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We prove that in the Cayley graph of any braid group modulo its center $B_n/Z(B_n)$, equipped with Garside’s generating set, the axes of all pseudo-Anosov braids are strongly contracting. More generally, we consider a Garside group G of finite type with cyclic center. We prove that in the Cayley graph of $G/Z(G)$, equipped with the Garside generators, the axis of any Morse element is strongly contracting. As a consequence, we prove that Morse elements act loxodromically on the additional length graph of G .

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1 Introduction

For a finitely generated group G , equipped with some fixed finite generating set, and an element $g \in G$ of infinite order, one can study the axis $\text{axis}(g) = \langle g \rangle$, seen as a set of vertices in the Cayley graph $\Gamma(G)$. There are many different ways of formalizing the idea that this axis might “look like a geodesic in a hyperbolic space”.

A particularly weak notion is that $\langle g \rangle$ is quasi-isometrically embedded in G . A particularly strong condition is that the axis is *strongly contracting*, which is equivalent to being *strongly constricting* [Arzhantseva et al. 2015]. There are many intermediate notions — for instance the axis could be hyperbolically embedded [Dahmani et al. 2017; Osin 2016; 2018], it could be rank one [Hamenstädt 2009; Sisto 2018], it could be Morse [Dahmani et al. 2017; Sisto 2016], it could be contracting in the sense of [Abbott et al. 2021; Arzhantseva et al. 2015], it could have various other contraction and divergence properties [Arzhantseva et al. 2017], or constriction properties [Arzhantseva et al. 2015].

In this paper we will be interested in two of these properties, namely the Morse property and the strong contraction property. Precise definitions can be found in Sections 3 and 5.

An element g (or its axis in the Cayley graph $\Gamma(G)$) is said to be *Morse* if this axis is quasi-isometrically embedded in $\Gamma(G)$, and if for each pair of constants (K, L) , there exists a constant $M_g^{(K,L)}$ such that every (K, L) -quasigeodesic between two points of the axis travels in an $M_g^{(K,L)}$ -neighborhood of the axis. A remarkable example of the Morse property is the result of Behrstock [2006] that pseudo-Anosov elements in mapping class groups are Morse — see also [Sisto 2016; Dahmani et al. 2017]. (In fact, [Duchin and Rafi 2009, Theorem 4.2] implies that their axes satisfy the stronger condition of being contracting.)

Saying that the axis of an element g (or indeed any other subset A of the Cayley graph $\Gamma(G)$) has the *strong contraction* property means, roughly speaking, that there is a constant C such that taking any ball in $\Gamma(G)$ disjoint from A , and projecting this ball to A via a closest-point projection, yields a subset of A of diameter at most C .

One first crucial observation about these two properties is that the Morse property is invariant under quasi-isometry (eg when looking at the axis of an element in the Cayley graphs of G with respect to two different generating sets), whereas the strong contraction/constriction property is not. One reason for the failure of quasi-isometry invariance is that these strong properties make reference to actual distances and geodesics, *not* to quasigeodesics.

There is one well-known family of groups with a very natural family of geodesics between any pair of points in the Cayley graphs, namely the Garside groups. The notions of Garside theory needed in this paper will be recalled in Section 2. We will be interested specifically in Δ -pure Garside groups of finite type.

A Garside group of finite type G is generated by a finite lattice \mathcal{D} with a top element called Δ . Garside groups are bi-automatic — in particular, every element $g \in G$ is represented by a unique word in a certain normal form, with letters in $\mathcal{D}^{\pm 1}$; these normal form words represent *geodesics* in the Cayley graph of G with respect to \mathcal{D} . We will also require that our Garside groups are Δ -pure, or equivalently Zappa–Szép indecomposable — this condition means in particular that the center of G is infinite cyclic and is generated by some power Δ^e .

The most famous examples of Garside groups of finite type are the braid groups, and more generally the Artin–Tits groups of spherical type [Brieskorn and Saito 1972; Charney 1992; Deligne 1972]. (In this setting, the Δ -purity condition is equivalent to the defining Coxeter graph being connected.)

Since the infinite subgroup $\langle \Delta^e \rangle$ of G is central, there cannot be any elements whose axes in the Cayley graph $\Gamma(G, \mathcal{D})$ are Morse or strongly contracting. Instead, we will study the axes of elements in the Cayley graph of G modulo its center. We will let $\bar{\Gamma} = \Gamma(G/Z(G), \mathcal{D})$ and we will say that an element of a Garside group G is Morse if its axis in $\bar{\Gamma}$ is Morse.

Actually, for our proof it will be technically convenient not to work with the model space $\bar{\Gamma} = \Gamma(G, \mathcal{D})/\langle \Delta^e \rangle$, but with the following quasi-isometric variation. We define \mathcal{X} to be the quotient of the Cayley graph $\Gamma(G, \mathcal{D})$ under the right $\langle \Delta \rangle$ -action: $\mathcal{X} = \Gamma(G, \mathcal{D})/\langle \Delta \rangle$. We note that this graph \mathcal{X} is the 1-skeleton of the simplicial complex previously considered in [Bestvina 1999; Charney et al. 2004]. The graph \mathcal{X} was also studied in [Dehornoy et al. 2015, VIII.3.2], under the name \mathcal{G}^0 , and in [Calvez and Wiest 2017a; 2017b].

Our main result is as follows (see Theorem 5.5 for a precise version):

Theorem 1.1 *Suppose G is a Δ -pure Garside group of finite type. Suppose g is an element of G whose axis in \mathcal{X} (or, equivalently, in $\bar{\Gamma}$) is Morse. Then this axis is strongly contracting, both as a subset of \mathcal{X} and as a subset of $\bar{\Gamma}$.*

Corollary 1.2 *Consider the braid group B_n , equipped with the generating set $\mathcal{D}_{\text{classical}}$ or $\mathcal{D}_{\text{dual}}$ coming from its classical or dual Garside structure. Then in the Cayley graphs $\Gamma(B_n/Z(B_n), \mathcal{D}_{\text{classical}})$ and $\Gamma(B_n/Z(B_n), \mathcal{D}_{\text{dual}})$, the axis of any pseudo-Anosov braid is strongly contracting.*

Remark 1.3 Here is some context for these results:

(i) Rafi and Verberne [2021] have constructed a pseudo-Anosov element of the mapping class group of the five-punctured sphere (which contains $B_4/\langle \Delta^2 \rangle$ as a subgroup of index 5), and a generating set for this mapping class group, such that the axis of this pseudo-Anosov element in the corresponding Cayley graph is not strongly contracting. Our Corollary 1.2 supports the idea that examples such as that of [Rafi and Verberne 2021] can only exist under a pathological choice of generating set.

(ii) Let S be a surface of finite type, and $\mathcal{T}(S)$ its Teichmüller space, equipped with the Teichmüller metric and the $\text{Mod}(S)$ -action. Minsky [1996] proved that the axis of every pseudo-Anosov element of $\text{Mod}(S)$ has the strong contraction property in $\mathcal{T}(S)$.

(iii) There is a hierarchy of contraction properties: strongly contracting implies contracting which in turn implies sublinearly contracting, which is equivalent to being Morse [Arzhantseva et al. 2017]; neither of the implications is an equivalence [Brady and Tran 2022; Rafi and Verberne 2021]. However, under a $\text{CAT}(0)$ -hypothesis there is a strong converse: Morse geodesics in $\text{CAT}(0)$ -spaces are strongly contracting [Cashen 2020; Sultan 2014] (as are axes of hyperbolic isometries not bounding a half-flat [Bestvina and Fujiwara 2009, Theorem 5.4]). Thus Theorem 1.1 is an indicator that Garside groups have $\text{CAT}(0)$ -like behavior; this gives further evidence that the answer to the following question may be affirmative.

Question 1.4 Are all Garside groups $\text{CAT}(0)$? (Note that it is not even known whether all braid groups are $\text{CAT}(0)$ [Brady and McCammond 2010; Haettel et al. 2016; Jeong 2023].)

Our Theorem 1.1 begs the question which elements of well-known Garside groups are Morse. For braid groups, we know the answer from [Behrstock 2006]: it's the pseudo-Anosov elements. For other irreducible Artin-Tits groups of spherical type, however, the question is open. The authors [Calvez and Wiest 2017b] hand-constructed some elements in each such group whose axes are strongly contracting, and in particular Morse. We believe, however, that all plausible candidates for being Morse really are Morse:

Conjecture 1.5 *An element a of an irreducible Artin-Tits groups of spherical type A is Morse if and only if its image in $A/Z(A)$ has virtually cyclic centralizer.*

An application of our results (indeed, the authors' original motivation for this research) concerns the additional length graph $\mathcal{C}_{\text{AL}}(G)$ [Calvez and Wiest 2017a; 2017b; 2021]. To any Garside group G

one can associate a δ -hyperbolic graph $\mathcal{C}_{\text{AL}}(G)$ whose relation to G is loosely analogous to the curve graph's relation to the mapping class group. Indeed, if B_n is the n -strand braid group, then $\mathcal{C}_{\text{AL}}(B_n)$ is conjectured to be quasi-isometric to the curve graph of the $(n+1)$ -punctured sphere.

Using Theorem 1.1 we can prove (see Theorem 6.5 and Corollary 6.7)

Proposition 1.6 *Suppose G is a Δ -pure Garside group of finite type. If g is a Morse element of G , then the action of g on $\mathcal{C}_{\text{AL}}(G)$ is loxodromic and weakly properly discontinuous. In particular, if G contains a Morse element then $\mathcal{C}_{\text{AL}}(G)$ has infinite diameter.*

The plan of this paper is as follows. In Section 2 we review some, mostly standard, elements from Garside theory which we will need. In Section 3 we recall the definition of the Morse property and prove a first contraction property for the axes of Morse elements in Garside groups. In Section 4 we define, in a Garside-theoretical fashion, a projection to the axis of any element satisfying a Garside-theoretical rigidity condition. For elements satisfying both the rigidity and the Morse condition, we strengthen our previous contraction result, and deduce that our projection is uniformly close to any closest-point projection. Section 5 contains a precise definition of the strong contraction property and the proof of Theorem 1.1. In Section 6 we present the applications of the main result to the additional length graph.

2 Garside groups

The notion of a Garside group stems from Garside's approach to solving the conjugacy problem in the braid groups [Garside 1969]. Soon generalized to Artin-Tits groups of spherical type [Brieskorn and Saito 1972; Deligne 1972], this approach was first axiomatized in [Dehornoy 2002; Dehornoy and Paris 1999] and thoroughly studied over the first decade of the 2000s. The book [Dehornoy et al. 2015] provides a comprehensive account of what is now called "Garside theory".

Definition 2.1 Let G be a group; G is a *Garside group* with *Garside structure* (G^+, Δ) if G^+ is a submonoid of G such that $G^+ \cap G^{+^{-1}} = \{1\}$ and there exists an element $\Delta \in G^+$ with the following properties:

- (1) The partial order relations \preceq and \succeq on G defined by
 - $x \preceq y$ (x is a *prefix* of y) if and only if $x^{-1}y \in G^+$,
 - $x \succeq y$ (y is a *suffix* of x) if and only if $xy^{-1} \in G^+$
 are lattice orders on G ; that is, all $x, y \in G$ admit a unique greatest common prefix $x \wedge y$, a unique greatest common suffix $x \wedge^{\uparrow} y$, a unique least common right multiple $x \vee y$ and a unique least common left multiple $x \vee^{\uparrow} y$.
- (2) The set $\mathcal{D} = \{x \in G^+ \mid x \preceq \Delta\} = \{x \in G^+ \mid \Delta \succeq x\}$ generates G^+ as a monoid and G as a group.
- (3) For all $x \in G^+ \setminus \{1\}$,

$$\|x\| = \sup\{k \mid \exists a_1, \dots, a_k \in G^+ \setminus \{1\} \text{ such that } x = a_1 \cdots a_k\} < \infty.$$

The elements of G^+ are called *positive*, Δ is the *Garside element* and the elements of \mathfrak{D} are called *simple*. The elements x of G^+ such that $\|x\| = 1$ are called *atoms* and they form a subset of \mathfrak{D} .

Given a simple element s , its *right-complement* $\partial(s)$ is defined by $\partial(s) = s^{-1}\Delta$ and its *left-complement* is defined by $\partial^{-1}(s) = \Delta s^{-1}$; both $\partial(s)$ and $\partial^{-1}(s)$ belong to \mathfrak{D} . Conjugation by Δ will be denoted by τ (that is, for $g \in G$, $\tau(g) = \Delta^{-1}g\Delta$); notice that for every simple element s , $\partial^2(s) = \partial(\partial(s)) = \tau(s)$.

We shall make the additional assumption that G is of *finite type*, ie that the set of simple elements \mathfrak{D} is finite. In this case, it follows that τ has finite order, and we will denote this order by e . The element Δ^e is then central in G . We shall further assume that G is Δ -*pure*. This property was defined in [Picantin 2001], and shown in [Gebhardt and Tawn 2016, Theorem 39] to be equivalent to indecomposability as a Zappa–Szépp product. All the reader needs to know about Δ -pure Garside groups of finite type is that their center is cyclic, generated by Δ^e , ie $Z(G) = \langle \Delta^e \rangle$ [Picantin 2001]. For instance, Artin–Tits groups of spherical type are Garside groups, and they are Δ -pure if and only if the defining Coxeter graph is connected [Picantin 2001, Proposition 4.7].

Notation 2.2 Throughout this paper, G denotes a Δ -pure Garside group of finite type. We denote by e the positive integer such that $Z(G) = \langle \Delta^e \rangle$. When we talk about Cayley graphs, it is always understood that the generating set is obtained from the set of simple elements \mathfrak{D} . We will use the notation Γ for the Cayley graph $\Gamma(G, \mathfrak{D})$. Also, we define $\bar{\Gamma} = \Gamma(G/Z(G))$, the Cayley graph of G modulo its center, with respect to the generators which are the images of \mathfrak{D} in $G/Z(G)$. The corresponding graph metrics will be denoted by d_Γ and $d_{\bar{\Gamma}}$ respectively.

To each element of G we associate three integer numbers as follows.

Definition 2.3 [El-Rifai and Morton 1994, Section 1] Let $g \in G$. The *infimum* of g is defined by $\text{inf}(g) = \max\{r \in \mathbb{Z} \mid \Delta^r \preceq g\}$, the *supremum* of g is defined by $\text{sup}(g) = \min\{s \in \mathbb{Z} \mid g \preceq \Delta^s\}$ and the *canonical length* of g is defined by $\ell(g) = \text{sup}(g) - \text{inf}(g)$.

It is well known that each element of G can be written uniquely as an irreducible fraction involving elements of G^+ —the letters D and N in the following stand for “denominator” and “numerator” respectively.

Lemma 2.4 [Charney 1999, Lemma 4.4] Let $g \in G$.

- (i) **Left-fraction** There is a unique pair of positive elements $(D_l(g), N_l(g))$ such that

$$D_l(g) \wedge N_l(g) = 1 \quad \text{and} \quad g = D_l(g)^{-1} N_l(g).$$

In particular, if c is any positive element such that cg is positive, then $c \succcurlyeq D_l(g)$.

- (ii) **Right-fraction** There is a unique pair of positive elements $(D_r(g), N_r(g))$ such that

$$D_r(g) \wedge^{\uparrow} N_r(g) = 1 \quad \text{and} \quad g = N_r(g) D_r(g)^{-1}.$$

In particular, if c is any positive element such that gc is positive, then $D_r(g) \preceq c$.

(iii) We have the equalities

$$\begin{aligned}\inf(D_l(g)) &= \inf(D_r(g)), & \sup(D_l(g)) &= \sup(D_r(g)), \\ \inf(N_l(g)) &= \inf(N_r(g)), & \sup(N_l(g)) &= \sup(N_r(g)).\end{aligned}$$

Moreover, every element of G can be decomposed as follows. Recall that a pair of simple elements $(s, t) \in \mathcal{D}^2$ is *left-weighted* if $\partial(s) \wedge t = 1$ and *right-weighted* if $\partial^{-1}(t) \wedge^{\uparrow} s = 1$.

Proposition 2.5 [Adyan 1984; Dehornoy 2002, Section 3] *Let $g \in G$. Let $p = \inf(g)$ and $r = \ell(g)$.*

- (i) *There exists a unique decomposition $g = \Delta^p s_1 \cdots s_r$, where $s_1, \dots, s_r \in \mathcal{D} \setminus \{1, \Delta\}$ and for every $1 \leq i \leq r - 1$, (s_i, s_{i+1}) is left-weighted.*
- (ii) *Similarly, there exists a unique decomposition $g = s'_r \cdots s'_1 \Delta^p$, where $s'_1, \dots, s'_r \in \mathcal{D} \setminus \{1, \Delta\}$ and for every $1 \leq i \leq r - 1$, (s'_{i+1}, s'_i) is right-weighted.*

*These decompositions are called **left normal form** and **right normal form** of g , respectively.*

Considering the latter normal forms together with the fractional decompositions of Lemma 2.4 yields a slightly different notion of normal form.

Definition 2.6 [Dehornoy 2002, Proposition 3.9] *Let $g \in G$. If $\inf(g) < 0 < \sup(g)$ and $D_l(g) = a_1 \cdots a_r$ and $N_l(g) = b_1 \cdots b_s$ are left normal forms, then the *left mixed normal form* of g is the representation $g = a_r^{-1} \cdots a_1^{-1} b_1 \cdots b_s$; similarly, the *right mixed normal form* is the representation $g = b'_s \cdots b'_1 a'^{-1}_1 \cdots a'^{-1}_r$, where $a'_r \cdots a'_1$ and $b'_s \cdots b'_1$ are the respective right normal forms of $D_r(g)$ and $N_r(g)$. If $\inf(g) \geq 0$, then the left (resp. right) mixed normal form of g coincides with the left (resp. right) normal form of g given by Proposition 2.5. If $\sup(g) \leq 0$, then the left (resp. right) mixed normal form of g is the formal inverse of the left (resp. right) normal form of g^{-1} given by Proposition 2.5.*

These mixed normal forms have an important geometric meaning:

Lemma 2.7 [Charney and Meier 2004, Lemma 3.1] *In Γ , the Cayley graph of G with respect to \mathcal{D} , mixed normal forms are geodesics.*

Finally, we shall need one more Garside-theoretical definition:

Definition 2.8 *Let $x \in G$ with right normal form $x = x_r \cdots x_1 \Delta^p$. We say that x is *right-rigid* if its preferred simple suffix $\mathfrak{p}^{\uparrow}(x) := \tau^p(x_1) \wedge^{\uparrow} \partial^{-1}(x_r)$ is trivial. In particular, if $\inf(x) = 0$, then for $k \geq 1$, the right normal form of x^k consists of the concatenation of k copies of the right normal form of x .*

Our aim is, of course, to study the geometry of G modulo its center $Z(G) = \langle \Delta^e \rangle$. However, it is technically far more convenient not to study the Cayley graph $\bar{\Gamma} = \Gamma(G/Z(G))$ directly, but a very closely related graph, which we will denote by \mathcal{X} . This graph \mathcal{X} is the 1-skeleton of “Bestvina’s normal

form complex” considered in [Bestvina 1999; Charney et al. 2004], and it has been described previously in [Calvez and Wiest 2017a; Dehornoy et al. 2015, Chapter VIII, Section 3.2]. We recall the definition:

Notation 2.9 We denote by \mathcal{X} the quotient of $\Gamma = \Gamma(G)$ by the right-action of $\langle \Delta \rangle$,

$$\mathcal{X} = \Gamma(G, \mathcal{D}) / \langle \Delta \rangle.$$

- The vertices of \mathcal{X} are the left-cosets of G modulo $\langle \Delta \rangle$, $\{g\langle \Delta \rangle \mid g \in G\}$. Each vertex $g\langle \Delta \rangle$ of \mathcal{X} possesses a unique *distinguished representative* \underline{g} , which is by definition the representative satisfying $\text{inf}(\underline{g}) = 0$: given $g \in G$, we have $\underline{g} = g\Delta^{-\text{inf}(g)}$. We denote by $*$ the vertex $\langle \Delta \rangle$, whose distinguished representative is the trivial element of G .
- Two vertices $g\langle \Delta \rangle$ and $h\langle \Delta \rangle$ of \mathcal{X} are connected by an edge if there is a proper simple element $s \in \mathcal{D}$ such that $\underline{g}s \in h\langle \Delta \rangle$; this is equivalent to the existence of a proper simple element t such that $\underline{h}t \in g\langle \Delta \rangle$.

The following provides more precise information about adjacent vertices of \mathcal{X} :

Lemma 2.10 [Bestvina 1999, Lemma 3.4] *Suppose that $g\langle \Delta \rangle$ and $h\langle \Delta \rangle$ are adjacent vertices of \mathcal{X} . Then there exists a proper simple element $s \in \mathcal{D} \setminus \{1, \Delta\}$ such that one of the following holds:*

- $\underline{g}s = \underline{h}$ (and in this case we have $\underline{h}\partial s = \underline{g}\Delta$), or
- $\underline{h}s = \underline{g}$ (and in this case we have $\underline{g}\partial s = \underline{h}\Delta$).

Notation 2.11 We denote by $d_{\mathcal{X}}$ the graph metric on the graph \mathcal{X} ; for $g, h \in G$, we sometimes write $d_{\mathcal{X}}(g, h)$ for $d_{\mathcal{X}}(g\langle \Delta \rangle, h\langle \Delta \rangle)$. Note that the groups G and $G/Z(G)$ act isometrically by left-translations on \mathcal{X} by $g \cdot (g'\langle \Delta \rangle) = (gg')\langle \Delta \rangle$.

The spaces \mathcal{X} and $\bar{\Gamma}$ are very closely related:

Proposition 2.12 *There is an isometric embedding $\iota: \mathcal{X} \hookrightarrow \bar{\Gamma}$ with $\lfloor \frac{1}{2}e \rfloor$ -dense image (ie every vertex of $\bar{\Gamma}$ is at distance at most $\lfloor \frac{1}{2}e \rfloor$ from a vertex belonging to $\iota(\mathcal{X})$). In particular, ι is a quasi-isometry.*

Proof If $g, h \in G$ represent adjacent vertices of \mathcal{X} , then we know from Lemma 2.10 that \underline{g} and \underline{h} represent adjacent vertices in the Cayley graph Γ of G , and thus also in the Cayley graph $\bar{\Gamma}$ of $G/\langle \Delta^e \rangle$. This means that the map

$$\iota: \{\text{vertices of } \mathcal{X}\} \hookrightarrow \{\text{vertices of } \bar{\Gamma}\}, \quad g\langle \Delta \rangle \mapsto \underline{g}\langle \Delta^e \rangle$$

sends adjacent vertices to adjacent vertices, and thus induces a well-defined and 1-Lipschitz map of graphs $\iota: \mathcal{X} \hookrightarrow \bar{\Gamma}$.

In the other direction, there is a natural projection map

$$p: \{\text{vertices of } \bar{\Gamma}\} \rightarrow \{\text{vertices of } \mathcal{X}\}, \quad g\langle \Delta^e \rangle \mapsto g\langle \Delta \rangle,$$

which induces a well-defined map of graphs $p: \bar{\Gamma} \rightarrow \mathcal{X}$. Both ι and p are 1-Lipschitz, and $p \circ \iota = \text{id}_{\mathcal{X}}$. This implies that ι is an isometric embedding.

Now we look at the opposite composition

$$\iota \circ p: \bar{\Gamma} \rightarrow \bar{\Gamma}, \quad g \langle \Delta^e \rangle \mapsto \underline{g} \langle \Delta^e \rangle.$$

We observe that $d_{\bar{\Gamma}}(g \langle \Delta^e \rangle, \underline{g} \langle \Delta^e \rangle) < \lfloor \frac{1}{2}e \rfloor$, which means that $\iota \circ p$ is at distance $\lfloor \frac{1}{2}e \rfloor$ from $\text{id}_{\bar{\Gamma}}$; thus the image of ι is $\lfloor \frac{1}{2}e \rfloor$ -dense. □

We now recall from [Calvez and Wiest 2017a] the notion of a preferred path between two vertices in \mathcal{X} .

Definition 2.13 Given any vertex $g \langle \Delta \rangle$ of \mathcal{X} , let $s_1 \cdots s_r$ be the left normal form of \underline{g} ; the *preferred path* between $*$ and $g \langle \Delta \rangle$ is the path

$$*, s_1 \langle \Delta \rangle, \dots, (s_1 \cdots s_r) \langle \Delta \rangle = g \langle \Delta \rangle.$$

Then, for $g, h \in G$, we denote by $A(g, h)$ the g -left translate of the preferred path between $*$ and $g^{-1}h \langle \Delta \rangle$ — so $A(g, h)$ is a path in \mathcal{X} with starting point $g \langle \Delta \rangle$ and end point $h \langle \Delta \rangle$.

Remark 2.14 If $g' \in g \langle \Delta \rangle$ and $h' \in h \langle \Delta \rangle$, then the paths $A(g, h)$ and $A(g', h')$ coincide, so we have a preferred path $A(g \langle \Delta \rangle, h \langle \Delta \rangle)$ between any pair of points $g \langle \Delta \rangle, h \langle \Delta \rangle$ in \mathcal{X} .

Here is a proof of this coincidence of the two paths: Suppose the left normal form of $\underline{g^{-1}h}$ is $s_1 \cdots s_n$. Then by definition, the i^{th} vertex on the path $A(g, h)$ in \mathcal{X} is represented by $g \cdot s_1 \cdots s_i$. Let us calculate, for comparison, the i^{th} vertex on the path $A(g\Delta^k, h\Delta^\ell)$. We have

$$\underline{(g\Delta^k)^{-1}h\Delta^\ell} = \underline{\Delta^{-k} \cdot g^{-1}h} = \underline{\tau^k(g^{-1}h)} = \tau^k(s_1) \cdots \tau^k(s_n).$$

The i^{th} vertex of the $g\Delta^k$ -translate of $A(*, \underline{(g\Delta^k)^{-1}h\Delta^\ell})$ is thus represented by

$$g\Delta^k \cdot \tau^k(s_1) \cdots \tau^k(s_i) = g \cdot s_1 \cdots s_i \cdot \Delta^k,$$

which represents the same point of \mathcal{X} .

We record some basic properties of the preferred paths (recall that for $g, h \in G$, $g \wedge h$ is the unique greatest common prefix of g and h):

Proposition 2.15 (i) Let $g, h \in G$; let $p = \underline{g \wedge h}$. The preferred path $A(g, h)$ is the concatenation of the preferred paths $A(g, p)$ and $A(p, h)$.

(ii) Preferred paths are symmetric: for all $g, h \in G$, $A(g, h)$ is the reverse of $A(h, g)$.

(iii) Preferred paths are preserved by left-translation: for all $g, h, k \in G$, $A(kg, kh) = kA(g, h)$.

(iv) Preferred paths are geodesics in \mathcal{X} and for all $g, h \in G$, $d_{\mathcal{X}}(g, h) = d_{\Gamma}(\underline{g}, \underline{h})$.

(v) Balls in \mathcal{X} are convex: if $g, h \in G$ and $k \langle \Delta \rangle \in A(g, h)$, then

$$d_{\mathcal{X}}(*, k \langle \Delta \rangle) \leq \max(d_{\mathcal{X}}(*, g \langle \Delta \rangle), d_{\mathcal{X}}(*, h \langle \Delta \rangle)).$$

Proof Parts (i) and (ii) correspond to [Calvez and Wiest 2017a, Lemma 4] and [Calvez and Wiest 2017a, Lemma 5], respectively. For (iii), write the left normal form of $\underline{g}^{-1}h$ as $z_1 \cdots z_r$, so that $A(g, h)$ is by definition the path

$$g\langle\Delta\rangle = \underline{g}\langle\Delta\rangle, \underline{g}z_1\langle\Delta\rangle, \dots, \underline{g}z_1 \cdots z_r\langle\Delta\rangle = h\langle\Delta\rangle.$$

Note that $\underline{kg} = k\underline{g}\Delta^j$ (with $j = \inf(g) - \inf(kg)$); therefore the left normal form of $\underline{(kg)^{-1}kh}$ is $\tau^j(z_1) \cdots \tau^j(z_r)$. So by definition, $A(kg, kh)$ is the path

$$kg\langle\Delta\rangle = \underline{kg}\langle\Delta\rangle, \underline{kg}\tau^j(z_1)\langle\Delta\rangle, \dots, \underline{kg}\tau^j(z_1 \cdots z_r)\langle\Delta\rangle,$$

but for all $1 \leq i \leq r$, we have

$$\underline{kg}\tau^j(z_1 \cdots z_i) = k\underline{g}\Delta^j \Delta^{-j} z_1 \cdots z_i \Delta^j = k\underline{g}z_1 \cdots z_i \Delta^j.$$

Thus $A(kg, kh)$ is the path

$$kg\langle\Delta\rangle, k\underline{g}z_1\langle\Delta\rangle, \dots, k\underline{g}z_1 \cdots z_r\langle\Delta\rangle,$$

the k -left translate of $A(g, h)$, as claimed.

To see (iv), recall first that mixed normal forms are geodesics in Γ (Lemma 2.7). Let $g, h \in G$. It is shown in the proof of [Calvez and Wiest 2017a, Lemma 4] that the path $A(g, h)$ in \mathcal{X} has the exact length of the mixed normal form of $\underline{g}^{-1}h$, say r . Suppose that there was a shorter path in \mathcal{X} between $g\langle\Delta\rangle$ and $h\langle\Delta\rangle$, that is a sequence of vertices $g\langle\Delta\rangle = g_0\langle\Delta\rangle, g_1\langle\Delta\rangle, \dots, g_k\langle\Delta\rangle = h\langle\Delta\rangle$, with $k < r$. Then by Lemma 2.10 there are simple elements s_1, \dots, s_k such that for $1 \leq i \leq k$, we have $\underline{g}_i = \underline{g}_{i-1}s_i$ or $\underline{g}_i = \underline{g}_{i-1}s_i^{-1}$; setting t_i to be s_i or s_i^{-1} accordingly, we obtain that $\underline{h} = \underline{g}_k = \underline{g}_0 t_1 \cdots t_k = \underline{g} t_1 \cdots t_k$, where each t_i is either a simple element or the inverse of a simple element. This contradicts the fact that mixed normal forms are geodesics in Γ .

Now, let us prove (v). In view of (iv), the distances involved are the respective lengths of the left normal forms of \underline{k} , \underline{g} and \underline{h} . Let $p = \underline{g} \wedge \underline{h}$; write $\underline{g} = pa$ and $\underline{h} = pb$ (with $a, b \in G^+$), and let $a = a_1 \cdots a_r$ and $b = b_1 \cdots b_s$ be the respective left normal forms. By the proof of [Calvez and Wiest 2017a, Lemma 4], the distinguished representatives of the vertices along the path $A(g, h)$ are (in this order)

$$\underline{g} = pa_1 \cdots a_r, \dots, pa_1, p, pb_1, \dots, pb_1 \cdots b_s = \underline{h}.$$

Any of these is a prefix of \underline{g} or \underline{h} ; thus its left normal form is at most as long as that of \underline{g} or \underline{h} and the claim is proved. \square

Lemma 2.16 (fellow traveler property) *Suppose that $g, g', h, h' \in G$ are such that $d_{\mathcal{X}}(g, g') = 1$ and $d_{\mathcal{X}}(h, h') = 1$. Then the set of vertices along the path $A(g, h)$ and the set of vertices along the path $A(g, h')$ in \mathcal{X} are at Hausdorff distance 1. Also, the analogous statement holds for the paths $A(g, h)$ and $A(g', h)$.*

Proof By symmetry of preferred paths (Proposition 2.15(ii)), the second statement follows from the first. After a left translation, we may assume $g = 1$, so we must show the claim for $A(*, h\langle\Delta\rangle)$ and $A(*, h'\langle\Delta\rangle)$.

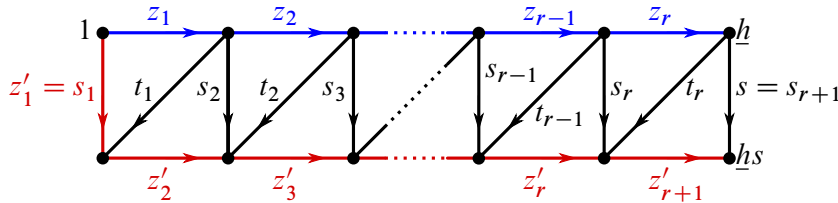


Figure 1: The normal form paths of \underline{h} (in blue) and of $\underline{h}s$ (in red) stay at Hausdorff distance 1 in the Cayley graph of G .

We may assume from the hypothesis (and Lemma 2.10) that there exists a simple element s such that $\underline{h}s = \underline{h}'$. Let $z_1 \cdots z_r$ be the left normal form of \underline{h} . The vertices along $A(*, h\langle\Delta\rangle)$ are $z_1 \cdots z_i\langle\Delta\rangle$, for $0 \leq i \leq r$.

Now, the left normal form of $\underline{h}' = \underline{h}s$ may be obtained, for instance, from the algorithm given in [Gebhardt and González-Meneses 2010, Proposition 1.2], which is illustrated in Figure 1. It goes as follows. Set $s_{r+1} = s$; for $i = r, \dots, 1$, define recursively the simple elements $t_i = \partial(z_i) \wedge s_{i+1}$, $s_i = z_i t_i$ and $z'_{i+1} = t_i^{-1} s_{i+1}$. Finally define $z'_1 = s_1$. Then $z'_1 \cdots z'_r z'_{r+1}$ (or $z'_1 \cdots z'_r$, if $z'_{r+1} = 1$) is the left normal form of \underline{h} .

We then see that for $i = 1, \dots, r$, we have $z'_1 \cdots z'_i = z_1 \cdots z_i t_i$; hence the $(i + 1)^{\text{st}}$ vertex along $A(*, h'\langle\Delta\rangle)$ is at distance 1 from the $(i + 1)^{\text{st}}$ vertex along $A(*, h\langle\Delta\rangle)$. □

Lemma 2.17 (concatenation of normal form paths) *Let $g, h, k \in G$ be such that $g \preceq \underline{h} \preceq \underline{k}$. Then:*

- (i) *The concatenation of the paths $A(g, h)$ and $A(h, k)$ is a $(2, 0)$ -quasigeodesic connecting $g\langle\Delta\rangle$ to $k\langle\Delta\rangle$.*
- (ii) *Suppose in addition that the paths $A(g, h)$ and $A(g, k)$ have the same length; then the concatenation of $A(h, k)$ and $A(k, g)$ is a $(2, 0)$ -quasigeodesic connecting $h\langle\Delta\rangle$ to $g\langle\Delta\rangle$.*

Proof Let $a = g^{-1}\underline{h}$, $b = \underline{h}^{-1}\underline{k}$ and $c = g^{-1}\underline{k}$. Under our hypothesis, $\inf(a) = \inf(b) = \inf(c) = 0$ and $c = ab$. Denote by $a = a_1 \cdots a_r$ and $b = b_1 \cdots b_s$ the respective left normal forms.

(i) Denote by α the concatenation of paths under consideration. Let $p\langle\Delta\rangle$ be the k^{th} vertex on $A(g, h)$, for some $1 \leq k \leq r$; let $q\langle\Delta\rangle$ be the $(l + 1)^{\text{st}}$ vertex on $A(h, k)$, for some $1 \leq l \leq s$. The distance $d_{\mathcal{X}}(p, q)$ is the length of $A(p, q)$, that is $\sup(\underline{p}^{-1}q) = \sup(a_k \cdots a_r b_1 \cdots b_l)$. As the subwords $a_k \cdots a_r$ and $b_1 \cdots b_l$ are in left normal form, we obtain $\overline{d_{\mathcal{X}}}(p, q) \geq \max\{r - k + 1, l\}$. Note that the portion of α between $p\langle\Delta\rangle$ and $q\langle\Delta\rangle$ has length $r - k + 1 + l$. We then obtain

$$\frac{1}{2}(r - k + 1 + l) \leq \max\{r - k + 1, l\} \leq d_{\mathcal{X}}(p, q) \leq r - k + 1 + l \leq 2(r - k + 1 + l);$$

hence α is a $(2, 0)$ -quasigeodesic.

(ii) The extra hypothesis is saying that $\sup(a) = \sup(c)$. Consider the vertices $g'\langle\Delta\rangle = \underline{h}^{-1}g\langle\Delta\rangle$, $h'\langle\Delta\rangle = \underline{h}^{-1}h\langle\Delta\rangle = *$ and $k'\langle\Delta\rangle = \underline{h}^{-1}k\langle\Delta\rangle$; our goal is to show that the concatenation of $A(h', k')$

and $A(k', g')$ is a $(2, 0)$ -quasigeodesic connecting $h'\langle\Delta\rangle$ to $g'\langle\Delta\rangle$. Let us compute the respective distinguished representatives: we have $\underline{h}' = 1$, $\underline{k}' = b$ and

$$\underline{g}' = \underline{h}^{-1}g = \underline{a}^{-1} = a^{-1}\Delta^{\text{sup}(a)}.$$

Moreover, $a^{-1}\Delta^{\text{sup}(a)} = b(c^{-1}\Delta^{\text{sup}(a)}) = b(c^{-1}\Delta^{\text{sup}(c)})$ and $c^{-1}\Delta^{\text{sup}(c)}$ is positive, so $b \preceq \underline{g}'$. We thus have $\underline{h}' \preceq \underline{k}' \preceq \underline{g}'$ and we are in a position to apply Lemma 2.17(i) to obtain the desired claim. \square

3 Morse elements in Garside groups

In this section we will define Morse elements in Garside groups, and prove some preliminary results on them. First, we set some notation and we recall the definition. We recall that throughout, G denotes a Δ -pure Garside group of finite type.

Notation 3.1 Let $g \in G$.

- The *axis* of g in \mathcal{X} is the set of vertices $\{g^t\langle\Delta\rangle \mid t \in \mathbb{Z}\}$.
- The *axis* of g in $\bar{\Gamma}$ is the set of vertices $\{g^t\langle\Delta\rangle \mid t \in \mathbb{Z}\}$.

Remark 3.2 We caution the reader that two elements g and $g\Delta$ may have completely different axes in $G/Z(G)$, and thus in \mathcal{X} . For instance in braid groups, it may very well happen that some element g is pseudo-Anosov while $g\Delta$ is reducible.

Definition 3.3 (a) Let (X, d) be a metric space, and let $\gamma: \mathbb{Z} \rightarrow X$ be a map. We say that γ (or equivalently its image) is *Morse* if

- (i) γ is a quasi-isometric embedding, and
- (ii) for every pair (Λ, K) with $\Lambda \geq 1$ and $K \geq 0$, there is a number $M^{(\Lambda, K)}$ (the Morse constant) such that every (Λ, K) -quasigeodesic in X connecting two points, $\gamma(i)$ and $\gamma(j)$ (for $i, j \in \mathbb{Z}$), of the image of γ remains in the $M^{(\Lambda, K)}$ -neighborhood of the image of γ in X .

- (b) Let H be a group generated by a finite set S . An infinite order element $h \in H$ is *Morse* if the map $\gamma: \mathbb{Z} \rightarrow \text{Cay}(H, S)$, $t \mapsto h^t$, is Morse in the sense of (a). In this situation, given $\Lambda \geq 1$ and $K \geq 0$ we denote by $M_h^{(\Lambda, K)}$ the associated Morse constant.

Notation 3.4 Since many quasigeodesics in this paper will be $(2, 0)$ -quasigeodesics, we will use the simplified notation $M_h = M_h^{(2, 0)}$.

Since a Δ -pure Garside group of finite type has an infinite-cyclic center, it cannot contain any Morse elements. We adapt the definition as follows, keeping in mind Proposition 2.12 which says that the projection $\bar{\Gamma} \rightarrow \mathcal{X}$, $g\langle\Delta^e\rangle \mapsto g\langle\Delta\rangle$, is a quasi-isometry and the fact that the Morse property is invariant under quasi-isometry.

Definition 3.5 We say that an element g of a Δ -pure Garside group of finite type G is *Morse* if any of the following equivalent condition holds:

- (i) the image of g in $G/Z(G)$ is Morse in the sense of Definition 3.3(b);
- (ii) the axis of g in $\bar{\Gamma}$ is Morse in the sense of Definition 3.3(a);
- (iii) the axis of g in \mathcal{X} is Morse in the sense of Definition 3.3(a).

Example 3.6 Pseudo-Anosov braids are Morse; indeed, their projections to the group $B_n/Z(B_n)$, which is a finite-index subgroup of the mapping class group of an $(n+1)$ -times punctured sphere, are pseudo-Anosov mapping classes in this group. By [Behrstock 2006], these are Morse.

The following is a key technical result; recall that the notion of right-rigidity is introduced in Definition 2.8.

Proposition 3.7 *Every Morse element of G has a power which is conjugate to a right-rigid element. Moreover, this right-rigid element can be required to be of the form $\Delta^{e \cdot m} x$, with $m \in \mathbb{Z}$, $\text{inf}(x) = 0$, and x right-rigid.*

In order to prepare the proof of Proposition 3.7, we need the following lemma. It is well known to experts (see eg [Tran 2019, Corollary 4.16]), but for the sake of completeness we will give an elementary proof.

Lemma 3.8 *Let H be a group generated by a finite set S . If $h \in H$ is a Morse element in the sense of Definition 3.3(b), then $\langle h \rangle$ has finite index in the centralizer $Z(h)$ in H .*

Proof Denote by d the word distance associated to S in H . Let $A \geq 1$ and $B \geq 0$ such that $t \mapsto h^t$ is an (A, B) -quasi-isometric embedding of \mathbb{Z} in $\text{Cay}(H, S)$. Let $M = M_h^{(4A, B)}$.

We claim that if $zh = hz$, then $d(z, \langle h \rangle) \leq M$. Granted this claim, we then see that for all $z \in Z(h)$, there is some $m \in \mathbb{Z}$ such that $d(z, h^m) = d(1, z^{-1}h^m) \leq M$, that is the coset $z^{-1}\langle h \rangle$ has a representative of word length at most M . It follows that each coset of $Z(h)$ modulo $\langle h \rangle$ has a representative of word length at most M , so there are only finitely many cosets.

To prove the claim, assume on the contrary that $d(z, \langle h \rangle) = K > M$, for some $z \in Z(h)$. Let r be such that $d(z, h^r) = K$; up to replacing z by $h^{-r}z$, we may assume that $K = d(z, \langle h \rangle) = d(z, 1)$. Choose N big enough that $d = d(h^{-N}, h^N) > 4K$. Then we piecewise define a map $\gamma: \{0, \dots, 2K + 2N\} \rightarrow \text{Cay}(H, S)$ as follows:

- For $0 \leq t \leq K$, $\gamma(t) = \gamma_1(t)$ follows a geodesic from h^{-N} to zh^{-N} .
- For $K \leq t \leq K + 2N$, $\gamma(t) = \gamma_2(t) = zh^{t-K-N}$.
- For $K + 2N \leq t \leq 2K + 2N$, $\gamma(t) = \gamma_3(t)$ follows a geodesic from zh^N to h^N .

To conclude, we shall observe that γ defines a $(4A, B)$ -quasigeodesic with endpoints h^{-N} and h^N on the axis of h , hence contradicting the hypothesis that h is Morse (as the vertex $z = \gamma(K + N)$ does not lie in the M -neighborhood of $\langle h^n, n \in \mathbb{Z} \rangle$).

Let $0 \leq s, t \leq 2K + 2N$. The upper bound $d(\gamma(s), \gamma(t)) \leq A|s - t| + B$ follows immediately from the fact that γ is a juxtaposition of geodesics and an (A, B) -quasi-geodesic. To obtain the lower bound, we consider two cases.

First, let $0 \leq s \leq K$ and $K \leq t \leq K + 2N$ (the situation is symmetric for $K + 2N \leq s \leq 2K + 2N$). Because $zh^{-N} = \gamma(K)$ realizes the minimum possible distance between h^{-N} and $\{zh^n, n \in \mathbb{Z}\}$, we have $d(\gamma(s), \gamma(t)) \geq d(\gamma(s), \gamma(K))$. Then we have

$$d(\gamma(s), \gamma(K)) + d(\gamma(K), \gamma(t)) \leq 2d(\gamma(s), \gamma(K)) + d(\gamma(s), \gamma(t)) \leq 3d(\gamma(s), \gamma(t))$$

and the lower bound $d(\gamma(s), \gamma(t)) \geq \frac{1}{3A}|s - t| - \frac{1}{3}B$ follows.

Second, let $0 \leq s \leq K$ and $K + 2N \leq t \leq 2K + 2N$. By our choice of N , we have

$$d(\gamma(s), \gamma(t)) \geq \frac{1}{2}d \geq \frac{1}{4}d + K \geq \frac{1}{4}d + \frac{1}{2}K \geq \frac{1}{4}d + \frac{1}{4}d(\gamma(s), zh^{-N}) + \frac{1}{4}d(\gamma(t), zh^N)$$

and we obtain the lower bound $d(\gamma(s), \gamma(t)) \geq \frac{1}{4A}|t - s| - \frac{1}{4}B$. □

Proof of Proposition 3.7 For the proof of the first sentence, we recall the following result, proven (although not stated in this form) in [Birman et al. 2007, Theorem 3.23 and Corollary 3.24]: Suppose that a certain element of a Garside group is not a root of a central element (infinite order condition), and that every element commuting with it has a common power with it, up to multiplication by a central power of Δ (small centralizer condition). Then the element has a power which is conjugate to a right-rigid element. This result from [Birman et al. 2007] implies the first sentence, because any Morse element satisfies the infinite order condition, and it satisfies the small centralizer condition by Lemma 3.8.

For the proof of the second sentence, we simply remark that for a *rigid* element g and an integer k ,

$$\inf(g^k) = k \cdot \inf(g).$$

Thus by taking a further power, we can achieve that the infimum of the rigid conjugate is a multiple of e . □

Remark 3.9 There is an alternative proof of Proposition 3.7, which does not use Lemma 3.8 or the paper [Birman et al. 2007], but which is rather reminiscent of the “pumping lemma” [Epstein et al. 1992, Theorem 1.2.13]. The idea is that the right normal form of g^n has to stay close to the axis of the Morse element g . Now right normal forms belong to a language recognized by a finite state automaton; thus for large enough n , this right normal form has a middle segment with periodic behavior. This periodic segment is a right-rigid conjugate of a power of g . We leave the details as an exercise.

Notation 3.10 From here on, all diagrams in this paper will take place in \mathcal{X} (not in the Cayley graph $\Gamma(G)$ or in $\bar{\Gamma} = \Gamma(G/Z(G))$). Also, in the diagrams, we simplify the notation, labeling a vertex g if it is represented by a group element g — strictly speaking, it should be labeled $g\langle\Delta\rangle$.

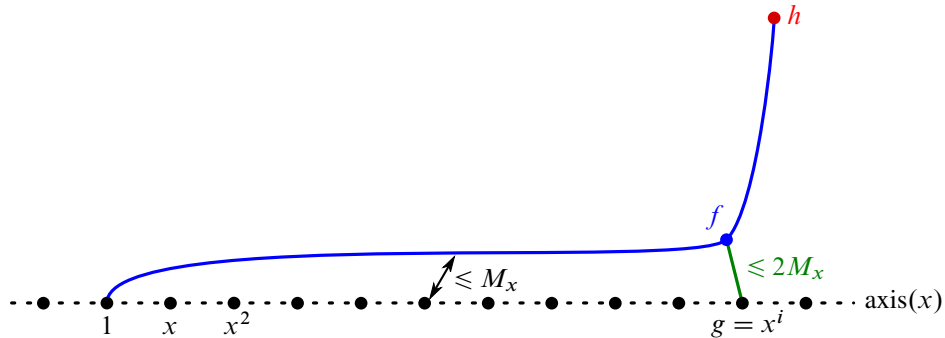


Figure 2: The statement of Proposition 3.11. Here the segment from $*$ to $f\langle\Delta\rangle$, with $f = \underline{h} \wedge \Delta^{\ell_i}$, has the same length as the segment from $*$ to $g\langle\Delta\rangle$, namely ℓ_i .

Proposition 3.11 (preferred paths stay close to the axis) *Suppose x is a Morse element of G satisfying $\inf(x^k) = 0$ for every positive integer k . For $i \in \mathbb{N}$, define $\ell_i = \ell(x^i)$. Let $h \in G$ and suppose that there exists an integer $i \geq 0$ such that $x^i \preceq \underline{h}$. Consider the initial segment of the path $A(1, h)$ in \mathcal{X} of length ℓ_i (the same length as x^i). Then this segment stays within distance M_x from the axis of x :*

$$\text{if } 0 \leq k \leq \ell_i \text{ then } d_{\mathcal{X}}((\underline{h} \wedge \Delta^k)\langle\Delta\rangle, \text{axis}(x)) \leq M_x.$$

Moreover,

$$d_{\mathcal{X}}((\underline{h} \wedge \Delta^{\ell_i})\langle\Delta\rangle, x^i\langle\Delta\rangle) \leq 2M_x.$$

Proof Let $f = \underline{h} \wedge \Delta^{\ell_i}$ and $g = x^i$. Of course, $\underline{f} = f$, and our first hypothesis on x says that $\underline{g} = g$. Because $x^i \preceq \underline{h}$, we have $1 \preceq \underline{g} \preceq \underline{f}$ and by construction of f , the paths $A(1, f)$ and $A(1, g)$ have the same length. By Lemma 2.17(ii), the concatenation of paths $A(1, f)$, followed by $A(f, g)$ is a $(2, 0)$ -quasigeodesic. This yields the first statement, recalling that $M_x = M_x^{(2,0)}$ is the Morse constant.

For the second statement, we notice that in particular, the vertex $f\langle\Delta\rangle$ is at distance at most M_x from some point $x^k\langle\Delta\rangle$ on the axis. By the triangle inequality,

$$\ell_i - M_x \leq d_{\mathcal{X}}(*, x^k\langle\Delta\rangle) \leq \ell_i + M_x,$$

and therefore $x^k\langle\Delta\rangle$ lies in an M_x -neighborhood of $g\langle\Delta\rangle$. We conclude that

$$d_{\mathcal{X}}(f\langle\Delta\rangle, g\langle\Delta\rangle) \leq d_{\mathcal{X}}(f\langle\Delta\rangle, x^k\langle\Delta\rangle) + d_{\mathcal{X}}(x^k\langle\Delta\rangle, g\langle\Delta\rangle) \leq 2M_x. \quad \square$$

4 Projection to the axis

In this section, we will define, in Garside-theoretical terms, a projection from \mathcal{X} to the axis of any element x of G , provided that $\inf(x) = 0$ and that x satisfies the additional hypothesis of being *right-rigid* (Definition 2.8). If, moreover, x is Morse, then this projection satisfies a contraction property (Proposition 4.8) which extends Proposition 3.11. This will be sufficient for deducing that our projection coincides, up to a bounded error, with any closest point projection (Corollary 4.9).

Remark 4.1 We remark that the occurrence of the condition of right-rigidity in our context is quite surprising, as this is a condition on the *right* normal form of x , whereas otherwise, we are generally using *left* normal forms throughout this paper.

Lemma 4.2 *Let $x \in G \setminus \{1_G\}$ be such that $\inf(x^k) = 0$ for all $k \in \mathbb{N}$. Let $h \in G$. Then the set $\{k \in \mathbb{Z} \mid x \not\leq \underline{x^k h}\}$ is nonempty and bounded above.*

In other words, we are claiming that there are indices k with $x \not\leq \underline{x^k h}$, but if k is sufficiently large then $x \leq \underline{x^k h}$.

Proof We will study the bi-infinite sequence $(\inf(x^k h))_{k \in \mathbb{Z}}$. This sequence is nondecreasing: indeed, by [El-Rifai and Morton 1994, Section 1], for any group elements g and h , we have $\inf(g \cdot h) \geq \inf(g) + \inf(h)$; in particular, $\inf(x^{k+1} h) \geq \inf(x) + \inf(x^k h) \geq \inf(x^k h)$.

The same argument yields the inequality $\inf(x^k) \geq \inf(x^k h) + \inf(h^{-1}) = \inf(x^k h) - \sup(h)$, which we will be using in the equivalent form

$$\inf(x^k h) \leq \inf(x^k) + \sup(h).$$

Let us now prove that the set $\{k \in \mathbb{Z} \mid x \not\leq \underline{x^k h}\}$ is nonempty. The preceding inequality implies that $\lim_{k \rightarrow -\infty} \inf(x^k h) = -\infty$, and in particular that there exists some index k with $\inf(x^{k-1} h) < \inf(x^k h)$.

Now if an index k has the property that $x \leq \underline{x^k h}$ then $\inf(x^{-1} \underline{x^{-k} h}) = 0$. However,

$$x^{-1} \underline{x^{-k} h} = x^{-1} \cdot x^{-k} h \cdot \Delta^{-\inf(x^{-k} h)} = x^{-k-1} h \cdot \Delta^{-\inf(x^{-k} h)}$$

and thus

$$\inf(x^{-k-1} h) = \inf(x^{-1} \underline{x^{-k} h}) + \inf(x^{-k} h) = \inf(x^{-k} h).$$

As seen above, this cannot be true for all indices k , so there must exist an index k with $x \not\leq \underline{x^k h}$.

We now turn to the proof that, by contrast, $x \leq \underline{x^k h}$ for sufficiently large values of k . In fact, we will prove the stronger result that there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, $x^{k-k_0} \leq \underline{x^k h}$.

We look at positive indices k , for which, we recall, $\inf(x^k) = 0$. As seen above, we have

$$\inf(x^k h) \leq \inf(x^k) + \sup(h) = \sup(h) \text{ for } k \geq 0.$$

Thus the increasing integer sequence $(\inf(x^k h))_{k \in \mathbb{N}}$ is bounded above by $\sup(h)$, and hence it is eventually constant: there exist $k_0 \in \mathbb{N}$ and $n_0 \in \mathbb{Z}$, with $n_0 \leq \sup(h)$, such that for all $k \geq k_0$, $\inf(x^k h) = n_0$. Now, for $k \geq k_0$,

$$\underline{x^k h} = x^k h \Delta^{-n_0} = x^{k-k_0} x^{k_0} h \Delta^{-n_0} = x^{k-k_0} \underline{x^{k_0} h}$$

which finishes the proof. □

Definition 4.3 (Garside-theoretical projection to the axis of x) Let $x \in G$ be such that $\inf(x^k) = 0$ for all $k \in \mathbb{N}$.

(a) Let $h \in G$. We define the integer

$$\lambda(h) = -\max\{k \in \mathbb{Z}, x \not\leq \underline{x^k h}\}$$

and we note that for any $t \in \mathbb{Z}$, $\lambda(h\Delta^t) = \lambda(h)$.

(b) We define a map $\pi: G \rightarrow G$ as follows: for any $h \in G$, we set $\pi(h) = x^{\lambda(h)}$. For any $t \in \mathbb{Z}$, we have $\pi(h\Delta^t) = \pi(h)$ so we can define the *Garside-theoretical projection to the axis of x* to be the map (which we denote by the same letter)

$$\pi: \mathcal{X} \rightarrow \mathcal{X}, \quad h\langle\Delta\rangle \mapsto x^{\lambda(h)}\langle\Delta\rangle.$$

From now on, for the rest of the paper, we make the stronger hypothesis that x satisfies $\inf(x) = 0$ and is *right-rigid* (see Definition 2.8). This implies in particular that $\inf(x^k) = 0$ for all $k \in \mathbb{N}$.

Lemma 4.4 *Let $x \in G$ with $\inf(x) = 0$ be a **right-rigid** element. Let $h \in G$. The following are equivalent:*

- (i) $x \leq \underline{h}$;
- (ii) for all $k \geq 0$, $x^{k+1} \leq \underline{x^k h}$.

Proof Only (i) \implies (ii) needs a proof. Suppose that $x \leq \underline{h}$, and let $k > 0$. We claim that $\underline{x^k h} = x^k \underline{h}$ (ie that $\inf(x^k \underline{h}) = 0$). In order to prove this claim, we write $\underline{h} = xa$ with a positive. Since x is right-rigid, the condition $\inf(xa) = 0$ implies $\inf(x^{k+1}a) = 0$ (see [Calvez and Wiest 2017b, Lemma 1]). This means that $\inf(x^k \underline{h}) = 0$, proving the claim. Now (ii) is an immediate consequence. \square

When x is right-rigid, Lemma 4.4 yields a clean interpretation of the Garside-theoretical projection to the axis of x (see Figure 3):

Lemma 4.5 *Let $x \in G$ with $\inf(x) = 0$ be a right-rigid element. Let $h \in G$. For $m \in \mathbb{Z}$, the following conditions are equivalent:*

- (i) $m = -\lambda(h)$;
- (ii) (a) $x \not\leq \underline{x^m h}$, and
(b) $x \leq \underline{x^{m+1} h}$;
- (iii) for every $k \geq 0$,
(a) $x^{k+1} \not\leq \underline{x^{m+k} h}$, and
(b) $x^k \leq \underline{x^{m+k} h}$.

In particular, whenever $\lambda(h) > 0$, we have $x^{\lambda(h)} \leq \underline{h}$. Also, for every $k \in \mathbb{Z}$, $\lambda(x^k) = k$.

Proof Implications (i) \implies (ii) and (iii) \implies (i) are clear by definition of λ . Assume (ii); this yields immediately statement (iii) for $k = 0$. Suppose $k > 0$. For (iii)(a), suppose on the contrary that

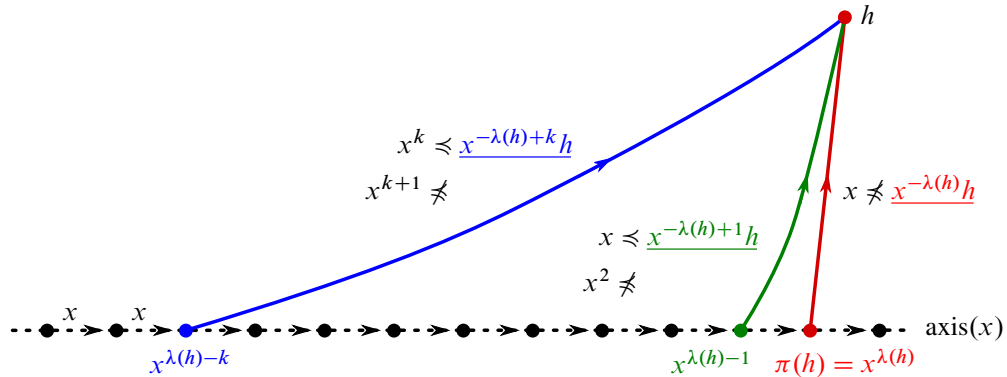


Figure 3: The definition of the projection π from \mathcal{O} to the axis, for $h \in G$, and its properties (supposing that x is right-rigid).

$x^{k+1} \preceq \underline{x^{m+k}h}$. Then $\underline{x^{m+k}h} = x^{k+1}a$ for some positive a and $\underline{x^m h} = xa$, contradicting (ii)(a). For (iii)(b), we use the hypothesis (ii)(b) that $x \preceq \underline{x^{m+1}h}$; the conclusion then follows immediately from Lemma 4.4. \square

Lemma 4.6 *Let $x \in G$ with $\text{inf}(x) = 0$ be a right-rigid element. Let $z \in G$ with $\text{inf}(z) = 0$, and let s be a simple element. Suppose that $x \not\preceq z$. Then $x^2 \not\preceq zs$.*

Proof Let $x_r \cdots x_2 x_1$ be the right normal form of x . Assume, contrary to the claim, that $x^{-2}zs$ is positive. Then by Lemma 2.4(ii), $D_r(x^{-2}z) \preceq s$. We deduce (Lemma 2.4(iii)) that $D_l(x^{-2}z)$ is also simple. On the other hand, by Lemma 2.4(i), $x^2 \succcurlyeq D_l(x^{-2}z)$ and, as x is right-rigid, $x_1 \succcurlyeq D_l(x^{-2}z)$. It then follows that $xx_r \cdots x_2 \preceq x^2 D_l(x^{-2}z)^{-1} \preceq z$, a contradiction. \square

Proposition 4.7 (π is $\ell(x)$ -Lipschitz) *Let $x \in G$ be a right-rigid element with $\text{inf}(x) = 0$ and canonical length $\ell = \ell(x)$. If $g, h \in G$ satisfy $d_{\mathcal{O}}(g, h) = 1$, then $|\lambda(g) - \lambda(h)| \leq 1$ and $d_{\mathcal{O}}(\pi(g), \pi(h)) \leq \ell$.*

Proof Let $k_g = -\lambda(g)$ and $k_h = -\lambda(h)$. Apply Lemma 2.10 to the adjacent vertices $x^{k_g}g\langle\Delta\rangle$ and $x^{k_g}h\langle\Delta\rangle$: for some simple element s , either $(x^{k_g}g)s = x^{k_g}h$ or $(x^{k_g}h)s = x^{k_g}g$ holds. Let us consider the first case. By definition of k_g , $x \not\preceq \underline{x^{k_g}g}$ and by Lemma 4.6, $x^2 \not\preceq (x^{k_g}g)s = \underline{x^{k_g}g}$. From Lemma 4.4, we deduce that $x \not\preceq \underline{x^{k_g-1}h}$. This means that $k_g - 1 \leq k_h$. In the second case, we obtain that $x \not\preceq \underline{x^{k_g}h}$, whence $k_g \leq k_h$. A similar reasoning applied to the adjacent vertices $x^{k_h}g\langle\Delta\rangle$ and $x^{k_h}h\langle\Delta\rangle$ shows that either $k_h - 1 \leq k_g$ or $k_h \leq k_g$. In either case, we obtain the desired claim for $\lambda(g) = -k_g$ and $\lambda(h) = -k_h$.

Finally, the inequality $d_{\mathcal{O}}(\pi(g), \pi(h)) \leq \ell$ is an immediate consequence. \square

Let us now combine the rigidity and the Morse hypothesis on x :

Proposition 4.8 *Let $x \in G$ be a right-rigid Morse element with $\text{inf}(x) = 0$ and canonical length ℓ . Let π be the Garside-theoretical projection to $\text{axis}(x)$. Then there exists a $D \in \mathbb{N}$ such that for any $h \in G$ and*

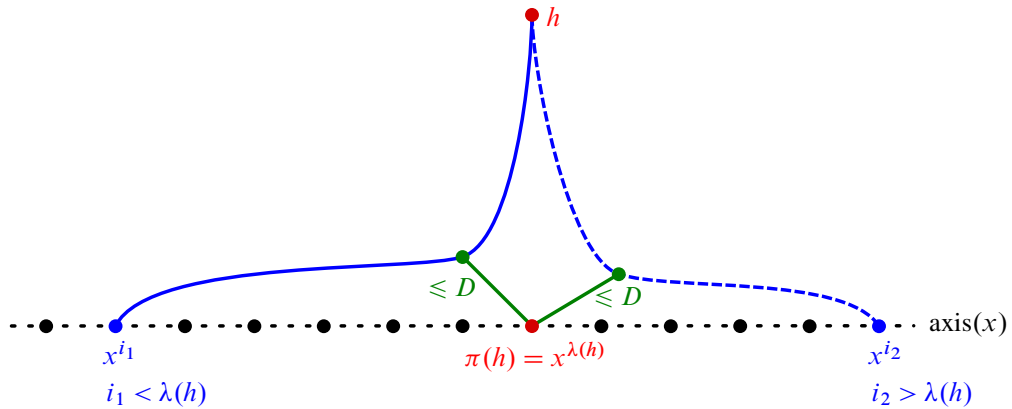


Figure 4: The statement of Proposition 4.8.

for any $i \in \mathbb{Z}$, the preferred geodesic $A(x^i, h)$ in \mathcal{X} passes at distance at most D from $\pi(h\langle\Delta\rangle)$: there exists some $h' \in G$ with $h'\langle\Delta\rangle$ belonging to $A(x^i, h)$ such that

$$d_{\mathcal{X}}(h', \pi(h)) \leq D.$$

Specifically, we can take $D = (\ell + 1) \cdot M_x$.

Proof We treat the cases $i < \lambda(h)$ and $i > \lambda(h)$ separately.

If $i < \lambda(h)$, then after the action of x^{-i} , we can assume without loss of generality that $i = 0$. Noting that $x^{\lambda(h)} \preceq \underline{h}$, we are then precisely in the situation of Proposition 3.11. Note that by the rigidity hypothesis, $\ell(x^{\lambda(h)}) = \lambda(h) \cdot \ell$. Thus, if we define $h' = \underline{h} \wedge \Delta^{\lambda(h) \cdot \ell}$, we have

$$d_{\mathcal{X}}(h', \pi(h)) \leq 2 \cdot M_x.$$

If $i > \lambda(h)$, then this time we will assume, again without loss of generality (after the action of $x^{-\lambda(h)+1}$), that $\lambda(h) = 1$. Thus $\pi(h) = x$ and $x \preceq \underline{h}$ but $x^2 \not\preceq \underline{h}$ (Lemma 4.5).

Let $h' = x^i \wedge \underline{h}$ and note that $\underline{h}' = h'$. We know from Proposition 2.15(i) that the vertex $h'\langle\Delta\rangle$ lies on the preferred geodesic $A(h, x^i)$, and our aim now is to bound its distance from $\pi(h\langle\Delta\rangle)$.

We make two observations about the vertex $h'\langle\Delta\rangle$. The first observation is that it lies in the M_x -neighborhood of $\text{axis}(x)$, where we recall that M_x is the Morse constant for $(2, 0)$ -quasigeodesics with endpoints on $\text{axis}(x)$. This follows from Lemma 2.17(i) and the fact that $x \preceq \underline{h}' \preceq x^i$. (For later reference — see Remark 5.6 — we observe that we even have $x \preceq \underline{h}' \preceq x^i \wedge \Delta^{\text{sup}(\underline{h}'})$.)

The second observation about the vertex $h'\langle\Delta\rangle$ is that it has the same projection to the axis as $h\langle\Delta\rangle$,

$$\pi(h') = \pi(h).$$

Here is a proof of this fact. We have to prove that $\lambda(x^i \wedge \underline{h}) = \lambda(h) = 1$. By Lemma 4.5, it suffices to prove that

$$x \preceq x^i \wedge \underline{h} \quad \text{but} \quad x^2 \not\preceq x^i \wedge \underline{h}.$$

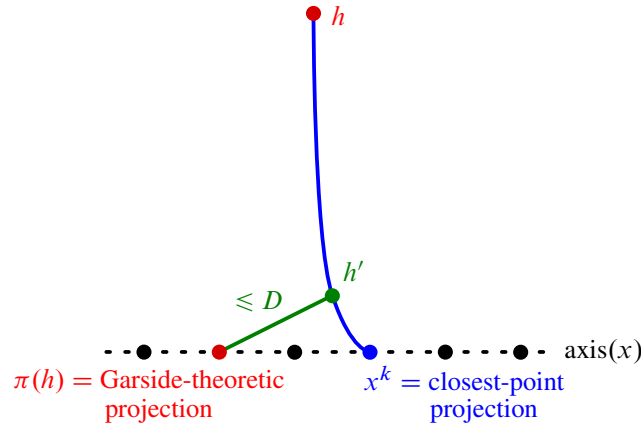


Figure 5: The projection π is uniformly close to any closest-point projection.

Keeping in mind the hypothesis that $i \geq 2$, this follows immediately from the analogous condition on \underline{h} . This completes the proof of the second observation about $h' \langle \Delta \rangle$.

By the first observation, there exists an integer k with

$$d_{\mathcal{X}}(h', x^k) \leq M_x.$$

Since the projection π is ℓ -Lipschitz (Proposition 4.7), we have

$$d_{\mathcal{X}}(\pi(h'), x^k) = d_{\mathcal{X}}(\pi(h'), \pi(x^k)) \leq \ell \cdot M_x.$$

Applying the triangle-inequality we obtain

$$d_{\mathcal{X}}(h', \pi(h')) \leq \ell \cdot M_x + M_x = (\ell + 1) \cdot M_x.$$

Also, by the second observation above we have $\pi(h') = \pi(h)$, so

$$d_{\mathcal{X}}(h', \pi(h)) \leq (\ell + 1) \cdot M_x.$$

The proof of Proposition 4.8 is complete, with

$$D = \max(2 \cdot M_x, (\ell + 1) \cdot M_x) = (\ell + 1) \cdot M_x. \quad \square$$

We deduce that π is uniformly close to the closest point projection:

Corollary 4.9 *Let $x \in G$ with $\inf(x) = 0$ be a right-rigid Morse element, and π be the Garside-theoretical projection to $\text{axis}(x)$. Let $h \in G$ and let $x^k \langle \Delta \rangle$ be any point of the axis such that*

$$d_{\mathcal{X}}(h, x^k) = \min_{i \in \mathbb{Z}} d_{\mathcal{X}}(h, x^i).$$

Then

$$d_{\mathcal{X}}(\pi(h), x^k) \leq 2D,$$

where D is the constant promised by Proposition 4.8.

Proof By Proposition 4.8, there is a point $h' \langle \Delta \rangle$ on the geodesic $A(h, x^k)$ such that $d_{\mathcal{X}}(\pi(h), h') \leq D$. Since $x^k \langle \Delta \rangle$ is a point on the axis as close as possible to $h \langle \Delta \rangle$, we must have

$$d_{\mathcal{X}}(x^k, h') \leq d_{\mathcal{X}}(\pi(h), h') \leq D.$$

By the triangle inequality, we conclude

$$d_{\mathcal{X}}(\pi(h), x^k) \leq 2D. \quad \square$$

5 The strong contraction property

In this section we recall the definition of the strong contraction property and the strong constriction property. Then we prove the main result of this paper: in a Δ -pure Garside group of finite type G , the axis of any Morse element is strongly contracting.

The following two definitions and a proof of their equivalence can be found in [Arzhantseva et al. 2015].

Definition 5.1 Let (X, d) be a metric space and let \mathcal{A} be any subset of X . A map $\rho : X \rightarrow \mathcal{A}$ is *C-strongly contracting* for $C \geq 0$ if the following hold:

- (i) ρ is coarsely equivalent to $\text{id}_{\mathcal{A}}$ on \mathcal{A} : for every $a \in \mathcal{A}$, $d(\rho(a), a) \leq C$;
- (ii) for all $x \in X$, $d(x, \rho(x)) - d(x, \mathcal{A}) \leq C$;
- (iii) for all $u, v \in X$, $d(u, v) < d(v, \mathcal{A}) - C$ implies that $d(\rho(u), \rho(v)) \leq C$ (ie if a ball in \mathcal{X} is disjoint from a C -neighborhood of \mathcal{A} , then its image under ρ is contained in a ball of radius C).

The map ρ is *strongly contracting* if there exists a nonnegative integer C such that ρ is C -strongly contracting. The subset $\mathcal{A} \subset X$ is *strongly contracting* if there exists a strongly contracting map $X \rightarrow \mathcal{A}$.

Note [Arzhantseva et al. 2015, Lemma 2.8] that a strongly contracting map $\rho : X \rightarrow \mathcal{A}$ satisfies in fact a strengthened version of clause (ii) in Definition 5.1; namely, ρ is *coarsely a closest point-projection* to \mathcal{A} , meaning that for all $x \in X$, there exists an $a \in \mathcal{A}$ with $d(x, \mathcal{A}) = d(x, a)$ such that $d(\rho(x), a)$ is uniformly bounded.

As proven in [Arzhantseva et al. 2015, Proposition 2.9], a map ρ is strongly contracting if and only if it is *strongly constricting*; this alternative characterization will be useful in Lemma 5.3 and Section 6:

Definition 5.2 Let (X, d) be a metric space and let \mathcal{A} be any subset of X . A map $\rho : X \rightarrow \mathcal{A}$ is *C-strongly constricting* for $C \geq 0$ if the following hold:

- (i) ρ is coarsely equivalent to $\text{id}_{\mathcal{A}}$ on \mathcal{A} : for every $a \in \mathcal{A}$, $d(\rho(a), a) \leq C$;
- (ii) for every geodesic γ in X with endpoints x_0 and x_1 , if $d(\rho(x_0), \rho(x_1)) > C$, then $d(\rho(x_i), \gamma) < C$ for $i = 0, 1$.

The map ρ is *strongly constricting* if there exists a nonnegative integer C such that ρ is C -strongly constricting.

Lemma 5.3 *Let (X, d) be a metric space and let \mathcal{A} be a subset of X . Let $\rho: X \rightarrow \mathcal{A}$ be a strongly contracting map.*

- (i) *Suppose that $\mathcal{B} \subset X$ is another subset of X with $d_{\text{Hausdorff}}(\mathcal{A}, \mathcal{B}) < \infty$. Then there is a strongly contracting map $\rho': X \rightarrow \mathcal{B}$.*
- (ii) *Let (X', d') be another metric space. Suppose there is an isometric and quas surjective embedding $\iota: X \hookrightarrow X'$. Then there is a strongly contracting map $\rho': X' \rightarrow \iota(\mathcal{A})$.*

Proof For (i), let $\delta = d_{\text{Hausdorff}}(\mathcal{A}, \mathcal{B})$. We construct ρ' by choosing, for any $x \in X$, a point $b \in \mathcal{B}$ with $d(b, \rho(x)) \leq \delta$, and declaring that $\rho'(x) = b$. Thus ρ and ρ' are δ -coarsely equivalent. Now it is an easy exercise to show that if ρ is C -strongly contracting then ρ' is $(C + 2\delta)$ -strongly contracting.

For (ii), let ε be such that the ε -neighborhood of $\iota(X)$ in X' is all of X' . We define ρ' by choosing, for every $x' \in X'$, a point $x \in X$ with $d'(\iota(x), x') \leq \varepsilon$, and declaring that $\rho'(x') = \iota(\rho(x))$. We have to prove that ρ' is strongly contracting. More precisely, supposing that $\rho: X \rightarrow \mathcal{A}$ is C -strongly contracting, our aim is to prove that ρ' is $(C + 3\varepsilon)$ -strongly contracting. We shall prove only part (iii) of Definition 5.1; the other two clauses can be checked easily.

For any point $v' \in X'$, consider the ball B' centered in v' and of radius $d'(v', \mathcal{A}) - C - 3\varepsilon$. If we choose a point of $\iota(X)$ at distance at most ε from each point of B' , we obtain a subset of X which is contained in a ball in X centered at some point v (with $d'(\iota(v), v') \leq \varepsilon$) and of radius $d'(v', \mathcal{A}) - C - \varepsilon$. Since $d'(v', \mathcal{A}) - C - \varepsilon \leq d(v, \mathcal{A}) - C$, the projection $\rho'(B')$ is contained in $\iota(\rho(B))$, where B is the ball in X centered in v and of radius $d(v, \mathcal{A}) - C$. By hypothesis, $\text{diam}(\rho(B)) \leq C < C + 3\varepsilon$, which is what we wanted to prove. □

Proposition 5.4 *Let G be a Δ -pure Garside group of finite type. Let $x \in G$ with $\text{inf}(x) = 0$ be a right-rigid Morse element. The Garside-theoretical projection π to $\text{axis}(x)$ is $5D$ -strongly contracting, where D is the constant promised by Proposition 4.8.*

Proof The first and second conditions of Definition 5.1 follow respectively from the last statement of Lemma 4.5 and Corollary 4.9, which asserts the stronger condition that π is coarsely a closest-point projection. Let us prove that condition (iii) is satisfied. Let $h, g \in G$ be such that $d_{\mathcal{X}}(h, g) \leq d_{\mathcal{X}}(h\langle\Delta\rangle, \text{axis}(x))$ (that is, $g\langle\Delta\rangle$ lies in a ball in \mathcal{X} centered at $h\langle\Delta\rangle$ and disjoint from the axis of x).

Let us write $r_* = d_{\mathcal{X}}(h, \pi(h))$. Now,

$$d_{\mathcal{X}}(h, g) \leq d_{\mathcal{X}}(h\langle\Delta\rangle, \text{axis}(x)) \leq r_*.$$

By Proposition 4.8, the preferred geodesic $A(g, \pi(h))$ contains a point $g'\langle\Delta\rangle$ at distance at most D from $\pi(g\langle\Delta\rangle)$. By Proposition 2.15(v) (convexity of balls), we have $d_{\mathcal{X}}(h, g') \leq r_*$.

Let us now study the preferred geodesic $A(h, g')$. We have just seen that it is of length at most r_* . Moreover, by Lemma 2.16, it is at Hausdorff distance at most D from $A(h, \pi(g))$, which in turn passes

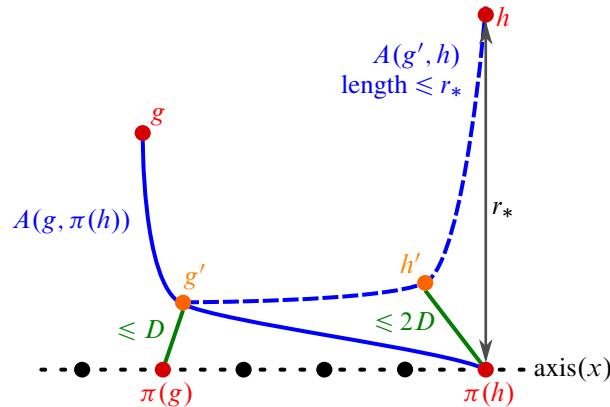


Figure 6: The proof of Proposition 5.4.

at distance at most D from $\pi(h\langle\Delta\rangle)$ (by Proposition 4.8 again). Thus $A(h, g')$ contains a point $h'\langle\Delta\rangle$ at distance at most $2D$ from $\pi(h\langle\Delta\rangle)$.

Now $d_{\mathcal{X}}(h', h) \geq r_* - 2D$ by the triangle inequality. Therefore,

$$d_{\mathcal{X}}(g', h') = d_{\mathcal{X}}(g', h) - d_{\mathcal{X}}(h', h) \leq 2D,$$

and we obtain the desired conclusion

$$d_{\mathcal{X}}(\pi(g), \pi(h)) \leq d_{\mathcal{X}}(\pi(g), g') + d_{\mathcal{X}}(g', h') + d_{\mathcal{X}}(h', \pi(h)) \leq D + 2D + 2D = 5D. \quad \square$$

The following is the main result of this paper:

Theorem 5.5 (strong contraction property of axes) *Let G be a Δ -pure Garside group of finite type. Let $g \in G$ be a Morse element. Then:*

- (i) *In $\mathcal{X} = \Gamma(G)/\langle\Delta\rangle$, the axis $\{g^k\langle\Delta\rangle \mid k \in \mathbb{Z}\} \subset \mathcal{X}$ is strongly contracting.*
- (ii) *In $\bar{\Gamma} = \Gamma(G/Z(G))$, the axis $\{g^k Z(G) \mid k \in \mathbb{Z}\} \subset \bar{\Gamma}$ is strongly contracting.*

Proof of Theorem 5.5 First we recall that the axis of g being Morse in \mathcal{X} or in $\bar{\Gamma}$ are equivalent properties, because the property of being Morse is invariant under quasi-isometry.

Now, by Proposition 3.7, there is an element $x \in G$ with $\text{inf}(x) = 0$ which is right-rigid, and which is obtained from g by taking a power, conjugating by some element $a \in G$, and multiplying by a central element. Thus in both spaces, \mathcal{X} and $\bar{\Gamma}$, taking the axis of x and translating it by the action of a yields a subset which is at finite Hausdorff distance from the axis of g . By Proposition 5.4, the axis of x is strongly contracting in \mathcal{X} , and so is its image under the a -action; by Lemma 5.3(i), the axis of g is strongly contracting in \mathcal{X} .

For Theorem 5.5(ii) we recall from Proposition 2.12 that there is an isometric embedding $\iota: \mathcal{X} \hookrightarrow \bar{\Gamma}$ with $\lfloor \frac{1}{2}e \rfloor$ -dense image. The vertices of the image are those which are represented by elements g with

$\inf(g) \equiv 0 \pmod{e}$. In particular, the axis of x in $\bar{\Gamma}$ is the image under ι of the axis of x in \mathcal{X} . By Proposition 5.4 and Lemma 5.3(ii), the axis of x in $\bar{\Gamma}$ is strongly contracting (and so is its a -translate). By Lemma 5.3(i), the axis of g in $\bar{\Gamma}$ is also strongly contracting. \square

Remark 5.6 The proof of our main theorem (Theorem 5.5) did not use the full strength of the Morse hypothesis. Let us state carefully what is required in order to prove that the axis of an element g of a Δ -pure Garside group of finite type G is strongly contracting.

First, we need g to have some power which is conjugate to a rigid element, which we will call x , with $\inf(x)$ divisible by e . By the result of [Birman et al. 2007] cited in the proof of Proposition 3.7, this is the case if the centralizer of g in $G/Z(G)$ is virtually cyclic. (Alternatively, as seen in Proposition 3.7, it follows if g is Morse.) We note that the axis of g is strongly contracting if and only if the axis of the rigid element x is, as seen in the proof of Theorem 5.5.

Second, we need x to satisfy a Morse hypothesis: let us say that a rigid element with $\inf(x) = 0$ is *weakly Garside Morse* if there exists an integer M such that, for any integer i ,

- (a) if $h \in G$ with $x^i \preceq h$, $\inf(h) = \inf(x^i) = 0$ and $\sup(h) = \sup(x^i)$, then $\sup(g^{-1}h) \leq M$, and similarly
- (b) if $h \in G$ with $h \preceq x^i$, $\inf(h) = \inf(x^i) = 0$ and $\sup(h) = \sup(x^i)$, then $\sup(h^{-1}g) \leq M$.

Roughly speaking, the weak Garside version of the Morse condition requires that in any isosceles triangle in \mathcal{X} made up of three Garside normal form paths, and where one of the two equal sides is part of the axis of x , the third side must be of universally bounded length.

Our proof of Theorem 5.5 shows that for any element g of any Δ -pure Garside group of finite type, the above two conditions imply strong contractibility of the axis of g , and in particular the full Morse property.

Indeed, there are three places in the proof where the Morse property was used: first, in Proposition 3.11, but this proposition is an immediate consequence of part (a) of the weak Garside Morse property. Second, the Morse property was used in the proof of Proposition 4.8, which has two cases; we observe that the first case follows again from part (a) of the weak Garside Morse property, and the second case only uses part (b) of this property, not the full Morse property. Third, the proof of Proposition 5.4 uses the Morse property, but only indirectly, by citing Proposition 4.8.

Remark 5.7 We conjecture that arbitrary bi-infinite axes in \mathcal{X} satisfying the Morse property (not just the periodic ones) are strongly contracting.

6 Consequences for the additional length graph

In this section, we record a consequence of Theorem 5.5 for the study of the *additional length graph* $\mathcal{C}_{\text{AL}}(G)$ of a Garside group G . This graph was introduced in [Calvez and Wiest 2017a], and further

studied in [Calvez and Wiest 2017b; 2021] (see also [Calvez 2022]). We briefly recall the definition and the main results from [Calvez and Wiest 2017a]:

Definition 6.1 Let G be a Garside group of finite type.

- (a) An element $h \in G$ is *absorbable* if it satisfies two conditions:
 - (i) $\inf(h) = 0$ or $\sup(h) = 0$;
 - (ii) there exists an element $g \in G$ which “absorbs” h , meaning that $\inf(g) = \inf(gh)$ and $\sup(g) = \sup(gh)$.
- (b) The *additional length graph* $\mathcal{C}_{\text{AL}}(G)$ of G is the (usually locally infinite) graph with the same set of vertices and edges as \mathcal{X} , but with, additionally, a new edge between vertices $g\langle\Delta\rangle$ and $h\langle\Delta\rangle$ whenever there is an absorbable element $s \in G$ so that $gs \in h\langle\Delta\rangle$. The graph metric of $\mathcal{C}_{\text{AL}}(G)$ is denoted by d_{AL} : for vertices $g\langle\Delta\rangle, h\langle\Delta\rangle$ of \mathcal{C}_{AL} , we sometimes write $d_{\text{AL}}(g, h) = d_{\text{AL}}(g\langle\Delta\rangle, h\langle\Delta\rangle)$.

Proposition 6.2 [Calvez and Wiest 2017a, Lemmas 1–3] (i) *An element $h \in G$ is absorbable if and only if h^{-1} is.*

- (ii) *If $h = h_1 \cdot h_2 \cdot h_3$, with $\inf(h) = \inf(h_1) = \inf(h_2) = \inf(h_3) = 0$ is absorbable, then h_1, h_2 and h_3 are also absorbable.*
- (iii) *Suppose that $h \in G$ is absorbable; then there exists an absorbing element g with $\inf(g) = 0$ and $\sup(g) = \ell(h)$.*

Since there is a natural inclusion $\mathcal{X} \hookrightarrow \mathcal{C}_{\text{AL}}(G)$, we can interpret the family of paths $A(g, h)$ from Definition 2.13 as a family of paths in $\mathcal{C}_{\text{AL}}(G)$.

Proposition 6.3 (properties of \mathcal{C}_{AL} [Calvez and Wiest 2017a, Theorem 1]) *Let G be a Garside group of finite type.*

- (i) *The additional length graph $\mathcal{C}_{\text{AL}}(G)$ is 60–hyperbolic.*
- (ii) *The paths $A(g, h)$ form a uniform family of unparametrized quasigeodesics in $\mathcal{C}_{\text{AL}}(G)$.*

Remark 6.4 (a) If G is the braid group B_n , equipped with the classical or dual Garside structure, then $\mathcal{C}_{\text{AL}}(G)$ is conjectured to be quasi-isometric to the curve graph of the $(n+1)$ –times punctured sphere.

- (b) Note that Proposition 6.3 is *not* claiming that $\text{diam}(\mathcal{C}_{\text{AL}}(G)) = \infty$. For instance, the group $G = \mathbb{Z}^3$ carries a Garside structure with $\Delta = (1, 1, 1)$ (see [Dehornoy et al. 2015, Chapter 1.1.1]), for which all elements $(k, 0, 0)$, $(0, k, 0)$ and $(0, 0, k)$ (with $k \in \mathbb{Z}$) are absorbable, so that $\text{diam}(\mathcal{C}_{\text{AL}}(\mathbb{Z}^3)) = 3$. By contrast, for any Artin group of spherical type A we do have $\text{diam}(\mathcal{C}_{\text{AL}}(A)) = \infty$; the proof of this fact in [Calvez and Wiest 2017b] involved an explicit Garside-theoretical construction of elements with (very) strongly constricting axes.

Theorem 6.5 For a Δ -pure Garside group of finite type G we consider the additional length graph $\mathcal{C}_{AL}(G)$, equipped with the $G/Z(G)$ -action. For any Morse element g , the action of g on $\mathcal{C}_{AL}(G)$ is loxodromic and WPD.

Here WPD is the weak proper discontinuity condition of [Bestvina and Fujiwara 2002]:

Definition 6.6 The action of $g \in G/Z(G)$ on $\mathcal{C}_{AL}(G)$ is *weakly properly discontinuous* (WPD) if for every (equivalently, for any) $k \in G$, and for every $\kappa > 0$, there exists $N > 0$ such that for all $n \geq N$, the set

$$\{h \in G/Z(G) \mid d_{AL}(k, hk) \leq \kappa, d_{AL}(g^N k, hg^n k) \leq \kappa\}$$

is finite.

Corollary 6.7 If G contains a Morse element then $\text{diam}(\mathcal{C}_{AL}(G)) = \infty$.

Corollary 6.8 Pseudo-Anosov braids act loxodromically and WPD on $\mathcal{C}_{AL}(B_n)$.

Proof of Theorem 6.5 Let g a Morse element of G ; by Proposition 3.7, g has a power which is conjugate to a right-rigid element of the form $\Delta^{em}x$ with x right-rigid and $\text{inf}(x) = 0$. Thus it suffices to prove the theorem for a right-rigid Morse element x with $\text{inf}(x) = 0$.

We know from Proposition 5.4 and from [Arzhantseva et al. 2015, Proposition 2.9] that there is a constant $C \in \mathbb{N}$ such that the Garside-theoretical projection $\pi : \mathcal{X} \rightarrow \text{axis}(x)$ is C -strongly constricting.

Lemma 6.9 There is a constant $F \in \mathbb{N}$ with the following property: suppose that we have $h_1, h_2 \in G$ and an absorbable element $s \in G$ such that $\underline{h}_1 s \in h_2 \langle \Delta \rangle$. Then

$$d_{\mathcal{X}}(\pi(h_1), \pi(h_2)) < F.$$

Proof After exchanging the roles of h_1 and h_2 , if necessary, we can suppose that $\text{inf}(s) = 0$ (rather than $\text{sup}(s) = 0$). We are going to prove that the bound $F = 2M_x^{(2,C)} + 6C$ works, where $M_x^{(2,C)}$ is the Morse constant for $(2, C)$ -quasigeodesics with endpoints on $\text{axis}(x)$.

If $d_{\mathcal{X}}(\pi(h_1), \pi(h_2)) \leq C$, then we are done. If $d_{\mathcal{X}}(\pi(h_1), \pi(h_2)) > C$, then the C -strong constriction property implies that the Garside normal form of s (as a word in the letters \mathcal{D}) can be cut into three pieces, yielding a factorization $s = s_1 \cdot s_2 \cdot s_3$ with $\text{inf}(s_1) = \text{inf}(s_2) = \text{inf}(s_3) = 0$, and such that (see Figure 7)

$$d_{\mathcal{X}}(\underline{h}_1 s_1, \pi(h_1)) \leq C \quad \text{and} \quad d_{\mathcal{X}}(\underline{h}_1 s_1 s_2, \pi(h_2)) \leq C.$$

By Proposition 6.2(ii), all three factors s_1, s_2 and s_3 are absorbable. In particular, s_2 is. Let us denote by ℓ_2 the Garside length of s_2 — thus $\text{inf}(s_2) = 0$ and $\text{sup}(s_2) = \ell_2$.

As seen in [Calvez and Wiest 2021], absorbability of s_2 means that there is a geodesic triangle in \mathcal{X} which is equilateral of side length ℓ_2 , and one of whose sides is the geodesic $A(\underline{h}_1 s_1, \underline{h}_1 s_1 s_2)$. Moreover, for any two points in two different sides of this triangle, with distances d_1 and d_2 from the shared

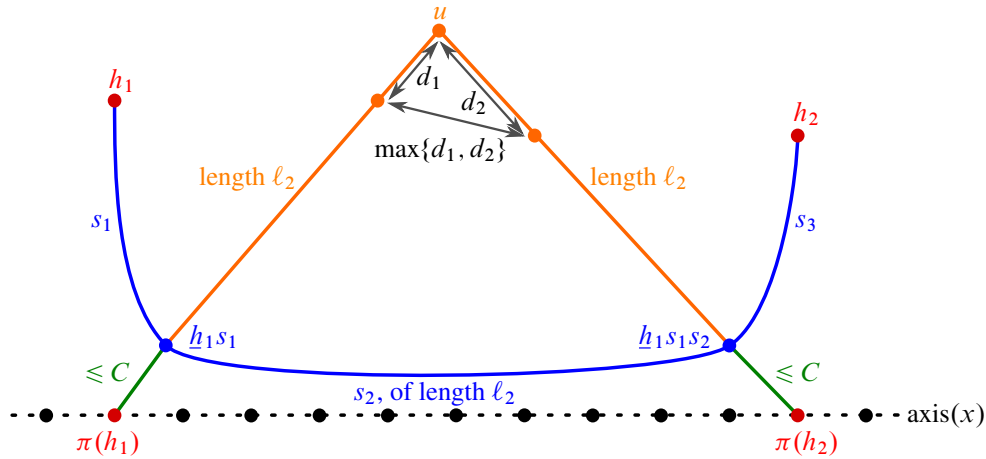


Figure 7: The proof of Lemma 6.9.

corner of the triangle, the distance of the two points in \mathcal{X} is $\max(d_1, d_2)$. In particular, the triangle is $(2, 0)$ -quasi-isometrically embedded in \mathcal{X} (compare Lemma 2.17).

Let $u \in G$ be such that $u\langle\Delta\rangle$ is the corner of the triangle furthest from the axis of x . We claim that the distance of $u\langle\Delta\rangle$ from the axis is at least $\frac{1}{2}l_2 - 2C$. Indeed,

$$l_2 - C \leq d_{\mathcal{X}}(u, \pi(h_i)) \leq l_2 + C$$

for $i = 1, 2$. Moreover, for any $k \in \mathbb{Z}$ (not necessarily between $\lambda(h_1)$ and $\lambda(h_2)$) we have by the triangle inequality

$$d_X(u, x^k) \geq \max(d_{\mathcal{X}}(x^k, \pi(h_2)) - l_2 - C, l_2 - C - d_{\mathcal{X}}(x^k, \pi(h_1)), l_2 - C - d_{\mathcal{X}}(x^k, \pi(h_2)), d_{\mathcal{X}}(x^k, \pi(h_1)) - l_2 - C).$$

Using the fact that $l_2 - 2C \leq d_{\mathcal{X}}(\pi(h_1), \pi(h_2)) \leq l_2 + 2C$, one can calculate that, depending on k , one of these four values is always at least $\frac{1}{2}l_2 - 2C$. This completes the proof of the claim.

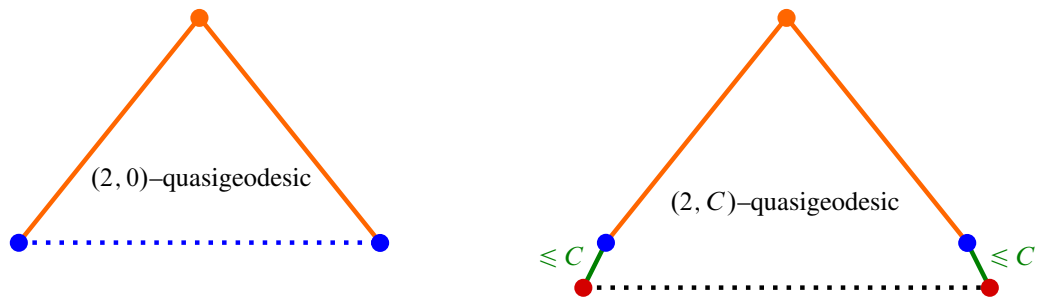


Figure 8: Left: the unit-speed parametrization of this path is a $(2, 0)$ -quasigeodesic. Right: γ , which coincides with the previous path except for jumps of size at most C at the starting and endpoint, is a $(2, C)$ -quasigeodesic.

Now consider the path $\gamma : [0, 2\ell_2] \rightarrow \mathcal{X}$

- with $\gamma(0) = \pi(h_1\langle\Delta\rangle)$,
- which for $t \in]0, \ell_2[$ follows a unit speed parametrization of $A(\underline{h}_1 s_1, u)$,
- which for $t \in [\ell_2, 2\ell_2[$ follows a unit speed parametrization of $A(u, \underline{h}_1 s_1 s_2)$, and
- with $\gamma(2\ell_2) = \pi(h_2\langle\Delta\rangle)$.

We see that γ is a $(2, C)$ -quasigeodesic (because, apart from the jumps of size at most C at the starting and end points it is a $(2, 0)$ -quasigeodesic).

Thus, the Morse condition for the axis of x implies that $\frac{1}{2}\ell_2 - 2C \leq M_x^{(2,C)}$, or equivalently,

$$\ell_2 \leq 2M_x^{(2,C)} + 4C.$$

Therefore,

$$d_{\mathcal{X}}(\pi(h_1), \pi(h_2)) \leq \ell_2 + 2C \leq 2M_x^{(2,C)} + 6C. \quad \square$$

Coming back to the proof of Theorem 6.5, suppose that the action of x is not loxodromic. This means that $t_n := d_{\text{AL}}(1, x^n)$ grows sublinearly with n . Consider elements $1 = h_0, h_1, \dots, h_{t_n} = x^n$ of G and a geodesic in $\mathcal{C}_{\text{AL}}(G)$ between $*$ and $x^n\langle\Delta\rangle$ through the vertices $h_i\langle\Delta\rangle$. The sublinear growth means that for sufficiently large values of n , there must be an $i \in \{1, 2, \dots, t_n\}$ such that

$$d_{\mathcal{X}}(h_{i-1}, h_i) \geq \max\{F, \ell(x)\}.$$

This contradicts either Lemma 6.9 or the Lipschitz property of π (Proposition 4.7). This completes the proof that the x -action on $\mathcal{C}_{\text{AL}}(G)$ is loxodromic.

We now turn to the proof that the x -action on $\mathcal{C}_{\text{AL}}(G)$ is WPD. Fix $\kappa > 0$. Define

$$S_x^{(\kappa,n)} = \{h \in G/Z(G) \mid d_{\text{AL}}(1, h) < \kappa \text{ and } d_{\text{AL}}(x^n, hx^n) < \kappa\}.$$

We look at the situation in \mathcal{X} : letting $E = \max\{F, \ell(x)\}$, Lemma 6.9 and Proposition 4.7 tell us that for $h \in S_g^{(\kappa,n)}$,

$$d_{\mathcal{X}}(1, \pi(h)) \leq E \cdot \kappa \quad \text{and} \quad d_{\mathcal{X}}(x^n, \pi(hx^n)) \leq E \cdot \kappa.$$

We now choose N sufficiently large that $d_{\mathcal{X}}(1, x^N) > C + 2 \cdot E \cdot \kappa$ — then for all integers n with $n \geq N$ we also have $d_{\mathcal{X}}(1, x^n) > C + 2 \cdot E \cdot \kappa$, and by the triangle inequality

$$d_{\mathcal{X}}(\pi(h), \pi(hx^n)) > C.$$

The strong constriction property of π then guarantees that the geodesic $A(h, hx^n)$ passes through points $a\langle\Delta\rangle$ and $b\langle\Delta\rangle$ at distance at most C from $\pi(h\langle\Delta\rangle)$ and $\pi(hx^n\langle\Delta\rangle)$ respectively, and hence at distance at most $C + E \cdot \kappa$ from $*$ and $x^n\langle\Delta\rangle$ respectively. Therefore, $d_{\mathcal{X}}(a, b) \geq d_{\mathcal{X}}(1, x^n) - 2 \cdot C - 2 \cdot E \cdot \kappa$.

On the other hand, the geodesic $A(h, hx^n)$ has the same length as the segment of the axis $A(1, x^n)$ (as it is its image under left-translation by h). Thus,

$$d_{\mathcal{X}}(h, a) = d_{\mathcal{X}}(1, x^n) - d_{\mathcal{X}}(a, b) - d_{\mathcal{X}}(b, hx^n) \leq d_{\mathcal{X}}(1, x^n) - d_{\mathcal{X}}(a, b) \leq 2 \cdot C + 2 \cdot E \cdot \kappa.$$

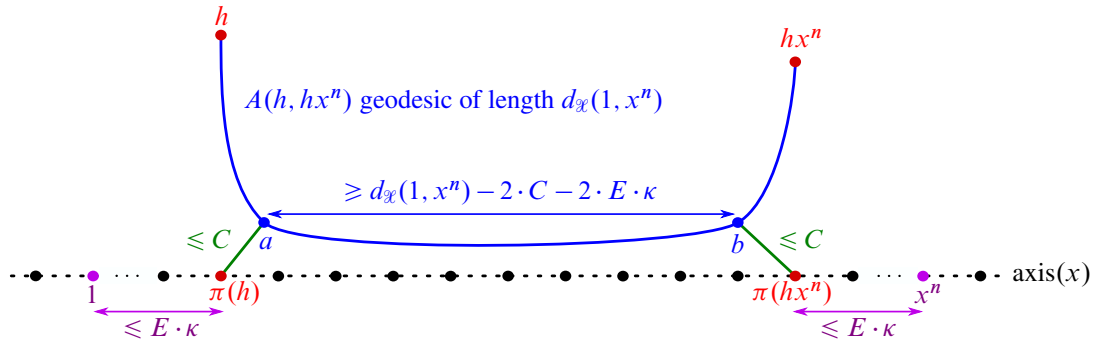


Figure 9: The proof that the action of x is WPD.

We conclude that $d_x(1, h) \leq 3 \cdot (C + E \cdot \kappa)$. There are only finitely many elements $h \in G/Z(G)$ with this property. This completes the proof that the action of g is WPD. In fact we have proven something slightly stronger than what was required: we found a bound on the size of the set $S_g^{(r,n)}$ which does not depend on n , as long as $n \geq N$. \square

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
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