cannot admit a positive product of Dehn twists preserving T up to compactly supported symplectic isotopy.

In Section 6.3, we explore the analogous questions for projective twists, by means of the tools developed in Section 4. After setting the necessary conditions to ensure the existence of the Hopf correspondence, we use Theorem A to prove a comparable result for real projective twists.

6.1 Alternative proof of Theorem A

In this section, we reprove the following theorem:

Theorem A [Barth et al. 2019, Theorem 1.4] *Let (M, ω) be a Liouville manifold, and let $L_1, \dots, L_m \subset M$ be Lagrangian spheres. Let $\phi = \prod_{i=1}^k \tau_{L_{j_i}} \in \mathrm{Symp}_{\mathrm{ct}}(M)$, $j_i \in \{1, \dots, m\}$ be a positive word of Dehn twists. Then ϕ is not compactly supported isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(M)$.*

Example 6.1 The exactness condition of Theorem A is necessary, as the following examples show:

(a) Consider the 2–torus $M := T^2$, and let $a, b \subset M$ represent the longitude and meridian of M. Then the associated Dehn twists satisfy $(\tau_a \tau_b)^6 = \mathrm{Id}$ in $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(M))$. This is a classical result; see for example [Farb and Margalit 2012] (see [Auroux 2003, Section 3.1] for the same example in a symplectic setting).

(b) Let $(M := S^2 \times S^2, \omega_{S^2} \oplus \omega_{S^2})$, and consider the antidiagonal $\bar{\Delta} := \{(x, y) \in S^2 \times S^2 \mid x + y = 0\} \subset M$. Then the Dehn twist $\tau_{\bar{\Delta}}$ is symplectically isotopic to an involution $(x, y) \mapsto (y, x)$, which implies $\tau_{\bar{\Delta}}^2 = \mathrm{Id}$ in $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(M))$ (see [Seidel 2008b, Example 2.9]). ◁

Remark 6.2 (1) The two-dimensional case of Theorem A (for a product of Dehn twists in a Riemann surface) is a consequence of [Smith 2001, Theorem 1.3].

(2) The outcome of Theorem A is strictly geometric, and may not hold for the compact Fukaya category: we are not able to obtain information about the functors associated to the Dehn twists. Consider a punctured torus $M := T^2 \setminus \{*\}$ (the same applies to a punctured genus g surface), and the two (Lagrangian) circles a and b, representatives of the homological generators. In the closed case, the composition $(\tau_a \tau_b)^6$ is isotopic to the identity by the example above. In the punctured torus, there is an isotopy $(\tau_a \tau_b)^6 \simeq \tau_d$, where τ_d is the Dehn twist along the boundary curve d encircling the puncture (this is a consequence of the *chain relation*; see [Farb and Margalit 2012, Proposition 4.12]). But, since the support of τ_d is disjoint from any exact compact circle in M, the product $(\tau_a \tau_b)^6$ still acts as the identity on objects of the *compact* Fukaya category $\mathcal{F}\mathrm{uk}(M)$. ◁

The original proof of [Barth et al. 2019] relies on the theory of open book decompositions, whereas the proof below uses Picard–Lefschetz theory. To simplify notation we prove the version of the theorem where $(M, \omega = d\lambda_M)$ is a Liouville domain.

Let $\phi = \prod_{i=1}^{k} \tau_{L_{j_i}} \in \mathrm{Symp}_{\mathrm{ct}}(M)$ with $j_i \in \{1, \dots, m\}$ be the word in positive powers of Dehn twists in the given collection, and assume by contradiction that this product is compactly supported Hamiltonianly isotopic to the identity (recall ϕ is an exact symplectomorphism).

Let $\pi : (E, \Omega_E, \lambda_E) \to (\mathbb{C}, \lambda_{\mathbb{C}})$ be the exact Lefschetz fibration determined by the data

$$\{(M, \lambda_M), (L_{j_1}, \dots, L_{j_k})\}$$

as in Section 2.2. Let $z_* \in \mathbb{C}$ be the basepoint, so that $\pi^{-1}(z_*) \cong M$, and $\phi \in \mathrm{Symp}_{\mathrm{ct}}(M)$ be the total monodromy of π.

Let $j_{\mathbb{C}}$ be the standard complex structure on $\mathbb{C}$.

By assumption, the monodromy of π is isotopic to the identity via a compactly supported Hamiltonian isotopy $(\phi_t)_{t \in [0,1]}$ with $\phi_0 = \phi$ and $\phi_1 = \mathrm{Id}$. Then π can be extended to a fibration $\hat{\pi} : \widehat{E} \to \mathbb{CP}^1$ as follows. Let $D_R \subset \mathbb{C}$ be a large circle of radius $R > 0$ passing through z_* and containing all the critical values. Define a fibration $E' \to D_{R+1}$ by extending $E|_{D_R}$ to a larger disc D_{R+1} such that, for $t \in [0,1]$, the monodromy around D_{R+t} is $\phi_t \in \mathrm{Symp}_{\mathrm{ct}}(M)$. Then $\hat{E}$ is obtained after gluing E' to a trivial fibration with fibre (M,ω) over a disc neighbourhood of the point at "infinity", $\hat{z} \in \mathbb{C} \cup \{\infty\} \simeq \mathbb{CP}^1$.

Moreover, as the symplectic connection around the fibre $\hat{\pi}^{-1}(\hat{z})$ is trivial, $\hat{\pi}\colon \hat{E} \to \mathbb{CP}^1$ has the following properties:

(1) There is a closed (possibly degenerate) two-form $\widehat{\Omega}_{\hat{E}}$ on $\hat{E}$ satisfying $\widehat{\Omega}_{\hat{E}}|_{\hat{\pi}^{-1}(z)} = \Omega_E|_{\pi^{-1}(z)}$ for all $z \in \mathbb{CP}^1 \setminus \hat{z}$,

(2) A neighbourhood of the horizontal boundary $V \supset \partial^h \hat{E}$ can be trivialised as $V \cong \mathbb{CP}^1 \times M^{\mathrm{out}}$, where $M^{\mathrm{out}} \subset M$ is an open neighbourhood of the boundary of the smooth fibre.

Definition 6.3 The set of almost complex structures compatible with $\hat{\pi}$, denoted by $\mathcal{J}(\hat{E},\hat{\pi},j)$, is defined as follows. An element $\hat{J} \in \mathcal{J}(\hat{E},\hat{\pi},j)$ satisfies:

- $D\hat{\pi} \circ \hat{J} = j \circ D\hat{\pi}$, where j is the standard complex structure on $\mathbb{CP}^1$.
- There is an integrable almost complex structure J_0 such that $\hat{J} = J_0$ in a neighbourhood of $\mathrm{Crit}(\hat{\pi})$.
- For all $z \in \mathbb{CP}^1$, the restriction $J^{vv} := \hat{J}|_{\hat{\pi}^{-1}(z)}$ is an almost complex structure of contact type compatible with the Liouville form λ_M, and its restriction to V is isomorphic to a product $j \times J^{vv}$.
- $\widehat{\Omega}_{\hat{E}}(\cdot, \hat{J}\cdot)$ is symmetric and positive definite. ◁

The form $\widehat{\Omega}_{\hat{E}}$ can be modified to a symplectic form $\widehat{\Omega} := \widehat{\Omega}_E + \hat{\pi}^*(\beta)$ that tames $\hat{J}$ for $\beta \in \Omega^2(\mathbb{CP}^1)$ (similar to [Seidel 2003, Lemma 2.1; McDuff and Salamon 2017, Theorem 6.1.4]).

From now onwards, we fix a generic element $\hat{J} \in \mathcal{J}(\hat{E},\hat{\pi},j)$, so that, by the same arguments as in [Seidel 2003, Lemma 2.4], all the moduli spaces we encounter satisfy the necessary regularity conditions.

Consider the moduli space of closed $(\hat{J}, j)$–holomorphic sections

$$\mathcal{M}_{\hat{J}} = \{u\colon \mathbb{CP}^1 \to \hat{E} \mid \hat{\pi} \circ u = \mathrm{id}_{\mathbb{CP}^1},\ \hat{J} \circ Du = Du \circ j\}. \tag{46}$$

The moduli space has a nonempty boundary, but, as we explain below, this does not cause compactness issues, as the only sections reaching the boundary must be trivial.

Lemma 6.4 *The space $\mathcal{M}_{\hat{J}}$ is not empty. Moreover, there is a compact subset $K \subset \hat{E} \setminus \partial^h \hat{E}$ such that, for all $u \in \mathcal{M}_{\hat{J}}$, either $\mathrm{Im}(u) \subset K$ or u is a trivial (constant) section.*

Proof Let $q \in V$ a point in a neighbourhood $\partial^h \hat{E} \subset V$ of the horizontal boundary as in (2) above. Via the trivialisation of this neighbourhood, one obtains a trivial section $s\colon \mathbb{CP}^1 \to \hat{E}$ with $s(z) = q$ for all $z \in \mathbb{CP}^1$, which is a regular $(\hat{J}, j)$–holomorphic section, so $\mathcal{M}_{\hat{J}}$ is not empty. The rest of the proof follows from a maximum principle as in [Seidel 2003, Lemma 2.2]. □

We can adapt the argument of [Seidel 2003, Lemma 2.3] to the case of closed curves to show that, for our choice of almost complex structure $\hat{J}$, the moduli space $\mathcal{M}_{\hat{J}}$ is a compact smooth manifold with boundary. The only issue that could possibly occur is a loss of compactness for the component containing sections outside the compact part K, which, by Lemma 6.4, can only be trivial sections. These elements have bounded energy, as they are all in the same homology class. By the Gromov compactness theorem, the only noncompact phenomenon that can occur in this case is sphere bubbling. The next lemma shows how to discard bubbles.

Lemma 6.5 *Let u_∞ be the limit of a (sub)sequence of pseudoholomorphic sections $(u_n)_{n\in\mathbb{N}}$ of the Lefschetz fibration $\hat{\pi}$. A component of u_∞ is either an element in the class $[u_i]$ (for $i \in \mathbb{N}$) or is contained in a single fibre. In the latter case, the component is a bubble.*

Proof Let $v_1, v_2, \ldots, v_k$ be the components of u_∞. The limiting curve u_∞ is assumed to be $(\hat{J}, j)$–holomorphic and nonconstant, so it has to have degree one, as $\sum_{j=1}^{k}[\pi \circ v_i] = [\pi \circ u_\infty] = [\mathbb{CP}^1]$. It follows that the degrees of its components sum up to one. All degrees are nonnegative, so there is only one component with degree one. If in addition there were a bubble, it would be represented in a degree zero component and therefore would have to be entirely contained in a fibre (note that, by positivity of intersections, the bubble cannot intersect other fibres).

Since the fibres are exact, there can be no bubbling of the type of Lemma 6.5, so the moduli space $\mathcal{M}_{\hat{J}}$ is compact. □

Lemma 6.6 *Through each point of the smooth fibre M there is at least one holomorphic section $s \in \mathcal{M}_{\hat{J}}$.*

Proof As in the proof of Lemma 6.4, we consider a neighbourhood of the horizontal boundary $V \supset \partial^h \hat{E}$ and $q \in V$ such that $\hat{\pi}(q) =: z_{\text{gen}} \in \mathbb{CP}^1 \setminus \operatorname{Crit} v(\hat{\pi})$ and the trivial section through q is $s \colon \mathbb{CP}^1 \to \hat{E}$. Consider

$$\mathcal{M}(\hat{J}, q) := \{u \in \mathcal{M}_{\hat{J}} \mid q \in \operatorname{Im}(u)\} \subset \mathcal{M}_{\hat{J}}.$$

It is a smooth compact manifold (by the same arguments as for $\mathcal{M}_{\hat{J}}$). Moreover, by Lemma 6.4, the only element in $\mathcal{M}(\hat{J}, q)$ is the trivial $\hat{J}$–holomorphic section $s \colon \mathbb{CP}^1 \to \hat{E}$ through q.

Let $p \in \hat{\pi}^{-1}(z_{\text{gen}})$ be any other point in the fibre of q, and consider a path $\alpha \colon [0,1] \to M$ with $\alpha(0) = p$ and $\alpha(1) = q$. For every point $\alpha(t)$, $t \in [0,1]$, define $\mathcal{M}(\hat{J}, \alpha(t), [s]) := \{u \in \mathcal{M}_{\hat{J}} \mid \alpha(t) \in \operatorname{Im}(u) \text{ and } [u] = [s]\}$. Clearly, $\mathcal{M}(\hat{J}, \alpha(1), [s]) = \mathcal{M}(\hat{J}, q)$.

Consider

$$\mathcal{M}_{\text{cob}} := \bigcup_{t \in [0,1]} \mathcal{M}(\hat{J}, \alpha(t), [s]) \subset \mathcal{M}_{\hat{J}}. \tag{47}$$

The boundary components of (47) are given by $\partial \mathcal{M}_{\text{cob}} = \mathcal{M}(\hat{J}, p, [s]) \sqcup \mathcal{M}(\hat{J}, q)$. We want to show that the space $\mathcal{M}_{\text{cob}}$ is compact, so that it defines a one-dimensional cobordism between $\mathcal{M}(\hat{J}, p, [s])$ and $\mathcal{M}(\hat{J}, q)$. As before, since $\hat{J} \in \mathcal{J}(\hat{E}, \hat{\pi}, j)$ is chosen to be generic, $\mathcal{M}_{\text{cob}}$ is a smooth manifold.

To show that $\mathcal{M}_{\text{cob}}$ is compact, the same strategy applies as in the case of $\mathcal{M}_{\hat{J}}$. In particular, consider a sequence $t_i \subset [0,1]$ and for each i a section $u_i \in \mathcal{M}(\hat{J}, \alpha(t_i), [s]) \subset \mathcal{M}_{\text{cob}}$.

All the sections of the sequence we are considering belong to the same homology class, by definition. In particular, they have the same area, so Gromov's theorem applies. Consequently, as t_i tends to a limit value t_∞, the sequence u_i converges to a stable map u_∞. As before (in the proof of Lemma 6.5), if sphere bubbling occurred, bubbles would have to be "vertical" (meaning entirely contained in the fibres), which is impossible by the exactness of the fibres.

It follows that $\mathcal{M}_{\text{cob}}$ is compact, and hence $\mathcal{M}(\hat{J}, p, [s])$ and $\mathcal{M}(\hat{J}, q)$ are indeed cobordant. Since the signed count of the boundary components of a one-dimensional compact manifold is zero, the zero-dimensional components of the two spaces have the same cardinality. In particular, for any $p \in \hat{\pi}^{-1}(z_{\text{gen}})$, $\mathcal{M}(\hat{J}, p, [s])$ is not empty, which means there is at least one element in $\mathcal{M}_{\hat{J}}$ that passes through p. □

Corollary 6.7 *The map induced by the evaluation map*

$$\text{ev}\colon \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1 \to \hat{E}, \quad (u,z) \mapsto \text{ev}_z(u) = u(z), \tag{48}$$

is surjective.

Proof By Lemma 6.6, the image of the map (48) is dense, since each point on a smooth fibre has a preimage. As $\mathcal{M}_{\hat{J}}$ is compact and the mapping is continuous, the result extends to all points of $\hat{E}$ and hence (48) is surjective. □

Proof of Theorem A Assume by contradiction that the product $\phi = \tau_{L_{j_1}} \cdots \tau_{L_{j_k}}$ is isotopic to the identity, and build the fibration $\hat{\pi}\colon \hat{E} \to \mathbb{CP}^1$ and the moduli space $\mathcal{M}_{\hat{J}}$ as above. By Corollary 6.7, the evaluation map $\text{ev}\colon \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1 \to \hat{E}$ is surjective. Consider the commuting diagram

$$\begin{array}{ccc} \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1 & \xrightarrow{\ \text{ev}\ } & \hat{E} \\ & \searrow^{\text{pr}_2} & \downarrow{\scriptstyle \hat{\pi}} \\ & & \mathbb{CP}^1 \end{array}$$

where $\text{pr}_2\colon \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1 \to \mathbb{CP}^1$ is the projection to the second factor.

Let $x \in \text{Crit}(\hat{\pi}) \subset \hat{E}$ be any point in the critical set. By the surjectivity of ev, there is a pair $(u,w) \in \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1$ such that $u(w) = x$, so that $w \in \mathbb{CP}^1$ is the critical value associated to x. From the diagram, we obtain

$$D_{(u,w)}(\text{pr}_2) = D_{(u,w)}(\hat{\pi} \circ \text{ev}), \quad D_{(u,w)}(\text{pr}_2) = D_x\hat{\pi}\, D_{(u,w)}(\text{ev}).$$

As x is a critical point, $D_x\hat{\pi} = 0$, which forces $D_{(u,w)}(\text{pr}_2)$ to be the zero map. But this is in contradiction with $D_{(u,w)}(\text{pr}_2)$ being surjective. □

Corollary 6.8 *There is no exact Lefschetz fibration with global monodromy symplectically isotopic to the identity, except for the trivial fibration.* □

6.2 Relative version

Let

(49) $$M := T^*S^m \#_{\text{pt}} T^*S^m \#_{\text{pt}} T^*S^m \#_{\text{pt}} \cdots \#_{\text{pt}} T^*S^m$$

be a "multiplumbing" of m spheres (an iterated construction of transverse plumbing of spheres; see Section 5.1 for the definition of plumbing). By Theorem A, we know that no product $\phi \in \operatorname{Symp}_{\text{ct}}(M)$ of Dehn twists along the core spheres can be compactly supported symplectically isotopic to the identity. However, the theorem, a priori, doesn't prevent such a product from acting trivially on some Lagrangian submanifolds of M. Is it possible to tell whether there are Lagrangians that detect the nontriviality of ϕ? Let T be a cotangent fibre of the j^{th} T^*S^n–summand for $j \in \{1, \dots, m\}$. The theorem we prove in this section shows that any product of positive Dehn twists along Lagrangian cores and involving the j^{th} sphere does not preserve T up to compactly supported symplectic isotopy.

Theorem C *Let (M^{2n}, ω) be a Liouville manifold containing embedded Lagrangian spheres $L_1, \dots, L_m$ and a conical Lagrangian disc T intersecting one of the spheres L_j transversely in a point. Let $\phi := \prod_{i=1}^k \tau_{L_{j_i}} \in \operatorname{Symp}_{\text{ct}}(M)$ with $j_i \in \{1, \dots, m\}$ be a positive word of Dehn twists involving τ_{L_j}. Then the Lagrangians T and $\phi(T)$ are not isotopic via a compactly supported Lagrangian isotopy.*

We prove the statement of Theorem C in the equivalent version where $(M, \omega = d\lambda_M)$ is a Liouville domain and $T \subset M$ is a Lagrangian disc preserved by the Liouville flow near the boundary ∂M (so that $\partial T \subset \partial M$). This is only chosen so that the Lefschetz fibrations involved have exact compact fibres.

As in the statement, write $\phi = \prod_{i=1}^k \tau_{L_{j_i}}$ with $j_i \in \{1, \dots, m\}$. By assumption, there is at least one index $l \in \{1, \dots, k\}$ such that $j_l = j$. Assuming $\phi(T) \simeq T$ via a compactly supported isotopy, we arrive at the contradictory statement $j \notin \{j_1, \dots, j_k\}$.

From the data $(M, (L_{j_1}, \dots, L_{j_l}, \dots, L_{j_k}))$, build an exact Lefschetz fibration $\pi' \colon (E', \Omega_{E'}, \lambda_{E'}) \to (\mathbb{C}, \lambda_{\mathbb{C}})$ with smooth fibre the Liouville domain $(M, d\lambda)$, basepoint $z_* \in \mathbb{R}$ with $z_* \gg 0$ such that the k critical values $\operatorname{Crit} v(\pi') = \{w_{j_1}, \dots, w_{j_l}, \dots, w_{j_k}\}$ are ordered vertically on the imaginary line, $\operatorname{Crit} v(\pi) \subset i\mathbb{R}$ with a basis of vanishing paths $(\gamma_{j_1}, \dots, \gamma_{j_k})$ [Seidel 2008a, (16e)].

Let $(\Delta_{\gamma_{j_1}}, \dots, \Delta_{j_k})$ be the corresponding basis of Lefschetz thimbles and $V_{j_i} := \pi^{-1}(z_*) \cap \Delta_{\gamma_{j_i}}$ for $i = 1, \dots, k$, be the associated vanishing cycles, which, under the identification $\pi^{-1}(z_*) = M$, correspond to L_{j_i}. Let $\sigma \colon S^1 \to \mathbb{C}$ be a loop encircling all critical values.

Build a new exact fibration $\pi \colon (E, \Omega_E, \lambda_E) \to (\mathbb{C}, \lambda_{\mathbb{C}})$ associated to the data

$$(M, (V_{j_1}, \dots, V_{j_l}, \dots, V_{j_k}, V_{j_l})),$$

with basepoint $z_* \in \mathbb{C}$, an extra critical value $w_{j+1} \in \operatorname{Crit} v(\pi) \subset i\mathbb{R}$ and an extra vanishing path $\gamma_{j_{k+1}}$ such that $\operatorname{Im}(\gamma_{j_{k+1}}) \cap \operatorname{Im}(\sigma) = \varnothing$ (all the other choices are the same as for π').

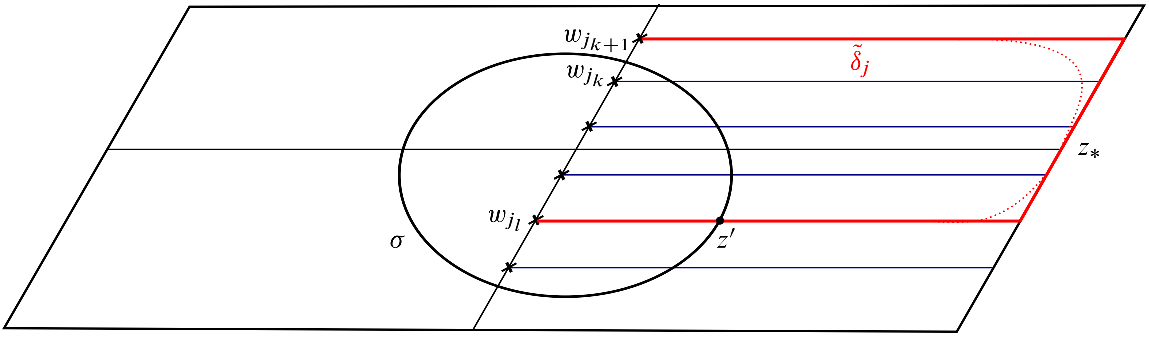

Figure 2: The new fibration π has an extra critical value $w_{j_{k+1}}$ and a matching sphere that fibres over the smoothing δ_j of the red arc $\tilde{\delta}_j$.

Compared to π', there are now two critical points w_{j_l} and $w_{j_{k+1}}$ associated to the same vanishing cycle V_{j_l}. Therefore, there is a matching path $\delta_j : [0,1] \to \mathbb{C}$ with $\delta_j(0) = w_{j_l}$ and $\delta\bigl(\frac{1}{2}\bigr) = z_*$, $\delta_j(1) = w_{j_{k+1}}$ whose parallel transport is a Lagrangian matching sphere $S_j \cong S^{n+1} \subset E$ (Section 2.2 and [Seidel 2008a, (16g)]) fibred by Lagrangians isomorphic to L_j (see Figure 2). Let $z' \in \mathrm{Im}(\delta_j) \cap \mathrm{Im}(\sigma)$, and, via parallel transport, identify $T \subset \pi^{-1}(z_*)$ with a copy of the Lagrangian in $\pi^{-1}(z')$.

By construction, the monodromy around σ is given by the product ϕ. By assumption, there is an isotopy $\phi(T) \simeq T$, so parallel transport of T along σ yields a well-defined Lagrangian $P_\sigma \subset E$. For $z \in \mathrm{Im}(\sigma)$, let $T_z \subset \pi^{-1}(z)$ be the exact fibres of P_σ. Then $\Omega_E|_{P_\sigma} = df_\sigma + \pi^*(\kappa_\sigma)$ for a function $f_\sigma \in C^\infty(P_\sigma, \mathbb{R})$ such that, for every $z \in \mathrm{Im}(\sigma)$, $f_\sigma|_{\pi^{-1}(z)}$ makes T_z exact and $\kappa_\sigma \in \Omega^1(\mathrm{Im}(\sigma))$ [Seidel 2003, Lemma 1.3].

Lemma 6.9 *The Lagrangian P_σ defines a nontrivial class in $H_{n+1}(E, \partial E; \mathbb{Z})$.*

Proof The matching sphere S_j and the disc bundle P_σ are properly embedded Lagrangian submanifolds meeting transversely at the point $y \in L_j$ lying over the intersection between σ and the matching path associated to S_j. Their homological intersection, which is the image of a nondegenerate pairing

$$H_{n+1}(E; \mathbb{Z}) \times H_{n+1}(E, \partial E; \mathbb{Z}) \to \mathbb{Z},$$

is one, so, in particular, P_σ represents a nontrivial homology class in $H_{n+1}(E, \partial E; \mathbb{Z})$. □

6.2.1 Proof of Theorem C Let $D \subseteq \mathbb{C}$ be the disc bounded by the loop σ in the base of π. The idea for the proof of Theorem C is based on a section count which follows the same principles as Section 6. In this context, however, we consider pseudoholomorphic sections defining boundary conditions for $E|_D$ on P_σ.

Let $\mathcal{J}(\pi, E, j_{\mathbb{C}})$ be the set of almost complex structures compatible with π (see Definition 2.15), where $j_{\mathbb{C}}$ is the standard complex structure on $\mathbb{C}$. For a generic element $J \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$, let

(50) $$\mathcal{M}(J, P_\sigma) := \{u : (D, \partial D) \to (E, P_\sigma) \mid \pi \circ u = \mathrm{id}_D,\ J \circ Du = Du \circ j_{\mathbb{C}}|_D\}$$

be the moduli space of pseudoholomorphic sections with boundary condition on P_σ.

The Lagrangian P_σ is fibred by copies of the exact Lagrangian $T \subset M$, and therefore $P_\sigma \cap \partial^h E \neq \varnothing$, ie it is not disjoint from the horizontal boundary. As a result, the moduli space $\mathcal{M}(J, P_\sigma)$ is not compact, but fortunately its noncompact ends are very well behaved.

Below, we show that, for a generic almost complex structure in $\mathcal{J}(\pi, E, j_{\mathbb{C}})$, the "noncompact" elements (those sections reaching the horizontal boundary) of the moduli space (50) are regular. We do this by showing that such sections must be trivial — and the trivial section can be made regular, as the almost complex structure is product-like near $\partial^h E$. For all the other holomorphic sections, which are entirely contained in the compact region, the same regularity arguments as in [Seidel 2003] apply.

Lemma 6.10 *There is $\hat{J} \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$ with the following property: there is no $\hat{J}$–holomorphic section $v\colon (D, \partial D) \to (E, P_\sigma)$ with boundary condition on P_σ such that there are $z_1, z_2 \in \partial D$ with $v(z_1) \in P_\sigma \setminus (P_\sigma \cap \partial^h E)$ and $v(z_2) \in P_\sigma \cap \partial^h E$.*

Proof We show that any generic element $J \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$ can be deformed to an almost complex structure $\hat{J} \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$ as in the statement. To do that we use a *reverse isoperimetric inequality* from [Groman and Solomon 2014] that applies to the Liouville completion of E.

Identify a collar neighbourhood of $\partial^h E$ with $C(\partial^h E) := \mathbb{C} \times ((-\varepsilon, 0] \times \partial M))$, and consider the Liouville completion of E, $(\bar{E}, \omega_{\bar{E}})$, obtained by gluing a cylindrical end $U^h := \mathbb{C} \times ([0, \infty) \times \partial M)$ along a collar neighbourhood of the horizontal boundary $\partial^h E$, such that $\omega_{\bar{E}}|_{U^h} = d(\lambda_{\mathbb{C}} + e^t \lambda_M|_{\partial M})$ for the coordinate t on $[0, \infty)$.

Let $(\bar{M}, \bar{\omega})$ be the generic smooth fibre of $\bar{E}$, and $\bar{T} \subset \bar{M}$ the Lagrangian obtained from T by gluing a conical end at the boundary. Accordingly, let $\bar{P}_\sigma \subset \bar{E}$ be the "completion" of $P_\sigma \subset E$ in $\bar{E}$. This Lagrangian can be trivialised outside of a compact set as $\partial D \times U^\infty \subset \mathbb{C} \times U^\infty \subset \bar{E}$, where $U^\infty \subset \bar{M}$ is a neighbourhood of the cylindrical end of $\bar{T}$. Extend J to a cylindrical almost complex structure $\bar{J}$ on $\bar{E}$ (see Definition 2.12).

By [Ganatra et al. 2020, Lemma 2.43], $(\bar{E}, \omega_{\bar{E}})$ has *bounded geometry* in the sense of [Ganatra et al. 2020, Definition 2.42], which is equivalent to the notion of bounded geometry of [Groman and Solomon 2014, Section 1.4]; see [Ganatra et al. 2020, page 104]. The same holds for the Lagrangian $\bar{P}_\sigma \subset \bar{E}$, as it is compact in the base direction, and conical in the fibre direction (see the proof of [Ganatra et al. 2020, Lemma 2.43]). Bounded geometry implies that for any $\bar{J}$–holomorphic section $u\colon (D, \partial D) \to (\bar{E}, \bar{P}_\sigma)$ there is a reverse isoperimetric inequality [Groman and Solomon 2014, Theorem 1.4]

$$\ell(u|_{\partial D}) \leqslant a(u) \cdot C, \tag{51}$$

where ℓ is the length function associated to a $\bar{J}$–compatible metric $g_{\bar{J}}$, $C > 0$ is a constant depending on $\bar{E}$, and $a(u)$ is the area of the curve.

Let $A := \int_{\partial D} \kappa_\sigma$ (for $\kappa_\sigma \in \Omega^1(\mathrm{Im}(\sigma))$ as above), and set $R := A \cdot C$. For $R > 0$, consider a piece of symplectisation $(E_{R+1} := E \cup \mathbb{C} \times ([0, R+1] \times \partial M), \omega_{R+1})$ with $\omega_{R+1}|_E = \omega_E$ and $\omega_{R+1}|_{\mathbb{C} \times ([0,R+1] \times \partial M)} = d(\lambda_{\mathbb{C}} + e^t \lambda_M)$, and a compatible almost complex structure $J_{R+1} = \bar{J}|_{E_{R+1}}$ of contact type. Clearly

$E \subset E_{R+1} \subset \overline{E}$, and there is a diffeomorphism $\psi\colon E_{R+1} \to E$ that is the identity on $E \setminus C(\partial^h E)$ and compresses $\mathbb{C} \times ((-\varepsilon, R+1] \times \partial M)$ to $\mathbb{C} \times ((-\varepsilon, 0] \times \partial M)$ via the negative Liouville flow.

Every $\overline{J}$–holomorphic curve $u\colon (D, \partial D) \to (\overline{E}, \overline{P}_\sigma)$ such that there are $z_1, z_2 \in \partial D$ with $u(z_1) \in \operatorname{Int}(E)$ and $u(z_2) \in \overline{E} \setminus E_{R+1}$ satisfies $d(u(z_1), u(z_2)) > A \cdot C$ and the inequality (51)

Now set $\widehat{J} := \psi_*(J_{R+1})$. This satisfies the requirements of the lemma.

Namely, let $v\colon (D, \partial D) \to (E, P_\sigma)$ be a $\widehat{J}$–holomorphic section as in the statement, ie such that there are $z_1, z_2 \in \partial D$ with $v(z_1) \in P_\sigma \setminus (P_\sigma \cap \partial^h E)$ and $v(z_2) \in P_\sigma \cap \partial^h E$. Then we certainly have $d(v(z_1), v(z_2)) < \ell(v|_{\partial D})$ for the distance function d and the length ℓ associated to a compatible metric $g_{\widehat{J}}$. On the other hand, the area of v is bounded by a fixed upper bound since $a(v) = \int_D v^*\Omega_E = \int_D d(v^*\lambda_E) = \int_{\partial D} v^*(\lambda_E) = \int_{\partial D} \kappa_\sigma = A$ by exactness of Ω_E and fibrewise exactness of P_σ.

By stretching the neck in a neighbourhood of the boundary of E to E_{R+1}, the pullback $\psi^*(v)$ produces a contradiction, since $d\big(\psi^*(v(z_1)), \psi^*(v(z_2))\big) < \ell(\psi^*(v|_{\partial D})) < a(\psi^*(v)) \cdot C = A \cdot C$, but also $d\big(\psi^*(v(z_1)), \psi^*(v(z_2))\big) > A \cdot C$ by construction of E_{R+1}. □

From now onwards, fix an almost complex structure $\widehat{J} \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$ as in Lemma 6.10. The above results imply that the only possible scenario left to consider in the case of a nonconstant section with boundary condition on P_σ intersecting $\partial^h E$ is to be entirely contained in the horizontal boundary of the fibration.

Lemma 6.11 *Let $u\colon D \to E$ be a $\widehat{J}$–holomorphic section such that* $\operatorname{Im}(u) \subset \partial^h E$. *Then u is a constant section.*

Proof Assume there is a nonconstant section $u\colon D \to E$ such that $\operatorname{Im}(u) \subset \partial^h E$. Identify (via a trivialisation as in (8)) a neighbourhood of $\partial^h E$ as $U^\partial \cong \mathbb{C} \times M^{\text{out}} \subset \mathbb{C} \times M$ for a collar neighbourhood $M^{\text{out}} \subset M$ of ∂M. Then the projection of $\operatorname{Im}(u)$ to M defines a nonconstant $\widehat{J}|_M$–holomorphic disc $u\colon (D, \partial D) \to (M, T)$, which, by the exactness assumptions on M, cannot exist. Therefore, u must be a constant section. □

We now prove that there are no compactness issues. The moduli space $\mathcal{M}(\widehat{J}, P_\sigma)$ has noncompact end, but by the regularity discussion above, the only sections reaching it are the constant ones, and all elements of $\mathcal{M}(\widehat{J}, P_\sigma)$ have bounded energy so that the Gromov compactness theorem applies. The bubbles in the Gromov limit of a sequence of $(\widehat{J}, j_{\mathbb{C}})$–holomorphic sections in $\mathcal{M}(\widehat{J}, P_\sigma)$ are either spheres in the fibres over D, or discs in the fibres $\pi^{-1}(z)$ for $z \in \operatorname{Im}(\sigma)$ with boundary condition on T_z. Both options can be discarded by exactness of E and fibrewise exactness of P_σ.

Lemma 6.12 *The evaluation map*

$$\operatorname{ev}\colon \mathcal{M}(\widehat{J}, P_\sigma) \times D \to E, \quad (u, z) \mapsto u(z), \tag{52}$$

(i) *is proper;*

(ii) *restricts to a surjective map $\mathcal{M}(\widehat{J}, P_\sigma) \times \partial D \to P_\sigma$ of degree one.*

Proof (i) To prove this property is enough to show that every sequence of sections $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{M}(\widehat{J}, P_\sigma)$ whose image under ev lies in a relatively compact set of E has a convergent subsequence. Consider such a sequence. If its image under (52) lie in a compact set, then, by exactness, there is an upper bound to the energy of all elements in the sequence (which is bounded by a finite value determined by the maximum among all areas of the curves). Then, by the Gromov compactness theorem, $\{u_k\}_k$ admits a subsequence converging to a stable map, which, in the absence of bubbles, can only be another section.

(ii) To prove the second point, we show that the algebraic count of sections through every point of P_σ is one. Let $U^\partial \supset \partial^h E$ be a neighbourhood of the horizontal boundary as in the proof of the previous lemma and $q \in U^\partial \cap \pi^{-1}(\sigma)$. Since ϕ is compactly supported in a neighbourhood of the vanishing cycles, the monodromy around σ preserves q. By Lemmas 6.10 and 6.11, the moduli space $\mathcal{M}(\widehat{J}, q) := \{u \in \mathcal{M}(\widehat{J}, P_\sigma) \mid q \in \mathrm{Im}(u)\} \subset \mathcal{M}(\widehat{J}, P_\sigma)$ is compact and only contains the constant section $s\colon D \to E$ through q.

Given another point $p \in P_\sigma$, consider the path $\alpha\colon [0,1] \to M$ with $\alpha(0) = p$ and $\alpha(1) = q$, and define

$$\mathcal{M}(\widehat{J}, P_\sigma, \alpha(t), [s]) := \{u \in \mathcal{M}(\widehat{J}, P_\sigma) \mid \alpha(t) \in \mathrm{Im}(u) \text{ and } [u] = [s]\}.$$

Clearly, $\mathcal{M}(\widehat{J}, P_\sigma, \alpha(1), [s]) = \mathcal{M}(\widehat{J}, q)$.

Consider

$$\mathcal{M}_{\mathrm{cob}} := \bigcup_{t\in[0,1]} \mathcal{M}(\widehat{J}, P_\sigma, \alpha(t), [s]) \subset \mathcal{M}(\widehat{J}, P_\sigma). \tag{53}$$

All elements in $\mathcal{M}_{\mathrm{cob}}$ are in the same homology class so that the same compactness arguments apply as above. Compactness implies that, for every $p \in P_\sigma$, the moduli space $\mathcal{M}(\widehat{J}, P_\sigma, p, [s])$ is cobordant to the moduli space $\mathcal{M}(\widehat{J}, q)$. Therefore, by the same reasoning as in the proof of Lemma 6.6, through each point of P_σ there is algebraically a unique section in $\mathcal{M}(\widehat{J}, P_\sigma)$, so the restriction $\mathcal{M}(\widehat{J}, P_\sigma) \times \partial D \to P_\sigma$ is surjective and of degree one. □

Proof of Theorem C Under the assumption that $\phi(T) \simeq T$, we have proved that P_σ represents a nontrivial class in $H_{n+1}(E, \partial E)$ (Lemma 6.9). The same assumption however also yields Lemma 6.12, which in particular implies that $\mathrm{ev}_*(\mathcal{M}_{\widehat{J}}(E, P_\sigma) \times \partial D) = [P_\sigma] \in H_{n+1}(E, \partial E)$ is realised as the boundary of the chain $\mathrm{ev}_*(\mathcal{M}_{\widehat{J}}(E, P_\sigma) \times D) \in C_{n+2}(E, \partial E)$. This is a contradiction. □

6.3 Product of projective twists

We continue the investigation on positive products of twists in Liouville manifolds, this time focussing on projective twists. Ideally, one would try to generalise as many results from the previous sections to this situation.

The previous section heavily relied on the link between Dehn twists and Lefschetz fibrations, and many constructions we used depended on section-count invariants of Lefschetz fibrations.

Perutz [2007] showed that any *fibred twist* admits a representation as the local monodromy of a Morse–Bott–Lefschetz (MBL) fibration. Projective twists can be thought of as an example of S^1–fibred twists, so we could envisage extending the mechanisms behind the proof for the spherical case to the setting of MBL fibrations (following [Perutz 2007; Wehrheim and Woodward 2016]) to show the analogous statement for projective twists.

Question 2 Let $\varphi \in \mathrm{Symp}_{\mathrm{ct}}(W)$ be a nonempty composition of positive powers of projective twists on a Liouville manifold (W, ω) of dimension at least four. Can φ be isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(W)$?

Unfortunately, the section-count strategy presents a route filled with obstacles, the central problem being the lack of compactness of moduli spaces of sections of MBL fibrations. The critical locus $\mathrm{Crit}(\pi)$ of such a fibration is a compact symplectic submanifold of the total space, and in general contains rational curves. The total space of a MBL fibration $\pi : E \to \mathbb{C}$ associated to a projective twist cannot be made into an exact symplectic manifold, so bubbling phenomena can become an issue when considering moduli spaces of pseudoholomorphic sections.

Instead, the idea remains, as in Section 5, to use the Hopf correspondence to translate a situation involving projective twists into one involving Dehn twists.

Theorem D *Let (W^{2n}, ω) be a Liouville manifold containing Lagrangian real projective spaces $K_1, \dots, K_m$ with $K_i \cong \mathbb{RP}^n$. Suppose that there is a class $\alpha \in H^1(W; \mathbb{Z}/2\mathbb{Z})$ such that, for every $i = 1, \dots, m$, $\alpha|_{K_i}$ generates $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$. Let $\varphi \in \mathrm{Symp}_{\mathrm{ct}}(W)$ be a positive word in the subset of projective twists $\{\tau_{K_i}\}_{i \in \{1,\dots,m\}}$. Then φ is not isotopic to the identity in $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(W))$.*

Proof As in Section 4.3.2, let $q : (\widetilde{W}, \widetilde{\omega}) \to (W, \omega)$ be the symplectic double cover given by the class α and $L_1, \dots, L_m \subset \widetilde{W}$ Lagrangian spheres obtained as double cover of $K_1, \dots, K_m \subset W$. The composition of projective twists $\varphi \in \mathrm{Symp}_{\mathrm{ct}}(W)$ lifts to a composition of spherical Dehn twists $\phi \in \mathrm{Symp}_{\mathrm{ct}}(\widetilde{W})$. Assume there is an isotopy $(\varphi_t)_{0 \leqslant t \leqslant 1}$ connecting the composition of projective twists $\varphi_0 = \varphi$ to the identity $\varphi_1 = \mathrm{Id}$. The isotopy lifts to a family of compactly supported maps $(\phi_t)_{0 \leqslant t \leqslant 1}$ in the double cover $\widetilde{W}$, where $\phi_0 = \phi$ is the lift of φ. Then ϕ_1 covers the identity and can therefore only be either the identity or a deck transformation. The latter type would define a noncompactly supported symplectomorphism, hence ϕ_1 must be the identity. It follows that $\phi \in \mathrm{Symp}_{\mathrm{ct}}(\widetilde{W})$ is a composition of Dehn twists in a Liouville domain which is isotopic to the identity, contradicting Theorem A. □

Remark 6.13 A similar argument fails when applied to complex projective twists. Let (W^{4n}, ω) be a symplectic manifold with complex projective Lagrangians $K_1, \dots, K_m$ satisfying Assumption (CX). The fibration $(Y, \Omega) \to (W, \omega)$ constructed from the cohomological condition is not proper, so an isotopy in $\mathrm{Symp}_{\mathrm{ct}}(W)$ cannot be lifted to an isotopy in $\mathrm{Symp}_{\mathrm{ct}}(Y)$. ◁

7 Epilogue: framings of projective twists, homotopy projective Lagrangians

As a last application of the Hopf correspondence, we examine homotopy projective Lagrangians. We prove two nonembedding results for Lagrangian projective spaces in nonstandard homeomorphism/diffeomorphism classes (Theorems E and F), and, for $n \geqslant 19$, the existence of projective twists obtained from a nonstandard choice of framing that are not Hamiltonian isotopic to the standard $\tau_{\mathbb{CP}^n} \in \mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$:

Theorem G *The $\mathbb{CP}^n$–twist depends on the framing when $n = 19, 23, 25, 29$.*

Embedding theorems are obtained in Section 7.1 using homotopy theory results combined with the existing state of the art of the nearby Lagrangian conjecture, and the use of the Hopf correspondence.

We subsequently investigate the question of framings for projective twists in Section 7.2. For that purpose, we utilise the current literature on framing of Dehn twists, a pairing constructed by Bredon, and the Hopf correspondence. This enables us to obtain instances in which the (Hamiltonian isotopy class of the) local projective twist does depend on a choice of framing of the associated Lagrangian projective space. With the additional use of topological modular forms, we explain why there should be infinitely many such examples.

7.1 Lagrangian nonembeddings of projective spaces

The *nearby Lagrangian conjecture* states that, given a closed smooth manifold Q, any closed exact Lagrangian submanifold of $(T^*Q, d\lambda_Q)$ is Hamiltonian isotopic to the zero section. If this conjecture were true, the existence of another closed exact Lagrangian embedding $L \hookrightarrow T^*Q$ would yield a diffeomorphism $L \cong Q$. By Weinstein's neighbourhood theorem, the latter version of the statement can also be read as: if $(T^*L, d\lambda_{T^*L})$ is symplectomorphic to $(T^*Q, d\lambda_{T^*Q})$, then L is diffeomorphic to Q.

The conjecture has been verified for some specific examples (T^*S^2 and $T^*\mathbb{RP}^2$ by [Hind 2012; Li and Wu 2012] and T^*T^2 by Dimitroglou Rizell, Goodman and Ivrii [Dimitroglou Rizell et al. 2016]), and weaker versions of it have been proved. Currently, the most general feature one can deduce from an exact Lagrangian embedding in $(T^*Q, d\lambda_{T^*Q})$ is (simple) homotopy equivalence:

Theorem 7.1 [Abouzaid 2012b; Kragh 2013; Abouzaid and Kragh 2018] *If $L \subset T^*Q$ is a closed, exact Lagrangian embedding, then the projection $L \subset T^*Q \xrightarrow{p} Q$ is a (simple) homotopy equivalence.*

Remark 7.2 If $L \subset T^*Q \xrightarrow{p} Q$ is a Lagrangian as in the above statement, then $TL \otimes \mathbb{C} \cong p^*(TQ \otimes \mathbb{C})$. The Pontryagin classes $p_i \in H^{4k}(\cdot)$ satisfy $2p_i(L) = 2p_i(Q)$. Moreover, the (rational) Pontryagin classes p_i are homeomorphism invariants [Novikov 1965]. ◁

Equipped with the connected sum operation, the set of h–cobordism classes of homotopy m–spheres Θ_m has an abelian group structure (where the standard sphere plays the role of neutral element). We will always assume $m > 5$, in which case the elements of Θ_m correspond to diffeomorphism classes of m–spheres.

The group Θ_m fits in an exact sequence [Kervaire and Milnor 1963]

$$0 \to \mathrm{bP}_{m+1} \to \Theta_m \xrightarrow{\psi} \operatorname{coker}(J_m) \to \mathrm{bP}_m. \tag{54}$$

Here $\mathrm{bP}_{m+1} = \ker(\psi) \subset \Theta_m$ denotes the subgroup of homotopy m–spheres bounding an $(m+1)$–dimensional parallelisable manifold, and $J_m\colon \pi_m(O) \to \pi_m(S)$ is a map from the m^{th} stable homotopy group $\pi_m(O) = \lim_{l\to\infty} \pi_m(\mathrm{SO}(l))$ to the m^{th} stable homotopy group of spheres $\pi_m(S) := \lim_{l\to\infty} \pi_{m+l}(S^l)$ (see eg [Levine 1985, Section 3]). This group is also called the *m^{th} stable stem.*

Throughout the section, we will repeatedly use the following fact about the sequence (54):

Theorem 7.3 [Kervaire and Milnor 1963, Theorem 5.1] *If m is an odd integer,* $\mathrm{bP}_m = 0$. *Consequently, for any odd m, $\psi\colon \Theta_m \to \operatorname{coker}(J_m)$ is surjective.*

In the symplectic setting, homotopy spheres are good candidates to test the nearby Lagrangian conjecture.

Theorem 7.4 [Abouzaid 2012a] (extended by [Ekholm et al. 2016]) *Let $m > 4$ odd. If $\Sigma, \Sigma' \in \Theta_m$ and $T^*\Sigma$ is symplectomorphic to $T^*\Sigma'$, then $[\Sigma] = \pm[\Sigma'] \in \Theta_m/\mathrm{bP}_{m+1}$.*

It will be practical to paraphrase the above theorem as follows:

Corollary 7.5 *If $m > 4$ is odd and $\Sigma \in \Theta_m \setminus \mathrm{bP}_{m+1}$, then Σ does not admit a Lagrangian embedding into T^*S^m.*

Definition 7.6 We choose to depart from the classic terminology of *exotic* manifolds. Instead, we will call a smooth manifold that is homeomorphic, but not diffeomorphic, to the standard sphere an *AD* sphere (AD stands for alternative differentiable structure). Correspondingly, a smooth manifold that is homeomorphic, but not diffeomorphic, to the standard $\mathbb{CP}^n$ will be called an *AD* projective space. Finally, a smooth manifold that is homotopy equivalent, but not homeomorphic, to the standard projective space will be called an *AT* projective space (where AT stands for alternative topological structure). ◁

7.1.1 Results The results of this section hinge on the existence of homotopy projective spaces that are obtained as the reduced space of a circle action on an AD sphere. It is not always possible to relate an n–dimensional AD/AT projective space to a $(2n+1)$–dimensional AD sphere in this way. Below, we start by exploring a few facts about AD/AT projective spaces, after which we can discuss three interesting examples where the desired phenomenon is observed (the spaces of Theorems E and F).

Definition 7.7 [Kawakubo 1969] The inertia group $I(M)$ of an oriented closed smooth manifold M is the subgroup of Θ_m consisting of homotopy spheres $S \in \Theta_m$ such that the connected sum $M \# S$ is in the same diffeomorphism class as M. ◁

If $I(\mathbb{CP}^n)=0$ and Θ_{2n} is nontrivial, one can build an AD projective space as follows. Given an AD sphere $\Sigma \in \Theta_{2n}$ the connected sum $\mathbb{CP}^n \# \Sigma$ (a zero-dimensional surgery) is another manifold homeomorphic to $\mathbb{CP}^n$ but not diffeomorphic to it. For $n \geqslant 8$, there are examples for which the inertia group $I(\mathbb{CP}^n)$ is nontrivial (see [Kawakubo 1969]); in those cases the smooth structure of the resulting manifold is not automatically distinct from the standard smooth structure on $\mathbb{CP}^n$. In dimension four, we know:

Theorem 7.8 [Kasilingam 2016] *There are two possible distinct smooth structures on a manifold homeomorphic to $\mathbb{CP}^4$: the standard $\mathbb{CP}^4$–structure, and the one on $\mathbb{CP}^4 \# \Sigma^8$, where $\Sigma^8 \in \Theta_8$ is the unique AD 8–sphere.*

In contrast, it is known that there is an abundance of AT projective spaces: for even integers $n \geqslant 4$, there are infinitely many AT projective spaces, distinguished by the first Pontryagin class $p_1 \in H^4(\mathbb{CP}^n;\mathbb{Z})$ [Hsiang 1966].

Is there a way to associate an AD sphere to an AD/AT projective space? Given an AD/AT projective space K, the unit bundle of the line bundle $\mathcal{L} \to K$ satisfying $c_1(\mathcal{L}) = \alpha_K$ (where $\alpha_K \in H^2(K;\mathbb{Z})$ is the cohomology generator) could still be diffeomorphic to a standard sphere. Note that, in the special case where the projective space is a surgery of the form $K = \mathbb{CP}^n \# \Sigma$, for an AD sphere $\Sigma \in \Theta_{2n}$, the $(2n+1)$–sphere obtained as the unit bundle of $\mathcal{L} \to K$ is given by $\mathrm{stab}(\Sigma) \in \Theta_{2n+1}$ (where stab is the map constructed in Section 7.2; see Remark 7.17).

On the other hand, one could examine S^1–quotients of AD spheres $\widetilde{S} \in \Theta_{2n+1}$. A priori this is not always a successful strategy, as not all homotopy spheres admit a smooth free circle action. But, if such an action exists, then the quotient $P := \widetilde{S}/S^1$ resulting from it is an AD or AT projective space. Namely, this reduced space is necessarily homotopy equivalent to a projective space [Hsiang 1966], but it is at least not diffeomorphic to the standard $\mathbb{CP}^n$ (since circle bundles over P are classified by elements of $H^2(P;\mathbb{Z})$, and, if P were the standard projective space, then the total space of the line bundle would have to be a standard sphere).

Theorem 7.9 [James 1980, Sections 2–3] *There is a homotopy 9–sphere $\widetilde{S}$ such that*:

(i) $\widetilde{S} \notin \mathrm{bP}_{10} \cong \mathbb{Z}/2\mathbb{Z}$.

(ii) *$\widetilde{S}$ admits a free action of S^1.*

(iii) *The quotient $P := \widetilde{S}/S^1$ is not homeomorphic to $\mathbb{CP}^4$.*

(iv) *P and the standard $\mathbb{CP}^4$ have the same tangent bundles.*

Remark 7.10 James [1980, Section 3] notes that there is another S^1–action on $\widetilde{S}$ with quotient space $P \# \Sigma^8$. The latter is an AT projective space that is not diffeomorphic to P. ◁

We now have enough material to state and prove the results of this section.

Theorem E *There is a manifold P homotopy equivalent to $\mathbb{CP}^4$ and with the same first Pontryagin class such that neither P nor $P \# \Sigma^8$ admits an exact Lagrangian embedding into $T^*\mathbb{CP}^4$.*

Proof Consider the homotopy 9–sphere $\widetilde{S}$ admitting a free S^1–action of Theorem 7.9. We let $P := \widetilde{S}/S^1$ and prove it is the right candidate to satisfy the claim. The quotient P is homotopy equivalent to $\mathbb{CP}^4$, but, by Theorem 7.9(iii), it is not homeomorphic to it. The first Pontryagin classes of P and $\mathbb{CP}^4$ coincide by Theorem 7.9(iv). Assume there is a Lagrangian embedding $P \hookrightarrow T^*\mathbb{CP}^4$. The Hopf correspondence (see Lemma 3.1) lifts P to $\widetilde{S}$, giving an exact Lagrangian embedding $\widetilde{S} \hookrightarrow T^*S^9$. However, by Theorem 7.9, $\widetilde{S} \in \Theta_9 \setminus \mathrm{bP}_{10}$, so the existence of the Lagrangian embedding contradicts Corollary 7.5.

The same argument applies to prove that $P \# \Sigma^8$ does not embed as Lagrangian into $T^*\mathbb{CP}^4$. Namely, the Hopf correspondence would, in that case too, lift (via the S^1–action of Remark 7.10) $P \# \Sigma^8$ to $\widetilde{S}$ [James 1980, Section 3]. □

Remark 7.11 Our techniques do not allow to prove whether the AD projective space $\mathbb{CP}^4 \# \Sigma^8$ of Theorem 7.8 does admit a Lagrangian embedding into $T^*\mathbb{CP}^4$ or not. ◁

Theorem F *There is an element Σ^{14} in the group of homotopy 14–spheres Θ_{14} such that $\mathbb{CP}^7 \# \Sigma^{14}$ does not admit an exact Lagrangian embedding into $T^*\mathbb{CP}^7$.*

Proof First note that $\Theta_{14} \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{bP}_{15} = 0$ [Kervaire and Milnor 1963], so there is a unique AD 14–sphere. We define Σ^{14} to be this AD 14–sphere and prove it is the right candidate to satisfy the claim. By [Bredon 1967, Theorem 4.6], there is an AD sphere $\Sigma^{15} \in \Theta_{15} \setminus \mathrm{bP}_{16}$ admitting a free S^1–action, with quotient $P := \mathbb{CP}^7 \# \Sigma^{14}$. If P admitted a Lagrangian embedding in $T^*\mathbb{CP}^7$, the Hopf correspondence would yield a Lagrangian embedding $\Sigma^{15} \hookrightarrow T^*S^{15}$. But $\Sigma^{15} \notin \mathrm{bP}_{16}$, which would contradict Corollary 7.5 (for the same reasons as in the proof of Theorem E). □

7.2 Framing of projective twists

The background material that we use to examine the question of framing of projective twists is based on [Dimitroglou Rizell and Evans 2015], in which it is proved that the Hamiltonian isotopy class of a Dehn twist does in general depend on a choice of framing.

Let (M, ω) be a symplectic manifold. Given a framing of a Lagrangian sphere $L \subset M$, ie a diffeomorphism $S^n \to L$ (see Section 2.1), the precomposition with an element $F \in \mathrm{Diff}(S^m)$ yields another framing.

Consider the symplectomorphism $F^* \colon T^*S^m \to T^*S^m$ induced by the lift of F to the cotangent bundle T^*S^m. The standard model twist $\tau_{S^m}^{\mathrm{loc}} \in \mathrm{Symp}_{\mathrm{ct}}(T^*S^m)$ can be replaced by $F^* \circ \tau_{S^m}^{\mathrm{loc}} \circ (F^{-1})^*$, and the latter can be implanted in a Weinstein neighbourhood as in Definition 2.8 to produce a new element in $\mathrm{Symp}_{\mathrm{ct}}(M)$. To study framings of twists, we can then restrict to these parametrisations of the standard model twist $\tau_{S^m} := \tau_{S^m}^{\mathrm{loc}} \in \mathrm{Symp}_{\mathrm{ct}}(T^*S^m)$.

A core fact for the study of parametrisations of twists is the isomorphism $\pi_0(\mathrm{Diff}^+(S^m)) \cong \Theta_{m+1}$ [Kervaire and Milnor 1963; Cerf 1970]. In particular, given a nontrivial diffeomorphism $F \in \mathrm{Diff}^+(S^m)$, there is an $(m+1)$–dimensional AD sphere constructed as follows:

Definition 7.12 Let $F \in \mathrm{Diff}(S^m)$ be a diffeomorphism not isotopic to the identity. Then $\Sigma_F := D^{m+1} \cup_F D^{m+1} \in \Sigma_{m+1}$ is an $(m+1)$–dimensional homotopy sphere obtained by gluing two $(m+1)$–discs along their boundary S^m twisted by F. In the notation of [Dimitroglou Rizell and Evans 2015, Definition 1.4] (which is more apt to visualise the Lagrangian suspension we utilise in Section 7.2.2), this is equivalent to

$$\Sigma_F := (D^{m+1} \times S^0) \cup_\Phi S^m \times [0,1]$$

glued along $S^m \times S^0$ via $\Phi\colon S^m \times S^0 \to S^m \times S^0$, $\Phi(x,y) = (F(x), y)$. ◁

Also recall there is an isomorphism $\pi_0(\mathrm{Diff}^+(S^m)) \cong \pi_0(\mathrm{Diff}^+_{\mathrm{ct}}(D^m))$ induced by a map $\mathrm{Diff}^+_{\mathrm{ct}}(D^m) \to \mathrm{Diff}^+(S^m)$ which extends all elements of $\mathrm{Diff}^+_{\mathrm{ct}}(D^n)$ over a capping disc.

Dimitroglou Rizell and Evans proved the existence of Dehn twists, whose Hamiltonian isotopy class depends on the choice of framing.

Definition 7.13 [Dimitroglou Rizell and Evans 2015, Definition 1.1] Fix a cotangent fibre $\Lambda \subset T^*S^m$ and let $\mathcal{L}_m \subset \Theta_m$ be the subset of homotopy spheres which admit a Lagrangian embedding into T^*S^m with the additional requirement that the embedding intersects Λ transversely in exactly one point. ◁

Theorem 7.14 [Dimitroglou Rizell and Evans 2015, Theorem A] *Let $F \in \mathrm{Diff}^+(S^m)$ be such that $\Sigma_F \notin \mathcal{L}_{m+1}$. Then $\tau_{S^m}^{-1} \circ (F^* \circ \tau_{S^m} \circ (F^{-1})^*)$ is not trivial in $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(T^*S^m))$.*

In the rest of the section, we analyse the analogous problem for reparametrisations $f \in \mathrm{Diff}(\mathbb{CP}^n)$ of projective twists. We prove that there exist $n \in \mathbb{N}$ such that the twist $\tau_f := f^* \circ \tau_{\mathbb{CP}^n} \circ (f^{-1})^*$ is not isotopic to the standard projective twist in $\pi_0(\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n))$, where $f^*\colon T^*\mathbb{CP}^n \to T^*\mathbb{CP}^n$ is the symplectomorphism induced by the lift of f to the cotangent bundle. We will not directly use Theorem 7.14 but an intermediary result (Proposition 7.15 below) that Dimitroglou Rizell and Evans proved (using [Abouzaid 2012a; Abouzaid and Kragh 2018; Ekholm and Smith 2014]) to support their arguments.

Proposition 7.15 [Dimitroglou Rizell and Evans 2015, Proposition 1.2] *There is an inclusion $\mathcal{L}_m \subset \mathrm{bP}_{m+1}$.*

Remark 7.16 There is a slight abuse of terminology in the entirety of the section. A *framing* will be employed (as in the rest of the paper) in the nonstandard sense à la Seidel to signify a smooth parametrisation of a sphere. The classical topological notion of framing (as a trivialisation of the normal bundle) is also needed in this section, and, in order to avoid a conflict of nomenclature, we call the latter a *normal framing*. ◁

7.2.1 Bredon's pairing We begin by introducing an essential component of the arguments of this section: a map

$$\mathrm{stab}\colon \Theta_m \to \Theta_{m+1} \tag{55}$$

obtained as a special case of a homomorphism $\Theta_m \otimes \pi_l(S) \to \Theta_{m+l}$ studied in [Bredon 1967].

Consider the linear action of $\mathrm{SO}(2) \simeq S^1$ on $S^{m+1} \subset \mathbb{R}^{m+2}$ via the representation

$$\mathrm{SO}(2) \to \mathrm{SO}(m+2), \quad A \mapsto \varphi(A) = \begin{bmatrix} A & & \\ & A & \\ & & \ddots \end{bmatrix},$$

with 1 in the right-hand bottom corner if m is odd. This is the linear S^1–action on S^{m+1}, which is free if m even (in which case it is the standard Hopf action), and whose fixed-point set is S^0 if m odd.

For $\Sigma \in \Theta_m$, Bredon's construction [1967, Sections 1 and 4] yields a homotopy $(m+1)$–sphere as follows. Let $V \subset \Sigma$ be an open neighbourhood of a point $p \in \Sigma$, and $g\colon (V, p) \to (\mathbb{R}^m, 0)$ an orientation-reversing diffeomorphism. Let $B := g^{-1}(D^m) \subset V \subset \Sigma$, where $D^m \subset \mathbb{R}^m$ is the unit disc.

Let $\mathcal{C} \cong S^1 \subset S^{m+1}$ be a principal orbit of the SO(2)–action on S^{m+1}, equipped with a normal framing $\mathcal{F}\colon \mathcal{C} \times \mathbb{R}^m \to S^{m+1}$.

Define

$$\mathrm{stab}(\Sigma) := S^{m+1} \setminus (\mathcal{F}(\mathcal{C} \times D^m)) \cup \mathcal{C} \times (\Sigma \setminus B), \tag{56}$$

where the two pieces are glued along their boundaries, which can be identified via a diffeomorphism $\mathcal{F}(\mathcal{C} \times (\mathbb{R}^m \setminus \{0\})) \cong \mathcal{C} \times (V \setminus \{p\})$ as in [Bredon 1967, page 435].

The normally framed orbit $(\mathcal{C}, \mathcal{F})$ represents an element $\gamma \in \pi_{m+1}(S^m) \cong \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ via the Thom–Pontryagin construction (see [Milnor 1965, Section 7]). With this identification in mind, the map stab is derived from a pairing $\Theta_m \times \pi_{m+1}(S^m) \to \Theta_{m+1}$, $(\Sigma, \gamma) \mapsto \mathrm{stab}(\Sigma) = \langle \Sigma, \gamma \rangle$ (see [Bredon 1967, (1)]). The latter induces a homomorphism [Bredon 1967, (2)]

$$\Theta_m \otimes \pi_1(S) \to \Theta_{m+1}. \tag{57}$$

To determine the class γ, we follow [Bredon 1967, (4.1)] and find that $\gamma = \eta^j$, where $\eta \in \pi_1(S) := \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ is the nontrivial element in the stable stem $\pi_1(S)$ and

$$j = \begin{cases} \frac{1}{2}m & \text{if } m \text{ even (ie the action on } S^{m+1} \text{ is free)}, \\ \frac{1}{2}(m-1) & \text{if } m \text{ odd (ie the action } S^{m+1} \text{ is not free)}. \end{cases}$$

Intuitively, if a normally framed Hopf circle in S^3 represents the class $\eta \in \pi_4(S^3)$, then γ is determined by the number of times (mod 2) that this normal framing fits in the normal bundle to $\mathcal{C} \subset S^{m+1}$.

For $m+1 = 2n+1$ and $m+1 = 2n+2$, we have $j = n$ and

$$\gamma = \eta^n = \begin{cases} \eta & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases} \tag{58}$$

Remark 7.17 For an even-dimensional homotopy sphere $\Sigma \in \Theta_{2n}$, the image $\mathrm{stab}(\Sigma)$ can also be described as follows (this remark is relevant for Section 7.1). Consider the surgery $\mathbb{CP}^n \# \Sigma$ and the complex line bundle $\mathcal{L} \to \mathbb{CP}^n \# \Sigma$ associated to the generator of $H^2(\mathbb{CP}^n \# \Sigma; \mathbb{Z})$. Then $\mathrm{stab}(\Sigma)$ is the homotopy sphere obtained as the unit circle bundle of $\mathcal{L}$. ◁

We now focus on the case $m + 1 = 2n + 2$.

Lemma 7.18 *The map $\Theta_{2n+1} \to \Theta_{2n+2}/\mathrm{bP}_{2n+3}$ is nontrivial for $n = 19, 23, 25, 29$.*

Proof There is a commuting diagram (see [Bredon 1967, Corollary 2.2]) obtained from the exact sequence (54),

$$\begin{array}{ccc} \Theta_{2n+1} & \xrightarrow{\ \mathrm{stab}\ } & \Theta_{2n+2} \\ \downarrow \psi & & \downarrow \psi \\ \mathrm{coker}(J_{2n+1}) & \xrightarrow{(-)\cdot\eta^n} & \mathrm{coker}(J_{2n+2}) \end{array} \tag{59}$$

where $(-)\cdot\eta^n : \mathrm{coker}(J_{2n+1}) \to \mathrm{coker}(J_{2n+2})$, is a map descending from the multiplication $\pi_{2n+1}(S) \times \pi_1(S) \to \pi_{2n+2}(S)$ with the class $\eta \in \pi_1(S) \cong \mathbb{Z}/2\mathbb{Z}$, which is well defined since, for $l + 1 < m$, $\mathrm{Im}(J_m) \cdot \mathrm{Im}(J_l) \subseteq \mathrm{Im}(J_{m+l})$, the image of the J–homomorphism is preserved under multiplication with elements of the stable stems.

By (58), we know that a necessary requirement for the map stab to be nontrivial is to have $n = 2k + 1$ for some $k \in \mathbb{N}$ so that $\eta^n = \eta$ is nontrivial. In that case, we get

$$\begin{array}{ccc} \Theta_{4k+3} & \xrightarrow{\ \mathrm{stab}\ } & \Theta_{4(k+1)} \\ \downarrow \psi & & \downarrow \psi \\ \mathrm{coker}(J_{4k+3}) & \xrightarrow{(-)\cdot\eta} & \mathrm{coker}(J_{4(k+1)}) \end{array} \tag{60}$$

The vertical maps are both surjective since ψ is always surjective in odd dimensions and when $m \equiv 0 \bmod 4$ (see [Levine 1985, Theorem 5.4]).

The exact sequence (54) implies that $\mathrm{coker}(J_{4k+4}) \cong \Theta_{4k+4}/\ker(\psi) \cong \Theta_{4k+4}$, and the nontriviality of the composition $\psi \circ \mathrm{stab}\colon \Theta_{4k+3} \to \Theta_{4k+4}$ is equivalent to the nontriviality of the multiplication $(-)\cdot\eta\colon \mathrm{coker}(J_{4k+3}) \to \mathrm{coker}(J_{4k+4})$. This amounts to looking for elements in the stable stems whose η–multiples are not in the image of J. As η is of order two, this information can be found in the "two-primary part" of the stable stems, the subgroups obtained after quotienting all elements of odd order. These are tabulated in a diagram in [Hatcher 2002, page 385], where the elements of interest appear to be in degrees $2n + 1 \in \{39, 47, 51, 59\}$, which means that $n \in \{19, 23, 25, 29\}$. □

The rest of the section is dedicated to explaining how to relate a parametrisation $f \in \mathrm{Diff}^+(\mathbb{CP}^n)$ of the standard projective twist to a parametrisation $F \in \mathrm{Diff}^+(S^{2n+1})$ of the standard Dehn twist.

Lemma 7.19 *Let n be an odd integer and $f \in \mathrm{Diff}^+(\mathbb{CP}^n)$ an orientation-preserving diffeomorphism. There exists a diffeomorphism $F \in \mathrm{Diff}^+(S^{2n+1})$ satisfying $h \circ F = f \circ h$, ie F is the lift of f by the Hopf bundle map.*

Proof Let $h\colon S^{2n+1} \to \mathbb{CP}^n$ the Hopf bundle map. A diffeomorphism $f\colon \mathbb{CP}^n \to \mathbb{CP}^n$ induces a continuous function $F\colon f^*(S^{2n+1}) \to S^{2n+1}$ covering f such that the diagram

$$\begin{array}{ccc} f^*(S^{2n+1}) & \xrightarrow{\;F\;} & S^{2n+1} \\ \big\downarrow h' & & \big\downarrow h \\ \mathbb{CP}^n & \xrightarrow{\;f\;} & \mathbb{CP}^n \end{array} \tag{61}$$

commutes, where $h'\colon f^*(S^{2n+1}) \to \mathbb{CP}^n$ is the pullback bundle of h by f. The map induced by f on the second cohomology $\bar{f}\colon H^2(\mathbb{CP}^n) \to H^2(\mathbb{CP}^n)$ is $\pm\mathrm{Id}$. Therefore, the Euler classes of $h\colon S^{2n+1} \to \mathbb{CP}^n$ and $h'\colon f^*(S^{2n+1}) \to \mathbb{CP}^n$ coincide up to sign, so these principal S^1–bundles must have diffeomorphic total spaces. It follows that $F\colon f^*(S^{2n+1}) \to S^{2n+1}$ is in fact a diffeomorphism $F\colon S^{2n+1} \to S^{2n+1}$ covering f satisfying $h \circ F = f \circ h$. □

Lemma 7.20 *Let n be an odd integer and $f \in \mathrm{Diff}^+(\mathbb{CP}^n)$ be an orientation-preserving diffeomorphism supported in an open chart, ie f is induced by an element of $\mathrm{Diff}^+_{\mathrm{ct}}(D^{2n})$, and let $\Sigma_f \in \Theta_{2n+1}$ be the homotopy $(2n+1)$–sphere associated to f.*

Let $F \in \mathrm{Diff}^+(S^{2n+1})$ be the S^1–equivariant lift of f of Lemma 7.19 and $\Sigma_F \in \Theta_{2n+2}$ the corresponding homotopy $(2n+2)$–sphere. Then $\mathrm{stab}(\Sigma_f) = \Sigma_F$.

Proof The lift $F \in \mathrm{Diff}^+(S^{2n+1})$ is supported in a tubular neighbourhood of a Hopf circle in S^{2n+1}. To build Σ_F, identify S^{2n+1} with an equator in S^{2n+2}, and consider the above circle as a normally framed circle $\mathcal{C} \subset S^{2n+2}$. That requires a choice of trivialisation of the normal bundle to a Hopf circle, a normal framing $\mathcal{F}\colon \mathcal{C} \times \mathbb{R}^{2n+1} \to S^{2n+2}$ that defines the support of the gluing map for the construction of Σ_F: on $\mathcal{F}(\mathcal{C} \times D^{2n+1})$, F acts as $\mathrm{id} \times f$. By the same arguments as in the beginning of this section, the normal framing of this Hopf circle corresponds to the class $\eta^n \in \pi_1(S)$, so $\Sigma_F = \mathrm{stab}(\Sigma_f)$ by the construction (56). □

7.2.2 Results We first mention an auxiliary result from [Dimitroglou Rizell and Evans 2015] we will need in the proof of Theorem G.

Lemma 7.21 [Dimitroglou Rizell and Evans 2015, Proposition 2.5] *Consider $(T^*S^{2n+1}, d\lambda_{T^*S^{2n+1}})$ equipped with the well-known structure of a Lefschetz fibration $T^*S^{2n+1} \to \mathbb{C}$ with smooth fibre $(T^*S^{2n}, d\lambda_{T^*S^{2n}})$ and two singular fibres. Let $L \subset T^*S^{2n+1}$ be the standard Lagrangian embedding of the zero section. There is an open symplectic embedding*

$$e\colon T^*S^{2n+1} \times T^*[0,1] \to T^*S^{2n+2} \tag{62}$$

such that:

- *$L\times[0,1]$ is sent to a subset of the zero section $S^{2n+2}\subset T^*S^{2n+2}$ (the matching sphere).*
- *The image of the embedding is disjoint from a particular cotangent fibre $\Lambda\subset T^*S^{2n+2}$ (a Lefschetz thimble).*

Proposition 7.22 *If the map $\mathrm{stab}\colon \Theta_{2n+1}\to\Theta_{2n+2}$ is nontrivial and n is odd, then the $\mathbb{CP}^n$–twist depends on a choice of framing.*

Proof Choose a framing $f\in \mathrm{Diff}^+(\mathbb{CP}^n)$, coming from an element of $\mathrm{Diff}^+_{\mathrm{ct}}(D^{2n})$ extended by the identity on the projective space. Let $F\in\mathrm{Diff}^+(S^{2n+1})$ be the S^1–equivariant lift of f as in Lemma 7.19, supported in a tubular neighbourhood of a Hopf circle $\mathcal{F}\colon\cong S^1\times D^{2n}\subset S^{2n+1}$. Let $\Sigma_f\in\Theta_{2n+1}$ be the sphere associated to f, and $\Sigma_F\in\Theta_{2n+2}$ that associated to F. By Lemma 7.20, $\Sigma_F=\mathrm{stab}(\Sigma_f)=\langle\Sigma_f,\eta^n\rangle\in\Theta_{2n+2}$. Since n is odd, $\eta^n=\eta$ and $\Sigma_F=\mathrm{stab}(\Sigma_f)\in\Theta_{2n+2}$ is nontrivial.

The map f^* induced by f on the cotangent bundle is not compactly supported, but can be used to define the compactly supported conjugation

$$\tau_f := f^*\circ\tau_{\mathbb{CP}^n}\circ(f^{-1})^*\colon T^*\mathbb{CP}^n\to T^*\mathbb{CP}^n \tag{63}$$

of the projective twist $\tau_{\mathbb{CP}^n}\in\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$.

We next show below that τ_f belongs to a Hamiltonian class distinct from that of the standard projective twist:

Lemma 7.23 *The twist τ_f defined in (63) is not isotopic to the standard twist $\tau_{\mathbb{CP}^n}\in\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$.*

Proof Assume by contradiction that $\tau^{-1}_{\mathbb{CP}^n}\circ\tau_f$ is (Hamiltonian) isotopic to the identity in $\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$. Let $(\phi_t)_{t\in[0,1]}$ be an isotopy connecting the two symplectomorphisms in $\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$ such that there are $s'>s\in(0,1)$ with

$$\phi_t=\begin{cases}\tau^{-1}_{\mathbb{CP}^n}\circ\tau_f & \text{if } t\leqslant s,\\ \mathrm{Id} & \text{if } t\geqslant s'.\end{cases} \tag{64}$$

Let $H\colon T^*\mathbb{CP}^n\times[0,1]\to\mathbb{R}$ be the generating Hamiltonian function. Define the Lagrangian embedding

$$\psi'\colon K\times[0,1]\to T^*\mathbb{CP}^n\times T^*[0,1],\quad (x,t)\mapsto\big(\phi_t(x),t,-H(\phi_t(x),t)\big), \tag{65}$$

where $K\subset T^*\mathbb{CP}^n$ is the standard Lagrangian embedding of the zero section. By construction, near the ends of the interval, $K\times[0,1]$ is preserved by (65) in the sense that $\psi'(K\times[0,s])\subset K\times[0,s]$ and $\psi'(K\times[s',1])\subset K\times[s',1]$.

On each fibre $T^*\mathbb{CP}^n$ of $T^*\mathbb{CP}^n\times T^*[0,1]$, apply the Hopf correspondence to lift the image of (65) to a Lagrangian embedding

$$\Psi\colon L\times[0,1]\to T^*S^{2n+1}\times T^*[0,1] \tag{66}$$

(where $L\subset T^*S^{2n+1}$ is the standard Lagrangian embedding of the zero section) such that $L\times I$ is preserved by Ψ near the ends of the interval.

By Lemma 7.21, we can replace $e(L \times [0,1]) \subset T^*S^{2n+2}$ by the Lagrangian suspension $\Psi(L \times [0,1])$, so that the ends of $\Psi(L \times [0,1])$ are "capped" into a $(2n+2)$–dimensional sphere diffeomorphic to $\Sigma_F \in \Theta_{2n+2}$ (see [Dimitroglou Rizell and Evans 2015, Section 3.3]) which intersects a cotangent fibre once transversely and is therefore contained in $\mathcal{L}_{2n+2}$. By Proposition 7.15, $\mathcal{L}_{2n+2} \subset \mathrm{bP}_{2n+3}$ and since $\mathrm{bP}_{2n+3} = 0$ (this holds for all odd integers; see [Kervaire and Milnor 1963]), Σ_F has to be the standard sphere. However, as we have seen above, $\Sigma_F = \langle \eta^n, \Sigma_f \rangle \in \Theta_{2n+2}$ is nontrivial as n is odd. This is a contradiction, which proves Lemma 7.23; τ_f cannot be isotopic to the standard projective twist in $\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$. □

This also concludes the proof of Proposition 7.22. □

The above results are sufficient to prove the following:

Theorem G *The $\mathbb{CP}^n$–twist depends on the framing when $n = 19, 23, 25, 29$.*

Proof The statement is proved by combining Lemma 7.18 with Proposition 7.22. □

Proposition 7.24 *The $\mathbb{CP}^n$–twist depends on the choice of framing for infinitely many dimensions n.*

Proof One way to obtain infinite families of nontrivial multiples of η which are not contained in the image of J is by detecting them in topological modular forms, denoted by tmf (we refer to [Henriques 2014] for a survey on the subject). There is a "Hurewicz homomorphism" $\pi_*(\mathbb{S}) \to \pi_*(\mathrm{tmf})$ between the ring of stable homotopy groups of spheres and the homotopy ring of tmf, and the two primary components of the ring of homotopy groups have a certain kind of periodicity of degree 192. Therefore, if we can identify an element in one of the homotopy groups $\pi_{4k+3}(\mathrm{tmf})$ that is also in the image of the Hurewicz homomorphism and arises as a product of η, we obtain a periodic family of elements to which the argument of Lemma 7.18 applies.

A (partially conjectural) diagram depicting the two-primary components can be found in [Henriques 2014] and it is helpful to first identify a potential candidate. Degree $39 = 4 \cdot 9 + 3$ presents an element which has been confirmed to be the image of a nontrivial multiple of η (see [Hopkins and Mahowald 2014, Corollary 11.2], there the element in question is called u and arises as image of a product of $\bar{\kappa}$, ν, η and κ; all of these are standard names of generators of stable homotopy groups stems). It follows that, in every dimension $m \equiv 39 \bmod 192$, there is an element for which the map $(-) \cdot \eta \colon \mathrm{coker}(J_m) \to \mathrm{coker}(J_{m+1})$ and hence $\mathrm{stab}\colon \Theta_m \to \Theta_{m+1}$ are not trivial. Recall that $m = 4k+3 = 2n+1$, so that, by Proposition 7.22, the projective twist depends on the framing for $n \equiv 19 \bmod 96$. Further scrutiny of the literature would provide other such elements, eg for $m = 59$ ($n = 29$). □

Remark 7.25 It is very likely that a version of Theorem G holds for $\mathbb{HP}^n$–twists as well. Bredon [1967, page 446] computes the class that would be associated to a framing of $S^3 \subset S^{4n+3}$, which is a power of $\nu \in \pi_3(S) = \lim_m \pi_{m+3}(S^m) \cong \pi_8(S^5) \cong \mathbb{Z}_{24}$. Nontriviality results for the map stab in this case would not only depend on the parity of n, so a nonvanishing criterion would be harder to obtain.

But such a criterion could then be combined with the existence of smooth semifree actions of S^3 on homotopy $(4k+3)$–spheres explicitly computed in [Bredon 1967, Theorems 4.4 and 4.7] (also note that there are infinitely many inequivalent free S^3–actions on homotopy S^{4k+3}–spheres, by [Hsiang 1966, Theorem 3]). Then the above strategy could be applied to obtain infinitely many dimensions in which the $\mathbb{HP}^n$–twist would depend on the framing. ◁

Corollary 7.26 *In the above dimensions*, $\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n) \not\simeq \mathbb{Z}$.

Proof If $\tau_{\mathbb{CP}^n} \in \pi_0(\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n))$ is the standardly framed twist along the zero section, then we claim that $\mathbb{Z}\langle\tau_{\mathbb{CP}^n}\rangle \subsetneq \pi_0(\mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n))$. Let $f \in \mathrm{Diff}^+_{\mathrm{ct}}(\mathbb{CP}^n)$ be a framing such that the projective twist $\tau_f \in \mathrm{Symp}_{\mathrm{ct}}(T^*\mathbb{CP}^n)$ defined using f is not isotopic to $\tau_{\mathbb{CP}^n}$, as in Theorem G. Then $\tau_f^{-1} \circ \tau_{\mathbb{CP}^n}$ cannot be isotopic to any power $\tau^k_{\mathbb{CP}^n}$ for any $k \in \mathbb{Z}$. This is because $\tau_{\mathbb{CP}^n}$, viewed as a graded symplectomorphism, acts nontrivially on the grading of the zero section, viewed as a graded Lagrangian (see [Seidel 2000, Lemma 5.7]), whereas $\tau_f^{-1} \circ \tau_{\mathbb{CP}^n}$ acts trivially on the grading (see also [Dimitroglou Rizell and Evans 2015, Remark 1.5]). □

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