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Upper bounds for the Lagrangian cobordism relation on Legendrian links

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Lagrangian cobordism induces a preorder on the set of Legendrian links in any contact 3–manifold. We show that any finite collection of null-homologous Legendrian links in a contact 3–manifold with a common rotation number has an upper bound with respect to the preorder. In particular, we construct an exact Lagrangian cobordism from each element of the collection to a common Legendrian link. This construction allows us to define a notion of minimal Lagrangian genus between any two null-homologous Legendrian links with a common rotation number.

57K33; 53D12, 57K10

1 Introduction

The relation $\preceq$ defined by (exact, orientable) Lagrangian cobordism between Legendrian submanifolds in the symplectization of the contact manifold raises a host of surprisingly subtle structural questions. While the Lagrangian cobordism relation is trivially a preorder (ie is reflexive and transitive), it is not symmetric [Baldwin and Sivek 2018; Chantraine 2010; Cornwell et al. 2016]; it is unknown whether the relation is a partial order. Further, not every pair of Legendrians is related by Lagrangian cobordism, with the first obstructions coming from the classical invariants: for links $\Lambda_\pm$ in $\mathbb{R}^3$, if $\Lambda_- \preceq \Lambda_+$ via the Lagrangian $L \subset \mathbb{R} \times \mathbb{R}^3$, then $\mathrm{r}(\Lambda_+) = \mathrm{r}(\Lambda_-)$ and $\mathrm{tb}(\Lambda_+) - \mathrm{tb}(\Lambda_-) = -\chi(L)$ [Chantraine 2010]. A growing toolbox of nonclassical obstructions has been developed to detect this phenomenon; see, just to begin, [Baldwin et al. 2022; Baldwin and Sivek 2018; Ekholm et al. 2016; Golla and Juhász 2019; Pan 2017; Sabloff and Traynor 2013].

If two Legendrians are not related by a Lagrangian cobordism, one may still ask if they have a common upper or lower bound with respect to $\preceq$. Implicit in the work of Boranda, Traynor and Yan [Boranda et al. 2013] is that any finite collection of Legendrian links in the standard contact $\mathbb{R}^3$ with the same rotation number has a lower bound with respect to $\preceq$. In another direction, Lazarev [2020] has shown that any finite collection of formally isotopic Legendrians in a contact $(2n+1)$–manifold with $n \geq 2$ has an upper bound with respect to a moderate generalization of $\preceq$.

The goal of this paper is to find both lower and upper bounds for finite collections of Legendrian links in any contact 3–manifold. On one hand, in contrast to the diagrammatic methods of [Boranda et al. 2013],

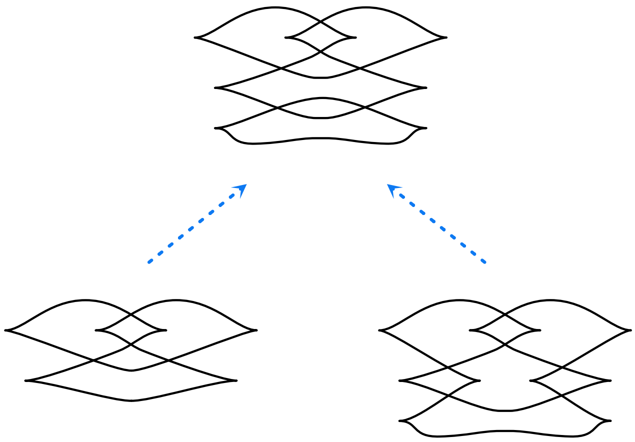

Figure 1: An upper bound for the maximal right-handed trefoil and an $m(5_2)$ knot.

our topological techniques allow us to find lower bounds in any contact 3–manifold, though we also present a refinement of the proof in [Boranda et al. 2013] that better suits our goal of constructing upper bounds. On the other hand, in contrast to Lazarev's use of an h–principle, which restricts his results to higher dimensions, our direct constructions of upper bounds work for Legendrian links in dimension 3.

Theorem 1.1 *Let Λ and Λ' be oriented Legendrian links in a contact 3–manifold (Y,ξ), and suppose that there exist Seifert surfaces Σ and Σ' for which $\mathrm{r}_{[\Sigma]}(\Lambda) = \mathrm{r}_{[\Sigma']}(\Lambda')$. Then there exist oriented Legendrian links $\Lambda_\pm \subset (Y,\xi)$ such that $\Lambda_- \preceq \Lambda \preceq \Lambda_+$ and $\Lambda_- \preceq \Lambda' \preceq \Lambda_+$.*

Remark 1.2 For Legendrian links in $\mathbb{R}^3$, the rotation number may be defined without reference to Seifert surfaces, and the hypotheses merely require $\mathrm{r}(\Lambda) = \mathrm{r}(\Lambda')$.

Remark 1.3 If Λ_- and Λ_+ are connected, then all Lagrangians constructed in the proof of Theorem 1.1 will be connected as well.

Example 1.4 In Figure 1, we display an upper bound for the maximal Legendrian right-handed trefoil and a Legendrian $m(5_2)$ knot. These two Legendrian knots are not related by Lagrangian cobordism. To see why, note that any Lagrangian cobordism between them must be a concordance since they have the same Thurston–Bennequin number, but no such concordance exists even topologically.

Example 1.5 In Figure 2, we display an upper bound for the maximal Legendrian unknot and the maximal Legendrian figure-eight knot. Once again, these two Legendrian knots are not related by Lagrangian cobordism. The fact that the figure-eight has lower Thurston–Bennequin number shows that there cannot be a cobordism from the unknot to the figure-eight; the fact that the figure-eight has two normal rulings shows that there cannot be a cobordism from the figure-eight to the unknot [Cornwell et al. 2016, Theorem 2.7].

In fact, we prove the following strengthened version of Theorem 1.1.

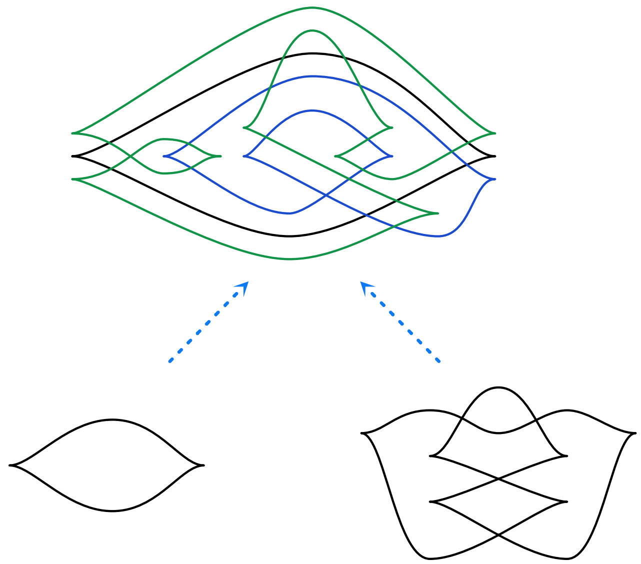

Figure 2: An upper bound for the maximal unknot and the maximal figure-eight knot. The colors in the diagram of the upper bound are only meant to distinguish components of the link to improve readability.

Proposition 1.6 *Under the same hypotheses of Theorem 1.1, there exist oriented Legendrian links $\Lambda_-, \Lambda_+ \subset Y$ and oriented exact decomposable Lagrangian cobordisms L and L' from Λ_- to Λ_+, such that*

- *the Legendrian link Λ appears as a collared slice of L;*
- *the Legendrian link Λ' appears as a collared slice of L'; and*
- *L and L' are exact-Lagrangian isotopic.*

Remark 1.7 There are statements analogous to Theorem 1.1 and Proposition 1.6 that hold for unoriented Legendrian links and unoriented (and possibly nonorientable) exact Lagrangian cobordisms, for which there are no requirements on the rotation number.

The main theorem has several interesting consequences. First, we recall that not every Legendrian knot has a Lagrangian filling. The figure-eight knot in Figure 2 is one such example. By transitivity, this implies that not every Legendrian knot lies at the top of a Lagrangian cobordism from a fillable Legendrian. On the other hand, we have the following corollary of the main theorem:

Corollary 1.8 *For any Legendrian link Λ, there exists a Legendrian link Λ_+ with a Lagrangian filling and a Lagrangian cobordism from Λ to Λ_+.*

The proof simply requires us to apply Theorem 1.1 with Λ being the given Legendrian and Λ' being the maximal Legendrian unknot. The upper bound Λ_+ is Lagrangian fillable since there is a cobordism to it from the unknot.

A second consequence of the main theorem is that we are able to define a notion of the minimal genus of a Lagrangian cobordism between *any* two Legendrian links with the same rotation number. Roughly speaking, we define a Lagrangian zigzag-cobordism between Λ and Λ' to be a sequence $\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_n = \Lambda'$ of Legendrian links together with upper (or lower) bounds between each of Λ_i and Λ_{i+1}. The genus of the zigzag-cobordism is the genus of the (smooth) composition of the underlying Lagrangian cobordisms between the Λ_i and their bounds; we may then define $g_L(\Lambda, \Lambda')$ to be the minimal genus of such a Lagrangian zigzag-cobordism. When there is a Lagrangian cobordism from Λ to Λ' and Λ is fillable, $g_L(\Lambda, \Lambda')$ agrees with the relative smooth genus $g_s(\Lambda, \Lambda')$; see Lemma 6.7.

The remainder of the paper is organized as follows. In Section 2, we review key ideas in the definition and construction of Lagrangian cobordisms between Legendrian links. We also define the notion of a Legendrian handle graph, which will form the basis of our later constructions. In Sections 3 and 4, we prove that any two Legendrians in a contact 3–manifold have a lower bound with respect to $\preceq$, and encode the Lagrangian cobordisms involved with Legendrian handle graphs. We present two approaches to this goal: in Section 3, we prove the claim for general contact 3–manifolds using convex surface theory, while in Section 4, we provide a diagrammatic proof in $\mathbb{R}^3$, refining a proof of [Boranda et al. 2013]. We then proceed in Section 5 to prove Proposition 1.6, and hence Theorem 1.1. We end the paper in Section 6 by beginning an exploration of Lagrangian zigzag-cobordisms and their genera, finishing with some open questions.

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2 A description of Lagrangian cobordisms

In this section, we describe Lagrangian cobordisms, how to construct them, and how to keep track of those constructions.

2.1 Lagrangian cobordisms

We begin with the formal definition of a Lagrangian cobordism between Legendrian links.

Definition 2.1 Let Λ_- and Λ_+ be Legendrian links in the contact manifold (Y,ξ), where $\xi = \ker(\alpha)$ for a contact 1–form α. An (exact, orientable) *Lagrangian cobordism* L from Λ_- to Λ_+ is an exact, orientable, properly embedded Lagrangian submanifold $L \subset (\mathbb{R}\times Y, d(e^t\alpha))$ that satisfies the following:

- there exists $T_+ \in \mathbb{R}$ such that $L \cap ([T_+,\infty)\times Y) = [T_+,\infty)\times\Lambda_+$;
- there exists $T_- < T_+$ such that $L \cap ((-\infty,T_-]\times Y) = (-\infty,T_-]\times\Lambda_-$; and
- the primitive of $(e^t\alpha)|_L$ is constant (rather than locally constant) at each cylindrical end of L.

Note that the last condition enables us to concatenate Lagrangian cobordisms while preserving exactness.

We will use three constructions of Lagrangian cobordisms in this paper, which we will call the *elementary Lagrangian cobordisms*:

- **0–handle** Adding a disjoint, unlinked maximal Legendrian unknot Υ to Λ induces an exact Lagrangian cobordism from Λ to $\Lambda \sqcup \Upsilon$ [Bourgeois et al. 2015; Ekholm et al. 2016].
- **Legendrian isotopy** A Legendrian isotopy from Λ to Λ' induces an exact Lagrangian cobordism from Λ to Λ', though the construction is more complicated than simply taking the trace of the isotopy [Bourgeois et al. 2015; Ekholm et al. 2016; Eliashberg and Gromov 1998].
- **Legendrian ambient surgery** We describe this construction in more detail in Section 2.2, and we will develop a method for keeping track of a set of ambient surgeries in Section 2.3.

2.2 Legendrian ambient surgery

Our next step is to explain Dimitroglou Rizell's [2016] Legendrian ambient surgery construction in the 3–dimensional setting. Similar constructions appear in [Bourgeois et al. 2015; Ekholm et al. 2016], though Dimitroglou Rizell's more flexible language is best suited for our purposes. In dimension 3, Legendrian ambient surgery begins with the data of an oriented Legendrian link $\Lambda \subset (Y,\xi)$ and an embedded Legendrian curve D with endpoints on Λ that is, in a sense to be defined, compatible with the orientation of Λ. The construction then produces a Legendrian Λ_D, contained in an arbitrarily small neighborhood of $\Lambda \cup D$, that is obtained from Λ by ambient surgery along D. Further, the construction produces an exact Lagrangian cobordism from Λ to Λ_D.

More precisely, given $\Lambda \subset (Y,\xi)$ with contact 1–form α, a *surgery disk* is an embedded Legendrian arc $D \subset Y$ such that

(1) $D \cap \Lambda = \partial D$;

(2) the intersection $D \cap \Lambda$ is transverse; and

(3) the vector field $H \subset T_p\Lambda$ defined for all $p \in \partial D$ (up to scaling) by $d\alpha(G, H(p)) > 0$ for all outward-pointing vectors G in T_pD either completely agrees with or completely disagrees with the framing on ∂D induced by the orientation of Λ.

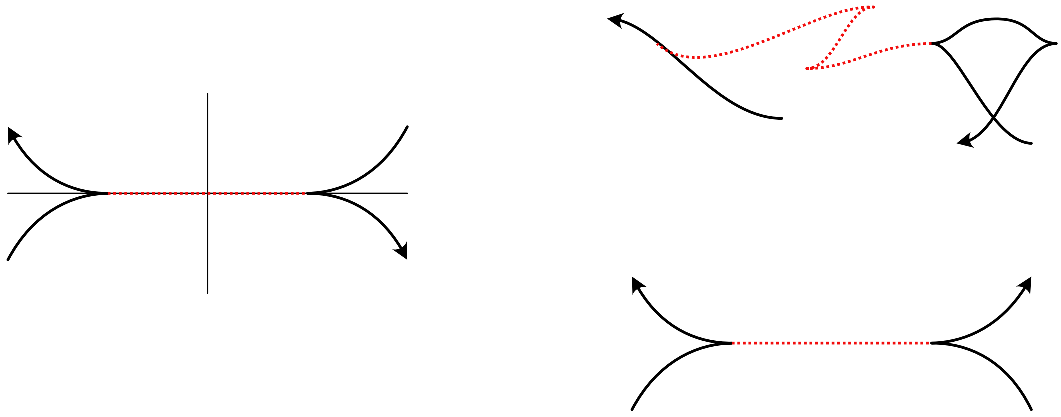

Figure 3: Left: the standard model in $(\mathbb{R}^3, \alpha_0)$ of a surgery disk D_0 with endpoints on a Legendrian Λ_0. Top right: another example of a surgery disk. Bottom right: a disk that fails condition (3).

For an unoriented surgery, we need not specify a framing for ∂D, and the last condition is no longer relevant.

The standard model for such a surgery disk appears in Figure 3, left. In fact, up to an overall orientation reversal on Λ, there is a neighborhood U of D in Y that is contactomorphic to a neighborhood of the standard model for Λ_0 and D_0 [Dimitroglou Rizell 2016, Section 4.4.1]. Working in the standard model, we may replace Λ_0 by the Legendrian arcs Λ_1 as in Figure 4, a process that realizes the ambient surgery on Λ_0 along D_0. Pulling this construction back to the neighborhood of D in Y, we call the resulting link *Legendrian ambient surgery* on Λ along D.

Theorem 2.2 [Dimitroglou Rizell 2016] *Given an oriented Legendrian link Λ and a surgery disk D, let Λ_D be the Legendrian link obtained from Λ by Legendrian ambient surgery along D. Then there exists an exact Lagrangian cobordism from Λ to Λ_D arising from the attachment of a 1–handle to $(-\infty, T] \times \Lambda$.*

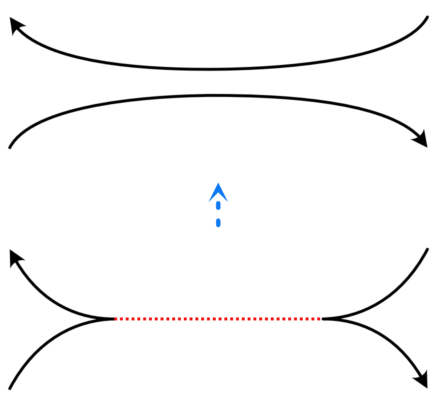

Figure 4: Surgery on the standard model $\Lambda_0 \cup D$ yields a new Legendrian Λ_1.

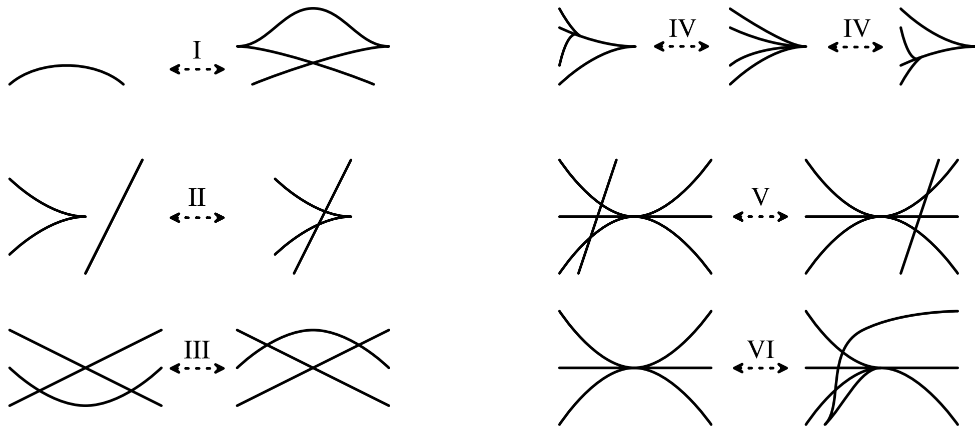

Figure 5: Reidemeister moves for Legendrian graphs in $\mathbb{R}^3$.

Remark 2.3 The construction of Legendrian ambient surgery and the associated Lagrangian cobordism is local. In particular, for a small neighborhood U of D, the surgery construction does not alter $\Lambda \cap (Y \setminus U)$, and the cobordism L outside of $\mathbb{R} \times U$ is cylindrical over $\Lambda \cap (Y \setminus U)$.

2.3 Legendrian handle graphs

In this section, we introduce a structure for keeping track of independent ambient surgeries. We use the notion of a Legendrian graph, following the conventions in [O'Donnol and Pavelescu 2012].

Before we begin, recall from eg [O'Donnol and Pavelescu 2012] that two Legendrian graphs in $(\mathbb{R}^3, \xi_0)$ are Legendrian isotopic if and only if their front diagrams are related by planar isotopy and six Reidemeister moves, as seen in Figure 5.

Definition 2.4 A *Legendrian handle graph* is a pair (G, Λ), where $G \subset (Y, \xi)$ is a trivalent Legendrian graph and $\Lambda \subset (Y, \xi)$ is a Legendrian link (called the *underlying link*), such that

- $\Lambda \subset G$;
- the vertices of G lie on Λ; and
- $G \setminus \Lambda$ is the union of a finite collection of pairwise disjoint Legendrian arcs $\gamma_1, \dots, \gamma_m$ whose closures satisfy the conditions of surgery disks for Λ.

We also say that G is a *Legendrian handle graph on* Λ. The set of closures of the components of $G \setminus \Lambda$ is denoted by $\mathscr{H}$.

See the bottom of Figure 6 for an example of a Legendrian handle graph whose underlying Legendrian link is a Legendrian Hopf link in $(\mathbb{R}^3, \xi_0)$.

Definition 2.5 Let (G, Λ) be a Legendrian handle graph and let $\mathscr{H}_0$ be a subset of the arcs in $\mathscr{H}$. The *Legendrian ambient surgery* $\mathrm{Surg}(G, \Lambda, \mathscr{H}_0)$ is the Legendrian handle graph (G', Λ') resulting from performing Legendrian ambient surgery along each arc in $\mathscr{H}_0$, as described in Section 2.2.

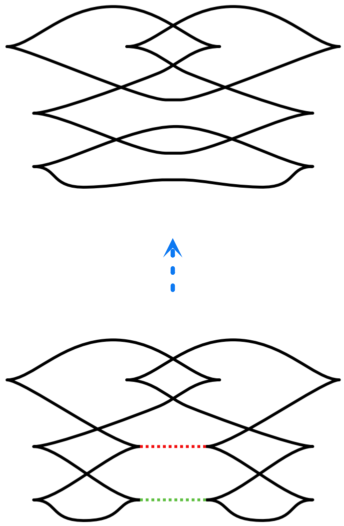

Figure 6: The Legendrian link at the top of the figure is the Legendrian ambient surgery on the Legendrian handle graph (G, Λ) at the bottom.

We will, at times, abuse notation and refer to the underlying Legendrian link Λ' by $\mathrm{Surg}(G, \Lambda, \mathcal{H}_0)$; we will also use $\mathrm{Surg}(G, \Lambda)$ when $\mathcal{H}_0 = \mathcal{H}$. For example, in Figure 6, the Legendrian link at the top is $\mathrm{Surg}(G, \Lambda)$ for the Legendrian handle graph (G, Λ) at the bottom.

By the work of Dimitroglou Rizell [2016] as described in Section 2.2, Legendrian ambient surgery on any given Legendrian arc corresponds to an exact Lagrangian cobordism. This implies that, given an order $\mathfrak{o} = (\gamma_{j_1}, \dots, \gamma_{j_m})$ of the components of $\mathcal{H}_0$, one obtains an exact Lagrangian cobordism $L(G, \Lambda, \mathcal{H}_0, \mathfrak{o})$ from Λ to $\mathrm{Surg}(G, \Lambda, \mathcal{H}_0)$ by performing Legendrian ambient surgery in the order given by $\mathfrak{o}$. The order, in fact, does not matter.

Proposition 2.6 *Suppose (G, Λ) is a Legendrian handle graph, and $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are orders of the components of $\mathcal{H}_0$. The Lagrangian cobordisms $L(G, \Lambda, \mathcal{H}_0, \mathfrak{o}_1)$ and $L(G, \Lambda, \mathcal{H}_0, \mathfrak{o}_2)$ are exact-Lagrangian isotopic.*

Proof It suffices to consider the case where $\mathfrak{o}_1$ and $\mathfrak{o}_2$ differ by an adjacent transposition

$$(\gamma_{j_1}, \gamma_{j_2}) \to (\gamma_{j_2}, \gamma_{j_1}).$$

The cobordism $L(G, \Lambda, \mathcal{H}_0, \mathfrak{o})$ is defined by composing the elementary Lagrangian cobordisms associated to the arcs $\gamma_1, \dots, \gamma_m$. Since there are finitely many of these, by shrinking the neighborhoods of γ_j as in Remark 2.3, we may assume the neighborhoods to be pairwise disjoint. This implies that the elementary Lagrangian cobordisms associated to γ_{j_1} and γ_{j_2} may be constructed simultaneously and shifted past each other along the cylindrical parts of the cobordism. Thus, the parameter given by the relative heights of these two cobordisms gives an exact Lagrangian isotopy between $L(G, \Lambda, \mathcal{H}_0, \mathfrak{o}_1)$ and $L(G, \Lambda, \mathcal{H}_0, \mathfrak{o}_2)$. □

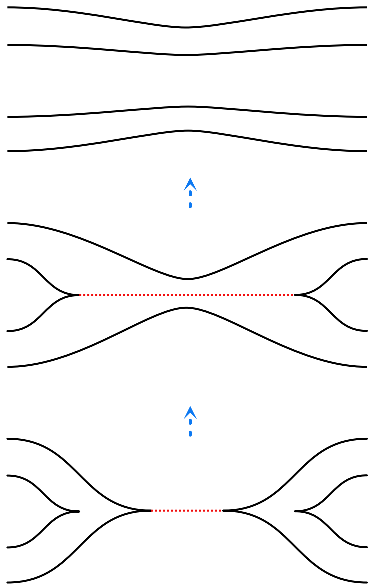

Figure 7: The surgery joining the two inner cusps cannot be performed until after the surgery joining the two outer cusps.

Proposition 2.6 allows us to associate an *isotopy class* $L(G, \Lambda, \mathcal{H}_0)$ of exact Lagrangian cobordisms to a Legendrian handle graph (G, Λ) and $\mathcal{H}_0 \subset \mathcal{H}$.

Remark 2.7 It would be extremely surprising if every decomposable cobordism can be described using a Legendrian handle graph. As shown in Figure 7, one may need to perform one ambient surgery in order for another to be possible; this would violate Proposition 2.6. We emphasize here the particularity of those decomposable cobordisms that can be described by a Legendrian handle graph, as much of Sections 3 and 4 revolves around ensuring the cobordisms we are building belong to this class.

3 Lower bounds via contact topology

In this section, for a pair of Legendrian links with the same rotation number, we construct a pair of exact Lagrangian cobordisms from a common lower bound, encoded by Legendrian handle graphs with the same underlying link.

Proposition 3.1 *Let Λ and Λ' be oriented Legendrian links in a contact manifold (Y, ξ) and suppose that there exist Seifert surfaces Σ and Σ' for which $\mathrm{r}_{[\Sigma]}(\Lambda) = \mathrm{r}_{[\Sigma']}(\Lambda')$. Then there exists an oriented Legendrian link $\Lambda_- \subset (Y, \xi)$ and Legendrian handle graphs G and G' on Λ_- such that $\mathrm{Surg}(G, \Lambda_-)$ (resp. $\mathrm{Surg}(G', \Lambda_-)$) is Legendrian isotopic to Λ (resp. Λ').*

Our proof of Proposition 3.1 relies on convex surface theory applied to the Seifert surfaces Σ and Σ'. To accomplish this, we require two basic results. The first is a lemma extending the work of Boranda, Traynor,

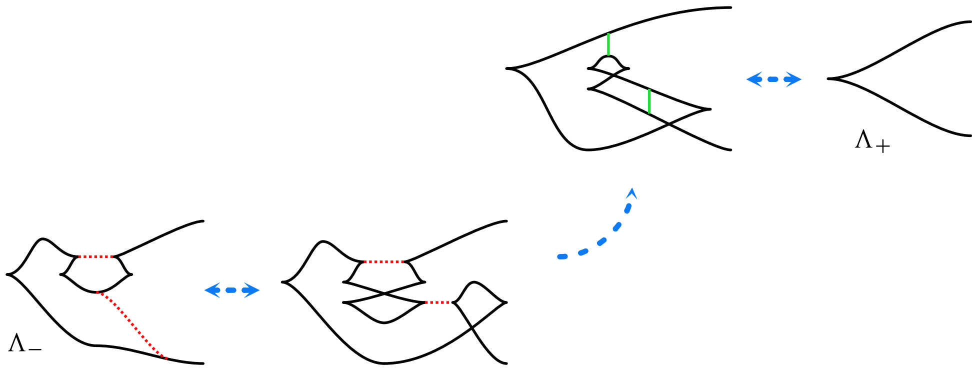

Figure 8: A handle graph giving rise to a Lagrangian cobordism from $S_+ \circ S_-(\Lambda)$ to Λ.

and Yan [Boranda et al. 2013] by placing their result in the context of Legendrian handle graphs. The second translates Dimitroglou Rizell's [2016] Legendrian ambient surgery into a convex surface-theoretic model.

Lemma 3.2 (cf [Boranda et al. 2013, Lemma 3.3]) *Let Λ be an oriented Legendrian link in a contact manifold (Y, ξ), and $S_+ \circ S_-(\Lambda)$ the result of successive negative and positive stabilization on a component of Λ. Then there is a Legendrian handle graph G on $S_+ \circ S_-(\Lambda)$ such that* $\mathrm{Surg}(G, S_+ \circ S_-(\Lambda))$ *is Legendrian isotopic to Λ.*

Proof The proof is essentially contained in Figure 8, which explicitly identifies a local model for the desired Legendrian handle graph. □

Our next task is to describe an explicit, convex surface-theoretic local model for Legendrian ambient surgery. In service of this goal, consider the Legendrian graph G depicted in the left part of Figure 9, which will serve as our local model below. The graph G contains three distinguished subsets:

(1) a max-tb unlink Λ of two components, consisting of the two blue arcs and the two black cusps at the two ends;

(2) a dotted red arc D joining the two components of Λ; and

(3) a large, black max-tb unknot Λ'.

Importantly, one can identify Λ' with $\mathrm{Surg}(G, \Lambda)$ in this local model.

The right part of Figure 9 illustrates a convex disk bounded by Λ' and containing the Legendrian graph G as an explicit subset. The key observation is that the above actually yields a convex surface-theoretic local model of the Legendrian ambient surgery operation.

A more general situation is illustrated in Figure 10. On its right, this figure depicts a portion of a convex surface Σ, bounded by a Legendrian link Λ', and containing a Legendrian graph G. The graph G is the union of Λ', the two blue arcs, and the dotted red arc D, and we let Λ be the union of the two blue arcs

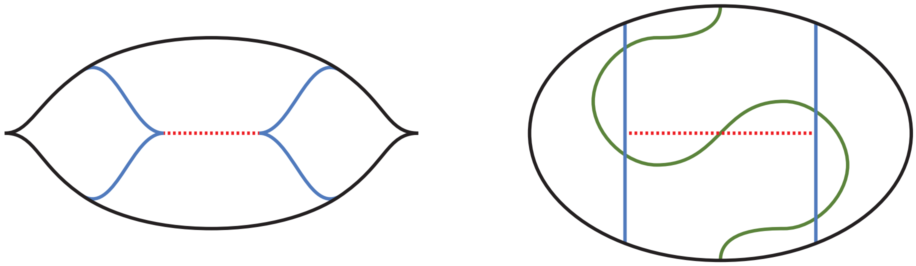

Figure 9: Left: a planar Legendrian graph depicting the Legendrian ambient surgery operation performed on a max-tb unlink. Right: a corresponding convex surface-theoretic interpretation; here, the green arc represents the dividing set.

and Λ', minus the two black boundary arcs between the blue arcs. Then, as in the simpler case above, we have a Legendrian graph G that lies on a convex surface Σ, together with distinguished subsets Λ, D, and Λ', colored blue/black, red, and black respectively. We now claim that this convex surface-theoretic picture corresponds to Legendrian ambient surgery as illustrated on the right of Figure 10.

Lemma 3.3 *Let Σ, G, Λ, Λ', and D be as described in the paragraph above. Then Λ' can be identified with* $\operatorname{Surg}(G, \Lambda, \{D\})$.

Proof This follows immediately from the observation that Legendrian ambient surgery is itself a local operation [Dimitroglou Rizell 2016]. In other words, since Lemma 3.3 is true for a single example — where Λ is a max-tb unlink of two components, D is a trivial arc joining them, and Λ' is a max-tb unknot) — it must be true in general. □

Remark 3.4 While the configuration depicted in Figure 10 provides one possible convex surface-theoretic local model for the Legendrian ambient surgery operation, it is not necessarily unique.

Lemma 3.5 *Let Λ be an oriented, null-homologous Legendrian link in a contact manifold (Y, ξ). Then there exists an oriented Legendrian unknot $\Lambda_U \subset (Y, \xi)$ and a Legendrian handle graph G on Λ_U such that* $\operatorname{Surg}(G, \Lambda_U)$ *is Legendrian isotopic to Λ.*

Proof Suppose that Σ is a Seifert surface for Λ. Applying Lemma 3.2 to successively double-stabilize each component of Λ if necessary, we obtain a Legendrian handle graph (G_1, Λ_1) and a Seifert surface

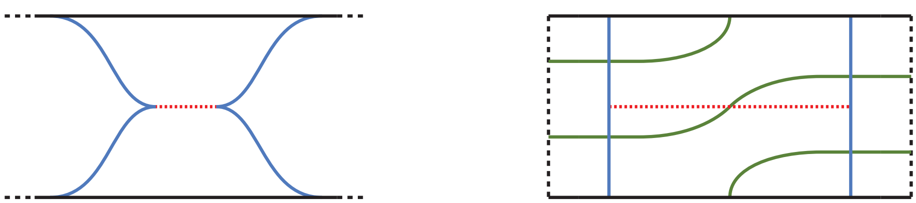

Figure 10: A convex surface-theoretic local model for Legendrian ambient surgery. Again, the green arcs represent the dividing set.

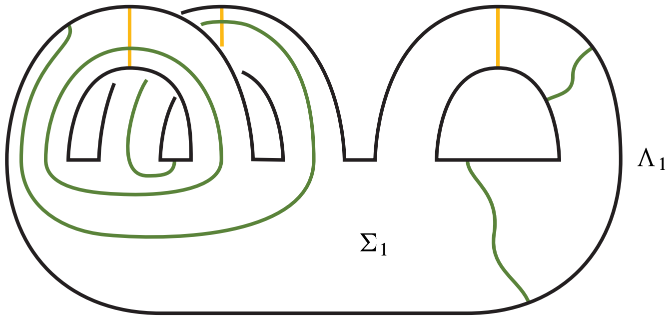

Figure 11: The convex Seifert surface Σ_2, dividing set Γ_Σ, and arc basis, viewed in disk-band form.

Σ_1 for Λ_1 isotopic to Σ, such that the twisting of ξ relative to Σ_1 along each component of $\partial\Sigma_1 = \Lambda_1$ is negative, and $\mathrm{Surg}(G_1, \Lambda_1)$ is Legendrian isotopic to Λ. Below, we will denote this first condition by the shorthand notation $\mathrm{tw}(\xi, \Sigma_1) < 0$, and similarly for other surfaces.

By work of Kanda [1998], since $\mathrm{tw}(\xi, \Sigma_1) < 0$, there is an isotopy of Σ_1 relative to $\partial\Sigma_1 = \Lambda_1$ such that the resulting surface Σ_2 is convex. (While we will not use this, we may assume that this isotopy is a C^0 perturbation near the boundary, followed by a C^∞ perturbation of the interior.) Further, by possibly Legendrian-isotoping the handle arcs of G_1, we obtain a Legendrian handle graph $(G_2, \Lambda_2 = \Lambda_1)$ whose handle arcs $G_2 \setminus \Lambda_2$ intersect Σ_2 transversely in a finite number of points.

To aid the discussion to follow, we picture the convex Seifert surface Σ_2 in disk-band form, meaning that we view it as the union of a 0–handle (disk) and a number g of 1–handles (bands); see Figure 11. Below, we shall fix a particular choice of disk-band decomposition. The cocores $a_1, \dots, a_g$ of the 1–handles form an *arc basis* for Σ_2. (Note that Figure 11 is an abstract diagram of Σ_2; as Σ_2 is embedded in Y, the bands may be "linked".) Since $\mathrm{tw}(\xi, \Sigma_2) < 0$, the dividing set must intersect each component of Λ.

To obtain the desired Legendrian handle graph (G, Λ_U), our strategy is to cut the bands of Σ_2. More precisely, let $\{a_1, \dots, a_g\}$ be an arc basis for Σ_2 consisting of a collection of properly embedded arcs in Σ_2, such that the intersection of each a_i with $G_2 \setminus \Lambda_2$ is empty. Figure 12 depicts a band of Σ_2 and a corresponding basis arc a_i.

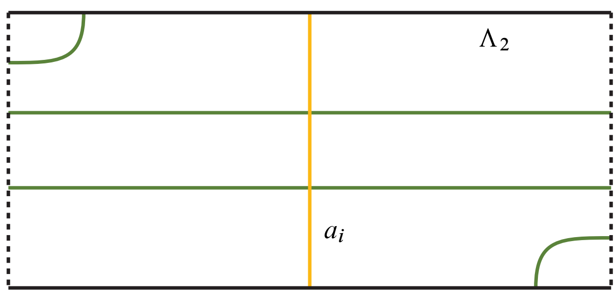

Figure 12: A band of the convex Seifert surface Σ_2 and a corresponding basis arc a_i.

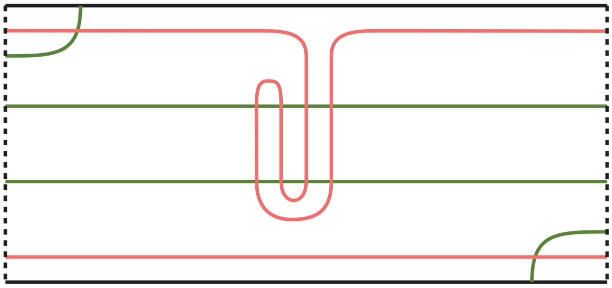

Figure 13: A double-stabilization Λ_3 of Λ_2 whose new Seifert surface Σ_3 contains an arc basis disjoint from its dividing set.

Now construct a (not necessarily Legendrian) link $\Lambda_{2.5}$ as follows: Take a parallel push-off of Λ_2 in Σ_2 and, for each basis arc a_i that intersects the dividing set Γ_{Σ_2}, perform a finger move of the pushoff across each of the dividing curves involved and back, as shown in Figure 13. We may choose to perform this finger move from either end of a_i — and it is sufficient to perform it on only one side — and we may choose to turn either left or right on the way back; these choices are immaterial. The goal of the finger move is to obtain a curve that is smoothly isotopic to Λ_2, such that instead of a_i (whose ends are on Λ_2), we may choose a cocore a_i' (with ends on the new curve) that does not intersect the dividing set. Since $a_i \cap (G_2 \setminus \Lambda_2) = \varnothing$ for each i, we may assume that the finger moves avoid all intersection points between $G_2 \setminus \Lambda_2$ and Σ_2.

Recall that the Legendrian realization principle (LeRP) states that, given a convex surface S and a multicurve $C \subset S$ that is transverse to Γ_S, if each component of $S \setminus C$ intersects Γ_S, then S can be isotoped to another convex surface $\phi(S)$ such that $\phi(C)$ is Legendrian (see [Kanda 1998; Honda 2000, Section 3]). Here, since $\operatorname{tw}(\xi, \Sigma_2) < 0$, each component of $\Sigma_2 \setminus \Lambda_{2.5}$ intersects Γ_{Σ_2}, and so we can apply the LeRP to $\Lambda_{2.5} \subset \Sigma_2$ to obtain an isotopy of Σ_2 to a convex surface $\Sigma_{2.5}$ with $\partial\Sigma_{2.5} = \Lambda_2$, such that the image Λ_3 of $\Lambda_{2.5}$ under the isotopy is Legendrian, and $\Gamma_{\Sigma_{2.5}}$ is the image of Γ_{Σ_2}.

A closer look at the proof of [Kanda 1998; Honda 2000, Section 3] reveals that the LeRP in fact applies more generally to graphs $G \subset S$ satisfying the complement condition. We will need this fact towards the end of this proof.

Since the finger moves giving rise to $\Lambda_{2.5}$ — and hence Λ_3 — each involved isotoping across elements of the dividing set an even number of times, we have that the Legendrian link Λ_3 is necessarily an iterated double-stabilization of Λ_2. In fact, Λ_3 is Legendrian isotopic to Λ_2 outside of a tubular neighborhood U_a of the a_i's. We now construct a Legendrian handle graph on Λ_3 as follows: First, extend the Legendrian isotopy between $\Lambda_2 \setminus U_a$ and $\Lambda_3 \setminus U_a$ to a local contact isotopy, and apply the local contact isotopy to the Legendrian handle arcs in $G_2 \setminus \Lambda_2$, obtaining Legendrian handle arcs that are attached to Λ_3. Second, by Lemma 3.2, we may add a collection $\mathscr{H}_3$ of Legendrian handle arcs to this collection to obtain a Legendrian handle graph (G_3, Λ_3) such that $\operatorname{Surg}(G_3, \Lambda_3, \mathscr{H}_3)$ is Legendrian isotopic to (G_2, Λ_2). In particular, this means that $\operatorname{Surg}(G_3, \Lambda_3)$ is Legendrian isotopic to Λ. As before, by possibly Legendrian-isotoping the

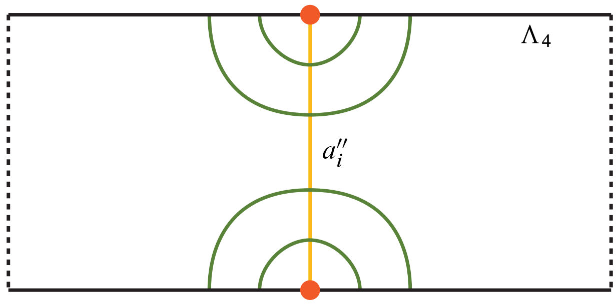

Figure 14: The result Λ_4 of iteratively doubly stabilizing Λ_3, whose Seifert surface Σ_4 contains an arc basis, each component of which intersects the dividing set in exactly four points.

handle arcs of G_3, we may assume that the Legendrian handle arcs in (G_3, Λ_3) intersect $\Sigma_{2.5}$ transversely in a finite number of points. (Note that, by Figure 8 in the proof of Lemma 3.2, the Legendrian handle arcs of $\mathcal{H}_3$ can be taken to be contained in an arbitrarily small tubular neighborhood of the a_i's, implying that the complication in Remark 2.7 does not arise, since the Legendrian handle arcs in $\mathcal{H}_3$ are contained in a neighborhood disjoint from $G_{2.5} \setminus \Lambda_3$.)

Let Σ_3 be the closure of the component of $\Sigma_{2.5} \setminus \Lambda_3$ that does not intersect $\partial\Sigma_{2.5}$. Then we have obtained a Legendrian link Λ_3 bounding a convex Seifert surface Σ_3 that contains an arc basis $\{a'_1, \dots, a'_g\}$ that does not intersect the dividing set Γ_{Σ_3}, and a Legendrian handle graph G_3 on Λ_3 such that $\mathrm{Surg}(G_3, \Lambda_3)$ is Legendrian isotopic to Λ.

We have one final preparatory step before we construct the desired unknot and the accompanying Legendrian handle graph. In this step, we double-stabilize Λ_3 at each point where it intersects the arc basis $\{a'_1, \dots, a'_g\}$. The result is a Legendrian link Λ_4 bounding a convex Seifert surface Σ_4 whose dividing set differs from that of Σ_3 by a collection of nested pairs of boundary-parallel dividing curves, as shown in Figure 14. We again produce an arc basis $\{a''_1, \dots, a''_g\}$ which now intersects each of the newly added dividing curves exactly once. As in two paragraphs above, we obtain a Legendrian handle graph G_4 on Λ_4 by applying a local contact isotopy to the Legendrian handle arcs in $G_3 \setminus \Lambda_3$, and then adding the handle arcs required to perform the double-stabilizations.

We are now ready to construct the unknot Λ_U and the desired Legendrian handle graph G on Λ_U. We do this in two steps. First, see Figure 15. Let $G_{4.5}$ be the graph consisting of

(1) the Legendrian link Λ_4;

(2) the curves b_i^1 and b_i^2 for $i \in \{1, \dots, g\}$, which are topologically parallel to the arc basis elements a''_i but have endpoints shifted as in Figure 15; and

(3) the arcs D_i for $i \in \{1, \dots, g\}$, each joining b_i^1 to b_i^2 and intersecting the dividing set once.

Since each component of the complement $\Sigma_4 \setminus G_{4.5}$ contains elements of the dividing set, we can apply the LeRP to isotope Σ_4 rel boundary to obtain a convex surface Σ_5 containing $\Lambda_{4.5}$ as a Legendrian graph G.

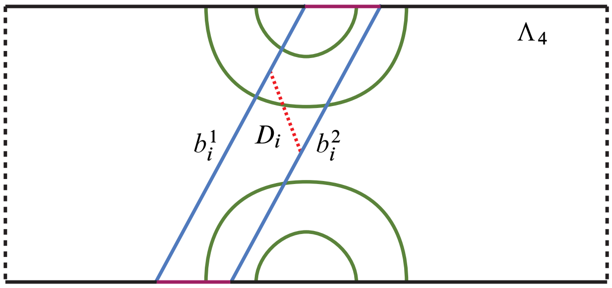

Figure 15: An identification of the convex surface-theoretic local model in Σ_5 for Legendrian ambient surgery.

Finally, we build $\Lambda_U = \Lambda_5$ by taking the segments of Λ_4 in the complement of the short arcs joining the b_i arcs (black in Figure 15) together with the b_i arcs. By construction, this is a Legendrian knot which is topologically trivial.

The key observation is that the Legendrian graph G satisfies the hypotheses of Lemma 3.3 — the "parallelogram" in the center of Figure 15 between the b_i arcs is isotopic to Figure 10. Thus, Legendrian ambient surgery of Λ_U along the collection of arcs $\{D_1, \dots, D_g\}$ is precisely $\partial\Sigma_4 = \Lambda_4$.

We add to the collection $\{D_1, \dots, D_g\}$ the handles in G_4 to obtain $G = G_5$. Thus, we obtain a Legendrian handle graph (G, Λ_U) such that, by construction, $\mathrm{Surg}(G, \Lambda_U)$ is Legendrian isotopic to Λ, completing the proof. □

We are now ready to prove the main result of this section.

Proof of Proposition 3.1 According to Lemma 3.5, there are oriented Legendrian unknots Λ_U and Λ'_U and Legendrian handle graphs G and G', such that $\mathrm{Surg}(G, \Lambda_U)$ (resp. $\mathrm{Surg}(G', \Lambda'_U)$) is Legendrian isotopic to Λ (resp. Λ').

Since $\mathrm{r}_{[\Sigma]}(\Lambda) = \mathrm{r}_{[\Sigma']}(\Lambda')$, it follows that $\mathrm{r}(\Lambda_U) = \mathrm{r}(\Lambda'_U)$. This also implies that $\mathrm{tb}(\Lambda_U)$ and $\mathrm{tb}(\Lambda'_U)$ differ by a multiple of 2. Without loss of generality, assume that $\mathrm{tb}(\Lambda_U) \geq \mathrm{tb}(\Lambda'_U)$; then by successively applying Lemma 3.2 to Λ_U if necessary, we obtain a Legendrian handle graph $(\overline{G}, \overline{\Lambda}_U)$ such that $\mathrm{tb}(\overline{\Lambda}_U) = \mathrm{tb}(\Lambda'_U)$ and $\mathrm{Surg}(\overline{G}, \overline{\Lambda}_U)$ is Legendrian isotopic to G. Again, by Figure 8 in the proof of Lemma 3.2, the Legendrian handle arcs of $\overline{G}$ can be taken to be contained in an arbitrarily small neighborhood of a point; thus, we may combine these Legendrian handle arcs with those of G, as in the proof of Lemma 3.5 to obtain a Legendrian handle graph $(\widetilde{G}, \overline{\Lambda}_U)$ such that $\mathrm{Surg}(\widetilde{G}, \overline{\Lambda}_U)$ is Legendrian isotopic to Λ.

Now $\overline{\Lambda}_U$ and Λ'_U are unknots in the contact 3–manifold (Y, ξ) with the same Thurston–Bennequin and rotation numbers. We claim that, possibly after further applying Lemma 3.2 until the Thurston–Bennequin numbers of both unknots are negative, there exists a contact isotopy ϕ_t of (Y, ξ) taking $\overline{\Lambda}_U$ to Λ'_U. If (Y, ξ) is tight, the existence of such an isotopy follows from Eliashberg and Fraser's [2009,

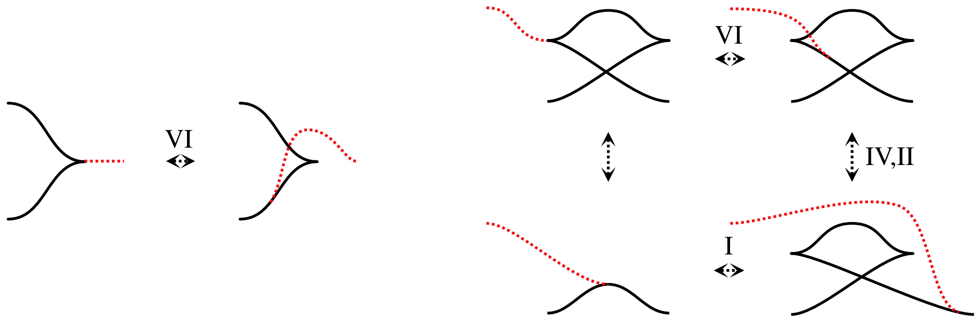

Figure 16: Left: moving a triple point off of a cusp using a Reidemeister VI move from [O'Donnol and Pavelescu 2012]. Right: clearing a cusp of Λ for a Reidemeister I move.

Theorem 1.5] classification of Legendrian unknots in tight contact manifolds. If (Y,ξ) is overtwisted, then our assumption that the Thurston–Bennequin numbers are negative allows us to apply [Eliashberg and Fraser 2009, Proposition 4.12] to find the desired isotopy.

We now apply the isotopy ϕ_t to the Legendrian handle graph $\widetilde{G}$ and perturb the result so that the attached Legendrian handles are disjoint from those of G'. We then obtain a pair of Legendrian handle graphs for the unknot Λ'_U, surgery along which yields Legendrian links isotopic to Λ and Λ' respectively. □

4 Lower bounds via diagrams

In this section, we reprove Lemma 3.5 for Legendrian links in the standard contact $\mathbb{R}^3$ using diagrammatic techniques rather than convex surface theory. This proof refines that of [Boranda et al. 2013] to produce a handle graph as well as a Lagrangian cobordism from an unknot. While this section is not logically necessary for the proof of Theorem 1.1 given our work in the previous section, the techniques introduced herein are essential for the understanding and practical application of the main ideas of this paper, as justified in Examples 5.1 and 6.3 below.

We begin with a sequence of lemmas that reduce the number of crossings of the front diagram of the Legendrian link in a Legendrian handle graph at the expense of increasing the number of handles. But first, we state a technical general position result.

Lemma 4.1 *For any Legendrian handle graph (G,Λ), there exists a C^0–close, isotopic Legendrian handle graph (G',Λ') such that all singular points of the front diagram of G have distinct x–coordinates.*

Proof While this lemma simply expresses general position for the graph G, we note in Figure 16, left, that moving a triple point off of a cusp of Λ is tantamount to using a Reidemeister VI move. □

First, we remove negative crossings.

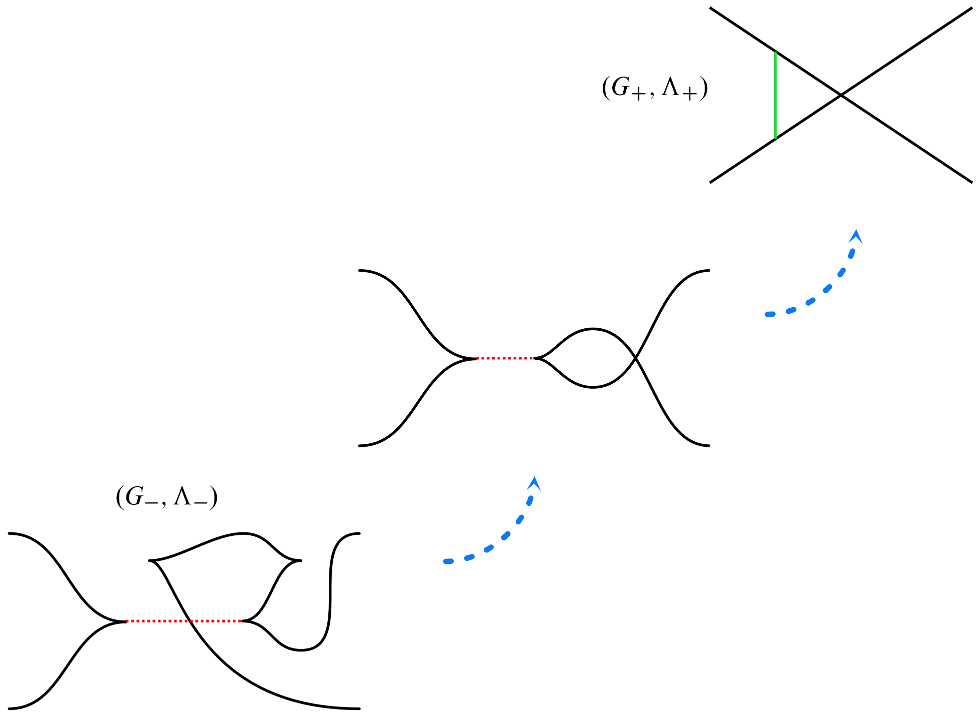

Figure 17: The Legendrian handle graph (G_-, Λ_-) has one fewer negative crossing than (G_+, Λ_+). Red curves represent surgery disks, ie cores of handles, while green curves represent cocores.

Lemma 4.2 *Given a Legendrian link Λ_+, whose front diagram has a negative crossing, and a Legendrian handle graph G_+ on Λ_+, there exists a Legendrian handle graph (G_-, Λ_-) and a subset $\mathcal{H}_0$ of handles of G_-, such that $\mathrm{Surg}(G_-, \Lambda_-, \mathcal{H}_0)$ is Legendrian isotopic to (G_+, Λ_+), and the front diagram of Λ_- has one fewer negative crossing than that of Λ_+.*

After applying Lemma 4.1 to isolate negative crossings, the proof of Lemma 4.2 is contained in Figure 17.

Next, we remove positive crossings.

Lemma 4.3 *Given a Legendrian link Λ_+, the leftmost crossing of whose front diagram is positive, and a Legendrian handle graph G_+ on Λ_+, there exists a Legendrian handle graph (G_-, Λ_-) and a subset $\mathcal{H}_0$ of handles of G_-, such that $\mathrm{Surg}(G_-, \Lambda_-, \mathcal{H}_0)$ is Legendrian isotopic to (G_+, Λ_+) and the front diagram of Λ_- has one fewer positive crossing than that of Λ_+.*

Proof Apply Lemma 4.1 to isolate crossings and cusps of Λ_+ from handles of G_+. Consider the leftmost crossing X_0 of Λ_+. Without loss of generality, we may assume that Λ_+ is oriented from right to left on both strands of X_0. The upper-left strand incident to X_0 must thus next return to X_0; the same is true for the bottom-left strand. Since there are no crossings of Λ_+ to the left of X_0, either the upper left strand must next cross the x–coordinate of X_0 above X_0 or the lower left strand must next cross the

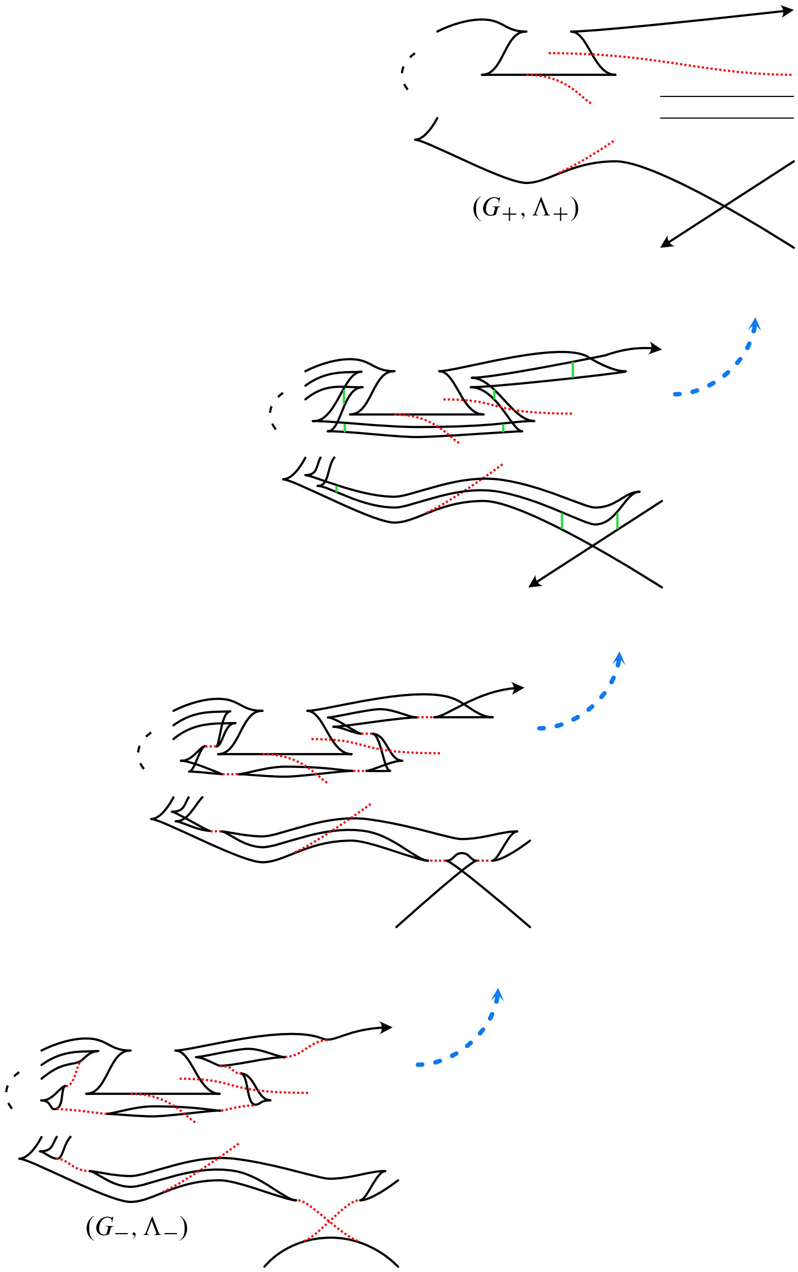

Figure 18: The Legendrian handle graph (G_-, Λ_-) has one fewer positive crossing than (G_+, Λ_+). Red curves represent cores of handles, while green curves represent cocores.

x–coordinate of X_0 below X_0. Without loss of generality, assume that this holds for the upper left strand as in the upper-right portion of Figure 18. Let $-\eta \subset \Lambda_+$ be the compact 1–manifold that starts at X_0 and traverses along the upper-left strand of X_0 until returning to the same x–coordinate, and let η be $-\eta$ with the orientation reversed, so that X_0 is at the end of η.

As in the second diagram down in Figure 18, create a finger of Λ_+ parallel to η using a Reidemeister I move at the initial point of η and next to every cusp of η along with Reidemeister II moves to pass the lead cusp of the finger through handles of G_+ that are incident to η. Move the end of the finger just to the right of X_0. Place a cocore of a handle inside the finger just below each crossing created by a Reidemeister I move. Place two additional cocores from the finger to the original link on either side of the crossing X_0.

Finally, replace the cocores by surgery disks to create a new Legendrian handle graph as in the third row of Figure 18. Isotope the new Legendrian handle graph as at the bottom of Figure 18, using a combination of the move in Figure 16, right, to move the handles away and Reidemeister I moves to remove the crossings. The result is a Legendrian handle graph that has many more surgery disks, but whose underlying Legendrian link has one fewer crossing than before. □

The procedure above may produce a disconnected Legendrian link. We next see how to join these components.

Lemma 4.4 *Let (G_+, Λ_+) be a Legendrian handle graph, where Λ_+ has $n \geq 2$ components, which are mutually disjoint in the front diagram. Suppose that there exists a path γ in the front diagram of G_+ that starts on component $\Lambda'_+ \subset \Lambda_+$, ends on $\Lambda''_+ \subset \Lambda_+$, and does not intersect Λ_+ otherwise. Then there exists a Legendrian handle graph (G_-, Λ_-) and a subset $\mathscr{H}_0$ of handles of G_-, such that $\mathrm{Surg}(G_-, \Lambda_-, \mathscr{H}_0)$ is Legendrian isotopic to (G_+, Λ_+), one component of Λ_- is topologically the connected sum of Λ'_+ and Λ''_+, the other components of Λ_- match the remaining components of Λ_+, and none of the components of Λ_- intersect in the front diagram.*

Proof We may assume that γ intersects Λ'_+ and Λ''_+ away from triple points, crossings, and cusps. Create a finger of Λ'_+ that follows γ, starting with a Reidemeister I move and using Reidemeister II moves to cross handles of G_+ and additional Reidemeister I moves when γ has a vertical tangent; see the middle diagram of Figure 19. Stop the finger just before γ intersects Λ''_+, performing an additional Reidemeister I move if necessary to ensure that the orientations of parallel strands of the finger and Λ''_+ are opposite. Place a cocore of a handle between those two parallel strands. Finally, replace the cocore by a core of a handle to create a new Legendrian handle graph (G_-, Λ_-) as in the bottom-left portion of Figure 19.

That the new component of Λ_- is the connect sum of Λ'_+ and Λ''_+ comes from the facts that the diagrams of Λ'_+ and Λ''_+ are disjoint and that γ is disjoint from the diagram of Λ_+ on its interior. The final two conclusions of the lemma follow immediately from the construction. □

With the tools above in place, we are ready to reprove Lemma 3.5 using the diagrammatic techniques of this section.

Diagrammatic proof of Lemma 3.5 in $(\mathbb{R}^3, \xi_0)$ Given a Legendrian Λ, use Lemma 4.2 repeatedly, and then Lemma 4.3 repeatedly, to obtain a Legendrian handle graph (G_1, Λ_1) such that the front diagram of

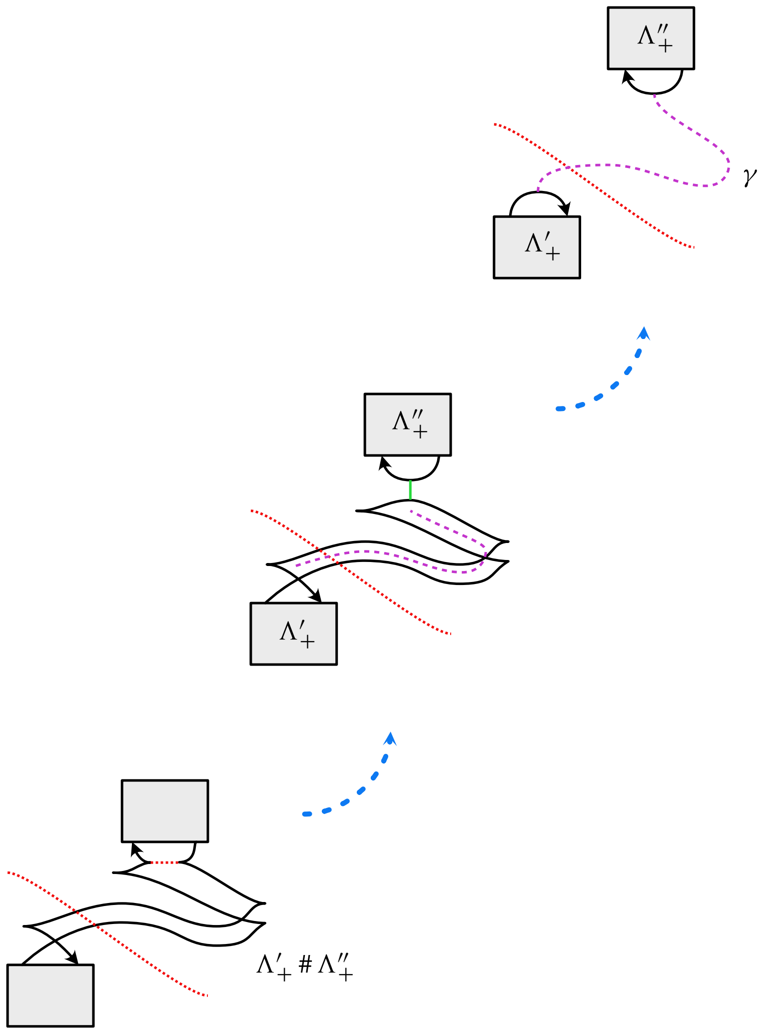

Figure 19: The Legendrian link Λ_- has a component that is topologically the connected sum of two components of Λ_+.

Λ_1 has no crossings, and $\mathrm{Surg}(G_1, \Lambda_1)$ is Legendrian isotopic to Λ. Use Lemma 4.4 to find a Legendrian handle graph (G_2, Λ_2) with a subset $\mathscr{H}_0$ of handles, such that Λ_2 is connected, and $\mathrm{Surg}(G_2, \Lambda_2, \mathscr{H}_0)$ is Legendrian isotopic to $\mathrm{Surg}(G_1, \Lambda_1)$, which implies that $\mathrm{Surg}(G_2, \Lambda_2)$ is Legendrian isotopic to Λ. Finally, note that Λ_2 is a smooth unknot since it is the connected sum of smooth unknots. □

5 Upper bounds

With constructions of a common lower bound and corresponding handle graphs for Λ and Λ' in hand, we are ready to find an upper bound. The structure of the following proof parallels that of Lazarev [2020] in higher dimensions.

Proof of Proposition 1.6 Given oriented Legendrian links in Λ and Λ' in (Y, ξ), Proposition 3.1 implies that there exist an oriented Legendrian link Λ_- and Legendrian handle graphs (G, Λ_-) and (G', Λ_-) such that $\mathrm{Surg}(G, \Lambda_-)$ (resp. $\mathrm{Surg}(G', \Lambda_-)$) is Legendrian isotopic to Λ (resp. Λ').

We Legendrian isotope the handles $\mathscr{H}'$ of G' to be in general position with respect to the handles $\mathscr{H}$ of G. In particular, we may assume that the Legendrian handle graph (G', Λ_-) has $\mathscr{H}'$ is disjoint from $\mathscr{H}$, with $\mathrm{Surg}(G', \Lambda_-)$ still Legendrian isotopic to Λ'.

Define the Legendrian graph $G_+ = G \cup G'$; it is clear that (G_+, Λ_-) is a Legendrian handle graph. Note that $\mathrm{Surg}(G_+, \Lambda_-, \mathscr{H})$ is Legendrian isotopic to a Legendrian handle graph $(G_{+,1}, \Lambda)$; similarly, $\mathrm{Surg}(G_+, \Lambda_-, \mathscr{H}')$ is Legendrian isotopic to a Legendrian handle graph $(G_{+,2}, \Lambda')$.

Observe that both $\mathrm{Surg}(G_{+,1}, \Lambda)$ and $\mathrm{Surg}(G_{+,2}, \Lambda')$ are Legendrian isotopic to $\mathrm{Surg}(G_+, \Lambda_-)$, which we denote by Λ_+. Let $L\colon \Lambda_- \to \Lambda_+$ be the concatenation of $L(G, \Lambda_-)$ with $L(G_{+,1}, \Lambda)$; similarly, let $L'\colon \Lambda_- \to \Lambda_+$ be the concatenation of $L(G', \Lambda_-)$ with $L(G_{+,2}, \Lambda')$. Then it is clear that Λ (resp. Λ') appears as a collared slice of L (resp. L'). At the same time, Proposition 2.6 implies that L and L' are exact-Lagrangian isotopic, since they are both obtained from the same Legendrian handle graph (G_+, Λ_-) by Legendrian ambient surgery, only in a different order — in other words, they both belong to the isotopy class $L(G_+, \Lambda_-)$. □

Example 5.1 Figures 20 and 21 display the full process of creating the upper bounds in Figures 1 and 2, respectively.

6 The Lagrangian cobordism genus

In this section, we use the construction of upper and lower bounds for a pair of Legendrian knots to define a new quantity, the relative Lagrangian genus, and a new relation, Lagrangian zigzag-concordance. We explore foundational properties and immediate examples, leaving deeper explorations, as embodied in the list of open questions at the end, for future work. For ease of notation, we work with Legendrian links in the standard contact $\mathbb{R}^3$, though our definitions may easily be adapted to Legendrians in any contact 3–manifold.

6.1 Lagrangian quasicobordism

We begin with a definition that undergirds the two concepts referred to above.

Definition 6.1 A *Lagrangian zigzag-cobordism* between Legendrian knots Λ and Λ' consists of an ordered set of $n+1$ Legendrian links

$$\boldsymbol{\Lambda} = (\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_n = \Lambda'),$$

another ordered set of n nonempty Legendrian links

$$\boldsymbol{\Lambda}^* = (\Lambda_1^*, \dots, \Lambda_n^*),$$

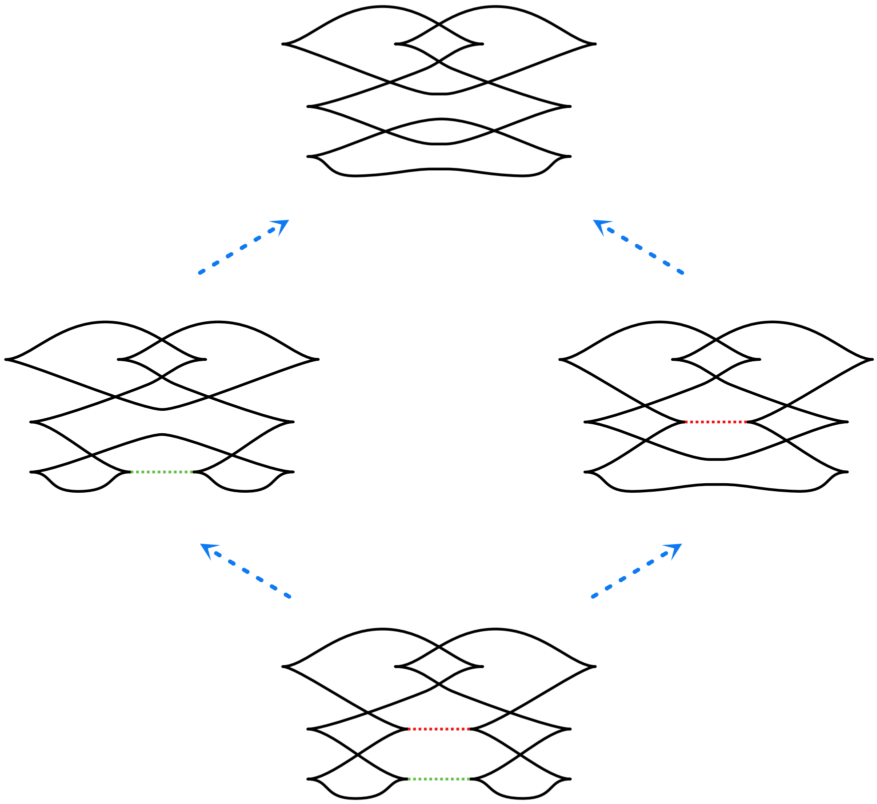

Figure 20: The handle graph at the bottom of the figure is used to create the upper bound of the trefoil and an $m(5_2)$ knot that appeared in Figure 1.

such that Λ_i^* is an upper or lower bound for the pair $(\Lambda_{i-1}, \Lambda_i)$, and connected Lagrangian cobordisms

$$\mathbf{L} = (L_1^<, L_1^>, L_2^<, L_2^>, \dots, L_n^<, L_n^>)$$

that realize the upper or lower bound constructions.

There are several quantities associated to a Lagrangian zigzag-cobordism.

Definition 6.2 Given a Lagrangian zigzag-cobordism $(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L})$, its *length* is one less than the number of elements in $\mathbf{\Lambda}$, while its *Euler characteristic* $\chi(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L})$ is the sum of the Euler characteristics of the Lagrangians in $\mathbf{L}$ and its *genus* $g(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L})$ defined, as usual, in terms of the Euler characteristic.

Further, we define the *relative Lagrangian genus* $g_L(\Lambda, \Lambda')$ between the Legendrian knots Λ and Λ' as the minimum genus of any Lagrangian zigzag-cobordism between them. Two Legendrian knots Λ and Λ' are *Lagrangian zigzag-concordant* if $g_L(\Lambda, \Lambda') = 0$.

Example 6.3 Let Υ be the maximal Legendrian unknot, and let Λ be a maximal Legendrian representative of $m(6_2)$. Note that both Υ and Λ have Thurston–Bennequin number -1 and that the smooth 4–genus of

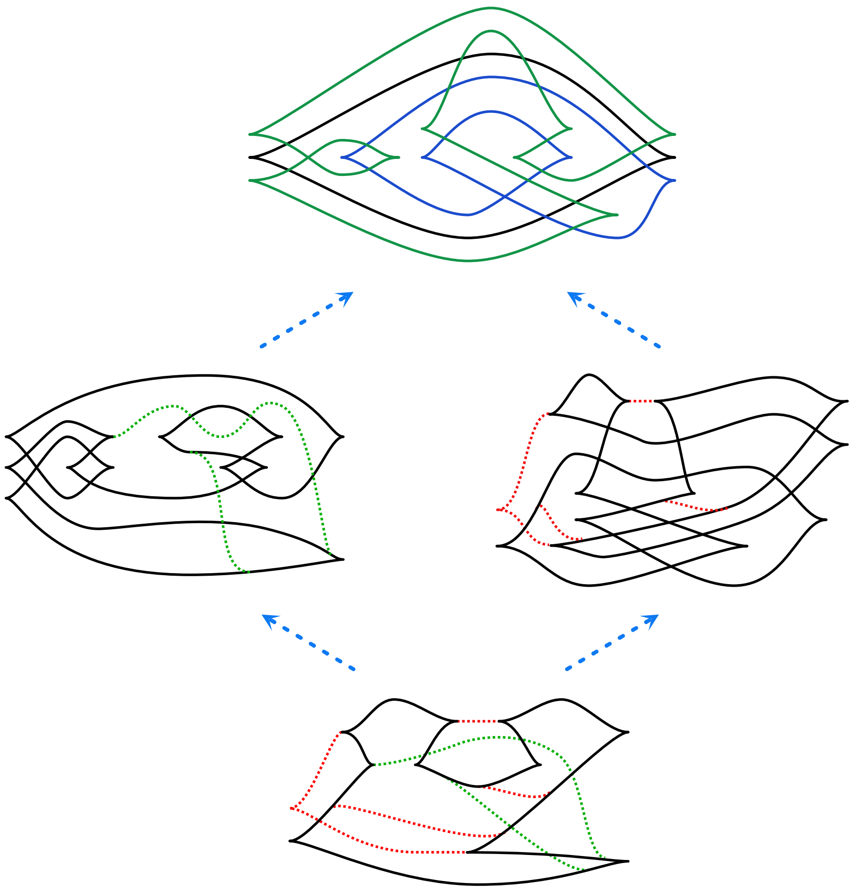

Figure 21: The handle graph at the bottom of the figure is used to create the upper bound of the figure eight knot and the unknot that appeared in Figure 2. Note that the Legendrian knots in the handle graphs in the middle level are isotopic to the unknot (left) and the figure eight (right).

6_2 is equal to 1 [Livingston and Moore 2021]. It follows from the behavior of the Thurston–Bennequin invariant under Lagrangian cobordism that there cannot be a Lagrangian cobordism joining Υ and Λ in either direction. Nevertheless, there is a genus-1 Lagrangian zigzag-cobordism between the two; see Figure 22.

Lagrangian zigzag-cobordism induces an equivalence relation on the set of isotopy classes of Legendrian links. As in the smooth case, this equivalence relation is uninteresting, as shown by the following immediate corollary of Theorem 1.1 or Proposition 3.1, together with Remark 1.3:

Corollary 6.4 *Any two Legendrian knots with the same rotation number are Lagrangian zigzag-cobordant. In fact*, *the zigzag-cobordism may be chosen to have length* 1.

The corollary shows that the relative Lagrangian genus is defined for any two Legendrian knots of the same rotation number.

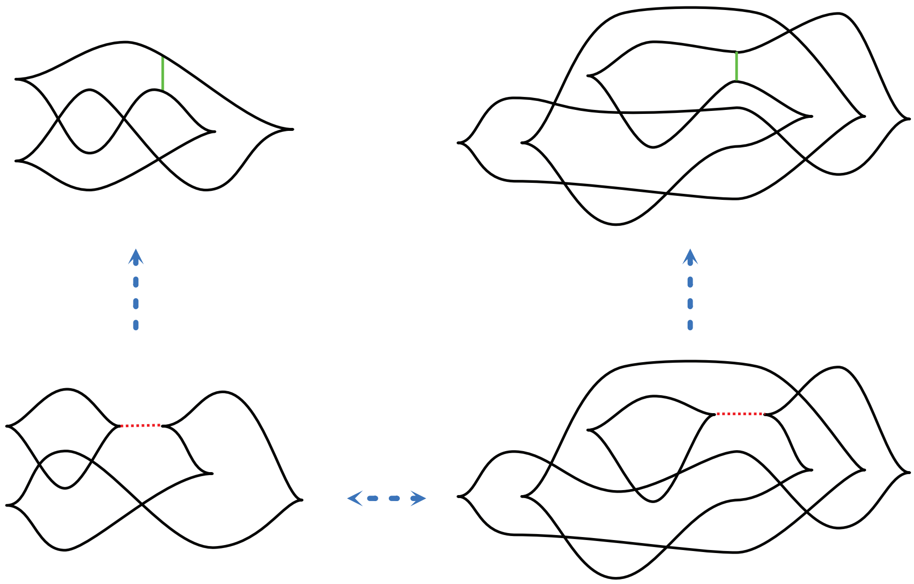

Figure 22: A genus 1 Lagrangian quasicobordism between the maximal unknot Υ and a maximal representative Λ of the mirror of the 6_2 knot. The zigzag-cobordism was produced using the ideas in [Boranda et al. 2013, Section 5], especially Figures 25 and 27.

On the other hand, Lagrangian zigzag-concordance also clearly induces an equivalence relation on the set of isotopy classes of Legendrian knots. The relative Lagrangian genus descends to Lagrangian zigzag-concordance classes. Using [Chantraine 2010] and the connectedness of the Lagrangians, we see that both the rotation number and the Thurston–Bennequin number are invariants of Lagrangian zigzag-concordance, though nonclassical invariants coming from Legendrian contact homology or Heegaard Floer theory will have a more complicated relationship with zigzag-concordance.

6.2 Relation to smooth genus

To connect the relative Lagrangian genus to smooth constructions, note that we may define the smooth cobordism genus between two smooth knots K_1 and K_2 to be the minimum genus of all cobordisms between them; we denote this by $g_s(K_1, K_2)$. Chantraine [2010] proved that Lagrangian *fillings* minimize the smooth 4–ball genus of a Legendrian knot, and so one might ask if this minimization property extends to g_L. We begin with a simple lemma.

Lemma 6.5 *Given Legendrian knots Λ and Λ', we have $g_s(\Lambda, \Lambda') \leq g_L(\Lambda, \Lambda')$.*

Proof Let $(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L})$ be a Lagrangian zigzag-cobordism between Λ and Λ'. Assume for ease of notation that each $\Lambda^* \in \mathbf{\Lambda}^*$ is an upper bound. Let $\bar{L}_i^>$ be the smooth cobordism from Λ_i^* to Λ_i obtained from reversing $L_i^>$; note that $\bar{L}_i^>$ is not, in general, a Lagrangian cobordism. Since Euler characteristic is additive under gluing, the smooth cobordism $L_1^< \circ \bar{L}_1^> \circ L_2^< \circ \cdots \circ \bar{L}_n^>$ has genus $g(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L})$, and hence $g_s(\Lambda, \Lambda') \leq g_L(\Lambda, \Lambda')$. □

It is natural to ask under what conditions on Λ_1 and Λ_2 — as Legendrian or as smooth knots — is the inequality in Lemma 6.5 an equality? On one hand, we cannot expect to achieve equality in all cases.

Example 6.6 Let Λ be any Legendrian knot, and let Λ' be a double stabilization of Λ with the same rotation number as Λ. Since Λ and Λ' have the same underlying smooth knot type, we have $g_s(\Lambda, \Lambda') = 0$. On the other hand, let $(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L})$ be a Lagrangian zigzag-cobordism between Λ and Λ'. Note that $\chi(L_i^<), \chi(L_I^>) \leq 0$ for all i, since each of $L_i^<$ and $L_i^>$ is connected and has at least two boundary components. Since $\mathrm{tb}(\Lambda) > \mathrm{tb}(\Lambda')$, some pair Λ_i, Λ_{i+1} in $\mathbf{\Lambda}$ must have different Thurston–Bennequin numbers. In particular, the bound Λ_i^* must have a different Thurston–Bennequin number than at least one of Λ_i or Λ_{i+1}. It follows that $\chi(L_i^<) + \chi(L_i^>) < 0$, and hence that $\chi(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L}) < 0$. Since Λ and Λ' are knots, this implies that $g(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L}) > 0$. In particular, we have $g_L(\Lambda, \Lambda') > 0$ even though $g_s(\Lambda, \Lambda') = 0$.

On the other hand, there is a simple sufficient condition for equality in the lemma above.

Lemma 6.7 *If the Legendrian knot Λ has a Lagrangian filling, and there exists a Lagrangian cobordism from Λ to Λ', then $g_s(\Lambda, \Lambda') = g_L(\Lambda, \Lambda')$.*

Proof We begin by setting notation. Let L_0 be the Lagrangian filling of Λ and let $L_1^<$ be the Lagrangian cobordism from Λ to Λ'. Taking $L_1^>$ to be the trivial cylindrical Lagrangian cobordism from Λ' to itself, and taking $\Lambda_1^* = \Lambda'$, we see that

$$g_L(\Lambda, \Lambda') \leq g(L_1^<). \tag{1}$$

Let Σ be the smooth cobordism from Λ to Λ' that minimizes the smooth cobordism genus. We know that $L_0 \circ L_1^<$ is a Lagrangian filling of Λ', and hence that $g(L_0 \circ L_1^<) \leq g(L_0 \circ \Sigma)$. Since Λ is a knot, the genus is additive under composition of cobordisms, and we obtain

$$g(L_1^<) \leq g(\Sigma). \tag{2}$$

Combining (1) and (2), we obtain

$$g_L(\Lambda, \Lambda') \leq g(\Sigma) = g_s(\Lambda, \Lambda').$$

The lemma now follows from Lemma 6.5. □

6.3 Open questions

We end with a list of questions about Lagrangian zigzag-cobordism and zigzag-concordance beyond the motivating question above about the relationship between the relative Lagrangian genus and the relative smooth genus.

(1) Building off of Example 6.3, is there an example of a pair Λ and Λ' that are Lagrangian zigzag-concordant but not Lagrangian concordant?

(2) Taking the previous question further, for two Lagrangian zigzag-concordant Legendrians Λ and Λ', what is the minimal length of any Lagrangian zigzag-concordance between them? Are there examples for which this minimal length is arbitrarily high?

(3) Even more generally, define $g_L(\Lambda, \Lambda', n)$ to be the minimal genus of any Lagrangian zigzag-cobordism between Λ and Λ' of length at most n. The sequence $(g_L(\Lambda, \Lambda', n))_{n=1}^{\infty}$ decreases to and stabilizes at $g_L(\Lambda, \Lambda')$. Are there examples for which the number of steps it takes the sequence to stabilize is arbitrarily long?

(4) Can $g_L(\Lambda, \Lambda') - g_s(\Lambda, \Lambda')$ be arbitrarily large when Λ and Λ both have maximal Thurston–Bennequin invariant?

(5) Can the hypotheses of Lemma 6.7 be weakened to Λ having only an augmentation instead of a filling?

(6) Is there a version of this theory for Maslov 0 Lagrangians, which would better allow the use of Legendrian contact homology, especially the tools in [Pan 2017]?

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