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On the profinite distinguishability of hyperbolic Dehn fillings of finite-volume 3–manifolds

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We use the Culler–Shalen machine and tools from model theory to study the profinite rigidity of residually finite groups, especially 3–manifold groups. We borrow a transfer principle from model theory to apply to $\mathbb{C}$–character varieties in order to study cofinite collections of $\mathbb{F}_p$–character varieties and prove that under certain finiteness conditions weaker than non-Hakenness, they all have the same (finite) cardinality. We prove that residually finite groups satisfying a niceness property are almost relatively profinitely distinguishable within a geometrically relevant class, and we finish up by applying that result to knot complements in S^3 in particular.

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1 Introduction

One of the foundational lines of research in modern geometry has been the study of 3–manifold topology; some relevant recent progress has been made primarily through teasing out differences between different manifolds through appeal to their fundamental groups; see for example Wise [34], Agol [1], Wilton and Zalesskii [32] and Przytycki and Wise [22].

It is well known that fundamental groups of finite-volume hyperbolic 3–manifolds are finitely presented, and additionally that they are residually finite; see Maltsev [18]. Thus it arises as a natural line of inquiry to try to distinguish the fundamental groups of finite-volume 3–manifolds by looking at their finite quotients. This leads us to ideas of profinite equivalence and distinguishability; compare Noskov, Remeslennikov and Romankov [21], Reid [23] and Wilton and Zalesskii [33], among many others. In particular, Wilton and Zalesskii [32, Theorem 8.4] show that the profinite completion of a geometric 3–manifold group determines its geometry, and further, in [33], they show that the profinite completion of a 3–manifold group also determines the JSJ decomposition of the manifold.

The study of finite-volume hyperbolic 3–manifolds is itself a central area within 3–manifold topology: among 3–manifolds, the possession of a hyperbolic structure is the "generic" case, as we see by Thurston's hyperbolic Dehn surgery theorem [30, Theorem 2.6], and which is explored in more formal and precise detail by Maher in [17]. We recall further that the study of Dehn fillings is fundamental to 3–manifold topology, and that the volumes of the Dehn fillings $M_{p/q}$ — notation which is defined later on in Definition 2.6 — of a hyperbolic 3–manifold M are strictly less than and converge to vol M,[1] by [10, Thurston's theorem]. Additionally, Mostow's rigidity theorem says that geometric invariants of a complete and finite-volume hyperbolic 3–manifold M, such as volume, are topological invariants, and thus give invariants of their fundamental groups. Accordingly, we have even further reason to suspect that the study of finite-volume hyperbolic 3–manifolds might be fruitful. Mostow rigidity implies that the fundamental groups $\pi_1(M_{p/q})$ are different, and it becomes a natural question whether the profinite completions of these groups are also distinguishable. This is a question that we answer in the affirmative in cofinitely many cases.

It is an open question as to whether all finite-volume hyperbolic 3–manifold groups are absolutely profinitely rigid, and furthermore, it is also currently unknown whether there exists some pair of nonisometric hyperbolic manifolds that are not profinitely distinguishable. In particular, we have the following theorem, proved in Section 4, with the precise definition of the notation $|\chi^I_{\mathbb{C}}(\Gamma)|$ we use for character varieties in the theorem below found in Definition 2.12:

Theorem A *Let Γ be any finitely generated residually finite group with $|\chi^I_{\mathbb{C}}(\Gamma)| < \infty$, and let M be an oriented finite-volume hyperbolic 3–manifold with a single cusp. Then $\Lambda = \pi_1(M_{m/n})$ has $\widehat{\Gamma} \ncong \widehat{\Lambda}$ for all but finitely many choices of orbifold surgery coefficient m/n with hyperbolic Dehn filling $M_{m/n}$.*

Our journey takes us through representation theory, as well: it turns out to be easier to look at the $\mathrm{SL}(2,k)$–representations of 3–manifold groups over careful choices of field k rather than at the groups themselves, and in turn at the character variety of a given representation rather than at the representation itself. By a result of Culler and Shalen [8, Proposition 1.5.2], which here is Proposition 2.13, a point in the character variety of a 3–manifold group picks out an irreducible representation up to conjugacy, and by a result of [8] — Theorem 2.14 here — the character variety of a non-Haken 3–manifold group is in fact finite.

The "special sauce" here is our use of model theory as described by Marker [19], employing a Lefschetzian transfer principle rather than algebraic geometry in order to transport statements between the zero- and positive-characteristic cases of algebraically closed fields. We carefully construct model-theoretic predicates which we use to represent matrices, representations, and even character varieties, permitting us to bring otherwise totally unfamiliar compactness results from logic to bear on questions more at home in geometric group theory. In particular, we follow Culler and Shalen in observing that the conjugacy classes of representations of $\mathrm{Hom}_I(\Gamma, \mathrm{SL}(2,k))$ correspond to the points of $V_\Gamma(k)$, and then noting that the defining equations for $V_\Gamma(k)$ arise from the defining relations for Γ, of which we may assume that there are only finitely many; this gives us a robust, mostly bidirectional link between definable sets and affine algebraic varieties.

Most importantly, this framework ensures that we can pass back and forth between $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\overline{\mathbb{F}}_p)$–representations as needed, which is crucial, as both have desirable properties: $\mathrm{SL}(2,\overline{\mathbb{F}}_p)$ is

[1] In fact, the order type of the set of all volumes of Dehn fillings of all 3–manifolds is ω^ω!

locally finite, which we find useful in controlling profinite extensions of maps in both Lemmas 2.19 and 4.6, but by contrast, representations into $\mathrm{SL}(2,\mathbb{C})$ are both much better understood and more directly connected to more concrete geometric applications. In particular, any finite-volume hyperbolic manifold corresponds naturally to a finite-covolume lattice within $\mathrm{(P)SL}(2,\mathbb{C})$, and this gives us a canonical representation. This model-theoretic approach allows us to give a much cleaner and more elementary proof of the following result than existing ones, and to prove a result like the one that follows it:

Theorem 1.1 *The equality $|\chi^I_{\mathbb{C}}(\Gamma)| = n$ holds if and only if $|\chi^I_p(\Gamma)| = n$ for cofinitely many p as well.*

Corollary 1.1.1 *Let M be a one-cusped, finite-volume, hyperbolic 3–manifold. Suppose $M_{m/n}$ is a hyperbolic Dehn filling of M with (orbifold) surgery coefficients m/n and with finite character variety (for instance, a non-Haken such filling), and let $\Gamma = \pi_1(M_{m/n})$. Let M_* be the set of all Dehn fillings $M_{m'/n'}$, and let Λ_* be the set of all fundamental groups of those manifolds. Then Γ is profinitely almost distinguishable within Λ_*.*

Acknowledgements We owe a debt of gratitude to Bridson, McReynolds, Reid and Spitler [5], who have used representation theory to think about profinite distinguishability of specific 3–manifold groups in a different way. Additionally, while this paper was still in thesis form, Liu's [14] appeared on the arXiv, and proves a more general version of Corollary 1.1.1 using sophisticated methods more closely hewing to orthodox geometric group theory, and which as a consequence concerns itself purely with 3–manifold groups; by contrast, our borrowing from model theory has the major advantage of being a more elementary approach, which also permits the study of more general objects.

We thank the referee for an extremely thorough referee report, greatly improving this paper, and for pointing out the applicability of several of our results to orbifold surgeries; Daniel Groves, who was the author's advisor when this paper was still a thesis and who provided copious editing feedback; and Alan Reid, who provided key insights for the strengthening of the main theorem to discuss more general residually finite groups, along with the initial proof sketch of how to extend it from the original result.

2 Preliminaries

2.1 Hyperbolic geometry and Dehn surgery

For a very readable treatment of foundational concepts in 3–manifold topology, see Hatcher [12].

Definition 2.1 Let M be a 3–manifold, S a compact surface properly embedded in M. Suppose there exists some disk $D \subseteq M$ such that $D \cap S = \partial D$, with the intersection being transverse. If ∂D bounds no disk in S, then we call D a *nontrivial compressing disk*, and S is a *compressible surface*. Otherwise, if S neither has a nontrivial compressing disk nor is an embedded 2–sphere, then S is an *incompressible surface*.

Definition 2.2 Let M be a connected 3–manifold. We call M *prime* if there exist no 3–manifolds $N_1, N_2 \not\cong S^3$ such that $N_1 \# N_2 \cong M$. We call M *irreducible* if every $S^2 \subseteq M$ bounds a 3–ball.

Lemma 2.3 [20, Lemma 1] *Let M be a prime 3–manifold. Then either M is irreducible or $M \cong S^2 \times S^1$.*

Definition 2.4 Let M be a compact, orientable, prime 3–manifold. Then we say that M is *Haken* if it has at least one properly embedded two-sided incompressible surface; if it has none, we call it *non-Haken*.

Definition 2.5 Let M be a connected Riemannian manifold. We say that M is a *hyperbolic 3–manifold* if it is complete and everywhere locally isometric to the hyperbolic 3–space $\mathbb{H}^3$.

We may note here that many knot complements $S^3 \setminus K$ have hyperbolic structure; a notable family of exceptions is the set of torus knots. For example, the figure-eight knot has a complement with hyperbolic structure; on the other hand, the trefoil knot does not. Additionally, we note (after Benedetti and Petronio [4, Chapter D]) that the boundary components of finite-volume hyperbolic 3–manifolds always comprise zero or more tori.

Definition 2.6 Let M be a 3–manifold such that ∂M consists of a single torus $T \cong S^1 \times S^1$, with $H_1(\partial T)$ generated by choices of longitude l and meridian m. For $p/q \in \mathbb{Q} \cup \infty$, with p coprime to q, the *Dehn filling of M along T with slope p/q* is given by $M \cup T'_{p/q}$ for T' a solid torus $T' \cong B^2 \times S^1$, where the union is a gluing along T such that the meridian of T' maps to a corresponding curve in ∂T homotopic to $q \cdot [m] + p \cdot [l]$, and where $T'_\infty = \varnothing$. We denote the resulting manifold by $M_{p/q}$.

Remark If $\gcd(p,q) \neq 1$, then we instead have the *orbifold Dehn surgery of M along T with slope p/q*. The result will be similar to the manifold case above, except that the nature of $T'_{p/q}$ is much more complex, in general depending primarily on $\gcd(p,q)$.

For a more complete description and characterization of manifold Dehn fillings as a special case of orbifold Dehn surgeries, we recommend referring to Cooper and Futer [7, Section 2.4] or to Thurston's notes [29].

Theorem 2.7 (one-cusp case of Thurston's hyperbolic Dehn surgery theorem [30, Theorem 2.6]) *Let M be a hyperbolic 3–manifold with a single cusp, and let $M(p/q)$ be the manifold obtained through applying a hyperbolic Dehn filling with surgery coefficient p/q to that cusp. Then if p/q differs from finitely many exceptional slopes, $M_{p/q}$ is also hyperbolic.*

Proposition 2.8 (adapted from the orbifold version of Thurston's hyperbolic Dehn surgery theorem [9, Theorem 5.3]) *Let M be a compact 3–manifold whose interior admits a complete hyperbolic structure such that ∂M consists of a single torus. Then there is a neighborhood U of ∞ in S^2 such that for all $p/q \in U$, the manifold $M_{p/q}$ also admits a hyperbolic structure.*

We note that the groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$ show up frequently in this paper. This is due to the fact that $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$: for an orientable finite volume hyperbolic 3–manifold M, $\Gamma = \pi_1(M)$, we can define M as $\mathbb{H}^3/\Gamma$, where Γ is a subgroup of $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$, so that since $\widetilde{M} = \mathbb{H}^3$, $\mathrm{PSL}(2,\mathbb{C})$ itself acts by deck transformations on M. What is more, the inclusion representation $\rho\colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ lifts to the discrete faithful representation $\hat{\rho}\colon \Gamma \to \mathrm{SL}(2,\mathbb{C})$, while other hyperbolic Dehn surgeries yield representations of Γ that are discrete but no longer faithful; for more on this, refer to MacLachlan and Reid in [16] on pages 111–112. In particular:

Proposition 2.9 *Let M be an orientable finite volume hyperbolic 3–manifold with a single cusp, with $\Gamma = \pi_1(M)$. Let $\Gamma_{p/q} = \pi_1(M_{p/q})$. Then the discrete faithful representation of $\Gamma_{p/q}$ into $\mathrm{SL}(2,\mathbb{C})$ is also a nonfaithful but still discrete $\mathrm{SL}(2,\mathbb{C})$–representation of Γ.*

2.2 Representation theory

Definition 2.10 Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$. The *trace field* of Γ, $\mathrm{TF}(\Gamma)$, is the field generated by all traces of all elements of Γ, which can also be written equivalently as $\mathbb{Q}(\operatorname{tr}\Gamma)$. The *degree* of the trace field is the degree of the extension of $\mathrm{TF}(\Gamma)$ over $\mathbb{Q}$.

We'll write $\mathbb{Q}(\operatorname{tr}\Gamma)$ whenever we want to emphasize the trace field's nature as a number field, and $\mathrm{TF}(\Gamma)$ otherwise.

Theorem 2.11 [16, Theorem 3.1.2] *Let M be a finite-volume orientable hyperbolic 3–manifold, so that $\Gamma = \pi_1(M)$ is Kleinian. Then $\mathbb{Q}(\operatorname{tr}\Gamma)$ is a finite-degree extension of $\mathbb{Q}$.*

Next, we define notation to be used for the sake of clarity and concision for the rest of the paper.

Definition 2.12 Let G be a group, and k a field. Let $\mathrm{Hom}_{\mathrm{Irr}}(G,\mathrm{SL}(2,k))$ be the set of irreducible representations of G in $\mathrm{SL}(2,k)$. We define

$$\chi_k^I(\Gamma) := \mathrm{Hom}_{\mathrm{Irr}}(\Gamma,\mathrm{SL}(2,k))/{\sim},$$

where $\sim$ denotes the conjugacy relation. In particular,

$$\chi_{\mathbb{C}}^I(\Gamma) := \mathrm{Hom}_{\mathrm{Irr}}(\Gamma,\mathrm{SL}(2,\mathbb{C}))/{\sim} \quad\text{and}\quad \chi_{\overline{\mathbb{F}}_p}^I(\Gamma) := \mathrm{Hom}_{\mathrm{Irr}}(\Gamma,\mathrm{SL}(2,\overline{\mathbb{F}}_p))/{\sim}.$$

This last we write as $\chi_p^I(\Gamma)$ for the sake of further concision.

Proposition 2.13 [8, Proposition 1.5.2] *Let Π be a finitely generated group, and let ρ and σ be representations of Π into $\mathrm{SL}(2,\mathbb{C})$ with corresponding characters χ_ρ and χ_σ. Let $\chi_\rho = \chi_\sigma$, and assume that ρ is an irreducible representation. Then ρ and σ are conjugate.*

Proposition 2.13 gives us a bijection between the set of irreducible characters, and the set of irreducible representations up to conjugacy. Since traces (and thus characters) are invariant under conjugation, trace is in fact a complete invariant of conjugacy classes of irreducible representations. It thus suffices to consider the trace field of any choice of character within each conjugacy class. Accordingly, we call $\chi^I_{\mathbb{C}}(\Gamma)$ the $\mathbb{C}$–character variety of Γ. A natural way to think about the image of the character is therefore to look at its trace field, and the degree of the trace field over $\mathbb{Q}$ is one of several natural measures of complexity. Keeping this framing in mind, we make use of the following corollary:

Corollary 2.13.1 [8, Corollary 1.4.5] *$\chi^I_{\mathbb{C}}(\Gamma)$ is a closed algebraic set.*

We remark that this corollary illuminates an important part of our approach: not only are we fine with reducible varieties, but we explicitly anticipate that our character varieties might be reducible, and that its irreducible components give us important information about the structure of $\mathrm{Hom}_{\mathrm{Irr}}(G, \mathrm{SL}(2,\mathbb{C}))$. In particular, the number of irreducible components (in fact, the number of points) is the key invariant.

The Culler–Shalen machine associates locally separating incompressible surfaces of a manifold to positive-dimensional components of its character variety; consequently whenever a manifold has a positive-dimensional character variety, we know it to be Haken. This gives us the following corollary, which is an important finiteness result:

Theorem 2.14 [25, Lemma 2.2] *Let M be a non-Haken 3–manifold of finite volume, with $\Gamma = \pi_1(M)$. Then $|\chi^I_{\mathbb{C}}(\Gamma)|$ is finite and Γ has finitely many representations into $\mathrm{SL}(2,\mathbb{C})$ up to conjugacy.*

2.3 Profinite groups

We now introduce the other aspect of geometric group theory that we make use of in this paper. For a more complete and formal treatment of the elementary characterizing properties of profinite completions of groups, we recommend reading through [23].

Keeping in mind the fact that any group can be made into a topological group by endowing it with the discrete topology, we remind the reader of the following definition:

Definition 2.15 A *profinite* group is a topological group isomorphic to some inverse limit of an inverse system of finite groups with the discrete topology. Equivalently, profinite groups are compact, Hausdorff, totally disconnected topological groups, as per Ribes and Zalesskii in [26, Theorem 1.1.12].

The *profinite completion* of a group G, which we denote by $\widehat{G}$, is that profinite group whose choice of finite groups is the set of all G/N, where N ranges over the normal subgroups of G of finite index, and the homomorphisms are given by the partial ordering of reverse containment of normal subgroups.

We care about the profinite completion of a residually finite group G primarily because it packages together all data on maps from G to its quotient groups of finite order.

We may think of residual finiteness as the capacity for at least one of the finite-index (normal) subgroups of G to tell an arbitrary $g \in G\backslash\{1\}$ apart from the identity, and we may think of the profinite completion of a group $\widehat{G}$ to be the packaging together of all of this finite-index information.

Lemma 2.16 [26, adapted from Lemma 1.1.7] *Let G be a group, with $\widehat{G}$ its profinite completion. Let $\iota\colon G \to \widehat{G}$ given by $g \mapsto ([g]_i)_{i\in I}$ be the canonical map sending g to the I–indexed tuple of equivalence classes of g under quotienting by the normal subgroups $\{N_i\}_{i\in I}$. Then ι has dense image.*

The following proposition is well-known: see for example Reid [24, Section 2.2] as well as Ribes and Zalesskii [26, page 78] .

Proposition 2.17 *Given a group G, denote by $\mathcal{N}(G)$ the set of all finite-index normal subgroups of G. Then the following are equivalent*:

- *G is residually finite.*
- $\displaystyle\bigcap_{N\in\mathcal{N}(G)} N = \{1\}$.
- *The natural homomorphism $\iota\colon G \to \widehat{G}$ is injective.*

Lemma 2.18 [26, Lemma 3.2.1, page 79] *This canonical map ι satisfies the universal property that for any profinite group H and group map $f\colon G \to H$, there exists a unique $g\colon \widehat{G} \to H$ such that $g \circ \iota = f$.*

Lemma 2.19 *The group $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$ is locally finite. Furthermore, for G a finitely generated group, $\rho\colon G \to \mathrm{SL}(2, \overline{\mathbb{F}}_p)$ a representation, $\operatorname{im}\rho$ is also finite, and ρ extends uniquely to a map $\widehat{\rho}\colon \widehat{G} \to \mathrm{SL}(2, \overline{\mathbb{F}}_p)$.*

Proof It suffices to show that every finitely generated subgroup H of $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$ has $|H|$ finite. To see this, we note that some generator of H must have an entry h such that the minimal $\mathbb{F}_{p^k}$ it belongs to is maximal among all entries of all generators of H, and that neither addition nor matrix multiplication can increase that k for any element of H; finally, every group of the form $\mathrm{SL}(2, \mathbb{F}_{p^k})$ is finite. □

Definition 2.20 We say that two groups G, H are *profinitely equivalent* if $\widehat{G} \cong \widehat{H}$.

Definition 2.21 Let G be a residually finite group, and S a set of groups. We say that a residually finite group G is *almost profinitely distinguishable* within S if there are at most finitely many residually finite groups $H \in S$ such that $\widehat{G} \cong \widehat{H}$, we have $G \cong H$.

3 Model theory and its uses

What could model theory be doing in a paper on geometric group theory and representation theory? Well, we use it as a means to prove the following theorem, whose proof can be found in Section 4.1:

Theorem 1.1 *The equality $|\chi^I_{\mathbb{C}}(\Gamma)| = n$ holds if and only if $|\chi^I_p(\Gamma)| = n$ for cofinitely many p as well.*

Where model theory comes in is its ability to permit us to pass back and forth between working over $\mathbb{C}$ and over cofinite collections of $\overline{\mathbb{F}}_p$ — this is an example of a transfer principle. For early examples of this, see Lefschetz [13], Weil [31] or Seidenberg [28]; for their context in a more general logical framework, see Ax [2] in his proof of the Ax–Grothendieck theorem, Barwise and Eklof [3], or Robinson and Tarski [27]; and for the earliest modern form of the principle as used here, see Chapter 2 of Cherlin [6]. Subsequently, the aim of Section 4.1 is to apply techniques as found in Marker [19, Section 1] to weld together *algebraic varieties over fields* and the *definable sets they coincide with*, and then use these techniques to interpret character varieties $\chi^I_{\mathbb{C}}(\Gamma)$ in terms of the first-order theory of $\mathbb{C}$, also called ACF_0.[2] We remark more formally on this in a remark in Section 4.1 but for now, the motivating slogan to keep in mind is this: "The relations of our groups are equally validly affine algebraic conditions, and conjugacy classes of representations correspond to points of the resulting variety".

In any case, if for whatever reason you want to blackbox this, you can simply use Theorem 1.1. Otherwise, however, we first need a few closely related definitions and a pair of results from model theory, which we detail later on in Section 4.1; for a background reference not only for the model-theoretic techniques we use here but also introductory model theory as a whole, refer to [19]. First, though, a few of the important basics of the language and notation of model theory:

Definition 3.1 The *arity* of a function or predicate is the number of arguments (zero or more) that that function or predicate accepts. For example, addition is a function of arity 2; we describe it as *being* 2*–ary* or as *having arity* 2. A nullary (0–ary) function is a constant, and a nullary predicate is either truth ($\top$) or falsehood ($\bot$).

The *alphabet* of first-order logic consists of the quantifiers $\forall, \exists$, the logical symbols $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$, disambiguating punctuation like parentheses and brackets, infinitely many variables (which we may notate as we choose, as long as it is clear that they are variables), the equals symbol $=$, any number of predicates of any arity, and any number of functions of any arity.

A *term*, which represents an object, is always either a variable or a function of any arity, and its arguments can themselves can be either variables or functions, to finite depth.

A *formula*, which represents a statement, is always finitely long, and consists of one or more predicates of any arity (including binary equality), possibly modified or joined by logical symbols, with zero or more of its variables quantified, or *bound*. An unbound variable is said to be *free*. A *sentence in first-order logic*, or $\mathscr{L}_1$–sentence, is a formula with no free variables.

A *signature* of a first-order theory T is the set of zero or more functions or relations, each of any arity, that we choose to represent important constants, functions, operations, and relations of our structure of interest.

A *first-order theory* is a set of axioms, which are sentences whose symbols come from the ordinary alphabet of first-order logic along with symbols from the theory's signature. A sentence S is said to *hold*

[2] This is not, strictly speaking, true, as we're about to see. However, $\mathbb{C}$ is, up to isomorphism, the unique algebraically closed field of characteristic 0, and transcendence degree over $\mathbb{Q}$ (and thus also cardinality) given by $\mathfrak{c}$.

in a first-order theory T, or equivalently that T *models* S, if S can be proven from the axioms of T. We use the symbol $\models$ to represent that, that is, $T \models S$.

A first-order theory T is said to be *consistent* if there exists no statement s for which $T \models s$ and also $T \models \sim s$, that is, no false statement can be proven from its axioms.

A first-order theory T is said to be *semantically complete*, or just *complete*, if for every statement s, $(T \models s) \rightarrow (T \vdash s)$, that is, every statement true within T is also provable in T. In fact, by Gödel's completeness theorem, every first-order theory is complete.

Definition 3.2 The *first-order theory of fields* has signature given by the constants $0, 1$ and the binary functions $+, \times$. It has the axioms that addition makes the set into an abelian group, multiplication is associative, commutative and distributive with identity 1, $\neg 0 = 1$, and removing 0 makes the set into an abelian group under multiplication.

Definition 3.3 The *first-order theory of algebraically closed fields*, ACF, extends the first-order theory of fields by appending countably many axioms, one for each natural number, each of the form that every nontrivial polynomial of degree n has at least one root.

Definition 3.4 The *first-order theory of algebraically closed fields of characteristic p*, ACF_p, extends ACF by appending the additional axiom that

$$\overbrace{1 + 1 + \cdots + 1}^{p \text{ copies of } 1} = 0.$$

Definition 3.5 The *first-order theory of algebraically closed fields of characteristic* 0, ACF_0, extends ACF by appending countably many axioms, one for each prime p, each of the form

$$\neg\, \overbrace{1 + 1 + \cdots + 1}^{p \text{ copies of } 1} = 0.$$

Remark ACF models all and only those statements true of (or in) every algebraically closed field. Because all of the ACF_p and ACF_0 both extend the axioms defining ACF, anything that ACF models, every ACF_p and ACF_0 also model, though the reverse need not be true — for instance, trivially

$$\mathrm{ACF}_5 \models 1 + 1 + 1 + 1 + 1 = 0,$$

but

$$\mathrm{ACF}, \mathrm{ACF}_0 \not\models 1 + 1 + 1 + 1 + 1 = 0.$$

Similarly, if ACF models a statement, then that statement also holds of any specific algebraically closed field, but the reverse need not be true.

Lemma 3.6 [19, Corollary 1.2] *Let* ACF_0 *be the first-order theory of algebraically closed fields of characteristic* 0, *and for rational prime* p, *let* ACF_p *be the first-order theory of algebraically complete fields of characteristic* p. *Let* Σ *be an* $\mathscr{L}_1$*–sentence. Then the following are equivalent*:

(i) $\mathrm{ACF}_0 \models \Sigma$.

(ii) $\mathrm{ACF}_p \models \Sigma$ *for cofinitely many choices of* p.

(iii) $\mathrm{ACF}_p \models \Sigma$ *for infinitely many choices of* p.

(iv) $\mathbb{C} \models \Sigma$.

Lemma 3.7 *Let* $S(k)$ *be a first-order statement in the theory of a field* k. *Then* $S(\overline{\mathbb{F}}_p)$ *holds for infinitely* (*in fact*, *cofinitely*) *many choices of* p *if and only if* $S(\mathbb{C})$ *holds.*

Proof Assume that $S(\mathbb{C})$ holds. By Lemma 3.6, this means that $\mathrm{ACF}_0 \models S$, and thus also that $\mathrm{ACF}_p \models S$ for cofinitely many p, that is, $S(\overline{\mathbb{F}}_p)$ holds for those p. On the other hand, assume that $S(\mathbb{C})$ does not hold. Then by Lemma 3.6 again we must have $\mathrm{ACF}_0 \models \neg S$, and thus also that $\mathrm{ACF}_p \models \neg S$ for cofinitely many p, that is, $S(\overline{\mathbb{F}}_p)$ does not hold for those p. □

4 Main result

4.1 More model theory

With the preliminaries well in hand, we can begin to discuss the specific way we apply tools from model theory to the study of profinite rigidity.

Whenever we want to leave the decision of which field we're working over until later, we will just write ACF. Fixing k to be some arbitrary algebraically closed field,[3] we start by looking at how we can use ACF to talk about matrices in $\mathrm{SL}(2,k)$. Let x_1, x_2, x_3, x_4 be variables in ACF. Then we define the predicate

$$M(x_1, x_2, x_3, x_4)$$

to be

$$x_1 \cdot x_4 - x_2 \cdot x_3 = 1.$$

The attentive reader may notice that this is exactly the defining relation for the determinant of a matrix in $\mathrm{M}(2,k)$ to be 1 in terms of its elements. More subtly, and perhaps more powerfully, one may note a tactic that will be used throughout this section: namely, that we will make our predicates complex and full of equations, so that they can do the heavy lifting that a mere abstract tuple cannot do. More simply, though, we bundle 4–tuples of variables that satisfy M and notate them as matrices $A \in \mathrm{SL}(2,\mathbb{C})$ unless we really do need access to the entries.

[3] This still works for k an arbitrary ring instead, but in that case, many of the properties below might be much weaker or have otherwise misleading names.

To write that a given matrix is the identity is actually even easier. We define the predicate

$$\mathrm{Id}(x_1, x_2, x_3, x_4)$$

to be

$$x_1 = 1 \wedge x_2 = 0 \wedge x_3 = 0 \wedge x_4 = 1.$$

We can also just write $\mathrm{Id}(A)$, when we have a 4–tuple as mentioned above.

Before we can look at how to extend our method for talking about matrices of $\mathrm{SL}(2,k)$ in ACF to a method for talking about representations $\Gamma \to \mathrm{SL}(2,k)$, we are going to need to be able to write predicates that verify that each relation of Γ is satisfied. Consider how, for some finitely presented group

$$\Gamma = \langle L | R \rangle,$$

one might describe a new representation $\rho\colon \Gamma \to \mathrm{SL}(2,k)$. It suffices to specify what the map does to a given choice of generators of Γ, $(l_i) \mapsto \rho(l_i)$. It is worth noting that this gives us a natural map $\mathrm{Hom}(\Gamma, \mathrm{SL}(2,k)) \to k^{4l}$ determined by which element of $\mathrm{SL}(2,k)$ that particular $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(2,k))$ sends the ordered l–tuple of generators of Γ to, interpreted by reading off the matrix entries. This is certainly injective — different elements of $\mathrm{Hom}(\Gamma, \mathrm{SL}(2,k))$ send at least one generator of Γ to different matrices. What [8] gives us is the conceptualization of the image as also some vanishing set $V_\Gamma(k)$, and then additionally, using the next few results (when we have proven them), that we can also can recognize which points of k^{4l} are in $V_\Gamma(k)$ and are thus true representations by understanding $V_\Gamma(k)$ as the vanishing set of the polynomial relations in R, along with the polynomials ensuring that the generators map to elements of $\mathrm{SL}(2,k)$. But to make use of all this, we have to tackle the challenge of how to communicate all of the machinery used here in ACF first. Bringing this to ACF, let $(A_i) := A_1, \ldots, A_l$ be 4–tuples such that

$$\bigwedge_{i=1}^{l} M(A_i).$$

That is, the vector in k^{4l} can be thought of more helpfully as a l–tuple of 4–tuples, each of which we conceptualize as a matrix. As such, we write the l–tuple $(A_i)_{i=1}^{l}$ as $\vec{A}$. A couple of lemmata on the relation between the entries of matrices and those of their products and inverses mean we can use ACF to talk about the satisfaction of relations:

Lemma 4.1 *Let $A, B \in \mathrm{SL}(2,k)$. Then the entries of AB and A^{-1} are polynomial in the entries of A and B.*

Corollary 4.1.1 *Let $F\langle L \rangle$ be the set of freely reduced words on some finite set of letters and their inverses, and let $l = |L|$ and $r \in F\langle L \rangle$. Let $f_r\colon k^{4l} \to k^4$ be the map treating successive 4–tuples of the argument as matrix elements of a generating set, interpreting concatenation as matrix multiplication, and inverses of letters as inverses of generators, to take r to its image under this choice of assignment. Then for all $\vec{z} \in k^{4l}$, $f_r(\vec{z})$ is polynomial in the z_i.*

In particular, we use this in the case where the z_i represent elements of $\mathrm{SL}(2,\mathbb{C})$.

Now that we have ensured that nothing goes horribly wrong when we talk about matrix inversions, matrix products, and free reduction in $\mathrm{SL}(2,k)$, we can now figure out how to use ACF to write that a relationship $r \in R$ is satisfied by some l–tuple of matrices. For $r \in R$, we define the predicate $\mathrm{SAT}_r(\vec{A})$ to be

$$\mathrm{Id}(f_r(\vec{A})).$$

That is, the f_r map reads off the word r and takes the l–tuple it spells out from the A_i to the identity, as is required by the fact that the $r \in R$ are the relations of Γ.

Perhaps the most important immediate idea of this section is that we can use these previous predicates to write a statement in first-order logic that says whether or not a given $4l$–tuple is a representation of Γ into $\mathrm{SL}(2,k)$. In particular, we define $\mathrm{REP}_\Gamma(\vec{A})$ to be

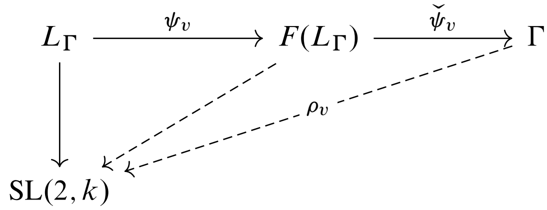
$$\Bigl(\bigwedge_{i=1}^{l} M(A_i)\Bigr) \wedge \Bigl(\bigwedge_{r\in R} \mathrm{SAT}_r(\vec{A})\Bigr),$$

where the Γ subscript reminds us that we started by fixing Γ and a presentation for Γ once and for all, and where the assumption of finite presentation ensures that the sentence is of finite length. Half-translating towards prose, this says that a $4l$–tuple A corresponds to a representation of Γ if all l of the 4–tuples are matrices of $\mathrm{SL}(2,k)$, and that all of the relations $r \in R$ of Γ are satisfied.

Proposition 4.2 [8, Corollary to Proposition 1.4.1] *Let Γ be a finitely presented group, with finite presentation $\Gamma = \langle L|R\rangle$, and let $V_\Gamma(k) = \{\vec{A} \mid \mathrm{REP}_\Gamma(\vec{A})\} = \{\vec{A} \mid V(f_r(\vec{A}) - I_2), M(A_i)_{i=1}^{l}\} \subseteq k^{4l}$. Then the map $\Phi_\Gamma\colon V_\Gamma \to \mathrm{Hom}(\Gamma, \mathrm{SL}(2,k))$ taking points in V_Γ to the representations that they determine is well-defined and is a bijection.*

Remark The Hilbert basis theorem permits us to assume finite generation without loss of generality, rather than the stronger condition of finite presentation, and, in proofs, to pass to the case of finite presentation rather than finite generation.

Proof We start by noting that the first part of REP_Γ uses the M predicate to check that the $4l$–tuple genuinely is an l–tuple of matrices in $\mathrm{SL}(2,k)$. Thinking once more of $\vec{A}$ as a $4l$–tuple v, this defines a map $\psi_v\colon L \to \mathrm{SL}(2,k), a_i \mapsto A_i$. Then by the universal property of free groups, this extends to a map $\check{\psi}_v\colon F(L) \to \mathrm{SL}(2,k)$ under multiplication and inverses:



Subsequently, the second part of REP_Γ uses each $\mathrm{SAT}(r)$ to check that each relation of Γ is satisfied, and by the universal property of groups defined by presentations, this uniquely determines a representation $\rho_v\colon \Gamma \to \mathrm{SL}(2,k)$. But then that means that $\Phi_\Gamma(v) = \rho_v$. □

Remark After [8, Section 1], we call $V_\Gamma(k) \subseteq k^{4l}$ the *definable k–points of the* $\mathrm{SL}(2,k)$*–representation variety of* Γ, or (trading precision for readability by abusing notation once again) simply the $\mathrm{SL}(2,k)$*–representation variety of* Γ. We justify this deliberate confusion as Culler and Shalen do: by observing that the conjugacy classes of representations of $\mathrm{Hom}_I(\Gamma, \mathrm{SL}(2,k))$ correspond to the points of $V_\Gamma(k)$ in the natural[4] way, and subsequently recalling that the defining equations for $V_\Gamma(k)$ arise from the defining relations for Γ, of which (by the Hilbert basis theorem) we may assume without loss of generality that there are only finitely many.

Now that we have established that we can talk about whether a given $4l$–tuple corresponds to a representation of Γ, we can talk about whether that representation is irreducible. As it turns out, though, it is much easier to start with reducibility. We recall that a representation into $\mathrm{SL}(2,k)$ is *reducible* if the action by all of the images of the generators on k^2 fix some line through the origin: more formally, for some generator-dependent $\lambda \in k$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ ranges over the images of all generators of Γ. We can therefore define the predicate $\mathrm{RED}(\vec{A})$ to be

$$\mathrm{REP}_\Gamma(\vec{A}) \wedge \exists a\, \exists b \bigwedge_{i=1}^{l} \exists \lambda_i : ((a,b) \neq (0,0)) \wedge A_i \cdot \langle a,b\rangle = \lambda_i \langle a,b\rangle,$$

where we treat $\langle a,b\rangle$ as a column vector, and matrix and scalar multiplication are accordingly appropriately defined.

Proposition 4.3 *For all $4l$–tuples (A_i), $\mathrm{ACF} \models \mathrm{RED}(\vec{A})$ if and only if $\Phi(\vec{A})$ is a reducible representation.*

We can then talk about irreducibility, defining the predicate $\mathrm{IRREP}(\vec{A})$ to be

$$\mathrm{REP}_G(\vec{A}) \wedge \neg\, \mathrm{RED}(\vec{A}).$$

With a little more work, we can also talk about whether two representations are conjugate. We recall that two representations $\rho, \sigma : \Gamma \to \mathrm{SL}(2,k)$ are conjugate if there exists some matrix M such that for all i, $M\rho(x_i)M^{-1} = \sigma(x_i)$, where x_i is the i^{th} generator of G under some fixed choice of ordering. We can represent this in model theory by defining the predicate $\mathrm{CONJ}(\vec{A}, \vec{B})$ to be

$$\mathrm{REP}_\Gamma(\vec{A}) \wedge \mathrm{REP}_\Gamma(\vec{B}) \wedge \left(\exists C : M(C) \wedge \bigwedge_{j=1}^{l} (CA_jC^{-1} = B_j)\right).$$

Remark If $\mathrm{ACF} \models \mathrm{CONJ}(\vec{A}, \vec{B})$, then $\mathrm{ACF} \models \mathrm{REP}_\Gamma(\vec{A})$ and $\mathrm{ACF} \models \mathrm{REP}_\Gamma(\vec{B})$, so that $\Phi(\vec{A})$ and $\Phi(\vec{B})$ are both defined.

[4]By taking each point in $V_\Gamma(k)$ to correspond to the representation whose images under the representation are exactly the successive 4–tuples of coordinates of that point.

Proposition 4.4 *For all pairs $\vec{A}, \vec{B}$ of $4l$–tuples*, $\mathrm{ACF} \models \mathrm{CONJ}(\vec{A}, \vec{B})$ *if and only if* $\Phi(\vec{A}) \sim \Phi(\vec{B})$, *that is*, *there exists some* $C \in \mathrm{SL}(2,k)$ *with* $\tau_C \circ \Phi(\vec{A}) = \Phi(\vec{B})$, *where* τ_C *is the inner automorphism of* $\mathrm{SL}(2,k)$ *that* C *defines.*

The whole point of this section, of course, was to be able to talk about the number of irreducible representations of a given finitely generated group G into $\mathrm{SL}(2,k)$, up to conjugacy. However, this certainly is not a proper sentence in first-order logic. We might think of the "half-translated" version of $\Sigma_{\Gamma,n}$ as the following:

> There exist n irreducible representations up to conjugacy of Γ into $\mathrm{SL}(2,k)$, they are not conjugate to each other, and any other irreducible representation of G into $\mathrm{SL}(2,k)$ must be conjugate to one of the n representations previously mentioned.

We now have all the tools we need to write the previously mentioned sentence $\Sigma_{\Gamma,n}$; this will be almost exactly a predicate-by-predicate, symbol-by-symbol calque of what was written above as the half-translation, making use of the predicates we have constructed here. We write the first-order sentence $\Sigma_{\Gamma,n}$ as

$$\bigwedge_{j=1}^{n} \exists \vec{A}^{(j)} : \mathrm{IRREP}(\vec{A}^{(j)}) \wedge \Bigg(\bigwedge_{\substack{j,j'=1 \\ j \neq j'}}^{n} \neg\, \mathrm{CONJ}(\vec{A}^{(j)}, \vec{A}^{(j')}) \Bigg) \wedge \forall \vec{B} : \Bigg(\mathrm{IRREP}(\vec{B}) \Rightarrow \bigvee_{j=1}^{n} \mathrm{CONJ}(\vec{B}, \vec{A}^{(j)}) \Bigg).$$

Theorem 4.5 *Let* Γ *be a finitely generated group. Then the above sentence*, $\Sigma_{\Gamma,n}$, *is a sentence in* ACF *saying that for all algebraically closed fields* k, $\Sigma_{\Gamma,n}(k)$ *is true if and only if* $|\chi_k^I(\Gamma)| = n$.

Proof Assume first that Γ is finitely presented.

($\Leftarrow$) Suppose that $|\chi_k^I(\Gamma)| = n$. Then by completeness of first-order logic, to verify that $\mathrm{ACF} \models \Sigma_{\Gamma,n}$, it suffices to carefully step through the sentence itself to verify its meaning. We cut the sentence $\Sigma_{\Gamma,n}$ into three parts along the two conjunctions. The first clause asserts that there exists some family of n tuples of appropriate length $\{\vec{A}^{(i)}\}_{j=1}^{n}$, each of which corresponds to an irreducible representation in itself. The second clause asserts that any distinct pair of those representations is nonconjugate. The final clause asserts that for all tuples $\vec{B}$ of the same length as the $\vec{A}^{(j)}$, if B corresponds to an irreducible representation, then that representation must be conjugate to the representation corresponding to one of the $\vec{A}^{(j)}$. Taken in sum, the sentence asserts that $|\chi_k^I(\Gamma)| = n$, and since we know it to be the case, ACF models it.

($\Rightarrow$) Suppose that $\mathrm{ACF} \models \Sigma_{\Gamma,n}$. Since from the previous part we know that $\Sigma_{\Gamma,n}$ asserts that $|\chi_k^I(\Gamma)| = n$ within ACF, by consistency of first-order logic we know that $|\chi_k^I(\Gamma)| = n$.

Now if Γ is merely finitely generated and not finitely presented, we must invoke the Hilbert basis theorem: using it, we have that there exists a finitely presented $\widetilde{\Gamma}$, $\eta \colon \widetilde{\Gamma} \to \Gamma$ such that the induced map $\eta^* \colon \chi_k^I(\Gamma) \to \chi_k^I(\widetilde{\Gamma})$ is a bijection. This $\widetilde{\Gamma}$ is generated by any generating set for Γ, and its relations are the finitely many relations that correspond to the finitely many equations that the Hilbert basis theorem gives us. Then the sentence $\Sigma_{\widetilde{\Gamma},n}$ for $\widetilde{\Gamma}$ also works for Γ. □

Theorem 1.1 *The equality $|\chi^I_{\mathbb{C}}(\Gamma)| = n$ holds if and only if $|\chi^I_p(\Gamma)| = n$ for cofinitely many p as well.*

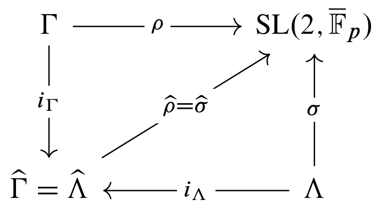
Proof This follows immediately from Lemma 3.6 and Theorem 4.5. We recall that we can use $\Sigma_{\Gamma,n}$ in ACF_0 and ACF_p to encode the finiteness property we care about. Using this, we can now apply transfer principles: since $|\chi^I_{\mathbb{C}}(\Gamma)| = n$, $|\chi^I_p(\Gamma)| = n$ for infinitely many p, and thus by Lemma 3.6, for cofinitely many p. □

4.2 Constraints on profinite completions

Now that we have established that we can relate (and thus constrain) information about representations into $\mathrm{SL}(2,\mathbb{C})$ and into $\mathrm{SL}(2,\overline{\mathbb{F}}_p)$, we can start to get a sense of what this means for the profinite distinguishability of groups.

Lemma 4.6 *Let Γ and Λ be two finitely generated groups such that $\widehat{\Gamma} \cong \widehat{\Lambda}$. Suppose that $|\chi^I_p(\Gamma)| = n$ for cofinitely many p. Then $|\chi^I_p(\Gamma)| = |\chi^I_p(\Lambda)| = n$ for those p.*

Proof Consider the commutative diagram



We note that by the universal property of profinite completions, any representation $\rho\colon \Gamma \to \mathrm{SL}(2,\overline{\mathbb{F}}_p)$ extends profinitely to a representation $\hat{\rho}\colon \widehat{\Gamma} \to \mathrm{SL}(2,\overline{\mathbb{F}}_p)$ by the local finiteness of $\mathrm{SL}(2,\overline{\mathbb{F}}_p)$, as in Lemma 2.19. By the commutativity of the diagram, any representation from $\chi^I_p(\Gamma)$ must factor as a composition of the canonical injection into $\widehat{\Gamma}$ and a representation from $\chi^I_p(\widehat{\Gamma})$, so that $|\chi^I_p(\Gamma)| = |\chi^I_p(\widehat{\Gamma})|$. Looking at the other half of the diagram, we note that since $\widehat{\Gamma} = \widehat{\Lambda}$, the same argument applies in reverse: compositions of the canonical injection of Λ into $\widehat{\Lambda}$ with representations from $\chi^I_p(\widehat{\Lambda})$ must yield all of $\chi^I_p(\Lambda)$, so that $|\chi^I_p(\Lambda)| = |\chi^I_p(\widehat{\Lambda})|$. □

Theorem 4.7 *Let Γ and Λ be two finitely generated groups with $\widehat{\Gamma} \cong \widehat{\Lambda}$ and $|\chi^I_{\mathbb{C}}(\Gamma)| = n$ for some $n \in \mathbb{N}$. Then $|\chi^I_{\mathbb{C}}(\Lambda)| = |\chi^I_{\mathbb{C}}(\Gamma)| = n$.*

Proof Since $|\chi^I_{\mathbb{C}}(\Gamma)| = n$, by Theorem 1.1 we know that $|\chi^I_p(\Gamma)| = n$ for cofinitely many p. By profinite equivalence and Lemma 4.6, we know that $|\chi^I_p(\Lambda)| = |\chi^I_p(\Gamma)| = n$. Finally, by another application of Theorem 1.1, $|\chi^I_{\mathbb{C}}(\Lambda)| = n$. □

Theorem 4.8 *Let $\Gamma = \pi_1(M)$, where M is a compact hyperbolic 3–manifold. If* $\deg(\mathrm{TF}(\Gamma)) \geq d$ *for some $d \in \mathbb{N}$, then $|\chi^I_{\mathbb{C}}(\Gamma)| \geq d$.*

Proof Let Γ be the fundamental group of a finite-volume hyperbolic 3–manifold. Let $k = \mathbb{Q}(\operatorname{tr}\Gamma)$ be its trace field, so that $A_0\Gamma = \left\{\sum_{i<n} a_i\gamma_i \mid n \in \mathbb{N}, a_i \in \mathbb{Q}(\operatorname{tr}\Gamma), \gamma_i \in \Gamma\right\}$ is the quaternion algebra over Γ that k generates. Let $\theta\colon \Gamma \hookrightarrow A_0\Gamma$ be the natural inclusion, let $\phi\colon A_0\Gamma \to \mathrm{SL}(1, A_0\Gamma)$ be any irreducible representation, and let $\{\sigma_i\}\colon k \hookrightarrow \mathbb{C}$ be a family of distinct embeddings of the trace field. Then we may extend the scalars of ϕ by tensoring over the different images of k, taking $A_0\Gamma \otimes_{\sigma_i(k)} \mathbb{C}$ and the corresponding $\widehat{\phi}_i\colon A_0\Gamma \to \mathrm{SL}(1, A_0\Gamma \otimes_{\sigma_i(k)} \mathbb{C})$; we know that $A_0\Gamma \otimes_{\sigma_i(k)} \mathbb{C}$ also extends a quaternion algebra over $\mathbb{C}$ and is thus split, so that $\mathrm{SL}(1, A_0\Gamma \otimes_{\sigma_i(k)} \mathbb{C})$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})$.

Then the result will follow if $\rho_i = \widehat{\phi}_i \circ \theta$ is a representation $\rho_i\colon \Gamma \to \mathrm{SL}(2,\mathbb{C})$, and if for $i \neq j$, ρ_i and ρ_j are nonconjugate.

To see this, recall that k is a number field by [16, Theorem 3.1.2], and that k, being the trace field of Γ, is generated by traces. Denote its degree over $\mathbb{Q}$ as $d = \deg(k)$, so that the $\{\sigma_i\}_{i=1}^d\colon k \hookrightarrow \mathbb{C}$ are its d embeddings. Then since the maps $\{\sigma_i\}\colon k \to \mathbb{C}$ are all different, and are all embeddings, they cannot agree on every $\gamma \in \Gamma$: there must exist some $\gamma \in \Gamma$ such that $\widehat{\sigma_i} \circ \theta(\gamma) \neq \widehat{\sigma_j} \circ \theta(\gamma)$.

But then given that $\rho_i = \widehat{\sigma_i} \circ \theta$ and $\rho_j = \widehat{\sigma_j} \circ \theta$, for $\operatorname{tr}\colon \mathrm{SL}(2,\mathbb{C}) \to \mathbb{C}$ the trace map, $\operatorname{tr} \circ \rho_i(\gamma) \neq \operatorname{tr} \circ \rho_j(\gamma)$. Thus γ represents a witnessing element of Γ on which the representations ρ_i and ρ_j have different traces, which by Proposition 2.13 tells us that the two representations cannot be conjugate. □

Having shown that profinite equivalence means that the number of representations up to conjugacy into $\mathrm{SL}(2,k)$ (if finite) are the same between the profinitely equivalent groups, the goal is now to attack the main theorem.

5 Main theorem

We need one last result from Long and Reid in [15].[5]

Theorem 5.1 [15, Theorem 3.2] *Let M be an orientable hyperbolic 3–manifold of finite volume and with a single cusp, and $d \in \mathbb{N}$. Then there are only finitely many surgery coefficients m/n such that* $\operatorname{tr}(\rho(\pi_1(M_{m/n})) \in k$, *where* $\deg(k/\mathbb{Q}) \leq d$ *as an extension.*

Theorem A *Let Γ be any finitely generated residually finite group with $|\chi^I_{\mathbb{C}}(\Gamma)| < \infty$, and let M be an oriented finite-volume hyperbolic 3–manifold with a single cusp. Then for all but finitely many choices of orbifold surgery coefficient m/n with hyperbolic Dehn filling $M_{m/n}$, $\Lambda = \pi_1(M_{m/n})$ has $\widehat{\Gamma} \not\cong \widehat{\Lambda}$.*

[5] We may remark that p, q need not be coprime; that is, orbifold surgeries are covered by the quoted result.

Proof It suffices to show that $\widehat{\Gamma} \cong \widehat{\Lambda}$ only for finitely many Λ; we thus assume that $\widehat{\Gamma} \cong \widehat{\Lambda}$. By assumption, $|\chi^I_{\mathbb{C}}(\Gamma)| < \infty$; let $|\chi^I_{\mathbb{C}}(\Gamma)| = d$. However, by Theorem 4.7, we know that since $\widehat{\Gamma} \cong \widehat{\Lambda}$, $|\chi^I_{\mathbb{C}}(\Gamma)| = |\chi^I_{\mathbb{C}}(\Lambda)|$. Now, by Theorem 5.1, we know that there are at most finitely many choices of surgery coefficient resulting in a manifold with degree of trace field of fundamental group with at most a given degree $d+1$, and by Theorem 4.8, we know that if $\deg(\mathrm{TF}(\Lambda)) > d$, then $|\chi^I_{\mathbb{C}}(\Lambda)| > d$ as well, so that it is exactly these finitely many choices of surgery coefficient where it is even possible for us to have $|\chi^I_{\mathbb{C}}(\Lambda)| = d$. Finally, we recall that by Proposition 2.13, irreducible representations with the same trace are always conjugate, so we know that it suffices to check that the characters of the two groups differ. □

The above result thus lends itself to the following more geometrically focused corollary:

Corollary 1.1.1 *Let M be a one-cusped, finite-volume, hyperbolic 3–manifold. Suppose $M_{m/n}$ is a hyperbolic Dehn filling of M with (orbifold) surgery coefficients m/n and with finite character variety (for instance, a non-Haken such filling), and let $\Gamma = \pi_1(M_{m/n})$. Let M_* be the set of all Dehn fillings $M_{m'/n'}$, and let Λ_* be the set of all fundamental groups of those manifolds. Then Γ is profinitely almost distinguishable within Λ_*.*

Proof We may start by passing without loss of generality to the case where $M_{m'/n'}$ has hyperbolic structure, thanks to [32, Theorems A and 8.4]. By assumption, $|\chi^I_{\mathbb{C}}(\Gamma)| < \infty$, so Theorem A applies. □

Remark Liu in [14] proves a more general version of this case using completely different methods.

We can extend this to the question that actually motivated this entire line of inquiry, with a little help from [11].

Definition 5.2 A knot K is *small* if its complement $S^3 \backslash K$ contains no closed incompressible surface.

Theorem 5.3 [11, unnumbered theorem from pages 373–374] *Let K be a small knot. Then all but finitely many of its Dehn fillings $M_{m/n} = \{S^3 \backslash K\}_{m/n}$ are non-Haken.*

This allows us to narrow in on the trailhead for this line of thought: that we can use all of this to say something interesting about hyperbolic Dehn fillings of small knots.

Corollary 5.3.1 *Let K be a small knot such that $S^3 \backslash K = M$ is a one-cusped, finite-volume, hyperbolic 3–manifold. Let $\Gamma = \pi_1(M_{m/n})$ be the fundamental group of a non-Haken hyperbolic Dehn filling of M with (orbifold) surgery coefficients m/n. Let M_* be the set of all Dehn fillings $M_{m'/n'}$, and let Λ_* be the set of all fundamental groups of those manifolds. Then Γ is profinitely almost distinguishable within Λ_*.*

Proof This follows from Theorems 2.14 and 5.3 as a special case of Theorem A. □

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