AGT

Algebraic & Geometric Topology

Volume 24 (2024)

Strong topological rigidity of noncompact orientable surfaces

Sumanta Das

msp

Algebraic & Geometric Topology **24**:8 (2024) 4423–4469
DOI: 10.2140/agt.2024.24.4423
Published: 17 December 2024

Strong topological rigidity of noncompact orientable surfaces

SUMANTA DAS

We show that every orientable infinite-type surface is properly rigid as a consequence of a more general result. Namely, we prove that if a homotopy equivalence between any two noncompact orientable surfaces is a proper map, then it is properly homotopic to a homeomorphism, provided the surfaces are neither the plane nor the punctured plane. Thus all noncompact orientable surfaces, except the plane and the punctured plane, are *topologically rigid in a strong sense*.

57K20; 55S37

1 Introduction

All manifolds will be assumed to be second countable and Hausdorff. A surface is a 2–dimensional manifold with an empty boundary. All surfaces will be considered connected and orientable. We say a surface is of *infinite type* if its fundamental group is not finitely generated; otherwise, we say it is of *finite type*.

A fundamental question in topology is whether two closed n–manifolds that are homotopy equivalent to each other are homeomorphic. This has a positive answer in dimension 2, as two closed surfaces with isomorphic fundamental groups are homeomorphic. But the same doesn't happen in other dimensions; for example, there are homotopy equivalent lens spaces (a particular type of spherical 3–manifolds) that are not homeomorphic. A closed topological n–manifold M is said to be *topologically rigid* if any homotopy equivalence $N \to M$ with a closed topological n–manifold N as the source is homotopic to a homeomorphism. The *Borel conjecture* (see Rosenberg [34, Conjecture (A Borel)]) asserts that every closed aspherical (ie $\pi_k = 0$ if $k \neq 1$) manifold is topologically rigid. In dimension 2, every closed surface is topologically rigid. This is known as the *Dehn–Nielsen–Baer theorem*; see Dehn [12, Appendix]. The Borel conjecture is known to be true in other dimensions under some additional hypotheses; for example, see Waldhausen [39, Theorem 6.1] and Gabai, Meyerhoff, and Thurston [21, Theorem 0.1(i)] for dimension 3, and for high dimensions see Farrell and Jones [18, proof of Theorem 3.2].

Though noncompact manifolds are not rigid in the above sense, for example in McMillan [29, Theorem 2], the author has constructed (generalizing a construction given by J H C Whitehead) uncountably many contractible open subsets of $\mathbb{R}^3$ such that any two of them are not homeomorphic. Similarly, for noncompact surfaces, we have several examples. In the case of finite-type surfaces we may consider the once-punctured torus and thrice-punctured sphere, which are homotopy equivalent but nonhomeomorphic, as any homomorphism preserves the cardinality of the puncture set as well as the genus. On the other hand, up to homotopy equivalence, there is precisely one infinite-type surface, but up to homeomorphism there are $2^{\aleph_0}$–many infinite-type surfaces (see Proposition 3.1.11). We consider only noncompact surfaces and discuss their topological rigidity in the *proper category*. Here proper category means the category of spaces with proper maps (recall that a map from a space X to a space Y is called a *proper map* if the inverse image of each compact subset of Y is a compact subset of X). We first define the analogs of homotopy, homotopy equivalence, etc in the proper category.

If a homotopy $\mathcal{H}\colon X\times[0,1]\to Y$ is a proper map, then we call $\mathcal{H}$ a *proper homotopy*. Two proper maps from X to Y are said to be *properly homotopic* if there is a proper homotopy between them. We say that a proper map $f\colon X\to Y$ is a *proper homotopy equivalence* if there exists a proper map $g\colon Y\to X$ such that both $g\circ f$ and $f\circ g$ are properly homotopic to the identity maps (when such a g exists, g is a *proper homotopy inverse* of f). Two spaces X and Y are said to have the same *proper homotopy type* if there is a proper homotopy equivalence between them. It is worth noting that homotopy through proper maps is a weaker notion than proper homotopy. For example, consider $H\colon\mathbb{C}\times[0,1]\to\mathbb{C}$ given by $H(z,t):=tz^2-z$. Being a polynomial, each $H(-,t)$ is proper. But H itself is not proper as $H(n,1/n)=0$ for all integers $n\geq 1$.

The analog of topological rigidity in the proper category is defined as follows: a noncompact topological manifold M without boundary is said to be *properly rigid* if, whenever N is another boundaryless topological manifold of the same dimension and $h\colon N\to M$ is a proper homotopy equivalence, h is properly homotopic to a homeomorphism. The analog of the Borel conjecture in the proper category, often called the *proper Borel conjecture* (see Chang and Weinberger [11, Conjecure 3.1]), asserts that every noncompact aspherical topological manifold without boundary is properly rigid.

It is known that noncompact finite-type surfaces are properly rigid. Further, using the algebraic tools of classification of noncompact surfaces [23, Theorem 4.1], Goldman showed that two noncompact surfaces of the same proper homotopy type are homeomorphic; see [22, Corollary 11.1]. We show that infinite-type surfaces are also properly rigid. In fact, we show the rigidity of all noncompact surfaces, except for the plane and the punctured plane, under a weaker assumption, namely only assuming the existence of a homotopy inverse, which a priori may or may not be proper. For brevity, define a weaker version of proper homotopy equivalence:

Definition A homotopy equivalence is said to be a *pseudoproper homotopy equivalence* if it is proper.

Indeed, a proper homotopy equivalence is a pseudoproper homotopy equivalence, though not conversely: a pseudoproper homotopy equivalence has an "ordinary" homotopy inverse but may not have a proper homotopy inverse. For example, consider the φ and ψ below. Our main theorem is the following:

Theorem *Let $f:\Sigma'\to\Sigma$ be a pseudoproper homotopy equivalence between two noncompact surfaces. Then Σ' is homeomorphic to Σ. If we further assume that Σ is homeomorphic to neither the plane nor the punctured plane, then f is a proper homotopy equivalence, and there exists a homeomorphism $g_{\text{homeo}}\colon\Sigma\to\Sigma'$ as a proper homotopy inverse of f.*

The reason for the exclusion of the plane and the punctured plane from the hypothesis is almost immediate. Consider $\varphi\colon \mathbb{C} \ni z \mapsto z^2 \in \mathbb{C}$ and $\psi\colon \mathbb{S}^1 \times \mathbb{R} \ni (z,x) \mapsto (z,|x|) \in \mathbb{S}^1 \times \mathbb{R}$. Both of these proper maps are homotopy equivalences, but neither is a proper homotopy equivalence as the degree of a proper homotopy equivalence is ± 1 (see Section 2.6), though $\deg(\varphi) = \pm 2$ (as φ is a twofold branched covering) and $\deg(\psi) = 0$ (as ψ is not surjective; see Lemma 3.6.4.1).

In general, additional assumptions must be imposed on a pseudoproper homotopy equivalence to become a proper homotopy equivalence. For example, using the binary symmetry of the Cantor tree $\mathcal{T}_{\text{Cantor}}$, we have a twofold branched covering $f_{\text{Cantor}}\colon \mathcal{T}_{\text{Cantor}} \to \mathcal{T}_{\text{Cantor}}$ which is undoubtedly a pseudoproper homotopy equivalence (trees are contractible) but not a proper homotopy equivalence (the induced map on $\text{Ends}(\mathcal{T}_{\text{Cantor}})$ by f_{Cantor} is noninjective; see parts (1) and (3) of Proposition 2.3.1). Here is another example: Let M be a connected noncompact contractible boundaryless manifold of dimension $n \geq 2$, and let $f\colon M \to M$ be the composition of a proper map $M \to [0,\infty)$ (using partition of unity) and a nonsurjective proper map $[0,\infty) \to M$ corresponding to an end of M (using compact exhaustion by connected codimension 0–submanifolds; see Guilbault [24, Exercise 3.3.18]). Then f is a pseudoproper homotopy equivalence (M is contractible) but not a proper homotopy equivalence (a proper homotopy equivalence is a surjective map as its degree is ± 1; see Lemma 3.6.4.1).

Brown showed that a pseudoproper homotopy equivalence between two connected finite-dimensional locally finite simplicial complexes is a proper homotopy equivalence if and only if it induces a homeomorphism on the spaces of ends and isomorphisms on all proper homotopy groups [7, Whitehead theorem]. Farrell, Taylor, and Wagoner [19, Corollary 4.10] showed that if $f\colon M \to N$ is a pseudoproper homotopy equivalence between two simply connected noncompact boundaryless n–dimensional smooth manifolds, where both M and N both are simply connected at infinity, then f is a proper homotopy equivalence if and only if $\deg(f) = \pm 1$. Another interesting statement in this context is that a proper map $f\colon X \to Y$ between two locally finite infinite connected 1–dimensional CW–complexes is a proper homotopy equivalence if $\text{Ends}(f)$ is a homeomorphism and f is an extension of a proper homotopy equivalence $X_g \to Y_g$ (where X_g (resp. Y_g) denotes the smallest connected subcomplex of X (resp. Y) that contains all immersed loops of X (resp. Y)); see Algom-Kfir and Bestvina [1, Corollary 3.7].

We conclude this section by citing a few more related results of two different flavors: when does a proper homotopy equivalence exist, and if it does exist, does it determine the space up to homeomorphism. Similar to Kerékjártó's classification theorem (see Theorem 2.4.1), there exists a classification of graphs up to proper homotopy type: two locally finite infinite connected 1–dimensional CW–complexes X and Y have the same proper homotopy type if and only if $\text{rank}(\pi_1(X)) = \text{rank}(\pi_1(Y))$ and there exists a homeomorphism $\varphi\colon \text{Ends}(X) \to \text{Ends}(Y)$ with $\varphi(\text{Ends}(X_g)) = \text{Ends}(Y_g)$; see Ayala, Dominguez, Márquez, and Quintero [3, Theorem 2.7].

As stated earlier, any two noncompact surfaces of the same proper homotopy type are homeomorphic. Sometimes this also happens in other dimensions; for instance, a boundaryless topological manifold of dimension $n \geq 3$ with the same proper homotopy type as $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$; see Edwards [14, Theorem 1] for $n = 3$, Freedman [20, Corollary 1.2] for $n = 4$, and Siebenmann [36, Corollary 1.4] for $n \geq 5$. In contrast, there are exotic pairs: two noncompact connected boundaryless manifolds N and M of dimension $n \geq 5$ exist, where N is smoothable and M is a nonuniform arithmetic manifold, such that M and N have the same proper homotopy type but M is not homeomorphic to N; see Chang and Weinberger [10, Theorem 2.6; 11, pages 137 and 138].

1.1 Main results

The analog of Farb and Margalit's [17, first proof of Theorem 8.9] in the proper category is Theorem 2, which follows almost directly from our main result Theorem 1. Indeed, Theorem 1 is more general.

Theorem 1 (strong topological rigidity) *Let $f : \Sigma' \to \Sigma$ be a pseudoproper homotopy equivalence between two noncompact surfaces. Suppose Σ is homeomorphic to neither $\mathbb{R}^2$ nor $\mathbb{S}^1 \times \mathbb{R}$. Then Σ' is homeomorphic to Σ and f is properly homotopic to a homeomorphism.*

Theorem 2 (proper rigidity) *If $f : \Sigma' \to \Sigma$ is a proper homotopy equivalence between two noncompact surfaces, then Σ' is homeomorphic to Σ and f is properly homotopic to a homeomorphism.*

A theorem of Edmonds [13, Theorem 3.1] says that any π_1–injective map of degree 1 between two closed surfaces is homotopic to a homeomorphism. The analogous fact for noncompact surfaces is Theorem 3, which classifies all π_1–injective degree 1 maps between two noncompact surfaces and also follows almost directly from Theorem 1.

Theorem 3 (classification of π_1–injective degree 1 maps) *Let Σ and Σ' be any two noncompact oriented surfaces. Suppose there exists a π_1–injective proper map $f : \Sigma' \to \Sigma$ of degree ± 1. Then Σ is homeomorphic to Σ' and f is properly homotopic to a homeomorphism.*

Proofs of Theorems 1, 2, and 3 can be found in Section 4. A statement equivalent to Theorem 1 is claimed by Brittenham [6, Proposition 2.1(b)] referencing his unpublished work [5], and where the proof of [5] is claimed to be in the spirit of a result of Brown and Tucker [9].

1.2 Outline of the proof of Theorem 1

Let $f : \Sigma' \to \Sigma$ be a pseudoproper homotopy equivalence between two noncompact oriented surfaces. Suppose Σ is homeomorphic to neither $\mathbb{R}^2$ nor $\mathbb{S}^1 \times \mathbb{R}$.

1.2.1 Decomposition and transversality Let $\mathscr{C}$ be a locally finite pairwise-disjoint collection of smoothly embedded circles on Σ such that $\mathscr{C}$ decomposes Σ into bordered subsurfaces, and a complementary component of this decomposition is homeomorphic to the one-holed torus, the pair of pants, or the punctured disk (see Theorem 3.1.5).

Properly homotope f to make it smooth as well as transverse to $\mathscr{C}$. Thus $f^{-1}(\mathcal{C})$ is either *empty* or a pairwise-disjoint finite collection of smoothly embedded circles on Σ' for each component $\mathcal{C}$ of $\mathscr{C}$; see Theorem 3.2.3.

1.2.2 Removing unnecessary circles Following the steps below, we properly homotope f further so that for each component of $\mathcal{C}$ of $\mathscr{C}$, either $f^{-1}(\mathcal{C})$ is empty or $f|f^{-1}(\mathcal{C}) \to \mathcal{C}$ is a homeomorphism.

(1) Notice that $f^{-1}(\mathscr{C})$ may have infinitely many disk-bounding components. But, in such a case, an arbitrarily large disk in Σ' bounded by a component of the locally finite collection $f^{-1}(\mathscr{C})$ is not possible as $\Sigma' \not\cong \mathbb{R}^2$ (see Lemma 3.3.1), ie there always exists an "outermost disk" bounded by some component of $f^{-1}(\mathscr{C})$. Now, properly homotope f to remove all disk bounding components of $f^{-1}(\mathscr{C})$ upon considering all these outermost disks simultaneously; see Theorem 3.3.5.

(2) Thereafter, using π_1–bijectivity of f, properly homotope f to map each (primitive) component of $f^{-1}(\mathscr{C})$ onto a component of $\mathscr{C}$ homeomorphically; see Theorem 3.4.3.

(3) Since f has homotopy left inverse, any two components of $f^{-1}(\mathscr{C})$ cobound an annulus in Σ' if and only if their f–images are the same, ie an arbitrarily large annulus in Σ' cobounded by two components of $f^{-1}(\mathscr{C})$ is impossible. So, considering all these "outermost annuli" simultaneously, we complete the goal, as stated in the beginning; see Theorem 3.5.3.

1.2.3 Showing f is a degree ± 1 map (see Theorem 3.6.3.1) To rule out the possibility that $f^{-1}(\mathcal{C})$ is empty, where $\mathcal{C}$ is a component of $\mathscr{C}$, we prove $\deg(f) = \pm 1$. This is because $\deg(f)$ remains the same after any proper homotopy of f, and a map of nonzero degree is surjective; see Lemmas 3.6.4.1 and 3.6.4.3. Our aim is to properly homotope f to obtain a closed disk $\mathcal{D} \subseteq \Sigma$ such that $f|f^{-1}(\mathcal{D}) \to \mathcal{D}$ becomes a homeomorphism, and thus we show $\deg(f) = \pm 1$; see Theorem 2.6.1. The argument is based on finding a smoothly embedded finite-type bordered surface $\boldsymbol{S}$ in Σ such that, for each component c of $\partial \boldsymbol{S}$, we have $f^{-1}(c) \neq \varnothing$, even after any proper homotopy of f. Depending on the nature of $\boldsymbol{S}$, we consider two cases:

(1) If Σ is either an infinite-type surface or any $S_{g,0,p}$ with high complexity ($g + p \geq 4$ or $p \geq 6$), then using π_1–surjectivity of f, we can choose $\boldsymbol{S}$ as a smoothly embedded pair of pants in Σ such that $\Sigma \setminus \boldsymbol{S}$ has at least two components and every component of $\Sigma \setminus \boldsymbol{S}$ has a nonabelian fundamental group; see Lemmas 3.6.1.2 and 3.6.1.4. Properly homotope f so that it becomes transverse to $\partial \boldsymbol{S}$. Then remove unnecessary components from the transverse preimage $f^{-1}(\partial \boldsymbol{S})$. Thus after a proper homotopy, we may assume $f|f^{-1}(c) \to c$ is a homeomorphism for each component c of $\partial \boldsymbol{S}$. Now, since f is π_1–injective, by the rigidity of the pair of pants (see Theorem 3.6.1.9), after a proper homotopy one can show that $f|f^{-1}(\boldsymbol{S}) \to \boldsymbol{S}$ is a homeomorphism; see Lemma 3.6.1.10. Therefore the required $\mathcal{D}$ can be any disk in $\operatorname{int}(\boldsymbol{S})$.

(2) If Σ is a finite-type surface, then we choose a smoothly embedded punctured disk $\boldsymbol{S}$ in Σ so that the puncture of $\boldsymbol{S}$ is an end $e \in \operatorname{im}(\operatorname{Ends}(f)) \subseteq \operatorname{Ends}(\Sigma)$. By Theorem 3.6.2.1, this means every deleted neighborhood of e in Σ intersects $\operatorname{im}(f)$, even after any proper homotopy of f. Now, properly

homotope f so that it becomes transverse to ∂S. Then remove unnecessary components from the transverse preimage $f^{-1}(\partial S)$, ie after a proper homotopy we may assume $f|f^{-1}(\partial S)\to\partial S$ is a homeomorphism (as $\Sigma\not\cong\mathbb{S}^1\times\mathbb{R}$, the fundamental group of $\Sigma\setminus S$ is nonabelian, and so π_1–surjectivity of f says $f^{-1}(\partial S)\neq\varnothing$, even after any proper homotopy of f). Since f is π_1–injective, by the proper rigidity of the punctured disk (see Theorem 3.6.2.4), after a proper homotopy, one can show that $f|f^{-1}(S)\to S$ is a homeomorphism; see Lemma 3.6.2.3. Therefore the required $\mathcal{D}$ can be any disk in $\operatorname{int}(S)$.

1.2.4 Inverse decomposition By the last three parts, after a proper homotopy, removing unnecessary components from the transverse preimage $f^{-1}(\mathscr{C})$, we may assume that $f|f^{-1}(\mathcal{C})\to\mathcal{C}$ is a homeomorphism for each component $\mathcal{C}$ of $\mathscr{C}$. Thus $\mathscr{C}$ and $f^{-1}(\mathscr{C})$ decompose Σ and Σ', respectively, and there is a shape-preserving bijective correspondence between these two collections of complementary components (see Lemmas 3.6.1.10 and 3.6.2.3). On each complementary component, apply either the rigidity of compact bordered surfaces (see Theorem 3.6.1.9) or the proper rigidity of the punctured disk (see Theorem 3.6.2.4). Thus, we have a collection of boundary-relative proper homotopies such that by pasting them, a proper homotopy from f to a homeomorphism $\Sigma'\to\Sigma$ can be constructed; see the proof of Theorem 1 in Section 4.

2 Background

2.1 Conventions

A *bordered surface* (resp. *surface*) is a connected orientable 2–dimensional manifold with a nonempty (resp. an empty) boundary. For integers $g\geq 0$, $b\geq 0$, and $p\geq 0$, denote the connected orientable 2–manifold of genus g with b boundary components by $S_{g,b}$, and let $S_{g,b,p}$ be the 2–manifold after removing p points from $\operatorname{int}(S_{g,b})$. Note that for a manifold M, we use $\operatorname{int}(M)$ to denote the interior of M. Sometimes $S_{0,1}$, $S_{0,2}$, $S_{0,3}$, $S_{1,2}$, and $S_{0,1,1}$ will be called a disk, an annulus, a pair of pants, a two-holed torus, and a punctured disk, respectively.

We say a connected 2–manifold with or without boundary is of *infinite type* if its fundamental group is not finitely generated; otherwise, we say it is of *finite type*.

2.2 Simple closed curves on 2–manifolds

Definition 2.2.1 Let S be a connected orientable 2–dimensional manifold with or without boundary. A *circle* (resp. *smoothly embedded circle*) on S is the image of an embedding (resp. a smooth embedding) of $\mathbb{S}^1$ into S. We say a circle $\mathcal{C}$ on S is a *trivial circle* if there is an embedded disk $\mathcal{D}$ in S such that $\partial\mathcal{D}=\mathcal{C}$, and we say a circle $\mathcal{C}$ on S is a *primitive circle* if it is not a trivial circle.

The following theorem justifies naming a nondisk bounding circle a primitive circle: a primitive circle represents a primitive element of the fundamental group. Recall that an element g of a group G is *primitive* if there does not exist any $h\in G$ such that $g=h^k$, where $|k|>1$.

Theorem 2.2.2 [15, Theorems 1.7. and 4.2] *Let S be a connected orientable 2–dimensional manifold with or without boundary. Let $\mathcal{C}$ be a primitive circle on S, and let $f : \mathbb{S}^1 \hookrightarrow S$ be an embedding with $f(\mathbb{S}^1) = \mathcal{C}$. Then $[f] \in \pi_1(S)$ is a primitive element. In particular, $[f]$ is a nontrivial element of $\pi_1(S)$.*

Recall that for a path-connected space X, there is a bijective correspondence between the set of all conjugacy classes of $\pi_1(X, *)$ and the set of all free homotopy classes of maps $\mathbb{S}^1 \to X$. The next theorem says that two pairwise-disjoint freely homotopic primitive circles on a 2–manifold cobound an annulus.

Theorem 2.2.3 [15, Lemma 2.4] *Let S be a connected orientable 2–dimensional manifold with or without boundary. Let $\ell_0, \ell_1 : \mathbb{S}^1 \hookrightarrow S$ be two embeddings such that $\ell_0(\mathbb{S}^1)$ is a smoothly embedded submanifold of S and $\ell_0(\mathbb{S}^1) \cap \ell_1(\mathbb{S}^1) = \varnothing$. If ℓ_0 and ℓ_1 represent the same nontrivial conjugacy class in $\pi_1(S, *)$, then there is an embedding $\mathcal{L} : \mathbb{S}^1 \times [0,1] \hookrightarrow S$ such that $\mathcal{L}(-,0) = \ell_0$ and $\mathcal{L}(-,1) = \ell_1$.*

2.3 Ends of spaces

Let X be a connected separable locally compact locally connected Hausdorff ANR (absolute neighborhood retract) space. For example, X can be any connected topological manifold. We say X admits an *efficient exhaustion by compacta* if there is a nested sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact connected subsets of X such that $\bigcup_i K_i = X$, $K_i \subseteq \operatorname{int}(K_{i+1})$, $\bigcap_i (X \setminus K_i) = \varnothing$, and the closure of each component of any $X \setminus K_i$ is noncompact. For the existence of efficient exhaustion of X by compacta, see [24, Exercise 3.3.4].

Let $\operatorname{Ends}(X)$ be the set of all sequences $(V_1, V_2, \dots)$, where V_i is a component of $X \setminus K_i$ and $V_1 \supseteq V_2 \supseteq \cdots$. Set $X^\dagger := X \cup \operatorname{Ends}(X)$ with the topology generated by the basis consisting of all open subsets of X, and all sets $V_i^\dagger$, where

$$V_i^\dagger := V_i \cup \{(V_1', V_2', \dots) \in \operatorname{Ends}(X) : V_i' = V_i\}.$$

Then $X^\dagger$ is separable, compact, and metrizable, so X is an open dense subset of $X^\dagger$; it is known as the *Freudenthal compactification* of X (recall that we say a space $X_{\boldsymbol{c}}$ is a *compactification* of X if $X_{\boldsymbol{c}}$ is compact Hausdorff space and X is a dense subset of $X_{\boldsymbol{c}}$). The subspace $\operatorname{Ends}(X)$ of $X^\dagger$ is a totally disconnected space; hence $\operatorname{Ends}(X)$ is a closed subset of the Cantor set.

The Freudenthal compactification *dominates* any other compactification: If $\widetilde{X}$ is a compactification of X such that $\widetilde{X} \setminus X$ is totally disconnected, then there exists a map $f : X^\dagger \to \widetilde{X}$ extending Id_X.

Also, the Freudenthal compactification is *unique* in the following sense: If $X^{\dagger\dagger}$ is a compactification of X such that $X^{\dagger\dagger} \setminus X$ is totally disconnected and $X^{\dagger\dagger}$ dominates any other compactification, then there exists a homeomorphism $X^{\dagger\dagger} \to X^\dagger$ extending Id_X; see [22, Theorem 3.1]. Thus the definition of $\operatorname{Ends}(X)$ is independent of the choice of efficient exhaustion of X by compacta.

Now we consider a relationship between Ends and proper maps:

Proposition 2.3.1 [24, Proposition 3.3.12] *Let X and Y be two connected separable locally compact locally connected Hausdorff ANRs. Then we have the following:*

(1) *Every proper map $f\colon X \to Y$ induces a map $\mathrm{Ends}(f)\colon \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$ that can be used to extend $f\colon X \to Y$ to a map $f^{\dagger}\colon X^{\dagger} \to Y^{\dagger}$ between the Freudenthal compactifications.*

(2) *If two proper maps $f_0, f_1\colon X \to Y$ are properly homotopic, then $\mathrm{Ends}(f_0) = \mathrm{Ends}(f_1)$.*

(3) *If $f\colon X \to Y$ is a proper homotopy equivalence, then $\mathrm{Ends}(f)\colon \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$ is a homeomorphism.*

More about the ends of spaces and proper homotopy can be found in [31; 26].

2.4 Kerékjártó's classification theorem and Ian Richards' representation theorem

Let Σ be a noncompact surface with an efficient exhaustion $\{K_i\}_1^{\infty}$. Let $e := (V_1, V_2, \dots) \in \mathrm{Ends}(\Sigma)$ be an end, where V_i is a component of $X \setminus K_i$. We say e is a *planar end* if V_i is embeddable in $\mathbb{R}^2$ for some positive integer i. An end is said to be *nonplanar* if it is not planar. Denote the subspace of $\mathrm{Ends}(\Sigma)$ consisting of all planar (resp. nonplanar) ends by $\mathrm{Ends}_{\mathrm{p}}(\Sigma)$ (resp. $\mathrm{Ends}_{\mathrm{np}}(\Sigma)$). Note that $\mathrm{Ends}_{\mathrm{p}}(\Sigma)$ is an open subset of $\mathrm{Ends}(\Sigma)$. Define the *genus* of Σ as $g(\Sigma) := \sup g(\mathcal{S})$, where $\mathcal{S}$ is a compact bordered subsurface of Σ. Therefore, the genus counts the number of handles of a surface, ie the number of embedded copies of $S_{1,1}$ in a surface, which may be any nonnegative integer or countably infinite.

Theorem 2.4.1 (Kerékjártó's classification of noncompact surfaces [33, Theorem 1]) *Let Σ and Σ' be noncompact surfaces of genus g and g', respectively. Then Σ is homeomorphic to Σ' if and only if $g = g'$ and there is a homeomorphism $\varphi\colon \mathrm{Ends}(\Sigma) \to \mathrm{Ends}(\Sigma')$ with $\varphi(\mathrm{Ends}_{\mathrm{np}}(\Sigma)) = \mathrm{Ends}_{\mathrm{np}}(\Sigma')$.*

Theorem 2.4.2 (realization of ends and representation of a noncompact surface [33, Theorems 2 and 3]) *Let $\mathscr{E}_{\mathrm{np}} \subseteq \mathscr{E}$ be two closed totally disconnected subsets of $\mathbb{S}^1$, and let $\mathscr{G}$ be an at most countable set such that $\mathscr{E} \neq \varnothing$, and $\mathscr{E}_{\mathrm{np}} \neq \varnothing$ if and only if $\mathscr{G}$ is infinite. Define $\mathbb{D} := \{z \in \mathbb{C} : 0 \le |z| \le 1\}$. Then there exists a pairwise-disjoint collection $\{\mathcal{D}_i : i \in \mathscr{G}\}$ of disks in $\mathrm{int}(\mathbb{D})$ such that a point $p \in \mathbb{D}$ is an element of $\mathscr{E}_{\mathrm{np}}$ if and only if every neighborhood of p in $\mathbb{D}$ contains infinitely many elements of $\{\mathcal{D}_i : i \in \mathscr{G}\}$. Moreover, $\boldsymbol{S} := (\mathbb{D} \setminus \mathscr{E}) \setminus \bigcup_{i \in \mathscr{G}} \mathrm{int}(\mathcal{D}_i)$ is a noncompact bordered surface, and*

$$D\boldsymbol{S} := \frac{(\boldsymbol{S} \times 0) \sqcup (\boldsymbol{S} \times 1)}{(p, 0) \sim (p, 1)} \quad \text{for } p \in \partial \boldsymbol{S}$$

is a genus-$|\mathscr{G}|$ noncompact surface with $\mathrm{Ends}(D\boldsymbol{S}) \cong \mathscr{E}$ and $\mathrm{Ends}_{\mathrm{np}}(D\boldsymbol{S}) \cong \mathscr{E}_{\mathrm{np}}$.

Thus, given any noncompact surface Σ, in this procedure, if we assume $\mathscr{E}_{\mathrm{np}} \subseteq \mathscr{E}$ is homeomorphic to the pair $\mathrm{Ends}_{\mathrm{np}}(\Sigma) \subseteq \mathrm{Ends}(\Sigma)$, and $|\mathscr{G}|$ is equal to $g(\Sigma)$, then $D\boldsymbol{S} \cong \Sigma$ by Theorem 2.4.1.

Remark 2.4.3 The classification of noncompact bordered surfaces is also possible: When the boundary is compact, it follows from Theorem 2.4.1 with [38, Proposition A.3]. When each boundary component

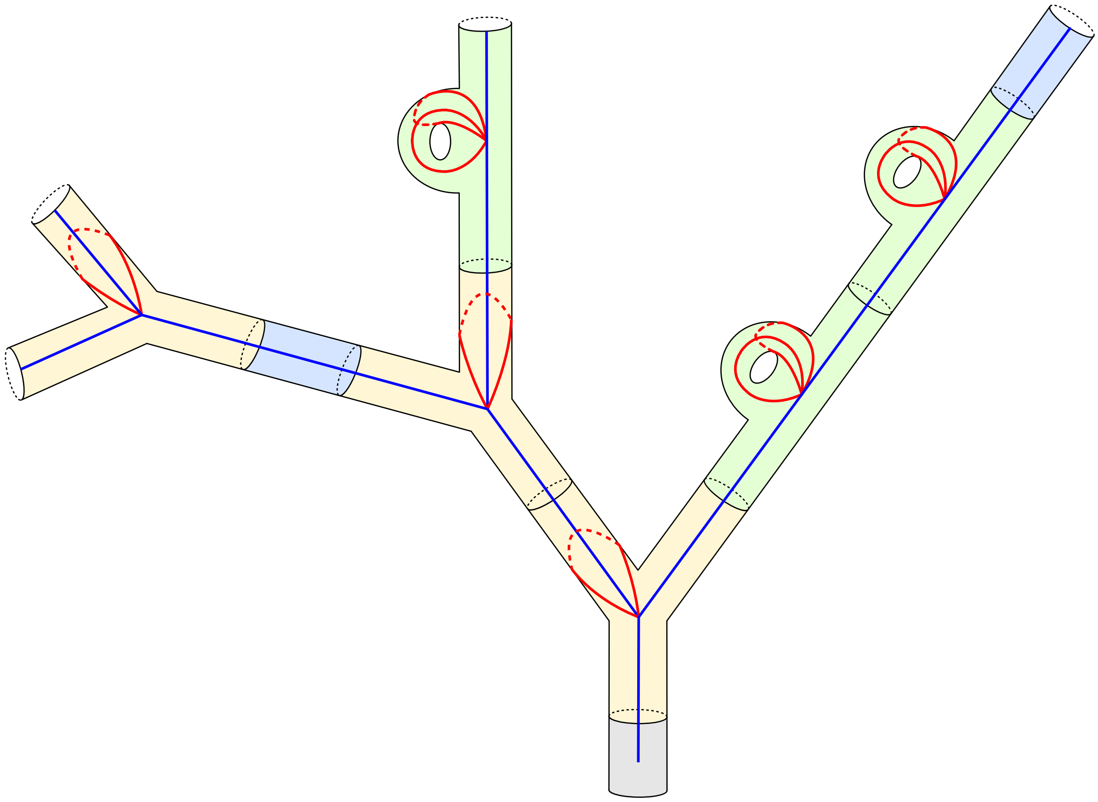

Figure 1: The inductive construction of any noncompact surface and its spine uses four compact bordered surfaces: the disk, annulus, pair of pants, and torus with two holes.

is compact, this follows from [4] (based on the classification of their interiors) or [38, Theorem A.7] (based on the classification of noncompact surfaces obtained from gluing a disk along each boundary component). For arbitrary boundary, see [8, Theorem 2.2].

2.5 Goldman's inductive procedure of constructing all noncompact surfaces

A noncompact surface Σ_{std} is said to be in *standard form* if it is built up from four building blocks, $S_{0,1}$, $S_{0,2}$, $S_{0,3}$, and $S_{1,2}$, in the following inductive manner: Start with $S_{0,1}$. Suppose the i^{th} step of the induction has already been done. Let K_i be the compact bordered subsurface of Σ_{std} after the i^{th} step of induction. In particular, $K_1 \cong S_{0,1}$. Now, to obtain K_{i+1} from K_i, consider one of the last three building blocks, say $\mathcal{S}$ (homeomorphic to $S_{0,2}$, $S_{0,3}$, or $S_{1,2}$); finally, suitably identify one boundary circle of $\mathcal{S}$ with a boundary circle of K_i; see Figure 1.

Theorem 2.5.1 [23, Section 2.6; 27, page 173] *Let Σ be a noncompact surface. Then Σ is homeomorphic to a noncompact surface Σ_{std} in standard form. Thus every noncompact surface is homeomorphic to a noncompact surface constructed using an inductive procedure as above, though two noncompact surfaces obtained from two different inductive procedures may be homeomorphic.*

Theorem 2.5.2 [23, Section 2.6. and Section 7.3] *The graph in Figure 1 consisting of blue straight line segments and red circles is a deformation retract of the surface Σ. Thus Σ is homotopy equivalent to the wedge of at most countably many circles. In particular, $\pi_1(\Sigma)$ is free.*

Remark 2.5.3 An alternative proof of the last two sentences of Theorem 2.5.2 is [35, Lemma 3.2.2].

2.6 The degree of a proper map

We use singular cohomology with compact support to define the notion of the degree of a proper map. Recall that for a topological manifold X, the r^{th} singular cohomology with compact support $H_c^r(X, \partial X; \mathbb{Z})$ is equal to the direct limit $\varinjlim H^r(X, \partial X \cup (X \setminus K); \mathbb{Z})$, where K is a compact subset of X and various maps to define this direct system are inclusion induced maps. Hence, for a compact subset K of X, the definition of the direct limit yields an *obvious map* $H^r(X, \partial X \cup (X \setminus K); \mathbb{Z}) \to H_c^r(X, \partial X; \mathbb{Z})$. It is worth noting that when X is a compact topological manifold, $H_c^r(X, \partial X; \mathbb{Z}) = H^r(X, \partial X; \mathbb{Z})$ for all r.

Let X and Y be two topological manifolds. If $f: X \to Y$ is a proper map with $f(\partial X) \subseteq \partial Y$, then for each r, f induces a map $H_c^r(f): H_c^r(Y, \partial Y; \mathbb{Z}) \to H_c^r(X, \partial X; \mathbb{Z})$ such that H_c^r becomes a functor in the following sense: the induced map of the identity is the identity, and the induced map of a (well-defined) composition of two proper maps (each of which sends boundary into boundary) is the composition of their induced maps. Moreover, if $\mathcal{H}: X \times [0,1] \to Y$ is a proper homotopy such that $\mathcal{H}(\partial X, t) \subseteq \partial Y$ for each $t \in [0,1]$, then $H_c^r(\mathcal{H}(-,0)) = H_c^r(\mathcal{H}(-,1))$ for all r. For more details see [37, pages 320, 322, 323, 339, and 341].

Let M be a connected orientable topological n–manifold. Then $H_c^n(M, \partial M; \mathbb{Z})$ is an infinite cyclic group; see [37, page 342]. If we choose an orientation of M (ie M is oriented), then there exists a unique element $[M] \in H_c^n(M, \partial M; \mathbb{Z})$ such that $[M]$ generates $H_c^n(M, \partial M; \mathbb{Z})$, and for each $x \in M \setminus \partial M$, the unique generator of $H^n(M, M \setminus x; \mathbb{Z})$, which comes from the chosen orientation of M, is sent to $[M]$ by the obvious isomorphism $H^n(M, M \setminus x; \mathbb{Z}) \to H_c^n(M, \partial M; \mathbb{Z})$; see [16, proof of Lemma 2.1]. Thus if $f: M \to N$ is a proper map between two connected oriented topological n–manifolds with $f(\partial M) \subseteq \partial N$, the (*compactly supported cohomological*) *degree* of f is the unique integer $\deg(f)$ defined by $H_c^n(f)([N]) = \deg(f)[M]$.

By the previous two paragraphs, we have the following:

(i) When manifolds are compact, the notion of compactly supported cohomological degree agrees with the notion of the usual degree defined by singular cohomology.

(ii) The degree is proper homotopy invariant: if $f, g: M \to N$ are proper maps between two connected oriented topological n–manifolds with $f(\partial M) \cup g(\partial M) \subseteq \partial N$ such that there is a proper homotopy $\mathcal{H}: M \times [0,1] \to N$ with $\mathcal{H}(\partial M \times [0,1]) \subseteq \partial N$ from f to g, then $\deg(f) = \deg(g)$.

(iii) The degree is multiplicative: the degree of the (well-defined) composition of two proper maps (each of which sends boundary into boundary) is the product of their degrees.

Therefore the degree of a proper homotopy equivalence between two oriented connected boundaryless n–manifolds is ± 1 due to (ii) and (iii) above. We use the following well-known characterizations of a map of degree ± 1. In the below two theorems, "D is a disk in a smooth n–manifold X" means D is the image of $\{z \in \mathbb{R}^n : |z| \leq 1\}$ under a smooth embedding $\{z \in \mathbb{R}^n : |z| \leq 2\} \hookrightarrow X$.

Theorem 2.6.1 [16, Lemma 2.1b] *Let $f : M \to N$ be a proper map between two connected oriented smooth manifolds of the same dimension such that $f^{-1}(\partial N) = \partial M$. Suppose for a disk D in $\operatorname{int}(N)$, $f^{-1}(D)$ is a disk in $\operatorname{int}(M)$ such that f maps $f^{-1}(D)$ homeomorphically onto D. Then $\deg(f) = +1$ or -1 according to whether $f | f^{-1}(D) \to D$ is orientation preserving or orientation reversing.*

The following theorem is due to Hopf, and says that for a degree 1 map, we can achieve such a disk with nice properties, as mentioned in Theorem 2.6.1, after a proper homotopy:

Theorem 2.6.2 [16, Theorems 3.1 and 4.1] *Let $f : M \to N$ be a proper map between two connected oriented smooth manifolds of the same dimension such that $f^{-1}(\partial N) \subseteq \partial M$. Suppose $\deg(f) = \pm 1$. Then there is a proper map $g : M \to N$ with $g(\partial M) \subseteq \partial N$ and a homotopy $\mathcal{H} : M \times [0,1] \to N$ from f to g with the following properties:*

- *There exists a compact subset $K \subseteq \operatorname{int}(M)$ such that $\mathcal{H}(x,t) = f(x)$ for all $(x,t) \in (M \setminus K) \times [0,1]$. In particular, H is a proper homotopy and $\mathcal{H}(\partial M, t) \subseteq \partial N$ for all $t \in [0,1]$.*
- *There exists a disk $D \subseteq \operatorname{int}(N)$ such that $g^{-1}(D)$ is a disk in $\operatorname{int}(M)$ and $g | g^{-1}(D) \to D$ is a homeomorphism.*

The theorem below is due to Olum, and roughly says that when there is a degree 1 map, the domain is more massive than the codomain.

Theorem 2.6.3 [16, Corollary 3.4] *Let $f : M \to N$ be a proper map between two connected oriented topological manifolds of the same dimension such that $f(\partial M) \subseteq \partial N$. If $\deg(f) = \pm 1$, then $\pi_1(f) : \pi_1(M) \to \pi_1(N)$ is surjective.*

3 Ingredients for proving Theorem 1

3.1 Decomposition of a noncompact surface into pairs of pants and punctured disks

Every compact surface of genus $g \geq 2$ is the union (with pairwise-disjoint interiors) of $(2g-2)$–many copies of the pair of pants, but the same thing doesn't happen for noncompact surfaces. For example, the thrice punctured sphere is not a union (with pairwise-disjoint interiors) of copies of the pair of pants; we need copies of the punctured disk. The main aim of this section is to prove that every noncompact surface, except the plane and the once punctured torus, decomposes into copies of the pair of pants and copies of the punctured disk when we cut it along a collection of circles, where each circle of this collection has an open neighborhood that does not intersect with any other circles of this collection.

First, we define some terminology:

Definition 3.1.1 Let X be a space, and let $\{X_\alpha : \alpha \in \mathscr{I}\}$ be a collection of subsets of X. We say $\{X_\alpha : \alpha \in \mathscr{I}\}$ is a *locally finite collection* and write $X_\alpha \to \infty$ if, for each compact subset K of X, $X_\alpha \cap K = \varnothing$ for all but finitely many $\alpha \in \mathscr{I}$.

Definition 3.1.2 Let $\mathscr{A}$ be a pairwise-disjoint collection of smoothly embedded circles on a surface Σ. We say $\mathscr{A}$ is a *locally finite curve system* (in short, LFCS) on Σ if $\mathscr{A}$ is a locally finite collection.

Remark 3.1.3 Let $\mathscr{A}$ be an LFCS on a surface Σ. Notice that $\bigcup \mathscr{A}$ (ie the union of all elements of $\mathscr{A}$) is a closed subset of Σ as well as a smoothly embedded submanifold of Σ such that the set of all components of $\bigcup \mathscr{A}$ is $\mathscr{A}$. But to simplify notation, whenever needed we will think of $\mathscr{A}$ and $\bigcup \mathscr{A}$ as the same without any harm.

Definition 3.1.4 Let $\mathscr{A}$ be an LFCS on a surface Σ. Suppose there exists an at most countable collection $\{\Sigma_n\}$ of bordered subsurfaces of Σ such that

(1) each Σ_n is a closed subset of Σ,
(2) $\operatorname{int}(\Sigma_n) \cap \operatorname{int}(\Sigma_m) = \varnothing$ if $n \neq m$,
(3) $\bigcup_n \Sigma_n = \Sigma$, and
(4) $\bigcup_n \partial\Sigma_n = \bigcup \mathscr{A}$.

In this case, we say $\mathscr{A}$ *decomposes* Σ *into bordered subsurfaces*, where *complementary components* are $\{\Sigma_n\}$. Also, we call each component of $\mathscr{A}$ a *decomposition circle*.

The following theorem asserts that any noncompact surface other than the plane has a decomposition, where each complementary part is either a pair of pants, a one-holed torus, or a punctured disk. This decomposition of the codomain of a pseudoproper homotopy equivalence will be used in all cases.

Theorem 3.1.5 *Let Σ be a noncompact surface not homeomorphic to $\mathbb{R}^2$. Then there is an LFCS $\mathscr{C}$ on Σ such that $\mathscr{C}$ decomposes Σ into bordered subsurfaces, and a complementary component of this decomposition is homeomorphic to either $S_{1,1}$ (used at most once), $S_{0,3}$, or $S_{0,1,1}$.*

Proof It is enough to find a collection $\{\Sigma_n\}$ of bordered subsurfaces of Σ with the four properties, as mentioned in Definition 3.1.4, so that each Σ_n is homeomorphic to either $S_{0,3}$, $S_{1,1}$, or $S_{0,1,1}$. For that, consider an inductive construction of Σ; see Theorem 2.5.1. Now, a finite sequence of annuli, when added to the compact bordered surface used just before it, can be ignored. Thus we may assume $S_{0,3}$ or $S_{1,2}$ is used after $S_{0,1}$ without loss of generality because $\Sigma \not\approx \mathbb{R}^2$, and hence, pushing $S_{0,1}$ into $\operatorname{int}(S_{0,3})$ or $\operatorname{int}(S_{1,2})$, we end up with $S_{0,2}$ (which can be ignored) or $S_{1,1}$. We complete the proof by observing that $S_{1,2}$ can be decomposed into two copies of $S_{0,3}$, and $S_{0,1,1}$ is the union (with pairwise-disjoint interiors) of countably many copies of $S_{0,2}$. □

Remark 3.1.6 A statement closely related to Theorem 3.1.5 is [2, Theorem 1.1], which says that *"every surface except for the sphere, the plane, and the torus is the union (with pairwise-disjoint interiors) of copies of the pair of pants and copies of the punctured disk"*. But due to Definition 3.1.4(4), if we want

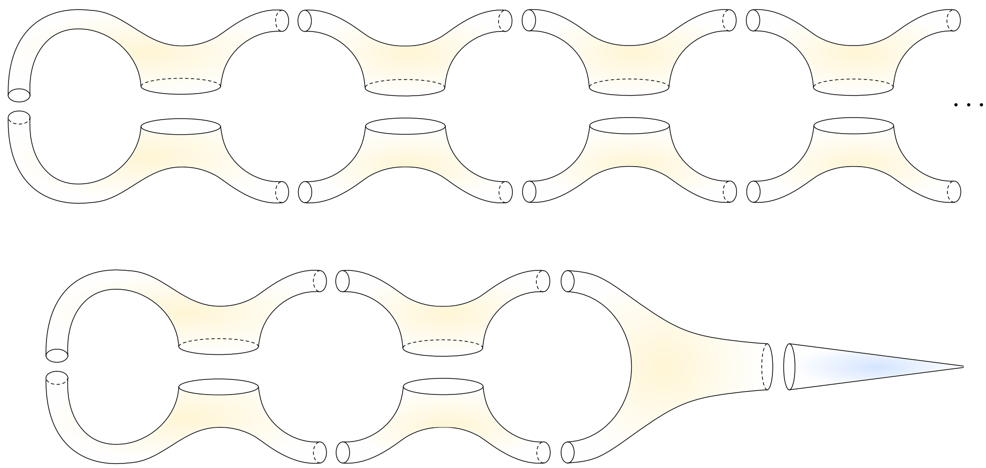

Figure 2: Top: decomposition of the Loch Ness monster into countably infinitely many copies of the pair of pants. Bottom: decomposition of $S_{3,0,1}$ into five copies of the pair of pants and a copy of the punctured disk.

that any complementary component is homeomorphic to only either $S_{0,3}$ or $S_{0,1,1}$, then we also need to assume that $\Sigma \not\cong S_{1,0,1}$; see Figure 2 and Theorem 3.1.7.

Theorem 3.1.7 *Let Σ be a noncompact surface that is not homeomorphic to either $\mathbb{R}^2$ or $S_{1,0,1}$. Then there is an LFCS $\mathscr{C}'$ on Σ such that $\mathscr{C}'$ decomposes Σ into bordered subsurfaces, and a complementary component of this decomposition is homeomorphic to either $S_{0,3}$ or $S_{0,1,1}$.*

Proof It is enough to find a collection $\{\Sigma_n\}$ of bordered subsurfaces of Σ with the four properties of Definition 3.1.4 such that each Σ_n is homeomorphic to either $S_{0,3}$ or $S_{0,1,1}$. For that, consider an inductive construction of Σ; see Theorem 2.5.1. We will divide the whole proof into two cases, depending on whether Σ has at least two ends.

At first, suppose the number of ends of Σ is at least two. Now, the definition of the space of ends tells us that we need to use at least one pair of pants in the inductive construction of Σ. By Lemma 3.1.8, we may assume that in this inductive construction, a pair of pants is used just after the disk. An argument similar to before (see the proof of Theorem 3.1.5) concludes this case.

Next, consider the case when the number of ends of Σ is precisely one. That is, Σ can be either the Loch Ness monster (the infinite-genus surface with one end) or $S_{g,0,1}$ with $g \geq 2$. The Loch Ness monster decomposes into countably infinitely many copies of the pair of pants, and $S_{g,0,1}$ with $g \geq 2$ decomposes into $(2g-1)$–many copies of the pair of pants and a copy of the punctured disk; see Figure 2. □

To prove Theorem 3.1.7 we used Lemma 3.1.8, which says that in an inductive construction of a noncompact surface, interchanging the positions of the compact bordered surfaces used in the first few inductive steps doesn't change the homeomorphism type; its proof is based on the observation that the portions outside compact subsets determine the space of ends.

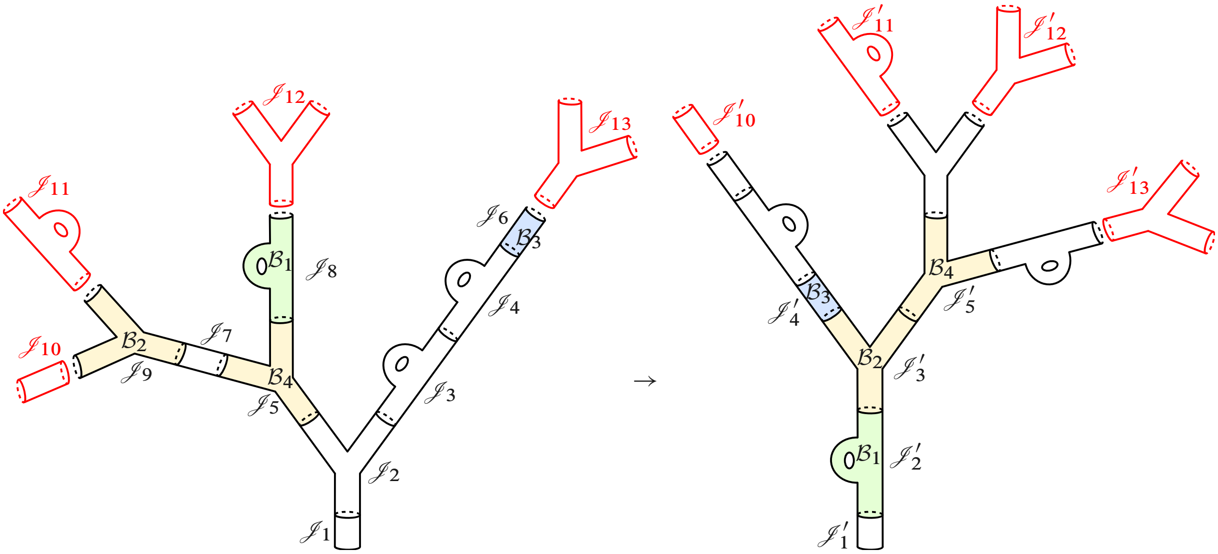

Figure 3: $\mathscr{I}_r$ (resp. $\mathscr{I}'_r$) denotes the r^{th} step of $\mathscr{I}$ (resp. $\mathscr{I}'$). Here $n_0 = 9$ and $n = 4$. Also, the red-colored compact bordered surfaces are the portions of $\boldsymbol{S}$, and the inductive construction of $\boldsymbol{S}$ given by $\mathscr{I}'$ is inherited from the inductive construction of $\boldsymbol{S}$ given by $\mathscr{I}$.

Lemma 3.1.8 *Let Σ be a noncompact surface with some inductive construction $\mathscr{I}$. Denote the compact bordered subsurface of Σ after the i^{th} step of $\mathscr{I}$ by K_i. Suppose*

$$\{\mathcal{B}_1, \dots, \mathcal{B}_n : \text{ each } \mathcal{B}_l \text{ is homeomorphic to } S_{0,2}, S_{0,3}, \text{ or } S_{1,2}\}$$

is a finite collection of compact bordered surfaces such that $\mathcal{B}_l$ is used to construct K_{i_l+1} from K_{i_l} for each $l = 1, \dots, n$. Then there exists a noncompact surface Σ' with an inductive construction $\mathscr{I}'$ such that $\Sigma' \cong \Sigma$ and $\mathcal{B}_l$ is used to construct K'_{l+1} from K'_l for each $l = 1, \dots, n$, where K'_i denotes the compact bordered subsurface of Σ after the i^{th} step of $\mathscr{I}'$.

Proof Let n_0 be a positive integer such that K_{n_0} contains each $\mathcal{B}_l$. Define $\boldsymbol{S} := \Sigma \setminus \text{int}(K_{n_0})$. Thus $\boldsymbol{S}$ is a bordered subsurface of Σ with $\partial \boldsymbol{S} = \partial K_{n_0}$. Now consider all copies of different building blocks used up to the n_0^{th} step of $\mathscr{I}$, and inside K_{n_0} interchange them so that $\mathcal{B}_1, \dots, \mathcal{B}_n$ comes just after the initial disk K_1 one by one following the increasing order of their indices. Denote the result of this interchange process by K'_{n_0}. So $K_{n_0} \cong K'_{n_0}$ as $g(K_{n_0}) = g(K'_{n_0})$ and $\partial K_{n_0} \cong \partial K'_{n_0}$. Define a noncompact surface Σ' as $\Sigma' := K'_{n_0} \cup_{\partial K'_{n_0} \equiv \partial \boldsymbol{S}} \boldsymbol{S}$. Therefore $\Sigma \setminus K_{n_0} = \text{int}(\boldsymbol{S}) = \Sigma' \setminus K'_{n_0}$ (notice that we are thinking of $\boldsymbol{S}$ as a subset of Σ' using the obvious embedding $\boldsymbol{S} \hookrightarrow \Sigma'$).

Choose an inductive construction $\mathscr{I}'_{\le n_0}$ of K'_{n_0} such that the i^{th} element of the ordered sequence $K_1, \mathcal{B}_1, \dots, \mathcal{B}_l$ is used in the i^{th} step of $\mathscr{I}'_{\le n_0}$. Then $\mathscr{I}$ gives a truncated inductive construction $\mathscr{I}|\boldsymbol{S}$ on $\boldsymbol{S}$ starting from the $(n_0+1)^{\text{th}}$ step. Now $\mathscr{I}'_{\le n_0}$ followed by $\mathscr{I}|\boldsymbol{S}$ together gives an inductive construction $\mathscr{I}'$ of Σ'. Roughly it means $\mathscr{I}'$ is the same as the inductive construction of Σ, except for the first few steps.

Denote the compact bordered subsurface of Σ' after the i^{th} step of $\mathscr{I}'$ by K_i'. To complete the proof, we show $\Sigma' \cong \Sigma$ using Theorem 2.4.1.

Consider the efficient exhaustion $\{K_i\}$ (resp. $\{K_i'\}$) of Σ (resp. Σ') by compacta to define $\mathrm{Ends}(\Sigma)$ (resp. $\mathrm{Ends}(\Sigma')$). Recall that the space of ends remains the same up to homeomorphism even if we choose a different efficient exhaustion by compacta; see Section 2.3. By $\Sigma \setminus K_{n_0} = \mathrm{int}(\boldsymbol{S}) = \Sigma' \setminus K'_{n_0}$, for every sequence $(V_1, V_2, \dots) \in \mathrm{Ends}(\Sigma)$, there exists a unique sequence $(V_1', V_2', \dots) \in \mathrm{Ends}(\Sigma')$ such that $V_i = V_i'$ for all integers $i \geq n_0$, and conversely. Thus there exists a homeomorphism $\varphi\colon \mathrm{Ends}(\Sigma) \to \mathrm{Ends}(\Sigma')$ with $\varphi(\mathrm{Ends}_{\mathrm{np}}(\Sigma)) = \mathrm{Ends}_{\mathrm{np}}(\Sigma')$. Also, $\Sigma \setminus K_{n_0} = \mathrm{int}(\boldsymbol{S}) = \Sigma' \setminus K'_{n_0}$ and $K_{n_0} \cong K'_{n_0}$ together imply $g(\Sigma) = g(\Sigma')$. Therefore $\Sigma' \cong \Sigma$ by Theorem 2.4.1. □

The spine construction of Goldman's inductive procedure shows that every noncompact surface Σ (possibly of infinite type) is the interior of a bordered surface: consider the graph $\mathcal{G}$ consisting of blue straight line segments and red circles, as given in Figure 1. Any thickening [23, Definition 7.2] of $\mathcal{G}$ in Σ is the interior of a bordered subsurface $\boldsymbol{S}$ of Σ. Now [23, Corollary 7.2. and Section 7.3] says that $\mathrm{int}(\boldsymbol{S}) \cong \Sigma$. When Σ is of finite type, we prove the same thing differently in the following theorem:

Theorem 3.1.9 *A noncompact finite-type surface is the interior of a compact bordered surface. In particular, if a noncompact surface has infinite cyclic (resp. trivial) fundamental group, then it is homeomorphic to* $\mathbb{S}^1 \times \mathbb{R}$ *(resp.* $\mathbb{R}^2$*).*

Proof Let Σ be a finite-type noncompact surface. Consider an inductive construction of Σ; see Theorem 2.5.1. Since $\pi_1(\Sigma)$ is finitely generated, Theorem 2.5.2 says that Σ is homotopy equivalent to $\bigvee^{2r+s}\mathbb{S}^1$, where in this inductive construction $r \in \mathbb{N}$ is the total number of copies of $S_{1,2}$, and $s \in \mathbb{N}$ is the total number of copies of $S_{0,3}$; see Figure 1. Thus there is an integer n such that $\overline{\Sigma \setminus K_n}$ (where K_n is the compact bordered subsurface of Σ after n^{th} inductive step) is a finite collection of punctured disks. Now $g(\Sigma) = g(\mathrm{int}(K_n))$. Also, each end of Σ (resp. $\mathrm{int}(K_n)$) is planar, and the total number of ends of Σ (resp. $\mathrm{int}(K_n)$) is the same as the number of components of ∂K_n. By Theorem 2.4.1, $\Sigma \cong \mathrm{int}(K_n)$.

If Σ is a noncompact surface with infinite-cyclic fundamental group, then any inductive construction of Σ contains no copy of $S_{1,2}$ but precisely one copy of $S_{0,3}$, ie $\Sigma \cong \mathbb{S}^1 \times \mathbb{R}$. Similarly, if Σ is a noncompact surface with trivial fundamental group, then any inductive construction of Σ has no copy of $S_{0,3}$ as well as no copy of $S_{1,2}$, ie $\Sigma \cong \mathbb{R}^2$. □

The proposition below follows directly from Goldman's inductive construction, so we quote it without proof. It says that an infinite-type surface has a finite genus only if it has infinitely many ends. On the other hand, Theorem 2.4.2 guarantees the existence of an infinite-type surface of infinite genus with infinitely many ends.

Proposition 3.1.10 *A noncompact surface is of a finite genus if and only if the total number of copies of* $S_{1,2}$ *used in any inductive construction of* Σ *is finite. Thus if an infinite-type surface has a finite genus, it must have infinitely many ends.*

This section's final fact (as promised in the introduction) says that the fundamental group alone can't detect the homeomorphism type of an infinite-type surface.

Proposition 3.1.11 *Up to homotopy equivalence, there is exactly one infinite-type surface, but up to homeomorphism, there are $2^{\aleph_0}$–many infinite-type surfaces.*

Proof Any infinite-type surface is homotopy equivalent to the wedge of countably infinitely many circles; see Theorem 2.5.2. Thus, any two infinite-type surfaces are homotopy equivalent.

Now, we prove that up to homeomorphism, there are $2^{\aleph_0}$–many infinite-type surfaces. Notice that except for the first step, in each step of Goldman's inductive procedure, we use $S_{0,3}$, or $S_{0,2}$, or $S_{1,2}$. Thus we have at most $3^{\aleph_0} = 2^{\aleph_0}$–many noncompact surfaces, up to homeomorphism. Therefore it is enough to show that this upper bound is reachable. Let τ be a nonempty closed subset of the Cantor set. By Theorem 2.4.2, there exists an infinite-genus surface Σ_τ such that $\mathrm{Ends}(\Sigma_\tau) = \mathrm{Ends}_{\mathrm{np}}(\Sigma_\tau) \cong \tau$. Therefore, if τ_1 and τ_2 are two nonhomeomorphic nonempty closed subsets of the Cantor set, then Σ_{τ_1} is not homeomorphic to Σ_{τ_2} by Theorem 2.4.1. Now [32, Theorem 2] says that up to homeomorphism, there are $2^{\aleph_0}$–many closed subsets of the Cantor set. □

3.2 Transversality of a proper map with respect to all decomposition circles

In the previous section, the codomain of a pseudoproper homotopy equivalence has been decomposed into finite-type bordered surfaces by a locally finite pairwise-disjoint collection of circles. This section aims to properly homotope the pseudoproper homotopy equivalence to make it transverse to each decomposition circle.

The theorem below follows from the theory developed in the appendix. We aim to use it to impose a 1–dimensional submanifold structure on the inverse image of each decomposition circle.

Theorem 3.2.1 *Let $f : \Sigma' \to \Sigma$ be a proper map between noncompact surfaces, and let $\mathscr{A}$ be an LFCS on Σ. Then f can be properly homotoped to make it smooth as well as transverse to the manifold $\mathscr{A}$.*

Proof Using Theorem A.1.1, after a proper homotopy, we may assume that f is a smooth proper map. After that, using Theorem A.1.2, properly homotope f so that it becomes transverse to $\mathscr{A}$. □

Remark 3.2.2 Note that in Theorem 3.2.1, we have no control over those proper homotopies, which make the proper map f smooth as well as transverse to $\mathscr{A}$, ie after these proper homotopies, $f^{-1}(\mathscr{A})$ can be empty, even if these proper homotopies start with a surjective proper map. A remedy for this is to assume $\deg(f) \neq 0$. This is because the degree is invariant under proper homotopy, and a map of nonzero degree is surjective; see Lemmas 3.6.4.1 and 3.6.4.3. If f is a proper homotopy equivalence, then f has a proper homotopy inverse; hence $\deg(f) \neq 0$ (see Section 2.6). But if f is a pseudoproper homotopy equivalence, then we don't know (at least till this stage) whether f has a proper homotopy inverse or not (though it has a homotopy inverse). Later in Section 3.6, using π_1–bijectivity, we will show that most pseudoproper homotopy equivalence is a map of degree ± 1.

The following theorem says that the transverse preimage of an LFCS under a proper map is an LFCS.

Theorem 3.2.3 *Let $f:\Sigma' \to \Sigma$ be a smooth proper map between two noncompact surfaces, and let $\mathscr{A}$ be an LFCS on Σ such that $f \pitchfork \mathscr{A}$. Then for each component $\mathcal{C}$ of $\mathscr{A}$, either $f^{-1}(\mathcal{C})$ is empty or a pairwise-disjoint finite collection of smoothly embedded circles on Σ'. Therefore $f^{-1}(\mathscr{A})$ is an LFCS on Σ'.*

Proof By the definition of transversality, $f \pitchfork \mathscr{A}$ implies $f \pitchfork \mathcal{C}$ for each component $\mathcal{C}$ of $\mathscr{A}$. Thus $f^{-1}(\mathcal{C})$ is either empty or is a compact (since f is proper) 1–dimensional boundaryless smoothly embedded submanifold of Σ'. By the classification of closed 1–dimensional manifolds, this completes the first part.

Next, if possible, let K' be a compact subset of Σ' such that K' intersects infinitely many components of $f^{-1}(\mathscr{A})$. By the first part, this means the compact set $f(K')$ intersects infinitely many components of $\mathscr{A}$, which contradicts the fact that $\mathscr{A}$ is a locally finite collection. □

3.3 Disk removal

As previously observed, after a proper homotopy, the number of components in the collection of transverse preimages of all decomposition circles can be infinite, and many components (possibly infinitely many) of this collection, may be trivial circles. Here our goal is to group all these trivial circles in terms of the size of the disk bounded by a trivial circle and then remove all groups of trivial circles simultaneously by a proper homotopy.

Our intended grouping requires a technical lemma, which asserts that on a nonsimply connected surface, an LFCS consisting of concentric trivial circles doesn't exist. Roughly, it means, on a nonsimply connected surface, arbitrarily large disks bounded by components of an LFCS don't exist.

Lemma 3.3.1 *Let Σ be a surface, and let $\mathscr{A} := \{\mathcal{C}_i : i \in \mathbb{N}\}$ be an LFCS on Σ such that for each i the circle $\mathcal{C}_i$ bounds a disk $\mathcal{D}_i \subset \Sigma$ with $\mathcal{C}_i \subset \operatorname{int}(\mathcal{D}_{i+1})$. Then Σ is homeomorphic to $\mathbb{R}^2$.*

Proof At first, notice that Σ must be noncompact as $\mathscr{A}$ is a locally finite pairwise-disjoint infinite collection of circles. Using inductive construction (see Theorem 2.5.1), we have a sequence $\{\boldsymbol{S}_j : j \in \mathbb{N}\}$ of compact bordered subsurfaces of Σ such that $\bigcup_j \boldsymbol{S}_j = \Sigma$ and $\boldsymbol{S}_j \subset \operatorname{int}(\boldsymbol{S}_{j+1})$ for each $j \in \mathbb{N}$. Consider any $p \in \Sigma$. A $j_0 \in \mathbb{N}$ exists such that $p \in \boldsymbol{S}_{j_0}$ and $\boldsymbol{S}_{j_0} \cap (\cup_i \mathcal{C}_i) \neq \varnothing$. Since $\mathscr{A}$ is a locally finite collection, only finitely many components of $\mathscr{A}$ intersect the compact set $\boldsymbol{S}_{j_0}$. Let $\mathcal{C}_{i_1}, \dots, \mathcal{C}_{i_l}$ be the only components of $\mathscr{A}$ intersecting $\boldsymbol{S}_{j_0}$, where $i_1 < \cdots < i_l$. Pick an integer $i_0 > i_l$. Then $\mathcal{C}_{i_0} \cap \boldsymbol{S}_{j_0} = \varnothing$. Now, since $\mathcal{C}_{i_l} \subset \operatorname{int}(\mathcal{D}_{i_0})$, $\boldsymbol{S}_{j_0}$ is connected, and Σ is locally Euclidean, we can say that $\boldsymbol{S}_{j_0} \subseteq \operatorname{int}(\mathcal{D}_{i_0})$. Thus every point $x \in \Sigma$ has an open neighborhood $\mathcal{U}_x$ in Σ such that $\mathcal{U}_x \subseteq \mathcal{D}_i$ for some $i \in \mathbb{N}$. Therefore every loop on Σ is contained in a disk $\mathcal{D}_i$ for some large $i \in \mathbb{N}$, ie Σ is simply connected. By Theorem 3.1.9, $\Sigma \cong \mathbb{R}^2$. □

The following lemma is the primary tool for showing that a homotopy is proper. It tells how a proper map can be properly homotoped so that it changes on infinitely many pairwise-disjoint compact sets.

Lemma 3.3.2 *Let $f:\Sigma'\to\Sigma$ be a proper map between two noncompact surfaces, and let $\{\Sigma'_n : n\in\mathbb{N}\}$ be a pairwise-disjoint collection of compact bordered subsurfaces of Σ'. For each $n\in\mathbb{N}$, suppose $H_n:\Sigma'_n\times[0,1]\to\Sigma$ is a homotopy relative to $\partial\Sigma'_n$ such that $H_n(-,0)=f|\Sigma'_n$ and $\operatorname{im}(H_n)\to\infty$. Then $\mathcal{H}:\Sigma'\times[0,1]\to\Sigma$ defined by*

$$\mathcal{H}(p,t):=\begin{cases} H_n(p,t) & \text{if } p\in\Sigma'_n \text{ and } t\in[0,1],\\ f(p) & \text{if } p\in\Sigma'\setminus\left(\bigcup_{n\in\mathbb{N}}\Sigma'_n\right) \text{ and } t\in[0,1],\end{cases}$$

is a proper map.

Proof Let $\mathcal{K}$ be a compact subset of Σ. By continuity of $\mathcal{H}$, $\mathcal{H}^{-1}(\mathcal{K})$ is closed in Σ'. Since $\operatorname{im}(H_n)\to\infty$, there exists $n_0\in\mathbb{N}$ such that $\operatorname{im}(H_l)\cap\mathcal{K}=\varnothing$ for all integers $l\ge n_0+1$. Now, $f^{-1}(\mathcal{K})$ is compact as f is proper. Also, the domain of each H_n is compact. Hence, the closed subset $\mathcal{H}^{-1}(\mathcal{K})$ of Σ' is contained in the compact set $f^{-1}(\mathcal{K})\cup\bigcup_{l=1}^{n_0} H_n^{-1}(\mathcal{K})$. □

To remove trivial components from the transverse preimage of an LFCS with infinitely many components, we need to impose some conditions on this LFCS. One such preferred LFCS is given in Theorem 3.1.5, but we will require other kinds of LFCS on the codomain, so here is the list of different preferred LFCS:

Definition 3.3.3 Let Σ be a noncompact surface such that $\Sigma\not\cong\mathbb{R}^2$. Suppose $\mathscr{A}$ is a given LFCS on Σ. We say $\mathscr{A}$ is a *preferred LFCS* on Σ if either

- (i) $\mathscr{A}$ is a finite collection of primitive circles on Σ, or
- (ii) $\mathscr{A}$ decomposes Σ into bordered subsurfaces, and a complementary component of this decomposition is homeomorphic to $S_{1,1}$, $S_{0,3}$, $S_{0,2}$, or $S_{0,1,1}$.

Remark 3.3.4 The only use of case (i) of Definition 3.3.3 is in Section 3.6, where we consider the process of removing unnecessary circles from the transverse preimage of the boundary of an essential pair of pants or an essential punctured disk. It is worth noting that by a finite LFCS, one can't decompose an infinite-type surface into finite-type bordered surfaces.

In the theorem below, we construct a proper homotopy which removes all trivial components keeping a neighborhood of each primitive component stationary from the transverse preimage of a preferred LFCS. Recall that a homotopy $H:X\times[0,1]\to Y$ is said to be *stationary on a subset A of X* if $H(a,t)=H(a,0)$ for all $(a,t)\in A\times[0,1]$.

Theorem 3.3.5 *Let $f:\Sigma'\to\Sigma$ be a smooth proper map between two noncompact surfaces, where $\Sigma'\not\cong\mathbb{R}^2\not\cong\Sigma$, and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f\pitchfork\mathscr{A}$. Then we can properly homotope f to remove all trivial components of the LFCS $f^{-1}(\mathscr{A})$ such that each primitive component of $f^{-1}(\mathscr{A})$ has an open neighborhood on which this proper homotopy is stationary.*

Proof Since $\Sigma' \not\cong \mathbb{R}^2$ and $f^{-1}(\mathscr{A})$ is an LFCS (see Theorem 3.2.3), by Lemma 3.3.1 there don't exist infinitely many components $\mathcal{C}'_1, \mathcal{C}'_2, \dots$ of $f^{-1}(\mathscr{A})$ bounding the disks $\mathcal{D}'_1, \mathcal{D}'_2, \dots$, respectively, such that $\mathcal{C}'_n$ is contained in the interior of $\mathcal{D}'_{n+1}$ for each n. Thus, if $f^{-1}(\mathscr{A})$ has a trivial component, we can introduce the notion of an outermost disk bounded by a component of $f^{-1}(\mathscr{A})$ in the following way: A disk $\mathcal{D}' \subset \Sigma'$ bounded by a component of $f^{-1}(\mathscr{A})$ is called an *outermost disk* if, given another disk $\mathcal{D}'' \subset \Sigma$ bounded by a component of $f^{-1}(\mathscr{A})$, then either $\mathcal{D}'' \subseteq \mathcal{D}'$ or $\mathcal{D}' \cap \mathcal{D}'' = \varnothing$.

Let $\{\mathcal{D}'_n\}$ be the pairwise-disjoint collection (which may be an infinite collection) of all outermost disks. Assume $\mathcal{C}_n$ represents that component of $\mathscr{A}$ for which $f(\partial\mathcal{D}'_n) \subseteq \mathcal{C}_n$. Note $\mathcal{C}_n$ may equal to $\mathcal{C}_m$ even if $m \neq n$.

For each integer n, we will construct a compact bordered subsurface $\mathcal{Z}_n$ with $f(\mathcal{D}'_n) \subseteq \mathcal{Z}_n$ such that $\mathcal{Z}_n \to \infty$. Roughly, $\mathcal{Z}_n$ will be obtained from taking the union of all those complementary components of Σ (if a punctured disk appears, truncate it) which are hit by $f(\mathcal{D}'_n)$.

Fix an integer n. Let $\mathcal{X}'_{n,1}, \dots, \mathcal{X}'_{n,k_n}$ be the all connected components of $\mathcal{D}'_n \setminus f^{-1}(\mathscr{A})$. By continuity of f, for each $\mathcal{X}'_{n,l}$, there exists a complementary component $\mathcal{Y}_{n,l}$ of Σ decomposed by $\mathscr{A}$ such that $f(\mathcal{X}'_{n,l}) \subseteq \mathcal{Y}_{n,l}$ and $\partial\mathcal{X}'_{n,l} \subseteq f^{-1}(\partial\mathcal{Y}_{n,l})$; see Figure 4. For each l, define a compact bordered subsurface $\mathcal{Z}_{n,l}$ of Σ as follows: If $\mathcal{Y}_{n,l}$ is homeomorphic to $S_{1,1}$, $S_{0,3}$, or $S_{0,2}$; define $\mathcal{Z}_{n,l} := \mathcal{Y}_{n,l}$. On the other hand, if $\mathcal{Y}_{n,l}$ is homeomorphic to $S_{0,1,1}$, then let $\mathcal{Z}_{n,l}$ be an annulus in $\mathcal{Y}_{n,l}$ such that $\partial\mathcal{Z}_{n,l} \cap \partial\mathcal{Y}_{n,l} = \partial\mathcal{Y}_{n,l}$ and $f(\overline{\mathcal{X}'_{n,l}}) \subseteq \mathcal{Z}_{n,l}$. Finally, define $\mathcal{Z}_n := \mathcal{Z}_{n,1} \cup \dots \cup \mathcal{Z}_{n,k_n}$.

Now, we show $\mathcal{Z}_n \to \infty$, so consider a compact subset $\mathcal{K}$ of Σ. Let $\boldsymbol{S}_1, \dots, \boldsymbol{S}_m$ be a collection of complementary components of Σ decomposed by $\mathscr{A}$ such that $\mathcal{K} \subseteq \operatorname{int}\big(\bigcup_{l=1}^m \boldsymbol{S}_l\big)$. Define $\boldsymbol{S} := \bigcup_{l=1}^m \boldsymbol{S}_l$. Thus, for an integer n, $f(\mathcal{D}'_n) \cap \boldsymbol{S} \neq \varnothing$ if and only if $\mathcal{D}'_n$ contains at least one component of $\bigcup_{l=1}^m f^{-1}(\partial\boldsymbol{S}_l)$. This is due to the construction of each $\mathcal{Z}_n$; see Figure 4. For each component $\mathcal{C}$ of $\mathscr{A}$, Theorem 3.2.3 says that $f^{-1}(\mathcal{C})$ has only finitely many components. So $\mathcal{D}'_n$ doesn't contain any component of $\bigcup_{l=1}^m f^{-1}(\partial\boldsymbol{S}_l)$ for all sufficiently large n, ie $f(\mathcal{D}'_n) \cap \boldsymbol{S} = \varnothing$ for all sufficiently large n. Since $\mathcal{K} \subseteq \operatorname{int}(\boldsymbol{S})$ and each $\mathcal{Z}_n$ is obtained from taking the union of all those complementary components of Σ (if a punctured disk appears, truncate it) which are hit by $f(\mathcal{D}'_n)$, we can say that $\mathcal{Z}_n \cap \mathcal{K} = \varnothing$ for all sufficiently large n. Therefore $\mathcal{Z}_n \to \infty$ as $\mathcal{K}$ is an arbitrary compact subset of Σ.

For each n, adding a small external collar to one of the boundary components of $\mathcal{Z}_n$ (if needed), we can construct a compact bordered surface Σ_n with $\mathcal{C}_n \subseteq \operatorname{int}(\Sigma_n)$, $f(\mathcal{D}'_n) \subseteq \Sigma_n$ such that $\{\Sigma_n\}$ is a locally finite collection, ie $\Sigma_n \to \infty$; see Figure 4.

For each n, write $\mathcal{C}'_n := \partial\mathcal{D}'_n$. Thus $f(\mathcal{C}'_n) \subseteq \mathcal{C}_n$. Since $\mathcal{C}_n \subseteq \operatorname{int}(\Sigma_n)$, using Theorem A.2.1, choose a one-sided tubular neighborhood $\mathcal{C}_n \times [0, \varepsilon_n]$ of $\mathcal{C}_n$ in Σ with $\mathcal{C}_n \times 0 \equiv \mathcal{C}_n$ such that $f \pitchfork (\mathcal{C}_n \times t_n)$ for each $t_n \in [0, \varepsilon_n]$ and $\mathcal{C}_n \times [0, \varepsilon_n] \subseteq \Sigma_n$. Without loss of generality, we may further assume that $f(x') \in \mathcal{C}_n \times [0, \varepsilon_n]$ for each $x' \in \Sigma' \setminus \mathcal{D}'_n$ sufficiently near to $\mathcal{C}'_n$. Next, since $f^{-1}(\mathscr{A})$ is an LFCS, for each n, Theorem A.2.3 gives a one-sided compact tubular neighborhood $\mathcal{U}'_n$ of $\mathcal{C}'_n$ such that $\mathcal{U}'_n \cap \mathcal{D}'_n = \mathcal{C}'_n = \mathcal{U}'_n \cap f^{-1}(\mathscr{A})$, $f(\mathcal{U}'_n) \subseteq \mathcal{C}_n \times [0, \varepsilon_n]$ for each n, and $\mathcal{U}'_n \cap \mathcal{U}'_m = \varnothing$ for $m \neq n$. Finally, Theorem A.2.5

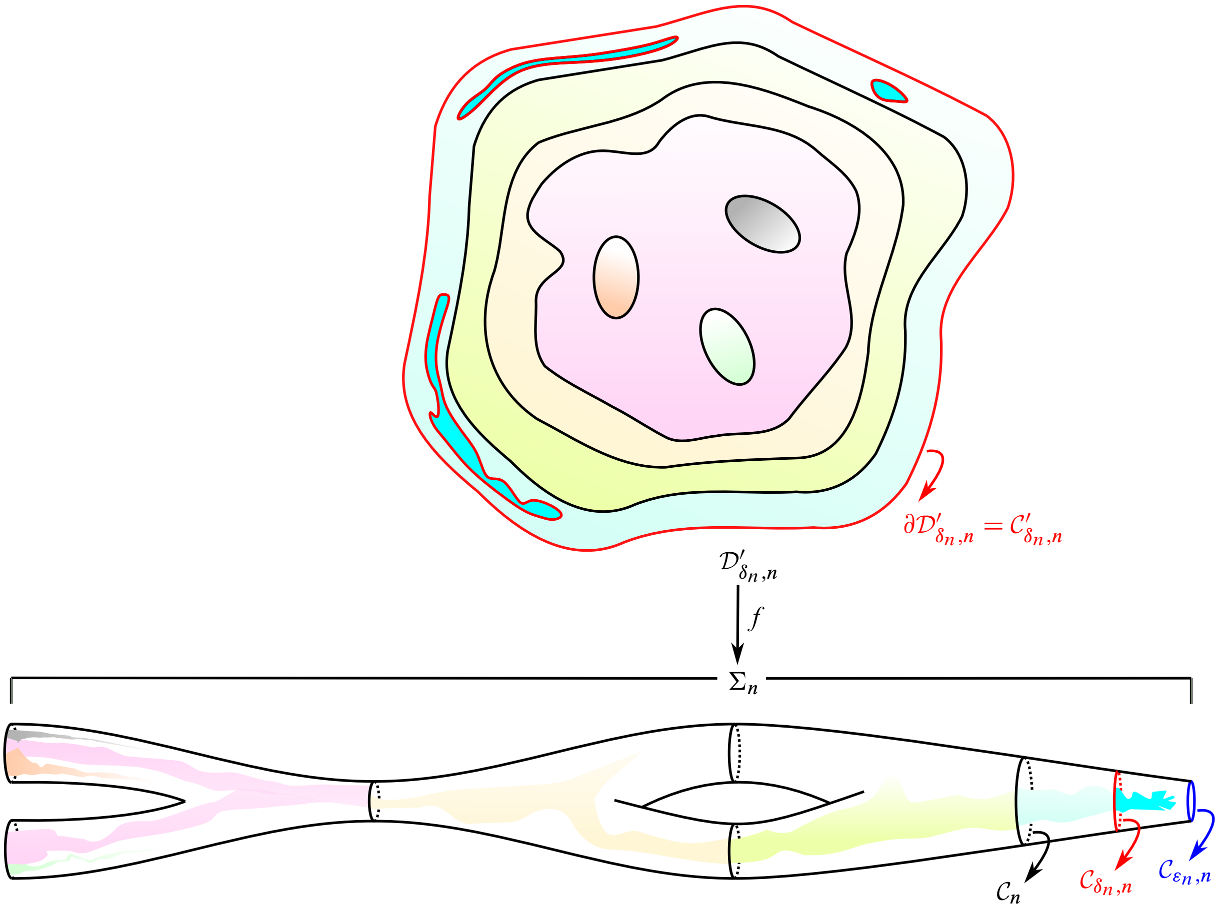

Figure 4: Each component of $\mathcal{D}'_n \setminus f^{-1}(\mathscr{A})$ maps into a component of $\Sigma \setminus \mathscr{A}$. This fact, together with Theorem A.2.5, provides Σ_n. A black circle denotes a component of either $\mathscr{A}$ or a component of $f^{-1}(\mathscr{A})$.

gives $\delta_n \in (0, \varepsilon_n)$ and a component $\mathcal{C}'_{\delta_n,n}$ of $f^{-1}(\mathcal{C}_{\delta_n,n})$ such that $\mathcal{C}'_{\delta_n,n}$ bounds a disk $\mathcal{D}'_{\delta_n,n}$ in Σ' with $(\mathcal{U}'_n \cup \mathcal{D}'_n) \supseteq \mathcal{D}'_{\delta_n,n} \supset \operatorname{int}(\mathcal{D}'_{\delta_n,n}) \supset \mathcal{D}'_n$ (equivalently, $\mathcal{U}'_n$ contains the annulus cobounded by $\mathcal{C}'_{\delta_n,n}$ and $\mathcal{C}'_n$) and $f(\mathcal{D}'_{\delta_n,n} \setminus \operatorname{int}(\mathcal{D}'_n)) \subseteq \mathcal{C}_n \times [0, \varepsilon_n]$. Thus $\mathcal{D}'_{\delta_n,n} \cap f^{-1}(\mathscr{A}) = \mathcal{D}'_n \cap f^{-1}(\mathscr{A})$, $f(\mathcal{D}'_{\delta_n,n}) \subseteq \Sigma_n$ for each n, and $\mathcal{D}'_{\delta_n,n} \cap \mathcal{D}'_{\delta_m,m} = \varnothing$ when $m \neq n$.

Since $\mathcal{C}_{\delta_n,n}$ cobounds an annulus with the primitive circle $\mathcal{C}_n$, the inclusion $\mathcal{C}_{\delta_n,n} \hookrightarrow \Sigma_n$ is π_1–injective (see Theorem 2.2.2). Also, Σ_n is homotopy equivalent to $\bigvee_{\text{finite}} \mathbb{S}^1$, which implies that the universal cover of Σ_n is contractible, and thus $\pi_2(\Sigma_n) = 0$. Therefore, exactness of

$$\cdots \to \pi_2(\Sigma_n) \to \pi_2(\Sigma_n, \mathcal{C}_{\delta_n,n}) \to \pi_1(\mathcal{C}_{\delta_n,n}) \to \pi_1(\Sigma_n) \to \cdots$$

gives $\pi_2(\Sigma_n, \mathcal{C}_{\delta_n,n}) = 0$, ie we have a homotopy $H_n\colon \mathcal{D}'_{\delta_n,n} \times [0,1] \to \Sigma_n$ relative to $\mathcal{C}'_{\delta_n,n}$ from $f|(\mathcal{D}'_{\delta_n,n}, \mathcal{C}'_{\delta_n,n}) \to (\Sigma_n, \mathcal{C}_{\delta_n,n})$ to a map $\mathcal{D}'_{\delta_n,n} \to \mathcal{C}_{\delta_n,n}$ for each n; see [25, Lemma 4.6]. Now, to conclude, apply Lemma 3.3.2 on $\{H_n\}$. □

Remark 3.3.6 In Theorem 3.3.5, the number of components of $\mathscr{A}$ can be infinite; thus the number of trivial components of $f^{-1}(\mathscr{A})$ can be infinite. That's why we have removed all trivial components of $f^{-1}(\mathscr{A})$ by a single proper homotopy upon considering all outermost disks simultaneously. This process is in contrast to the finite-type surface theory, where the number of decomposition circles is finite, and therefore all trivial circles in the collection of transverse preimages of all decomposition circles can be removed one by one, considering the notion of an innermost disk.

3.4 Homotope a degree 1 map between circles to a homeomorphism

Previously, we have removed all trivial components keeping a neighborhood of each primitive component stationary from the transverse preimage $f^{-1}(\mathscr{A})$ of a preferred LFCS $\mathscr{A}$. In this section, we properly homotope our pseudoproper homotopy equivalence $f\colon \Sigma' \to \Sigma$ to send each component $\mathcal{C}'$ of $f^{-1}(\mathscr{A})$ homeomorphically onto a component $\mathcal{C}$ of $\mathscr{A}$ so that the restriction of f to a small one-sided tubular neighborhood $\mathcal{C}' \times [1,2]$ of $\mathcal{C}'$ (on either side of $\mathcal{C}'$) can be described by the following homeomorphism:

$$\mathcal{C}' \times [1,2] \ni (z,t) \mapsto (f(z),t) \in \mathcal{C} \times [1,2].$$

First, we fix some notation. Define $\partial_\varepsilon := \mathbb{S}^1 \times \varepsilon$ for $\varepsilon \in \mathbb{R}$ and $\boldsymbol{I} := [0,1]$. Let $p\colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ be the projection. The following lemma roughly says that a self-map of $\mathbb{S}^1 \times [0,2]$ can be homotoped rel $\mathbb{S}^1 \times 0$ to map $\mathbb{S}^1 \times [1,2]$ into itself by the product $\theta \times \mathrm{Id}_{[1,2]}$, where θ is a self-map of $\mathbb{S}^1$.

Lemma 3.4.1 *Let Φ be a self-map of $A := \mathbb{S}^1 \times [0,2]$ such that $\Phi^{-1}(\partial_b) = \partial_b$ for each $b \in \{0,2\}$. If we are given a map $\varphi_2\colon \partial_2 \to \partial_2$ and a homotopy $h_{(2)}\colon \partial_2 \times \boldsymbol{I} \to \partial_2$ from $\Phi|\partial_2 \to \partial_2$ to φ_2, then Φ can be homotoped relative to ∂_0 to map $\mathbb{S}^1 \times [0,1]$ into $\mathbb{S}^1 \times [0,1]$ and to satisfy $\Phi(-,r) = (p \circ \varphi_2(-,2), r)$ for each $r \in [1,2]$.*

Remark 3.4.2 In Lemma 3.4.1, up to homotopy, φ_2 is either a constant map or a covering map.

Proof Homotope Φ relative to $\partial_0 \cup \partial_2$ so that $\Phi(\mathbb{S}^1 \times [0,1]) \subseteq \mathbb{S}^1 \times [0,1]$ and $\Phi(z,r) = (p \circ \Phi(z,2), r)$ for all $(z,r) \in \mathbb{S}^1 \times [1,2]$. For each $r \in [1,2]$, $h_{(2)}$ provides a homotopy $h_{(r)}\colon \partial_r \times \boldsymbol{I} \to \partial_r$. Let $H\colon (\partial_0 \cup \partial_1) \times \boldsymbol{I} \to \partial_0 \cup \partial_1$ be the homotopy defined by $H|\partial_1 \times \boldsymbol{I} = h_{(1)}$ and $H(-,t)|\partial_0 = \Phi|\partial_0$ for any $t \in [0,1]$. The homotopy extension theorem gives a homotopy $\widetilde{H}\colon \mathbb{S}^1 \times [0,1] \times \boldsymbol{I} \to \mathbb{S}^1 \times [0,1]$ such that $\widetilde{H}|(\partial_0 \cup \partial_1) \times \boldsymbol{I} = H$. Finally, paste $\widetilde{H}$ with the collection $h_{(r)}, 1 \le r \le 2$. □

The following theorem is the simple modification (in the proper category) of the analogous theorem for closed surfaces:

Theorem 3.4.3 *Let $f\colon \Sigma' \to \Sigma$ be a smooth pseudoproper homotopy equivalence between two noncompact surfaces where $\Sigma' \not\approx \mathbb{R}^2 \not\approx \Sigma$, and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f \pitchfork \mathscr{A}$. Then f can be properly homotoped to remove all trivial components of $f^{-1}(\mathscr{A})$ as well as to map each primitive component of $f^{-1}(\mathscr{A})$ homeomorphically onto a component of $\mathscr{A}$. Moreover, after this proper homotopy, near each component of $f^{-1}(\mathscr{A})$, the map f can be described as follows:*

Let $\mathcal{C}'_p$ (resp. $\mathcal{C}$) be a component of $f^{-1}(\mathscr{A})$ (resp. $\mathscr{A}$) such that $f|\mathcal{C}'_p \to \mathcal{C}$ is a homeomorphism. Then $\mathcal{C}'_p$ (resp. $\mathcal{C}$) has two one-sided tubular neighborhoods $\mathcal{M}'$ and $\mathcal{N}'$ (resp. $\mathcal{M}$ and $\mathcal{N}$) with the specific identifications $(\mathcal{M}', \mathcal{C}'_p) \cong (\mathcal{C}'_p \times [1,2], \mathcal{C}'_p \times 2) \cong (\mathcal{N}', \mathcal{C}'_p)$ (resp. $(\mathcal{M}, \mathcal{C}) \cong (\mathcal{C} \times [1,2], \mathcal{C} \times 2) \cong (\mathcal{N}, \mathcal{C})$) such that the following hold:

- *$\mathcal{M}' \cup \mathcal{N}'$ is a (two-sided) tubular neighborhood of $\mathcal{C}'_p$,*
- *$f|\mathcal{M}' \to \mathcal{M}$ and $f|\mathcal{N}' \to \mathcal{N}$ are homeomorphisms given by $\mathcal{C}'_p \times [1,2] \ni (z,t) \mapsto (f(z), t) \in \mathcal{C} \times [1,2]$.*

Remark 3.4.4 In Theorem 3.4.3, though $\mathcal{M}' \cup \mathcal{N}'$ is a (two-sided) tubular neighborhood of $\mathcal{C}'_p$, both $\mathcal{M}$ and $\mathcal{N}$ may lie on the same side of $\mathcal{C}$, ie $\mathcal{M} \cup \mathcal{N}$ may not be a two-sided tubular neighborhood of $\mathcal{C}$.

Proof Let $\{\mathcal{C}'_{pn}\}$ be the collection of all primitive components of $f^{-1}(\mathscr{A})$. Assume $\mathcal{C}_n$ represents that component of $\mathscr{A}$ for which $f(\mathcal{C}'_{pn}) \subseteq \mathcal{C}_n$. Note $\mathcal{C}_n$ may equal to $\mathcal{C}_m$ even if $m \neq n$.

Claim 3.4.4.1 *There are one-sided compact tubular neighborhoods $\mathcal{U}'_n, \mathcal{V}'_n (\subseteq \Sigma')$ of $\mathcal{C}'_{pn}$, and there are one-sided compact tubular neighborhoods $\mathcal{U}_n, \mathcal{V}_n (\subseteq \Sigma)$ of $\mathcal{C}_n$ such that after defining $\mathcal{T}'_n := \mathcal{U}'_n \cup \mathcal{V}'_n$, the following hold:*

(1) *$\widetilde{\mathscr{A}} := \mathscr{A} \cup \{(\partial\mathcal{U}_n \cup \partial\mathcal{V}_n) \setminus \mathcal{C}_n\}_n$ is an LFCS and $f \pitchfork \widetilde{\mathscr{A}}$,*

(2) *$\partial\mathcal{U}'_n \setminus \mathcal{C}'_{pn}$ (resp. $\partial\mathcal{V}'_n \setminus \mathcal{C}'_{pn}$) is the only component of $f^{-1}(\partial\mathcal{U}_n \setminus \mathcal{C}_n) \cap \mathcal{U}'_n$ (resp. $f^{-1}(\partial\mathcal{V}_n \setminus \mathcal{C}_n) \cap \mathcal{V}'_n$) that cobounds an annulus with $\mathcal{C}'_{pn}$ (see Figure 5),*

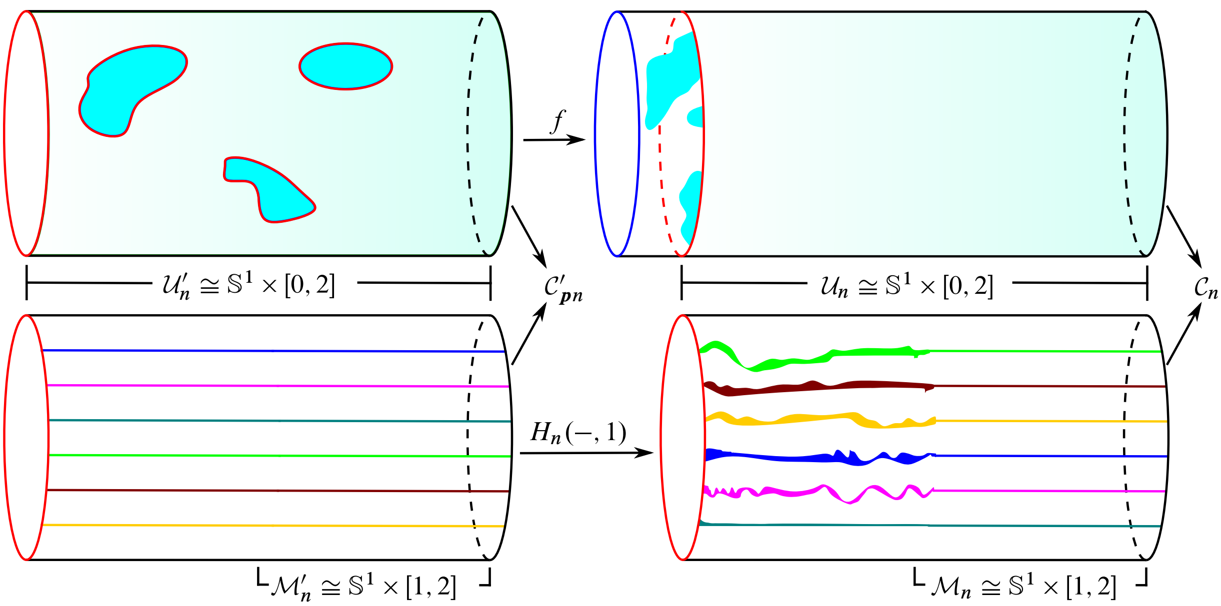

Figure 5: Top: description of $f|\mathcal{U}'_n \to \mathcal{U}_n$ using Theorem A.2.5. Bottom: after removing all trivial components of $f^{-1}(\partial\mathcal{U}_n \setminus \mathcal{C}_n)$ from $\mathcal{U}'_n$ and then applying Lemma 3.4.1 to $f|\mathcal{U}'_n \to \mathcal{U}_n$, we obtain $H_n(-,1)|\mathcal{U}'_n \to \mathcal{U}_n$.

(3) *each point of* $\mathrm{int}(\mathcal{U}'_n)$ *(resp.* $\mathrm{int}(\mathcal{V}'_n)$*) that is sufficiently near to* $\mathcal{C}'_{pn}$ *is mapped into* $\mathrm{int}(\mathcal{U}_n)$ *(resp.* $\mathrm{int}(\mathcal{V}_n)$*),*

(4) $\mathcal{T}'_n$ *is a two-sided tubular neighborhood of* $\mathcal{C}'_{pn}$ *with* $f^{-1}(\mathscr{A}) \cap \mathcal{T}'_n = \mathcal{C}'_{pn}$*, and*

(5) $\mathcal{T}'_n \cap \mathcal{T}'_m = \varnothing$ *if* $m \neq n$*, and* $(\mathcal{U}_n \cup \mathcal{V}_n) \to \infty$*.*

Proof For any positive integer n_0, Theorem 3.2.3 says that the set $\{m \in \mathbb{N} : \mathcal{C}_m = \mathcal{C}_{n_0}\}$ is finite. Also, $\mathscr{A}$ is locally finite. Thus $\{\mathcal{C}_n : n \in \mathbb{N}\}$ is locally finite. So, for each n, there exists a two-sided tubular neighborhood $\mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n]$ of $\mathcal{C}_n$ with $\mathcal{C}_n \times 0 \equiv \mathcal{C}_n$ such that $\{\mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n] : n \in \mathbb{N}\}$ is a locally finite collection. Further, for each $n \in \mathbb{N}$, we may assume that $f \,\overline{\pitchfork}\, (\mathcal{C}_n \times t_n)$ whenever $t_n \in [-\varepsilon_n, \varepsilon_n]$ by Theorem A.2.1.

Since $f^{-1}(\mathscr{A})$ is a locally finite collection, for each n there are one-sided compact tubular neighborhoods $\mathcal{U}'_n$ and $\mathcal{V}'_n$ of $\mathcal{C}'_{pn}$ in Σ' such that after defining $\mathcal{T}'_n := \mathcal{U}'_n \cup \mathcal{V}'_n$, the following hold: $\mathcal{T}'_n$ is a two-sided tubular neighborhood of $\mathcal{C}'_{pn}$, $f^{-1}(\mathscr{A}) \cap \mathcal{T}'_n = \mathcal{C}'_{pn}$, and $\mathcal{T}'_n \cap \mathcal{T}'_m = \varnothing$ if $m \neq n$. Moreover, using Theorem A.2.3, $f(\mathcal{T}'_n) \subseteq \mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n]$ can also be assumed for each n.

Next, by Theorem A.2.5, we may further assume $\partial\mathcal{U}'_n \setminus \mathcal{C}'_{pn}$ (resp. $\partial\mathcal{V}'_n \setminus \mathcal{C}'_{pn}$) is a component of $f^{-1}(\mathcal{C}_n \times x_n)$ (resp. $f^{-1}(\mathcal{C}_n \times y_n)$) for some $x_n, y_n \in (-\varepsilon_n, 0) \cup (0, \varepsilon_n)$ such that after defining $\mathcal{U}_n$ (resp. $\mathcal{V}_n$) as the annulus in $\mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n]$ cobounded by $\mathcal{C}_n \times 0$ and $\mathcal{C}_n \times x_n$ (resp. $\mathcal{C}_n \times y_n$), Claim 3.4.4.1(2)–(3) hold. Finally $\mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n] \to \infty$ implies $(\mathcal{U}_n \cup \mathcal{V}_n) \to \infty$. □

Using Theorem 3.3.5, keeping stationary a neighborhood of each primitive component of $f^{-1}(\widetilde{\mathscr{A}})$, we can properly homotope f to remove all trivial components from $f^{-1}(\widetilde{\mathscr{A}})$. So, after this proper homotopy, Claim 3.4.4.1(2)–(3) imply that $f(\mathcal{U}'_n) \subseteq \mathcal{U}_n$, $f^{-1}(\partial\mathcal{U}_n) \cap \mathcal{U}'_n = \partial\mathcal{U}'_n$, $f(\mathcal{V}'_n) \subseteq \mathcal{V}_n$, and $f^{-1}(\partial\mathcal{V}_n) \cap \mathcal{V}'_n = \partial\mathcal{V}'_n$. Notice the abuse of notation: the initial and final maps of this proper homotopy both are denoted by f.

Now, let $h_n : \mathcal{C}'_{pn} \times [0,1] \to \mathcal{C}_n$ be a homotopy from $f|\mathcal{C}'_{pn} \to \mathcal{C}_n$ such that $h_n(-,1)$ is either a constant map or a covering map between two circles. Applying Lemma 3.4.1 on $f|\mathcal{U}'_n \to \mathcal{U}_n$ and $f|\mathcal{V}'_n \to \mathcal{V}_n$ separately upon considering h_n, a homotopy $H_n : \mathcal{T}'_n \times [0,1] \to \mathcal{U}_n \cup \mathcal{V}_n$ relative to $\partial\mathcal{T}'_n$ exists such that $H_n(-,0) = f|\mathcal{T}'_n$, $(H_n(-,1))^{-1}(\mathcal{C}_n) = \mathcal{C}'_{pn}$, and $H_n(-,1)|\mathcal{C}'_{pn} \to \mathcal{C}_n$ is the same as $h_n(-,1)$; see Figure 5.

Next, Claim 3.4.4.1(5) tells us that we can apply Lemma 3.3.2 on $\{H_n\}$ to obtain a proper homotopy $\mathcal{H}\colon \Sigma' \times [0,1] \to \Sigma$ starting from f. Next, being an isomorphism, $\pi_1(f) = \pi_1(\mathcal{H}(-,1))$ preserves primitiveness, ie $h_n(-,1) = \mathcal{H}(-,1)|\mathcal{C}'_{pn} \to \mathcal{C}_n$ must be a homeomorphism. Thus $\mathcal{H}$ is our ultimate required homotopy.

Finally, we need to describe f near each component of $f^{-1}(\mathscr{A})$ after the proper homotopy $\mathcal{H}$. Abusing notation, the final map of $\mathcal{H}$ will be denoted by f. Since Lemma 3.4.1 is being used, we have $\mathcal{M}'_n \subseteq \mathcal{U}'_n$ and $\mathcal{M}_n \subseteq \mathcal{U}_n$ with the identifications $(\mathcal{M}'_n, \mathcal{C}'_{pn}) \cong (\mathcal{C}'_{pn} \times [1,2], \mathcal{C}'_{pn} \times 2)$ and $(\mathcal{M}_n, \mathcal{C}_n) \cong (\mathcal{C}_n \times [1,2], \mathcal{C}_n \times 2)$ such that after the proper homotopy $\mathcal{H}\colon \Sigma' \times [0,1] \to \Sigma$, the map f sends $\mathcal{C}'_{pn} \times r$ onto $\mathcal{C}_n \times r$ using the homeomorphism $f|\mathcal{C}'_{pn} \to \mathcal{C}_n$ for all $r \in [1,2]$; see Figure 5. The reasoning is similar for $f|\mathcal{V}'_n \to \mathcal{V}_n$. □

The following proposition, *which we don't need to use anywhere*, tells what happens if we drop the phrase "homotopy equivalence" in the statement of Theorem 3.4.3. Its proof is almost the same.

Proposition 3.4.5 *Let $f : \Sigma' \to \Sigma$ be a smooth proper map between two noncompact surfaces, where $\Sigma' \not\cong \mathbb{R}^2 \not\cong \Sigma$, and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f \pitchfork \mathscr{A}$. Then f can be properly homotoped to remove all trivial components of $f^{-1}(\mathscr{A})$ as well as to map each primitive component of $f^{-1}(\mathscr{A})$ into a component of $\mathscr{A}$ so that for any component $\mathcal{C}$ of $\mathscr{A}$ and any primitive component $\mathcal{C}'_p$ of $f^{-1}(\mathcal{C})$, after this proper homotopy, $f|\mathcal{C}'_p \to \mathcal{C}$ is either a constant map or a covering map.*

3.5 Annulus removal

In the previous two sections, after removing all trivial components from the transverse preimage of a decomposition circle, the remaining primitive circles have been mapped homeomorphically to that decomposition circle. This section aims to remove all these primitive circles except one from the inverse image of each decomposition circle using the following three steps: annulus bounding, then annulus compression, and finally annulus pushing.

We start with annulus bounding. Consider the collection of inverse images of all decomposition circles. The following lemma says that any two circles in this collection cobound an annulus in the domain if and only if their images are the same. In other words, in the domain, by pasting all small annuli, we get the outermost annulus corresponding to a decomposition circle.

Lemma 3.5.1 *Let $f : \Sigma' \to \Sigma$ be a homotopy equivalence between two noncompact surfaces, and let $\mathscr{A}'$ and $\mathscr{A}$ be two LFCS on Σ' and Σ, respectively, such that f maps each component of $\mathscr{A}'$ homeomorphically onto a component of $\mathscr{A}$. Suppose each component of $\mathscr{A}$ is primitive, and any two distinct components of $\mathscr{A}$ don't cobound an annulus in Σ. Let $\mathcal{C}'_0$ and $\mathcal{C}'_1$ be two distinct components of $\mathscr{A}'$. Then $\mathcal{C}'_0$ and $\mathcal{C}'_1$ cobound an annulus in Σ' if and only if $f(\mathcal{C}'_0) = f(\mathcal{C}'_1)$.*

Proof To prove the only if part, let $\Phi\colon \mathbb{S}^1 \times [0,1] \hookrightarrow \Sigma'$ be an embedding such that $\Phi(\mathbb{S}^1, k) = \mathcal{C}'_k$ for $k = 0, 1$. Note that f maps each component of $\mathscr{A}'$ homeomorphically onto a component of $\mathscr{A}$, and each component of $\mathscr{A}$ is a primitive circle on Σ. Thus the embeddings $f\Phi(-,0), f\Phi(-,1)\colon \mathbb{S}^1 \hookrightarrow \Sigma$ are freely homotopic, and hence $f\Phi(-,0), f\Phi(-,1)\colon \mathbb{S}^1 \hookrightarrow \Sigma$ represent the same nontrivial conjugacy class in $\pi_1(\Sigma, *)$. Since any two distinct components of $\mathscr{A}$ don't cobound an annulus in Σ, by Theorem 2.2.3, $f(\mathcal{C}'_0) = f(\mathcal{C}'_1)$.

To prove the if part, let $g\colon \Sigma \to \Sigma'$ be a homotopy inverse of f, and let $\mathcal{C}$ be the component of $\mathscr{A}$ defined by $\mathcal{C} := f(\mathcal{C}'_0) = f(\mathcal{C}'_1)$. Now, $f|\mathcal{C}'_k \to f(\mathcal{C}'_k)$ is a homeomorphism for $k = 0, 1$. Thus, for a homeomorphism $j : \mathbb{S}^1 \xrightarrow{\cong} \mathcal{C}$, there are homeomorphisms $\ell_0\colon \mathbb{S}^1 \xrightarrow{\cong} \mathcal{C}'_0$ and $\ell_1\colon \mathbb{S}^1 \xrightarrow{\cong} \mathcal{C}'_1$ such that $f\ell_0 = j = f\ell_1$. Since $\ell_0 \simeq gf\ell_0 = gj = gf\ell_1 \simeq \ell_1$, applying Theorem 2.2.3 to ℓ_0, ℓ_1, we are done. □

The following theorem, which will be used to compress each annulus bounded by two primitive circles of the domain, roughly says that most homotopies of a circle embedded in a surface are trivial:

Theorem 3.5.2 [35, Lemma 4.9.15] *Let $\boldsymbol{S}$ be a compact bordered surface other than the disk, and let Φ be a map from $\boldsymbol{A} := \mathbb{S}^1 \times [0,1]$ to $\boldsymbol{S}$ such that $\Phi(\operatorname{int}(\boldsymbol{A})) \subseteq \operatorname{int}(\boldsymbol{S})$ and there is a boundary component $\boldsymbol{C}$ of $\boldsymbol{S}$ for which $\Phi(-,0), \Phi(-,1)\colon \mathbb{S}^1 \xrightarrow{\cong} \boldsymbol{C}$ are the same homeomorphisms. Then Φ can be homotoped relative to $\partial \boldsymbol{A}$ to map $\boldsymbol{A}$ onto $\boldsymbol{C}$.*

The following theorem considers the last two steps — annulus compressing and annulus pushing. At first, by a proper homotopy, each outermost annulus will be mapped onto its decomposition circle; after that, by another proper homotopy, each outermost annulus will be pushed into a one-sided tubular neighborhood of one of its boundary components.

Theorem 3.5.3 *Let $f\colon \Sigma' \to \Sigma$ be a smooth pseudoproper homotopy equivalence between two noncompact surfaces, where $\Sigma' \ncong \mathbb{R}^2 \ncong \Sigma$, and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f \pitchfork \mathscr{A}$. Suppose any two distinct components of $\mathscr{A}$ don't cobound an annulus in Σ. In that case, f can be properly homotoped to a proper map g such that for each component $\mathcal{C}$ of $\mathscr{A}$, either $g^{-1}(\mathcal{C})$ is empty or $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{A})$ that is mapped homeomorphically onto $\mathcal{C}$ by g.*

Proof Using Theorem 3.4.3, we may assume each component of $f^{-1}(\mathscr{A})$ is primitive and also mapped homeomorphically onto a component of $\mathscr{A}$. So for simplicity, we may drop the subscript $\boldsymbol{p}$ to indicate a primitive component of $f^{-1}(\mathscr{A})$. Let $\{\mathcal{C}_n\}$ be the pairwise disjoint collection of all those components of $\mathscr{A}$ such that for each n, $f^{-1}(\mathcal{C}_n)$ has more than one component. By Lemma 3.5.1, for each n, an annulus $\mathcal{A}'_n$ (say the n^{th} *outermost annulus*) exists with the following properties: $\partial \mathcal{A}'_n \subseteq f^{-1}(\mathcal{C}_n)$, and $\mathcal{A}'_n$ is not contained in the interior of an annulus bounded by any two components of $f^{-1}(\mathscr{A})$. Thus $\mathcal{A}'_n \cap f^{-1}(\mathscr{A}) = f^{-1}(\mathcal{C}_n)$ and $\mathcal{A}'_n \cap \mathcal{A}'_m = \varnothing$ for $m \neq n$. Now, using Theorem 2.2.3, find a parametrization $\tau_n\colon \mathbb{S}^1 \times [0,k_n] \xrightarrow{\cong} \mathcal{A}'_n$ for some integer $k_n \geq 1$ such that $\tau_n(\mathbb{S}^1 \times \{0,\dots,k_n\}) = f^{-1}(\mathcal{C}_n)$ and $f\tau_n(-,l)\colon \mathbb{S}^1 \xrightarrow{\cong} \mathcal{C}_n$ represents the same homeomorphism of $\mathcal{C}_n$ for each $l = 0,\dots,k_n$.

Claim 3.5.3.1 *The proper map $f\colon \Sigma' \to \Sigma$ can be properly homotoped relative to $\Sigma' \setminus \bigcup_n \operatorname{int}(\mathcal{A}'_n)$ so that $f(\mathcal{A}'_n) = \mathcal{C}_n$ for each n.*

Proof For each integer n, we will construct a compact bordered subsurface Σ_n of Σ with $f(\mathcal{A}'_n) \subseteq \operatorname{int}(\Sigma_n)$ such that $\Sigma_n \to \infty$. Roughly, Σ_n will be obtained from taking the union of all those complementary components of Σ (if a punctured disk appears, truncate it) which are hit by $f(\mathcal{A}'_n)$.

Using continuity of $f|\Sigma' \setminus f^{-1}(\mathscr{A}) \to \Sigma \setminus \mathscr{A}$, we can say that $f(\mathcal{A}'_n) \subseteq \mathcal{X}_n \cup \mathcal{Y}_n$, where $\mathcal{X}_n$ and $\mathcal{Y}_n$ are complementary components of Σ decomposed by $\mathscr{A}$ such that $\mathcal{C}_n \subseteq \partial \mathcal{X}_n \cap \partial \mathcal{Y}_n$.

(1) We define Σ_n as $\Sigma_n := \mathcal{X}_n \cup \mathcal{Y}_n$ if $\mathcal{X}_n \cong S_{0,3} \cong \mathcal{Y}_n$, $\mathcal{X}_n \cong S_{1,1}$ and $\mathcal{Y}_n \cong S_{0,3}$, or $\mathcal{Y}_n \cong S_{1,1}$ and $\mathcal{X}_n \cong S_{0,3}$; see Figure 6.

(2) If $\mathcal{X}_n \cong S_{0,1,1} \cong \mathcal{Y}_n$ (in this case Σ is homeomorphic to the punctured plane), then using compactness of $f(\mathcal{A}'_n)$, let Σ_n be an annulus in $\mathcal{X}_n \cup \mathcal{Y}_n$ such that $f(\mathcal{A}'_n) \subseteq \operatorname{int}(\Sigma_n)$.

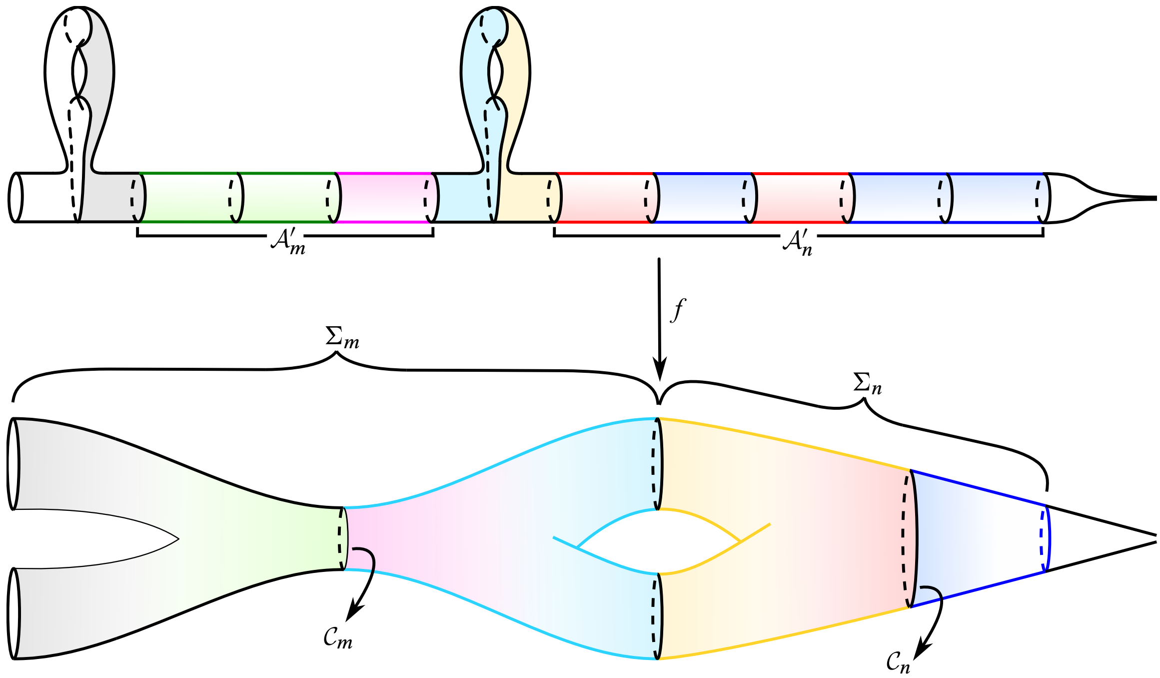

Figure 6: Illustration of parts (1) and (3) of the definition of Σ_n given in the proof of Claim 3.5.3.1. Only black circles denote a component of either $\mathscr{A}$ or a component of $f^{-1}(\mathscr{A})$.

(3) If $\mathcal{X}_n \cong S_{0,1,1}$ and $\mathcal{Y}_n$ is homeomorphic to either $S_{0,3}$ or $S_{1,1}$, then using compactness of $f(\mathcal{A}'_n)$, find an annulus $\mathcal{A}_n$ in $\mathcal{X}_n$ such that $f(\mathcal{A}'_n) \subseteq \operatorname{int}(\mathcal{A}_n \cup \mathcal{Y}_n)$. Define $\Sigma_n := \mathcal{A}_n \cup \mathcal{Y}_n$; see Figure 6.

(4) If $\mathcal{Y}_n \cong S_{0,1,1}$ and $\mathcal{X}_n$ is homeomorphic to either $S_{0,3}$ or $S_{1,1}$, define Σ_n similarly to (3).

Thus $f(\mathcal{A}'_n) \subseteq \operatorname{int}(\Sigma_n)$ for each n. Now we show $\Sigma_n \to \infty$, so consider a compact subset $\mathcal{K}$ of Σ. Let $\boldsymbol{S}_1, \dots, \boldsymbol{S}_m$ be a collection of complementary components of Σ decomposed by $\mathscr{A}$ such that $\mathcal{K} \subseteq \operatorname{int}\bigl(\bigcup_{l=1}^m \boldsymbol{S}_l\bigr)$. Define $\boldsymbol{S} := \bigcup_{l=1}^m \boldsymbol{S}_l$. Notice that for an integer n, $f(\mathcal{A}'_n) \cap \boldsymbol{S} \neq \varnothing$ if and only if $\mathcal{C}_n$ is a component of $\bigcup_{l=1}^m \partial \boldsymbol{S}_l$. This is due to the construction of each Σ_n; see Figure 6. Since $\mathcal{C}_n \to \infty$ and $\bigcup_{l=1}^m \partial \boldsymbol{S}_l$ is compact, $f(\mathcal{A}'_n) \cap \boldsymbol{S} = \varnothing$ for all sufficiently large n. Now, $\mathcal{K} \subseteq \operatorname{int}(\boldsymbol{S})$ and each Σ_n is obtained from taking the union of all those complementary components of Σ (if a punctured disk appears, truncate it) which are hit by $f(\mathcal{A}'_n)$. Thus $\Sigma_n \cap \mathcal{K} = \varnothing$ for all sufficiently large n. Therefore, $\Sigma_n \to \infty$, as $\mathcal{K}$ is an arbitrary compact subset of Σ.

Next, for each $l \in \{1, \dots, k_n\}$, applying Theorem 3.5.2 to each $f\tau_n|\mathbb{S}^1 \times [l-1, l] \to \mathcal{Z}_n$ where $\mathcal{Z}_n$ can be either $\Sigma_n \cap \mathcal{X}_n$ or $\Sigma_n \cap \mathcal{Y}_n$, we have a homotopy $H_n\colon \mathcal{A}'_n \times [0,1] \to \Sigma_n$ relative to $\partial \mathcal{A}'_n$ such that $H_n(-,0) = f|\mathcal{A}'_n$ and $H_n(\mathcal{A}'_n, 1) = \mathcal{C}_n$. Finally, apply Lemma 3.3.2 on $\{H_n\}$. □

Consider Figure 7, where $\mathcal{M}'_{n\alpha}$ and $\mathcal{M}_n$ are provided by Theorem 3.4.3 so that after defining $\mathcal{A}'_{\varepsilon n}$ as $\mathcal{A}'_n \cup \mathcal{M}'_{n\alpha}$,

$$(\mathcal{A}'_{\varepsilon n}, \mathcal{M}'_{n\alpha}, \mathcal{A}'_n) \cong (\mathbb{S}^1 \times [1,3], \mathbb{S}^1 \times [1,2], \mathbb{S}^1 \times [2,3]) \quad \text{and} \quad (\mathcal{M}_n, \mathcal{C}_n) \cong (\mathbb{S}^1 \times [1,2], \mathbb{S}^1 \times 2),$$

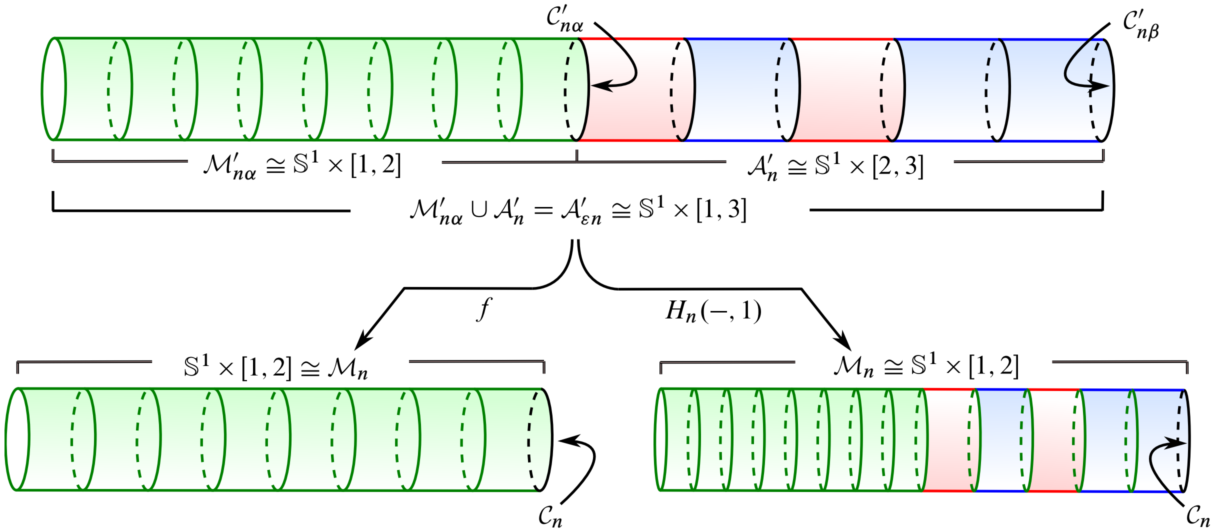

Figure 7: The description of $f|\mathcal{A}'_{\varepsilon n} \to \mathcal{M}_n$ (resp. $H_n(-,1)\colon \mathcal{A}'_{\varepsilon n} \to \mathcal{M}_n$) using Theorem 3.4.3 and Claim 3.5.3.1 (resp. Lemma 3.5.4). Only black circles denote a component of either $\mathscr{A}$ or a component of $f^{-1}(\mathscr{A})$.

resulting in the following description of f: if $\theta\colon \mathbb{S}^1 \to \mathbb{S}^1$ describes the homeomorphism $f|\mathcal{C}'_{n\alpha} \to \mathcal{C}_n$ under the above identification, then $f(z,t) = (\theta(z),t)$ for $z \in \mathbb{S}^1 \times [1,2]$ and $f(z,t) \in \mathbb{S}^1 \times 2$ for $(z,t) \in \mathbb{S}^1 \times [2,3]$. Consider Claim 3.5.3.1 to see why $f(\mathbb{S}^1 \times [2,3]) = \mathbb{S}^1 \times 2$.

Now, use Lemma 3.5.4 to construct a homotopy $H_n\colon \mathcal{A}'_{\varepsilon n} \times [0,1] \to \mathcal{M}_n$ relative to $\partial\mathcal{A}'_{\varepsilon n}$ from $f|\mathcal{A}'_{\varepsilon n} \to \mathcal{M}_n$ to the map $H_n(-,1)$ so that $(H_n(-,1))^{-1}(\mathcal{C}_n) = \mathcal{C}'_{n\beta}$ and $H_n(-,1)|\mathcal{C}'_{n\beta} \to \mathcal{C}_n$ is a homeomorphism.

Notice that we are using the setup of the proof of Theorem 3.4.3. By Claim 3.4.4.1(4)–(5), $\mathcal{A}'_{\varepsilon n} \cap f^{-1}(\mathscr{A}) = f^{-1}(\mathcal{C}_n)$, $\mathcal{A}'_{\varepsilon n} \cap \mathcal{A}'_{\varepsilon m} = \varnothing$ if $m \neq n$, and $\mathcal{M}_n \to \infty$. Now consider Lemma 3.3.2 with $\{H_n\}$ to obtain the desired homotopy. □

We prove the annulus-pushing lemma used in the proof of the previous theorem:

Lemma 3.5.4 *Any map $\varphi\colon \mathbb{S}^1 \times [1,3] \to \mathbb{S}^1 \times [1,2]$ which sends $\mathbb{S}^1 \times r$ into $\mathbb{S}^1 \times r$ for $1 \le r \le 2$ and sends $\mathbb{S}^1 \times r$ into $\mathbb{S}^1 \times 2$ for $2 \le r \le 3$ can be homotoped relative to $\mathbb{S}^1 \times \{1,3\}$ to satisfy $\varphi^{-1}(\mathbb{S}^1 \times 2) = \mathbb{S}^1 \times 3$.*

Proof Let $\varphi_1\colon \mathbb{S}^1 \times [1,3] \to \mathbb{S}^1$ and $\varphi_2\colon \mathbb{S}^1 \times [1,3] \to [1,2]$ be the components of φ. Consider a homeomorphism $\ell\colon [1,3] \to [1,2]$ with $\ell(1) = 1$ and $\ell(3) = 2$. Now $H\colon \mathbb{S}^1 \times [1,3] \times [0,1] \to \mathbb{S}^1 \times [1,2]$ defined by

$$H((z,s),t) := (\varphi_1(z,s), (1-t)\varphi_2(z,s) + t\ell(s)) \quad \text{for } (z,s) \in \mathbb{S}^1 \times [1,3] \text{ and } t \in [0,1]$$

is our required homotopy. □

Remark 3.5.5 In Theorem 3.5.3, the number of components of $\mathscr{A}$ can be infinite; thus, the number of outermost annuli (one outermost annulus for each component of $\mathscr{A}$, if any) can be infinite. That's why we have removed all outermost annuli simultaneously by a single proper homotopy, not one by one. Also, to prove the topological rigidity of closed surfaces, one may ignore the annulus removal process considering induction on the genus; see [13, Theorem 3.1] or [35, Theorems 4.6.2 and 4.6.3]. But, since the genus of a noncompact surface can be infinite, we can't ignore the annulus removal process here.

3.6 Is pseudoproper homotopy equivalence a map of degree ±1?

Let $f: \Sigma' \to \Sigma$ be a pseudoproper homotopy equivalence between two noncompact oriented surfaces, where surfaces are homeomorphic to neither the plane nor the punctured plane. Our aim in this section is to properly homotope f to obtain a closed disk $\mathcal{D} \subseteq \Sigma$ so that $f | f^{-1}(\mathcal{D}) \to \mathcal{D}$ becomes a homeomorphism, and thus show $\deg(f) = \pm 1$; see Theorem 2.6.1. Having got this and then using Lemma 3.6.4.1, it can be said that f is surjective, which further implies that after a proper homotopy for removing unnecessary components from the transverse preimage of a decomposition circle $\mathcal{C}$, a single circle will still be left that can be mapped onto $\mathcal{C}$ homeomorphically; see Theorem 3.6.4.4.

The argument for finding such a disk $\mathcal{D}$ is based on finding a finite-type bordered surface $\boldsymbol{S}$ in Σ such that for each component $\mathcal{C}$ of $\partial \boldsymbol{S}$, we have $f^{-1}(\mathcal{C}) \neq \varnothing$, even after any proper homotopy of f. Once we get $\boldsymbol{S}$, after a proper homotopy, we may assume that $f | f^{-1}(\partial \boldsymbol{S}) \to \partial \boldsymbol{S}$ is a homeomorphism; see Theorem 3.5.3. Now, since f is π_1–injective, by the topological rigidity of the pair of pants together with the proper rigidity of the punctured disk, after a proper homotopy, one can show that $f | f^{-1}(\boldsymbol{S}) \to \boldsymbol{S}$ is a homeomorphism. Therefore the required $\mathcal{D}$ can be any disk in $\operatorname{int}(\boldsymbol{S})$.

Now, to find such an $\boldsymbol{S}$, we consider two cases: If Σ is either an infinite-type surface or any $S_{g,0,p}$ with high complexity (to us, high complexity always means $g + p \geq 4$ or $p \geq 6$), then using π_1–surjectivity of f, we can choose $\boldsymbol{S}$ as a pair of pants in Σ so that $\Sigma \setminus \boldsymbol{S}$ has at least two components and every component of $\Sigma \setminus \boldsymbol{S}$ has a nonabelian fundamental group. On the other hand, if Σ is a finite-type surface, then we choose a punctured disk in Σ as $\boldsymbol{S}$ so that the puncture of $\boldsymbol{S}$ is an end $e \in \operatorname{im}(\operatorname{Ends}(f)) \subseteq \operatorname{Ends}(\Sigma)$.

We can recall our earlier two examples to show that the plane and the punctured plane are the only surfaces for which our theory fails: consider the pseudoproper homotopy equivalences $\varphi: \mathbb{C} \ni z \mapsto z^2 \in \mathbb{C}$ and $\psi: \mathbb{S}^1 \times \mathbb{R} \ni (z, x) \mapsto (z, |x|) \in \mathbb{S}^1 \times \mathbb{R}$. The local homeomorphism φ is a map of $\deg = \pm 2$ by [16, Lemma 2.1b] (note that for any local homeomorphism $p: X \to Y$ between two manifolds, where Y is orientable, an orientation of Y can be pulled back to induce an orientation on X so that p becomes an orientation-preserving map). On the other hand, $\deg(\psi) = 0$ as ψ is not surjective; see Lemma 3.6.4.1.

3.6.1 Essential pairs of pants and the degree of a pseudoproper homotopy equivalence

Definition 3.6.1.1 A smoothly embedded pair of pants $\boldsymbol{P}$ in a surface Σ is said to be an *essential pair of pants* of Σ if $\Sigma \setminus \boldsymbol{P}$ has at least two components and every component of $\Sigma \setminus \boldsymbol{P}$ has a nonabelian fundamental group.

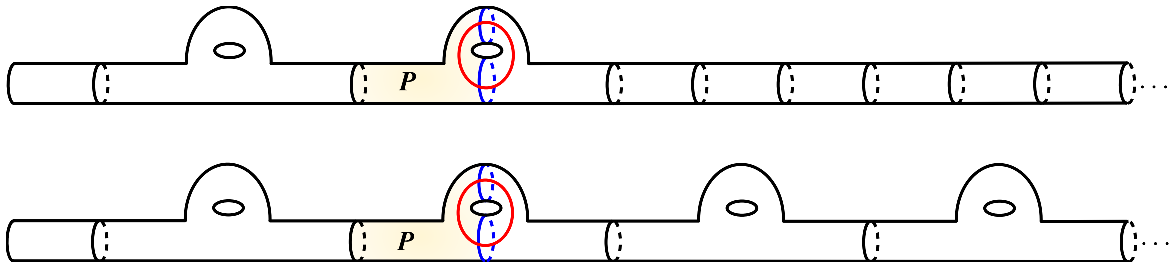

Figure 8: Finding an essential pair of pants $\boldsymbol{P}$ in each of $S_{2,0,1}$ and the Loch Ness monster by decomposing a two-holed torus into two copies of the pair of pants.

Finding an essential pair of pants in a noncompact surface will be divided into two cases: when the genus is at least two and when the space of ends has at least six elements.

Definition 3.6.1 Let $\boldsymbol{P}$ be a smoothly embedded copy of the pair of pants in a two-holed torus $\mathcal{S}$ ($\mathcal{S}$ is a copy of $S_{1,2}$). We say $\boldsymbol{P}$ *is obtained from decomposing* $\mathcal{S}$ *into two copies of the pair of pants* if there exists another smoothly embedded copy $\widetilde{\boldsymbol{P}}$ of the pair of pants in $\mathcal{S}$ such that $\boldsymbol{P} \cup \widetilde{\boldsymbol{P}} = \mathcal{S}$ and $\boldsymbol{P} \cap \widetilde{\boldsymbol{P}} = \partial \boldsymbol{P} \cap \partial \widetilde{\boldsymbol{P}}$ is the union of two smoothly embedded disjoint circles in the interior of $\mathcal{S}$ ($\partial \boldsymbol{P}$ shares exactly two of its components with $\partial \widetilde{\boldsymbol{P}}$).

The following lemma says that every noncompact surface with a genus of at least two has an essential pair of pants with some additional properties:

Lemma 3.6.1.2 *Let* Σ *be a noncompact surface of genus at least two. Then* Σ *has an essential pair of pants* $\boldsymbol{P}$ *with the following additional properties*: Σ *contains a smoothly embedded copy* $\mathcal{S}$ *of* $S_{1,2}$ *such that* $\Sigma \setminus \mathcal{S}$ *has precisely two components and each component of* $\Sigma \setminus \mathcal{S}$ *has a nonabelian fundamental group*, *and* $\boldsymbol{P}$ *is a smoothly embedded copy of the pair of pants in* $\mathcal{S}$ *obtained by decomposing* $\mathcal{S}$ *into two copies of the pair of pants.*

Proof Consider an inductive construction of Σ; see Theorem 2.5.1. Since $g(\Sigma) \geq 2$, at least two smoothly embedded copies of $S_{1,2}$ are used in this inductive construction. By Lemma 3.1.8, without loss of generality, we may assume that two smoothly embedded copies of $S_{1,2}$ are used successively just after the initial disk; see Figure 8. Among these two copies of $S_{1,2}$, breaking the last one (that copy of $S_{1,2}$ which we just used to construct K_3 from K_2) into two copies of the pair of pants, as illustrated in Figure 8, we get the required essential pair of pants. □

Lemma 3.6.1.3 *Let* $f : \Sigma' \to \Sigma$ *be a* π_1*–surjective map between two noncompact surfaces*, *where* Σ *has genus at least two. Consider an essential pair of pants* $\boldsymbol{P}$ *in* Σ *given by Lemma 3.6.1.2. Then* $f^{-1}(\operatorname{int} \boldsymbol{P}) \neq \varnothing$ *and* $f^{-1}(c) \neq \varnothing$ *for each component* c *of* $\partial \boldsymbol{P}$.

Proof Let $\mathcal{S}$ be a smoothly embedded copy of $S_{1,2}$ in Σ such that $\boldsymbol{P}$ is obtained from decomposing $\mathcal{S}$ into two copies of the pair of pants. If possible, let $f^{-1}(\operatorname{int} \boldsymbol{P}) \neq \varnothing$. By continuity of f, the image of f

is contained in precisely one of the two components of $\Sigma \setminus \mathrm{int}(\boldsymbol{P})$. But each component of $\Sigma \setminus \mathrm{int}(\boldsymbol{P})$ has a nonabelian fundamental group, ie $\pi_1(f)\colon \pi_1(\Sigma') \to \pi_1(\Sigma)$ is not surjective, a contradiction. Therefore $f^{-1}(\mathrm{int}\,\boldsymbol{P})$ must be nonempty.

To prove the second part, let c_1, c_2, and c_3 denote all three components of $\boldsymbol{P}$ such that both $\Sigma \setminus c_1$ and $\Sigma \setminus (c_2 \cup c_2)$ are disconnected, but neither $\Sigma \setminus c_2$ nor $\Sigma \setminus c_3$ is disconnected. In Figure 8, c_2 and c_3 are blue circles, and c_1 is black. Notice that we have a smoothly embedded primitive circle $\mathcal{C} \subseteq \mathrm{int}(\mathcal{S})$ (in Figure 8, each red circle denotes $\mathcal{C}$) so that for each $k = 2, 3$, $c_k \cap \mathcal{C}$ is a single point, where c_k intersects $\mathcal{C}$ transversally. Therefore, for each $k = 2, 3$, using the bigon criterion [17, Proposition 1.7], any loop belonging to class $[\mathcal{C}] \in \pi_1(\Sigma)$ must intersect c_k. That is, if $f^{-1}(c_2)$ or $f^{-1}(c_3)$ were empty, then $[\mathcal{C}]$ would not belong to the image of $\pi_1(f)\colon \pi_1(\Sigma') \to \pi_1(\Sigma)$. But f is π_1–surjective. Thus $f^{-1}(c_2) \neq \varnothing \neq f^{-1}(c_3)$. On the other hand, $\Sigma \setminus c_1$ has precisely two components, and each component of $\Sigma \setminus c_1$ has a nonabelian fundamental group, ie by continuity and π_1–surjectivity of f, we can say that $f^{-1}(c_1) \neq \varnothing$. □

Now we consider the second case of finding an essential pair of pants in a noncompact surface, namely when the space of ends has at least six elements:

Lemma 3.6.1.4 *Let Σ be a noncompact surface with at least six ends. Then Σ has an essential pair of pants $\boldsymbol{P}$ such that $\Sigma \setminus \boldsymbol{P}$ has precisely three components and each component of $\Sigma \setminus \boldsymbol{P}$ has a nonabelian fundamental group.*

Proof Consider an inductive construction of Σ; see Theorem 2.5.1. Since $|\mathrm{Ends}(\Sigma)| \geq 6$, at least five smoothly embedded copies of $S_{0,3}$ are used in this inductive construction. By Lemma 3.1.8, without loss of generality, we may assume that five smoothly embedded copies of $S_{0,3}$ are used successively just after the initial disk. Let $\boldsymbol{P}$ be the copy that shares all three boundary components with three other copies of this sequence of five copies of $S_{0,3}$; see Figure 9. Thus $\Sigma \setminus \boldsymbol{P}$ has precisely three components, and each component of $\Sigma \setminus \boldsymbol{P}$ has a nonabelian fundamental group.

In Figure 9, inductive constructions (up to a sufficient number of steps) of two surfaces have been given: the surface at the top contains a copy of $\{z \in \mathbb{C} : |z| \geq 1, z \notin \mathbb{N} \times 0\}$, and the bottom is the Cantor tree surface (the planar surface whose space of ends is homeomorphic to the Cantor set). In each surface, an essential pair of pants $\boldsymbol{P}$ is contained in the shaded compact bordered subsurface. □

We can prove the following lemma by a similar argument as in the proof of Lemma 3.6.1.3:

Lemma 3.6.1.5 *Let $f\colon \Sigma' \to \Sigma$ be a π_1–surjective proper map between two noncompact surfaces, where Σ has at least six ends. Consider an essential pair of pants $\boldsymbol{P}$ in Σ given by Lemma 3.6.1.4. Then $f^{-1}(\mathrm{int}\,\boldsymbol{P}) \neq \varnothing$ and $f^{-1}(c) \neq \varnothing$ for each component c of $\partial \boldsymbol{P}$.*

The following theorem completes the whole process of finding an essential pair of pants, which will be used to find the degree of a pseudoproper homotopy equivalence:

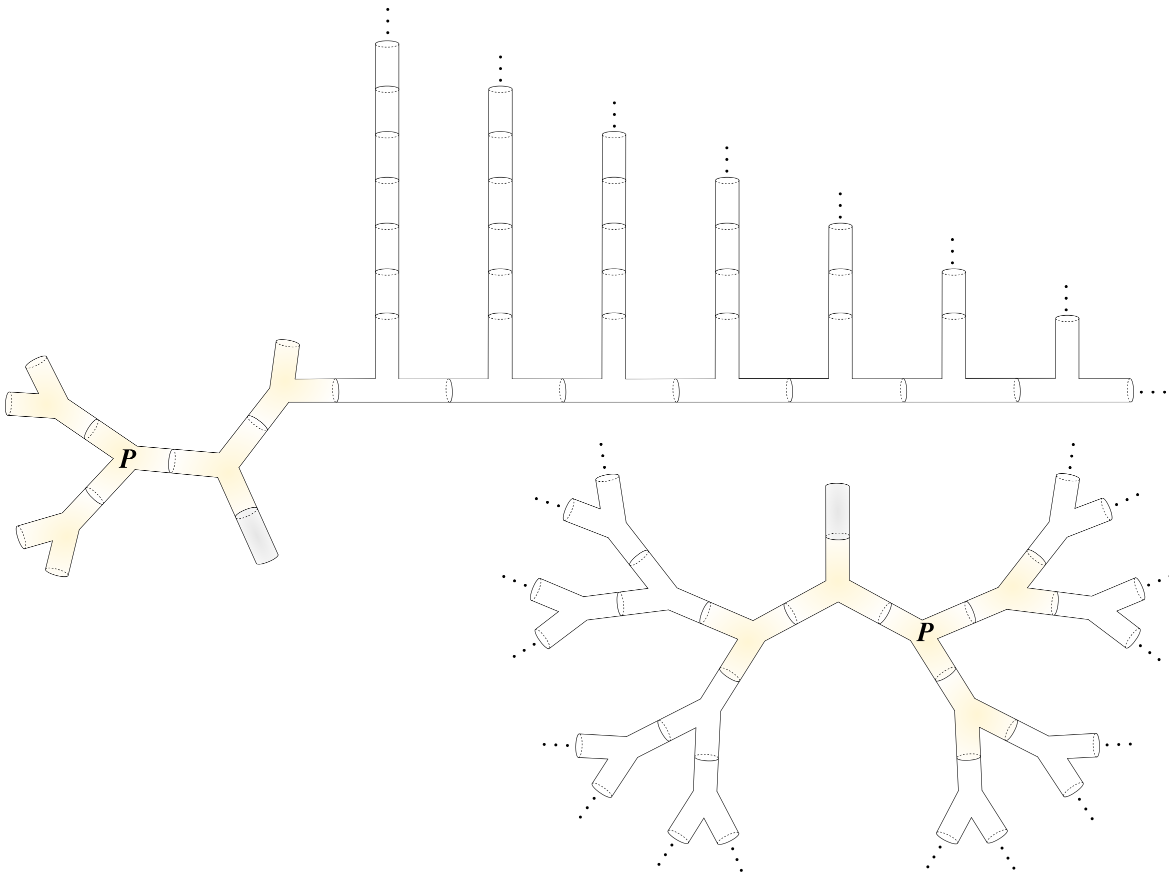

Figure 9: Finding an essential pair of pants $\boldsymbol{P}$ in a noncompact surface with at least six ends.

Theorem 3.6.1.6 *Let $f : \Sigma' \to \Sigma$ be a π_1–surjective proper map between two noncompact surfaces. Suppose Σ is either an infinite-type surface or a finite-type surface $S_{g,0,p}$ with high complexity (ie $g + p \geq 4$ or $p \geq 6$). Then Σ contains an essential pair of pants $\boldsymbol{P}$ such that $f^{-1}(\operatorname{int} \boldsymbol{P}) \neq \varnothing$ and $f^{-1}(c) \neq \varnothing$ for each component c of $\partial \boldsymbol{P}$.*

Proof If an infinite-type surface has a finite genus, then it must have infinitely many ends; see Proposition 3.1.10. Thus using Lemmas 3.6.1.2, 3.6.1.3, 3.6.1.4, and 3.6.1.5, the proof is complete in all cases, except when Σ is homeomorphic to $S_{1,0,3}$, $S_{1,0,4}$, or $S_{1,0,5}$. We consider the case when $\Sigma \cong S_{1,0,3}$; the other cases are similar.

Define an inductive construction of $S_{1,0,3}$ by starting with a copy of $S_{0,1}$, consecutively adding two copies of $S_{0,3}$, then a copy of $S_{1,2}$, and finally three sequences of annuli to obtain three planar ends; see Figure 1. Therefore, in this inductive construction, K_4 is obtained from K_3, adding a copy $\mathcal{S}$ of $S_{1,2}$. Let $\boldsymbol{P}$ be a smoothly embedded copy of the pair of pants in $\mathcal{S}$ such that $\boldsymbol{P}$ is obtained from decomposing $\mathcal{S}$ into two copies of the pair of pants and $\boldsymbol{P} \cap K_3 \neq \varnothing$. Now, an argument similar to that given in Lemma 3.6.1.3 completes the proof. □

At this stage, we need a couple of lemmas. The first one, Lemma 3.6.1.7, is well known; its proof has been given for reader convenience.

Lemma 3.6.1.7 *Let Σ be a surface, and let S be a smoothly embedded bordered subsurface of Σ. Then the inclusion-induced map $\pi_1(S) \to \pi_1(\Sigma)$ is injective if either of the following is satisfied*:

(1) *∂S is a separating primitive circle on Σ and S is one of the two sides of ∂S in Σ.*

(2) *S is compact and each component of ∂S is a primitive circle on Σ.*

Proof (1) Since $\pi_1(\Sigma) \cong \pi_1(S) *_{\pi_1(\partial S)} \pi_1(\Sigma \setminus \operatorname{int} S)$ by the Seifert–Van Kampen theorem and the inclusions $\partial S \hookrightarrow S, \Sigma \setminus \operatorname{int}(S)$ are π_1–injective, we are done.

(2) It is enough to construct a sequence $\Sigma = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n = S$ of subsurfaces of Σ, where n is the number of components of ∂S, such that for each $k = 1, \dots, n$, the following hold:

(i) S_k is a bordered subsurface of S_{k-1} and the inclusion map $S_k \hookrightarrow S_{k-1}$ is π_1–injective.

(ii) $\partial S_k \setminus \partial S_{k-1}$ is either a component of ∂S_k or union of two components of ∂S_k. In either case, $\partial S_k \setminus \partial S_{k-1}$ shares only one component with ∂S.

We construct this sequence inductively as follows: To construct S_k from S_{k-1}, pick a component c of $\partial S \setminus \partial S_{k-1}$. If c separates S_{k-1}, define S_k as that side of c in S_{k-1} which contains S; then consider an argument similar to the proof of Lemma 3.6.1.7(1). If c doesn't separate S_{k-1}, pick a smoothly embedded annulus $A \subset \operatorname{int}(S_{k-1})$ such that $A \cap S = c$. Define $S_k := S_{k-1} \setminus \operatorname{int}(A)$. Now, S_{k-1} is obtained from S_k identifying c with $\partial A \setminus c$ by an orientation-reversing diffeomorphism $\varphi\colon c \to \partial A \setminus c$. By the HNN–Seifert–Van Kampen theorem, $\pi_1(S_{k-1}) \cong \pi_1(S_k) *_{\pi_1(\varphi)}$, where the map $\pi_1(S_k) \to \pi_1(S_{k-1})$ (which is inclusion induced) is injective due to Britton's lemma. □

The following lemma roughly says that the degree of a map between two compact bordered surfaces can be determined from the degree of its restriction on the boundaries:

Lemma 3.6.1.8 *Let $\varphi\colon S_{g_1,b_1} \to S_{g_2,b_2}$ be a map between two compact bordered surfaces such that $\varphi|\partial S_{g_1,b_1} \hookrightarrow \partial S_{g_2,b_2}$ is an embedding. Then $\varphi(\partial S_{g_1,b_1}) = \partial S_{g_2,b_2}$ and $\deg(\varphi) = \pm 1$.*

Proof Notice that φ maps each component of $\partial S_{g_1,b_1}$ homeomorphically onto a component of $\partial S_{g_2,b_2}$, and any two distinct components of $\partial S_{g_1,b_1}$ have distinct φ–images. Now, naturality of homology long exact sequences of $(S_{g_1,b_1}, \partial S_{g_1,b_1})$ and $(S_{g_2,b_2}, \partial S_{g_2,b_2})$ give the following commutative diagram:

$$\begin{array}{ccc} H_2(S_{g_1,b_1}, \partial S_{g_1,b_1}) \cong \mathbb{Z} & \xrightarrow{1 \mapsto \bigoplus^{b_1} 1} & \bigoplus^{b_1} \mathbb{Z} \cong H_1(\partial S_{g_1,b_1}) \\ \times\deg(\varphi) \big\downarrow & & \big\downarrow \bigoplus^{b_1} 1 \mapsto \bigoplus^{b_1}(\pm 1) \oplus \bigoplus^{b_2 - b_1} 0 \\ H_2(S_{g_2,b_2}, \partial S_{g_2,b_2}) \cong \mathbb{Z} & \xrightarrow{1 \mapsto \bigoplus^{b_2} 1} & \bigoplus^{b_2} \mathbb{Z} \cong H_1(\partial S_{g_2,b_2}) \end{array}$$

The horizontal maps are the connecting homomorphisms for homology long exact sequences; for their description see [25, Exercise 31 of Section 3.3]. Commutativity of this diagram gives $b_2 = b_1$ (the integer $\deg(\varphi)$ can't be simultaneously 0 and ± 1), and thus $\deg(\varphi) = \pm 1$. □

The proof of Theorem 3.6.1.9 can be found in [35, Theorem 4.6.2]. It also follows from the much more general result [13, Theorem 3.1]. Since compact bordered surfaces are aspherical, an application of the Whitehead theorem says that the assumption "$\varphi\colon S' \to S$ is a homotopy equivalence" in Theorem 3.6.1.9 is equivalent to the assumption "$\pi_1(\varphi)$ is an isomorphism".

Theorem 3.6.1.9 (rigidity of compact bordered surfaces) *Let $\varphi\colon S' \to S$ be a homotopy equivalence between two compact bordered surfaces such that $\varphi^{-1}(\partial S) = \partial S'$. If $\varphi|\partial S' \to \partial S$ is a homeomorphism, then φ is homotopic to a homeomorphism relative to $\partial S'$.*

The following lemma gives some sufficient conditions that ensure the preimage of a compact bordered subsurface under a proper map becomes a compact bordered subsurface of the same homeomorphism type. Its usage is twofold: firstly, in Theorem 3.6.1.11, to find the degree of a pseudoproper homotopy equivalence, and secondly in the proof of Theorem 1.

Lemma 3.6.1.10 *Let $f\colon \Sigma' \to \Sigma$ be a π_1–injective proper map between two noncompact oriented surfaces, and let S be a smoothly embedded compact bordered subsurface of Σ with $f^{-1}(\operatorname{int} S) \neq \varnothing$. Suppose $f^{-1}(\partial S)$ is a pairwise-disjoint collection of smoothly embedded primitive circles on Σ' such that f sends $f^{-1}(\partial S)$ homeomorphically onto ∂S. Then $f^{-1}(S)$ is a copy of S in Σ' with $\partial f^{-1}(S) = f^{-1}(\partial S)$, and $\deg(f) = \pm 1$.*

Proof Since $f^{-1}(\operatorname{int} S) \neq \varnothing$ and f is proper, the continuity of $f|\Sigma' \setminus f^{-1}(\partial S) \to \Sigma \setminus \partial S$ tells us that $\Sigma' \setminus f^{-1}(\partial S)$ is disconnected. Let $S' \subset \Sigma'$ be a bordered subsurface obtained as a complementary component of the decomposition of Σ' by $f^{-1}(\partial S)$ such that $f(S') \subseteq S$. That is, S' is the closure of one of the components of $\Sigma' \setminus f^{-1}(\partial S)$ and S' is contained in the compact set $f^{-1}(S)$. So S' is a compact bordered subsurface of Σ', and each component of $\partial S'$ is a component of $f^{-1}(\partial S)$. In the following few lines, we will show that each component of $f^{-1}(\partial S)$ is also a component of $\partial S'$. Since $f|f^{-1}(\partial S) \to \partial S$ is a homeomorphism, we can say that $f|\partial S' \hookrightarrow \partial S$ is an embedding. Now, by Lemma 3.6.1.8, $\partial S' = f^{-1}(\partial S)$ and $\deg(f|S' \to S) = \pm 1$. Next, by Theorem 2.6.3, $f|S' \to S$ is π_1–surjective. Since the inclusion $S' \hookrightarrow \Sigma'$ and f are π_1–injective, $f|S' \to S$ is also so; see Lemma 3.6.1.7(2). Thus $f|S' \to S$ is π_1–bijective, and so Theorem 3.6.1.9 tells that $S' \cong S$. Finally, if S'' is another bordered subsurface obtained as a complementary component of decomposition of Σ' by $f^{-1}(\partial S)$ with $f(S'') \subseteq S$, then similarly, $S'' \cong S$. Since $f|f^{-1}(\partial S) \to \partial S$ is a homeomorphism and Σ' is connected, $S'' = S'$ (otherwise Σ' would be the compact surface $S' \cup_{\partial S' = \partial S''} S''$). Therefore $f^{-1}(S) = S' \cong S$, and thus the proof of the first part is completed.

Now we will prove that $\deg(f) = \pm 1$. Since $\deg(f)$ remains invariant after any proper homotopy of f, we can properly homotope f as we want. So apply Theorem 3.6.1.9 to $f|S' \to S$. Thus $f\colon \Sigma' \to \Sigma$ can be properly homotoped relative to $\Sigma' \setminus \operatorname{int}(S')$ to map $S' = f^{-1}(S)$ homeomorphically onto S. Then by Theorem 2.6.1, $\deg(f) = \pm 1$. □

We are now ready to prove that a pseudoproper homotopy equivalence is a map of degree ± 1 if the codomain contains an essential pair of pants.

Theorem 3.6.1.11 *Let $f:\Sigma'\to\Sigma$ be a pseudoproper homotopy equivalence between two noncompact oriented surfaces, where Σ is either an infinite-type surface or a finite-type noncompact surface $S_{g,0,p}$ with high complexity (to us, high complexity means $g+p\geq 4$ or $p\geq 6$). Then $\deg(f)=\pm 1$.*

Proof Since $\deg(f)$ remains invariant after any proper homotopy of f, we can properly homotope f as we want. Theorem 3.6.1.6 gives an essential pair of pants $\boldsymbol{P}$ in Σ such that $f^{-1}(\operatorname{int}(\boldsymbol{P}))\neq\varnothing$ and $f^{-1}(c)\neq\varnothing$ for each component c of $\partial\boldsymbol{P}$, even after any proper homotopy of f. Using Theorem 3.2.1 and then Theorem 3.5.3, after a proper homotopy, we may assume that $f^{-1}(\operatorname{int}\boldsymbol{P})\neq\varnothing$ and $f^{-1}(\partial\boldsymbol{P})$ is a pairwise-disjoint collection of three smoothly embedded circles on Σ such that $f\,|\,f^{-1}(\partial\boldsymbol{P})\to\partial\boldsymbol{P}$ is a homeomorphism.

Now, if possible, let c' be a component of $f^{-1}(\partial\boldsymbol{P})$ such that there is an embedding $i':\mathbb{D}^2\hookrightarrow\Sigma'$ with $c'=i'(\mathbb{S}^1)$. Then the embedding $f\circ i'|\mathbb{S}^1\hookrightarrow\Sigma$ is nullhomotopic and $c:=f\circ i'(\mathbb{S}^1)$ is a component of $\partial\boldsymbol{P}$. But $\boldsymbol{P}$ is an essential pair of pants in Σ, and so each component of $\partial\boldsymbol{P}$ is a primitive circle on Σ. Theorem 2.2.2 tells us we have reached a contradiction. Hence each component of $f^{-1}(\partial\boldsymbol{P})$ is a primitive circle on Σ'. Finally, applying Lemma 3.6.1.10, we complete the proof. □

3.6.2 An essential punctured disk of a proper map and the degree of a pseudoproper homotopy equivalence We first build up notation for Section 3.6.2. Let Σ be a noncompact surface. Since $\operatorname{Ends}(\Sigma)$ is independent of the choice of efficient exhaustion of Σ by compacta, we will use Goldman's inductive construction to define $\operatorname{Ends}(\Sigma)$; see Section 2.3. So consider an inductive construction of Σ. For each $i\geq 1$, define K_i to be the compact bordered subsurface of Σ after the i^{th} step of the induction. Then $\{K_i\}_{i=1}^{\infty}$ is an efficient exhaustion of Σ by compacta. Also, notice that $\operatorname{int}(K_1)\subseteq\operatorname{int}(K_2)\subseteq\cdots$ is an increasing sequence of open subsets of Σ such that $\bigcup_{i=1}^{\infty}\operatorname{int}(K_i)=\Sigma$, and thus every compact subset of Σ is contained in some $\operatorname{int}(K_i)$.

Suppose Σ' is another noncompact surface and $f:\Sigma'\to\Sigma$ is a proper map. Let $(V_1,V_2,\dots)$ be an end of Σ, ie V_i is a component of $\Sigma\setminus K_i$ and $V_1\supseteq V_2\supseteq\cdots$. With this setup, we are now ready to state a lemma that is more or less related to Proposition 2.3.1:

Theorem 3.6.2.1 *Assume that $f^{-1}(V_i)\neq\varnothing$ for each $i\geq 1$. Then for every proper map $g:\Sigma'\to\Sigma$ which is properly homotopic to f, we have $g^{-1}(V_i)\neq\varnothing$ for each $i\geq 1$.*

Proof Let $g:\Sigma'\to\Sigma$ be a proper map and let $\mathcal{H}:\Sigma'\times[0,1]\to\Sigma$ be a proper homotopy from f to g. Notice that $V_i\to\infty$: if $\mathcal{X}$ is a compact subset of Σ, then $\mathcal{X}\subseteq\operatorname{int}(K_{i_0})$ for some positive integer i_0, ie $\mathcal{X}\cap V_i=\varnothing$ for all $i\geq i_0$. Therefore $f^{-1}(V_i)\to\infty$: if $\mathcal{X}'$ is a compact subset of Σ', then $f(\mathcal{X}')$ is compact, so $f(\mathcal{X}')\cap V_i=\varnothing$ for all but finitely many i, ie $\mathcal{X}'\cap f^{-1}(V_i)=\varnothing$ for all but finitely many i.

Let n be any positive integer. Consider the compact subset $p(\mathcal{H}^{-1}(K_n))$ of Σ', where $p\colon \Sigma'\times[0,1]\to\Sigma'$ is the projection. Since $f^{-1}(V_i)\to\infty$, we have an integer $i_n>n$ such that $f^{-1}(V_{i_n})\subseteq\Sigma'\setminus p(\mathcal{H}^{-1}(K_n))$. Now consider any $x_{i_n}\in f^{-1}(V_{i_n})$. Then $\mathcal{H}(x_{i_n}\times[0,1])\subseteq\Sigma\setminus K_n$, that is, the connected set $\mathcal{H}(x_{i_n}\times[0,1])$ is contained in one of the components of $\Sigma\setminus K_n$. But $\mathcal{H}(x_{i_n},0)=f(x_{i_n})\in V_{i_n}\subseteq V_n$, meaning $\mathcal{H}(x_{i_n}\times[0,1])\subseteq V_n$. In particular, this means $g(x_{i_n})=\mathcal{H}(x_{i_n},1)\in V_n$. Since n is an arbitrary positive integer, we are done. □

Definition 3.6.2.2 Let $e=(V_1,V_2,\dots)$ be an end of Σ such that for some nonnegative integer i_e, $\overline{V}_i\cong S_{0,1,1}$ for all $i\geq i_e$ (e is an isolated planar end of Σ). If $f^{-1}(V_i)\neq\varnothing$ for all $i\geq 1$, then for each integer $i\geq i_e$, we say $\overline{V}_i$ is an *essential punctured disk* of f.

Theorem 3.6.2.1 says that the notion of an essential punctured disk is invariant under proper homotopy. In Theorem 3.6.2.6, we show that, after a proper homotopy, the preimage of the boundary of an essential punctured disk under a pseudoproper homotopy equivalence bounds a planar end of the domain. Before proving this, we need the following lemma, which gives some sufficient conditions so that the preimage of a punctured disk in the codomain under a proper map becomes a punctured disk in the domain.

Lemma 3.6.2.3 *Let $f:\Sigma'\to\Sigma$ be a π_1–injective proper map between two noncompact oriented surfaces, and let $\mathcal{C}$ be a smoothly embedded separating circle on Σ such that one of the two sides of $\mathcal{C}$ in Σ is a punctured disk $\mathcal{D}_*$. Also, let Σ' be homeomorphic to neither $\mathbb{S}^1\times\mathbb{R}$ nor $\mathbb{R}^2$. If $f^{-1}(\mathcal{C})$ is a smoothly embedded primitive circle on Σ' such that $f|f^{-1}(\mathcal{C})\to\mathcal{C}$ is a homeomorphism and $f^{-1}(\operatorname{int}\mathcal{D}_*)\neq\varnothing$, then $f^{-1}(\mathcal{D}_*)$ is a copy of the punctured disk in Σ' with $\partial f^{-1}(\mathcal{D}_*)=f^{-1}(\mathcal{C})$ and $\deg(f)=\pm1$.*

Proof Notice that $\Sigma'\ncong\mathbb{R}^2,\mathbb{S}^1\times\mathbb{R}$, ie $\pi_1(\Sigma')$ is nonabelian by Theorem 3.1.9. Since $f^{-1}(\operatorname{int}\mathcal{D}_*)\neq\varnothing$ and $\pi_1(f)(\pi_1(\Sigma'))$ is nonabelian, by continuity of $f|\Sigma'\setminus f^{-1}(\mathcal{C})\to\Sigma\setminus\mathcal{C}$ we can say that $\Sigma'\setminus f^{-1}(\mathcal{C})$ is disconnected. Let $\boldsymbol{S}'$ be a side of $f^{-1}(\mathcal{C})$ in Σ' for which $f(\boldsymbol{S}')\subseteq\mathcal{D}_*$. Since f is π_1–injective, by Lemma 3.6.1.7(1), $f|\boldsymbol{S}'\to\mathcal{D}_*$ is also so. Thus $\pi_1(\boldsymbol{S}')$ is a subgroup of $\mathbb{Z}$. Now, $\operatorname{int}(\boldsymbol{S}')$ is homotopy equivalent to $\boldsymbol{S}'$ and bounded by the primitive circle $f^{-1}(\mathcal{C})$ on Σ'; so, using Theorem 3.1.9, $\boldsymbol{S}'\cong S_{0,1,1}$. Next, if $\boldsymbol{S}''$ is another side of $f^{-1}(\mathcal{C})$ in Σ' for which $f(\boldsymbol{S}'')\subseteq\mathcal{D}_*$, then similarly, $\boldsymbol{S}''\cong S_{0,1,1}$. Since $f|f^{-1}(\mathcal{C})\to\mathcal{C}$ is a homeomorphism and Σ' is connected, $\boldsymbol{S}''=\boldsymbol{S}'$; otherwise, Σ' would be $\boldsymbol{S}'\cup_{f^{-1}(\mathcal{C})}\boldsymbol{S}''\cong\mathbb{S}^1\times\mathbb{R}$. Therefore $f^{-1}(\mathcal{D}_*)=\boldsymbol{S}'\cong\mathcal{D}_*$, and thus the proof of the first part is completed.

Since $\deg(f)$ remains invariant after any proper homotopy of f, we can properly homotope f as we want. So apply Theorem 3.6.2.4 to $f|\boldsymbol{S}'\to\mathcal{D}_*$. Thus $f:\Sigma'\to\Sigma$ can be properly homotoped relative to $\Sigma'\setminus\operatorname{int}(\boldsymbol{S}')$ to map $\boldsymbol{S}'=f^{-1}(\mathcal{D}_*)$ homeomorphically onto $\mathcal{D}_*$. Now, by Theorem 2.6.1, $\deg(f)=\pm1$. □

We prove a well-known theorem used in the previous lemma:

Theorem 3.6.2.4 (proper rigidity of the punctured disk) *Let $\boldsymbol{D}_*$ be a punctured disk and let $\varphi:\boldsymbol{D}_*\to\boldsymbol{D}_*$ be a proper map such that $\varphi^{-1}(\partial\boldsymbol{D}_*)=\partial\boldsymbol{D}_*$ and $\varphi|\partial\boldsymbol{D}_*\to\partial\boldsymbol{D}_*$ is a homeomorphism. Then φ is properly homotopic to a homeomorphism $\boldsymbol{D}_*\to\boldsymbol{D}_*$ relative to the boundary $\partial\boldsymbol{D}_*$.*

Proof Without loss of generality we may assume $\boldsymbol{D}_*=\{z\in\mathbb{C}:0<|z|\le 1\}$. Define $\mathcal{H}:\boldsymbol{D}_*\times[0,1]\to\boldsymbol{D}_*$ by

$$\mathcal{H}(z,t):=\begin{cases}(1-t)\varphi(z/(1-t)) & \text{if } 0<|z|\le 1-t,\\ |z|\varphi(z/|z|) & \text{if } 1-t<|z|\le 1.\end{cases}$$

Notice that $\varphi\simeq\mathcal{H}(-,1)$ relative to $\partial\boldsymbol{D}_*$, and $\mathcal{H}(-,1)\colon\boldsymbol{D}_*\to\boldsymbol{D}_*$ is a homeomorphism.

Now we prove $\mathcal{H}$ is a proper map, so let $\{(z_n,t_n)\}$ be a sequence in $\boldsymbol{D}_*\times[0,1]$ with $z_n\to 0$. We need to show that $\mathcal{H}(z_n,t_n)\to 0$. Define $\mathscr{A}:=\{n\in\mathbb{N}:1-t_n<|z_n|\}$ and $\mathscr{B}:=\{n\in\mathbb{N}:|z_n|\le 1-t_n\}$. Then $\mathbb{N}=\mathscr{A}\cup\mathscr{B}$. Therefore it is enough to show $\{\mathcal{H}(z_n,t_n):n\in\mathscr{A}\}\to 0$ (resp. $\{\mathcal{H}(z_n,t_n):n\in\mathscr{B}\}\to 0$) whenever $\mathscr{A}$ (resp. $\mathscr{B}$) is infinite.

If $\mathscr{A}$ is infinite, then $\{\mathcal{H}(z_n,t_n):n\in\mathscr{A}\}\to 0$, since $|\mathcal{H}(z_n,t_n)|=|z_n|\cdot|\varphi(z_n/|z_n|)|\le|z_n|$ for all $n\in\mathscr{A}$.

Next, assume $\mathscr{B}$ is infinite. We will prove that $\{\mathcal{H}(z_n,t_n):n\in\mathscr{B}\}\to 0$, so consider any $\varepsilon>0$. We need to show $|\mathcal{H}(z_n,t_n)|<\varepsilon$ for all but finitely many $n\in\mathscr{B}$. Let $\mathscr{B}'_\varepsilon:=\{n\in\mathscr{B}:1-t_n<\varepsilon\}$. Therefore $|\mathcal{H}(z_n,t_n)|=(1-t_n)|\varphi(z_n/(1-t_n))|\le(1-t_n)<\varepsilon$ for all $n\in\mathscr{B}'_\varepsilon$. Also, if $\mathscr{B}\setminus\mathscr{B}'_\varepsilon$ is infinite, then $\{z_n/(1-t_n):n\in\mathscr{B}\setminus\mathscr{B}'_\varepsilon\}\to 0$, which implies $\{\varphi(z_n/(1-t_n)):n\in\mathscr{B}\setminus\mathscr{B}'_\varepsilon\}\to 0$ (as φ is proper), and thus $|\mathcal{H}(z_n,t_n)|\le|\varphi(z_n/(1-t_n))|<\varepsilon$ for all but finitely many $n\in\mathscr{B}\setminus\mathscr{B}'_\varepsilon$. The previous two lines together imply that $|\mathcal{H}(z_n,t_n)|<\varepsilon$ for all but finitely many $n\in\mathscr{B}$. □

Remark 3.6.2.5 Theorem 3.6.2.4 is obtained from a straightforward modification of the Alexander trick [17, Lemma 2.1].

Theorem 3.6.2.6 *Let $f:\Sigma'\to\Sigma$ be a pseudoproper homotopy equivalence between two noncompact oriented surfaces. Suppose $\pi_1(\Sigma)$ is a finitely generated nonabelian group (equivalently $\Sigma\cong S_{g,0,p}$ for some $(g,p)\ne(0,1),(0,2)$). Then* $\deg(f)=\pm 1$.

Proof Since $\deg(f)$ remains invariant after any proper homotopy of f, we can properly homotope f as we want. The fact that Σ is a finite-type noncompact surface implies each end of Σ is an isolated planar end, that is, for every $e=(V_1,V_2,\dots)\in\mathrm{Ends}(\Sigma)$, we have an integer i_e such that $\overline{V}_i$ is homeomorphic to the punctured disk for each $i\ge i_e$. Next, since f is proper there exists $\mathcal{E}=(\mathcal{W}_1,\mathcal{W}_2,\dots)\in\mathrm{Ends}(\Sigma)$ such that $f^{-1}(\mathcal{W}_i)\ne\varnothing$ for each $i\ge 1$. Notice that $\overline{\mathcal{W}}_{i_\mathcal{E}}$ is an essential punctured disk and $\mathcal{C}_{i_\mathcal{E}}:=\partial\overline{\mathcal{W}}_{i_\mathcal{E}}$ is a smoothly embedded separating circle on Σ. Also, $\mathcal{C}_{i_\mathcal{E}}$ is a primitive circle on Σ as $\mathcal{C}_{i_\mathcal{E}}$ bounds the punctured disk $\overline{\mathcal{W}}_{i_\mathcal{E}}$ on $\Sigma\ncong\mathbb{R}^2$.

We aim to use Lemma 3.6.2.3, but some observations are needed before that. Let $g:\Sigma'\to\Sigma$ be a proper map such that g is properly homotopic to f (note that f is properly homotopic to itself, ie g can be f). If possible, assume $g^{-1}(\mathcal{C}_{i_\mathcal{E}})=\varnothing$. Then continuity of g implies $g(\Sigma')$ is contained in one of the two components of $\Sigma\setminus\mathcal{C}_{i_\mathcal{E}}$. By Theorem 3.6.2.1, $g(\Sigma')$ must be contained in $\mathcal{W}_{i_\mathcal{E}}$. But then $\pi_1(f)=\pi_1(g)$ is nonsurjective as $\pi_1(\Sigma\setminus\mathcal{W}_{i_\mathcal{E}})=\pi_1(\Sigma)$ is nonabelian. Therefore $g^{-1}(\mathcal{C}_{i_\mathcal{E}})\ne\varnothing$. Also, by Theorem 3.6.2.1, $g^{-1}(\mathcal{W}_i)\ne\varnothing$ for each $i\ge 1$, and thus $g^{-1}(\mathcal{W}_{i_\mathcal{E}})\ne\varnothing$.

We are ready to apply Lemma 3.6.2.3 after the observation given in the previous paragraph. At first, notice that Σ' is homeomorphic to neither the plane nor the punctured plane as $\pi_1(\Sigma') = \pi_1(\Sigma)$ is nonabelian. After a proper homotopy of f, we may assume that $f \pitchfork \mathcal{C}_{i_\varepsilon}$; see Theorem 3.2.1. By the previous paragraph, $f^{-1}(\mathcal{C}_{i_\varepsilon})$ is a pairwise-disjoint nonempty collection of finitely many smoothly embedded circles on Σ'. By Theorem 3.5.3 and the previous paragraph, after a proper homotopy of f, we may further assume that $\mathcal{C}'_{i_\varepsilon} := f^{-1}(\mathcal{C}_{i_\varepsilon})$ is a (single) smoothly embedded circle on Σ' and $f|\mathcal{C}'_{i_\varepsilon} \to \mathcal{C}_{i_\varepsilon}$ is a homeomorphism. The previous paragraph also tells that after all these proper homotopies, $f^{-1}(\mathcal{W}_{i_\varepsilon})$ remains nonempty.

We show that $\mathcal{C}'_{i_\varepsilon}$ is a primitive circle on Σ'. On the contrary, let there be an embedding $i' \colon \mathbb{D}^2 \hookrightarrow \Sigma'$ with $\mathcal{C}'_{i_\varepsilon} = i'(\mathbb{S}^1)$. Then the embedding $f \circ i'|\mathbb{S}^1 \hookrightarrow \Sigma$ is nullhomotopic and $\mathcal{C}_{i_\varepsilon} = f \circ i'(\mathbb{S}^1)$. But $\mathcal{C}_{i_\varepsilon}$ is a primitive circle on Σ. Now, Theorem 2.2.2 tells us we have reached a contradiction. Finally, applying Lemma 3.6.2.3, we can say that $\deg(f) = \pm 1$. □

3.6.3 Most pseudoproper homotopy equivalences between noncompact surfaces are of degree ±1

Theorem 3.6.3.1 *Let $f : \Sigma' \to \Sigma$ be a pseudoproper homotopy equivalence between two noncompact oriented surfaces. If $\Sigma \ncong \mathbb{S}^1 \times \mathbb{R}, \mathbb{R}^2$ (equivalently $\Sigma' \ncong \mathbb{S}^1 \times \mathbb{R}, \mathbb{R}^2$), then $\deg(f) = \pm 1$.*

Proof Combining Theorems 3.6.1.11 and 3.6.2.6, we complete the proof. □

The following proposition, *which we don't need to use anywhere*, says that if either of the integers 1 or -1 appears as the degree of a pseudoproper homotopy equivalence between two noncompact oriented surfaces, then the other also appears.

Proposition 3.6.3.2 *Let $f : \Sigma' \to \Sigma$ be a pseudoproper homotopy equivalence between two noncompact oriented surfaces. Then there exists another pseudoproper homotopy equivalence $\bar{f} : \Sigma' \to \Sigma$ such that $\deg(\bar{f}) = -\deg(f)$.*

Proof Write Σ as the double of a bordered surface $\boldsymbol{S}$; see Theorem 2.4.2. Define a homeomorphism $\varphi \colon \Sigma \to \Sigma$ by sending $[p,t] \in \Sigma$ to $[p, 1-t] \in \Sigma$ for all $(p,t) \in \boldsymbol{S} \times \{0,1\}$. Then φ is an orientation-reversing homeomorphism. Therefore the degree of $\bar{f} := \varphi \circ f$ is $-\deg(f)$ as the degree is multiplicative; see Section 2.6. □

3.6.4 An application of the nonvanishing degree of a pseudoproper homotopy equivalence Consider a nonsurjective map $\varphi \colon \mathscr{M} \to \mathscr{N}$ between two closed oriented connected n–manifolds. Then for any $p \in \mathscr{N} \setminus \operatorname{im}(\varphi)$, the map $H^n(\varphi)$ factors through the inclusion-induced zero map $H^n(\mathscr{N}) \cong \mathbb{Z} \to 0 \cong H^n(\mathscr{N} \setminus p)$ (recall that the top integral singular cohomology of any connected noncompact boundaryless manifold is zero), ie $\deg(\varphi) = 0$. The lemma below generalizes this phenomenon in the proper category:

Lemma 3.6.4.1 *Let $\Phi \colon M \to N$ be a proper map between two connected oriented boundaryless smooth k–dimensional manifolds. If $\deg(\Phi) \neq 0$, then Φ is surjective.*

Proof Being a proper map between two manifolds, Φ is a closed map; see [30]. Now, if possible, let Φ be nonsurjective. Therefore $N \setminus \Phi(M)$ is a nonempty open subset of N. Pick a point $y \in N \setminus \Phi(M)$. Since N is locally Euclidean, there is a smoothly embedded closed ball $B \subset N$ such that $B \subseteq N \setminus \Phi(M)$. Notice that $N \setminus \operatorname{int}(B)$ is a smoothly embedded codimension-zero submanifold of N with $\partial(N \setminus \operatorname{int}(B)) = \partial B$. By Poincaré duality (see [25, Exercise 35 of Section 3.3]), $H_{\boldsymbol{c}}^k(N \setminus \operatorname{int}(B); \mathbb{Z}) \cong H_0(N \setminus \operatorname{int}(B), \partial B; \mathbb{Z})$. Also, $H_0(N \setminus \operatorname{int}(B), \partial B; \mathbb{Z}) = 0$ as N is path connected; see [25, Exercise 16(a) of Section 2.1]. Now, $\Phi\colon M \to N$ can be thought as the composition $M \xrightarrow{\Phi^\dagger} N \setminus \operatorname{int}(B) \xhookrightarrow{i} N$, where i is the inclusion map and $\Phi^\dagger(m) := \Phi(m)$ for all $m \in M$. Certainly, $\Phi^\dagger$ and i are both proper maps. Therefore $H_{\boldsymbol{c}}^k(\Phi)$ is the composition

$$H_{\boldsymbol{c}}^k(N; \mathbb{Z}) \xrightarrow{H_c^k(i)} H_{\boldsymbol{c}}^k(N \setminus \operatorname{int}(B); \mathbb{Z}) = 0 \xrightarrow{H_c^k(\Phi^\dagger)} H_{\boldsymbol{c}}^k(M; \mathbb{Z}),$$

ie $H_{\boldsymbol{c}}^k(\Phi) = 0$, which contradicts $\deg(\Phi) \neq 0$. Thus Φ must be a surjective map. □

The above lemma, together with Theorem 3.6.3.1, gives the following corollary:

Corollary 3.6.4.2 *A pseudoproper homotopy equivalence between two noncompact surfaces is a surjective map, provided the surfaces are homeomorphic to neither the plane nor the punctured plane.*

The following lemma tells that one way to achieve the surjectivity throughout a proper homotopy is to assume that the initial map of this proper homotopy is a map of nonzero degree. Note that any proper map $f\colon X \to Y$ is properly homotopic to itself due to the proper homotopy $X \times [0,1] \ni (x,t) \mapsto f(x) \in Y$.

Lemma 3.6.4.3 *Let $\Phi\colon M \to N$ be a proper map of nonzero degree between two connected oriented boundaryless smooth k–dimensional manifolds, and let $\Psi\colon M \to N$ be a proper map such that Ψ is properly homotopic to Φ. Then Ψ is a surjective map.*

Proof Since Ψ is properly homotopic to Φ, $\deg(\Psi) = \deg(\Phi) \neq 0$; see Section 2.6. Now, to conclude, consider Lemma 3.6.4.1. □

Here is the main application of the nonvanishing degree of a pseudoproper homotopy equivalence:

Theorem 3.6.4.4 *Let $f\colon \Sigma' \to \Sigma$ be a smooth pseudoproper homotopy equivalence between two noncompact surfaces, where $\mathbb{S}^1 \times \mathbb{R} \ncong \Sigma \ncong \mathbb{R}^2$, and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f \pitchfork \mathscr{A}$. Suppose any two distinct components of $\mathscr{A}$ don't cobound an annulus in Σ. In that case, f can be properly homotoped to a proper map g such that for each component $\mathcal{C}$ of $\mathscr{A}$, $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{A})$ that is mapped homeomorphically onto $\mathcal{C}$ by g.*

Proof Theorem 3.5.3 gives a proper map $g\colon \Sigma' \to \Sigma$ such that g is properly homotopic to f, and for each component $\mathcal{C}$ of $\mathscr{A}$, if $g^{-1}(\mathcal{C}) \neq \varnothing$, then $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{A})$ such that $g|g^{-1}(\mathcal{C}) \to \mathcal{C}$ is a homeomorphism. But $\deg(f) = \pm 1$ by Theorem 3.6.3.1. Thus the map g is surjective since it is properly homotopic to the nonzero degree map f; see Lemma 3.6.4.3. So, for each component $\mathcal{C}$ of $\mathscr{A}$, $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{A})$ such that $g|g^{-1}(\mathcal{C}) \to \mathcal{C}$ is a homeomorphism. □

Remark 3.6.4.5 For closed surfaces, the analog of Theorem 3.6.4.4 can be stated far before, in the "annulus removal" section, as every homotopy equivalence between two closed manifolds has a homotopy inverse, so is a map of degree ± 1, and hence is surjective. But, before Section 3.6, we didn't know the degree of a pseudoproper homotopy equivalence; even at this stage, we don't know whether a pseudoproper homotopy equivalence has a proper homotopy inverse or not.

4 Finishing the proofs of Theorems 1, 2 and 3

Proof of Theorem 1 Consider an LFCS $\mathscr{C}$ on Σ provided by Theorem 3.1.5. Using Theorem 3.2.1, assume f is smooth as well as $f \pitchfork \mathscr{C}$. Thus $f^{-1}(\mathscr{C})$ is a nonempty LFCS on Σ'; see Corollary 3.6.4.2 and Theorem 3.2.3. By Theorem 3.6.4.4, f can be properly homotoped to a proper map g such that for each component $\mathcal{C}$ of $\mathscr{C}$, $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{C})$ that is mapped homeomorphically onto $\mathcal{C}$ by g. Thus $g^{-1}(\mathscr{C})$ decomposes Σ' into bordered subsurfaces and each component of $\Sigma \setminus \mathscr{C}$ has nonempty preimage; see Corollary 3.6.4.2. Let $\boldsymbol{S} \subset \Sigma$ be a bordered subsurface obtained as a complementary component of the decomposition of Σ by $\mathscr{C}$. Now, $\boldsymbol{S} \cong g^{-1}(\boldsymbol{S})$ by Lemmas 3.6.1.10 (see its proof also) and 3.6.2.3. Since g sends $\operatorname{int}(g^{-1}(\boldsymbol{S}))$ onto $\operatorname{int}(\boldsymbol{S})$ and $\partial g^{-1}(\boldsymbol{S})$ homeomorphically onto $\partial \boldsymbol{S}$, we can properly homotope $g|g^{-1}(\boldsymbol{S}) \to \boldsymbol{S}$ relative to $\partial g^{-1}(\boldsymbol{S})$ to a homeomorphism $g^{-1}(\boldsymbol{S}) \to \boldsymbol{S}$; see Theorems 3.6.1.9 and 3.6.2.4. Finally, vary $\boldsymbol{S}$ over different complementary components of Σ decomposed by $\mathscr{C}$ to collect these boundary-relative proper homotopies and then paste them to get a proper homotopy from g to a homeomorphism $\Sigma' \to \Sigma$. Since g is properly homotopic to f, we are done. □

The proof of Theorem 1 shows that we are using the nonzero degree assumption of the pseudoproper homotopy equivalence (which is given by Theorem 3.6.3.1) to ensure surjectivity after each proper homotopy. Thus, by a similar argument, we can prove Theorem 4.1.

Theorem 4.1 *Let $f\colon \Sigma' \to \Sigma$ be a pseudoproper homotopy equivalence between two noncompact oriented surfaces. Suppose Σ is not homeomorphic to $\mathbb{R}^2$ and $\deg(f) \neq 0$. Then Σ' is homeomorphic to Σ and f is properly homotopic to a homeomorphism.*

Theorem 4.2 *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a proper map of degree ± 1. Then f is properly homotopic to a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$.*

Proof By Theorem 2.6.2, f can be properly homotoped to get smoothly embedded closed disks $\boldsymbol{D}, \boldsymbol{D}' \subseteq \mathbb{R}^2$ such that $\boldsymbol{D}' = f^{-1}(\boldsymbol{D})$ and $f|\boldsymbol{D}' \to \boldsymbol{D}$ is a homeomorphism. Using the Jordan–Schönflies theorem, $f|\overline{\mathbb{R}^2 \setminus \boldsymbol{D}'} \to \overline{\mathbb{R}^2 \setminus \boldsymbol{D}}$ resembles a map between two punctured disks, on which we apply Theorem 3.6.2.4. □

Proof of Theorem 3 Since $\deg(f) = \pm 1$, by Theorem 2.6.3, $\pi_1(f)$ is surjective. Thus $\pi_1(f)$ is bijective. Both Σ' and Σ are homotopy equivalent to $\bigvee_{\mathscr{I}} \mathbb{S}^1$ for some index set $\mathscr{I}$ with $|\mathscr{I}| \leq \aleph_0$, ie $\pi_k(\Sigma') = 0 = \pi_k(\Sigma)$ for all $k \geq 2$. So, by the Whitehead theorem, f is a homotopy equivalence (note that each surface has a CW–complex structure due to its C^∞–smooth structure). Now, a simply connected noncompact surface is homeomorphic to $\mathbb{R}^2$; see Theorem 3.1.9. So, combining Theorems 4.1 and 4.2, we are done. □

Proof of Theorem 2 A proper homotopy equivalence is a π_1–injective map of degree ± 1. Now apply Theorem 3. □

The following proposition is an application of Theorem 1:

Proposition 4.3 *Let Σ be a noncompact surface such that $\mathbb{S}^1 \times \mathbb{R} \ncong \Sigma \ncong \mathbb{R}^2$. Suppose $f, g : \Sigma \to \Sigma$ are two pseudoproper homotopy equivalences. If f is homotopic to g, then f is properly homotopic to g.*

Proof By applying Theorem 1 up to proper homotopy, we may assume both f and g are homeomorphisms without loss of generality. Since $\mathbb{S}^1 \times \mathbb{R} \ncong \Sigma \ncong \mathbb{R}^2$ and $f^{-1}g$ is homotopic to Id_Σ, by [15, Theorem 6.4], there exists a level-preserving homeomorphism $\mathcal{H}\colon \Sigma \times [0,1] \to \Sigma \times [0,1]$ which agrees with $f^{-1}g$ on $\Sigma \times 0$ and with Id_Σ on $\Sigma \times 1$. The fact that the projection $\Sigma \times [0,1] \to \Sigma$ is proper implies $f^{-1}g$ is properly homotopic to Id_Σ, so we are done. □

Appendix

A.1 Approximation and transversality in the proper category

Throughout Section A.1, M and N will denote two smooth boundaryless manifolds, possibly noncompact. Let $F\colon N \to M$ be a smooth map, and let X be a smoothly embedded boundaryless submanifold of M. We say F *is transverse to* X, and write $F \pitchfork X$, if for every $p \in F^{-1}(X)$ we have $T_{F(p)}X + dF_p(T_pN) = T_{F(p)}M$. If F is transverse to X and $F(N) \cap X \neq \varnothing$, then $F^{-1}(X)$ is a smoothly embedded boundaryless submanifold of N such that $\dim(N) - \dim(F^{-1}(X)) = \dim(M) - \dim(X)$; see [28, Theorem 6.30(a)].

The Whitney approximation theorem [28, Theorem 6.26] says that any continuous map $N \to M$ is homotopic to a smooth map. The transversality homotopy theorem [28, Theorem 6.36] says that for any smooth map $F\colon N \to M$ and for any smoothly embedded boundaryless submanifold X of M, the smooth map F can be homotoped to another smooth map $\widetilde{F}\colon N \to M$ such that $\widetilde{F} \pitchfork X$. We modify these two theorems in the proper category. Our interest is in the properness of homotopies; the extra stuff not related to properness is in [28, Theorems 6.26 and 6.36].

Theorem A.1.1 (proper Whitney approximation theorem) *Let $f : N \to M$ be a continuous proper map. Then f is properly homotopic to a smooth proper map.*

Theorem A.1.2 (proper transversality homotopy theorem) *Let $f : N \to M$ be a smooth proper map, and let X be a smoothly embedded boundaryless submanifold of M. Then f is properly homotopic to a smooth proper map $g\colon N \to M$ which is transverse to X.*

We start by summarizing key facts in and around the tubular neighborhood theorem. Let $M \hookrightarrow \mathbb{R}^l$ be a smooth proper embedding; see [28, Theorems 6.15]. For each $x \in M$, define the *normal space* $\mathcal{N}_x M$ to M at x as $\mathcal{N}_x M := \{v \in \mathbb{R}^l : v \perp T_x M\}$. Then $\mathcal{N}M := \{(x, v) \in \mathbb{R}^l \times \mathbb{R}^l : x \in M,\ v \perp T_x M\}$ is a smoothly embedded l–dimensional submanifold of $\mathbb{R}^l \times \mathbb{R}^l$ and $\pi\colon \mathcal{N}M \ni (x, v) \mapsto x \in M$ is vector bundle of rank $l - \dim(M)$, called the *normal bundle* of M in $\mathbb{R}^l$; see [28, Corollary 10.36].

Consider the smooth map $E\colon \mathcal{N}M \ni (x, v) \mapsto x + v \in \mathbb{R}^l$. One can show that $dE_{(x,0)}$ is bijective for each $x \in M$. Thus, for each $x \in M$, we have $\delta > 0$ such that E maps

$$V_\delta(x) := \{(x', v') \in \mathcal{N}M : |x - x'| < \delta, |v'| < \delta\}$$

diffeomorphically onto an open neighborhood of x in $\mathbb{R}^l$. Now, the map $\rho\colon M \to (0, 1]$ defined by

$$\rho(x) := \sup\{\delta \le 1 : E \text{ maps } V_\delta(x) \text{ diffeomorphically onto an open neighborhood of } x \text{ in } \mathbb{R}^l\}$$

is continuous. Further, $V := \left\{(x, v) \in \mathcal{N}M : |v| < \frac{1}{2}\rho(x)\right\}$ is an open subset of NM and E maps V diffeomorphically onto an open subset U of $\mathbb{R}^l$ with $M \subseteq U$, ie U is a tubular neighborhood of M in $\mathbb{R}^l$; see [28, Theorem 6.24]. Note that the map $r\colon U \to M$ defined by $r := \pi \circ (E|V \to U)^{-1}$ is a retraction and submersion; see [28, Proposition 6.25]. Denote $\{y \in \mathbb{R}^l : |y - x| < \varepsilon\}$ by $B_\varepsilon(x)$. By an argument similar to showing the continuity of ρ, one can prove that $\delta\colon M \to (0, 1]$, defined by $\delta(x) := \sup\{\varepsilon \le 1 : B_\varepsilon(x) \subseteq U\}$ for any $x \in M$, is also continuous.

With this setup, we are now ready to state a crucial lemma, which in particular says that if two points are at most unit distance apart, then the distance between their images under the tubular neighborhood retraction can be at most 2.

Lemma A.1.3 *Let $\varepsilon > 0$. If $y, y' \in U$ with $|y - y'| < \varepsilon$, then $|r(y) - r(y')| \le \varepsilon + 1$.*

Proof Notice $|r(y) - r(y')| - |y - y'| \le |y - r(y)| + |y' - r(y')| \le \frac{1}{2}\rho \circ r(y) + \frac{1}{2}\rho \circ r(y')$. □

Consider another smooth proper embedding $N \hookrightarrow \mathbb{R}^k$ for the proof of the following three facts. The following lemma says that a homotopy lying in a λ–neighborhood (where λ is a fixed positive number) of a proper map is a proper homotopy:

Lemma A.1.4 *Let $h\colon N \to M$ be a continuous proper map, and let $\mathcal{H}\colon N \times [0, 1] \to M$ be a homotopy. If there exists a constant λ such that $|\mathcal{H}(p, t) - h(p)| \le \lambda$ for each $(p, t) \in N \times [0, 1]$, then $\mathcal{H}$ is proper.*

Proof Note that the embeddings $M \hookrightarrow \mathbb{R}^l$ and $N \hookrightarrow \mathbb{R}^k$ are closed maps as they are proper maps; see [30]. Consider the induced metric d_M on M inherited from $\mathbb{R}^l$, ie $d_M(m, m') = |m - m'|$ for all $m, m' \in M$. Also, we have the induced metric $d_{N\times[0,1]}$ on $N \times [0, 1]$ inherited from $\mathbb{R}^k \times [0, 1]$, ie $d_{N\times[0,1]}((n, t), (n', t')) = |n - n'| + |t - t'|$ for all $(n, t), (n', t') \in N \times [0, 1]$. Thus a subset of $N \times [0, 1]$ (resp. M) is compact if and only if it is closed and bounded in $N \times [0, 1]$ (resp. M).

Let C be a compact subset of M. Continuity of $\mathcal{H}$ implies $\mathcal{H}^{-1}(C)$ is closed in $N \times [0,1]$. Also, if there were an unbounded sequence $\{(n_i, t_i)\} \subseteq \mathcal{H}^{-1}(C)$, then $\{n_i\}$, and hence $\{h(n_i)\}$, would be unbounded (as h is proper); thus the unbounded set $\{h(n_i)\}$ would be inside the λ–neighborhood of the bounded set C, a contradiction. Therefore $\mathcal{H}^{-1}(C)$ is closed and bounded in $N \times [0,1]$, and hence $\mathcal{H}^{-1}(C)$ is compact. Since C is an arbitrary compact subset of M, we are done. □

Now we are ready to prove the analogs of the Whitney approximation theorem and transversality homotopy theorem in the proper category:

Proof of Theorem A.1.1 The Whitney approximation theorem gives a smooth function $\tilde{f}: N \to \mathbb{R}^l$ such that $|\tilde{f}(y) - f(y)| < \delta(f(y))$ for each $y \in N$; see [28, Theorem 6.21]. Now define $\mathcal{H}: N \times [0,1] \to M$ as $\mathcal{H}(p,t) := r((1-t)f(p) + t\tilde{f}(p))$ for all $(p,t) \in N \times [0,1]$. If $(p,t) \in N \times [0,1]$, then

$$|(1-t)f(p) + t\tilde{f}(p) - f(p)| \le t|\tilde{f}(p) - f(p)| \le 1.$$

Therefore, for all $(p,t) \in N \times [0,1]$, we have that $|\mathcal{H}(p,t) - f(p)| = |\mathcal{H}(p,t) - r \circ f(p)| \le 2$ by Lemma A.1.3. Now Lemma A.1.4 tells us that $\mathcal{H}$ is proper. Therefore $\mathcal{H}(-,1) = r \circ \tilde{f}$ is a smooth proper map that is properly homotopic to f (recall that r is a smooth retraction). □

Proof of Theorem A.1.2 The Whitney approximation theorem gives a smooth function $e: N \to (0,\infty)$ with $0 < e < \delta \circ f$; see [28, Corollary 6.22]. Let $\mathbb{B}^l := \{\boldsymbol{s} \in \mathbb{R}^l : |\boldsymbol{s}| < 1\}$. Define $F: N \times \mathbb{B}^l \to M$ as $F(p,\boldsymbol{s}) := r(f(p) + e(p)\boldsymbol{s})$ for any $(p,\boldsymbol{s}) \in N \times \mathbb{B}^l$. If $p \in N$, the restriction of F to $\{p\} \times \mathbb{B}^l$ is the composition of the local diffeomorphism $\boldsymbol{s} \mapsto f(p) + e(p)\boldsymbol{s}$ with the smooth submersion r, so F is a smooth submersion and hence transverse to X.

By the parametric transversality theorem [28, Theorem 6.35], $F(-,\boldsymbol{s}_0)$ is transverse to X for some $\boldsymbol{s}_0 \in \mathbb{B}^l$. Now define $\mathcal{H}: N \times [0,1] \to M$ as $\mathcal{H}(p,t) := r(f(p) + te(p)\boldsymbol{s}_0)$ for all $(p,t) \in N \times [0,1]$. If $(p,t) \in N \times [0,1]$, then

$$|(f(p) + te(p)\boldsymbol{s}_0) - f(p)| \le te(p)|\boldsymbol{s}_0| < \delta(f(p)) \le 1.$$

Therefore $|\mathcal{H}(p,t) - f(p)| = |\mathcal{H}(p,t) - r \circ f(p)| \le 2$ for all $(p,t) \in N \times [0,1]$ by Lemma A.1.3. Lemma A.1.4 tells that $\mathcal{H}$ is proper. Define $g := \mathcal{H}(-,1)$, ie $g = r(f(-) + e(-)\boldsymbol{s}_0) = F(-,\boldsymbol{s}_0)$ is properly homotopic to f (recall that r is a smooth retraction) as well as transverse to X. □

A.2 Transversality of a proper map between two surfaces with respect to a circle

Here is some notation that will be used throughout Section A.2. Let $f: \Sigma' \to \Sigma$ be a smooth *proper* map between two surfaces, and let $\mathcal{C}$ be a smoothly embedded circle on Σ such that f is transverse to $\mathcal{C}$. Also, let $\varphi: \mathcal{C} \times [-1,1] \hookrightarrow \Sigma$ be a smooth embedding with $\varphi(\mathcal{C}, 0) = \mathcal{C}$, that is, $\operatorname{im}(\varphi)$ is a *two-sided* (*trivial*) *tubular neighborhood* of $\mathcal{C}$. We call each of $\varphi(\mathcal{C} \times [-1,0])$ and $\varphi(\mathcal{C} \times [0,1])$ a *one-sided tubular neighborhood of* $\mathcal{C}$ (in short, *a side of* $\mathcal{C}$). By scaling, we may replace $[-1,0]$ and $[0,1]$ with other closed intervals.

The following theorem says that f is transverse to all circles which are parallel to and sufficiently near $\mathcal{C}$.

Theorem A.2.1 *There exists $\varepsilon_0 \in (0,1)$ such that f is transverse to $\mathcal{C}_\varepsilon := \varphi(\mathcal{C}, \varepsilon)$ for each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Thus for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $f^{-1}(\mathcal{C}_\varepsilon)$ is either empty or a pairwise-disjoint collection of finitely many smoothly embedded circles on Σ'.*

First we need a lemma:

Lemma A.2.2 *Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map and $x_n \to x$ in $\mathbb{R}^2$ with $r_n := |g(x_n)| \to 1$. Write $S_r := \{z \in \mathbb{R}^2 : |z| = r\}$ and assume $\operatorname{im}(dg_{x_n}) = T_{g(x_n)}(S_{r_n})$ for all n. If $dg_x \neq 0$, then $\operatorname{im}(dg_x) = T_{g(x)}(S_1)$.*

Proof The derivative map $dg\colon \mathbb{R}^2 \to L(\mathbb{R}^2, \mathbb{R}^2)$ is continuous so $dg_{x_n} \to dg_x$, and this convergence can be thought as convergence of (2×2)–matrices. In particular, if $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{R}^2$ are two perpendicular unit vectors, then $dg_{x_n}(\boldsymbol{i}) \to dg_x(\boldsymbol{i})$ and $dg_{x_n}(\boldsymbol{j}) \to dg_x(\boldsymbol{j})$.

Recall that the tangent space at any point of a circle is the vector space of all points perpendicular to this point. So $\langle dg_{x_n}(\boldsymbol{i}), g(x_n)\rangle = 0 = \langle dg_{x_n}(\boldsymbol{j}), g(x_n)\rangle$ by hypothesis, and now $\langle dg_x(\boldsymbol{i}), g(x)\rangle = 0 = \langle dg_x(\boldsymbol{j}), g(x)\rangle$ by the convergence of the inner product. Hence $\operatorname{im}(dg_x) \subseteq T_{g(x)}(S_1)$. Since $dg_x \neq 0$ and $\dim T_{g(x)}(S_1) = 1$, we are done. □

Proof of Theorem A.2.1 Suppose not. So, a sequence $\varepsilon_n \to 0$ and points $x_n \in f^{-1}(\mathcal{C}_{\varepsilon_n})$ exist such that $\operatorname{im}(df_{x_n}) + T_{f(x_n)}\mathcal{C}_{\varepsilon_n} \subsetneqq T_{f(x_n)}\Sigma$ for all n. Hence $\operatorname{im}(df_{x_n}) \subseteq T_{f(x_n)}\mathcal{C}_{\varepsilon_n}$ as $T_{f(x_n)}\mathcal{C}_{\varepsilon_n} \oplus \mathcal{N}_{f(x_n)}\mathcal{C}_{\varepsilon_n} = T_{f(x_n)}\Sigma$ for all n. Now, $\{x_n\}$ is contained in the compact set $f^{-1}(\operatorname{im}(\varphi))$ (recall that f is a proper map), ie passing to subsequence, if needed, assume $x_n \to x \in f^{-1}(\mathcal{C})$.

The continuity of the derivative map says $df_{x_n} \to df_x$. After discarding the first few terms, we may assume $df_{x_n} \neq 0$ for all n (otherwise $df_x = 0$, which would mean $T_{f(x)}\mathcal{C} + \operatorname{im}(df_x) = T_{f(x)}\mathcal{C}$ wouldn't be equal to $T_{f(x)}\Sigma$ and so f wouldn't be transverse to $\mathcal{C}$). So $\operatorname{im}(df_{x_n}) = T_{f(x_n)}(\mathcal{C}_{\varepsilon_n})$ for all n (a nonzero vector subspace of a 1–dimensional vector space is equal to the whole space).

Now, restricting f to a coordinate ball containing x and then postcomposing with φ^{-1}, we can consider Lemma A.2.2, which gives $\operatorname{im}(df_x) = T_{f(x)}(\mathcal{C})$, a contradiction to the assumption $f \pitchfork \mathcal{C}$. □

The previous theorem guarantees transversality near $\mathcal{C}$. In the rest of Section A.2, we aim to prove that every small one-sided tubular neighborhood of a component of $f^{-1}(\mathcal{C})$ maps into a small one-sided tubular neighborhood of $\mathcal{C}$.

We first fix some notation. Let $\mathcal{C}'$ be a component of $f^{-1}(\mathcal{C})$. Also, consider an $\varepsilon_0 \in (0,1)$ such that $f \pitchfork \mathcal{C}_\varepsilon$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$; see Theorem A.2.1.

Theorem A.2.3 *Let $\varepsilon \in (0, \varepsilon_0]$, and let $\mathcal{T}'$ be a two-sided compact tubular neighborhood of $\mathcal{C}'$ in Σ'. Then there exist two one-sided compact tubular neighborhoods $\mathcal{U}_l'$ and $\mathcal{U}_r'$ of $\mathcal{C}'$ in Σ' such that $\mathcal{U}_l' \cup \mathcal{U}_r'$ is a two-sided tubular neighborhood of $\mathcal{C}'$ with $\mathcal{U}_l' \cup \mathcal{U}_r' \subseteq \mathcal{T}'$, and for each $s \in \{l, r\}$ the following hold: $f^{-1}(\mathcal{C}) \cap \mathcal{U}_s' = \mathcal{C}'$, and either $f(\mathcal{U}_s') \subseteq \varphi(\mathcal{C} \times [0, \varepsilon])$ or $f(\mathcal{U}_s') \subseteq \varphi(\mathcal{C} \times [-\varepsilon, 0])$.*

Proof By Theorem A.2.1, $f^{-1}(\mathcal{C}_{-\varepsilon}) \cup f^{-1}(\mathcal{C}) \cup f^{-1}(\mathcal{C}_{\varepsilon})$ is a pairwise-disjoint collection of finitely many smoothly embedded circles on Σ'. Now, consider two one-sided compact tubular neighborhoods $\mathcal{U}'_l$ and $\mathcal{U}'_r$ of $\mathcal{C}'$ in Σ' such that $\mathcal{U}'_l \cup \mathcal{U}'_r$ is a two-sided tubular neighborhood of $\mathcal{C}'$ with $\mathcal{U}'_l \cup \mathcal{U}'_r \subseteq \mathcal{T}'$, and for each $s \in \{l, r\}$ the following hold: $f^{-1}(\mathcal{C}) \cap \mathcal{U}'_s = \mathcal{C}'$, and $\mathcal{U}'_s \cap f^{-1}(\mathcal{C}_{\varepsilon}) = \varnothing = \mathcal{U}'_s \cap f^{-1}(\mathcal{C}_{-\varepsilon})$.

Now fix $s \in \{l, r\}$. Since $\mathcal{U}'_s \setminus \mathcal{C}'$ is connected and f is continuous, $f(\mathcal{U}'_s \setminus \mathcal{C}')$ is contained in one of the components of $\Sigma \setminus (\mathcal{C}_{-\varepsilon} \cup \mathcal{C} \cup \mathcal{C}_{\varepsilon})$. But $f(\mathcal{C}') \subseteq \mathcal{C}$ implies either $f(\mathcal{U}'_s) \subseteq \varphi(\mathcal{C} \times [0, \varepsilon])$ or $f(\mathcal{U}'_s) \subseteq \varphi(\mathcal{C} \times [-\varepsilon, 0])$. □

Remark A.2.4 In Theorem A.2.3, it is possible that $f(\mathcal{U}'_l \cup \mathcal{U}'_r)$ is contained in either $\varphi(\mathcal{C} \times [0, \varepsilon])$ or $\varphi(\mathcal{C} \times [-\varepsilon, 0])$, ie f may map both sides of $\mathcal{C}'$ into one of the two sides of $\mathcal{C}$.

Consider the one-sided compact tubular neighborhoods $\mathcal{U}'_l$ and $\mathcal{U}'_r$ of $\mathcal{C}'$ in Σ' given by Theorem A.2.3. Notice that for some $s \in \{l, r\}$, it is possible that $f((\partial \mathcal{U}'_s) \setminus \mathcal{C}') \not\subseteq \varphi(\mathcal{C} \times t)$ for any $t \in [-\varepsilon, \varepsilon]$. A remedy for this is given in the following theorem:

Theorem A.2.5 *Let $\varepsilon \in (0, \varepsilon_0]$, and let $\mathcal{U}'$ be a one-sided compact tubular neighborhood of $\mathcal{C}'$ such that $f^{-1}(\mathcal{C}) \cap \mathcal{U}' = \mathcal{C}'$ and $f(\mathcal{U}') \subseteq \varphi(\mathcal{C} \times [0, \varepsilon])$. Then there is a $\delta \in (0, \varepsilon)$ and a component $\mathcal{C}'_\delta$ of $f^{-1}(\mathcal{C}_\delta)$ such that the following hold*:

(1) *$\mathcal{C}'_\delta$ together with $\mathcal{C}'$ cobound an annulus $\mathcal{A}' \subseteq \mathcal{U}'$ such that any other component of $f^{-1}(\mathcal{C}_\delta)$ in $\operatorname{int}(\mathcal{A}')$, if any, bounds a disk inside $\mathcal{A}'$.*

(2) *The map f sends $\mathcal{A}'$ into $\varphi(\mathcal{C} \times [0, \varepsilon])$. Also, after removing the interiors of all disks bounded by components of $f^{-1}(\mathcal{C}_\delta)$ from $\mathcal{A}'$, we can send it to $\varphi(\mathcal{C} \times [0, \delta])$ by f.*

Proof of Theorem A.2.5(1) Choose a $\delta \in (0, \varepsilon)$ such that $\varphi(\mathcal{C} \times [0, \delta]) \cap f((\partial \mathcal{U}') \setminus \mathcal{C}') = \varnothing$. Note that such a δ exists; otherwise, using the compactness of $(\partial \mathcal{U}') \setminus \mathcal{C}'$, we would have a sequence $\{x'_n\} \subseteq (\partial \mathcal{U}') \setminus \mathcal{C}'$ converging to some $x' \in (\partial \mathcal{U}') \setminus \mathcal{C}'$ such that $f(x'_n) \in \varphi(\mathcal{C} \times [0, 1/n])$, ie $f(x')$ would belong to $\mathcal{C}$, a contradiction to the assumption $f^{-1}(\mathcal{C}) \cap \mathcal{U}' = \mathcal{C}'$. Define an open set $\mathcal{W}'$ by

$$\mathcal{W}' := \operatorname{int}(\mathcal{U}') \cap f^{-1}(\varphi(\mathcal{C} \times (0, \delta))).$$

Notice that no sequence in $\mathcal{W}'$ converges to some point of $(\partial \mathcal{U}') \setminus \mathcal{C}'$. Otherwise, if we assume $\mathcal{W}'_n \ni w'_n \to x' \in (\partial \mathcal{U}') \setminus \mathcal{C}'$, then $\varphi(\mathcal{C} \times (0, \delta)) \ni f(w'_n) \to f(x')$. Since $\varphi(\mathcal{C} \times [0, \delta])$ is a closed set containing the sequence $\{f(w'_n)\}$, we can say that $f(x') \in f((\partial \mathcal{U}') \setminus \mathcal{C}') \cap \varphi(\mathcal{C} \times [0, \delta])$, which is impossible by our choice of δ.

So $\overline{\mathcal{W}'} \subseteq \mathcal{U}'$ (as $\mathcal{U}'$ is compact) but $((\partial \mathcal{U}') \setminus \mathcal{C}') \cap \overline{\mathcal{W}'} = \varnothing$. In particular, $\partial \mathcal{W}' \subseteq \mathcal{U}'$ but $((\partial \mathcal{U}') \setminus \mathcal{C}') \cap \partial \mathcal{W}' = \varnothing$.

Claim A.2.5.1 *We have $\partial \mathcal{W}' \subseteq \mathcal{C}' \cup f^{-1}(\mathcal{C}_\delta)$. Thus $\partial \mathcal{W}'$ is contained in a finite union of pairwise-disjoint circles.*

Proof of Claim A.2.5.1 Let $y' \in \partial\mathcal{W}'$ and consider a sequence $\{y'_n\} \subseteq \mathcal{W}'$ converging to y'. Then $\varphi(\mathcal{C} \times (0,\delta)) \ni f(y'_n) \to f(y') \in \varphi(\mathcal{C} \times [0,\delta])$. If $f(y') \in \varphi(\mathcal{C} \times \{0,\delta\}) = \mathcal{C} \cup \mathcal{C}_\delta$, then we are done since $f^{-1}(\mathcal{C}) \cap \mathcal{U}' = \mathcal{C}'$. On the other hand, if $f(y') \in \varphi(\mathcal{C} \times (0,\delta))$, then the definition of $\mathcal{W}'$ and $\mathcal{W}' \cap \partial\mathcal{W}' = \varnothing$ (as $\mathcal{W}'$ is open) together imply $y' \in \mathcal{U}' \setminus \mathrm{int}(\mathcal{U}') = \partial\mathcal{U}'$, ie $y' \in \mathcal{C}'$ as $((\partial\mathcal{U}') \setminus \mathcal{C}') \cap \partial\mathcal{W}' = \varnothing$. Since $y' \in \partial\mathcal{W}'$ is arbitrary, we are done. □

The definition of $\mathcal{W}'$ tells us that each point of $\mathrm{int}(\mathcal{U}')$ that is sufficiently near to $\mathcal{C}'$ must belong to $\mathcal{W}'$. Now, using Claim A.2.5.1, we can say that there is at least one component of $f^{-1}(\mathcal{C}_\delta)$ which cobounds an annulus with $\mathcal{C}'$ inside $\mathcal{U}'$. Of all the $\mathcal{C}'$–parallel components of $f^{-1}(\mathcal{C}_\delta)$, we consider the closest to $\mathcal{C}'$ as $\mathcal{C}'_\delta$. □

Proof of Theorem A.2.5(2) Certainly $f(\mathcal{A}') \subseteq f(\mathcal{U}') \subseteq \varphi(\mathcal{C} \times [0,\varepsilon])$. The rest follows, once we observe that, removing the interiors of all disks bounded by components of $f^{-1}(\mathcal{C}_\delta)$ from $\mathcal{A}'$, $\mathcal{A}'$ remains connected, so by continuity of $f|\Sigma' \setminus f^{-1}(\mathcal{C} \cup \mathcal{C}_\delta) \to \Sigma \setminus (\mathcal{C} \cup \mathcal{C}_\delta)$ it maps into $\varphi(\mathcal{C} \times (0,\delta))$. □

Acknowledgments

I want to thank my thesis adviser Siddhartha Gadgil for suggesting the proper rigidity problem, lots of helpful discussions and considerable improvements to the first draft of this paper. I also want to thank Ajay Kumar Nair for several fruitful discussions. I am grateful to the referee for the careful reading of the paper and the comments and detailed suggestions, which helped considerably in improving the manuscript. This work is supported by a scholarship from the National Board for Higher Mathematics reference number 0203/14/2019/R&D-II/7481.

References

[1] **Y Algom-Kfir**, **M Bestvina**, *Groups of proper homotopy equivalences of graphs and Nielsen realization* (2021) arXiv 2109.06908 To appear in Contemp. Math., Amer. Math. Soc.

[2] **V Álvarez**, **J M Rodríguez**, *Structure theorems for Riemann and topological surfaces*, J. Lond. Math. Soc. 69 (2004) 153–168 MR Zbl

[3] **R Ayala**, **E Dominguez**, **A Márquez**, **A Quintero**, *Proper homotopy classification of graphs*, Bull. Lond. Math. Soc. 22 (1990) 417–421 MR Zbl

[4] **M Barros**, *Classificação das superfícies não compactas cujo bordo é reunião de curvas de Jordan*, Anais da Faculdade de Ciências 57, Univ. Porto (1974) 33–41 Zbl

[5] **M Brittenham**, *π_1–injective, proper maps of open surfaces*, unpublished (1989)

[6] **M Brittenham**, *Essential laminations and deformations of homotopy equivalences: from essential pullback to homeomorphism*, Topology Appl. 60 (1994) 249–265 MR Zbl

[7] **E M Brown**, *Proper homotopy theory in simplicial complexes*, from "Topology Conference" (R F Dickman, Jr, P Fletcher, editors), Lecture Notes in Math. 375, Springer (1974) 41–46 MR Zbl

[8] **E M Brown**, **R Messer**, *The classification of two-dimensional manifolds*, Trans. Amer. Math. Soc. 255 (1979) 377–402 MR Zbl

[9] **E M Brown**, **T W Tucker**, *On proper homotopy theory for noncompact 3–manifolds*, Trans. Amer. Math. Soc. 188 (1974) 105–126 MR Zbl

[10] **S Chang**, **S Weinberger**, *Modular symbols and the topological nonrigidity of arithmetic manifolds*, Comm. Pure Appl. Math. 68 (2015) 2022–2051 MR Zbl

[11] **S Chang**, **S Weinberger**, *A course on surgery theory*, Ann. of Math. Stud. 211, Princeton Univ. Press (2021) MR Zbl

[12] **M Dehn**, *Papers on group theory and topology*, Springer (1987) MR Zbl

[13] **A L Edmonds**, *Deformation of maps to branched coverings in dimension two*, Ann. of Math. 110 (1979) 113–125 MR Zbl

[14] **C H Edwards, Jr**, *Open 3–manifolds which are simply connected at infinity*, Proc. Amer. Math. Soc. 14 (1963) 391–395 MR Zbl

[15] **D B A Epstein**, *Curves on 2–manifolds and isotopies*, Acta Math. 115 (1966) 83–107 MR Zbl

[16] **D B A Epstein**, *The degree of a map*, Proc. Lond. Math. Soc. 16 (1966) 369–383 MR Zbl

[17] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Math. Ser. 49, Princeton Univ. Press (2012) MR Zbl

[18] **F T Farrell**, **L E Jones**, *Topological rigidity for compact non-positively curved manifolds*, from "Differential geometry: riemannian geometry" (R Greene, S-T Yau, editors), Proc. Sympos. Pure Math. 54.3, Amer. Math. Soc., Providence, RI (1993) 229–274 MR Zbl

[19] **F T Farrell**, **L R Taylor**, **J B Wagoner**, *The Whitehead theorem in the proper category*, Compos. Math. 27 (1973) 1–23 MR Zbl

[20] **M H Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982) 357–453 MR Zbl

[21] **D Gabai**, **G R Meyerhoff**, **N Thurston**, *Homotopy hyperbolic 3–manifolds are hyperbolic*, Ann. of Math. 157 (2003) 335–431 MR Zbl

[22] **M E Goldman**, *Open surfaces and an algebraic study of ends*, PhD thesis, Yale University (1967) MR Available at `https://www.proquest.com/docview/288355577`

[23] **M E Goldman**, *An algebraic classification of noncompact 2–manifolds*, Trans. Amer. Math. Soc. 156 (1971) 241–258 MR Zbl

[24] **C R Guilbault**, *Ends, shapes, and boundaries in manifold topology and geometric group theory*, from "Topology and geometric group theory" (M W Davis, J Fowler, J-F Lafont, I J Leary, editors), Springer Proc. Math. Stat. 184, Springer (2016) 45–125 MR Zbl

[25] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[26] **B Hughes**, **A Ranicki**, *Ends of complexes*, Cambridge Tracts in Math. 123, Cambridge Univ. Press (1996) MR Zbl

[27] **B Kerékjártó**, *Vorlesungen über Topologie, I: Flächentopologie*, Grundl. Math. Wissen. 8, Springer (1923) JFM

[28] **J M Lee**, *Introduction to smooth manifolds*, 2nd edition, Graduate Texts in Math. 218, Springer (2013) MR Zbl

[29] **D R McMillan, Jr**, *Some contractible open* 3*–manifolds*, Trans. Amer. Math. Soc. 102 (1962) 373–382 MR Zbl

[30] **R S Palais**, *When proper maps are closed*, Proc. Amer. Math. Soc. 24 (1970) 835–836 MR Zbl

[31] **T Porter**, *Proper homotopy theory*, from "Handbook of algebraic topology" (I M James, editor), North-Holland, Amsterdam (1995) 127–167 MR Zbl

[32] **M Reichbach**, *The power of topological types of some classes of* 0*–dimensional sets*, Proc. Amer. Math. Soc. 13 (1962) 17–23 MR Zbl

[33] **I Richards**, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. 106 (1963) 259–269 MR Zbl

[34] **J Rosenberg**, *C^*–algebras, positive scalar curvature, and the Novikov conjecture, III*, Topology 25 (1986) 319–336 MR Zbl

[35] **P Scott**, *An introduction to* 3*–manifolds*, course notes, University of Maryland (1974) Available at `https://www.ams.org/open-math-notes/omn-view-listing?listingId=110690`

[36] **L C Siebenmann**, *On detecting Euclidean space homotopically among topological manifolds*, Invent. Math. 6 (1968) 245–261 MR Zbl

[37] **E H Spanier**, *Algebraic topology*, McGraw-Hill, New York (1966) MR Zbl

[38] **R J Tondra**, *Homeotopy groups of surfaces whose boundary is the union of* 1*–spheres*, Fund. Math. 105 (1980) 79–85 MR Zbl

[39] **F Waldhausen**, *On irreducible* 3*–manifolds which are sufficiently large*, Ann. of Math. 87 (1968) 56–88 MR Zbl

Department of Mathematics, Indian Institute of Science
Bangalore, India

`sumantadas@iisc.ac.in`

Received: 26 June 2022 Revised: 21 May 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 8 (pages 4139–4730) 2024