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# Interleaving Mayer–Vietoris spectral sequences

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We discuss the Mayer–Vietoris spectral sequence as an invariant in the context of persistent homology. In particular, we introduce the notion of  $\varepsilon$ -acyclic carriers and  $\varepsilon$ -acyclic equivalences between filtered regular CW-complexes and study stability conditions for the associated spectral sequences. We also look at the Mayer–Vietoris blowup complex and the geometric realization, finding stability properties under compatible noise; as a result we prove a version of an approximate nerve theorem. Adapting work by Serre, we find conditions under which  $\varepsilon$ -interleavings exist between the spectral sequences associated to two different covers.

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## 1 Introduction

One of the benefits of homology as a topological invariant over, for example, the homotopy groups, is its computability via long exact sequences. The classical Mayer–Vietoris exact sequence has been used in countless examples to compute  $H_k(X)$  from a decomposition of a space  $X$  into two open subsets  $U$  and  $V$ . When we generalise this concept to open covers  $(U_i)_{i \in I}$  consisting of more than just two subsets, the relations between the parts  $H_k(U_i)$  become more intricate and are encoded in the Mayer–Vietoris spectral sequence. These sequences first appeared in work of Leray and later Serre, and they proved to be one of the most powerful tools in pure algebraic topology. Applications of spectral sequences in applied algebraic topology, however, is still a young subject.

In [Torras-Casas 2023] it was proven that the persistence Mayer–Vietoris spectral sequence can be used to compute persistent homology. The starting point is a filtered simplicial complex  $X$  together with a cover by subcomplexes  $\mathcal{U}$ . Then, one computes  $\text{PH}_i(\mathcal{U}_\sigma)$  for all  $i \geq 0$  and  $\sigma \in N_{\mathcal{U}}$ . Here, notice that  $N_{\mathcal{U}}$  is the nerve of the cover  $\mathcal{U}$  whose simplices  $\sigma \in N_{\mathcal{U}}$  are subsets from  $\mathcal{U}$ ; this leads to the notation  $\mathcal{U}_\sigma = \bigcap_{U \in \sigma} U$ . The Mayer–Vietoris spectral sequence starts from these groups and the morphisms induced by inclusions and converges to  $\text{PH}_i(X)$ . As pointed out in [Yoon and Ghrist 2020], the additional insight gained from the cover  $\mathcal{U}$  can be used, for example, for multiscale feature detection. Similar information was also explored much earlier in [Zomorodian and Carlsson 2008] in the form of *localized homology*.

Motivated by these results, we study the spectral sequence  $E_{p,q}^*(X, \mathcal{U})$  and answer the following questions:

- Let a pair  $(X, \mathcal{U})$  consist of a space,  $X$ , and a cover for  $X$ ,  $\mathcal{U}$ . The Mayer–Vietoris spectral sequence  $E_{p,q}^*(X, \mathcal{U})$  converges to  $\text{PH}_*(\Delta^{\mathcal{U}}(X))$ . Is  $\text{PH}_*(\Delta^{\mathcal{U}}(X))$  stable? Can this result be generalised?
- Suppose that the data in each covering set  $\mathcal{U}_\sigma$  for  $\sigma \in N_{\mathcal{U}}$  is modified slightly. If the underlying cover  $\mathcal{U}$  is ignored, then we would not expect  $E_{p,q}^*(X, \mathcal{U})$  to be stable. Are there natural coherence conditions between changes in the sets  $\mathcal{U}_\sigma$  that imply stability? If so, what do we mean by stability of spectral sequences?
- Let  $\mathcal{U}$  and  $\mathcal{V}$  be covers of the same space  $X$ . Can we compare  $E_{p,q}^*(X, \mathcal{U})$  and  $E_{p,q}^*(X, \mathcal{V})$  up to  $\varepsilon$ -interleavings?

To explain why the first question is important and how it is linked to spectral sequences, we note that  $E_{p,q}^*(X, \mathcal{U})$  converges to the target persistent homology  $\text{PH}_*(\Delta^{\mathcal{U}}(X))$  (this is usually denoted by  $E_{p,q}^*(X, \mathcal{U}) \Rightarrow \text{PH}_*(\Delta^{\mathcal{U}}(X))$ ). The blowup complex  $\Delta^{\mathcal{U}}(X)$  already appeared in the context of topological data analysis in [Lewis and Morozov 2015] and [Zomorodian and Carlsson 2008]. It is homotopy equivalent to a *homotopy colimit*, and therefore enjoys good properties with respect to local homotopy equivalences. For example, if we assume that  $\mathcal{U}_\sigma$  is contractible for all  $\sigma \in N_{\mathcal{U}}$ , then we can use [Hatcher 2002, Proposition 4G.2] to recover Leray’s nerve theorem. That is, there are homotopy equivalences

$$X \simeq \Delta^{\mathcal{U}}(X) \simeq \Delta^{\mathcal{U}}(*) = N(\mathcal{U}),$$

where  $*$  denotes the constant *complex of spaces* on  $\mathcal{U}$ ; see [Hatcher 2002, Appendix 4.G]. The fundamental importance of this result in applied topology is underlined by the persistent nerve lemma presented in [Chazal and Oudot 2008]. It is worth mentioning the approximate nerve theorem [Govc and Skraba 2018] and the generalised nerve theorem [Cavanna 2019], which are approximate versions of the Leray theorem within the context of persistence. In particular, in [Govc and Skraba 2018] the spectral sequence  $E_{p,q}^*(X, \mathcal{U}) \Rightarrow \text{PH}_*(X)$  is examined, and it is studied how much it differs from another spectral sequence  $E_{p,q}^*(*, \mathcal{U}) \Rightarrow \text{PH}_*(N(\mathcal{U}))$ , by careful inspection of all pages as well as the extension problem.

Throughout the paper we focus on the category **RCW-cpx** of *regularly filtered regular CW-complexes* as well as the subcategory **FCW-cpx** of *filtered regular CW-complexes*; see Section 2.1. Instead of restricting our attention to a space  $X$  together with a cover  $\mathcal{U}$ , we look at regular diagrams  $\mathcal{D}$  in **RCW-cpx** over a simplicial complex  $K$ . There is a natural replacement for the Mayer–Vietoris blowup complex in this setting, denoted by  $\Delta_K(\mathcal{D})$ , as explained in [Hatcher 2002, Appendix 4.G]. This object also appears in the context of semisimplicial spaces, where it is called the *geometric realization* [Ebert and Randal-Williams 2019]; in fact, it has an associated spectral sequence [Ebert and Randal-Williams 2019, Section 1.4]. As we explain in Section 3, there are good reasons why it is worth taking this more general perspective. In particular, we consider the spectral sequence

$$E_{p,q}^2(\mathcal{D}) \Rightarrow \text{PH}_{p+q}(\Delta_K \mathcal{D}).$$

In order to address the first two questions, we introduce the notion of acyclic carriers to define  $\varepsilon$ -acyclic equivalences. Using the acyclic carrier theorem we show the following: Let  $X$  and  $Y$  be two objects

in **RCW-cpx**. If there exists an  $\varepsilon$ -acyclic equivalence  $F^\varepsilon: X \rightrightarrows Y$ , then  $\mathrm{PH}_*(X)$  is  $\varepsilon$ -interleaved with  $\mathrm{PH}_*(Y)$  (see Corollary 4.7 and Proposition 4.2 for a stronger statement in **FCW-cpx**). These equivalences provide a very flexible notion that works in different contexts as Examples 4.5, 4.6 and 4.8 show.

We address the first question in the following way. Let  $\mathcal{D}$  and  $\mathcal{L}$  be two diagrams over the same simplicial complex  $K$  and assume that for all  $\sigma \in K$  there are  $\varepsilon$ -acyclic equivalences  $F_\sigma^\varepsilon: \mathcal{D}(\sigma) \rightrightarrows \mathcal{L}(\sigma)$  which satisfy a compatibility condition with respect to composition in the poset category associated to  $K$ ; see Theorem 5.2 for details. Then, there is an  $\varepsilon$ -acyclic equivalence  $F^\varepsilon: \Delta_K(\mathcal{D}) \rightrightarrows \Delta_K(\mathcal{L})$ . This result implies stability in the targets of convergence of the spectral sequences. We use this result to show a “strong approximate multinerve theorem” in Corollary 5.3. Later, in Section 6, we introduce  $(\varepsilon, n)$ -interleavings, which are given by spectral sequence morphisms that start at some page  $n$  together with a shift by a persistence parameter  $\varepsilon > 0$ . Assuming the same conditions as in the geometric realization case, we can obtain a  $(\varepsilon, 1)$ -interleaving between  $E_{p,q}^*(\mathcal{D})$  and  $E_{p,q}^*(\mathcal{L})$ ; see Theorem 5.2. This result appears in Theorem 6.5 and a specialised strong statement for covers of spaces in **FCW-cpx** is given in Proposition 6.4.

As for the third question about the comparison of the spectral sequences associated to two covers  $\mathcal{U}$  and  $\mathcal{V}$  of a space  $X$ , we rely on work of Serre [1955], in which he studied the relation between the Čech cohomology of two different covers; here we adapt this work in the context of cosheaves and cosheaf homology. Take a cosheaf  $\mathcal{F}$  of abelian groups on  $X$  and assume that there is a refinement  $\mathcal{V} < \mathcal{U}$ . Serre showed that the refinement morphism induced on Čech homology  $\rho^{\mathcal{U}\mathcal{V}}: \check{\mathcal{H}}_*(\mathcal{V}, \mathcal{F}) \rightarrow \check{\mathcal{H}}_*(\mathcal{U}, \mathcal{F})$  is independent of the particular choice of morphism in the cochains. In [Serre 1955] it was also shown that  $\rho^{\mathcal{U}\mathcal{V}}$  can be factored through a construction that uses a double complex associated to both covers  $C_{p,q}(\mathcal{U}, \mathcal{V}; \mathcal{F})$ , see [Serre 1955, Proposition 4, Section 29]. This construction introduces two double complex spectral sequences  ${}^I E_{p,q}^*(\mathcal{U}, \mathcal{V}; \mathcal{F})$  and  ${}^{II} E_{p,q}^*(\mathcal{U}, \mathcal{V}; \mathcal{F})$ , both of which converge to  $\check{\mathcal{H}}_*(\mathcal{U} \cap \mathcal{V}; \mathcal{F}) \simeq \check{\mathcal{H}}_*(\mathcal{V}; \mathcal{F})$ . Here one might study conditions on  ${}^{II} E_{p,q}^*(\mathcal{U}, \mathcal{V}; \mathcal{F})$  to find when an inverse of  $\rho^{\mathcal{U}\mathcal{V}}$  exists. As an application, Serre [1955, Theorem 1, Section 29] obtained an analogous result to the Leray theorem in the context of sheaves.

We start our analysis of the third question in Section 7. In case  $\mathcal{V} < \mathcal{U}$  there is a unique morphism induced by the refinement map on the second page

$$\rho^{\mathcal{U}\mathcal{V}}: E_{p,q}^*(X, \mathcal{V}) \rightarrow E_{p,q}^*(X, \mathcal{U}).$$

On the other hand, Theorem 7.10 tells us under what conditions there exists an  $\varepsilon$ -shifted morphism  $\psi: E_{p,q}^*(X, \mathcal{U}) \rightarrow E_{p,q}^*(X, \mathcal{V})[\varepsilon]$  such that  $\rho^{\mathcal{U}\mathcal{V}}$  and  $\psi$  form an  $(\varepsilon, 2)$ -interleaving between  $E_{p,q}^*(X, \mathcal{U})$  and  $E_{p,q}^*(X, \mathcal{V})$ . Finally, in Proposition 7.12 we give a means of obtaining an  $(\varepsilon, 2)$ -interleaving between  $E_{p,q}^*(X, \mathcal{U})$  and  $E_{p,q}^*(X, \mathcal{V})$  through the computation of local Mayer–Vietoris spectral sequences  $E_{p,q}^*(\mathcal{U}_\sigma, \mathcal{V}|_{\mathcal{U}_\sigma})$  for all  $\sigma \in N_{\mathcal{U}}$ . Since the open regions  $\mathcal{U}_\sigma$  are assumed to be “small” in comparison to  $X$ , this gives a means of using local calculations to deduce the interleaving. As Corollary 7.14 we present the case when  $\mathcal{V}$  does not need to refine  $\mathcal{U}$ .

## 2 Background

### 2.1 Regular CW-complexes with filtrations

Recall the definition of CW-complex from [Hatcher 2002, Chapter 0]. In contrast to the usual treatment of CW-complexes, but in line with the structure we are dealing with in TDA, we consider the cell decomposition as part of the data of our CW-complexes. For a CW-complex  $X$ , if  $c$  is an open cell in  $X$  we follow the notation from [Cooke and Finney 1967] and denote this by  $c \in X$ . We denote by  $X^n$  the set of  $n$ -dimensional cells from  $X$  and we denote by  $X^{\leq n}$  the  $n$ -skeleton from  $X$ . Recall that  $X$  has a natural filtration given by its skeleta  $X^0 \subseteq X^{\leq 1} \subseteq \dots \subseteq X^{\leq N} \subseteq \dots$ , and a *cellular morphism*  $f: X \rightarrow Y$  respects this filtration, in the sense that it restricts to morphisms  $f^m: X^{\leq m} \rightarrow Y^{\leq m}$  for all  $m \geq 0$ . We work with regular CW-complexes, which are CW-complexes where the attaching maps are homeomorphisms. It is recommended to consult [Cooke and Finney 1967; Massey 1991] for properties and results related to regular CW-complexes. An intuitive way of understanding incidences of cells in regular complexes is through the barycentric subdivision, as explained in [Ellis 2019, Section 2.1]. Given a pair of cells  $a \in X^n$  and  $b \in X^{n-1}$ , we denote by  $[b : a]$  the degree of attaching map  $\partial a \rightarrow \bar{b}/\partial b$ .

**Definition 2.1** A cellular morphism  $f: X \rightarrow Y$  is a *regular morphism* whenever the closure  $\overline{f(a)}$  is a subcomplex of  $Y$  for all cells  $a \in X$ . For such a morphism and a pair  $a \in X^n$  and  $b \in Y^n$ , we denote by  $[b : f(a)]$  the degree of the morphism  $f$  restricted to the open cell  $a$  and mapping into the open cell  $b$ .

We write **CW-cpx** to denote the category of finite regular CW-complexes and regular morphisms. Denote by  $\mathbf{R}$  the ordered category  $(\mathbb{R}, \leq)$  of real numbers. We focus on functors  $X: \mathbf{R} \rightarrow \mathbf{CW-cpx}$  which we call *regularly filtered CW-complexes*, and we denote their category by **RCW-cpx**. We say that an object  $X \in \mathbf{RCW-cpx}$  is *tame*, whenever  $X$  is constant along a finite number of right open intervals decomposing the poset  $\mathbf{R}$ . For  $X \in \mathbf{RCW-cpx}$ , we write  $X_r$  for the regular CW-complex  $X(r)$  for all  $r \in \mathbf{R}$ . On the other hand we write  $X(r \leq s)$  to denote the morphisms  $X_r \rightarrow X_s$  for all  $r \leq s$  in  $\mathbf{R}$ ; we call such morphisms *structure maps*. The reader might find an example of such a regularly filtered complex in the appendix. If the morphisms  $X(r \leq s): X_r \rightarrow X_s$  are injections preserving the cellular structure for all  $r \leq s$  in  $\mathbf{R}$ , then we call  $X$  a *filtered CW-complex*, denoting by **FCW-cpx** the corresponding subcategory of **RCW-cpx**. Notice that objects in **FCW-cpx** can be seen as a pair  $(\text{colim } X_*, f)$  where  $\text{colim } X_*$  is a regular CW-complex and  $f: \text{colim } X_* \rightarrow \mathbb{R}$  is a filtration function.

Throughout this text, we work with a fixed field  $\mathbb{F}$ . Given  $X \in \mathbf{RCW-cpx}$ , we define the persistent homology in degree  $n$  as the functor  $\text{PH}_n(X): \mathbf{R} \rightarrow \mathbf{vect}$  given by computing cellular homology  $\text{PH}_n(X)_r = H_n^{\text{cell}}(X_r; \mathbb{F})$  for all  $r \in \mathbf{R}$ . As  $X_r$  is finite, the vector space  $\text{PH}_n(X)_r$  is finite dimensional for all  $r \in \mathbf{R}$ . If in addition  $X$  is tame,  $\text{PH}_n(X)$  only changes at a finite number of points  $r \in \mathbf{R}$ . We call the category of functors  $\mathbf{R} \rightarrow \mathbf{vect}_{\mathbb{F}}$  *persistence modules* and denote it by **PMod**. Given  $a \in (0, \infty)$  and  $X \in \mathbf{RCW-cpx}$ , we write  $X[a]$  for the element of **RCW-cpx** such that  $X[a]_r = X_{r+a}$  for all  $r \in \mathbf{R}$ . We use  $\Sigma^\varepsilon$  to denote the  $\varepsilon$ -shift functor  $\Sigma^\varepsilon: \mathbf{RCW-cpx} \rightarrow \mathbf{Hom}(\mathbf{RCW-cpx})$  which sends  $X \in \mathbf{RCW-cpx}$  to

$\Sigma^\varepsilon X: X \rightarrow X[\varepsilon]$ , where  $\varepsilon \geq 0$ . Also, for any morphism of filtered CW–complexes  $f: A \rightarrow B$ , one can check that  $f[\varepsilon] \circ \Sigma^\varepsilon A = \Sigma^\varepsilon B \circ f$ , where we use  $f[\varepsilon]: A[\varepsilon] \rightarrow B[\varepsilon]$ . Similarly, there are shift functors for persistence modules  $\Sigma^\varepsilon: \mathbf{PMod} \rightarrow \mathbf{Hom}(\mathbf{PMod})$  for  $\varepsilon \geq 0$ .

**Remark 2.2** Notice that the standard algorithm for the computation of persistent homology cannot be applied to objects in **RCW-cpx**. However, if  $X$  is tame and one successfully computes the coefficients for the morphisms  $C_*^{\text{cell}}(X_r) \rightarrow C_*^{\text{cell}}(X_s)$  for all  $r \leq s$  in  $\mathbf{R}$ , then one can use [Torrás-Casas 2023, Algorithm 2 image\_kernel1] to obtain a *barcode basis* for the filtered cellular complex  $C_*^{\text{cell}}(X)$ . Then we compute homology of the persistence morphisms given by the differentials  $d_n: C_n^{\text{cell}}(X) \rightarrow C_{n-1}^{\text{cell}}(X)$  by the use of image\_kernel1. See [Torrás-Casas 2023] for an explanation.

## 2.2 Acyclic carriers

Fix a field  $\mathbb{F}$ . We say that  $X \in \mathbf{CW-cpx}$  is  $\mathbb{F}$ –acyclic if the reduced homology  $\tilde{H}^*(X; \mathbb{F})$  with  $\mathbb{F}$ –coefficients vanishes in all dimensions; as the field is understood from the context, we just say that  $X$  is *acyclic*. Consider two objects  $\Phi$  and  $\Gamma$  from **CW-cpx** with their respective pairs of chains and differentials  $(C_*^{\text{cell}}(\Phi), \delta^\Phi)$  and  $(C_*^{\text{cell}}(\Gamma), \delta^\Gamma)$ . Let  $\langle \cdot, \cdot \rangle_\Phi$  and  $\langle \cdot, \cdot \rangle_\Gamma$  denote the inner products on  $C_*^{\text{cell}}(\Phi)$  and  $C_*^{\text{cell}}(\Gamma)$ , where the cells form an orthonormal basis. We define a relation  $\prec$  on  $\Phi$  by setting  $\tau \prec \sigma$  if  $\langle \tau, \delta^\Phi(\sigma) \rangle_\Phi \neq 0$  and by taking the transitive closure. We denote by  $\preceq$  the partial order generated by  $\prec$ . Thus,  $\tau \prec \sigma$  does not necessarily imply  $\dim(\tau) + 1 = \dim(\sigma)$ . Also, notice that  $\langle \tau, \delta^\Phi(\sigma) \rangle_\Phi = [\tau : \sigma]$ ; see the cellular boundary formula from [Hatcher 2002, Section 2.2].

**Definition 2.3** A carrier  $F: \Phi \rightrightarrows \Gamma$  is a map from the set of cells of  $\Phi$  to subcomplexes of  $\Gamma$  that is semicontinuous in the sense that for any pair  $\tau \prec \sigma$  in  $\Phi$ ,  $F(\tau) \subseteq F(\sigma)$ . A carrier  $F: \Phi \rightrightarrows \Gamma$  is called *acyclic*, if for every  $\sigma \in \Phi$ ,  $F(\sigma)$  is a nonempty acyclic subcomplex of  $\Gamma$ .

Given a chain map  $w_p: C_p^{\text{cell}}(\Phi) \rightarrow C_{p+r}^{\text{cell}}(\Gamma)$  of degree  $r = 0, 1$ , we say that it is carried by  $F$  if for all cells  $\sigma \in \Phi_p$ ,

$$\{\gamma \in \Gamma_{p+r} \mid \langle w_p(\sigma), \gamma \rangle_\Gamma \neq 0\} \subseteq F(\sigma),$$

where we followed the notation from [Nanda 2012].

The next statement is an application of [Munkres 1984, Theorem 13.4]. In Proposition 4.2 we prove a version of this statement that applies to filtered CW–complexes. Notice that this theorem works for carriers which are  $\mathbb{F}$ –acyclic and which do not necessarily need to be  $\mathbb{Z}$ –acyclic; see the proof of Proposition 4.2.

**Theorem 2.4** Let  $F: \Phi \rightrightarrows \Gamma$  be an acyclic carrier between CW–complexes  $\Phi$  and  $\Gamma$ . Then:

- **Existence** There is a chain map carried by  $F$ .
- **Equivalence** If  $F$  carries two chain maps  $\phi$  and  $\varphi$ , then  $F$  carries a chain homotopy between  $\phi$  and  $\varphi$ .

Given two acyclic carriers  $F, G: \Phi \rightrightarrows \Gamma$ , we write  $F \subseteq G$  whenever  $F(\sigma) \subseteq G(\sigma)$  for all  $\sigma \in \Phi$ . Given another pair of acyclic carriers  $F': \Phi \rightrightarrows \Gamma$  and  $G': \Gamma \rightrightarrows \Psi$ , we also define the composition carrier  $G' \circ F': \Phi \rightrightarrows \Psi$ , where each  $\sigma \in \Phi$  is sent to

$$G' \circ F'(\sigma) := \bigcup_{\tau \in F(\sigma)} H(\tau).$$

In particular, notice that if  $f$  is carried by  $F'$  and  $g$  is carried by  $G'$ , then  $g \circ f$  is “carried” by  $G' \circ F'$ . However, this composition does not need to be acyclic.

**Example 2.5** Consider a regular morphism  $f: \Phi \rightarrow \Gamma$ . We define the (not necessarily acyclic) carrier  $F_f: \Phi \rightrightarrows \Gamma$  induced by  $f$  that sends  $\sigma \in \Phi$  to  $\overline{f(\sigma)}$ . By continuity of  $f$ , for any pair  $\tau < \sigma$  in  $\Phi$ , we have that  $\overline{f(\tau)} \subseteq \overline{f(\sigma)}$ . Also,  $\overline{f(\sigma)} \neq \emptyset$  since it must contain at least a point. Given an acyclic carrier  $G: \Gamma \rightrightarrows \Psi$ , we denote by  $G(f(\sigma))$  the composition of carriers  $G \circ F_f(\sigma)$  for all  $\sigma \in \Phi$ . This comes up very often in this text and whenever we are looking at the composition  $G \circ F_f$  we assume that it is acyclic. Note that  $F_f$  is acyclic if  $f$  is an embedding of the regular CW-complex  $\Phi$  as a subcomplex of  $\Gamma$ . The hypothesis that  $f$  is regular is key to define the carrier  $F_f$ . If we considered a more general continuous morphism  $f: \Phi \rightarrow \Gamma$ , a possible strategy would be to use outer approximations [Kaczynski et al. 2004; Nanda 2012]. However, for simplicity, we restrict to regular morphisms in this article.

### 2.3 Regular diagrams of filtered complexes

First, recall a few gluing constructions that one can perform in algebraic topology. For a brief introduction to these, see [Hatcher 2002, Appendix 4.G]. They are also relevant in Kozlov’s approach [2008], where diagrams of spaces over trips are studied.

Let  $K$  be a simplicial complex. We view  $K$  as a category whose objects are given by the simplices  $\sigma \in K$ . For any pair of simplices  $\tau, \sigma \in K$  such that  $\tau \preceq \sigma$ , there is a unique arrow  $\tau \rightarrow \sigma$  in  $K$ . We are particularly interested in  $K^{\text{op}}$ , the *opposite category* of  $K$  whose arrows are given by reversing the arrows of  $K$ . The example one should have in mind here is the case where  $K$  is the nerve of a cover of a cellular complex. Splitting the input data up by the cover then provides a diagram over the nerve where higher intersections of covering sets are included into smaller degree intersections. We formalise these constructions in the following definition.

**Definition 2.6** Let  $K$  be a simplicial complex. A functor  $\mathfrak{D}: K^{\text{op}} \rightarrow \mathbf{CW}\text{-cpx}$  is called a *regular diagram of CW-complexes* and its category is denoted by  $\mathbf{RDiag}(K)$ ; notice here that, for any pair of simplices  $\tau \preceq \sigma$  of  $K$ , the morphism  $\mathfrak{D}(\tau \preceq \sigma): \mathfrak{D}(\sigma) \rightarrow \mathfrak{D}(\tau)$  is regular; we call such morphisms *face maps*. Given a pair of diagrams  $\mathfrak{D}, \mathfrak{L} \in \mathbf{RDiag}(K)$ , a morphism of diagrams  $\varphi: \mathfrak{D} \rightarrow \mathfrak{L}$  is a natural transformation; ie the commutativity relation

$$\mathfrak{D}(\tau \preceq \sigma) \circ \varphi(\sigma) = \varphi(\tau) \circ \mathfrak{D}(\tau \preceq \sigma)$$

holds for any pair  $\tau \preceq \sigma$  of simplices in  $K$ .

**Example 2.7** Let  $L$  be a simplicial complex and suppose that it is covered by a pair of nontrivial subcomplexes  $L_0$  and  $L_1$ . Consider a pair of vertices  $v, w \in L_0 \cap L_1$  and suppose that both are connected by a pair of paths  $\gamma_0$  and  $\gamma_1$  within the respective 1–skeletons of  $L_0$  and  $L_1$ . Further, we ask that these paths are simple, in the sense that they do not self intersect. Now, consider a diagram  $\mathcal{D} \in \mathbf{RDiag}(\Delta^1)$  given by the closures of the paths on the vertices  $\mathcal{D}(0) = \bar{\gamma}_0$  and  $\mathcal{D}(1) = \bar{\gamma}_1$ , while  $\mathcal{D}([0, 1]) = \Delta^1$ , the standard one-simplex. We define the face maps of  $\mathcal{D}$ , for  $i = 0, 1$ , as the regular morphism mapping  $0 \mapsto v$  and  $1 \mapsto w$ , while  $[0, 1]$  is sent to  $\gamma_i$ . On the other hand, we consider a diagram  $\mathcal{L} \in \mathbf{RDiag}(\Delta^1)$  which is given by the cover  $\{L_0, L_1\}$ ; that is, we define  $\mathcal{L}(0) = L_0$  and  $\mathcal{L}(1) = L_1$ , while  $\mathcal{L}([0, 1]) = L_0 \cap L_1$ ; also, the face maps of  $\mathcal{L}$  are given by inclusions. Then, we might consider a morphism of diagrams  $\varphi: \mathcal{D} \rightarrow \mathcal{L}$  given by inclusions  $\mathcal{D}(0) \hookrightarrow \mathcal{L}(0)$  and  $\mathcal{D}(1) \hookrightarrow \mathcal{L}(1)$ , while  $\mathcal{D}([0, 1]) = \Delta^1$  is sent to some path within  $L_0 \cap L_1$  so that  $\varphi$  is well defined. In fact,  $\varphi$  can only be well defined whenever  $\gamma_0 = \gamma_1$ . Later, in Definition 5.1, we introduce  $(\varepsilon, K)$ –acyclic carriers; in this case, one might be able to consider such a carrier  $F^\varepsilon: \mathcal{D} \rightrightarrows \mathcal{L}$  so that  $\gamma_0$  and  $\gamma_1$  are only required to lie within some acyclic complex.

The main object of study in this work are diagrams of filtered CW–complexes. These arise naturally in topological data analysis, for example whenever point clouds come equipped with a cover. We therefore make the following definition:

**Definition 2.8** A regularly filtered regular diagram of CW–complexes  $\mathcal{D}$  over  $K$  is a functor

$$\mathcal{D}: K^{\text{op}} \rightarrow \mathbf{RCW-cpx};$$

we denote this category by  $\mathbf{RRDiag}(K)$ . As in  $\mathbf{RDiag}(K)$ , morphisms in  $\mathbf{RRDiag}(K)$  are given by natural transformations. We might consider the subcategory of  $\mathbf{RRDiag}(K)$  given by functors

$$\mathcal{D}: K^{\text{op}} \rightarrow \mathbf{FCW-cpx},$$

which we call *filtered regular diagrams of CW–complexes*, denoting the corresponding category by  $\mathbf{FRDiag}(K)$ . If for a diagram  $\mathcal{D} \in \mathbf{FRDiag}(K)$  the face maps  $\mathcal{D}(\tau \prec \sigma)$  are inclusions respecting the cellular structures for all  $\tau \prec \sigma$  from  $K$ , then we call  $\mathcal{D}$  a *fully filtered diagram of CW–complexes*, whose category we denote by  $\mathbf{FFDiag}(K)$ .

**Example 2.9** Consider a filtered CW–complex  $X$  covered by filtered subcomplexes  $\mathcal{U}$ . We define  $X^{\mathcal{U}}$  over the nerve  $N_{\mathcal{U}}$  as  $X^{\mathcal{U}}(\sigma) = \mathcal{U}_\sigma$  for all  $\sigma \in N_{\mathcal{U}}$ . This diagram  $X^{\mathcal{U}}$  is part of  $\mathbf{FFDiag}(N_{\mathcal{U}})$  since all morphisms  $X^{\mathcal{U}}(\tau \preceq \sigma)$  are actually embeddings of subcomplexes. On the other hand, we can define the constant diagram  $*^{\mathcal{U}}$  as  $*^{\mathcal{U}}(\sigma)_r = *$  if  $X^{\mathcal{U}}(\sigma)_r \neq \emptyset$  or  $*^{\mathcal{U}}(\sigma)_r = \emptyset$  otherwise; for all  $\sigma \in N_{\mathcal{U}}$  and all  $r \in \mathbf{R}$ . We also have that  $*^{\mathcal{U}}$  is in  $\mathbf{FFDiag}(N_{\mathcal{U}})$ . Then, there is an obvious epimorphism of diagrams  $X^{\mathcal{U}} \rightarrow *^{\mathcal{U}}$ . Continuing with the same example, we can also define the complex of spaces  $\pi_0^{\mathcal{U}}$  given by  $\pi_0^{\mathcal{U}}(\sigma) = \pi_0(U_\sigma)$  for all  $\sigma \in N_{\mathcal{U}}$ ; where for each  $r \in \mathbf{R}$ ,  $\pi_0(U_\sigma(r))$  denotes the discrete topological space given by the connected components of  $U_\sigma(r)$ . Thus, each  $\pi_0(U_\sigma)$  is a disjoint union of points that are identified with each other as the filtration value increases and so it cannot be an element in  $\mathbf{FCW-cpx}$ ,

but rather an element from **RCW-cpx**. Thus, in this case  $\pi_0^{\text{ql}} \in \mathbf{RRDiag}(K)$ . For all  $r \in \mathbf{R}$ , there is an epimorphism  $X^{\text{ql}}(r) \rightarrow \pi_0^{\text{ql}}(r)$  sending each cell from  $X^{\text{ql}}(r)$  to its respective connected component from  $\pi_0^{\text{ql}}(r)$ ; these morphisms are consistent along  $\mathbf{R}$ . Altogether we have a sequence of epimorphisms  $X^{\text{ql}} \rightarrow \pi_0^{\text{ql}} \rightarrow *^{\text{ql}}$ .

### 2.4 Geometric realization

For an abstract simplicial complex  $K$ , we denote by  $|K|$  its underlying topological space. Given a simplex  $\sigma \in K$ , we write  $|\sigma|$  to denote the number of vertices of  $\sigma$ . We use  $\dim(\sigma)$  for the dimension of a simplex  $\sigma$ , that is  $\dim(\sigma) = |\sigma| - 1$ . We denote by  $\Delta^n$  the topological space associated to the standard  $n$ -simplex. Given a simplex  $\sigma \in K$ , we use the notation  $\Delta^\sigma := \Delta^{\dim(\sigma)}$  for simplicity. Given a pair  $\tau < \sigma$  in  $K$ , we have a corresponding inclusion  $\Delta^\tau \hookrightarrow \Delta^\sigma$ . As a special case of a CW-complex, we denote by  $K^n$  and  $K^{\leq n}$  the set of  $n$ -cells and the  $n$ -skeleton respectively.

**Definition 2.10** Let  $\mathcal{D} \in \mathbf{RDiag}(K)$ . The *geometric realization*  $\Delta_K \mathcal{D}$  of  $\mathcal{D}$  is the object in **CW-cpx** defined as

$$\Delta_K \mathcal{D} = \bigsqcup_{\sigma \in K} \Delta^\sigma \times \mathcal{D}(\sigma) / \sim$$

where, for any pair  $\tau \leq \sigma$  in  $K$ , the relation identifies a pair of points

$$(\Delta^\tau \hookrightarrow \Delta^\sigma)(x) \times y \sim x \times \mathcal{D}(\tau \leq \sigma)(y)$$

for each pair of points  $x \in \Delta^\tau$  and  $y \in \mathcal{D}(\sigma)$ . This  $\Delta_K \mathcal{D}$  has a natural filtration given by

$$F^p \Delta_K \mathcal{D} = \bigcup_{\sigma \in K^{\leq p}} \Delta^\sigma \times \mathcal{D}(\sigma)$$

for all  $p \geq 0$ . A cell  $\tau \times c$  is a face of another cell  $\sigma \times a$  if and only if  $\tau \leq \sigma$  and also  $c \in \overline{\mathcal{D}(\tau \leq \sigma)(a)}$ . If the underlying simplicial complex  $K$  is clear from the context, we write  $\Delta \mathcal{D}$  instead of  $\Delta_K \mathcal{D}$ .

Notice that Definition 2.10 also applies to diagrams  $\mathcal{D} \in \mathbf{RRDiag}(K)$ . We define  $\Delta_K \mathcal{D}$  by setting  $(\Delta_K \mathcal{D})_r := \Delta_K(\mathcal{D}_r)$  for all  $r \in \mathbf{R}$ . Notice that our gluing conditions are consistent in this case as

$$\mathcal{D}(\tau \leq \sigma) \circ \Sigma^t \mathcal{D}(\sigma)(y) = \Sigma^t \mathcal{D}(\tau) \circ \mathcal{D}(\tau \leq \sigma)(y)$$

for any pair  $\tau \leq \sigma$  from  $K$  and all  $t > 0$  and all points  $y \in \mathcal{D}(\sigma)$ . Altogether we obtain  $\Delta_K(\mathcal{D}) \in \mathbf{RCW-cpx}$ . Given a regular morphism  $\mathcal{F}: \mathcal{D} \rightarrow \mathcal{L}$  of diagrams in  $\mathbf{RRDiag}(K)$ , there is an induced morphism on the geometric realization which we denote by  $\Delta \mathcal{F}$ . Denote by  $*^{\mathcal{D}}$  the diagram given by

$$*^{\mathcal{D}}(\sigma)_r = \begin{cases} * & \text{if } \mathcal{D}(\sigma)_r \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

and note that there is a homotopy equivalence  $\Delta(*^{\mathcal{D}})_r \simeq |K_r^{\mathcal{D}}|$ , where  $K^{\mathcal{D}}$  is the filtered simplicial complex with the same underlying vertex set as  $K$  and  $\sigma \in K_r^{\mathcal{D}}$  if and only if  $\mathcal{D}(\sigma)_r \neq \emptyset$ . The projection onto the simplex coordinates gives a *base projection*  $p_b: \Delta \mathcal{D} \rightarrow \Delta(*^{\mathcal{D}}) \simeq |K^{\mathcal{D}}|$ .

**Example 2.11** Let  $\mathcal{D} \in \mathbf{FRDiag}(K)$ . We define the *multinerve* of  $\mathcal{D}$  as

$$\mathrm{MNerv}(\mathcal{D}) = \Delta(\pi_0(\mathcal{D})).$$

This object was first introduced in [Colin de Verdière et al. 2014] in the case of  $\pi_0^{\mathcal{U}}$  for a space  $X$  covered by  $\mathcal{U}$ . In [Colin de Verdière et al. 2014] it was defined as a simplicial poset, a notion that is equivalent to that of a  $\Delta$ -complex. There are epimorphisms  $\Delta\mathcal{D} \rightarrow \mathrm{MNerv}(\mathcal{D}) \rightarrow \Delta(*^{\mathcal{D}}) \simeq |K|$ .

**Remark 2.12** Let  $\mathcal{D}$  be a diagram of CW-complexes over the simplicial complex  $K$ . We can extend  $\mathcal{D}$  to a diagram  $\mathcal{D}'$  on the barycentric subdivision  $\mathrm{Bd}(K)$  by defining  $\mathcal{D}'(\tau_0 \prec \cdots \prec \tau_n) = \mathcal{D}(\tau_n)$  on an  $n$ -simplex  $\tau_0 \prec \tau_1 \prec \cdots \prec \tau_n$  in  $\mathrm{Bd}(K)$ . A nonidentity morphism in  $\mathrm{Bd}(K)$  that has  $\tau_0 \prec \tau_1 \prec \cdots \prec \tau_n$  as its codomain must have the same flag with one of the  $\tau_k$  left out as its domain. The diagram  $\mathcal{D}'$  maps such a morphism to the identity in case  $k \neq n$  or the morphism  $\mathcal{D}(\tau_{n-1} \prec \tau_n)$  in case  $k = n$ . It is clear from the definition of the homotopy colimit via the simplicial replacement that the geometric realization  $\Delta(\mathcal{D}')$  coincides with the definition of  $\mathrm{hocolim} \mathcal{D}$ ; see [Dugger 2008, Section 4] and also [Kozlov 2008, Definition 15.8]. Notice that in the category  $K$ , each flag is to be interpreted as a sequence of arrows  $\tau_0 \leftarrow \tau_1 \leftarrow \cdots \leftarrow \tau_n$ . A modified version of the homotopy equivalence  $|K| \simeq |\mathrm{Bd}(K)|$  shows that  $\Delta(\mathcal{D}) \simeq \Delta(\mathcal{D}')$ . Hence, we could have worked with homotopy colimits all throughout, but we chose to work with the geometric realization since it is technically easier to handle and because in some instances it is the Mayer–Vietoris blowup complex, which has already appeared before in TDA [Zomorodian and Carlsson 2008]. An instance of a homotopy colimit in TDA can be found in [Cavanna et al. 2017, Appendix B].

**Proposition 2.13** Let  $\mathcal{F}: \mathcal{D} \rightarrow \mathcal{L}$  be a morphism of diagrams in  $\mathbf{RDiag}(K)$ . If  $\mathcal{F}(\sigma)$  is a homotopy equivalence for all  $\sigma \in K$ , then  $\Delta\mathcal{F}: \Delta\mathcal{D} \rightarrow \Delta\mathcal{L}$  is a homotopy equivalence.

One way to see this is to view  $\Delta\mathcal{D}$  as a homotopy colimit (see Remark 2.12), which is a homotopy invariant functor on diagrams. Also, a proof of this result in the more general context of diagrams of spaces can be found in [Hatcher 2002, Proposition 4G.1].

**Example 2.14** Let  $X \in \mathbf{CW-cpx}$  covered by  $\mathcal{U}$  and recall the diagram  $X^{\mathcal{U}}$  from Example 2.9. In this case  $\Delta(X^{\mathcal{U}})$  is the Mayer–Vietoris blowup complex associated to the pair  $(X, \mathcal{U})$  and it can be described as a subspace of the product  $X \times |N_{\mathcal{U}}|$ . This leads to the *fiber projection*  $p_f: \Delta(X^{\mathcal{U}}) \rightarrow X$  and to the *base projection*  $p_b: \Delta(X^{\mathcal{U}}) \rightarrow |N_{\mathcal{U}}|$ . As shown in [Hatcher 2002, Proposition 4G.2],  $p_f$  is a homotopy equivalence  $\Delta(X^{\mathcal{U}}) \simeq X$ . If each  $X^{\mathcal{U}}(\sigma)$  is contractible for all  $\sigma \in N_{\mathcal{U}}$ , then  $p_b$  is also a homotopy equivalence by Proposition 2.13.

An interesting direction of research would be to use Proposition 2.13 to define compatible *collapses*, such as in discrete Morse theory (see [Bauer 2011; Nanda 2012; Sköldbberg 2006]) and end up with a diagram of regular CW-complexes. This motivates the study of spectral sequences associated to such diagrams. We see further reasons in Section 3. On the other hand, given the importance of Proposition 2.13, we

would like to adapt it to an approximate version in the context of diagrams in  $\mathbf{RRDiag}(K)$ . Instead of studying homotopy equivalences, we consider equivalences induced by acyclic carriers. This is done in Section 5.

## 2.5 Spectral sequences of bounded filtrations

Let  $A_*$  be a graded module with differentials  $d_n: A_n \rightarrow A_{n-1}$  for all  $n \geq 1$ , and such that  $A_m = 0$  for all  $m < 0$ . Assume that there is a filtration  $0 = F^{-1}A_* \subseteq F^0A_* \subseteq F^1A_* \subseteq \dots \subseteq F^N A_* = A_*$  of  $A_*$  that is preserved by the differentials  $d_*$  in the sense that  $d_n(F^p A) \subseteq F^p A$  for all  $p \geq 0$ . We say that  $A_*$  is a *filtered differential graded module* and denote this by the triple  $(A, d, F)$ . Then there is a spectral sequence

$$E_{p,q}^1 = H_q(F^p A_* / F^{p-1} A_*) \Rightarrow H_{p+q}(A_*)$$

for all  $p, q \geq 0$ ; see [McCleary 2001, Theorem 2.6]. A morphism of spectral sequences is a sequence of bigraded morphisms  $f^r: E_{p,q}^r \rightarrow \bar{E}_{p,q}^r$  that commute with the spectral sequence differentials, ie  $d_r \circ f^r = f^r \circ d_r$  for all  $r \geq 0$ . Apart from that, these morphisms satisfy  $f^{r+1} = H(f^r)$  for all  $r \geq 0$ .

Suppose that  $(\bar{A}_*, \bar{d}, \bar{F})$  is another filtered differential graded module together with its corresponding spectral sequence  $\bar{E}_{p,q}^r$ . Consider a morphism  $f: A_* \rightarrow B_*$  that commutes with the differential  $f \circ d = \bar{d} \circ f$  and also preserves filtrations  $f(F^p A_*) \subseteq \bar{F}^p(\bar{A}_*)$  for all  $p \geq 0$ . This induces a morphism of spectral sequences

$$E_{p,q}^r \rightarrow \bar{E}_{p,q}^r$$

by [McCleary 2001, Theorem 3.5]. We denote by  $\mathbf{SpSq}$  the category of spectral sequences, and we denote by  $\mathbf{PSpSq}$  the category of functors  $F: \mathbf{R} \rightarrow \mathbf{SpSq}$ .

## 3 Spectral sequences for geometric realizations

Recall the persistent Mayer–Vietoris spectral sequence [Torras-Casas 2023] associated to a pair  $(X, \mathcal{U})$  of a space with a cover:

$$(1) \quad E_{p,q}^1(X, \mathcal{U}) = \bigoplus_{\sigma \in N_{\mathcal{U}}^p} \mathrm{PH}_q(X^{\mathcal{U}}(\sigma)) \Rightarrow \mathrm{PH}_{p+q}(\Delta X^{\mathcal{U}}) \simeq \mathrm{PH}_{p+q}(X).$$

For the details about this spectral sequence in the persistent case we refer the reader to [Torras-Casas 2023]. There are some limitations to the applicability of this spectral sequence to Vietoris–Rips complexes that were already pointed out in [Yoon and Ghrist 2020]: if we choose a cover of a point cloud  $\mathbb{X}$  and then deduce a cover  $\mathcal{U}$  of the associated Vietoris–Rips complex  $\mathrm{VR}_*(\mathbb{X})$  by subcomplexes, then we can only recover  $\mathrm{PH}_k(\mathrm{VR}(\mathbb{X}))$  from  $\mathrm{PH}_k(\Delta \mathrm{VR}_*(\mathbb{X})^{\mathcal{U}})$  for filtration parameters below an upper bound  $R$  determined by the overlaps of the covering sets. In this section we present a regular diagram of CW-complexes that avoids this upper limit problem completely; see Example 3.6.

Before we solve our problem, we need to introduce some chain complexes. We come back to the case of filtrations later, but for now we focus on regular diagrams instead. Given a diagram  $\mathcal{D}$  in  $\mathbf{RDiag}(K)$ , we denote by  $\mathcal{D}(\tau \preceq \sigma)_*$  the induced morphism of cellular chain complexes  $C_*^{\text{cell}}(\mathcal{D}(\sigma)) \rightarrow C_*^{\text{cell}}(\mathcal{D}(\tau))$ . The cellular chain complex  $C_*^{\text{cell}}(\Delta \mathcal{D}, \delta^\Delta)$  associated to  $\Delta \mathcal{D}$  is defined as follows: For all  $m \geq 0$  we have that  $C_m^{\text{cell}}(\Delta \mathcal{D})$  is a vector space generated by cells  $\sigma \times c$  with  $\dim(\sigma) = p$  and  $c \in \mathcal{D}(\sigma)_q$  so that  $p + q = m$ . On such a cell  $\sigma \times c$  the differential  $\delta^\Delta$  is given by

$$\sum_{\sigma_i \prec \sigma} (-1)^i \left( \sum_{a \in \overline{\mathcal{D}(\sigma_i \preceq \sigma)(c)}} [a : \mathcal{D}(\sigma_i \preceq \sigma)(c)] \sigma_i \times a \right) + (-1)^{\dim(\sigma)} \sum_{b \in \bar{c} \setminus c} [b : c] \sigma \times b$$

where the first sum runs over the faces  $\sigma_i$  of  $\sigma$ . As shown in the proof of Lemma 3.1, the map  $\delta^\Delta$  is indeed a differential. In addition, notice that the filtration of  $\Delta_K(\mathcal{D})$  carries over to  $C_*^{\text{cell}}(\Delta_K \mathcal{D})$  by taking  $F^p C_*(\Delta_K \mathcal{D}) := C_*(F^p \Delta_K \mathcal{D})$  for all  $p \geq 0$ .

Now, consider the double complex  $(C_{p,q}(\mathcal{D}), d^V, d^H)$  given by

$$C_{p,q}(\mathcal{D}) = \bigoplus_{\sigma \in K^p} C_q^{\text{cell}}(\mathcal{D}(\sigma))$$

for all  $p, q \geq 0$ . The vertical differential is defined by the direct sum of chain differentials

$$d_{p,q}^V = (-1)^p \bigoplus_{\sigma \in K^p} d_q^\sigma$$

where  $d_*^\sigma$  denotes the differential from  $C_*^{\text{cell}}(\mathcal{D}(\sigma))$  for all  $\sigma \in K^p$ ; of course  $d^V \circ d^V = 0$ . The horizontal differential is given by the Čech differential  $d_{p,q}^H$  which is defined for a cell  $a \in \mathcal{D}(\sigma)$  as

$$\sum_{\sigma_i \prec \sigma} (-1)^i \mathcal{D}(\sigma_i \prec \sigma)_*(a),$$

where  $\mathcal{D}(\sigma_i \prec \sigma)_*$  denotes the induced chain morphism  $C_*^{\text{cell}}(\mathcal{D}(\sigma)) \rightarrow C_*^{\text{cell}}(\mathcal{D}(\sigma_i))$  for all faces  $\sigma_i$  from  $\sigma$ . Also,  $d^H \circ d^H = 0$  by functoriality of  $C_*^{\text{cell}}(\cdot)$  and the fact that  $\mathcal{D}(\rho \prec \tau)_* \mathcal{D}(\tau \prec \sigma)_* = \mathcal{D}(\rho \prec \sigma)_*$  for any three simplices  $\rho \prec \tau \prec \sigma$ . Note that for each pair of indices  $i < j$ , the face map  $\mathcal{D}(\sigma_{ij} \preceq \sigma)_*$  appears twice with respective coefficients  $(-1)^i (-1)^j$  and  $(-1)^i (-1)^{j-1}$ ; which have opposite sign and cancel out. On the other hand, anticommutativity  $d^V \circ d^H = -d^H \circ d^V$  follows since  $\mathcal{D}(\tau \prec \sigma)_*$  is a chain morphism for all  $\tau \prec \sigma$  from  $K$ .

Now, we consider the double complex spectral sequence from [McCleary 2001, Section 2.4]. Given  $\mathcal{D}$  in  $\mathbf{RDiag}(K)$  there is a spectral sequence

$$E_{p,q}^1(\mathcal{D}) = \bigoplus_{\sigma \in K^p} H_q(\mathcal{D}(\sigma)) \Rightarrow H_{p+q}(S_*^{\text{Tot}}(\mathcal{D}))$$

where  $S^{\text{Tot}}(\mathcal{D})$  is the *total complex* defined as

$$S_n^{\text{Tot}}(\mathcal{D}) = \bigoplus_{p+q=n} C_{p,q}(\mathcal{D})$$

together with a differential  $d^{\text{Tot}} = d^V + d^H$ . Also, recall that the total complex has a filtration induced by the vertical filtration on  $C_{p,q}(\mathcal{D})$  given by

$$F^m S_*^{\text{Tot}}(\mathcal{D}) = \bigoplus_{\substack{p+q=n \\ p \leq m}} C_{p,q}(\mathcal{D})$$

for all integers  $m \geq 0$ ; see [Torras-Casas 2023] for an explanation. Next, we relate this total complex to the geometric realization from Definition 2.10.

**Lemma 3.1** *There is an isomorphism  $C_*^{\text{cell}}(\Delta \mathcal{D}, \delta^\Delta) \simeq S_*^{\text{Tot}}(\mathcal{D})$  which preserves filtration. That is,  $F^p C_*^{\text{cell}}(\Delta \mathcal{D}, \delta^\Delta) \simeq F^p S_*^{\text{Tot}}(\mathcal{D})$  for all  $p \geq 0$ .*

**Proof** First we define a chain morphism  $\psi : C_m^{\text{cell}}(\Delta \mathcal{D}) \rightarrow S_m^{\text{Tot}}(\mathcal{D})$  generated by the assignment: a cell  $\sigma \times c \in (\Delta \mathcal{D})_m$  with  $\sigma \in K^p$  and  $c \in \mathcal{D}(\sigma)^q$  for integers  $p + q = m$ , is sent to  $\psi(\sigma \times c) = (c)_\sigma \in S_m^{\text{Tot}}(\mathcal{D})$ ; where by  $(c)_\sigma$  we refer to the vector from  $S_m^{\text{Tot}}(\mathcal{D})$  which is zero in all components except at the component indexed by  $\sigma$ , where it is equal to  $c$ . On the other hand,  $\psi$  is a chain morphism since we have the equality

$$\begin{aligned} \psi(\delta^\Delta(\sigma \times c)) &= \sum_{\sigma_i < \sigma} (-1)^i \left( \sum_{a \in \overline{\mathcal{D}(\sigma_i \preceq \sigma)(c)}} ([a : \mathcal{D}(\sigma_i \preceq \sigma)(c)]a)_{\sigma_i} \right) + (-1)^{\dim(\sigma)} \sum_{b \in \bar{c} \setminus c} ([b : c]b)_\sigma \\ &= \sum_{\sigma_i < \sigma} (-1)^i (\mathcal{D}(\sigma_i \preceq \sigma)_*(c))_{\sigma_i} + (-1)^{\dim(\sigma)} (d_q^\sigma(c))_\sigma \\ &= (d^H + d^V)((c)_\sigma) \\ &= d^{\text{Tot}}((c)_\sigma). \end{aligned}$$

One can see that  $\psi$  is injective, and admits an inverse  $\psi^{-1} : S_m^{\text{Tot}}(\mathcal{D}) \rightarrow C_m^{\text{cell}}(\Delta \mathcal{D})$  that sends  $(\sigma)_c$  to  $\sigma \times c$ . Notice that by definition  $\psi$  sends a chain in  $F^p C_n^{\text{cell}}(\Delta \mathcal{D})$  to a chain in  $F^p S_n^{\text{Tot}}(\mathcal{D})$  for all  $p \geq 0$  and in particular it preserves filtration.  $\square$

**Remark 3.2** Continuing with Remark 2.12, as both  $\Delta_{\text{Bd}(K)} \mathcal{D}'$  and  $\text{hocolim}(\mathcal{D})$  refer to the same space, we could have considered the homotopy colimit spectral sequence

$$E_{p,q}^1(\text{Bd}(K), \mathcal{D}') = \bigoplus_{\sigma \in \text{Bd}(K)^p} H_q(\mathcal{D}'(\sigma)) \Rightarrow H_{p+q}(\text{hocolim } \mathcal{D}).$$

Let us construct a diagram of spaces whose geometric realization is homeomorphic to  $|K|$  for any finite simplicial complex  $K$ . We start by taking a finite partition  $\mathcal{P}$  of the vertex set  $V(K)$  and denote by  $K(U)$  the maximal subcomplex of  $K$  with vertices in  $U \in \mathcal{P}$ . We denote by  $\Delta^\mathcal{P}$  the standard simplex with vertices in  $\mathcal{P}$ . For a simplex  $\tau \in K$ , we define  $\mathcal{P}(\tau) \in \Delta^\mathcal{P}$  to be the simplex consisting of all partitioning sets  $U \in \mathcal{P}$  such that  $\tau \cap U \neq \emptyset$ . In particular if  $U \in \mathcal{P}(\tau)$ , then it determines a standard simplex  $\tau(U) \in K(U)$  of dimension  $|\tau \cap U| - 1 \geq 0$  whose vertices are precisely those from  $\tau \cap U$ , so that there is an inclusion  $\Delta^{\tau(U)} \hookrightarrow |K(U)|$ . For a vertex  $v \in K$ , we denote by  $\mathcal{P}(v)$  the partitioning set from  $\mathcal{P}$  which contains  $v$ .

We define the  $(K, \mathcal{P})$ -join diagram  $\mathcal{F}_{\mathcal{P}}^K : (\Delta^{\mathcal{P}})^{\text{op}} \rightarrow \mathbf{FCW}\text{-cpx}$  for all  $\sigma \subseteq \mathcal{P}$  by assigning the subspace formed by the union of products of images

$$\mathcal{F}_{\mathcal{P}}^K(\sigma) = \bigcup_{\substack{\rho \in K \\ \mathcal{P}(\rho) = \sigma}} \prod_{U \in \sigma} \text{Im}(\Delta^{\rho(U)} \hookrightarrow |K(U)|)$$

for all  $\sigma \in \Delta^{\mathcal{P}}$ ; by definition, notice that  $\mathcal{F}_{\mathcal{P}}^K(\sigma) \subseteq \prod_{U \in \sigma} |K(U)|$ . Additionally,  $\mathcal{F}_{\mathcal{P}}^K(U) = |K(U)|$  for all  $U \in \mathcal{P}$ . However,  $\mathcal{F}_{\mathcal{P}}^K(\sigma)$  could even be empty for  $\sigma \in \Delta^{\mathcal{P}}$  with  $\dim(\sigma) > 0$ . For any pair  $\tau \preceq \sigma$  in  $\Delta^{\mathcal{P}}$ , we consider the projection  $\pi_{\tau \preceq \sigma} : \prod_{U \in \sigma} |K(U)| \rightarrow \prod_{U \in \tau} |K(U)|$  that forgets all product components which are indexed by vertices of  $\sigma$  that are not vertices of  $\tau$ . We claim that  $\pi_{\tau \preceq \sigma}$  restricts to a well-defined face map  $\mathcal{F}_{\mathcal{P}}^K(\tau \preceq \sigma) : \mathcal{F}_{\mathcal{P}}^K(\sigma) \rightarrow \mathcal{F}_{\mathcal{P}}^K(\tau)$ . In order to show this, we consider an arbitrary simplex  $\rho \in K$  such that  $\mathcal{P}(\rho) = \sigma$ . Next, we consider the face  $\lambda(\tau) \preceq \rho$  which is spanned by the vertices from  $\rho \cap U$  for all  $U \in \tau$ , so that  $\mathcal{P}(\lambda(\tau)) = \tau$  and also  $\lambda(\tau)(U) = \rho(U)$  for all  $U \in \mathcal{P}$ . Then, we obtain the equality

$$\pi_{\tau \preceq \sigma} \left( \prod_{U \in \sigma} \text{Im}(\Delta^{\rho(U)} \hookrightarrow |K(U)|) \right) = \prod_{U \in \tau} \text{Im}(\Delta^{\lambda(\tau)(U)} \hookrightarrow |K(U)|),$$

so the face maps are well defined, as claimed.

**Lemma 3.3** *Let  $K$  be a simplicial complex together with a partition  $\mathcal{P}$  of its vertex set  $V(K)$ . There is a CW-complex homeomorphism  $\Delta(\mathcal{F}_{\mathcal{P}}^K) \simeq |K|$ .*

**Proof** Consider the continuous map  $f : \Delta(\mathcal{F}_{\mathcal{P}}^K) \rightarrow |K|$  defined by mapping a point

$$\left( \sum_{U \in \mathcal{P}(\tau)} y_U U, \left( \sum_{v \in U} x_v v \right)_{U \in \mathcal{P}(\tau)} \right) \in \Delta^{\mathcal{P}(\tau)} \times \prod_{U \in \mathcal{P}(\tau)} \Delta^{\tau(U)} / \sim$$

to  $\sum_{v \in \tau} y_{\mathcal{P}(v)} x_v v$  in  $\Delta^{\tau}$  for all  $\tau \in K$ , where we have values  $0 \leq y_U \leq 1$  and  $0 \leq x_v \leq 1$  for all  $U \in \mathcal{P}(\tau)$  and all  $v \in U$ , and such that  $\sum_{U \in \mathcal{P}(\tau)} y_U = 1$  and  $\sum_{v \in U} x_v = 1$  for all  $U \in \mathcal{P}$ . On the other hand, let  $\sum_{v \in \tau} x_v v \in \Delta^{\tau}$  be a point such that  $0 \leq x_v \leq 1$  for all  $v \in \Delta^{\tau}$  and such that  $\sum_{v \in \tau} x_v = 1$ . Then we can define the inverse continuous map

$$f^{-1} \left( \sum_{v \in \tau} x_v v \right) = \left( \sum_{U \in \mathcal{P}(\tau)} \left( \sum_{v \in U} x_v \right) U, \left( \psi_U \left( \sum_{v \in \tau} x_v v \right) \right)_{U \in \mathcal{P}(\tau)} \right),$$

where we consider a map  $\psi_U : \Delta^{\tau} \rightarrow \Delta^{\tau(U)}$  given by

$$\psi_U \left( \sum_{v \in \tau} x_v v \right) = \begin{cases} \sum_{v \in \tau(U)} \left( \frac{x_v}{\sum_{v \in \tau(U)} x_v} \right) v & \text{if } \sum_{v \in \tau(U)} x_v \neq 0, \\ * \in \Delta^{\tau(U)} & \text{otherwise, where } * \text{ denotes any point (see below).} \end{cases}$$

By the equivalence relation used to define  $\Delta(\mathcal{F}_{\mathcal{P}}^K)$ , the product factor  $\Delta^{\tau(U)}$  is collapsed to a single point for the subset of points whose  $U$ -coordinate in  $\Delta^{\mathcal{P}(\tau)}$  vanishes. If  $\sum_{v \in \tau(U)} x_v = 0$ , then  $x_v = 0$  for all  $v \in \tau(U)$  and the  $U$ -coordinate of the point  $\sum_{v \in \tau} x_v v$  in  $\Delta^{\mathcal{P}(\tau)}$  is 0. It is straightforward to check that  $f$  and  $f^{-1}$  are well defined and consistent along  $K$ . □

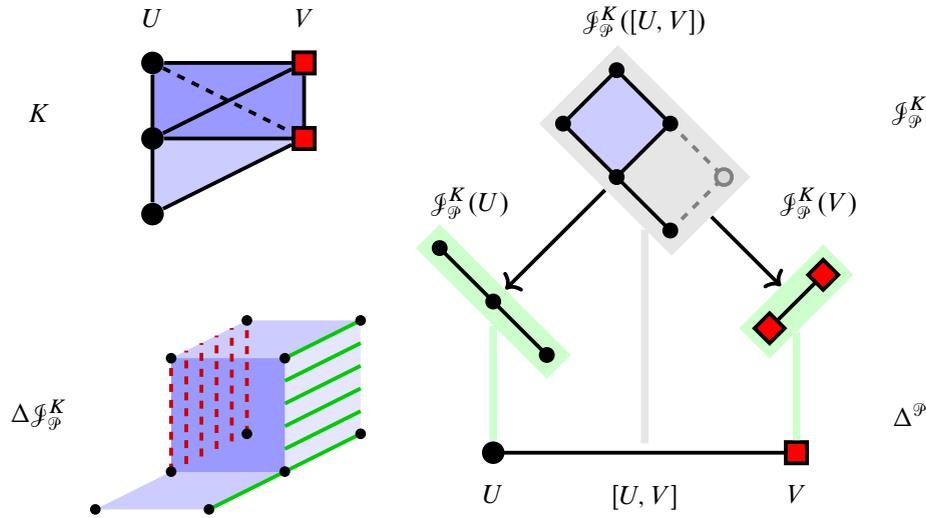


Figure 1: Depiction of  $K$ ,  $\mathcal{J}_\varphi^K$  and  $\Delta \mathcal{J}_\varphi^K$  from Example 3.4. Over the edge  $[U, V]$ , we consider  $\mathcal{J}_\varphi^K([U, V]) \subset |K(U)| \times |K(V)|$ , where we add dashed lines to illustrate the embedding into the product. On the bottom left we depict  $\Delta \mathcal{J}_\varphi^K$ , where each red dashed line and each green line is collapsed to a single point.

**Example 3.4** Consider the simplicial complex  $K$  depicted in the top left part of Figure 1, formed by gluing a 2–simplex to a 4–simplex along an edge. We consider a partition of the vertex set of  $K$  into the two subsets  $\mathcal{P} = \{U, V\}$ , where points in  $U$  are indicated by black circles and points in  $V$  are indicated by red squares. In the top right of Figure 1, we depict the standard 1–simplex  $\Delta^\mathcal{P}$  together with the diagram  $\mathcal{J}_\varphi^K$  over it. In particular, notice that  $\mathcal{J}_\varphi^K([U, V])$  is a subset of the product  $|K(U)| \times |K(V)|$  and that the morphisms  $\mathcal{J}_\varphi^K([U, V]) \rightarrow \mathcal{J}_\varphi^K(V) = |K(V)|$  and  $\mathcal{J}_\varphi^K([U, V]) \rightarrow \mathcal{J}_\varphi^K(U) = |K(U)|$  are both projections. In addition, notice that  $\mathcal{J}_\varphi^K([U, V])$  has five vertices corresponding to the five different edges connecting vertices from  $U$  to  $V$ , five edges corresponding to five 2–simplices containing vertices in both  $U$  and  $V$  and a single 2–cell corresponding to the unique 4–simplex in  $K$ . Finally, the bottom left of Figure 1 shows the geometric realization  $\Delta \mathcal{J}_\varphi^K$ .

Observe that  $\mathcal{J}_\varphi^K$  is a diagram of *prodsimplicial* complexes as in [Kozlov 2008, Definition 2.43], which are in particular regular CW–complexes. By the observations above we can therefore consider the associated double complex spectral sequence

$$E_{p,q}^1(\mathcal{J}_\varphi^K) = \bigoplus_{\sigma \in \Delta^\mathcal{P}} H_q(\mathcal{J}_\varphi^K(\sigma)) \Rightarrow H_{p+q}(\Delta \mathcal{J}_\varphi^K) \simeq H_{p+q}(K).$$

Next, we show that the “size” of  $K$  is the same as the “size” of the diagram  $\mathcal{J}_\varphi^K$ . For this, recall that each simplex  $\sigma \in K$  corresponds to a unique simplex  $\mathcal{P}(\sigma) \in \Delta^\mathcal{P}$ . This is different to the case of a cover  $\mathcal{U}$  for  $K$ , where a simplex in  $K$  might correspond to several simplices from the nerve  $N_\mathcal{U}$ . Here, we write  $\#L$  for the number of cells in a complex  $L$ .

**Proposition 3.5**

$$\#K = \sum_{\sigma \in \Delta^{\mathcal{P}}} \#\mathcal{F}_{\mathcal{P}}^K(\sigma).$$

**Proof** Consider an assignment  $\psi: K \rightarrow \bigsqcup_{\sigma \in \Delta^{\mathcal{P}}} \mathcal{F}_{\mathcal{P}}^K(\sigma)$  given by sending  $\rho \in K$  to  $(\rho(U))_{U \in \mathcal{P}(\rho)}$  in  $\mathcal{F}_{\mathcal{P}}^K(\mathcal{P}(\rho))$ , where  $(\rho(U))_{U \in \mathcal{P}(\rho)} \in \prod_{U \in \mathcal{P}(\rho)} |K(U)|$ . By the definition of  $\mathcal{F}_{\mathcal{P}}^K$ ,  $\psi$  is well defined and surjective. Also,  $\psi$  is injective as the vertex set from  $\rho \in K$  is uniquely determined by the simplices  $\rho(U)$  for all  $U \in \mathcal{P}(\rho)$ .  $\square$

Now, let us consider a filtered simplicial complex  $K_* \in \mathbf{FCW}\text{-cpx}$  such that its vertex set  $V(K_*)$  is fixed throughout all values of  $\mathbf{R}$ . Let  $\mathcal{P}$  be a partition of  $V(K_*)$ . We define the filtered regular diagram  $\mathcal{F}_{\mathcal{P}}^K \in \mathbf{FRDiag}(\mathcal{P})$  by sending  $r \in \mathbf{R}$  to  $\mathcal{F}_{\mathcal{P}}^{K_r}$ . These diagrams inherit the shift morphisms  $\Sigma K_*$  from  $K_*$  in the following way: Let  $\sigma \in \Delta^{\mathcal{P}}$  and notice that we have restrictions  $\Sigma^{s-r} K|_U: |K_r(U)| \rightarrow |K_s(U)|$  for all  $U \in \sigma$ , so that we have induced morphisms

$$\prod_{U \in \sigma} \Sigma^{s-r} K|_U: \mathcal{F}_{\mathcal{P}}^{K_r}(\sigma) \rightarrow \mathcal{F}_{\mathcal{P}}^{K_s}(\sigma)$$

for all  $\sigma \in \Delta^{\mathcal{P}}$ . In turn, these induce a shift morphism on  $\Delta \mathcal{F}_{\mathcal{P}}^K$  which respect filtrations, so that we have a commutative diagram

$$\begin{array}{ccccc} E_{p,q}^*(\mathcal{F}_{\mathcal{P}}^{K_r}) & \implies & \Delta \mathcal{F}_{\mathcal{P}}^{K_r} & \xrightarrow{\simeq} & K_r \\ \downarrow & & \downarrow & & \downarrow \\ E_{p,q}^*(\mathcal{F}_{\mathcal{P}}^{K_s}) & \implies & \Delta \mathcal{F}_{\mathcal{P}}^{K_s} & \xrightarrow{\simeq} & K_s \end{array}$$

and thus  $\text{PH}_*(\Delta \mathcal{F}_{\mathcal{P}}^K) \simeq \text{PH}_*(K_*)$ . For each simplex  $\sigma \in \Delta^{\mathcal{P}}$  one can see  $\mathcal{F}_{\mathcal{P}}^K(\sigma)$  as a filtered simplicial complex, so that

$$E_{p,q}^1(\mathcal{F}_{\mathcal{P}}^K) = \bigoplus_{\sigma \in (\Delta^{\mathcal{P}})^p} \text{PH}_q(\mathcal{F}_{\mathcal{P}}^K(\sigma)) \Rightarrow \text{PH}_{p+q}(K).$$

**Example 3.6** Consider a point cloud  $\mathbb{X}$ , a partition  $\mathcal{P}$  and consider its Vietoris Rips complex  $\text{VR}_*(\mathbb{X})$  in  $\mathbf{FCW}\text{-cpx}$ . In this case we have a fixed partition of the vertex set of  $\text{VR}_*(\mathbb{X})$ , which allows us to consider the spectral sequence

$$E_{p,q}^1(\mathcal{F}_{\mathcal{P}}^{\text{VR}_*(\mathbb{X})}) = \bigoplus_{\sigma \in \Delta^{\mathcal{P}}} \text{PH}_q(\mathcal{F}_{\mathcal{P}}^{\text{VR}_*(\mathbb{X})}(\sigma)) \Rightarrow \text{PH}_{p+q}(\text{VR}_*(\mathbb{X})).$$

This is very convenient as it avoids the main difficulties with the Mayer–Vietoris blowup complex associated to a cover. Namely, one recovers  $\text{PH}_*(K)$  completely without any bounds depending on the cover overlaps. In addition, notice that  $\Delta \mathcal{F}_{\mathcal{P}}^{\text{VR}_*(\mathbb{X})}$  has the same number of cells as  $\text{VR}_*(\mathbb{X})$ , contrary to the Mayer–Vietoris blowup complex, whose number of cells is much larger, as shown in Proposition 3.5.

The  $(K, \mathcal{P})$ -join diagram is related to [Robinson 2020, Example 4]. There the motivation behind the filtrations is given by a consistency radius and a filtration based on the differences between local measurements. The same example appears (without a filtration) as one of the opening examples in [Hatcher 2002, Appendix 4.G].

## 4 $\varepsilon$ -acyclic carriers

The following definition encodes our notion of “noise”.

**Definition 4.1** Let  $X, Y \in \mathbf{RCW-cpx}$ . An  $\varepsilon$ -acyclic carrier  $F_*^\varepsilon: X_* \rightrightarrows Y[\varepsilon]_*$  is a family of acyclic carriers  $F_a^\varepsilon: X_a \rightrightarrows Y_{a+\varepsilon}$  for all  $a \in \mathbf{R}$  such that

$$Y(a + \varepsilon \leq b + \varepsilon)F_a^\varepsilon(c) \subseteq F_b^\varepsilon(X(a \leq b)(c))$$

for all cells  $c$  of  $X_a$  and  $a, b \in \mathbf{R}$  with  $a \leq b$ .

The proposition below is an adaptation of [Munkres 1984, Theorem 13.4] or [Cooke and Finney 1967, Lemma 2.4] to the context of tame filtered CW-complexes.

**Proposition 4.2** Let  $X_*, Y_* \in \mathbf{FCW-cpx}$  be tame, and assume that there exists an  $\varepsilon$ -acyclic carrier

$$F_*^\varepsilon: X_* \rightrightarrows Y[\varepsilon]_*.$$

Then there exist chain morphisms  $f_a^\varepsilon: C_*(X_a) \rightarrow C_*(Y_{a+\varepsilon})$  carried by  $F_a^\varepsilon$  for all  $a \in \mathbf{R}$ , so that  $Y(a + \varepsilon \leq b + \varepsilon) \circ f_a^\varepsilon = f_b^\varepsilon \circ X(a \leq b)$ . Furthermore, given another such sequence of morphisms  $g_a^\varepsilon: C_*(X_a) \rightarrow C_*(Y_{a+\varepsilon})$ , there exist chain homotopy equivalences  $H_a^\varepsilon: g_a^\varepsilon \simeq f_a^\varepsilon$  which are carried by  $F_a^\varepsilon$  for all  $a \in \mathbf{R}$ .

**Proof** Let  $b \in \mathbf{R}$  and assume that  $f_a^\varepsilon$  has already been defined for all values  $a < b$ , where we allow for  $b = -\infty$ . We first define  $f_b^\varepsilon$  on all cells which are in the image of  $X(a < b)$  for any  $a < b$  using the definition

$$f_b^\varepsilon \circ X(a < b) = Y(a + \varepsilon < b + \varepsilon) \circ f_a^\varepsilon.$$

Notice that the assumption that  $X_a \subseteq X_b$  is crucial for this to work. By hypotheses, given a cell  $c \in \text{Im}(X(a < b))$ , its image  $f_b^\varepsilon(c)$  is then contained in

$$Y(a + \varepsilon < b + \varepsilon)F_a^\varepsilon(\tilde{c}) \subseteq F_b^\varepsilon(X(a < b)(\tilde{c})),$$

where  $\tilde{c} \in X_a$  is such that  $c = X(a < b)(\tilde{c})$ . Hence,  $f_b^\varepsilon$  satisfies the carrier condition. Next we define  $f_b^\varepsilon$  on the remaining cells in

$$\tilde{X}_b = X_b \setminus \left( \bigcup_{a < b} X(a < b) \right).$$

We proceed to prove this by induction. First, choose a 0-cell  $f_b^\varepsilon(v) \in F_b^\varepsilon(v)$  for each remaining 0-cell  $v \in \tilde{X}_b$ , and notice that  $d_* f_b^\varepsilon(v) = 0 = f_b^\varepsilon(d_* v)$ , where we use  $d_*$  for the chain complex differentials. Next, by induction, assume that for a fixed  $p \geq 0$ , the  $p$ -cells  $s \in X_b$  have image  $f_b^\varepsilon(s)$  carried by  $F_b^\varepsilon(s)$  and such that  $d_* \circ f_b^\varepsilon(s) = f_b^\varepsilon \circ d_*(s)$ . We would like to extend  $f_b^\varepsilon$  to the  $(p+1)$ -cells. By semicontinuity, given such a cell  $c \in X_b$ , its boundary  $d_* c$  is contained in  $F_b^\varepsilon(c)$ . On the other hand, notice that by linearity and the induction hypotheses  $d_* f_b^\varepsilon(d_* c) = f_b^\varepsilon(d_* d_* c) = 0$ ; thus  $f_b^\varepsilon(d_* c)$  is a

cycle in  $C_*(F_b^\varepsilon(c))$ . By acyclicity, there exists  $h \in C_*(F_b^\varepsilon(c))$  such that  $d_*h = f_b^\varepsilon(d_*c)$  and thus we can define  $f_b^\varepsilon(c) = h$ . Altogether, we have defined a chain morphism  $f_b^\varepsilon$  which is carried by  $F_b^\varepsilon$ .

Since  $X_*$  is tame, there exist a finite sequence of values  $a_1 < a_2 < \dots < a_N$  such that  $X_s = X_{a_i}$  for all  $s \in (a_{i-1}, a_i)$  where we define  $a_0 = -\infty$  and  $a_{N+1} = \infty$ . We apply the construction of  $f_b^\varepsilon$  for all values  $b$  ranging over  $a_i$  from  $i = 1$  up to  $i = N$ . This determines the chain morphism  $f_*^\varepsilon: C_*(X_*) \rightarrow C_*(Y[\varepsilon]_*)$ , where we set  $f_s^\varepsilon = f_{a_i}^\varepsilon$  for all  $s \in (a_{i-1}, a_i]$  where  $i = 1, 2, \dots, N$  and also  $f_t^\varepsilon = f_{a_N}^\varepsilon$  for all  $t > a_N$ .

Now, assume that  $g_b^\varepsilon$  is also carried by  $F_b^\varepsilon$  for all  $b \in \mathbb{R}$ . Following [May 1999, Section 12.3], we define the chain complex  $\mathcal{F}$  given by  $\mathcal{F}_0 = \langle [0], [1] \rangle$  and  $\mathcal{F}_1 = \langle [0, 1] \rangle$  and  $\mathcal{F}_k = 0$  for  $k \geq 2$ . This is the cellular chain complex of the unit interval  $I$  decomposed into two 0-cells and one 1-cell. A chain homotopy  $h_b^\varepsilon: f_b^\varepsilon \simeq g_b^\varepsilon$  corresponds to a chain map  $h_b^\varepsilon: C_*^{\text{cell}}(X_b) \otimes \mathcal{F} \rightarrow C_*^{\text{cell}}(Y_b)$  such that  $h_b^\varepsilon(x, [0]) = f_b^\varepsilon(x)$  and  $h_b^\varepsilon(x, [1]) = g_b^\varepsilon(x)$  for all  $x \in X_b$ . Let  $H_b^\varepsilon(c, i) = F_b^\varepsilon(c)$  for a cell  $(c, i) \in X \times I$ . By assumption,  $H^\varepsilon: X \times I \rightrightarrows Y$  is an  $\varepsilon$ -acyclic carrier. Note that  $C_*^{\text{cell}}(X_b) \otimes \mathcal{F} \cong C_*^{\text{cell}}(X_b \times I)$ . Replicating the first part of the proof we can now extend any map  $h_b^\varepsilon: C_*^{\text{cell}}(X_b) \otimes \mathcal{F}_0 \rightarrow C_*^{\text{cell}}(Y_b)$  with the above properties to all cells of  $X \times I$ . □

**Definition 4.3** Let  $X_*, Y_* \in \mathbf{RCW-cpx}$ . A *shift carrier* is an  $\varepsilon$ -acyclic carrier  $I_X^\varepsilon: X_* \rightrightarrows X_{*+\varepsilon}$  carrying the standard shift  $\Sigma^\varepsilon X_*$ . Let two  $\varepsilon$ -acyclic carriers

$$F^\varepsilon: X_* \rightrightarrows Y_{*+\varepsilon}, \quad G^\varepsilon: Y_* \rightrightarrows X_{*+\varepsilon},$$

together with shift carriers  $I_X^{2\varepsilon}$  and  $I_Y^{2\varepsilon}$ . We say that  $X_*$  and  $Y_*$  are  $\varepsilon$ -acyclic equivalent whenever we have inclusions  $G^\varepsilon \circ F^\varepsilon \subseteq I_X^{2\varepsilon}$  and  $F^\varepsilon \circ G^\varepsilon \subseteq I_Y^{2\varepsilon}$ .

The motivation for the definition of  $\varepsilon$ -acyclic equivalences is the following lemma:

**Proposition 4.4** Let  $X_*$  and  $Y_*$  be two tame elements from  $\mathbf{FCW-cpx}$  which are  $\varepsilon$ -acyclic equivalent. Then  $\text{PH}(X_*)$  and  $\text{PH}(Y_*)$  are  $\varepsilon$ -interleaved.

**Proof** By Proposition 4.2 we know that there exist two chain maps  $f_*^\varepsilon: C_*(X_*) \rightarrow C_*(Y_{*+\varepsilon})$  and  $g_*^\varepsilon: C_*(Y_*) \rightarrow C_*(X_{*+\varepsilon})$  carried by  $F^\varepsilon$  and  $G^\varepsilon$  respectively. By hypothesis the compositions  $g_*^\varepsilon \circ f_*^\varepsilon$  and  $f_*^\varepsilon \circ g_*^\varepsilon$  are carried by corresponding shift carriers  $I_X^{2\varepsilon}$  and  $I_Y^{2\varepsilon}$ . Thus, using the second part of Proposition 4.2 we obtain chain homotopies  $g_*^\varepsilon \circ f_*^\varepsilon \simeq \Sigma^{2\varepsilon} C_*(X)$  and  $f_*^\varepsilon \circ g_*^\varepsilon \simeq \Sigma^{2\varepsilon} C_*(Y)$ . Altogether, in homology these compositions are equal to the corresponding shifts, and  $\text{PH}_*(X_*)$  and  $\text{PH}_*(Y_*)$  are  $\varepsilon$ -interleaved. □

**Example 4.5** Consider two finite metric spaces  $\mathbb{X}$  and  $\mathbb{Y}$ . Let  $d_H(\mathbb{X}, \mathbb{Y})$  be their Hausdorff distance and set  $\varepsilon = 2d_H(\mathbb{X}, \mathbb{Y})$ . Given a subcomplex  $K \subseteq \text{VR}(\mathbb{X})$ , we denote its vertex set by  $\mathbb{X}(K) \subseteq \mathbb{X}$ . Likewise for a simplex  $\sigma \in \text{VR}(\mathbb{X})$ , we write  $\mathbb{X}(\sigma) \subseteq \mathbb{X}$  for the vertices spanning  $\sigma$ . Define a carrier  $F^\varepsilon: \text{VR}(\mathbb{X}) \rightrightarrows \text{VR}(\mathbb{Y})$  by mapping a simplex  $\sigma \in \text{VR}(\mathbb{X})_a$  to

$$F^\varepsilon(\sigma) = \left| \sup\{K \subseteq \text{VR}(\mathbb{Y})_{a+\varepsilon} \mid d_H(\mathbb{X}(\sigma), \mathbb{Y}(K)) \leq \varepsilon/2\} \right|.$$

This is clearly semicontinuous. If  $v_0, \dots, v_n$  are vertices in  $F^\varepsilon(\sigma)$ , then by definition  $\{v_0, \dots, v_n\}$  is an  $n$ -simplex in  $F^\varepsilon(\sigma)$ . Therefore we have  $F^\varepsilon(\sigma) \simeq \Delta^N$  for some  $N \in \mathbb{Z}_{\geq 0}$ , which is acyclic. In particular,  $F^\varepsilon$  is an  $\varepsilon$ -acyclic carrier. Interchanging the roles of  $\mathbb{X}$  and  $\mathbb{Y}$  we also obtain an  $\varepsilon$ -acyclic carrier  $G^\varepsilon: \text{VR}(\mathbb{Y}) \rightrightarrows \text{VR}(\mathbb{X})$ . Similarly, we define for a simplex  $\sigma \in \text{VR}(\mathbb{X})_a$  the shift carrier

$$I_{\mathbb{X}}^{2\varepsilon}(\sigma) = |\text{sup}\{K \subseteq \text{VR}(\mathbb{X})_{a+2\varepsilon} \mid d_H(\mathbb{X}(\sigma), \mathbb{X}(K)) \leq \varepsilon\}|.$$

Analogously one defines  $I_{\mathbb{Y}}^{2\varepsilon}$ . Since  $G^\varepsilon \circ F^\varepsilon \subseteq I_{\mathbb{X}}^{2\varepsilon}$  and  $F^\varepsilon \circ G^\varepsilon \subseteq I_{\mathbb{Y}}^{2\varepsilon}$ , Proposition 4.4 implies that  $\text{PH}_*(\text{VR}(\mathbb{X}))$  and  $\text{PH}_*(\text{VR}(\mathbb{Y}))$  are  $\varepsilon$ -interleaved. This is similar to the proof using *correspondences*; see [Oudot 2015, Proposition 7.8, Section 7.3].

**Example 4.6** Consider  $\mathbb{R}^N$  together with a 1-Lipschitz function  $f: \mathbb{R}^N \rightarrow \mathbb{R}$  with constant  $\varepsilon > 0$ . On the other hand, consider the lattices  $\mathbb{Z}^N$  and  $r\mathbb{Z}^N + l$  for a pair of vectors  $r, l \in \mathbb{R}^N$  such that the coordinates of  $r$  satisfy  $0 < r_i \leq 1$  for all  $1 \leq i \leq N$ . Then we take their corresponding cubical complexes  $\mathcal{C}(\mathbb{Z}^N)$  and  $\mathcal{C}(r\mathbb{Z}^N + l)$  thought as embedded in  $\mathbb{R}^N$ . The function  $f$  induces a natural filtration for these cubical complexes: a vertex  $v \in \mathcal{C}(\mathbb{Z}^N)$  is contained in  $\mathcal{C}(\mathbb{Z}^N)_{f(v)}$ , while a cell  $a \in \mathcal{C}(\mathbb{Z}^N)$  appears at the maximum filtration value on its vertices. There is an  $\varepsilon$ -acyclic carrier  $F^\varepsilon: \mathcal{C}(\mathbb{Z}^N) \rightrightarrows \mathcal{C}(r\mathbb{Z}^N + l)$  sending each cell  $a \in \mathcal{C}(\mathbb{Z}^N)$  to the smallest subcomplex  $F^\varepsilon(a)$  containing all  $b \in \mathcal{C}(r\mathbb{Z}^N + l)$  such that  $\bar{b} \cap a \neq \emptyset$ . In an analogous way the inverse acyclic carrier can be defined, and the compositions  $F^\varepsilon \circ G^\varepsilon$  and  $G^\varepsilon \circ F^\varepsilon$  define the shift carriers. Thus, using Proposition 4.4, one shows that  $\text{PH}_*(\mathcal{C}(\mathbb{Z}^N))$  and  $\text{PH}_*(\mathcal{C}(r\mathbb{Z}^N + l))$  are  $\varepsilon$ -interleaved.

An important assumption of Proposition 4.2 is that we are dealing with tame filtered CW-complexes. However, what if we considered a pair of elements  $X_*, Y_* \in \mathbf{RCW-cpx}$  instead? In this context, we notice that given an  $\varepsilon$ -acyclic carrier  $F^\varepsilon: X_* \rightarrow Y_*[\varepsilon]$ , it is not necessarily true that the compositions

$$Y(a + \varepsilon \leq b + \varepsilon)F_a^\varepsilon(c) \quad \text{and} \quad F_b^\varepsilon(X(a \leq b)(c))$$

are still acyclic for all pairs  $a \leq b$  from  $\mathbf{R}$ . Thus, whenever we talk about  $\varepsilon$ -acyclic carriers  $F^\varepsilon: X_* \rightarrow Y_*[\varepsilon]$  in this context we assume that  $F_b^\varepsilon(X(a \leq b)(c))$  is acyclic for all pairs  $a, b \in \mathbf{R}$  with  $a \leq b$  and all cells  $c \in X(a)$ .

**Corollary 4.7** *Let  $X_*, Y_* \in \mathbf{RCW-cpx}$  be a pair of elements such that both are  $\varepsilon$ -acyclic equivalent in the above sense. Then  $d_I(\text{PH}_*(X_*), \text{PH}_*(Y_*)) \leq \varepsilon$ .*

**Proof** For each persistence value  $a \in \mathbf{R}$ , we use Theorem 2.4 twice to obtain a pair of chain morphisms  $f_a: C_a^{\text{cell}}(X) \rightarrow C_{a+\varepsilon}^{\text{cell}}(Y)$  and  $g_{a+\varepsilon}: C_{a+\varepsilon}^{\text{cell}}(Y) \rightarrow C_{a+2\varepsilon}^{\text{cell}}(X)$ . In a similar way we obtain a pair of chain homotopies  $g_{a+\varepsilon} \circ f_a \simeq (\Sigma^{2\varepsilon} C_*^{\text{cell}}(X))_a$  and  $f_{a+\varepsilon} \circ g_a \simeq (\Sigma^{2\varepsilon} C_*^{\text{cell}}(Y))_a$  so that we have equalities between the induced homology morphisms  $[g_{a+\varepsilon}] \circ [f_a] = [(\Sigma^{2\varepsilon} C_*^{\text{cell}}(X))_a]$  and  $[f_{a+\varepsilon}] \circ [g_a] = [(\Sigma^{2\varepsilon} C_*^{\text{cell}}(Y))_a]$  for all  $a \in \mathbf{R}$ . Now, for a pair of values  $a \leq b$  from  $\mathbf{R}$ , it is not necessarily true that

$$Y(a + \varepsilon \leq b + \varepsilon) \circ f_a = f_b \circ X(a \leq b).$$

However, since  $Y(a + \varepsilon \leq b + \varepsilon) \circ f_a$  and  $f_b \circ X(a \leq b)$  are both included in  $F_b^\varepsilon(X(a \leq b)(c))$  by hypotheses, then by applying Theorem 2.4 again there is a chain homotopy equivalence

$$Y(a + \varepsilon \leq b + \varepsilon) \circ f_a \simeq f_b \circ X(a \leq b),$$

which implies

$$[Y(a + \varepsilon \leq b + \varepsilon)] \circ [f_a] = [f_b] \circ [X(a \leq b)],$$

and we have defined a persistence morphism  $[f_*]: \text{PH}_*(X_*) \rightarrow \text{PH}_*(Y_*[\varepsilon])$ . Similarly, we can also put together the  $g_a$  for all  $a \in \mathbf{R}$  so that we obtain a morphism  $[g_*]: \text{PH}_*(Y_*) \rightarrow \text{PH}_*(X_*[\varepsilon])$ . This leads to the claimed  $\varepsilon$ -interleaving.  $\square$

**Example 4.8** In the appendix, we describe a filtered CW-complex  $X$ , a regularly filtered CW-complex  $Y$ , together with a pair of 0-acyclic carriers (ie  $\varepsilon = 0$ )  $F: Y \rightrightarrows X$  and  $G: X \rightrightarrows Y$  which, together with the compositions  $G \circ F$  and  $G \circ F$  as shift carriers, define a 0-acyclic equivalence between  $Y$  and  $X$ . Therefore, by Corollary 4.7 we obtain isomorphisms  $\text{PH}_n(X) \cong \text{PH}_n(Y)$  for all  $n \geq 0$ . In this case, notice that  $Y$  is much smaller than  $X$ ; thus it is worth considering the regularly filtered complex  $Y$  in place of  $X$ . Next, we briefly describe how one could use  $\varepsilon$ -equivalences. In this case, one could have considered a filtered complex  $\tilde{X}$  which is equal to  $X_*$  outside the intervals  $(i - \varepsilon, i + \varepsilon)$  for values  $i = 1, 2, 3, 4$  and for some  $\varepsilon < 1/2$ . Notice that in this case one should be able to obtain an  $\varepsilon$ -acyclic equivalence between  $\tilde{X}$  and  $Y$ , so that by Corollary 4.7  $\text{PH}_n(\tilde{X})$  and  $\text{PH}_n(Y)$  are  $\varepsilon$ -interleaved for all  $n \geq 0$ .

**Remark 4.9** Notice that our notion of acyclicity is different from that in [Cavanna 2019] and [Govc and Skraba 2018]. In [Govc and Skraba 2018] a filtered complex  $K_*$  is called  $\varepsilon$ -acyclic whenever the induced homology maps  $H_*(K_r) \rightarrow H_*(K_{r+\varepsilon})$  vanish for all  $r \in \mathbb{R}$ . In this case, one can still (trivially) define acyclic carriers between  $*$  and  $K_*$ . The problem arises when defining the shift carrier  $I_K^{A\varepsilon}$  for some constant  $A > 0$ , which does not exist in general. One can however, adapt the proof of Proposition 4.2 so that there is a chain morphism  $\psi^{\varepsilon(\dim(K_r)+1)}: C_*(K_r) \rightarrow C_*(K_{r+\varepsilon(\dim(K_r)+1)})$ ; and that this coincides up to chain homotopy with the composition through  $C_*(*)$ . One does this by following the same proof as in Proposition 4.2, but increasing the filtration value by  $\varepsilon$  each time we assume that some cycle lies in an acyclic carrier. Thus, if we have  $\dim(K) = \sup_{r \in \mathbb{R}}(\dim(K_r)) < \infty$ , then one could say that there is an  $\varepsilon(\dim(K)+1)$ -approximate chain homotopy equivalence between  $C(*)$  and  $C(K_*)$ .

## 5 Interleaving geometric realizations

Next, we focus on acyclic carrier equivalences between a pair of diagrams  $\mathcal{D}, \mathcal{L} \in \mathbf{RRDiag}(K)$ . We start by taking  $\varepsilon$ -acyclic carriers  $F_\sigma^\varepsilon: \mathcal{D}(\sigma) \rightrightarrows \mathcal{L}(\sigma)$  for all  $\sigma \in K$  which have to be compatible in the sense that for any pair  $\tau \preceq \sigma$  and any cell  $c \in \mathcal{D}(\sigma)$ , there is an inclusion

$$(2) \quad \mathcal{L}(\tau \preceq \sigma)(F_\sigma^\varepsilon(c)) \subseteq F_\tau^\varepsilon(\mathcal{D}(\tau \preceq \sigma)(c))$$

and we assume in addition that  $F_\tau^\varepsilon(\mathfrak{D}(\tau \preceq \sigma)\Sigma^r\mathfrak{D}(\sigma)(c))$  is acyclic for all  $r \geq 0$ . This compatibility leads to “local” diagrams of spaces. That is, given a pair of values  $a \in \mathbf{R}$  and  $r \geq 0$  and a cell  $c \in \mathfrak{D}(\sigma)_a$ , we consider an object  $F_{\sigma \times c}^{r,\varepsilon} \in \mathbf{RDiag}(\Delta^\sigma)$ . It is given by the space  $F_{\sigma \times c}^{r,\varepsilon}(\tau) = F_\tau^\varepsilon(\mathfrak{D}(\tau \preceq \sigma)\Sigma^r\mathfrak{D}(\sigma)(c))$  for all faces  $\tau \preceq \sigma$ . For any sequence  $\rho \preceq \tau \preceq \sigma$  in  $K$ , there are morphisms in  $F_{\sigma \times c}^{r,\varepsilon}$  given by restricting morphisms from  $\mathcal{L}$ :

$$\begin{array}{ccc} \tau & \longrightarrow & F_{\sigma \times c}^{r,\varepsilon}(\tau) \equiv F_\tau^\varepsilon(\mathfrak{D}(\tau \preceq \sigma)\Sigma^r\mathfrak{D}(\sigma)(c)) \\ \uparrow \preceq & & \downarrow \\ \rho & \longrightarrow & F_{\sigma \times c}^{r,\varepsilon}(\rho) \equiv F_\rho^\varepsilon(\mathfrak{D}(\rho \preceq \sigma)\Sigma^r\mathfrak{D}(\sigma)(c)) \end{array} \quad \begin{array}{c} \\ \\ \downarrow \mathcal{L}(\rho \preceq \tau) \end{array}$$

Using condition (2) repeatedly on the cells from  $L = \mathfrak{D}(\tau \preceq \sigma)\Sigma^r\mathfrak{D}(\sigma)(c)$ , we see that we have an inclusion

$$\mathcal{L}(\rho \preceq \tau)(F_\tau^\varepsilon(L)) \subseteq F_\sigma^\varepsilon(\mathfrak{D}(\rho \preceq \tau)(L)).$$

Thus the diagram  $F_{\sigma \times c}^{r,\varepsilon}$  is indeed well defined, and we may consider the geometric realization  $\Delta F_{\sigma \times c}^{r,\varepsilon}$ . By hypothesis, each  $F_{\sigma \times c}^{r,\varepsilon}(\tau)$  is acyclic for all  $\tau \preceq \sigma$ , so the first page of the spectral sequence  $E_{p,q}^*(F_{\sigma \times c}^{r,\varepsilon}) \Rightarrow H_{p+q}(\Delta F_{\sigma \times c}^{r,\varepsilon})$  is equal to

$$E_{p,q}^1(F_{\sigma \times c}^{r,\varepsilon}) = \bigoplus_{\tau \in (\Delta^\sigma)^p} H_q(F_{\sigma \times c}^{r,\varepsilon}(\tau)) = \begin{cases} \bigoplus_{\tau \in (\Delta^\sigma)^p} \mathbb{F} & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, computing the homology with respect to the horizontal differentials on the first page corresponds to computing the homology of  $\Delta^\sigma$ . Thus,  $E_{p,q}^2(F_{\sigma \times c}^{r,\varepsilon})$  is zero everywhere except at  $p = q = 0$  where it is equal to  $\mathbb{F}$ . Thus, the spectral sequence collapses on the second page, and  $\Delta F_{\sigma \times c}^{r,\varepsilon}$  is acyclic. We use the notation  $F_{\sigma \times c}^\varepsilon = F_{\sigma \times c}^{0,\varepsilon}$ .

**Definition 5.1** Let  $\mathfrak{D}$  and  $\mathcal{L}$  be two diagrams in  $\mathbf{RRDiag}(K)$ . Suppose that there are  $\varepsilon$ -acyclic carriers  $F_\sigma^\varepsilon: \mathfrak{D}(\sigma) \rightrightarrows \mathcal{L}(\sigma)$  for all  $\sigma \in K$ , and that

$$\mathcal{L}(\tau \preceq \sigma)(F_\sigma^\varepsilon(c)) \subseteq F_\sigma^\varepsilon(\mathfrak{D}(\tau \preceq \sigma)(c))$$

for all  $c \in \mathfrak{D}(\sigma)$  and in addition  $F_\tau^\varepsilon(\mathfrak{D}(\tau \preceq \sigma)\Sigma^r\mathfrak{D}(\sigma)(c))$  is acyclic for all  $r \geq 0$ . Then we say that the set  $\{F_\sigma^\varepsilon\}_{\sigma \in K}$  is an  $(\varepsilon, K)$ -acyclic carrier between  $\mathfrak{D}$  and  $\mathcal{L}$ . We denote this by  $F^\varepsilon: \mathfrak{D} \rightrightarrows \mathcal{L}$ .

**Theorem 5.2** Let  $\mathfrak{D}$  and  $\mathcal{L}$  be two diagrams in  $\mathbf{RRDiag}(K)$ . Suppose that there are  $(\varepsilon, K)$ -acyclic carriers  $F^\varepsilon: \mathfrak{D} \rightrightarrows \mathcal{L}$  and  $G^\varepsilon: \mathcal{L} \rightrightarrows \mathfrak{D}$ , together with a pair of shift  $(\varepsilon, K)$ -acyclic carriers  $I_{\mathfrak{D}}^{2\varepsilon}: \mathfrak{D} \rightrightarrows \mathfrak{D}$  and  $I_{\mathcal{L}}^{2\varepsilon}: \mathcal{L} \rightrightarrows \mathcal{L}$ , and such that these restrict to acyclic equivalences

$$G_\tau^\varepsilon \circ F_\tau^\varepsilon \subseteq (I_{\mathfrak{D}}^{2\varepsilon})_\tau \quad \text{and} \quad F_\tau^\varepsilon \circ G_\tau^\varepsilon \subseteq (I_{\mathcal{L}}^{2\varepsilon})_\tau$$

for each simplex  $\tau \in K$ . Then there is an  $\varepsilon$ -acyclic equivalence  $F^\varepsilon: \Delta\mathfrak{D} \rightrightarrows \Delta\mathcal{L}$  which preserves filtration. That is, there are  $\varepsilon$ -acyclic equivalences  $F^p F^\varepsilon: F^p \Delta\mathfrak{D} \rightrightarrows F^p \Delta\mathcal{L}$  for all  $p \geq 0$ .

**Proof** Let  $\sigma \times c \in \Delta\mathcal{D}$  be a cell, where  $c$  is an  $m$ -cell in  $\mathcal{D}(\sigma)$ . Define an acyclic carrier  $F^\varepsilon: \Delta\mathcal{D} \Rightarrow \Delta\mathcal{L}$  by sending  $\sigma \times c$  to the acyclic carrier  $\Delta F_{\sigma \times c}^\varepsilon$ , which is a subcomplex of  $\Delta\mathcal{L}$ . Let us first check semicontinuity. For any pair of cells  $\tau \times a \preceq \sigma \times c$  in  $\Delta\mathcal{D}$ , the cell  $a$  is contained in the subcomplex  $\overline{\mathcal{D}(\tau \preceq \sigma)(c)}$ , and by continuity of  $\mathcal{D}(\rho \preceq \tau)$  we have that  $\mathcal{D}(\rho \preceq \tau)(a) \subseteq \overline{\mathcal{D}(\rho \preceq \sigma)(c)}$ . Thus there are inclusions

$$F_\rho^\varepsilon(\mathcal{D}(\rho \preceq \tau)(a)) \subseteq F_\rho^\varepsilon(\overline{\mathcal{D}(\rho \preceq \sigma)(c)}) = F_\rho^\varepsilon(\mathcal{D}(\rho \preceq \sigma)(c))$$

for all  $\rho \preceq \tau$ . More concisely,  $F_{\tau \times a}^\varepsilon(\rho) \subseteq F_{\sigma \times c}^\varepsilon(\rho)$  for all  $\rho \preceq \tau$ . As a consequence  $\Delta F_{\tau \times a}^\varepsilon \subseteq \Delta F_{\sigma \times c}^\varepsilon$  and semicontinuity holds.

Next, notice that  $F^\varepsilon(\Sigma^r \Delta\mathcal{D}(\sigma \times c)) = F^\varepsilon(\sigma \times \Sigma^r \mathcal{D}(\sigma)(c)) = \Delta F_{\sigma \times c}^{r, \varepsilon}$  which is an acyclic carrier. In order for  $F^\varepsilon$  to be an  $\varepsilon$ -acyclic carrier, it remains to show the inclusion  $\Sigma^r \Delta\mathcal{L} \circ F^\varepsilon \subseteq F^\varepsilon \circ \Sigma^r \Delta\mathcal{D}$  for all  $r \geq 0$ . For this, take  $\sigma \times c \in \Delta\mathcal{D}$  and see that

$$\begin{aligned} \Sigma^r \Delta\mathcal{L} \circ F^\varepsilon(\sigma \times c) &= \Sigma^r \Delta\mathcal{L} \left( \bigcup_{\tau \preceq \sigma} \tau \times F_\tau^\varepsilon(\mathcal{D}(\tau \preceq \sigma)(c)) \right) \\ &= \bigcup_{\tau \preceq \sigma} \tau \times \Sigma^r \mathcal{L}(\tau)(F_\tau^\varepsilon(\mathcal{D}(\tau \preceq \sigma)(c))) \\ &\subseteq \bigcup_{\tau \preceq \sigma} \tau \times F_\tau^\varepsilon(\Sigma^r \mathcal{D}(\tau) \mathcal{D}(\tau \preceq \sigma)(c)) \\ &= \bigcup_{\tau \preceq \sigma} \tau \times F_\tau^\varepsilon(\mathcal{D}(\tau \preceq \sigma) \Sigma^r \mathcal{D}(\sigma)(c)) \\ &= F^\varepsilon(\sigma \times \Sigma^r \mathcal{D}(\sigma)(c)) = F^\varepsilon \circ \Sigma^r \Delta\mathcal{D}(\sigma \times c). \end{aligned}$$

Similarly, one can define an  $\varepsilon$ -acyclic carrier  $G^\varepsilon: \Delta\mathcal{L} \Rightarrow \Delta\mathcal{D}$  sending  $\sigma \times c \in \Delta\mathcal{L}$  to  $\Delta G_{\sigma \times c}^\varepsilon$ . In addition, we define respective shift  $\varepsilon$ -acyclic carriers  $I_{\mathcal{D}}^{2\varepsilon}: \Delta\mathcal{D} \Rightarrow \Delta\mathcal{D}$  and  $I_{\mathcal{L}}^{2\varepsilon}: \Delta\mathcal{L} \Rightarrow \Delta\mathcal{L}$  sending, respectively,  $\sigma \times c \in \Delta\mathcal{D}$  to  $\Delta(I_{\mathcal{D}}^{2\varepsilon})_{\sigma \times c}$  and  $\tau \times a \in \Delta\mathcal{L}$  to  $\Delta(I_{\mathcal{L}}^{2\varepsilon})_{\tau \times a}$ . Then we have

$$\begin{aligned} G^\varepsilon \circ F^\varepsilon(\sigma \times c) &= G^\varepsilon(\Delta F_{\sigma \times c}^\varepsilon) \\ &= G^\varepsilon \left( \bigcup_{\tau \preceq \sigma} \tau \times F_\tau^\varepsilon(\mathcal{D}(\tau \preceq \sigma)(c)) \right) \\ &= \bigcup_{\rho \preceq \tau \preceq \sigma} \rho \times G_\rho^\varepsilon(\mathcal{L}(\rho \preceq \tau) F_\tau^\varepsilon(\mathcal{D}(\tau \preceq \sigma)(c))) \\ &\subseteq \bigcup_{\rho \preceq \sigma} \rho \times G_\rho^\varepsilon F_\rho^\varepsilon(\mathcal{D}(\rho \preceq \sigma)(c)) \subseteq \Delta(I_{\mathcal{D}}^{2\varepsilon})_{\sigma \times c} = I_{\mathcal{D}}^{2\varepsilon}(\sigma \times c), \end{aligned}$$

where we have used the commutativity condition and equivalence of  $F_\rho^\varepsilon$  and  $G_\rho^\varepsilon$ . Consequently,  $G^\varepsilon \circ F^\varepsilon \subseteq I_{\mathcal{D}}^{2\varepsilon}$ ; the other inclusion  $F^\varepsilon \circ G^\varepsilon \subseteq I_{\mathcal{L}}^{2\varepsilon}$  follows by symmetry. Altogether, we have obtained an  $\varepsilon$ -equivalence  $F^\varepsilon: \Delta\mathcal{D} \Rightarrow \Delta\mathcal{L}$ . Finally, notice that for all  $p \geq 0$  and for each cell  $\sigma \times c \in F^p \Delta\mathcal{D}$ , its carrier  $\Delta F_{\sigma \times c}^\varepsilon$  is contained in  $F^p \Delta\mathcal{D}$  and so it preserves filtration. The same follows for the other acyclic carriers.  $\square$

Let  $X \in \mathbf{FCW}\text{-cpx}$  together with a cover  $\mathcal{U}$ . Recall the definitions of the diagrams  $X^{\mathcal{U}}$  and  $\pi_0^{\mathcal{U}}$  over  $N_{\mathcal{U}}$  from Example 2.9. Let  $d_I(\mathrm{PH}_*(X^{\mathcal{U}}(\sigma)), \mathrm{PH}_*(\pi_0^{\mathcal{U}}(\sigma))) \leq \varepsilon$  for all  $\sigma \in N_{\mathcal{U}}$ . This example has been of interest before; see for example [Govc and Skraba 2018] or [Cavanna 2019]. As mentioned in Remark 4.9, our notion of  $\varepsilon$ -acyclicity is much stronger than that from [Govc and Skraba 2018]. This is why we obtain a result closer to the *persistence nerve theorem* from [Chazal and Oudot 2008] than to the *approximate nerve theorem* from [Govc and Skraba 2018].

Given a diagram  $\mathcal{D} \in \mathbf{FRDiag}(K)$ , recall the diagram  $\pi_0\mathcal{D}$  from Example 2.11. We may define an  $(\varepsilon, K)$ -acyclic carrier  $\pi_0^{\varepsilon}\mathcal{D}: \mathcal{D} \rightrightarrows \pi_0\mathcal{D}$  where we send cells to their corresponding connected component classes. The compatibility condition  $\pi_0(\mathcal{D}(\tau \leq \sigma))(\pi_0^{\varepsilon}\mathcal{D}(\mathcal{D}(\sigma))) \subseteq \pi_0^{\varepsilon}\mathcal{D}(\mathcal{D}(\tau))$  also follows for any pair of simplices  $\tau \leq \sigma$  from  $K$ .

**Corollary 5.3** (strong approximate multinerve theorem) *Consider a diagram  $\mathcal{D}$  in  $\mathbf{FRDiag}(K)$ . Assume that there is a  $(\varepsilon, K)$ -acyclic carrier  $F^{\varepsilon}: \pi_0\mathcal{D} \rightrightarrows \mathcal{D}$  such that the composition  $F^{\varepsilon} \circ \pi_0^{\varepsilon}\mathcal{D}_{\sigma}$  carries the shift morphism  $\Sigma^{2\varepsilon}\mathcal{D}_{\sigma}$  for all  $\sigma \in K$ . Then, there is an  $\varepsilon$ -acyclic equivalence  $F^{\varepsilon}: \mathrm{MNerv}(\mathcal{D}) \rightrightarrows \Delta\mathcal{D}$ . Consequently,*

$$d_I(\mathrm{PH}_*(\mathrm{MNerv}(\mathcal{D})), \mathrm{PH}_*(\Delta\mathcal{D})) \leq \varepsilon.$$

**Proof** The shift  $(2\varepsilon, K)$ -carrier  $I_{\pi_0\mathcal{D}}^{2\varepsilon}$  sends points to points, while the other  $I_{\mathcal{D}}^{2\varepsilon}$  is defined as the composition  $F^{\varepsilon} \circ \pi_0^{\varepsilon}\mathcal{D}_{\sigma}$ , which is a  $(2\varepsilon, K)$ -acyclic carrier by hypotheses. Thus, by Theorem 5.2 there exists an  $\varepsilon$ -acyclic equivalence  $F^{\varepsilon}: \mathrm{MNerv}(\mathcal{D}) \rightrightarrows \Delta\mathcal{D}$ .  $\square$

**Example 5.4** Consider a filtered simplicial complex  $L_*$  together with a partition of its vertex set  $\mathcal{P}$ . Assume that the  $(L_*, \mathcal{P})$ -join diagram  $\mathcal{F}_{\mathcal{P}}^{L_*}$  is such that there exists a  $(\varepsilon, K)$ -acyclic carrier  $F^{\varepsilon}: \pi_0\mathcal{F}_{\mathcal{P}}^{L_*} \rightrightarrows \mathcal{F}_{\mathcal{P}}^{L_*}$  such that  $F^{\varepsilon} \circ \pi_0^{\varepsilon}\mathcal{F}_{\mathcal{P}}^{L_*}(\sigma)$  is a carrier for  $\Sigma^{2\varepsilon}\mathcal{F}_{\mathcal{P}}^{L_*}(\sigma)$  for all  $\sigma \in \Delta^P$ . Then, by Corollary 5.3, there is an  $\varepsilon$ -acyclic equivalence  $\Delta\pi_0(\mathcal{F}_{\mathcal{P}}^{L_*}) \rightrightarrows \Delta\mathcal{F}_{\mathcal{P}}^{L_*}$  such that

$$d_I(\mathrm{PH}_*(\mathrm{MNerv}(\mathcal{F}_{\mathcal{P}}^{L_*})), \mathrm{PH}_*(L_*)) \leq \varepsilon.$$

Acyclic carriers have been used in [Kaczynski et al. 2004] and in [Nanda 2012] for approximating continuous morphisms by means of simplicial maps. Here we have used the same tools to obtain an approximate homotopy colimit theorem. The acyclic carrier theorem is an instance of the more general acyclic model theorem; see [Eilenberg and MacLane 1953, Section 2]. An interesting future research direction would be to see how that general result can bring new insights into applied topology.

## 6 Interleaving spectral sequences

**Definition 6.1** Let  $\mathcal{A}$  and  $\mathcal{B}$  be from  $\mathbf{SpSq}$ . A  $n$ -spectral sequence morphism  $f: \mathcal{A} \rightarrow \mathcal{B}$  is a spectral sequence morphism  $f: \mathcal{A} \rightarrow \mathcal{B}$  which is defined from page  $n$ .

**Definition 6.2** Given two objects  $\mathcal{A}$  and  $\mathcal{B}$  in  $\mathbf{PSpSq}$ . We say that  $\mathcal{A}$  and  $\mathcal{B}$  are  $(\varepsilon, n)$ –interleaved whenever there exist two  $n$ –morphisms  $\psi : \mathcal{A} \rightarrow \mathcal{B}[\varepsilon]$  and  $\varphi : \mathcal{B} \rightarrow \mathcal{A}[\varepsilon]$  such that the diagram

$$(3) \quad \begin{array}{ccc} \mathcal{A} & & \mathcal{B} \\ \Sigma^\varepsilon \mathcal{A} \downarrow & \swarrow \psi & \searrow \varphi \downarrow \Sigma^\varepsilon \mathcal{B} \\ \mathcal{A}[\varepsilon] & & \mathcal{B}[\varepsilon] \\ \Sigma^\varepsilon \mathcal{A}[\varepsilon] \downarrow & \swarrow \psi[\varepsilon] & \searrow \varphi[\varepsilon] \downarrow \Sigma^\varepsilon \mathcal{B}[\varepsilon] \\ \mathcal{A}[2\varepsilon] & & \mathcal{B}[2\varepsilon] \end{array}$$

commutes for all pages  $r \geq n$ . This interleaving defines a pseudometric in  $\mathbf{PSpSq}$ ,

$$d_I^n(\mathcal{A}, \mathcal{B}) := \inf\{\varepsilon \mid \mathcal{A} \text{ and } \mathcal{B} \text{ are } (\varepsilon, n)\text{–interleaved}\}.$$

**Proposition 6.3** Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are  $(\varepsilon, n)$ –interleaved. Then these are  $(\varepsilon, m)$ –interleaved for all  $m \geq n$ . In particular, we have that

$$d_I^m(\mathcal{A}, \mathcal{B}) \leq d_I^n(\mathcal{A}, \mathcal{B})$$

for any pair of integers  $m \geq n$ .

**Proof** This follows directly from the definitions. □

We start now by considering Mayer–Vietoris spectral sequences. Under some conditions which are a special case of Theorem 5.2, one can obtain one-page stability. In fact, this stability is due to morphisms directly defined on the underlying double complexes, which is a very strong property.

**Proposition 6.4** Let  $X$  and  $Y$  be two tame elements in  $\mathbf{FCW-cpx}$  together with a pair of respective finite covers  $\mathcal{U}$  and  $\mathcal{V}$  by subcomplexes such that  $K = N_{\mathcal{U}} = N_{\mathcal{V}}$ . Suppose that there are  $(\varepsilon, K)$ –acyclic carriers  $F^\varepsilon : X^{\mathcal{U}} \rightrightarrows Y^{\mathcal{V}}$  and  $G^\varepsilon : Y^{\mathcal{V}} \rightrightarrows X^{\mathcal{U}}$ , together with a pair of shift  $(\varepsilon, K)$ –acyclic carriers  $I_{X^{\mathcal{U}}}^{2\varepsilon} : X^{\mathcal{U}} \rightrightarrows X^{\mathcal{U}}$  and  $I_{Y^{\mathcal{V}}}^{2\varepsilon} : Y^{\mathcal{V}} \rightrightarrows Y^{\mathcal{V}}$ , and such that these restrict to acyclic equivalences

$$G_\tau^\varepsilon \circ F_\tau^\varepsilon \subseteq (I_{X^{\mathcal{U}}}^{2\varepsilon})_\tau \quad \text{and} \quad F_\tau^\varepsilon \circ G_\tau^\varepsilon \subseteq (I_{Y^{\mathcal{V}}}^{2\varepsilon})_\tau$$

for each simplex  $\tau \in K$ . Then there is a pair of double complex morphisms

$$\phi^\varepsilon : C_{*,*}(X, \mathcal{U}) \rightarrow C_{*,*}(Y, \mathcal{V})[\varepsilon] \quad \text{and} \quad \psi^\varepsilon : C_{*,*}(Y, \mathcal{V}) \rightarrow C_{*,*}(X, \mathcal{U})[\varepsilon]$$

inducing a first page interleaving between  $E_{*,*}^*(X, \mathcal{U})$  and  $E_{*,*}^*(Y, \mathcal{V})$ .

**Proof** Unpacking the definitions, this means we have to give chain homomorphisms

$$(\phi_\sigma^\varepsilon)_r : C_*(X^{\mathcal{U}}(\sigma)_r) \rightarrow C_*(Y^{\mathcal{V}}(\sigma)_{r+\varepsilon}), \quad (\psi_\sigma^\varepsilon)_r : C_*(Y^{\mathcal{V}}(\sigma)_r) \rightarrow C_*(X^{\mathcal{U}}(\sigma)_{r+\varepsilon})$$

that are natural in  $\sigma \in K$  and in  $r \in \mathbf{R}$ . Since  $K$  is a poset category, these can be constructed inductively as follows: As in Proposition 4.2 we may define  $\phi_\sigma^\varepsilon$  on all simplices  $\sigma \in K$  of dimension  $\dim(\sigma) = \dim(K)$ .

Note that  $(\phi_\sigma^\varepsilon)_r$  is carried by  $(F_\sigma^\varepsilon)_r$  for all  $r \in \mathbf{R}$ . Assume by (reverse) induction that  $\phi_\tau^\varepsilon$  are defined and carried by  $F_\tau^\varepsilon$  for all  $\tau \in K$  with  $n \leq \dim(\tau) \leq \dim(K)$  in such a way that for all cofaces  $\tau \preceq \sigma$  the naturality condition  $\phi_\tau^\varepsilon \circ X^{\mathcal{U}}(\tau \prec \sigma) = Y^{\mathcal{V}}(\tau \prec \sigma)[\varepsilon] \circ \phi_\sigma^\varepsilon$  holds. Now let  $\tau \in K$  have dimension  $\dim(\tau) = n - 1 \geq 0$ . The naturality condition on the simplices fixes  $\phi_\tau^\varepsilon$  on the filtered subcomplex  $X^\tau = \bigcup_{\tau \prec \sigma} \text{Im}(X^{\mathcal{U}}(\tau \prec \sigma))$ , where the union is taken over all cofaces  $\sigma$  of  $\tau$ . Here notice that we can assume that  $\phi_\tau^\varepsilon$  is well defined since the previous choices of  $\phi_\sigma^\varepsilon$  for all cofaces  $\tau \prec \sigma$  are consistent due to the fact that for each cell  $c \in X^\tau$  there exists a unique maximal simplex  $\sigma \in N_{\mathcal{U}}$  such that  $c \in X^{\mathcal{U}}(\sigma)$ . In addition, notice that by hypotheses  $Y^{\mathcal{V}}(\tau \prec \sigma)((F_\sigma^\varepsilon)(c)) \subseteq F_\tau^\varepsilon(X^{\mathcal{U}}(\tau \prec \sigma)(c))$  for all  $a \in \mathbf{R}$  and  $c \in X^{\mathcal{U}}(\sigma)$ , so that our definition of  $\phi_\tau^\varepsilon$  in  $X^\tau$  is indeed carried by  $F_\tau^\varepsilon$ . We then proceed as in Proposition 4.2 to define  $(\phi_\tau^\varepsilon)_a$  on all simplices in the subset  $X^{\mathcal{U}}(\tau)_a \setminus X_a^\tau$  for all  $a \in \mathbf{R}$ . The resulting chain map  $(\phi_\tau^\varepsilon)_a$  is carried by  $(F_\tau^\varepsilon)_a$  for all  $a \in \mathbf{R}$ . Since  $X^{\mathcal{U}}$  is tame, we only need finitely many steps to obtain a morphism  $\phi_\tau^\varepsilon: C_*(X^{\mathcal{U}}(\tau)) \rightarrow C_*(Y^{\mathcal{V}}(\tau)[\varepsilon])$  that satisfies the induction hypotheses.

Thus, we obtain double complex morphisms  $\phi_{p,q}^\varepsilon: C_{p,q}(X, \mathcal{U}) \rightarrow C_{p,q}(Y, \mathcal{V})[\varepsilon]$  for all  $p, q \geq 0$  by adding up our defined local morphisms

$$\phi_{p,q}^\varepsilon: \bigoplus_{\sigma \in K^p} \phi_\sigma^\varepsilon: \bigoplus_{\sigma \in K^p} C_q(X^{\mathcal{U}}(\sigma)) \longrightarrow \bigoplus_{\sigma \in K^p} C_q(Y^{\mathcal{V}}(\sigma))[\varepsilon].$$

Notice that the  $\phi_{p,q}^\varepsilon$  commute both with horizontal and vertical differentials since we assumed that each  $\phi_\sigma^\varepsilon$  is a chain morphism and these satisfy a naturality condition with respect to  $K$ . Thus, this double complex morphism induces a spectral sequence morphism  $\phi_{p,q}^\varepsilon: E_{p,q}^*(X^{\mathcal{U}}) \rightarrow E_{p,q}^*(Y^{\mathcal{V}})[\varepsilon]$ . By doing the same construction, we can obtain local chain morphisms  $\psi_\sigma^\varepsilon: C_*(Y^{\mathcal{V}}(\sigma)) \rightarrow C_*(X^{\mathcal{U}}(\sigma))[\varepsilon]$  so that by Proposition 4.2 we have equalities  $[\psi_\sigma^\varepsilon] \circ [\phi_\sigma^\varepsilon] = [\Sigma^{2\varepsilon} C_*(X^{\mathcal{U}}(\sigma))]$  and also  $[\phi_\sigma^\varepsilon] \circ [\psi_\sigma^\varepsilon] = [\Sigma^{2\varepsilon} C_*(Y^{\mathcal{V}}(\sigma))]$  for all  $\sigma \in K$ . Then we can construct a double complex morphism  $\psi_{p,q}^\varepsilon: C_{p,q}(Y, \mathcal{V}) \rightarrow C_{p,q}(X, \mathcal{U})[\varepsilon]$  inducing an “inverse” spectral sequence morphism  $\psi_{p,q}^\varepsilon: E_{p,q}^*(Y, \mathcal{V}) \rightarrow E_{p,q}^*(X, \mathcal{U})[\varepsilon]$ . These are such that from the first page,  $\phi_{*,*}^\varepsilon$  and  $\psi_{*,*}^\varepsilon$  form a  $(\varepsilon, 1)$ -interleaving of spectral sequences.  $\square$

Notice that the proof of Proposition 6.4 relies heavily on the fact that the diagrams we are considering come from a cover. This allows us to define a pair of double complex morphisms that are compatible along the common indexing nerve. However, in Theorem 5.2 we observed that, under some conditions, the geometric realizations of regularly filtered regular diagrams are stable. Does this stability carry over to the associated spectral sequences? The next theorem shows that this is indeed the case.

**Theorem 6.5** *Let  $\mathcal{D}$  and  $\mathcal{L}$  be two diagrams in  $\mathbf{RRDiag}(K)$ . Suppose that there are  $(\varepsilon, K)$ -acyclic carriers  $F^\varepsilon: \mathcal{D} \rightrightarrows \mathcal{L}$  and  $G^\varepsilon: \mathcal{L} \rightrightarrows \mathcal{D}$ , together with a pair of shift  $(\varepsilon, K)$ -acyclic carriers  $I_{\mathcal{D}}^{2\varepsilon}: \mathcal{D} \rightrightarrows \mathcal{D}$  and  $I_{\mathcal{L}}^{2\varepsilon}: \mathcal{L} \rightrightarrows \mathcal{L}$ , and such that these restrict to acyclic equivalences*

$$G_\tau^\varepsilon \circ F_\tau^\varepsilon \subseteq (I_{\mathcal{D}}^{2\varepsilon})_\tau \quad \text{and} \quad F_\tau^\varepsilon \circ G_\tau^\varepsilon \subseteq (I_{\mathcal{L}}^{2\varepsilon})_\tau$$

for each simplex  $\tau \in K$ . Then

$$d_I^1(E(\mathcal{D}, K), E(\mathcal{L}, K)) \leq \varepsilon.$$

**Proof** Recall from Theorem 5.2 that there is a filtration-preserving  $\varepsilon$ -acyclic carrier  $F^\varepsilon : \Delta_K \mathcal{D} \rightrightarrows \Delta_K \mathcal{L}$ . Given  $r \in \mathbf{R}$ , this implies that there is a chain complex morphism  $f_r^\varepsilon : C_*(\Delta \mathcal{D})_r \rightarrow C_*(\Delta \mathcal{L})_{r+\varepsilon}$  carried by  $F_r^\varepsilon$  and which respects filtrations in the sense that  $f_r^\varepsilon(F^p C_*(\Delta \mathcal{D})_r) \subseteq F^p C_*(\Delta \mathcal{L})_{r+\varepsilon}$  for all  $p \geq 0$ . By Lemma 3.1 this defines a morphism  $f_r^\varepsilon : S_*^{\text{Tot}}(\mathcal{D})_r \rightarrow S_*^{\text{Tot}}(\mathcal{L})_{r+\varepsilon}$  which respects filtrations. Altogether we deduce that  $f_r^\varepsilon$  determines a morphism of spectral sequences  $f_r^\varepsilon : E_{p,q}^*(\mathcal{D})_r \rightarrow E_{p,q}^*(\mathcal{L})_{r+\varepsilon}$ . Similarly as in Corollary 4.7, the commutativity

$$(4) \quad \Sigma^s E_{p,q}^*(\mathcal{L})_{r+\varepsilon} \circ f_r^\varepsilon = f_{r+s}^\varepsilon \circ \Sigma^s E_{p,q}^*(\mathcal{D})_r$$

does not need to hold for all  $r \in \mathbf{R}$  and all  $s \geq 0$ . However, by definition of  $\varepsilon$ -acyclic carrier, there is an inclusion  $\Sigma^s \Delta \mathcal{L} \circ F^\varepsilon \subseteq F^\varepsilon \circ \Sigma^s \Delta \mathcal{D}$  where the superset is acyclic, so  $\Sigma^s C_*(\Delta \mathcal{L})_{r+\varepsilon} \circ f_r^\varepsilon$  and  $f_{r+s}^\varepsilon \circ \Sigma^s C_*(\Delta \mathcal{D})_r$  are both carried by the filtration preserving acyclic carrier  $F^\varepsilon \circ \Sigma^s \Delta \mathcal{D}_r$ . This implies that there exist chain homotopies  $h_r^\varepsilon : C_n(\Delta \mathcal{D})_r \rightarrow C_{n+1}(\Delta \mathcal{L})_{r+s+\varepsilon}$  which respect filtrations and satisfy

$$f_{r+s}^\varepsilon \circ \Sigma^s C_*(\Delta \mathcal{D})_r - \Sigma^s C_*(\Delta \mathcal{L})_{r+\varepsilon} \circ f_r^\varepsilon = \delta^\Delta \circ h_r^\varepsilon + h_r^\varepsilon \circ \delta^\Delta.$$

for all  $r \in \mathbf{R}$  and all  $s \geq 0$ . Recall that the zero page terms are given as quotients on successive filtration terms  $E_{p,q}^0(\mathcal{D})_r = F^p S_{p+q}^{\text{Tot}}(\mathcal{D})_r / F^{p-1} S_{p+q}^{\text{Tot}}(\mathcal{D})_r$ , for all  $r \in \mathbf{R}$  and all integers  $p, q \geq 0$ . Thus, by Lemma 3.1, these chain homotopies carry over to  $S_*^{\text{Tot}}(\mathcal{D})_r$  and the commutativity relation from (4) holds from the first page onwards.

Similarly, we can define spectral sequence morphisms  $g_r^\varepsilon : E_{p,q}^*(\mathcal{L})_r \rightarrow E_{p,q}^*(\mathcal{D})_{r+\varepsilon}$  for all  $r \in \mathbf{R}$  which commute with the shift morphisms from the first page. Also, by inspecting the shift carriers, we can obtain equalities of 1-spectral sequence morphisms  $g_{r+\varepsilon}^\varepsilon \circ f_r^\varepsilon = \Sigma^{2\varepsilon} E_{p,q}^*(\mathcal{D})_r$  and also  $f_{r+\varepsilon}^\varepsilon \circ g_r^\varepsilon = \Sigma^{2\varepsilon} E_{p,q}^*(\mathcal{L})_r$  for all  $r \in \mathbf{R}$ , and the result follows.  $\square$

**Example 6.6** Consider a pair of point clouds  $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^N$ , together with partitions  $\mathcal{P}$  and  $\mathcal{Q}$  for  $\mathbb{X}$  and  $\mathbb{Y}$  respectively. Also, assume that there is an isomorphism  $\phi : \Delta^\mathcal{P} \rightarrow \Delta^\mathcal{Q}$  such that  $d_H(\mathbb{X} \cap V, \mathbb{Y} \cap \phi(V)) < \varepsilon$  for all  $V \in \mathcal{P}$ . As defined in Example 4.5, there are  $\varepsilon$ -acyclic carrier equivalences

$$F_V^\varepsilon : \text{VR}_*(\mathbb{X} \cap V) \rightrightarrows \text{VR}_*(\mathbb{Y} \cap V)$$

for all  $V \in \mathcal{Q}$ . Now suppose that, for some  $\eta > 0$ , if  $\mathcal{F}_\mathcal{P}^{\text{VR}_*(\mathbb{X})}(\sigma)_r \neq \emptyset$  then  $\mathcal{F}_\mathcal{Q}^{\text{VR}_*(\mathbb{Y})}(\phi(\sigma))_{r+\eta} \neq \emptyset$  for all  $\sigma \in \Delta^\mathcal{P}$  and all  $r \in \mathbf{R}$ . For any  $\sigma \in \Delta^\mathcal{P}$ , one can define  $(\varepsilon+\eta)$ -acyclic carriers

$$\tilde{F}_\sigma^{(\varepsilon+\eta)} : \mathcal{F}_\mathcal{P}^{\text{VR}_*(\mathbb{X})}(\sigma) \rightrightarrows \mathcal{F}_\mathcal{Q}^{\text{VR}_*(\mathbb{Y})}(\sigma)$$

by sending a cell  $\prod_{V \in \sigma} \tau_V \in \mathcal{F}_\mathcal{P}^{\text{VR}_*(\mathbb{X})}(\sigma)_r$  to  $\prod_{V \in \sigma} \Sigma^\eta \text{VR}_*(\mathbb{Y} \cap V)(F_V^\varepsilon(\tau_V)) \in \mathcal{F}_\mathcal{Q}^{\text{VR}_*(\mathbb{Y})}(\sigma)_{r+(\varepsilon+\eta)}$  for all  $r \in \mathbf{R}$ . Similarly, we assume the converse that  $\mathcal{F}_\mathcal{Q}^{\text{VR}_*(\mathbb{Y})}(\tilde{\sigma})_r \neq \emptyset$  implies  $\mathcal{F}_\mathcal{P}^{\text{VR}_*(\mathbb{X})}(\phi^{-1}(\tilde{\sigma}))_{r+\eta} \neq \emptyset$  for all  $\tilde{\sigma} \in \Delta^\mathcal{Q}$  and all  $r \in \mathbf{R}$ . With an analogous definition to that of  $\tilde{F}_\sigma^{(\varepsilon+\eta)}$ , we obtain “inverses” for the carriers  $\tilde{F}_\sigma^{(\varepsilon+\eta)}$ , so that these become  $(\varepsilon+\eta)$ -acyclic equivalences. One can check that these are compatible along  $\Delta^\mathcal{P}$  and  $\Delta^\mathcal{Q}$ , so by Theorem 6.5

$$d_I^1(E_{*,*}^*(\mathcal{F}_\mathcal{P}^{\text{VR}_*(\mathbb{X})}, \Delta^\mathcal{P}), E_{*,*}^*(\mathcal{F}_\mathcal{Q}^{\text{VR}_*(\mathbb{Y})}, \Delta^\mathcal{Q})) \leq \varepsilon + \eta.$$

## 7 Interleavings with respect to different covers

### 7.1 Refinement induced interleavings

In the previous sections we considered general diagrams in  $\mathbf{FRDiag}(K)$  for some simplicial complex  $K$ . We now focus on the situation where we have a filtered complex  $X$  together with a cover  $\mathcal{U}$ , which provides a diagram  $X^{\mathcal{U}}: N_{\mathcal{U}} \rightarrow \mathbf{FCW}\text{-}\mathbf{cpx}$ . The associated spectral sequence is denoted by  $E_{*,*}^*(X, \mathcal{U})$ , as done at the start of Section 3. We want to measure how  $E_{*,*}^*(X, \mathcal{U})$  changes depending on  $\mathcal{U}$  and follow ideas from [Serre 1955] to achieve this. First we consider a refinement  $\mathcal{V} < \mathcal{U}$ , which means that for all  $V \in \mathcal{V}$ , there exists  $U \in \mathcal{U}$  such that  $V \subseteq U$ . In particular, one can choose a morphism  $\rho^{\mathcal{U}, \mathcal{V}}: N_{\mathcal{V}} \rightarrow N_{\mathcal{U}}$  such that  $\mathcal{V}_{\sigma} \subseteq \mathcal{U}_{\rho\sigma}$  for all  $\sigma \in N_{\mathcal{V}}$ . This choice is of course not necessarily unique. We would like to compare the Mayer–Vietoris spectral sequences of both covers. For this, we recall the definition of the Čech chain complex outlined in the introduction of [Torras-Casas 2023], which leads to the following isomorphism on the terms from the 0–page:

$$(5) \quad E_{p,q}^0(X, \mathcal{U}) = \check{\mathcal{C}}_p(\mathcal{U}; C_q^{\text{cell}}) := \bigoplus_{\sigma \in N_{\mathcal{U}}^p} C_q^{\text{cell}}(\mathcal{U}_{\sigma}) \simeq \bigoplus_{s \in X^q} f_*^{\sigma(s, \mathcal{U})} (C_p^{\text{cell}}(\Delta^{\sigma(s, \mathcal{U})})).$$

Here,  $\sigma(s, \mathcal{U})$  is the simplex of maximal dimension in  $N_{\mathcal{U}}$  such that  $s \in X^{\mathcal{U}}(\sigma(s, \mathcal{U}))$ , and

$$f^{\sigma(s, \mathcal{U})}: \Delta^{\sigma(s, \mathcal{U})} \hookrightarrow N_{\mathcal{U}}$$

denotes the inclusion. The isomorphism in (5) is given by sending a generator

$$(a)_{\sigma} \in \bigoplus_{\sigma \in N_{\mathcal{U}}^p} C_q^{\text{cell}}(\mathcal{U}_{\sigma})$$

to its transpose  $(\sigma)_a$ , for all cells  $a \in X$  and all  $\sigma \in N_{\mathcal{U}}$ .

Returning to a refinement  $\mathcal{V} < \mathcal{U}$  and a morphism  $\rho^{\mathcal{U}, \mathcal{V}}: N_{\mathcal{V}} \rightarrow N_{\mathcal{U}}$ , there is an induced double complex morphism  $\rho_{p,q}^{\mathcal{U}, \mathcal{V}}: C_{p,q}(X, \mathcal{V}) \rightarrow C_{p,q}(X, \mathcal{U})$  given by

$$\rho_{p,q}^{\mathcal{U}, \mathcal{V}}((\sigma)_a) = \begin{cases} (\rho^{\mathcal{U}, \mathcal{V}} \sigma)_a & \text{if } \dim(\rho^{\mathcal{U}, \mathcal{V}} \sigma) = p, \\ 0 & \text{otherwise,} \end{cases}$$

for all generators  $(\sigma)_a \in C_{p,q}(X, \mathcal{V})$  with  $\sigma \in N_{\mathcal{V}}^p$  and  $a \in X^q$ .

**Lemma 7.1**  $\rho_{*,*}^{\mathcal{U}, \mathcal{V}}$  is a morphism of double complexes. Thus, it induces a morphism of spectral sequences

$$\rho_{p,q}^{\mathcal{U}, \mathcal{V}}: E_{p,q}^*(X, \mathcal{V}) \rightarrow E_{p,q}^*(X, \mathcal{U})$$

dependent on the choice of  $\rho^{\mathcal{U}, \mathcal{V}}$ .

**Proof** Let  $\delta^{\mathcal{V}}$  and  $\delta^{\mathcal{U}}$  denote the respective Čech differentials from  $\check{\mathcal{C}}_p(\mathcal{V}; C_q^{\text{cell}})$  and  $\check{\mathcal{C}}_p(\mathcal{U}; C_q^{\text{cell}})$ . The refinement  $\rho^{\mathcal{U}, \mathcal{V}}: N_{\mathcal{V}} \rightarrow N_{\mathcal{U}}$  induces a chain morphism  $\rho_{*,*}^{\mathcal{U}, \mathcal{V}}: C_{*,*}^{\text{cell}}(N_{\mathcal{V}}) \rightarrow C_{*,*}^{\text{cell}}(N_{\mathcal{U}})$ , so that we have commutativity  $\rho_{*,*}^{\mathcal{U}, \mathcal{V}} \circ \delta^{\mathcal{V}} = \delta^{\mathcal{U}} \circ \rho_{*,*}^{\mathcal{U}, \mathcal{V}}$ . This implies that  $\rho_{*,*}^{\mathcal{U}, \mathcal{V}}$  commutes with the horizontal differential  $d^H$ .

For commutativity with  $d^V$ , we consider a generating chain  $(\sigma)_a \in E_{p,q}^0(X, \mathcal{V})$  with  $\sigma \in N_{\mathcal{V}}^p$  and  $a \in X^q$ . Then, if  $\dim(\rho^{\mathcal{U}, \mathcal{V}} \sigma) = p$ ,

$$\begin{aligned} \rho_{p,q-1}^{\mathcal{U}, \mathcal{V}} \circ d^V((\sigma)_a) &= \rho_{p,q-1}^{\mathcal{U}, \mathcal{V}} \left( (-1)^p \sum_{b \leq \bar{a}} ([b : a] \sigma)_b \right) \\ &= (-1)^p \sum_{b \leq \bar{a}} ([b : a] \rho^{\mathcal{U}, \mathcal{V}} \sigma)_b = (-1)^p d_q^{\text{cell}}((\rho^{\mathcal{U}, \mathcal{V}} \sigma)_a) = d^V \circ \rho_{p,q}^{\mathcal{U}, \mathcal{V}}((\sigma)_a) \end{aligned}$$

and for  $\dim(\rho^{\mathcal{U}, \mathcal{V}} \sigma) < p$  commutativity follows since both terms vanish.

A morphism of double complexes gives rise to a morphism of the vertical filtration. By [McCleary 2001, Theorem 3.5] this induces a morphism of spectral sequences  $\rho_{*,*}^{\mathcal{U}, \mathcal{V}}$ .  $\square$

Since  $\rho^{\mathcal{U}, \mathcal{V}} : N_{\mathcal{V}} \rightarrow N_{\mathcal{U}}$  is not unique, the induced morphism  $\rho_{*,*}^{\mathcal{U}, \mathcal{V}}$  on the 0–page does not need to be unique either. We have, however, the following:

**Proposition 7.2** *The 2–morphism obtained by restricting  $\rho_{*,*}^{\mathcal{U}, \mathcal{V}}$  is independent of the particular choice of refinement map  $\rho^{\mathcal{U}, \mathcal{V}} : N_{\mathcal{V}} \rightarrow N_{\mathcal{U}}$ .*

**Proof** We have to show that  $\rho_{*,*}^{\mathcal{U}, \mathcal{V}}$  is independent of the particular choice of the refinement morphism. First, define a carrier  $R : N_{\mathcal{V}} \rightrightarrows N_{\mathcal{U}}$  by the assignment

$$\sigma \mapsto R(\sigma) = \{v \in N_{\mathcal{U}} \mid V_{\sigma} \subseteq U_v\}.$$

The geometric realization of the subcomplex  $R(\sigma)$  is homeomorphic to a standard simplex, in particular contractible, so  $R$  is acyclic. Note that  $\rho_{*,*}^{\mathcal{U}, \mathcal{V}}$  is carried by  $R$ . Hence, by Theorem 2.4 for any pair of refinement maps  $\rho^{\mathcal{U}, \mathcal{V}}, \tau^{\mathcal{U}, \mathcal{V}} : N_{\mathcal{V}} \rightarrow N_{\mathcal{U}}$ , there exists a chain homotopy  $k_* : C_n(N_{\mathcal{V}}) \rightarrow C_{n+1}(N_{\mathcal{U}})$  carried by  $R$  such that

$$k_* \delta^{\mathcal{V}} + \delta^{\mathcal{U}} k_* = \tau_*^{\mathcal{U}, \mathcal{V}} - \rho_*^{\mathcal{U}, \mathcal{V}}$$

for all  $n \geq 0$  and where  $\tau_*^{\mathcal{U}, \mathcal{V}}$  and  $\rho_*^{\mathcal{U}, \mathcal{V}}$  are induced morphisms of chain complexes  $C_*(N_{\mathcal{V}}) \rightarrow C_*(N_{\mathcal{U}})$ . In particular, using the same notation, this translates into chain homotopies  $k_* : E_{p,q}^0(X, \mathcal{V}) \rightarrow E_{p+1,q}^0(X, \mathcal{U})$  on the 0–page such that

$$k_* \delta^{\mathcal{V}} + \delta^{\mathcal{U}} k_* = \tau_{*,*}^{\mathcal{U}, \mathcal{V}} - \rho_{*,*}^{\mathcal{U}, \mathcal{V}}.$$

Thus,  $\tau_{*,*}^{\mathcal{U}, \mathcal{V}} = \rho_{*,*}^{\mathcal{U}, \mathcal{V}}$  from the second page onward.  $\square$

**Example 7.3** Consider a filtered cubical complex  $\mathcal{C}_*$ . At value 0,  $\mathcal{C}_*$  is given by the vertices on  $\mathbb{R}^2$  at the coordinates  $a = (0, 0)$ ,  $b = (1, 0)$ ,  $c = (2, 0)$ ,  $d = (3, 0)$ ,  $e = (0, 1)$ ,  $f = (1, 1)$ ,  $g = (2, 1)$  and  $h = (3, 1)$ , together with all edges contained in the boundary of the rectangle  $adhe$ . Then, at value 1 there appears the edge  $bf$  with the face  $abfe$ . At value 2 the edge  $gc$  with the face  $fgcb$ , and finally at value 3 the face  $ghdc$  appears. This is depicted on Figure 2. Then, consider the cover  $\mathcal{U}_0$  by three subcomplexes on the squares  $A = (a, b, f, e)$ ,  $B = (b, c, g, f)$  and  $C = (c, d, h, g)$ . Also, we consider the cover  $\mathcal{U}_1$  given by  $A$  and  $C \cup B$ , and  $\mathcal{U}_2$  given by all  $\mathcal{C}_*$ . The induced morphisms on second-page terms at different filtration values are either null or the identity, as illustrated on Figure 3.

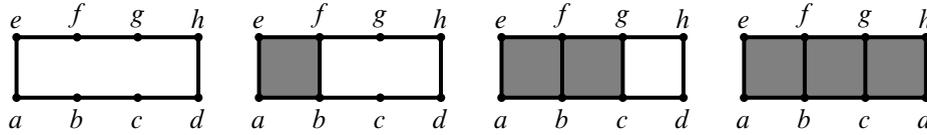


Figure 2: Cubical complex  $\mathcal{C}_*$  at values 0, 1, 2 and 3.

A consequence of Proposition 7.2 is that if we have a space  $X$  together with covers  $\mathcal{U} \prec \mathcal{V} \prec \mathcal{W}$ , then by uniqueness the morphism on the second page induced by the consecutive inclusions coincides with the identity. This gives rise to the next result.

**Proposition 7.4** *Suppose a pair of covers  $\mathcal{U}$  and  $\mathcal{V}$  of  $X$  are a refinement of one another. Then there is a 2-spectral sequence isomorphism  $E_{*,*}^2(X, \mathcal{U}) \simeq E^2(X, \mathcal{V})$ .*

This implies that for any cover  $\mathcal{U}$  of  $X$ , the cover  $\mathcal{U} \cup X$  obtained by adding the extra covering element  $X$  is such that the second page  $E_{p,q}^2(X, \mathcal{U} \cup X)$  has only the first column nonzero.

**Lemma 7.5** *Consider a cover  $\mathcal{U}$  of a space  $X$ , and suppose that  $X \in \mathcal{U}$ . Then  $E_{p,q}^2(X, \mathcal{U}) = 0$  for all  $p > 0$ .*

**Proof** This follows from the observation that the cover  $\{X\}$  consisting of a single element satisfies  $\{X\} \prec \mathcal{U} \prec \{X\}$ . Using Proposition 7.4 we therefore obtain isomorphisms  $E_{p,q}^2(X, \mathcal{U}) \simeq E_{p,q}^2(X, \{X\})$ , and the result follows.  $\square$

Suppose that none of the two covers  $\mathcal{V}$  and  $\mathcal{U}$  refines the other. One can still compare them using the common refinement  $\mathcal{V} \cap \mathcal{U} = \{V \cap U\}_{V \in \mathcal{V}, U \in \mathcal{U}}$  which is a cover of  $X$ . Thus, there are two refinement morphisms

$$(6) \quad E_{p,q}^2(X, \mathcal{U}) \xleftarrow{\rho_{p,q}^{\mathcal{U}, \mathcal{V} \cap \mathcal{U}}} E_{p,q}^2(X, \mathcal{V} \cap \mathcal{U}) \xrightarrow{\rho_{p,q}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}}} E_{p,q}^2(X, \mathcal{V}).$$

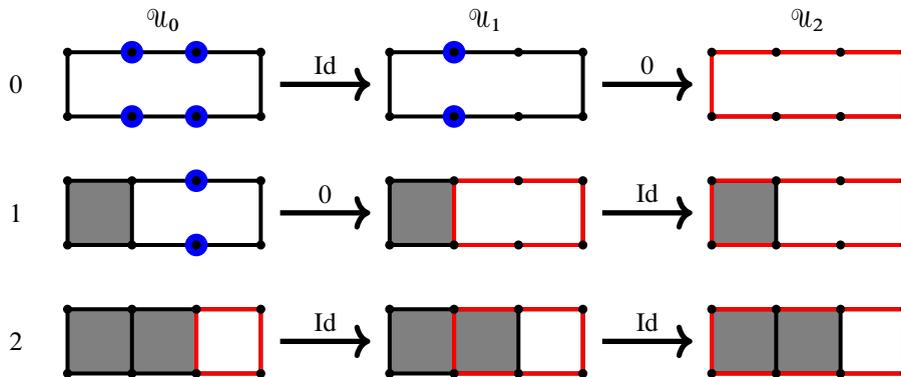


Figure 3: Cubical complex  $\mathcal{C}_*$  with covers  $\mathcal{U}_0, \mathcal{U}_1$  and  $\mathcal{U}_2$ , and with filtration values 0, 1 and 2. Blue dots represent classes in  $E_{1,0}^2(\mathcal{C}, \mathcal{U}_i)$  and red loops represent classes on  $E_{0,1}^2(\mathcal{C}, \mathcal{U}_i)$ , for  $i = 0, 1, 2$ .

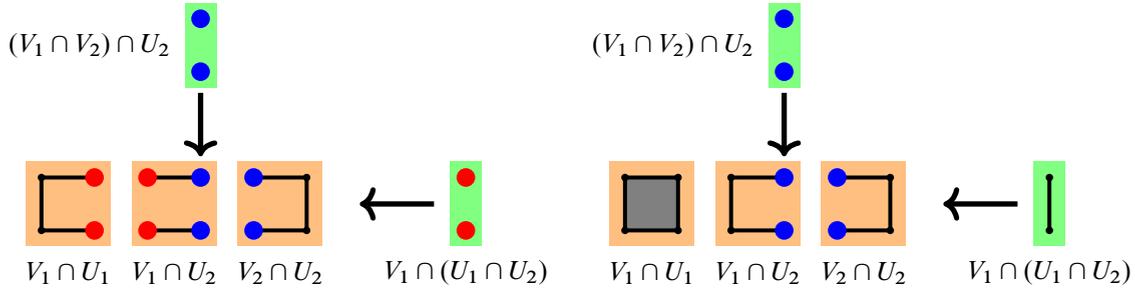


Figure 4:  $C_{p,q}(\mathcal{V}, \mathcal{U}, \text{PH}_k)$  at filtration values 0 and 1.

Following [Serre 1955, Section 28] we can now build the double complex  $C_{p,q}(\mathcal{V}, \mathcal{U}, \text{PH}_k)$  which, for each  $k \geq 0$ , is given by

$$\begin{array}{ccc}
 \bigoplus_{\substack{\sigma \in N_{\mathcal{V}}^{p+1} \\ \tau \in N_{\mathcal{U}}^q}} \text{PH}_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau) & \xleftarrow{(-1)^{p+1} \delta^{\mathcal{U}}} & \bigoplus_{\substack{\sigma \in N_{\mathcal{V}}^{p+1} \\ \tau \in N_{\mathcal{U}}^{q+1}} \text{PH}_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau) \\
 \downarrow \delta^{\mathcal{V}} & & \downarrow \delta^{\mathcal{V}} \\
 \bigoplus_{\substack{\sigma \in N_{\mathcal{V}}^p \\ \tau \in N_{\mathcal{U}}^q}} \text{PH}_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau) & \xleftarrow{(-1)^p \delta^{\mathcal{U}}} & \bigoplus_{\substack{\sigma \in N_{\mathcal{V}}^p \\ \tau \in N_{\mathcal{U}}^{q+1}} \text{PH}_k(\mathcal{V}_\sigma \cap \mathcal{U}_\tau)
 \end{array}$$

for any pair of integers  $p, q \geq 0$ . From this double complex we can study the two associated spectral sequences

$$\begin{aligned}
 {}^I E_{p,q}^1(\mathcal{V}, \mathcal{U}; \text{PH}_k) &= \bigoplus_{\sigma \in N_{\mathcal{V}}^p} \check{\mathcal{H}}_q(\mathcal{V}_\sigma \cap \mathcal{U}; \text{PH}_k), \\
 {}^{II} E_{p,q}^1(\mathcal{V}, \mathcal{U}; \text{PH}_k) &= \bigoplus_{\tau \in N_{\mathcal{U}}^q} \check{\mathcal{H}}_p(\mathcal{V} \cap \mathcal{U}_\tau; \text{PH}_k),
 \end{aligned}$$

whose common target of convergence is  $\check{\mathcal{H}}_n(\mathcal{V} \cap \mathcal{U}; \text{PH}_k)$  with  $p + q = n$ . For details about the spectral sequence associated to a double complex, the reader is recommended to look at [McCleary 2001, Theorem 2.15].

**Example 7.6** Consider the cubical complex  $\mathcal{C}_*$  from Example 7.3. Set  $\mathcal{U} = \mathcal{U}_1$ , that is,  $\mathcal{U}$  is the cover by the sets  $U_1 = A$  and  $U_2 = B \cup C$ . On the other hand, consider  $\mathcal{V}$  to be formed of  $V_1 = A \cup B$  and  $V_2 = C$ . The double complex  $C_{p,q}(\mathcal{V}, \mathcal{U}, \text{PH}_k)$  is illustrated on Figure 4 for filtration values 0 and 1, and for  $k = 0$ . We encourage the reader to work out the refinement morphisms from (6) and see that these are actually projections.

Consider the nerve  $N_{\mathcal{V} \cap \mathcal{U}}$  as a subset of the product of nerves  $N_{\mathcal{V}} \times N_{\mathcal{U}}$ . We have then two projections  $\pi^{\mathcal{V}}: N_{\mathcal{V} \cap \mathcal{U}} \rightarrow N_{\mathcal{V}}$  and  $\pi^{\mathcal{U}}: N_{\mathcal{V} \cap \mathcal{U}} \rightarrow N_{\mathcal{U}}$ , both of which induce chain morphisms  $\pi_*^{\mathcal{V}}: C_*(N_{\mathcal{V} \cap \mathcal{U}}) \rightarrow C_*(N_{\mathcal{V}})$

and  $\pi_*^{\mathcal{U}} : C_*(N_{\mathcal{V} \cap \mathcal{U}}) \rightarrow C_*(N_{\mathcal{U}})$ . For example,  $\pi_*^{\mathcal{V}}$  is given by  $\pi_*^{\mathcal{V}}(\sigma \times \tau) = \sigma$  if  $\dim(\tau) = 0$  or  $\pi_*^{\mathcal{V}}(\sigma \times \tau) = 0$  otherwise, for all  $\sigma \in N_{\mathcal{V}}$  and  $\tau \in N_{\mathcal{U}}$ . These induce a pair of morphisms

$$\bigoplus_{\sigma \in N_{\mathcal{V}}^p} C_k^{\text{cell}}(\mathcal{V}_{\sigma}) \xleftarrow{\pi_{p,k}^{\mathcal{V}}} \bigoplus_{\substack{\sigma \in N_{\mathcal{V}}^p \\ \tau \in N_{\mathcal{U}}^q}} C_k^{\text{cell}}(\mathcal{V}_{\sigma} \cap \mathcal{U}_{\tau}) \xrightarrow{\pi_{q,k}^{\mathcal{U}}} \bigoplus_{\tau \in N_{\mathcal{U}}^q} C_k^{\text{cell}}(\mathcal{U}_{\tau}),$$

for any pair of integers  $p, q \geq 0$ . The induced map  $\pi_{p,k}^{\mathcal{V}}$  on  $C_k(\mathcal{V}_{\sigma} \cap \mathcal{U}_{\tau})$  satisfies

$$\pi_{p,k}^{\mathcal{V}}((\sigma \times \tau)_a) = (\pi_*^{\mathcal{V}}(\sigma \times \tau))_a$$

for all  $\sigma \in N_{\mathcal{V}}^p, \tau \in N_{\mathcal{U}}^q$  and all  $a \in (\mathcal{V}_{\sigma} \cap \mathcal{U}_{\tau})^k$ . The map  $\pi_{*,*}^{\mathcal{U}}$  acts similarly. By definition  $\pi_{*,*}^{\mathcal{U}}$  and  $\pi_{*,*}^{\mathcal{V}}$  both commute with the Čech differentials  $\delta^{\mathcal{U}}$  and  $\delta^{\mathcal{V}}$  respectively. Let  $\sigma \in N_{\mathcal{V}}^p$  and  $\tau \in N_{\mathcal{U}}^q$ . Then we have

$$\begin{array}{ccc} (\sigma \times \tau)_a & \xrightarrow{\pi_{*,*}^{\mathcal{V}}} & (\sigma)_a \\ \downarrow d_n & & \downarrow d_n \\ \sum_{b \in \bar{a}} ([b : a] \sigma \times \tau)_b & \xrightarrow{\pi_{*,*}^{\mathcal{V}}} & \sum_{b \in \bar{a}} ([b : a] \sigma)_b \end{array}$$

for all cells  $a \in (\mathcal{V}_{\sigma} \cap \mathcal{U}_{\tau})^k$ . This implies that  $\pi_{*,*}^{\mathcal{V}}$  commutes with  $d_n$  and the same holds for  $\pi_{*,*}^{\mathcal{U}}$ . We obtain a morphism  $\pi_{p,k}^{\mathcal{V}} : \check{\mathcal{C}}_p(\mathcal{V} \cap \mathcal{U}; C_k^{\text{cell}}) \rightarrow \check{\mathcal{C}}_p(\mathcal{V}; C_k^{\text{cell}})$  commuting with  $d_*$  and  $\delta^{\mathcal{V} \cap \mathcal{U}}$  and  $\delta^{\mathcal{V}}$ . This induces  $\kappa_{p,k}^{\mathcal{V}} : \check{\mathcal{C}}_p(\mathcal{V} \cap \mathcal{U}; \text{PH}_k) \rightarrow \check{\mathcal{C}}_p(\mathcal{V}; \text{PH}_k)$  and, in turn, this induces

$$\theta_{p,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}} : \check{\mathcal{H}}_p(\mathcal{V} \cap \mathcal{U}; \text{PH}_k) \rightarrow \check{\mathcal{H}}_p(\mathcal{V}; \text{PH}_k).$$

There is a very natural way of understanding how much  $\theta_{p,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}}$  fails to be an isomorphism. To start, notice that  $\kappa_{p,k}^{\mathcal{V}}$  is equal to the composition

$$\check{\mathcal{C}}_p(\mathcal{V} \cap \mathcal{U}; \text{PH}_k) \twoheadrightarrow {}^1E_{p,0}^0(\mathcal{V}, \mathcal{U}; \text{PH}_k) \xrightarrow{{}^1\pi_{p,k}^{\mathcal{V}}} \check{\mathcal{C}}_p(\mathcal{V}; \text{PH}_k),$$

where the first morphism forgets the summands with  $\tau \notin N_{\mathcal{U}}^0$ ; the second morphism is the restriction of  $\kappa_{p,k}^{\mathcal{V}}$  to the remaining terms. Next, we take for each simplex  $\sigma \in N_{\mathcal{V}}^p$ , the Mayer–Vietoris spectral sequence for  $\mathcal{V}_{\sigma}$  covered by  $\mathcal{V}_{\sigma} \cap \mathcal{U}$

$$M_{q,k}^2(\mathcal{V}_{\sigma} \cap \mathcal{U}) \rightrightarrows \text{PH}_{q+k}(\mathcal{V}_{\sigma}),$$

where we changed the notation from  $E_{q,k}^2(\mathcal{V}_{\sigma}, \mathcal{V}_{\sigma} \cap \mathcal{U})$  to  $M_{q,k}^2(\mathcal{V}_{\sigma} \cap \mathcal{U})$  as it helps distinguishing this spectral sequence from  ${}^1E_{p,q}^*$ . Then, we write more compactly

$${}^1E_{p,q}^1(\mathcal{V}, \mathcal{U}; \text{PH}_k) = \bigoplus_{\sigma \in N_{\mathcal{V}}^p} M_{q,k}^2(\mathcal{V}_{\sigma} \cap \mathcal{U}).$$

Taking  ${}^1E_{p,0}^1(\mathcal{V}, \mathcal{U}; \text{PH}_k)$  as a chain complex,  ${}^1\pi_{p,k}^{\mathcal{V}}$  induces a chain morphism

$${}^1\pi_{p,k}^{\mathcal{V}} : {}^1E_{p,0}^1(\mathcal{V}, \mathcal{U}; \text{PH}_k) \rightarrow \check{\mathcal{C}}_p(\mathcal{V}; \text{PH}_k)$$

for all  $p \geq 0$ . In particular, the restriction of  ${}^I\pi_{p,k}^{\mathcal{V}}$  to the summand  $M_{0,k}^2(\mathcal{V}_\sigma \cap \mathcal{U})$  equals the composition

$$M_{0,k}^2(\mathcal{V}_\sigma \cap \mathcal{U}) \twoheadrightarrow M_{0,k}^\infty(\mathcal{V}_\sigma \cap \mathcal{U}) \hookrightarrow \text{PH}_k(\mathcal{V}_\sigma).$$

Notice that  $\text{PH}_0$  is a cosheaf, and in this case  $M_{0,0}^2(\mathcal{V}_\sigma \cap \mathcal{U}) = \text{PH}_0(\mathcal{V}_\sigma)$  for all  $\sigma \in N_{\mathcal{V}}$ . This implies that  ${}^I\pi_{p,0}^{\mathcal{V}}$  is an isomorphism for all  $p \geq 0$ . By the same argument, there is another chain morphism for all  $q \geq 0$ ,

$${}^{II}\pi_{q,k}^{\mathcal{U}} : {}^{II}E_{0,q}^1(\mathcal{V}, \mathcal{U}; \text{PH}_k) \rightarrow \check{\mathcal{C}}_q(\mathcal{U}; \text{PH}_k).$$

Going back to the morphism  $\theta_{p,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}}$ , it is given by the composition

$$\check{\mathcal{H}}_p(\mathcal{V} \cap \mathcal{U}; \text{PH}_k) \twoheadrightarrow {}^I E_{p,0}^\infty(\mathcal{V}, \mathcal{U}; \text{PH}_k) \hookrightarrow {}^I E_{p,0}^2(\mathcal{V}, \mathcal{U}, \text{PH}_k) \xrightarrow{{}^I\pi_{p,k}^{\mathcal{V}}} \check{\mathcal{H}}_p(\mathcal{V}; \text{PH}_k).$$

Using Lemma 7.5, if  $\mathcal{V} < \mathcal{U}$  then  $M_{q,k}^2(\mathcal{V}_\sigma \cap \mathcal{U}) = 0$  for all  $q > 0$  and  ${}^I\pi_{p,k}^{\mathcal{V}}$  becomes an isomorphism. In addition,  ${}^I E_{p,q}^1 = 0$  for all  $q > 0$  and the first two arrows in the above factorisation of  $\theta_{p,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}}$  are isomorphisms. Altogether, the inverse  $(\theta_{p,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}})^{-1}$  is well defined, and by composition we define morphisms  $\theta_{p,k}^{\mathcal{U}, \mathcal{V}} = \theta_{p,k}^{\mathcal{U}, \mathcal{V} \cap \mathcal{U}} \circ (\theta_{p,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}})^{-1}$ . Here notice that  $\theta_{p,k}^{\mathcal{U}, \mathcal{V} \cap \mathcal{U}}$  is defined in an analogous way to  $\theta_{p,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}}$ , but it factors through  ${}^{II}\pi_{q,k}^{\mathcal{U}}$  instead of  ${}^I\pi_{p,k}^{\mathcal{V}}$ . The following proposition should also follow from applying an appropriate version of the universal coefficient theorem to [Serre 1955, Proposition 4.4]. Instead, we prove the dual statement of this proposition by means of acyclic carriers.

**Proposition 7.7** *Suppose that  $\mathcal{V} < \mathcal{U}$ , and let  $\rho^{\mathcal{U}, \mathcal{V}}$  denote a refinement map. The morphism*

$$\theta_{p,k}^{\mathcal{U}, \mathcal{V}} : E_{p,k}^2(X, \mathcal{V}) \rightarrow E_{p,k}^2(X, \mathcal{U})$$

*coincides with the standard morphism induced by  $\rho^{\mathcal{U}, \mathcal{V}}$ .*

**Proof** Since  $\mathcal{V} < \mathcal{U}$ , the morphism  $\theta_{p,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}} : \check{\mathcal{H}}_p(\mathcal{V} \cap \mathcal{U}, \text{PH}_k) \rightarrow \check{\mathcal{H}}_p(\mathcal{V}, \text{PH}_k)$  is an isomorphism. Now consider the diagram

$$\begin{array}{ccc} \check{\mathcal{H}}_p(\mathcal{V}; \text{PH}_k) & \xrightarrow{\rho_{p,k}^{\mathcal{U}, \mathcal{V}}} & \check{\mathcal{H}}_p(\mathcal{U}; \text{PH}_k) \\ \uparrow \simeq & & \uparrow {}^{II}\pi_{p,k}^{\mathcal{U}} \\ \check{\mathcal{H}}_p(\mathcal{V} \cap \mathcal{U}; \text{PH}_k) & \twoheadrightarrow {}^{II}E_{0,p}^\infty(\mathcal{V}, \mathcal{U}; \text{PH}_k) \hookrightarrow {}^{II}E_{0,p}^2(\mathcal{V}, \mathcal{U}; \text{PH}_k) & \end{array}$$

To check that it commutes we study triangles of acyclic carriers

$$\begin{array}{ccc} & N_{\mathcal{V} \cap \mathcal{U}} & \\ F \nearrow & & \searrow P_{\mathcal{U}} \\ N_{\mathcal{V}} & \xrightarrow{R} & N_{\mathcal{U}} \end{array}$$

where  $R$  is defined in Proposition 7.2. The carrier  $F$  is given for every  $\sigma \in N_{\mathcal{V}}$  by  $F(\sigma) = \Delta^\sigma \times |R(\sigma)|$ . In fact,  $F$  defines an acyclic equivalence by considering the inverse carrier  $P_{\mathcal{V}} : N_{\mathcal{V} \cap \mathcal{U}} \rightrightarrows N_{\mathcal{V}}$  sending  $\sigma \times \tau$  to  $\Delta^\sigma$ . In this case the shift carrier  $I_{\mathcal{V}} : N_{\mathcal{V}} \rightrightarrows N_{\mathcal{V}}$  is given by the assignment  $\sigma \mapsto \Delta^\sigma$ , and

$I_{V \cap \mathcal{U}}: N_{V \cap \mathcal{U}} \rightrightarrows N_{V \cap \mathcal{U}}$  is given by  $\sigma \times \tau \mapsto \Delta^\sigma \times \Delta^{\tau \cup \tau'}$ , where  $\tau' \in N_{\mathcal{U}}$  is such that  $|R(\sigma)| = \Delta^{\tau'} \subseteq N_{\mathcal{U}}$ . Here, we need to show that  $\Delta^\sigma \times \Delta^{\tau \cup \tau'}$  is a subcomplex of  $N_{V \cap \mathcal{U}}$ . First notice that, by hypotheses,  $V_\sigma \cap \mathcal{U}_\tau \neq \emptyset$  and, by definition of  $R(\sigma)$ , we have  $V_\sigma \subseteq \mathcal{U}_{\tau'}$ . Consequently  $V_\sigma \cap (\mathcal{U}_\tau \cap \mathcal{U}_{\tau'}) \neq \emptyset$ , which accounts for  $\Delta^\sigma \times \Delta^{\tau \cup \tau'}$  being a subcomplex of  $N_{V \cap \mathcal{U}}$ .

Since  $F$  is acyclic, there exists  $\nu_*: C_*(N_V) \rightarrow C_*(N_{V \cap \mathcal{U}})$  carried by  $F$  and inducing a chain morphism  $f_*: \check{C}_p(V, C_k^{\text{cell}}) \rightarrow \check{C}_p(V \cap \mathcal{U}, C_k^{\text{cell}})$  by the assignment  $(\sigma)_s \mapsto (\nu_*(\sigma))_s$  for all cells  $s \in X$  and all  $\sigma \in N_V$ . On the other hand, recall that  $\theta_{p,k}^{V, V \cap \mathcal{U}}$  is induced by  $\pi_{p,k}^V$ , which is given as an assignment

$$(\sigma \times \tau)_s \rightarrow (\pi_*^V(\sigma \times \tau))_s.$$

As  $\pi_*^V$  is carried by  $P_V$  and, as noted earlier,  $F$  defines an acyclic equivalence, it follows that  $\pi_*^V \circ \nu_*$  is the identity in  $C_*(N_V)$  up to boundary. Thus,  $\pi_{p,k}^V \circ f_*$  is the identity in  $\check{C}_p(V, C_k^{\text{cell}})$  up to the Čech boundary  $\check{\delta}_V$ . This implies that  $f_* = (\theta_{p,k}^{V, V \cap \mathcal{U}})^{-1}$  as morphisms  $\check{\mathcal{H}}_p(V, \text{PH}_k) \rightarrow \check{\mathcal{H}}_p(V \cap \mathcal{U}, \text{PH}_k)$ . Consequently,  $\theta_{p,k}^{\mathcal{U}, V}$  is induced by the assignment  $(\sigma)_s \mapsto (\pi_*^{\mathcal{U}} \circ \nu_*(\sigma))_s$  for all  $\sigma \in N_V$  and all  $s \in X$ , where  $\pi_*^{\mathcal{U}} \circ \nu_*$  is carried by  $P_{\mathcal{U}}F = R$ . Altogether, as  $\rho^{\mathcal{U}, V}$  is carried by  $R$ , we obtain the equality  $\theta_{p,k}^{\mathcal{U}, V} = \rho_{p,k}^{\mathcal{U}, V}$  as morphisms  $\check{\mathcal{H}}_p(V, \text{PH}_k) \rightarrow \check{\mathcal{H}}_p(\mathcal{U}, \text{PH}_k)$ .  $\square$

Still assuming that  $V < \mathcal{U}$ , we now look for conditions for the existence of an inverse of  $\theta_{p,k}^{\mathcal{U}, V}$ ,

$$\varphi_{p,k}^{V, \mathcal{U}}: E_{p,k}^2(X, \mathcal{U}) \rightarrow E_{p,k}^2(X, V).$$

**Proposition 7.8** *Suppose that  $V < \mathcal{U}$ . If  $M_{p,k}^2(V \cap \mathcal{U}_\tau) = 0$  for all  $p > 0, k \geq 0$  and all  $\tau \in N_{\mathcal{U}}^q$ , then the maps  $\theta_{*,*}^{\mathcal{U}, V}$  induce a 2-isomorphism of spectral sequences*

$$E_{*,*}^{\geq 2}(X, \mathcal{U}) \simeq E_{*,*}^{\geq 2}(X, V).$$

**Proof** By Propositions 7.2 and 7.7 we can choose a refinement map  $\rho^{\mathcal{U}, V}: N_V \rightarrow N_{\mathcal{U}}$  giving a morphism of spectral sequences

$$\rho_{*,*}^{\mathcal{U}, V}: E_{*,*}^{\geq 2}(X, V) \rightarrow E_{*,*}^{\geq 2}(X, \mathcal{U})$$

that coincides with  $\theta_{*,*}^{\mathcal{U}, V}$ . Our assumption about  $M_{p,k}^2$  implies  ${}^{\text{II}}E_{p,q}^2(V, \mathcal{U}; \text{PH}_k) = 0$  for all  $p > 0$ , which in turn, gives

$$(7) \quad \text{Ker}(\check{\mathcal{H}}_q(V \cap \mathcal{U}; \text{PH}_k) \twoheadrightarrow {}^{\text{II}}E_{0,q}^\infty(V, \mathcal{U}; \text{PH}_k)) = 0$$

and

$$(8) \quad \text{Coker}({}^{\text{II}}E_{0,q}^\infty(V, \mathcal{U}; \text{PH}_k) \hookrightarrow {}^{\text{II}}E_{0,q}^2(V, \mathcal{U}, \text{PH}_k)) = 0.$$

Now note that  ${}^{\text{II}}\pi_{q,k}^{\mathcal{U}}$  yields an isomorphism  ${}^{\text{II}}E_{0,q}^2(V, \mathcal{U}, \text{PH}_k) \simeq \check{\mathcal{H}}_q(\mathcal{U}, \text{PH}_k)$ . This shows that  $\theta_{q,k}^{\mathcal{U}, V}$  is a composition of isomorphisms; thus the statement follows.  $\square$

We now relax the conditions in Proposition 7.8 and use the relations of *left-interleaving* and *right-interleaving* of persistence modules (denoted by  $\sim_L^\varepsilon$  and  $\sim_R^\varepsilon$ , respectively) to achieve this (see [Govc and Skraba 2018, Section 4]). We have to adapt [Govc and Skraba 2018, Proposition 4.14].

**Lemma 7.9** Suppose that we have persistence modules  $A, B$  and  $C$ , and a parameter  $\varepsilon \geq 0$  such that  $A \sim_R^\varepsilon B$  and  $B \sim_L^\varepsilon C$ . Denote by  $\Phi$  the morphism  $\Phi: A \rightarrow C$  given by the composition  $A \twoheadrightarrow B \hookrightarrow C$ . Then there exists  $\Psi: C \rightarrow A[2\varepsilon]$  such that  $\Phi$  and  $\Psi$  define a  $2\varepsilon$ -interleaving  $A \sim^{2\varepsilon} C$ .

**Proof** By hypothesis, we have a sequence

$$\mathcal{C}_1 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow \mathcal{C}_2$$

which is exact in  $A$  and  $C$  and where  $\mathcal{C}_1 \sim^\varepsilon 0$  and  $\mathcal{C}_2 \sim^\varepsilon 0$ . Then, let  $v \in C$  and notice that  $\Sigma^\varepsilon C(v) \in \text{Im}(g)$ . Thus, there exists a unique vector  $w \in B$  such that  $g(w) = \Sigma^\varepsilon C(v)$ . On the other hand, there exists  $z \in A$ , not necessarily unique, such that  $f(z) = w$ . This defines a unique element  $\Sigma^\varepsilon A(z) \in A$ . To see this, suppose that another  $z' \in A$  is such that  $f(z') = w$ . Then  $f(z - z') = 0$  and  $z - z' \in \text{Ker}(f)$ , which implies  $0 = \Sigma^\varepsilon A(z - z') = \Sigma^\varepsilon A(z) - \Sigma^\varepsilon A(z')$ , and then  $\Sigma^\varepsilon A(z) = \Sigma^\varepsilon A(z')$ . Altogether, we set  $\Psi = \Sigma^\varepsilon A \circ \Phi^{-1} \circ \Sigma^\varepsilon C$ , which is well defined.  $\square$

Recall that for  $\mathcal{V} < \mathcal{U}$  we have that  $\check{\mathcal{H}}_q(\mathcal{V}; \text{PH}_k) \simeq \check{\mathcal{H}}_q(\mathcal{V} \cap \mathcal{U}; \text{PH}_k)$  for all  $k \geq 0$  and  $q \geq 0$ . There is a natural way to relax (7) and (8) to the persistent case. We assume that for  $\varepsilon \geq 0$ , there are right and left interleavings

$$(9) \quad \check{\mathcal{H}}_q(\mathcal{V} \cap \mathcal{U}; \text{PH}_k) \sim_R^\varepsilon \text{II}E_{0,q}^\infty(\mathcal{V}, \mathcal{U}; \text{PH}_k) \sim_L^\varepsilon \text{II}E_{0,q}^2(\mathcal{V}, \mathcal{U}, \text{PH}_k).$$

If we define  $\Phi_{q,k}: \check{\mathcal{H}}_q(\mathcal{V} \cap \mathcal{U}; \text{PH}_k) \rightarrow \text{II}E_{0,q}^2(\mathcal{V}, \mathcal{U}, \text{PH}_k)$  to be the composition of the associated persistence morphisms as in Lemma 7.9, then there exists

$$\Psi_{q,k}: \text{II}E_{0,q}^2(\mathcal{V}, \mathcal{U}, \text{PH}_k) \rightarrow \check{\mathcal{H}}_q(\mathcal{V} \cap \mathcal{U}; \text{PH}_k)[2\varepsilon]$$

such that  $\Phi_{q,k}$  and  $\Psi_{q,k}$  define a  $2\varepsilon$ -interleaving. We repeat this argument for the local Mayer–Vietoris spectral sequences. Assume that for some  $\nu \geq 0$  there are interleavings

$$(10) \quad \text{II}E_{0,q}^1(\mathcal{V}, \mathcal{U}, \text{PH}_k) \sim_R^\nu \bigoplus_{\tau \in N_{\mathcal{U}}^q} M_{k,0}^\infty(\mathcal{V} \cap \mathcal{U}_\tau) \sim_L^\nu \bigoplus_{\tau \in N_{\mathcal{U}}^q} \text{PH}_k(\mathcal{U}_\tau).$$

Let  $\Pi_{q,k}: \text{II}E_{0,q}^1(\mathcal{V}, \mathcal{U}, \text{PH}_k) \rightarrow \bigoplus_{\tau \in N_{\mathcal{U}}^q} \text{PH}_k(\mathcal{U}_\tau)$  be the composition of the associated morphisms. By Lemma 7.9 there exists  $\Xi_{q,k}$  such that  $\Pi_{q,k}$  and  $\Xi_{q,k}$  define a  $2\nu$ -interleaving. By slight abuse of notation we continue to denote the induced  $2\nu$ -interleaving between  $\text{II}E_{0,q}^2(\mathcal{V}, \mathcal{U}, \text{PH}_k)$  and  $\check{\mathcal{H}}_q(\mathcal{U}; \text{PH}_*)$  by  $\Pi_{q,k}$  and  $\Xi_{q,k}$ . Altogether we have that

$$\theta_{q,k}^{\mathcal{U}, \mathcal{V}} = \Pi_{q,k} \circ \Phi_{q,k} \circ (\theta_{q,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}})^{-1}$$

and in this situation there is an “inverse”  $\psi_{q,k}^{\mathcal{V}, \mathcal{U}} = \theta_{q,k}^{\mathcal{V}, \mathcal{V} \cap \mathcal{U}} \circ \Psi_{q,k} \circ \Xi_{q,k}$ , which increases the persistence values by  $2(\varepsilon + \nu)$ .

**Theorem 7.10** Suppose that  $\mathcal{V} < \mathcal{U}$  and for  $\varepsilon \geq 0$  and  $\nu \geq 0$  the interleavings in (9) and (10) hold. Then

$$\psi_{p,q}^{\mathcal{V}, \mathcal{U}}: E_{p,q}^*(X, \mathcal{U}) \rightarrow E_{p,q}^*(X, \mathcal{V})[2(\varepsilon + \nu)]$$

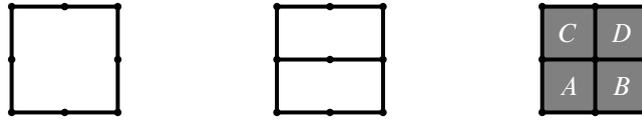
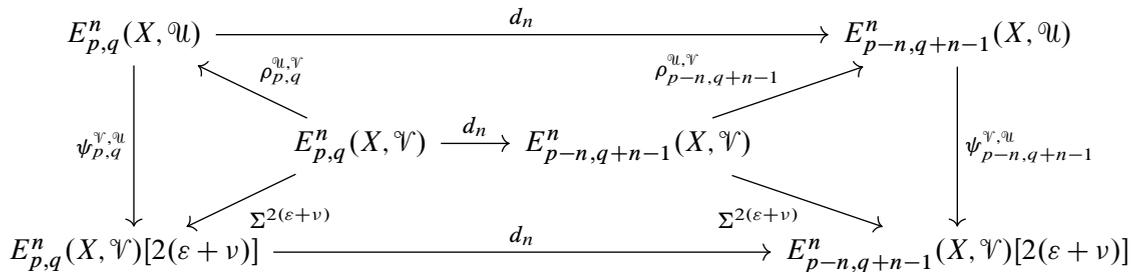


Figure 5: Cubical complex  $\mathcal{C}_*$  at values 0, 1 and  $1 + \varepsilon$ .

is a 2-morphism of spectral sequences such that  $\theta_{p,q}^{\mathcal{U},\mathcal{V}}$  and  $\psi_{p,q}^{\mathcal{V},\mathcal{U}}$  define a second page  $2(\varepsilon + \nu)$ -interleaving between  $E_{p,q}^*(X, \mathcal{U})$  and  $E_{p,q}^*(X, \mathcal{V})$ .

**Proof** The only thing that remains to be proved is that  $\psi_{p,q}^{\mathcal{V},\mathcal{U}}$  commutes with the spectral sequence differentials  $d_n$  for all  $n \geq 2$ . Since these differentials commute with the shift morphisms  $\Sigma^{2(\varepsilon + \nu)}$ , this follows from considering the diagram



in which the two trapeziums and the two triangles commute. □

**Example 7.11** Consider a cubical complex  $\mathcal{C}_*$  as shown in Figure 5, together with the covers

$$\mathcal{V} = \{\bar{A}, \bar{B}, \bar{C}, \bar{D}\} \quad \text{and} \quad \mathcal{U} = \{\overline{A \cup B}, \overline{C \cup D}\};$$

see Figure 5 for the cells  $A, B, C$  and  $D$ . In this case, we have

$$\check{\mathcal{H}}_1(\mathcal{V}; \text{PH}_0) \simeq \check{\mathcal{H}}_1(\mathcal{V} \cap \mathcal{U}; \text{PH}_0) \simeq \text{I}(0, 1 + \varepsilon) \oplus \text{I}(1, 1 + \varepsilon) \sim^\varepsilon \text{I}(0, 1) \simeq {}^{\text{II}}E_{0,1}^2(\mathcal{V}, \mathcal{U}, \text{PH}_0)$$

and also

$${}^{\text{II}}E_{0,0}^1(\mathcal{V}, \mathcal{U}, \text{PH}_1) \simeq 0 \sim^\varepsilon \text{I}(1, 1 + \varepsilon) \oplus \text{I}(1, 1 + \varepsilon) \simeq \bigoplus_{\dim(\tau)=0} \text{PH}_1(\mathcal{U}_\tau).$$

These interleavings are shown in Figure 6. Theorem 7.10 implies that there is a  $4\varepsilon$ -interleaving between  $E_{p,q}^*(X, \mathcal{U})$  and  $E_{p,q}^*(X, \mathcal{V})$ . Notice that in this example, the nontrivial interleaved terms are in different positions of the spectral sequences. Therefore we can improve the upper bound to  $2\varepsilon$ . We use this observation later in Proposition 7.12.

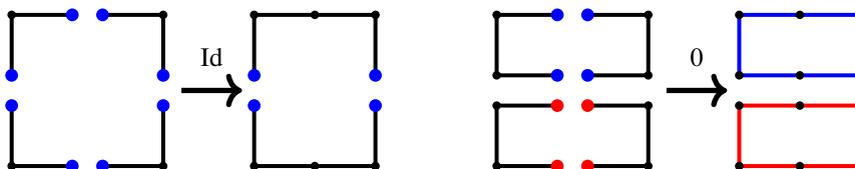


Figure 6: Morphisms  $\theta_{1,0}^{\mathcal{U},\mathcal{V}}$  along  $[0, 1)$  and along  $[1, 1 + \varepsilon)$ .

### 7.2 Interpolating covers and spectral sequence interleavings

Consider  $X \in \mathbf{FCW}\text{-cpx}$ , together with a pair of covers  $\mathcal{W}$  and  $\mathcal{U}$  such that  $\mathcal{W} < \mathcal{U}$ . Motivated by the interleaving constructed in Theorem 7.10 we take a closer look at the following finite sequence of covers interpolating between  $\mathcal{W}$  and a cover that both refines and is refined by  $\mathcal{U}$ . Let the strict  $r^{\text{th}}$  intersections of  $\mathcal{U}$  be the family of sets  $\mathcal{U}^r = \{\mathcal{U}_\tau\}_{\tau \in N_{\mathcal{U}}^r}$  for all  $r \geq 0$ . We define the  $(r, \mathcal{W}, \mathcal{U})$ -interpolation as the covering set  $\mathcal{W}^r = \mathcal{W} \cup \mathcal{U}^r$ . In particular, note that the  $(0, \mathcal{W}, \mathcal{U})$ -interpolation has the property that  $\mathcal{W}^0 < \mathcal{U} < \mathcal{W}^0$ , and consequently  $E_{p,q}^2(X, \mathcal{U}) \simeq E_{p,q}^2(X, \mathcal{W}^0)$ . In addition if  $\mathcal{U}$  is a finite cover, then we have  $\mathcal{U}^N = \emptyset$  for  $N \geq 0$  sufficiently large and consequently  $\mathcal{W}^N = \mathcal{W}$ .

**Proposition 7.12** (local checks) *Let  $\mathcal{W} < \mathcal{U}$  be a pair of covers for  $X$ , where  $\mathcal{U}$  is finite. Let  $N \geq 0$  be such that  $\mathcal{U}^N = \emptyset$ . For every  $0 \leq r \leq N$ , we assume that there exist  $\varepsilon_r \geq 0$  and  $\nu_r \geq 0$  such that for all  $\tau \in N_{\mathcal{U}}^r$ ,*

$$E_{0,q}^2(\mathcal{U}_\tau, \mathcal{W}_{|\mathcal{U}_\tau}^{r+1}) \sim_R^{\nu_r} E_{0,q}^\infty(\mathcal{U}_\tau, \mathcal{W}_{|\mathcal{U}_\tau}^{r+1}) \sim_L^{\nu_r} \text{PH}_q(\mathcal{U}_\tau),$$

and also

$$d_I(E_{p,q}^2(\mathcal{U}_\tau, \mathcal{W}_{|\mathcal{U}_\tau}^{r+1}), 0) \leq \varepsilon_r$$

for all  $p > 0$  and  $q \geq 0$ . Then we have that

$$d_I^2(E_{p,q}^*(X, \mathcal{W}^r), E_{p,q}^*(X, \mathcal{W}^{r+1})) \leq 2 \max(\varepsilon_r, \nu_r).$$

Therefore, by using the triangle inequality, we obtain

$$d_I^2(E_{p,q}^*(X, \mathcal{U}), E_{p,q}^*(X, \mathcal{W})) \leq \sum_{r=0}^N 2 \max(\varepsilon_r, \nu_r).$$

**Proof** We need to consider the spectral sequence  ${}^{\text{II}}E_{p,q}^2(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k)$ . Note that, by the construction of  $\mathcal{W}^r$ , for each  $\tau \in N_{\mathcal{U}^r}$  with  $\dim(\tau) > 0$  the set  $\mathcal{W}_\tau^r$  is contained in one of the open sets from  $\mathcal{W}^{r+1}$ . By Lemma 7.5 this implies that  ${}^{\text{II}}E_{p,q}^1(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k) = 0$  for all  $p > 0, q > 0$  and  $k \geq 0$ . Moreover, we have that  ${}^{\text{II}}E_{0,q}^1(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k) = \bigoplus_{\tau \in N_{\mathcal{W}^r}^q} \text{PH}_k(\mathcal{W}_\tau^r)$  for all  $q > 0$  and  $k \geq 0$ . The resulting spectral sequence is shown in Figure 7.

As a consequence of these observations condition (10) holds for these indices with  $\nu = 0$ . In addition,  ${}^{\text{II}}E_{0,q}^2(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k) = E_{q,k}^2(X, \mathcal{W}^r)$  holds for all  $q \geq 2$  and  $k \geq 0$  (see Figures 7 and 8). In particular,

$$\begin{array}{ccccccc} {}^{\text{II}}E_{2,0}^1(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k) & 0 & 0 & \cdots & & & \\ {}^{\text{II}}E_{1,0}^1(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k) & 0 & 0 & 0 & & & \\ {}^{\text{II}}E_{0,0}^1(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k) & \xleftarrow{d_1} \bigoplus_{\tau \in N_{\mathcal{W}^r}^1} \text{PH}_k(\mathcal{W}_\tau^r) & \longleftarrow \bigoplus_{\tau \in N_{\mathcal{W}^r}^2} \text{PH}_k(\mathcal{W}_\tau^r) & \longleftarrow \bigoplus_{\tau \in N_{\mathcal{W}^r}^3} \text{PH}_k(\mathcal{W}_\tau^r) & & & \end{array}$$

Figure 7: First page of  ${}^{\text{II}}E_{p,q}^*(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k)$ .

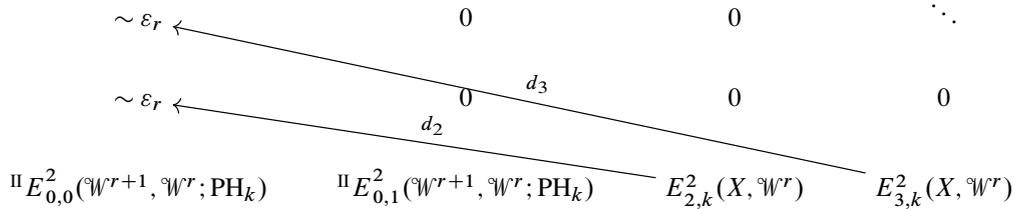


Figure 8: Second page of  ${}^{\text{II}}E_{p,q}^*(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k)$  together with higher differentials.

there is only one possible nontrivial differential for each entry in the bottom row as indicated in Figure 8. Note that our hypothesis  $d_I(E_{p,q}^2(\mathcal{U}_\tau, \mathcal{W}_{|\mathcal{U}_\tau}^{r+1}), 0) \leq \epsilon_r$  applies to the entries in the first column with  $p > 0$  and gives left and right interleavings of the form

$$\check{\mathcal{H}}_q(\mathcal{W}^{r+1} \cap \mathcal{W}^r; \text{PH}_k) \sim_R^{\epsilon_r} {}^{\text{II}}E_{0,q}^\infty(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k) \sim_L^{\epsilon_r} {}^{\text{II}}E_{0,q}^2(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k)$$

for all  $q > 0$  and  $k \geq 0$ . Hence, condition (9) holds with value  $\epsilon_r$ .

Let us look now at the case  $q = 0$ . Here we have  $\check{\mathcal{H}}_0(\mathcal{W}^{r+1} \cap \mathcal{W}^r; \text{PH}_k) = {}^{\text{II}}E_{0,0}^2(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k)$  and consequently (9) holds with value  $\epsilon = 0$ . Next, by hypothesis, for all  $k \geq 0$  we have right and left interleavings

$$M_{0,k}^2(\mathcal{U}_\tau \cap \mathcal{W}^{r+1}) \sim_R^{\nu_r} M_{0,k}^\infty(\mathcal{U}_\tau \cap \mathcal{W}^{r+1}) \sim_L^{\nu_r} \text{PH}_k(\mathcal{U}_\tau),$$

for all  $\tau \in N_{\mathcal{U}}^r$ . Thus by taking the direct sum of these interleavings we obtain

$${}^{\text{II}}E_{0,0}^1(\mathcal{W}^{r+1}, \mathcal{W}^r; \text{PH}_k) \sim_R^{\nu_r} \bigoplus_{\tau \in N_{\mathcal{W}^r}^0} M_{0,k}^\infty(\mathcal{W}_\tau^r \cap \mathcal{W}^{r+1}) \sim_L^{\nu_r} E_{0,k}^1(X, \mathcal{W}^r).$$

and condition (10) also holds for  $q = 0$ . The result now follows from Theorem 7.10.

Notice that we can slightly improve the statement of Theorem 7.10 here: for each term in the bottom row of the spectral sequence in this particular example only one of the two conditions (9) and (10) is nontrivial, and the proof of Theorem 7.10 carries over with  $2 \max(\epsilon_r, \nu_r)$  replacing  $2(\epsilon_r + \nu_r)$ .  $\square$

**Remark 7.13** Notice that for reasonable cases the parameters  $\nu_r$  are bounded above by  $K\epsilon_r$  for some constant  $K > 0$  by a result from [Govc and Skraba 2018]. Nevertheless, we would like to keep  $\nu_r$  and  $\epsilon_r$  separated here, since we hope to compute it from  $M_{p,k}^*(\mathcal{U}_\tau, \mathcal{W}_{|\mathcal{U}_\tau}^{r+1})$  for  $\tau \in N_{\mathcal{U}}^r$  hereby get more accurate estimates. Intuitively, asking for  $\epsilon_r$  and  $\nu_r$  to be small is equivalent to asking for cycle representatives in covers from  $\mathcal{W}^r$  to be approximately contained in covering sets from  $\mathcal{W}^{r+1}$ .

Finally, we would like to compare two separate covers  $\mathcal{U}$  and  $\mathcal{V}$  and have an estimate for the interleaving distance between the associated spectral sequences. The main idea of Proposition 7.12 is to translate this comparison problem into a few local checks that can be run in parallel. We formalize this in the following corollary.

**Corollary 7.14** (stability of covers) Consider two pairs  $(X, \mathcal{U})$  and  $(X, \mathcal{V})$ , where  $X$  is a space and  $\mathcal{U}$  and  $\mathcal{V}$  are covers. Let  $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$  and denote by  $\mathcal{W}_{\mathcal{U}}^r$  and  $\mathcal{W}_{\mathcal{V}}^r$  the respective  $(r, \mathcal{W}, \mathcal{U})$  and  $(r, \mathcal{W}, \mathcal{V})$

interpolations. For every  $0 \leq r \leq N$ , we assume that there exist  $\varepsilon_r, \varepsilon'_r \geq 0$  and  $\nu_r, \nu'_r \geq 0$  such that for all  $\tau \in N_{\mathcal{U}}^r$  and  $\sigma \in N_{\mathcal{V}}^r$

$$\begin{aligned} E_{0,q}^2(\mathcal{U}_\tau, \mathcal{W}_{\mathcal{U}}^{r+1}) &\sim_R^{\nu_r} E_{0,q}^\infty(\mathcal{U}_\tau, \mathcal{W}_{\mathcal{U}}^{r+1}) \sim_L^{\nu_r} \text{PH}_q(\mathcal{U}_\tau), \\ E_{0,q}^2(\mathcal{V}_\sigma, \mathcal{W}_{\mathcal{V}}^{r+1}) &\sim_R^{\nu'_r} E_{0,q}^\infty(\mathcal{V}_\sigma, \mathcal{W}_{\mathcal{V}}^{r+1}) \sim_L^{\nu'_r} \text{PH}_q(\mathcal{V}_\sigma), \end{aligned}$$

for all  $r \geq 0$ , and also

$$d_I(E_{p,q}^2(\mathcal{U}_\tau, \mathcal{W}_{\mathcal{U}}^{r+1}), 0) \leq \varepsilon_r, \quad d_I(E_{p,q}^2(\mathcal{V}_\sigma, \mathcal{W}_{\mathcal{V}}^{r+1}), 0) \leq \varepsilon'_r$$

for all  $p > 0$ , and  $q \geq 0$ . Then we have that

$$d_I^2(E_{p,q}^*(X, \mathcal{U}), E_{p,q}^*(X, \mathcal{V})) \leq R(\mathcal{U}, \mathcal{V})$$

where  $R(\mathcal{U}, \mathcal{V}) = \max(\sum_{r=0}^N 2 \max(\varepsilon_r, \nu_r), \sum_{r=0}^N 2 \max(\varepsilon'_r, \nu'_r))$ .

**Proof** By Lemma 7.1 there are double complex morphisms given by the refinement maps

$$\check{\mathcal{C}}_p(\mathcal{U}, C_q^{\text{cell}}) \xleftarrow{\rho_{p,q}^{\mathcal{U}, \mathcal{W}}} \check{\mathcal{C}}_p(\mathcal{W}, C_q^{\text{cell}}) \xrightarrow{\rho_{p,q}^{\mathcal{V}, \mathcal{W}}} \check{\mathcal{C}}_p(\mathcal{V}, C_q^{\text{cell}}).$$

In turn, these induce 2–morphisms of spectral sequences

$$E_{p,q}^2(X, \mathcal{U}) \xleftarrow{\rho_{p,q}^{\mathcal{U}, \mathcal{W}}} E_{p,q}^2(X, \mathcal{W}) \xrightarrow{\rho_{p,q}^{\mathcal{V}, \mathcal{W}}} E_{p,q}^2(X, \mathcal{V}).$$

Let  $\psi_{p,q}^{\mathcal{U}, \mathcal{W}}$  and  $\psi_{p,q}^{\mathcal{V}, \mathcal{W}}$  be the “inverses” of  $\rho_{p,q}^{\mathcal{U}, \mathcal{W}}$  and  $\rho_{p,q}^{\mathcal{V}, \mathcal{W}}$ , respectively, witnessing the interleavings of the two spectral sequences (see Theorem 7.10 and Proposition 7.12). The result follows from considering the commutative diagram

$$\begin{array}{ccccc} E_{p,q}^2(X, \mathcal{U}) & \xleftarrow{\rho_{p,q}^{\mathcal{U}, \mathcal{W}}} & E_{p,q}^2(X, \mathcal{W}) & \xrightarrow{\rho_{p,q}^{\mathcal{V}, \mathcal{W}}} & E_{p,q}^2(X, \mathcal{V}) \\ \downarrow \Sigma^{R(\mathcal{V}, \mathcal{U})} & \searrow \psi_{p,q}^{\mathcal{W}, \mathcal{U}} & \downarrow \Sigma^{R(\mathcal{V}, \mathcal{U})} & \swarrow \psi_{p,q}^{\mathcal{W}, \mathcal{V}} & \downarrow \Sigma^{R(\mathcal{V}, \mathcal{U})} \\ E_{p,q}^2(X, \mathcal{U})[R(\mathcal{V}, \mathcal{U})] & \xleftarrow{\rho_{p,q}^{\mathcal{U}, \mathcal{W}}} & E_{p,q}^2(X, \mathcal{W})[R(\mathcal{V}, \mathcal{U})] & \xrightarrow{\rho_{p,q}^{\mathcal{V}, \mathcal{W}}} & E_{p,q}^2(X, \mathcal{V})[R(\mathcal{V}, \mathcal{U})] \end{array}$$

where all arrows are 2–morphisms of spectral sequences. □

## 8 Outlook

We expect spectral sequences associated to the geometric realizations of diagrams of CW–complexes to have a natural use in the distributed computation of persistent homology. The first future research direction is to develop further examples and use cases that benefit from the theory developed in this article.

The  $\varepsilon$ –acyclic carriers and equivalences which we introduced here in the context of persistent homology are of course based on acyclic carriers, which are similar to the ones used for example in [Björner 2003, Theorem 6] to prove a generalisation of the nerve theorem. A possible future research direction might

be to ask for conditions on the acyclic carriers with the goal of obtaining similar results as those from [Björner 2003] within the category of regularly filtered diagrams.

The bounds obtained in Section 7 for the interleavings between the second pages of two spectral sequences can certainly be improved; one possible direction is to explore similar examples as those in [Govc and Skraba 2018, Section 9] where the authors found sharp bounds.

In general, we think that spectral sequences deserve a more prominent role in applied algebraic topology and hope that the tools we developed here will motivate further study.

## Appendix Example of acyclic equivalence in RCW-cpx

Consider a filtered regular CW-complex  $X$  which is constant along  $\mathbf{R}$ , except at values 1, 2, 3 and 4, where it changes; see Figure 9. In order to describe  $X$ , we use the notation  $(CD)_1$  for the edge between  $C$  and  $D$ ,  $(FGIJ)_2$  for a two cell whose vertices are  $F, G, I$  and  $J$ , and so on. By regularity of  $X$ , and since we do not define multiple edges between the same pair of vertices,  $X$  is determined by

$$X_1 = \{A, B, C, D, E, F, H\} \cup \{(AH)_1, (BC)_1, (CD)_1, (EF)_1\},$$

$$X_2 = X_1 \cup \{G\} \cup \{(AB)_1, (DE)_1, (FG)_1, (GH)_1\},$$

$$X_3 = X_2 \cup \{I, J\} \cup \{(BI)_1, (CJ)_1, (FJ)_1, (GI)_1, (IJ)_1\} \cup \{(FGIJ)_2\},$$

$$X_4 = X_3 \cup \{K\} \cup \{(AK)_1, (CK)_1, (EK)_1, (GK)_1\} \cup \{(ABCK)_2, (CDEK)_2, (EFGK)_2, (AKGH)_2\},$$

where  $X_0 = \emptyset$ ; this is shown in Figure 9, which illustrates  $X$ . Of course, as  $X$  is a filtered complex, the structure maps of  $X$  are given by inclusions  $X_s \hookrightarrow X_t$  for all  $s < t$  from  $\mathbf{R}$ . Next, we describe the regularly filtered CW-complex  $Y$ , which is constant along  $\mathbf{R}$ , except at values 1, 2, 3 and 4, where it changes; this is also depicted in Figure 9. We define  $Y_*$  by

$$Y_1 = \{\alpha, \beta, \gamma\},$$

$$Y_2 = Y_1 \cup \{(\alpha\beta)_1, (\alpha\gamma)_1, (\beta\gamma)_1\},$$

$$Y_3 = (Y_2 \setminus \{(\alpha\gamma)_1\}) \cup \{\delta, \tau\} \cup \{(\gamma\tau)_1, (\tau\delta)_1, (\alpha\delta)_1, (\beta\delta)_1, (\beta\tau)_1\},$$

$$Y_4 = Y_3 \setminus \{\alpha, (\alpha\beta)_1, (\alpha\delta)_1\},$$

and  $Y_0 = \emptyset$ .

The structure maps of  $Y$  are defined as follows, where we use the overline notation  $\bar{*}$  to denote the closure of some cell:

- $Y(1 \leq 2)$  is an inclusion,
- $Y(2 \leq 3)$  restricts to an inclusion in the subcomplex  $\overline{(\alpha\beta)_1} \cup \overline{(\beta\gamma)_1}$ , while  $\overline{(\alpha\gamma)_1}$  is sent to  $\overline{(\alpha\delta)_1} \cup \overline{(\delta\tau)_1} \cup \overline{(\tau\gamma)_1}$ .
- $Y(3 \leq 4)$  restricts to the identity in  $Y_3 \setminus \{(\alpha\beta)_1, \alpha, (\alpha\delta)_1\}$  while it maps the vertex  $\alpha$  to  $\gamma$ , the edge  $(\alpha\beta)_1$  to  $(\beta\gamma)_1$  and the edge  $(\alpha\delta)_1$  to  $\{(\gamma\tau)_1, \tau, (\tau\delta)_1\}$ .

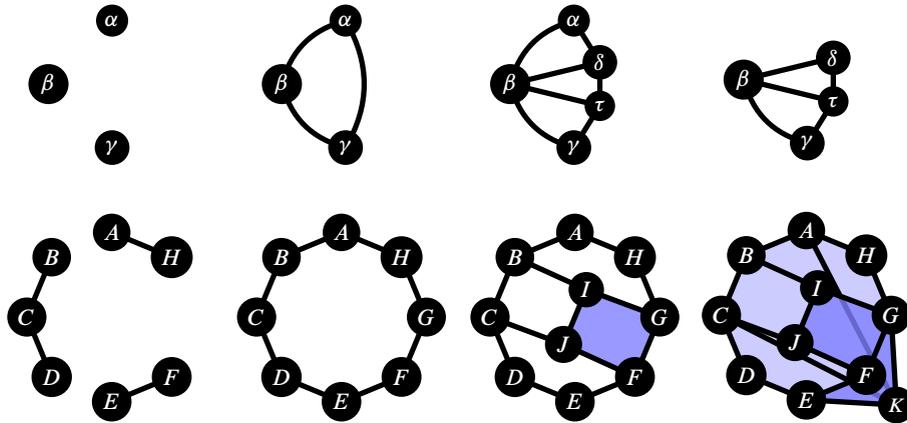


Figure 9: The spaces  $Y_i$  are shown at the top and  $X_i$  are at the bottom for values  $i = 1, 2, 3, 4$ . In filtration value 4, a cone with vertex in  $K$  is attached along the octahedron at the boundary of  $X_3$ ; notice that we used 2-cells which are not 2-simplices.

One might check that  $Y$  is well defined according to Section 2.1. Next, we proceed to define an acyclic carrier  $F: Y \rightrightarrows X$ , which we depict in Figure 10, as follows:

- $F_1(\alpha) = \overline{(AH)}_1, F_1(\beta) = \overline{(BC)}_1 \cup \overline{(CD)}_1, F_1(\gamma) = \overline{(EF)}_1,$
- $F_2((\alpha\beta)_1) = F_1(\alpha) \cup F_1(\beta) \cup \{(AB)_1\}, F_2((\alpha\gamma)_1) = F_1(\alpha) \cup F_1(\gamma) \cup \{(HG)_1, G, (FG)_1\},$   
 $F_2((\beta\gamma)_1) = F_1(\beta) \cup F_1(\gamma) \cup \{(DE)_1\},$
- $F_3(\delta) = G, F_3(\tau) = F, F_3((\alpha\delta)_1) = \overline{(AH)}_1 \cup \overline{(HG)}_1, F_3((\delta\tau)_1) = \overline{(IJFG)}_2, F_3((\gamma\tau)_1) = \overline{(EF)}_1,$   
 $F_3((\beta\delta)_1) = \overline{(BC)}_1 \cup \overline{(CD)}_1 \cup \overline{(BI)}_1 \cup \overline{(IG)}_1, F_3((\beta\tau)_1) = \overline{(BC)}_1 \cup \overline{(CD)}_1 \cup \overline{(CJ)}_1 \cup \overline{(JF)}_1,$
- $F_4(\gamma) = F_4((\beta\gamma)_1) = F_4((\gamma\tau)_1) = \text{St}(K).$

If we did not define a carrier, this is because we assume it is continued from an earlier definition. On the other hand, we define the carrier  $G: X \rightrightarrows Y$  as follows:

- $G_1(A) = G_1(H) = G_1((AH)_1) = \alpha, G_1(E) = G_1(F) = G_1((EF)_1) = \gamma, G_1(B) = G_1(C) =$   
 $G_1(D) = G_1((BC)_1) = G_1((CD)_1) = \beta,$
- $G_2((AB)_1) = \overline{(\alpha\beta)}_1, G_2((DE)_1) = \overline{(\beta\gamma)}_1, G_2((HG)_1) = G_2(G) = G_2((GF)_1) = \overline{(\alpha\gamma)}_1,$
- Define  $A_3 = \{I, J, G, (IJ)_1, (GI)_1, (FJ)_1, (HG)_1, (GF)_1, (FGIJ)_2\}$ ; then for all  $\sigma \in A_3$ , we have  $G_3(\sigma) = \overline{(\alpha\delta)}_1 \cup \overline{(\delta\tau)}_1 \cup \overline{(\tau\gamma)}_1, G_3((BI)_1) = \overline{(\beta\delta)}_1, G_3((CJ)_1) = \overline{(\beta\tau)}_1,$
- for all  $\sigma \in X_4 \setminus \{(BI)_1, (CJ)_1\}, G_4(\sigma) = \overline{(\beta\gamma)}_1 \cup \overline{(\gamma\tau)}_1 \cup \overline{(\tau\delta)}_1.$

We define the shift carriers on  $X$  and  $Y$  by composition, that is,  $I_X^0 = G \circ F$  and  $I_Y^0 = F \circ G$ , which in this particular case lead to well-defined acyclic carriers as one can check; to illustrate this, we write a couple of compositions:

$$G_3 \circ F_3((\beta\tau)_1) = \overline{(\alpha\delta)}_1 \cup \overline{(\delta\tau)}_1 \cup \overline{(\tau\gamma)}_1 \cup \overline{(\beta\tau)}_1,$$

$$F_3 \circ G_3((IJ)_1) = \overline{(AH)}_1 \cup \overline{(HG)}_1 \cup \overline{(IJFG)}_2 \cup \overline{(EF)}_1.$$

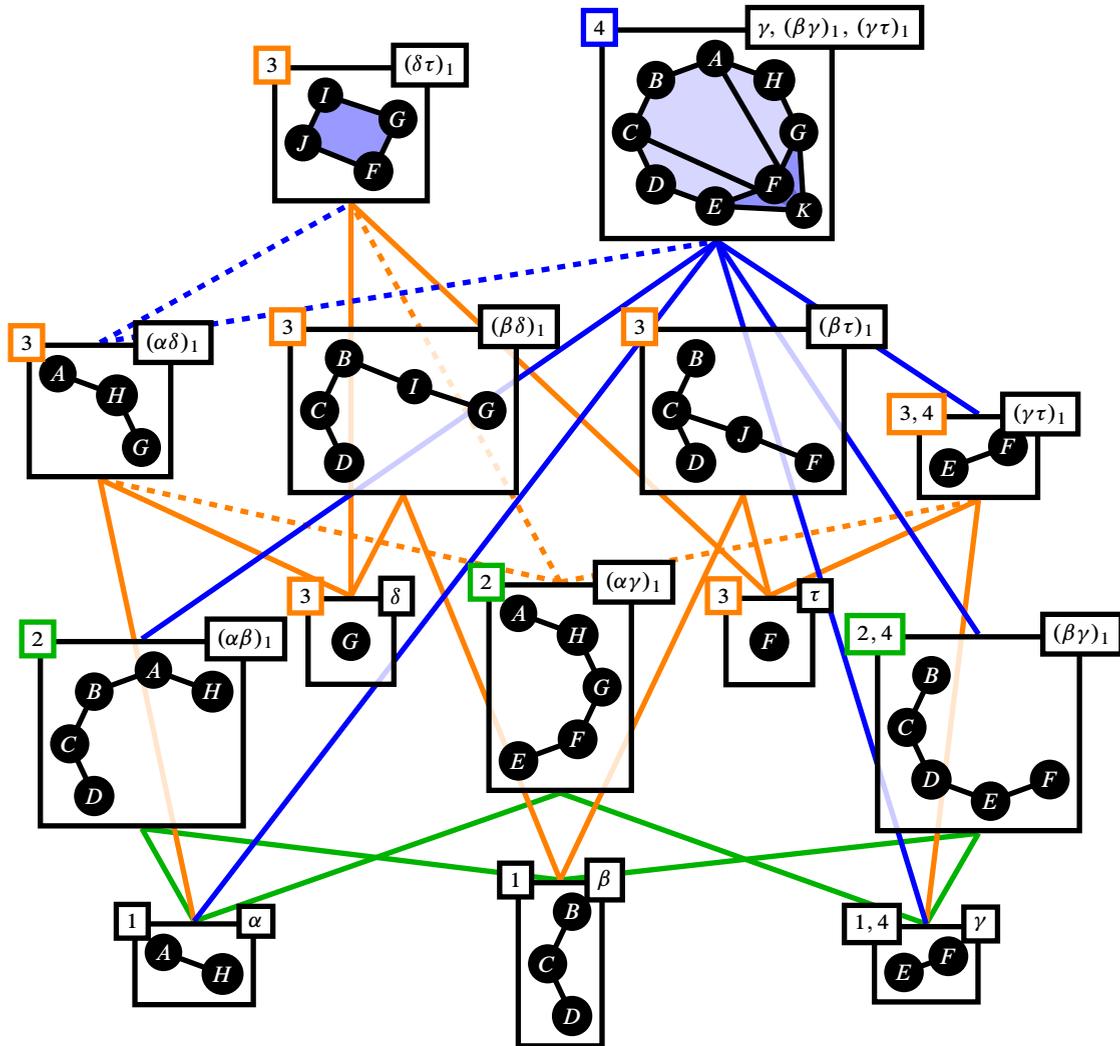


Figure 10: We depict the acyclic carriers from  $F$ . For each acyclic carrier we include its initial filtration value within a square on the top left while we write the cell(s) it corresponds to within a square on the top right; sometimes we write a pair of numbers  $a, b$  to indicate that the carrier applies for the filtration values in  $[a, b)$  and that a new carrier is defined at  $b$ . Solid lines connecting the middle top of a box to the middle bottom of another box indicate that the containment relation must hold, where the carrier in the lower box needs to be embedded into the carrier on the upper box. We use dashed lines for containment relations involving a union of carriers, eg  $F_3((\alpha\delta)_1) \subseteq F_4((\gamma\tau)_1) \cup F_4((\delta\tau)_1)$ .

One can check that the conditions from Definition 4.3 are satisfied and so by Corollary 4.7 we obtain isomorphisms  $\text{PH}_*(X) \cong \text{PH}_*(Y)$ .

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