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Luigi Caputi
Carlo Collari
Sabino Di Trani

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Multipath cohomology of directed graphs

LUIGI CAPUTI
CARLO COLLARI
SABINO DI TRANI

This work is part of a series of papers focusing on multipath cohomology of directed graphs. Multipath cohomology is defined as the (poset) homology of the path poset — ie the poset of disjoint simple paths in a graph — with respect to a certain functor. This construction is essentially equivalent, albeit more computable, to taking the higher limits of said functor on (a certain modification of) the path poset. We investigate the functorial properties of multipath cohomology. We provide a number of sample computations, show that multipath cohomology does not vanish on trees, and show that, when evaluated at the coherently oriented polygon, it recovers Hochschild homology. Finally, we use the same techniques employed to study the functoriality to investigate the connection with the chromatic homology of (undirected) graphs introduced by L Helme-Guizon and Y Rong.

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1 Introduction

Directed graphs are ubiquitous objects in mathematics and science in general. Due to their simplicity and flexibility, (directed) graphs find application in a wide range of fields: physics, computer science, complex systems, engineering, biology, neuroscience, medicine, robotics, etc, encompassing and embracing most scientific domains. Extracting topological and combinatorial information from directed graphs is, therefore, not only interesting, but also particularly important from different perspectives.

Cohomological invariants of directed graphs have been extensively studied in the last decades, with prominent work in combinatorial topology — see [Wachs 2003; Kozlov 2008; Jonsson 2008] — and have deep connections with other areas of mathematics — see [Jonsson 2008, Chapter 1] for an overview. One of the common strategies is to construct suitable simplicial complexes — eg matching complexes, independence complexes, complexes of directed trees, etc [Jonsson 2008] — associated to a (directed) graph, and then to analyse the associated homology groups. In this work we follow a similar approach; we first represent a graph using a suitable poset, called the *path poset* [Turner and Wagner 2012], and then apply a cohomology theory of posets known as *poset homology* — see eg [Chandler 2019] — to get cohomological invariants of directed graphs. We call these invariants *multipath cohomology groups*, as they are constructed using the combinatorial information of multipaths (ie the elements of the path poset) in a directed graph.

Roughly speaking, poset homology associates to a poset P and a functor $\mathcal{F}$ on P a graded module. As, in our case, the poset P (and the functor $\mathcal{F}$) depend on a directed graph $\mathtt{G}$, we obtain a (cohomology) theory for directed graphs. This idea is not novel; for instance Helme-Guizon and Rong [2005], using a different poset, defined chromatic homology. On a different note, one might use a different homology theory of posets, for instance, the classical functor homology groups, an approach pursued by Turner and Wagner [2012]. Comparisons between multipath cohomology, Helme-Guizon and Rong's chromatic homology, Turner–Wagner's homology, and other theories obtained using different (co)homologies for posets will be carried out in Sections 6 and 7.

Our main goal is to investigate structural and combinatorial properties of digraphs through the lenses of multipath cohomology. In this work, we are interested in the definition and general properties of multipath cohomology, such as functoriality, and in its relationship with similar theories. The investigation of combinatorial properties, and the computations of multipath cohomology groups for various families of directed graphs are the subject of [Caputi et al. 2023; 2024b] and forthcoming papers. We believe that the framework developed here can be helpful both in answering theoretical questions, as well as solving problems in the applied setting — see Section 8.

1.1 Other approaches

The development and investigation of homology theories for directed graphs (shortly, *digraphs*) is a very active research field. To define such theories, a first approach comes from the observation that a (directed) graph can naturally be seen as a topological space (a 1–dimensional CW–complex) on which ordinary homology can be applied. However, in this case, the homology groups in degree $i > 1$ would vanish. To sidestep this issue, there are various ways that can be pursued. For instance, one can define (higher-dimensional) simplicial complexes from a graph — eg by constructing the (directed) flag complex (also known as clique complex) [Ivashchenko 1994; Chen et al. 2001; Aharoni et al. 2005; Govc et al. 2021], or the matching complexes, independence complexes — see [Jonsson 2008] — and hence, compute their ordinary (simplicial) homology. Alternatively, one can construct the so-called path complex (see eg [Grigoryan et al. 2016]), whose homology is called path homology. In a third approach, one can associate to a digraph the so-called path algebra. Then, homology groups of digraphs can be defined as the Hochschild homology groups of the path algebras — see [Happel 1989; Caputi and Riihimäki 2024]. A pitfall of most of these homology theories is that they vanish when evaluated on trees; this hints to the fact that they might be discarding an important part of the combinatorics of the input digraph, including their directionality information.

Turner and Wagner [2012] move in a different direction. Given a graph $\mathtt{G}$ they consider its path poset $P(\mathtt{G})$, that is, the collection of all the unions of disjoint simple paths in $\mathtt{G}$, partially ordered by inclusion. Since posets can be seen as categories in a natural way, one can apply homology with functor coefficients [Gabriel and Zisman 1967] to the path poset and obtain topological invariants of the directed graph. On the one hand, this homology is nontrivial. In particular, for a given algebra A, the Turner–Wagner homology of the coherently oriented polygon with n edges is isomorphic, up to degree n, to the Hochschild homology groups of A — see [Turner and Wagner 2012, Theorem 1]. On the other hand, the homology of a category with coefficients in a functor is generally difficult to compute. Computations can be done with relative ease if one considers the constant functor, ie the functor which associates to each element of the path poset the base ring. However, in this case, the result is trivial since the path poset has a minimum: the empty multipath.

Close to Turner–Wagner homology sits the so-called chromatic homology, introduced in [Helme-Guizon and Rong 2005]. Chromatic homology is a homology theory for unoriented graphs, inspired by Khovanov homology [2000], and with the remarkable property that it categorifies the chromatic polynomial. Przytycki [2010] has shown that a version of the chromatic homology (further extended to incorporate the orientation in the case of linear and polygonal graphs) can recover (a truncation of) the Hochschild homology. This fact was later used to prove [Turner and Wagner 2012, Theorem 1] by showing that for coherently oriented polygons their homology is in fact isomorphic to Przytycki's version of chromatic homology.

In this work, inspired by the approaches of Turner and Wagner and of Helme-Guizon and Rong, we follow a certain modification of Turner and Wagner's functorial framework; instead of directly applying functors to the path poset $P(\mathtt{G})$ of $\mathtt{G}$, we use poset homology [Chandler 2019], a suitable adaptation of Helme-Guizon and Rong's construction to this context. Alternatively, instead of the naive poset homology, one can use the so-called cellular cohomology, introduced in [Everitt and Turner 2015]. Cellular cohomology extends poset homology to arbitrary finite (ranked) posets; this yields, after some minor modifications on the path poset, an essentially equivalent theory — see Section 6. Nonetheless, the advantage of poset homology over cellular homology is its computability, which is essential in view of possible applications — see Question 91 and the computations developed in [Caputi et al. 2023].

1.2 Statement of results

We construct a cochain complex $(C^*_{\mathcal{F}}(P), d^*)$; this depends on the datum of a poset P associated to a graph $\mathtt{G}$, and a covariant functor $\mathcal{F}$, defined on (the category associated to) P with values in an additive category $\boldsymbol{A}$. Roughly, $C^n_{\mathcal{F}}(P)$ is given by a directed sum of $\mathcal{F}(x)$ for all $x \in P$ of "level" n. The differential d^* is induced by the functor $\mathcal{F}$ applied to the covering relations in P. In Section 4.1, we specialise this construction to obtain multipath cohomology. First, we fix a ring R, an R–algebra A, and an (A, A)–bimodule M. Then the role of the poset P is played by the path-poset $P(\mathtt{G})$, and the part of the functor $\mathcal{F}$ is taken by $\mathcal{F}_{A,M}$. The latter assigns a tensor product of copies of M and A to each $\mathtt{H} \in P(\mathtt{G})$. We finally denote by $(C^*_\mu(\mathtt{G}; A, M), d^*)$ the cochain complex $(C^*_{\mathcal{F}_{A,M}}(P(\mathtt{G})), d^*)$. The main definition of multipath cohomology as homology of this complex is given in Definition 46.

Unless otherwise specified, for the rest of the Introduction we set $M = A$. In particular, we drop M from the notation of multipath cohomology, writing $\mathrm{H}^*_\mu(\mathtt{G}; A)$ instead of $\mathrm{H}^*_\mu(\mathtt{G}; A, M)$. Some computations of multipath cohomology, for $A = R = \mathbb{K}$ a field, are collected in Table 1.

A key property of cohomology theories is that they are functorial. One of the main results of this paper is that functoriality for multipath cohomology holds once we fix the number of vertices in our graphs.

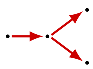
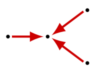
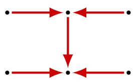
Digraph $\mathtt{G}$	$\mathrm{H}^0_\mu(\mathtt{G};\mathbb{K})$	$\mathrm{H}^1_\mu(\mathtt{G};\mathbb{K})$	$\mathrm{H}^2_\mu(\mathtt{G};\mathbb{K})$	$\mathrm{H}^i_\mu(\mathtt{G};\mathbb{K}),\ i>2$
•	$\mathbb{K}$	0	0	0
	0	0	0	0
	0	$\mathbb{K}$	0	0
	0	0	0	0
	0	$\mathbb{K}^2$	0	0
	0	0	$\mathbb{K}^2$	0

Table 1: Some digraphs and their respective multipath cohomologies.

Theorem 1 *Let* R–**Alg** *be the category of* R*–algebras, let* **Digraph**(n) *be the category of digraphs with* n *vertices, and let* R–**Mod**$^{\mathrm{gr}}$ *be the category of graded* R*–modules. Then, multipath cohomology*

$$\mathrm{H}_\mu(-;-)\colon \mathbf{Digraph}^{\mathrm{op}}(n)\times R\text{–}\mathbf{Alg}\to R\text{–}\mathbf{Mod}^{\mathrm{gr}}$$

is a bifunctor for all n*.*

We need to restrict to the category **Digraph**(n) for a purely technical reason; intuitively, the issue is due to the tensor products involved in the definition of multipath cohomology. More formally, the functor $\mathcal{F}_{A,A}$ is not a coefficients system — see Remark 59. This technical issue is solved when either we fix the number of vertices, or we take $A=R$. In the latter case, we have a stronger result (see Theorem 65).

Hochschild homology is a homology theory for pairs (A,M) with A an algebra and M an (A,A)–bimodule [Loday 1992]. It has been proven in [Przytycki 2010] that (a suitable modification of) the chromatic homology of the coherently oriented n–polygon recovers the Hochschild homology up to degree n. It turns out that multipath cohomology shares the same property — see Corollary 82.

A consequence of Corollary 82 and Theorem 1 is that, once one fixes a digraph $\mathtt{G}$, the functor $\mathrm{H}_\mu(\mathtt{G};-)$, which is a functor between R–**Alg** and R–**Mod**$^{\mathrm{gr}}$, can be seen as a homology theory of algebras. From this viewpoint, we can rephrase Corollary 82 by stating that the family of homologies for algebras $\{\mathrm{H}_\mu(\mathtt{G};-)\}_{\mathtt{G}}$ generalises Hochschild homology (compare also with [Turner and Wagner 2012]). In light of these results, one can expect chromatic homology and multipath cohomology to be, in some sense, related. Despite being defined on different categories (of undirected and directed graphs, respectively) we obtain a short exact sequence featuring the two theories — see Proposition 84.

A great advantage of multipath cohomology is that it is amenable to computations. We postpone the (combinatorial) analysis, as well as the description of an algorithm to calculate the multipath cohomology of certain graphs, to [Caputi et al. 2023; 2024b] and forthcoming papers. In the present work, we limit our computations to a restricted number of cases (with coefficients in a field $\mathbb{K} = R = A = M$); see Section 4.2 and Table 1. Such computations hint to the fact that multipath cohomology might be sensible to some combinatorial properties of graphs. We observe that multipath cohomology does not vanish, nor is it concentrated in degree 0, in the case of trees.

Another consequence of the computations collected in Table 1 is that chromatic, multipath, and Turner–Wagner (co)homologies are not isomorphic. We conclude with the following observation. Although not isomorphic "on the nose", Turner–Wagner and multipath cohomology are related, if $A = R$, by the universal coefficients short exact sequence. On the one hand, Turner–Wagner homology computes the higher colimits of the functor $\mathcal{F}_{R,R}$. On the other hand, multipath cohomology computes the associated higher limits; then, the short exact sequence gives the relation between the two, with a correcting Ext term. We refrain from giving a more detailed account of this case here, inviting the interested reader to Section 6.

Conventions

Typewriter font, eg $\mathtt{G}$, $\mathtt{H}$, etc, will be used to denote graphs (both directed and unoriented). Calligraphic font, eg $\mathcal{F}$, $\mathcal{G}$, etc, will be used to denote functors. Bold capital letters, eg $\boldsymbol{A}$, $\boldsymbol{C}$, etc will be used to denote categories. Depending on the context, $\boldsymbol{A}$ will denote an *Abelian*, or more generally, an *additive* category, and, for a given poset P, we will denote with the same letter in roman and bold, that is $\boldsymbol{P}$, its associated category. All rings are assumed to be unital and commutative, and algebras are assumed to be associative. Unless otherwise stated, R will denote a base ring, A will denote an R–algebra, M will denote an (A, A)–bimodule, and all tensor products $\otimes$ are assumed to be over the base ring. Given a (co)chain complex C^*, we will denote by $C^*[i]$ the shifted complex C^{*+i}.

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2 Basic notions

In this section we introduce and provide the basic notions and conventions about graphs and posets that will be used throughout the paper. In particular, in Section 2.2, we introduce the path poset, one of the main ingredients in the construction of multipath homology (Section 4.1).

2.1 Digraphs and posets

We will only consider *finite* graphs and digraphs. For a set V, let $\wp(V)$ denote its power set. We recall the definitions of various types of graphs — see, for instance, [West 1996].

An *unoriented graph* G is a pair (V, E), where V is the set of *vertices* and E (the *edges*) are unordered pairs of distinct vertices. A graph is a *directed graph*, or a *digraph*, when the edges are ordered pairs, ie $E \subseteq V \times V \setminus \{(v, v) \mid v \in V\}$. Finally, a graph is *oriented* if it is a digraph and for any vertices v, w at most one among (v, w) and (w, v) are in E.

The main object of interest in this work is digraphs. Unless otherwise stated, we will refer to digraphs simply as *graphs*. When dealing with (un)oriented graphs the adjective "(*un*)*oriented*" will be explicitly stated. Two vertices v and w in a digraph can share at most two edges: (v, w) and (w, v). There are no multiple edges between two vertices in oriented and unoriented graphs.

For the sake of simplicity, we restrict to the case of digraphs. Everything in this paper can be carried out verbatim in the more general case of directed multigraphs.

By definition, an edge of a digraph is an ordered set of two distinct vertices, say $e = (v, w)$. The vertex v is called the *source* of e, while the vertex w is called the *target* of e. The source and target of an edge e will be denoted by $s(e)$ and $t(e)$, respectively. If a vertex v is either a source or a target of an edge e, we will say that e is incident to v.

Later we shall also deal with more than one graph at the time. In such cases, the sets of vertices and edges of a digraph $\mathtt{G}$ will be denoted by $V(\mathtt{G})$ and $E(\mathtt{G})$, respectively. A morphism of digraphs from $\mathtt{G}_1$ to $\mathtt{G}_2$ is a function $\phi\colon V(\mathtt{G}_1) \to V(\mathtt{G}_2)$ such that

$$e = (v, w) \in E(\mathtt{G}_1) \implies \phi(e) := (\phi(v), \phi(w)) \in E(\mathtt{G}_2).$$

A morphism of digraphs sends directed edges to directed edges; in particular, it does not allow collapsing — that is, $(v, w) \in E(\mathtt{G}_1) \Rightarrow \phi(v) \neq \phi(w)$. A morphism of digraphs is called *regular* if it is injective as a function; digraphs and regular morphisms of digraphs form a category that we denote by **Digraph**. If $\mathtt{G}_1$ and $\mathtt{G}_2$ are isomorphic in **Digraph**, we write $\mathtt{G}_1 \cong \mathtt{G}_2$.

A subgraph $\mathtt{H}$ of a graph $\mathtt{G}$ is a graph such that $V(\mathtt{H}) \subseteq V(\mathtt{G})$ and $E(\mathtt{H}) \subseteq E(\mathtt{G})$, and in such case we write $\mathtt{H} \leq \mathtt{G}$. If $\mathtt{H} \leq \mathtt{G}$ and $\mathtt{H} \neq \mathtt{G}$ we say that $\mathtt{H}$ is a proper subgraph of $\mathtt{G}$, and write $\mathtt{H} < \mathtt{G}$. If $\mathtt{H} \leq \mathtt{G}$ and $V(\mathtt{H}) = V(\mathtt{G})$ we say that $\mathtt{H}$ is a *spanning subgraph* of $\mathtt{G}$. Given a proper spanning subgraph $\mathtt{H} \leq \mathtt{G}$, we can always find an edge e in $E(\mathtt{G}) \setminus E(\mathtt{H})$. We use the following notation:

Notation 2 The spanning subgraph of $\mathtt{G}$ obtained from $\mathtt{H}$ by adding an edge e is denoted by $\mathtt{H} \cup e$.

Let S be a set. Recall that a (strict) partial order on S is a transitive binary relation $\lhd$ such that, for each $x, y \in S$, at most one of the following is true: $x \lhd y$, $y \lhd x$, or $x = y$. As a matter of notation, we will write $x \unlhd y$ in place of "$x \lhd y$ or $x = y$".

Given a partial order, there is an associated covering relation given by $x \,\tilde{\lhd}\, y$ if, and only if, $x \lhd y$ and there is no z such that $x \lhd z,\ z \lhd y$. A partial order can be also seen as the transitive closure[1] of its associated covering relation. Moreover, the associated covering relation is the smallest relation whose transitive closure is the given partial order. A partially ordered set, or simply *poset*, is a pair $(S, \lhd)$ consisting of a set S and a partial order $\lhd$ on S. A morphism of posets $f\colon (S, \lhd) \to (S', \lhd')$ is a monotonic map of sets, that is, a function $f\colon S \to S'$ such that $x \lhd y$ implies $f(x) \lhd' f(y)$. Posets and morphisms of posets form a category, which will be denoted by **Poset**.

Remark 3 Each poset $P = (S, \lhd)$ can be seen as a (small) category $\boldsymbol{P}$ in a straightforward manner; the set of objects of $\boldsymbol{P}$ is the set S, and the set of morphisms between x and y contains a single element if, and only if, $x \lhd y$ or $x = y$, and it is empty otherwise.

Let $P = (S, \lhd)$ be a poset. An element $m \in P$ is a maximal element if there are no elements of P strictly greater than m, ie if $m \unlhd s$ with $s \in P$, then $m = s$. A maximum of P is an element $M \in S$ which is greater than any other element, ie $s \unlhd M$ for all $s \in S$.

The following two facts are standard:

(M.i) If $P = (S, \lhd)$ is a finite poset — that is, S is finite — then for each $s \in S$ there exists a maximal element $m \in S$ such that $s \unlhd m$.

(M.ii) A poset has a unique maximal element if, and only if, said element is a maximum.

Minimal elements and minima are defined analogously by exchanging the role of s with m and M in the definitions of maximum and maximal elements, respectively. Moreover, the obvious translations of facts (M.i) and (M.ii) hold.

A poset is called a *Boolean poset* if it is isomorphic to the power set $\wp(S)$ — ie the set of all subsets — of a finite set S with partial order $\subset$ given by inclusion. The standard Boolean poset (of size 2^n) is by definition the poset $\mathbb{B}(n) = (\wp(\{0, 1, 2, \dots, n-1\}), \subset)$.

Example 4 Let $\mathtt{G}$ be a (possibly unoriented) graph. Among others, we can specifically consider two posets: the *poset of subgraphs* $\mathrm{SG}(\mathtt{G})$ and the *poset of spanning subgraphs* $\mathrm{SSG}(\mathtt{G})$. The elements of these posets are all the subgraphs and all the spanning subgraphs of $\mathtt{G}$, respectively. In both cases the order relation $<$ is given by the property of being a proper subgraph. The covering relation $\prec$ of $<$ in $\mathrm{SSG}(\mathtt{G})$ is easily checked to be

$$\mathtt{H} \prec \mathtt{H}' \iff \text{there exists } e \in E(\mathtt{H}') \setminus E(\mathtt{H}) \text{ such that } \mathtt{H}' = \mathtt{H} \cup e.$$

Equivalently, $\mathtt{H} \prec \mathtt{H}'$ if, and only if, $E(\mathtt{H}') \setminus E(\mathtt{H}) = \{e\}$ and $E(\mathtt{H}) \setminus E(\mathtt{H}') = \varnothing$. The covering relation on $\mathrm{SG}(\mathtt{G})$ is slightly different from $\prec$; we need to consider, in addition to the case above, also the case where $E(\mathtt{H}') = E(\mathtt{H})$ and $V(\mathtt{H}') = V(\mathtt{H}) \cup \{v\}$ for a certain $v \notin V(\mathtt{H})$.

[1]The transitive closure of a relation $R \subset S \times S$ is a relation R' such that $(s, s') \in R'$ if, and only if, either $(s, s') \in R$, or there exists $s'' \in S$ such that $(s, s''), (s'', s') \in R'$.

Note that $\mathrm{SSG}(\mathtt{G})$ is a Boolean poset; in fact, we have natural isomorphisms of posets

$$(\mathrm{SSG}(\mathtt{G}), <) \cong (\wp(E(\mathtt{G})), \subset)$$

given by $\mathtt{H} \mapsto E(\mathtt{H})$. On the contrary, the poset $\mathrm{SG}(\mathtt{G})$ is generally not isomorphic to a Boolean poset; a counterexample is given by the 1–step graph — see Figure 1. However, $\mathrm{SG}(\mathtt{G})$ is a subposet of a Boolean poset, namely the poset $(\wp(V(\mathtt{G}) \cup E(\mathtt{G})), \subset)$.

Definition 5 Given a poset $(S, \lhd)$ a *subposet* is a subset $S' \subseteq S$ with the order $\lhd_{|S'\times S'}$ induced by $\lhd$. A subposet $(S', \lhd_{|S'\times S'})$ is called *downward closed* (resp *upward closed*) with respect to $(S, \lhd)$ if for every $h \in S$ such that $h \lhd h'$ (resp $h' \lhd h$) for some $h' \in S'$, we have $h \in S'$.

The poset of spanning subgraphs $\mathrm{SSG}(\mathtt{G})$ is a subposet of the subgraphs poset $\mathrm{SG}(\mathtt{G})$, but it is easily checked *not* to be downward closed. Nonetheless, it is upward closed.

Furthermore, observe the complement of an upward closed subposet is downward closed, and vice versa.

We conclude the subsection with the definition of two properties which will be essential to define multipath cohomology.

Definition 6 Let $(S, \lhd)$ be a poset and $(S', \lhd_{|S'\times S'})$ be a subposet of $(S, \lhd)$.

(1) We say that $(S, \lhd)$ is *squared* if for each triple $x, y, z \in S'$ such that z covers y and y covers x, there is a unique $y' \neq y$ such that z covers y' and y' covers x. Such elements x, y, y', and z will be called a *square* in S.

(2) We say that $(S', \lhd_{|S'\times S'})$ is *faithful* if the covering relation in S' induced by $\lhd_{|S'\times S'}$ is the restriction of the covering relation in S induced by $\lhd$.

Observe that square posets have also been called *thin posets* in the literature; see eg [Björner 1984, Section 4] or [Chandler 2019]. Prime examples of squared posets are Boolean posets.

Example 7 Downward and upward closed subposets are faithful. Furthermore, each downward or upward closed subposet of a squared poset is squared.

The following proposition is straightforward:

Proposition 8 *Let $(S, \lhd)$ be a poset. Given the subposets $S', S'' \subset S$, we have:*

(1) *If $S'' \subset S'$ is faithful and $S' \subset S$ is faithful, then S'' is faithful in S.*

(2) *If S' and S'' are faithful in S (resp squared), then $S' \cap S''$ is faithful in S (resp squared).*

2.2 Path posets

We now define one of the main ingredients in the construction of multipath cohomology: the path poset.

Let $\mathtt{G}$ be a graph and let $|\mathtt{G}|$ denote its geometric realisation as a CW–complex. A connected component of $\mathtt{G}$ is a subgraph $\mathtt{H}$ of $\mathtt{G}$ whose realisation $|\mathtt{H}|$ is connected. A *simple path* of $\mathtt{G}$ is a sequence of edges

Figure 1: The n–step graph I_n.

$e_1, \dots, e_n$ of G such that $s(e_{i+1}) = t(e_i)$ for $i = 1, \dots, n-1$, and no vertex is encountered twice, ie if $s(e_i) = s(e_j)$ or $t(e_i) = t(e_j)$, then $i = j$, and it is not a cycle, ie $s(e_1) \neq t(e_n)$.

Remark 9 If a connected graph G admits an ordering of all its edges with respect to which it is a simple path, then it is isomorphic to the graph $\mathtt{I}_n$ shown in Figure 1. The explicit isomorphism is given by the morphism of digraphs $\phi\colon V(\mathtt{G}) \to V(\mathtt{I}_n),\ s(e_i) \mapsto v_{i-1},\ t(e_n) \mapsto v_n$.

We are interested in taking disjoint sets of simple paths; following [Turner and Wagner 2012], we call them multipaths.

Definition 10 A *multipath* of G is a spanning subgraph such that each connected component is either a vertex or its edges admit an ordering such that it is a simple path.

Remark 11 Every spanning subgraph of a multipath is still a multipath. In particular, the set of multipaths of a graph G — denoted by $P(\mathtt{G})$ — forms a downward closed subposet of SSG(G) (with the induced order). Moreover, there is a unique minimum in both $P(\mathtt{G})$ and SSG(G), which is the spanning subgraph with no edges.

With the definition of multipath in place we can present the main actor of the section.

Definition 12 The *path poset* of G is the poset $(P(\mathtt{G}), <)$ associated to G, that is, the set of multipaths of G ordered by the relation of "being a subgraph".

When the partial order on $P(\mathtt{G})$ is not specified, we will always implicitly assume it to be the order relation $<$. Moreover, with abuse of notation, we will also write $P(\mathtt{G})$ instead of $(P(\mathtt{G}), <)$.

We now provide some examples of path posets.

Example 13 Consider the coherently oriented linear graph $\mathtt{I}_n$ with n edges, illustrated in Figure 1. In this case all spanning subgraphs are multipaths, that is $P(\mathtt{I}_n) = \mathrm{SSG}(\mathtt{I}_n)$. In particular, it follows that $(P(\mathtt{I}_n), <)$ is a Boolean poset.

Example 14 Consider the coherently oriented polygonal graph $\mathtt{P}_n$ with $n+1$ edges, illustrated in Figure 2. Note that, according to our definition, also the digon $\mathtt{P}_1$, which is shown explicitly in Figure 3, is a digraph. In this case all spanning subgraphs but the polygon itself are multipaths. Equivalently, we have $(P(\mathtt{P}_n) \cup \{\mathtt{P}_n\}, <) = (\mathrm{SSG}(\mathtt{P}_n), <)$. In particular, $(P(\mathtt{P}_n), <)$ for $n \in \mathbb{N} \setminus \{0\}$ is *not* a Boolean poset (as it is missing the maximum).

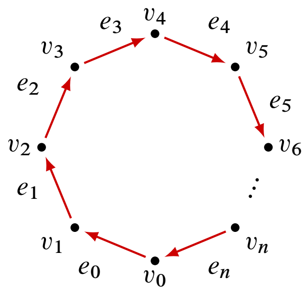

Figure 2: The coherently oriented polygonal graph $\mathtt{P}_n$ with a fixed ordering of vertices.

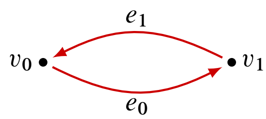

Figure 3: The digon graph $\mathtt{P}_1$.

Recall that the symbol $\tilde{\lhd}$ denotes a covering relation. In order to visually represent path posets associated to digraphs, we use the Hasse diagrams. The Hasse digraph $\mathtt{Hasse}(S, \lhd)$ of a poset $(S, \lhd)$ is the graph whose vertices are the elements of S and such that (x, y) is an edge if, and only if, $x \tilde{\lhd} y$. Note that the Hasse graph of a poset S completely encodes the covering relation of S and hence, by transitivity, the order relation.

Example 15 Consider the Y–shaped graphs in Figure 4. Their associated path posets, up to isomorphism, are shown in Figure 5; the figures show the covering relation in the posets or, alternatively, the Hasse digraph of the path poset. Note that the path poset of the graph in Figure 4(b) is isomorphic to the path poset of the graph in Figure 4(a); in fact, these two graphs are isomorphic up to reversing the orientation in all arcs of one of the two. However, the path poset of the graph in Figure 4(b) is not isomorphic to the path poset of the graph in Figure 4(c) (eg there are no multipaths of with two edges in the latter).

Example 16 Consider the H–shaped digraph of Figure 6. The associated path poset, which is illustrated in Figure 7, has multipaths with at most two edges.

The following remark will be essential in the functorial applications — see Section 5.

Remark 17 A morphism $f\colon \mathtt{G}_1 \to \mathtt{G}_2$ in **Digraph** (which is regular by definition) induces a morphism of posets $Pf\colon P(\mathtt{G}_1) \to P(\mathtt{G}_2)$; more precisely, to a multipath $\mathtt{H} \subset P(\mathtt{G}_1)$ we associate the spanning subgraph $Pf(\mathtt{H})$ of $\mathtt{G}_2$ defined by $E(Pf(\mathtt{H})) = \{f(e) \mid e \in E(\mathtt{H})\}$. This association yields a functor $P\colon$ **Digraph** $\to$ **Poset**. Note that $Pf(P(\mathtt{G}_1))$ is a faithful subposet of $P(\mathtt{G}_2)$.

We conclude the section by noting that, in favourable cases, the path poset determines the graph.

Proposition 18 *Let $\mathtt{G}$ be a connected graph with n edges. If $P(\mathtt{G})$ has a maximum then we have that $P(\mathtt{G}) \cong \mathbb{B}(n)$ and $\mathtt{G} \cong \mathtt{I}_n$. In particular, a connected graph has a Boolean path poset if, and only if, it is isomorphic to $\mathtt{I}_n$.*

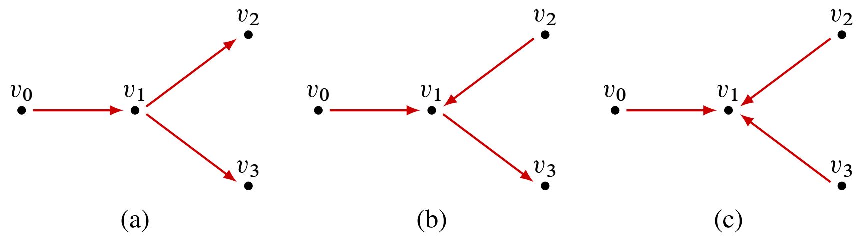

Figure 4: Three nonisomorphic Y–shaped digraphs.

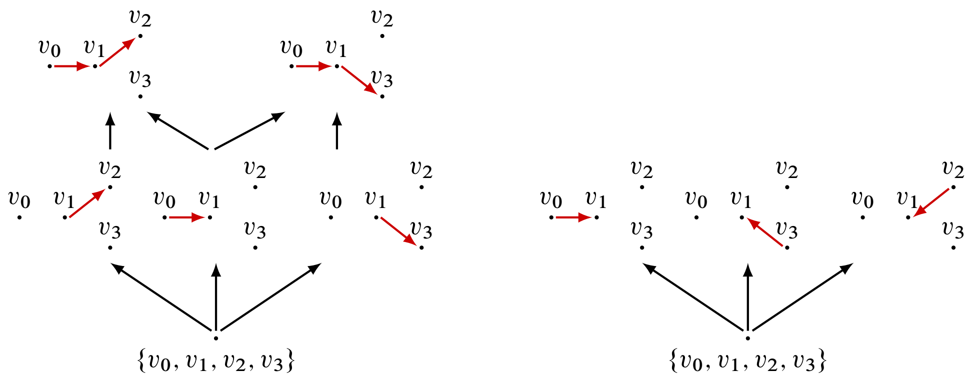

Figure 5: The path posets of the Y–shaped digraphs in Figure 4(a), left, and (c), right.

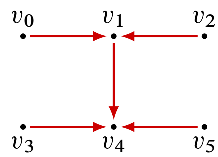

Figure 6: A figure H–type digraph.

Proof Denote by $\mathtt{M}$ the maximum, which is the unique maximal element, of $P(\mathtt{G})$. We shall now prove that $\mathtt{M} = \mathtt{G}$. That being true, $\mathtt{G}$ would be a connected graph admitting an ordering of the edges with respect to which is a simple path, since $\mathtt{M}$ is a connected multipath. The statement would then follow from Remark 9.

Since $P(\mathtt{G})$ is finite poset, for every $x \in P(\mathtt{G})$ there is a maximal element $m \in P(\mathtt{G})$ such that $x \leq m$. Assume, for the sake of contradiction, that $\mathtt{M} \neq \mathtt{G}$. Then, there exists an edge $e \in E(\mathtt{G}) \setminus E(\mathtt{M})$. Consider the (multi-)path $\mathtt{e}$ defined by $E(\mathtt{e}) = \{e\}$. Then, as stated above, we have a maximal multipath $\mathtt{M}'$ such that $\mathtt{e} \leq \mathtt{M}'$. In particular, $\mathtt{M} \neq \mathtt{M}'$; this is not possible as we have a unique maximal element in $P(\mathtt{G})$. □

We pointed out in Example 14 that the coherently oriented polygonal graphs have a path poset which is almost a Boolean poset; more precisely, the path poset of $\mathtt{P}_n$ is a Boolean poset minus the maximum.

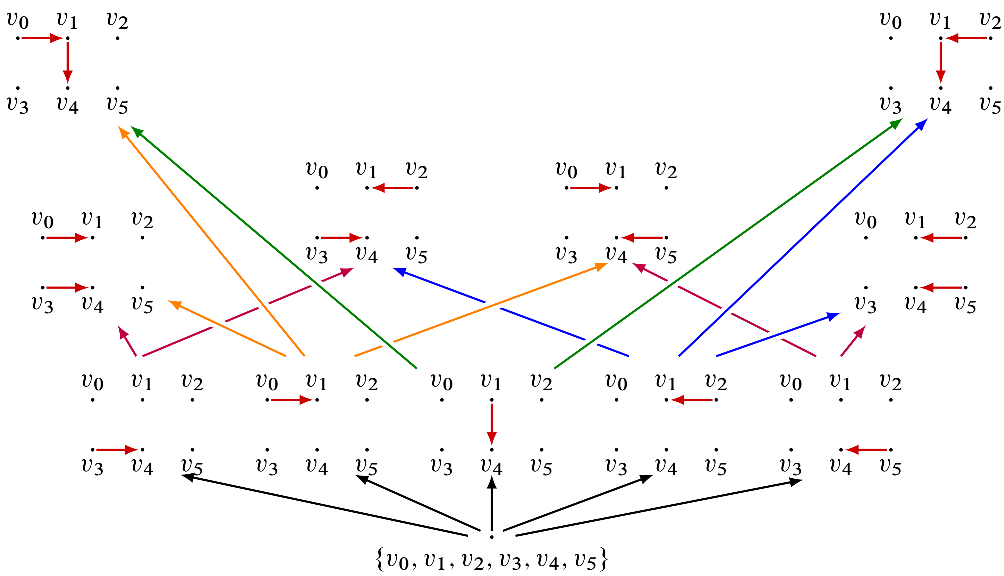

Figure 7: The path poset of the H–shaped digraph of Figure 6.

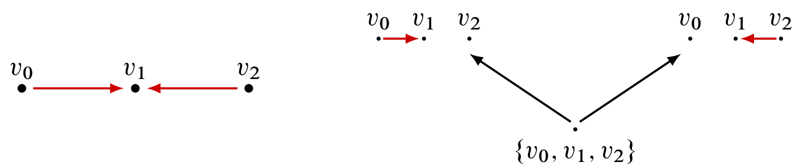

Figure 8: A noncoherent linear digraph with two edges and its path poset.

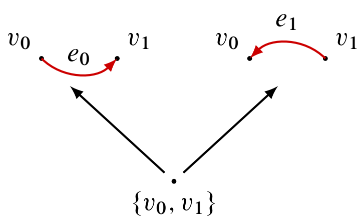

Figure 9: The path poset of the digon graph $\mathtt{P}_1$ in Figure 3.

Example 19 Consider the digon graph $\mathtt{P}_1$ illustrated in Figure 3. As depicted in Figure 9, its associated path poset consists of a minimum together with two elements corresponding to the two edges of the digon. It is easy to see that this poset is equivalent to the path poset associated to the linear digraph with two edges and noncoherent orientation illustrated in Figure 8.

We claim that, aside from the graph in Figure 8, the only connected graphs whose path poset is a Boolean poset minus its maximum are the coherently oriented polygonal graphs. The key observation to prove our claim is the following;

Remark 20 If $\mathtt{G}$ is a graph with n edges, and $P(\mathtt{G})$ is isomorphic to $\mathbb{B}(n)$ minus its maximum, then all the subgraphs of $\mathtt{G}$ but $\mathtt{G}$ itself must be multipaths. In fact, $\mathrm{SSG}(\mathtt{G})$ has exactly 2^n elements, which is the same number of elements in $\mathbb{B}(n)$. It follows that only one subgraph $\mathtt{H}$ of $\mathtt{G}$ does not belong to $P(\mathtt{G})$. Since $P(\mathtt{G})$ is downward closed in $\mathrm{SSG}(\mathtt{G})$, we must have $\mathtt{H} = \mathtt{G}$.

Proposition 21 *Let* $\mathtt{G}$ *be a connected graph with* n *edges. If* $P(\mathtt{G})$ *is isomorphic to* $\mathbb{B}(n)$ *minus its maximum, then* $\mathtt{G} \cong \mathtt{P}_{n-1}$.

Proof Take one of the n maximal elements in $P(\mathtt{G})$, say $\mathtt{M}$. Note that $|E(\mathtt{M})| = n-1$. Moreover, since $\mathtt{M}$ differs from $\mathtt{G}$ by a single edge, and $\mathtt{G}$ is connected, then $\mathtt{M}$ has at most two connected components.

Assume, for the sake of contradiction, that $\mathtt{M}$ is not connected. Each component of $\mathtt{M}$ is a simple path. It follows that $\mathtt{G}$ is either $\mathtt{I}_n$ — which is absurd by Proposition 18 — or contains the graph in Figure 8 (up to orientation reversal of both edges) as a subgraph. Note that the graph in Figure 8 cannot be $\mathtt{G}$ since $|E(\mathtt{G})| = n > 2$. It follows that any proper spanning subgraph containing a copy of the graph in Figure 8 is a subgraph different than $\mathtt{G}$ which belongs to $\mathrm{SSG}(\mathtt{G})$ but not to $P(\mathtt{G})$. This contradicts Remark 20.

From our argument above, it follows that $\mathtt{M}$ is connected, and thus isomorphic to $\mathtt{I}_n$ (since it is a multipath). So either $\mathtt{G} \cong \mathtt{P}_{n-1}$, $\mathtt{G} \cong \mathtt{I}_n$ (absurd), or again it contains a copy of the graph in Figure 8. The latter case can be excluded with the same argument as above. □

3 Digraph (co)homologies

The goal of this section is to outline a rather general framework within which to define cohomology theories of directed graphs, using *poset homology* [Chandler 2019] as main tool — see also Remark 27. For the sake of being self-contained, and also for clarity, we provide a fairly detailed exposition of the construction of poset homology for posets of subgraphs. This is carried out in the first subsection. As the aforementioned construction might, a priori, depend upon the choice of a sign assignment on the considered posets, we further explore such dependence in the second subsection. We point out here that more general cohomology theories of posets can be used to obtain similar digraph cohomologies; we explore it in Section 6.

3.1 A poset homology

In this subsection we define, given a special type of poset coherently assigned to each digraph, and a choice of a sign assignment (see Definition 22), a cohomology theory for directed graphs; the cohomology theory depends on many choices and the functorial discussion is postponed to Section 5. The construction presented here has been inspired by [Turner and Wagner 2012], on one side, and by [Helme-Guizon and Rong 2005], on the other side. In the former paper homologies for digraphs were defined using the path poset functor — see Definition 12 and Remark 17; while in [Helme-Guizon and Rong 2005] a homology for nonoriented graphs was obtained via a construction similar to [Khovanov 2000]. Poset homology [Chandler 2019] interpolates between the two constructions.

Recall from Definition 6(1) that a square in a poset $(S, \lhd)$ is given by elements x, y, y', and z such that $y \neq y'$, $x \mathrel{\tilde{\lhd}} y \mathrel{\tilde{\lhd}} z$, and $x \mathrel{\tilde{\lhd}} y' \mathrel{\tilde{\lhd}} z$, where $\tilde{\lhd}$ denotes the covering relation in S. Let $\mathbb{Z}_2$ be the cyclic group on two elements.

Definition 22 A *sign assignment* on a poset $(S, \lhd)$ is an assignment of elements $\epsilon_{x,y} \in \mathbb{Z}_2$ to each pair of elements $x, y \in S$ with $x \mathrel{\tilde{\lhd}} y$ such that the equation

$$\epsilon_{x,y} + \epsilon_{y,z} \equiv \epsilon_{x,y'} + \epsilon_{y',z} + 1 \bmod 2 \tag{1}$$

holds for each square $x \mathrel{\tilde{\lhd}} y, y' \mathrel{\tilde{\lhd}} z$.

Observe that the restriction of a sign assignment to a subposet is a sign assignment. In general the existence of a sign assignment on a given poset is not clear. However, for the spanning subgraphs poset — or, better, for Boolean posets — and their subposets, there is an easy sign assignment:

Example 23 Let $\mathtt{G}$ be a graph with a fixed total ordering $\lhd$ on the set of edges $E(\mathtt{G})$. Recall from Notation 2 and Example 4 that $\mathtt{H} \prec \mathtt{H}'$ in $\mathrm{SSG}(\mathtt{G})$ if, and only if, $\mathtt{H}' = \mathtt{H} \cup e$. Then, we can define a sign assignment on the poset $\mathrm{SSG}(\mathtt{G})$ as

$$\epsilon(\mathtt{H}, \mathtt{H}') := \#\{e' \in E(\mathtt{H}) \mid e' \lhd e\} \bmod 2,$$

where $\mathtt{H}' = \mathtt{H} \cup e$. The verification is straightforward, but the reader may consult eg [Khovanov 2000].

The following definition will be used to define the cochain complexes.

Definition 24 Let $P \subseteq \mathrm{SG}(\mathtt{G})$ be a faithful subposet. We define the *level* of an element $\mathtt{H} \in P$ as

$$\ell(\mathtt{H}) = \#E(\mathtt{H}) + \#V(\mathtt{H}) - \min\{\#E(\mathtt{H}') + \#V(\mathtt{H}') \mid \mathtt{H}' \in P\}.$$

Note that the level of an element $\mathtt{H} \in P \subseteq \mathrm{SG}(\mathtt{G})$ if P has a minimum is just the difference between the distances of $\mathtt{H}$ and the minimum of P, respectively, from the minimum of $\mathrm{SG}(\mathtt{G})$ in $\mathtt{Hasse}(\mathrm{SG}(\mathtt{G}))$. Note also that if $P = \mathrm{SG}(\mathtt{G})$, $P(\mathtt{G}) \subseteq \mathrm{SSG}(\mathtt{G})$ then ℓ is the number of edges.

Recall from Remark 3 that a poset $(S, \lhd)$ can be seen as a category $\boldsymbol{S}$ with set of objects S, and the set of morphisms between x and y containing a single element if and only if $x \lhd y$ or $x = y$.

Remark 25 Let $\boldsymbol{C}$ be a small category. For each square $x \mathrel{\tilde{\lhd}} y, y' \mathrel{\tilde{\lhd}} z$ in $(S, \lhd)$ and any covariant functor $\mathcal{F}\colon \boldsymbol{S} \to \boldsymbol{C}$, we have

$$\mathcal{F}(y \mathrel{\tilde{\lhd}} z) \circ \mathcal{F}(x \mathrel{\tilde{\lhd}} y) = \mathcal{F}(x \lhd z) = \mathcal{F}(y' \mathrel{\tilde{\lhd}} z) \circ \mathcal{F}(x \mathrel{\tilde{\lhd}} y').$$

In other words, all functors preserve the commutativity of the squares in $(S, \lhd)$.

Let $\boldsymbol{A}$ be an additive category, $P \subseteq \mathrm{SG}(\mathtt{G})$ squared and faithful and ϵ a sign assignment on P. Given a covariant functor $\mathcal{F}\colon \boldsymbol{P} \to \boldsymbol{A}$, we can define the cochain groups

$$C^n_{\mathcal{F}}(P) := \bigoplus_{\substack{\mathtt{H} \in P \\ \ell(\mathtt{H}) = n}} \mathcal{F}(\mathtt{H})$$

and the differentials

$$d^n = d^n_{\mathcal{F}} := \sum_{\substack{\mathtt{H}\in P\\ \ell(\mathtt{H})=n}} \sum_{\substack{\mathtt{H}'\in P\\ \mathtt{H}\prec\mathtt{H}'}} (-1)^{\epsilon(\mathtt{H},\mathtt{H}')}\mathcal{F}(\mathtt{H}\prec\mathtt{H}').$$

Note that the differentials d^n, and therefore the cochain complexes, depend, a priori, upon the choice of the sign assignment ϵ. However, in the cases we are interested in, this choice does not affect the isomorphism type of the complexes — see Corollary 36. We will further discuss this topic in Section 3.2 below. We now give the proof that the defined complexes are indeed cochain complexes.

In the proof of the theorem below, P being squared plays an essential role; otherwise, the square of the differential might not be zero.

Theorem 26 *Let $\boldsymbol{A}$ be an additive category, $P\subseteq \mathrm{SG}(\mathtt{G})$ a squared and faithful poset, and ϵ a sign assignment on P. Then, for any $n\in\mathbb{N}$ and any covariant functor $\mathcal{F}\colon \boldsymbol{P}\to\boldsymbol{A}$, we have $d^n\circ d^{n-1}\equiv 0$. In particular, $(C^*_{\mathcal{F}}(P),d^*)$ is a cochain complex.*

Proof Fix a natural number n; then

$$C^n_{\mathcal{F}}(P) = \bigoplus_{\substack{\mathtt{H}\in P\\ \ell(\mathtt{H})=n}} \mathcal{F}(\mathtt{H}).$$

Let $\pi_{\mathtt{H}}\colon C^*_{\mathcal{F}}(P)\to\mathcal{F}(\mathtt{H})$ and $\iota_{\mathtt{H}}\colon\mathcal{F}(\mathtt{H})\to C^*_{\mathcal{F}}(P)$ be the projection onto $\mathcal{F}(\mathtt{H})$ and the inclusion of $\mathcal{F}(\mathtt{H})$ in $C^*_{\mathcal{F}}(P)$ as direct summand, respectively. Note that the composition $d^n\circ d^{n-1}$ equals 0 if, and only if, the composition $\pi_{\mathtt{H}''}\circ d^n\circ d^{n-1}$ is trivial for all $\mathtt{H}''\in P$ such that $\ell(\mathtt{H}'')=n+1$. In particular, $d^n\circ d^{n-1}\equiv 0$ if there are no $\mathtt{H}''\in P$ with $\ell(\mathtt{H}'')=n+1$.

Every element of $C^{n-1}_{\mathcal{F}}(P)$ is a linear combination of elements of $\mathcal{F}(\mathtt{H})$ for $\mathtt{H}$ ranging in P with $\ell(\mathtt{H}) = n-1$, and d is linear. Thus, if the composition $\pi_{\mathtt{H}''}\circ d^n\circ d^{n-1}\circ\iota_{\mathtt{H}}$ equals 0 for all $\mathtt{H},\mathtt{H}''$ such that $\ell(\mathtt{H})+2=\ell(\mathtt{H}'')=n+1$, then $d^n\circ d^{n-1}\equiv 0$. We can factor the map $\pi_{\mathtt{H}''}\circ d^n\circ d^{n-1}\circ\iota_{\mathtt{H}}$ through $C^n_{\mathcal{F}}(P)$, and write

$$\pi_{\mathtt{H}''}\circ d^n\circ d^{n-1}\circ\iota_{\mathtt{H}} = \sum_{\mathtt{H}\prec\mathtt{H}'\prec\mathtt{H}''} (\pi_{\mathtt{H}''}\circ d^n\circ\iota_{\mathtt{H}'})\circ(\pi_{\mathtt{H}'}\circ d^{n-1}\circ\iota_{\mathtt{H}}).$$

The right-hand side of the above equation vanishes if there is no $\mathtt{H}'$ such that $\mathtt{H}\prec\mathtt{H}'\prec\mathtt{H}''$. It follows that it is sufficient to check this case.

Since P is squared, if there is $\mathtt{H}\prec\mathtt{H}'_1\prec\mathtt{H}''$, then there a unique $\mathtt{H}'_2$ ($\neq\mathtt{H}'_1$) such that $\mathtt{H}\prec\mathtt{H}'_2\prec\mathtt{H}''$. In other words, $\mathtt{H},\mathtt{H}'_1,\mathtt{H}'_2,\mathtt{H}''$ form a square in P. Thus, we obtain

$$\begin{aligned}
&\pi_{\mathtt{H}''}\circ d^n\circ d^{n-1}\circ\iota_{\mathtt{H}}\\
&\qquad= (\pi_{\mathtt{H}''}\circ d^n\circ\iota_{\mathtt{H}'_1})\circ(\pi_{\mathtt{H}'_1}\circ d^{n-1}\circ\iota_{\mathtt{H}}) + (\pi_{\mathtt{H}''}\circ d^n\circ\iota_{\mathtt{H}'_2})\circ(\pi_{\mathtt{H}'_2}\circ d^{n-1}\circ\iota_{\mathtt{H}})\\
&\qquad= (-1)^{\epsilon(\mathtt{H},\mathtt{H}'_1)+\epsilon(\mathtt{H}'_1,\mathtt{H}'')}\mathcal{F}(\mathtt{H}'_1\prec\mathtt{H}'')\circ\mathcal{F}(\mathtt{H}\prec\mathtt{H}'_1) + (-1)^{\epsilon(\mathtt{H},\mathtt{H}'_2)+\epsilon(\mathtt{H}'_2,\mathtt{H}'')}\mathcal{F}(\mathtt{H}'_2\prec\mathtt{H}'')\circ\mathcal{F}(\mathtt{H}\prec\mathtt{H}'_2)\\
&\qquad= ((-1)^{\epsilon(\mathtt{H},\mathtt{H}'_1)+\epsilon(\mathtt{H}'_1,\mathtt{H}'')} + (-1)^{\epsilon(\mathtt{H},\mathtt{H}'_2)+\epsilon(\mathtt{H}'_2,\mathtt{H}'')})\mathcal{F}(\mathtt{H}'_2\prec\mathtt{H}'')\circ\mathcal{F}(\mathtt{H}\prec\mathtt{H}'_2),
\end{aligned}$$

where the last equality is due to the fact the functor $\mathcal{F}$ preserves the commutative squares in P — see Remark 25. The result now follows immediately as ϵ is a sign assignment on P. □

Remark 27 The definition of the cochain complex $C_{\mathcal{F}}(P)$ relies on the structure of the input graph $\mathtt{G}$, via the associated squared and faithful poset $P \subseteq \mathrm{SG}(\mathtt{G})$, on the choice of the functor $\mathcal{F}$, and on a sign assignment ϵ on P. More in general, the same machinery can be applied without graphs but dealing only with a certain type of posets. This viewpoint was taken by Chandler, and we refer the reader to [Chandler 2019] for a more comprehensive discussion. For completeness we pursue our independently developed approach. In particular, we shall provide an independent proof of functoriality with respect to graphs, extending the generality to include also coefficients systems, in Section 5.

We conclude the section by observing that the general discussion of this section can be applied to the case of $\mathcal{G}\colon \boldsymbol{P} \to \boldsymbol{A}$ a contravariant functor. All proofs are straightforward adaptations of the proofs in the case of covariant functors.

3.2 Existence and uniqueness of sign assignments

The cochain complexes defined in the previous subsection may depend on the choice of the sign assignment. In this subsection, we see that this is actually not the case for a quite general class of posets, including path posets.

A sign assignment on $(S, \lhd)$ can be seen as a map $\epsilon\colon E(\mathtt{Hasse}(S, \lhd)) \to \mathbb{Z}_2$ such that (1) holds for each square $x \mathrel{\tilde{\lhd}} y, y' \mathrel{\tilde{\lhd}} z$ of S. Consider the Hasse graph $\mathtt{Hasse}(S, \lhd)$ of a poset $(S, \lhd)$ as a CW–complex (formally, by taking its geometric realisation).

Definition 28 Given a poset $(S, \lhd)$ define $\mathscr{K}(S, \lhd)$ as the CW–complex obtained from $(S, \lhd)$ by attaching to the (geometric realisation of the) Hasse graph $\mathtt{Hasse}(S, \lhd)$ a 2–cell $e_{x,y,y',z}$ for each square $x \mathrel{\tilde{\lhd}} y, y' \mathrel{\tilde{\lhd}} z$ in $(S, \lhd)$.

We will now show that the existence and uniqueness of a sign assignment on a poset $(S, \lhd)$ depends only upon the topological structure of the CW–complex $\mathscr{K}(S, \lhd)$. Denote by $(C^*_{\mathrm{CW}}(\mathscr{K}(S, \lhd); \mathbb{Z}_2), d^*_{\mathrm{CW}})$ the CW–cochain complex of $\mathscr{K}(S, \lhd)$, with respect to the given CW–structure, with coefficients in $\mathbb{Z}_2$. We can interpret the sign assignments as cochains in the CW–cochain complex associated to $\mathscr{K}(S, \lhd)$:

Lemma 29 *Let ϵ be a sign assignment on a poset $(S, \lhd)$, and denote by ψ the 2–cochain which associates $1 \in \mathbb{Z}_2$ to each 2–cell in $\mathscr{K}(S, \lhd)$. Then, ϵ defines a cochain $a(\epsilon) \in C^1_{\mathrm{CW}}(\mathscr{K}(S, \lhd); \mathbb{Z}_2)$ such that $d_{\mathrm{CW}}\, a(\epsilon) = \psi$. Moreover, for each $a \in C^1_{\mathrm{CW}}(\mathscr{K}(S, \lhd); \mathbb{Z}_2)$ such that $d_{\mathrm{CW}}\, a = \psi$ there is a unique sign assignment ϵ such that $a = a(\epsilon)$.*

Proof A 1–cocycle a with values in $\mathbb{Z}_2$ is a map from the set of 1–cells (which are the edges of the Hasse graph, in our case) to $\mathbb{Z}_2$. Since the edges of the Hasse graph correspond to the pairs in the covering relations, this is equivalent to the assignment of an element $\mathbb{Z}_2$ for each pair (x, y) such that

$x \mathrel{\tilde{\lhd}} y$. It is left to show that the differential of a is ψ if, and only if, (1) holds for each square. Note that $d_{\mathrm{CW}}\, a(e) = \sum_{(x,y)\in\partial e} a(x,y)$ for 2–cells e; therefore $d_{\mathrm{CW}}\, a = \psi$ if, and only if,

$$a(x,y) + a(y,z) + a(x,y') + a(y',z) \equiv 1 \bmod 2$$

for each square $x \mathrel{\tilde{\lhd}} y, y' \mathrel{\tilde{\lhd}} z$, concluding the proof. □

It is easy to see that a poset $(S,\lhd)$ admits a sign assignment if the CW–complex $\mathscr{K}(S,\lhd)$ has trivial second homology group:

Proposition 30 *Let $(S,\lhd)$ be a poset. If $\mathrm{H}^2_{\mathrm{CW}}(\mathscr{K}(S,\lhd);\mathbb{Z}_2) = 0$, then there exists a sign assignment on $(S,\lhd)$.*

Proof Consider the cochain $\psi\colon C_2^{\mathrm{CW}}(\mathscr{K}(S,\lhd);\mathbb{Z}_2) \to \mathbb{Z}_2$ assigning $1 \in \mathbb{Z}_2$ to each 2–cell of $\mathscr{K}(S,\lhd)$. Since $\mathscr{K}(S,\lhd)$ has no 3–cells, $d_{\mathrm{CW}}(\psi) \equiv 0$ and hence ψ is a cocycle. Since, by assumption, we have $\mathrm{H}^2_{\mathrm{CW}}(\mathscr{K}(S,\lhd);\mathbb{Z}_2) = 0$, every 2–cocycle is a coboundary. Thus, there is $a \in C^1_{\mathrm{CW}}(\mathscr{K}(S,\lhd);\mathbb{Z}_2)$ such that $d_{\mathrm{CW}} a = \psi$. The statement now follows directly from the second part of Lemma 29. □

The above proposition provides a condition for a poset to admit a sign assignment. We now describe when also the uniqueness is satisfied. First, we introduce the notion of isomorphisms of sign assignments.

Definition 31 Let ϵ, ϵ' be sign assignments on a poset $(S,\lhd)$. An *isomorphism of sign assignments between ϵ and ϵ'* is a map $\eta\colon S = V(\mathtt{Hasse}(S,\lhd)) \to \mathbb{Z}_2$ such that

(2) $$\eta(x) + \epsilon'_{x,y} = \epsilon_{x,y} + \eta(y) \bmod 2$$

holds for all $x \mathrel{\tilde{\lhd}} y$.

Roughly speaking, an isomorphism of sign assignments is a map $\eta\colon S \to \mathbb{Z}_2$ such that the elements of $\mathbb{Z}_2$ on the edges of the square

$$\begin{array}{ccc} x & \xrightarrow{\epsilon_{x,y}} & y \\ {\scriptstyle \eta_x}\big\downarrow & & \big\downarrow{\scriptstyle \eta_y} \\ x & \xrightarrow[\epsilon'_{x,y}]{} & y \end{array}$$

add up to $0 \in \mathbb{Z}_2$. Intuitively this condition encodes the "commutativity" of such squares. We can now provide a uniqueness result for sign assignments on posets — compare with [Putyra 2014, Lemma 5.7].

Proposition 32 *Let ϵ and ϵ' be two sign assignments on a poset $(S,\lhd)$. If $\mathrm{H}^1_{\mathrm{CW}}(\mathscr{K}(S,\lhd);\mathbb{Z}_2) = 0$, then there is an isomorphism η of sign assignments from ϵ to ϵ'.*

Proof Let $a(\epsilon), a(\epsilon')$ be the 1–cochains corresponding to ϵ, ϵ' as in Lemma 29. Notice that

$$d_{\mathrm{CW}}(a(\epsilon) - a(\epsilon')) = d_{\mathrm{CW}}(a(\epsilon)) - d_{\mathrm{CW}}(a(\epsilon')) = \psi - \psi = 0,$$

Figure 10: Two (isomorphic) sign assignments on the poset $(\wp(\{0,1\}),\subseteq)$.

where ψ is the usual 2–cocycle assigning $1\in\mathbb{Z}_2$ to each face of $\mathscr{K}(S,\lhd)$. Since, by assumption, $\mathrm{H}^1_{\mathrm{CW}}(\mathscr{K}(S,\lhd);\mathbb{Z}_2)=0$, we must have $a(\epsilon)-a(\epsilon')=d_{\mathrm{CW}}(\eta)$ for some $\eta\in C^0_{\mathrm{CW}}(\mathscr{K}(S,\lhd);\mathbb{Z}_2)$. We can see η as a map

$$\eta\colon\{0\text{-cells of }\mathscr{K}(S,\lhd)\}=V(\mathtt{Hasse}(S,\lhd))\to\mathbb{Z}_2.$$

Moreover, the equality $a(\epsilon)-a(\epsilon')=d_{\mathrm{CW}}(\eta)$ applied to each edge of the Hasse graph gives precisely the condition of isomorphisms of sign assignment in (2), concluding the proof. □

Example 33 Consider the two sign assignments on the Boolean poset $(\wp(\{0,1\}),\subseteq)$ illustrated in Figure 10. By definition $\mathscr{K}(\wp(\{0,1\}),\subseteq)$ is a disk.

Thus, $\mathrm{H}^1_{\mathrm{CW}}(\mathscr{K}(\wp(\{0,1\}),\subseteq);\mathbb{Z}_2)=0$. This implies the uniqueness of the sign assignment up to isomorphism in this case. It is not difficult, in this case, to produce a concrete isomorphism:

$$\eta\colon V(\mathtt{Hasse}(\wp(\{0,1\}),\subseteq))\to\mathbb{Z}_2,\qquad v\mapsto\begin{cases}1 & \text{if } v=\{1\},\\ 0 & \text{otherwise.}\end{cases}$$

Recall the definition of downward closed subposet $(S',\lhd_{|S'\times S'})$ of a poset $(S,\lhd)$ — see Definition 5. As a consequence of Proposition 32, we get the following:

Theorem 34 *Let P be a downward (or upward) closed subposet of* SSG(G). *Then, any two sign assignments ϵ and ϵ' on P are isomorphic.*

Proof The poset SSG(G) is a Boolean poset — see Example 4. It follows that its Hasse diagram is the 1–skeleton of an n–dimensional cube, with $n=|E(\mathtt{G})|$. This implies that $\mathscr{K}(\mathrm{SSG}(\mathtt{G}),<)$, which for $n\geq 2$ is the union of the 1– and 2–skeletons of an n–dimensional cube, has trivial homology groups in degree $i=1$ — this being trivial for $n=1$. Therefore, the statement follows in this case from Proposition 32. Note that there always exists a sign assignment on Boolean posets; this is true for homological reasons for $n\neq 3$, see Proposition 30, and we can even define it explicitly for all n — see for instance [Khovanov 2000, Section 3].

In the general case in which P is a downward or upward closed proper subposet of a cube, observe that it is squared (see Example 7) and it contains either the minimum or the maximum of the cube. Furthermore, the sub-CW–complex $\mathscr{K}(P,<)\subset\mathscr{K}(\mathrm{SSG}(\mathtt{G}),<)$ retracts onto the minimum (or the maximum); hence, again by Proposition 32, the uniqueness of the sign assignment up to isomorphism follows. For a detailed proof of this fact we refer the interested reader to [Chandler 2019, Theorem 4.5] (to be read in conjunction with Theorems 2.9 (3) and 5.14 of that work). □

In particular, if $P = P(\mathtt{G})$ is the path poset of a digraph $\mathtt{G}$ — see Remark 11 — we get:

Corollary 35 *Any two sign assignments ϵ and ϵ' on $P(\mathtt{G})$ are isomorphic.*

We conclude the section with an application to the cohomology theories defined in Section 3.1.

Corollary 36 *Let $\mathtt{G}$ be a digraph, $P \subseteq \mathrm{SSG}(\mathtt{G})$ be a downward (or upward) closed subposet, and $\mathcal{F}\colon \boldsymbol{P} \to \boldsymbol{A}$ a covariant functor to an additive category $\boldsymbol{A}$. Then, the cochain complex $(C^*_{\mathcal{F}}(P), d^*)$ does not depend, up to isomorphism, on the choice of the sign assignment on P.*

Proof Let ϵ and ϵ' be two sign assignments on P. Denote by $(C^*_{\mathcal{F}}(P), d^*_\epsilon)$ and $(C^*_{\mathcal{F}}(P), d^*_{\epsilon'})$ the associated cochain complexes defined using ϵ and ϵ', respectively. By Theorem 34, any two sign assignments on P are isomorphic. Let η be an isomorphism between them and define the map

$$\Phi\colon (C^*_{\mathcal{F}}(P), d^*_\epsilon) \to (C^*_{\mathcal{F}}(P), d^*_{\epsilon'}),$$

as $\Phi := \bigoplus_{\mathtt{H}} \Phi_{\mathtt{H}}$, where $\Phi_{\mathtt{H}} = (-1)^{\eta(\mathtt{H})}\,\mathrm{Id}_{\mathcal{F}(\mathtt{H})}$. Observe that this clearly gives an isomorphism of modules. Furthermore, the commutativity of Φ with the differentials is immediate by the definition of isomorphisms of sign assignments (see (2)); hence it provides an isomorphism of chain complexes, concluding the proof. □

4 Multipath cohomology

The goal of this section is to define *multipath cohomology* of directed graphs using poset homology. This will be achieved in the first subsection, whereas the second subsection is devoted to providing some computations. In particular, we will see that the multipath cohomology may be nontrivial when evaluated on trees.

4.1 Multipath cohomology

In this subsection we specialise the general construction described in Section 3.1 by taking as poset the path poset $P(\mathtt{G})$ (Definition 12), and defining an explicit functor $\mathcal{F}_{A,M}\colon \boldsymbol{P}(\mathtt{G}) \to R\text{–}\mathbf{Mod}$. In order to define $\mathcal{F}_{A,M}$ and an explicit sign assignment on $P(\mathtt{G})$, we need some auxiliary data, more precisely, an ordering on the vertices of $\mathtt{G}$. An *ordered digraph* is a digraph with a fixed well-ordering[2] of the vertices. Note that the order of the vertices induces an order of the edges of $\mathtt{G}$; this order is given by ordering the pairs source-target lexicographically. We can use the ordering on the vertices of an ordered graph to index the connected components of any subgraph $\mathtt{H} < \mathtt{G}$; the order being given according to the minimum of the vertices belonging to each component.

[2]Every nonempty subset has a minimal element.

Notation 37 Given a subgraph $\mathtt{H}$ of an ordered graph $\mathtt{G}$, we will denote by $\mathrm{index}_{\mathtt{H}}(c)$ the position of a connected component c of $\mathtt{H}$ with respect to the aforementioned order — we start the count at 0. More precisely, if the ordered connected components of $\mathtt{H}$ are $c_0 < c_1 < \cdots < c_k$, then $\mathrm{index}_{\mathtt{H}}(c) = i$ if $c = c_i$. Note that the definition of index is well-posed. Whenever $\mathtt{H}$ is clear from the context, we will remove it from the notation of the index.

Definition 38 Consider $\mathtt{H} \in \mathrm{SG}(\mathtt{G})$ and $e \in E(\mathtt{G}) \setminus E(\mathtt{H})$ such that $s(e), t(e) \in V(\mathtt{H})$. The *source* (resp the *target*) *index of e with respect to* $\mathtt{H}$ is defined as

$$s(e, \mathtt{H}) = \mathrm{index}_{\mathtt{H}}(c) \text{ such that } s(e) \in c \quad (\text{resp. } t(e, \mathtt{H}) = \mathrm{index}_{\mathtt{H}}(c) \text{ such that } t(e) \in c).$$

The naming is motivated by the following facts: $\mathrm{index}(s(e)) = s(e, \mathtt{G}_{\varnothing})$ and $\mathrm{index}(t(e)) = t(e, \mathtt{G}_{\varnothing})$, where $\mathtt{G}_{\varnothing}$ denotes the spanning subgraph of $\mathtt{G}$ with no edges.

With this notation in place we are now ready to define a sign assignment σ_{e} on $P(\mathtt{G})$:

$$\sigma_{\mathrm{e}}(\mathtt{H}, \mathtt{H}') = \begin{cases} t(e, \mathtt{H}) + 1 & \text{if } \mathtt{H}' = \mathtt{H} \cup e \text{ and } t(e, \mathtt{H}) > s(e, \mathtt{H}), \\ s(e, \mathtt{H}) & \text{if } \mathtt{H}' = \mathtt{H} \cup e \text{ and } s(e, \mathtt{H}) > t(e, \mathtt{H}) \end{cases} \mod 2. \tag{3}$$

Lemma 39 *The function* σ_{e} *in* (3) *gives a sign assignment on* $P(\mathtt{G})$.

For the sake of presentation we moved the proof of the lemma to the Appendix.

Remark 40 More generally, observe that, for each faithful and squared poset $P \subseteq P(\mathtt{G})$, the restriction $\sigma_{\mathrm{e}|P}$ is a sign assignment. Here we are using the faithfulness of P in $P(\mathtt{G})$ (and, by Proposition 8, in $\mathrm{SSG}(\mathtt{G})$) to be sure that the covering relation only amounts to the addition of a single edge.

We now construct an explicit functor $\mathcal{F}_{A,M} : \boldsymbol{P}(\mathtt{G}) \to R$–**Mod**. From now on, R will denote a commutative ring with identity, A an associative unital R–algebra and M an (A, A)–bimodule, ie M is both a left and a right A–module, and the two actions are compatible.

Let $\mathtt{G}$ be an ordered graph and let $v_0 \in V(\mathtt{G})$ be the minimum with respect to the given ordering. Given a multipath $\mathtt{H} < \mathtt{G}$, to each connected component of $\mathtt{H}$ but the one containing the vertex v_0 we associate a copy of A, and to the component containing v_0 we associate a copy of M. Then we take the ordered tensor product. More concretely, if $c_0 < \cdots < c_k$ is the set of ordered connected components of $\mathtt{H}$, we define

$$\mathcal{F}_{A,M}(\mathtt{H}) := M_{c_0} \otimes_R A_{c_1} \otimes_R \cdots \otimes_R A_{c_k}, \tag{4}$$

where all the modules are labelled by the respective component.

Assume $\mathtt{H}' = \mathtt{H} \cup e$. Denote by $c_0, \dots, c_k$ the ordered components of $\mathtt{H}$, denote by $c'_0, \dots, c'_{k-1}$ the ordered components of $\mathtt{H}'$, and assume that the addition of e merges c_i and c_j. Then, for each $h = 0, \dots, k-1$, there is a natural identification

$$c'_h = \begin{cases} c_h & \text{if } 0 \le h < i \text{ or } i < h < j, \\ c_i \cup e \cup c_j & \text{if } h = i, \\ c_{h+1} & \text{if } j \le h < k \end{cases} \tag{5}$$

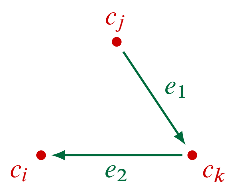

Figure 11: A schematic description of case (A) subcase (b).

for some $0 \le i < j \le k$. Using this identification, we define $\mu_{\mathtt{H} \prec \mathtt{H}'} \colon \mathcal{F}_{A,M}(\mathtt{H}) \to \mathcal{F}_{A,M}(\mathtt{H}')$ as

$$\mu_{\mathtt{H} \prec \mathtt{H}'}(a_0 \otimes \cdots \otimes a_k) = a_0 \otimes \cdots \otimes a_{s(e,\mathtt{H})-1} \otimes a_{s(e,\mathtt{H})} \cdot a_{t(e,\mathtt{H})} \otimes a_{s(e,\mathtt{H})+1} \otimes \cdots \otimes \widehat{a_{t(e,\mathtt{H})}} \otimes \cdots \otimes a_{k-1} \otimes a_k,$$

where $\widehat{a_{t(e,\mathtt{H})}}$ indicates the $a_{t(e,\mathtt{H})}$ is missing. We set

$$\mathcal{F}_{A,M}(\mathtt{H} \preceq \mathtt{H}') := \begin{cases} \mu_{\mathtt{H} \prec \mathtt{H}'} & \text{if } \mathtt{H} \prec \mathtt{H}', \\ \mathrm{Id}_{\mathcal{F}_{A,M}(\mathtt{H})} & \text{if } \mathtt{H} = \mathtt{H}'. \end{cases} \tag{6}$$

Equations (4) and (6) describe a functor

$$\mathcal{F}_{A,M} \colon \boldsymbol{P}(\mathtt{G}) \to R\text{–}\mathbf{Mod} \tag{7}$$

from the category $\boldsymbol{P}(\mathtt{G})$ associated to the path poset $P(\mathtt{G})$ to the additive category R-**Mod** of left R–modules. In fact, we have the following:

Lemma 41 *Let* $\mathtt{G}$ *be an ordered digraph. The assignment* $\mathcal{F}_{A,M}(\mathtt{H} \prec \mathtt{H}') := \mu_{\mathtt{H} \prec \mathtt{H}'}$ *in* (6) *preserves all the commutative squares in* $\boldsymbol{P}(\mathtt{G})$ *— see Remark 25.*

Proof The possible configurations of squares in $P(G)$ are described in the proof of Lemma 39, contained in the Appendix — see Figures 16 and 17. We leave the full checking to the dedicated reader and present here only one case, namely case (A), subcase (b). The remaining checks can be dealt with similarly. In the case at hand we have $\mathtt{H}' = \mathtt{H} \cup \{e_1, e_2\}$ with

$$t(e_2, \mathtt{H}) = i < j = s(e_1, \mathtt{H}) < s(e_2, \mathtt{H}) = t(e_1, \mathtt{H}) = k.$$

The schematic description of this configuration is shown in Figure 11. Now, we compute the two compositions directly, and we obtain

$$\mathcal{F}_{A,M}(\mathtt{H} \cup e_1 \prec \mathtt{H}') \circ \mathcal{F}_{A,M}(\mathtt{H} \prec \mathtt{H} \cup e_1)(a_0 \otimes \cdots \otimes a_h) = a_0 \otimes \cdots \otimes (a_j a_k) a_i \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes \hat{a}_k \otimes \cdots \otimes a_h,$$
$$\mathcal{F}_{A,M}(\mathtt{H} \cup e_2 \prec \mathtt{H}') \circ \mathcal{F}_{A,M}(\mathtt{H} \prec \mathtt{H} \cup e_2)(a_0 \otimes \cdots \otimes a_h) = a_0 \otimes \cdots \otimes a_j (a_k a_i) \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes \hat{a}_k \otimes \cdots \otimes a_h.$$

The statement follows in this case by the associativity of A (and the definition of (A, A)–bimodule if $i = 0$). □

Proposition 42 *Let* $\mathtt{G}$ *be an ordered digraph. The assignment* $\mathcal{F}_{A,M} \colon \boldsymbol{P}(\mathtt{G}) \to R$–**Mod** *defines a covariant functor.*

Proof It is clear that $\mathcal{F}_{A,M}$ preserves the identities. Let $f_{\mathtt{H},\mathtt{H}'}\colon \mathtt{H} \to \mathtt{H}'$ be a morphism in $\boldsymbol{P}(\mathtt{G})$. The morphism $f_{\mathtt{H},\mathtt{H}'}$ can be written as the composition of the covering morphisms $f_{\mathtt{H}_i,\mathtt{H}_{i+1}}$ for any given chain $\mathtt{H} = \mathtt{H}_0 \prec \mathtt{H}_1 \prec \cdots \prec \mathtt{H}_{n-1} = \mathtt{H}'$ in $P(\mathtt{G})$ — this is well-defined since we have a unique morphism between two related objects in $\boldsymbol{P}(\mathtt{G})$; see Remark 3. We only have to show that the composition

$$\mathcal{F}_{A,M}(\mathtt{H}_{n-2} \prec \mathtt{H}_{n-1}) \circ \cdots \circ \mathcal{F}_{A,M}(\mathtt{H}_0 \prec \mathtt{H}_1)$$

depends only on $\mathtt{H} = \mathtt{H}_0$ and $\mathtt{H}' = \mathtt{H}_{n-1}$ and not on the chosen chain.

Note that $\mathtt{H}' = \mathtt{H} \cup \{e_1, \dots, e_{n-1}\}$, and each chain corresponds to a choice of the order in which we add the edges $e_1, \dots, e_{n-1}$ to $\mathtt{H}$. Therefore, the proof boils down to showing that we can switch the order in which we add two edges to $\mathtt{H}$. This is equivalent to showing that $\mathcal{F}_{A,M}$ preserves the commutative squares in $\boldsymbol{P}(\mathtt{G})$. Thus, the proposition follows directly from Lemma 41. □

The above proof shows that both the poset $P(\mathtt{G})$ and its squared faithful subposets are, in the language of [Chandler 2019], diamond transitive. For a more general proof of this fact in the case of downward or upward closed subposets of $\mathrm{SSG}(\mathtt{G})$, or even more in general, the reader can consult [Chandler 2019].

We conclude this section with the following theorem which is an immediate consequence of Theorem 26, Lemma 39, and Proposition 42.

Theorem 43 *Given a graph* $\mathtt{G}$ *the graded* R*–module* $C^*_\mu(\mathtt{G}; A, M) := C^*_{\mathcal{F}_{A,M}}(P(\mathtt{G}))$ *endowed with the map* $d^* := d^*_{\mathcal{F}_{A,M},\sigma}$ *is a cochain complex.*

By Corollary 36, up to isomorphism of chain complexes, $(C^*_\mu(\mathtt{G}; A, M), d^*)$ does not depend on the choice of the sign σ_e.

Assume now that M is isomorphic to A as an (A, A)–bimodule. Then, the chain complex does not depend, up to isomorphism, on the given ordering of the vertices of the graph. In other words, the isomorphism class of $(C^*_\mu(\mathtt{G}; A, A), d^*)$ depends only on the underlying graph $\mathtt{G}$ and on the algebra A:

Proposition 44 *Let* $\mathtt{G}$ *be an ordered digraph. Then, the cochain complex* $(C^*_\mu(\mathtt{G}; A, A), d^*)$ *does not depend on the choice of the ordering on* $V(\mathtt{G})$.

Proof A permutation of the ordering on the vertices of $\mathtt{G}$ induces for each $\mathtt{H} \in P(\mathtt{G})$ a permutation on the factors appearing in $\mathcal{F}(\mathtt{H})$. There is an induced natural isomorphism of modules induced by the latter, which extends to an isomorphism of chain complexes; the commutativity with the differentials is clear up to sign. The statement now follows by Corollary 36. □

Remark 45 More generally, the cochain complex $(C^*_\mu(\mathtt{G}; A, M), d^*)$ does not depend, up to isomorphisms, on the choice of the order on $V(\mathtt{G})$ preserving the minimum — which can be considered as a base vertex.

We are ready to give the main definition of the paper:

Definition 46 The *multipath cohomology* $\mathrm{H}^*_\mu(\mathtt{G}; A, M)$ of a digraph $\mathtt{G}$ with (A, M)–coefficients is the homology of the cochain complex $(C^*_\mu(\mathtt{G}; A, M), d^*)$. When $A = M$ we simply write $\mathrm{H}^*_\mu(\mathtt{G}; A)$.

Consider the category $\mathbf{Digraph}_*$ of pointed digraphs, ie digraphs with the choice of a base vertex, and morphisms of pointed digraphs, ie morphisms of digraphs that preserve the base vertex. Then, we can define multipath cohomology of a pointed digraph $(\mathtt{G}, v_0)$ with (A, M)–coefficients as the homology of the cochain complex $(C^*_\mu(\mathtt{G}; A, M), d^*)$. Note that in the case $M \neq A$ we need to keep track of the base vertex because the associated cohomology groups $\mathrm{H}^*_\mu(\mathtt{G}; A, M)$ may depend upon this choice — see Remark 51.

We conclude this subsection by observing that the sign assignment on $\mathrm{SSG}(\mathtt{G})$, given in Example 23, induces, by restriction, a sign assignment on the path poset $P(\mathtt{G})$. The cochain complex obtained from this sign assignment and the one obtained from σ_{e} are isomorphic. However, this is not true for more general subposets of $P(\mathtt{G})$ (as it depends on their topology) and the two constructions may lead to nonisomorphic cohomology theories of digraphs.

4.2 Computations and examples

In this section we provide some computations of multipath cohomology — see Table 1. Further calculations, new computational tools and more general results concerning the structure of multipath homology, are developed in [Caputi et al. 2023; 2024b].

For the whole section, unless otherwise specified, we will always implicitly assume both M and A to be the ground ring R, and $R = \mathbb{K}$ to be a field. Tensor products $\otimes$ will always be tensor products over $\mathbb{K}$.

We remark here that, from our computations, see Table 1, it follows that there exist trees with nontrivial multipath cohomology. Most digraph homology theories known to the authors — as path homology, clique homology and Hochschild homology of digraphs, or Turner–Wagner homology with constant coefficients — vanish on trees.

Our first example is the noncoherently oriented linear digraph on three vertices. In this case we provide the explicit computation of the multipath cohomology both with constant coefficients, that is, $M = A = \mathbb{K}$, and in a nonconstant setting, namely $M = A = \mathbb{K}[x]/(x^2)$. As we will see, these two coefficients provide different cohomologies, showing that the multipath cohomology actually depends on the choices of A and M. Note also that, being the base ring a field, this example additionally shows that the classical universal coefficients theorem is not sufficient to recover the cohomology computed using A from the cohomology computed using R.

Example 47 Let $\mathtt{G}$ be the noncoherent linear digraph on three vertices v_0, v_1, v_2 — see Figure 8. Application of the functor $\mathcal{F}_{A,A}$ on (the category associated to) its path poset $P(\mathtt{G})$ gives the following diagram of $\mathbb{K}$–modules:

$$A_{v_0} \otimes_{\mathbb{K}} A_{v_1} \otimes_{\mathbb{K}} A_{v_2} \xrightarrow{(m \otimes \mathrm{Id}_A) \oplus (\mathrm{Id}_A \otimes m)} A_{(v_0,v_1)} \otimes_{\mathbb{K}} A_{v_2} \oplus A_{v_0} \otimes_{\mathbb{K}} A_{(v_2,v_1)},$$

where we have decorated the modules with the components of the corresponding multipaths, and the arrows with the induced signs σ_e as by (3). The map on the left sends the elementary tensor product $a_0 \otimes a_1 \otimes a_2 \in A_{v_0} \otimes_{\mathbb{K}} A_{v_1} \otimes_{\mathbb{K}} A_{v_2}$ to the element $(a_0 \cdot a_1) \otimes a_2 \in A_{(v_0,v_1)} \otimes_{\mathbb{K}} A_{v_2}$, whereas the map on the right sends the same element to $a_0 \otimes (a_2 \cdot a_1) \in A_{v_0} \otimes_{\mathbb{K}} A_{(v_2,v_1)}$. If $A = \mathbb{K}$, using the identification $\mathbb{K} \otimes_{\mathbb{K}} \mathbb{K} \cong \mathbb{K}$ and the commutativity of $\mathbb{K}$, we get the cochain complex

$$0 \to \mathbb{K} \xrightarrow{d^0 = (\mathrm{Id}_{\mathbb{K}}, \mathrm{Id}_{\mathbb{K}})} \mathbb{K}^2 \xrightarrow{d^1 = 0} 0.$$

It is now straightforward that the homology of such a cochain complex is concentrated in degree 1 and is of dimension 1.

Now, take A to be the $\mathbb{K}$–algebra $\mathbb{K}[x]/(x^2)$. Fix the basis $e_0 = 1$ and $e_1 = x$ for A as a $\mathbb{K}$–vector space. The basis for a tensor product of copies of A will be given by elementary tensors of e_0 and e_1 ordered lexicographically. We can now write explicitly the matrix associated to the differential d^0 with respect to these bases, which yields a matrix M_{d^0} of rank 6 (over any field). Therefore, we have that $\dim(H^0_\mu(\mathtt{G}; A, A)) = \dim(\mathrm{Ker}(d^0)) = 2$, and that $\dim(H^1_\mu(\mathtt{G}; A, A)) = 8 - \dim(\mathrm{Img}(d^1)) = 2$, concluding our computations.

To facilitate the calculations in the remaining examples, we will use some basic notions of algebraic Morse theory; a general reference is [Kozlov 2008, Chapter 11, Section 3]. Roughly speaking, algebraic Morse theory gives a way to reduce a (co)chain complex by eliminating acyclic summands via changes of bases.

The theory works as follows: Consider a finitely generated complex of $\mathbb{K}$–vector spaces, say (C^*, d^*), and a basis $B_i = \{b^i_j\}_j$ of C^i as a $\mathbb{K}$–vector space for each i. With respect to these bases, the differential can be expressed as

$$d(b^i_j) = \sum_h c^{i+1}_{j,h} b^{i+1}_h$$

for some $c^{i+1}_{j,h} \in \mathbb{K}$. One can now construct a digraph $\mathtt{C}$ by taking $V(\mathtt{C}) = \bigcup_i B_i$, and $(b^i_k, b^j_h) \in E(\mathtt{C})$ if, and only if, $i = j - 1$ and the coefficient $c^{i+1}_{k,h}$ is nontrivial.

An *acyclic matching* M on a graph $\mathtt{C}$ is a subset of pairwise disjoint[3] edges of $\mathtt{C}$ such that the graph obtained from $\mathtt{C}$ by changing the orientations of the edges in M has no cycles, ie there are no embedded copies of $\mathtt{P}_n$ in $\mathtt{C}$.

The main result in algebraic Morse theory (see [Kozlov 2008, Theorem 11.24]) is that, given an acyclic matching M on $\mathtt{C}$, the complex (C^*, d^*) is quasi-isomorphic to a complex (C^*_M, d^*_M), where C^i_M is generated by all the b^i_j's that are not incident to the edges in M.

Remark 48 If M is an acyclic matching and $\{v \in V(\mathtt{C}) \mid v = s(e) \text{ or } v = t(e),\ e \in M\} = V(\mathtt{C})$, then the complex (C^*, d^*) has trivial homology.

[3]Two edges e and e' are said to be disjoint if the sets $\{s(e), t(e)\}$ and $\{s(e'), t(e')\}$ are disjoint.

Remark 49 If M is an acyclic matching and

$$V(\mathtt{C}) \setminus \{v \in V(\mathtt{C}) \mid v = s(e) \text{ or } v = t(e),\ e \in M\} \subseteq B_i$$

for a fixed i, then (C^*_M, d^*_M) is concentrated in degree i. Therefore, (C^*_M, d^*_M) has a trivial differential. Hence the homology of (C^*, d^*) is concentrated in degree i, and it is isomorphic to C^i_M.

In the following examples, for each digraph $\mathtt{G}$, we can take the graph $\mathtt{C}$ to be the Hasse graph of the path poset $P(\mathtt{G})$. This is due to the following two facts:

- All tensor products are taken over $\mathbb{K}$ and $A = M = \mathbb{K}$; hence $\mathcal{F}_{A,M}(\mathtt{H}) \cong \mathbb{K}$ has a single generator for each multipath $\mathtt{H}$ in the path poset.
- For each pair of multipaths $\mathtt{H}, \mathtt{H}'$ such that $\mathtt{H} \prec \mathtt{H}'$, the map $\mathcal{F}_{A,M}(\mathtt{H} \prec \mathtt{H}')$, under the identifications $\mathcal{F}_{A,M}(\mathtt{H}) \cong \mathbb{K}$ and $\mathcal{F}_{A,M}(\mathtt{H}') \cong \mathbb{K}$, can be taken to be the identity up to a sign.

We can now proceed with the computation of the multipath cohomology of the n–step graph $\mathtt{I}_n$.

Example 50 Let $\mathtt{I}_n$ be the n–step graph in Figure 1. We claim that $\mathrm{H}^*_\mu(\mathtt{I}_n; \mathbb{K}) = 0$ for all $n > 0$.

If $n = 0$, we have the degenerate case where $\mathtt{I}_n$ is just a vertex with no edges. By definition, the cochain complex $(C^*_\mu(\mathtt{I}_0; \mathbb{K}), d^*)$ is just a copy of $\mathbb{K}$ in degree 0 and has trivial differential. Hence, we have $\mathrm{H}^*_\mu(\mathtt{I}_0; \mathbb{K}) = \mathrm{H}^0_\mu(\mathtt{I}_0; \mathbb{K}) = \mathbb{K}$.

Let us turn back to the general computation. Notice that the path poset $P(\mathtt{I}_n)$ is a Boolean poset — see Example 13. Since $\mathcal{F}_{A,M}(\mathtt{H}) \cong \mathbb{K}$ for each multipath $\mathtt{H} \in P(\mathtt{I}_n)$, it follows that

$$C^k_\mu(\mathtt{I}_n; \mathbb{K}, \mathbb{K}) = \bigoplus_{\substack{\mathtt{H} \in P \\ \ell(\mathtt{H}) = k}} \mathcal{F}_{A,M}(\mathtt{H}) \cong \mathbb{K}^{\binom{n}{k}}$$

for each $k = 0, \dots, n$. In other words, the resulting cochain complex $C^*_\mu(\mathtt{I}_n; \mathbb{K}, \mathbb{K})$ is of the form

$$0 \to \mathbb{K} \xrightarrow{d^0} \mathbb{K}^n \xrightarrow{d^1} \cdots \to \mathbb{K}^{\binom{n}{k}} \xrightarrow{d^{\binom{n}{k}}} \mathbb{K}^{\binom{n}{k+1}} \to \cdots \to \mathbb{K}^n \xrightarrow{d^n} \mathbb{K} \xrightarrow{d^{n+1}} 0.$$

An acyclic matching (the check of the nonexistence of cycles is left to the reader) on the Hasse graph of $P(\mathtt{I}_n) \cong \wp(\{0, \dots, n-1\})$ is given by all edges $(s, s \cup \{0\})$ with $s \in \wp(\{1, \dots, n-1\})$. Since each $s \in \wp(\{0, \dots, n-1\})$ either contains 0 or does not, this matching touches all vertices of $\mathtt{Hasse}(P(\mathtt{I}_n))$, and our claim follows from Remark 48.

In the general case $A \neq \mathbb{K}$, the computation that $\mathrm{H}^*_\mu(\mathtt{I}_n; A) = 0$ is more convoluted. In Corollary 81 we will prove the claim for every unital algebra A and positive degrees. In [Caputi et al. 2023] we prove a more general result on the vanishing of multipath cohomology for $A = \mathbb{K}$.

In degree 0, the multipath cohomology is possibly not trivial — eg $\mathrm{H}^*_\mu(\mathtt{I}_n; A) \neq 0$; see Corollary 81. In the next remark we see that $\mathrm{H}^*_\mu(-; A, M)$, when $M \neq A$, depends on the choice of the base vertex.

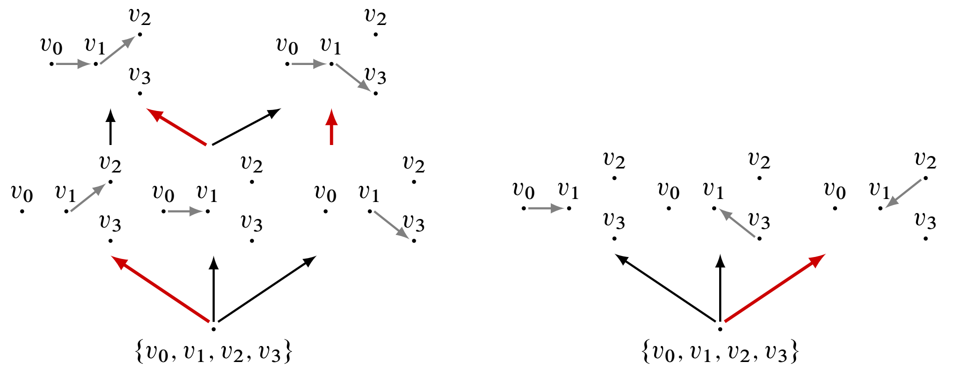

Figure 12: Acyclic matchings (in red and thicker) in the path posets of the graphs $\mathtt{Y}_1$(left) and $\mathtt{Y}_2$ (right) depicted in Figure 4(a) and (c).

Remark 51 When the bimodule M is not the R–algebra A itself, the multipath cohomology of a digraph $\mathtt{G}$ may depend upon the choice of the base vertex. As an example, let $\mathtt{I}_2$ be the 2–step graph on vertices v and w and the only directed edge (v, w). Choose first v to be the base vertex; then, the associated cochain complex $\mathrm{C}_\mu(\mathtt{G}; A, M)$ is

$$0 \to M \otimes A \xrightarrow{d^0} M \to 0,$$

where d_0 is induced by the left action: $(m \otimes a) \mapsto m \cdot a$. If we choose the base vertex to be w, we get

$$0 \to A \otimes M \xrightarrow{d^0} M \to 0,$$

where now d_0 is induced by the right action: $(a \otimes m) \mapsto a \cdot m$. Therefore, if left and right action do not agree, then the homology groups may differ, in this case.

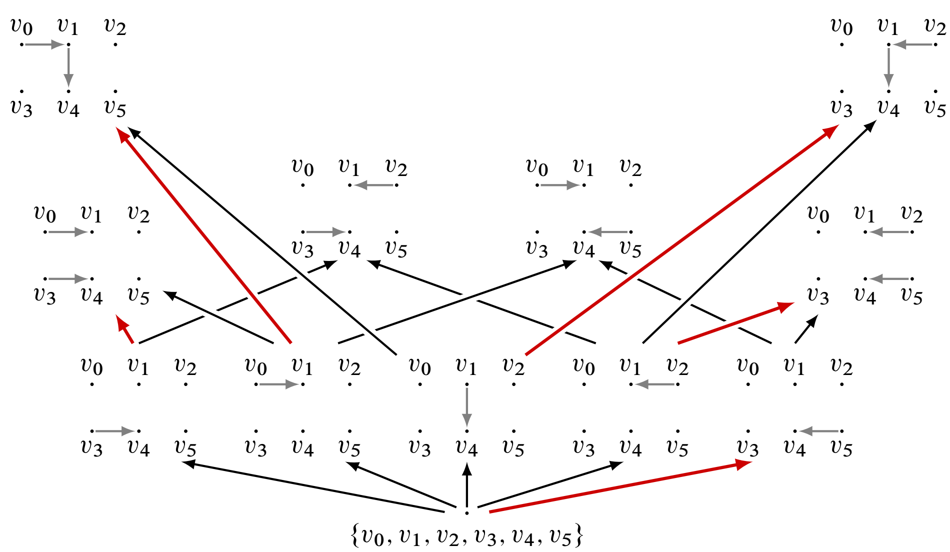

Figure 13: An acyclic matching (in red and thicker) in the path poset of the H–shaped digraph in Figure 6.

We proceed with the computation of the multipath cohomology groups of the examples in Table 1.

Example 52 Consider the graphs $\mathtt{Y}_1$ and $\mathtt{Y}_2$ depicted in Figure 4(a) and (c), respectively. Let us start with $\mathtt{Y}_1$. In this case we have an acyclic matching on $\mathtt{Hasse}(P(\mathtt{Y}_1))$ which touches all vertices (see Figure 12). It follows from Remark 48 that $\mathrm{H}^*_\mu(\mathtt{Y}_1;\mathbb{K}) = 0$.

Moving on to the graph $\mathtt{Y}_2$, all (nonempty) acyclic matchings on $\mathtt{Hasse}(P(\mathtt{Y}_2))$ consist of a single edge going from $(\mathtt{Y}_2)_\varnothing$ (ie the multipath with no edges) to a multipath with a single edge (eg see Figure 12). This leaves only two vertices unmatched (ie not incident to the edges in the matching), both corresponding to multipaths with a single edge (and thus representing two generators in cohomological degree 1). It follows from Remark 49 that $\mathrm{H}^*_\mu(\mathtt{Y}_2;\mathbb{K}) = H^1_\mu(\mathtt{Y}_2;\mathbb{K}) \cong \mathbb{K}^2$.

Example 53 Let $\mathtt{G}$ be the digraph illustrated in Figure 6. An acyclic matching M on the Hasse graph associated to $P(\mathtt{G})$ is shown in Figure 13. There are only two multipaths not incident to the edges in M, both with two edges. It follows from Remark 49 that $\mathrm{H}^*_\mu(\mathtt{G};\mathbb{K}) = H^2_\mu(\mathtt{G};\mathbb{K}) \cong \mathbb{K}^2$.

5 Functorial properties and exact sequences

The aim of the following section is to better understand the functorial properties of multipath cohomology. The machinery developed here can be adapted also to other contexts and to the general framework described in Section 3. As an application, in Section 7.2 we will clarify the relationship between multipath cohomology and chromatic homology.

Let **Digraph**(n) be the subcategory of the category **Digraph** consisting of digraphs with precisely n vertices, and morphisms of digraphs. In this section, among others, we prove the following functoriality result, which is one of the main results of the paper.

Theorem 54 (Theorem 1) *Let R–**Alg** be the category of unital R–algebras, **Digraph**$^{\mathrm{op}}(n)$ the opposite category of **Digraph**(n), and R–**Mod**$^{\mathrm{gr}}$ the category of graded R–modules. Then, multipath cohomology*

$$\mathrm{H}_\mu\colon \mathbf{Digraph}^{\mathrm{op}}(n) \times R\text{–}\mathbf{Alg} \to R\text{–}\mathbf{Mod}^{\mathrm{gr}}$$

is a bifunctor for all $n \in \mathbb{N}$.

We start by discussing the functoriality of multipath cohomology with respect to the algebras.

Proposition 55 *Let $\mathtt{G}$ be a graph, and let P be a squared and faithful subposet of $\mathrm{SSG}(\mathtt{G})$ with a fixed sign assignment. Then*

$$\mathrm{H}^*_{\mathcal{F}_{-,-}}(P)\colon R\text{–}\mathbf{Alg} \to R\text{–}\mathbf{Mod}^{\mathrm{gr}},$$

which associates to A the graded R–module $\mathrm{H}^*_{\mathcal{F}_{A,A}}(P)$, is a covariant functor. In particular, the multipath cohomology of a fixed graph is covariant with respect to morphisms of R–algebras.

Proof Let A be an R–algebra, and let $f\colon A \to B$ be a homomorphism of R–algebras. Recall the definition of the functor $\mathcal{F}_{A,A}$ from Section 4.1; we have $\mathcal{F}_{A,A}(\mathtt{H}) := A_{c_1} \otimes_R \cdots \otimes_R A_{c_k}$ for each $\mathtt{H} \in P$, and $\mathcal{F}_{A,A}(\mathtt{H} \prec \mathtt{H}')$ is induced by the multiplication. Since $f\colon A \to B$ is an R–algebra homomorphism, it induces maps between the tensor powers

$$f \otimes \cdots \otimes f\colon A_{c_1} \otimes_R \cdots \otimes_R A_{c_k} = \mathcal{F}_{A,A}(\mathtt{H}) \to \mathcal{F}_{B,B}(\mathtt{H}) = B_{c_1} \otimes_R \cdots \otimes_R B_{c_k}.$$

For each $n \in \mathbb{N}$, these extend to a map

$$C^n_{\mathcal{F}_{A,A}}(P) = \bigoplus_{\substack{\mathtt{H}\in P\\ \ell(\mathtt{H})=n}} \mathcal{F}_{A,A}(\mathtt{H}) \to \bigoplus_{\substack{\mathtt{H}\in P\\ \ell(\mathtt{H})=n}} \mathcal{F}_{B,B}(\mathtt{H}) = C^n_{\mathcal{F}_{B,B}}(P)$$

because f extends linearly to directed sums. Note that the sign assignment is the same on both complexes. Since f commutes with the multiplication, the induced map commutes with the differentials. Thus the map induced by f is a map of cochain complexes. The fact that this construction respects compositions is straightforward since $f^{\otimes k} \circ g^{\otimes k} = (f \circ g)^{\otimes k}$ for any composable morphisms of R–algebras f and g. □

We remark here that an R–algebra homomorphism $f\colon A \to B$ provides a natural transformation between the two functors $\mathcal{F}_{A,A}\colon \boldsymbol{P} \to R$–**Mod** and $\mathcal{F}_{B,B}\colon \boldsymbol{P} \to R$–**Mod**; this follows since f extends to tensor powers and directed sums. A natural transformation $\eta\colon \mathcal{F} \to \mathcal{G}$ between two functors $\mathcal{F}, \mathcal{G}\colon \boldsymbol{P} \to \boldsymbol{A}$, which preserves the biproducts in $\boldsymbol{A}$, induces a morphism of cochain complexes $\eta^*\colon C_{\mathcal{F}}(P) \to C_{\mathcal{G}}(P)$.

Before turning back to multipath cohomology, we consider the behaviour of the cohomology $\mathrm{H}_{\mathcal{F}}$ under a change of graph. First, we need a "coherent" way to choose, for each graph, a squared subposet of $\mathrm{SSG}(\mathtt{G})$. Recall that for a poset P we denote by $\boldsymbol{P}$ the associated category — see Remark 3.

Definition 56 Let $S\colon$ **Digraph** $\to$ **Poset** be a covariant functor. The functor S is called *path-like* if the following properties hold for each regular morphism of digraphs $\phi\colon \mathtt{G}' \to \mathtt{G}$:

- $S(\mathtt{G}) \subseteq \mathrm{SSG}(\mathtt{G})$ is a faithful subposet.
- $S(\phi)(S(\mathtt{G}'))$ is a downward closed subposet of $S(\mathtt{G})$.
- $S(\phi)$, seen as a functor between the associated categories $\boldsymbol{S}(\mathtt{G}')$ and $\boldsymbol{S}(\mathtt{G})$, is faithful[4] as a functor.

Example 57 The functors $\mathrm{SSG}\colon$ **Digraph** $\to$ **Poset** and $P\colon$ **Digraph** $\to$ **Poset** associating to a digraph $\mathtt{G}$ the poset of spanning subgraphs and the path poset, respectively, are path-like functors. This follows from Remark 17 in the case of the functor P; in a similar way, this is also true for the functor SSG.

Observe that, if S is a path-like functor, then $S(\mathtt{G}) = S(\mathrm{Id}_{\mathtt{G}})(S(\mathtt{G}))$ is squared.

The second ingredient needed is a way to fix $\mathcal{F}$ for each graph. Let $S\colon$ **Digraph** $\to$ **Poset** be a covariant functor and $\boldsymbol{A}$ an Abelian category (eg R–**Mod**).

[4] A functor is called *faithful* if, for each pair of objects, it is injective on the sets of morphisms between them.

Definition 58 A *coefficients system* for S is family of functors $\{\mathcal{F}_{S,\mathtt{G}}\colon \boldsymbol{S}(\mathtt{G}) \to \boldsymbol{A}\}_{\mathtt{G}}$ such that, given a regular morphism of digraphs $\phi\colon \mathtt{G}' \to \mathtt{G}$, the associated functor $S(\phi)\colon \boldsymbol{S}(\mathtt{G}') \to \boldsymbol{S}(\mathtt{G})$ makes the following diagram commute:

$$\begin{array}{ccc} \boldsymbol{S}(\mathtt{G}') & \xrightarrow{\;S(\phi)\;} & \boldsymbol{S}(\mathtt{G}) \\ & {\scriptstyle \mathcal{F}_{S,\mathtt{G}'}} \searrow \quad \swarrow {\scriptstyle \mathcal{F}_{S,\mathtt{G}}} & \\ & \boldsymbol{A} & \end{array}$$

Remark 59 The functor $\mathcal{F}_{A,A}$ is not a coefficients system for the functor path poset P unless either we restrict to $\mathbf{Digraph}(n) \subset \mathbf{Digraph}$, or we work with constant coefficients — ie $A = R$.

Notation 60 For $\phi\colon \mathtt{G}' \to \mathtt{G}$ a regular map of digraphs and $S\colon \mathbf{Digraph} \to \mathbf{Poset}$ a functor, we denote by $S_\phi(\mathtt{G}', \mathtt{G})$ the poset

$$S_\phi(\mathtt{G}', \mathtt{G}) := S(\mathtt{G}) \setminus S(\phi)(S(\mathtt{G}')).$$

We are ready to compare, under mild hypotheses, the cochain complexes associated to two graphs.

Remark 61 Recall that the complex $C^*_{\mathcal{F}}(P)$ depends also on a sign assignment ϵ on P and should have been denoted by $C^*_{\mathcal{F}}(P,\epsilon)$. By Theorem 34, if $P \subseteq \mathrm{SSG}(\mathtt{G})$ is upward or downward closed, then $C^*_{\mathcal{F}}(P,\epsilon) \cong C^*_{\mathcal{F}}(P,\epsilon')$ for any two sign assignments ϵ, ϵ' on P. This fact motivated the removal of the sign assignment from the notation.

When comparing complexes associated to different graphs, their subcomplexes, or their quotient complexes, we need to be more careful; it is often the case that we have a chain map

$$f_0\colon C^*_{\mathcal{F}}(P,\epsilon_0) \to C^*_{\mathcal{F}'}(P',\epsilon'_0),$$

while we might need a chain map

$$f_1\colon C^*_{\mathcal{F}}(P,\epsilon_1) \to C^*_{\mathcal{F}'}(P',\epsilon'_1).$$

In our case, P and P' will be either upward or downward closed. Hence, to obtain f_1 it is sufficient to compose f_0 with isomorphisms associated to the change of sign assignments, say η and η', such that the diagram

$$\begin{array}{ccc} C^*_{\mathcal{F}}(P,\epsilon_1) & \xrightarrow{\;f_1\;} & C^*_{\mathcal{F}'}(P',\epsilon'_1) \\ {\scriptstyle \eta}\big\downarrow & & \big\uparrow{\scriptstyle \eta'} \\ C^*_{\mathcal{F}}(P,\epsilon_0) & \xrightarrow[\;f_0\;]{} & C^*_{\mathcal{F}'}(P',\epsilon'_0) \end{array}$$

is commutative. Formally, in order to prove functoriality, one needs to find a coherent way to fix the isomorphisms η, η' once and for all. One approach would be to extend the category of posets to pairs of poset-sign assignments, and expand the notion of coefficient systems to this setting. This can be done formally — compare with [Chandler 2019, Sections 6 and 7], where a similar approach is pursued. Nonetheless, for the sake of simplicity and to ease the notation, signs and the induced isomorphisms will be treated naively in this section. We do not fix them nor require compatibility; instead we just make use of the existence of such isomorphisms.

From now on, given any $P \subseteq \mathrm{SSG}(\mathtt{G})$, for some digraph $\mathtt{G}$, we fix a sign assignment on P as a restriction of a (fixed) sign assignment on $\mathrm{SSG}(\mathtt{G})$. This choice is immaterial, up to isomorphism of the complex $C^*_{\mathcal{F}}(P)$, when assuming P to be a downward (or upward) closed subposet of $\mathrm{SSG}(\mathtt{G})$ by Theorem 34.

Recall that ℓ_P denotes the level in a faithful subposet $P \subset \mathrm{SG}(\mathtt{G})$; see Definition 24.

Proposition 62 *Let S : **Digraph** → **Poset** be a path-like functor, and $\mathcal{F}_{S,-}$ be a coefficient system for S. Then, we have the following short exact sequence of cochain complexes:*

$$0 \to C^*_{\mathcal{F}_{S,\mathtt{G}}}(S_\phi(\mathtt{G}',\mathtt{G}))\big[-\min_{x\in S_\phi(\mathtt{G}',\mathtt{G})}\{\ell_{S(\mathtt{G})}(x)\}\big] \to C^*_{\mathcal{F}_{S,\mathtt{G}}}(S(\mathtt{G})) \to C^*_{\mathcal{F}_{S,\mathtt{G}'}}(S(\mathtt{G}')) \to 0.$$

Proof By definition, we have

$$C^n_{\mathcal{F}_{S,\mathtt{G}}}(S(\mathtt{G})) = \bigoplus_{\substack{\mathtt{H}\in S(\mathtt{G})\\ \ell_{S(\mathtt{G})}(\mathtt{H})=n}} \mathcal{F}_{S,\mathtt{G}}(\mathtt{H}),$$

$$C^n_{\mathcal{F}_{S,\mathtt{G}}}(S_\phi(\mathtt{G}',\mathtt{G}))\big[-\min_{x\in S_\phi(\mathtt{G}',\mathtt{G})}\{\ell_{\mathtt{G}}(x)\}\big] = \bigoplus_{\substack{\mathtt{H}\in S_\phi(\mathtt{G}',\mathtt{G})\\ \ell_{S(\mathtt{G})}(\mathtt{H})=n}} \mathcal{F}_{S,\mathtt{G}}(\mathtt{H}),$$

where we used $\ell_{S(\mathtt{G})}(\mathtt{H}) = \ell_{S_\phi(\mathtt{G}',\mathtt{G})}(\mathtt{H}) + \min\{\ell_{\mathtt{G}}(x) \mid x \in S_\phi(\mathtt{G}',\mathtt{G})\}$ for $\mathtt{H} \in S_\phi(\mathtt{G}',\mathtt{G})$. As a consequence, we get a natural inclusion of cochain complexes. Note that the inclusion commutes with the differential due to the fact that the poset $S_\phi(\mathtt{G}',\mathtt{G})$ is upward closed, and the sign assignment on $S_\phi(\mathtt{G}',\mathtt{G})$ is induced by $\mathrm{SSG}(\mathtt{G})$.

We need to identify the quotient, with respect to this inclusion, with the cochain complex associated to $S(\mathtt{G}')$. At the level of modules, we have

$$\frac{C^n_{\mathcal{F}_{S,\mathtt{G}}}(S(\mathtt{G}))}{C^n_{\mathcal{F}_{S,\mathtt{G}}}(S_\phi(\mathtt{G}',\mathtt{G}))\big[-\min_{x\in S_\phi(\mathtt{G}',\mathtt{G})}\{\ell_{\mathtt{G}}(x)\}\big]} = \bigoplus_{\substack{\mathtt{H}\in S(\mathtt{G})\setminus S_\phi(\mathtt{G}',\mathtt{G})\\ \ell_{S(\mathtt{G})}(\mathtt{H})=n}} \mathcal{F}_{S,\mathtt{G}}(\mathtt{H}).$$

Since $S(\phi)(S(\mathtt{G}'))$ and $S_\phi(\mathtt{G}',\mathtt{G})$ are, by definition, complementary in $S(\mathtt{G})$, we can identify the above quotient with $C^*_{\mathcal{F}_{S,\mathtt{G}}}(S(\phi)(S(\mathtt{G}')))$. Now, the components of the differentials corresponding to the coverings $\mathtt{H}' \prec \mathtt{H}$ with $\mathtt{H} \notin S(\phi)(S(\mathtt{G}'))$ are set to 0 in the quotient. Thus, the above identification commutes with the differentials, hence inducing an isomorphism of cochain complexes, since the sign assignment on the poset $S(\phi)(S(\mathtt{G}'))$ is induced by $\mathrm{SSG}(\mathtt{G})$.

To conclude the proof, we need to identify $C^*_{\mathcal{F}_{S,\mathtt{G}}}(S(\phi)(S(\mathtt{G}')))$ with $C^*_{\mathcal{F}_{S,\mathtt{G}'}}(S(\mathtt{G}'))$. The functor S is path-like. Therefore, by definition, we have that

$$\mathcal{F}_{S,\mathtt{G}'}(\mathtt{H}) = \mathcal{F}_{S,\mathtt{G}}(S(\phi)(\mathtt{H})),$$

and similarly for the maps associated to the covering relations. This gives us an identification of $C^*_{\mathcal{F}_{S,\mathtt{G}}}(S(\phi)(S(\mathtt{G}')))$ with $C^*_{\mathcal{F}_{S,\mathtt{G}'}}(S(\mathtt{G}'))$ as graded R–modules. Observe that there is no shift in the identification because $S(\phi)(S(\mathtt{G}'))$ is downward closed. This identification commutes with the differentials up to an isomorphism induced by a change of sign assignment in one of the complexes. Composing the quotient map with such isomorphism gives us the desired short exact sequence. □

We now consider the functor S to be either the path poset functor P or SSG, and the functor $\mathcal{F}$ to be the functor $\mathcal{F}_{A,A}$ for A an R–algebra.

Proposition 63 *Let $\phi\colon \mathtt{G}' \to \mathtt{G}$ be a regular morphism of digraphs. The inclusion of $S(\mathtt{G}')$ in $S(\mathtt{G})$ induces the following short exact sequence of complexes:*

$$0 \to C^*_{\mathcal{F}_{A,A}}(S_\phi(\mathtt{G}',\mathtt{G}))\Big[-\min_{x\in S_\phi(\mathtt{G}',\mathtt{G})}\{\ell(x)\}\Big] \to C^*_{\mathcal{F}_{A,A}}(S(\mathtt{G})) \to C^*_{\mathcal{F}_{A,A}}(S(\mathtt{G}')) \otimes A^{\otimes \#(V(\mathtt{G})\setminus V(\mathtt{G}'))} \to 0.$$

In particular, if $\mathtt{G}'$ is a spanning subgraph of $\mathtt{G}$, we have the short exact sequence

$$0 \to C^*_{\mathcal{F}_{A,A}}(S_\phi(\mathtt{G}',\mathtt{G}))\Big[-\min_{x\in S_\phi(\mathtt{G}',\mathtt{G})}\{\ell(x)\}\Big] \to C^*_{\mathcal{F}_{A,A}}(S(\mathtt{G})) \xrightarrow{\pi_{\mathtt{G},\mathtt{G}'}} C^*_{\mathcal{F}_{A,A}}(S(\mathtt{G}')) \to 0. \tag{8}$$

Proof The proof proceeds exactly as the proof of Proposition 62, until the identification of the complexes $C^*_{\mathcal{F}_{A,A}}(S(\mathtt{G}'))$ and $C^*_{\mathcal{F}_{A,A}}(S(\phi)(S(\mathtt{G}')))$. At this point, we need to use that the family of functors $\mathcal{F}_{S,-} = \mathcal{F}_{A,A}$ is a coefficient system; however, this is not true — see Remark 59. Nonetheless, we have

$$\mathcal{F}_{S,\mathtt{G}'}(\mathtt{H}) = \mathcal{F}_{S,\mathtt{G}}(S(\phi)(\mathtt{H})) \otimes A^{\otimes \#(V(\mathtt{G})\setminus V(\mathtt{G}'))},$$

and the identification extends to the maps associated to the covering relations by tensoring with the opportune tensor power of Id_A. The proof now continues exactly as in Proposition 62. We conclude the proof by observing that if $\mathtt{G}' \in \mathrm{SSG}(\mathtt{G})$, we have $A^{\otimes \#(V(\mathtt{G})\setminus V(\mathtt{G}'))} = R$, and the statement follows. □

With the same notation, we can now consider compositions of morphisms of digraphs:

Lemma 64 *If $\mathtt{G}'' \in \mathrm{SSG}(\mathtt{G})$ and $\mathtt{G}'' \subseteq \mathtt{G}' \subseteq \mathtt{G}$, then $\pi_{\mathtt{G},\mathtt{G}'} \circ \pi_{\mathtt{G}',\mathtt{G}''} = \pi_{\mathtt{G},\mathtt{G}''}$, where $\pi_{\mathtt{G},\mathtt{G}'}$ is the induced morphism in* (8).

Proof We can explicitly write the maps:

$$\begin{array}{ccc}
C^n_{\mathcal{F}_{A,A}}(S(\mathtt{G})) = \bigoplus_{\substack{\mathtt{H}\in S(\mathtt{G})\\ \ell_{\mathtt{G}}(\mathtt{H})=n}} \mathcal{F}_{A,A}(\mathtt{H}) & \xrightarrow{\pi_{\mathtt{G},\mathtt{G}''}} & \bigoplus_{\substack{\mathtt{H}\in S(\mathtt{G}'')\\ \ell_{\mathtt{G}}(\mathtt{H})=n}} \mathcal{F}_{A,A}(\mathtt{H}) = C^n_{\mathcal{F}_{A,A}}(S(\mathtt{G}'')) \\
\downarrow \pi_{\mathtt{G},\mathtt{G}'} & \nearrow \pi_{\mathtt{G}',\mathtt{G}''} & \\
C^n_{\mathcal{F}_{A,A}}(S(\mathtt{G}')) = \bigoplus_{\substack{\mathtt{H}\in S(\mathtt{G}')\\ \ell_{\mathtt{G}'}(\mathtt{H})=n}} \mathcal{F}_{A,A}(\mathtt{H}) & &
\end{array}$$

Since each of the maps above restricts to the identity for $\mathtt{H}$ appearing in the summands, and is zero otherwise, we get a commutative diagram of cochain complexes. Note that we are implicitly using the fact that the (family of) functor(s) $\mathcal{F}_{A,A}$ is a coefficients system (since $\mathtt{G}', \mathtt{G}''$ are spanning subgraphs of $\mathtt{G}$) for $S = \mathrm{SSG}$ or $S = P$, and Remark 61. □

We are ready to conclude the proof of the functoriality.

Proof of Theorem 1 The statement follows from Lemma 64, giving the functoriality with respect to maps of digraphs, and Proposition 55, giving the functoriality with respect to maps of R–algebras. □

We conclude the section with the result of functoriality with respect to change of base rings:

Theorem 65 *Let* **Ring** *be the category of unital rings and* $\mathbf{Ab}^{\mathrm{gr}}$ *be the category of graded Abelian groups. Then, the multipath cohomology*

$$\mathrm{H}_\mu(-;-)\colon \mathbf{Digraph}^{\mathrm{op}} \times \mathbf{Ring} \to \mathbf{Ab}^{\mathrm{gr}}$$

is a bifunctor.

Proof For a homomorphism $f: R \to S$ of rings, there is an extension-of-scalars functor along f defined as $S \otimes_R (-)\colon R\text{–}\mathbf{Mod} \to S\text{–}\mathbf{Mod}$, where the tensor product in S is regarded as an R–module via the map f. In this way, we get natural isomorphisms $S \otimes_R R \cong S$ (more generally, it is true that if R is commutative and M an R–module, then $M \otimes_R R \cong M$), and, for each product $R \otimes_R \cdots \otimes_R R$, isomorphisms $S \otimes_R R \otimes_R \cdots \otimes_R R \cong S \otimes_R R \cong S$. Reasoning as in Lemma 64 and Theorem 1 gives the functoriality with respect to all regular maps of digraphs (with any finite number of vertices). □

6 Other poset (co)homologies and Turner and Wagner's approach

The definition of multipath cohomology given in Section 4 uses a certain homology of posets which we referred to as poset homology. After application of the path poset functor P : **Digraph** → **Poset** — see Remark 17 — other (co)homology theories of posets can also be used to get similar graph (co)homology theories, for example, the general *functor homology* (of categories) — see eg [Gabriel and Zisman 1967; Mac Lane 1971] — or the *cellular cohomology* (of posets) introduced in [Everitt and Turner 2015]. In this section, we provide a brief review of these (co)homology theories, and compare them with poset homology on (suitable modifications of) path posets. In particular, we argue that, after mild modifications, we can interpret multipath cohomology groups as (cellular and hence) functor cohomology groups, shedding light on the nature of multipath cohomology.

6.1 Functor homology (of posets)

For a poset P, recall that $\boldsymbol{P}$ denotes its associated category — see Remark 3. Given a functor $\mathcal{F}\colon \boldsymbol{P} \to \boldsymbol{A}$, where $\boldsymbol{A}$ is a complete and cocomplete Abelian category, we can define the *functor homology* (resp *cohomology*) groups $\mathrm{H}_*(\boldsymbol{P};\mathcal{F})$ (resp $\mathrm{H}^*(\boldsymbol{P};\mathcal{F})$) as the associated higher colimits (resp higher limits). For the sake of completeness, we spell out the definition. Denote by $\mathbf{1}$ the category with a single object and a single morphism. Then, there is a unique functor $\mathcal{T}\colon \boldsymbol{P} \to \mathbf{1}$. Since $\boldsymbol{A}$ is complete and cocomplete, both left and right Kan extensions of $\mathcal{F}$ exist. In particular, the left Kan extension $\operatorname{Lan}_{\mathcal{T}} \mathcal{F}$ of $\mathcal{F}$ along $\mathcal{T}$ exists, and it yields the colimit functor of $\mathcal{F}$.

Definition 66 [Mac Lane 1971] The *functor homology* $\mathrm{H}_n(\boldsymbol{P};\mathcal{F})$ *of* $\boldsymbol{P}$ *with coefficients in* $\mathcal{F}$ is the n^{th} left derived functor of $\operatorname{Lan}_{\mathcal{T}} \mathcal{F}$.

Analogously, the right Kan extension along $\mathcal{T}$ yields the limit of $\mathcal{F}$; thus, $\mathrm{H}^n(\boldsymbol{P};\mathcal{F})$ is given by the n^{th} derived functor of $\lim \mathcal{F}$. Definition 66 is rather abstract; more concretely, $\mathrm{H}_*(\boldsymbol{P};\mathcal{F})$ can be computed (see [Gabriel and Zisman 1967]) as the homology groups of the chain complex

$$\cdots \xrightarrow{\partial_n} \bigoplus_{c_0\to\cdots\to c_n} \mathcal{F}(c_0) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \bigoplus_{c_0\to c_1\to c_2} \mathcal{F}(c_0) \xrightarrow{\partial_1} \bigoplus_{c_0\to c_1} \mathcal{F}(c_0) \xrightarrow{\partial_0} \bigoplus_{c_0\in \boldsymbol{P}} \mathcal{F}(c_0) \to 0$$

with differential

$$\partial_n(f(c_0\to\cdots\to c_{n+1})) = \mathcal{F}(c_0\to c_1) f(c_1\to\cdots\to c_{n+1}) + \sum_{i=1}^{n+1} (-1)^i f(c_0\to\cdots\to \hat{c}_i \to\cdots\to c_{n+1}),$$

where $\hat{c}_i$ means that c_i is missing, and $(c_0\to\cdots\to c_n)$ denotes the inclusion of $f\in\mathcal{F}(c_0)$ into the summand corresponding to the sequence $c_0\to\cdots\to c_{n+1}$. Dually, the Roos complex [1961] computes the functor cohomology groups $\mathrm{H}^n(\boldsymbol{P};\mathcal{F})$. More precisely, $\mathrm{H}^*(\boldsymbol{P};\mathcal{F})$ is the cohomology of the cochain complex

$$0 \to \prod_{c_0\in\boldsymbol{P}} \mathcal{F}(c_0) \xrightarrow{d^0} \prod_{c_0\to c_1} \mathcal{F}(c_1) \xrightarrow{d^1} \prod_{c_0\to c_1\to c_2} \mathcal{F}(c_2) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} \prod_{c_0\to\cdots\to c_n} \mathcal{F}(c_n) \xrightarrow{d^n} \cdots$$

endowed with differential d^n, whose evaluation on $f\in\prod_{c_0\to\cdots\to c_n}\mathcal{F}(c_n)$, is given by

$$d^n(f)(c_0\to\cdots\to c_{n+1}) = (-1)^{n+1}\mathcal{F}(c_n\to c_{n+1}) f(c_0\to\cdots\to c_n) + \sum_{i=0}^{n}(-1)^i f(c_0\to\cdots\to\hat{c}_i\to\cdots\to c_{n+1}).$$

Note that here $(c_0\to\cdots\to c_n)$ denotes the projection onto the factor corresponding to the sequence $c_0\to\cdots\to c_{n+1}$. In other words, functor (co)homology groups are defined as the (co)homology groups of a suitable (co)simplicial replacement. We also point out that similar constructions can be performed using contravariant functors instead of covariant.

The homology of a category with coefficients in a functor has been extensively studied and the literature on it is very rich. When restricting to constant functors, the functor (co)homology groups depend only on the geometric realisation of the source category — see [Quillen 1973]. In particular, by Corollary 2 of that work, every poset with an initial element has with respect to the constant functor the homology of a point. We now provide an example.

Example 67 Consider the path poset associated to the digon digraph — see Figure 9. Its associated category is the pushout category $1 \leftarrow 0 \rightarrow 2$, where the initial object 0 corresponds to the empty multipath. For an Abelian category $\boldsymbol{A}$ and functor $\mathcal{F}$, set $f := \mathcal{F}(0 \to 1)$ and $g := \mathcal{F}(0 \to 2)$. The corresponding functor homology groups are the homology groups of the chain complex

$$0 \to A_0 \oplus A_0 \to A_0 \oplus A_1 \oplus A_2 \to 0,$$

where A_0, A_1, A_2 are objects of $\boldsymbol{A}$ with $\mathcal{F}(i) = A_i$, and the only nontrivial map is given by

$$(a,b) \mapsto (-(a+b), f(a), g(b)).$$

The homology groups of the complex are therefore $\mathrm{H}_0(\boldsymbol{P};\mathcal{F}) = \operatorname{colim}\mathcal{F}$, $\mathrm{H}_1(\boldsymbol{P};\mathcal{F}) \simeq \ker(f) \cap \ker(g)$, and they are 0 in higher degrees. Note that the functor cohomology groups are trivial in all degrees but the 0^{th} (in which it agrees with A_0), because the category has an initial object. Note also that the poset homology groups as defined in Section 3 would be given by the kernel and image of $f - g$.

Assume now that $\mathcal{F}$ takes values in $\boldsymbol{A} = \mathbf{Ab}$, the category of Abelian groups, and assume that $\mathcal{F}$ sends every morphism in $\boldsymbol{P}$, ie every $x \leq y$ in P, to an isomorphism of $\boldsymbol{A}$. Then, $\mathcal{F}$ induces a local coefficient system on the classifying space[5] $B\boldsymbol{P}$ of $\boldsymbol{P}$, ie on the order complex of P. Quillen [1973] has shown that there is an isomorphism

$$\mathrm{H}_*(\boldsymbol{P},\mathcal{F}) \cong \mathrm{H}_*(B\boldsymbol{P},\mathcal{F})$$

between the homology groups of the category $\boldsymbol{P}$ and the classical homology groups of the space $B\boldsymbol{P}$, with local coefficients (here for simplicity denoted with the same symbol $\mathcal{F}$). In order to show it, one considers the skeleton filtration

$$B\boldsymbol{P}^{(0)} \subseteq B\boldsymbol{P}^{(1)} \subseteq \cdots$$

and the associated spectral sequence with E^1–term $E^1_{p,q} = \mathrm{H}_{p+q}(B\boldsymbol{P}^{(p)}, B\boldsymbol{P}^{(p-1)}, \mathcal{F})$. When $q = 0$, the E^1–term yields the homology groups $\mathrm{H}_p(\boldsymbol{P},\mathcal{F})$. The spectral sequence converges to $\mathrm{H}_p(B\boldsymbol{P},\mathcal{F})$, providing the isomorphism. In a similar fashion, Turner and Everitt have defined the so-called cellular cohomology groups of posets, as we shall recall in the next subsection.

6.2 Cellular poset cohomology

Cellular poset (co)homology is a rather general (co)homology theory of posets introduced in [Everitt and Turner 2015]. The cellular poset (co)chain groups are defined using a relative version of functor (co)homology and, for a rather large class of posets, it agrees with functor (co)homology, providing a tool to the computation of the higher (co)limits of functors on posets. We now proceed by reviewing its definition in the cohomological case (the homological case is analogous).

[5] The geometric realisation of the nerve.

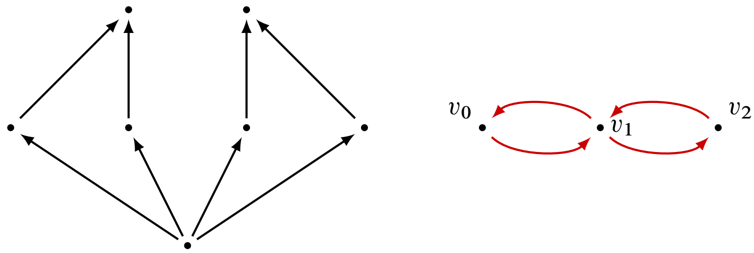

Figure 14: The poset P (left) and a graph realising P as its path poset.

In what follows, we assume that P is a finite and ranked poset, with rank function $\mathrm{rk}\colon P \to \mathbb{N}$. Let $r := \max\{\mathrm{rk}\, x \mid x \in P\}$ be the maximum rank; then one can filter P with subposets

$$P^k := \{x \in P \mid \mathrm{rk}(x) \geq r-k\},$$

yielding a filtration $P^0 \subseteq P^1 \subseteq \cdots \subseteq P^r = P$. Let $\mathcal{F}$ be a *contravariant* functor (a presheaf) on (the associated category of) P.

Definition 68 [Everitt and Turner 2015, Definition 2.1] The cellular cochain complex has cochain groups

$$\mathrm{C}^i_{\mathrm{cell}}(P;\mathcal{F}) := \mathrm{H}^n(\boldsymbol{P}^i, \boldsymbol{P}^{i-1}, \mathcal{F}),$$

where $\mathrm{H}^n(\boldsymbol{P}^i, \boldsymbol{P}^{i-1}, \mathcal{F})$ are the relative functor cohomology groups.

The differentials are also induced from functor cohomology — see [Everitt and Turner 2015] for a description of the differential. Observe that, as taking the classifying space of a category is natural, the relative cohomology groups appearing in the definition can be interpreted as the usual relative cohomology groups (of the associated classifying spaces). One can compute explicitly this complex via the formulae

$$\mathrm{C}^i_{\mathrm{cell}}(P;\mathcal{F}) = \begin{cases} \bigoplus_{\mathrm{rk}(x)=n} \mathcal{F}(x), & i=0, \\ \bigoplus_{\mathrm{rk}(x)=n-i} \widetilde{\mathrm{H}}^{i-1}(|NP_{>x}|, \mathcal{F}(x)), & i>0, \end{cases}$$

where N denotes the nerve, $|\cdot|$ denotes the geometric realisation, and $\widetilde{\mathrm{H}}^*$ denotes the usual reduced singular cohomology — see [Everitt and Turner 2015, Propositions 3, 4 and 5].

Note that the functors appearing in the definition of cellular cohomology are contravariant, and hence defined on $\boldsymbol{P}^{\mathrm{op}}$. The constant functor can be seen both as a covariant and as a contravariant functor; hence computations can be carried on in both cases. We now proceed with an example of calculation, computing the cellular cohomology of a path poset, with respect to the contravariant constant functor.

Example 69 Consider the poset P and the graph $\mathtt{G}$ represented in Figure 14. The path poset $P(\mathtt{G})$ is isomorphic to P. For a fixed field $\mathbb{K}$, consider the constant functor $\mathbb{K}$ on the category associated to the poset P.

We now compute the cellular cochain groups of the poset P. First, observe that the poset P is ranked with rank function rk given by the distance from the minimum; this function is bounded with maximum value

$r=2$, which is achieved by the maximal elements. The corank function is defined to be $|x| := 2-\mathrm{rk}(x)$. In degree 0, the cellular cochain complex is generated by the (evaluation of the constant) functor $\mathbb{K}$ at the maxima, obtaining

$$C^0_{\mathrm{cell}}(P;\mathbb{K}) \cong \mathbb{K}^2.$$

In order to analyse the higher degrees, we use [Everitt and Turner 2015, Proposition 3]

$$C^n_{\mathrm{cell}}(P;\mathbb{K}) \cong \prod_{|x|=n} \mathrm{H}^n(P_{\geq x}, P_{>x};\mathbb{K}),$$

with the convention that, $\mathrm{H}^n(P_{\geq x}, \varnothing; k) = \mathrm{H}^n(P_{\geq x}; k)$ — see [loc cit, page 140] — and where H^* denotes functor cohomology. Then, for the elements x in P of corank 1, we get

$$\mathrm{H}^1(P_{\geq x}, P_{>x};\mathbb{K}) \cong \widetilde{\mathrm{H}}^0(P_{>x};\mathbb{K})$$

by [loc cit, Proposition 4]. As $P_{>x}$ consists of a single point, we get $\widetilde{\mathrm{H}}^0(P_{>x};\mathbb{K}) \cong \widetilde{\mathrm{H}}^0(\{*\};\mathbb{K}) \cong 0$ (see [loc cit, page 140]). Therefore, we have

$$C^1(P;\mathbb{K}) \cong 0.$$

We conclude the computation of the cellular cohomology groups by analysing $C^2(P;\mathbb{K})$, as there are no elements of corank ≥ 3. There is only a single element m of corank 2, given by the minimum of P, and the geometric realisation of $P_{>m}$ consists of two intervals. By [loc cit, Propositions 3 and 4],

$$C^2_{\mathrm{cell}}(P;\mathbb{K}) \cong \widetilde{\mathrm{H}}^1(P_{>m};\mathbb{K}) \cong 0.$$

Therefore, it follows that the cellular cohomology, in this case, is concentrated in degree 0, where its dimension is 2.

Arguing as in Example 69, we have the analogue of Example 67:

Example 70 Consider the path poset associated to the digon digraph — see Figure 9. Then, P has a unique element m of rank 0 and two elements of rank 1. Then,

$$C^0_{\mathrm{cell}}(P;\mathbb{K}) \cong \mathbb{K}^2$$

generated by the elements of rank 1. The group $C^1_{\mathrm{cell}}(P;\mathbb{K})$, instead, is isomorphic to $\widetilde{\mathrm{H}}^1(P_{>m};\mathbb{K}) \cong \mathbb{K}$. The differential acts by $(x,y) \mapsto x-y$, giving $\mathrm{H}^1_{\mathrm{cell}}(P;\mathbb{K}) \cong \mathbb{K}$ and 0 in other degrees. When passing to arbitrary coefficients, as in Example 67, let $A_i := \mathcal{F}(i)$ and set $f^* := \mathcal{F}(0 \to 1)$, $g^* := \mathcal{F}(0 \to 2)$. Then, the cellular cochain complex becomes

$$0 \to A_1 \oplus A_2 \to A_0 \to 0,$$

with unique differential $(a,b) \mapsto f^*(a) - g^*(b)$.

Using a spectral sequences argument, one can prove that, for certain ranked and finite posets, cellular cohomology groups compute the higher limits of (a contravariant functor) $\mathcal{F}$. We first recall — see [Everitt and Turner 2015, Definition 3.1] — that a ranked poset is *cellular* if, and only if, for every contravariant functor $\mathcal{F}$ on $\boldsymbol{P}$, the relative functor cohomology groups $\mathrm{H}^i(\boldsymbol{P}^n, \boldsymbol{P}^{n-1}, \mathcal{F})$ are 0 for all $i \neq n$. For example, for X a regular CW–complex, the face poset $P(X)^{\mathrm{op}}$ with reversed inclusion (hence, $x \leq y$ if, and only if, $y \subseteq x$) is cellular — see [Everitt and Turner 2015, Section 4.1]. By [loc cit, Theorem 1], when P is a cellular poset and $\mathcal{F}\colon \boldsymbol{P} \to \mathbf{Ab}$ is a contravariant functor, there are isomorphisms $\mathrm{H}^*_{\mathrm{cell}}(\boldsymbol{P}, \mathcal{F}) \cong \mathrm{H}^*(\boldsymbol{P}, \mathcal{F})$ between cellular cohomology groups and the functor cohomology groups, showing that for a large class of posets cellular (co)chain groups compute the higher (co)limits.

6.3 Comparisons on path posets

In this subsection we restrict to posets arising as path posets of digraphs. The idea of defining graph homologies using the path poset is, to the best of the authors' knowledge, due to Turner and Wagner, and inspired this work. In [Turner and Wagner 2012], they make use of functor homology to define a graph homology, as the functor homology groups of the (category associated to the) path poset. In the special case $\mathcal{F} = \mathcal{F}_{A,M}$, that is, the functor defined in (7) (or, better, a symmetrised version of it, see [loc. cit.]), we get what we call the *Turner–Wagner homology* TW of $\mathtt{G}$:

$$\mathrm{TW}_*(\mathtt{G}; A, M) := \mathrm{H}_*(\boldsymbol{P}(\mathtt{G}); \mathcal{F}_{A,M}).$$

Here we point out a small technical issue; if the module M is different from A, we have to fix a base vertex, and the theory provides a homology for *based digraphs*, ie graphs with a base vertex, exactly as in our case — see Remark 45. As every category with an initial element, with respect to the constant functor, has the homology of a point, we obtain the following:

Remark 71 We have $\mathrm{TW}_0(\mathtt{G}; R, R) \cong R$ and $\mathrm{TW}_i(\mathtt{G}; R, R) = 0$ for $i > 0$.

An immediate consequence of the previous remark and of the examples in Section 4.2, along with Example 67, is the following result.

Remark 72 The (co)homologies TW and H_μ are not isomorphic nor dual to each other.

In order to understand the precise relation between the multipath cohomology of a graph and the Turner–Wagner homology, we use cellular cohomology as an intermediate theory. In the following, we aim to show that, after some mild modifications of the path poset, all these theories agree. However, despite the similarities, it is easy to see that these are different "on the nose":

Example 73 Consider the path poset $\mathtt{P}_1$ of the digon graph — see Figure 3. Note that the associated category is the pushout category $1 \leftarrow 0 \rightarrow 2$. As shown in Example 67, for an algebra A and the functor $\mathcal{F}_{A,A}\colon \boldsymbol{P} \to \boldsymbol{A}$ described in (7), we have $\mathcal{F}_{A,A}(0) = A \otimes A$ and $\mathcal{F}_{A,A}(1) = \mathcal{F}_{A,A}(2) = A$. The functor homology groups are the homology groups of the complex

$$0 \to A \otimes A \oplus A \otimes A \to A \otimes A \oplus A \oplus A \to 0$$

whose differential is given by

$$(a_0 \otimes b_0, a_1 \otimes b_1) \mapsto (a_0 \otimes b_0 + a_1 \otimes b_1, -a_0 b_0, -a_1 b_1).$$

Note that, as the path poset has a minimum, the functor *cohomology* groups are all trivial in higher degree, and isomorphic to $A \otimes A$ in degree 0. The functor $\mathcal{F}_{A,A}$ is not directly defined on $\boldsymbol{P}^{\mathrm{op}}$, so we cannot directly compute the associated cellular cohomology groups. However, $\mathcal{F}_{A,A}$ can be seen as a contravariant functor on $\boldsymbol{P}^{\mathrm{op}}$, in which case the cellular cohomology groups can be computed. Observe that the only nontrivial cellular cochain group in this case is $C^0_{\mathrm{cell}}(\boldsymbol{P}^{\mathrm{op}}, \mathcal{F}_{A,A}) \cong A \otimes A$. Note also that the cellular *homology* groups would be trivial because of the analogue of [Everitt and Turner 2015, Theorem 1] in this context. When considering the multipath cohomology cochain complex, we get

$$0 \to A \otimes A \to A \oplus A \to 0,$$

with unique differential

$$a \otimes b \mapsto (ab, -ba).$$

To be concrete, when $A = \mathbb{K}$ we get that functor homology and cellular cohomology are both of dimension 1 concentrated in degree 0, whereas multipath cohomology is of dimension 0 concentrated in degree 1.

The previous example shows that the poset homology theories described in this section, when evaluated at the path poset, are not the same on the nose. However, they become all equivalent after some mild modification of the path poset, as we now shall explain.

Let $\mathtt{G}$ be a digraph and let $\boldsymbol{P}(\mathtt{G})^{\mathrm{op}}$ be the opposite category (with the same objects as $\boldsymbol{P}(\mathtt{G})$ but reversed arrows) of $\boldsymbol{P}(\mathtt{G})$. Consider the category $\boldsymbol{Q}(\mathtt{G}) := \boldsymbol{P}(\mathtt{G})^{\mathrm{op}} \setminus \{\varnothing\}$ obtained from $\boldsymbol{P}(\mathtt{G})^{\mathrm{op}}$ by removing the empty multipath — ie the terminal object in $\boldsymbol{P}(\mathtt{G})^{\mathrm{op}}$. Note that $\mathcal{F}_{A,M}$ is a functor on $\boldsymbol{P}(\mathtt{G})$, and hence a presheaf on $\boldsymbol{P}(\mathtt{G})^{\mathrm{op}}$. Then, the cellular cochain groups $\mathrm{C}^i_{\mathrm{cell}}(\boldsymbol{Q}(\mathtt{G}); \mathcal{F}_{A,M})$ and $C^{i+1}_{\mu}(\mathtt{G}; \mathcal{F}_{A,M})$ are isomorphic for all $i \geq 0$. Furthermore, this isomorphism is an isomorphism of chain complexes $\mathrm{C}^*_{\mathrm{cell}}(\boldsymbol{Q}(\mathtt{G}); \mathcal{F}_{A,M}) \cong C^{*\geq 1}_{\mu}(\mathtt{G}; \mathcal{F}_{A,M})$. Therefore we obtain the following remark.

Remark 74 Although $P(\mathtt{G})$ is not cellular in the sense of [Everitt and Turner 2015], $Q(\mathtt{G})$ is — see [loc cit, Section 4.1]; thus the previous isomorphism of cochain complexes, together with [loc cit, Theorem 1], provides isomorphisms

$$\mathrm{H}^i_{\mu}(\mathtt{G}; A, M) \cong \mathrm{H}^{i-1}_{\mathrm{cell}}(\boldsymbol{Q}(\mathtt{G}); \mathcal{F}_{A,M}) \cong \mathrm{H}^{i-1}(\boldsymbol{Q}(\mathtt{G}); \mathcal{F}_{A,M})$$

of cohomology groups between $\mathrm{H}^i_{\mu}(\mathtt{G}; A, M)$, the cellular cohomology $\mathrm{H}^{i-1}_{\mathrm{cell}}(\boldsymbol{Q}(\mathtt{G}); \mathcal{F}_{A,M})$ and the functor cohomology groups $\mathrm{H}^{i-1}(\boldsymbol{Q}(\mathtt{G}); \mathcal{F}_{A,M})$ for all $i > 1$.

In light of Remark 74, one can wonder if the graded module obtained by removing the minimum from the path poset in the Turner–Wagner construction and multipath cohomology become related. However, this is not generally the case, as shown by the next example. Before that, recall that the face poset of a simplicial complex X is the poset on the set of simplices of X, ordered by containment. The augmented face poset of X is its face poset together with a minimum element $\varnothing$ corresponding to the empty simplex.

Example 75 The path poset $P(\mathtt{G})$ is the augmented face poset of a topological space $X = X(\mathtt{G})$ — see [Caputi et al. 2023, Section 6]. Note that the geometric realisation of an augmented face poset is always contractible (since it is the cone on the geometric realisation of the face poset). In particular, the geometric realisation $B(P(\mathtt{G}))$ is the cone over $B(P(\mathtt{G}) \setminus \{\varnothing\})$. For $A = R$ the base ring, the functor homology of (the category associated to) $P(\mathtt{G}) \setminus \{\varnothing\}$ with coefficients in $\mathcal{F}_{R,R}$ agrees with the simplicial homology of $X \simeq B(P(\mathtt{G}) \setminus \{\varnothing\})$ with coefficients in R. On the other hand, it is not difficult to see — see [loc cit, Theorem 6.8] — that multipath cohomology is simplicial, ie

$$\widetilde{\mathrm{H}}^n(X; R) \cong \mathrm{H}^{n+1}_{\mu}(\mathtt{G}; R),$$

where $\widetilde{\mathrm{H}}^*$ denotes the reduced simplicial cohomology. Therefore, although the Turner–Wagner homology

$$\mathrm{TW}_*(\mathtt{G}; \mathcal{F}_{R,R}) = \mathrm{H}_*(\boldsymbol{P}(\mathtt{G}); \mathcal{F}_{R,R})$$

is always trivial (as $P(\mathtt{G})$ is an augmented face poset), after removing the minimum element, the associated functor homology $\mathrm{H}_i(\boldsymbol{P}(\mathtt{G}) \setminus \{\varnothing\}; \mathcal{F}_{R,R})$ is not. In fact, the functor homology $\mathrm{H}_i(\boldsymbol{P}(\mathtt{G}) \setminus \{\varnothing\}; \mathcal{F}_{R,R})$ and the multipath cohomology groups $\mathrm{H}^{i+1}_{\mu}(\mathtt{G}; R)$ are related, for $i \geq 2$, by the standard universal coefficients theorem. The induced short exact sequence

$$0 \to \mathrm{Ext}^1_R(\mathrm{H}_{i-1}(\boldsymbol{P}(\mathtt{G}) \setminus \{\varnothing\}; \mathcal{F}_{R,R})) \to \mathrm{H}^{i+1}_{\mu}(\mathtt{G}; R) \to \mathrm{Hom}_R(\mathrm{H}_i(\boldsymbol{P}(\mathtt{G}) \setminus \{\varnothing\}; \mathcal{F}_{R,R})) \to 0$$

features an Ext functor, which is nontrivial in general. For instance, taking $A = R = \mathbb{Z}$, the multipath cohomology of the bipartite complete graph $\mathtt{K}_{5,5}$ has 3–torsion [Caputi et al. 2024a, Proposition 4.5].

The connection shown in the previous example between functor homology and multipath cohomology is given by two facts; the first, that functor (co)homology of a category, for nice functors, agrees with the usual (co)homology of the classifying space (with local coefficients as in [Quillen 1978, Section 7]), and the second, that the classifying space of the opposite category is naturally homeomorphic to the classifying space of the category itself. As shown in Remark 74, multipath cohomology and functor cohomology agree (in degree $i \geq 2$) when we pass to the opposite category associated to the path poset. Then, in the Turner–Wagner approach, which uses functor homology, one computes the higher colimits of $\mathcal{F}$, whereas multipath cohomology provides a way to compute the higher limits of $\mathcal{F} \circ \mathrm{op}$; when $\mathcal{F}$ is a coefficient system in the sense of Quillen, as in the case of constant functors, higher limits and colimits are computed as usual cohomology on the classifying spaces; then, as the op functor does not change the homotopy type of the classifying spaces, the assertion follows. Note that this does not provide a precise relation for nonlocal coefficients (eg $\mathcal{F}_{A,M}$, $A \neq \mathbb{K}$). In particular, this reasoning does not provide a precise relationship between the Turner–Wagner and multipath cohomologies.

Remark 76 All said above provides an alternative way to define multipath cohomology; ie after passing to the path poset and removing the minimum, one can take the opposite associated category and compute (equivalently) either the higher limits of $\mathcal{F}_{A,M}$ or the associated cellular cohomology groups (as the obtained poset is now cellular). However, functor and cellular cohomologies are not directly computable from the definitions, whereas poset homology happens to be quite computable, also algorithmically. The approach has shown to be fruitful in computing the multipath cohomology of all linear graphs — see [Caputi et al. 2023].

To conclude the comparisons, we point out that, in special cases, like the linear graph $\mathtt{I}_n$ and the polygonal graph $\mathtt{P}_n$, the difference between the multipath and Turner–Wagner homologies is controlled. This is also due to the fact that both homologies provide roughly the same amount of information as the chromatic homology — see [Everitt and Turner 2009; Turner and Wagner 2012] for relations between TW and the chromatic homology. In the next subsection we recall the definition of the latter homology, and prove a comparison result for the graphs $\mathtt{I}_n$ and $\mathtt{P}_n$.

7 Comparison with chromatic homology

We now compare multipath cohomology with chromatic homology of unoriented graphs [Helme-Guizon and Rong 2005; Przytycki 2010]. The latter can be seen as a special case of the construction in Section 3; in light of this observation, we can interpret multipath cohomology as an extension of chromatic homology to the directed setting.

In the first subsection, we briefly revise the construction of the chromatic homology (both in its original version [Helme-Guizon and Rong 2005] and in Przytycki's variant [2010]). We argue that the multipath cohomology of a graph differs from either of these theories computed for the underlying unoriented graph. This uses the fact that multipath cohomology is sensible to orientations. Nonetheless, in the special case of coherently oriented polygonal graphs and linear graphs, we prove that the two (co)homology theories contain the same amount of information — see Theorem 80. As a consequence, see Corollary 82, we obtain that the multipath cohomology of the coherently oriented polygon recovers (a truncated version of) the Hochschild homology of its coefficients. As an application of the functoriality, in the second subsection we clarify the relationship between multipath cohomology and chromatic homology, providing the long exact sequence relating multipath and chromatic cohomologies.

7.1 Chromatic homologies

In this subsection we review the construction of two graph homology theories. The first of these homologies goes under the name of chromatic homology and was introduced in [Helme-Guizon and Rong 2005]. The second homology is a variation of the chromatic homology, and it is due to Przytycki [2010].

Let $\mathtt{G}$ denote a *unoriented* graph with ordered edges and a base vertex v_0. Let A be a *commutative* unital R–algebra, and M be an (A, A)–bimodule. Assume that the A–action on M is symmetric — that is, $a \cdot m = m \cdot a$ for all $m \in M$ and $a \in A$. To each spanning subgraph $\mathtt{H} \in \mathrm{SSG}(\mathtt{G})$ we associate the module

$$M(\mathtt{H}) = M \otimes \bigotimes_{c \not\ni v_0} A_c,$$

where c ranges among the connected components of $\mathtt{H}$ — ordered arbitrarily. If $\mathtt{H} \prec \mathtt{H}'$ then $\mathtt{H} \cup e = \mathtt{H}'$ for some edge e. We can define a map $\mathtt{d}_{\mathtt{H} \prec \mathtt{H}'} \colon M(\mathtt{H}) \to M(\mathtt{H}')$ (see [Helme-Guizon and Rong 2005]). There are two cases to consider depending on the number of components merged by e:

(i) The edge e is incident to two distinct components of $\mathtt{H}$. We have a natural identification of the components of $\mathtt{H}$ and $\mathtt{H}'$ that do not share vertices with e. Furthermore, precisely two distinct components, say c_1 and c_2, of $\mathtt{H}$ are merged into a single component of $\mathtt{H}'$, say c'. The map $\mathtt{d}_{\mathtt{H}\prec\mathtt{H}'}\colon M(\mathtt{H}) \to M(\mathtt{H}')$ is defined as the identity on all factors but those corresponding to c_1 and c_2, where it behaves as follows:

$$A_{c_1} \otimes A_{c_2} \to A_{c'}, \quad a \otimes b \mapsto ab = ba,$$

or if c_{3-i} for $i \in \{1,2\}$ contains the marked vertex,

$$M \otimes A_{c_i} \to A_{c'}, \quad a \otimes b \mapsto a \cdot b;$$

(ii) The edge e is incident to a single component of $\mathtt{H}$. There is a natural identification of *all* components of $\mathtt{H}$ and $\mathtt{H}'$, and the map $\mathtt{d}_{\mathtt{H}\prec\mathtt{H}'}\colon M(\mathtt{H}) \to M(\mathtt{H}')$ is taken to be the corresponding identification of the associated modules.

Similarly, Przytycki [2010] defines the map $\hat{\mathtt{d}}_{\mathtt{H}\prec\mathtt{H}'}\colon M(\mathtt{H}) \to M(\mathtt{H}')$ as above, but setting it to be the zero map instead of the identity in case (ii). The cochain complexes

$$(C^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \mathtt{d}^*) \quad \text{and} \quad (\hat{C}^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \hat{\mathtt{d}}^*)$$

are defined as

$$C^i_{\mathrm{Chrom}}(\mathtt{G}; A, M) = \hat{C}^i_{\mathrm{Chrom}}(\mathtt{G}; A, M) = \bigoplus_{\substack{\mathtt{H}\subset\mathtt{G} \\ \#E(\mathtt{H})=i}} M(\mathtt{H}),$$

and, for $x \in M(\mathtt{H})$,

$$\mathtt{d}(x) = \sum_{\mathtt{H}\prec\mathtt{H}'} (-1)^{\zeta(\mathtt{H}\prec\mathtt{H}')} \mathtt{d}_{\mathtt{H}\prec\mathtt{H}'}(x) \quad \text{and} \quad \hat{\mathtt{d}}(x) = \sum_{\mathtt{H}\prec\mathtt{H}'} (-1)^{\zeta(\mathtt{H}\prec\mathtt{H}')} \hat{\mathtt{d}}_{\mathtt{H}\prec\mathtt{H}'}(x),$$

where ζ is defined as

$$\zeta(\mathtt{H} \prec \mathtt{H} \cup e) = \begin{cases} 0 & \text{if an even number of edges preceding } e \text{ belong to } \mathtt{H}, \\ 1 & \text{otherwise.} \end{cases} \tag{9}$$

Remark 77 The chain complexes $(C^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \mathtt{d}^*)$ and $(\hat{C}^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \hat{\mathtt{d}}^*)$ do not depend on the ordering of the edges up to isomorphism — see [Helme-Guizon and Rong 2005; Przytycki 2010].

Recall that $\mathtt{I}_n$ denotes the n–step graph in Figure 1, and $\mathtt{P}_n$ denotes the polygonal graph in Figure 2.

Remark 78 In the special case of the coherently oriented line graph $\mathtt{I}_n$ and of the polygon $\mathtt{P}_n$, the cochain complexes $(C^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \mathtt{d}^*)$ and $(\hat{C}^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \hat{\mathtt{d}}^*)$ can be extended, using the orientation of $\mathtt{I}_n$ and of $\mathtt{P}_n$, to the noncommutative context — see [Przytycki 2010, Remark 2.3 (ii)] — and this extension is perfectly identical to our definition of μ — see Section 4.1.

Observe that the chromatic homology theories can be recovered from the framework in Section 3.1.

Remark 79 Consider an unoriented graph $\mathtt{G}$ and the poset $P = \mathrm{SSG}(\mathtt{G}) \supseteq P(\mathtt{G})$. Recall that $\boldsymbol{P}$ denotes the category associated to P. Consider the covariant functor $\mathcal{F}\colon \boldsymbol{P} \to R\text{–}\mathbf{Mod}$ defined by extending the functor $\mathcal{F}_{A,M}\colon \boldsymbol{P}(\mathtt{G}) \to R\text{–}\mathbf{Mod}$, see (7), to the whole $\mathrm{SSG}(\mathtt{G})$. This extension is defined as follows: when the covering relation $\mathtt{H} \prec \mathtt{H}'$ is as in case (ii) above, $\mathcal{F}$ is either the identity or the 0–map, depending whether we want to recover $(C^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \mathtt{d}^*)$ or $(\widehat{C}^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \hat{\mathtt{d}}^*)$. These constructions do not depend on signs by Corollary 36.

The following theorem establishes a first relation between multipath and chromatic (co)homologies.

Theorem 80 *Let A be a unital R–algebra, and M an (A,A)–bimodule. Then, we have the following isomorphisms of chain complexes (of R–modules):*

$$(C^*_\mu(\mathtt{I}_n; A, M), d) \cong (\widehat{C}^*_{\mathrm{Chrom}}(\mathtt{I}_n; A, M), \hat{\mathtt{d}}) \cong (C^*_{\mathrm{Chrom}}(\mathtt{I}_n; A, M), \mathtt{d}), \tag{10}$$

$$(C^*_\mu(\mathtt{P}_n; A, M), d) \oplus (M[n+1], 0) \cong (\widehat{C}^*_{\mathrm{Chrom}}(\mathtt{P}_n; A, M), \hat{\mathtt{d}}), \tag{11}$$

where $(M[n+1], 0)$ is the cochain complex consisting of a copy of M in degree $n+1$.

Proof By Remark 78, the cochain complexes $(C^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \mathtt{d}^*)$ and $(\widehat{C}^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \hat{\mathtt{d}}^*)$ can be defined for arbitrary unital R–algebras, using the orientation of the coherently oriented n–step graph $\mathtt{I}_n$ or the polygon $\mathtt{P}_n$. The proof follows directly from Remark 79 by noticing that $\mathrm{SSG}(\mathtt{I}_n) = P(\mathtt{I}_n)$ and $\mathrm{SSG}(\mathtt{P}_n)$ is the poset $P(\mathtt{P}_n) \cup \{\mathtt{P}_n\}$ obtained from the path poset $P(\mathtt{P}_n)$ by adding the $\mathtt{P}_n$ as the maximum. □

Corollary 81 *Let A be a unital R–algebra and R a principal ideal domain. Then, for all $n \in \mathbb{N}$, we have $\mathrm{H}^i_\mu(\mathtt{I}_n; A) = 0$ for all $i \in \mathbb{N} \setminus \{0\}$, and*

$$\mathrm{rank}_R(\mathrm{H}^0_\mu(\mathtt{I}_n; A)) = \begin{cases} \mathrm{rank}_R(A)(\mathrm{rank}_R(A) - 1)^n, & n > 0, \\ \mathrm{rank}_R(A), & n = 0. \end{cases}$$

Proof By (10), the statement follows directly from [Przytycki 2010, Lemma 3.3]. □

For a unital R–algebra A and an (A,A)–bimodule M, denote by $\mathrm{HH}_*(A, M)$ the *Hochschild homology* of A with coefficients in the bimodule M — see, for instance, [Loday 1992, Section 1.1.3] for the definition. Let $\widehat{\mathrm{H}}^*_{\mathrm{Chrom}}(\mathtt{G}; A, M)$ denote the homology of the complex $(\widehat{C}^*_{\mathrm{Chrom}}(\mathtt{G}; A, M), \hat{\mathtt{d}})$. We conclude the section showing that the multipath cohomology groups of the polygon agree with the Hochschild homology of A with coefficients in the bimodule M:

Corollary 82 *Let A be a flat unital R–algebra, and M an (A,A)–bimodule and let $\mathtt{P}_n$ be the polygon* (see Figure 2). *Then, we have the following chain of isomorphisms of homology groups:*

$$\mathrm{H}^i_\mu(\mathtt{P}_n; A, M) \cong \widehat{\mathrm{H}}^i_{\mathrm{Chrom}}(\mathtt{P}_n; A, M) \cong \mathrm{HH}_{n-i}(A, M) \quad \text{for } i = 1, \dots, n.$$

Proof The result follows directly from [Przytycki 2010, Theorem 3.1] and (11). □

Note that, by [Turner and Wagner 2012, Theorem 1], we have $\mathrm{TW}_i(\mathtt{P}_n; A, M) \cong \widehat{\mathrm{H}}^{n-i}_{\mathrm{Chrom}}(\mathtt{P}_n; A, M)$ for i in the set $\{1, \dots, n\}$. From which follows the isomorphism with the multipath cohomology in this case.

We conclude this section by remarking that in general the chromatic and the multipath homologies are distinct also in the case where A is commutative.

Proposition 83 *The cohomologies* $\mathrm{H}_{\mathrm{Chrom}}$ *and* H_μ *are not isomorphic.*

Proof The multipath homology of the noncoherent 3–step graph is different from the homology of $\mathtt{I}_3$. Since the chromatic homology does not distinguish orientations, the statement follows. □

7.2 Short exact sequences and chromatic homology

Here we apply the machinery developed in Section 5 to obtain a long exact sequence featuring both multipath and chromatic homologies. This clarifies the relationship between the two homology theories. As an application we recover the isomorphisms in the case of the linear graph and polygonal graph, when A is a commutative R–algebra.

For an oriented graph $\mathtt{G}$, let $\widehat{C}^*_{\mathrm{Chrom}}(\mathtt{G}; A)$ be the chromatic cochain complex of the underlying unoriented graph. From the results in the previous section, it follows immediately:

Proposition 84 *Let* $\mathtt{G}$ *be an oriented graph, and let* A *be a commutative* R*–algebra. Then, we have the following short exact sequence of complexes*

$$0 \to \widetilde{C}_\mu(\mathtt{G}; A) \to \widehat{C}_{\mathrm{Chrom}}(\mathtt{G}; A) \to C_\mu(\mathtt{G}; A) \to 0,$$

where we set

$$\widetilde{C}_\mu(\mathtt{G}; A) := C_{\mathcal{F}_{A,A}}(\mathrm{SSG}(\mathtt{G}) \setminus P(\mathtt{G}))\Big[-\min_{x \in \mathrm{SSG}(\mathtt{G}) \setminus P(\mathtt{G})} \{\ell(x)\}\Big]$$

and we extended $\mathcal{F}_{A,A}(\mathtt{H} \prec \mathtt{H} \cup e)$ *to be zero if the number of components of* $\mathtt{H}$ *and* $\mathtt{H} \cup e$ *is the same.*

Proof Fix a graph $\mathtt{G}$ and consider $\mathcal{F}\colon \mathbf{SSG}(\mathtt{G}) \to \boldsymbol{A}$. Following the proof of Proposition 62 almost verbatim, we obtain that if P is a downward closed subposet of $\mathrm{SSG}(\mathtt{G})$, then we have the following short exact sequence of chain complexes:

$$0 \to C_{\mathcal{F}|_{\mathrm{SSG}(\mathtt{G}) \setminus P}}(\mathrm{SSG}(\mathtt{G}) \setminus P)\Big[-\min_{x \in \mathrm{SSG}(\mathtt{G}) \setminus P} \{\ell(x)\}\Big] \to C_{\mathcal{F}}(\mathrm{SSG}(\mathtt{G})) \to C_{\mathcal{F}|_P}(P) \to 0,$$

where the sign assignments are induced by any sign assignment on $\mathrm{SSG}(\mathtt{G})$. The statement now follows by taking $P = P(\mathtt{G})$ and $\mathcal{F} = \mathcal{F}_{A,A}$. □

As a consequence we (partially) recover one of the main results of this paper:

Corollary 85 *Let A be a commutative unital R–algebra. Then, we have the following isomorphisms of chain complexes (of R–modules):*

$$(C^*_\mu(\mathtt{I}_n; A), d) \cong (\hat{C}^*_{\mathrm{Chrom}}(\mathtt{I}_n; A), \hat{\mathtt{d}}) \cong (C^*_{\mathrm{Chrom}}(\mathtt{I}_n; A), \mathtt{d}), \tag{12}$$

$$(C^*_\mu(\mathtt{P}_n; A), d) \oplus (A[n+1], 0) \cong (\hat{C}^*_{\mathrm{Chrom}}(\mathtt{P}_n; A), \hat{\mathtt{d}}), \tag{13}$$

where $(A[n+1], 0)$ indicates the cochain complex consisting of a copy of A in degree $n+1$.

Proof It is sufficient to notice that the poset $\mathrm{SSG}(\mathtt{G}) \setminus P(\mathtt{G})$ is either empty (if $\mathtt{G} = \mathtt{I}_n$) or a single point (if $\mathtt{G} = \mathtt{P}_n$). The corollary is an immediate consequence of Proposition 84. □

8 Open questions

In this section we gather some open questions.

Question 86 (full functoriality) We have shown in Section 5 that multipath cohomology is a bifunctor when restricting either to the category of rings or to the category of graphs with same number of vertices. Is it possible to lift this result simultaneously to the full categories **Digraph** of directed graphs and R-**Alg** of R–algebras? If not, what are the obstructions to this extension?

Question 87 (cyclic homology theories and extensions) One of the main properties of $\mathrm{H}_\mu(-; A)$ (for a fixed A) is that it recovers (a truncation of) the Hochschild homology of A. To the best of the authors' knowledge, it is still an open question by [Przytycki 2010] whether or not it is possible to recover, in a similar fashion, also the cyclic homology groups of A — see [Loday 1992] for the definition. Moreover, the construction in Section 3.1 can be generalised, by application of the nerve functor and a suitable adaptation, to the realm of ∞–categories — see [Lurie 2009]. In particular, this generalisation should hold for functors in the module categories over commutative ring spectra. A topological enhancement of the cyclic homology theories is given by the so-called *topological Hochschild homology* (or *topological cyclic homology*) — see [Nikolaus and Scholze 2018]. Do we have for topological Hochschild homology, cyclic homology, negative homology, or periodic homology, a result similar to Corollary 82?

Question 88 (categorification of graph invariants) The chromatic homology is named after the chromatic polynomial, which can be obtained as the graded Euler characteristic of the chromatic homology. In other terms, we can say that the chromatic homology is a categorification of the chromatic polynomial. This holds, of course, for a specific choice of the (commutative) algebra A (eg it must be graded or filtered, and its graded dimension should be the chromatic polynomial of a vertex). The first question is: are there natural choices of the algebra A such that the appropriate Euler characteristic of $C^*_\mu(\mathtt{G}; A)$ is a known invariant of the graph $\mathtt{G}$? In general, what are the combinatorial properties of the graded Euler characteristic of the multipath cohomology of a graph with coefficients in a graded algebra?

Question 89 (relationship with Turner–Wagner theory) We showed that the chromatic homology and the multipath cohomology, when both are defined (ie A commutative), fit into a long exact sequence. Does there exist a long exact sequence, or a spectral sequence, featuring both the multipath and Turner–Wagner homologies with general coefficients?

Question 90 (spectral sequences and applications) A very interesting and deep feature of Khovanov homology is that it admits a spectral sequence which abuts to a very simple homology called Lee homology [2005]. From this and similar spectral sequences one can extract numerical invariants with interesting applications to low-dimensional topology and knot theory. More importantly, these spectral sequences provide structural information on Khovanov homology. Chromatic homology mimics Khovanov homology. Hence, it is not surprising to find similar spectral sequences and invariants in the context of chromatic homology [Chmutov et al. 2008]. This spectral sequence has been used to prove structural properties of chromatic homology. Are there similar spectral sequences for multipath cohomology? If yes, which kind of information can be extracted from them?

Question 91 (persistent multipath cohomology) Persistent homology [Edelsbrunner et al. 2002; Zomorodian and Carlsson 2005] is nowadays one of the main tools adopted in topological data analysis, with applications in several domains. One usually starts with a fixed number of data points, joined by (weighted) edges representing the connections between them. These edges are typically added gradually; that is, we have a filtration of the resulting (unoriented) graph $\mathtt{G}$. This filtration is a sequence $\mathtt{G}_0 \subset \cdots \subset \mathtt{G}_n$ of spanning subgraphs of $\mathtt{G}$. Then, one uses the functorial properties of the classical homology to obtain information in the form of persistent homology groups. Within this framework, one usually works with unoriented graphs, but in concrete applications, graphs are often directed; it is also interesting to compare the undirected versus the directed information (see [Caputi et al. 2021]). Multipath cohomology is a cohomology theory of directed graphs and it is functorial with respect to morphisms of digraphs with the same number of vertices. It is hence natural to define a persistent multipath cohomology for filtrations of digraphs. Which information of the input data can multipath cohomology capture? How does it compare with the analysis using unoriented graphs?

Appendix Proof of Lemma 39

Lemma 92 *The function σ_{e} in* (3) *gives a sign assignment on $P(\mathtt{G})$.*

Proof Consider a square $\mathtt{H} \prec \mathtt{H}'_1, \mathtt{H}'_2 \prec \mathtt{H}''$ in $P(\mathtt{G})$. Then, there exist two edges e_1 and e_2 of $\mathtt{G}$ such that $\mathtt{H}'_1 = \mathtt{H} \cup e_1$, $\mathtt{H}'_2 = \mathtt{H} \cup e_2$, and $\mathtt{H}'' = \mathtt{H}'_2 \cup e_1 = \mathtt{H}'_1 \cup e_2$ (see Example 4 and Figure 15).

The proof is split in cases, according to the number of components of $\mathtt{H}$ which are merged by adding the edges e_1 and e_2. First, adding both e_1 and e_2 to $\mathtt{H}$ decreases the number of connected components by at most 2. Second, the result of the addition of e_1 and e_2 must still be a multipath — submultipath of $\mathtt{H}''$ to be precise. In particular, observe that cycles are not allowed. It follows that there are two cases:

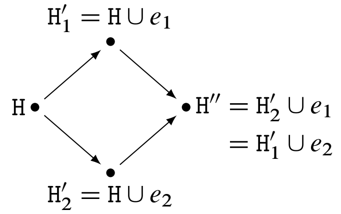

Figure 15: A square in $P(\mathtt{G})$: four multipaths such that $\mathtt{H} \prec \mathtt{H}'_1, \mathtt{H}'_2 \prec \mathtt{H}''$.

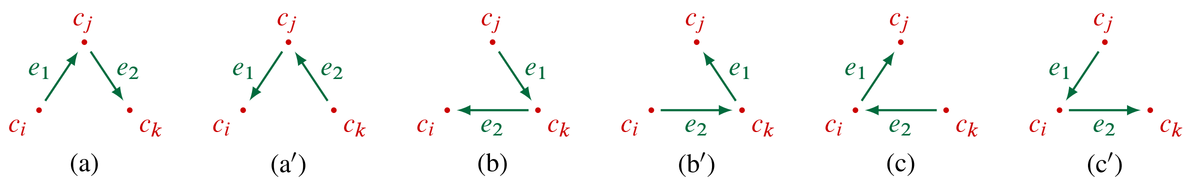

Figure 16: A schematic description of the subcases of case (A). Note that, since the merge of the components c_i, c_j, and c_k must be a path, all possible orientations of e_1 and e_2 are precisely those illustrated.

(A) Three connected components of $\mathtt{H}$ merge into a single connected component of $\mathtt{H}''$.

(B) Four connected components of $\mathtt{H}$ merge into two connected components of $\mathtt{H}''$.

All cases are divided into subcases depending on the indices of the components involved (to be more precise, on the relative order of said indices), and on the orientations of the edges e_1 and e_2 — see Figures 16 and 17. We now proceed with the core of the proof.

(A) Three connected components, c_i, c_j, and c_k, of the multipath $\mathtt{H}$ are merged into a single component of $\mathtt{H}''$. Without loss of generality, up to a permutation of the labels of the components, we may assume that $i < j < k$. Note that e_1 and e_2 cannot be incident to the same pair of components; otherwise $\mathtt{H}''$ would contain a loop. We have six subcases in total — see Figure 16. Since the result of merging the components c_i, c_j, and c_k must be a unique simple path, the orientations of e_1 and e_2 must be coherent; that is, the source of an edge has to be the target of the previous one in the resulting path, and the edges e_1 and e_2 cannot have same sources or targets — eg if the source of e_1 lies in c_k, then the source of e_2 cannot lie in c_k. We report in Table 2 the result of the computation of the signs of σ_{e} in this case.

(B) Four connected components of $\mathtt{H}$, say c_i, c_j, c_k and c_h, are pairwise merged to obtain exactly two connected components of $\mathtt{H}''$. Without loss of generality we may assume $i < j < k < h$. We have twelve relevant cases, but we can reduce them to six; in fact, a change in the orientation of the edges induces a change of the parity of the index. As a consequence, a simultaneous change in the orientations of both e_1 and e_2 affect our computation by a global sign. All six cases are shown in Figure 17 and the results are summarised in Table 3.

It follows from (A) and (B) that σ_{e} is a sign assignment on $P(\mathtt{G})$, which concludes the proof. □

subcase	$\sigma_e(H, H'_1)$	$\sigma_e(H'_1, H'')$	$\sigma_e(H, H'_2)$	$\sigma_e(H'_2, H'')$
(a)	$j+1$	k	$k+1$	$j+1$
(a′)	j	$k-1$	k	j
(b)	$k+1$	j	k	j
(b′)	k	$j+1$	$k+1$	$j+1$
(c)	$j+1$	$k-1$	k	$j+1$
(c′)	j	k	$k+1$	j

Table 2: Computations for all subcases of case (A).

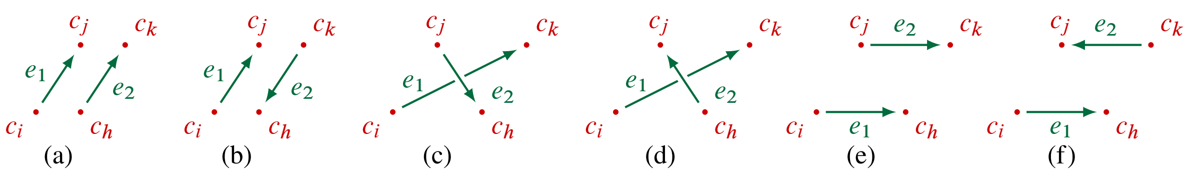

Figure 17: A schematic description of the subcases of case (B) up to a global change in the orientations of e_1 and e_2.

subcase	$\sigma_e(H, H'_1)$	$\sigma_e(H'_1, H'')$	$\sigma_e(H, H'_2)$	$\sigma_e(H'_2, H'')$
(a)	$j+1$	k	$k+1$	$j+1$
(b)	$j+1$	$k-1$	k	$j+1$
(c)	$k+1$	h	$h+1$	$k+1$
(d)	$k+1$	$h-1$	h	$k+1$
(e)	$h+1$	$k+1$	$k+1$	h
(f)	$h+1$	k	k	h

Table 3: Computations for all relevant subcases of case (B).

References

[Aharoni et al. 2005] **R Aharoni**, **E Berger**, **R Meshulam**, *Eigenvalues and homology of flag complexes and vector representations of graphs*, Geom. Funct. Anal. 15 (2005) 555–566 MR Zbl

[Björner 1984] **A Björner**, *Posets, regular CW complexes and Bruhat order*, European J. Combin. 5 (1984) 7–16 MR Zbl

[Caputi and Riihimäki 2024] **L Caputi**, **H Riihimäki**, *Hochschild homology, and a persistent approach via connectivity digraphs*, J. Appl. Comput. Topol. 8 (2024) 1121–1170 MR

[Caputi et al. 2021] **L Caputi**, **A Pidnebesna**, **J Hlinka**, *Promises and pitfalls of topological data analysis for brain connectivity analysis*, NeuroImage 238 (2021) art. id. 118245

[Caputi et al. 2023] **L Caputi**, **C Collari**, **S Di Trani**, *Combinatorial and topological aspects of path posets, and multipath cohomology*, J. Algebraic Combin. 57 (2023) 617–658 MR Zbl

[Caputi et al. 2024a] **L Caputi**, **D Celoria**, **C Collari**, *Monotone cohomologies and oriented matchings*, Homology Homotopy Appl. 26 (2024) 137–161 Zbl

[Caputi et al. 2024b] **L Caputi**, **C Collari**, **S Di Trani**, **J P Smith**, *On the homotopy type of multipath complexes*, Mathematika 70 (2024) art. id. e12235 MR Zbl

[Chandler 2019] **A Chandler**, *Thin posets, CW posets, and categorification*, preprint (2019) arXiv 1911.05600

[Chen et al. 2001] **B Chen**, **S-T Yau**, **Y-N Yeh**, *Graph homotopy and Graham homotopy*, Discrete Math. 241 (2001) 153–170 MR Zbl

[Chmutov et al. 2008] **M Chmutov**, **S Chmutov**, **Y Rong**, *Knight move in chromatic cohomology*, European J. Combin. 29 (2008) 311–321 MR Zbl

[Edelsbrunner et al. 2002] **H Edelsbrunner**, **D Letscher**, **A Zomorodian**, *Topological persistence and simplification*, Discrete Comput. Geom. 28 (2002) 511–533 MR Zbl

[Everitt and Turner 2009] **B Everitt**, **P Turner**, *Homology of coloured posets: a generalisation of Khovanov's cube construction*, J. Algebra 322 (2009) 429–448 MR Zbl

[Everitt and Turner 2015] **B Everitt**, **P Turner**, *Cellular cohomology of posets with local coefficients*, J. Algebra 439 (2015) 134–158 MR Zbl

[Gabriel and Zisman 1967] **P Gabriel**, **M Zisman**, *Calculus of fractions and homotopy theory*, Ergebnisse der Math. 35, Springer (1967) MR Zbl

[Govc et al. 2021] **D Govc**, **R Levi**, **J P Smith**, *Complexes of tournaments, directionality filtrations and persistent homology*, J. Appl. Comput. Topol. 5 (2021) 313–337 MR Zbl

[Grigoryan et al. 2016] **A A Grigoryan**, **Y Lin**, **Y V Muranov**, **S Yau**, *Path complexes and their homologies*, Fundam. Prikl. Mat. 21 (2016) 79–128 MR Zbl In Russian; translated in J. Math. Sci. 248 (2020) 564–599

[Happel 1989] **D Happel**, *Hochschild cohomology of finite-dimensional algebras*, from "Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin", Lecture Notes in Math. 1404, Springer (1989) 108–126 MR Zbl

[Helme-Guizon and Rong 2005] **L Helme-Guizon**, **Y Rong**, *A categorification for the chromatic polynomial*, Algebr. Geom. Topol. 5 (2005) 1365–1388 MR Zbl

[Ivashchenko 1994] **A V Ivashchenko**, *Contractible transformations do not change the homology groups of graphs*, Discrete Math. 126 (1994) 159–170 MR Zbl

[Jonsson 2008] **J Jonsson**, *Simplicial complexes of graphs*, Lecture Notes in Math. 1928, Springer (2008) MR Zbl

[Khovanov 2000] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426 MR Zbl

[Kozlov 2008] **D Kozlov**, *Combinatorial algebraic topology*, Algor. Computat. Math. 21, Springer (2008) MR Zbl

[Lee 2005] **E S Lee**, *An endomorphism of the Khovanov invariant*, Adv. Math. 197 (2005) 554–586 MR Zbl

[Loday 1992] **J-L Loday**, *Cyclic homology*, Grundl. Math. Wissen. 301, Springer (1992) MR Zbl

[Lurie 2009] **J Lurie**, *Higher topos theory*, Ann. of Math. Stud. 170, Princeton Univ. Press (2009) MR Zbl

[Mac Lane 1971] **S Mac Lane**, *Categories for the working mathematician*, Graduate Texts in Math. 5, Springer (1971) MR Zbl

[Nikolaus and Scholze 2018] **T Nikolaus**, **P Scholze**, *On topological cyclic homology*, Acta Math. 221 (2018) 203–409 MR Zbl

[Przytycki 2010] **J H Przytycki**, *When the theories meet: Khovanov homology as Hochschild homology of links*, Quantum Topol. 1 (2010) 93–109 MR Zbl

[Putyra 2014] **K K Putyra**, *A 2–category of chronological cobordisms and odd Khovanov homology*, from "Knots in Poland, III: Part 3", Banach Center Publ. 103, Polish Acad. Sci. Inst. Math., Warsaw (2014) 291–355 MR Zbl

[Quillen 1973] **D Quillen**, *Higher algebraic K–theory, I*, from "Algebraic K–theory, I: Higher K–theories" (H Bass, editor), Lecture Notes in Math. 341, Springer (1973) 85–147 MR Zbl

[Quillen 1978] **D Quillen**, *Homotopy properties of the poset of nontrivial p–subgroups of a group*, Adv. Math. 28 (1978) 101–128 MR Zbl

[Roos 1961] **J-E Roos**, *Sur les foncteurs dérivés de* $\varprojlim$*: applications*, C. R. Acad. Sci. Paris 252 (1961) 3702–3704 MR Zbl

[Turner and Wagner 2012] **P Turner**, **E Wagner**, *The homology of digraphs as a generalization of Hochschild homology*, J. Algebra Appl. 11 (2012) art. id. 1250031 MR Zbl

[Wachs 2003] **M L Wachs**, *Topology of matching, chessboard, and general bounded degree graph complexes*, Algebra Universalis 49 (2003) 345–385 MR Zbl

[West 1996] **D B West**, *Introduction to graph theory*, Prentice Hall, Upper Saddle River, NJ (1996) MR Zbl

[Zomorodian and Carlsson 2005] **A Zomorodian**, **G Carlsson**, *Computing persistent homology*, Discrete Comput. Geom. 33 (2005) 249–274 MR Zbl

Institute of Mathematics, University of Aberdeen
Aberdeen, United Kingdom

New York University Abu Dhabi
Abu Dhabi, United Arab Emirates

Dipartimento di Matematica "G Castelnuovo", Sapienza Università di Roma
Rome, Italy

luigi.caputi@unibo.it, carlo.collari@dm.unipi.it, sabino.ditrani@uniroma1.it

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