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Automorphisms of some variants of fine graphs

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Recently, Bowden, Hensel and Webb defined the *fine curve graph* for surfaces, extending the notion of curve graphs for the study of homeomorphism or diffeomorphism groups of surfaces. Later, Long, Margalit, Pham, Verberne and Yao proved that for a closed surface of genus $g \geqslant 2$, the automorphism group of the fine graph is naturally isomorphic to the homeomorphism group of the surface. We extend this result to the torus case $g = 1$; in fact our method works for more general surfaces, compact or not, orientable or not. We also discuss the case of a smooth version of the fine graph.

37E30, 57K20

1 Introduction

1.1 Context and results

For a connected, compact surface Σ_g of genus $g \geqslant 1$, Bowden, Hensel and Webb [2] recently introduced the *fine curve graph* $\mathcal{C}^\dagger(\Sigma)$, as the graph whose vertices are all the essential closed curves on Σ, with an edge between two vertices a and b whenever $a \cap b = \varnothing$, if $g \geqslant 2$, and whenever $|a \cap b| \leqslant 1$ if $g = 1$. They proved that for every $g \geqslant 1$, the graph $\mathcal{C}^\dagger(\Sigma)$ is hyperbolic, and derived a construction of an infinite dimensional family of quasimorphisms on $\mathrm{Homeo}_0(\Sigma)$, thereby answering long standing questions of Burago, Ivanov and Polterovich.

The ancestor of the fine graph is the usual curve complex of a surface Σ, ie the complex whose vertices are the isotopy classes of essential curves, with an edge (or a simplex, more generally) between some vertices if and only if they have disjoint representatives. Since its introduction by Harvey [5], the curve complex of a surface has been an extremely useful tool for the study of the mapping class group $\mathrm{Mod}(\Sigma)$ of that surface, as it acts on it naturally. In particular, the fact that this complex is hyperbolic, discovered by Masur and Minsky [13; 14], has greatly improved the understanding of the mapping class groups. The result of Bowden, Hensel and Webb, promoting the hyperbolicity of the curve complex to that of the fine curve graph, opens the door both to the study of what classical properties of usual curve complexes have counterparts in the fine curve graph, and to the use of this graph to derive properties of homeomorphism groups. A first step in this direction was taken by Bowden, Hensel, Mann, Militon and Webb [1], who explored the metric properties of the action of $\mathrm{Homeo}(\Sigma)$ on this hyperbolic graph.

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A classical theorem by Ivanov (see [7], as well as Korkmaz [8] and Luo [12] for other surfaces) states that, when Σ is a closed surface of genus $g \geqslant 2$, every automorphism of $\mathcal{C}(\Sigma)$ is realized by a homeomorphism. Recently, Long, Margalit, Pham, Verberne and Yao [11] proved the following natural counterpart of Ivanov's theorem for fine graphs: provided Σ is a compact orientable surface of genus $g \geqslant 2$, the natural map

$$\mathrm{Homeo}(\Sigma) \to \mathrm{Aut}(\mathcal{C}^\dagger(\Sigma))$$

is an isomorphism. They also suggested that this map (with the appropriate version of $\mathcal{C}^\dagger$) may also be an isomorphism when $g = 1$, and conjectured that the automorphism group of the fine curve graph of smooth curves should be nothing more than $\mathrm{Diff}(\Sigma)$.

In this article, we address both these questions. Our motivation originates from the case of the torus: excited by [1], we wanted to understand more closely the relation between the rotation set of homeomorphisms isotopic to the identity and the metric properties of their actions on the fine graph. This subject will be treated in another article, joint with Passeggi and Sambarino [10]. The methods developed in the present article are valid not only for the torus but for a large class of surfaces.

We work on nonspherical surfaces (ie surfaces not embeddable in the 2–sphere, or equivalently, containing at least one nonseparating simple closed curve), orientable or not, compact or not. We consider the graph $\mathcal{NC}^\dagger_{\pitchfork}(\Sigma)$, whose vertices are the nonseparating simple closed curves, and with an edge between two vertices a and b whenever they are either disjoint, or have exactly one topologically transverse intersection point (see the beginning of Section 2 for more detail). Our first result answers a problem raised in [11].

Theorem 1 *Let Σ be a connected, nonspherical surface, without boundary. Then the natural map $\mathrm{Homeo}(\Sigma) \to \mathrm{Aut}(\mathcal{NC}^\dagger_{\pitchfork}(\Sigma))$ is an isomorphism.*

Our second result concerns the smooth version of fine graphs. We consider the graph $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$ whose vertices are the smooth nonseparating curves in Σ, with an edge between a and b if they are disjoint or have one transverse intersection point, in the differentiable sense (in particular, $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$ is not the subgraph of $\mathcal{NC}^\dagger_{\pitchfork}(\Sigma)$ induced by the vertices corresponding to smooth curves; it has fewer edges). The following result partially confirms the conjecture of [11]; here we restrict to the case of orientable surfaces for simplicity.

Theorem 2 *Let Σ be a connected, orientable, nonspherical surface, without boundary. Then all the automorphisms of $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$ are realized by homeomorphisms of Σ.*

In other words, if we denote by $\mathrm{Homeo}_{\infty\pitchfork}(\Sigma)$ the subgroup of $\mathrm{Homeo}(\Sigma)$ preserving the collection of smooth curves and preserving transversality, then the natural map

$$\mathrm{Homeo}_{\infty\pitchfork}(\Sigma) \to \mathrm{Aut}(\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma))$$

is an isomorphism. We were surprised to realize however that $\mathrm{Homeo}_{\infty\pitchfork}(\Sigma)$ is strictly larger than $\mathrm{Diff}(\Sigma)$.

Figure 1: A necklace (left) and a bouquet (right) of three circles.

Proposition 3 *Every surface Σ admits a homeomorphism f such that f and f^{-1} preserve the set of smooth curves, and preserve transversality, but such that neither f nor f^{-1} is differentiable. In particular, the natural map*

$$\mathrm{Diff}(\Sigma) \to \mathrm{Aut}(\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma))$$

is not surjective.

1.2 Idea of the proof of Theorem 1

The main step in this proof is the following.

Proposition 4 *If $\{a,b\}$ or $\{a,b,c\}$ is a 2–clique or a 3–clique of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ then, from the graph structure of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, we can tell the type of the clique.*

If $\{a_1,\dots,a_n\}$ and $\{b_1,\dots,b_n\}$ are two n–cliques in the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, we say they have the same *type* if the unions $\bigcup_{j=1}^n a_j$ and $\bigcup_{j=1}^n b_j$ are homeomorphic. Note that we do not require that such a homeomorphism extends to a homeomorphism of the surface (as we do not consider how the cliques are embedded, Figure 1 does not show the transversality at the intersection points). Let us describe the different types of 2– and 3–cliques. A 2–clique $\{a,b\}$, ie an edge of the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, may be of two distinct types: the intersection $a\cap b$ may be empty or not. For a 3–clique $\{a,b,c\}$, up to permuting the curves a, b and c, the cardinals of the intersections $a\cap b$, $a\cap c$ and $b\cap c$, respectively, may be $(1,1,1)$, $(1,1,0)$, $(1,0,0)$, or $(0,0,0)$. This determines the type of the 3–clique, except in the case $(1,1,1)$, where the intersection points $a\cap b$, $a\cap c$ and $b\cap c$ may be pairwise distinct, in which case we will speak of a 3–clique of type *necklace*, or these intersection points may be equal, in which case we will speak of a 3–clique of type *bouquet*; see Figure 1.

The main bulk of the proof of Proposition 4 consists in distinguishing the 3–cliques of type necklace from any other 3–clique of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$. Here, the key is that among all the 3–cliques, the cliques $\{a,b,c\}$ of type necklace are exactly those such that the union $a\cup b\cup c$ contains nonseparating simple closed curves other than a, b and c. In terms of the graph structure, this leads to the following property, denoted by $N(a,b,c)$, which turns out to characterize these cliques:

There exists a finite set F of at most 8 vertices of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, all distinct from a, b and c, such that every vertex d connected to a, b and c in this graph, is connected to at least one element of F.

From this, we will easily characterize all the configurations of 2–cliques and 3–cliques in terms of similar statements in the first order logic of the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$.

Now, let $T(\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma))$ denote the set of edges $\{a,b\}$ of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ satisfying $|a\cap b|=1$. Then we have a map

$$\mathrm{Point}\colon T(\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma))\to\Sigma,$$

which to each edge $\{a,b\}$ of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, associates the intersection point $a\cap b$. The next step in the proof of Theorem 1 now consists in characterizing the equality $\mathrm{Point}(a,b)=\mathrm{Point}(c,d)$ in terms of the structure of the graph. This characterization shows that every automorphism of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ is realized by some bijection of Σ; then we prove that such a bijection is necessarily a homeomorphism (see Proposition 20).

In order to characterize the equality $\mathrm{Point}(a,b)=\mathrm{Point}(c,d)$, we introduce on $T(\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma))$ a relation $\diamondsuit$. This relation is generated, essentially, by $(a,b)\diamondsuit(b,c)$ if (a,b,c) is a 3–clique of type bouquet (see Section 3.2 for details). If $(a,b)\diamondsuit(c,d)$ then $\mathrm{Point}(a,b)=\mathrm{Point}(c,d)$. Interestingly, the converse is false, but we can still use this idea in order to characterize the points of Σ in terms of the graph structure of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$.

This subtlety between the relation $\diamondsuit$ and the equality of points is related to the nonsmoothness of the curves involved, and more precisely, to the fact that a curve may spiral infinitely with respect to another curve in a neighborhood of a common point. We think that this phenomenon is of independent interest and we investigate it in Section 4. In particular, we can easily state, in terms of the graph structure of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, an obstruction for a homeomorphism to be conjugate to a C^1–diffeomorphism; see Section 4.6.

1.3 Ideas of the proof of Theorem 2

In the smooth case, the adaptation of our proof of Theorem 1 fails from the start: indeed, the closed curves contained in the union $a\cup b\cup c$ of a necklace, and distinct from a, b and c, are not smooth. This suggests the idea to use sequences of curves (at the expense of losing the characterizations of configurations in terms of first order logic).

This time it is easiest to first characterize disjointness of curves (see Lemma 43), and then recover the different types of 3–cliques. Then the strategy follows the C^0 case.

Once we start to work with sequences, it is natural to say that a sequence (f_n) of curves not escaping to infinity converges to a in some weak sense, if for every vertex d such that $\{a,d\}$ is an edge of the graph, $\{f_n,d\}$ is also an edge for all n large enough. As it turns out, this property implies convergence in the C^0–sense to a, and is implied by convergence in the C^1–sense. But it is not equivalent to the convergence in the C^1–sense, and it is precisely this default of C^1–convergence that enables us to distinguish between disjoint or transverse pairs of curves.

Interestingly, this simple criterion for disjointness has no counterpart in the C^0–setting. Indeed, in that setting, no (noneventually constant) sequence of curves converges in this weak sense: given a curve a, and a sequence (f_n) of curves with, say, some accumulation point x in a, we can build a curve d intersecting a once transversally (topologically) at x, but oscillating so much that it intersects several times each f_n that does not contain x. From this perspective, none of our approaches in the C^0–setting and in the C^∞–setting are directly adaptable to the other.

1.4 Further comments

We can imagine many variants of fine graphs. For example, in the arXiv version of [2], for the case of the torus they worked with the graph $\mathcal{NC}^\dagger_{\pitchfork}(\Sigma)$ on which we are working here, whereas in the published version, they changed to a fine graph in which two curves a and b are still related by an edge when they have one intersection, not necessarily transverse.

More generally, in the spirit of Ivanov's metaconjecture, we expect that the group of automorphisms should not change from any reasonable variant to another. And indeed, using the ideas of [11, Section 2] and those presented here, we can navigate between various versions of fine graphs, and recover, from elementary properties of one version, the configurations defining the edges in another version, thus proving that their automorphism groups are naturally isomorphic. From this perspective, it seems satisfying to recover the group of homeomorphisms of the surface as the automorphism group of any reasonable variant of the fine graph. In this vein, we should mention that the results of [11, Section 2] directly yield a natural map $\mathrm{Aut}(\mathcal{C}^\dagger(\Sigma)) \to \mathrm{Aut}(\mathcal{NC}^\dagger_{\pitchfork}(\Sigma))$, and from there, our proof of Theorem 1 may be used as an alternative proof of their main result.

All reasonable variants of the fine graphs should be quasi-isometric, and a unifying theorem (yet out of reach today, as it seems to us) would certainly be a counterpart of the theorem by Rafi and Schleimer [15], which states that every quasi-isometry of the usual curve graph is bounded distance from an isometry.

1.5 Organization of the article

Section 2 is devoted to the recognition of the 3–cliques in the C^0–setting, and of some other configurations regarding the nonorientable case. We encourage the reader to skip, at first reading, everything that concerns the nonorientable case; these points should be easily identified, and this halves the length of the proof. In Section 3 we prove Theorem 1. In Section 4 we characterize, from the topological viewpoint, the relation $\diamondsuit$ introduced above in terms of the graph structure, and deduce our obstruction to differentiability. Finally in Section 5 we prove Theorem 2 and Proposition 3.

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2 Recognizing configurations of curves

2.1 Standard facts and notation

We will use, often without mention, the following easy or standard facts for curves on surfaces.

The first is the classification of connected, topological surfaces with boundary (not necessarily compact). In particular, every topological surface admits a smooth structure. Given a closed curve a in a surface Σ, we can apply this classification to $\Sigma \smallsetminus a$ and understand all possible configurations of simple curves; this is the so-called *change of coordinates principle* in the vocabulary of the book of Farb and Margalit [3].

In particular, every closed curve a has a neighborhood homeomorphic to an annulus or a Möbius strip in which a is the "central curve". Very often in this article, we will consider the curves a' obtained by deforming a in such a neighborhood, so that a' is disjoint from a in the first case, or intersects it once, transversely, in the second, as in Figure 2. We will say that a' is obtained by *pushing a aside*.

If two curves a and b have a unique intersection point, we say that the intersection is *transverse*, or *essential*, if there is a homeomorphism of the surface which maps a and b to smooth curves intersecting transversely in the usual smooth sense. Otherwise we say the intersection is *inessential*.

The change of coordinates principle also applies to finite graphs embedded in Σ: there is a homeomorphism of Σ that sends any given graph to a smooth graph, such that all edges connected to a given vertex leave it in distinct directions. In the simple case when the graph is the union of two or three simple closed curves that pairwise intersect at most once, this observation justifies the description of the possible configurations of cliques in the introduction.

Here are two other useful facts.

Fact 5 *A simple closed curve a in a surface is nonseparating if and only if there exists a closed curve b such that $a \cap b$ is a single point and this intersection is transverse.*

Fact 6 *Let p and q be two distinct points, and x, x' and x'' three simple arcs, each with endpoints p and q, such that*

$$x \cap x' = x' \cap x'' = x \cap x'' = \{p, q\}.$$

If two of the three curves $x \cup x'$, $x' \cup x''$, $x'' \cup x$ are separating, then the third one is also separating.

Proof Denote $y = x \smallsetminus \{p, q\}$, the arc x without its ends, and similarly, define y' and y''. Suppose $x \cup x'$ and $x \cup x''$ are separating. Denote by Σ_1 and Σ_2 (resp. Σ_3 and Σ_4) the components of $\Sigma \smallsetminus (x \cup x')$ (resp. $\Sigma \smallsetminus (x \cup x'')$) where Σ_2 contains y'' and Σ_4 contains y'. By looking at neighborhoods of p and q

Figure 2: Left: pushing a two-sided curve a. Right: pushing a one-sided curve a.

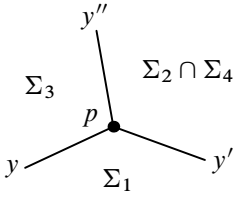

Figure 3: A neighborhood of p.

(see Figure 3), we see that $\Sigma' = \Sigma_2 \cap \Sigma_4$ is nonempty, and that the arc y bounds Σ_1 on one side, and Σ_3 on the other, so $\Sigma'' = \Sigma_1 \cup y \cup \Sigma_3$ is a surface. Now, $\Sigma \smallsetminus (x' \cup x'') = \Sigma' \cup \Sigma''$, and Σ' and Σ'' are disjoint by construction. □

2.2 Properties characterizing geometric configurations

Now we list the properties, in terms of the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, that will be used as characterizations of certain configurations of curves. This allows us to specify the statement of Proposition 4, which will be proved in the next paragraph, and define the relation $\diamondsuit$ in terms of the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$.

In the following, the letters N D, T and B respectively stand for necklace, disjoint, transverse and bouquet. If a, b and c are vertices of this graph, we will denote by

- $N(a,b,c)$ the property that $\{a,b,c\}$ is a 3–clique of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ and there exists a finite set F of at most 8 vertices of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, all distinct from a, b and c, such that for every vertex d such that $\{a,b,c,d\}$ is a 4–clique, there is an edge from d to at least one element of F;
- $D(a,b)$ the property that $\{a,b\}$ is an edge of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ and there does not exist a vertex d such that $N(a,b,d)$ holds,
- $T(a,b)$ the property that $\{a,b\}$ is an edge of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ and $D(a,b)$ does not hold;
- $B(a,b,c)$ the property that $T(a,b)$, $T(a,c)$ and $T(b,c)$ all hold but $N(a,b,c)$ does not.

The following proposition is the main part of Proposition 4.

Proposition 7 *Let a, b and c be vertices of the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$. Property $N(a,b,c)$ holds if and only if $\{a,b,c\}$ is a 3–clique of type necklace of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$.*

The proof of Proposition 7 will occupy Section 2.3 below. The following corollary complements Proposition 7 and provides a precise version of Proposition 4.

Corollary 8

- *Property $D(a,b)$ holds if and only if the curves a and b are disjoint.*
- *Property $T(a,b)$ holds if and only if a and b have a unique intersection point and the intersection is transverse.*
- *Property $B(a,b,c)$ holds if and only if $\{a,b,c\}$ is a 3–clique of type bouquet.*

Figure 4: Completing (a,b) to a 3–clique of type necklace.

Proof Let a and b be neighbors in the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$. Of course, if a and b are disjoint, then $D(a,b)$ holds: there does not exist a curve d such that $N(a,b,d)$ holds, since this would mean that $\{a,b,d\}$ is of type necklace and then, by definition, a and b would intersect. Conversely, suppose that a and b are not disjoint, and let us prove that $D(a,b)$ does not hold, ie let us find a curve c such that $\{a,b,c\}$ is a 3–clique of type necklace. In the case when one of a or b is one-sided, up to exchanging the two, suppose a is one-sided. Then we may push a in order to find a curve c which makes a 3–clique of type necklace with a and b; see Figure 4, left. In the case when both a and b are two-sided, then by the change of coordinates principle, a regular neighborhood of $a \cup b$ is homeomorphic to a one-holed torus, embedded in Σ, with a choice of meridian and longitude coming from a and b. In this torus, a curve c with slope 1 will form a 3–clique of type necklace with a and b; see Figure 4, right. This proves the first point.

The second point is a straightforward consequence of the first, and the third simply follows from the second point together with Proposition 7. □

2.3 Proof of Proposition 7

The following lemma is a key step in the proof of the direct implication in Proposition 7.

Lemma 9 *Let $\{a,b,c\}$ be a 3–clique of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ which is not of type necklace. Then there exists a vertex d of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ such that $\{a,b,c,d\}$ is a 4–clique of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, and such that d meets every connected component of $\Sigma \smallsetminus (a \cup b \cup c)$.*

Before entering the proof, we note that we cannot remove the hypothesis that $\{a,b,c\}$ is not of type necklace. Indeed, in the flat torus $\Sigma = \mathbb{R}^2/\mathbb{Z}^2$, consider three closed geodesics a, b and c respectively directed by $(1,0)$, $(0,1)$ and $(1,1)$. By pushing c aside if necessary, we obtain a 3–clique of type necklace. The complement of $a \cup b \cup c$ in Σ has three connected components, and there is no curve d satisfying the conclusion of the lemma.

Proof Throughout the proof, we will denote $\Sigma' = \Sigma \smallsetminus (a \cup b \cup c)$. Up to permuting the curves a, b and c, we may suppose that the triple of cardinals of intersections, $(|a \cap b|, |a \cap c|, |b \cap c|)$, equals $(1,1,1)$, $(1,1,0)$, $(1,0,0)$, or $(0,0,0)$. We will deal with these cases separately.

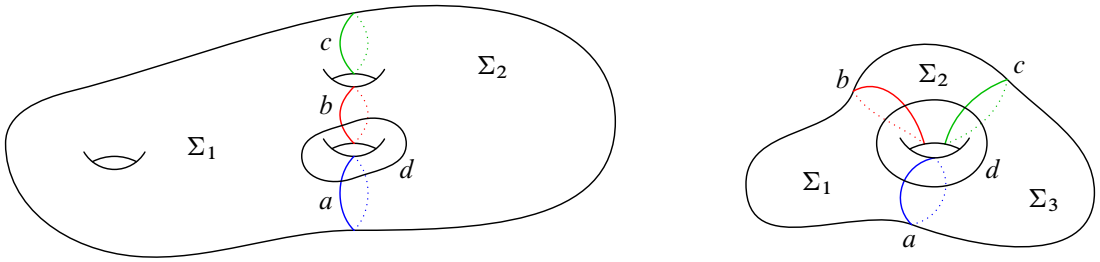

Figure 5: Finding d in the case $(0,0,0)$.

Let us begin with the case $(0,0,0)$. If Σ' is connected, then any curve d making a 4–clique with (a,b,c) satisfies the lemma. Such a curve can be found, for example, by pushing a aside. If Σ' has two connected components, denote them by Σ_1 and Σ_2. Since a, b and c are each nonseparating, at least two of the curves a, b, c (say, a and b) correspond to boundary components of both Σ_1 and Σ_2. Choose one point x_a in a and one point x_b in b. For $i=1,2$, there is an arc γ_i connecting x_a to x_b in Σ_i, and disjoint from the boundary of Σ_i except at its ends. Then the curve $d=\gamma_1\cup\gamma_2$ satisfies the lemma (see Figure 5, left). It may also happen that Σ' has three connected components, in which case we find a curve d exactly in the same way; see Figure 5, right.

Next we deal with the case of intersections $(1,0,0)$. In this case, a and b intersect transversely, once, and c is disjoint from $a\cup b$. By hypothesis, the curve c is (globally) nonseparating. Consider the union $a\cup b$. If a or b is two-sided, then $a\cup b$ does not disconnect its regular neighborhoods. This is seen by traveling along a small band on one side of $a\cup b$ (see Figure 6, left). In this case, Σ' cannot have more connected components than $\Sigma\smallsetminus c$; hence Σ' is connected, and any curve d obtained by pushing c, as in the preceding case, satisfies the lemma. If both a and b are one-sided, then $a\cup b$ is locally disconnecting, so Σ' may have up to two connected components. In this case, a curve d obtained by pushing a satisfies the lemma (see Figure 6, right).

Now assume we are in the case $(1,1,1)$ or $(1,1,0)$. Since $\{a,b,c\}$ is not of type necklace, in these cases b and c do not meet outside a. We first treat the subcase when a is two-sided. For this we consider any curve d obtained by pushing a aside, and we claim that d meets every connected component of Σ'. Indeed, let C be such a component. Of course the closure of C meets a, b or c. Since both b and c meet a, it actually has to meet a, as we can see by traveling along b or c in C. More precisely, by following b or c in both directions, we see that C meets any neighborhood of a from both sides. Thus it meets d.

Figure 6: Finding d in case $(1,0,0)$.

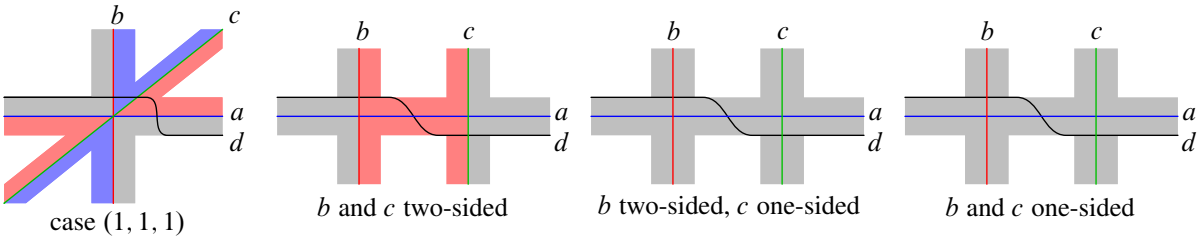

Figure 7: Finding d in cases $(1,1,1)$ and $(1,1,0)$.

It remains to treat the subcase when a is one-sided, first for the $(1,1,1)$ case, and then for the $(1,1,0)$ case. In the $(1,1,1)$ case, the curves a, b and c play symmetric roles, and by the above argument it just remains to consider the case when they are all one-sided. Then the situation is depicted on Figure 7, left: $a \cup b \cup c$ disconnects its regular neighborhoods into three connected components, and the figure shows a curve d, obtained by pushing a aside, which intersects all three components and such that $\{a,b,c,d\}$ is a 4–clique. In the remaining case the curves b and c play symmetric roles, and there are three different cases to consider, regarding whether b and c are one or two-sided. These three cases are pictured in Figure 7, and in each case, we obtain d by pushing a aside. □

We deduce the following.

Lemma 10 *Let $\{a,b,c\}$ be a 3–clique of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, not of type necklace. Let $(\alpha_1,\dots,\alpha_n)$ be a finite family of vertices of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, all distinct from a, b and c. Then there exists a vertex d of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, such that*

- *$\{a,b,c,d\}$ is a 4–clique of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$;*
- *for all $j \in \{1,\dots,n\}$, the intersection $d \cap \alpha_j$ is infinite; in particular, $\{d,\alpha_j\}$ is not an edge of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$.*

As a corollary, we get the direct implication in Proposition 7.

Corollary 11 *If $\{a,b,c\}$ is a 3–clique not of type necklace, then $N(a,b,c)$ does not hold.*

Proof of Lemma 10 The hypotheses that $\{a,b,c\}$ is not of type necklace and $\alpha_j \notin \{a,b,c\}$, impose that for every j, the curve α_j is not contained in the union $a \cup b \cup c$. Hence, there exists a small subarc $\beta_j \subset \alpha_j$ lying in the complement of $a \cup b \cup c$, and we may further suppose that these n arcs are pairwise disjoint, and choose a point x_j in β_j for each j.

Now, let d_0 be a vertex of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ as from Lemma 9. Since d_0 meets every component of

$$\Sigma' = \Sigma \smallsetminus (a \cup b \cup c),$$

we may perform a surgery on d_0, far from $a \cup b \cup c$, to obtain a new curve d_1 such that $\{a,b,c,d_1\}$ is still a 4–clique, and d_1 still meets every component of Σ', and d_1 passes through x_1. We may iterate this

process, to get a curve d_n which passes through x_j for every j, and such that $\{a, b, c, d_n\}$ is a 4–clique. Finally, we may perform a last surgery on d_n, in the neighborhood of all x_j, in order to obtain a curve d such that for each j, $|\beta_j \cap d|$ is infinite. □

It remains to prove the converse implication in Proposition 7, which we restate as Lemma 12.

Lemma 12 *Let $\{a, b, c\}$ be a 3–clique of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ of type necklace. Then there exists a finite set F, of at most 8 vertices of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ all distinct from a, b and c, and such that every d such that $\{a, b, c, d\}$ is a 4–clique of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ is connected by an edge to some element of F.*

In fact this set F can be chosen explicitly, with cardinal at most 8, as follows. If $\{a, b, c\}$ is a 3–clique of type necklace, then there exists a family of arcs (x, X, y, Y, z, Z), all embedded in Σ, such that $a = x \cup X$, $b = y \cup Y$, $c = z \cup Z$, and such that these six arcs pairwise intersect at most at their ends. The union $a \cup b \cup c$ may be viewed as a graph embedded in Σ, and these six arcs are the edges of this embedded graph. We let F be the set of nonseparating curves, among the 8 curves $X \cup Y \cup Z$, $X \cup Y \cup z$, $X \cup y \cup Z$, etc (there is one choice of upper/lower case for each letter). In the course of the proof of Lemma 12, we will see that F is nonempty, and satisfies the lemma.

Proof Let $\{a, b, c\}$ be a 3–clique of type necklace. Let F be the set of nonseparating curves, as above, among all the 8 curves $x \cup y \cup z$, $X \cup y \cup z$, $X \cup Y \cup z$, etc. Let d be such that $\{a, b, c, d\}$ is a 4–clique. Up to permuting the curves a, b and c, we may suppose that $(|a \cap d|, |b \cap d|, |c \cap d|)$ equals $(1, 1, 1)$, $(1, 1, 0)$, $(1, 0, 0)$ or $(0, 0, 0)$; our proof proceeds case by case.

The easiest case is $(1, 0, 0)$. In this case, up to exchanging the arcs X and x, we may suppose that d intersects a at an interior point of X, and is disjoint from all the other arcs. Consider $f = X \cup y \cup z$. This curve intersects d at a unique point, transversely. It follows that f is nonseparating. Hence $f \in F$ and f satisfies the conclusion of the lemma.

Now let us deal simultaneously with the cases $(1, 1, 1)$ and $(1, 1, 0)$. Suppose first that the intersections of d with $a \cup b \cup c$ do not occur at the intersection points $a \cap b$, $a \cap c$ or $b \cap c$. Up to exchanging x with X, y with Y and z with Z, we may suppose that the intersections occur in the interior of the arcs X, Y and Z in the case $(1, 1, 1)$, and in the interior of the arcs X and Y in the case $(1, 1, 0)$. Now the curve $f = X \cup y \cup z$, for instance, satisfies the conclusion of the lemma.

Now suppose that d contains one of the points $a \cap b$, $a \cap c$ or $b \cap c$. In case $(1, 1, 1)$ we may suppose, up to permuting a, b and c, that d contains the point $a \cap b$, and in the case $(1, 1, 0)$, this is automatic, as d is disjoint from c. Now in any case, d cannot contain $a \cap c$ nor $b \cap c$, because it intersects a and b only once. Hence, up to exchanging z with Z, we may suppose $d \cap z = \varnothing$. In the neighborhood of the point $a \cap b$, up to homeomorphism, the configuration of our curves is as depicted in Figure 8, because all the intersections are supposed to be transverse. Then, up to exchanging X with x or Y with y, we can suppose that the arc $X \cup Y$ has a transverse intersection with d, and then the arc $f = X \cup Y \cup z$ satisfies the conclusion of the lemma.

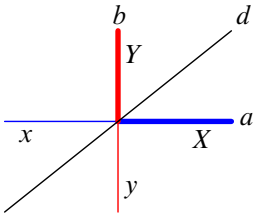

Figure 8: A suitable choice of X and Y.

We are left with the case $(0,0,0)$. In this case, any curve in F will satisfy the conclusion of the lemma, and hence all we have to do is to prove that F is nonempty. If $X \cup Y \cup Z$ and $X \cup Y \cup z$ were both separating, then so would be $c = z \cup Z$, by Fact 6. Hence, among these two curves, at least one is nonseparating, and F is nonempty (in fact, it contains at least 4 elements). □

2.4 One- or two-sided curves, and extra bouquets

In this last paragraph of this section, we will see how to recognize, from the graph structure of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, some additional configurations. We insist that the work in this paragraph is useful only in the case when Σ is nonorientable; it is needed in order to make our proof of Theorem 1 work in that case (see Remark 22 below).

We start with a simple characterization of one-sided and two-sided curves.

- Two(a), the property that for all b such that $T(a,b)$ holds, there exists c such that $T(b,c)$ and $D(a,c)$ both hold,
- One(a), the negation of Two(a): there exists b such that $T(a,b)$ and such that there does not exist c satisfying $T(b,c)$ and $D(a,c)$.

Observation 13 *Let a be a vertex of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$. Then the curve a is one-sided, if and only if* One(a) *holds.*

Proof If a is two-sided, and b satisfies $T(a,b)$, by pushing a aside we find another curve c as in the definition of Two(a). This proves the reverse implication.

If a is one-sided, let b be a curve obtained by pushing a aside. We have $T(a,b)$, and a and b bound a disk. Any curve c disjoint from a, and with $T(b,c)$, has to enter this disk; but then, it has to get out, which is impossible without touching a and without intersecting b another time. □

Our next objective is to characterize when two one-sided curves a and b meet exactly once, nontransversely. We will do this in several steps.

Lemma 14 *Let a and b be one-sided simple curves of Σ. Suppose the intersection $a \cap b$ is not connected. Then there exists a vertex c of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, distinct from a and b, such that for every neighbor d of both a and b in this graph, and such that $D(a,d)$ or $D(b,d)$ (or both), the vertices c and d are neighbors in this fine graph.*

Proof Let x be a subarc of b, whose endpoints lie in a, and disjoint from a otherwise. Since $a \cap b$ is disconnected, such an arc exists, and has two distinct end points, p and q. Let x' and x'' be the two subarcs of a whose ends are p and q. From Fact 6, we know that $x \cup x'$ or $x \cup x''$ (or both) is a nonseparating curve; denote it by c. By construction, we have $c \neq a$ and $c \neq b$.

Now let d be a curve satisfying the hypothesis of the lemma. If d is disjoint from a and b, then it is disjoint from c; otherwise d intersects exactly one of a or b, far away from the other. So the intersection between d and c, if any, is still transverse, and d is a neighbor of c in $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$. □

This contrasts with the situation we want to characterize, as we see now.

Lemma 15 *Let a and b be two one-sided curves, and suppose that $a \cap b$ consists in one, inessential intersection point. Then, for every nonseparating curve $c \notin \{a, b\}$, there exists d such that $T(a, d)$ and $D(b, d)$ hold but such that the intersection $c \cap d$ is infinite.*

Proof We first observe that $\Sigma' = \Sigma \smallsetminus (a \cup b)$ is connected. This is seen by following the curves a and b in both directions: the union $a \cup b$ does not disconnect its small neighborhoods. Let c be a curve as above. Then we may consider a first curve d_0, obtained by pushing a aside, in such a way that d_0 is disjoint from b (this is possible since the intersection $a \cap b$ is inessential). Since $c \notin \{a, b\}$, the curve c intersects Σ'. Since d_0 meets every component of Σ' (there is only one), we may deform it into a curve d which intersects c infinitely many times, exactly as in the proof of Lemma 10. □

After these two lemmas, we have a simple sentence in terms of the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, which holds when $a \cap b$ is a single inessential intersection point, and which guarantees that $a \cap b$ is connected. In order to upgrade this into a characterization of the first situation, we need to be able to exclude as well the cases when $a \cap b$ is a nondegenerate arc. These cases fall into two subcases: the intersection arc $a \cap b$ can be essential or inessential, exactly as an intersection point. One way to formalize this, is to say that the intersection $a \cap b$ is essential if a cuts a regular neighborhood of $a \cap b$ into two regions both containing a subarc of b, and inessential otherwise.

Lemma 16 *Let a and b be one-sided curves. Suppose that $a \cap b$ is a nondegenerate arc, and suppose this intersection is essential. Then there exist curves $\alpha, \alpha', \beta, \beta'$ obtained by pushing a aside, such that $B(a, \alpha, \beta)$, $B(b, \alpha, \beta)$, $B(a, \alpha', \beta')$, $B(b, \alpha', \beta')$, and $N(a, \alpha, \beta')$.*

Proof The curves α, β, α' and β' may be taken in a neighborhood of $a \cup b$, as pictured in Figure 9. □

Finally, we deal with inessential arcs.

Lemma 17 *Let a and b be one-sided curves, such that $a \cap b$ is an inessential arc or intersection point. Let c be such that $T(a, c)$. Then there exists d such that $B(a, c, d)$ and $D(b, d)$, if and only if the intersection point $a \cap c$ does not belong to b.*

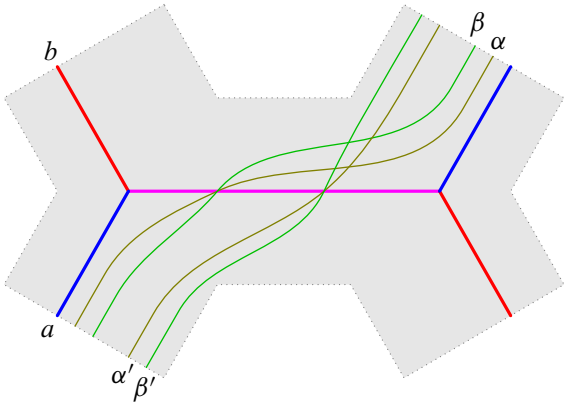

Figure 9: The curves α, β, α' and β'.

Proof If $a \cap c$ belongs to b, then any curve d satisfying $B(a,c,d)$ will meet b. Otherwise, we may push a aside, in order to obtain a curve d, as in Figure 10, left. □

Putting all together, this yields the following characterization of inessential intersection points between one-sided curves.

Corollary 18 *Let a and b be one-sided curves. Then $a \cap b$ consists of one inessential intersection point if and only if the following conditions are satisfied*:

(1) *a and b are not neighbors in $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$;*

(2) *for every $c \notin \{a,b\}$, there exists d such that d is a neighbor of a and b in $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, and $D(a,d)$ or $D(b,d)$ (or both), but d is not a neighbor of c in that graph*;

(3) *there do not exist α, β, α' and β' such that $B(a,\alpha,\beta)$, $B(b,\alpha,\beta)$, $B(a,\alpha',\beta')$, $B(b,\alpha',\beta')$ and $N(a,\alpha,\beta')$ all hold*;

(4) *there do not exist c_1 and c_2 with $D(c_1,c_2)$ and with the property that, for $i=1,2$, we have $T(a,c_i)$ and for all d, $B(a,c_i,d)$ and $D(b,d)$ do not both hold.*

This enumeration of conditions expressed only in terms of the graph structure of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, with the addition of the conditions $\mathrm{One}(a)$ and $\mathrm{One}(b)$, will be also denoted by $I(a,b)$, for *inessential intersection* (of one-sided curves).

Proof First, let us check that if a and b have one inessential intersection point then $I(a,b)$ holds. Condition (1) holds by definition, and (2) follows from Lemma 15. The negation of condition (3) would imply that the cardinality of $a \cap b$ is at least 2. Indeed, $B(a,\alpha,\beta)$ implies that $\alpha \cap \beta$ is a point lying in a. Thus, the bouquet conditions imply that both $\alpha \cap \beta$ and $\alpha' \cap \beta'$ lie in $a \cap b$. And the condition $N(a,\alpha,\beta')$ then implies that $a \cap \alpha$ and $a \cap \beta'$ are disjoint; hence the two points $\alpha \cap \beta$ and $\alpha' \cap \beta'$ are distinct. Finally, condition (4) follows from Lemma 17. Indeed, this lemma implies that the two curves c_1 and c_2 should both contain a point of $a \cap b$; hence they cannot be disjoint.

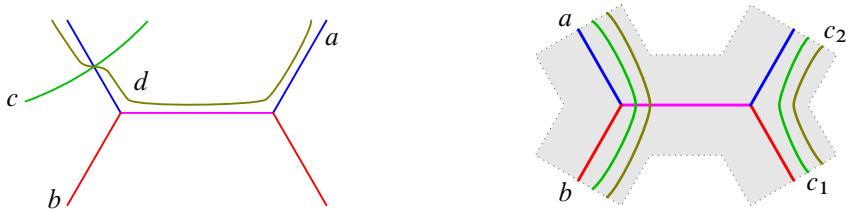

Figure 10: Some configurations of curves for properties I and xB.

Now, let a and b be any two nonseparating curves and suppose that $I(a,b)$ holds. By conditions (1) and (2), the intersection $a \cap b$ is nonempty, and connected. Along the lines of the proof of Lemma 16, we can see that condition (3) implies that $a \neq b$, so $a \cap b$ is an inessential intersection point, or an arc. Suppose for contradiction that it is a nondegenerate arc. By Lemma 16 and condition (3), this intersection arc cannot be essential. Now Figure 10, right, shows the desired contradiction with condition (4). □

Finally, we deal with extra bouquets. We denote by $xB(a,b,c)$ the property that $T(a,b)$, $T(a,c)$ and $I(b,c)$ all hold and, moreover, for all d such that $B(a,b,d)$ holds, $D(c,d)$ does not.

Lemma 19 *Let a, b and c be such that $T(a,b)$, $T(a,c)$ and $I(b,c)$, with b and c one-sided. Then $xB(a,b,c)$ holds if and only if the intersection points $a \cap b$, $a \cap c$ and $b \cap c$ coincide.*

Proof Of course if these points coincide, then property $xB(a,b,c)$ holds; every curve d such that $B(a,b,d)$ holds, must contain this point and hence cannot be disjoint from c.

Now suppose that these points do not coincide, hence, are three pairwise distinct points. Then, we may push b aside, in order to find a curve d which does not intersect c any more, as the intersection $b \cap c$ is not essential. This curve d, obtained by pushing b, can be made to satisfy $T(b,d)$, while crossing b precisely at the point $a \cap b$, and this intersection can be made transverse; the illustration of this situation is similar to Figure 10, left, and this time we leave it to the reader. This yields a curve d such that $B(a,b,d)$ holds and d disjoint from c. □

3 Proof of Theorem 1

Here as above, Σ is a connected surface admitting a nonseparating closed curve.

3.1 From bijections to homeomorphisms

In order to prove Theorem 1, it suffices to prove that every automorphism of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ is supported by a bijection of the surface, in virtue of the following observation.

Proposition 20 *Let $f : \Sigma \to \Sigma$ be a bijection. We suppose that for every nonseparating simple closed curve $\alpha \subset \Sigma$, the sets $f(\alpha)$ and $f^{-1}(\alpha)$ are also nonseparating simple closed curves in Σ. Then f is a homeomorphism.*

Proof If our hypothesis was that f and f^{-1} are closed (ie send closed sets to closed sets), then f would be a homeomorphism. So our strategy is to use our hypothesis here in a similar fashion. We need only prove that f is continuous, the argument for f^{-1} is symmetric.

Let $x \in \Sigma$ and suppose that f is not continuous at x. Then there exists a sequence $(x_n)_{n\geqslant 0}$ of distinct points converging to x, and a neighborhood V of $f(x)$, such that for all n we have $f(x_n) \notin V$. Notice that in the open unit disk of the plane, up to homeomorphism, there is only one sequence of distinct points converging to the origin. With this in mind, we may construct an embedded arc, in Σ, with one end at x, and which contains all the points x_n. Then we may construct a nonseparating simple closed curve α containing this arc. By hypothesis, $f(\alpha)$ is a nonseparating closed curve in Σ, which contains $f(x)$. We may perform a surgery of $f(\alpha)$ inside V, to obtain a nonseparating simple closed curve β, which coincides with $f(\alpha)$ outside V but which does not contain $f(x)$. Now $f^{-1}(\beta)$ is, by hypothesis, a closed subset of Σ, which contains all the points x_n but not x. This is a contradiction. □

3.2 The adjacency relation $\diamondsuit$

Let $E_T(\mathcal{NC}^\dagger_{\pitchfork}(\Sigma))$ denote the set of edges $\{a,b\}$ of $\mathcal{NC}^\dagger_{\pitchfork}(\Sigma)$ satisfying $T(a,b)$. Then we have a map

$$\text{Point}\colon E_T(\mathcal{NC}^\dagger_{\pitchfork}(\Sigma)) \to \Sigma,$$

which to each edge $\{a,b\}$ associates the intersection point $a \cap b$. The main part of proof of Theorem 1 consists in showing that we can express the equality

$$\text{Point}(a,b) = \text{Point}(\alpha,\beta)$$

in terms of the graph. For this we introduce the equivalence relation $\diamondsuit$ on $E_T(\mathcal{NC}^\dagger_{\pitchfork}(\Sigma))$ as follows. If $\{a,b,c\}$ is a 3–clique of $\mathcal{NC}^\dagger_{\pitchfork}(\Sigma)$ of type bouquet we set $\{a,b\} \diamondsuit \{a,c\}$. We also set $\{a,b\} \diamondsuit \{a,c\}$ if a, b and c satisfy the "extra bouquet" condition, denoted above by $xB(a,b,c)$; see Section 2.4 (this is void when Σ is orientable). Then $\diamondsuit$ is defined as the equivalence relation generated by these relations. In other words, $\diamondsuit$ is by definition the smallest equivalence relation that connects the pairs $\{a,b\}$ and $\{a,c\}$ whenever $B(a,b,c)$ or $xB(a,b,c)$ holds. We will gradually make this relation more explicit, in this section and in Section 4. When Σ is orientable, the relation $\diamondsuit$ corresponds to the equivalence relation on triangles, generated by adjacency, in the subgraphs of $\mathcal{NC}^\dagger_{\pitchfork}(\Sigma)$ induced by curves passing through a common point. This is what motivates our notation.

The relation $\{a,b\} \diamondsuit \{a',b'\}$ obviously implies $\text{Point}(a,b) = \text{Point}(a',b')$. We will see that the converse is not true, and describe geometrically the equivalence classes in Section 4, but for now we will only need the following partial statement.

Proposition 21 *Let a, b, a' and b' be such that $T(a,b)$ and $T(a',b')$ hold. Suppose that they have the same intersection point, $x = \text{Point}(a,b) = \text{Point}(a',b')$, and suppose that the germs of a and a' coincide, ie there exists a neighborhood V of x such that $a \cap V = a' \cap V$. Then $\{a,b\} \diamondsuit \{a',b'\}$.*

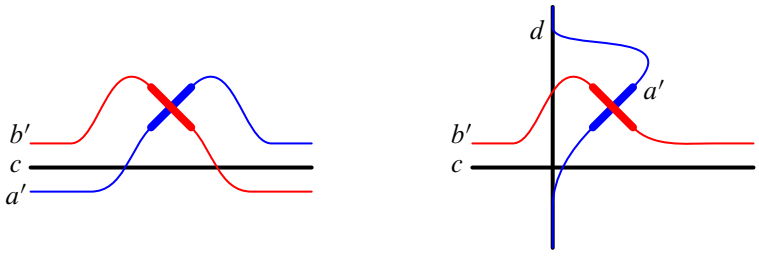

Figure 11: Completing the germs of a and b to form a nonbouquet 3–clique (a', b', c).

Remark 22 If Σ is a Klein bottle, there are no couples $\{a, b\}$ of two-sided curves such that $T(a, b)$ holds, and for every one-sided curve b, the curves c such that $T(b, c)$ holds fall into only two isotopy classes: that of b and that of a two-sided curve, prescribed by b. It follows that, without the extra bouquets in the definition of $\diamondsuit$, there would have been too many classes of $\diamondsuit$, as such a class would remember the isotopy class of a one-sided curve, and Proposition 21 would not be true in this special case. These extra bouquets will be used in the proof of Lemma 28 below.

We postpone the proof of the proposition to the end of this section; for now we will explain how it implies Theorem 1.

3.3 Proof of Theorem 1

If a, b and c are vertices of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, we denote by $F(a, b, c)$ the property that $T(a, b)$ holds, and there exists an edge $\{a', b'\}$ with $\{a, b\} \diamondsuit \{a', b'\}$ such that $\{a', b', c\}$ is a 3–clique which is not of type bouquet. Note that this property $F(a, b, c)$ implies that c does not contain the point $\mathrm{Point}(a', b') = \mathrm{Point}(a, b)$. The next lemma asserts that $F(a, b, c)$ actually characterizes this geometric property, and the letter F stands for "$a \cap b$ is far from c".

Lemma 23 *Let a, b and c be vertices of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, and suppose $T(a, b)$ holds. Then*

$$F(a, b, c) \iff \mathrm{Point}(a, b) \notin c.$$

Proof The direct implication follows directly from the definitions; we have to prove the converse implication. Suppose $\mathrm{Point}(a, b) \notin c$. Since Σ is connected, there exists a regular neighborhood of c containing the point $\mathrm{Point}(a, b)$. Depending on whether c is one-sided or two-sided, up to homeomorphism, this leads to only two distinct situations. In Figure 11, we represent in bold the germs of the curves a and b near the point $a \cap b$, and show how to complete these germs to new curves a' and b' such that $\{a', b', c\}$ is a 3–clique not of type bouquet. When c is one-sided (see Figure 11, left), we may use two curves a' and b' obtained by pushing c, while when c is two-sided (see Figure 11, right), we have to use a curve d which meets c once transversely.

Now, by Proposition 21, we have $\{a, b\} \diamondsuit \{a', b'\}$, and $\{a', b', c\}$ is a nonbouquet 3–clique, so we have $F(a, b, c)$ by definition. □

Corollary 24 *Suppose $T(a,b)$ and $T(\alpha,\beta)$ hold. Then* $\mathrm{Point}(a,b) \neq \mathrm{Point}(\alpha,\beta)$ *if and only if there exists a nonseparating closed curve c such that $F(\alpha,\beta,c)$ holds but not $F(a,b,c)$.*

The corollary is a direct consequence of Lemma 23. It follows that the equality $\mathrm{Point}(a,b) = \mathrm{Point}(\alpha,\beta)$ can be expressed in terms of the graph structure of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, because, as a consequence of Corollary 8, being a 3–clique not of type bouquet is also characterized in terms of this graph structure. Now we can conclude the proof of Theorem 1, provided Proposition 21 holds.

Proof of Theorem 1 Let φ be an automorphism of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$. Given a point x in Σ, we choose two nonseparating simple closed curves a and b intersecting exactly once, transversely, at x, and set $\varphi_\Sigma(x) = \mathrm{Point}(\varphi(a),\varphi(b))$. This formula is valid because, by Proposition 4, $\varphi(a)$ and $\varphi(b)$ are still nonseparating simple closed curves intersecting exactly once. The point $\varphi_\Sigma(x)$ does not depend on the choice of (a,b), because if (α,β) is another choice, the equalities $\mathrm{Point}(a,b) = \mathrm{Point}(\alpha,\beta)$ and $\mathrm{Point}(\varphi(a),\varphi(b)) = \mathrm{Point}(\varphi(\alpha),\varphi(\beta))$ can be all expressed in terms of the graph structure of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$, thanks to Corollary 24. Thus, the map φ_Σ is well defined, and by following the definitions we observe that the map $(\varphi^{-1})_\Sigma$ is its inverse; hence φ_Σ is a bijection of Σ. Finally, it follows from Lemma 23 that for any nonseparating simple closed curve α, the curve $\varphi(\alpha)$ coincides with the set of points $\varphi_\Sigma(x)$ as x describes α. In other words, the automorphism φ is realized by the bijection φ_Σ. Proposition 20 concludes. □

3.4 Connectedness of some arc graphs

In order to finally prove Proposition 21, we will first need a couple of elementary results on fine arc graphs.

Lemma 25 *Let S be a connected topological surface, with boundary, and let x and y be two distinct points of ∂S. Let $\mathcal{EA}^{\dagger}(S,x,y)$ be the graph whose vertices are the simple arcs joining x and y and which meet ∂S only at their ends, with an edge between two such arcs if and only if they are disjoint except at x and y. Then the graph $\mathcal{EA}^{\dagger}(S,x,y)$ is connected.*

Note that, when x and y are taken in the same connected component, we are not requiring that the arcs be nonseparating; this is the reason why we use the letter $\mathcal{E}$, for *extended*, in the same fashion as in [11].

Proof Let a and b be two vertices of this graph. As a first case we suppose that $a \cap b$ is made of a finite number of transverse intersection points; we will prove by induction on the cardinal of $a \cap b$ that, in this case, a and b are connected in $\mathcal{EA}^{\dagger}(S,x,y)$. If $a \cap b$ is as small as possible, ie is equal to $\{x,y\}$, then a and b are neighbors in this graph. Otherwise, if a and b intersect at other points than x and y, we may, in the spirit of [6], pick a *unicorn path* c, made of one subarc of a beginning at x, and one subarc of b ending at y. (For example, we may follow a until it first meets b after x, and then continue along b). Now we may push c aside while fixing its ends, at the appropriate side of c, to obtain a new arc c' with both $c' \cap a$ and $c' \cap b$ of cardinal strictly lower than that of $a \cap b$.

This proves that arcs intersecting at finitely many points are connected in $\mathcal{EA}^\dagger(S,x,y)$.

Now if the intersection $a\cap b$ is infinite, fix a differentiable structure on the surface S. We may consider a smooth curve a', neighbor of a, by pushing a aside while fixing its ends, and similarly, a smooth neighbor b' of b similarly. Up to perturbing b', we may suppose that b' is transverse to a'. By the step above, a' and b' are connected in $\mathcal{EA}^\dagger(S,x,y)$ and the lemma is proved. □

We will also need a version for nonseparating arcs.

Lemma 26 *Let S be a connected surface containing a nonseparating curve. Suppose S has boundary, and let x and y be two distinct points in the same boundary component of S. Let $\mathcal{NA}^\dagger(S,x,y)$ be the set of nonseparating simple arcs connecting x to y, with an edge when they are disjoint away from x and y. Then $\mathcal{NA}^\dagger(S,x,y)$ is connected.*

The points x and y add some technicality; let us first prove the following simpler statement.

Lemma 27 *Let S be a connected surface containing a nonseparating curve, and with at least one boundary component, denoted by C. Let $\mathcal{NA}^\dagger(S,C)$ be the graph whose vertices are the nonseparating arcs joining two distinct points of C, and with an edge between two such vertices whenever they are disjoint. Then this graph is connected.*

This lemma is a variation on [11, Corollary 3.2]; here we additionally require that the arcs end at C. In fact, [11, Corollary 3.3] is stated for surfaces with $b>0$ boundary components, but proved only in the case $b=1$, which is the case needed in the proof of their main theorem. Lemma 27 may be used to extend this corollary to any $b>0$.

Proof We begin with the observation that the graph $\mathcal{NA}^\dagger(S,C)$ has no isolated point. Indeed, if γ is a vertex of $\mathcal{NA}^\dagger(S,C)$, by definition it is nonseparating. So we may consider a simple closed curve u with one transverse intersection point with γ. This curve u is nonseparating; this follows from Fact 5, applied to u, and a curve v obtained by concatenation of γ with some arc of C. Now we can perform a surgery on u, and push its intersection point towards one end of γ until we hit C. This constructs an arc α, which is now disjoint from γ, and which is also nonseparating.

Next, we claim that we can suppose, without loss of generality, that the surface S is compact. Indeed, if γ_1 and γ_2 are vertices of $\mathcal{NA}^\dagger(S,C)$, and if u is a nonseparating curve intersecting γ_1 as above, consider the set $K=C\cup\gamma_1\cup\gamma_2\cup u$. This set is compact; hence there exists a compact topological subsurface S' of S containing K. This surface S' contains nonseparating curves, as it contains u and $\gamma_1\cup C$, which may be used as above to find two simple closed curves u and v with one essential intersection. Now a path joining γ_1 to γ_2 in S' is also a path joining γ_1 to γ_2 in S. So, until the end of the proof, S is now supposed to be compact.

Next, observe that if two vertices γ_1 and γ_2 of $\mathcal{NA}^\dagger(S,C)$ are isotopic (ie there exists a continuous map $H:[0,1]^2\to S$ such that $\gamma_1(t)=H(0,t)$ and $\gamma_2(t)=H(1,t)$ for all t, $H(s,0),H(s,1)\in C$ for all s and

the curve $H_s : t \to H(s,t)$ is injective for all s), then γ_1 and γ_2 are in the same component of $\mathcal{NA}^\dagger(S,C)$. This argument is borrowed from [2]: for all s, the arc H_s has at least a neighbor α_s (by the first observation above), and the set of s' such that $H_{s'}$ is still a neighbor of α_s is open in $[0,1]$. By compactness of $[0,1]$, there exist a finite number of arcs $\alpha_1, \dots, \alpha_n$, and a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that α_j is disjoint from H_t for all $t \in [t_{j-1}, t_j]$ for all j, and now $(\gamma_1, \alpha_1, H_{t_1}, \alpha_2, \dots, H_{t_{n-1}}, \alpha_n, \gamma_2)$ is a path of $\mathcal{NA}^\dagger(S,C)$ joining γ_1 to γ_2. As a result of this observation, we need only prove the connectedness of the graph $\mathrm{NA}(S,C)$, whose vertices are the isotopy classes of arcs between two distinct points of C, and with an edge between two vertices whenever the corresponding classes admit disjoint representatives.

We proceed with the observation that the graph $\mathrm{A}(S,C)$, defined exactly as $\mathrm{NA}(S,C)$ except we consider essential arcs, which may be separating, is connected. A simple way to do this is by using the idea of unicorn arcs exactly as in the proof of the preceding lemma: if two arcs a and b are in minimal position then their unicorn arcs are essential, and have fewer intersections with both a and b than the number of points of $a \cap b$.

We will promote the connectedness of $\mathrm{A}(S,C)$ to that of $\mathrm{NA}(S,C)$, by induction on the number of boundary components of S.

First, suppose that S has only one boundary component, C. Let γ_1 and γ_2 be two vertices of $\mathrm{NA}(S,C)$. We may connect them by a path $(\gamma_1, \alpha_1, \alpha_2, \dots, \alpha_n, \gamma_2)$ in $\mathrm{A}(S,C)$, where each of the α_j may be separating; consider such a path with minimal number of separating arcs. For contradiction, and up to some relabeling, suppose α_1 is separating. Then it cuts S in two components; denote by S_1 the one containing γ_1 and S_2 the other. Then α_2 is also contained in S_1, otherwise we may delete α_1 from our path. Since S has no boundary component other than C and since the curve α_1 is essential, the surface S_2 contains a nonseparating arc, α_1'. This arc may be used instead of α_1 in our initial path from γ_1 to γ_2, contradicting the minimality of the number of separating arcs. This proves that $\mathrm{NA}(S,C)$ is connected if S has no other boundary component.

Now, we suppose, for inductive hypothesis, that $\mathrm{NA}(S',C')$ is connected for every surface S' with less boundary components than S. Let γ_1 and γ_2 be two vertices of $\mathrm{NA}(S,C)$. As before, consider a path $(\gamma_1, \alpha_1, \alpha_2, \dots, \alpha_n, \gamma_2)$ in $\mathrm{A}(S,C)$ between them, with minimal number of separating arcs. For contradiction, and up to some relabeling, suppose α_1 is separating: it cuts S into two subsurfaces; let S_1 be the one containing γ_1, and, by hypothesis, must also contain α_2, and let S_2 be the other. If S_2 contains nonseparating arcs, we conclude as before. If not, then S_2 contains some of the boundary components of S; hence the surface with boundary $S' = S_1 \cup \alpha_1$ has strictly less boundary components than S. One is C', composed by an arc of C and the arc α_1, and there may be others.

If α_2 is nonseparating, then, by the induction hypothesis, there is a path $(\gamma_1, \beta_1, \dots, \beta_k, \alpha_2)$ of $\mathrm{NA}(S',C')$ connected them. The arcs $\beta_1, \dots, \beta_k$ may have end points in α_1, but we may perform a surgery in order to push all these points to C, and obtain arcs $\beta_1', \dots, \beta_k'$ which are also vertices of $\mathrm{NA}(S,C)$, and we are done in this case.

Finally, if α_2 is a separating arc (of S, or of S_1, equivalently), then we may find an arc α_2' of S_1 which is nonseparating and disjoint from α_2. By following the last case above, there exists a path $(\gamma_1, \beta_1, \dots, \beta_k, \alpha_2')$ in $\mathrm{NA}(S, C)$; hence the path $(\gamma_1, \beta_1, \dots, \beta_k, \alpha_2', \alpha_2, \dots, \alpha_n, \gamma_2)$ of $\mathrm{A}(S, C)$ has one less separating arc than the initial path. This contradiction ends the proof. □

Proof of Lemma 26 Let γ_1 and γ_2 be two vertices of $\mathcal{NA}^\dagger(S, x, y)$. First, we may construct a neighbor γ_2' in $\mathcal{NA}^\dagger(S, x, y)$ of γ_2, which, in a neighborhood of x (resp. y), touches γ_1 only at x (resp. y).

Indeed, there is a neighborhood U_x of x homeomorphic to the closed half unit disk

$$\{z \mid |z| \leqslant 1 \text{ and } \mathrm{Im}(z) \geqslant 0\},$$

where the middle ray ($\mathrm{Re}(z) = 0$) corresponds to the points of γ_1. On either side of this ray, we may find an arc disjoint from γ_1 and γ_2 except at 0, arbitrarily close to the boundary ($\mathrm{Im}(z) = 0$), and joining 0 to the unit circle, and then this small arc may be continued to construct a curve γ_2' which consists of pushing γ_2 aside.

So we may suppose that γ_1 and γ_2, close to x and y, intersect only at these points, and we may now find neighborhoods U_x and U_y as above, such that their intersections with γ_1 and γ_2 are along rays in this disk, in distinct directions around 0. Let S' be the surface obtained by removing the interiors of U_x and U_y from S. Then the path given by applying Lemma 27 to S, yields a path from γ_1 to γ_2 in $\mathcal{NA}^\dagger(S, x, y)$, just by adding some rays in U_x and U_y to the corresponding arcs. □

3.5 Proof of Proposition 21

Let us go back to the proof of Proposition 21. For the remainder of the section we fix a point $x \in \Sigma$. Let X denote the set of nonseparating simple closed curves passing through x.

Lemma 28 *Let $a, b, c \in X$. Suppose that $T(a, b)$ and $T(a, c)$ hold. Then $(a, b) \diamondsuit (a, c)$.*

Proof Let S be the surface obtained by cutting Σ along a; it is the surface with boundary obtained by gluing back two copies of the curve a to $\Sigma \smallsetminus a$. The point x of Σ yields two points, p and q, of ∂S, and the curves b and c define two arcs of S joining p and q. By Lemma 25, there exists a finite sequence $\gamma_0 = b, \dots, \gamma_n = c$, of arcs of S joining p and q, with γ_i and γ_{i+1} disjoint except at p and q. For each i, the arc γ_i defines a closed curve in Σ, which has precisely one, transverse intersection with a; we will still denote it by γ_i, abusively.

For every i, if $T(\gamma_i, \gamma_{i+1})$ holds, then we have $(a, \gamma_i) \diamondsuit (a, \gamma_{i+1})$, by definition. If $T(\gamma_i, \gamma_{i+1})$ does not hold, then either γ_i or γ_{i+1} are both one-sided, or one of them is two-sided. In the first case, the condition $xB(a, \gamma_i, \gamma_{i+1})$ holds, by definition, and hence $(a, \gamma_i) \diamondsuit (a, \gamma_{i+1})$. In the second, up to reversing the notation suppose γ_i is two-sided. Figure 12 shows how to insert a curve δ such that $B(a, \delta, \gamma_i)$ and $B(a, \delta, \gamma_{i+1})$ both hold, and hence we still have $(a, \gamma_i) \diamondsuit (a, \gamma_{i+1})$ in this case.

By transitivity, we deduce that $(a, b) \diamondsuit (a, c)$. □

Figure 12: Connecting the curves by common adjacency. The left shows the case when a is one-sided, and the right is when a is two-sided

The last ingredient for the proof of Proposition 21 is the following observation.

Observation 29 *Let a and a' be two nonseparating simple closed curves in Σ such that $a \cap a'$ is an arc. Then both sides of this arc lie in the same connected component of $\Sigma \smallsetminus (a \cup a')$.*

Proof A priori, the complement of $\Sigma \smallsetminus (a \cup a')$ may have up to four connected components, as suggested in Figure 13. Suppose first that the intersection $a \cap a'$ is essential. If a (resp. a') is one-sided, by following the curve a (resp. a') we see that $A = B$. If both a and a' are two-sided, by following a we see that $A = D$ and $C = B$, while by following a' we get $A = C$ and $B = D$, so $A = B$.

Now, suppose the intersection arc $a \cap a'$ is inessential. By following a, we see that $C = D$, regardless of a being one or two-sided. Thus, if $A \neq B$, then one of A or B, say A, is not connected from any of B, C or D. But this implies that a is separating, a contradiction. □

We are now in a position to prove Proposition 21, but instead we will prove the following stronger statement, which will be more convenient later in this article.

Proposition 30 *Let a, b, a' and b' be such that $T(a,b)$ and $T(a',b')$ hold, with intersection point $x = \mathrm{Point}(a,b) = \mathrm{Point}(a',b')$, and suppose that a and a' locally "half coincide" near x, ie $a \cap a'$ contains a nondegenerate arc with endpoint x. Then $\{a,b\} \diamondsuit \{a',b'\}$.*

Proof Suppose first that a and a' coincide along some arc with x as an endpoint, and are disjoint apart from this arc. By Observation 29, there exists a curve d passing through x such that $T(a,d)$ and $T(a',d)$. By Lemma 28, this implies $(a,d) \diamondsuit (a',d)$, and by the same lemma we also have $(a,b) \diamondsuit (a,d)$ and $(a',b') \diamondsuit (a',d)$. Hence $(a,b) \diamondsuit (a',b')$.

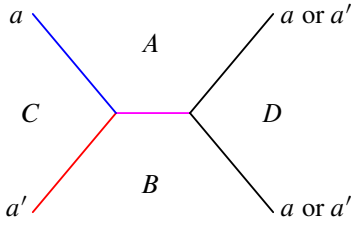

Figure 13: The arc $a \cap a'$ cannot disconnect.

Now we do not make the assumption any more that a and a' meet only along an arc. Still, thanks to the hypothesis of the proposition, we may choose a set V homeomorphic to a closed disk, with x on its boundary, and such that $a \cap V = a' \cap V$ is an arc whose endpoints are x and some other point y. Lemma 26, applied to the surface $\Sigma \smallsetminus \mathring{V}$, provides a sequence $a_0 = a, \dots, a_n = a'$, of nonseparating curves such that for all i, the curves a_i and a_{i+1} intersect only along the arc $a \cap V$; hence we may conclude by applying iteratively the reasoning above. □

4 Local subgraphs

In Section 3.2, we considered edges $\{a, b\}$ and $\{a', b'\}$ in the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ satisfying

$$|a \cap b| = |a' \cap b'| = 1,$$

and $\mathrm{Point}(a, b) = \mathrm{Point}(a', b')$. We defined and used the equivalence relation $\diamondsuit$. The aim of this section is to provide a geometric interpretation of the equivalence classes. The results here are not used anywhere else in the paper. In particular, this section is not used in the proof of our main results. Nevertheless, we think it may help the reader to get a clear picture of the situation.

4.1 The graph of germs

Let x be a marked point in the surface Σ. In this section the we will study the local geometry of curves near x, so we may assume that $(\Sigma, x) = (\mathbb{R}^2, 0)$ whenever this is convenient. Given two simple arcs $a, a' \colon [0, 1] \to \Sigma$ with $a(0) = a'(0) = x$, we say that a and a' *locally coincide* at x if there exists a neighborhood V of x such that $a([0, 1]) \cap V = a'([0, 1]) \cap V$. This is an equivalence relation, whose equivalence classes are called *germs of simple arcs at* x. The germ of a is denoted $[a]_x$. We say that a and a' *locally intersect only at* x if there exists a neighborhood V of x such that $a([0, 1]) \cap a'([0, 1]) \cap V = \{x\}$. This second relation depends only on the germs of a and a', and thus induces a relation on germs. Let us consider the graph $\mathcal{A}(x)$ whose vertices are the germs of simple arcs at x, with an edge between the germs of a and a' whenever a and a' locally intersect only at x.

This graph is *not* connected, in fact it has infinitely (uncountably) many connected components, as we will see below. We postpone the description of the connected components to explain the relation with the adjacency relation defined in Section 3.2. We say that two vertices α and α' of the graph $\mathcal{A}(x)$ are *comparable* if they belong to the same connected component of the graph.

4.2 Germs and adjacency

Given a point x in Σ and a simple closed curve a in Σ that contains x, we choose any one of the two germs of simple arc at x included in a and denote it by $\lfloor a \rfloor_x$. Which one of the two germs is chosen will not matter in what follows.

Proposition 31 *Let a, b, a' and b' be vertices in $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ such that $T(a,b)$ and $T(a',b')$ hold, and assume* $\mathrm{Point}(a,b) = \mathrm{Point}(a',b')$. *Then $\{a,b\} \diamondsuit \{a',b'\}$ if and only if the germs $\lfloor a \rfloor_x$ and $\lfloor a' \rfloor_x$ are comparable.*

Proposition 31 will be proved in Section 4.5 below. The aim of the next two sections is to provide a simple characterization of distance, and connected components, in the graph of germs; see Proposition 33 below.

4.3 Distance in local subgraphs

In this section, we give a geometric interpretation of the distance in three different graphs, which are very much like the graph of germs $\mathcal{A}(x)$.

Let Σ be one of the following surfaces:

(1) the compact annulus $\mathbb{S}^1 \times [0,1]$,

(2) the open annulus $\mathbb{S}^1 \times \mathbb{R}$, or

(3) the 2–torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

We consider nonoriented simple arcs in Σ; more precisely, simple curves connecting both sides of the annulus in case (1), properly embedded images of the real line connecting both ends of the open annulus in case (2), or simple closed curves in a fixed homotopy class, say homotopic to $\{0\} \times \mathbb{S}^1$ in case (3). Let $\mathcal{A}$ denote the graph whose vertices are one of the three above family of curves, with an edge between two curves whenever they are disjoint.

In order to express geometrically the distance in $\mathcal{A}$, let us consider the cyclic cover $p\colon \widetilde{\Sigma} \to \Sigma$, respectively in cases (1), (2) and (3),

$$p\colon \mathbb{R} \times [0,1] \to \mathbb{R}/\mathbb{Z} \times [0,1], \quad p\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}, \quad p\colon \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z} \times \mathbb{S}^1$$

given by the formula $p(x,y) = (x \bmod 1, y)$. Let T be the deck transformation $(x,y) \to (x+1, y)$.

Now consider two curves a, b which are vertices of the graph $\mathcal{A}$. Let $\tilde{a}$ and $\tilde{b}$ be respective lifts of a and b under the covering map p. Note that the set

$$\{k \in \mathbb{Z} \mid T^k(\tilde{a}) \cap \tilde{b} \neq \varnothing\}$$

is an interval of $\mathbb{Z}$, which is finite in the compact cases (1) and (3) but may be infinite in the open annulus case (2). We define the *relative width* $\mathrm{Width}(a,b)$ as the cardinal of this set. This is an element of $\{0, 1, \dots, +\infty\}$. The reader may check easily that $\mathrm{Width}(a,b) = \mathrm{Width}(b,a)$.

Proposition 32 *For every vertices $a \neq b$ of the graph $\mathcal{A}$, the distance in the graph is given by*

$$d(a,b) = \mathrm{Width}(a,b) + 1.$$

In cases (1) *and* (3), *the graph $\mathcal{A}$ is connected. In case* (2), *a and b are in the same connected component of $\mathcal{A}$ if and only if* $\mathrm{Width}(a,b) < +\infty$.

Proof Let a and b be as in the statement, and denote $w = \mathrm{Width}(a, b)$. We first assume that $w < +\infty$. By Schoenflies' theorem (in case (2), applied in the two-point compactification of the annulus, which is a sphere), we may assume that a is a vertical curve whenever this makes our life easier. We first note that if $w = 0$ then a and b admit lifts that are disjoint from every T–translate of each other, which shows that a and b are disjoint, and thus $d(a, b) = 1$. Let us now assume $w > 0$, and prove the two following key properties.

(i) For every vertex a' of $\mathscr{A}$ such that $d(a, a') = 1$,

$$\mathrm{Width}(a', b) \geq w - 1.$$

(ii) There exists a vertex a' of $\mathscr{A}$ such that $d(a, a') = 1$ and

$$\mathrm{Width}(a', b) \leq w - 1.$$

To prove the first property, consider a' such that $d(a, a') = 1$. By definition of the width w, we may find lifts $\tilde{a}$ and $\tilde{b}$ of a and b such that $\tilde{b}$ is disjoint from $\tilde{a}$ and $T^{w+1}\tilde{a}$ but meets $T(\tilde{a}), \dots, T^w(\tilde{a})$. Since a and a' are disjoint, there is a lift $\tilde{a}'$ of a' which is between $\tilde{a}$ and $T(\tilde{a})$. Then the curves

$$T(\tilde{a}'), \dots, T^{w-1}(\tilde{a}')$$

are between the two curves $T(\tilde{a})$ and $T^w(\tilde{a})$, and those two curves are not in the same connected component of $\widetilde{\Sigma} \smallsetminus T^i(\tilde{a}')$, for $i = 1, \dots, w-1$. Since the curve $\tilde{b}$ is connected and meets the two curves $T(\tilde{a})$ and $T^w(\tilde{a})$, it must meet all the $T^i(\tilde{a}')$. This proves that $\mathrm{Width}(a', b) \geq w - 1$.

Let us prove the second property. We consider $\tilde{a}$ and $\tilde{b}$ as above. Let S denote the compact strip or annulus bounded by $\tilde{a} \cup T(\tilde{a})$. Remember that $\tilde{b}$ meets $T(\tilde{a})$ but not $\tilde{a}$. Thus $\tilde{b} \cap S$ is included in a (maybe infinite) family of *bigons*, ie topological disks bounded by a simple closed curve made of a segment of the curve $T(\tilde{a})$ and a segment of the curve $\tilde{b}$. Let S^+ denote the union of these bigons. Symmetrically, the curve $\tilde{b}' := T^{-w}(\tilde{b})$ meets $\tilde{a}$ but not $T(\tilde{a})$. Thus $T^{-w}(\tilde{b}) \cap S$ is included in a union S^- of bigons formed by the curves $\tilde{a}$ and $\tilde{b}'$. A key point is that the sets S^- and S^+ are disjoint, because the curves $\tilde{b}$ and $\tilde{b}'$ are disjoint, since b is simple. Thus we may construct a homeomorphism H supported in S such that $H(S^-)$ is included in an arbitrarily small neighborhood of $\tilde{a}$, and $H(S^+)$ is included in an arbitrarily small neighborhood of $T(\tilde{a})$. In particular, we may find a curve $\tilde{a}'$, which is a lift of some element a' of $\mathscr{A}$, included in the interior of S and disjoint from both S^- and S^+ (to be more explicit, take $\tilde{a}' = H^{-1}(\{1/2\} \times [0, 1])$ in the annulus case, in coordinates for which a is the vertical curve $\{0\} \times [0, 1]$). Note that $\tilde{a}'$ is disjoint from $\tilde{b}$ and $T^{-w}(\tilde{b})$, and separate both curves, ie the first one is on the right-hand side of $\tilde{a}'$, and the second one is on the left-hand side. Thus the set

$$\{k \in \mathbb{Z} \mid T^k(\tilde{b}) \cap \tilde{a}' \neq \varnothing\}$$

has cardinality at most $w - 1$. Which proves that $\mathrm{Width}(a', b) \leq w - 1$, as wanted.

Using (i) and (ii), an induction on n shows that $d(a, b) = n$ if and only if $\mathrm{Width}(a, b) + 1 = n$, which completes the proof in the case when $\mathrm{Width}(a, b)$ is finite. When $\mathrm{Width}(a, b) = +\infty$, an argument

analogous to property (i) above shows that $\mathrm{Width}(a', b) = +\infty$ for every a' such that $d(a, a') = 1$. This shows that a and b are not in the same connected component of the graph. □

4.4 Distance in the graph of germs

Let us go back to the graph of germs $\mathcal{A}(x)$. Assume $(\Sigma, x) = (\mathbb{R}^2, 0)$. Given two vertices a and b of $\mathcal{A}(x)$, let us define their *local relative width* $\mathrm{Width}(a, b)$ as the number of turns b does around x in arbitrary small neighborhoods of x, in a coordinate system in which a goes straight to x. More precisely, the plane minus the origin is identified with the open annulus $\mathbb{S}^1 \times \mathbb{R}$, and we consider the graph $\mathcal{A}$ from the previous section in the open annulus case. Then $\mathrm{Width}(a, b)$ is defined as the infimum of the quantity $\mathrm{Width}(A, B)$, where A and B are vertices of $\mathcal{A}$ whose germs respectively equal a and b. Here is a more practical definition, which is easily seen to be equivalent. Consider the universal cover $p\colon \widetilde{\Sigma} \to \Sigma$ as above. Abuse the definition by still denoting $a, b\colon [0, 1] \to \Sigma$ two curves with $a(0) = b(0) = 0$ whose germs respectively equal a and b. Let $\tilde{a}$ and $\tilde{b}$ denote lifts of (the restrictions to $(0, 1]$ of) a and b in $\widetilde{\Sigma}$. Then the number $\mathrm{Width}(a, b) = w$ is characterized by the two following properties:

(i) for every $t_0 \in (0, 1]$, the restriction of $\tilde{a}$ to $(0, t_0]$ meets at least w integer translates of $\tilde{b}$;

(ii) there exists $t_0 \in (0, 1]$ such that the restriction of $\tilde{a}$ to $(0, t_0]$ meets exactly w integer translates of $\tilde{b}$;

Analogously to the previous section, the distance in the graph of germs is characterized by the local relative width.

Proposition 33 *Let $a \neq b$ be two vertices of the graph $\mathcal{A}(x)$. Then a and b are in the same connected component of $\mathcal{A}(x)$ if and only if $\mathrm{Width}(a, b) < +\infty$. In this case, the distance in the graph is given by*

$$d(a, b) = \mathrm{Width}(a, b) + 1.$$

The proof is very similar to the proof in the previous section. Details are left to the reader.

4.5 Proof of Proposition 31

Let a, b, a' and b' be vertices of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ such that $T(a, b)$ and $T(a', b')$ hold, and assume that $\mathrm{Point}(a, b) = \mathrm{Point}(a', b')$. Denote by x the common intersection point.

If c is another vertex such that $\{a, b, c\}$ is a 3–clique of type bouquet or extra bouquet, then the germs $\lfloor a \rfloor_x$ and $\lfloor c \rfloor_x$ are disjoint, thus comparable. This entails the direct implication in Proposition 31.

Let us prove the converse implication. We assume that the germs $\lfloor a \rfloor_x$ and $\lfloor a' \rfloor_x$ are comparable. In other words, there exists arcs $\alpha_0, \dots, \alpha_n$ with $\alpha_i(0) = x$ and whose sequence of corresponding germs is a path from $\lfloor a \rfloor_x$ to $\lfloor a' \rfloor_x$ in the graph of germs. Note that each germ α_i may be extended to a nonseparating curve a_i, and we can find another nonseparating curve b_i such that $T(a_i, b_i)$ holds. Thus the end of the proof is a direct consequence of the following lemma.

Lemma 34 *Let a, b, a' and b' be vertices in $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ such that $T(a,b)$ and $T(a',b')$ hold. Assume that for some choices $\lfloor a \rfloor_x$ and $\lfloor a' \rfloor_x$ of arcs at x included respectively in a and a', the germs $\lfloor a \rfloor_x$ and $\lfloor a' \rfloor_x$ intersect only at x. Then $\{a,b\} \diamondsuit \{a',b'\}$.*

Proof Let c be an arc that contains x in its interior and locally coincides with $\lfloor a \rfloor_x \cup \lfloor a' \rfloor_x$. Extend c into a nonseparating closed curve, still denoted by c, and consider any other nonseparating curve d such that $T(c,d)$ holds. Since c locally "half coincides" near x with both a and a', we may apply Proposition 30 twice, and get that $\{a,b\} \diamondsuit \{c,d\} \diamondsuit \{a',b'\}$. □

4.6 Curves and diffeomorphisms

In this short subsection we explain how one can use the fine curve graph to detect fundamental nondifferentiability.

Let Φ be an automorphism of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$. We introduce the following property $D(\Phi)$:

For all vertices a and b of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ such that $T(a,b)$ holds, if $\mathrm{Point}(\Phi(a),\Phi(b)) = \mathrm{Point}(a,b)$ *then there exists a' and b' such that $T(a',b')$ holds*, $\mathrm{Point}(a',b') = \mathrm{Point}(a,b)$ *and* $\{\Phi(a'),\Phi(b')\} \diamondsuit \{a',b'\}$.

Note that this property is invariant under conjugacy in the group of automorphisms; indeed, according to Lemma 23, this property is defined entirely in terms of the graph structure of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$. Let h be a homeomorphism of Σ, and denote $\Phi = \Phi_h$ the action of h on the graph $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$.

Observation 35 *If h is differentiable everywhere, then property $D(\Phi_h)$ holds.*

Indeed, the hypothesis $\mathrm{Point}(\Phi(a),\Phi(b)) = \mathrm{Point}(a,b)$ is equivalent to the fact that $x = \mathrm{Point}(a,b)$ is a fixed point of h. Since h is differentiable at x, it is easy to check that the germ of any smooth arc at x is comparable to its image. Take any two smooth curves a' and b' such that $T(a',b')$ and $\mathrm{Point}(a',b') = \mathrm{Point}(a,b)$; then Proposition 31 tells us that $\{\Phi(a'),\Phi(b')\} \diamondsuit \{a',b'\}$.

Now consider a particular homeomorphism h of Σ and assume that h admits a fixed point where, for some local polar coordinates, h is defined by

$$(r,\theta) \mapsto \Big(r, \theta + \frac{1}{r}\Big).$$

Observation 36 *Property $D(\Phi_h)$ does not hold.*

An easy proof of this is obtained by considering the *local rotation interval* of h at x, as defined in [9, Section 2.3]. Indeed, the local rotation interval of h at x equals $\{+\infty\}$, which accounts for the fact that orbits turn faster and faster around x, in the positive direction, as we get nearer and nearer to x (the quickest way to check this is to show that the *local rotation set* of h at x is $\{+\infty\}$, and then to apply [9, théorème 3.9] that relates the local rotation set and the local rotation interval). We argue by contradiction to show that property $D(\Phi_h)$ does not hold. Assuming property $D(\Phi_h)$ holds, consider curves a and b such that $T(a,b)$ holds and $\mathrm{Point}(a,b) = x$. Let a' and b' be given by property $D(\Phi_h)$, such that

$\{\Phi(a'), \Phi(b')\} \pitchfork \{a', b'\}$. The reverse direction of Proposition 31 tells us that the germs of $h(a')$ and a' are comparable at x. This entails easily, from the definition, that the local rotation interval of h at s is a bounded interval, a contradiction.

5 Fine graph of smooth curves

In this section we address the case of smooth curves, and prove Theorem 2 and Proposition 3. Throughout the section, Σ will be a connected, nonspherical surface without boundary, endowed with a smooth structure. In Section 5.3 we will restrict to the orientable case.

5.1 From bijections to higher regularity

One step in the proof of Theorem 1 was Proposition 20, in which we proved that if an automorphism of $\mathcal{NC}^{\dagger}_{\pitchfork}(\Sigma)$ is supported by a bijection of Σ, then that bijection is a homeomorphism of Σ.

We may ask the same question about automorphisms of $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$, and this paragraph is devoted to the proof of the following two statements. We denote by $\mathrm{Homeo}_{\infty\pitchfork}(\Sigma)$ the group of bijections of Σ which preserve the family of smooth, nonseparating closed curves, and preserve transversality between such curves. The first statement below justifies this notation. Here, for simplicity we restrict to the case of orientable surfaces.

Proposition 37 *Let Σ be a connected, nonspherical orientable surface. The group $\mathrm{Homeo}_{\infty\pitchfork}(\Sigma)$ is contained in $\mathrm{Homeo}(\Sigma)$.*

Proof Let $h \in \mathrm{Homeo}_{\infty\pitchfork}(\Sigma)$. We will prove that the image under h of any open set is an open set. This is the continuity of h^{-1}, and by applying the argument to h we also get the continuity of h.

To do this, we only need to consider the images of a family of sets that generates the topology. Given three nonseparating curves a, b and c, we denote by $V(a; b, c)$ the union of all the nonseparating curves d that meet a and are disjoint from b and c.

Observation 38 *The set $V(a; b, c)$ is the union of some of the connected components of the complement of $b \cup c$ that meet a. In particular, it is an open set.*

Indeed, let x be a point of $V(a; b, c)$. By definition there is a nonseparating curve d passing through x and meeting a but not b nor c. Consider another point y that belongs to the connected component V_x of the complement of $b \cup c$ that contains x. By modifying d using an arc connecting x to y in V_x, we find another curve d', isotopic to d, still meeting a but not b nor c, and passing through y. This proves that $V(a; b, c)$ contains V_x, and the observation follows.

Now let a be a nonseparating curve. Let a^+ and a^- be obtained by pushing a to both sides. Then $V(a; a^+, a^-)$ is a neighborhood of a, and by making a^+ and a^- vary we get a basis of neighborhoods $\mathcal{B}(a)$ of the curve a. The union of all these families $\mathcal{B}(a)$ generates the topology of Σ.

Thus it suffices to check that the image under h of each set $V(a;b,c)$ is an open set. But since h is a bijection, we have

$$h(V(a;b,c)) = V(h(a);h(b),h(c)).$$

By hypothesis, $h(a)$, $h(b)$ and $h(c)$ are nonseparating closed curves, and by the observation this set is open. □

We now prove Proposition 3 stated in the introduction, namely the existence of elements of $\mathrm{Homeo}_{\infty\pitchfork}(\Sigma)$ that are not smooth.

Proof of Proposition 3 We will construct a homeomorphism $F:\mathbb{R}^2\to\mathbb{R}^2$, which is not differentiable at the origin, but such that both F and F^{-1} send smooth curves to smooth curves. The construction can easily be modified to make F compactly supported, and then be transported on our surface Σ. It will be clear from the construction that this map preserves transversality.

Let $h:\mathbb{R}\to\mathbb{R}$ be a smooth diffeomorphism supported in the segment $[1/2,2]$. That is to say, $h(x)=x$ for all x outside $[1/2,2]$; we suppose however that $h(1)\neq 1$. We consider the map F defined by $F(x,y)=(x,xh(y/x))$ if $x\neq 0$, and $F(x,y)=(x,y)$ otherwise. We claim that this map has the desired property.

This map, as well as its inverse, is obviously smooth in restriction to $\mathbb{R}^2\smallsetminus\{(0,0)\}$. Direct computation shows that F has directional derivatives in all directions around the origin, but the "differential" fails to be linear: both partial derivatives are those of the identity, while the directional derivative in the direction $(1,1)$ is not. So F is not differentiable at the origin.

Now, let $\gamma:\mathbb{R}\to\mathbb{R}^2$ be a smooth, proper embedding. If $(0,0)$ is not in the image of γ, then of course, $F\circ\gamma$ is still smooth. So suppose, say, that $\gamma(0)=(0,0)$. If $\gamma'(0)$ lies outside the two (opposite) sectors of vectors of slopes between $1/2$ and 2, then $F\circ\gamma$ and γ have the same germ at 0. Otherwise, and up to reparametrization, we can write, near 0, $\gamma(t)=(t,\alpha(t))$ where α is a smooth map (satisfying $\alpha(0)=0$)), from a neighborhood of 0, to $\mathbb{R}$. This yields the formula

$$F\circ\gamma(t)=\left(t,th\left(\frac{\alpha(t)}{t}\right)\right).$$

Now, the smoothness of $F\circ\gamma$ follows from the following elementary observation.

Claim *Let $\alpha:\mathbb{R}\to\mathbb{R}$ be a smooth map satisfying $\alpha(0)=0$. Then the map $t\mapsto\alpha(t)/t$ when $t\neq 0$, and $\alpha'(0)$ when $t=0$, is smooth.*

Indeed, by the fundamental theorem of calculus, for all $t\in\mathbb{R}^*$ we have

$$\frac{\alpha(t)}{t}=\int_0^1\alpha'(ts)ds,$$

and this integral with parameter can be differentiated indefinitely.[1] □

[1] We borrow this elegant argument from [4].

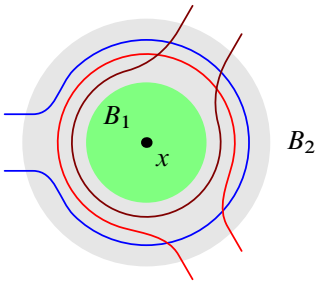

Figure 14: Three bottles. Any arc from B_1 to the outside of B_2 and disjoint from d_1 must meet d_2 and d_3.

5.2 A weak convergence for sequences of curves

In order to prove Theorem 2, we now explain how to recognize configurations of smooth curves. Given two vertices a and b of the graph $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$, we will denote by $a-b$ the property that they are neighbors in the graph. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of vertices, we denote by $(f_n)_{n\in\mathbb{N}}-a$ the property that for all n large enough, f_n-a. The first property of sequences we may recover from the graph is the distinction of what curves go to infinity.

Lemma 39 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of vertices of $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$. The following are equivalent*:

- *for all d, we have $(f_n)_{n\in\mathbb{N}}-d$;*
- *for every compact subset K of Σ, for every n large enough, $K\cap f_n=\varnothing$.*

Proof The second statement implies the first, as we may just take $K=d$. Let us prove the converse implication by contraposition. Suppose K intersects infinitely many f_n. Since K is compact, there is a point $x\in K$, such that every neighborhood of x intersects infinitely many f_n. We consider two open sets B_1 and B_2 with $x\in B_1$ and $\overline{B}_1\subset B_2$, and three bottle-shaped arcs, as in Figure 14. These arcs may be continued to form three nonseparating closed curves, d_1, d_2 and d_3. Now, let n be such that f_n enters B_1. If f_n does not enter nor leave B_2 through the neck of the bottle corresponding to d_1, then we cannot have f_n-d_1, since f_n and d_1 have to intersect at least twice. Hence, f_n passes through the neck of d_1, and in order to impose that f_n-d_2, another arc of f_n has to get out of the bottle corresponding of d_2 through its neck. But then f_n has to meet d_3 twice, and we cannot have f_n-d_3. In other words, for all n such that f_n enters B_1, we can't have f_n-d_1 and f_n-d_2 and f_n-d_3, hence the first statement is not true, and our implication is proved. □

Thus, we will say here that a sequence $(f_n)_{n\in\mathbb{N}}$ is *relevant* if it has no subsequence $(f_{\varphi(n)})_{n\in\mathbb{N}}$ such that for all d, $(f_{\varphi(n)})_{n\in\mathbb{N}}-d$. We now explore, for such sequences, the following notion of convergence. We say that a relevant sequence (f_n) *converges in a weak sense* to a curve a if for every d such that $a-d$, we have $(f_n)-d$. We denote this property by $W((f_n),a)$.

Lemma 40 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of vertices, and a be a vertex of $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$.*

- *If $W((f_n),a)$, then the sequence $(f_n)_{n\in\mathbb{N}}$ converges in the Hausdorff topology to a; for every neighborhood V of the curve a, for all n large enough, we have $f_n \subset V$.*
- *If the sequence $(f_n)_{n\in\mathbb{N}}$ of curves, with some appropriate parametrization, converges in the C^1–topology to a, then $W((f_n)_{n\in\mathbb{N}},a)$.*

Proof We first prove the first point. Let us first mention that the statement we wrote is indeed equivalent to the Hausdorff convergence, because any essential curve in a small enough neighborhood of a must pass close to every point of a, and thus is Hausdorff-close to a. Suppose for contradiction that for some neighborhood V of a, we have $f_n \not\subset V$ infinitely often. Then we may find a point x, not in $\overline{V}$, such that every neighborhood of x is visited by infinitely many f_n. We may construct three bottle-shaped arcs around x exactly as in the proof of the preceding lemma, and complete these arcs to nonseparating simple closed curves d_1, d_2 and d_3, which can be requested to satisfy $a - d_i$ for $i = 1,2,3$. Then we cannot have $f_n - d_i$ for all $i \in \{1,2,3\}$ for the same reason as in this preceding proof, and this contradicts the hypothesis that $W((f_n),a)$ holds.

The second point is the well known stability of transversality in the C^1–topology. □

Remark 41 In fact, the condition $W((f_n),a)$ implies C^0–convergence, in the following sense. Given a parametrization α of a, we can choose the parametrizations of the f_n's yielding a sequence of parametrized curves converging uniformly to α.

As we will not use this fact, we only sketch a quick argument. Let V be a small tubular neighborhood of a, and let $d_1,\dots,d_N$ be simple closed nonseparating curves, each meeting a transversely at one point, and cutting V in small chunks $V_1,\dots,V_N$ that are met by α in that cyclic order. Let n be large enough that $f_n \subset V$ and $f_n - d_i$ for each i. Then $f_n \cap V_i$ is connected, and f_n visits the pieces $V_1,\dots,V_N$ in that order. Hence we may choose a parametrization of f_n, say F_n, in such a way that for all $i \in \{1,\dots,N\}$ and all t, $F_n(t) \in V_i$ if and only if $\alpha(t) \in V_i$. This implies that F_n is uniformly close to α.

This notion is actually somewhere strictly in between C^0–convergence and C^1–convergence, as we remark in the following example. This construction will play a crucial role below in the proof of Theorem 2.

Example 42 Let a be a smooth nonseparating curve in Σ. We choose a point p in a, and a chart around one of its points, diffeomorphic to $\mathbb{R}^2$, in such a way that a corresponds to the axis of equation $y = 0$ in that plane. In this chart, we consider the functions $f_1 : x \mapsto 2/(1+x^2)$, and for all $n \geqslant 2$, $f_n : x \mapsto f_1(nx)/n$. Abusively, we still denote their graphs by the same letters, and then, we may extend these arcs, viewed in Σ, to simple closed curves (consisting of pushing a aside), that converges C^1 to a outside of the point p. Abusively we still use the same letters f_n to denote these closed curves.

The sequence $(f_n)_{n\geqslant 1}$ does not converge C^1 to a, because f_n has slope -1 at the point $(1/n, 1/n)$.

Nonetheless, we claim that $W((f_n), a)$ holds. Indeed, let d be such that $a - d$. Since the sequence (f_n) converges C^1 to a everywhere except at the origin of this $\mathbb{R}^2$ chart, the only case in which it is not already clear that $(f_n) - d$ is when d meets a transversely at the origin. If d has a strictly positive slope there, then for n large enough, the intersection $f_n \cap d$ will be transverse because the slopes of f_n are all negative in the region $x > 0$. The case when d has negative slope is symmetric, and if d has vertical slope, it will be transverse with f_n since these have bounded slopes.

5.3 Recognizing configurations of smooth curves

In this last section we assume that our surface Σ is orientable.

If a, b are vertices of $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$, we will denote by $D_\infty(a, b)$ the condition that $a - b$ and for all sequences (f_n) and (g_m) such that $W((f_n), a)$ and $W((g_m), b)$, we have $f_n - g_m$ for all m, n large enough.

Lemma 43 *Let a and b be smooth nonseparating curves. Then $D_\infty(a, b)$ holds if and only if a and b are disjoint.*

Proof Suppose first that a and b are disjoint. Then they admit disjoint neighborhoods, V_1 and V_2. For any sequences (f_n) and (g_m) with $W((f_n), a)$ and $W((g_m), b)$, for all m and n large enough we have $f_n \subset V_1$ and $g_m \subset V_2$, by Lemma 40. Hence, $f_n - g_m$ for all m and n large enough, and $D_\infty(a, b)$ holds indeed.

Now, suppose that a and b are not disjoint. Since $a - b$, the curves a and b have a transverse intersection, and in an appropriate chart diffeomorphic to $\mathbb{R}^2$, the curves a and b correspond respectively to the axes $y = 0$ and $x = 0$.

Then we may form a sequence (f_n) such that $W((f_n), a)$ exactly as in Example 42, and for (g_n) we just exchange coordinates x and y. For all n, the curves f_n and g_n have a nontransverse intersection point (at $(1/n, 1/n)$ in the chart of Example 42); hence the condition $D_\infty(a, b)$ does not hold. □

In the end of the proof, the curves f_n and g_n were tangent at their intersection point, hence not neighbors in the graph $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$. This tangency may look like an accidental reason for f_n and g_n not being neighbors in the graph $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$, but upon changing the formula of f_1 in Example 42 by a small C^1 perturbation we can get f_n and g_n to have, for example, infinitely many transverse intersection points.

From now on, we restrict ourselves to the case of orientable surfaces. One reason is that it would take more work to recover the extra bouquets and not only the bouquets; one other reason is that the next lemma works best when at least one of a, b or c is two-sided.

Lemma 44 *Suppose Σ is orientable. Let $\{a, b, c\}$ be a 3–clique of $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$, and suppose that these three curves pairwise intersect. Then the following are equivalent*:

(1) *This 3–clique is of type bouquet.*

(2) *There exists a relevant sequence (f_n) of vertices of $\mathcal{NC}^{\dagger\infty}_{\pitchfork}(\Sigma)$, which are all disjoint from a, and such that for all d disjoint from b and satisfying $c - d$, we have $(f_n) - d$.*

Proof Suppose $\{a, b, c\}$ is of type bouquet. Then the sequence $(f_n)_{n\in\mathbb{N}}$ can be constructed explicitly. Let $p = b \cap c$. Fix a (smooth) metric on Σ, we remove all points of the ball $B(p, 1/n)$ off the curves b and c, this gives two arcs. There is a natural way of adding smooth subarcs of $B(p, 1/n)$ in order to extend this union of two arcs, to a curve f_n which does not intersect a. In a one-holed torus neighborhood of $b \cup c$, with a choice of meridian and longitude coming from b and c, these curves f_n have slope 1, or -1; these are indeed nonseparating simple closed curves. Now if d is disjoint from b and satisfies $c - d$, then either d is disjoint from c, and then f_n is disjoint from d for all n large enough, or d has a transverse intersection with c at a point distinct from p, and we also have $f_n - d$ for all n large enough. Thus, (1) implies (2).

Conversely, suppose (2). We first claim that the sequence (f_n) then concentrates into neighborhoods of $b \cup c$. For contradiction, suppose that we can find a neighborhood V of $b \cup c$, such that $f_n \not\subset V$ for infinitely many n. Then, there exists a point x, with $x \notin b \cup c$, and such that every neighborhood of x meets infinitely many f_n. Then we may choose three bottle-shaped arcs around x, and complete them into curves d_1, d_2 and d_3 disjoint from b and satisfying $d_j - c$ for $j = 1, 2, 3$. Indeed, we may start with a curve d_0 obtained by pushing b aside, and then perform surgeries on d_0. The same reasoning as in the proof of Lemma 39 shows that $f_n \neq d_j$ for some $j \in \{1, 2, 3\}$ and for infinitely many n, contradicting the hypothesis (2).

Now, suppose for contradiction that $\{a, b, c\}$ is a necklace. Then, for a sufficiently small regular neighborhood V of $b \cup c$, we may observe that $V \smallsetminus a$ is contractible. Hence it cannot contain any nonseparating simple closed curve f_n, and the hypothesis (2) cannot be fulfilled. This proves that (2) implies (1). □

Now the proof of Theorem 2 is a straightforward adaptation of the proof of Theorem 1. The statements about connectedness of complexes of arcs, for example, are equivalent to their counterparts with regularity, because of the argument of homotopy recalled in the proof of Lemma 27 and borrowed from [2].

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