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Heegaard Floer homology, knotifications of links, and plane curves with noncuspidal singularities

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We describe a formula for the H_1–action on the knot Floer homology of knotifications of links in S^3. Using our results about knotifications, we are able to study complex curves with noncuspidal singularities, which were inaccessible using previous Heegaard Floer techniques. We focus on the case of a transverse double point, and give examples of complex curves of genus g which cannot be topologically deformed into a genus $g-1$ surface with a single double point.

14H50; 14B05, 57K18, 57R58

1 Introduction

1.1 General context

Let C be a complex curve in $\mathbb{C}P^2$. The curve C is called *rational* if C is irreducible and there exists a continuous degree one map from S^2 to C. The curve C is called *cuspidal* if all its singularities have one branch (ie their links have one component).

Fernandez de Bobadilla, Luengo, Melle-Hernandez and Némethi [Fernández de Bobadilla et al. 2006] indicated a connection between Seiberg–Witten invariants and rational cuspidal curves. As a consequence of these connections, they stated a conjecture binding coefficients of Alexander polynomials of singular points of a rational cuspidal curve. A variant of this conjecture was proved in [Borodzik and Livingston 2014]; the proof used the relation of semigroups of singular points with V_s–invariants of knots together with the Ozsváth–Szabó d–invariant inequality.

The methods of [Borodzik and Livingston 2014] were later generalized by Bodnár, Celoria and Golla [Bodnár et al. 2016] and Borodzik, Hedden and Livingston [Borodzik et al. 2017] to the case of nonrational cuspidal curves. Their result does not generalize immediately to the case where C has noncuspidal singularities. In this case, the boundary of a suitably defined tubular neighborhood of C can be presented as a surgery on a connected sum of links of cuspidal singularities and *knotifications* of links of noncuspidal singularities of C.

Knotification is an operation described by Ozsváth and Szabó [2008a], which transforms an n–component link L in S^3 into a knot $\widehat{L} \subset \#^{n-1} S^2 \times S^1$. The knot Floer homology $\mathrm{HFK}^-(\widehat{L})$ admits an action of the exterior algebra over $\mathbb{Z}$ on $n-1$ generators, which is identified with $\Lambda^* H_1\big(\#^{n-1} S^2 \times S^1\big)$. To apply the strategy of [Bodnár et al. 2016; Borodzik et al. 2017; Borodzik and Livingston 2014] to noncuspidal singularities, one must compute explicitly the action of $\Lambda^* H_1\big(\#^{n-1} S^2 \times S^1\big)$ on the knot Floer complex of the knotification. Performing explicit computations is often challenging, since computing the action of $\Lambda^* H_1\big(\#^{n-1} S^2 \times S^1\big)$ involves counting pseudoholomorphic curves in a symmetric product $\mathrm{Sym}^d(\Sigma)$ of a surface Σ in a Heegaard decomposition of $\#^{n-1} S^2 \times S^1$, which is used to compute the knot Floer complex. In this paper, we prove a general result which relates the homology action on the knotified link to counts of pseudoholomorphic curves on a Heegaard diagram for the original link in S^3. In many cases, this is more practical, since it allows us to compute pseudoholomorphic curves in a symmetric product of lower index d. For the links we consider, we are able to reduce the computations to $\mathrm{Sym}^1(S^2)$, which is completely combinatorial.

1.2 Main results

Given an n–component link $L \subset S^3$, we use Heegaard Floer TQFT to recover the knot Floer complex of the knotification $\widehat{L}$ of L together with the action of $\Lambda^* H_1\big(\#^{n-1} S^2 \times S^1\big)$ on it. This result builds on recent developments in the Heegaard Floer TQFT due to the third author as well as many others; see [Hendricks et al. 2018; Juhász 2016; Zemke 2015; 2017; 2019c; 2019b]. Our main result concerning knotifications is Proposition 2.10, which describes the action of $\Lambda^* H_1\big(\#^{n-1} S^2 \times S^1\big)$ on the knot Floer homology of a knotification in terms of a link diagram for L.

Using this general result, we compute the knot Floer complexes of the knotifications of the $(2, 2n)$–torus link and of its mirror, as well as the action of $H_1(S^2 \times S^1)$. In particular, we are able to compute the invariants V_s^{bot} and V_s^{top} of these knots. To the best of our knowledge, these computations have not appeared in the literature before. For the reader's convenience, we present the precise result for the knotification of the torus link $T_{2,2n}$. For more details about its mirror, see Proposition 2.41.

Proposition 2.40 *Let $\widehat{T}_{2,2n}$ be the knotification of the torus link $T_{2,2n}$. The pair*

$$(\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,2n}), A_\gamma)$$

has a model where $\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,2n})$ is equal to $\mathcal{S}^n\{\frac{1}{2}, \frac{1}{2}\} \oplus \mathcal{S}^{n-1}\{-\frac{1}{2}, -\frac{1}{2}\}$ and A_γ maps $\mathcal{S}^n$ to $\mathcal{S}^{n-1}$ on the chain level. Here we recall that $\{i, j\}$ denotes a shift in the $(\mathrm{gr}_w, \mathrm{gr}_z)$–grading by (i, j), and $\mathcal{S}^n$ and $\mathcal{S}^{n-1}$ are the chain complexes in Definition 2.28.

Our main application is concerned with general curves in $\mathbb{C}P^2$. To generalize the results of [Bodnár et al. 2016; Borodzik et al. 2017] to the setting of complex curves $C \subset \mathbb{C}P^2$ with noncuspidal singularities, we take a precisely defined "tubular" neighborhood N of C. The boundary $Y = \partial N$ can be described as a surgery on a link L in $\#^\rho\, S^2 \times S^1$, where L is a suitable connected sum of knotifications of links of singularities and Borromean knots, and ρ can be expressed in terms of the topology of C. As in [Bodnár et al. 2016; Borodzik et al. 2017], the manifold Y bounds a four-manifold $X = \mathbb{C}P^2 \setminus N$ with trivial intersection form. Using Ozsváth and Szabó's d–invariant inequality in the version proved by Levine and Ruberman [2014], we obtain restrictions on $V_s^{\mathrm{top}}(L)$ and $V_s^{\mathrm{bot}}(L)$.

The main case we focus on is curves C with some finite number of cuspidal singularities as well as singularities whose links are $(2, 2n)$–torus links. We obtain the following result:

Theorem 6.4 *Let C be a reduced curve of degree d and genus g. Suppose that C has cuspidal singular points $p_1, \dots, p_\nu$ whose semigroup counting functions are $R_1, \dots, R_\nu$, respectively. Assume that, apart from these ν points, the curve C has, for each $n \geq 1$, $m_n \geq 0$ singular points whose links are $(2, 2n)$–torus links and no other singularities. Define*

$$\eta_+ = \sum_{n=1}^{\infty} m_n \quad \text{and} \quad \kappa_+ = \sum_{n=1}^{\infty} n m_n.$$

For any $k = 1, \dots, d-2$,

$$\max_{0 \leq j \leq g} \min_{0 \leq i \leq \kappa_+ - \eta_+} (R(kd + 1 - \eta_+ - 2i - 2j) + i + j) \leq \tfrac{1}{2}(k+1)(k+2) + g,$$

$$\min_{0 \leq j \leq g + \kappa_+} (R(kd + 1 - 2j) + j) \geq \tfrac{1}{2}(k+1)(k+2).$$

Here R denotes the infimal convolution of the functions $R_1, \dots, R_\nu$.

Although complex curves cannot have singularities whose links are (nonalgebraic) $(2, -2n)$–torus links, our techniques also obstruct smooth (nonalgebraic) surfaces with these singularities. See Theorem 6.8.

The technical statement in Theorem 6.4 is best understood by comparing the obstruction in the case of a single transverse double point to the genus $g = 1$ obstruction from [Bodnár et al. 2016; Borodzik et al. 2017]. We do this in Proposition 6.14, which we now summarize. Let C be a degree d curve, and define the quantity $\upsilon_k = \frac{1}{2}(k+1)(k+2)$ for $k = 1, \dots, d-2$. Write R for the semigroup counting function.

If C has genus 1, then the genus bound from [Bodnár et al. 2016; Borodzik et al. 2017] implies that, for each $k \in \{1, \dots, d-2\}$,

$$R(kd-1) \in \{\upsilon_k - 1, \upsilon_k\} \quad \text{and} \quad R(kd+1) \in \{\upsilon_k, \upsilon_k + 1\}. \tag{1.1}$$

In this case, the only constraint on $R(kd)$ is that it lies between $R(kd-1)$ and $R(kd+1)$, and hence $R(kd) \in \{\upsilon_k - 1, \upsilon_k, \upsilon_k + 1\}$.

On the other hand, our bounds from Theorems 6.4 and 6.8 give a slightly stronger obstruction than the bound for genus 1 curves in (1.1), based on the value of $R(kd)$. Since double points may be smoothed topologically, (1.1) must also hold for genus 0 curves C with a single double point. If C is a genus 0 curve with a single positive double point, then our bound implies

$$R(kd) \le \upsilon_k.$$

If instead C is a smooth curve with a negative double point, then we prove that $R(kd) \ge \upsilon_k$.

We compare our obstruction with known examples, focusing on the question of deforming a genus 1 surface into a surface with one double point. In Section 6.5 we provide concrete obstructions. For existing curves (ie curves that we can construct), there are obstructions to trading genus for negative double points; see Example 6.15.

We also compare our obstruction to the obstruction for genus 1 curves from [Borodzik et al. 2017]. In [loc. cit., Theorem 9.1], there is a list of genus one curves with a singularity whose link is the (p,q)–torus knot with p and q coprime. The curves in the list pass the obstruction provided in [loc. cit.], but it is not known whether these complex curves exist. We apply our bound to this list of potential examples. There is a remarkable case of a degree 27 curve with a $(10,73)$ singularity, where the genus cannot be traded for either a positive or a negative double point; see Table 1. While the curve passes all known criteria, we do not have a recipe to construct it.

1.3 Further applications and perspectives

There has been recent interest in the question of "trading genus for double points". To be more precise, given a surface of genus g, one can ask whether it is possible to deform it to a genus $g-1$ surface with an extra positive or negative double point. In the context of the surfaces in a four-ball with fixed boundaries, this question is related to studying the difference between the clasp number and the smooth four-ball genus; see [Daemi and Scaduto 2024; Feller and Park 2022; Juhász and Zemke 2020; Kronheimer and Mrowka 2021; Owens and Strle 2016]. We deal with a variation of this question, which concerns trading genus of a closed surface in $\mathbb{C}P^2$ for double points, while preserving the remaining singularities.

In Section 6.6, we consider another infinite family of higher genus curves constructed by Bodnár, Celoria and Golla. We show that the genus cannot be traded for a negative double point for any member of the family.

As a perspective and a possibility for future research, we indicate that the methods can be used to study line arrangements in $\mathbb{C}P^2$. The only missing ingredient is the computation of the Heegaard Floer chain complex of the (d,d)–torus link for $d > 2$, and understanding the H_1–action on the knotification of these links.

Organization

Section 2 reviews Heegaard Floer theory. After recalling various known definitions and results, we show how to obtain the knot Floer chain complex of the knotification of links, as well as the H_1/Tors–action. A detailed construction of the Heegaard Floer chain complex of the Hopf link is presented in Section 2.5. The generalization to knotifications of arbitrary $(2,2n)$–torus links is given in Section 2.6. We conclude Section 2 with Section 2.7, where we recall the computations of the Heegaard Floer chain complex of the Borromean knot $\mathcal{B}_0$.

Section 3 is devoted to a detailed study of correction terms. We recall the Levine–Ruberman versions of d–invariants and recall definitions of V_s–invariants.

Section 4 contains some important computations that happen behind a scene. We recall the computation of the Heegaard Floer chain complex of L–space knots, and in particular of algebraic knots, in Section 4.2. We show how to recover the V_s–invariant of a product of positive and negative staircases. A precise statement is given in Proposition 4.18. We show that the assumptions in the second item of that proposition are necessary in Section 4.4.

Next we consider tensor products of knot Floer chain complexes in manifolds with $b_1 > 0$. It turns out that most of the knots that we encounter share a property, which greatly facilitates our computations, namely they have *split towers*; see Definition 4.29.

Section 5 constructs a tubular neighborhood N of a singular curve and presents the boundary Y of this neighborhood as a surgery on a link L in $\#^{\rho}\, S^2 \times S^1$, where ρ is the first Betti number of C. We then compute homological invariants of Y, N and $\mathbb{C}P^2 \setminus N$. In particular, we study which Spin^c structures on Y extend over $\mathbb{C}P^2 \setminus N$. These computations are slight generalizations of calculations of [Bodnár et al. 2016; Borodzik et al. 2017; Borodzik and Livingston 2014].

Section 6 contains the proofs of Theorems 6.4 and 6.8. The main technical result is Proposition 6.3, which computes the d–invariants of Y in terms of the semigroup counting functions of knots of cuspidal singularities. We also compare Theorems 6.4 and 6.8 with bounds for cuspidal curves of higher genus in Section 6.4. Sections 6.5 and 6.6 provide explicit examples of curves for which our obstruction can be applied.

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2 Review of Heegaard Floer theory

2.1 Heegaard Floer complexes with multiple basepoints

Definition 2.1 A *multipointed Heegaard diagram* for a 3–manifold Y is a quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ where:

- Σ is a genus g surface, which splits Y into two genus g handlebodies, U_α and U_β, and $\boldsymbol{w} = (w_1, \dots, w_n)$ is a nonempty set of basepoints in Σ.
- $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{g+n-1})$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{g+n-1})$ are collections of simple closed curves on Σ, where $n = |\boldsymbol{w}|$. Each curve in $\boldsymbol{\alpha}$ bounds a compressing disk in U_α, and each curve in $\boldsymbol{\beta}$ bounds a compressing disk in U_β. Furthermore, the curves in $\boldsymbol{\alpha}$ are pairwise disjoint, and similarly for $\boldsymbol{\beta}$.
- The curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are transverse.
- The curves in $\boldsymbol{\alpha}$ are linearly independent in $H_1(\Sigma \setminus \boldsymbol{w})$, and similarly for $\boldsymbol{\beta}$.

Let $\mathbb{T}_\alpha, \mathbb{T}_\beta \subset \mathrm{Sym}^{g+n-1}(\Sigma)$ be two half-dimensional tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+n-1} \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+n-1}.$$

Ozsváth and Szabó [2004b, Section 2.6] describe a map

$$\mathfrak{s}_{\boldsymbol{w}} \colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y).$$

Given a Heegaard diagram of Y with a Spin^c structure $\mathfrak{s}$, we define a Floer chain complex $\mathrm{CF}^-(Y, \boldsymbol{w}, \mathfrak{s})$ over $\mathbb{F}[U_1, \dots, U_n]$, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. The chain complex is generated over $\mathbb{F}[U_1, \dots, U_n]$ by intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ satisfying $\mathfrak{s}_{\boldsymbol{w}}(\boldsymbol{x}) = \mathfrak{s}$.

For any $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the differential is defined by

$$\partial \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathbb{R}) U_1^{n_{w_1}(\phi)} \cdots U_n^{n_{w_n}(\phi)} \boldsymbol{y}. \tag{2.2}$$

Here, $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ denotes the set of homotopy classes of maps of a complex unit disk $\mathbb{D}$ to $\mathrm{Sym}^{g+n-1}(\Sigma)$ such that point $-i$ is mapped to $\boldsymbol{x}$, the point i is mapped to $\boldsymbol{y}$, $\partial\mathbb{D} \cap \{\mathrm{Re}(z) < 0\}$ is mapped to $\mathbb{T}_\beta$ and $\partial\mathbb{D} \cap \{\mathrm{Re}(z) > 0\}$ is mapped to $\mathbb{T}_\alpha$. The quantity $\mu(\phi)$ is the Maslov index of the disk. The space $\mathcal{M}(\phi)$ is the moduli space of J_s–holomorphic disks representing ϕ (for some 1–parameter family of almost complex structures J_s on $\mathrm{Sym}^{g+n-1}(\Sigma)$). The condition that $\mu(\phi)=1$ implies that $\mathcal{M}(\phi)/\mathbb{R}$ is generically a finite set of points. The integers $n_{w_i}(\phi)$ are intersection numbers of $\{w_i\}\times\mathrm{Sym}^{g+n-2}(\Sigma)\subset\mathrm{Sym}^{g+n-1}(\Sigma)$ with the image of ϕ.

The homology group $\mathrm{HF}^-(Y,\boldsymbol{w},\mathfrak{s})$ of $\mathrm{CF}^-(Y,\boldsymbol{w},\mathfrak{s})$ has the structure of an $\mathbb{F}[U_1,\dots,U_n]$–module.

If $c_1(\mathfrak{s})$ is torsion, then $\mathrm{CF}^-(Y,\boldsymbol{w},\mathfrak{s})$ admits an absolute $\mathbb{Q}$–valued grading, which we denote by gr_{w}. The differential decreases the grading by 1, so the grading descends to $\mathrm{HF}^-(Y,\boldsymbol{w},\mathfrak{s})$. Multiplication by any of the U_i decreases the grading by -2.

Formally inverting the variables $U_1,\dots,U_n$ in $\mathrm{CF}^-(Y,\boldsymbol{w},\mathfrak{s})$ gives a chain complex $\mathrm{CF}^\infty(Y,\boldsymbol{w},\mathfrak{s})$ over $\mathbb{F}[U_1,U_1^{-1},\dots,U_n,U_n^{-1}]$. The associated homology group is denoted by $\mathrm{HF}^\infty(Y,\boldsymbol{w},\mathfrak{s})$.

2.2 The link Floer complex

For links in S^3, Ozsváth and Szabó [2008a] introduced the link Floer homology, which generalizes the knot Floer homology defined separately in [Rasmussen 2003; Ozsváth and Szabó 2004a]. We presently recall their construction.

Definition 2.3 An *oriented multipointed link* $\mathbb{L}=(L,\boldsymbol{w},\boldsymbol{z})$ in a closed 3–manifold Y is an oriented link L with two disjoint collections of basepoints $\boldsymbol{w}=\{w_1,\dots,w_n\}$ and $\boldsymbol{z}=\{z_1,\dots,z_n\}$ such that, as one traverses L, the basepoints alternate between $\boldsymbol{w}$ and $\boldsymbol{z}$. Furthermore, each component of L has a positive (necessarily even) number of basepoints, and each component of Y contains at least one component of L.

Analogously to Definition 2.1, we have the following:

Definition 2.4 A *multipointed Heegaard link diagram* for $\mathbb{L}=(L,\boldsymbol{w},\boldsymbol{z})$ in Y is a tuple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{z})$ satisfying the following:

- $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{w})$ and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z})$ are embedded Heegaard diagrams for $(Y,\boldsymbol{w})$ and $(Y,\boldsymbol{z})$, respectively, in the sense of Definition 2.1.
- $L\cap\Sigma=\boldsymbol{w}\cup\boldsymbol{z}$, and furthermore L intersects Σ positively at $\boldsymbol{z}$ and negatively at $\boldsymbol{w}$.
- $L\cap U_\alpha$ (resp. $L\cap U_\beta$) is a boundary-parallel tangle in U_α (resp. U_β).

Given a multipointed Heegaard link diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{z})$ for $(Y,\mathbb{L})$, the *link Floer chain complex* is defined as follows. Let

$$\mathscr{R}^-=\mathbb{F}[\mathscr{U},\mathscr{V}],\qquad \mathscr{R}^\infty=\mathbb{F}[\mathscr{U},\mathscr{U}^{-1},\mathscr{V},\mathscr{V}^{-1}].$$

Let $\mathfrak{s}$ be a Spin^c structure on Y. We define the chain complex $\mathcal{CFL}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z}, \mathfrak{s})$ to be the free $\mathscr{R}^-$–module generated by $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_{\boldsymbol{w}}(\boldsymbol{x}) = \mathfrak{s}$. The differential is given by

$$\partial \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \mathscr{U}^{n_{w_1}(\phi)+\cdots+n_{w_n}(\phi)} \mathscr{V}^{n_{z_1}(\phi)+\cdots+n_{z_n}(\phi)} \cdot \boldsymbol{y}, \tag{2.5}$$

extended $\mathscr{R}^-$–equivariantly. The differential ∂ squares to 0.

There is a larger version of the link Floer complex, which we call the *full link Floer complex*, denoted by $\mathcal{CFL}^-_{\mathrm{full}}(Y, \mathbb{L}, \mathfrak{s})$. As a module, $\mathcal{CFL}^-_{\mathrm{full}}(Y, \mathbb{L}, \mathfrak{s})$ is freely generated over the ring $\mathbb{F}[\mathscr{U}_1, \dots, \mathscr{U}_n, \mathscr{V}_1, \dots, \mathscr{V}_n]$ by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The differential is similar to (2.5), except we use the weight $n_{w_i}(\phi)$ for the variable $\mathscr{U}_i$, and the weight of $n_{z_i}(\phi)$ for the variable $\mathscr{V}_i$. In general, $\mathcal{CFL}^-_{\mathrm{full}}(Y, \mathbb{L}, \mathfrak{s})$ is a *curved chain complex*, ie $\partial^2 = \omega_{\mathbb{L}} \cdot \mathrm{id}$ for some $\omega_{\mathbb{L}} \in \mathbb{F}[\mathscr{U}_1, \dots, \mathscr{U}_n, \mathscr{V}_1, \dots, \mathscr{V}_n]$; see [Zemke 2017, Lemma 2.1].

2.3 Homological actions

Ozsváth and Szabó [2004b, Section 4.2.5] describe an action of $\Lambda^*(H_1(Y)/\mathrm{Tors})$ on the homology group $\mathrm{HF}^-(Y, \boldsymbol{w}, \mathfrak{s})$. For a multipointed 3–manifold $(Y, \boldsymbol{w})$, there is an analogous action of the relative homology group $H_1(Y, \boldsymbol{w})$ on $\mathrm{CF}^-(Y, \boldsymbol{w}, \mathfrak{s})$ [Zemke 2015]. In this section, we recall the construction and describe some analogs on link Floer homology.

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ is a multipointed Heegaard diagram, and λ is a path which connects two distinct basepoints $w_1, w_2 \in \boldsymbol{w}$, then there is a *relative homology action* A_λ, which is an endomorphism of $\mathrm{CF}^-(Y, \boldsymbol{w}, \mathfrak{s})$ and satisfies

$$A_\lambda \partial + \partial A_\lambda = U_1 + U_2. \tag{2.6}$$

See [Zemke 2015, Lemma 5.1].

The map A_λ is defined via the formula

$$A_\lambda(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi)=1}} a(\lambda, \phi) \#(\mathcal{M}(\phi)/\mathbb{R}) U_1^{n_{w_1}(\phi)} \cdots U_n^{n_{w_n}(\phi)} \cdot \boldsymbol{y}. \tag{2.7}$$

Here $a(\lambda, \phi) \in \mathbb{F}$ is a quantity determined as follows. Homotope the path λ so that it is an immersed curve in Σ, transverse to the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ curves. We write $D(\phi)$ for the *domain* of the class ϕ, which is a 2–chain on Σ with boundary in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$. We write $\partial D(\phi) = \partial_\alpha(\phi) + \partial_\beta(\phi)$. Then we set $a(\lambda, \phi) = \#(\partial_\alpha(\phi) \cap \lambda)$. Compare [Zemke 2015, Section 5.1]. Up to chain homotopy, the map A_λ only depends on the relative homology class of λ in Y, relative to its boundary. In particular, the map A_λ does not depend on the choice of representative on the surface Σ. See [Ni 2014, Lemma 2.4] for a proof in a related context, or [Zemke 2015, Lemma 5.6] for a similar proof in the present context.

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ is a multipointed Heegaard link diagram, and λ connects two basepoints w_1 and w_2, there is an analogous map A_λ on the link Floer homology. In contrast to (2.6), we have

$$A_\lambda \partial + \partial A_\lambda = \mathscr{U}_1 \mathscr{V}_1 + \mathscr{U}_2 \mathscr{V}_2, \tag{2.8}$$

where $\mathscr{V}_1$ denotes the variable for the basepoint z_1 which immediately follows w_1 with respect to the ordering of basepoints on the link, and similarly $\mathscr{V}_2$ is the variable for the basepoint z_2 which immediately follows w_2. The proof follows the same strategy as [Zemke 2015, Lemma 5.1]: One counts the ends of index 2 families of holomorphic disks. There are two types of ends: pairs of index 1 holomorphic disks as well as index 2 boundary degenerations. Pairs of index 1 holomorphic disks contribute to the left-hand side of (2.8), while the count of boundary degenerations, weighted by $a(\lambda, \phi)$, constitutes the right-hand side.

If $z_i \in \boldsymbol{z}$, then there is an endomorphism of $\mathcal{CFL}^-_{\mathrm{full}}(Y, \mathbb{L}, \mathfrak{s})$ defined by

$$\Psi_{z_i}(\boldsymbol{x}) = \mathscr{V}_i^{-1} \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi)=1}} n_{z_i}(\phi) \#(\mathcal{M}(\phi)/\mathbb{R}) \mathscr{U}_1^{n_{w_1}(\phi)} \cdots \mathscr{U}_n^{n_{w_n}(\phi)} \mathscr{V}_1^{n_{z_1}(\phi)} \cdots \mathscr{V}_n^{n_{z_n}(\phi)} \cdot \boldsymbol{y}.$$

We call Ψ_{z_i} the *basepoint action* of z_i. Note that, since the contribution of each disk class ϕ is multiplied by $n_{z_i}(\phi)$ in the sum, the additional factor of $\mathscr{V}_i^{-1}$ never results in negative powers of $\mathscr{V}_i$, and hence the formula induces a well-defined endomorphism of $\mathcal{CFL}^-_{\mathrm{full}}(Y, \mathbb{L}, \mathfrak{s})$.

Given $w_i \in \boldsymbol{w}$, there is an analogous endomorphism Φ_{w_i}. The map Ψ_{z_i} satisfies

$$\Psi_{z_i} \partial + \partial \Psi_{z_i} = \mathscr{U}_j + \mathscr{U}_{j+1},$$

where w_j and w_{j+1} are the $\boldsymbol{w}$ basepoints adjacent to z_i on the link. In particular, if we identify all of the $\mathscr{U}_i$ variables to a single $\mathscr{U}$, then Ψ_{z_i} is a chain map. See [Sarkar 2011, Lemma 4.1] or [Zemke 2017, Lemma 3.1]. Similarly, if z_i is on a link component which has only one other basepoint, then Ψ_{z_i} is also a chain map.

2.4 Heegaard Floer homology of a knotification

Definition 2.9 (knotification) Let $\mathcal{L} = L_1 \cup \cdots \cup L_n$ be a null-homologous link in a 3–manifold Y.

(1) A *partial knotification* of $\mathcal{L}$ with respect to components L_i and L_j is a $(n-1)$–component null-homologous link $\mathcal{L}_{ij}$ in $Y \# S^2 \times S^1$ obtained by connecting L_i and L_j with an oriented band going across the $S^2 \times S^1$ summand.

(2) A *knotification* of $\mathcal{L}$ is a knot $\widehat{\mathcal{L}}$ in $Y \# \#^{n-1} S^2 \times S^1$ obtained by consecutive partial knotifications.

The isotopy type $\widehat{\mathcal{L}}$ does not depend on the feet of the bands [Ozsváth and Szabó 2004a, Proposition 2.1].

Suppose $\mathbb{L} = (\mathcal{L}, \boldsymbol{w}, \boldsymbol{z})$ is an n–component link in $\#^m S^2 \times S^1$, equipped with $2n$ basepoints, and $\mathbb{L}'$ is a multipointed link in $\#^{m+1} S^2 \times S^1$, obtained by knotifying the components L_{n-1} and L_n of $\mathcal{L}$.

Furthermore, we assume that the basepoints on the link components $L_1, \dots, L_{n-2}$ are unchanged in $\mathbb{L}'$, and on L'_{n-1} we have only the two basepoints w_n and z_{n-1}. There are two natural maps

$$F : \mathcal{CFL}^-(\#^m S^2 \times S^1, \mathbb{L}) \to \mathcal{CFL}^-(\#^{m+1} S^2 \times S^1, \mathbb{L}'),$$
$$G : \mathcal{CFL}^-(\#^{m+1} S^2 \times S^1, \mathbb{L}') \to \mathcal{CFL}^-(\#^m S^2 \times S^1, \mathbb{L}).$$

The map F is the link cobordism map for a 4–dimensional 1–handle, followed by a saddle which crosses over the 1–handle. The decoration on the saddle consists of an arc, which connects the two link components of $\mathbb{L}$. Outside of the saddle region, the decoration consists of "vertical" arcs which connect $\mathbb{L}$ to $\mathbb{L}'$. See the left-hand side of [Zemke 2019a, Figure 5.1]. The map G is the map for the link cobordism obtained by reversing the orientation and turning around the above cobordism for F.

The following is a key lemma which we use to compute the H_1–action for knotified links:

Proposition 2.10 *Suppose $\mathbb{L}$, $\mathbb{L}'$, F and G are as above. Let λ be an arc in $\#^m S^2 \times S^1$ which connects the $\boldsymbol{w}$ basepoints of L_{n-1} and L_n. Let γ be the unique element of $H_1(\#^{m+1} S^2 \times S^1)$ obtained by joining the ends of λ across the 1–handle used in knotification. We have the following*:

(a) *F and G are homogeneously graded chain homotopy inverses.*

(b) *The map F satisfies*

$$F(A_\lambda + \mathscr{U}\Phi_{w_n}) \simeq F(A_\lambda + \mathscr{V}\Psi_{z_n}) \simeq A_\gamma F.$$

Proof To simplify the notation, we will describe the case when $\mathcal{L}$ is a link in S^3 with two components L_1 and L_2. We begin with claim (a). The proof is formally identical to the proof of [Zemke 2019a, Proposition 5.1] and follows from two 4–dimensional surgery relations [Zemke 2019a, Propositions 5.2 and 5.4].

We now move onto claim (b). We first show that

$$F(A_\lambda + \mathscr{V}\Psi_{z_2}) \simeq A_\gamma F. \tag{2.11}$$

By definition, we may take

$$F = S^-_{w_2, z_1} F^{\boldsymbol{w}}_B F_1, \tag{2.12}$$

where F_1 is the 1–handle map, $S^-_{w_2,z_1}$ is a quasidestabilization map, and $F^{\boldsymbol{w}}_B$ is a type-$\boldsymbol{w}$ saddle map; see [Zemke 2019c] for precise definitions of the relevant maps. Here B denotes the band (ie saddle) which crossed over the 1–handle used in the knotification operation.

We now have

$$F_1(A_\lambda + \mathscr{V}\Psi_{z_2}) = (A_\lambda + \mathscr{V}\Psi_{z_2}) F_1 \tag{2.13}$$

by the same argument as Ozsváth and Szabó's proof that the 1–handle is a chain map [Ozsváth and Szabó 2006, Section 4.3]. Analogously, the computation of the quasistabilized differential in [Zemke 2017, Proposition 5.3] implies that

$$A_\gamma S^-_{w_2,z_1} = S^-_{w_2,z_1} A_\gamma.$$

Hence, it is sufficient to show that

$$F^{\boldsymbol{w}}_B(A_\lambda + \mathscr{V}\Phi_{z_1}) = A_\gamma F^{\boldsymbol{w}}_B.$$

We recall the definition of the map $F^{\boldsymbol{w}}_B$. We pick a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$ subordinate to the band [Zemke 2019c, Defintion 6.2]. The diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$ contains two canonical intersection points, $\Theta^{\boldsymbol{w}}_{\beta,\beta'}$ and $\Theta^{\boldsymbol{z}}_{\beta,\beta'}$, where $\Theta^{\boldsymbol{o}}_{\beta,\beta'}$ is the top degree generator with respect to the $\mathrm{gr}_{\boldsymbol{o}}$–grading for $\boldsymbol{o} \in \{\boldsymbol{w}, \boldsymbol{z}\}$. By definition,

$$F^{\boldsymbol{w}}_B(\boldsymbol{x}) = F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}).$$

Counting the ends of Maslov index 1 families of holomorphic triangles, weighted by $a(\lambda, \psi)$, we obtain the relation

$$F_{\alpha,\beta,\beta'}(A_\lambda(\boldsymbol{x}), \Theta^{\boldsymbol{z}}_{\beta,\beta'}) + A_\lambda(F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'})) = F^A_\lambda(\partial \boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}) + F^A_\lambda(\boldsymbol{x}, \partial\Theta^{\boldsymbol{z}}_{\beta,\beta'}) + \partial F^A_\lambda(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'});$$

see [Zemke 2015, Lemma 5.2]. Here F^A_λ counts index 0 holomorphic triangles with an extra factor of $a(\lambda, \psi)$. Note that one might expect an extra term involving $F_{\alpha,\beta,\beta'}(\boldsymbol{x}, A_\lambda(\Theta^{\boldsymbol{z}}_{\beta,\beta'}))$; however, this term vanishes since A_λ weights disks based on their changes across the $\boldsymbol{\alpha}$ curves and $\Theta^{\boldsymbol{z}}_{\beta,\beta'} \in \mathbb{T}_\beta \cap \mathbb{T}_{\beta'}$. Since $\partial\Theta^{\boldsymbol{z}}_{\beta,\beta'} = 0$, we obtain that

$$F^{\boldsymbol{w}}_B \circ A_\lambda + A_\lambda \circ F^{\boldsymbol{w}}_B \simeq 0. \tag{2.14}$$

Similarly, counting the ends of index 1 families of holomorphic triangles, weighted by $n_{z_2}(\psi)$, we obtain

$$\begin{aligned} F_{\alpha,\beta,\beta'}(\mathscr{V}\Psi_{z_2}(\boldsymbol{x}), \Theta^{\boldsymbol{z}}_{\beta,\beta'}) + F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})) + \mathscr{V}\Psi_{z_2}(F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}))& \\ = F'(\partial\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}) + F'(\boldsymbol{x}, \partial\Theta^{\boldsymbol{z}}_{\beta,\beta'}) + \partial F'(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}),& \end{aligned}$$

where F' counts index 0 triangles weighted by a factor of $n_{z_1}(\psi)$. The above equation implies that

$$F^{\boldsymbol{w}}_B \circ \mathscr{V}\Psi_{z_2} + \mathscr{V}\Psi_{z_2} \circ F^{\boldsymbol{w}}_B \simeq F_{\alpha,\beta,\beta'}(-, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})). \tag{2.15}$$

We claim now that the map $F_{\alpha,\beta,\beta'}(-, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'}))$ is null-homotopic. To establish this, it is sufficient to show that

$$[\mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})] = 0, \tag{2.16}$$

where the brackets denote the induced element of homology. Indeed, assuming the existence of an $\eta \in \mathcal{CFL}^-(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$ such that $\partial\eta = \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})$, associativity of holomorphic triangles implies that

$$F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})) = \partial F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \eta) + F_{\alpha,\beta,\beta'}(\partial\boldsymbol{x}, \eta),$$

so

$$F_{\alpha,\beta,\beta'}(-, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})) \simeq 0. \tag{2.17}$$

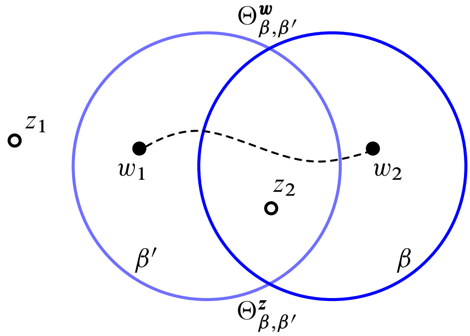

Figure 1: An unknot with four basepoints. The dashed arc is λ.

We will now demonstrate (2.16). We observe that the map Ψ_{z_2} commutes with the homotopy equivalences associated to changing Heegaard diagrams by [Zemke 2017, Lemma 3.2]. Furthermore, the homology class $[\Theta^{\boldsymbol{z}}_{\beta,\beta'}]$ is also preserved by these homotopy equivalences by [Zemke 2019c, Lemma 3.7], since it is the unique generator in its grading. In particular, we may verify (2.16) for any convenient choice of Heegaard diagram for an unknot with four basepoints. We perform the computation using the genus 0 Heegaard diagram shown in Figure 1. On this diagram, $\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})=0$.

Combining (2.14) and (2.15) with (2.17), we obtain

(2.18) $$F^{\boldsymbol{w}}_B(A_\lambda+\mathscr{V}\Psi_{z_2})\simeq(A_\lambda+\mathscr{V}\Psi_{z_2})F^{\boldsymbol{w}}_B.$$

Next, consider a path λ' from w_1 to w_2, which is a subarc of $\mathbb{L}'$. We choose λ' so that it is oriented from w_1 to w_2. There are two such subarcs of $\mathbb{L}'$, and we pick the one so that the portion of λ' nearest to w_1 is in the beta-handlebody (equivalently, we pick the one which goes over the band B before arriving at a $\boldsymbol{z}$ basepoint). Without loss of generality, we may assume that λ' crosses over z_2. See Figure 2. We define

$$\gamma:=\lambda*\lambda',$$

where $*$ denotes concatenation.

On the Heegaard diagram, we may choose λ' to cross only the alpha curves between w_1 and z_2, and only the beta curves between z_2 and w_2. Clearly,

$$a(\lambda',\phi)=n_{w_2}(\phi)-n_{z_2}(\phi).$$

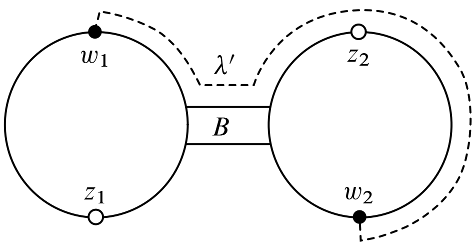

Figure 2: The configuration of the band B, the basepoints and the arc $\lambda'\subset\mathbb{L}'$.

Hence, $A_{\lambda'} = \mathscr{U}\Phi_{w_2} + \mathscr{V}\Psi_{z_2}$, or, equivalently,

(2.19)
$$\mathscr{V}\Psi_{z_2} = A_{\lambda'} + \mathscr{U}\Phi_{w_2}.$$

Combining (2.18) and (2.19), we obtain

(2.20)
$$\begin{aligned} F(A_\lambda + \mathscr{V}\Psi_{z_2}) &\simeq S^-_{w_2,z_1}(A_\lambda + A_{\lambda'} + \mathscr{U}\Phi_{w_2})F^{\boldsymbol{w}}_B F_1 \\ &\simeq S^-_{w_2,z_1}(A_\gamma + \mathscr{U}\Phi_{w_2})F^{\boldsymbol{w}}_B F_1 \\ &\simeq A_\gamma S^-_{w_2,z_1} F^{\boldsymbol{w}}_B F_1. \end{aligned}$$

The second line of (2.20) follows from the relation $A_\gamma \simeq A_\lambda + A_{\lambda'}$. The final line follows from (2.13), as well as the relation that $S^-_{w_2,z_1}\Phi_{w_2} \simeq S^-_{w_2,z_1}S^+_{w_2,z_1}S^-_{w_2,z_1} \simeq 0$ by [Zemke 2019c, Lemmas 4.11 and 4.13], completing the proof of (2.11).

Finally, to see that
$$F(A_\lambda + \mathscr{U}\Phi_{w_2}) \simeq A_\gamma F,$$
it is sufficient to show that $\mathscr{V}\Psi_{z_2} \simeq \mathscr{U}\Phi_{w_2}$ on $\mathcal{CFL}^-(\mathbb{L})$. To see this, we note that on a diagram for $\mathbb{L}$, we can consider a shadow of the link component L_2. The arc $L_2 \setminus \{w_2, z_2\}$ contains two subarcs, one of which intersects only the alpha curves, and one of which intersects only the beta curves. Hence $a(L_2, \phi) = n_{w_2}(\phi) - n_{z_2}(\phi)$ for any class of disks ϕ. On the other hand, this implies that the homology action associated to $0 = [L_2] \in H_1(S^3)$ satisfies
$$0 \simeq A_{L_2} = \mathscr{U}\Phi_{w_2} + \mathscr{V}\Psi_{z_2}. \qquad \square$$

The homology action on full knotifications may be computed by iterating the above result, via the following lemma:

Lemma 2.21 *Let $\mathbb{L}$, $\mathbb{L}'$, F and G be as in Proposition 2.10.*

1. *Suppose that $\gamma \in H_1(\#^m\, S^2 \times S^1)$. Write γ also for the induced element of $H_1(\#^{m+1}\, S^2 \times S^1)$. Then A_γ commutes with F and G up to chain homotopy.*
2. *If λ is an arc in $\#^m\, S^2 \times S^1$ which connects two components of $L_1, \dots, L_{n-2}$, then the relative homology map A_λ commutes with F and G up to chain homotopy.*
3. *If w and z are basepoints on one of the link components $L_1, \dots, L_{n-2}$, then Φ_w and Ψ_z commute with F and G up to chain homotopy.*

The proof of Lemma 2.21 is similar to the proof of Proposition 2.10 (though strictly easier), and hence we omit it. We refer the reader to [Zemke 2015, Section 5; 2019c, Section 4] for related results.

2.5 The Hopf link

Our next goal is to describe the $\mathcal{CFL}^-$–complexes for the $(2, 2n)$–torus links, denoted by $T_{2,2n}$, their mirrors and their knotifications. As the calculations are rather involved, we begin by describing the Floer chain complex for the link $T_{2,2}$ (ie. the positive Hopf link), leaving the general case to Section 2.6. While

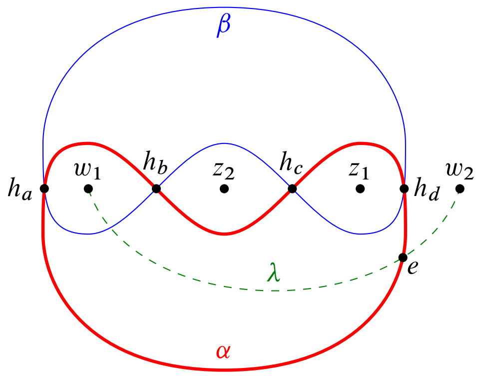

Figure 3: A genus 0 Heegaard diagram for the Hopf link. The thick (red) curve is the α curve, the thin (blue) curve is the β curve. The dotted curve is used to compute the action of $H_1(S^2 \times S^1; \mathbb{Z})$ on the knotification of the Hopf link.

the complex $\mathcal{CFL}^-(T_{2,2})$ is well known (it can be computed explicitly using a very simple diagram), to the best of our knowledge, the calculation of the action of $H_1(S^2 \times S^1)$ on the knot Floer chain complex of the knotification of $T_{2,2}$ is new.

As our main focus will eventually be the knotification of $T_{2,2}$, we restrict our attention to the link Floer complex over the ring $\mathscr{R}^- = \mathbb{F}[\mathscr{U}, \mathscr{V}]$, as opposed to the version with a variable for each basepoint.

Consider the diagram for the Hopf link, as in Figure 3. The complex $\mathcal{CFL}^-(T_{2,2})$ is generated over $\mathscr{R}^-$ by four elements, h_a, h_b, h_c and h_d, which correspond to the intersections of the α and β curves in Figure 3. The gradings are

$$\begin{aligned}(\mathrm{gr}_w(h_a), \mathrm{gr}_z(h_a)) &= \left(\tfrac{1}{2}, -\tfrac{3}{2}\right), \quad (\mathrm{gr}_w(h_b), \mathrm{gr}_z(h_b)) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right),\\ (\mathrm{gr}_w(h_c), \mathrm{gr}_z(h_c)) &= \left(-\tfrac{3}{2}, \tfrac{1}{2}\right), \quad (\mathrm{gr}_w(h_d), \mathrm{gr}_z(h_d)) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right).\end{aligned} \tag{2.22}$$

The differential in the complex is computed by counting holomorphic disks of Maslov index 1. Counting bigons shows that

$$\partial h_a = \partial h_c = 0, \quad \partial h_b = \partial h_d = \mathscr{U} h_a + \mathscr{V} h_c. \tag{2.23}$$

The homology of $\mathcal{CFL}^\infty(T_{2,2})$ is a direct sum of two copies of $\mathscr{R}^\infty$. One copy is spanned by $[h_b + h_d]$; the other copy is spanned by h_a or h_c.

We now describe the homology action A_γ on $\mathcal{CFK}^-(\widehat{T}_{2,2})$, where $\widehat{T}_{2,2}$ denotes the knotification of $T_{2,2}$, and γ is a generator of $H_1(S^2 \times S^1)$. We will use Proposition 2.10. The formula therein involves the relative homology action A_λ on $\mathcal{CFL}^-(T_{2,2})$, which we compute now. In our present case, the arc λ has only one intersection with an alpha curve, which occurs at a point labeled e in Figure 3. The map A_λ counts holomorphic disks of Maslov index 1, with weights corresponding to changes along the alpha boundary of a disk; see (2.7). Counting bigons with these weights, we obtain

$$A_\lambda(h_a) = \mathscr{V}(h_b + h_d), \quad A_\lambda(h_b) = 0, \quad A_\lambda(h_c) = \mathscr{U}(h_b + h_d), \quad A_\lambda(h_d) = \mathscr{U} h_a. \tag{2.24}$$

We recall that, in Section 2.4, we defined a knotification map

$$F : \mathcal{CFL}^-(T_{2,2}) \to \mathcal{CFK}^-(\hat{T}_{2,2}),$$

which is a homotopy equivalence. In Proposition 2.10, we showed that

$$F(A_\lambda + \mathscr{U}\Phi_{w_2}) \simeq A_\gamma F.$$

Hence, as a model for the pair $(\mathcal{CFK}^-(\hat{T}_{2,2}), A_\gamma)$, we may use $(\mathcal{CFL}^-(T_{2,2}), A_\lambda + \mathscr{U}\Phi_{w_2})$. Hereafter, by a model for a chain complex (possibly with extra structure) defined up to chain homotopy equivalence, we mean a concrete chain complex in the class of an appropriate (usually bifiltered) chain homotopy equivalence. Abusing notation slightly, we will write A_γ for the endomorphism of $\mathcal{CFL}^-(T_{2,2})$ given by $A_\gamma := A_\lambda + \mathscr{U}\Phi_{w_2}$. One easily computes

$$\Phi_{w_2}(h_d) = h_a,$$

and Φ_{w_2} vanishes on the other generators. Hence,

(2.25) $$A_\gamma(h_a) = \mathscr{V}(h_b + h_d), \quad A_\gamma(h_b) = \mathscr{U}h_a, \quad A_\gamma(h_c) = \mathscr{U}(h_b + h_d), \quad A_\gamma(h_d) = \mathscr{U}h_a.$$

With a change of basis $h'_d = h_b + h_d$, we obtain the following presentation of $(\mathcal{CFK}^-(\hat{T}_{2,2}), A_\gamma)$:

(2.26)
$$\begin{tikzcd}
& h_a \arrow[dl, "\mathscr{V}"'] & h_b \arrow[l, dashed, "\mathscr{U}"] \arrow[l, bend right, "\mathscr{U}"'] \arrow[d, dashed, "\mathscr{V}"] \\
h'_d & & h_c \arrow[ll, "\mathscr{U}"']
\end{tikzcd}$$

In (2.26), the dashed arrows denote differentials, and the solid arrows denote the action of A_γ.

We may obtain a simpler model of the homology action by replacing A_γ with $A_\gamma + [\partial, F]$, where F is the $\mathscr{R}^-$–equivariant map which satisfies

$$F(h_a) = h_a \quad \text{and} \quad F(h_b) = F(h_c) = F(h_d) = 0.$$

The resulting model for $(\mathcal{CFK}^-(\hat{T}_{2,2}), A_\gamma)$ is

(2.27)
$$\begin{tikzcd}
& h_a \arrow[dl, "\mathscr{V}"'] & h_b \arrow[l, dashed, "\mathscr{U}"'] \arrow[d, dashed, "\mathscr{V}"] \\
h'_d & & h_c \arrow[ll, "\mathscr{U}"']
\end{tikzcd}$$

2.6 The torus link $T_{2,2n}$

Before we start our computation of the Floer chain complex of the $(2,2n)$–torus link and its knotification, we introduce a family of complexes $\mathcal{S}_n$ for $n \in \mathbb{Z}$, which play a prominent role in the present paper.

Definition 2.28 Let $n \geq 1$. We write $\mathcal{S}^n$ for the complex generated by elements $x_0, y_1, \dots, y_{2n-1}, x_{2n}$ with differential $\partial(x_{2i}) = 0$ and

$$\partial(y_{2i+1}) = \mathscr{U}x_{2i} + \mathscr{V}x_{2i+2}.$$

The bigradings are given by $(\mathrm{gr}_w(x_j), \mathrm{gr}_z(x_j)) = (-j, j-2n)$ if j is even. The same formula holds for y_j if j is odd.

The complex $\mathcal{S}^{-n}$ is defined as the dual complex to $\mathcal{S}^n$. More specifically, it is generated by elements $\underline{x}_0, \underline{y}_1, \dots, \underline{y}_{2n-1}, \underline{x}_{2n}$ with differential $\partial(\underline{y}_{2i+1}) = 0$, $\partial(\underline{x}_{2i}) = \mathscr{V}\underline{y}_{2i-1} + \mathscr{U}\underline{y}_{2i+1}$, and the convention that $\underline{y}_{-1} = \underline{y}_{2n+1} = 0$. For j even, the grading of $\underline{x}_j$ is $(j, 2n-j)$, and an analogous formula holds for the grading of $\underline{y}_j$ if j is odd.

Remark 2.29 The complex $\mathcal{S}^n$ is the $\mathcal{CFK}^-$–complex of the positive torus knot $T_{2,2n+1}$, while $\mathcal{S}^{-n}$ is the complex for the negative torus knot $T_{2,-(2n+1)}$. Hence, we also call $\mathcal{S}^n$ a *staircase complex*. For details of staircase complexes, see Section 4.1.

Recall that $T_{2,2n} \subset S^3$ denotes a 2–component $(2,2n)$–torus link. In this subsection, we study the Floer chain complex $\mathcal{CFL}^-(T_{2,2n})$ as an $\mathscr{R}^-$–module. This gives the Floer chain complex $\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,2n})$, where $\widehat{T}_{2,2n}$ is the knotification of $T_{2,2n}$.

The Heegaard diagram of the link $T_{2,2n}$ in S^3 is shown in Figure 4 and the Floer chain complex is in Figure 5. The Heegaard diagram displayed therein is obtained from a doubly pointed open book whose page is a disk and whose monodromy is γ^n, where γ denotes a Dehn twist parallel to the boundary.

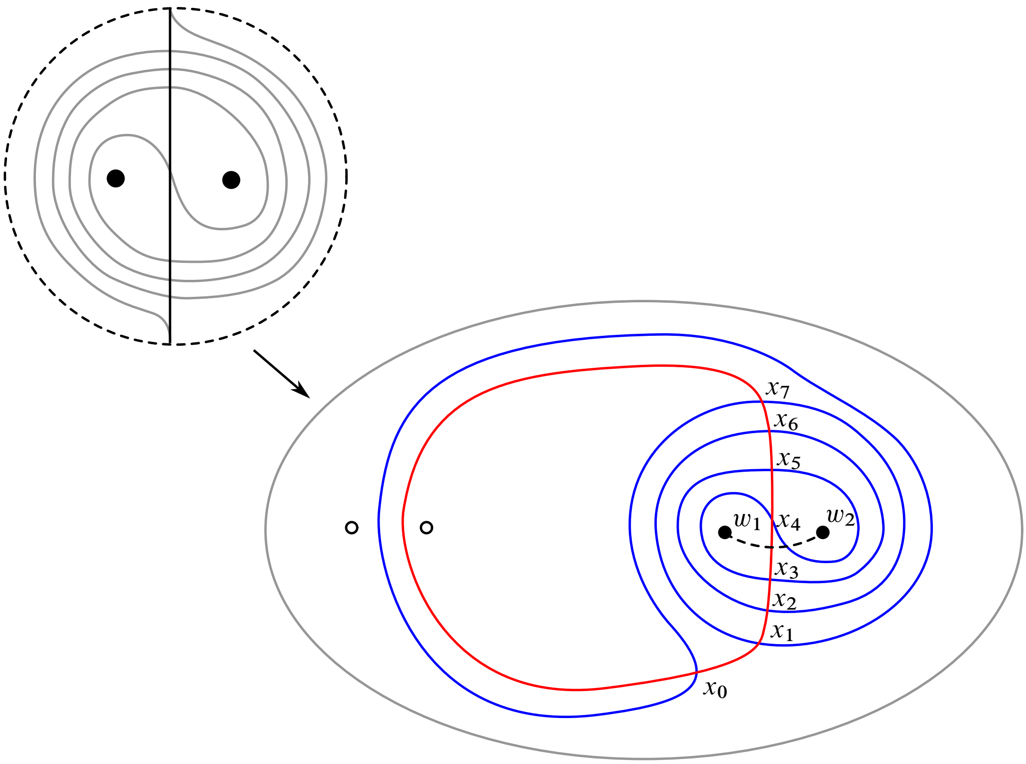

Figure 4: A Heegaard diagram for $T_{2,4}$ from a doubly pointed open book. The dashed line is an arc λ connecting w_1 and w_2.

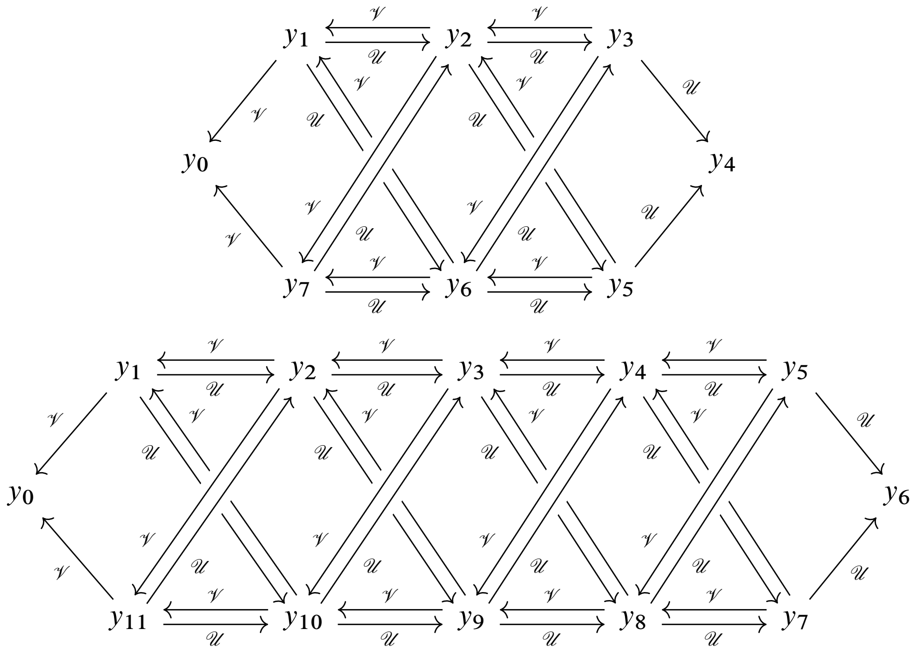

Figure 5: The chain complexes for $T_{2,4}$ (top) and $T_{2,6}$ (bottom).

It is easy to see that there are $4n$ generators $y_0, \dots, y_{4n-1}$ of the complex $\mathcal{CFL}^-(T_{2,2n})$. By counting bigons, one obtains formulas for the differential

$$
\begin{aligned}
\partial y_i = \partial y_{4n-i} &= \mathscr{V}(y_{i-1} + y_{4n-i+1}) + \mathscr{U}(y_{i+1} + y_{4n-i-1}) \quad \text{if } 2 \le i \le 2n-2, \\
\partial y_1 = \partial y_{4n-1} &= \mathscr{V} y_0 + \mathscr{U}(y_2 + y_{4n-2}), \\
\partial y_{2n-1} = \partial y_{2n+1} &= \mathscr{U} y_{2n} + \mathscr{V}(y_{2n-2} + y_{2n+2}), \\
\partial y_0 = \partial y_{2n} &= 0.
\end{aligned}
\tag{2.30}
$$

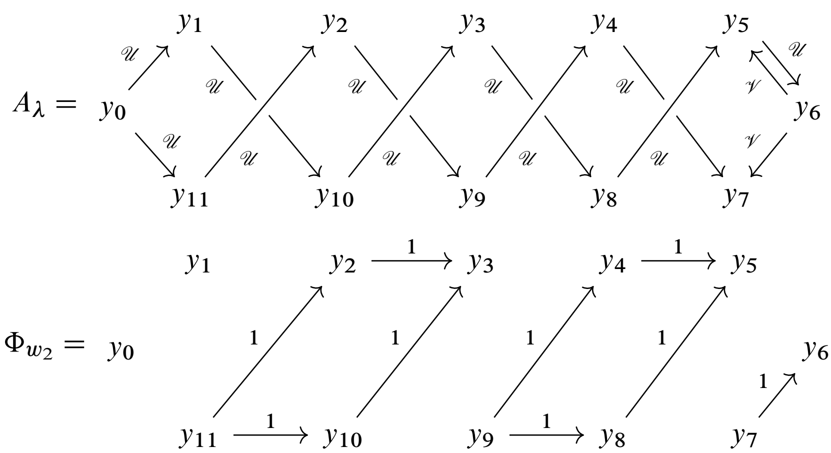

Figure 6: Figure 5 continued. The map A_λ on the complex for $T_{2,6}$ (top) and the map Φ_{w_2} (bottom).

It is convenient to do the following bigraded change of basis to the complex $\mathcal{CFL}^-(T_{2,2n})$. Namely we consider the basis $y_1, \dots, y_{2n-1}, x_0, \dots, x_{2n}$, where

$$x_i = y_i + y_{4n-i} \quad \text{if } 1 \le i \le 2n-1, \qquad x_0 = y_0, \qquad x_{2n} = y_{2n}. \tag{2.31}$$

With this change of basis, the differential takes the form

$$\partial y_i = \mathscr{V} x_{i-1} + \mathscr{U} x_{i+1} \quad \text{if } 1 \le i \le 2n-1, \qquad \partial x_i = 0. \tag{2.32}$$

The gradings of the generators in $\mathcal{CFL}^-(T_{2,2n})$ are summarized in the following lemma:

Lemma 2.33 *If* $1 \le i \le 2n-1$, *then*

$$(\mathrm{gr}_w(y_i), \mathrm{gr}_z(y_i)) = (\mathrm{gr}_w(x_i), \mathrm{gr}_z(x_i)) = \left(\tfrac{1}{2} - 2n + i, \tfrac{1}{2} - i\right).$$

If $i = 0$ *or* $i = 2n$, *then the same formula holds for* x_i.

Proof Recall that ∂ has $(\mathrm{gr}_w, \mathrm{gr}_z)$–bigrading $(-1,-1)$, and that $\mathscr{U}$ and $\mathscr{V}$ have bigradings $(-2, 0)$ and $(0,-2)$, respectively. Using the description in Figure 6, it is easy to check that the formula holds up to an overall additive constant. That is, the formula holds for the relative gr_w– and gr_z–gradings. Hence, it is sufficient to show the absolute gr_w–grading is correct for one of the generators, and similarly for the gr_z–grading. To check the absolute gradings, we note that, if we set $\mathscr{V} = 1$ and $\mathscr{U} = 0$, then we recover the Heegaard Floer complex for $\widehat{\mathrm{CF}}(S^3, w_1, w_2)$, which is homotopy equivalent to $\mathbb{F}_{1/2} \oplus \mathbb{F}_{-1/2}$ as a gr_w–graded chain complex. In this case, the complex has generators x_{2n-1} and x_{2n}, which pins down their gr_w–grading. A similar argument computes the gr_z–gradings. □

We now compute the homology action A_γ on the complex of the knotification of $T_{2,2n}$. In order to use Proposition 2.10, we need to compute A_λ and Φ_{w_2}. For a choice of arc on the Heegaard surface as in Figure 4, by counting bigons we obtain that A_λ has the form

$$\begin{aligned} A_\lambda(y_0) &= \mathscr{U}(y_1 + y_{4n-1}), & A_\lambda(y_i) &= \mathscr{U} y_{i+1} & &\text{if } 0 < i < 2n, \\ A_\lambda(y_{2n}) &= \mathscr{V}(y_{2n-1} + y_{2n+1}), & A_\lambda(y_i) &= \mathscr{U} y_{4n-i+1} & &\text{if } 2n+1 < i < 4n. \end{aligned} \tag{2.34}$$

By (2.31), we have

$$\begin{aligned} A_\lambda(x_0) &= \mathscr{U} x_1, & A_\lambda(x_i) &= \mathscr{U} x_{i+1} \quad \text{if } 0 < i < 2n-1 \\ A_\lambda(x_{2n}) &= \mathscr{V} x_{2n-1}, & A_\lambda(x_{2n-1}) &= 0. \end{aligned} \tag{2.35}$$

Next, we need to understand the map Φ_{w_2}. Counting bigons on diagrams like those shown in Figure 4 implies that Φ_{w_2} takes the form

$$\begin{aligned} \Phi_{w_2}(y_{2i}) &= y_{2i+1} & &\text{if } 0 < i < n, & \Phi_{w_2}(y_{2i}) &= y_{4n-2i+1} \quad \text{if } n < i < 2n, \\ \Phi_{w_2}(y_{2i+1}) &= y_{2i} + y_{4n-2i} & &\text{if } n < i < 2n, & \Phi_{w_2}(y_{2n+1}) &= y_{2n}, \end{aligned} \tag{2.36}$$

and Φ_{w_2} vanishes on all other generators.

Finally, we combine Proposition 2.10 with (2.35) and (2.36) to obtain the following formula for $A_\gamma \simeq A_\lambda + \mathscr{U}\Phi_{w_2}$ on the knotified complex, which we write in terms of the basis from (2.31):

$$\begin{aligned} A_\gamma(y_{2i+1}) &= \mathscr{U}x_{2i+2} + \mathscr{U}y_{2i+2} && \text{if } 0 \le i < n-1, \\ A_\gamma(y_{2i}) &= \mathscr{U}x_{2i+1} && \text{if } 0 < i < n-1, \\ A_\gamma(x_{2i}) &= \mathscr{U}x_{2i+1} && \text{if } 0 \le i < n, \\ A_\gamma(x_{2n}) &= \mathscr{V}x_{2n-1}, \end{aligned} \tag{2.37}$$

and A_γ vanishes on all other generators. The example of $T_{2,6}$ is shown below:

$$A_\gamma = \begin{tikzcd} & y_1 \arrow[dl, dashed, "\mathscr{V}"'] \arrow[dr, dashed, "\mathscr{U}"] \arrow[dr, bend right, "\mathscr{U}"'] \arrow[ddr, bend right, "\mathscr{U}"] & & y_3 \arrow[dl, dashed, "\mathscr{V}"] \arrow[dr, dashed, "\mathscr{U}"] \arrow[dr, bend right, "\mathscr{U}"'] \arrow[ddr, bend right, "\mathscr{U}"] & & y_5 \arrow[dl, dashed, "\mathscr{V}"] \arrow[dr, dashed, "\mathscr{U}"] \arrow[dr, bend right, "\mathscr{U}"'] & \\ x_0 \arrow[ddr, "\mathscr{U}"] & & x_2 \arrow[ddr, "\mathscr{U}"] & & x_4 \arrow[ddr, "\mathscr{U}"] & & x_6 \arrow[ddl, "\mathscr{V}"'] \\ & & y_2 \arrow[dl, dashed, "\mathscr{V}"] \arrow[dr, dashed, "\mathscr{U}"] \arrow[dr, bend right, "\mathscr{U}"'] & & y_4 \arrow[dl, dashed, "\mathscr{V}"] \arrow[dr, dashed, "\mathscr{U}"] \arrow[dr, bend right, "\mathscr{U}"'] & & \\ & x_1 & & x_3 & & x_5 & \end{tikzcd} \tag{2.38}$$

The dashed lines denote the differential and the solid lines denote the A_γ–action. It is convenient to modify the map A_γ by a further chain homotopy, so that it takes one staircase summand to the other, with no self-arrows, as follows. Define a function $\delta\colon \mathbb{N} \to \mathbb{F}$ by

$$\delta(n) = \tfrac{1}{2}n(n-1) \mod 2.$$

Conceptually, it is easier to think of $\delta(n)$ as the sequence $0, 0, 1, 1, 0, 0, 1, 1, \ldots$. We define a homotopy F as follows. On the first staircase summand, we define F via

$$F(x_{2i}) = \delta(2i)\cdot x_{2i} \quad \text{if } 0 \le i \le n, \qquad F(y_{2i+1}) = \delta(2i+1)\cdot y_{2i+1} \quad \text{if } 0 \le i < n.$$

On the second staircase summand, we define F via

$$F(x_{2i+1}) = \delta(2i)\cdot x_{2i+1} \quad \text{if } 0 \le i < n, \qquad F(y_{2i}) = \delta(2i-1)\cdot y_{2i} \quad \text{if } 0 < i < n.$$

Writing A'_γ for $A_\gamma + [\partial, F]$, we compute that

$$\begin{aligned} A'_\gamma(x_{2i}) &= \mathscr{U}x_{2i+1} && \text{if } 0 \le i < n, \\ A'_\gamma(y_{2i+1}) &= \mathscr{U}y_{2i+2} && \text{if } 0 \le i < n-1, \\ A_{\gamma'}(x_{2n}) &= \mathscr{V}x_{2n-1}. \end{aligned}$$

Continuing our running example of $T_{2,6}$, (2.38) becomes

$$A_\gamma + [\partial, F] = \begin{tikzcd} & y_1 \arrow[dl, dashed, "\mathscr{V}"'] \arrow[dr, dashed, "\mathscr{U}"] \arrow[ddr, "\mathscr{U}"] & & y_3 \arrow[dl, dashed, "\mathscr{V}"'] \arrow[dr, dashed, "\mathscr{U}"] \arrow[ddr, "\mathscr{U}"] & & y_5 \arrow[dl, dashed, "\mathscr{V}"'] \arrow[dr, dashed, "\mathscr{U}"] & \\ x_0 \arrow[ddr, "\mathscr{U}"] & & x_2 \arrow[ddr, "\mathscr{U}"] & & x_4 \arrow[ddr, "\mathscr{U}"] & & x_6 \arrow[ddl, "\mathscr{V}"'] \\ & & y_2 \arrow[dl, dashed, "\mathscr{V}"'] \arrow[dr, dashed, "\mathscr{U}"] & & y_4 \arrow[dl, dashed, "\mathscr{V}"'] \arrow[dr, dashed, "\mathscr{U}"] & & \\ & x_1 & & x_3 & & x_5 & \end{tikzcd} \tag{2.39}$$

We summarize the above computation as follows:

Proposition 2.40 *The pair $(\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,2n}), A_\gamma)$ has a model where $\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,2n})$ is equal to $\mathcal{S}^n\{\frac{1}{2}, \frac{1}{2}\} \oplus \mathcal{S}^{n-1}\{-\frac{1}{2}, -\frac{1}{2}\}$ and A_γ maps $\mathcal{S}^n$ to $\mathcal{S}^{n-1}$ on the chain level. Here, we recall that $\{i, j\}$ denotes a shift in the $(\mathrm{gr}_w, \mathrm{gr}_z)$–grading by (i, j), and $\mathcal{S}^n$ and $\mathcal{S}^{n-1}$ are the chain complexes in Definition 2.28.*

We now consider the mirror of the $(2, 2n)$–torus link, which we denote by $T_{2,-2n}$. We denote its knotification by $\widehat{T}_{2,-2n}$. On the level of Floer complexes, taking the mirror amounts to replacing the link Floer complex by the dual complex over the ring $\mathscr{R}^-$. In practice, this amounts to reversing all the arrows in the differential and multiplying the $(\mathrm{gr}_w, \mathrm{gr}_z)$–bigrading by an overall factor of -1. The homology action on the mirror is also the dual. We summarize this as follows:

Proposition 2.41 *The pair $(\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,-2n}), A_\gamma)$ has a model where $\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,-2n})$ is equal to $\mathcal{S}^{-n}\{-\frac{1}{2}, -\frac{1}{2}\} \oplus \mathcal{S}^{-(n-1)}\{\frac{1}{2}, \frac{1}{2}\}$ and A_γ maps $\mathcal{S}^{-(n-1)}$ to $\mathcal{S}^{-n}$ on the chain level.*

2.7 The Borromean knot $\mathcal{B}_0$

Let $\mathcal{B}_0 \subset \#^2\, S^2 \times S^1$ be the Borromean knot, that is, the knot obtained from the Borromean rings by a zero-framed surgery on two of its components. The Heegaard Floer chain complex of $\mathcal{B}_0$ is described in [Ozsváth and Szabó 2004a, Proposition 9.2]. We adapt the calculation of [Borodzik et al. 2017, Section 5; Bodnár et al. 2016, Section 4] to the present context.

The chain complex $\mathcal{CFK}^-(\mathcal{B}_0)$ is homotopy equivalent to $\mathbb{F}^4 \otimes_{\mathbb{F}} \mathscr{R}^-$, with vanishing differential. We write 1, x, y and xy for the generators of $\mathbb{F}^4$, which we can think of as being generators of $H^*(\mathbb{T}^2)$. The bigradings are

$$\begin{aligned} (\mathrm{gr}_w(1), \mathrm{gr}_z(1)) &= (1, -1), \\ (\mathrm{gr}_w(x), \mathrm{gr}_z(x)) = (\mathrm{gr}_w(y), \mathrm{gr}_z(y)) &= (0, 0), \\ (\mathrm{gr}_w(xy), \mathrm{gr}_z(xy)) &= (-1, 1). \end{aligned} \tag{2.42}$$

Up to an overall grading-preserving isomorphism, the $H_1(\#^2 S^2 \times S^1)$–module structure is uniquely determined by the formal properties of the action. In detail, if we write x^* and y^* for the two generators of $H_1(\#^2 S^2 \times S^1)$, then the module structure takes the form (up to overall isomorphism)

$$A_{y^*} = \begin{tikzcd} & xy \arrow[dl, "\mathscr{V}"'] & \\ x & & y \arrow[ul, "\mathscr{U}"'] \arrow[dl, "\mathscr{V}"] \\ & 1 \arrow[ul, "\mathscr{U}"] & \end{tikzcd} \qquad A_{x^*} = \begin{tikzcd} & xy \arrow[dr, "\mathscr{V}"] & \\ x \arrow[ur, "\mathscr{U}"] \arrow[dr, "\mathscr{V}"'] & & y \\ & 1 \arrow[ur, "\mathscr{U}"'] & \end{tikzcd}$$

For the explicit description of the top and bottom towers of $\mathcal{CFK}^-(\mathcal{B}_0)$, see [Borodzik et al. 2017, Section 5].

3 Correction terms

3.1 Generalized correction terms of Levine and Ruberman

Suppose Y is an oriented closed three-dimensional manifold. The module $\mathrm{HF}^\infty(Y)$ is *standard* if, for each torsion Spin^c structure $\mathfrak{s}$,

$$\mathrm{HF}^\infty(Y,\mathfrak{s}) \cong \Lambda^* H^1(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}[U,U^{-1}]$$

as $\Lambda^*(H_1(Y;\mathbb{Z})/\mathrm{Tors}) \otimes_{\mathbb{Z}} \mathbb{F}[U]$–modules. Any manifold Y for which the triple cup product vanishes is standard; see [Lidman 2013] (and also [Levine and Ruberman 2014, Theorem 3.2]). In particular, a connected sum of finitely many copies of $S^1 \times S^2$ has standard HF^∞. Hence, a large surgery on a null-homologous knot in $\# S^1 \times S^2$ has standard HF^∞; see [Ozsváth and Szabó 2003]. This means that essentially all 3–manifolds we are going to consider have standard HF^∞.

There is an action (up to homotopy) of $\Lambda^*(H_1(Y)/\mathrm{Tors})$ on $\mathrm{CF}^-(Y,\mathfrak{s})$. Expanding on work of Ozsváth and Szabó [2003], Levine and Ruberman [2014] associate a d–invariant to any primitive subspace G of $H_1(Y)/\mathrm{Tors}$ (recall that a *primitive subspace* is a free submodule whose quotient is free) and any Spin^c structure $\mathfrak{s}$ on Y whose first Chern class is torsion as long as $\mathrm{HF}^\infty(Y)$ is standard. We denote this invariant by $d(Y,\mathfrak{s},G)$. For our purposes, the two most important instances are the invariants

$$d_{\mathrm{bot}}(Y,\mathfrak{s}) := d(Y,\mathfrak{s},H_1(Y)/\mathrm{Tors}), \quad d_{\mathrm{top}}(Y,\mathfrak{s}) := d(Y,\mathfrak{s},\{0\}),$$

which correspond approximately to the kernel and cokernel, respectively, of the $H_1(Y)/\mathrm{Tors}$–action.

The key property of these invariants is the following inequality, generalizing the Ozsváth–Szabó inequality:

Theorem 3.1 [Levine and Ruberman 2014, Theorem 4.7] *Suppose X is a connected four-manifold such that $b_2^+(X) = 0$ and $\partial X = Y$. Suppose $\mathfrak{s}$ is a Spin^c structure on Y that extends to a Spin^c structure $\mathfrak{t}$ on X. Then*

$$d(Y,\mathfrak{s},G) \geq \tfrac{1}{4}(c_1^2(\mathfrak{t}) + b_2^-(X)) + \tfrac{1}{2}b_1(Y) - \operatorname{rk} G$$

if G contains the kernel of the inclusion map from $H_1(Y)/\mathrm{Tors}$ to $H_1(X)/\mathrm{Tors}$.

3.2 V–invariants

The aim of this section is to gather several definitions of V_s–invariants. In the context of Heegaard Floer theory, all these definitions lead to the same invariants.

The first definition recalls the classical V_s–invariant for knots. The assumptions on C_* in Definition 3.2 are modeled on a knot Floer complex CFK^-.

Definition 3.2 (V_s–invariants for complexes over $\mathbb{F}[U, U^{-1}]$) Suppose C_* is a filtered chain complex of free $\mathbb{F}[U]$–modules (with multiplication by U decreasing the filtration level by 1 and the grading by 2) such that the homology of the localized complex $U^{-1}C_*$ is equal to $\mathbb{F}[U, U^{-1}]$. For $s \in \mathbb{Z}$, the invariant $V_s(C_*)$ is such that $-2V_s(C_*)$ is the maximal grading of an element $x \in C_*$ at filtration level at most s such that the class of $U^k x$ is nonzero in $H_*(C_*)$ for all $k \geq 0$.

Next we define the V_s–invariants of a bigraded $\mathscr{R}^-$–module, where $\mathscr{R}^- = \mathbb{F}[\mathscr{U}, \mathscr{V}]$. The definition is essentially taken from [Zemke 2019b, equation (10.3)]. Suppose C_* is a bigraded chain complex over $\mathscr{R}^-$ such that multiplication by $\mathscr{U}$ changes the grading by $(-2, 0)$, multiplication by $\mathscr{V}$ changes the grading by $(0, -2)$, and the differential changes the grading by $(-1, -1)$. Let $(\mathrm{gr}_w, \mathrm{gr}_z)$ denote the bigrading. It is not hard to see that the differential and multiplication by $\mathscr{U}\mathscr{V}$ preserves the difference $\mathrm{gr}_w - \mathrm{gr}_z$.

Definition 3.3 (V_s–invariants over $\mathscr{R}^-$) Suppose C_* is a chain complex over $\mathscr{R}^-$ such that

$$(\mathscr{U}, \mathscr{V})^{-1} \cdot H_*(C_*) \cong \mathscr{R}^\infty = \mathbb{F}[\mathscr{U}, \mathscr{V}, \mathscr{U}^{-1}, \mathscr{V}^{-1}] \tag{3.4}$$

as bigraded groups. (Here $(\mathscr{U}, \mathscr{V})^{-1}\cdot$ denotes localization at the nonzero monomials of $\mathscr{R}^-$.) We write $\mathscr{A}_s(C_*)$ for the subcomplex of C_* which has $\mathrm{gr}_w - \mathrm{gr}_z = 2s$. We can view $\mathscr{A}_s(C_*)$ as a complex over $\mathbb{F}[U]$, where U acts by $\mathscr{U}\mathscr{V}$. We define $d(\mathscr{A}_s(C_*))$ for the maximal gr_w–grading of a homogeneously graded, $\mathbb{F}[U]$–nontorsion element of $H_*(\mathscr{A}_s(C_*))$. We define

$$V_s(C_*) = -\tfrac{1}{2} d(\mathscr{A}_s(C_*)).$$

Remark 3.5 Suppose M is a graded module over $\mathscr{R}^-$ such that $(\mathscr{U}^{-1}, \mathscr{V}^{-1}) \cdot M \cong \mathscr{R}^\infty$ as bigraded groups. We define $V_s(M)$ to be $V_s(C_*)$, with C_* the chain complex with the same underlying module structure as M but trivial differential.

Remark 3.6 If C_* is the chain complex $\mathcal{CFL}^-(S^3, K)$ for a knot $K \subset S^3$, $V_s(C_*)$ is the classical V–function of the knot K. In this case, we also denote it by $V_s(K)$ if the context is clear. See [Zemke 2019b, Section 1.5] for translating between the chain complex $\mathcal{CFL}^-(S^3, K)$ and $\mathrm{CFK}^-(S^3, K)$.

Suppose C_* is as in Definition 3.3. Let $a, b \in \mathbb{Z}$. The chain complex $C_*\{a, b\}$ is defined as the chain complex equal to C_*, but with grading shifted by (a, b). That is, if $x \in C_*$ has bigrading (c, d), then $x \in C_*\{a, b\}$ has bigrading $(a + c, b + d)$.

Lemma 3.7 *Suppose C_* is a bigraded chain complex over $\mathscr{R}^-$ and let $D_* = C_*\{a,b\}$ be the chain complex with shifted grading. Then $V_{s+(a-b)/2}(D_*) = V_s(C_*) - \frac{1}{2}a$.*

Proof We use the fact that $\mathscr{A}_s(C_*) = \mathscr{A}_{s+(a-b)/2}(D_*)$. □

In our computations, we will need to show that V_s–invariants of locally equivalent complexes are the same. We recall the relevant definition:

Definition 3.8 Two chain complexes C_* and D_* are *locally equivalent* if there exist grading-preserving, $\mathscr{R}^-$–equivariant chain maps $f\colon C_* \to D_*$, $g\colon D_* \to C_*$ such that both f and g induce the identity map on $(\mathscr{U},\mathscr{V})^{-1}\cdot C_* \cong (\mathscr{U},\mathscr{V})^{-1}\cdot D_*$.

As an example, we quote the following result of Hedden, Kim and Livingston (note that ν^+–equivalence is equivalent to local equivalence; see [Hom 2017, Proposition 3.11]):

Proposition 3.9 [Hedden et al. 2016, Theorem B.1] *The tensor product $\mathcal{S}^k \otimes \mathcal{S}^\ell$ is locally equivalent to $\mathcal{S}^{k+\ell}$ for any integers k and l.*

For the following result, see [Zemke 2019a, Section 2], [Hom 2017] or [Kim and Park 2018, Section 3]:

Proposition 3.10 (a) *If C_* is locally equivalent to D_*, then $V_s(C_*) = V_s(D_*)$ for all s.*

(b) *If C_* is locally equivalent to D_* and E_* is locally equivalent to F_*, then $C_* \otimes E_*$ is locally equivalent to $D_* \otimes F_*$.*

We now extend Definition 3.3 to the case of chain complexes with a group action. Suppose C_* is a bigraded chain complex over $\mathscr{R}^-$ and H is a free abelian group such that the ring $\Lambda^* H$ acts on $H_*(C_*)$, and the action of H has degree $(-1,-1)$. Let $\mathrm{Tors} \subset H_*(C_*)$ denote the $\mathscr{R}^-$–torsion submodule. Define

$$\mathcal{H}^{\mathrm{top}} = \mathrm{coker}\big(H \otimes (H_*(C_*)/\mathrm{Tors}) \to (H_*(C_*)/\mathrm{Tors})\big),$$
$$\mathcal{H}^{\mathrm{bot}} = \bigcap_{\gamma\in H} \ker\big(\gamma\colon (H_*(C_*)/\mathrm{Tors}) \to (H_*(C_*)/\mathrm{Tors})\big).$$

By analogy with (3.4), we require that

$$(\mathscr{U},\mathscr{V})^{-1}\cdot\mathcal{H}^{\mathrm{top}} \cong \mathscr{R}^\infty \cong (\mathscr{U},\mathscr{V})^{-1}\cdot\mathcal{H}^{\mathrm{bot}}$$

as relatively bigraded $\mathscr{R}^-$–modules. Let $\mathcal{H}^{\mathrm{top}}_s$ (resp. $\mathcal{H}^{\mathrm{bot}}_s$) denote the $\mathbb{F}[U]$–submodule generated by homogeneously graded elements $x \in \mathcal{H}^{\mathrm{top}}$ (resp. $x \in \mathcal{H}^{\mathrm{bot}}$) such that $\mathrm{gr}_w(x) - \mathrm{gr}_z(x) = 2s$ (recall U acts by $\mathscr{U}\mathscr{V}$). We define $d^{\mathrm{top}}_s(C_*)$ to be the maximal gr_w–grading of a homogeneously graded, $\mathbb{F}[U]$–nontorsion element of $\mathcal{H}^{\mathrm{top}}_s$, and we define $d^{\mathrm{bot}}_s(C_*)$ analogously.

Definition 3.11 We set

$$V^{\mathrm{top}}_s(C_*) := -\tfrac{1}{2}d^{\mathrm{top}}_s(C_*) \quad\text{and}\quad V^{\mathrm{bot}}_s(C_*) = -\tfrac{1}{2}d^{\mathrm{bot}}_s(C_*).$$

Remark 3.12 If K is a null-homologous knot in a closed, oriented connected 3–manifold Y with standard $\mathrm{HF}^\infty(Y)$, for simplicity we write $\mathscr{A}_s(K)$ for $\mathscr{A}_s(\mathcal{CFL}^-(Y,K))$, and $V_s^{\mathrm{top}}(K) = -\frac{1}{2}d_s^{\mathrm{top}}(K)$ and $V_s^{\mathrm{bot}}(K) = -\frac{1}{2}d_s^{\mathrm{bot}}(K)$ for $V_s^{\mathrm{top}}(\mathcal{CFL}^-(Y,K))$ and $V_s^{\mathrm{bot}}(\mathcal{CFL}^-(Y,K))$, respectively.

3.3 Large surgery formula

To set up the notation, we recall the large surgery formula [Ozsváth and Szabó 2004b, Section 4] and relate the d–invariants of the surgery on a knot to its V_s–invariants. We first recall the description of Spin^c structures on a surgery.

Definition 3.13 Suppose Y is a closed 3–manifold and $K \subset Y$ is a null-homologous knot. Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ and $q \in \mathbb{Z}_{>0}$. For any $m \in \left[-\frac{1}{2}q, \frac{1}{2}q\right) \cap \mathbb{Z}$ we denote by $\mathfrak{s}_m$ the unique Spin^c structure on $Y_q(K)$ such that $\mathfrak{s}_m$ extends to a Spin^c structure $\mathfrak{t}_m$ on W uniquely characterized by the properties that $\mathfrak{t}_m|_Y = \mathfrak{s}$ and $\langle c_1(\mathfrak{t}_m), F\rangle + q = 2m$, where W is the trace of the surgery on K and F is the generator of $H_2(W)$ obtained by gluing a Seifert surface for K with the core of the two-handle.

With this notation, we state Ozsváth and Szabó's large surgery theorem [2004b, Theorem 4.1]:

Theorem 3.14 *Suppose $K \subset Y$ is a null-homologous knot in a closed 3–manifold. Suppose $q > 2g_3(K)$ is an integer. For a Spin^c structure $\mathfrak{s}_m$ on Y as in Definition 3.13, there exists a quasi-isomorphism between $\mathrm{CF}^-(Y_q(K), \mathfrak{s}_m)$ and $\mathscr{A}_m$, where $\mathscr{A}_m$ is the $\mathbb{F}[U]$–subcomplex of $\mathcal{CFL}^-(Y,K,\mathfrak{s})$ of elements x with grading $\mathrm{gr}_w(x) - \mathrm{gr}_z(x) = 2m$. If $\mathfrak{s}$ is torsion, then the quasi-isomorphism shifts the grading (Maslov grading on $\mathrm{CF}^-(Y_q(K), \mathfrak{s}_m)$ and gr_w–grading on $\mathscr{A}_m$) by $((q-2m)^2 - q)/4q$.*

From this theorem we obtain the following well-known equalities:

Theorem 3.15 *Suppose $K \subset Y$ is as in Theorem 3.14 and $q > 2g_3(K)$.*

(a) *If Y is a rational homology sphere, then $d(Y_q(K), \mathfrak{s}_m) = ((q-2m)^2 - q)/4q - 2V_m(K)$;*

(b) *If $b_1(Y) > 0$ and $\mathrm{HF}^\infty(Y)$ is standard, then $d^{\mathrm{top}}(Y_q(K), \mathfrak{s}_m) = ((q-2m)^2 - q)/4q - 2V_m^{\mathrm{top}}(K)$ and $d^{\mathrm{bot}}(Y_q(K), \mathfrak{s}_m) = ((q-2m)^2 - q)/4q - 2V_m^{\mathrm{bot}}(K)$.*

4 Staircase complexes and their tensor products

In this section, we introduce staircase complexes. Next we compute the correction terms of certain tensor products of staircase complexes.

4.1 Staircase complexes

A *positive staircase complex* $\mathcal{P}$ is a bigraded chain complex over $\mathscr{R}^-$ with generators $x_0, y_1, x_2, \dots, y_{2n-1}, x_{2n}$ for some $n > 0$ with differential given by $\partial y_{2i+1} = \mathscr{U}^{\alpha_i} \cdot x_{2i} + \mathscr{V}^{\beta_i} \cdot x_{2i+2}$, $\partial x_{2j} = 0$, extended equivariantly over $\mathscr{R}^-$, for some positive integers α_i and β_i. We assume that ∂, $\mathscr{U}$ and $\mathscr{V}$ have bigradings $(-1,-1)$, $(-2,0)$ and $(0,-2)$, respectively. We assume that $\alpha_i = \beta_{n-i-1}$. Furthermore, we assume the gradings are normalized so that $H_*(\mathcal{P}/(\mathscr{U}-1)) \cong \mathbb{F}[\mathscr{V}]$ has generator with gr_z–grading 0, and $H_*(\mathcal{P}/(\mathscr{V}-1)) \cong \mathbb{F}[\mathscr{U}]$ has generator with gr_w–grading 0. A *negative staircase complex* is the dual complex of a positive staircase complex.

Example 4.1 The complex $\mathcal{S}^n$ of Definition 2.28 is a positive staircase complex for all $n > 0$. It is a negative staircase complex if $n < 0$.

Lemma 4.2 *Suppose that $\mathcal{P} = (P_1 \to P_0)$ is a positive staircase complex, viewed as a complex of free $\mathscr{R}^-$–modules, where P_1 is spanned by y_i and P_0 is spanned by x_i.*

(1) *$H_*(\mathcal{P})$ is torsion-free as an $\mathscr{R}^-$–module.*

(2) *There is a $(\mathrm{gr}_w, \mathrm{gr}_z)$–grading-preserving chain map*

$$F : \mathcal{P} \to \mathscr{R}^-$$

which sends $\mathscr{R}^-$–nontorsion cycles to $\mathscr{R}^-$–nontorsion cycles. Furthermore, F may be taken to map each generator of P_0 to a nonzero monomial in $\mathscr{R}^-$, and vanish on P_1.

Proof For the first claim, using the grading properties of $\mathcal{P}$ it is sufficient to show that $\mathscr{U}^i \mathscr{V}^j \cdot [x] \neq 0$ if $[x] \neq 0 \in H_*(\mathcal{P})$ when x is a homogeneously graded cycle in $\mathcal{P}$. Since the map from P_1 to P_0 is injective, there are no cycles with a nonzero summand in P_1. Hence, it is sufficient to see that, if $x \in P_0$ and $\mathscr{U}^i \mathscr{V}^j \cdot x \in \mathrm{im}(P_1)$, then $x \in \mathrm{im}(P_1)$. To see this, suppose that $y \in P_1$ is homogeneously graded and not a multiple of $\mathscr{U}$ or $\mathscr{V}$. We may write y as an $\mathscr{R}^-$ linear combination of $y_1, \dots, y_{2n-1}$. Let m (resp. M) be the minimal (resp. maximal) index which is supported by y. Hence, we may write $y = a_m y_m + \cdots + a_M y_M$ for $a_m, \dots, a_M \in \mathscr{R}^-$. We observe that

$$\mathrm{gr}_w(y_i) \geq \mathrm{gr}_w(y_{i+2}) \quad \text{and} \quad \mathrm{gr}_z(y_i) \leq \mathrm{gr}_z(y_{i+2}) \tag{4.3}$$

for all i. Since y is homogeneously graded, it follows that a_m is not a multiple of $\mathscr{V}$: if it were, then all other a_i would need to be a multiple of $\mathscr{V}$ for y to be homogeneously graded, which contradicts our assumption. Similarly, a_M is not a multiple of $\mathscr{U}$. We write $a_m = \mathscr{U}^{j_m}$ and $a_M = \mathscr{V}^{j_M}$ for some $j_m, j_M \in \mathbb{N}$. Then $\mathscr{U}^{j_m + \alpha_{(m-1)/2}} x_{m-1}$ and $\mathscr{V}^{j_M + \beta_{(M+1)/2}} x_{M+1}$ are summands of $\partial(y)$, and hence it is not a multiple of any element of $\mathscr{R}^-$.

For the second claim, if $x_i \in P_0$ is a generator, we define $F(x_i)$ to be the unique nonzero element of $\mathscr{R}^-$ in the same homogeneous grading as x. It follows from our normalization of the gradings of $H_*(\mathcal{P}/(\mathscr{U}-1)) \cong \mathbb{F}[\mathscr{V}]$ and $H_*(\mathcal{P}/(\mathscr{V}-1)) \cong \mathbb{F}[\mathscr{U}]$ as well as (4.3) that each generator of $\mathcal{P}$ has $(\mathrm{gr}_w, \mathrm{gr}_z)$–bigrading in $\mathbb{Z}^{\leq 0} \times \mathbb{Z}^{\leq 0}$, so this map is well defined. We leave it to the reader to verify that this map is a chain map and sends $\mathscr{R}^-$–nontorsion cycles to $\mathscr{R}^-$–nontorsion cycles. $\square$

Definition 4.4 We call a complex $\mathcal{P}$ a *positive multistaircase* if it is the tensor product of a nonzero number of positive staircase complexes. We call $\mathcal{N}$ a *negative multistaircase* if it is the tensor product of a nonzero number of negative staircases.

The dual of a positive multistaircase is a negative multistaircase, and vice versa.

By construction, a positive staircase $\mathcal{P}$ has a $\mathbb{Z}$–filtration with two levels, and we write $\mathcal{P}=(P_1\to P_0)$. Hence, a positive multistaircase with n factors has a $\mathbb{Z}$–filtration with $n+1$ nontrivial levels, for which we write

$$\mathcal{P}=(P_n\to P_{n-1}\to\cdots\to P_1\to P_0). \tag{4.5}$$

If $\mathcal{P}=(P_n\to\cdots\to P_0)$ is a positive multistaircase, we say that $\mathcal{P}$ is an *exact* multistaircase if the following sequence is exact:

$$0\to P_n\to\cdots\to P_0.$$

In particular, an exact multistaircase is a free resolution of its homology.

Remark 4.6 In general, the sequence in (4.5) will not be exact. As a concrete example, consider $\mathcal{C}=\mathcal{CFK}^-(T_{2,3})$ and the tensor product $\mathcal{P}=\mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C}$. Write $\mathcal{P}=(P_3\to P_2\to P_1\to P_0)$. Following our conventions, write x_0, y_1 and x_2 for the generators of the left-most factor of $\mathcal{C}$, where $\partial(y_1)=\mathscr{U}x_0+\mathscr{V}x_2$. One easily computes that

$$y_1|x_2|x_0+x_2|y_1|x_0+x_2|x_0|y_1+x_0|x_2|y_1+y_1|x_0|x_2+x_0|y_1|x_2\in P_1$$

is a cycle. In the above, bars denote tensor products. It is not a boundary, since the differential has image in $\operatorname{im}(\mathscr{U})+\operatorname{im}(\mathscr{V})$.

Lemma 4.7 (1) *Every positive staircase is exact.*

(2) *The tensor product of two positive staircases is exact.*

Proof Exactness of a positive staircase $\mathcal{P}=(P_1\to P_0)$ amounts to the claim that the map $P_1\to P_0$ is injective, which is easy to verify.

Next suppose $\mathcal{P}=(P_1\to P_0)$ and $\mathcal{D}=(D_1\to D_0)$ are staircases. We claim that their tensor product is also exact. Let $\mathcal{E}=(E_2\to E_1\to E_0)$ denote this tensor product. Clearly the map $E_2\to E_1$ is injective, so it is sufficient to show that $H_1(\mathcal{E})=0$. The homology $H_*(\mathcal{E})$ decomposes as the direct sum $H_2(\mathcal{E})\oplus H_1(\mathcal{E})\oplus H_0(\mathcal{E})$. Since every $\mathscr{R}^-$–nontorsion element contains a nonzero summand of $H_0(\mathcal{E})$, it follows that $H_1(\mathcal{E})$ consists only of $\mathscr{R}^-$–torsion elements. Since $\mathcal{E}$ is bigraded, each element $[x]\in H_1(\mathcal{E})$ satisfies $\mathscr{U}^i\mathscr{V}^j\cdot[x]=0$ for some i and j. In particular, if $x\in E_1$ is a cycle, then $\mathscr{U}^i\mathscr{V}^j\cdot x\in\operatorname{im}(E_2\to E_1)$ for some i, j. In order to show that $H_1(\mathcal{E})=0$ it is sufficient to show that, if $\mathscr{U}^i\mathscr{V}^j\cdot x\in\operatorname{im}(E_2\to E_1)$, then $x\in\operatorname{im}(E_2\to E_1)$. We argue as follows. Note first that the map from E_2 to E_1 is the sum of the maps $P_1 \otimes D_1 \to P_1 \otimes D_0$ and $P_1 \otimes D_1 \to P_0 \otimes D_1$. Suppose that $\mathscr{U}^i \mathscr{V}^j \cdot x \in \operatorname{im}(E_2 \to E_1)$. Write $\mathscr{U}^i \mathscr{V}^j \cdot x = \partial(y)$. We may assume that x and y are homogeneously graded. Write $x = x_{0,1} + x_{1,0}$, where $x_{1,0} \in P_1 \otimes D_0$ and $x_{0,1} \in P_0 \otimes D_1$. Then $\mathscr{U}^i \mathscr{V}^j \cdot x_{0,1} \in \operatorname{im}(P_1 \to P_0) \otimes D_1$. Since $\mathcal{P}$ is exact and D_1 is free, we conclude that $x_{0,1} \in \operatorname{im}(P_1 \to P_0) \otimes D_1$. Hence there is some $y' \in P_1 \otimes D_1$ such that the map from $P_1 \otimes D_1$ to $P_0 \otimes D_1$ maps y' to $x_{0,1}$. Since the map from $P_1 \otimes D_1$ to $P_0 \otimes D_1$ is injective, we conclude that $\mathscr{U}^i \mathscr{V}^j y' = y$, so $\partial(y') = x_{0,1} + x_{1,0}$ and $x_{0,1} + x_{1,0} \in \operatorname{im}(E_2 \to E_1)$. $\square$

4.2 The staircase complexes for L–space knots

A knot $K \subset S^3$ is called an *L–space knot* if there is a positive integer q such that $S^3_q(K)$ is an L–space, ie $\mathrm{HF}^-(S^3_q(K), \mathfrak{s}) \cong \mathbb{F}[U]$ for each $\mathfrak{s} \in \mathrm{Spin}^c(S^3_q(K))$. All algebraic knots are L–space knots; see [Hedden 2009, Theorem 1.10].

There is a simple description of Floer chain complexes of L–space knots, due to Ozsváth and Szabó [2005, Theorem 1.2]. (Note that, therein, only $\widehat{\mathrm{HFK}}(K)$ is described, but the algorithm actually produces a description of $\mathrm{CFK}^\infty(K)$.) We describe their algorithm presently. Let K be an L–space knot. Ozsváth and Szabó prove that the Alexander polynomial of K, which we denote by $\Delta_K(t)$, has the form

(4.8) $$\Delta_K(t) = t^{a_0} - t^{a_1} + \cdots + t^{a_{2r}},$$

where $0 = a_0 < a_1 < \cdots < a_{2r}$; that is, we use the normalization of Δ starting at degree 0. Define the gap function

$$\beta_i := a_i - a_{i-1}$$

for $1 \le i \le 2r$.

We now describe the complex $\mathcal{CFK}^-(K)$ over the ring $\mathscr{R}^-$. The complex $\mathcal{CFK}^-(K)$ is freely generated over $\mathscr{R}^-$ by elements

$$x_0, y_1, x_2, \dots, y_{2r-1}, x_{2r}.$$

The differential takes the form

(4.9) $$\partial(x_{2i}) = 0 \quad \text{and} \quad \partial(y_{2i+1}) = \mathscr{U}^{\beta_{2i+1}} x_{2i} + \mathscr{V}^{\beta_{2i+2}} x_{2i+2}.$$

The $(\mathrm{gr}_w, \mathrm{gr}_z)$–bigradings are determined by the normalization that $\mathrm{gr}_w(\boldsymbol{x}_0) = 0$ and $\mathrm{gr}_z(\boldsymbol{x}_{2r}) = 0$. Recall that the variable $\mathscr{U}$ has bigrading $(-2, 0)$ and the variable $\mathscr{V}$ has bigrading $(0, -2)$.

The gradings can be expressed in the following way. Write

$$\Delta_K = 1 + (t-1)(t^{m_1} + \cdots + t^{m_s})$$

for some positive integers $m_1 < \cdots < m_s$. Note that the integers β_i compute the number of consecutive integers or consecutive gaps (depending on i) of the sequence $m_1, \dots, m_s$; see [Borodzik and Livingston 2014, Lemma 4.2]. Define $S_K = \mathbb{Z}_{\ge 0} \setminus \{m_1, \dots, m_s\}$, and

(4.10) $$R_K(t) = \#S_K \cap [0, t) \quad \text{if } t \in \mathbb{Z}.$$

With this notation, the gradings of the generator x_{2i} are $\mathrm{gr}_w(x_{2i}) = -2R_K(a_{2i})$ and $\mathrm{gr}_z(x_{2i}) = 2R_K(a_{2i}) - 2g_3(K)$; compare [Borodzik and Livingston 2014, Section 4]. Note that, with our normalization, $2g_3(K) = a_{2r} = m_s + 1$. If the context is clear, we sometimes write R instead of R_K to simplify the notation.

Example 4.11 If K is the $(2, 2n+1)$–torus knot, then the above procedure produces the complex $\mathcal{S}^n$ of Definition 2.28.

Remark 4.12 If K is an algebraic knot, the set S_K turns out to be a semigroup (note that, if K is only an L–space knot, S_K need not be a semigroup). In fact, this is the semigroup of that singular point. The function R_K is the *semigroup counting function*. See [Wall 2004, Section 4] for details on semigroups.

The next corollary is a compilation of [Borodzik and Livingston 2014, Proposition 5.6 and Lemma 6.2]:

Corollary 4.13 *The V_s–invariants of an L–space knot satisfy $V_{-g_3(K)+j}(K) = R_K(j) - j + g_3(K)$.*

The Künneth formula for the knot Floer chain complex allows us to compute the V_j–invariants of a connected sum of L–space knots. The following result is given in [Borodzik and Livingston 2014, formula (6.3)]:

Proposition 4.14 *Let $K_1, \dots, K_n$ be L–space knots. Set $K = K_1 \# \cdots \# K_n$ and let $g = g_3(K)$. Then*

$$V_j(K) + j = R_K(g + j),$$

where $R_K = R_{K_1} \diamond \cdots \diamond R_{K_n}$ is the infimal convolution of $R_{K_1}, \dots, R_{K_n}$.

We recall that, if $I, J : \mathbb{Z} \to \mathbb{Z}$ are two functions bounded from above, their *infimal convolution* is given by $I \diamond J(m) = \min_{i+j=m} I(i) + J(j)$.

4.3 V_s–invariants of tensor products of staircases

In this subsection, we compute the V_s–invariants of certain tensor products of staircases. We wish to understand the V_s–invariants of tensor products of staircases where some factors are positive and some negative. Of course, we may group factors and write such a complex as a tensor product of $\mathcal{N} \otimes \mathcal{P}$, where $\mathcal{N}$ is a negative multistaircase and $\mathcal{P}$ is a positive multistaircase. Clearly,

$$\mathcal{N} \otimes \mathcal{P} \cong \mathrm{Hom}_{\mathscr{R}^-}(\mathcal{N}^\vee, \mathcal{P}),$$

where $\mathrm{Hom}_{\mathscr{R}^-}(N^\vee, \mathcal{P})$ denotes the chain complex of $\mathscr{R}^-$–module homomorphisms from $\mathcal{N}^\vee$ to $\mathcal{P}$. In particular, to understand the V_s–invariants of arbitrary tensor products of positive and negative staircases, it is sufficient to understand the morphism complex between two positive multistaircases.

It is also helpful to note that, if $\mathcal{N}$ and $\mathcal{P}$ are multistaircases (of either sign), then a cycle $\phi \in \operatorname{Hom}_{\mathscr{R}^-}(\mathcal{N}^\vee, \mathcal{P})$ is $\mathscr{R}^-$–nontorsion as a morphism if and only if ϕ maps $\mathscr{R}^-$–nontorsion cycles to $\mathscr{R}^-$–nontorsion cycles.

The following result is by now classical (see [Borodzik and Livingston 2014, Proposition 5.1]):

Proposition 4.15 *Let $\mathcal{P} = (P_n \to \cdots \to P_0)$ be a positive multistaircase and let $s \in \mathbb{Z}$. Then*

$$V_s(\mathcal{P}) = \min_{x \in \mathcal{G}(P_0)} \max(\alpha(x), \beta(x) - s),$$

where $\alpha(x) = -\frac{1}{2}\operatorname{gr}_w(x)$, $\beta(x) = -\frac{1}{2}\operatorname{gr}_z(x)$ and $\mathcal{G}(P_0)$ denotes the set of homogeneously graded generators of P_0.

Proof Lemma 4.2 implies that a homogeneously graded element $x \in \mathcal{P}$ is an $\mathscr{R}^-$–nontorsion cycle if and only if its summand in P_0 may be written as an $\mathscr{R}^-$–linear combination of an odd number of distinct elements in the generating set $\mathcal{G}(P_0)$ with nonzero, homogeneously graded coefficients in $\mathscr{R}^-$. In particular, the individual elements of $\mathcal{G}(P_0)$ determine the correction terms V_s. The expression $-2\max(\alpha(x), \beta(x) - s)$ is the maximal gr_w–grading of an element of the form $\mathscr{U}^m \mathscr{V}^n x$ such that $m, n \geq 0$ and $x \in \mathscr{A}_s$. Taking the minimum over all $x \in \mathcal{G}(P_0)$ gives the result. $\square$

We now pass to studying V_s–invariants of products of positive and negative multistaircases. We begin with the following statement, where we write $H_0(\mathcal{P})$ for $P_0/\operatorname{im} P_1$ for a multistaircase:

Proposition 4.16 *Suppose that $\mathcal{P} = (P_m \to \cdots \to P_0)$ and $\mathcal{Q} = (Q_n \to \cdots \to Q_0)$ are two positive multistaircases.*

(1) *In general*, $V_s(\operatorname{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q})) \geq V_s\big(\operatorname{Hom}_{\mathscr{R}^-}(H_*(\mathcal{P}), H_*(\mathcal{Q}))\big) = V_s\big(\operatorname{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q}))\big)$.

(2) *If $\mathcal{Q}$ is exact*, *then* $V_s(\operatorname{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q})) = V_s\big(\operatorname{Hom}_{\mathscr{R}^-}(H_*(\mathcal{P}), H_*(\mathcal{Q}))\big)$.

Proof There is a grading-preserving map of $\mathscr{R}^-$–modules

$$H_* \operatorname{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q}) \to \operatorname{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q})),$$

which sends $\mathscr{R}^-$–nontorsion elements to $\mathscr{R}^-$–nontorsion elements. Then the inequality of part (1) follows since the map sends $\mathscr{R}^-$–nontorsion elements in $\mathscr{A}_s(\operatorname{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q}))$ to $\mathscr{R}^-$–nontorsion elements in $\mathscr{A}_s\big(\operatorname{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q}))\big)$. The equality in part (1) follows since $H_*(\mathcal{P})$ decomposes as a direct sum

$$\bigoplus_{s=0}^{n} (\ker(P_i \to P_{i-1})/\operatorname{im}(P_{i+1} \to P_i)),$$

and $H_0(\mathcal{P}) = P_0/\operatorname{im} P_1$ is the only summand which contains $\mathscr{R}^-$–nontorsion elements.

We now consider the second claim. Suppose that $\mathcal{Q}$ is exact. We will show

$$V_s\big(\mathrm{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q}))\big) \geq V_s(\mathrm{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q})). \tag{4.17}$$

Suppose $\phi\colon H_0(\mathcal{P}) \to H_0(\mathcal{Q})$ is an $\mathscr{R}^-$–module homomorphism which maps $\mathscr{R}^-$–nontorsion elements to $\mathscr{R}^-$–nontorsion elements. It suffices to extend ϕ to obtain a commutative diagram

$$\begin{tikzcd}
P_m \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "\phi_2"] & P_1 \arrow[r] \arrow[d, dashed, "\phi_1"] & P_0 \arrow[r, two heads] \arrow[d, dashed, "\phi_0"] & H_0(\mathcal{P}) \arrow[d, "\phi"] \\
& \cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, two heads] & H_0(\mathcal{Q})
\end{tikzcd}$$

since this extension gives an $\mathscr{R}^-$–nontorsion element in $\mathscr{A}_s(\mathrm{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q}))$ corresponding to any $\mathscr{R}^-$–nontorsion element in $\mathscr{A}_s\big(\mathrm{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q}))\big)$. The construction of the maps ϕ_i follows from the same procedure as in [Weibel 1994, Theorem 2.2.6] and the discussion below it. We briefly summarize the construction. The map ϕ_0 may be chosen since P_0 is free, and hence projective, and $Q_0 \to H_0(\mathcal{Q})$ is surjective. Having constructed ϕ_0, we next construct ϕ_1. Using exactness of $\mathcal{Q}$, we may factor $\phi_0 \circ (P_1 \to P_0)$ into $\mathrm{im}(Q_1 \to Q_0)$. Using the fact that P_1 is projective and $Q_1 \to \mathrm{im}(Q_1 \to Q_0)$ is surjective, we obtain a map ϕ_1. We repeat this process until we exhaust $\mathcal{P}$. This gives (4.17). □

Proposition 4.18 *Suppose $\mathcal{N} = (N_0 \to \cdots \to N_n)$ is a negative multistaircase, and $\mathcal{P} = (P_m \to \cdots \to P_0)$ is a positive multistaircase. Write $\mathcal{G}(P_i)$ for the generators of P_i, and similarly for $\mathcal{G}(N_i)$.*

(1) *In general,*

$$V_s(\mathcal{N} \otimes \mathcal{P}) \geq -\tfrac{1}{2} \min_{x \in \mathcal{G}(N_0)} \max_{y \in \mathcal{G}(P_0)} \min(\mathrm{gr}_w(x) + \mathrm{gr}_w(y), \mathrm{gr}_z(x) + \mathrm{gr}_z(y) + 2s). \tag{4.19}$$

(2) *If $\mathcal{P} = (P_1 \to P_0)$ is a positive staircase, then (4.19) is an equality.*

Proof We dualize, and consider the isomorphism $\mathcal{N} \otimes \mathcal{P} \cong \mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P})$. For the first claim, suppose $\phi \in \mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P})$ is an $\mathscr{R}^-$–nontorsion cycle which is of homogeneous grading $(d, d-2s)$, where $d = d\big(\mathscr{A}_s(\mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P}))\big)$. Note $\phi \in \mathscr{A}_s(\mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P}))$. For each $x^\vee \in \mathcal{G}(N_0^\vee)$, $\phi(x^\vee)$ is an $\mathscr{R}^-$–nontorsion cycle, and hence must contain a summand of the form $f \cdot y$ for some nonzero monomial $f \in \mathscr{R}^-$ and $y \in \mathcal{G}(P_0)$. By the definition of the grading of a morphism, we have

$$\mathrm{gr}_w(y) - \mathrm{gr}_w(x^\vee) + \mathrm{gr}_w(f) = d \quad \text{and} \quad \mathrm{gr}_z(y) - \mathrm{gr}_z(x^\vee) + \mathrm{gr}_z(f) = d - 2s.$$

Since $\mathrm{gr}_w(f) \leq 0$ and $\mathrm{gr}_z(f) \leq 0$, and $(\mathrm{gr}_w(x^\vee), \mathrm{gr}_z(x^\vee)) = (-\mathrm{gr}_w(x), -\mathrm{gr}_z(x))$, for each x,

$$d\big(\mathscr{A}_s(\mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P}))\big) \leq \max_{y \in \mathcal{G}(P_0)} \min(\mathrm{gr}_w(x) + \mathrm{gr}_w(y), \mathrm{gr}_z(x) + \mathrm{gr}_z(y) + 2s).$$

Taking the minimum over $x \in \mathcal{G}(N_0)$ gives the statement.

We now consider the second claim. Suppose that $\mathcal{P} = (P_1 \to P_0)$ is a positive staircase. Using Lemma 4.7 and Proposition 4.16, we know that

$$V_s(\mathcal{N} \otimes \mathcal{P}) = V_s\big(\mathrm{Hom}_{\mathscr{R}^-}(H_0(\mathcal{N}^\vee), H_0(\mathcal{P}))\big).$$

Fix $s \geq 0$. Let δ_s denote the right-hand side of (4.19) without the factor of $-\frac{1}{2}$. For each $x^\vee$ in $\mathcal{G}(N_0^\vee)$, we pick a $y_x \in \mathcal{G}(P_0)$ so that

$$\operatorname{gr}_w(y_x) - \operatorname{gr}_w(x^\vee) \geq d \quad \text{and} \quad \operatorname{gr}_z(y_x) - \operatorname{gr}_z(x^\vee) \geq d - 2s.$$

We set $\phi_0 \colon N_0^\vee \to P_0$ to be the map which takes $x^\vee$ to $f_x \cdot y_x$, where $f_x \in \mathscr{R}^-$ is the unique monomial such that ϕ_0 has bigrading $(d, d-2s)$. By composition, we obtain a map $\phi' \colon N_0^\vee \to H_0(\mathcal{P})$.

Claim *The map ϕ' vanishes on* $\operatorname{im}(N_1^\vee)$.

Given the claim, we quickly conclude the proof. In fact, we obtain a map ϕ from $H_0(\mathcal{N})$ to $H_0(\mathcal{P})$. Hence, we may use the second part of Proposition 4.16 to conclude that

$$d\left(\mathscr{A}_s(\operatorname{Hom}(\mathcal{N}^\vee, \mathcal{P}))\right) \geq \delta_s,$$

which completes the proof modulo the claim.

It remains to prove the claim. Let $y_1 \in N_1^\vee$. We consider the element $v = \partial(y_1) \in N_0^\vee$. We can write v as a sum $\sum_{x^\vee \in \mathcal{G}(N_0^\vee)} f_x \cdot x^\vee$, where each f_x is a monomial. Tensoring the maps from the second part of Lemma 4.2, we obtain a chain map from $\mathcal{N}^\vee$ to $\mathscr{R}^-$, which is nonzero only on $N_0^\vee$ and, furthermore, maps each generator of $N_0^\vee$ to a monomial. Using the fact that this map is a chain map, we see that the number of $x^\vee \in \mathcal{G}(N_0^\vee)$ where f_x is nonzero is even. It follows immediately that $\phi_0(v)$ is an $\mathscr{R}^-$–torsion cycle. By Lemma 4.2, $H_*(\mathcal{P})$ is torsion-free, so $[\phi_0(v)] = 0 \in H_*(\mathcal{P}) = P_0/\operatorname{im}(P_1)$. This proves the claim and completes the proof of Proposition 4.18. □

4.4 A counterexample

We give an example indicating that the second statement of Proposition 4.18 need not hold if $\mathcal{P}$ is a product of more than one positive staircase, even if $\mathcal{P}$ is exact.

Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be the staircases of torus knots $T_{6,7}$ and $T_{4,5}$, respectively. As described in Section 4.2, the generators of $\mathcal{P}^1$ are at bigradings $(-30,0)$, $(-30,-2)$, $(-20,-2)$, $(-20,-6)$, $(-12,-6)$, $(-12,-12)$, $(-6,-12)$, $(-6,-20)$, $(-2,-20)$, $(-2,-30)$ and $(0,-30)$. We denote these generators by $x_0, y_1, \dots, x_{10}$. We have $\partial x_{2i} = 0$ and $\partial y_{2i+1} = \mathscr{U}^{\alpha_i} x_{2i+2} + \mathscr{V}^{\beta_i} x_{2i}$, where α_i and β_i are nonnegative integers determined by the condition that ∂ preserve the grading. In particular, the generators with odd index generate $\mathcal{P}_1^1$, while the generators with even index span $\mathcal{P}_0^1$.

Likewise, there are generators $x_0', y_1', \dots, x_6'$ for $\mathcal{P}^2$ with bigradings $(-12,0)$, $(-12,-2)$, $(-6,-2)$, $(-6,-6)$, $(-2,-6)$, $(-2,-12)$ and $(0,-12)$.

Lemma 4.20 *Let $\mathcal{P} = \mathcal{P}^1 \otimes \mathcal{P}^2$. The only elements x in $\mathcal{P}$ such that $\operatorname{gr}_w(x) = \operatorname{gr}_z(x) > -18$ are linear combinations of $\mathscr{U}^i \mathscr{V}^j x_4 \otimes x_4'$ with $(i,j) = (0,1), (1,2)$ and $\mathscr{U}^{i'} \mathscr{V}^{j'} x_6 \otimes x_2'$ with $(i',j') = (1,0), (2,1)$.*

Proof This is by direct inspection. □

Now let $\mathcal{N}$ be the negative staircase complex of the mirror of the trefoil. It is generated by elements c_0, c_1 and c_2 at bigradings $(2,0)$, $(2,2)$ and $(0,2)$, respectively. The differential is given by $\partial c_0 = \mathscr{V} c_1$, $\partial c_2 = \mathscr{U} c_1$ and $\partial c_1 = 0$. That is, $c_0, c_2 \in \mathcal{N}_0$ and $c_1 \in \mathcal{N}_{-1}$.

Lemma 4.21 *There is no cycle $z \in \mathscr{A}_0(\mathcal{N} \otimes \mathcal{P})$ such that $\mathrm{gr}_w(z) \geq -12$ and $z \neq 0$.*

Proof Any such cycle would be a linear combination of elements of type $\mathscr{U}^i \mathscr{V}^j \cdot x_k \otimes x'_\ell \otimes c_m$. By Lemma 4.20, unless $(k,\ell) = (4,4)$ or $(6,2)$, the gr_w–grading of such a combination is at most -14. Hence, if $z \in \mathscr{A}_0(\mathcal{N} \otimes \mathcal{P})$ and $z \neq 0$ has $\mathrm{gr}_w(z) \geq -12$, then z has to be a linear combination of

$$x_4 \otimes_4 \otimes c_0 \quad \text{and} \quad x_6 \otimes x'_2 \otimes c_2.$$

But then z is not a cycle. □

Corollary 4.22 *We have $V_0(\mathcal{N} \otimes \mathcal{P}) \geq 7$.*

The following result shows that the right-hand side of (4.19) is strictly smaller than 7:

Lemma 4.23 *The expression*

$$-\tfrac{1}{2} \min_{x \in G(\mathcal{N}_0)} \max_{y \in G(\mathcal{P}_0)} \min(\mathrm{gr}_w(x) + \mathrm{gr}_w(y), \mathrm{gr}_z(x) + \mathrm{gr}_z(y))$$

is equal to 6.

Proof For $x = c_0$, the expression

$$\max_{y \in G(\mathcal{P}_0)} \min(\mathrm{gr}_w(x) + \mathrm{gr}_w(y), \mathrm{gr}_z(x) + \mathrm{gr}_z(y)) \tag{4.24}$$

is equal to -12 with the equality attained at $y = x_4 \otimes x'_4$. For $x = c_2$, (4.24) attains its maximal value -12 for $y = x_6 \otimes x'_2$. □

4.5 More on the V_s–invariants of tensor products of staircases

In this subsection, we highlight some special cases of Propositions 4.15 and 4.18 which will be useful for our purposes.

Corollary 4.25 *Suppose $\mathcal{P}$ is a positive multistaircase and, for $i \in \{1, \dots, r\}$, let $\mathcal{S}^{n_i}$ denote the staircase complex of Definition 2.28 with $\sum n_i \geq 0$. Then*

$$V_s(\mathcal{P} \otimes \mathcal{S}^{n_1} \otimes \cdots \otimes \mathcal{S}^{n_r}) = \min_{0 \leq j \leq \sum n_i} (V_{s+2j-\sum n_i}(\mathcal{P}) + j).$$

Proof By Proposition 3.9, we know that $\mathcal{S}^{n_1} \otimes \cdots \otimes \mathcal{S}^{n_r}$ is locally equivalent to $\mathcal{S}^n$, where $n = \sum n_i$, so, by Proposition 3.10, it suffices to prove the result when $i = 1$. Write $a_1, \dots, a_m$ for the generators of C_0, and write $x_0, x_2, \dots, x_{2n}$ for the generators of $\mathcal{S}_0^n$. Then $a_i \otimes x_{2j}$ forms a basis of homogeneously graded elements of $(\mathcal{P} \otimes \mathcal{S}^n)_0$. By Proposition 4.16,

$$V_s(\mathcal{P} \otimes \mathcal{S}^n) = \min_{\substack{1 \le i \le m \\ 0 \le j \le n}} \max(\alpha(a_i) + \alpha(x_{2j}), \beta(a_i) + \beta(x_{2j}) - s).$$

We note that $\alpha(x_{2j}) = j$ and $\beta(x_{2j}) = n - j$, so we conclude that

$$\begin{aligned} V_s(\mathcal{P} \otimes \mathcal{S}^n) &= \min_{\substack{1 \le i \le m \\ 0 \le j \le n}} \max(\alpha(a_i) + j, \beta(a_i) + n - j - s) \\ &= \min_{0 \le j \le n} \min_{1 \le i \le m} \bigl(\max(\alpha(a_i), \beta(a_i) + n - 2j - s) + j\bigr) \\ &= \min_{0 \le j \le n} (V_{s+2j-n}(\mathcal{P}) + j). \end{aligned}$$ □

We have the following corollary of Proposition 4.18:

Corollary 4.26 *Suppose $\mathcal{P}$ is a positive staircase and, for $i \in \{1, \dots, r\}$, let $\mathcal{S}^{n_i}$ denote the staircase complexes of Definition 2.28. Assume $\sum n_i < 0$. Then*

$$V_s(\mathcal{P} \otimes \mathcal{S}^{n_1} \otimes \cdots \otimes \mathcal{S}^{n_r}) = \max_{0 \le j \le n} (V_{s-2j+n}(\mathcal{P}) - j),$$

where $n = -\sum n_i$.

Remark 4.27 In contrast to Corollary 4.25, where $\mathcal{P}$ was allowed to be a positive *multi*staircase (ie a tensor product of positive staircases), in Corollary 4.26 we require that $\mathcal{P}$ be a positive *staircase*.

Proof of Corollary 4.26 As in the proof of Corollary 4.25, $\mathcal{S}^{n_1} \otimes \cdots \otimes \mathcal{S}^{n_r}$ is locally equivalent to $\mathcal{S}^{-n}$ for some $n > 0$, so it is sufficient to consider the case when $i = 1$. Write $a_1, \dots, a_q$ for the generators of C_0, and $\tilde{x}_0, \tilde{x}_2, \dots, \tilde{x}_{2n}$ for the generators of the 0–level of $\mathcal{S}^{-n}$. According to Proposition 4.18,

$$\begin{aligned} V_s(\mathcal{P} \otimes \mathcal{S}^{-n}) &= \max_{0 \le i \le n} \min_{1 \le j \le q} \max(\alpha(a_j) + \alpha(\tilde{x}_{2i}), \beta(a_j) + \beta(\tilde{x}_{2i}) - s) \\ &= \max_{0 \le i \le n} \min_{1 \le j \le q} \max(\alpha(a_j) - i, \beta(a_j) - n + i - s) \\ &= \max_{0 \le i \le n} \min_{1 \le j \le q} \bigl(\max(\alpha(a_j), \beta(a_j) - n + 2i - s) - i\bigr) \\ &= \max_{0 \le i \le n} (V_{s-2i+n}(\mathcal{P}) - i). \end{aligned} \tag{4.28}$$ □

4.6 Knots with split towers

We now introduce the notion of a knot complex with *split towers*. The correction terms of a knot complex with split towers have a relatively simple form. An important example of a knot with split towers is connected sums of knotifications of positive and negative $(2, 2n)$–torus links.

Definition 4.29 (split towers) Let K be a knot in $Y = \#^m S^2 \times S^1$, and let $\mathcal{C}$ be a chain complex which is free and finitely generated over $\mathscr{R}^-$ and is homotopy equivalent to $\mathcal{CFK}^-(Y, K, \mathfrak{s}_0)$, where $\mathfrak{s}_0$ is the trivial Spin^c structure on Y. We say that $\mathcal{C}$ has *split towers* if there exists a basis $\gamma_1, \dots, \gamma_m$ of $H_1(\#^m S^2 \times S^1; \mathbb{Z})$ and subcomplexes $\mathcal{C}^I_* \subset \mathcal{C}$, indexed over subsets $I \subset \{\gamma_1, \dots, \gamma_m\}$, such that the following are satisfied:

(a) $\mathcal{C} = \bigoplus_{I \subset \{\gamma_1, \dots, \gamma_m\}} \mathcal{C}^I$.

(b) If $\gamma_i \notin I$, then A_{γ_i} takes $H_*(\mathcal{C}^I)$ to $H_*(\mathcal{C}^{I \cup \{\gamma_i\}})$, and becomes an isomorphism after inverting $\mathscr{U}$ and $\mathscr{V}$. If $\gamma_i \in I$, then A_{γ_i} vanishes on $H_*(\mathcal{C}^I)$, after inverting $\mathscr{U}$ and $\mathscr{V}$.

Abusing notation slightly, we say a knot K has split towers if there is a representative of $\mathcal{CFK}^-(Y, K)$ which has split towers. Note that, in many of our examples, the homology action actually respects the splitting on the chain level, ie A_{γ_i} maps $\mathcal{C}^I$ to $\mathcal{C}^{I \cup \{\gamma_i\}}$ if $\gamma_i \notin I$, and A_{γ_i} vanishes on $\mathcal{C}^I$ if $\gamma_i \in I$.

Example 4.30

- Any knot K in S^3 has split towers (trivially).
- The knotification of the $(2, 2n)$–torus link has split towers. See Proposition 2.40.
- The Borromean knot does not have split towers.

Lemma 4.31 *If K and K' have split towers, then $K \# K'$ has split towers.*

Proof This is a direct consequence of the Künneth formula. □

Proposition 4.32 *Suppose K is a knot in $\#^m S^2 \times S^1$ with split towers. Write*

$$\mathcal{C}^{\mathrm{top}} = \mathcal{C}^{\varnothing} \quad \text{and} \quad \mathcal{C}^{\mathrm{bot}} = \mathcal{C}^{\gamma_1, \dots, \gamma_m}.$$

Then

$$V_s^{\mathrm{top}}(K) = V_s(\mathcal{C}^{\mathrm{top}}) \quad \text{and} \quad V_s^{\mathrm{bot}}(K) = V_s(\mathcal{C}^{\mathrm{bot}}).$$

Suppose, additionally, that $n > 0$ and $\mathcal{B}_0$ is the Borromean knot. Then

$$V_s^{\mathrm{top}}\big(K \# \#^n \mathcal{B}_0\big) = -\tfrac{1}{2}n + \min_{0 \le j \le n} (V_{s+2j-n}(\mathcal{C}^{\mathrm{top}}) + j),$$
$$V_s^{\mathrm{bot}}\big(K \# \#^n \mathcal{B}_0\big) = -\tfrac{1}{2}n + \max_{0 \le j \le n} (V_{s+2j-n}(\mathcal{C}^{\mathrm{bot}}) + j).$$

Proof We consider first the proof that $V_s^{\mathrm{top}}(K) = V_s(\mathcal{C}^{\mathrm{top}})$. It is sufficient to show that

$$d_s^{\mathrm{top}}(K) = d(\mathcal{C}_s^{\mathrm{top}}), \tag{4.33}$$

where $\mathcal{C}_s^{\mathrm{top}}$ denotes the subcomplex of $\mathcal{C}^{\mathrm{top}}$ in Alexander grading s, and these d–invariants are defined in Definitions 3.3 and 3.11. By definition, $d_s^{\mathrm{top}}(K)$ is the maximal grading of a homogeneously graded element of $H_*(\mathscr{A}_s(K))$ which maps to an element of $U^{-1} H_*(\mathscr{A}_s(K))$ having nontrivial image in $\mathcal{H}^{\mathrm{top}}$.

Since K has split towers, by Definition 4.29 the cokernel $\mathcal{H}^{\mathrm{top}}$ is spanned by $U^{-1}H_*(\mathcal{C}_s^{\mathrm{top}})$, and $H_*(\mathcal{C}_s^I)$ has trivial image for $I \neq \varnothing$, equation (4.33) follows.

The claim about d^{bot} is similar. In this case, $d_s^{\mathrm{bot}}(K)$ is defined as the maximal grading of a homogeneous element in $H_*(\mathscr{A}_s(K))/\mathrm{Tors}$ which is in the image of $\mathcal{H}^{\mathrm{bot}}$. This is clearly $d(\mathcal{C}_s^{\mathrm{bot}})$.

We pass now to the second part of the proof. An analogous argument appeared in [Bodnár et al. 2016; Borodzik et al. 2017]; we recall it for completeness. The complex $\mathcal{CFK}^-(\mathcal{B}_0)$ is described in Section 2.7. Since $\mathcal{CFK}^-(\mathcal{B}_0)$ has vanishing differential, we obtain

$$H_*(\mathcal{CFK}^-(K)\otimes \mathcal{CFK}^-(\mathcal{B}_0)^{\otimes n}) \cong \mathcal{HFK}^-(K)\otimes_{\mathbb{F}} \mathbb{B}^{\otimes n},$$

where $\mathbb{B}$ is the 4–dimensional vector space spanned by 1, x, y and xy, whose bigradings are shown in (2.42).

We first consider the claim about V_s^{bot}. Using the H_1–action on $\mathcal{CFK}^-(\mathcal{B}_0)$ described in Section 2.7, one easily obtains the following: a cycle $x \in \mathscr{A}_s\big(K \,\#\, \#^n\, \mathcal{B}_0\big)$ is of homogeneous gr_w–grading d, is $\mathbb{F}[U]$–nontorsion, and maps to the kernel of the H_1–action in $U^{-1}H_*(\mathscr{A}_s(K \,\#\, B^{\# n}))$ if and only if it has the form

$$\sum_{\{a_1,\dots,a_n\}\in\{-1,1\}^n} x_{a_1,\dots,a_n}\otimes \epsilon_{a_1}\otimes\cdots\otimes\epsilon_{a_n},$$

where $\epsilon_{-1} = 1 \in \mathbb{B}$ and $\epsilon_1 = xy \in \mathbb{B}$ with $\mathrm{gr}_w = 1$ and -1, respectively. Moreover, each

$$x_{a_1,\dots,a_n} \in \mathcal{C}^{\mathrm{bot}}_{s+\sum a_i}(K)$$

is an $\mathbb{F}[U]$–nontorsion cycle of homogeneous gr_w–grading $d+\sum a_i$. Noting that $\sum a_i$ can be any integer of the form $n-2j$ for $0 \le j \le n$, we obtain that

$$d^{\mathrm{bot}}\big(\mathscr{A}_s\big(K \,\#\, \#^n\, \mathcal{B}_0\big)\big) = \min_{0\le j\le n}\,(d(\mathcal{C}^{\mathrm{bot}}_{s+n-2j}) - n + 2j).$$

Multiplying by $-\frac{1}{2}$ and switching j to $n-j$ yields the statement.

The proof for d^{top} is analogous. The cokernel of the H_1–action on $U^{-1}H_*\big(\mathscr{A}_s\big(K \,\#\, \#^n\, \mathcal{B}_0\big)\big)$ is spanned by any element of the form $x\otimes\epsilon_{a_1}\otimes\cdots\otimes\epsilon_{a_n}$ where ϵ_{a_i} are as above and $x \in \mathcal{C}^{\mathrm{top}}_{s+\sum a_i}(K)$ is a homogeneously graded, $\mathbb{F}[U]$–nontorsion element. Furthermore, any homogeneous element generating $U^{-1}H_*\big(\mathscr{A}_s\big(K \,\#\, \#^n\, \mathcal{B}_0\big)\big)$ is a sum of an odd number of such elements. The same argument as before shows that

$$d^{\mathrm{top}}\big(\mathscr{A}_s\big(K \,\#\, \#^n\, \mathcal{B}_0\big)\big) = \max_{0\le j\le n}\,(d(\mathcal{C}^{\mathrm{top}}_{s+n-2j}) - n + 2j).$$

Multiplying by $-\frac{1}{2}$ and switching j to $n-j$ yields the statement. □

5 Topology of complex curves and their neighborhoods

In this section we give a precise definition of the notion of a tubular neighborhood of a possibly singular curve in $\mathbb{C}P^2$. We describe the boundary of this neighborhood in terms of the surgery on a link. We perform several helpful algebrotopological computations.

As the main focus of our article is on algebraic curves, we present the construction using the language of complex geometry. In Section 5.4 we will show how to generalize our results to the smooth category.

5.1 "Tubular" neighborhood of a complex curve

Let $C \subset \mathbb{C}P^2$ be a reduced complex curve of degree d. We do not insist that C be irreducible. We write $C_1, \dots, C_e$ for the irreducible components of C and let $d_1, \dots, d_e$ (resp. $g_1, \dots, g_e$) denote their degrees (resp. genera). Hereafter, by the *genus* $g(C)$ of a complex curve, we mean the genus of its normalization, that is, the geometric genus. From the topological perspective, the geometric genus of a singular curve is the sum of genera of connected components of the smooth locus of the curve, regarded as an open surface. We set $g = g_1 + \dots + g_e$.

We denote by $p_1, \dots, p_u$ the singular points of C. For each such singular point p_i, we denote by r_i the number of branches. Here, recall that a branch of C at p_i is a connected component of $B_i \cap (C \setminus \{p_i\})$ for a sufficiently small ball $B_i \subset \mathbb{C}^2$ centered at p_i. We write $\mathcal{L}_i$ for the link of singularity at p_i, whose components are $L_{i1}, \dots, L_{ir_i}$. We choose once and for all pairwise disjoint closed balls $B_1, \dots, B_u$ with centers $p_1, \dots, p_u$, respectively, and such that $C \cap \partial B_i$ is the link $\mathcal{L}_i$ and $C \cap B_i$ is homeomorphic to a cone over $\mathcal{L}_i$.

As the curve C is not smoothly embedded at its singular points, the notion of a tubular neighborhood of C requires some clarification. The following is an extension of the construction of [Borodzik and Livingston 2014].

Take a tubular neighborhood N_0 in $\mathbb{C}P^2 \setminus (B_1 \cup \dots \cup B_u)$ of the smooth part $C_0 := C \setminus (B_1 \cup \dots \cup B_u)$. Note that all components $C_1, \dots, C_e$ intersect each other; hence, C is connected. On the other hand, the balls $B_1, \dots, B_u$ contain all the intersection points between various curves $C_1, \dots, C_e$. Hence, C_0 has e connected components, which are $C_i \setminus (B_1 \cup \dots \cup B_u)$ for $i = 1, \dots, e$. We define N to be the union of N_0 and $B_1, \dots, B_u$. With $g = g_1 + \dots + g_e$, set

$$\rho = 2g - e + 1 + \sum_{i=1}^{u} (r_i - 1) = b_1(C) = \dim H_1(C; \mathbb{Q}). \tag{5.1}$$

To see that $\dim H_1(C; \mathbb{Q}) = \rho$, we consider the normalization C' of C. It is a surface of genus g with e connected components. So $\chi(C') = 2e - 2g$. Next, C arises from C' by gluing r_i–tuples of points (corresponding to singular points of C) for $i = 1, \dots, u$. Hence $\chi(C) = 2e - 2g - \sum (r_i - 1)$. Now C is connected, and $\dim H_2(C; \mathbb{Q}) = e$. From this, we recover the formula for $\dim H_1(C; \mathbb{Q})$.

Observe that C_0 arises from the normalization C' by removing $\sum r_i$ disks. The first disk for each connected component of C' kills an element in H_2, and all of the subsequent disks create a basis element in H_1. That is to say, $\dim H_1(C_0;\mathbb{Q}) = 2g+\sum r_i - e = \rho+u-1$. By duality, $\dim H_1(C_0,\partial C_0;\mathbb{Q}) = \rho+u-1$.

We now provide a surgery-theoretical description of N and its boundary Y. We first define a 3–manifold Z containing a link $\mathcal{L}$, as follows. We begin with the disjoint union $\mathcal{L}_0 := \mathcal{L}_1 \sqcup \cdots \sqcup \mathcal{L}_u$ in $Z_0 := S^3 \sqcup \cdots \sqcup S^3$. Next, we pick a collection of pairwise disjoint and properly embedded arcs $\lambda_1,\dots,\lambda_{\rho+u-1}$ on C_0 which form a basis of $H_1(C_0,\partial C_0)$. Such a collection of arcs cuts C_0 into a union of e disks, one for every connected component of C_0. We let $Z = \#^{\rho} S^2\times S^1$ be the boundary of the 4–manifold Γ obtained by attaching $\rho+u-1$ 4–dimensional 1–handles to $\partial(B_1\cup\cdots\cup B_u) = Z_0$, each containing a 2–dimensional band (corresponding to a λ_i), which we attach to $\mathcal{L}_0$. We let $\mathcal{L}\subset Z$ be the resulting link. By construction, $\mathcal{L}$ is a link inside of the connected sum of ρ copies of $S^1\times S^2$. Furthermore, each component of $\mathcal{L}$ is null-homologous. The number of components of $\mathcal{L}$ is the number of disks $C_0\setminus(\lambda_1\cup\cdots\cup\lambda_{\rho+u-1})$. That is, $\mathcal{L}$ has e components, denoted henceforth by $L_1,\dots,L_e$, corresponding to connected components of C_0, ie to irreducible components of the complex curve C.

We have the following (compare [Borodzik et al. 2017, Theorem 3.1; Bodnár et al. 2016, Lemma 3.1]):

Proposition 5.2 *The 3–manifold $Y=\partial N$ is the surgery on $\mathcal{L}\subset Z$ with surgery coefficients $(d_1^2,\dots,d_e^2)$. The 4–manifold N is obtained by attaching e 2–handles to the boundary connected sum of ρ copies of $D^3\times S^1$.*

Proof The fact that N is obtained by attaching e 2–handles to Γ along $\mathcal{L}$ follows from the fact that the complement $C_0\setminus(\lambda_1,\dots,\lambda_{\rho+u-1})$ is a collection of disks $C_1',\dots,C_e'$ (we know that this complement has e components). The thickening of C_i' is a 2–handle in N. Upon renumbering, we may and will assume that C_i' is a subset of C_i and $\partial C_i' = L_i$, the component of $\mathcal{L}$. In particular, we know that N is the effect of a surgery on $\mathcal{L}$. It remains to determine the framing.

In order to do this, we recall that, if a 2–handle A is attached to B^4 along a knot $K\subset S^3=\partial B^4$, the framing of the 2–handle is determined as a self-intersection number of the surface F obtained by capping the core C of the 2–handle with a Seifert surface for K. We note that the self-intersection number does not depend on the choice of the Seifert surface. Moreover, instead of a Seifert surface, we can take any smooth compact surface in B^4 whose boundary is K.

The same procedure applies for surgeries on null-homologous knots in $\#^{\rho} S^2\times S^1$. In the present context, when we calculate the surgery coefficient at L_i, the role of the surface F is played by the union of C_i' and a surface in $\Gamma = \#^{\rho} B^3\times S^1$ bounding L_i. A possible choice for F is then a smoothing of C_i, which essentially replaces $C_i\cap\Gamma$ by a smooth compact surface in Γ with boundary L_i. That is to say, the self-intersection number of F is exactly the self-intersection number of C_i, which is d_i^2. □

Remark 5.3 If $e=1$, $\mathcal{L}$ is a knot. This knot can be obtained as a connected sum of $\widehat{\mathcal{L}}_1,\dots,\widehat{\mathcal{L}}_u$ and g copies of the Borromean knot. Here the hat denotes knotification.

5.2 Algebraic topology

In this section, we describe some basic algebrotopological facts about the tubular neighborhood N, and its boundary Y. Our description of Spin^c structures is similar to that in [Manolescu and Ozsváth 2010, Section 11.1].

Recall that, if N is a manifold obtained by gluing e handles along a null-homologous link to a four-manifold Γ with $H_2(\Gamma;\mathbb{Z})=0$, we can speak not only of a framing of handles, but of a framing matrix. An argument using the Mayer–Vietoris sequence reveals that $H_2(N;\mathbb{Z})=\mathbb{Z}^e$ is generated by the cores of the handles capped by Seifert surfaces of the components of the link. The *framing matrix*, denoted by Ξ, is the matrix of the intersection form $H_2(N;\mathbb{Z})\times H_2(N;\mathbb{Z})\to\mathbb{Z}$. In particular, the diagonal entries are surgery coefficients. The off-diagonal terms are linking numbers of the corresponding links (these are well defined as long as the components are null-homologous).

In the present situation, by Proposition 5.2, the surgery coefficients are $(d_1^2,\dots,d_e^2)$. The same argument shows that the off-diagonal terms are given by the intersection number of C_i with C_j. That is, the framing matrix has the form

$$\Xi=\{d_i d_j\}_{i,j=1}^e.$$

Note that this construction in particular reveals that $\operatorname{lk}(L_i,L_j)=d_i d_j$. We let $W_\Lambda(\mathcal{L})$ denote the 2–handle cobordism from Z to Y. Recall that N is the union of the 1–handlebody Γ and $W_\Lambda(\mathcal{L})$.

There is a map

$$\mathcal{F}\colon H^2(W_\Lambda(\mathcal{L}))\to\mathbb{Z}^e\oplus H^2(Z), \tag{5.4}$$

given by

$$\mathcal{F}(c)=(\langle c,[\widehat{F}_1]\rangle,\dots,\langle c,[\widehat{F}_e]\rangle,c|_Z).$$

Here $\widehat{F}_i$ is the surface obtained by capping a Seifert surface for L_i in Z with the core of the 2–handle. An easy argument involving the Mayer–Vietoris sequence on the handle attachment regions in Z shows that $\mathcal{F}$ is an isomorphism.

Dually, we may view $W_\Lambda(\mathcal{L})$ as being obtained by attaching 2–handles to a link $\mathcal{L}^*$ in Y. We consider the Mayer–Vietoris sequence obtained by viewing W_Λ as the union of $[0,1]\times Y$ and e 2–handles. A portion of this exact sequence reads

$$H^1(\mathcal{L}^*)\to H^2(W_\Lambda(Y))\to H^2(Y)\to 0.$$

In particular, $H^2(Y)$ is the quotient of $H^2(W_\Lambda(Y))$ by the image of $H^1(\mathcal{L}^*)$. Furthermore, from the definition of the coboundary map in the Mayer–Vietoris exact sequence, an element of $H^1(\mathcal{L}^*)$ acts by the Poincaré duals of the cores of the 2–handles attached along $\mathcal{L}$. Using the isomorphism $\mathcal{F}$ from (5.4), we thus obtain

$$H^2(Y)\cong(\mathbb{Z}^e/\operatorname{im}(\Xi))\oplus H^2(Z). \tag{5.5}$$

There are analogous descriptions for Spin^c structures on Y and $W_\Lambda(\mathcal{L})$, as follows. Consider the map

$$\mathcal{T}_W \colon \mathrm{Spin}^c(W_\Lambda(\mathcal{L})) \hookrightarrow \mathbb{Q}^e \times \mathrm{Spin}^c(Z), \tag{5.6}$$

given by

$$\mathcal{T}_W(\mathfrak{s}) = \left(\tfrac{1}{2}(\langle c_1(\mathfrak{s}), [\widehat{F}_1]\rangle - [\widehat{F}]\cdot[\widehat{F}_1]), \dots, \tfrac{1}{2}(\langle c_1(\mathfrak{s}), [\widehat{F}_e]\rangle - [\widehat{F}]\cdot[\widehat{F}_e]), \mathfrak{s}|_Z\right),$$

where $[\widehat{F}]$ is the sum of the $[\widehat{F}_i]$. Similar to the argument for cohomology, an easy application of Mayer–Vietoris shows that $\mathcal{T}_W$ is an isomorphism onto its image. Since $c_1(\mathfrak{s})$ is a characteristic vector, $\langle c_1(\mathfrak{s}), [\widehat{F}_i]\rangle - [\widehat{F}_i]^2$ is even as well. Using this, it is not hard to identify the image of $\mathcal{T}_W$ as $\mathbb{H}(\mathcal{L}) \times \mathrm{Spin}^c(Z)$, where $\mathbb{H}(\mathcal{L})$ is the affine lattice in $\mathbb{Q}^e$ generated by tuples $(a_1, \dots, a_e)$ where

$$a_i - \tfrac{1}{2}\,\mathrm{lk}(\mathcal{L}_i, \mathcal{L} \setminus \mathcal{L}_i) \in \mathbb{Z} \quad \text{for all } i.$$

The linking number is computed as

$$\mathrm{lk}(\mathcal{L}_i, \mathcal{L} \setminus \mathcal{L}_i) = d_i(d_1 + d_2 + \cdots + d_e) - d_i^2. \tag{5.7}$$

A similar argument as for cohomology implies $\mathrm{Spin}^c(Y)$ is isomorphic to the quotient of $\mathrm{Spin}^c(W_\Lambda(\mathcal{L}))$ by the action of the Poincaré duals of the cores of the 2–handles attached to $\mathcal{L}$. This translates into the isomorphism

$$\mathcal{T}_Y \colon \mathrm{Spin}^c(Y) \cong (\mathbb{H}(\mathcal{L})/\mathrm{im}(\Xi)) \times \mathrm{Spin}^c(Z). \tag{5.8}$$

With respect to the isomorphisms $\mathcal{F}$ and $\mathcal{T}_W$, the Chern class map takes the simple form

$$c_1(s_1, \dots, s_e, \mathfrak{t}) = (2s_1 + [\widehat{F}]\cdot[\widehat{F}_1], \dots, 2s_e + [\widehat{F}]\cdot[\widehat{F}_e], c_1(\mathfrak{t})).$$

Since $Z = \#^\rho\, S^2 \times S^1$ bounds the 1–handlebody $\Gamma \subset N$, we know that $\delta(H^1(Z)) = \{0\} \subset H^2(N)$. Hence, a Mayer–Vietoris argument identifies $\mathrm{Spin}^c(N)$ with the set of Spin^c structures on $W_\Lambda(\mathcal{L})$ which extend over Γ, or equivalently the ones which have torsion restriction to Z. Hence,

$$\mathrm{Spin}^c(N) \cong \mathbb{H}(\mathcal{L}).$$

The following is helpful for understanding $H^2(Y)$:

Lemma 5.9 *Suppose $\Xi = \{a_{ij}\}_{i,j=1}^e$ is a matrix such that $a_{ij} = d_i d_j$, for some nonzero integers d_i. Then $\mathbb{Z}^e/\mathrm{im}(\Xi) \cong \mathbb{Z}^{e-1} \oplus \mathbb{Z}/\theta^2$, where $\theta = \gcd(d_1, \dots, d_e)$.*

Proof Recall that

$$\Xi = \begin{pmatrix} d_1d_1 & d_1d_2 & \cdots & d_1d_e \\ d_2d_1 & d_2d_2 & \cdots & d_2d_e \\ \vdots & \vdots & \ddots & \vdots \\ d_ed_1 & d_ed_2 & \cdots & d_ed_e \end{pmatrix}.$$

It is clear that $\mathrm{im}(\Xi)$ is the span of $\theta(d_1,\dots,d_e)^T$, by considering the image of the standard basis in $\mathbb{R}^n$. By module theory over a principal ideal domain, $\mathbb{Z}^e/\mathrm{im}(\Xi) \cong \mathbb{Z}^{e-1} \oplus \mathrm{Tors}(\mathbb{Z}^e/\mathrm{im}(\Xi))$. By definition, $\mathrm{Tors}(\mathbb{Z}^e/\mathrm{im}(\Xi))$ is generated by the set of vectors v in $\mathbb{Z}^e$ such that $n[v] = m[\theta(d_1,\dots,d_e)^T]$ for some integers n and m. Clearly, $\mathrm{Tors}(\mathbb{Z}^e/\mathrm{im}(\Xi))$ is generated by the vector $(d_1/\theta,\dots,d_e/\theta)^T$, which has order θ^2. □

Combining Lemma 5.9 with (5.5), we conclude that

$$b_1(Y) = e - 1 + b_1(Z) = e - 1 + \rho. \tag{5.10}$$

If $j \in 2\mathbb{Z}+1$, let $\mathfrak{c}_j$ denote the Spin^c structure on $\mathbb{C}P^2$ which satisfies

$$\langle c_1(\mathfrak{c}_j), E\rangle = j, \tag{5.11}$$

where E is a complex line. In terms of the isomorphism in (5.8),

$$\mathcal{T}_Y(\mathfrak{c}_j|_Y) = \left(\tfrac{1}{2}(jd_1 - d_1(d_1+\cdots+d_e)),\dots,\tfrac{1}{2}(jd_e - d_e(d_1+\cdots+d_e)),0\right). \tag{5.12}$$

We now let X denote the complement of the interior of N in $\mathbb{C}P^2$.

Lemma 5.13 (1) *X has trivial intersection form.*

(2) *Suppose $\mathfrak{s}$ is a torsion Spin^c structure on Y. Then $\mathfrak{s}$ extends over X if and only if it extends over $\mathbb{C}P^2$.*

Proof The proof follows arguments identical to those in [Borodzik et al. 2017, Sections 3 and 4]; therefore, we provide only a sketch. Claim (1) follows from the fact that the inclusion map $H_2(X) \to H_2(\mathbb{C}P^2)$ vanishes, since all elements of $H_2(X)$ are disjoint from C.

Claim (2) is proven as follows. A Spin^c structure on Y always extends over $W_\Lambda(\mathcal{L})$. Furthermore, the isomorphisms in (5.6) and (5.8) are clearly compatible with the natural restriction maps from $\mathrm{Spin}^c(W_\Lambda(\mathcal{L}))$ to $\mathrm{Spin}^c(Y)$ and $\mathrm{Spin}^c(Z)$. A Spin^c structure on $W_\Lambda(\mathcal{L})$ extends over N if and only if it restricts to the torsion Spin^c structure on Z. Hence, a Spin^c structure on Y extends over N if and only if the Spin^c factor on $\mathrm{Spin}^c(Z)$ in (5.8) is torsion. In particular, any torsion Spin^c structure on Y extends over N. Since a Spin^c structure on Y extends over $\mathbb{C}P^2$ if and only if it extends over both X and N, the claim follows. □

5.3 d–invariant inequalities for the neighborhood of C

We are now in position to prove an inequality for the d–invariants of boundaries of neighborhoods of complex curves in $\mathbb{C}P^2$ as in Section 5.1. With the notation from that subsection, we have the following result:

Proposition 5.14 *For any Spin^c structure $\mathfrak{s}$ on Y that extends over X and whose first Chern class is torsion,*

$$d_{\mathrm{bot}}(Y,\mathfrak{s}) \geq -\tfrac{1}{2}(\rho + e - 1), \quad d_{\mathrm{top}}(Y,\mathfrak{s}) \leq \tfrac{1}{2}(\rho + e - 1).$$

Proof By (5.10), we know that $b_1(Y) = \rho + e - 1$. The intersection form on X is trivial by Lemma 5.13. From Theorem 3.1, we obtain

$$d_{\mathrm{bot}}(Y, \mathfrak{s}) = d(Y, \mathfrak{s}, H_1(Y)/\mathrm{Tors}) \geq -\tfrac{1}{2}(\rho + e - 1),$$

since the terms involving c_1^2 and $b_2^-(X)$ vanish.

Since the intersection form on X vanishes, we may reverse the orientation of X and Y and appeal to the same argument to get that

$$d_{\mathrm{bot}}(-Y, \mathfrak{s}) = d(-Y, \mathfrak{s}, H_1(Y)/\mathrm{Tors}) \geq -\tfrac{1}{2}(\rho + e - 1). \tag{5.15}$$

It follows from [Levine and Ruberman 2014, Proposition 4.2] and the fact that $d^*(Y, \mathfrak{s}, H_1(Y)/\mathrm{Tors}) = d_{\mathrm{top}}(Y, \mathfrak{s})$ (see [loc. cit., page 6]) that

$$d_{\mathrm{bot}}(-Y, \mathfrak{s}) = -d_{\mathrm{top}}(Y, \mathfrak{s}).$$

Combining this with (5.15), we conclude that

$$d_{\mathrm{top}}(Y, \mathfrak{s}) \leq \tfrac{1}{2}(\rho + e - 1). \qquad \square$$

5.4 Singular curves in smooth category

The methods we use in this article work in a smooth category. The term "smooth surface with singularities" might be misleading; therefore, we make precise our terminology. The definition we give is quite general.

Definition 5.16 A *singular curve in the smooth category* $C \subset \mathbb{C}P^2$ is a closed subset of $\mathbb{C}P^2$ such that there exist finitely pairwise disjoint closed balls $B_1, \dots, B_u$ in $\mathbb{C}P^2$ such that, with $C_0 = \overline{C \setminus (B_1 \cup \dots \cup B_u)}$,

- C is connected;
- the subset C_0 is a smoothly embedded surface whose boundary belongs to $B_1 \cup \dots \cup B_u$;
- the intersection $B_i \cap C$ is a link (we call it $\mathcal{L}_i$).

The definition means that we do not have to control any possible pathological behavior of C inside balls. We let $C_{01}, \dots, C_{0e}$ be the connected components of C_0. The quantity e plays the same role as the number of irreducible components of an algebraic curve.

Choose $j = 1, \dots, e$. For any $i = 1, \dots, u$ such that $\mathcal{L}_{ij} := B_i \cap C_{0j} \neq \varnothing$, let S_{ij} be a minimal genus surface in B_{ij} whose boundary is $\mathcal{L}_{ij}$. Let $\widetilde{C}_j$ be a closed surface obtained by removing $B_i \cap C_{0j}$, gluing S_{ij} and possibly smoothing corners. The surface $\widetilde{C}_j$ is called a *smooth model* of C_{0j}.

Note that $\widetilde{C}_j$ determines a class in $H_2(\mathbb{C}P^2; \mathbb{Z})$. If S_{ij} and S'_{ij} are two choices of minimal genus surfaces for $\mathcal{L}_{ij}$, then $S_{ij} \cup -S'_{ij}$ is homologically trivial (as a surface in the ball B_{ij}). Hence, the class of $\widetilde{C}_j$ does not depend on the particular choice of S_{ij}. We let d_j be the integer such that $[\widetilde{C}_j] = d_j \cdot 1 \in H_2(\mathbb{C}P^2; \mathbb{Z})$, where we write 1 for the class of a line. We call d_j the *smooth degree* of C_j.

Definition 5.17 A singular curve is the smooth category is called *adjunctive* if, for all $j = 1, \dots, e$, we have $g(\widetilde{C}_j) = \frac{1}{2}(d_j - 1)(d_j - 2)$.

Definition 5.18 Let C be an adjunctive singular curve in the smooth category.

- C is of *algebraic type* if all links $\mathcal{L}_i$ are algebraic links.
- C is of *weakly algebraic type* if all links $\mathcal{L}_i$ are either algebraic links or their mirrors.

Remark 5.19 The distinction between the requirements that $\mathcal{L}_i$ be an algebraic link or an L–space link is motivated by applications in algebraic geometry. In our paper, we never use the fact that the links $\mathcal{L}_i$ are algebraic links, instead of merely L–space links. We note that there are some nontrivial differences between L–space knots and algebraic knots. For example, the set S_K defined in Section 4.2 is not necessarily a semigroup if K is merely an L–space knot. We recall that S_K is used to define the function R_K, which is referred to as the *semigroup counting function*. In our theory, we never need S_K to be a semigroup, so the mathematical part of the theory goes through.

We now define the analogs of ρ, Y and N from Section 5.1 in the case of a singular curve in the smooth category. First set g_j to be the genus of C_{0j} (not of $\widetilde{C}_j$). Set $g=g_1+\cdots+g_e$ and $\rho=2g-e+1+\sum(r_i-1)$, where r_i is the number of components of $\mathcal{L}_i$.

We now repeat the procedure from Section 5.1, omitting the proofs if they are the same as in that subsection. We pick $\lambda_1,\dots,\lambda_{\rho+u-1}$ to be arcs on C_0 which form a basis of $H_1(C_0,\partial C_0;\mathbb{Z})$. We let Γ be the 4–manifold obtained by attaching $\rho+u-1$ 4–dimensional 1–handles to $\partial(B_1\cup\cdots\cup B_u)$ as in Section 5.1. We set $Z=\partial\Gamma$; then $Z=\#^{\rho}\, S^2\times S^1$. Finally, $\mathcal{L}=C\cap Z$. This is an e–component link. The set $C\setminus\Gamma$ is a disjoint union of e disks $C'_{01},\dots,C'_{0e}$. Reindexing these disks if necessary, we may and will assume that C'_{0i} is a subset of C_{0i}. Let N be the handlebody Γ with attached 2–handles whose cores are $C'_{01},\dots,C'_{0e}$. The manifold $Y=\partial N$ is the surgery on $\mathcal{L}$ with framings equal to $d_1^2,\dots,d_e^2$.

With these definitions, the results of Sections 5.2 and 5.3 hold for singular curves in smooth category.

6 Nonrational noncuspidal complex curves

6.1 General estimates

We now pass to the main applications of our paper. Suppose $C\subset\mathbb{C}P^2$ is a degree d curve. We mostly focus on the case where C is complex curve, but also consider the case where C is only a smooth surface, embedded away from a finite set of singular points, as in Definition 5.16. We further assume that the singularities of C are restricted to the following:

- There are ν cuspidal (unibranched) singular points $p_1,\dots,p_\nu$. We write $K_1,\dots,K_\nu$ for their links, and set $K=K_1\#\cdots\# K_\nu$.
- There are m_n singular points whose link is $T_{2,2n}$.
- There are $\underline{m}_n$ singular points whose link is $-T_{2,2n}$.

Define

$$\kappa_+ = \sum_n n m_n, \quad \kappa_- = \sum_n n \underline{m}_n, \quad \eta_+ = \sum_n m_n, \quad \eta_- = \sum_n \underline{m}_n.$$

Additionally, we assume that the curve is adjunctive (see Definition 5.17); that is, its genus g is given by

(6.1) $$g = g(C) = \tfrac{1}{2}(d-1)(d-2) - g_3(K) - (\kappa_+ + \kappa_-).$$

For algebraic curves, $\kappa_- = 0$ and (6.1) is the adjunction formula. If C is a singular curve in the smooth category of algebraic type (ie $\kappa_- = 0$; see Definition 5.18), the adjunction inequality implies that $g(C)$ is greater than or equal to the right-hand side of (6.1). If C is of weak algebraic type (see Definition 5.18), the relation between $g(C)$ and the right-hand side of (6.1) can be more involved, so the condition (6.1) is a significant restriction on $g(C)$.

We define

(6.2) $$K_+ = K \,\#\, \#^n\, m_n \widehat{T}_{2,2n}, \quad K_- = \#^n\, \underline{m}_n \widehat{T}_{2,-2n}, \quad \widetilde{K} = K_+ \,\#\, K_-, \quad \widehat{K} = \widetilde{K} \,\#\, \#^g\, \mathcal{B}_0,$$

where $\widehat{T}_{2,2n}$ denotes the knotification of the torus link $T_{2,2n}$ and $\widehat{T}_{2,-2n}$ denotes the knotification of its mirror.

Since the knots $K_1, \dots, K_\nu$ are algebraic knots and so, in particular, L–space knots, their knot Floer complexes are staircase complexes, which we denote by $\mathcal{C}(K_i)$. In particular,

$$\mathcal{CFK}^-(K) = \mathcal{C}(K_1) \otimes \cdots \otimes \mathcal{C}(K_\nu)$$

is a positive multistaircase complex. Note that, by Proposition 2.40 and Example 4.30, the knots K_+, K_- and $\widetilde{K}$ have split towers. The following relations follow from Proposition 2.40, the Künneth theorem for connected sums, and Proposition 3.9, where we write $\cong$ for homotopy equivalence of chain complexes and $\simeq_{\mathrm{loc}}$ for local equivalence, and the brackets denote an overall grading shift:

$$\begin{aligned}
\mathcal{C}^{\mathrm{top}}(K_+) &\cong \mathcal{C}^{\mathrm{top}}(K) \otimes \bigotimes_n (\mathcal{S}^n)^{\otimes m_n}\{\tfrac{1}{2}\eta_+, \tfrac{1}{2}\eta_+\},\\
\mathcal{C}^{\mathrm{bot}}(K_+) &\cong \mathcal{C}^{\mathrm{bot}}(K) \otimes \bigotimes_n (\mathcal{S}^{n-1})^{\otimes m_n}\{-\tfrac{1}{2}\eta_+, -\tfrac{1}{2}\eta_+\},\\
\mathcal{C}^{\mathrm{top}}(K_-) &\cong \bigotimes_n (\mathcal{S}^{-(n-1)})^{\otimes \underline{m}_n}\{\tfrac{1}{2}\eta_-, \tfrac{1}{2}\eta_-\},\\
\mathcal{C}^{\mathrm{bot}}(K_-) &\cong \bigotimes_n (\mathcal{S}^{-n})^{\otimes \underline{m}_n}\{-\tfrac{1}{2}\eta_-, -\tfrac{1}{2}\eta_-\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}^{\mathrm{top}}(\widetilde{K}) &\cong \mathcal{C}^{\mathrm{top}}(K_+) \otimes \mathcal{C}^{\mathrm{top}}(K_-) \simeq_{\mathrm{loc}} \mathcal{C}(K) \otimes \mathcal{S}^{\kappa_+ - (\kappa_- - \eta_-)}\{\tfrac{1}{2}(\eta_+ + \eta_-), \tfrac{1}{2}(\eta_+ + \eta_-)\},\\
\mathcal{C}^{\mathrm{bot}}(\widetilde{K}) &\cong \mathcal{C}^{\mathrm{bot}}(K_+) \otimes \mathcal{C}^{\mathrm{bot}}(K_-) \simeq_{\mathrm{loc}} \mathcal{C}(K) \otimes \mathcal{S}^{\kappa_+ - \eta_+ - \kappa_-}\{\tfrac{1}{2}(\eta_+ + \eta_-), \tfrac{1}{2}(\eta_+ + \eta_-)\}.
\end{aligned}$$

We set

$$\delta_1 := \kappa_+ - (\kappa_- - \eta_-), \quad \delta_2 := (\kappa_+ - \eta_+) - \kappa_-.$$

Whether the staircases in $\mathcal{C}^{\mathrm{top}}(\widetilde{K})$ and $\mathcal{C}^{\mathrm{bot}}(\widetilde{K})$ are positive or negative depends on the signs of δ_1 and δ_2. The following proposition is the main tool towards Theorems 6.4 and 6.8:

Proposition 6.3 *Suppose K, $\widetilde{K}$ and $\widehat{K}$ are as above and let $R = R_K$ be the infimal convolution of the semigroup counting functions for knots $K_1, \dots, K_\nu$.*

(a) *If $\delta_1 \geq 0$, then*

$$V_s^{\mathrm{top}}(\widetilde{K}) = -\tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0 \leq j \leq \delta_1} (V_{s+2j-\delta_1}(K) + j),$$

$$\begin{aligned} V_s^{\mathrm{top}}(\widehat{K}) &= -\tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0 \leq j \leq \delta_1 + g} (V_{s+2j-\delta_1-g}(K) + j) \\ &= -\tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0 \leq j \leq \delta_1 + g} \bigl(R(g_3(K) + s + 2j - \delta_1 - g) - (s + j - \delta_1 - g)\bigr). \end{aligned}$$

(b) *If $\delta_2 \geq 0$, then*

$$V_s^{\mathrm{bot}}(\widetilde{K}) = \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0 \leq j \leq \delta_2} (V_{s+2j-\delta_2}(K) + j),$$

$$\begin{aligned} V_s^{\mathrm{bot}}(\widehat{K}) &= \tfrac{1}{4}(\eta_+ + \eta_-) - \tfrac{1}{2}g + \max_{0 \leq i \leq g} \min_{0 \leq j \leq \delta_2} (V_{s+2j+2i-g-\delta_2}(K) + i + j) \\ &= -\tfrac{1}{2}g + \tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0 \leq i \leq g} \min_{0 \leq j \leq \delta_2} \bigl(R(g_3(K) + s + 2j + 2i - g - \delta_2) - (s + i + j - g - \delta_2)\bigr). \end{aligned}$$

(c) *If $\delta_1 < 0$ and $\mathcal{C}(K)$ is a positive staircase (not just a positive multistaircase), then*

$$V_s^{\mathrm{top}}(\widetilde{K}) = -\tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0 \leq j \leq -\delta_1} (V_{s-2j-\delta_1}(K) - j),$$

$$\begin{aligned} V_s^{\mathrm{top}}(\widehat{K}) &= \tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0 \leq i \leq g} \max_{0 \leq j \leq -\delta_1} (V_{s-2j-2i+g-\delta_1}(K) - i - j) \\ &= \tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0 \leq i \leq g} \max_{0 \leq j \leq -\delta_1} \bigl(R(g_3(K) + s - 2j - 2i + g - \delta_1) - (s - i - j + g - \delta_1)\bigr). \end{aligned}$$

(d) *If $\delta_2 < 0$ and $\mathcal{C}(K)$ is a positive staircase, then*

$$V_s^{\mathrm{bot}}(\widetilde{K}) = \tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0 \leq j \leq -\delta_2} (V_{s-2j-\delta_2}(K) - j),$$

$$\begin{aligned} V_s^{\mathrm{bot}}(\widehat{K}) &= \tfrac{1}{2}g + \tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0 \leq j \leq g - \delta_2} (V_{s-2j+g-\delta_2}(K) - j) \\ &= \tfrac{1}{2}g + \tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0 \leq j \leq g - \delta_2} \bigl(R(g_3(K) + s - 2j + g - \delta_2) - (s - j + g - \delta_2)\bigr). \end{aligned}$$

Proof The proof is similar in all cases and consists of gathering Corollaries 4.25 and 4.26, Propositions 4.32 and 4.14, and Lemma 3.7. For the reader's convenience, we present details of the computations of V^{top} for cases (a) and (c).

If $\delta_1 \geq 0$, then by Corollary 4.25 and Lemma 3.7,

$$V_s^{\mathrm{top}}(\widetilde{K}) = -\tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0 \leq j \leq \delta_1} (V_{s+2j-\delta_1}(K) + j).$$

Combining this with Proposition 4.32, we obtain

$$V_s^{\mathrm{top}}(\widehat{K}) = -\tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0 \leq j \leq \delta_1 + g} (V_{s+2j-\delta_1-g}(K) + j).$$

By Proposition 4.14,

$$V_s^{\mathrm{top}}(\widehat{K}) = -\tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\le i\le g}\ \max_{0\le j\le -\delta_1} \bigl(R(g_3(K)+s-2j-2i+g-\delta_1) - (s-i-j+g-\delta_1)\bigr).$$

This proves item (a). If $\delta_1 < 0$ and $\mathcal{C}(K)$ is a positive staircase, then, by Corollary 4.26,

$$V_s^{\mathrm{top}}(\widetilde{K}) = -\tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0\le j\le -\delta_1} (V_{s-2j-\delta_1}(K) - j).$$

Combining Propositions 4.32 and 4.14, we have

$$\begin{aligned} V_s^{\mathrm{top}}(\widehat{K}) &= \tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\le i\le g}\ \max_{0\le j\le -\delta_1} (V_{s-2j-2i+g-\delta_1}(K) - i - j) \\ &= \tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\le i\le g}\ \max_{0\le j\le -\delta_1} \bigl(R(g_3(K)+s-2j-2i+g-\delta_1) - (s-i-j+g-\delta_1)\bigr). \end{aligned}$$

This proves item (c). □

Proposition 6.3 allows us to express the d–invariants of the boundary $Y = \partial N$ of the tubular neighborhood of C in terms of the R_K–functions of singular points. In our applications, we will focus on two cases:

(1) **Algebraic case** We assume that C has only algebraic singularities; that is, $\underline{m}_n = 0$ for all $n > 0$. This corresponds to items (a) and (b) of Proposition 6.3.

(2) **Single knot case** We assume that $\nu = 1$, so K is a positive staircase and $m_n = 0$ for all $n > 0$. We will use items (c) and (d) of Proposition 6.3.

The first case is considered in Section 6.2. The second is addressed in Section 6.3.

6.2 Curves with no negative double points

For the reader's convenience, we repeat the statement from the introduction of the next result.

Theorem 6.4 *Let C be a reduced curve with degree d and genus g. Suppose that C has cuspidal singular points $p_1, \dots, p_\nu$ whose semigroup counting functions are $R_1, \dots, R_\nu$, respectively. Assume that, apart from these ν points, the curve C has, for each $n \ge 1$, $m_n \ge 0$ singular points whose links are $T_{2,2n}$ (A_{2n-1} singular points) and no other singularities. Define*

$$\eta_+ = \sum_n m_n \quad \text{and} \quad \kappa_+ = \sum_n n m_n.$$

For any $k = 1, \dots, d-2$,

$$\begin{aligned} &\max_{0\le j\le g}\ \min_{0\le i\le \kappa_+ - \eta_+} (R(kd+1-\eta_+ - 2i - 2j) + i + j) \le \tfrac{1}{2}(k+1)(k+2) + g, \\ &\min_{0\le j\le g+\kappa_+} (R(kd+1-2j) + j) \ge \tfrac{1}{2}(k+1)(k+2). \end{aligned} \tag{6.5}$$

Here R denotes the infimal convolution of the functions $R_1, \dots, R_\nu$.

Proof Let Y be the boundary of a tubular neighborhood of C. Then Y is the result of a d^2–surgery on $\widehat{K} \subset \#^\rho S^2 \times S^1$ obtained as in Section 6.2, where we readily compute from (5.1) that $\rho = 2g + \eta_+$. Note that, by (6.1), the genus $g_3(K)$ is less than or equal to $\frac{1}{2}(d-1)(d-2) < \frac{1}{2}d^2$. Hence, the surgery coefficient is greater than twice the genus of K. In particular, the large surgery formula can be applied [Ozsváth and Szabó 2008b, Theorem 4.10].

Let $\mathfrak{s}_j$ for $j \in [-\frac{1}{2}d^2, \frac{1}{2}d^2) \cap \mathbb{Z}$ denote the Spin^c structures on Y as in Definition 3.13. By Lemma 5.13, $\mathfrak{s}_j$ extends to $\mathbb{C}P^2 \setminus N$ if and only if $\mathfrak{s}_j$ is a restriction of $\mathfrak{c}_h$ for some h, where $\mathfrak{c}_h$ is as in (5.11). By (5.12), we infer that this holds if and only if $j = md$ with $m \in \mathbb{Z}$ if d is odd and $m \in \frac{1}{2} + \mathbb{Z}$ if d is even. Compare with [Borodzik and Livingston 2014, Lemma 3.1].

By Proposition 5.14, for any $md \in [-\frac{1}{2}d^2, \frac{1}{2}d^2)$ such that $m + \frac{1}{2}(d-1)$ is an integer,

$$d_{\mathrm{bot}}(Y, \mathfrak{s}_{md}) \geq -\tfrac{1}{2}\eta_+ - g, \quad d_{\mathrm{top}}(Y, \mathfrak{s}_{md}) \leq \tfrac{1}{2}\eta_+ + g. \tag{6.6}$$

By Theorem 3.15, (6.6) translates to the inequalities

$$\begin{aligned} V^{\mathrm{top}}_{md}(\hat{K}) &\geq \tfrac{1}{8}(d-2m+1)(d-2m-1) - \tfrac{1}{4}\eta_+ - \tfrac{1}{2}g, \\ V^{\mathrm{bot}}_{md}(\hat{K}) &\leq \tfrac{1}{8}(d-2m+1)(d-2m-1) + \tfrac{1}{4}\eta_+ + \tfrac{1}{2}g. \end{aligned} \tag{6.7}$$

We compute V^{top}_{md} and V^{bot}_{md} from Proposition 6.3. Using $g_3(K) = \frac{1}{2}(d-1)(d-2) - g - \kappa_+$, we rewrite the equations of Proposition 6.3(a)–(b) as

$$V^{\mathrm{top}}_{md}(\hat{K}) = -\tfrac{1}{2}g - \tfrac{1}{4}\eta_+ + \min_{0 \leq j \leq \kappa_+ + g} \Bigl(R\bigl(\tfrac{1}{2}(d-1)(d-2) + md + 2j - 2\kappa_+ - 2g\bigr) - (md + j - \kappa_+ - g)\Bigr),$$

$$\begin{aligned} V^{\mathrm{bot}}_{md}(\hat{K}) = -\tfrac{1}{2}g + \tfrac{1}{4}\eta_+ + \max_{0 \leq i \leq g} \min_{0 \leq j \leq \kappa_+ - \eta_+} \Bigl(&R\bigl(\tfrac{1}{2}(d-1)(d-2) + md + 2j + 2i - 2g - 2\kappa_+ + \eta_+\bigr) \\ &- (md + i + j - g - \kappa_+ + \eta_+)\Bigr). \end{aligned}$$

Comparing this with (6.7), we obtain

$$\min_{0 \leq j \leq \kappa_+ + g} R\bigl(\tfrac{1}{2}(d-1)(d-2) + md + 2j - 2\kappa_+ - 2g\bigr) - (md + j - \kappa_+ - g) \geq \tfrac{1}{8}(d-2m+1)(d-2m-1)$$

and

$$\begin{aligned} \max_{0 \leq i \leq g} \min_{0 \leq j \leq \kappa_+ - \eta_+} R\bigl(\tfrac{1}{2}(d-1)(d-2) + md + 2i + 2j - 2(\kappa_+ - \eta_+) - \eta_+ - 2g\bigr) - (md + j - \kappa_+ + \eta_+ - 2g) \\ \leq \tfrac{1}{8}(d-2m+1)(d-2m-1) + g. \end{aligned}$$

With a change $j \mapsto \kappa_+ + g - j$ in the first inequality and $i \mapsto g - i$ and $j \mapsto \kappa_+ - \eta_+ - j$ in the second, we obtain

$$\min_{0 \leq j \leq \kappa_+ + g} R\bigl(\tfrac{1}{2}(d-1)(d-2) + md - 2j\bigr) - md + j \geq \tfrac{1}{8}(d-2m+1)(d-2m-1),$$

$$\max_{0 \leq i \leq g} \min_{0 \leq j \leq \kappa_+ - \eta_+} R\bigl(\tfrac{1}{2}(d-1)(d-2) + md - 2i - 2j - \eta_+\bigr) - md + j \leq \tfrac{1}{8}(d-2m+1)(d-2m-1) + g.$$

With $m = k - \frac{1}{2}(d-3)$, after straightforward calculations we obtain

$$\min_{0 \leq j \leq g + \kappa_+} (R(kd + 1 - 2j) + j) \geq \tfrac{1}{2}(k+1)(k+2),$$

$$\max_{0 \leq j \leq g} \min_{0 \leq i \leq \kappa_+ - \eta_+} (R(kd + 1 - \eta_+ - 2i - 2j) + i + j) \leq \tfrac{1}{2}(k+1)(k+2) + g. \qquad \square$$

6.3 Negative double points

We now specialize to the case where C is a surface which has a single algebraic singularity and $\underline{m}_n \geq 0$ singular points whose links are $(2,-2n)$–torus links (which are not algebraic).

Theorem 6.8 *Suppose C is a genus g degree d singular curve in the smooth category as in Section 5.4 with a cuspidal singular point p, $\underline{m}_n$ singularities whose link is $-T_{2,2n}$ for each $n \geq 1$, and no other singular points. Suppose further that C is adjunctive.*

Then, for any $k = 1, \dots, d-2$,

$$\max_{0\leq j\leq g+\kappa_-}(R(kd+1-2j)+j) \leq \tfrac{1}{2}(k+1)(k+2)+g+\kappa_-,$$
$$\min_{0\leq i\leq g}\max_{0\leq j\leq \kappa_--\eta_-}(R(kd+1-2i-2j-\eta_-)+i+j) \geq \tfrac{1}{2}(k+1)(k+2)+\kappa_--\eta_-,$$

where R is the semigroup counting function for the singular point p and $\eta_- = \sum \underline{m}_n$ and $\kappa_- = \sum \underline{m}_n n$.

Remark 6.9 With the assumptions on the singularities of C, the condition that C be adjunctive (spelled out in Definition 5.17) is equivalent to saying that the genus of C is given by (6.1).

Proof The beginning of the proof is exactly the same as in the proof of Theorem 6.4. The boundary Y of the tubular neighborhood of C is a result of a surgery with coefficient d^2 on the knot $\widehat{K}$ in $\#^{2g+\eta_-} S^2\times S^1$. In particular, (6.7) holds with η_- replacing η_+:

$$\begin{aligned} V^{\mathrm{top}}_{md}(\widehat{K}) &\geq \tfrac{1}{8}(d-2m+1)(d-2m-1)-\tfrac{1}{4}\eta_--\tfrac{1}{2}g,\\ V^{\mathrm{bot}}_{md}(\widehat{K}) &\leq \tfrac{1}{8}(d-2m+1)(d-2m-1)+\tfrac{1}{4}\eta_-+\tfrac{1}{2}g. \end{aligned} \tag{6.10}$$

With $g_3(K) = \frac{1}{2}(d-1)(d-2)-g-\kappa_-$, the equations of Proposition 6.3(c)–(d) take the form

$$\begin{aligned} V^{\mathrm{top}}_{md}(\widehat{K}) = \tfrac{1}{2}g-\tfrac{1}{4}\eta_- + \min_{0\leq i\leq g}\max_{0\leq j\leq\kappa_--\eta_-}\Big(&R\big(\tfrac{1}{2}(d-1)(d-2)+md-2j-2i-\eta_-\big)\\ &-(md-i-j+g+\kappa_--\eta_-)\Big), \end{aligned}$$
$$V^{\mathrm{bot}}_{md}(\widehat{K}) = \tfrac{1}{2}g+\tfrac{1}{4}\eta_- + \max_{0\leq j\leq g+\kappa_-}\big(R\big(\tfrac{1}{2}(d-1)(d-2)+md-2j\big)-(md-j+g+\kappa_-)\big).$$

Comparing this with (6.10), after changes analogous to in Section 6.2, we arrive at

$$\max_{0\leq j\leq g+\kappa_-}(R(kd+1-2j)+j) \leq \tfrac{1}{2}(k+1)(k+2)+g+\kappa_-,$$
$$\min_{0\leq i\leq g}\max_{0\leq j\leq \kappa_--\eta_-}(R(kd+1-2i-2j-\eta_-)+i+j) \geq \tfrac{1}{2}(k+1)(k+2)+\kappa_--\eta_-. \qquad \square$$

6.4 Special cases of Theorems 6.4 and 6.8

The bounds in Theorems 6.4 and 6.8 are fairly general, but clarity is the price. To illustrate these bounds, we provide several special cases.

Corollary 6.11 (a) *Suppose C is a genus g, degree d curve with singular points $p_1, \dots, p_\nu$ and η_+ positive double points. Assume also that C has no other critical points. Then, for $k = 1, \dots, d-2$,*

$$\max_{0 \le j \le g} (R(kd + 1 - \eta_+ - 2j) + j) \le \tfrac{1}{2}(k+1)(k+2) + g,$$
$$\min_{0 \le j \le g+\eta_+} (R(kd + 1 - 2j) + j) \ge \tfrac{1}{2}(k+1)(k+2),$$

where R denotes the infimal convolution of the functions $R_{K_1}, \dots, R_{K_\nu}$.

(b) *Suppose C is a genus g, degree d curve with a singular point p and η_- negative double points. Assume that C has genus as in (6.1). Then, for $k = 1, \dots, d-2$,*

$$\max_{0 \le j \le g+\eta_-} (R(kd + 1 - 2j) + j) \le \tfrac{1}{2}(k+1)(k+2) + g + \eta_-,$$
$$\min_{0 \le j \le g} (R(kd + 1 - \eta_- - 2j) + j) \ge \tfrac{1}{2}(k+1)(k+2),$$

where R is the semigroup counting function for the singular point p.

Proof Items (a) and (b) follow from Theorems 6.4 and 6.8, respectively, noting that $\kappa_+ = \eta_+$ and $\kappa_- = \eta_-$. □

Specifying further $\eta_+ = 0$ in Corollary 6.11(a) recovers the following result of [Bodnár et al. 2016; Borodzik et al. 2017]:

Corollary 6.12 *Suppose C is a cuspidal curve of genus g and degree d. Let R be the convolution of semigroup counting functions of the singular points of C. Then*

$$\begin{aligned} &\max_{0 \le j \le g} (R(kd + 1 - 2j) + j) \le \tfrac{1}{2}(k+1)(k+2) + g, \\ &\min_{0 \le j \le g} (R(kd + 1 - 2j) + j) \ge \tfrac{1}{2}(k+1)(k+2). \end{aligned} \tag{6.13}$$

We now compare the cases $g = 0$ and $\eta_+ = 1$, $g = 0$ and $\eta_- = 1$, and $g = 1$ and $\eta_+ = \eta_- = 0$.

Proposition 6.14 *Let C be a degree d curve with one cuspidal singular point, whose semigroup counting function is denoted by R. Assume C has at most one ordinary double point ($\eta_+ + \eta_- \le 1$) and no other singularities. For all $k = 1, \dots, d-2$, set $\upsilon_k = \frac{1}{2}(k+1)(k+2)$.*

(a) *If $g = 1$ and $\eta_+ = \eta_- = 0$, then $R(kd-1) \in \{\upsilon_k - 1, \upsilon_k\}$ and $R(kd+1) \in \{\upsilon_k, \upsilon_k + 1\}$.*

(b) *If $g = 0$ and $\eta_+ = 1$, then $R(kd-1) \in \{\upsilon_k - 1, \upsilon_k\}$ and $R(kd+1) \in \{\upsilon_k, \upsilon_k + 1\}$, but also*
$$R(kd) \le \upsilon_k.$$

(c) *If $g = 0$ and $\eta_- = 1$, then $R(kd-1) \in \{\upsilon_k - 1, \upsilon_k\}$ and $R(kd+1) \in \{\upsilon_k, \upsilon_k + 1\}$, but also*
$$R(kd) \ge \upsilon_k.$$

Proof Item (a) is an immediate consequence of (6.13).

For item (b), note that Corollary 6.11(a) implies that $R(kd) \leq \upsilon_k$ and $R(kd+1) \geq \upsilon_k$, $R(kd-1) \geq \upsilon_k - 1$. Since $R(j+1) - R(j) \in \{0, 1\}$ for all j, the statement follows trivially.

The proof of item (c) is analogous. Corollary 6.11(c) implies that $R(kd+1) \leq \upsilon_k + 1$, $R(kd-1) \leq \upsilon_k$ and $R(kd) \geq \upsilon_k$. Again, the statement follows trivially. □

Proposition 6.14 can be interpreted as follows. Suppose C is a genus one curve with a single cuspidal singular point. Then the semigroup counting function R satisfies the constraints of item (a) of Proposition 6.14. If, for some $k = 1, \dots, d-2$, we have $R(kd) = \upsilon_k + 1$, then the function R does not satisfy the constraints of item (b). That is, C cannot be deformed to a curve with genus 0 and the same (topological type of) cuspidal singularity. That is, we cannot "trade genus for a positive double point".

If, for some k, we have $R(kd) = \upsilon_k - 1$, then the same argument shows that we cannot "trade genus for a negative double point".

6.5 Unicuspidal curves of genus 1

We will now check, for concrete examples, whether the genus can be traded for double points.

Example 6.15 Let $\phi_0 = 0$, $\phi_1 = 1$, $\phi_n = \phi_{n-1} + \phi_{n-2}$ be the Fibonacci sequence. Borodzik et al. [2017, Proposition 9.12], based on a construction of Orevkov [2002], constructed a family of genus 1 cuspidal curves C_n of degree ϕ_{4n} with a single singularity whose link is the $(\phi_{4n-2}, \phi_{4n+2})$–torus knot for $n = 2, 3, \dots$.

By Proposition 6.14(c), we deduce that the genus cannot be traded for negative double points. Indeed, a classical identity on Fibonacci numbers, $\phi_{k-2}+\phi_{k+2} = 3\phi_k$, shows that the semigroup generated by ϕ_{4n-2} and ϕ_{4n+2} has precisely nine elements in the interval $[0, 3\phi_{4n})$: $0, \phi_{4n-2}, \dots, 7\phi_{4n-2}$ and ϕ_{4n+2}. In fact, $7\phi_{4n-2} < 3\phi_{4n} < 8\phi_{4n-2}$ (we leave the proof of this to the reader) and $\phi_{4n+2} + \phi_{4n-2} = 3\phi_{4n}$. In particular, $R(3\phi_{4n}) = 9 < 10 = \upsilon_3 = \frac{1}{2}(3+1)(3+2)$.

Borodzik et al. [2017, Theorem 9.1] gave a complete list of candidates for curves of genus 1 with one singularity whose link is a torus link $T_{p,q}$. The list contains one infinite family (Orevkov curves) and a finite list of special cases. We apply our obstructions to these curves and obtain the following result:

Proposition 6.16 *Suppose C is a genus one, degree d curve, having a single singularity, whose link is a (p,q)–torus knot. Then either C is the Orevkov curve (of Example 6.15), or the values of (p,q) and d are one of*

(a) *$d = 4$ and $(p,q) = (2,5)$;*

(b) *$d = 5$ and $(p,q) = (2,11)$;*

case	(d,p,q)	positive	negative	existence
(a)	$(4,2,5)$	passes	passes	exists
(b)	$(5,2,11)$	passes	passes	exists
(c)	$(6,3,10)$	passes	$k=1$	
(d)	$(15,6,37)$	passes	$k=2$	
(e)	$(24,9,64)$	passes	$k=3$	
(f)	$(27,10,73)$	$k=12$	$k=8$	
(g)	$(33,12,91)$	$k=7$	$k=8$	
(h)	$(3p,p,9p+1)$	passes	fails if $p\geq 3$	

Table 1: Curves of Proposition 6.16 and the criteria of Proposition 6.14. "Positive" refers to item (b) of the proposition, "negative" refers to item (c). If the curve does not pass the criteria, we indicate the minimal k for which $R(kd)>\upsilon_k$ (case (b)) or $R(kd)<\upsilon_k$ (case (c)).

(c) *$d=6$ and $(p,q)=(3,10)$;*

(d) *$d=15$ and $(p,q)=(6,37)$;*

(e) *$d=24$ and $(p,q)=(9,64)$;*

(f) *$d=27$ and $(p,q)=(10,73)$;*

(g) *$d=33$ and $(p,q)=(12,91)$;*

(h) *$d=3p$ and $(p,q)=(p,9p+1)$ for $p=2,\dots,11$.*

By definition, all cases satisfy the statement of Proposition 6.14(a). We applied the criteria of Proposition 6.14(b)–(c). The results are in Table 1. We indicate that some of the examples predicted by Proposition 6.16 have not been either effectively constructed or obstructed by other means.

6.6 Generalized Orevkov curves

Bodnár et al. [2016] constructed a family of curves generalizing Orevkov's construction. Their work can be regarded as a generalization of the construction of [Borodzik et al. 2017, Proposition 9.12]. To begin with, fix $k\geq 2$. The *Lucas sequence* is the sequence L_i^k defined recursively via $L_0^k=k-1$, $L_1^k=1$, $L_{i+1}^k=L_i^k+L_{i-1}^k$. Here i is allowed to take all integer values.

Theorem 6.17 (BCG family; see [Bodnár et al. 2016, Theorem 1.7]) *For any $i\geq 2$, there exists a genus $\frac{1}{2}k(k-1)$ curve of degree L_{4i-1}^k with precisely one singularity whose link is the (L_{4i-3}^k, L_{4i+1}^k)–torus knot.*

For any $j\geq 1$, there exists a genus $\frac{1}{2}k(k-1)$ curve of degree $-L_{-4j-1}^k$ with singularity whose link is the $(-L_{-4j+1}^k,-L_{-4j-3}^k)$–torus knot.

Now we apply Corollary 6.11.

Proposition 6.18 *None of the curves of the BCG family can be transformed into a curve with genus one less and one negative double point.*

Proof We follow the same strategy as in Example 6.15. We begin with the first family. Suppose $i \geq 2$. Let S be the semigroup associated with the (L^k_{4i-3}, L^k_{4i+1})–torus knot, and let R be the counting function for it. The recursive formula for Lucas numbers implies that $L^k_s + L^k_{s+4} = 3L^k_{s+2}$ for all s. Moreover,

(6.19) $$L^k_{s+4} = L^k_{s+3} + L^k_{s+2} = 2L^k_{s+2} + L^k_{s+1} = 3L^k_{s+1} + 2L^k_s = 5L^k_s + 3L^k_{s-1} < 8L^k_s$$

as long as $s \geq 0$. In particular, $3L^k_{s+1} < 9L^k_s$. Therefore, all possible elements in $S \cap [0, 3L^k_{4j-1}]$ are $0, \dots, 8L^k_{4j-3}$ and L^k_{4j+1}. Hence, $R(3L^k_{4j-1}) \leq 9$, violating the second inequality in Corollary 6.11(b).

As for the second family, write $\widetilde{L}^k_i = (-1)^{i+1} L^k_{-i}$ for $i > 0$ and note that $\widetilde{L}^k_{i+1} = \widetilde{L}^k_i + \widetilde{L}^k_{i-1}$. Moreover, for $i > 0$, $\widetilde{L}^k_i$ is an increasing sequence of positive numbers. We have $\widetilde{L}^k_{s+4} + \widetilde{L}^k_s = 3\widetilde{L}^k_{s+2}$ and, for s odd, $\widetilde{L}^k_{s+4} < 8\widetilde{L}^k_s$ by the same argument as in (6.19). We conclude as in the first case. □

It is unknown whether it is possible to trade genus for *positive* double points in any curves in the BCG family.

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