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Homotopy ribbon discs with a fixed group

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In the topological category, the classification of homotopy ribbon discs is known when the fundamental group G of the exterior is $\mathbb{Z}$ and the Baumslag–Solitar group $\mathrm{BS}(1,2)$. We prove that if a group G is geometrically 2–dimensional and satisfies the Farrell–Jones conjecture, then a condition involving the fundamental group ensures that exteriors of aspherical homotopy ribbon discs with fundamental group G are s–cobordant rel boundary. When G is good, this leads to the classification of such discs. As an application, for any knot $J \subset S^3$ whose knot group $G(J)$ is good, we classify the homotopy ribbon discs for $J \# -J$ whose complement has group $G(J)$. A similar application is obtained for $\mathrm{BS}(m,n)$ when $|m-n| = 1$.

57K10, 57N35, 57N70, 57R67

1 Introduction

Given a knot $K \subset S^3$, we consider the problem of classifying locally flat discs $D \subset D^4$ with boundary K, up to topological ambient isotopy rel boundary. Naturally K need not bound such a disc (ie K need not be *slice*), but if it does, then it is conjectured that it necessarily bounds one for which the inclusion induced map $\pi_1(S^3 \setminus K) \to \pi_1(D^4 \setminus D)$ is surjective; such discs are called *homotopy ribbon*. For this reason, and for technical purposes, we restrict our attention to homotopy ribbon discs with boundary K. Additionally, observe that if D_1 and D_2 are two ambiently isotopic slice discs with boundary K, then their groups must be isomorphic: $\pi_1(D^4 \setminus D_1) \cong \pi_1(D^4 \setminus D_2)$. Our goal here is to study the following question:

Question 1.1 *Given a knot $K \subset S^3$ and a ribbon group G, can one describe the set of homotopy ribbon discs for K with group G, considered up to topological ambient isotopy rel boundary?*

Here a group is called *ribbon* if it arises as $\pi_1(D^4 \setminus D)$ for some (smoothly embedded) ribbon disc $D \subset D^4$ ($D \subset D^4$ is *ribbon* if the restriction of the radial function $D^4 \to \mathbb{R}$ to D is Morse and admits no local maxima). We work with ribbon groups instead of fundamental groups of locally flat disc exteriors for convenience: the former admit an algebraic characterisation [Friedl and Teichner 2005, Theorem 2.1], while no such description appears to be known for the latter [loc. cit., Question 1.7]. Examples of ribbon groups include $G = \mathbb{Z}$ and the Baumslag–Solitar group $G = \mathrm{BS}(1,2)$, and in those cases Question 1.1 has been fully resolved [Friedl and Teichner 2005; Conway and Powell 2021]. The answers, which will be partially recalled in Remark 1.11, both rely on Freedman's 5–dimensional s–cobordism theorem [1982] and therefore make use of the fact that $\mathbb{Z}$ and $\mathrm{BS}(1,2)$ are *good* groups. We refer to [Powell and Ray 2021, Definition 12.12] for the precise definition of a good group and to [Kim et al. 2021, Chapter 19] for a survey, but note that the class of good groups contains all groups of subexponential growth as well as all elementary amenable groups (eg solvable groups). At the time of writing, it is unknown whether all groups are good; this is equivalent to the question of whether the free group F_2 is good [loc. cit., Proposition 19.7].

Remark 1.2 The only elementary amenable ribbon groups are $\mathbb{Z}$ and $\mathrm{BS}(1,2)$, as can be seen by combining [Hillman 2002, Corollary 2.6.1] with the fact that ribbon groups have deficiency one and abelianise to $\mathbb{Z}$. As a consequence, if the class of good ribbon groups were eventually shown to coincide with the class of elementary amenable ribbon groups, then the current article would contain no new classification result. On the other hand, Theorem 1.7 contains criteria for certain disc exteriors to be s–cobordant rel boundary and holds regardless of the state of the art on the class of good groups. We also hope that the approach taken here will be of interest given the recent surge of activity around the topic of 2–discs in the 4–ball, both in the smooth and topological category [Juhász and Zemke 2020; Conway and Powell 2021; Hayden 2020; 2021; Sundberg and Swann 2022; Hayden et al. 2021; Hayden and Sundberg 2021; Lipshitz and Sarkar 2022; Dai et al. 2023].

In order to give a flavour of our results without listing technical assumptions this early on, we mention a corollary of our main theorems (Theorems 1.7 and 1.10). To state this result succinctly, we introduce some terminology. A *G–ribbon disc* refers to a homotopy ribbon disc $D \subset D^4$ with $\pi_1(D^4 \setminus D) \cong G$, and given a knot K, we write $\mathcal{D}_G(K)$ for the set of rel boundary topological ambient isotopy classes of G–ribbon discs with boundary K. We also write M_K for the result of 0–surgery on K and use $\mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K), G)$ to denote the set of epimorphisms $\pi_1(M_K) \twoheadrightarrow G$ that satisfy (FT) below. While this definition will be discussed in greater detail in the next couple of sections, for the moment we simply note that $\mathrm{Aut}(G)$ acts on $\mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K), G)$ by postcomposition, allowing us to consider the orbit set $\mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K), G)/\mathrm{Aut}(G)$. Mapping a G–ribbon disc $D \in \mathcal{D}_G(K)$ with aspherical complement to the inclusion induced homomorphism $\pi_1(M_K) \twoheadrightarrow \pi_1(D^4 \setminus D)$ determines an element $\Phi(D)$ in this orbit set.

Theorem *Fix a knot $K \subset S^3$.*

(i) *If G is a knot group (ie $G = \pi_1(S^3 \setminus J)$ for some knot J), then exteriors of G–ribbon discs $D_1, D_2 \in \mathcal{D}_G(K)$ are s–cobordant rel boundary if $\Phi(D_1) = \Phi(D_2)$. If G is good, then Φ induces a bijection $\mathcal{D}_G(K) \approx \mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K), G)/\mathrm{Aut}(G)$.*

(ii) *If $m, n \in \mathbb{Z}$ are such that $|m-n| = 1$ and G is the Baumslag–Solitar group $\mathrm{BS}(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$, then exteriors of aspherical G–ribbon discs $D_1, D_2 \in \mathcal{D}_G(K)$ are s–cobordant rel boundary if $\Phi(D_1) = \Phi(D_2)$. If G is good, then Φ induces a bijection $\mathcal{D}^a_G(K) \approx \mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K), G)/\mathrm{Aut}(G)$, where $\mathcal{D}^a_G(K) \subset \mathcal{D}_G(K)$ denotes the subset of G–ribbon discs with aspherical exterior.*

Examples 1.12 and 1.13 describe how this follows from Theorems 1.7 and 1.10. Additionally, as we explain in more detail in Remark 1.11, this theorem recovers the previously known classifications for $\mathrm{BS}(0,1)=\mathbb{Z}$ and $\mathrm{BS}(1,2)$ since, for these groups, homotopy ribbon disc exteriors are known to be aspherical.

1.1 Existence

We recall and motivate a sufficient condition for the existence of a G–ribbon disc with boundary K, which is due to Friedl and Teichner [2005, Theorem 1.9]. First, if K bounds a locally flat disc $D\subset D^4$, then $\partial N_D = M_K$, where $N_D := D^4\setminus \nu D$ is the exterior of D and M_K denotes the 3–manifold obtained by 0–framed surgery on K. Next, if $D\subset D^4$ is a G–ribbon disc for a knot K, then there is an epimorphism $\pi_1(M_K)\twoheadrightarrow \pi_1(N_D)\cong G$ and (N_D, M_K) satisfies Poincaré duality or, using surgery theory jargon, is a (4–dimensional) *Poincaré pair*. If, additionally, the disc exterior $N_D = D^4\setminus\nu D$ is aspherical, then we have a homotopy equivalence $N_D\simeq K(G,1)$ and we deduce that $(K(G,1), M_K)$ is a Poincaré pair.

Remark 1.3 It is expected that ribbon disc exteriors are aspherical [Gordon 1981, Conjecture 6.5]; see also [Howie 1985]. As noted in [Friedl and Teichner 2005, Section 2], this would imply the *ribbon group conjecture*: ribbon groups are geometrically 2–dimensional.[1] Here recall that a group G is called *geometrically* 2*–dimensional* if $K(G,1)$ is (homotopy equivalent to) a 2–complex. Both statements are in fact particular cases of the *Whitehead conjecture*, which states that every connected subcomplex of a 2–dimensional aspherical CW–complex is itself aspherical [Whitehead 1941]; see [Rosebrock 2007] for a nice overview. Howie [1982, Theorem 5.2] proved that locally indicable ribbon groups are geometrically 2–dimensional. On the other hand, to the best of our knowledge, the Whitehead conjecture is not known to imply that exteriors of homotopy ribbon discs are aspherical; see also Remark 1.11.

We argued that if D is a G–ribbon disc with aspherical exterior and boundary a knot K, then $\pi_1(M_K)\twoheadrightarrow \pi_1(N_D)\cong G$ is an epimorphism and $(K(G,1), M_K)$ is a Poincaré pair. On the other hand, if we start with an epimorphism $\pi_1(M_K)\twoheadrightarrow G$ onto a group G, then there is an embedding $\varphi\colon M_K\hookrightarrow K(G,1)=\mathrm{BG}$ that induces the given surjection on fundamental groups and, if G is geometrically 2–dimensional, then [Friedl and Teichner 2005, Lemma 3.2] shows that $(K(G,1), M_K)$ is a Poincaré pair if and only if the induced map satisfies

$$\text{(FT)}\qquad \varphi^*\colon H^i(\mathrm{BG};\mathbb{Z}[G])\to H^i(M_K;\mathbb{Z}[G]_\varphi) \text{ is an isomorphism for } i=1,2.$$

Under an additional condition on the group G, Friedl and Teichner [2005, Theorem 1.9 and Lemma 3.2] prove that this leads to a sufficient condition for K to bound a G–ribbon disc.

Theorem 1.4 (Friedl and Teichner) *Let $K\subset S^3$ be a knot and let G be a good geometrically 2–dimensional ribbon group such that $\widetilde{L}_4^h(\mathbb{Z}[G])=0$. If $\varphi\colon \pi_1(M_K)\twoheadrightarrow G$ is an epimorphism that satisfies* (FT), *then there exists a G–ribbon disc $D\subset D^4$ with aspherical exterior and boundary K such that the composition $\pi_1(M_K)\twoheadrightarrow\pi_1(N_D)\cong G$ agrees with φ.*

[1] Friedl and Teichner refer to geometrically 2–dimensional groups as *aspherical* groups.

Remark 1.5 • Friedl and Teichner actually prove a stronger result. Instead of asking for G to be geometrically 2–dimensional, they merely demand that $H_3(G)=0$ and $H^i(G;\mathbb{Z}[G])=0$ for $i>2$, and instead of assuming that G is ribbon, they only require that G be finitely presented and satisfy $H_1(G)=\mathbb{Z}$ and $H_2(G)=0$. Finally, they do not require G to be good, only that the surgery sequence (with h–decorations) be exact for all 4–dimensional Poincaré pairs (X,M) with $\pi_1(X)=G$.

• The fact that the disc exterior is aspherical is implicit in [loc. cit., proof of Theorem 1.9]: their surgery-theoretic argument yields a disc D whose exterior $N_D=D^4\setminus \nu D$ is homotopy equivalent to $K(G,1)$, which is aspherical.

• The groups $\mathbb{Z}$ and $\mathrm{BS}(1,2)$ satisfy all the assumptions of Theorem 1.4. Additionally, for those groups, condition (FT) simplifies considerably. Indeed if G is poly-(torsion-free abelian) (or PTFA for short), then (FT) reduces to

(Ext) $$\operatorname{Ext}^1_{\mathbb{Z}[G]}(H_1(M_K;\mathbb{Z}[G]_\varphi),\mathbb{Z}[G])=0,$$

and for $G=\mathbb{Z}$ it reduces further to the condition $\Delta_K=1$; all of this is explained in [loc. cit., Sections 1 and 4, and Lemma 3.3].

1.2 Uniqueness and classification

We now return to the set $\mathcal{D}_G(K)$ of rel boundary topological ambient isotopy classes of G–ribbon discs with boundary K. In fact, we will mostly be concerned with the subset $\mathcal{D}^a_G(K)\subset\mathcal{D}_G(K)$ of discs with aspherical exteriors. To that effect, inspired by [Hambleton et al. 2009, Definition 1.2], we describe some assumptions on the group G that we will require:

Definition 1.6 A group G *satisfies properties* W–AA if

(W) the Whitehead group $\mathrm{Wh}(G)$ vanishes,

(A4) the assembly map $A_4\colon H_4(\mathrm{BG};\boldsymbol{L}_\bullet)\to L_4(\mathbb{Z}[G])$ is an isomorphism,[2] and

(A5) the assembly map $A_5\colon H_5(\mathrm{BG};\boldsymbol{L}_\bullet)\to L_5(\mathbb{Z}[G])$ is surjective.

We will mostly use these conditions as a blackbox, but note that thanks to extensive work on the Farrell–Jones conjecture (see [Lück 2021] for a survey) they should not be thought of as insurmountable restrictions. We discuss all of this in more detail in Remark 1.11 and refer to [Ranicki 1992; Chang and Weinberger 2021; Lück 2020; 2021] for background on assembly maps in L–theory. Returning to our aim of describing $\mathcal{D}_G(K)$, we consider the set

(Epi) $$\mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K),G):=\{\varphi\colon\pi_1(M_K)\to G\mid \varphi \text{ is an epimorphism that satisfies (FT)}\},$$

[2] In the work of Hambleton, Kreck and Teichner [Hambleton et al. 2009] W–AA only requires A_4 to be injective.

and observe that it is acted upon (by postcomposition) by the group $\mathrm{Aut}(G)$ of automorphisms of G. Thanks to the discussion leading up to Theorem 1.4, note that sending a G–ribbon disc with aspherical exterior to an epimorphism $\pi_1(M_K) \twoheadrightarrow \pi_1(N_D) \cong G$ defines a map

$$\Phi\colon \mathcal{D}^a_G(K) \to \mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K), G)/\mathrm{Aut}(G)$$

which does not depend on the choice of the isomorphism $\pi_1(N_D) \cong G$. If G is a good geometrically 2–dimensional ribbon group such that $\widetilde{L}_4(\mathbb{Z}[G]) = 0$, then Theorem 1.4 ensures that Φ is surjective. Our main technical result gives conditions on G for Φ to be injective and, in the absence of the goodness condition on G, for exteriors of G–ribbon discs to be s–cobordant rel boundary.

Theorem 1.7 *Let K be a knot and let G be a geometrically 2–dimensional group that satisfies* (W) *and* (A5)*. If D_1 and D_2 are two G–ribbon discs with aspherical exteriors and boundary K such that $\Phi(D_1) = \Phi(D_2)$, then the disc exteriors N_{D_1} and N_{D_2} are s–cobordant rel boundary.*

If in addition to these conditions the group G is good, then the discs D_1 and D_2 are ambiently isotopic rel boundary.

We note that this result can alternatively be stated with normal subgroups instead of epimorphisms, as this is easier to verify in practice. To state this concisely, given a slice disc D for a knot K, we use $\iota_D\colon \pi_1(M_K) \to \pi_1(N_D)$ to denote the inclusion-induced map.

Corollary 1.8 *Let K be a knot and let G be a geometrically 2–dimensional group that satisfies* (W) *and* (A5)*. If D_1 and D_2 are two G–ribbon discs with aspherical exteriors and boundary K such that $\ker(\iota_{D_1}) = \ker(\iota_{D_2})$, then the disc exteriors N_{D_1} and N_{D_2} are s–cobordant rel boundary.*

If in addition to these conditions the group G is good, then the discs D_1 and D_2 are ambiently isotopic rel boundary.

For smoothly embedded discs, the hypotheses of these results can be relaxed:

Remark 1.9 If D_1 and D_2 are ribbon discs with aspherical exteriors and $\pi_1(N_{D_i}) \cong G$ for $i = 1, 2$, then the assumption that G be geometrically 2–dimensional can be omitted in both Theorem 1.7 and Corollary 1.8: in this case $K(G,1) \simeq N_{D_i}$ has the homotopy type of a 2–complex.

Combining Theorems 1.4 and 1.7 we obtain an answer to Question 1.1, provided we make some restrictions on the ribbon group G and require the ribbon disc exteriors to be aspherical.

Theorem 1.10 *Let $K \subset S^3$ be a knot and let G be a geometrically 2–dimensional good ribbon group that satisfies properties W–AA. Mapping a G–ribbon disc D to the epimorphism $\pi_1(M_K) \twoheadrightarrow \pi_1(N_D) \cong G$ defines a bijection Φ between*

(i) *the set $\mathcal{D}^a_G(K)$ of G–ribbon discs with aspherical exterior and boundary K, considered up to ambient isotopy rel boundary, and*

(ii) *the set $\mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K), G)/\mathrm{Aut}(G)$ defined in* (Epi).

Proof We argue in Remark 2.1 that since G is a geometrically 2–dimensional ribbon group with $\mathrm{Wh}(G)=0$, requiring G to satisfy condition (A4) is equivalent to asking for $\widetilde{L}_4(\mathbb{Z}[G])=0$. Thus the hypotheses of Theorem 1.4 are satisfied and so Φ is surjective. The injectivity of Φ follows from Theorem 1.7, which we can apply since G satisfies properties W–AA. □

Remark 1.11 • If the ribbon group conjecture (or more optimistically the Whitehead conjecture) were true, then requiring G to be geometrically 2–dimensional would be superfluous; recall Remark 1.3. It is also tempting to conjecture that exteriors of G–ribbon discs are aspherical, and in this case we would have $\mathcal{D}^a_G(K)=\mathcal{D}_G(K)$. This latter conjecture holds when G is PTFA [Conway and Powell 2021, Lemma 2.1] (eg when $G=\mathbb{Z}$ and $G=\mathrm{BS}(1,2)$) and is a consequence of the Whitehead conjecture if the disc exterior is homotopy equivalent to a 2–complex.

• The groups $\mathbb{Z}$ and $\mathrm{BS}(1,2)$ satisfy the hypotheses of Theorem 1.10, and in this case unpacking the definition of $\mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K),G)/\mathrm{Aut}(G)$ recovers [loc. cit., Theorems 1.5 and 1.6]. Instead of repeating those statements, we note that for $G=\mathbb{Z}$, $\mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K),G)/\mathrm{Aut}(G)$ has at most one element, while for $G=\mathrm{BS}(1,2)$ it has at most two [loc. cit., Section 4]. Estimating the cardinality of this set in general appears to be more challenging. Naturally, the set $\mathcal{D}_G(K)$ is often empty: for example, we refer to [Friedl and Teichner 2005, Corollary 3.4] for an obstruction (based on the Alexander polynomial) to a knot K bounding a G–ribbon disc.

• As we alluded to in Corollary 1.8, the classification result of Theorem 1.10 can be stated in terms of normal subgroups of $\pi_1(M_K)$ instead of epimorphisms originating from $\pi_1(M_K)$: to a G–ribbon disc D, one associates the normal subgroup $\ker(\pi_1(M_K)\twoheadrightarrow\pi_1(N_D))$ of $\pi_1(M_K)$. This was the perspective taken in [Conway and Powell 2021] where, using that $\mathrm{BS}(1,2)$ is metabelian, the results were then formulated using submodules of the Alexander module $H_1(M_K;\mathbb{Z}[t^{\pm1}])$; the details are in [loc. cit., Section 3].

• The requirement that the group be good is hard to verify in practice. On the other hand, G satisfies properties W–AA if it is geometrically 2–dimensional and satisfies the Farrell–Jones conjecture: if a group G is geometrically 2–dimensional, then $K(G,1)$ is a 2–complex and the claim now follows as in [Kasprowski and Land 2022, Lemma 2.3]. (The core of the argument will be recalled both in the proof of Theorem 1.7 and in Remark 2.1.) We treat the Farrell–Jones conjecture as a blackbox, but refer the interested reader to [Lück 2021] for a survey and to [Lück 2021, Chapter 15] for a list of groups for which the conjecture is known to hold.

Example 1.12 We argue that the group $G(J)=\pi_1(S^3\setminus J)$ of a classical knot $J\subset S^3$ is a geometrically 2–dimensional ribbon group that satisfies properties W–AA. Thus Theorem 1.7 provides a criterion for exteriors of $G(J)$–ribbon discs to be s–cobordant rel boundary and, if $G(J)$ is additionally assumed to be good, then Theorem 1.10 classifies $G(J)$–ribbon discs for $J\#-J$.

The group of $J\subset S^3$ is ribbon (the ribbon knot $J\#-J$ bounds a smoothly embedded ribbon disc with group $G(J)$ as explained in [Friedl and Teichner 2005, page 2135]). The sphere theorem ensures that

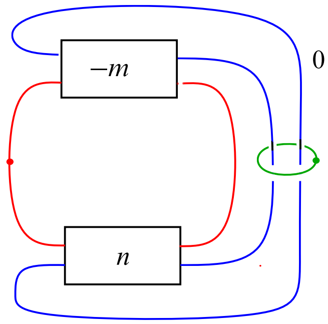

Figure 1: Assuming that $|m-n|=1$, this figure depicts a handle diagram of a ribbon disc exterior with fundamental for $\mathrm{BS}(m,n)$. Indeed, since $|m-n|=1$, the red and blue knots form a handle diagram for D^4 in which the green knot is sliced by a ribbon disc D with $\pi_1(N_D)=\mathrm{BS}(m,n)$.

$G(J)$ is geometrically 2–dimensional (the knot exterior is aspherical and has the homotopy type of a 2–complex; see eg [Lickorish 1997, Theorem 11.7]). The Farrell–Jones conjecture holds for $G(J)$ because it holds for the fundamental group of any 3–manifold with boundary [Lück 2021, Theorem 15.1(e)].

Since knot groups are PTFA by work of Strebel [1974], $G(J)$–ribbon discs are aspherical by [Conway and Powell 2021, Lemma 2.1] and thus $\mathcal{D}_{G(J)}(J\#-J)=\mathcal{D}^a_{G(J)}(J\#-J)$. Finally, as we noted in Remark 1.5, since $G(J)$ is PTFA we can use condition (Ext) instead of condition (FT).

Example 1.13 We argue that for $m,n\in\mathbb{Z}$ with $|m-n|=1$, the Baumslag–Solitar group $\mathrm{BS}(m,n)$ is a geometrically 2–dimensional ribbon group that satisfies properties W–AA. Thus Theorem 1.7 provides a criterion for exteriors of aspherical $\mathrm{BS}(m,n)$–ribbon discs to be s–cobordant rel boundary and, if $\mathrm{BS}(m,n)$ is additionally assumed to be good, then Theorem 1.10 classifies $\mathrm{BS}(m,n)$–ribbon discs with aspherical exteriors.

The fact that $\mathrm{BS}(m,n)$ is ribbon when $|m-n|=1$ can be seen by looking at the handle diagram depicted in Figure 1. Baumslag–Solitar groups are geometrically 2–dimensional: the universal cover of the presentation 2–complex for $\langle a,b \mid ba^mb^{-1}=b^n\rangle$ is homeomorphic to the product of $\mathbb{R}$ with a tree; see eg [Freden et al. 2011, Section 2]. Additionally, every Baumslag–Solitar group $\mathrm{BS}(m,n)$ satisfies the Farrell–Jones conjecture [Farrell and Wu 2015; Gandini et al. 2015].

We conclude with a brief final remark concerning asphericity. Our methods rely heavily on G–ribbon disc exteriors (conjecturally) being aspherical. Currently, nonaspherical 4–manifolds with boundary M_K and fundamental group G are poorly understood beyond the group $G=\mathbb{Z}$ [Conway and Powell 2023]. This is the reason why we only work in D^4 instead of in other 4–manifolds.

Conventions Throughout this article, we work in the topological category. Manifolds are assumed to be compact and oriented. Homeomorphisms, homotopy equivalences and isotopies are *rel boundary* if they fix the boundary pointwise. If M_1 and M_2 are two n–manifolds with boundary Y, a cobordism between M_1 and M_2 is *relative* Y if, when restricted to Y, it is the product $Y\times[0,1]$.

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2 Proof of the main technical result

We recall the statement of Theorem 1.7 and prove it. Let K be a knot and let G be a geometrically 2–dimensional group that satisfies (W) and (A5). The aim is to prove that if D_1 and D_2 are two G–ribbon discs with aspherical exteriors and boundary K such that $\Phi(D_1)=\Phi(D_2)\in \mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K),G)/\mathrm{Aut}(G)$, then the disc exteriors N_{D_1} and N_{D_2} are s–cobordant rel boundary and, if G is additionally assumed to be good, then D_1 and D_2 are ambiently isotopic rel boundary.

Proof of Theorem 1.7 Assume that D_1 and D_2 are two G–ribbon discs with aspherical exteriors and boundary K and that their epimorphisms agree in $\mathrm{Epi}^{\mathrm{FT}}(\pi_1(M_K),G)/\mathrm{Aut}(G)$. We must show that the exteriors N_{D_1} and N_{D_2} are s–cobordant rel boundary. If we additionally assume that G is good, then Freedman's 5–dimensional relative s–cobordism theorem will then ensure that N_{D_1} and N_{D_2} are in fact homeomorphic rel boundary. That D_1 and D_2 are ambiently isotopic rel boundary follows by applying Alexander's trick, as noted in [Conway and Powell 2021, Lemma 2.5]. Our strategy decomposes into two steps. The first uses the conditions on the epimorphisms to show that id_{M_K} extends to a homotopy equivalence $N_{D_1}\simeq N_{D_2}$. The second uses surgery theory to improve this homotopy equivalence to an s–cobordism rel boundary; here is where we rely on properties (W) and (A5) as well as on the fact that G is good.

We start with the first step. Since the epimorphisms of D_1 and D_2 agree, there exists an automorphism Ψ of G that makes the following diagram commute:

$$\begin{array}{ccc}
\pi_1(M_K) & \xrightarrow{\;=\;} & \pi_1(M_K) \\
{\scriptstyle \iota_{D_1}}\downarrow & & \downarrow{\scriptstyle \iota_{D_2}} \\
\pi_1(N_{D_1}) & & \pi_1(N_{D_2}) \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
G & \xrightarrow{\;\Psi,\cong\;} & G.
\end{array}$$

Since the bottom vertical maps in this diagram are isomorphisms, we deduce that there exists an isomorphism $g\colon \pi_1(N_{D_1})\cong \pi_1(N_{D_2})$ such that $g\circ\iota_{D_1}=\iota_{D_2}$; such isomorphisms were called *compatible* in [loc. cit., Section 2]. As the D_i have aspherical exteriors, the obstruction theory argument from [loc. cit., end of proof of Lemma 2.1] shows that the identity $\mathrm{id}_{M_K}\colon M_K\to M_K$ extends to a homotopy equivalence $f\colon N_{D_1}\to N_{D_2}$ which induces g on fundamental groups.

We now move on to the second step: we use surgery theory to improve the homotopy equivalence f to an s–cobordism $N_{D_1} \cong_{s\text{–cob}} N_{D_2}$ rel boundary. We describe the argument very briefly for readers that are familiar with surgery theory before giving some more details. Consider the surgery sequence, where we can ignore decorations thanks to condition (W),

$$\mathcal{N}(N_{D_2} \times [0,1], \partial(N_{D_2} \times [0,1])) \xrightarrow{\sigma_5} {}_5(\mathbb{Z}[G]) \to \mathcal{S}(N_{D_2}, \partial N_{D_2}) \xrightarrow{\eta} N(N_{D_2}, \partial N_{D_2}) \xrightarrow{\sigma_4} {}_4(\mathbb{Z}[G]).$$

We use that disc exteriors have trivial H_2 to deduce that η is the zero map. More concretely, we obtain a degree-1 normal map

$$(F', f, \mathrm{id}_{N_{D_2}}) \colon (W', N_{D_1}, N_{D_2}) \to (N_{D_2} \times [0,1], N_{D_2}, N_{D_2}) \tag{1}$$

that we can assume to be 2–connected by surgery below the middle dimension. We then use property (A5) and the fact that G is geometrically 2–dimensional to deduce that σ_5 is surjective. We infer that N_{D_1} and N_{D_2} are s–cobordant either by appealing to the exactness of the surgery sequence (which requires G to be good) or by using the surjectivity of σ_5 to replace F' by another degree-1 normal map with vanishing surgery obstruction (despite being slightly longer, this argument has the advantage of not requiring G to be good). Thus the fact that N_{D_1} and N_{D_2} are s–cobordant rel boundary can be proved without using that G is good. The homeomorphism classification result then follows from Freedman's 5–dimensional relative s–cobordism theorem, which we can apply if G is good.

We give more details. The set $\mathcal{N}(N_{D_2}, \partial N_{D_2})$ consists of equivalences classes of degree-1 normal maps $M \to N_{D_2}$ that restrict to a homeomorphism on the boundary. Two such degree-1 normal maps $f_i \colon M_i \to N_{D_2}$ for $i = 1, 2$ are equivalent if there exists a rel boundary cobordism (W, M_1, M_2) and a degree-1 normal map

$$(W, M_1, M_2) \to (N_{D_2} \times [0,1], N_{D_2}, N_{D_2})$$

that restricts to f_i on M_i for $i = 1, 2$. A homotopy equivalence $h \colon M \to N_{D_2}$ rel boundary is in particular a degree-1 normal map that we denote by $\eta(h) \in \mathcal{N}(N_{D_2}, \partial N_{D_2})$.

We claim that η is the zero map. Under the isomorphism

$$\mathcal{N}(N_{D_2}, \partial N_{D_2}) \cong H^4(N_{D_2}, \partial N_{D_2}) \oplus H^2(N_{D_2}, \partial N_{D_2}; \mathbb{Z}_2) = H^4(N_{D_2}, \partial N_{D_2}) \cong \mathbb{Z} \tag{2}$$

we have $\eta(h) = \frac{1}{8}(\sigma(M) - \sigma(N_{D_2}))$; this fact is well known to surgeons, but we refer to [Conway and Powell 2021, Proposition 2.2] in case the reader is curious about the details. Since the signature of a disc exterior vanishes and h is a homotopy equivalence, we deduce that $\eta(h) = 0$, as claimed.

We assert that the map σ_5 from the surgery sequence is surjective. This relies on surgery spectra and the algebraic theory of surgery. We treat this largely as a blackbox, but note that this part of surgery theory was developed by Quinn [1970; 1971] and Ranicki [1979; 1981]; we also refer to [Chang and Weinberger 2021, Section 4.4] for a nice overview of these topics and to [Cencelj et al. 2009, Section 4] for a helpful account of the rel boundary case. Using the relation between the assembly map and the surgery obstruction (as mentioned for example in [Chang and Weinberger 2021, page 158]) and the fact that N_{D_2} is a $K(G,1)$, the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{N}(N_{D_2}\times[0,1],\partial(N_{D_2}\times[0,1])) & & \xrightarrow{\quad\sigma_5\quad} & & L_5(\mathbb{Z}[G]) \\
\downarrow\cong & & & & \downarrow= \\
H_5(N_{D_2};\boldsymbol{L}\langle 1\rangle_\bullet) & \xrightarrow{\cong} & H_5(N_{D_2};\boldsymbol{L}_\bullet) & \longrightarrow & L_5(\mathbb{Z}[G]) \\
\downarrow\cong & & \downarrow\cong & & \downarrow= \\
H_5(\mathrm{BG};\boldsymbol{L}\langle 1\rangle_\bullet) & \xrightarrow{\cong} & H_5(\mathrm{BG};\boldsymbol{L}_\bullet) & \xrightarrow{A_5} & L_5(\mathbb{Z}[G]).
\end{array}$$

Here $\boldsymbol{L}_\bullet$ denotes the L–theory spectrum of the integers and $\boldsymbol{L}\langle 1\rangle_\bullet$ denotes its 1–connective cover. The fact that $H_5(\mathrm{BG};\boldsymbol{L}\langle 1\rangle_\bullet)\to H_5(\mathrm{BG};\boldsymbol{L}_\bullet)$ is an isomorphism follows because $K(G,1)$ admits a 2–dimensional CW–model (the Atiyah–Hirzebruch spectral sequence argument is the same as in [Kasprowski and Land 2022, proof of Lemma 2.3]), and the fact that the top left vertical map is an isomorphism is a fact from algebraic surgery theory; see eg [Cencelj et al. 2009, (27)]. Using this commutative diagram and property (A5) (which stipulates that the assembly map A_5 is surjective), one deduces that σ_5 is surjective.

There are now two closely related ways to conclude that N_{D_1} and N_{D_2} are s–cobordant rel boundary. The first way is shorter, but uses that the group G is good: since η is the zero map, σ_5 is surjective and the surgery sequence is exact (because G is good), the structure set $\mathcal{S}(N_{D_2},\partial N_{D_2})$ — to which f belongs — is trivial. The second argument (inspired by [Kasprowski and Land 2022]) is slightly longer but does not require that the group G be good: Since $\eta\equiv 0$, there is a rel boundary cobordism (W,N_{D_1},N_{D_2}) and a degree-1 normal map

$$(F,f,\mathrm{id}_{N_{D_2}})\colon (W,N_{D_1},N_{D_2})\to(N_{D_2}\times[0,1],N_{D_2},N_{D_2}).$$

Perform surgery below the middle dimension on the interior of W to obtain the 2–connected degree-1 normal map F' with surgery obstruction $x:=\sigma(F')\in L_5(\mathbb{Z}[G])$ that we alluded to in (1). Using the surjectivity of σ_5, one can find a degree-1 normal map

$$\Psi\colon (V,N_{D_2},N_{D_2})\to(N_{D_2}\times[0,1],N_{D_2},N_{D_2})$$

that restricts to the identity on both boundary components and with $-x$ as its surgery obstruction; stacking Ψ on top of F' leads to a degree-1 normal map F'' with vanishing surgery obstruction $\sigma(F'')\in L_5(\mathbb{Z}[G])$, and it follows that F'' is normal bordant rel $M_K\times[0,1]$ to a homotopy equivalence. Thus, we have two arguments for why N_{D_1} and N_{D_2} are s–cobordant rel boundary.

If G is good, we can apply Freedman's 5–dimensional relative s–cobordism theorem [Freedman and Quinn 1990, Theorem 7.1A] and it follows that N_{D_1} and N_{D_2} are homeomorphic rel boundary. As we already mentioned, [Conway and Powell 2021, Lemma 2.5] implies that the discs are ambiently isotopic rel boundary. □

We conclude by proving a statement that was used in the proof of Theorem 1.10:

Remark 2.1 Assume that G is a geometrically 2–dimensional ribbon group with vanishing Whitehead torsion (condition (W)). We claim that G satisfies $\widetilde{L}_4(\mathbb{Z}[G])=0$ if and only if it satisfies (A4), which stipulates that the assembly map $A_4\colon H_4(\mathrm{BG};\boldsymbol{L}_\bullet)\to L_4(\mathbb{Z}[G])$ is an isomorphism. Since G is a ribbon group, there is a (smoothly embedded) ribbon disc $D\subset D^4$ with $\pi_1(N_D)\cong G$. This time N_D might not be aspherical, but it is still a 2–complex with vanishing H_2. An Atiyah–Hirzebruch spectral sequence argument therefore shows that $H_4(N_D;\boldsymbol{L}\langle 1\rangle_\bullet)\to H_4(\mathrm{BG};\boldsymbol{L}\langle 1\rangle_\bullet)$ is an isomorphism. Here it is helpful to note that $H_2(G)=0$: use $H_2(N_D)=0$ together with the exact sequence $\pi_2(N_D)\to H_2(N_D)\to H_2(\pi_1(N_D))\to 0$; see eg [Brown 1982, (0.1)]. The same argument as above then produces the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{N}(N_D,M_K) & & \xrightarrow{\quad\sigma_4\quad} & & L_4(\mathbb{Z}[G]) \\
\downarrow\cong & & & & \downarrow = \\
H_4(N_D;\boldsymbol{L}\langle 1\rangle_\bullet) & \xrightarrow{\cong} & H_4(N_D;\boldsymbol{L}_\bullet) & \longrightarrow & L_4(\mathbb{Z}[G]) \\
\downarrow\cong & & \downarrow\cong & & \downarrow = \\
H_4(\mathrm{BG};\boldsymbol{L}\langle 1\rangle_\bullet) & \xrightarrow{\cong} & H_4(\mathrm{BG};\boldsymbol{L}_\bullet) & \xrightarrow{A_4} & L_4(\mathbb{Z}[G]).
\end{array}$$

As explained in (2) and [Freedman and Quinn 1990, Section 11.3B], the surgery obstruction σ_4 maps the set of normal invariants $\mathcal{N}(N_D,\partial N_D)\cong\mathbb{Z}$ isomorphically onto the $L_4(\mathbb{Z})\cong\mathbb{Z}$–summand of $L_4(\mathbb{Z}[G])=L_4(\mathbb{Z})\oplus\widetilde{L}_4(\mathbb{Z}[G])$. The claim now follows by combining this fact with the commutativity of the diagram.

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