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A cubical Rips construction

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Given a finitely presented group Q and a compact special cube complex X with nonelementary hyperbolic fundamental group, we produce a nonelementary torsion-free cocompactly cubulated hyperbolic group Γ that surjects onto Q, with kernel isomorphic to a quotient of $G = \pi_1 X$ and such that $\max\{\mathrm{cd}(G), 2\} \geq \mathrm{cd}(\Gamma) \geq \mathrm{cd}(G) - 1$.

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1 Introduction

The Rips exact sequence, first introduced by Rips [1982], is a useful tool for producing examples of groups satisfying combinations of properties that are not obviously compatible. It works by taking as input an arbitrary finitely presented group Q, and producing as output a hyperbolic group Γ that maps onto Q with finitely generated kernel. The "output group" Γ is crafted by adding generators and relations to a presentation of Q in such a way that these relations create enough "noise" in the presentation to ensure hyperbolicity. One can then lift pathological properties of Q to (some subgroup of) Γ. For instance, Rips used his construction to produce the first examples of incoherent hyperbolic groups, hyperbolic groups with unsolvable generalised word problem, hyperbolic groups having finitely generated subgroups whose intersection is not finitely generated, and hyperbolic groups containing infinite ascending chains of r–generated groups.

Our purpose here is to present a new variation of the Rips exact sequence. Our main result is:

Theorem 1.1 (Theorem 4.1) *Let Q be a finitely presented group and G be the fundamental group of a compact special* (in the sense of [Haglund and Wise 2008]) *cube complex X. If G is hyperbolic and nonelementary, then there is a short exact sequence*

$$1 \to N \to \Gamma \to Q \to 1,$$

where:

- (i) *Γ is a hyperbolic cocompactly cubulated group.*
- (ii) *$N \cong G/K$ for some $K < G$.*
- (iii) $\max\{\mathrm{cd}(G), 2\} \geq \mathrm{cd}(\Gamma) \geq \mathrm{cd}(G) - 1$. *In particular, Γ is torsion-free.*

Remark 1.2 By Agol's theorem [2013], the group Γ obtained in Theorem 4.1 is in fact virtually special.

Many variations of Rips' original construction have been produced over the years by a number of authors, including Arzhantseva and Steenbock [2023], Barnard, Brady and Dani [Barnard et al. 2007], Baumslag, Bridson, Miller and Short [Baumslag et al. 2000], Belegradek and Osin [2008], Bridson and Haefliger [1999], Bumagin and Wise [2005], Haglund and Wise [2008], Ollivier and Wise [2007], and Wise [2003; 1998]. Below is a sample of their corollaries:

- There exist non-Hopfian groups with Kazhdan's property (T) [Ollivier and Wise 2007].
- Every countable group embeds in the outer automorphism group of a group with Kazhdan's property (T) [Ollivier and Wise 2007; Belegradek and Osin 2008].
- Every finitely presented group embeds in the outer automorphism group of a finitely generated residually finite group [Wise 2003].
- There exists an incoherent group that is the fundamental group of a compact negatively curved 2–complex [Wise 1998].
- There exist hyperbolic special groups that contain (nonquasiconvex) nonseparable subgroups [Haglund and Wise 2008].
- Properties (T) and FA are not recursively recognisable among hyperbolic groups [Belegradek and Osin 2008].

The groups in Rips' original constructions are cubulable by [Wise 2004], as are the groups in [Haglund and Wise 2008]; on the other extreme, the groups produced in [Ollivier and Wise 2007] and in [Belegradek and Osin 2008] can have property (T), and so will not be cubulable in general; see [Niblo and Reeves 1997].

A notable limitation of all available Rips-type techniques is that the hyperbolic group Γ surjecting onto Q will have cohomological dimension at most equal to 2. This is unsurprising, since, in a precise sense, "most" hyperbolic groups are 2–dimensional; see [Gromov 1993; Ollivier 2005]. Moreover, examples of hyperbolic groups having large cohomological dimension are scarce: Gromov [1987] conjectured that all constructions of high-dimensional hyperbolic groups must utilise number-theoretic techniques, and later on, Bestvina [2000] made this precise by asking whether for every $K > 0$ there is an $N > 0$ such that all hyperbolic groups of (virtual) cohomological dimension $\geq N$ contain an arithmetic lattice of dimension $\geq K$. Both of these questions have been answered in the negative by work of a number of people, including Mosher and Sageev [1997], Januszkiewicz and Świątkowski [2003], and later on Fujiwara and Manning [2010] and Osajda [2013], but flexible constructions are still difficult to come by.

Theorem 4.1 produces cocompactly cubulated hyperbolic groups containing quotients of arbitrary special hyperbolic groups. While our construction particularises to Rips' original result, it can also produce groups with large cohomological dimension. Thus, it serves to exhibit a collection of examples of hyperbolic groups that is new and largely disjoint from that produced by all other Rips-type theorems.

Most versions of Rips' construction, including the original, rely on some form of small cancellation. This is what imposes a bound on the dimension of the groups thus obtained: group presentations are inherently 2–dimensional objects, and one can prove that the presentation complexes associated to (classical and graphical) small cancellation presentations are aspherical.

We rely instead on *cubical* presentations and *cubical* small cancellation theory, which are intrinsically higher-dimensional. Roughly speaking, cubical small cancellation theory considers higher-dimensional analogues of group presentations: pieces in this setting are overlaps between higher-dimensional subcomplexes, and cubical small cancellation conditions measure these overlaps. This viewpoint allows for the use of nonpositively curved cube complexes and their machinery, and has proved fruitful in many contexts, most notably in Agol's proofs [2013; 2008] of the virtual Haken and virtual fibred conjectures, which build on work of Wise [2021] and his collaborators [Bergeron and Wise 2012; Haglund and Wise 2012; Hsu and Wise 2015].

While many groups have convenient cubical presentations, producing these, or proving that they do satisfy useful cubical small cancellation conditions, is difficult in general. Some examples are discussed in [Wise 2021; Arzhantseva and Hagen 2022; Jankiewicz and Wise 2022]. Other than these, we are not aware of instances where explicit examples of nontrivial cubical small cancellation presentations are given, nor of many results producing families of examples with some given list of properties. This note can be viewed as one such construction, and can be used to produce explicit examples that are of a fundamentally different nature to those already available.

Structure In Section 2 we present the necessary background on hyperbolicity, quasiconvexity and cubical small cancellation theory. In Section 3 we state and prove Theorem 3.2, which is the main technical result, and also state and prove some auxiliary lemmas. In Section 4 we give the proof of Theorem 4.1. Finally, in Section 5 we review some standard material on the cohomological dimension of groups, and analyse the cohomological dimension of Γ.

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2 Background

We utilise the following theorem of Arzhantseva [2001]:

Theorem 2.1 *Let G be a nonelementary torsion-free hyperbolic group and H a quasiconvex subgroup of G of infinite index. Then there exist infinitely many $g \in G$ for which the subgroup $\langle H, g\rangle$ is isomorphic to $H * \langle g\rangle$ and is quasiconvex in G.*

Since cyclic subgroups of a hyperbolic group G are necessarily quasiconvex, one can repeatedly apply Theorem 2.1 to produce quasiconvex free subgroups of any finite rank:

Corollary 2.2 *If G is a nonelementary torsion-free hyperbolic group, then for every $n \in \mathbb{N}$ there exists a quasiconvex subgroup $F_n < G$.*

Recall that for a graph B, a subgraph $A \subset B$ is *full* if, whenever vertices $a_1, a_2 \in A$ are joined by an edge e of B, $e \subset A$. In other words, A is the subgraph of B induced by A^0. A map $X \to Y$ between cell complexes is *combinatorial* if it maps cells to cells of the same dimension. An *immersion* is a local injection.

Definition 2.3 A *local isometry* $\varphi\colon Y \to X$ between nonpositively curved cube complexes is a combinatorial map such that, for each $y \in Y^0$ and $x = \varphi(y)$, the induced map $\varphi\colon \operatorname{link}(y) \to \operatorname{link}(x)$ is an injection of a full subgraph.

A more visual way to think about local isometries is the following: an immersion φ is a local isometry if, whenever two edges $\varphi(e)$ and $\varphi(f)$ form the corner of a square in X, we have that e and f already formed the corner of a square in Y.

A key property of local isometries is that they are π_1–injective. It is then natural to ask *which* subgroups of the fundamental group of a nonpositively curved cube complex can be realised by local isometries of compact nonpositively curved cube complexes. In the setting of nonpositively curved cube complexes with hyperbolic fundamental group, one large class of subgroups having this property is that of quasiconvex subgroups. This is proved in [Haglund 2008], and collected in [Wise 2021, Proposition 2.31 and Lemma 2.38], as presented below.

Definition 2.4 A subspace $\widetilde{Y} \subset \widetilde{X}$ is *superconvex* if it is convex and for every bi-infinite geodesic line L, if $L \subset N_r(\widetilde{Y})$ for some $r > 0$ then $L \subset \widetilde{Y}$. A map $Y \to X$ is *superconvex* if the induced map between universal covers $\widetilde{Y} \to \widetilde{X}$ is an embedding onto a superconvex space.

Proposition 2.5 *Let X be a compact nonpositively curved cube complex with $\pi_1 X$ hyperbolic. Let $H < \pi_1 X$ be a quasiconvex subgroup and let $C \subset \widetilde{X}$ be a compact subspace. Then there exists a superconvex H–cocompact subspace $\widetilde{Y} \subset \widetilde{X}$ with $C \subset \widetilde{Y}$.*

Proposition 2.6 *Let X be a compact nonpositively curved cube complex with $\pi_1 X$ hyperbolic. Let $H < \pi_1 X$ be a quasiconvex subgroup. Then there exists a local isometry $Y \to X$ with $\pi_1 Y = H$.*

2.1 Cubical small cancellation theory

Cubical presentations, cubical small cancellation theory, and many related notions were introduced in [Wise 2021]. We recall them below.

Definition 2.7 A *cubical presentation* $\langle X \mid \{Y_i\}\rangle$ consists of a connected nonpositively curved cube complex X together with a collection of local isometries of connected nonpositively curved cube complexes $Y_i \xrightarrow{\varphi_i} X$. In this setting, we shall think of X as a "generator" and of the $Y_i \to X$ as "relators". The fundamental group of a cubical presentation is defined as $\pi_1 X / \langle\langle \{\pi_1 Y_i\} \rangle\rangle$.

Associated to a cubical presentation $\langle X \mid \{\varphi_i : Y_i \to X\}\rangle$ there is a *coned-off space* X^* obtained from $(X \cup \{Y_i \times [0,1]\})/\{(y_i,1) \sim \varphi_i(y_i)\}$ by collapsing each $Y_i \times \{0\}$ to a point. By the Seifert–Van Kampen theorem, the group $\pi_1 X/\langle\langle\{\pi_1 Y_i\}\rangle\rangle$ is isomorphic to $\pi_1 X^*$. Thus, the coned-off space is a presentation complex of sorts for $\langle X \mid \{Y_i\}\rangle$. In practice, when discussing cubical presentations, we often have in mind the coned-off space X^*, rather than the abstract cubical presentation.

Remark 2.8 A group presentation $\langle a_1,\dots,a_s \mid r_1,\dots,r_m\rangle$ can be interpreted cubically by letting X be a bouquet of s circles and letting each Y_i map to the path determined by r_i. On the other extreme, for every nonpositively curved cube complex X there is a "free" cubical presentation $X^* = \langle X \mid \ \rangle$ with fundamental group $\pi_1 X = \pi_1 X^*$.

In the cubical setting, there are two types of pieces: *wall-pieces* and *cone-pieces*. Cone-pieces are very much like pieces in the classical sense — they measure overlaps between relators in the presentation. On the other hand, wall-pieces measure the overlaps between cone-cells and *rectangles* (hyperplane carriers) — wall-pieces are always trivial in the classical case, since the square part of X^* coincides with the 1–skeleton of the presentation complex.

Definition 2.9 (elevations) Let $Y \to X$ be a map and $\widehat{X} \to X$ a covering map. An *elevation* $\widehat{Y} \to \widehat{X}$ is a map such that

(i) the composition $\widehat{Y} \to Y \to X$ equals $\widehat{Y} \to \widehat{X} \to X$, and

(ii) assuming all maps involved are basepoint preserving, $\pi_1 \widehat{Y}$ equals the preimage of $\pi_1 \widehat{X}$ in $\pi_1 Y$.

Notation 2.10 In the entirety of this text, a path $\sigma \to X$ is assumed to be a combinatorial path mapping to the 1–skeleton of X.

Definition 2.11 (pieces) Let $\langle X \mid \{Y_i\}\rangle$ be a cubical presentation. An *abstract contiguous cone-piece* of Y_j in Y_i is an intersection $\widetilde{Y}_j \cap \widetilde{Y}_i$ where either $i \neq j$, or where $i = j$ but $\widetilde{Y}_j \neq \widetilde{Y}_i$. A *cone-piece* of Y_j in Y_i is a path $p \to P$ in an abstract contiguous cone-piece of Y_j in Y_i. An *abstract contiguous wall-piece* of Y_i is an intersection $N(H) \cap \widetilde{Y}_i$ where $N(H)$ is the carrier of a hyperplane H that is disjoint from $\widetilde{Y}_i$. To avoid having to deal with empty pieces, we shall assume that H is dual to an edge with an endpoint on $\widetilde{Y}_i$. A *wall-piece* of Y_i is a path $p \to P$ in an abstract contiguous wall-piece of Y_i.[1]

A *piece* is either a cone-piece or a wall-piece.

Remark 2.12 In Definition 2.11, two lifts of a cone Y are considered identical if they differ by an element of $\mathrm{Stab}_{\pi_1 X}(\widetilde{Y})$. This is in keeping with the conventions of classical small cancellation theory, where overlaps between a relator and any of its cyclic permutations are not regarded as pieces. This hypothesis lets us replace relators by their proper powers to achieve good small cancellation conditions in some cases.

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[1] The "abstract contiguous cone-piece" and "abstract contiguous wall-piece" terminology comes from the fact that it is also a priori necessary to consider "noncontiguous cone-pieces" and "noncontiguous wall-pieces". However, [Wise 2021, Lemma 3.7] shows that one can limit oneself to the analysis of contiguous pieces.

The $C(p)$ and $C'(1/p)$ conditions are now defined as in the classical case (making no distinction between the two types of pieces when counting them). Namely:

Definition 2.13 A cubical presentation X^* satisfies the $C(p)$ *small cancellation condition* if no essential closed path $\sigma \to X^*$ is the concatenation of fewer than p pieces, and the $C'(1/p)$ *condition* if, whenever $\mu \to X^*$ is a piece in an essential closed path $\sigma \to X^*$, we have that $|\mu| < (1/p)|\sigma|$, where $|\mu|$ is the distance between endpoints of $\tilde{\mu} \subset \widetilde{X}$.

As in the classical case, if the fundamental group of X in a cubical presentation $X^* = \langle X \mid \{Y_i\} \rangle$ is hyperbolic, sufficiently good small cancellation conditions lead to hyperbolicity. The following form of [Wise 2021, Theorem 4.7] follows immediately from the fact that a cubical presentation that is $C'(1/\alpha)$ for $\alpha \geq 12$ can be endowed with a nonpositively curved angling rule that satisfies the short innerpaths condition when $\alpha \geq 14$ [Wise 2021, Theorem 3.32 and Lemma 3.70]:

Theorem 2.14 *Let X^* be a cubical presentation satisfying the $C'(1/p)$ small cancellation condition for $p \geq \frac{1}{14}$. Suppose $\pi_1 X$ is hyperbolic and X^* is compact. Then $\pi_1 X^*$ is hyperbolic.*

Definition 2.15 A collection $\{H_1, \dots, H_r\}$ of subgroups of a group G is *malnormal* provided that $H_i^g \cap H_j = 1$ unless $i = j$ and $g \in H_i$.

Compactness, malnormality and superconvexity will together guarantee the existence of a uniform bound on the size of both cone-pieces and wall-pieces. This is the content of [Wise 2021, Lemmas 2.40 and 3.52], which we recall below:

Lemma 2.16 *Let X be a nonpositively curved cube complex with $\pi_1 X$ hyperbolic. For $1 \leq i \leq r$, let $Y_i \to X$ be a local isometry with Y_i compact, and assume the collection $\{\pi_1 Y_1, \dots, \pi_1 Y_r\}$ is malnormal. Then there is a uniform upper bound D on the diameters of intersections $g\widetilde{Y}_i \cap h\widetilde{Y}_j$ between distinct $(\pi_1 X)$–translates of their universal covers in $\widetilde{X}$.*

Lemma 2.17 *Let Y be a superconvex cocompact subcomplex of a* CAT(0) *cube complex X. There exists $D \geq 0$ such that for each $n \geq 0$, if $I_1 \times I_n \to X$ is a combinatorial strip whose base $0 \times I_n$ lies in Y and such that $d((0,0),(0,n)) \geq D$, then $I_1 \times I_n$ lies in Y.*

Recall that a *wallspace* is a set X together with a collection of *walls* $\{W_i\}_{i \in I} = \mathcal{W}$ where $W_i = \{\overleftarrow{W}_i, \overrightarrow{W}_i\}$ and $\overleftarrow{W}_i, \overrightarrow{W}_i \subset X$ for each $i \in I$, and such that

(i) $\overleftarrow{W}_i \cup \overrightarrow{W}_i = X$ and

(ii) $\overleftarrow{W}_i \cap \overrightarrow{W}_i = \varnothing$.

Moreover, $\mathcal{W}$ satisfies a *finiteness property*: for every $p, q \in X$, the number of walls separating p and q, denoted by $\#_{\mathcal{W}}(p,q)$, is finite. The $\overleftarrow{W}_i$ and $\overrightarrow{W}_i$ above are the *half-spaces* of W_i.

Once we have specified a cubical presentation, we will cubulate its fundamental group $\pi_1 X^*$ via Sageev's construction, which produces a CAT(0) cube complex that is dual to a wallspace. We will assume the reader is familiar with this procedure. Good references include [Sageev 1995; Chatterji and Niblo 2005; Bestvina et al. 2014]. Cocompactness of the action on the dual cube complex will readily follow from Proposition 2.18, which is a well-known result of Sageev [1997]. Properness is more delicate, and will follow from Theorem 2.27 once we know that $\pi_1 X^*$ is hyperbolic, since in that case $\pi_1 X^*$ has no infinite torsion subgroups.

Proposition 2.18 *Let G be hyperbolic and $\{H_1, \dots, H_n\}$ be a collection of quasiconvex subgroups. Then the action of G on the dual* CAT(0) *cube complex C is cocompact.*

Before stating Theorem 2.27, we need some definitions:

Definition 2.19 Let $Y \to X$ be a local isometry. $\operatorname{Aut}_X(Y)$ is the group of automorphisms $\psi\colon Y \to Y$ such that the diagram below is commutative:

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$$

If Y is simply connected, then $\operatorname{Aut}_X(Y)$ is equal to $\operatorname{Stab}_{\pi_1 X}(Y)$.

Definition 2.20 A cubical presentation $\langle X \mid Y_i \rangle$ satisfies the $B(6)$ condition if it satisfies:

(i) **Small cancellation** $\langle X \mid Y_i \rangle$ satisfies the $C'(1/\alpha)$ condition for $\alpha \geq 14$.

(ii) **Wallspace cones** Each Y_i is a wallspace where each wall in Y_i is the union $\bigsqcup U_j$ of a collection of disjoint embedded 2–sided hyperplanes in Y_i, and there is an embedding $\bigsqcup N(U_j) \to Y_i$ of the disjoint union of their carriers into Y_i. Each such collection separates Y_i. Each hyperplane in Y_i lies in a unique wall.

(iii) **Hyperplane convexity** If $P \to Y_i$ is a path that starts and ends on vertices on 1–cells dual to a hyperplane U of Y_i and P is the concatenation of at most seven pieces, then P is path homotopic in Y_i to a path $P \to N(U) \to Y_i$.

(iv) **Wall convexity** Let S be a path in Y_i that starts and ends with 1–cells dual to the same wall of Y_i. If S is the concatenation of at most seven pieces, then S is path-homotopic into the carrier of a hyperplane of that wall.

(v) **Equivariance** The wallspace structure on each cone Y is preserved by $\operatorname{Aut}_X(Y)$.

Historical Remark 2.21 In the setting of classical small cancellation theory, the $B(2n)$ condition was defined in [Wise 2004]. Specifically, the "classical" $B(2n)$ condition states that for each 2–cell R in a 2–complex X, and for each path $S \to \partial R$ which is the concatenation of at most n pieces in X, we have $|S| \leq \frac{1}{2}|\partial R|$. The classical $B(2n)$ condition is intermediate to the $C'(1/(2n))$ and $C(2n)$ conditions in the sense that $C'(1/(2n)) \Longrightarrow B(2n) \Longrightarrow C(2n)$. While not a perfect parallel with the notion considered here, this notation is meant to suggest the fact that, in the classical setting, the $B(6)$ condition is sufficient to guarantee the existence of a wallspace structure on X that leads to cocompact cubulability.

The $B(6)$ condition is extremely useful because it facilitates producing a wallspace structure on the coned-off space X^* by starting *only* with a wallspace structure (satisfying some extra conditions) on each of the cones. This is done by defining an equivalence relation $\sim$ on the hyperplanes of $\widetilde{X}^*$ as explained below.

Definition 2.22 Let U and U' be hyperplanes in $\widetilde{X}^*$. Then $U \sim U'$ provided that, for some translate of some cone Y_i in $\widetilde{X}^*$, the intersections $U \cap Y_i$ and $U' \cap Y_i$ lie in the same wall of Y_i. A *wall* of $\widetilde{X}^*$ is a collection of hyperplanes of $\widetilde{X}^*$ corresponding to an equivalence class.

That this equivalence relation does in fact define a wallspace structure on X^* when the $B(6)$ condition is satisfied is the content of [Wise 2021, Section 5.f].

Definition 2.23 A hyperplane U is *m–proximate* to a 0–cube v if there is a path $P = P_1, \dots, P_m$ such that each P_i is either a single edge or a piece, v is the initial vertex of P_1 and U is dual to an edge in P_m. A wall is *m–proximate* to v if it has a hyperplane that is m–proximate to v. A hyperplane is *m–far* from a 0–cube if it is not m'–proximate to it for any $m' \leq m$.

Definition 2.24 A hyperplane U of a cone Y is *piecefully convex* if, for any path $\tau\rho \to Y$ with endpoints on $N(U)$, if τ is a geodesic and ρ is trivial or lies in a piece of Y containing an edge dual to U, then $\tau\rho$ is path-homotopic in Y to a path $\mu \to N(U)$.

The following is noted in [Wise 2021, Remark 5.43]. We write $\widetilde{N}(U) := \widetilde{N(U)}$.

Proposition 2.25 *Let K be the maximal diameter of any piece of Y_i in X^*. Then a hyperplane U of Y_i is piecefully convex provided that its carrier $N(U)$ satisfies $d_{\widetilde{Y}_i}(g\widetilde{N}(U), \widetilde{N}(U)) > K$ for any translate $g\widetilde{N}(U) \neq \widetilde{N}(U) \subset \widetilde{Y}_i$.*

Definition 2.26 (cut by a wall) Let $g \in G$ be an element acting on $\widetilde{X}$. An *axis* $\mathbb{R}_g$ for g is a g–invariant copy of $\mathbb{R}$ in $\widetilde{X}$. An element g is *cut by a wall* W if $g^n W \cap \mathbb{R}_g = \{n\}$ for all $n \in \mathbb{Z}$.

The theorem below is a restatement of [Wise 2021, Theorem 5.44] with [Wise 2021, Corollary 5.45], and the fact that the short innerpaths condition is satisfied when $C'(1/\alpha)$ holds for $\alpha \geq 14$.

Theorem 2.27 *Suppose $X^* = \langle X \mid \{Y_i\}\rangle$ satisfies the following hypotheses*:

(i) *X^* satisfies the $B(6)$ condition.*

(ii) *Each hyperplane U of each cone Y_i is piecefully convex.*

(iii) *Let $k \to Y \in \{Y_i\}$ be a geodesic with endpoints p and q. Let U_1 and U_1' be distinct hyperplanes in the same wall w_1 of Y. Suppose k traverses a 1–cell dual to U_1, and either U_1' is 1–proximate to q*

or k *traverses a* 1*–cell dual to* U_1'. *Then there is a wall* w_2 *in* Y *that separates* p *and* q *but is not* 2*–proximate to* p *or* q.

(iv) *Each infinite-order element of* $\operatorname{Aut}(Y_i)$ *is cut by a wall.*

Then the action of $\pi_1 X^*$ *on* C *has torsion stabilisers.*

3 Cubical noise

In the classical setting, there are two essentially distinct strategies for producing group presentations satisfying good small cancellation conditions: taking large enough powers of the relators, and adding "noise" to the presentation by multiplying each relator by a sufficiently long, suitably chosen word. In the cubical setting, taking powers of relators translates to taking finite-degree covers of the cycles that represent the relators, and this method generalises to taking finite-degree covers of cube complexes with more complicated fundamental groups. This is the line of enquiry that has been most explored, and for which there exist useful theorems producing cubical small cancellation. This will, however, not be suitable for our applications, because once a cubical presentation $\langle X \mid \{\widehat{Y}_i \to X\}\rangle$ has been obtained by taking covers, any modifications of the cones (other than taking further covers) will dramatically affect the size of pieces, and possibly invalidate whatever small cancellation conclusions had been attained. Thus, we instead prove a cubical small cancellation theorem that builds on the idea of adding noise to a presentation. The procedure we describe will be more stable, in the sense that slightly perturbing the choice of cones will not affect the small cancellation conclusions.

Remark 3.1 We state and prove Theorem 3.2 in more generality than is needed for later applications. In practice the reader may take Y to be a bouquet of finitely many circles, as this is all that is required for the proof of Theorem 4.1. It is also worth noting that while the statement of Theorem 3.2(iii) requires that X and Y be special, the proof only uses that hyperplanes are embedded and 2–sided.

Theorem 3.2 *Let* X *and* Y *be compact nonpositively curved cube complexes with hyperbolic fundamental groups and let* $\mathcal{H}$ *be the set of hyperplanes of* X. *Let* $\{H_1, \dots, H_r\}$ *be a malnormal collection of free nonabelian quasiconvex subgroups of* $\pi_1 X$, *and suppose that* $H_i \cap \operatorname{Stab}(\widetilde{U})$ *is trivial or equal to* H_i *for all* $U \in \mathcal{H}$. *Let* $\gamma_1 \to X \vee Y, \dots, \gamma_r \to X \vee Y$ *be closed essential paths based at the wedge point and let* $y_1, \dots, y_r$ *be the words in* $\pi_1 X * \pi_1 Y$ *represented by the* γ_i. *Then for each* $\alpha \geq 1$ *there are cyclic subgroups* $\langle w_i\rangle \subset H_i * \langle y_i\rangle$ *such that* $w_i = w_i' y_i$ *where* $w_i' \in H_i$ *for each* $i \in \{1, \dots, r\}$ *and*:

(i) *The group* $\pi_1 X * \pi_1 Y / \langle\langle w_1, \dots, w_r\rangle\rangle$ *has a cubical presentation satisfying the* $C'(1/\alpha)$ *condition.*

(ii) *If* $\alpha \geq 14$, *then the group* $\pi_1 X * \pi_1 Y / \langle\langle w_1, \dots, w_r\rangle\rangle$ *is hyperbolic.*

(iii) *If* X *and* Y *are special, there is an* $\alpha_0 \geq 14$ *such that if* $\alpha \geq \alpha_0$, *then* $\pi_1 X * \pi_1 Y / \langle\langle w_1, \dots, w_r\rangle\rangle$ *acts properly and cocompactly on a* $\operatorname{CAT}(0)$ *cube complex.*

Remark 3.3 The reader might wish to compare Theorem 3.2 with [Wise 2021, Corollary 5.48], which is the analogous result for finite-degree coverings, and whose proof informs the proof below.

Proof **Obtaining small cancellation** Since $H_1, \dots, H_r$ are quasiconvex subgroups of $\pi_1 X * \pi_1 Y$ and $\pi_1 X * \pi_1 Y$ is hyperbolic, by Proposition 2.6 there are based local isometries $C_1 \to X \vee Y, \dots, C_r \to X \vee Y$ of superconvex subcomplexes with $\pi_1 C_i \cong H_i$ for each $i \in \{1, \dots, r\}$. So there is a cubical presentation $(X \vee Y)^* = \langle X \vee Y \mid \{C_i \to X \vee Y\}_{i=1}^r \rangle$ with fundamental group $\pi_1 X * \pi_1 Y / \langle\langle \pi_1 C_1, \dots, \pi_1 C_r \rangle\rangle$. By Lemmas 2.16 and 2.17, malnormality of $\{H_1, \dots, H_r\}$ and superconvexity of the $\{C_1, \dots, C_r\}$ ensures that there is a uniform upper bound K on the diameter of pieces.

By hyperbolicity, any cyclic subgroup of $\pi_1 X * \pi_1 Y$ is quasiconvex. So for any choice of cyclic subgroups $\langle w_i \rangle < \pi_1 X * \pi_1 Y$ with $i \in \{1, \dots, r\}$ there are local isometries $W_i \to X \vee Y$ with $\pi_1 W_i \cong \langle w_i \rangle$.

Let $\alpha \geq 1$, and choose each $\sigma_i \to X \vee Y$ so that $\sigma_i = \sigma_i' \gamma_i$ where

(i) σ_i' is a based closed path in $C_i \subset X$,

(ii) σ_i' is not a proper power and does not contain subpaths of length $\geq K\alpha$ that are proper powers,

(iii) $\sigma_i' \gamma_i$ does not have any backtracks, and

(iv) the W_i corresponding to $\langle w_i \rangle = \langle \sigma_i \rangle$ has diameter $\|W_i\| \geq K\alpha^2$.

For instance, one can choose σ_i' to be of the form $\sigma_i' = \lambda_1 \lambda_2 \lambda_1 \lambda_2^2 \cdots \lambda_1 \lambda_2^{K\alpha}$ where λ_1 and λ_2 are paths representing distinct generators of the corresponding $H_i < \pi_1 X$. Without loss of generality, we can assume that λ_1, λ_2 and $\gamma_1, \dots, \gamma_r$ have minimal length in their homotopy classes, and therefore that none of the λ_i or γ_i have any backtracks, so any backtracks in $\sigma_i' \gamma_i$ arise from cancellation between σ_i' and γ_i. If any cancellation happens, we can rechoose λ_1 and λ_2 to eliminate it (for instance, by shortening λ_1 and λ_2).

Pieces in each C_i have size bounded by K, and each $W_i \to X \vee Y$ factors through the corresponding C_i, so the size of pieces between different cone-cells or between cone-cells and rectangles is bounded by K; the size of pieces between a cone-cell and itself is bounded by $K\alpha - 1$ and $\|W_i\| \geq (K\alpha)! + K\alpha \geq K\alpha^2$, so $\langle X \vee Y \mid \{W_i \to X \vee Y\}_{i=1}^r \rangle$ satisfies the $C'(1/\alpha)$ condition.

Obtaining hyperbolicity As explained above, the $\langle w_i \rangle$ can be chosen so that

$$(X \vee Y)^* = \langle X \vee Y \mid \{W_i \to X \vee Y\}_{i=1}^r \rangle$$

satisfies the $C'(1/\alpha)$ condition for $\alpha \geq 14$. Since $\pi_1 X * \pi_1 Y$ is hyperbolic and $(X \vee Y)^*$ is compact, Theorem 2.14 then implies that $(X \vee Y)^*$ is hyperbolic.

Obtaining cocompact cubulability Define a wallspace structure on $(X \vee Y)^*$ as follows. By subdividing $X \vee Y$, we may assume that each W_i has an even number of hyperplanes cutting the generator of $\langle w_i \rangle$. The specialness hypothesis ensures that all hyperplanes of $X \vee Y$ and of each W_i are embedded and 2–sided. Moreover, since each W_i has cyclic fundamental group and $H_i \cap \operatorname{Stab}(\widetilde{U})$ is trivial or equal to $\operatorname{Stab}(\widetilde{U})$ for each $\widetilde{U}$, all the hyperplanes of each W_i are contractible or have the homotopy type of a circle representing the generator of $\pi_1 W_i$. Hence, we can define a wallspace structure on each of the cones by defining a wall to be either a single hyperplane if the hyperplane does not cut the generator of the corresponding $\langle w_i \rangle$, or by defining a wall to be an equivalence class consisting of two antipodal

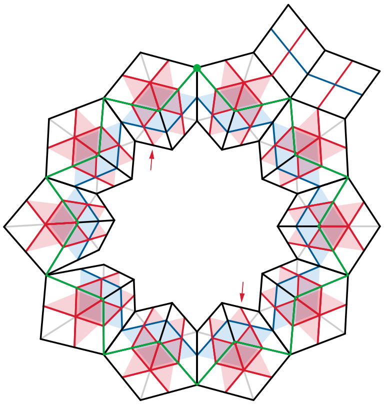

Figure 1: A potential cone-cell and its hyperplanes. The generator of its fundamental group is drawn in green, the hyperplanes that cross it are drawn in red, and the hyperplanes that do not cross it are drawn in blue. A pair of antipodal hyperplanes is indicated.

hyperplanes cutting the generator. Concretely, if the generator of $\langle w_i \rangle$ is a cycle $\sigma \to X \vee Y$ of length $2n$, then letting $\sigma = e_1 \cdots e_{2n}$, hyperplanes U and U' are in the same equivalence class if and only if U is dual to e_j and U' is dual to $e_{j+n} \pmod n$ for some $j \in \{1, \dots, 2n\}$. These choices are exemplified in Figure 1.

We now check condition (i) of Theorem 2.27, which will allow us to extend the wallspace structure on the cones to a wallspace structure for $(X \vee Y)^*$ using the equivalence relation in Definition 2.22.

Choose the $\langle w_i \rangle$ so that $(X \vee Y)^* = \langle X \vee Y \mid \{W_i \to X \vee Y\}_{i=1}^r \rangle$ satisfies the $C'(1/\alpha)$ condition for $\alpha \geq 14$. With the choice of walls described above, each cone is a wallspace satisfying condition (ii) of Definition 2.20. The $C'\bigl(\frac{1}{14}\bigr)$ condition is also sufficient to ensure that condition (iii) is met. Indeed, the only way for a path with endpoints on the carrier of a hyperplane U to not be homotopic into the carrier of the hyperplane is if the path is homotopic into a power of the generator of W_i, and such a path would have to traverse at least 14 pieces. Moreover, condition (iv) is met by rechoosing the cyclic subgroups so that the cubical presentation satisfies $C'\bigl(\frac{1}{16}\bigr)$. To wit, since pairs (U, U') of hyperplanes lying on the same wall W are antipodal and $(X \vee Y)^*$ satisfies $C'(1/\alpha)$, the number of pieces in a path $\sigma \to C_i$ with endpoints on distinct hyperplanes of W is at least $\frac{1}{2}\alpha$, and so choosing $\alpha \geq 16$ ensures that such a path traverses at least 8 pieces. The choice of wallspace on each cone also ensures that condition (v) is met: any automorphism of $X \vee Y$ sends a wall not cutting a generator to a wall not cutting a generator, and a wall cutting a generator to a wall cutting a generator.

Thus, condition (i) of Theorem 2.27 is satisfied. Since each wall arises from a quasiconvex subgroup, Proposition 2.18 ensures cocompactness of the action on the dual cube complex. To ensure properness of the action, we check the rest of the conditions of Theorem 2.27.

Similar modifications to X^* will ensure that conditions (ii) and (iii) of Theorem 2.27 are met. For condition (ii), by Proposition 2.25, it suffices to ensure that $d_{\widetilde{W}_i}(g\widetilde{N}(U), \widetilde{N}(U)) > K$ for any translate $g\widetilde{N}(U) \neq \widetilde{N}(U) \subset \widetilde{W}_i$. Since each piece of W_i contains at least one edge, this can be guaranteed by rechoosing the w_i so that X^* satisfies the $C'(1/K')$ condition, where $K' = \max\{K, 16\}$. Condition (iii) also follows, because any two hyperplanes in the same wall are at least 8–far, so there is a hyperplane V crossing the generator of $\langle w_i \rangle$ that is 2–far from both U and U', and one can ensure that the antipodal hyperplane V' is also 2–far from both U and U' by rechoosing X^* so that it satisfies the $C'(1/K'')$ condition, where $K'' = \max\{2K, 16\}$.

Finally, the choice of walls implies that condition (iv) also holds: since $\pi_1(W_i)$ is cyclic for each $i \in I$, every element $g \in \operatorname{Aut}(W_i)$ has an axis, which is cut by a wall of X^* crossing the generator of $\pi_1(W_i)$. □

Definition 3.4 (height) The *height* of $H \leq G$ is the maximal $n \in \mathbb{N}$ such that there exist distinct cosets $g_1 H, \dots, g_n H$ for which $H^{g_1} \cap \cdots \cap H^{g_n}$ is infinite.

In [Gitik et al. 1998] it was proven that:

Theorem 3.5 *Quasiconvex subgroups of hyperbolic groups have finite height.*

Definition 3.6 The *commensurator* $C_G(H)$ of a subgroup H of G is the set

$$C_G(H) = \{g \in G \mid [H : H^g \cap H] < \infty \text{ and } [H^g : H^g \cap H] < \infty\}.$$

Remark 3.7 If G is hyperbolic and the subgroup H is infinite and quasiconvex, then $[C_G(H) : H] < \infty$ by [Kapovich and Short 1996]. In particular, $C_G(H)$ is also a quasiconvex subgroup of G, and if G is torsion-free and H is free and nonabelian then so is $C_G(H)$.

The following result will be used in the proof of Lemma 3.9:

Lemma 3.8 [Wise 2021, Lemma 8.6] *Let G be hyperbolic and torsion-free and let $H_1, \dots, H_r$ be a collection of quasiconvex subgroups of G. Let $K_1, \dots, K_s$ be representatives of the finitely many distinct conjugacy classes of subgroups consisting of intersections of collections of distinct conjugates of $H_1, \dots, H_r$ in G that are maximal with respect to having infinite intersection. Then $\{C_G(K_1), \dots, C_G(K_s)\}$ is a malnormal collection of subgroups of G.*

The ensuing lemma surely exists in some form in the literature, but we include a proof for completeness:

Lemma 3.9 *Let G be a nonelementary torsion-free hyperbolic group. For every $k \in \mathbb{N}$, G contains a malnormal collection $\{H_1, \dots, H_k\}$ of infinite-index quasiconvex nonabelian free subgroups.*

Before proving Lemma 3.9, we make a few observations about malnormal subgroups of free groups. Recall that a subgroup $H < G$ is *isolated* if $g \in H$ whenever $g^n \in H$ for $g \in G$; a subgroup $H < G$ is *malnormal on generators* if $a_i^g \notin H$ for any $i \in \{1, \dots, n\}$, for any generating set $\{a_1, \dots, a_n\}$ for H, $g \in G$ and $g \notin H$.

Lemma 3.10 [Fine et al. 2002, Lemma 1] *Let F be a free group and $H \subset F$ a 2–generator subgroup. Then H is malnormal if and only if H is isolated and malnormal on generators.*

Claim 3.11 *Let F be a finite-rank nonabelian free group and let $\{h_1, \dots, h_k\}$ be a finite collection of nontrivial elements of F. Then there is a subgroup $J < F$ for which $\{J, \langle h_i \rangle\}$ is a malnormal collection for each $i \in \{1, \dots, k\}$.*

Proof Assume that a basis is given for F, and, abusing notation, also write $h_1, \dots, h_k$ to denote reduced words on the basis representing the finite set of elements $h_1, \dots, h_k$. We may also assume that $h_i \neq h_j^m$ whenever $m \in \mathbb{Z} - \{0\}$ and $i \neq j$. Let $J = \langle a, b \rangle$ where $a = h_1 \beta_1 \cdots h_k \beta_k$ and $b = \beta_1' h_1 \cdots \beta_k' h_k$, and where each β_i and β_i' is a reduced word on the basis satisfying that, for each $i \in \{1, \dots, k\}$:

(i) $\beta_i \neq \beta_j^m$, $\beta_i' \neq (\beta_j')^m$ and $\beta_i \neq (\beta_j')^m$ whenever $i \neq j$ and $m \in \mathbb{Z} - \{0\}$.

(ii) No β_i or β_i' is a product of h_i or their inverses.

(iii) For each β_i, its first letter is not equal to the last letter of h_i, and its last letter is not equal to the first letter of h_{i+1} (modulo k). Similarly, for each β_i', its last letter is not equal to the first letter of h_i, and its first letter is not equal to the last letter of h_{i+1} (modulo k). Moreover, the last letter of β_k is not equal to the inverse of the last letter of h_k, the first letter of β_1' is not equal to the inverse of the first letter of h_1, and the last letter of β_k is not equal to the first letter of β_1'.

As there is no cancellation, J is a rank-2 free group. By Lemma 3.10, to prove that J is malnormal it suffices to show that J is isolated and malnormal on generators. The choice of the β_i and β_i' implies in particular that a and b are not proper powers, and this implies in turn that J is isolated, since F is free. We now show that J is malnormal on generators. Consider a conjugate $a^g = (h_1 \beta_1 \cdots h_k \beta_k)^g$ where $g \notin J$ (the case of b^g is analogous). Since $g \notin J$, g cannot be written as a nontrivial product of powers of a and b and their inverses. If g cannot be written as a subword of a product of a's and b's and their inverses, then a^g cannot be an element of J as there will be no cancellation. The choice of a and b implies that no cyclic permutation of a, b, their product or their proper powers lies in J, so no conjugate of a by a subword of a product of a's and b's and their inverses lies in J.

Finally, consider a conjugate h_i^g of h_i. If $\langle h_i^g \rangle \cap J$ is nontrivial, then since F is free, it follows that g must be a subword of some $j \in J$, and even in this case h_i^g can only be a (nontrivial) cyclic permutation of a, b, their product or their proper powers, but no such cyclic permutation is an element of J, so J intersects all conjugates of the h_i trivially. □

Proof of Lemma 3.9 It suffices to show that G contains a malnormal quasiconvex free subgroup J of arbitrarily high rank, for then if $J = \langle a_1, \dots, a_k, b_1, \dots, b_k \rangle$, the collection $\{H_1, \dots, H_k\}$ where $H_i = \langle a_i, b_i \rangle$ is also malnormal and quasiconvex. For this, it suffices to show that G contains a malnormal quasiconvex free group J of *some* rank ≥ 2. Indeed, for any $n \in \mathbb{Z}$, J contains infinitely many subgroups of rank n, all of which are malnormal and quasiconvex.

By Theorem 2.1, G contains a free nonabelian quasiconvex subgroup J_0; we may assume further that J_0 has infinite index in G. Let $\mathcal{J}_0$ be the lattice of infinite intersections of conjugates of J_0. This lattice is finite by Theorem 3.5. If $\mathcal{J}_0$ contains a nonabelian free group J_1 then we replace J_0 with J_1, and we can repeat this process a finite number of times until we either reach a maximal intersection of conjugates of some J_i that is itself free nonabelian or until all subgroups in the lattice $\mathcal{J}_i$ are cyclic. In the former case, the commensurator $C_G(J_i)$ is malnormal and quasiconvex by Lemma 3.8. In the latter case, by Claim 3.11, J_i contains a free nonabelian subgroup J that forms a malnormal collection with each of these cyclic subgroups, and hence J is malnormal in G. □

Lemma 3.9 can be improved to control intersections with quasiconvex subgroups:

Corollary 3.12 *Let G be a nonelementary torsion-free hyperbolic group and let $\{S_1, \dots, S_l\}$ be a collection of quasiconvex subgroups of G. Then the collection $\{H_1, \dots, H_k\}$ from the conclusion of Lemma 3.9 can be chosen so that $H_i \cap S_j$ is either trivial or equal to H_i for each $i \in \{1, \dots, k\}$ and each $j \in \{1, \dots, l\}$.*

Remark 3.13 In particular, if G is the fundamental group of a compact nonpositively curved cube complex X, and $\mathcal{H}$ is the set of hyperplanes of X, then $\{H_1, \dots, H_k\}$ can be chosen so that $H_i \cap \operatorname{Stab}(\widetilde{U})$ is either trivial or equal to H_i for each $U \in \mathcal{H}$ and each $i \in \{1, \dots, k\}$. Indeed, since X is compact, it has finitely many hyperplanes, and hence finitely many hyperplane stabilisers. Each hyperplane stabiliser is quasi-isometrically embedded and $\pi_1 X$ is hyperbolic, so each hyperplane stabiliser is quasiconvex.

Proof of Corollary 3.12 It suffices to prove the result for a single malnormal quasiconvex nonabelian subgroup $J \leq G$. Indeed, as explained in the first paragraph of the proof of Lemma 3.9, the subgroups in the malnormal collection $\{H_1, \dots, H_k\}$ are produced as subgroups of a single nonabelian free subgroup J, so if we ensure that $J \cap S_j$ is either trivial or equal to J for each $j \in \{1, \dots, l\}$, then this will also be the case for $\{H_1, \dots, H_k\}$.

We now proceed by induction on l. Assume that $l = 1$ and consider the intersection $J \cap S_1$, where J is as provided in Lemma 3.9. If this intersection is trivial, then there is nothing to show, so suppose that $K_1 := J \cap S_1$ is nontrivial. So K_1 is either cyclic or free of rank ≥ 2. If K_1 is cyclic, say generated by k_1, then as J is free, by Claim 3.11 there exists a $J' \leq J$ such that $\{J', \langle k_1 \rangle\}$ is malnormal, so in particular $J' \cap S_1$ is trivial; if K_1 is free of rank ≥ 2, then since K_1 is quasiconvex, Lemma 3.9 implies that there exists a quasiconvex, nonabelian free subgroup $J'' \leq K_1$ that is malnormal in G, so that $J'' \cap S_1 = J''$.

Now assume that the result holds for $m = l - 1$ and let $\{S_1, \dots, S_l\}$ be a collection of quasiconvex free nonabelian subgroups. Then by the induction hypothesis and Lemma 3.9, there is a quasiconvex nonabelian subgroup $J < G$ such that $J \cap S_i$ is trivial or equal to J for each $i \in \{1, \dots, m\}$. As before, if $K_l := J \cap S_l = \{1\}$ then there is nothing to show, if K_l is cyclic then by Claim 3.11 there exists a $J' \leq J$ such that $J' \cap S_l$ is trivial, and since $J' \leq J$ then it is still the case that $J' \cap S_i$ is trivial or equal to J' for each $i \in \{1, \dots, m\}$. Finally, if K_l is nonabelian then since it is quasiconvex, Lemma 3.9 produces a new J'' inside K_l for which $J'' \cap S_l = J''$. Since $J'' < J$, then $J'' \cap S_i$ is trivial or equal to J'' for each $i \in \{1, \dots, m\}$, and the result follows. □

4 Main theorem

Theorem 4.1 *Let Q be a finitely presented group and G be the fundamental group of a compact special cube complex X. If G is hyperbolic and nonelementary, then there is a short exact sequence*

$$1 \to N \to \Gamma \to Q \to 1 \tag{1}$$

where:

(i) Γ *is a hyperbolic cocompactly cubulated group.*
(ii) $N \cong G/K$ *for some* $K < G$.
(iii) $\max\{\mathrm{cd}(G), 2\} \geq \mathrm{cd}(\Gamma) \geq \mathrm{cd}(G) - 1$. *In particular*, Γ *is torsion-free.*

In what follows, we prove all parts of Theorem 4.1 except for (iii), which we explain in the next section.

Remark 4.2 Cocompact cubulability of Γ hinges on the specialness of X as this is the hypothesis that is used in Theorem 3.2. However, as noted in Remark 3.1, in reality all that is needed is for the hyperplanes of X to be embedded and 2–sided.

Proof Choose a finite presentation $\langle a_1, \dots, a_s \mid r_1, \dots, r_k \rangle$ for Q, let B be a bouquet of s circles $a_1, \dots, a_s$, and let X be a compact nonpositively curved cube complex with $\pi_1 X = G = \langle x_1, \dots, x_m \rangle$. By Lemma 3.9 and Remark 3.13, there is a malnormal collection $\{H_l\} \cup \{H'_{ij}\} \cup \{H''_{ij}\}$ of quasiconvex free subgroups of rank ≥ 2 of $\pi_1(X \vee B)$, so we can apply Theorem 3.2 to X and B, where the y_i are given by

(i) $y_l = r_l$ for each $1 \leq l \leq k$,
(ii) $y'_{ij} = a_i x_j a_i^{-1}$ for each $1 \leq i \leq s$ and $1 \leq j \leq m$,
(iii) $y''_{ij} = a_i^{-1} x_j a_i$ for each $1 \leq i \leq s$ and $1 \leq j \leq m$.

Hence, there are words $w_1, \dots, w_l, w'_{1,1}, \dots, w'_{ij}, w''_{1,1}, \dots, w''_{ij} \in \pi_1 X$ for which the group

$$\Gamma = G * \pi_1 B / \langle\langle \{r_l w_l\}_{l=1}^{k}, \{a_i x_j a_i^{-1} w'_{ij}\}_{i=1,j=1}^{s,m}, \{a_i^{-1} x_j a_i w''_{ij}\}_{i=1,j=1}^{s,m} \rangle\rangle$$

is hyperbolic and acts properly and cocompactly on a CAT(0) cube complex.

There is a homomorphism $\Gamma \xrightarrow{\phi} \pi_1 B$ that sends every generator of $\pi_1 X$ to 1. Hence the relations $\{r_l w_l = 1\}_l$ map exactly to the relations $\{r_l = 1\}_l$ in $\pi_1 B$, and so we see that the image of the homomorphism is Q.

The relations $\{a_i x_j a_i^{-1} w'_{ij} = 1\}_{i \in S, j \in M}$, and $\{a_i^{-1} x_j a_i w''_{ij} = 1\}_{i \in S, j \in M}$ ensure that $\langle x_1, \dots, x_m \rangle$ is a normal subgroup of Γ, so $N = \mathrm{Ker}\, \phi = \langle x_1, \dots, x_m \rangle$ and $N \cong \pi_1 X / K$ for some subgroup $K < G$. □

5 Cohomological dimension

We briefly recall some standard facts about the cohomological dimension of groups. We refer the reader to [Brown 1982] for more details and proofs.

A *resolution* for a module M over a ring R is a long exact sequence of R–modules

$$\cdots \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M \to 0.$$

A resolution is *finite* if only finitely many of the M_i are nonzero. The *length* of a finite resolution is the maximum integer n such that M_n is nonzero. A resolution is *projective* if each M_i is a projective module.

Definition 5.1 The *cohomological dimension* $\mathrm{cd}(G)$ of a group G is the length of the shortest projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$–module.

There is a natural topological analogue of cohomological dimension:

Definition 5.2 The *geometric dimension* $\mathrm{gd}(G)$ of a group G is the least dimension of a classifying space for G.

Remark 5.3 The cellular chain complex of a classifying space for a group G yields a free (in particular, projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, the length of which is equal to the dimension of the classifying space. This implies immediately that $\mathrm{cd}(G) \leq \mathrm{gd}(G)$ for any group G. In particular, if G is free, then $\mathrm{cd}(G) = 1$.

Remark 5.4 The universal covers of nonpositively curved cube complexes are CAT(0) spaces, and hence are contractible, so every nonpositively curved cube complex X is a classifying space for its fundamental group. Therefore, if $G = \pi_1 X$ for a compact nonpositively curved cube complex X, then the cohomological dimension of G is bounded above by the dimension of X.

Proposition 5.5 *The following hold for any group* G:

(i) *If* $G' < G$ *then* $\mathrm{cd}(G') \leq \mathrm{cd}(G)$, *and equality holds provided that* $\mathrm{cd}(G) \leq \infty$ *and* $[G : G'] < \infty$.

(ii) *If* $1 \to G' \to G \to G'' \to 1$ *is exact*, *then* $\mathrm{cd}(G) \leq \mathrm{cd}(G') + \mathrm{cd}(G'')$.

(iii) *If* $G = G_1 * G_2$, *then* $\mathrm{cd}(G) = \max\{\mathrm{cd}(G_1), \mathrm{cd}(G_2)\}$.

The following result is a consequence of Corollary 5.11, stated below.

Proposition 5.6 *Let* G, Q, B *and* Γ *be as in Theorem 4.1 and let* $q\colon G * \pi_1 B \to \Gamma$ *be the natural quotient. Then* $\mathrm{Ker}(q)$ *is free.*

Proposition 5.7 Γ *can be chosen so that* $\mathrm{cd}(\Gamma) \geq \mathrm{cd}(G) - 1$.

Proof There is a short exact sequence

$$1 \to \mathrm{Ker}(q) \to G * \pi_1 B \to \Gamma \to 1.$$

Since $\mathrm{Ker}(q)$ is a free group and $\mathrm{cd}(G * \pi_1 B) = \max\{\mathrm{cd}(G), 1\} = \mathrm{cd}(G)$, we have that $\mathrm{cd}(G) \leq \mathrm{cd}(\Gamma) + \mathrm{cd}(\mathrm{Ker}(q)) = \mathrm{cd}(\Gamma) + 1$. □

The torsion-freeness of Γ will follow from having a finite upper bound on its cohomological dimension:

Proposition 5.8 Γ *can be chosen so that* $\max\{\mathrm{cd}(G), 2\} \geq \mathrm{cd}(\Gamma)$.

Before proving Proposition 5.8, we state some auxiliary results:

Definition 5.9 (the Cohen–Lyndon property) Let G be a group, $\{H_i\}_{i\in I}$ a family of subgroups and $N_i \lhd H_i$ for each $i \in I$. The triple $(G, \{H_i\}, \{N_i\})$ has the *Cohen–Lyndon property* if for each $i \in I$ there exists a left transversal T_i of $H_i \langle\!\langle \bigcup_{i\in I} N_i \rangle\!\rangle$ in G such that $\langle\!\langle \bigcup_{i\in I} N_i \rangle\!\rangle$ is the free product of the subgroups N_i^t for $t \in T_i$, so

$$\left\langle\!\!\left\langle \bigcup_{i\in I} N_i \right\rangle\!\!\right\rangle = \mathop{\huge *}_{i\in I, t\in T_i} N_i^t.$$

The Cohen–Lyndon property was first defined and studied in [Cohen and Lyndon 1963], where it was proven to hold for triples $(F, C, \langle c\rangle)$ where F is free, C is a maximal cyclic subgroup of F and $c \in C - \{1\}$. This was later generalised in [Edjvet and Howie 1987] to the setting of free products of locally indicable groups. Most remarkably, it was recently proven in [Sun 2020] that triples $(G, \{H_i\}, \{N_i\})$ have the Cohen–Lyndon property when the H_i are "hyperbolically embedded" subgroups of G and the N_i avoids a finite set of "bad" elements depending only on the H_i. We will not define hyperbolically embedded subgroups here, and instead state only the particular case of the theorem that is required for our applications:

Theorem 5.10 [Sun 2020] *Let G be hyperbolic, $\{H_i\}$ be malnormal and quasiconvex subgroups of G, and $N_i \lhd H_i$ for each i. Then there exists a finite set of elements $\{g_1, \dots, g_n\} \in \bigcup_i H_i - \{1\}$ such that the triple $(G, \{H_i\}, \{N_i\})$ has the Cohen–Lyndon property provided that $N_i \cap \{g_1, \dots, g_n\} = \varnothing$ for all i.*

To simplify notation, let $\{H_l\} \cup \{H'_{ij}\} \cup \{H''_{ij}\} := \boldsymbol{H}$ and write $H_\iota \in \boldsymbol{H}$. Let $S \subset \boldsymbol{H}$ be a finite set. It is clear from the constructions in Theorems 4.1 and 3.2 (say, by applying Claim 3.11 to S before producing the cyclic subgroups in the proof of Theorem 3.2) that the elements $\{r_l w_l\}_{l=1}^k$, $\{a_i x_j a_i^{-1} w'_{ij}\}_{i=1,j=1}^{s,m}$ and $\{a_i^{-1} x_j a_i w''_{ij}\}_{i=1,j=1}^{s,m}$ can be chosen so that for each

$$c_\iota \in \{r_l w_l\}_{l=1}^k \cup \{a_i x_j a_i^{-1} w'_{ij}\}_{i=1,j=1}^{s,m} \cup \{a_i^{-1} x_j a_i w''_{ij}\}_{i=1,j=1}^{s,m},$$

where $c_\iota \in H_\iota$, the intersection $\langle\!\langle c_\iota \rangle\!\rangle_{H_\iota} \cap S$ is empty, and hence:

Corollary 5.11 Γ *can be chosen so that the triple $(G * \pi_1 B, \{H_\iota\}, \{\langle\!\langle c_\iota \rangle\!\rangle_{H_\iota}\})$ has the Cohen–Lyndon property.*

The following is proven in [Petrosyan and Sun 2024]:

Proposition 5.12 *If $(G, \{H_i\}, \{N_i\})$ has the Cohen–Lyndon property, then*

$$\mathrm{cd}(G/\langle\!\langle \cup_{i\in I} N_i \rangle\!\rangle) \leq \max\bigl\{\mathrm{cd}(G), \sup\{\mathrm{cd}(H_i) + 1\}, \sup\{\mathrm{cd}(H_i/N_i)\}\bigr\}.$$

Definition 5.13 A *graphical* small cancellation presentation is a 1–dimensional cubical small cancellation presentation, namely, a cubical presentation $X^* = \langle X \mid \{Y_i\}\rangle$ where X is a graph and $Y_i \to X$ are graph immersions.

In the particular setting of graphical presentations, it is well known that the coned-off space X^* is aspherical. Concretely, the following theorem holds. A proof is given in [Gruber 2015], though we caution the reader that the language utilised there differs from ours.

Theorem 5.14 *Let $X^* = \langle X \mid \{Y_i\}\rangle$ be a $C(6)$–graphical small cancellation presentation. Then X^* is aspherical.*

Remark 5.15 As in Corollary 5.11, we simplify the notation so that $\{H_l\} \cup \{H'_{ij}\} \cup \{H''_{ij}\} := \boldsymbol{H}$, for $c_\iota \in H_\iota$, equals the corresponding $r_l w_l$, $a_i x_j a_i^{-1} w'_{ij}$ or $a_i^{-1} x_j a_i w''_{ij}$, and $C_\iota \to X \vee B$ and $W_\iota \to X \vee B$ are the corresponding local isometries defined in the proof of Theorem 3.2 having $\pi_1(C_\iota) = H_\iota$ and $\pi_1(W_\iota) = \langle c_\iota\rangle$ for each ι.

Proof of Proposition 5.8 By the Cohen–Lyndon property for $(G * \pi_1 B, \{H_\iota\}, \{\langle\langle c_\iota\rangle\rangle_{H_\iota}\})$, we have that $\mathrm{cd}\big(G * \pi_1 B / \langle\langle \bigcup_\iota c_\iota \rangle\rangle\big) \le \max\big\{\mathrm{cd}(G) * \pi_1 B, \sup\{\mathrm{cd}(H_\iota)+1\}, \sup\{\mathrm{cd}(H_\iota/\langle\langle c_\iota\rangle\rangle_{H_\iota}\})\big\}$, and since each H_ι is free, $\mathrm{cd}(H_\iota) = 1$ for all $H_\iota \in \boldsymbol{H}$. We claim that each of the quotients $H_\iota/\langle\langle c_\iota\rangle\rangle_{H_\iota}$ has a $C(6)$–graphical small cancellation presentation, and so $\mathrm{cd}(H_\iota/\langle\langle c_\iota\rangle\rangle_{H_\iota}) \le 2$ for all $H_\iota \in \boldsymbol{H}$.

To see this, consider the cubical presentation $\langle X \vee B \mid \{W_\iota \to X \vee B\}_\iota\rangle$ constructed in the proof of Theorem 4.1. As explained in the proof of Theorem 3.2, each H_ι is carried by a local isometry $C_\iota \to X \vee B$. Since each C_ι is itself a nonpositively curved cube complex and $\pi_1 C_\iota \cong H_\iota$ is free, each C_ι is homotopy equivalent to a graph $\overline{C}_\iota$ in $C_\iota^{(1)}$. Similarly, each W_ι is homotopy equivalent to a cycle $\overline{W}_\iota$ in W_ι, and we can further assume that each $\overline{W}_\iota$ lies in $\overline{C}_\iota$. The intersections between pieces of each $\overline{W}_\iota$ are contained in the corresponding intersections between pieces of W_ι, and the length of each $\overline{W}_\iota$ is bounded below by the diameter of the corresponding W_ι, so each $\overline{W}_\iota$ has at least as many pieces as the corresponding W_ι. Since the W_ι are chosen to satisfy at least $C'\big(\frac{1}{16}\big)$, and in particular $C'\big(\frac{1}{16}\big) \implies C(6)$, each $\langle \overline{C}_\iota \mid \overline{W}_\iota \to \overline{C}_\iota\rangle$ satisfies the $C(6)$ condition, and the proof is complete. □

This finishes the proof of Theorem 4.1.

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