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Shadows of 2–knots and complexity

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We introduce a new invariant for a 2–knot in S^4 , called the shadow-complexity, based on the theory of Turaev shadows, and we give a characterization of 2–knots with shadow-complexity at most 1. Specifically, we show that the unknot is the only 2–knot with shadow-complexity 0 and that there exist infinitely many 2–knots with shadow-complexity 1.

57K40, 57K45; 57R65

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1 Introduction

A 2–knot is a smoothly embedded 2–sphere in S^4 . The first example of a nontrivial 2–knot was given by Artin [1925], called a spun knot. The nontriviality is of fundamental interest in knot theory. In the classical case, namely the 1–knot case, the trefoil knot can be said to be the simplest nontrivial knot in terms of some numerical invariants: the crossing number, the bridge number, the tunnel number, and so forth. Then one naturally wonders which 2–knot is the simplest nontrivial one. The answer to this question should be based on as many criteria as possible that measures, in some sense, how different a given 2–knot is from the trivial one.

There are several studies on numerical invariants for 2–knots. For examples, we refer the reader to [Satoh 2000; Satoh and Shima 2004; Yajima 1964] for the triple point number and to [Saito and Satoh 2005; Satoh 2009] for the sheet number. These two invariants are defined with broken surface diagrams [Carter

et al. 1997; Roseman 1998], which is a natural analogue of classical knot diagrams. Yoshikawa [1994] introduced the ch-index by using ch-diagrams (also called marked graph diagrams), and he gave the table of 2–knots with ch-index up to 10. These invariants measure how complicated the descriptions of a given 2–knot must be, like the crossing numbers for 1–knots. On the one hand, for example, the bridge number and the tunnel number for classical knots seem to be the complexity of the embedding or the complement. Recently, bridge positions of knotted surfaces, called bridge trisections, were introduced by Meier and Zupan [2017; 2018] using a trisection (see [Gay and Kirby 2016] for the details) of the ambient 4–manifold, and the bridge numbers for knotted surfaces were defined by using this notion. Incidentally, Kirby and Thompson introduced an integer-valued invariant, called the \mathcal{L} –invariant or the Kirby–Thompson length, for closed 4–manifolds, and this notion was adapted to the knotted surfaces in [Blair et al. 2022].

In the present paper, we propose to study 2–knots using a Turaev *shadow*; a notion introduced by Turaev [1994]. A shadow is a simple polyhedron embedded in a closed 4–manifold as a 2–skeleton, in other words, a simple polyhedron such that the complement of its neighborhood is diffeomorphic to a 4–dimensional 1–handlebody. Turaev showed that regions of a shadow are naturally equipped with half-integers such as “relative self-intersection numbers”, which has sufficient information to reconstruct the ambient 4–manifold. It is known as Turaev’s reconstruction theorem. Thus, a shadow can be treated as a description of a 4–manifold, which brings about several studies: Stein structures, spin^c structures and complex structures, stable maps and hyperbolic structures on 3–manifolds, Lefschetz fibrations, and so on; see [Costantino 2006b; 2008; Costantino and Thurston 2008; Ishikawa and Koda 2017; Ishikawa and Naoe 2020] for examples. Moreover, as another benefit of the theory of shadows, we can define a complexity for 4–manifolds, called the *shadow-complexity*. Costantino [2006a] introduced this notion. The shadow-complexity of a 4–manifold is defined as the minimum number of specific points called true vertices contained in a shadow of the 4–manifold. He also gave the classification of closed 4–manifolds with complexity up to 1 in a special case. Martelli [2011] gave a characterization of all the closed 4–manifolds with shadow-complexity 0. He also showed that a closed simply connected 4–manifold with shadow-complexity 0 is diffeomorphic to S^4 or the connected sum of some copies of $\mathbb{C}\mathbb{P}^2$, $\overline{\mathbb{C}\mathbb{P}^2}$ and $S^2 \times S^2$. This implies that the shadow-complexity is a diffeomorphism invariant. Note that exotic smooth structures on $\mathbb{C}\mathbb{P}^2 \# k\overline{\mathbb{C}\mathbb{P}^2}$ have been found for some k . Koda, Martelli and the present author [Koda et al. 2022] introduced a new complexity, called the connected shadow-complexity, and gave a characterization of all the closed 4–manifolds with connected shadow-complexity at most 1.

If a surface F is embedded in a shadow of a 4–manifold, then such an F in the 4–manifold is smoothly embedded and generally knotted. In view of this situation, we define a shadow of a 2–knot K as a shadow X of the ambient 4–manifold S^4 such that K is embedded in X . Of course, we can define a shadow for a knotted surface as well (see Remark 4.5), but the focus in this paper is 2–knots.

Every closed 4–manifold admits a shadow, which follows from the existence of a handle decomposition. We can also show the following.

Theorem 3.2 *Every 2-knot admits a shadow.*

The above theorem allows us to define a complexity for 2-knots

$$\{2\text{-knots in } S^4\} \rightarrow \mathbb{Z}_{\geq 0}$$

as the minimum number of true vertices in a shadow of K . We call this number the *shadow-complexity* of K and write it as $\text{sc}(K)$.

The aim of this paper is to give the classification of 2-knots with shadow-complexity at most 1. First, one might expect that the unknot is the only 2-knot with shadow-complexity 0 as well as some other numerical invariants for 1- or 2-knots, which is indeed true.

Theorem 6.4 *A 2-knot has shadow-complexity 0 if and only if it is unknotted.*

The unknotted 2-knot admits a special shadow with no true vertices, which implies that the unknot is also the only 2-knot with special shadow-complexity 0. The condition for a shadow to be special is fairly strong, and any special polyhedron with one true vertex cannot be a shadow of any 2-knot except for the unknot.

Theorem 7.2 *There are no 2-knots with special shadow-complexity 1.*

It is noteworthy that the special shadow-complexity for closed 4-manifolds is a finite-to-one invariant [Martelli 2005, Corollary 2.7]. However, that for 2-knots is not finite-to-one as noted in Remark 8.11, where we find infinitely many 2-knots with special shadow-complexity at most 5. We actually have not determined the special shadow-complexity of any nontrivial 2-knot. Note that all the special polyhedra with complexity up to 2 are listed in [Koda and Naoe 2020, Appendix B], so we already have possible candidates of shadows of 2-knots with special shadow-complexity 2 if such a 2-knot exists.

Before stating the theorem on the complexity 1 case, we introduce a series of 2-knots. For $n \in \mathbb{Z}$, let K_n be a 2-knot presented by a banded unlink diagram shown in Figure 1. See [Hughes et al. 2020] and Section 4 for the definition and the details of banded unlink diagrams. Note that K_0 is trivial. The knot K_{-1} is the spun trefoil, and K_1 is the knot 9_1 in Yoshikawa's table [1994]. For any $n \in \mathbb{Z}$, the knot K_n is a ribbon 2-knots. As stated in Proposition 8.9, two 2-knots K_n and $K_{n'}$ are not equivalent unless $n = n'$, which can be distinguished by their Alexander polynomials.

The following is the main theorem for 2-knots with shadow-complexity 1.

Theorem 8.10 *A 2-knot K whose knot group is not infinite cyclic has shadow-complexity 1 if and only if K is diffeomorphic to K_n for some nonzero integer n .*

The unknotting conjecture says that a 2-knot is unknotted if its knot group is infinite cyclic, which is known to be true in the topological category in [Freedman and Quinn 1990] and is also studied in the

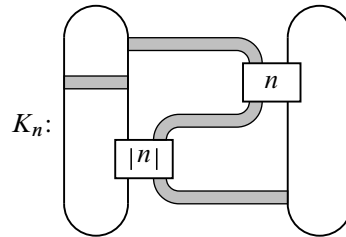


Figure 1: Banded unlink diagram of K_n for $n \in \mathbb{Z}$. Note that the number written in the lower-left box is the absolute value $|n|$ of n .

smooth category in [Kawauchi 2021] (see also [Kawauchi 2023]). Supposing the unknotting conjecture is true in the smooth category, we obtain the complete classification of all 2–knots with shadow-complexity at most 1.

One interpretation of the shadow-complexity for 2–knots is an analogue to the tunnel number for 1–knots. We recall that the tunnel number of a 1–knot is the minimum number of 1–cells such that the complement of the neighborhood of the union of the 1–knot and the 1–cells is a 3–dimensional 1–handlebody. The procedure to make the complement trivial is similar to a construction of a shadow from a 2–knot. By definition, a shadow of a 2–knot K can be obtained from K by attaching 1–cells and 2–cells so that the complement of the neighborhood is diffeomorphic to a 4–dimensional 1–handlebody. In this sense, the shadow-complexity can seem to measure the nontriviality of the complement of a given 2–knot.

Organization

In Section 2, we review the theory of Turaev shadows and encoding graphs. In Section 3, we define a shadow of a 2–knot, and give a presentation of the knot group using a shadow. In Section 4, we introduce a banded unlink diagram, from which we construct a shadow of a 2–knot. In Section 5, we introduce two important modifications: compressing disk addition and connected-sum and reduction. In Section 6, we give the proof for the complexity zero case. A large part of Section 7 is devoted to compute the knot groups of shadows having one true vertex. In Section 8, we give the proofs for the complexity one case by describing the 2–knots in banded unlink diagrams.

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2 Preliminaries

For polyhedral spaces $A \subset B$, let $\text{Nbd}(A; B)$ denote a regular neighborhood of A in B . If B collapses onto A , we use the notation $B \searrow A$.

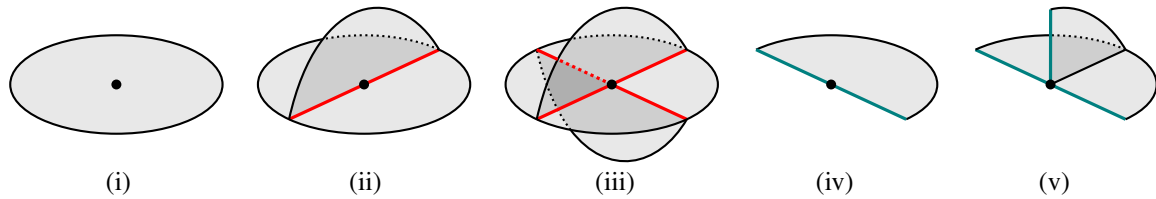


Figure 2: Local models of simple polyhedra.

For integers $0 \leq k \leq n$, an n -dimensional k -handlebody is an n -dimensional manifold admitting a handle decomposition consisting of handles whose indices are at most k .

Throughout this paper, we assume that any manifold is compact, connected, oriented and smooth unless otherwise noted.

2.1 Simple polyhedra and shadows of 4-manifolds with boundary

A *simple polyhedron* is a compact connected space whose every point has a regular neighborhood homeomorphic to one of (i)–(v) in Figure 2. Let X be a simple polyhedron. The *singular set* $S(X)$ of X is the set of points of type (ii), (iii) or (v) in Figure 2. The point of type (iii) is called a *true vertex*. Let $c(X)$ denote the number of true vertices contained in X , and this number is called the *complexity* of X . A connected component consisting of points of type (ii) is called a *triple line*. The *boundary* ∂X of X is the set of points of type (iv) or (v). It is clear that the boundary of a simple polyhedron is a (possibly nonconnected) trivalent graph. If $\partial X = \emptyset$, we say that X is *closed*. A *region* of X is a connected component of $X \setminus S(X)$. A region is called a *boundary region* if it contains a point of ∂X (or equivalently, a point of type (iv)), otherwise it is called an *internal region*. If every region is simply connected, then X is called a *special polyhedron*.

We then define a shadow of a 4-manifold with boundary.

Definition 2.1 Let M be a 4-manifold with boundary. A simple polyhedron X properly embedded in M is called a *shadow* of M if X is locally flat in M and $M \searrow X$.

Note that X is *locally flat* in M if for any $x \in X$, there exists a local chart (U_x, ϕ_x) around x in M such that $X \cap U_x$ is contained in a smooth 3-ball in U_x .

The notion of a shadow was introduced by Turaev [1994], and he proved the following.

Theorem 2.2 [Turaev 1994] *Any 4-dimensional 2-handlebody admits a shadow.*

We next define *gleams* of regions of a shadow. Let R be an internal region of X , and set $X_S = \text{Nbd}(S(X); X)$. Then there exists a (possibly nonorientable) 3-dimensional 1-handlebody H_S in M such that X_S is properly embedded in H_S and $H_S \searrow X_S$. Set $\bar{R} = R \setminus \text{Int } H_S$. Its boundary $\partial \bar{R} = R \cap \partial H_S$ forms a disjoint union of circles in the surface ∂H_S . Set $B = \text{Nbd}(\bar{R}; \partial H_S)$, which is a disjoint union

of some annuli or Möbius bands. Let \bar{R}' be a small perturbation of \bar{R} in M such that $\partial\bar{R}' \subset B$ and $|\bar{R} \cap \bar{R}'| < \infty$. Then the *gleam* $gl(R)$ is defined as

$$(1) \quad gl(R) = \#(\text{Int } \bar{R} \cap \text{Int } \bar{R}') + \frac{1}{2}\#(\partial\bar{R} \cap \partial\bar{R}'),$$

where $\#$ is the algebraic intersection number.

The number of the Möbius bands in B is actually determined only by the combinatorial structure of X , that is, it does not depend on the embedding $X \subset M$. If it is even, the region R is said to be *even*, otherwise *odd*. The gleam $gl(R)$ is an integer if and only if R is even.

2.2 Shadowed polyhedra and shadows of closed 4–manifolds

For a simple polyhedron X , we assign a half integer to each internal region R of X so that it is an integer if and only if R is even. Such a polyhedron is called a *shadowed polyhedron*.

The following theorem is known as Turaev’s reconstruction theorem.

Theorem 2.3 [Turaev 1994] *There is a canonical way to construct a 4–manifold M_X with boundary from a shadowed polyhedron X such that X is a shadow of M_X . Moreover, the gleam of an internal region of X coincides with that coming from (1).*

The gleam is a kind of “local self-intersection number” as one can see in (1). Indeed, the intersection form for the 4–manifold M_X , reconstructed from a shadowed polyhedron X , can be calculated with the gleam (see [Turaev 1994] and the next subsection). Especially, if a closed surface F is embedded in a shadowed polyhedron X , the sum of all the gleams of regions contained in F coincides with the self-intersection number of F in M_X .

We then define a shadow for a closed 4–manifold.

Definition 2.4 Let W be a closed 4–manifold. A simple polyhedron X embedded in W is called a *shadow* of W if it is locally flat in W and $W \setminus \text{Int Nbd}(X; W)$ is diffeomorphic to a 4–dimensional 1–handlebody.

By definition, $\partial \text{Nbd}(X; W)$ must be diffeomorphic to the connected-sum of some copies of $S^1 \times S^2$. Thus, if X is a shadowed polyhedron and ∂M_X is diffeomorphic to the connected-sum of some copies of $S^1 \times S^2$, then X is a shadow of some closed 4–manifold by [Laudenbach and Poénaru 1972].

We then define the complexities of 4–manifold that was introduced by Costantino [2006a].

Definition 2.5 For a 4–manifold W , the *shadow-complexity* $sc(W)$ and the *special shadow-complexity* $sc^{\text{SP}}(W)$ of W are the minimum number of true vertices of all shadows of W and that of all special shadows of W , respectively.

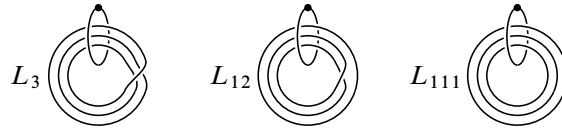


Figure 3: Links in $S^1 \times B^3$. The polyhedra Y_3, Y_{12} and Y_{111} can be constructed from these links.

2.3 Intersection forms

Let X be a shadow of a closed 4-manifold W . Since X can be considered as a 2-skeleton of W , the inclusion $\iota: X \hookrightarrow W$ induces an epimorphism $\iota_*: H_2(X) \rightarrow H_2(W)$.

We equip each orientable region R_i of X with an orientation arbitrarily. Then any element in $H_2(X)$ is represented by a sum $\sum a_i R_i$ of oriented regions R_1, \dots, R_n with integer coefficients $a_1, \dots, a_n \in \mathbb{Z}$. Defining the intersection form Q_X on $H_2(X)$ as

$$Q_X\left(\sum a_i R_i, \sum b_i R_i\right) = \sum a_i b_i \text{gl}(R_i),$$

we can calculate the intersection form Q_W on $H_2(W)$ as

$$Q_W(\alpha, \beta) = Q_X\left(\sum a_i R_i, \sum b_i R_i\right),$$

where α and β are the images of $\sum a_i R_i$ and $\sum b_i R_i$ by ι_* , respectively. See [Turaev 1994] for the details.

2.4 Topological types of neighborhoods of singular sets with $c \leq 1$

Let X be a simple polyhedron and S be a connected component of $S(X)$. Here we review the topological types of $\text{Nbd}(S; X)$ in the cases $c(\text{Nbd}(S; X)) = 0$ and 1. Note that $\text{Nbd}(S; X)$ itself is a simple polyhedron.

First suppose $c(\text{Nbd}(S; X)) = 0$, that is, S is a circle. There are three possibilities, Y_3, Y_{12} and Y_{111} , for topological types of $\text{Nbd}(S; X)$. These simple polyhedra are interpreted as follows. Let $\pi: S^1 \times B^3 \rightarrow S^1$ be the canonical projection, and let L_3, L_{12} and L_{111} be the links in $S^1 \times B^3$ given in Figure 3. Then Y_3, Y_{12} and Y_{111} are the mapping cylinders of π restricted to L_3, L_{12} and L_{111} , respectively.

Next we suppose $c(\text{Nbd}(S; X)) = 1$. Then S is an 8-shaped graph, that is, the wedge sum $S^1 \vee S^1$ of two circles. In this case, there are 11 possible topological types X_1, \dots, X_{11} of $\text{Nbd}(S; X)$, which are explained as follows. Let π be a natural projection from $(S^1 \times B^3) \natural (S^1 \times B^3)$ to $S^1 \vee S^1$, and let L_i be the link in $(S^1 \times B^3) \natural (S^1 \times B^3)$ given in Figure 4 for $i \in \{1, \dots, 11\}$. Then X_i is the mapping cylinder of π restricted to L_i .

Note that the over/under information of the links in Figures 3 and 4 does not matter for defining the polyhedra $Y_3, Y_{12}, Y_{111}, X_1, \dots, X_{11}$ since they are links in $S^1 \times B^3$ or $(S^1 \times B^3) \natural (S^1 \times B^3)$.

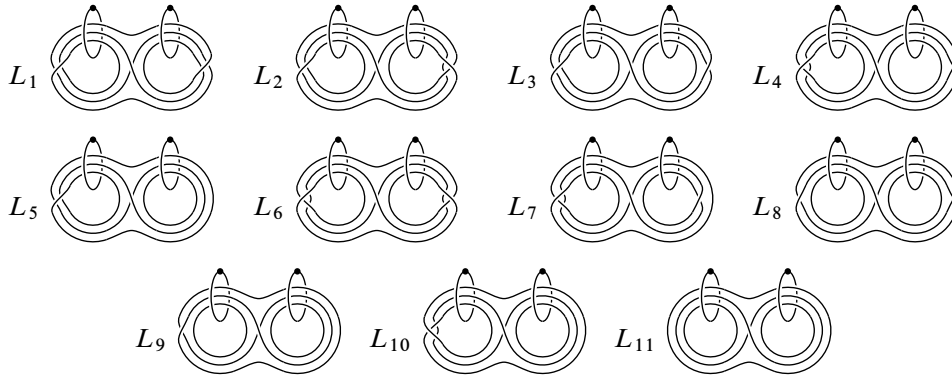


Figure 4: Links in $(S^1 \times B^3) \natural (S^1 \times B^3)$. The polyhedra X_1, \dots, X_{11} can be constructed from these links.

2.5 Encoding graphs

Here we explain a presentation of a simple polyhedron X using a graph consisting of some special vertices. This notion was introduced by Martelli [2011] for the case $c(X) = 0$ and generalized in [Koda et al. 2022] to the case where each connected component of $S(X)$ contains at most one true vertex.

Let X be a simple polyhedron whose boundary consists of circles and suppose that $c(X) \leq 1$. We first give a decomposition of X into some fundamental portions. Each connected component of $S(X)$ is homeomorphic to S^1 or $S^1 \vee S^1$. As reviewed in the previous subsection, a connected component of $\text{Nbd}(S(X); X)$ is homeomorphic to one of $Y_3, Y_{12}, Y_{111}, X_1, \dots, X_{11}$. Then each component of $X \setminus \text{Int Nbd}(S(X); X)$ is a compact surface corresponding to a region of X . Note that such a surface is possibly nonorientable, and hence it is decomposed into some disks, pair of pants and Möbius bands. Thus, we conclude that X is decomposed (along circles contained in regions) into some copies of $D, P, Y_2, Y_3, Y_{12}, Y_{111}, X_1, \dots, X_{11}$, where D is a 2-disk, P is a pair of pants, and Y_2 is a Möbius band.

The above decomposition induces an *encoding graph* G of X that has one vertex for each portion $D, P, Y_2, Y_3, Y_{12}, Y_{111}, X_1, \dots, X_{11}$ or boundary component of X . Two vertices are connected by an edge if the corresponding portions in X are adjacent. Hence each edge e of G corresponds to a simple closed curve contained in a region of X , which is determined up to isotopy. This simple closed curve

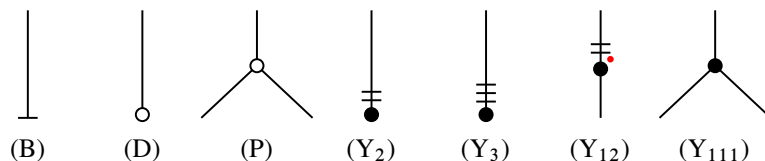


Figure 5: The vertices of types (B), (D), (P), (Y_2) , (Y_3) , (Y_{12}) , and (Y_{111}) , which correspond to boundary components of X and portions D, P, Y_2, Y_3, Y_{12} and Y_{111} , respectively.

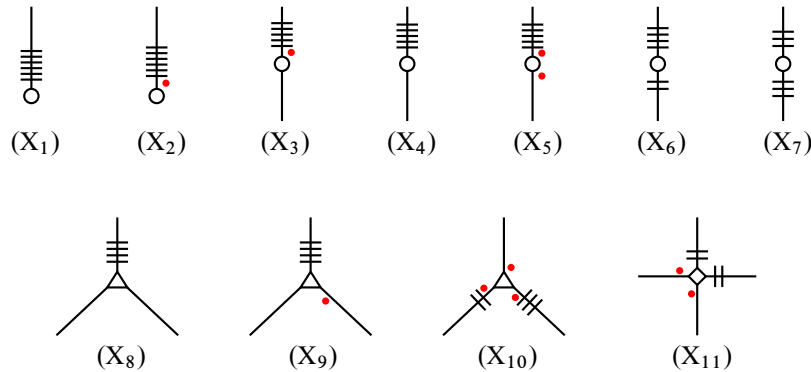


Figure 6: The vertices of types $(X_1), \dots, (X_{11})$, which correspond to portions X_1, \dots, X_{11} , respectively.

is called a *lift* of e . The vertices corresponding to $D, P, Y_2, Y_3, Y_{12}, Y_{111}, X_1, \dots, X_{11}$ are said to be of types (D), (P), $(Y_2), (Y_3), (Y_{12}), (Y_{111}), (X_1), \dots, (X_{11})$, respectively. These are shown in Figures 5 and 6, where the vertex of type (B) indicates a boundary component of X . See also [Koda et al. 2022; Martelli 2011].

If an encoding graph G is a tree, the polyhedron X is uniquely reconstructed from G up to homeomorphism. In such a case, we say that G *encodes* X . Hence, in this case, the fundamental group of X can be computed from G by using van Kampen’s theorem. The necessary information is summarized in Tables 1 and 2, which exhibit encoding graphs of the portions $D, P, Y_2, Y_3, Y_{12}, Y_{111}, X_1, \dots, X_{11}$, presentations of their fundamental groups and the homotopy classes of their boundaries. Here each vertex of type (B) is denoted by v_i for some $i \in \{1, \dots, 4\}$, and γ_i is the corresponding component of the boundary of a portion.

portion	graph	π_1	boundary classes in π_1
D		$\{1\}$	$\gamma_1 = 1$
P		$\langle x, y, z \mid xyz \rangle$	$\gamma_1 = x, \gamma_2 = y, \gamma_3 = z$
Y_2		$\langle x \rangle$	$\gamma_1 = x^2$
Y_3		$\langle x \rangle$	$\gamma_1 = x^3$
Y_{12}		$\langle x \rangle$	$\gamma_1 = x, \gamma_2 = x^2$
Y_{111}		$\langle x \rangle$	$\gamma_1 = \gamma_2 = \gamma_3 = x$

Table 1: Encoding graphs of D, P, Y_2, Y_3, Y_{12} and Y_{111} , their fundamental groups, and the homotopy classes of the boundary components.

portion	graph	π_1	boundary classes in π_1
X_1		$\langle x, y \rangle$	$\gamma_1 = xyx^{-2}y^{-2}$
X_2		$\langle x, y \rangle$	$\gamma_1 = xyx^2y^{-2}$
X_3		$\langle x, y \rangle$	$\gamma_1 = y, \gamma_2 = xyx^{-2}y^{-1}$
X_4		$\langle x, y \rangle$	$\gamma_1 = y, \gamma_2 = xyx^{-2}y$
X_5		$\langle x, y \rangle$	$\gamma_1 = y, \gamma_2 = xyx^2y^{-1}$
X_6		$\langle x, y \rangle$	$\gamma_1 = xy, \gamma_2 = x^2y^{-2}$
X_7		$\langle x, y \rangle$	$\gamma_1 = xy^2, \gamma_2 = x^2y^{-1}$
X_8		$\langle x, y \rangle$	$\gamma_1 = xyx^{-1}y^{-1}, \gamma_2 = x, \gamma_3 = y$
X_9		$\langle x, y \rangle$	$\gamma_1 = xyxy^{-1}, \gamma_2 = x, \gamma_3 = y$
X_{10}		$\langle x, y \rangle$	$\gamma_1 = y, \gamma_2 = xy, \gamma_3 = x^2y^{-1}$
X_{11}		$\langle x, y \rangle$	$\gamma_1 = x, \gamma_2 = y, \gamma_3 = xy, \gamma_4 = xy^{-1}$

Table 2: Encoding graphs of X_1, \dots, X_{11} , their fundamental groups, and the homotopy classes of the boundary components.

Each boundary component γ of $Y_3, Y_{12}, Y_{111}, X_1, \dots, X_{11}$ is represented by a word in $\langle x \rangle$ or $\langle x, y \rangle$ as in Tables 1 and 2, and its length coincides with the number of triple lines along which γ goes, counted with multiplicity. This number is called the *length* of γ as well.

Several edges of G are decorated with some dashes and red dots near the vertices; see Figure 5 and 6. The number of dashes indicates the length of the corresponding boundary, and a red dot indicates the parity of the corresponding region of X . Note that the length for a Möbius band Y_2 has not defined, but the incident edge of a vertex of type (Y_2) is also decorated with two dashes for consistency with the other notations. We also note that a red dot of a vertex of type (Y_{12}) is sometimes omitted if no confusion arises.

Notice that an encoding graph is not uniquely determined from X . Two moves on encoding graphs are shown in Figure 7: the left is a *YV-move* and the right is an *IH-move*. These moves correspond to giving another decomposition of a region, so they do not change the homeomorphism types of the corresponding polyhedra.

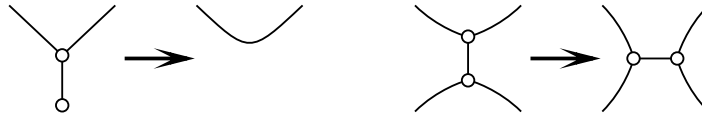


Figure 7: YV-move and IH-move.

Suppose that G is a tree and let $G' \subsetneq G$ be a subgraph. Let $N(G')$ denote the neighborhood of G' , that is, $N(G') \subset G$ is obtained from G' by adding all the vertices adjacent to vertices of G' and all the edges between them. Then we replace each vertex in $N(G') \setminus G'$ with a vertex of type (D), and the obtained graph is called the (D)–closure of G' , denoted by \widehat{G}' . See Figure 8 for an example. The left of the figure shows an encoding graph G and subgraphs G_0, \dots, G_4 of G , and the right one shows the (D)–closure \widehat{G}_0 of G_0 .

3 Shadows of 2–knots and knot groups

3.1 Shadows of 2–knots

A smoothly embedded surface K in a 4–manifold W is called a *knotted surface*. If K and W are diffeomorphic to S^2 and S^4 , respectively, then K is called a *2–knot*. A 2–knot is said to be *unknotted* (or *trivial*) if it bounds a smooth 3–ball in S^4 .

We now define a shadow of a 2–knot as follows.

Definition 3.1 Let K be a 2–knot. A shadow X of S^4 is called a *shadow* of K if K is embedded in X .

We can define a shadow also for a knotted surface in a similar manner, but it is not our focus in this paper.

Note that an unknotted 2–knot is a shadow of itself with gleam 0. In general, a 2–knot is unknotted if and only if it admits a shadow without true vertices, which will be shown in Theorem 6.4.

Theorem 3.2 *Every 2–knot admits a shadow.*

By considering the handle decomposition relative to $\text{Nbd}(K; S^4)$, we can prove the above as an application of Theorem 2.2. In Section 4, we will give a recipe for making a shadow of a 2–knot from a *banded unlink diagram*, which gives an alternative proof of Theorem 3.2.

Then we define complexities for 2–knots as well as for 4–manifolds.

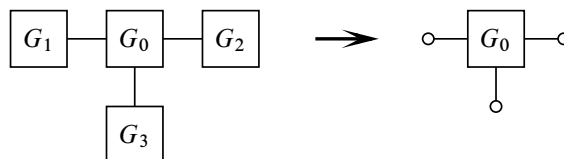


Figure 8: The (D)–closure of a subgraph G_0 .

Definition 3.3 For a 2–knot K , the *shadow-complexity* $\text{sc}(K)$ and the *special shadow-complexity* $\text{sc}^{\text{sp}}(K)$ of K are defined as the minimum number of true vertices of all shadows of K and that of all special shadows of K , respectively.

3.2 Knot groups

Let K be a 2–knot. The *knot group* $G(K)$ of K is the fundamental group of the complement of K . We will give a presentation of $G(K)$ in Proposition 3.4. To state this proposition, we first give some notation.

Let X be a shadow of K and X_K be a regular neighborhood $\text{Nbd}(K; X)$ of K in X . Choose a regular neighborhood $\text{Nbd}(K; S^4)$ such that X_K is proper in $\text{Nbd}(K; S^4)$, and let it be denoted by M_K . Set

$$\begin{aligned} X' &= X \setminus \text{Int } X_K, \\ T &= \partial X' \cap \partial X_K \subset \partial X', \\ M_{X'} &= \text{Nbd}(X'; S^4 \setminus \text{Int } M_K). \end{aligned}$$

We assume that X' and T are connected for simplicity. Note that this can always be assumed by applying some $(0 \rightarrow 2)$ –moves (see [Costantino 2004; Turaev 1994]) in advance; see also Remark 3.5. The gluing map $\partial M_{X'} \cap M_K \rightarrow \partial M_K$ will be written as f . Note that T is a graph and the valency of each vertex of T is 3. By the definition of shadows of 2–knots, the knot complement $S^4 \setminus K$ admits a decomposition

$$(M_K \setminus K) \cup_f M_{X'} \cup (3\text{– and } 4\text{–handles}).$$

We can easily see that $M_K \setminus K \cong (S^2 \times D^2) \setminus (S^2 \times \{0\})$ and it retracts onto $\partial M_K (\cong S^2 \times S^1)$. We also see that $M_{X'}$ retracts onto X' with keeping T . Thus, the knot group $G(K)$ can be computed from $\partial M_K \cup_{f|_T} X'$.

Choose a basepoint $t \in T$ and a presentation

$$\pi_1(X', t) = \langle S \mid R \rangle.$$

Since T is a graph, its fundamental group is freely generated by some loops

$$w_1, \dots, w_m \in \pi_1(X', t).$$

We assign an orientation to K arbitrarily. Then the fundamental group of ∂M_K has a presentation

$$\pi_1(\partial M_K, t) = \langle \mu \rangle,$$

where μ is the meridian of K whose orientation agrees with those of K and S^4 .

For each $i \in \{1, \dots, m\}$, there is a 2–chain $D_i = \sum a_{i,j} R_j$ in K with

$$\partial D_i = [w_i] \in H_1(\text{Cl}(X_K \setminus K)) \cong H_1(S(X) \cap K),$$

where R_j is a region contained in K with an orientation induced from that of K . Set

$$(2) \quad \text{gl}(w_i) = \sum a_{i,j} \text{gl}(R_j).$$

This number is equal to the algebraic intersection number of \tilde{D}_i and K in M_K , where \tilde{D}_i is a smooth oriented surface bounded by $f(w_i)$ in M_K . Hence we have $f(w_i) = \mu^{\text{gl}(w_i)}$ in $\pi_1(\partial M_K, *)$. Therefore, by van Kampen’s theorem, we obtain the following.

Proposition 3.4 Under the above settings,

$$G(K) \cong \langle S, \mu \mid R, w_1\mu^{-\text{gl}(w_1)}, \dots, w_m\mu^{-\text{gl}(w_m)} \rangle.$$

Remark 3.5 The assumption that X' and T are connected is not essential. The case where X' and T are not connected is as follows. Let X'_1, \dots, X'_k and T_1, \dots, T_k denote the connected components of X' and T , respectively. Then the knot group $G(K)$ is presented as

$$G(K) \cong \langle S_1, \dots, S_k, \mu \mid R_1, \dots, R_k, w_1\mu^{-\text{gl}(w_1)}, \dots, w_{m'}\mu^{-\text{gl}(w_{m'})} \rangle,$$

where $\pi_1(X'_j) \cong \langle S_j \mid R_j \rangle$ for $j \in \{1, \dots, k\}$ and $w_1, \dots, w_{m'}$ are loops in $\pi_1(X'_1) * \dots * \pi_1(X'_k)$ such that they generate $\pi_1(T_1) * \dots * \pi_1(T_k)$.

4 Banded unlink diagrams

In this section, we give a review of a description called a *banded unlink diagram* for a knotted surface. See [Hughes et al. 2020] for the details. We start with the definition of banded links in a 3–manifold.

4.1 Banded links

For a link L in a 3–manifold N , the image $b = \text{Im}\beta$ of an embedding $\beta: [0, 1] \times [0, 1] \rightarrow N$ with $L \cap b = \beta(\{0, 1\} \times [0, 1])$ is called a *band*. The *core* of b is defined as $\beta([0, 1] \times \{1/2\})$. The pair (L, \mathbf{b}) of L and mutually disjoint bands $\mathbf{b} = b_1 \cup \dots \cup b_n$ is called a *banded link*. The *negative resolution* and the *positive resolution* of (L, \mathbf{b}) , respectively, are defined as the links L and $L_{\mathbf{b}}$, where

$$L_{\mathbf{b}} = \left(L \setminus \left(\bigcup_{i=1}^n \beta_i(\{0, 1\} \times [0, 1]) \right) \right) \cup \left(\bigcup_{i=1}^n \beta_i([0, 1] \times \{0, 1\}) \right).$$

4.2 Banded unlink diagrams

Let W be a closed 4–manifold, and fix a handle decomposition of W having a single 0–handle. For $i \in \{0, \dots, 4\}$, let W_i denote the handlebody consisting of all the handles with indices at most i . Clearly, $W_0 \cong B^4$ and $W_4 = W$. Suppose that this handle decomposition is described by a Kirby diagram $\mathcal{K} = \mathcal{L}_1 \sqcup \mathcal{L}_2 \subset S^3$, where \mathcal{L}_1 is a dotted unlink indicating the 1–handles and \mathcal{L}_2 is a framed link indicating the 2–handles. Note that S^3 in which \mathcal{K} is drawn is considered as the boundary ∂W_0 of the 0–handle, and we can identify the complement $S^3 \setminus \nu\mathcal{K}$ of a tubular neighborhood $\nu\mathcal{K}$ of \mathcal{K} with a subset in ∂W_2 .

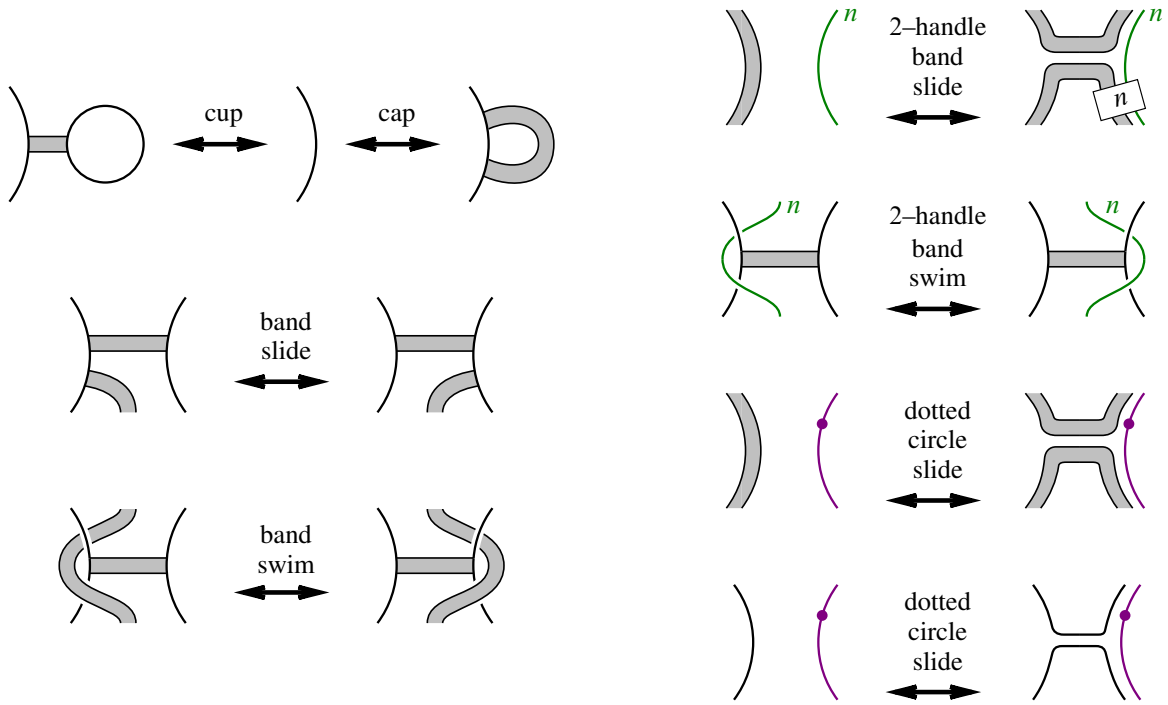


Figure 9: Band moves of banded unlink diagrams.

Let (L, \mathbf{b}) be a banded link in $S^3 \setminus \nu\mathcal{K}$. Note that (L, \mathbf{b}) is considered as a banded link in ∂W_2 and also in ∂W_1 . Suppose that the negative resolution L and the positive resolution $L_{\mathbf{b}}$ are unlinks in ∂W_1 and ∂W_2 , respectively. Then we call the triple $(\mathcal{K}, L, \mathbf{b})$ a *banded unlink diagram* in \mathcal{K} .

A banded unlink diagram $(\mathcal{K}, L, \mathbf{b})$ can be interpreted as a presentation of a knotted surface in the 4-manifold W in the following way. By the definition of a banded unlink diagram, there exist collections $\mathcal{D}_1 \subset \partial W_1$ and $\mathcal{D}_2 \subset \partial W_2$ of 2-disks bounded by L and $L_{\mathbf{b}}$, respectively. We push the interiors of \mathcal{D}_1 and \mathcal{D}_2 into $\text{Int } W_1$ and $W \setminus W_2$, respectively, with keeping the boundaries. Then set $K = \mathcal{D}_1 \cup \mathbf{b} \cup \mathcal{D}_2$. It forms a knotted surface in W , and $(\mathcal{K}, L, \mathbf{b})$ is also said to be a *banded unlink diagram* for K .

If a Kirby diagram for S^4 consists of no dotted circles and no framed knots, we will say that such a diagram is the *trivial Kirby diagram*. By Kawauchi, Shibuya and Suzuki [Kawauchi et al. 1982] and also by Lomanaco [1981], it was shown that any 2-knot admits a banded unlink diagram in the trivial Kirby diagram. In the general case, Hughes, Kim and Miller [Hughes et al. 2020] showed that any knotted surface in any closed 4-manifold admits a banded unlink diagram.

See Figure 9. The three moves shown in the left of the figure (in the trivial Kirby diagram) were introduced by Yoshikawa [1994], and it was shown that these moves are sufficient to relate any two banded unlink diagrams describing the same knotted surface by Swenton [2001] and also by Kearton and Kurlin [2008]. The other moves in Figure 9 were introduced by Hughes, Kim and Miller [Hughes et al. 2020] for the general case. The seven kinds of moves exhibited in the figure are called *band moves*.

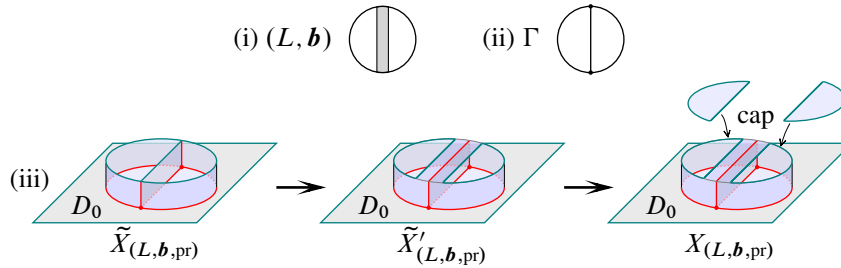


Figure 10: An example of a banded unlink diagram (L, \mathbf{b}) , Γ , and polyhedra $\tilde{X}_{(L, \mathbf{b}, \text{pr})}$, $\tilde{X}'_{(L, \mathbf{b}, \text{pr})}$ and $X_{(L, \mathbf{b}, \text{pr})}$.

Theorem 4.1 (Hughes, Kim and Miller [Hughes et al. 2020]) *Two banded unlink diagrams $(\mathcal{K}, L, \mathbf{b})$ and $(\mathcal{K}, L', \mathbf{b}')$ representing the same knotted surface are related by a finite sequence of band moves.*

4.3 Shadows from banded unlink diagrams

We again focus on the case of 2–knots. In this subsection, we give a construction of a shadow of a 2–knot from a banded unlink diagram.

Let K be a 2–knot and $(\mathcal{K}, L, \mathbf{b})$ a banded unlink diagram for K , where $L = L_1 \sqcup \dots \sqcup L_m$ and $\mathbf{b} = b_1 \sqcup \dots \sqcup b_n$. For simplicity, we suppose \mathcal{K} is the trivial Kirby diagram, and we will write (L, \mathbf{b}) instead of $(\mathcal{K}, L, \mathbf{b})$. In this case, the ambient 4–manifold S^4 is decomposed into one 0–handle W_0 and one 4–handle, and (L, \mathbf{b}) is assumed to be in ∂W_0 . As explained in Section 4.2, the 2–knot K lies in S^4 so that $K \cap \partial W_0 = L \cup \mathbf{b}$, and we recall the notations $\mathfrak{D}_1 = (K \cap \text{Int } W_0) \cup L$ and $\mathfrak{D}_2 = (K \cap S^4 \setminus W_0) \cup L_{\mathbf{b}}$.

Step 1 Let Γ be the union of L and the cores of b_1, \dots, b_n , which is a trivalent graph in $S^3 = \partial B^4$. Let π be a regular projection from Γ to a 2–disk D_0 such that the images of L_1, \dots, L_m bound mutually disjoint 2–disks $D_1, \dots, D_m \subset D_0$, respectively. Then consider (abstractly) the mapping cylinder

$$\tilde{X}_{(L, \mathbf{b}, \text{pr})} = (D_0 \cup (\Gamma \times [0, 1])) / \sim$$

of π , where \sim is defined by $\pi(x) \sim (x, 0)$ for $x \in \Gamma$. Since π is chosen so that the diagram is regular, $\tilde{X}_{(L, \mathbf{b}, \text{pr})}$ is a simple polyhedron. This polyhedron can be embedded in the 4–ball W_0 as a shadow since $\tilde{X}_{(L, \mathbf{b}, \text{pr})}$ can collapse onto the disk D_0 . Actually, there is a natural gleam on $\tilde{X}_{(L, \mathbf{b}, \text{pr})}$ determined from the diagram of Γ such that it corresponds to a 4–ball in which $\tilde{X}_{(L, \mathbf{b}, \text{pr})}$ is embedded as a shadow. See Remark 4.3. Then $\tilde{X}_{(L, \mathbf{b}, \text{pr})}$ will be considered as a shadow of W_0 , and we can naturally identify

- $\partial \tilde{X}_{(L, \mathbf{b}, \text{pr})} \setminus \partial D_0$ with $\Gamma \subset \partial W_0$, and
- $\bigcup_{i=1}^m (D_i \cup (L_i \times [0, 1]))$ with $\mathfrak{D}_1 \subset W_0$.

See Figure 10 for an example. The banded link (L, \mathbf{b}) shown in Figure 10(i) consists of one unknot L and one band $\mathbf{b} = b$, and it is a presentation of the trivial 2–knot. Figure 10(ii) shows a diagram of the graph $\Gamma = L \cup (\text{core of } b)$. Then $\tilde{X}_{(L, \mathbf{b}, \text{pr})}$ is a polyhedron as shown in the leftmost of Figure 10(iii).

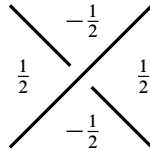


Figure 11: The local contribution to the gleams.

Step 2 The graph Γ lies in ∂W_0 as the boundary of $\tilde{X}_{(L, \mathbf{b}, \text{pr})}$, and the whole banded link (L, \mathbf{b}) is also embedded in ∂W_0 . Set

$$\tilde{X}'_{(L, \mathbf{b}, \text{pr})} = \tilde{X}_{(L, \mathbf{b}, \text{pr})} \cup \mathbf{b}$$

and then push $\text{Nbd}(\mathbf{b}; \tilde{X}'_{(L, \mathbf{b}, \text{pr})})$ into $\text{Int } W_0$ so that $\tilde{X}'_{(L, \mathbf{b}, \text{pr})}$ is properly embedded in W_0 . See the center of Figure 10(iii). Note that $\tilde{X}'_{(L, \mathbf{b}, \text{pr})}$ is also a shadow of W_0 .

Step 3 The boundary $\partial \tilde{X}'_{(L, \mathbf{b}, \text{pr})} \subset \partial W_0$ is the positive resolution $L_{\mathbf{b}}$ of (L, \mathbf{b}) , which is the m' -component unlink, where $m' = 2 + n - m$. We attach m' 2–handles to W_0 along $L_{\mathbf{b}}$ with 0–framing, and we set

$$X_{(L, \mathbf{b}, \text{pr})} = \tilde{X}'_{(L, \mathbf{b}, \text{pr})} \cup D'_1 \cup \dots \cup D'_{m'},$$

where $D'_1, \dots, D'_{m'}$ are the core disks of the 2–handles. See the right of Figure 10(iii). These disks $D'_1, \dots, D'_{m'}$ correspond to \mathcal{D}_2 . The 4–manifold W_0 with the 2–handles attached is diffeomorphic to $\natural_{m'}(S^2 \times D^2)$, and $X_{(L, \mathbf{b}, \text{pr})}$ is its shadow. We can obtain S^4 from this 4–manifold by attaching m' 3–handles and a 4–handle in a canonical way [Laudenbach and Poénaru 1972]. Hence $X_{(L, \mathbf{b}, \text{pr})}$ is a shadow of S^4 , and the 2–knot K is realized in S^4 as

$$\left(\bigcup_{i=1}^m (D_i \cup (L_i \times [0, 1])) \right) \cup \mathbf{b} \cup \left(\bigcup_{j=1}^{m'} D'_j \right)$$

in $X_{(L, \mathbf{b}, \text{pr})}$. Thus we have the following.

Proposition 4.2 *The polyhedron $X_{(L, \mathbf{b}, \text{pr})}$ is a shadow of K .*

Remark 4.3 The gleams of regions of $X_{(L, \mathbf{b}, \text{pr})}$ can be easily calculated. For regions on the disk D_0 , we can use the rule shown in Figure 11: the gleam of an internal region contained in D_0 is given as the sum of the local contribution shown in the figure at each crossing of the diagram of Γ adjacent to the region [Costantino and Thurston 2008; Turaev 1994]. The gleam of the region forming $(\text{core of } b_i) \times [0, 1]$ is given as the number of times b_i twists with respect to D_0 on the diagram. Each of the remaining regions is a part of K and contains a core disk D'_j of a 2–handle. The gleams of them are the minus of the writhe number of L'_j on D_0 , where L'_j is the component of $L_{\mathbf{b}}$ to which D'_j is attached.

Remark 4.4 All the true vertices of $X_{(L, \mathbf{b}, \text{pr})}$ lie on D_0 . Each of them derives from a crossing of the diagram of Γ or a trivalent vertex of Γ . Thus, we can estimate the shadow-complexity of K from the diagram of Γ . Examples will be studied in the next subsection.

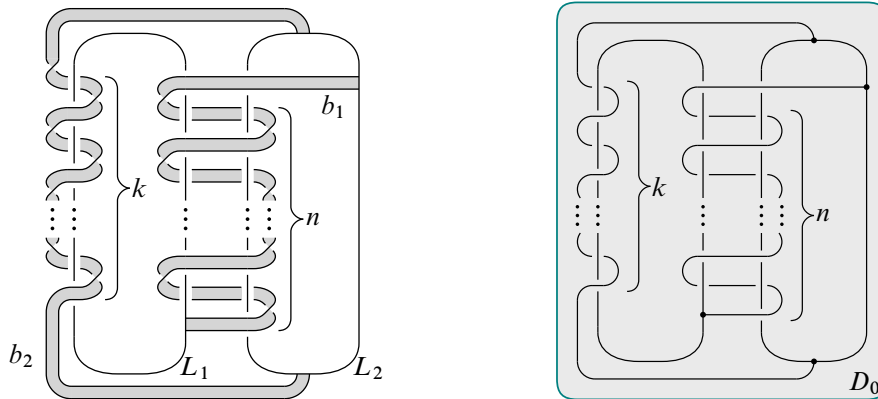


Figure 12: Left: a banded unlink diagram of the k -twist spun knot of $T(2, 2n + 1)$. Right: a diagram of the graph Γ that is the union of L_1, L_2 and the cores of b_1, b_2 .

Remark 4.5 Even if \mathcal{H} is not trivial, we also can construct a shadow of K by considering a shadow of W_2 instead of the disk D_0 .

4.4 Examples

Let $T(2, 2n + 1)^k$ denote the k -twist spun of the classical torus knot $T(2, 2n + 1)$. Figure 12, left, shows a banded unlink diagram $(L, \mathbf{b}) = (L_1 \sqcup L_2, b_1 \cup b_2)$ for $T(2, 2n + 1)^k$ that was given in [Jabłonowski 2016]. Considering a natural projection $\text{pr}: \Gamma \rightarrow D_0$ from the graph $\Gamma = L_1 \cup L_2 \cup (\text{cores of } b_1, b_2)$ to a 2-disk D_0 , we draw a regular diagram of Γ as in Figure 12, right. This diagram has $4n + 2k$ crossings, and Γ has 4 trivalent vertices. Hence $X_{(L, \mathbf{b}, \text{pr})}$, a shadow of $T(2, 2n + 1)^k$, has $4n + 2k + 4$ true vertices. The polyhedron $X_{(L, \mathbf{b}, \text{pr})}$ has a single boundary region, which is adjacent to $2k + 3$ true vertices. These true vertices are eliminated by collapsing, and the resulting polyhedron is still a shadow of $T(2, 2n + 1)^k$. Therefore, we obtain an upper bound of the shadow-complexity of the twist spun knot $T(2, 2n + 1)^k$.

Proposition 4.6
$$\text{sc}(T(2, 2n + 1)^k) \leq 4n + 1.$$

Remark 4.7 The 1-twist spun of any 1-knot is trivial [Zeeman 1965]. As we will see in Theorem 6.4, the shadow-complexity of an unknotted 2-knot is 0. The 2-knot $T(2, 2n + 1)^k$ with $n = 1$ and $k = 0$ is the spun trefoil, which is K_{-1} in our notation; see Figure 1. We will show that its shadow-complexity is 1 in Theorem 8.10.

5 Modifications and fundamental groups

A subspace Y of a simple polyhedron X is called a *subpolyhedron* if there exist simple closed curves $\gamma_1 \sqcup \dots \sqcup \gamma_n$ in $X \setminus (S(X) \cup \partial X)$ such that Y is the closure of a connected component of $X \setminus (\gamma_1 \sqcup \dots \sqcup \gamma_n)$. It is obvious that Y itself is a simple polyhedron. Each simple closed curve γ_i is a boundary component

of Y , and it is called a *cut end* of Y in X . In other words, a boundary component of Y but not of X is called a cut end. If X is a shadowed polyhedron, Y can also be assigned with gleams canonically [Naoe 2017].

Henceforth, we fix the following notation:

- K is a 2-knot, and
- X is a shadow of K .

Note that X is simply connected since it is a shadow of S^4 .

5.1 Compressing disk addition

Let γ be a simple closed curve contained in a region of X . Let $\pi : M_X \rightarrow X$ be the projection, where M_X is the 4-manifold obtained from X by Turaev's reconstruction. Since X is also a shadow of S^4 , we have $\partial M_X \cong \#_h(S^1 \times S^2)$ for some $h \in \mathbb{Z}_{\geq 0}$. Hence $(\pi|_{\partial M_X})^{-1}(\gamma)$ is an embedded torus in $\#_h(S^1 \times S^2)$. Every embedded torus in $\#_h(S^1 \times S^2)$ has a compressing disk by Dehn's lemma, so let D_γ be such a disk for the torus $(\pi|_{\partial M_X})^{-1}(\gamma)$. We consider a 2-disk D'_γ enlarged from D_γ such that $D'_\gamma \setminus D_\gamma \subset \pi^{-1}(\gamma)$ and $\partial D'_\gamma \subset X$, and then modify the disk D'_γ near its boundary so that $\partial D'_\gamma$ is a generically immersed curve in $\text{Nbd}(\gamma; X)$ by a small perturbation. This can be done without creating self-intersections of $\text{Int } D'_\gamma$. We thus obtain a new simple polyhedron $X \cup D'_\gamma$ embedded in S^4 .

Proposition 5.1 [Koda et al. 2022] *Under the above settings, $X \cup D'_\gamma$ is a shadow of S^4 and also of K .*

The disk D'_γ is called a *compressing disk* of γ . The addition of D'_γ corresponds to attaching a 2-handle that is canceled with a 3-handle.

Note that the image of $\partial D'_\gamma$ by π is contained in γ . Then we can define a map $\rho : \partial D'_\gamma \rightarrow \gamma$ by $\rho(x) = \pi(x)$. There are two important cases:

- (i) $\deg(\rho) = 0$, and
- (ii) $|\deg(\rho)| = 1$.

In other words,

- (i) $\partial D'_\gamma$ is null-homotopic in $\text{Nbd}(\gamma; X)$, and
- (ii) $\partial D'_\gamma$ is homotopic to γ in $\text{Nbd}(\gamma; X)$.

The disk D'_γ is said to be *vertical* if (i), and *horizontal* if (ii). Figure 13 shows the modification of X to $X \cup D'_\gamma$ in the cases (i) and (ii).

Remark 5.2 If γ is a small circle bounding a disk in a region and has a vertical compressing disk, then we often say that the region has a vertical compressing disk. Clearly, any such γ has a horizontal compressing disk.

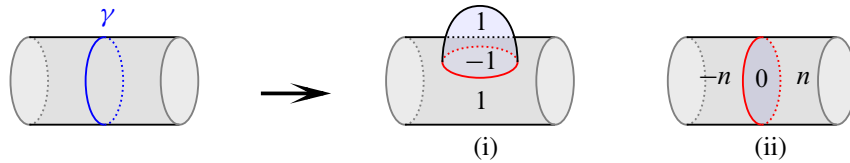


Figure 13: A vertical compressing disk (i) and a horizontal compressing disk (ii).

5.2 Connected-sum and reduction

Suppose that X has a disk region D such that $\text{Nbd}(\partial D; X)$ is homeomorphic to Y_{111} and $\text{gl}(D) = 0$. The neighborhood $\text{Nbd}(D; X)$ is shown in the left of Figure 14. Note that there exists a smooth 3-ball B_D in S^4 such that $\text{Nbd}(D; X) \subset B_D$ since $\text{gl}(D) = 0$. We consider a modification as shown in Figure 14. Precisely, we first remove $\text{Int Nbd}(D; X)$ from X and then cap each of the resulting boundary circles off with a 2-disk. This modification can be performed locally in B_D as in the figure. Since X is simply connected, this modification produces two new simple polyhedra X'_1 and X'_2 . We suppose that X'_1 contains the whole K , and we say that X'_1 is obtained from X by the *connected-sum reduction* along D .

Proposition 5.3 *Suppose $c(X) \leq 1$ and that X'_1 is obtained from X by the connected-sum reduction along a disk region D . Then X'_1 is a shadow of K .*

Proof Let W denote the 4-sphere in which K and X are embedded. By [Martelli 2011, Proposition 4.1], the 4-sphere W can be decomposed as $W_1 \# W_2$, where W_1 and W_2 are closed 4-manifolds admitting shadows X'_1 and X'_2 , respectively. Since either X'_1 or X'_2 has no true vertices by $c(X) \leq 1$, the corresponding 4-manifold, namely W_1 or W_2 , is diffeomorphic to S^4 by [Martelli 2011, Corollary 1.8]. Thus, both W_1 and W_2 are diffeomorphic to S^4 . Then $W \setminus \text{Int Nbd}(X'_1; W)$ is diffeomorphic to a 4-dimensional 1-handlebody, and hence X'_1 is a shadow of K . \square

5.3 Lemmas on encoding graphs

Here we prepare some lemmas about the shape and the types of the vertices of an encoding graph of X .

Lemma 5.4 [Martelli 2011, Claim 1] *Suppose that a loop $\gamma \subset X$ separates X into two connected components V_1 and V_2 . Then both V'_1 and V'_2 are simply connected, where $V'_1 = V_1 \cup_\gamma D^2$ and $V'_2 = V_2 \cup_\gamma D^2$.*

Proof The quotient space X/γ is homeomorphic to the wedge sum $V'_1 \vee V'_2$. Then we have a surjection from $\pi_1(X) = \{1\}$ onto $\pi_1(V'_1) * \pi_1(V'_2)$. \square

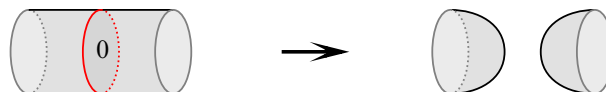


Figure 14: Connected sum.



Figure 15: Graphs that are not contained in a graph encoding a simply connected polyhedron.

We rephrase the above lemma in terms of encoding graphs as below.

Lemma 5.5 *Let G be a tree encoding X and $G' \subsetneq G$ be a subgraph. Then a simple polyhedron encoded by the (D)–closure \widehat{G}' is also simply connected.*

Lemma 5.6 *Suppose $c(X) \leq 1$. Let G be a graph encoding X .*

- (1) G is a tree.
- (2) G does not have a vertex of type (Y_2) , (Y_3) , (X_1) , (X_2) , (X_5) , (X_6) or (X_7) .
- (3) G does not contain either subgraph shown in Figure 15.

Proof (1) We can embed G in X as a retract, and hence there is a surjection $\pi_1(X) \rightarrow \pi_1(G)$.

- (2) Assume that G has a vertex v of type (Y_2) , (Y_3) , (X_1) , (X_2) , (X_5) , (X_6) or (X_7) . Then the polyhedron encoded by the (D)–closure \widehat{v} is not simply connected, which is a contradiction to Lemma 5.5.
- (3) The proof is similar to that of (2). □

Remark 5.7 (1) The fundamental groups of the special polyhedra encoded by the (D)–closures of vertices of types (Y_2) , (Y_3) , (X_1) , (X_2) , (X_5) , (X_6) and (X_7) are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\langle x, y \mid xyx^{-2}y^{-2} \rangle$, $\langle x, y \mid xyx^2y^{-2} \rangle$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$, respectively.

- (2) The two polyhedra encoded by the (D)–closures of the subgraphs shown in Figure 15 are homeomorphic, and their fundamental groups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

5.4 Lemmas on the fundamental groups of subpolyhedra

In this subsection, we present four lemmas on the fundamental group of subpolyhedron in X . Lemmas 5.9 and 5.10 treat a subpolyhedron with one cut end, and Lemmas 5.11 and 5.12 treat a subpolyhedron with two cut ends. Note that we will assume that X is closed in these lemmas, which actually does not matter for our main theorems due to Lemma 6.1.

Definition 5.8 Let G be a tree graph encoding a simple polyhedron. Let v and v' be vertices of G and v' is of type (Y_{12}) . If the edge incident to v' with no dashes is contained in the shortest path from v to v' , then v' is said to be *one-sided to v* . Otherwise we say that v' is *two-sided to v* .

Lemma 5.9 *Suppose X is closed, and let V be a subpolyhedron of X with a single cut end γ and $c(V) = 0$. Then there exists a nonnegative integer k such that $\pi_1(V) \cong \langle \gamma \mid \gamma^{2^k} \rangle$.*

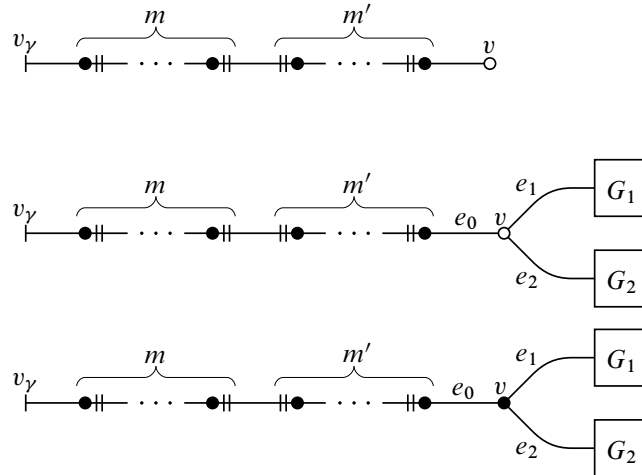


Figure 16: The possible cases of a graph G encoding V .

Proof Let G be an encoding graph of V . From Lemma 5.4, the polyhedron $V \cup_{\gamma} D^2$ is simply connected, and G is a tree from Lemma 5.6(1). This graph G has exactly one vertex of type (B) corresponding to γ ; let v_{γ} denote it. Other vertices in G are of types (D), (Y_{12}) , (P) or (Y_{111}) by Lemma 5.6(2). Note that the valencies of types (D), (Y_{12}) , (P) and (Y_{111}) are 1, 2, 3 and 3, respectively.

We assign an orientation to γ arbitrarily. Let v be the nearest vertex from v_{γ} among those of types (D), (P) and (Y_{111}) . By Lemma 5.6(3), the possible cases are shown in Figure 16, where $m, m' \in \mathbb{Z}_{\geq 0}$. We prove the argument by induction on the number of vertices of types (D), (P) and (Y_{111}) in G , so it is enough to consider the following (i)–(iii).

- (i) Suppose that G is as shown in Figure 16, top. Then we easily obtain $\pi_1(V) \cong \langle \gamma \mid \gamma^{2^m} \rangle$.
- (ii) Suppose that G is as shown in Figure 16, middle, and that $\pi_1(V_i) \cong \langle \gamma_i \mid \gamma_i^{2^{k_i}} \rangle$ for $i \in \{1, 2\}$, where V_i is the polyhedron encoded by the subgraph G_i and $\gamma_i = \partial V_i$. Let γ_0 be a lift of e_0 . Note that γ_i is a lift of e_i for $i \in \{1, 2\}$. By Lemma 5.5 applied to the subgraph $v \cup G_1 \cup G_2$, the group $\langle \gamma_1, \gamma_2 \mid \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_1 \gamma_2 \rangle$ must be trivial, and hence we have $k_1 = 0$ or $k_2 = 0$. Suppose $k_2 = 0$. Then

$$\begin{aligned} \pi_1(V) &\cong \langle \gamma, \gamma_0, \gamma_1, \gamma_2 \mid \gamma^{2^m} \gamma_0^{2^{m'}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 \gamma_1 \gamma_2 \rangle \\ &\cong \langle \gamma, \gamma_0, \gamma_1 \mid \gamma^{2^m} \gamma_0^{2^{m'}}, \gamma_1^{2^{k_1}}, \gamma_0 \gamma_1 \rangle \\ &\cong \langle \gamma, \gamma_0 \mid \gamma^{2^m} \gamma_0^{2^{m'}}, \gamma_0^{2^{k_1}} \rangle. \end{aligned}$$

It becomes trivial by adding a relation $\gamma = 1$ since $\pi_1(V \cup_{\gamma} D^2) = \{1\}$. Hence $m' = 0$ or $k_1 = 0$, and the lemma follows in either case.

- (iii) Suppose that G is as shown in Figure 16, bottom, and that $\pi_1(V_i) \cong \langle \gamma_i \mid \gamma_i^{2^{k_i}} \rangle$ for $i \in \{1, 2\}$, where V_i is the polyhedron encoded by the subgraph G_i and $\gamma_i = \partial V_i$. Let γ_0 be a lift of e_0 . We have

$$\pi_1(V) \cong \langle \gamma, \gamma_0, \gamma_1, \gamma_2 \mid \gamma^{2^m} \gamma_0^{2^{m'}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 = \gamma_1 = \gamma_2 \rangle \cong \langle \gamma, \gamma_0 \mid \gamma^{2^m} \gamma_0^{2^{m'}}, \gamma_0^{2^{k_3}} \rangle,$$

where $k_3 = \min\{k_1, k_2\}$. It becomes trivial by adding a relation $\gamma = 1$ since $\pi_1(V \cup_\gamma D^2) = \{1\}$. Hence $m' = 0$ or $k_3 = 0$, and the lemma follows in either case. \square

For each vertex v of type (D), let $k(v)$ denote the number of vertices of type (Y_{12}) one-sided to v_γ contained in the geodesic from v_γ to v . By the proof of Lemma 5.9, the integer k stated in the lemma is given as the minimum of $k(v)$ for any vertex v of type (D). Thus we also have proved the following.

Lemma 5.10 *Under the same notation as in Lemma 5.9 and its proof, if $\pi_1(V) = \{1\}$, then there exists a leaf such that all the vertices of type (Y_{12}) contained in the geodesic (in G) from v_γ to the leaf are two-sided to v_γ .*

Lemma 5.11 *Suppose X is closed, and let U be a subpolyhedron of X with exactly two cut ends $\gamma_1 \sqcup \gamma_2$ and $c(U) = 0$. Suppose that $[\gamma_1]$ is not a torsion element in $H_1(U)$.*

- (1) $[\gamma_2]$ is also not a torsion element in $H_1(U)$.
- (2) $\pi_1(U) \cong \langle \gamma_1, \gamma_2 \mid (\gamma_1^{2m} \gamma_2^{2l})^{2k} \rangle$ for some $k, l, m \in \mathbb{Z}_{\geq 0}$.

Proof (1) We have $c(U \cup_{\gamma_2} D^2) = 0$. Hence $H_1(U \cup_{\gamma_2} D^2)$ is a finite cyclic group generated by $[\gamma_1]$ by Lemma 5.9. Suppose, to derive a contradiction, that $[\gamma_2]$ is a torsion element in $H_1(U)$. Then the normal subgroup $\langle [\gamma_2] \rangle$ generated by $[\gamma_2]$ is contained in the torsion subgroup $t(H_1(U))$ of $H_1(U)$. Since the free part $H_1(U)/t(H_1(U))$ has the nontrivial element $[\gamma_1]$ and $H_1(U \cup_{\gamma_2} D^2) \cong H_1(U)/\langle [\gamma_2] \rangle$, the free part of $H_1(U \cup_{\gamma_2} D^2)$ is also nontrivial, which is a contradiction.

(2) Let G be a tree encoding U , and let v_1 and v_2 be the vertices of type (B) corresponding to γ_1 and γ_2 , respectively. Let ℓ be the geodesic from v_1 to v_2 . The vertices in G other than v_1 or v_2 are of types (D), (Y_{12}) , (P) or (Y_{111})

We assume that ℓ contains a vertex v of type (Y_{111}) and lead to a contradiction. Recall that there are three edges incident to v . See Figure 17(i-1). Let e be the edge such that it is incident to v and between v and v_2 , and let e' be the edge incident to v and not on ℓ . Let G_1, G_2 and G_3 be subgraphs of G as indicated in Figure 17(i-1). Now let U' be one of the components containing γ_1 obtained by cutting U along a lift of e , which is encoded in Figure 17(i-2). This subpolyhedron U' has two boundary components: one is γ_1 and the other, namely a lift of e , will be denoted by γ'_2 . Since U' itself satisfies the assumption of the lemma, the cycle $[\gamma'_2]$ is not a torsion element in $H_1(U')$ by (1). It is homologous to a cycle represented by a lift of e' , which is a torsion element in the subpolyhedron encoded by G_3 by Lemma 5.9. It is a contradiction. Therefore, the vertices between v_1 and v_2 are of types (Y_{12}) or (P).

If ℓ does not contain vertices of type (P), the graph G is as shown in Figure 17(ii) by Lemma 5.6(3). We then have $\pi_1(U) \cong \langle \gamma_1, \gamma_2 \mid \gamma_1^{2m} \gamma_2^{2l} \rangle$.

We then suppose that G has a vertex v of type (P), and G is as shown in Figure 17(iii-1). We can assume that each vertex of type (Y_{12}) in ℓ is two-sided to v by Lemma 5.6(3). Suppose that there is a subgraph of

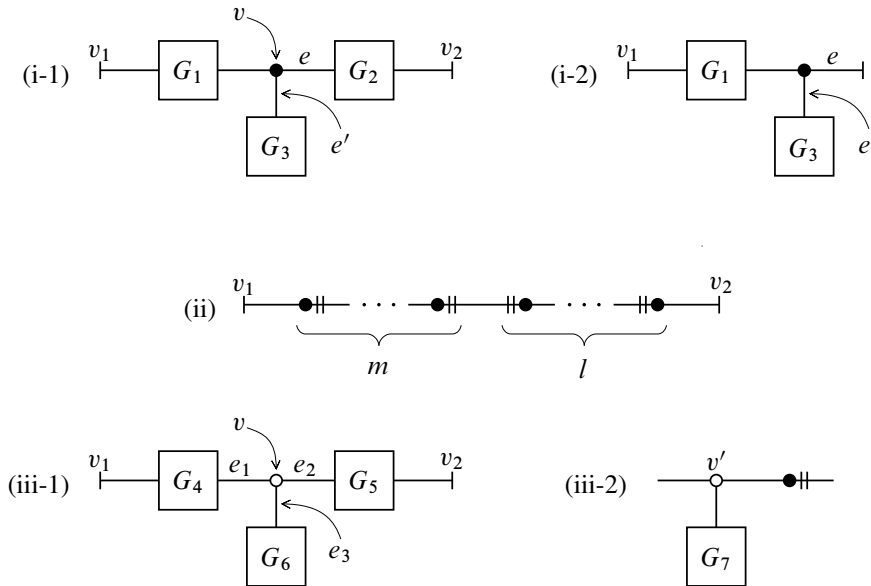


Figure 17: Encoding graphs of U used in the proof of Lemma 5.11.

G_4 as shown in Figure 17(iii-2), where v' is a vertex of type (P) contained in ℓ . The fundamental group of the subpolyhedron encoded by G_7 is isomorphic to $\langle \gamma' \mid \gamma'^{2^{k'}} \rangle$ for some $k' \in \mathbb{Z}_{\geq 0}$ by Lemma 5.9, where γ' is the boundary of the subpolyhedron. The simple polyhedron encoded by the (D)-closure of the graph shown in Figure 17(iii-2) is simply connected by Lemma 5.5. Hence the subpolyhedron encoded by G_7 must be simply connected as well. Therefore, the graph G_4 in Figure 17(iii-1) encodes a polyhedron whose fundamental group is presented by $\langle \gamma_1, u_1 \mid u_1 \gamma_1^{2^m} \rangle$, where u_1 is a lift of e_1 and m is the number of vertices of type (Y_{12}) in $G_4 \cap \ell$. Similarly, the graph G_5 in Figure 17(iii-1) encodes a polyhedron whose fundamental group is presented by $\langle \gamma_2, u_2 \mid u_2 \gamma_2^{2^l} \rangle$, where u_2 is a lift of e_2 and l is the number of vertices of type (Y_{12}) in $G_5 \cap \ell$. By Lemma 5.9, the graph G_6 in Figure 17(iii-1) encodes a polyhedron whose fundamental group is presented by $\langle u_3 \mid u_3^{2^k} \rangle$ for some $k \in \mathbb{Z}_{\geq 0}$, where u_3 is a lift of e_3 . Thus, we obtain a presentation

$$\pi_1(U) \cong \langle \gamma_1, \gamma_2, u_1, u_2, u_3 \mid u_1 \gamma_1^{2^m}, u_2 \gamma_2^{2^l}, u_3^{2^k}, u_1 u_2 u_3 \rangle \cong \langle \gamma_1, \gamma_2 \mid (\gamma_1^{2^m} \gamma_2^{2^l})^{2^k} \rangle. \quad \square$$

From the above proof, we immediately obtain the following lemma, which will be used in the proof of Theorem 8.6.

Lemma 5.12 *Under the same notation as in Lemma 5.11 and its proof, if $\pi_1(U) \cong \langle \gamma_1, \gamma_2 \mid \gamma_1^{2^m} \gamma_2 \rangle$, then*

- any vertex lying in ℓ is of type (Y_{12}) or (P),
- m is the number of the vertices of type (Y_{12}) lying in ℓ , all of which are one-sided to v_1 , and
- the subpolyhedron corresponding to each connected component of $G \setminus \ell$ is simply connected.

6 2–Knots with complexity zero

From now on, we discuss the classification of 2–knots according to the shadow-complexity, and we give the proof of the theorem for the case $\text{sc}(K) = 0$ in this section.

Let us start with the following lemma. It is an analogue of [Koda and Naoe 2020, Lemma 1.3], and the original statement in the paper is for shadows of “4–manifolds”. Roughly speaking, the shadow-complexity of any 2–knot is always attained by a closed shadow.

Lemma 6.1 *If $\text{sc}(K) = n$, then K admits a closed shadow with complexity exactly n .*

Proof The proof is almost the same as that of [Koda and Naoe 2020, Lemma 1.3], so we only sketch the proof.

Let X be a shadow of K with $c(X) = n$ and $\partial X \neq \emptyset$. Then X collapses onto an almost-simple polyhedron Y (see [Matveev 2003] for the definition and the details) that is minimum with respect to collapsing and has at most n true vertices. Note that the collapsing is done in a regular neighborhood $\text{Nbd}(X; S^4)$, which is also a regular neighborhood of Y in S^4 . Since K is a 2–sphere embedded in X , it is kept by collapsing, that is, K is also embedded in Y . The polyhedron Y is either a closed simple polyhedron or the union of a closed simple polyhedron and a graph. If the former, this Y is what we required. If the latter, as in the proof of [Koda and Naoe 2020, Lemma 1.3], Y can be modified to a closed simple polyhedron Y' such that Y and Y' have the same regular neighborhood in S^4 and also that no new true vertices are created by the modification. \square

As well as in Lemma 5.9, we consider a subpolyhedron having one cut end in the next lemma. However, unlike in Lemma 5.9, Lemma 6.2 gives a homological condition, and a subpolyhedron can have true vertices and boundary components other than the cut end. Recall the notation $\text{gl}(\gamma)$ defined in formula (2).

Lemma 6.2 *Set $X_K = \text{Nbd}(K; X)$, and suppose that ∂X_K has a circle component γ . Let X' be a connected component of $X \setminus \text{Int } X_K$ with $X_K \cap \partial X' = \gamma$. Give orientations to K and γ arbitrarily. Then at least one of the following holds:*

- $H_1(X')$ is an infinite cyclic group generated by $[\gamma]$, or
- $\text{gl}(\gamma) = 0$.

Proof We have $H_1(\gamma) \cong \mathbb{Z}\langle\gamma\rangle$ and $H_1(X) \cong 0$. From the Mayer–Vietoris sequence

$$H_1(\gamma) \rightarrow H_1(X \setminus \text{Int } X') \oplus H_1(X') \rightarrow H_1(X),$$

$H_1(X')$ admits a surjection from \mathbb{Z} generated by $[\gamma]$.

If $H_1(X') = \mathbb{Z}/k\mathbb{Z}\langle\gamma\rangle$ for some $k \in \mathbb{Z}_{>0}$, there is a 2–chain C in X' such that $\partial C = k[\gamma]$. Let

$$D = \sum a_j R_j$$

be a 2-chain in X_K with $\partial D = [\gamma]$, where R_j is a region contained in K . Then $C - kD$ is a homology cycle in X , and we have

$$Q([K], C - kD) = -k \sum a_j \text{gl}(R_j) = -k \text{gl}(\gamma),$$

where Q is the intersection form on $H_2(X)$. Since the intersection form of S^4 is trivial, $\text{gl}(\gamma) = 0$. \square

Lemma 6.3 *Suppose X is closed and that $S(X) \cap K$ has a circle component γ bounding a disk region D on K . Let X' be a connected component of $X \setminus K$ with $\partial X' = \gamma$. If X' does not contain true vertices, then $X \setminus X'$ is a shadow of K .*

Proof From Lemma 5.9, $H_1(X') \cong \mathbb{Z}/2^k\mathbb{Z}$ for some $k \in \mathbb{Z}_{\geq 0}$. Then $\text{gl}(D) = 0$ by Lemma 6.2, and $X \setminus X'$ is a shadow of K by Proposition 5.3. \square

Theorem 6.4 *A 2-knot K is unknotted if and only if $sc(K) = 0$.*

Proof Let X be a shadow of a 2-knot K with $c(X) = 0$. By Lemma 6.1, we can assume $\partial X = \emptyset$. Since $c(X) = 0$, each component of $S(X) \cap K$ is a circle, and each component of $X \setminus K$ does not contain true vertices. By iterating Lemma 6.3, K admits itself as a shadow. Hence K is unknotted. The converse is obvious. \square

7 Existence of 2-knots with complexity one

From here, we always assume that

- X is a closed shadow of a 2-knot K , and
- $c(X) = 1$.

Note that $K \cap S(X)$ is not empty. Then there are two cases:

- (i) the true vertex is contained in a component of $K \cap S(X)$, which is an 8-shaped graph, and
- (ii) the true vertex does not lie on K and every component of $K \cap S(X)$ is a circle.

Therefore, we can also assume the following by Lemma 6.3:

- $K \cap S(X)$ is connected, and it is a circle or an 8-shaped graph.

Moreover, we fix the notation

- $X' = X \setminus \text{Int Nbd}(K; X)$,

which is also connected by the above assumption.

7.1 True vertex lies on K

Suppose that the true vertex of X lies on K . Let us first consider the case of special shadow-complexity 1. We need the following lemma.

Lemma 7.1 *Let K be a 2–knot. Suppose that S^4 admits a decomposition consisting of $\text{Nbd}(K; S^4)$, one 2–handle, two 3–handles and one 4–handle. Then K is unknotted.*

Proof We regard $\text{Nbd}(K; S^4)$ as the union of a 0–handle h_0 and a 2–handle h_2 . This 2–handle h_2 is attached along the 0–framed unknot L lying on the boundary of h_0 since $\text{Nbd}(K; S^4) \cong S^2 \times D^2$. Let h'_2 be the other 2–handle in the decomposition, and let L' be its attaching circle. This framed knot L' is contained in $\partial h_0 \setminus \nu L$, and hence $L \sqcup L'$ forms a 2–component link in $\partial h_0 \cong S^3$. By [Gompf et al. 2010, Proposition 3.2], we can modify the link $L \sqcup L'$ into the unlink only by using handle-slides of L' over L . Then L' is contained in a small 3–ball in the boundary of $h_0 \cup h_2 (= \text{Nbd}(\mathcal{K}; S^4))$, and the 2–sphere K can be pushed to the boundary of $h_0 \cup h_2 \cup h'_2$ by isotopy. The resulting 2–sphere plays a role of the attaching sphere of a 3–handle by [Laudenbach and Poénaru 1972]. Hence K bounds a 3–ball in S^4 , which is the definition of K to be unknotted. \square

Theorem 7.2 *There are no 2–knots with special shadow-complexity 1.*

Proof Let us suppose that X is a special shadow X of K with $c(X) = 1$. Then $S(X)$ is connected and $S(X) \subset K$. Moreover, it is homeomorphic to an 8–shaped graph, and $\text{Nbd}(S(X); K)$ is homeomorphic to a pair of pants. Hence $\text{Nbd}(S(X); X)$ is homeomorphic to X_{11} (see Section 2.4), and X' is a 2–disk. Thus, S^4 is decomposed as in Lemma 7.1, and K is unknotted and $\text{sc}^{\text{sp}}(K) = 0$. \square

We next consider the nonspecial case.

Proposition 7.3 *If the true vertex of X lies on K , then $G(K)$ is an infinite cyclic group.*

Proof Let S be the component of $K \cap S(X)$ containing the true vertex. As well as in the proof of Theorem 7.2, $\text{Nbd}(S; X)$ is homeomorphic to X_{11} , and $\partial X' = X' \cap \text{Nbd}(K; X)$ is a circle. By Lemma 5.9, $\pi_1(X') \cong \langle \gamma \mid \gamma^{2^k} \rangle$ for some $k \in \mathbb{Z}_{\geq 0}$, where $\gamma = \partial X'$. Then we have $G(K) \cong \langle \gamma, \mu \mid \gamma^{2^k}, \gamma \mu^{-\text{gl}(\gamma)} \rangle$ by Proposition 3.4. Since $\text{gl}(\gamma) = 0$ by Lemma 6.2, $G(K)$ is an infinite cyclic group. \square

7.2 True vertex does not lie on K

Hereafter we suppose that the true vertex of X does not lie on K , that is, it is contained in X' . In this subsection, we investigate the knot group of K .

The part $S(X) \cap K$ of the singular set separates K into two disk regions, and their gleams are g and $-g$ for some $g \in \mathbb{Z}_{\geq 0}$ since the self-intersection number of K is 0.

If $g = 0$, K is a shadow of itself by Proposition 5.3. Then K is unknotted.

We henceforth suppose $g > 0$. Let us orient K arbitrarily, and then an oriented meridian μ of K is defined. Set $\gamma = \partial X'$, and let G be a graph encoding X' . This graph G has exactly one vertex of type (B), which corresponds to γ and will be denoted by v . By Lemma 5.6(2), G have exactly one vertex v_0 of type (X₃),

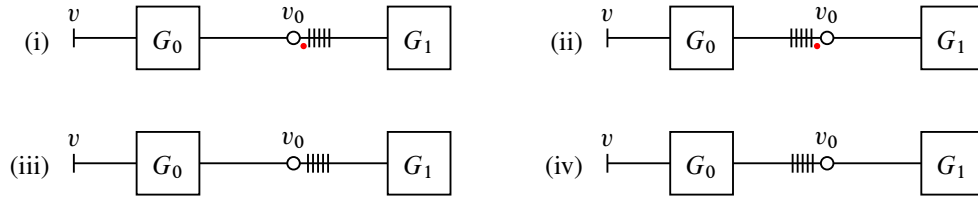


Figure 18: The possible cases of encoding graph G of X such that v_0 is of type (X_3) or (X_4) .

(X_4) , (X_8) , (X_9) , (X_{10}) or (X_{11}) , and hence G is one of those shown in Figures 18, 19 and 20. Note that X_8 and X_{11} have symmetries that interchange two boundary components with the same length. Let X_{v_0} be the connected component of $\text{Nbd}(S(X); X)$ corresponding to v_0 , which is homeomorphic to one of $X_3, X_4, X_8, X_9, X_{10}$ or X_{11} .

Let a denote the abelianization map of a group. In the following, we discuss what kind of 2-knot admits a shadow encoded by a graph in Figures 18, 19 and 20.

Case 1 (v_0 is of type (X_3) or (X_4)) The graph G is one of those shown in Figure 18. Let G_0 and G_1 be subgraphs of G as indicated in the figure. The subpolyhedron X' is decomposed into U, X_{v_0} and V , where U and V are the subpolyhedron corresponding to G_0 and G_1 , respectively. Note that U has two cut ends $\gamma \sqcup \gamma_0$, and also note that V has one cut end γ_1 . Then we have $\pi_1(V) \cong \langle \gamma_1 \mid \gamma_1^{2^{k_1}} \rangle$ by Lemma 5.9 for some $k_1 \in \mathbb{Z}_{\geq 0}$. We also have $H_1(X') \cong \mathbb{Z}\langle \gamma \rangle$ by Lemma 6.2. Then we can apply Lemma 5.11 to U , and we have $\pi_1(U) \cong \langle \gamma, \gamma_0 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}} \rangle$ for some $k_0, l, m \in \mathbb{Z}_{\geq 0}$.

Lemma 7.4 *The following hold.*

- (1) *If G is shown in Figure 18(i), then $\pi_1(U) \cong \langle \gamma, \gamma_0 \mid \gamma^{2^m} \gamma_0 \rangle$ and V is simply connected. Moreover, $G(K) \cong \langle x, \mu \mid x^2 \mu^n x^{-1} \mu^{-n} \rangle$, where $n = 2^m g$.*
- (2) *If G is shown in Figure 18(ii), then $G(K)$ is an infinite cyclic group.*
- (3) *If G is shown in Figure 18(iii), then $\pi_1(U) \cong \langle \gamma, \gamma_0 \mid \gamma^{2^m} \gamma_0 \rangle$ and V is simply connected. Moreover, $G(K) \cong \langle x, \mu \mid x^2 \mu^n x^{-1} \mu^n \rangle$, where $n = 2^m g$.*
- (4) *If G is shown in Figure 18(iv), then $G(K)$ is an infinite cyclic group.*

Proof (1) The polyhedron X' is decomposed as $U \cup X_3 \cup V$, and U and V are glued with X_3 along the boundary components of X_3 with length 1 and 5, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_0 = y, \gamma_1 = xyx^{-2}y^{-1} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, (xyx^{-2}y^{-1})^{2^{k_1}} \rangle \\ &\xrightarrow{a} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle 2^{m+k_0} \gamma + 2^{l+k_0} y \rangle \oplus \mathbb{Z}\langle 2^{k_1} x \rangle \\ &\cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$. Hence $k_0 = k_1 = l = 0$. Thus,

$$\begin{aligned} G(K) &\cong \langle x, y, \gamma, \mu \mid (\gamma^{2^m} y^{2^l})^{2^{k_0}}, (xyx^{-2}y^{-1})^{2^{k_1}}, \gamma\mu^{-g} \rangle \\ &\cong \langle x, \mu \mid x^2\mu^n x^{-1}\mu^{-n} \rangle, \end{aligned}$$

where $n = 2^m g$.

(2) The polyhedron X' is decomposed as $U \cup X_3 \cup V$, and U and V are glued with X_3 along the boundary components of X_3 with length 5 and 1, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1 \mid y = \gamma_1, xyx^{-2}y^{-1} = \gamma_0, (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} (xyx^{-2}y^{-1})^{2^l})^{2^{k_0}}, y^{2^{k_1}} \rangle \\ &\xrightarrow{a} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle 2^{m+k_0}\gamma + 2^{l+k_0}x \rangle \oplus \mathbb{Z}\langle 2^{k_1}y \rangle \\ &\cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$. Hence $k_0 = k_1 = l = 0$. Thus,

$$\begin{aligned} G(K) &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \mu \mid y = \gamma_1, xyx^{-2}y^{-1} = \gamma_0, (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma\mu^{-g} \rangle \\ &\cong \langle \mu \rangle. \end{aligned}$$

(3) The polyhedron X' is decomposed as $U \cup X_4 \cup V$, and U and V are glued with X_4 along the boundary components of X_4 with length 1 and 5, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1 \mid y = \gamma_0, xyx^{-2}y = \gamma_1, (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} y^{2^l})^{2^{k_0}}, (xyx^{-2}y)^{2^{k_1}} \rangle \\ &\xrightarrow{a} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle 2^{m+k_0}\gamma + 2^{l+k_0}y \rangle \oplus \mathbb{Z}\langle 2^{k_1}(2y-x) \rangle \\ &\cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$. Hence $k_0 = k_1 = l = 0$. Thus,

$$\begin{aligned} G(K) &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \mu \mid y = \gamma_0, xyx^{-2}y = \gamma_1, (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma\mu^{-g} \rangle \\ &\cong \langle x, \mu \mid x^2\mu^n x^{-1}\mu^n \rangle, \end{aligned}$$

where $n = 2^m g$.

(4) The polyhedron X' is decomposed as $U \cup X_4 \cup V$, and U and V are glued with X_4 along the boundary components of X_4 with length 5 and 1, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1 \mid y = \gamma_1, xyx^{-2}y = \gamma_0, (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} (xyx^{-2}y)^{2^l})^{2^{k_0}}, y^{2^{k_1}} \rangle \\ &\xrightarrow{a} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle 2^{m+k_0}\gamma + 2^{l+k_0}(2y-x) \rangle \oplus \mathbb{Z}\langle 2^{k_1}y \rangle \\ &\cong H_1(X'). \end{aligned}$$

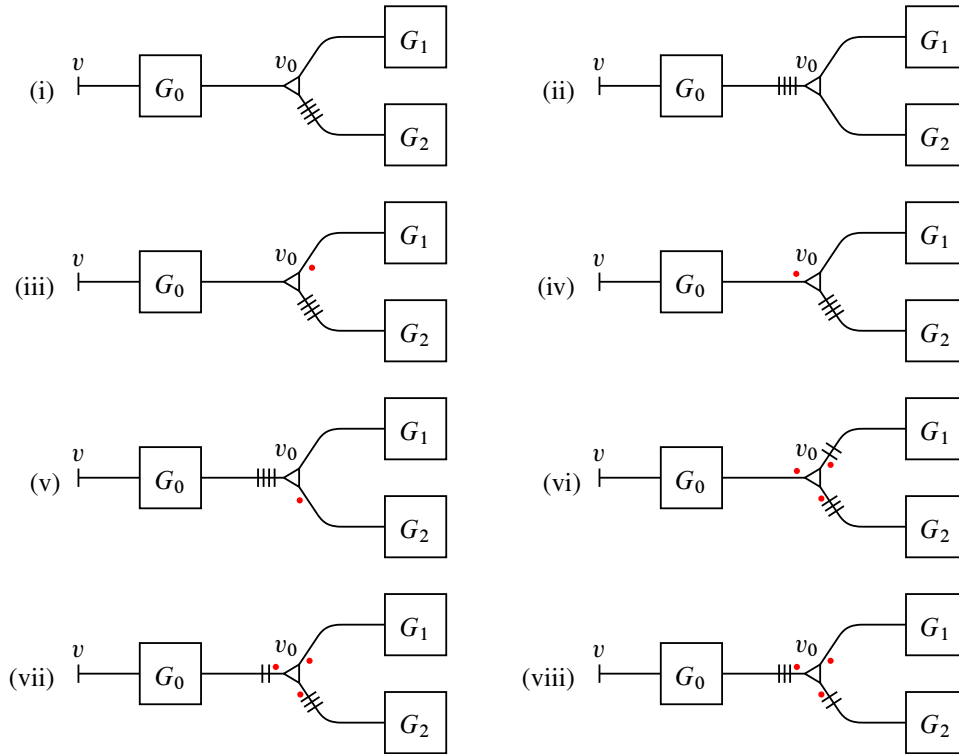


Figure 19: The possible cases of encoding graph G of X such that v_0 is of type (X_8) , (X_9) or (X_{10}) .

It must be an infinite cyclic group generated by $[\gamma]$. Hence $k_0 = k_1 = l = 0$. Thus,

$$G(K) \cong \langle x, y, \gamma, \mu \mid (\gamma^{2^m} (xyx^{-2}y)^{2^l})^{2^{k_0}}, y^{2^{k_1}}, \gamma\mu^{-g} \rangle \cong \langle \mu \rangle. \quad \square$$

Case 2 (v_0 is of type (X_8) , (X_9) or (X_{10})) The graph G is one of those shown in Figure 19. Let G_0, G_1 and G_2 be subgraphs of G as indicated in the figure. The subpolyhedron X' is decomposed into U, X_{v_0}, V_1 and V_2 , where U, V_1 and V_2 are the subpolyhedron corresponding to G_0, G_1 and G_2 , respectively. Note that U has two cut ends $\gamma \sqcup \gamma_0$, and also note that V_i has one cut end γ_i for $i \in \{1, 2\}$. Then we have $\pi_1(V_i) \cong \langle \gamma_i \mid \gamma_i^{2^{k_i}} \rangle$ by Lemma 5.9 for some $k_i \in \mathbb{Z}_{\geq 0}$. We also have $H_1(X') \cong \mathbb{Z}\langle \gamma \rangle$ by Lemma 6.2. Then we can apply Lemma 5.11 to U , and we have $\pi_1(U) \cong \langle \gamma, \gamma_0 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}} \rangle$ for some $k_0, l, m \in \mathbb{Z}_{\geq 0}$.

Lemma 7.5 *The following hold.*

- (1) *If G is shown in Figure 19(i), then $G(K)$ is an infinite cyclic group.*
- (2) *Figure 19(ii) does not encode a shadow of any 2–knot.*
- (3) *Figure 19(iii) does not encode a shadow of any 2–knot.*
- (4) *If G is shown in Figure 19(iv), then $G(K)$ is an infinite cyclic group.*
- (5) *Figure 19(v) does not encode a shadow of any 2–knot.*

- (6) Figure 19(vi) does not encode a shadow of any 2–knot.
- (7) Figure 19(vii) does not encode a shadow of any 2–knot.
- (8) Figure 19(viii) does not encode a shadow of any 2–knot.

Proof (1) Suppose that G is as shown in Figure 19(i). The polyhedron X' is decomposed as

$$U \cup X_8 \cup V_1 \cup V_2,$$

and U , V_1 and V_2 are glued with X_8 along the boundary components of X_8 with length 1, 1 and 4, respectively. The fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 = x, \gamma_1 = y, \gamma_2 = xyx^{-1}y^{-1} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} x^{2^l})^{2^{k_0}}, y^{2^{k_1}}, (xyx^{-1}y^{-1})^{2^{k_2}} \rangle \\ &\xrightarrow{a} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle 2^{m+k_0}\gamma + 2^{l+k_0}x \rangle \oplus \mathbb{Z}\langle 2^{k_1}y \rangle \\ &\cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$. Hence $k_0 = k_1 = l = 0$. Thus,

$$G(K) \cong \langle x, y, \gamma, \mu \mid (\gamma^{2^m} x^{2^l})^{2^{k_0}}, y^{2^{k_1}}, (xyx^{-1}y^{-1})^{2^{k_2}}, \gamma\mu^{-g} \rangle \cong \langle \mu \rangle.$$

(2) Suppose G is as shown in Figure 19(ii). The polyhedron X' is decomposed as $U \cup X_8 \cup V_1 \cup V_2$, and U , V_1 and V_2 are glued with X_8 along the boundary components of X_8 with length 4, 1 and 1, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 = xyx^{-1}y^{-1}, \gamma_1 = x, \gamma_2 = y \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} (xyx^{-1}y^{-1})^{2^l})^{2^{k_0}}, x^{2^{k_1}}, y^{2^{k_2}} \rangle \\ &\xrightarrow{a} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle 2^{m+k_0}\gamma \rangle \oplus \mathbb{Z}\langle 2^{k_1}x \rangle \oplus \mathbb{Z}\langle 2^{k_2}y \rangle \\ &\cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$, which is impossible.

(3) Suppose G is as shown in Figure 19(iii). The polyhedron X' is decomposed as $U \cup X_9 \cup V_1 \cup V_2$. One of the boundary components of X_9 has length 4 and the other two have 1. Note that, however, X_9 does not have a symmetry such as X_8 . The boundary components of X_9 are represented by words $xyxy^{-1}$, x and y . Here U , V_1 and V_2 are glued with X_9 along the boundary components of X_9 corresponding to x , y and $xyxy^{-1}$, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 = x, \gamma_1 = y, \gamma_2 = xyxy^{-1} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} x^{2^l})^{2^{k_0}}, y^{2^{k_1}}, (xyxy^{-1})^{2^{k_2}} \rangle \\ &\xrightarrow{a} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle 2^{m+k_0}\gamma + 2^{l+k_0}x \rangle \oplus \mathbb{Z}\langle 2^{k_1}y \rangle \oplus \mathbb{Z}\langle 2^{k_2+1}x \rangle \\ &\cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$, which is impossible.

(4) Suppose that G is as shown in Figure 19(iv). The polyhedron X' is decomposed as $U \cup X_9 \cup V_1 \cup V_2$, and U , V_1 and V_2 are glued with X_9 along the boundary components of X_9 corresponding to y , x and $xyxy^{-1}$, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 = y, \gamma_1 = x, \gamma_2 = xyxy^{-1} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} y^{2^l})^{2^{k_0}}, x^{2^{k_1}}, (xyxy^{-1})^{2^{k_2}} \rangle \\ &\xrightarrow{a} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle 2^{m+k_0} \gamma + 2^{l+k_0} y \rangle \oplus \mathbb{Z}\langle 2^{k_1} x \rangle \oplus \mathbb{Z}\langle 2^{k_2+1} x \rangle \cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$. Hence $k_0 = k_1 = l = 0$. Thus,

$$G(K) \cong \langle x, y, \gamma, \mu \mid (\gamma^{2^m} y^{2^l})^{2^{k_0}}, x^{2^{k_1}}, (xyxy^{-1})^{2^{k_2}}, \gamma \mu^{-g} \rangle \cong \langle \mu \rangle.$$

(5) Suppose G is as shown in Figure 19(v). The polyhedron X' is decomposed as $U \cup X_9 \cup V_1 \cup V_2$, and U , V_1 and V_2 are glued with X_9 along the boundary components of X_9 corresponding to $xyxy^{-1}$, x and y , respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 = xyxy^{-1}, \gamma_1 = x, \gamma_2 = y \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} (xyxy^{-1})^{2^l})^{2^{k_0}}, x^{2^{k_1}}, y^{2^{k_2}} \rangle \\ &\xrightarrow{a} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle 2^{m+k_0} \gamma + 2^{l+k_0+1} x \rangle \oplus \mathbb{Z}\langle 2^{k_1} x \rangle \oplus \mathbb{Z}\langle 2^{k_2} y \rangle \cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$, which is impossible.

(6) Suppose G is as shown in Figure 19(vi). The polyhedron X' is decomposed as $U \cup X_{10} \cup V_1 \cup V_2$, and U , V_1 and V_2 are glued with X_{10} along the boundary components of X_{10} with length 1, 2 and 3, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 = x, \gamma_1 = xy, \gamma_2 = xy^{-2} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} x^{2^l})^{2^{k_0}}, (xy)^{2^{k_1}}, (xy^{-2})^{2^{k_2}} \rangle \\ &\xrightarrow{a} \frac{\mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle}{\mathbb{Z}\langle 2^{m+k_0} \gamma + 2^{l+k_0} x \rangle \oplus \mathbb{Z}\langle 2^{k_1} (x+y) \rangle \oplus \mathbb{Z}\langle 2^{k_2} (x-2y) \rangle} \cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$, which is impossible.

(7) Suppose G is as shown in Figure 19(vii). The polyhedron X' is decomposed as $U \cup X_{10} \cup V_1 \cup V_2$, and U , V_1 and V_2 are glued with X_{10} along the boundary components of X_{10} with length 2, 1 and 3, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 = xy, \gamma_1 = x, \gamma_2 = xy^{-2} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} (xy)^{2^l})^{2^{k_0}}, x^{2^{k_1}}, (xy^{-2})^{2^{k_2}} \rangle \\ &\xrightarrow{a} \frac{\mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle}{\mathbb{Z}\langle 2^{m+k_0} \gamma + 2^{l+k_0} (x+y) \rangle \oplus \mathbb{Z}\langle 2^{k_1} x \rangle \oplus \mathbb{Z}\langle 2^{k_2} (x-2y) \rangle} \cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$, which is impossible.

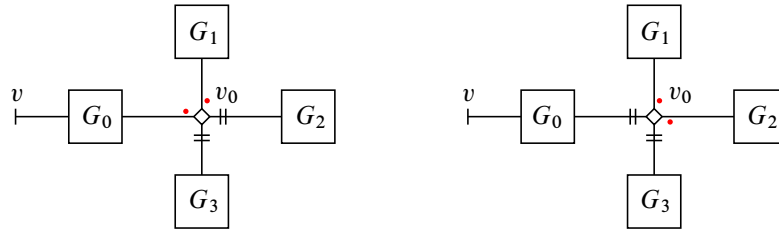


Figure 20: The possible cases of encoding graph G of X such that v_0 is of type (X_{11}) .

(8) Suppose G is as shown in Figure 19(viii). The polyhedron X' is decomposed as $U \cup X_{10} \cup V_1 \cup V_2$, and U , V_1 and V_2 are glued with X_{10} along the boundary components of X_{10} with length 3, 1 and 2, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_0 = xy^{-2}, \gamma_1 = x, \gamma_2 = xy \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} (xy^{-2})^{2^l})^{2^{k_0}}, x^{2^{k_1}}, (xy)^{2^{k_2}} \rangle \\ &\xrightarrow{\alpha} \frac{\mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle}{\mathbb{Z}\langle 2^{m+k_0}\gamma + 2^{l+k_0}(x-2y) \rangle \oplus \mathbb{Z}\langle 2^{k_1}x \rangle \oplus \mathbb{Z}\langle 2^{k_2}(x+y) \rangle} \cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$, which is impossible. □

Case 3 (v_0 is of type (X_{11})) The graph G is one of those shown in Figure 20. Let G_0, G_1, G_2 and G_3 be subgraphs of G as indicated in the figure. The subpolyhedron X' is decomposed into U, X_{v_0}, V_1, V_2 and V_3 , where U, V_1, V_2 and V_3 are the subpolyhedron corresponding to G_0, G_1, G_2 and G_3 . Note that U has two cut ends $\gamma \sqcup \gamma_0$ as a subpolyhedron of X , and also note that V_i has one cut end γ_i for $i \in \{1, 2, 3\}$. Then we have $\pi_1(V_i) \cong \langle \gamma_i \mid \gamma_i^{2^{k_i}} \rangle$ by Lemma 5.9 for some $k_i \in \mathbb{Z}_{\geq 0}$. We also have $H_1(X') \cong \mathbb{Z}\langle \gamma \rangle$ by Lemma 6.2. Then we can apply Lemma 5.11 to U , and we have $\pi_1(U) \cong \langle \gamma, \gamma_0 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}} \rangle$ for some $k_0, l, m \in \mathbb{Z}_{\geq 0}$.

Lemma 7.6 *The following hold.*

- (1) *Figure 20, left, does not encode a shadow of any 2-knot.*
- (2) *Figure 20, right, does not encode a shadow of any 2-knot.*

Proof (1) Suppose G is as shown in Figure 20, left. The polyhedron X' is decomposed as $U \cup X_{11} \cup V_1 \cup V_2 \cup V_3$, and U, V_1, V_2 and V_3 are glued with X_{11} along the boundary components of X_{11} with length 1, 1, 2 and 2, respectively. The fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2, \gamma_3 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_3^{2^{k_3}}, \gamma_0 = x, \gamma_1 = y, \gamma_2 = xy, \gamma_3 = xy^{-1} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} x^{2^l})^{2^{k_0}}, y^{2^{k_1}}, (xy)^{2^{k_2}}, (xy^{-1})^{2^{k_3}} \rangle \\ &\xrightarrow{\alpha} \frac{\mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle}{\mathbb{Z}\langle 2^{m+k_0}\gamma + 2^{l+k_0}x \rangle \oplus \mathbb{Z}\langle 2^{k_1}y \rangle \oplus \mathbb{Z}\langle 2^{k_2}(x+y) \rangle \oplus \mathbb{Z}\langle 2^{k_3}(x-y) \rangle} \cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$, which is impossible.

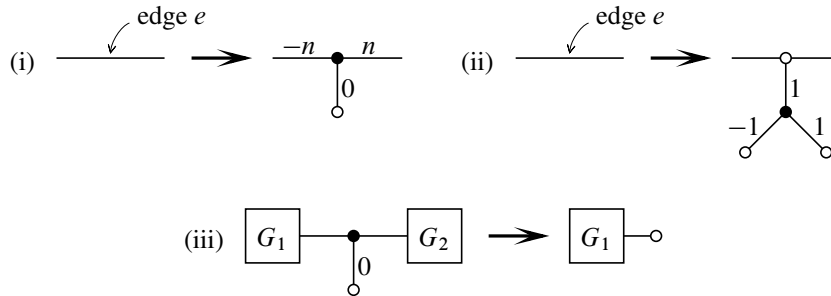


Figure 21: Adding a horizontal compressing disk (i), adding a vertical compressing disk (ii), and connected-sum reduction (iii).

(2) Suppose G is as shown in Figure 20, right. The polyhedron X' is decomposed as

$$U \cup X_{11} \cup V_1 \cup V_2 \cup V_3,$$

and U, V_1, V_2 and V_3 are glued with X_{11} along the boundary components of X_{11} with length 2, 1, 1 and 2, respectively. Then the fundamental group of X' and its abelianization are obtained as follows:

$$\begin{aligned} \pi_1(X') &\cong \langle x, y, \gamma, \gamma_0, \gamma_1, \gamma_2, \gamma_3 \mid (\gamma^{2^m} \gamma_0^{2^l})^{2^{k_0}}, \gamma_1^{2^{k_1}}, \gamma_2^{2^{k_2}}, \gamma_3^{2^{k_3}}, \gamma_0 = xy, \gamma_1 = x, \gamma_2 = y, \gamma_3 = xy^{-1} \rangle \\ &\cong \langle x, y, \gamma \mid (\gamma^{2^m} (xy)^{2^l})^{2^{k_0}}, x^{2^{k_1}}, y^{2^{k_2}}, (xy^{-1})^{2^{k_3}} \rangle \\ &\xrightarrow{a} \frac{\mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}\langle \gamma \rangle}{\mathbb{Z}\langle 2^{m+k_0} \gamma + 2^{l+k_0} (x+y) \rangle \oplus \mathbb{Z}\langle 2^{k_1} x \rangle \oplus \mathbb{Z}\langle 2^{k_2} y \rangle \oplus \mathbb{Z}\langle 2^{k_3} (x-y) \rangle} \cong H_1(X'). \end{aligned}$$

It must be an infinite cyclic group generated by $[\gamma]$, which is impossible. □

8 Classification of 2-knots with complexity one

8.1 Lemmas on decorated graphs

We now define a *decoration* of an edge e of an encoding graph G as a half-integer such that it is an integer if and only if the number of red dots appended to e is even (actually, zero or two). If every edge of G is assigned with a decoration, G is called a *decorated graph*. A decoration corresponds to a gleam, and a decorated tree encodes a shadowed polyhedron.

We can easily describe how a decorated graph G changes by adding a compressing disk and a connected-sum reduction. See Figure 21. If a lift of an edge e of G has a horizontal (resp. vertical) compressing disk, we can replace the edge e as shown in Figure 21(i) (resp. (ii)). If a decorated graph is as shown in the left of Figure 21(iii) and if the subpolyhedron corresponding to the subgraph G_1 contains K , we can adopt a decorated graph shown in the right of the figure.

In this subsection, we provide some modifications of shadows and decorated graphs not changing a 2-knot.

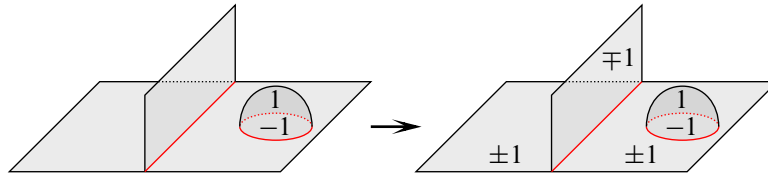


Figure 22: A move on a shadow: if a region R has a vertical compressing disk, then we can modify the gleams of R , R' and R'' as in the figure, where R' and R'' are regions adjacent to a common triple line with R .

Lemma 8.1 Suppose that X' has a part as shown in the left of Figure 22. Then the move shown in the figure and its inverse modify X to another shadow of K .

See [Koda et al. 2022] for the proof of the above lemma.

We next introduce eight moves on decorated graphs as shown in Figure 23; moves (a), (b), (c), (d), (e), (f), (g) and (h). Note that the decoration r in a move (g) is not $\pm \frac{1}{2}$.

Lemma 8.2 Let G be a decorated graph of X and G' be the subgraph corresponding to X' . Then the moves shown in Figure 23 that are performed on G' modify G to another decorated graph encoding a shadow of K .

Proof Moves (a) and (b) are obtained by a move in Figure 22.

A move (c) is explained in [Martelli 2011, Figure 34(7)].

A move (d) is a obtained by a move (a), a connected-sum reduction, and a YV-move.

A move (e) is a kind of *propagation principle* [Koda et al. 2022]: if two of the three regions adjacent to a triple line have vertical compressing disks, then the other also has.

A move (f) is explained in [Martelli 2011, Figure 34(4)].

A move (g) is explained in Figure 24. Let $\pi : M_X \rightarrow X$ be a natural projection, where $M_X = \text{Nbd}(X; S^4)$. Then the preimage of the subpolyhedron corresponding to the leftmost graph of Figure 24 by $\pi|_{\partial M_X}$ is homeomorphic to the complement of the $(2, 2r)$ -torus knot in S^3 ; see [Ishikawa and Koda 2017, Figure 11]. Recall that $r \neq \frac{1}{2}$. Let e be an edge as indicated in the figure, then a lift of e has a vertical compressing disk by Property P and Property R. Hence we can add a vertical compressing disk as in Figure 24(i). The move in Figure 24(ii) is obtained by performing a move (b) as many times as necessary. The move in Figure 24(iii) is done by a move (f) and a YV-move.

A move (h) is explained in Figure 25. The move in Figure 25(i) is obtained by performing moves (c) as many times as necessary. The move in Figure 25(ii) can be done by using [Martelli 2011, Figure 34(3)]. Here we need the following claim:

Claim 1 A lift of e' has a vertical compressing disk, where e' is an edge indicated in the figure.

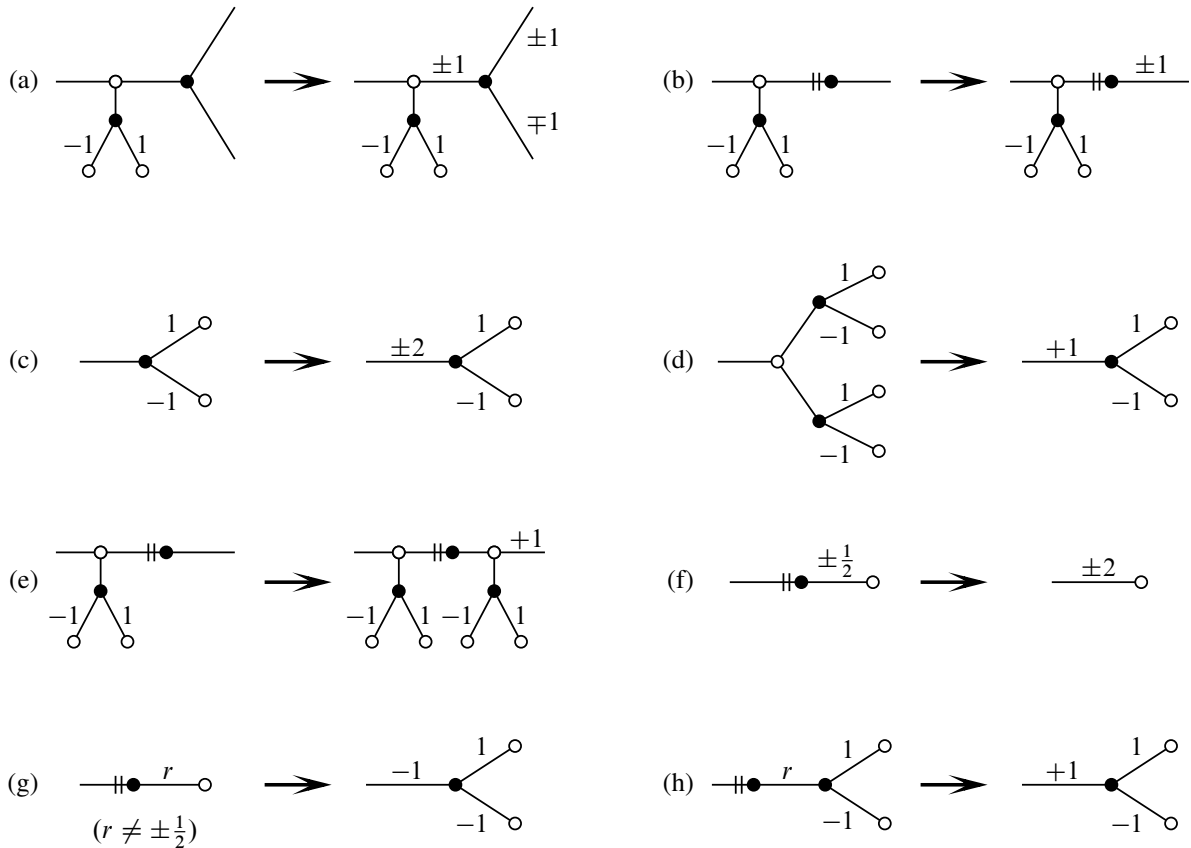


Figure 23: Eight moves on decorated graphs. The decoration r in (g) is not $\pm\frac{1}{2}$.

Proof Let $\pi : M_X \rightarrow X$ be a natural projection. The preimage of a subpolyhedron homeomorphic to Y_{111} by $\pi|_{\partial M_X}$ is homeomorphic to $P \times S^1$. Hence the subgraph after the move in Figure 24(ii) corresponds to a 3-manifold homeomorphic to the Seifert fibered space $(D^2; (2, 1), (2, -1))$, which has one torus boundary. The Dehn filling of this manifold along the (p, q) -slope is $(S^2; (2, 1), (2, -1), (p, q))$. Note that the slope with $(p, q) = (p, 1)$ is sent to a lift of the edge e' by π injectively, and the slope with $(p, q) = (1, 0)$ is sent to one point by π . The 3-manifold $(S^2; (2, 1), (2, -1), (p, q))$ is not homeomorphic to $\#_h(S^1 \times S^2)$ for any $h \in \mathbb{Z}_{\geq 0}$ unless $q = 0$. If $q = 0$, then we have $p = 1$ and $(S^2; (2, 1), (2, -1), (1, 0)) \cong (S^2; (2, 1), (2, -1))$.

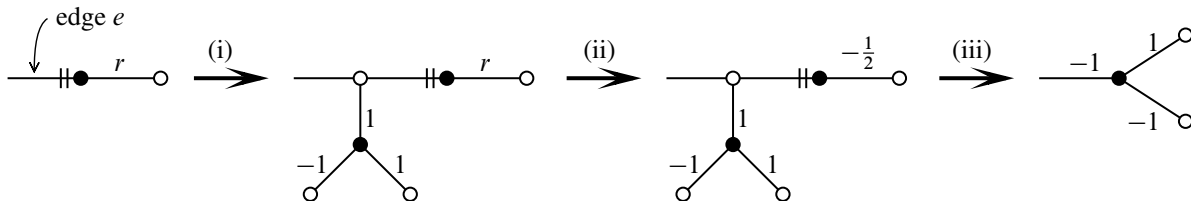


Figure 24: The proof of a move (g).

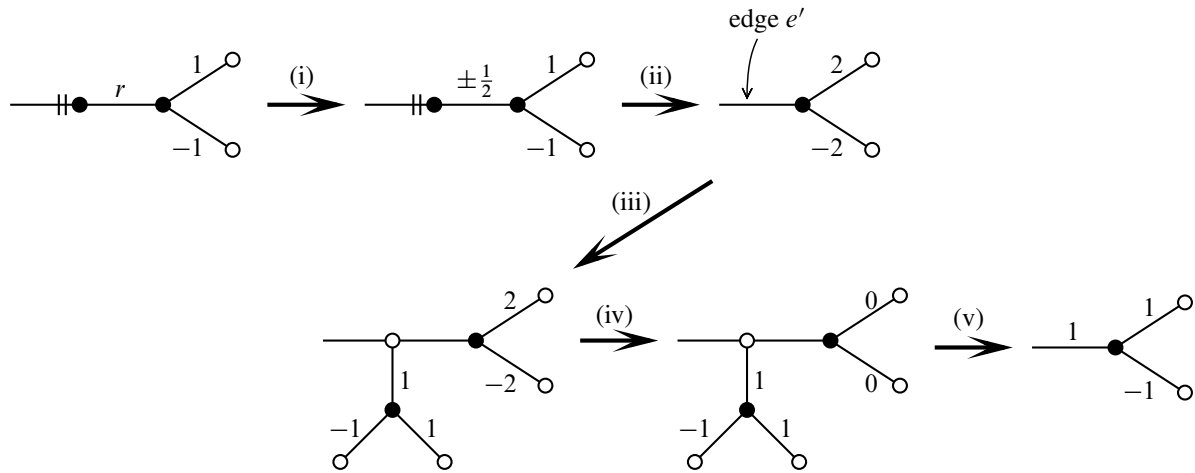


Figure 25: The proof of a move (h).

This space can be understood as a 3–manifold obtained by gluing two solid tori along their boundaries so that the meridians are identified, and hence it is homeomorphic to $S^1 \times S^2$. It follows that a lift of e' has a vertical compressing disk. □

We then continue the proof for the move (h). The move of Figure 25(iii) is the addition of a compressing disk of a lift of e' . The move in Figure 25(iv) is done by applying move (a) twice. The move in Figure 25(v) is done by a connected-sum reduction and a YV-move. □

Lemma 8.3 *Let V be a subpolyhedron of X with a single cut end γ and $c(V) = 0$. Let G be a graph encoding V and v_γ be the vertex of type (B) corresponding to γ . Suppose that G has a vertex v of type (D) that is adjacent to a vertex v' of type (Y_{111}) . Let R be the disk region of X corresponding to v . If G has another vertex of type (Y_{111}) between v_γ and v' , that is, if G is as shown in Figure 26, then $gl(R) = 0$.*

Proof We give an orientation to R arbitrarily, and we define edges e_1, e_2, e'_2 and e_3 and subgraphs G_1, G_2 and G_3 of G as indicated in Figure 26. Let V_1, U_2 and V_3 be subpolyhedra of V encoded by G_1, G_2 and G_3 , respectively. Note that each of V_1 and V_3 has one cut end and U_2 has two. Set $\gamma_i = \partial V_i$ for $i \in \{1, 3\}$, and note that it is a lift of e_i . Let γ_2 and γ'_2 denote the cut ends of U_2 , and also note that γ_2 and γ'_2 are lifts of e_2 and e'_2 , respectively. By Lemma 5.9, we have $\pi_1(V_i) \cong \langle \gamma_i \mid \gamma_i^{2k_i} \rangle$ for some

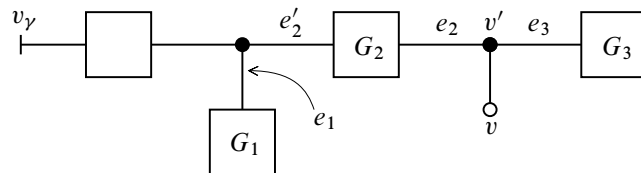


Figure 26: An encoding graph restricting the gleam of a disk region.

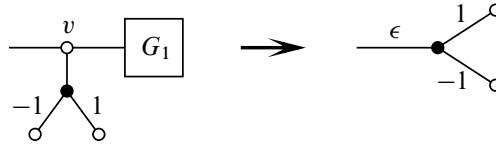


Figure 27: The modification of a decorated graph as in Lemma 8.4.

$k_i \in \mathbb{Z}_{\geq 0}$, and hence there is a 2-chain c_i in $V_i \subset V$ such that $\partial c_i = 2^{k_i}[\gamma_i]$ for $i \in \{1, 3\}$. We then define a 2-chain C_1 according to the order of $[\gamma_2]$;

- in the case $p_2[\gamma_2] = 0$ in $H_1(U_2)$ for some $p_2 \in \mathbb{Z}_{>0}$, there exists a 2-chain c_2 in U_2 with $\partial c_2 = p_2[\gamma_2]$, and then set $C_1 = c_2 - p_2[R]$;
- in the case where $[\gamma_2]$ is not a torsion element in $H_1(U_2)$, we have $p'_2[\gamma'_2] = p_2[\gamma_2]$ in $H_1(U_2)$ for some $p_2, p'_2 \in \mathbb{Z} \setminus \{0\}$ by Lemma 5.11, and then set $C_1 = p'_2 c_1 - p_2[R]$.

Define another 2-chain as $C_3 = c_3 - 2^{k_3}[R]$. These 2-chains C_1 and C_3 are homology cycles in $H_2(V)$ since $[\gamma_1] = [\gamma'_2]$ and $[\gamma_2] = [\gamma_3] = \partial[R]$. Then we have $Q(C_1, C_3) = p_2 2^{k_3} \text{gl}(R)$, which must be 0. Hence $\text{gl}(R) = 0$. □

Lemma 8.4 *Suppose that X' contains a simply connected subpolyhedron V with one cut end such that a lift of the cut end has a vertical compressing disk. Then K admits a shadow obtained from X by replacing V with a 2-disk, and the gleam of the newly formed region containing the 2-disk is one of $-\frac{1}{2}, 0, \frac{1}{2}$ or 1.*

Proof We give the proof by using decorated graphs. The assumption in the statement implies that a decorated graph of X' has a subgraph G as shown in the left of Figure 27, where the subgraph G_1 in the figure corresponds to V . It is enough to modify the graph as in the figure. Note that the decoration ϵ in the right of the figure can be replaced with $\epsilon \pm 2$ by applying move (c), and hence the gleam of the corresponding region can be chosen to be one of $-\frac{1}{2}, 0, \frac{1}{2}$ or 1 by applying the move as many times as necessary.

Let v be the vertex of type (P) shown in the left of Figure 27. By Lemma 5.10, there exists a leaf in G_1 such that all the vertices of type (Y_{12}) contained in the geodesic from v to the leaf are two-sided to v . Taking the union of all such geodesics ℓ_1, \dots, ℓ_m , which we denote by $T_\ell = \bigcup_{i=1}^m \ell_i$, is a subtree of G whose leaves are of type (D) except for v . Note that a vertex of type (P) contained in $T_\ell \setminus v$ is a trivalent vertex even in T_ℓ . We divide the proof into the following three cases:

- (1) $T_\ell \setminus v$ does not contain a vertex of type (Y_{111}) or (P);
- (2) $T_\ell \setminus v$ contains vertices of type (Y_{111}) or (P), and farthest one from v among them is of type (Y_{111}) ;
- (3) $T_\ell \setminus v$ contains vertices of type (Y_{111}) or (P), and farthest one from v among them is of type (P).

(1) In this case, T_ℓ is a line, and all the vertices between v and the leaf are of type (Y_{12}) . We can modify G as in Figure 27 by using moves (f), (g) and (h).

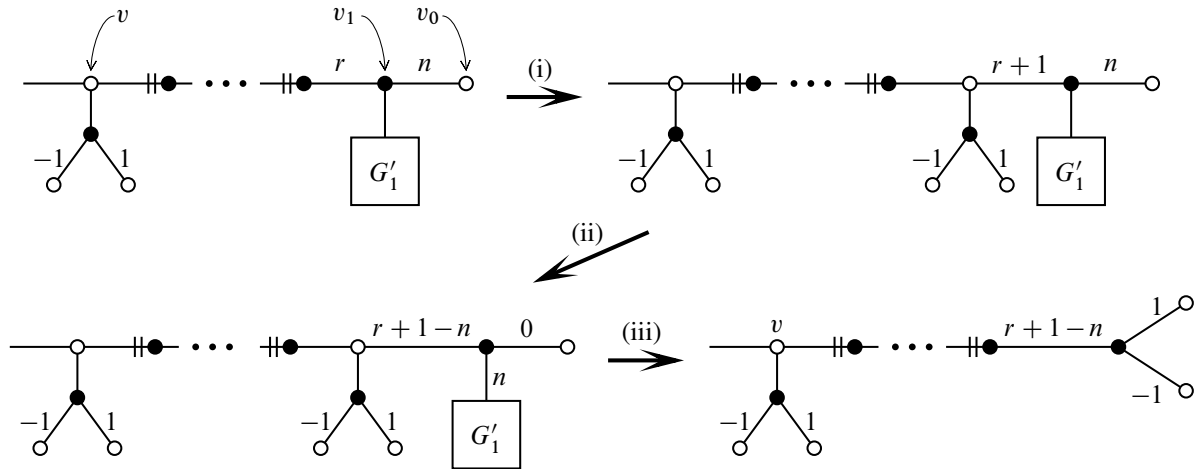


Figure 28: The case (2) in the proof of Lemma 8.4.

(2) Let v_1 be the vertex of type (Y_{111}) farthest from v . Then there is a geodesic ℓ_i containing v_1 , and let v_0 be the other endpoint than v . The vertices between v_0 and v_1 are of type (Y_{12}) . If the edge incident to v_0 is decorated by $r \neq \frac{1}{2}$, a move (g) can be applied, which is contrary to Lemma 8.3. Hence we can eliminate all the vertices of type (Y_{12}) between v_0 and v_1 using only moves (f), and then we can assume that v_0 and v_1 are connected by one edge. Let e denote this edge.

If ℓ_i has a vertex of type (Y_{111}) other than v_1 , the edge e is decorated with 0 by Lemma 8.3. Then the vertex v_1 can be eliminated by a connected-sum reduction.

If ℓ_i has no vertices of type (Y_{111}) other than v_1 , then G is as shown in the upper left of Figure 28. The modifications in Figure 28(i) and (ii) are done by moves (e) and (a), respectively. The move in Figure 28(iii) is a connected-sum reduction and a YV-move. The lower right graph in Figure 28 can be modified as required by moves (h), (d) and (c).

(3) Let v_2 be the vertex of type (P) farthest from v . Then G is as the uppermost graph in Figure 29. Let G'_1, G''_1 and G'''_1 be subgraphs of G_1 as defined in the figure. The subgraphs G''_1 and G'''_1 do not contain vertices of type (Y_{111}) or (P). Then we can apply moves (f), (g) or (h) as well as in (1) to these subgraphs, and G is modified as shown in one of Figure 29(i), (ii) or (iii). Moreover, the moves of Figure 29(iv) and (v) are obtained by YV-moves, and the move of Figure 29(vi) is done by a move (d). In either case, the vertex v_2 is eliminated, and we obtain the modification in Figure 27 inductively. \square

8.2 Existence of compressing disks

For $i \in \{1, \dots, 11\}$, the polyhedron X_i can be embedded in $\mathbb{H}_2(S^1 \times B^3)$ as a shadow, and the complement of ∂X_i in $\partial(\mathbb{H}_2(S^1 \times B^3)) (= \#_2(S^1 \times S^2))$ is a 3-manifold with tori boundary. Note that this 3-manifold $\#_2(S^1 \times S^2) \setminus \partial X_i$ admits a complete hyperbolic structure with finite volume [Costantino and Thurston

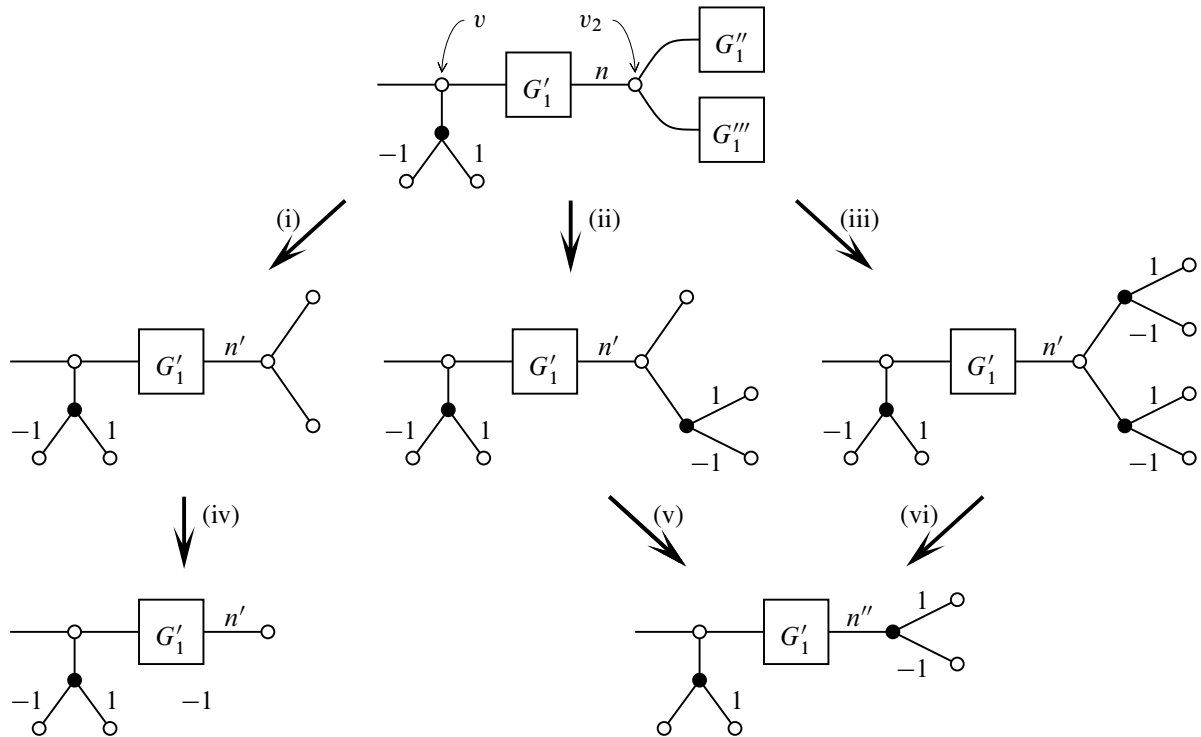


Figure 29: The case (3) in the proof of Lemma 8.4.

2008]. In [Koda et al. 2022], Dehn fillings on this 3-manifold giving $\#_h(S^1 \times S^2)$ for some $h \in \mathbb{Z}_{\geq 0}$ are studied, and it leads to the following.

Lemma 8.5 [Koda et al. 2022] *Suppose that X contains a subpolyhedron Y homeomorphic to X_3 or X_4 . Then at least one of the following holds:*

- (1) *both of the components of ∂Y have vertical compressing disks; or*
- (2) *the component of ∂Y with length 1 has a horizontal compressing disk.*

8.3 Banded unlink diagram of 2-knot with complexity one

Recall that, for $n \in \mathbb{Z}$, K_n is a 2-knot defined by the banded unlink diagram shown in Figure 1. We first prove the essential part of Theorem 8.10:

Theorem 8.6 *If a 2-knot K with $G(K) \not\cong \mathbb{Z}$ has shadow-complexity 1, then K is diffeomorphic to K_n for some nonzero integer n .*

Proof Let X be a shadow of K with $c(X) = 1$, and let G be a decorated tree graph for X . By Lemmas 7.4, 7.5 and 7.6, G has exactly one vertex v_0 of type (X_3) or (X_4) . Here we suppose that v_0 is of type (X_3) . Then G is as shown in Figure 30 by Lemma 7.4.

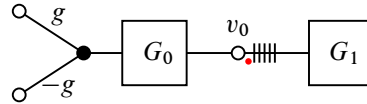


Figure 30: A decorated graph encoding a shadow of K having a subpolyhedron homeomorphic to X_3 .

If Lemma 8.5(2) holds, then the graph G can be assumed as shown in the left of Figure 31. Applying a connected-sum reduction, we obtain the right graph. This graph encodes a simple polyhedron without true vertex, which is not our focus here.

We then assume that Lemma 8.5(1) holds, and the graph G is as shown in the top of Figure 32. Let U and V be the subpolyhedra corresponding to the subgraphs G_0 and G_1 , respectively. By Lemmas 7.4, we have $\pi_1(U) \cong \langle \gamma, \gamma_1 \mid \gamma^{2^m} \gamma_1 \rangle$ and V is simply connected. We apply Lemmas 5.10 and 5.12 to U and V , respectively, and then the graph G can be assumed to be the second graph in Figure 32. Note that subgraphs $G_{0,0}, G_{0,1}, \dots, G_{0,m}$ in the figure encode simply connected subpolyhedra by Lemma 5.12. The move in Figure 32(ii) is done by iterating moves (e), and the move in Figure 32(iii) is done by that in Figure 27 (cf Lemma 8.4).

From the bottom graph in Figure 32, we obtain a banded unlink diagram shown in the top of Figure 33. We refer the reader to [Koda and Naoe 2020] for a translation of a shadow into a Kirby diagram; see also Remark 8.7 and [Costantino and Thurston 2008; Martelli 2005]. Though the framings $\epsilon_0, \epsilon_1, \dots, \epsilon_m$ and ϵ are determined from the gleams, each of them can be assumed to be 0 or 1 since there is a 0-framed knot as a meridian. The first move of Figure 33(i) is obtained by handle-slides and a cancellation of a 1-2 pair. We iterate the same process in Figure 33(ii). The move (iii) is obtained by handle-slides and a cancellation of a 1-2 pair, and we set $n = 2^m g$ and $\epsilon' = 0$ or 1. The move (iv) is done by a cup and 2-handle band swims. The move (v) is an isotopy, and (iv) is obtained by a 2-handle band slide and a cancellation of a 1-2 pair. The move (vii) is done by an isotopy if $\epsilon' = 1$, and we also need 2-handle band slides if $\epsilon' = 0$. The move (viii) is obtained by a cap and 2-handle band slides. Finally, applying a 2-handle band swim and a cancellation of a 2-3 pair, we obtain the diagram shown in Figure 1.

One can show the case where v_0 is of type (X_4) in a similar way to the above, so we skip the details. \square

Remark 8.7 The method of a translation of a shadow only to a Kirby diagram is treated in [Koda and Naoe 2020], and that to a banded unlink diagram is actually not discussed. However, we can draw a diagram as shown in Figure 33 by considering a decomposition $X = \text{Nbd}(K; X) \cup X'$ and using [Koda

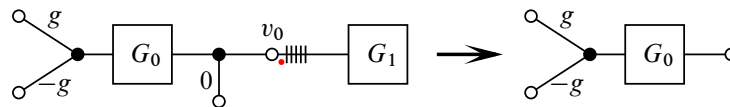


Figure 31: A decorated graph encoding a shadow of K having a subpolyhedron homeomorphic to X_3 such that the boundary component of the subpolyhedron with length 1 has a horizontal compressing disk.

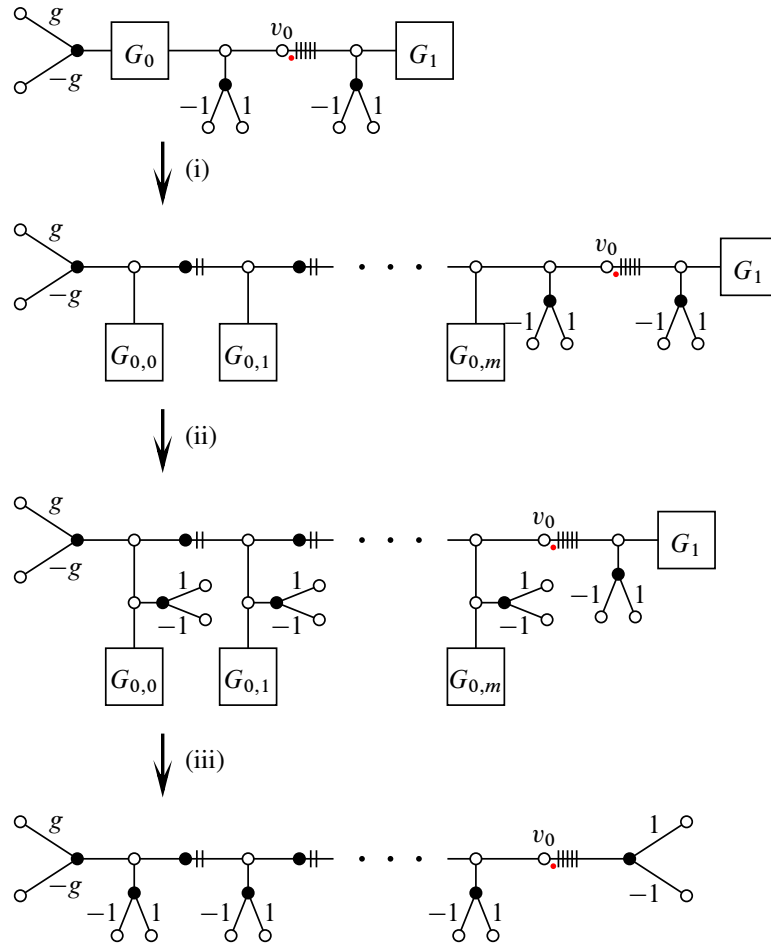


Figure 32: A decorated graph encoding a shadow of K having a subpolyhedron homeomorphic to X_3 such that the two boundary components of the subpolyhedron has vertical compressing disks.

and Naoe 2020, Lemmas 1.1 and 1.2]. Note that $\text{Nbd}(K; X)$ is a shadow of $\text{Nbd}(K; S^4) \cong S^2 \times D^2$ and $\partial \text{Nbd}(K; X)$ is a knot in $\partial \text{Nbd}(K; S^4) \cong S^2 \times S^1$ such that it winds g times along $\{\text{pt}\} \times S^1$.

Remark 8.8 If v_0 is of type (X_3) , the 2-knot K is diffeomorphic to K_n with $n = 2^m g > 0$. On the other hand, if v_0 is of type (X_4) , the 2-knot K is diffeomorphic to K_n with $n = -2^m g < 0$.

The following implies that there exist infinitely many 2-knots with shadow-complexity 1.

Proposition 8.9 *The 2-knots K_n and $K_{n'}$ are not equivalent unless $n = n'$.*

Proof From Lemma 7.4 and Remark 8.8, we have

$$G(K_n) \cong \langle x, y \mid x^2 y^{|n|} x^{-1} y^{-n} \rangle,$$

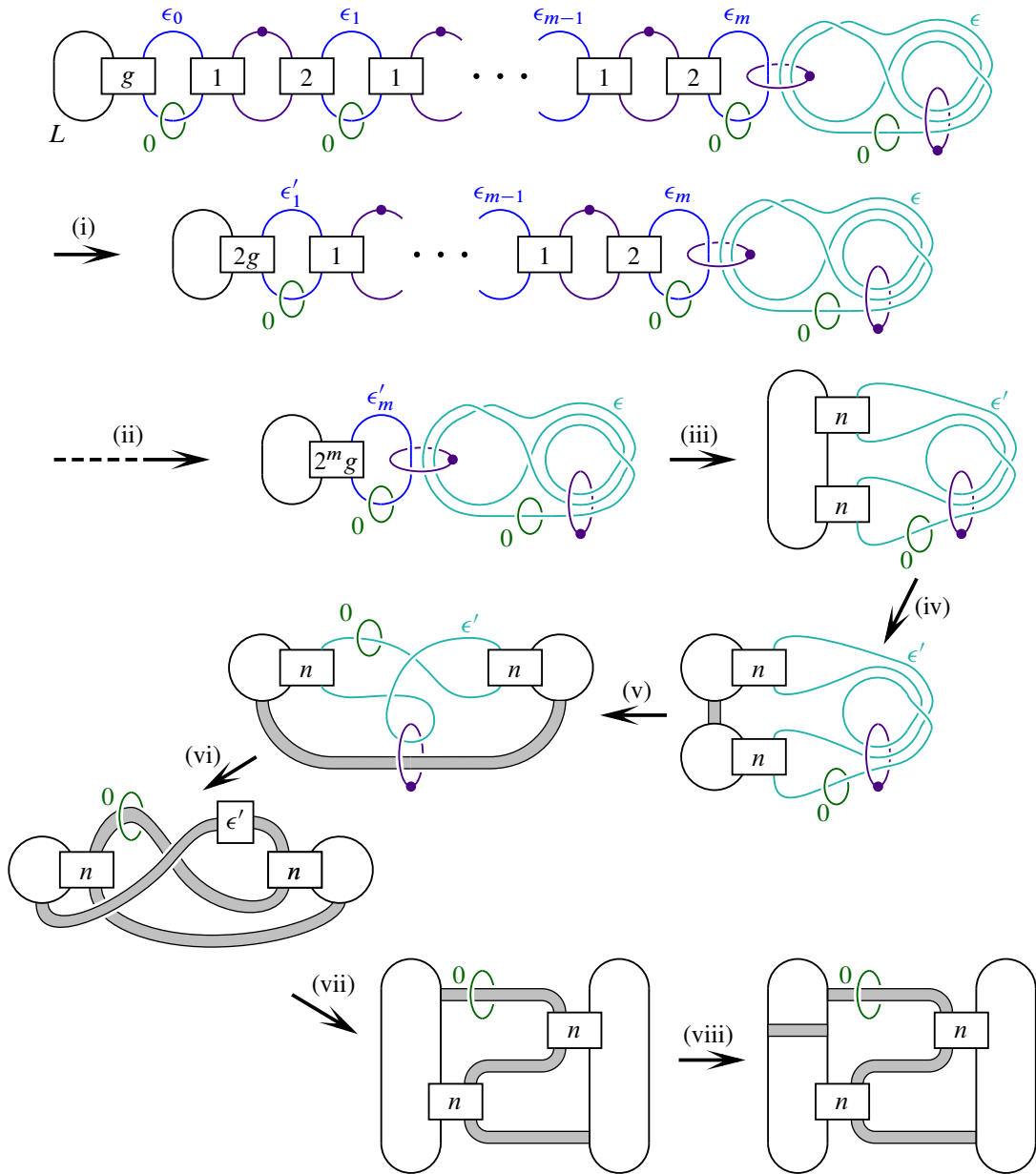


Figure 33: A banded unlink diagram of a 2-knot with complexity one and its modifications.

and its Alexander polynomial is

$$\Delta_{K_n}(t) = \begin{cases} 2 - t^n & \text{if } n \geq 0, \\ 1 - t^n + t^{2n} & \text{if } n < 0. \end{cases}$$

If $n \neq n'$, two polynomials $\Delta_{K_n}(t)$ and $\Delta_{K_{n'}}(t)$ are distinct. □

At last, we give the proof of the complexity one case.



Figure 34: A decorated graph of a shadow of a 2–knot with complexity one.

Theorem 8.10 A 2–knot K whose knot group is not infinite cyclic has shadow-complexity 1 if and only if K is diffeomorphic to K_n for some nonzero integer n .

Proof The only if part has been already discussed in Theorem 8.6.

Let n be an arbitrary nonzero integer. The banded unlink diagram of K_n shown in Figure 1 can be obtained from a shadow encoded in Figure 34, left or right, in the same way as in the proof of Theorem 8.6. Therefore, $\text{sc}(K_n) \leq 1$, and hence $\text{sc}(K_n) = 1$ by Proposition 8.9 and Theorem 6.4. \square

Remark 8.11 Let X be a shadow of K_n encoded by a decorated graph as shown in Figure 34. Its singular set $S(X)$ has 3 connected components: two circles and one 8–shaped graph. Then we can obtain a special shadow of K_n from X by applying $(0 \rightarrow 2)$ –moves twice [Costantino 2004; Turaev 1994], and hence we have $\text{sc}^{\text{sp}}(K_n) \leq 5$. This implies that the special shadow-complexity for 2–knots is not a finite-to-one invariant, while that for closed 4–manifolds is finite-to-one [Martelli 2005, Corollary 2.7].

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