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Instanton knot invariants with rational holonomy parameters and an application for torus knot groups

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There are several knot invariants in the literature that are defined using singular instantons. Such invariants provide strong tools to study the knot group and give topological applications, for instance, the topology of knots in terms of representations of fundamental groups. In particular, it has been shown that any traceless representation of the torus knot group can be extended to any concordance from the torus knot to another knot. Daemi and Scaduto proposed a generalization that is related to a version of the slice-ribbon conjecture for torus knots. Our results provide further evidence towards the positive answer to this question. We use a generalization of Daemi and Scaduto's equivariant singular instanton Floer theory following Echeverria's earlier work. We also determine the irreducible singular instanton homology of torus knots for all but finitely many rational holonomy parameters as $\mathbb{Z}/4$–graded abelian groups.

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1 Introduction

1.1 Background

Floer homology is an infinite-dimensional analog of Morse homology. In the context of gauge theory, instanton Floer homology (see Floer [14]), Heegaard Floer homology (see Ozsváth and Szabó [37]) and monopole Floer homology (Kronheimer and Mrowka [30]) have provided strong topological invariants for low-dimensional manifolds. Knot invariants have also been developed in Floer theories. This list of knot invariants includes knot Floer homology introduced by Ozsváth and Szabó [36] and Rasmussen [40] in Heegaard Floer theory, and Kronheimer and Mrowka [31] in monopole Floer theory. In the field of instanton Floer theory, invariants of knots were constructed by Floer [15] and Braam and Donaldson [1]

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via framed surgery of knots. It is conjectured that their instanton knot invariants are related to knot invariants in Ozsváth and Szabó [36] and Rasmussen [40] by Kronheimer and Mrowka [31]. Collin and Steer [3] and Kronheimer and Mrowka [32] developed other type invariants for knots. While knot invariants in [15; 1] are related to invariants of 3–manifolds via surgery along knots, knot invariants in [3; 32] are related to 3–manifold invariants via branched covering.

The advantage of instanton invariants is that they are directly related to fundamental groups of the knot complement. For example, Kronheimer and Mrowka [29] show that the knot group $\pi_1(S^3 \setminus K)$ for a nontrivial knot $K \subset S^3$ admits nonabelian representation $\pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$. This is a refinement of the result by Papakyriakopoulos [38] which states that $K \subset S^3$ is unknot if only if $\pi_1(S^3 \setminus K)$ is infinitely cyclic. A concordance analog of the result of Kronheimer and Mrowka [29] was given by Daemi and Scaduto [8] using a version of instanton Floer theory. Daemi and Scaduto [8] also show the following statement which is specific to torus knots:

Theorem 1 [8, Theorem 8] *Let $S\colon T_{p,q} \to K$ be a given smooth concordance. Then any traceless $\mathrm{SU}(2)$–representation of $\pi_1(S^3 \setminus T_{p,q})$ extends over the concordance complement.*

Here $T_{p,q}$ denotes the (p,q)–torus knot in S^3, where p and q are positive coprime integers. An $\mathrm{SU}(2)$–traceless representation of $\pi_1(S^3 \setminus K)$ is an $\mathrm{SU}(2)$–representation of $\pi_1(S^3 \setminus K)$ which sends a homotopy class of meridian μ_K of K to a traceless element in $\mathrm{SU}(2)$. The motivation of this theorem is related to a version of the slice-ribbon conjecture. A concordance $S\colon K \to K'$ is called ribbon concordance if the projection $S^3 \times [0,1] \supset S \to [0,1]$ is a Morse function without any local maximums. Consider a knot K which is concordant to the unknot U. The slice-ribbon conjecture proposed by Fox [16] states that there is a ribbon concordance from U to K under this assumption. A generalization of the slice-ribbon conjecture by Daemi and Scaduto [8] is:

Conjecture 2 [8, Question 2] *Let K be a knot which is concordant to the (p,q)–torus knot $T_{p,q}$. Then there is a ribbon concordance from $T_{p,q}$ to K.*

A necessary condition to show that a concordance $S\colon K \to K'$ is ribbon can be stated in terms of representations of knot groups. For a topological space X, we write $\mathcal{R}(X, \mathrm{SU}(2))$ for the $\mathrm{SU}(2)$–character variety of X (ie the space of conjugacy classes of $\mathrm{SU}(2)$–representations of $\pi_1(X)$).

Theorem 3 (Gordon [21, Lemma 3.1] and Daemi, Lidman, Vela-Vick and Wong [7, Proposition 2.1]) *Let $S\colon K \to K'$ be a ribbon concordance between two knots. Then the inclusion $i\colon S^3 \setminus K \to S^3 \times [0,1]$ induces a surjection $i^*\colon \mathcal{R}(S^3 \times [0,1] \setminus S, \mathrm{SU}(2)) \to \mathcal{R}(S^3 \setminus K, \mathrm{SU}(2))$.*

Hence Theorem 1 gives a piece of evidence towards Conjecture 2. The traceless condition on representations of $\pi_1(S^3 \setminus T_{p,q})$ arises from the specific type of knot invariants developed by Daemi and Scaduto [9]. In light of Theorem 3 and Conjecture 2, it is natural to ask the following question:

Question 4 *Can we drop the traceless condition in Theorem 1?*

We will affirmatively solve this question. To explain our strategy, let us describe the technical background of Daemi and Scaduto's work. It mainly consists of three ingredients: singular gauge theory, equivariant Floer theory and the Chern–Simons filtration.

Firstly, let us explain the notion of singular connections. Let $K \subset Y$ be a knot in a 3–manifold. Roughly speaking, an SU(2)–singular connection A is an SU(2)–connection defined over the knot complement with the holonomy condition

$$\lim_{r\to 0} \mathrm{Hol}_{\mu(r)}(A) \sim \begin{bmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{bmatrix}, \tag{1-1}$$

where $\mu(r)$ is a radius-r meridian of $K \subset Y$ and α is a fixed parameter in $\left(0, \frac{1}{2}\right)$. Here $\sim$ indicates that the two matrices are conjugate in SU(2). The parameter α is called the holonomy parameter of the singular connection A. In particular, a singular flat SU(2)–connection corresponds to an SU(2)–representation of $\pi_1(Y \setminus K)$ which sends the meridian μ_K of knot K to an element which is conjugate to the matrix in (1-1). Kronheimer and Mrowka developed a singular version of Yang–Mills gauge theory in [27; 28; 32]. These Floer homology theories constructed via singular connections are called singular instanton homology. Singular gauge theory has different features compared to nonsingular. In fact, singular Floer homology cannot be defined over the coefficient ring $\mathbb{Z}$ for a general holonomy parameter α. To be more precise, singular instanton Floer homology is defined over $\mathbb{Z}$ only for $\alpha = \frac{1}{4}$. This is called *the monotonicity condition*. Most of the works in singular instanton homology including [9; 8] impose the monotonicity condition. This is why the statement of Theorem 1 includes the traceless condition.

Next, we discuss the equivariant Floer theory. Frøyshov developed the homology cobordism invariant in [18; 19] based on the equivariant Floer theory for integral homology 3–spheres, which was introduced by Donaldson [10]. The equivariant Floer theory introduced by Daemi and Scaduto [9] produces invariants for a knot K in an integral homology 3–sphere Y, and this can be regarded as the counterpart of Frøyshov's work in singular gauge theory. Daemi and Scaduto's construction uses in a crucial way the $U(1)$–reducible singular flat connection θ which corresponds to the conjugacy class of the representation

$$\pi_1(Y \setminus K) \to H_1(Y \setminus K; \mathbb{Z}) \to \mathrm{SU}(2)$$

whose image of the meridian μ_K of $K \subset Y$ is trace-free. Here $\pi_1(Y \setminus K) \to H_1(Y \setminus K; \mathbb{Z})$ is the abelianization. In this situation, the construction which is similar to Floer's instanton homology [14] produces a chain complex $C_*(Y, K)$ for a knot in an integral homology 3–sphere. Its homology group $I_*(Y, K)$ can be interpreted as a categorification of the knot signature for the case $Y = S^3$. Daemi and Scaduto [9] also introduced chain complexes which have the form

$$\widetilde{C}_*(Y, K) := C_*(Y, K) \oplus C_{*-1}(Y, K) \oplus \mathbb{Z}.$$

Such objects are called $\mathcal{S}$–complexes. This can be interpreted as a version of S^1–equivariant Floer theory. Let $\mathcal{B}(Y, K)$ be the configuration space of singular connections over (Y, K) with a holonomy parameter $\alpha = \frac{1}{4}$. Then there is a configuration space $\mathcal{B}(Y, K)_0$ of framed connections. The Chern–Simons functional on $\mathcal{B}(Y,K)$ lifts to $\mathcal{B}(Y,K)_0$ in an equivariant way. An $\mathcal{S}$–complex $\widetilde{C}_*(Y,K)$ is related to the lifted S^1–equivariant Chern–Simons functional on $\mathcal{B}(Y,K)_0$.

Another feature of Daemi and Scaduto's construction is the Chern–Simons filtration of $\mathcal{S}$–complexes. While the usual instanton Floer theory is the analog of Morse theory on the configuration space, its filtered version can be seen the Morse theory on the universal covering of the configuration space. The Chern–Simons filtration gives more refined structures on $\mathcal{S}$–complexes. The counterpart idea in nonsingular instanton Floer theory was used by Daemi [5] and Nozaki, Sato and Taniguchi [35], which provided homology cobordism invariants.

Any of the above versions of singular instanton Floer theories can be extended to arbitrary holonomy parameters if the integer coefficient ring is replaced with a Novikov ring Λ by Echeverria's work [12]. To be more precise, the holonomy parameter should satisfy the technical condition $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$, where $\Delta_{(Y,K)}$ is the Alexander polynomial for $K \subset Y$. One of the flavors of Echeverria's Floer homology is a categorification of the Levine–Tristram signature when $Y = S^3$. For a knot K in an integral homology 3–sphere Y, the Levine–Tristram signature is given by

$$\sigma_\alpha(Y,K) := \operatorname{sign}[(1-e^{4\pi i\alpha})V + (1-e^{-4\pi i\alpha})V^T],$$

where V is a Seifert matrix form of $K \subset Y$. For the case $Y = S^3$, we omit Y from the notation.

Our strategy to drop the traceless condition from Theorem 1 is constructing a family of $\mathcal{S}$–complexes for general holonomy parameters.

1.2 Summary of results

First we state our main theorem, which gives the positive answer to Question 4:

Theorem 1.1 *For a given knot K and a smooth concordance $S\colon T_{p,q} \to K$, any* $\mathrm{SU}(2)$*–representation of* $\pi_1(S^3 \setminus T_{p,q})$ *extends to an* $\mathrm{SU}(2)$*–representation of* $\pi_1((S^3 \times [0,1]) \setminus S)$.

The proof of Theorem 1.1 requires the special property that all generators of singular instanton homology for torus knots have odd gradings. The outline of the proof is as follows. After extending the condition of Daemi and Scaduto [9], we define analogous knot Floer theory of [9] for all $\alpha \in \mathcal{I}$, where $\mathcal{I}$ is a dense subset of $\left[0, \frac{1}{2}\right]$. This means that all SU(2)–representations of $\pi_1(S^3 \setminus T_{p,q})$ with the holonomy parameter $\alpha \in \mathcal{I}$ extend to the concordance complement. The limiting argument shows that this extension property is true for all SU(2)–representations of $\pi_1(S^3 \setminus T_{p,q})$ with any holonomy parameter $\alpha \in \left[0, \frac{1}{2}\right]$.

As described above, singular instanton knot homology (see Echeverria [12]) and its equivariant counterparts are key tools for the proof, so we review the essential properties of these objects we use. We consider the Novikov ring $\Lambda^{\mathbb{Z}[T^{-1},T]]}$ which is given by

$$\Lambda^{\mathbb{Z}[T^{-1},T]]} := \left\{ \sum_{r\in\mathbb{R}} p_r \lambda^r \;\middle|\; p_r \in \mathbb{Z}[T^{-1},T]],\ \forall C > 0,\ \#\{p_r \neq 0\}_{r>C} < \infty \right\}.$$

Let α be a parameter in $\left(0, \frac{1}{2}\right)$. We introduce the subring $\mathscr{R}_\alpha$ of $\Lambda^{\mathbb{Z}[T^{-1},T]]}$

$$\mathscr{R}_\alpha := \begin{cases} \mathbb{Z}[\xi_\alpha^{\pm 1}][\![\lambda^{-1}, \lambda] & \text{if } \alpha \leq \frac{1}{4}, \\ \mathbb{Z}[\lambda^{\pm 1}][\![\xi_\alpha^{-1}, \xi_\alpha] & \text{if } \alpha > \frac{1}{4}, \end{cases}$$

where $\xi_\alpha = \lambda^{2\alpha} T^2$. The geometric aspect of the subring $\mathscr{R}_\alpha$ is described in Section 3.2.

Theorem 1.2 *Let $\mathscr{S}$ be an algebra over $\mathscr{R}_\alpha$. Let $K \subset Y$ be an oriented knot in an integral homology 3–sphere. Choose a holonomy parameter $\alpha \in \mathbb{Q} \cap \left(0, \frac{1}{2}\right)$ so that $\Delta_{(Y,K)}(e^{4\pi i \alpha}) \neq 0$. Then we can associate a $\mathbb{Z}/4$–graded module $I^\alpha_*(Y, K; \Delta_{\mathscr{S}})$ over $\mathscr{S}$ to this parameter. Moreover, if $\mathscr{S}$ is an integral domain, we can associate a $\mathbb{Z}/4$–graded $\mathcal{S}$–complex $(\widetilde{C}^\alpha_*(Y, K; \Delta_{\mathscr{S}}), \tilde{d}, \chi)$ to a given triple (Y, K, α) with $\Delta_{(Y,K)}(e^{4\pi i \alpha}) \neq 0$, up to $\mathcal{S}$–chain homotopy equivalence.*

The precise definition of an $\mathcal{S}$–complex can be seen in Section 3.1. We call $I^\alpha_*(Y, K; \Delta_{\mathscr{S}})$ *the irreducible singular instanton knot homology over $\mathscr{S}$ with the holonomy parameter α*. For the case $Y = S^3$, we drop Y from the notation. The difference between the construction of our Floer homology $I^\alpha_*(Y, K; \Delta_{\mathscr{S}})$ and $I_*(Y, K, \alpha)$ introduced by [12] is the choice of local coefficients. The construction of $(C^\alpha_*(Y, K), d)$ and $(\widetilde{C}^\alpha_*(Y, K), \tilde{d}, \chi)$ depends on additional data (metric and perturbation), however their chain homotopy classes in the sense of $\mathcal{S}$–complexes are independent of such choices.

Remark 1.3 To be more precise, we need to specify the choice of positive integer $\nu \in \mathbb{Z}_{>0}$, called the cone angle, to define the invariant $I^\alpha_*(Y, K, \Delta_{\mathscr{S}})$. The details are included in Remark 3.9. As conjectured in [12], we expect that the invariant $I^\alpha_*(Y, K, \Delta_{\mathscr{S}})$ does not depend on the choice of cone angle $\nu \in \mathbb{Z}_{>0}$, and hence it is reasonable to drop ν from the notation. Similar remarks are applied to the dependence of invariants $\widetilde{C}^\alpha_*(Y, K, \Delta_{\mathscr{S}})$ and $h^\alpha_{\mathscr{S}}(Y, K)$ which appear later. For a given holonomy parameter α, we always assume that the cone angle ν is a large enough integer.

Remark 1.4 For the coefficient $\mathscr{S} = \mathscr{R}_\alpha$, we consider underlying groups of $C^\alpha_*(Y, K; \Delta_{\mathscr{S}})$ and $\widetilde{C}^\alpha(Y, K, \Delta_{\mathscr{S}})$ as $\mathbb{Z}$–modules. Then if we fix the choice of auxiliary data, there exists a functional giving the $(\mathbb{Z}\times\mathbb{R})$–bigraded structure on sets of generators of these underlying groups. Moreover, they have a filtered structure induced from the $\mathbb{R}$–grading. The precise descriptions of the $(\mathbb{Z}\times\mathbb{R})$–bigrading and the filtered structure are contained in Sections 3.2 and 3.4.

The following statement describes the behavior of $\mathcal{S}$–complexes under the connected sum:

Theorem 1.5 *Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. Let $K \subset Y$ and $K' \subset Y'$ be two oriented knots in integral homology 3–spheres. Fix a holonomy parameter $\alpha \in \mathbb{Q} \cap \left(0, \frac{1}{2}\right)$ such that*

$$\Delta_{(Y,K)}(e^{4\pi i \alpha}) \Delta_{(Y',K')}(e^{4\pi i \alpha}) \neq 0.$$

Then there is a chain homotopy equivalence of $\mathcal{S}$–complexes

$$\widetilde{C}^\alpha_*(Y \# Y', K \# K'; \Delta_{\mathscr{S}}) \simeq \widetilde{C}^\alpha_*(Y, K; \Delta_{\mathscr{S}}) \otimes_{\mathscr{S}} \widetilde{C}^\alpha_*(Y', K'; \Delta_{\mathscr{S}}).$$

The precise definition of *a chain homotopy equivalence of $\mathcal{S}$–complexes* can be seen in Definition 3.3. This is a generalization of the connected sum theorem by Daemi and Scaduto [9]. The method of the proof of [9, Theorem 6.1] cannot be directly adapted to prove Theorem 1.5 since we have to deal with the nonmonotonicity situation which arises for general holonomy parameters.

As described in [9], we can associate an integer-valued invariant which is called the Frøyshov type invariant to a given $\mathcal{S}$–complex. Our construction of $\mathcal{S}$–complexes provides an integer-valued invariant $h^{\alpha}_{\mathscr{S}}(Y,K)$ for a knot in a homology 3–sphere (Y,K). We call $h^{\alpha}_{\mathscr{S}}(Y,K)$ *the Frøyshov invariant for* (Y,K) *over* $\mathscr{S}$ *with the holonomy parameter* α . We drop Y from the notation when $Y=S^3$. Note that Echeverria [12] also introduced the Frøyshov type invariant denoted by $h(Y,K,\alpha)$, which is constructed from singular instanton Floer homology with a different local coefficient system from our setting. The invariant $h^{\alpha}_{\mathscr{S}}(Y,K)$ satisfies the following properties:

Theorem 1.6 *Let* (Y,K) *and* (Y',K') *be two pairs of integral homology* 3*–spheres and knots. Assume that* $\alpha\in\mathbb{Q}\cap\left(0,\tfrac{1}{2}\right)$ *satisfies* $\Delta_{(Y,K)}(e^{4\pi i\alpha})\neq 0$ *and* $\Delta_{(Y',K')}(e^{4\pi i\alpha})\neq 0$. *Then*

$$h^{\alpha}_{\mathscr{S}}(Y\#Y',K\#K')=h^{\alpha}_{\mathscr{S}}(Y,K)+h^{\alpha}_{\mathscr{S}}(Y',K').$$

Moreover, if (Y,K) *and* (Y',K') *are homology concordant, then*

$$h^{\alpha}_{\mathscr{S}}(Y,K)=h^{\alpha}_{\mathscr{S}}(Y',K').$$

Let us consider a knot in S^3. It has been shown that the Frøyshov type invariant in [9] reduces to knot signature (see Daemi and Scaduto [8, Theorem 7]). The invariant $h^{\alpha}_{\mathscr{S}}$ reduces to the Levine–Tristram signature as follows:

Theorem 1.7 *Let* $\mathscr{S}$ *be an integral domain over* $\mathscr{R}_{\alpha}$. *For any knot* $K\subset S^3$ *and for a holonomy parameter* $\alpha\in\left(0,\tfrac{1}{2}\right)\cap\mathbb{Q}$ *with* $\Delta_K(e^{4\pi i\alpha})\neq 0$, *the following equality holds*:

$$h^{\alpha}_{\mathscr{S}}(K)=-\tfrac{1}{2}\sigma_{\alpha}(K).$$

For a given knot $K\subset S^3$ and integer l, we define a knot $lK\subset S^3$ so that

$$lK:=\begin{cases}\#_l K & \text{if } l>0,\\ U \text{ (unknot)} & \text{if } l=0,\\ \#_{-l}(-K) & \text{if } l<0,\end{cases}$$

where $-K$ is the mirror of K with the reverse orientation. More strongly, $\mathcal{S}$–complexes have the following structure theorem:

Theorem 1.8 *Let* $\mathscr{S}$ *be an integral domain over* $\mathscr{R}_{\alpha}$. *Then for a knot* K *in* S^3 *and for a holonomy parameter* $\alpha\in\mathbb{Q}\cap\left(0,\tfrac{1}{2}\right)$ *with* $\Delta_K(e^{4\pi i\alpha})\neq 0$, *there is a two-bridge torus knot* $T_{2,2n+1}$ *such that* $\Delta_{T_{2,2n+1}}(e^{4\pi i\alpha})\neq 0$, $\sigma_{\alpha}(T_{2,2n+1})=-2$ *and the relation*

$$\widetilde{C}^{\alpha}_{*}(K;\Delta_{\mathscr{S}})\simeq\widetilde{C}^{\alpha}_{*}(lT_{2,2n+1};\Delta_{\mathscr{S}})$$

holds, where $l=-\tfrac{1}{2}\sigma_{\alpha}(K)$.

For the proof of Theorem 1.8, it is essential to observe behaviors of morphisms of $\mathcal{S}$–complexes induced from cobordisms between pairs (Y, K) and (Y', K'). In [8], techniques demonstrated by Kronheimer [25] are used to describe behaviors of morphisms of $\mathcal{S}$–complexes for the case $\alpha = \frac{1}{4}$. However, such techniques do not directly adapt to prove Theorem 1.8 because of the lack of the monotonicity condition.

Theorems 1.5 and 1.8 imply the Euler characteristic formula

$$\chi(I^{\alpha}(K, \Delta_{\mathscr{S}})) = \tfrac{1}{2}\sigma_{\alpha}(K) \tag{1-2}$$

(see Section 5.1). Since our grading convention of generators coincides with that of Echeverria [12], the above argument also gives an alternative proof of the Euler characteristic formula in [12, Theorem 17] for the case $Y = S^3$.

Next, we focus on (p, q)–torus knot $T_{p,q}$ in the 3–sphere. We always assume that p and q are positive coprime integers. The following is a characteristic property of the torus knot and a key lemma for the proof of Theorem 1.1. Let $\mathcal{R}_{\alpha}(Y \setminus K, \mathrm{SU}(2))$ be the space of conjugacy classes of SU(2)–representations of $\pi_1(Y \setminus K)$ with the holonomy parameter α. Let $\mathcal{R}^*_{\alpha}(Y \setminus K, \mathrm{SU}(2))$ be its irreducible part.

Theorem 1.9 *For any $\alpha \in \left[0, \frac{1}{2}\right]$ with $\Delta_{T_{p,q}}(e^{4\pi i \alpha}) \neq 0$,*

$$|\mathcal{R}^*_{\alpha}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))| = -\tfrac{1}{2}\sigma_{\alpha}(T_{p,q}).$$

Here $|S|$ for a set S denotes the size of this set. In [23], Herald introduced the *signed count* of elements in the character variety $\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$ for a general knot K with a fixed holonomy parameter. One first perturbs $\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$ into a discrete set $\mathcal{R}^{*,h}_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$ and then associates a sign to each element of this set. The sum of these signs is Herald's signed count of $\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$, which we denote by $\#\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$. In general:

$$\#\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2)) = -\tfrac{1}{2}\sigma_{\alpha}(K).$$

See Herald [23, Corollary 0.2] and Lin [33] for the case $\alpha = \frac{1}{4}$. In the case $K = T_{p,q}$, the character variety $\mathcal{R}^*_{\alpha}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ is already discrete and one does not make any perturbation. Theorem 1.9 implies that all elements of $\mathcal{R}^*_{\alpha}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ have positive signs.

Theorem 1.9 implies that $C^{\alpha}_*(T_{p,q}; \Delta_{\mathscr{S}})$ is supported only on the odd graded components. In particular, its homology groups are isomorphic to chain complexes,

$$I^{\alpha}_*(T_{p,q}; \Delta_{\mathscr{S}}) \cong C^{\alpha}_*(T_{p,q}; \Delta_{\mathscr{S}}),$$

since all differentials of chain complexes are trivial. This can be interpreted as the counterpart of the computation of instanton homology of Brieskorn homology 3–spheres; see Fintushel and Stern [13].

Theorem 1.7 implies that $\operatorname{rank} C^{\alpha}_*(T_{p,q}; \Delta_{\mathscr{S}}) = h^{\alpha}_{\mathscr{S}}(T_{p,q})$, and by the definition of the invariant $h^{\alpha}_{\mathscr{S}}$:

Theorem 1.10 *Let* $\mathscr{S}$ *be an algebra over* $\mathscr{R}_\alpha$. *For* $\alpha \in \left(0, \frac{1}{2}\right) \cap \mathbb{Q}$ *with* $\Delta_{T_{p,q}}(e^{4\pi i \alpha}) \neq 0$, *there is an isomorphism*

$$I_*^\alpha(T_{p,q}; \Delta_{\mathscr{S}}) \cong \mathscr{S}_{(1)}^{\lceil -\sigma_\alpha(T_{p,q})/4 \rceil} \oplus \mathscr{S}_{(3)}^{\lfloor -\sigma_\alpha(T_{p,q})/4 \rfloor}$$

as a $\mathbb{Z}/4$*–graded abelian group.*

Theorem 1.10 describes the grading of generators of the Floer chain and it is independent of the choice of local coefficient system. A similar structure theorem holds for singular instanton knot homology introduced by Echeverria [12].

Theorem 1.8 implies that $\mathcal{S}$–complexes for knots are determined by the Levine–Tristram signature without the $(\mathbb{Z}\times\mathbb{R})$–grading structure. On the other hand, the $\mathbb{R}$–grading from the Chern–Simons filtration can be expected to have stronger information on the knot concordance. In upcoming work of Daemi, Sato, Scaduto, Taniguchi and the author [6], relying on the results here, we will introduce a generalization of the Γ–invariant of Daemi and Scaduto [9] for rational holonomy parameters, which can be regarded as a gauge-theoretic refinement of the Levine–Tristram signature. Our techniques are also used in the future work of Daemi and Scaduto to construct families of hyperbolic knots that are minimal with respect to the ribbon partial order; see Gordon [21, Conjecture 1.1].

1.3 Outline

In Section 2, we review the background of SU(2)–singular gauge theory for rational holonomy parameters. We also introduce the generalized definition of negative definite cobordism. In Section 3, we construct Floer chain groups and $\mathcal{S}$–complexes parametrized by holonomy parameters α, and introduce the Frøyshov type invariant. The argument is almost parallel to [9; 8], however, we need a careful choice of local coefficient system if we introduce the bigraded structure on the Floer chain complex. We also prove Theorem 1.6. In Section 4, we prove the Levine–Tristram signature formula for torus knots (Theorem 1.9). In the proof of Theorem 1.9, we use the correspondence of singular flat connections and nonsingular flat connections over the branched covering space. We also use the pillowcase picture of the SU(2)–character variety for the knot complement space. We prove Theorems 1.7, 1.8 and 1.10 in Section 5.1, and finally, we give the proof of our main theorem (Theorem 1.1) in Section 5.2. The bigraded structure of $\mathcal{S}$–complexes plays an important role in the proof of the main theorem. The appendix consists of the proof of the connected sum theorem (Theorem 1.5).

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2 Background on singular instantons

In this section, we review singular gauge theory, mainly developed by Kronheimer and Mrowka [32]. We also give a generalization of the setting of singular SU(2)–gauge theory adopted by Daemi and Scaduto [9; 8].

2.1 The space of singular connections

We review the construction of singular instantons over a closed pair of a 4–manifold and a surface. Let X be a closed and oriented smooth 4–manifold and S be a closed and oriented embedded surface in X. Let N be a tubular neighborhood of S in X. We identify N with a disk bundle over S and ∂N with a circle bundle over S. Let η be a connection 1–form on a circle bundle ∂N. This means that η is $U(1)$–invariant. We fix a decomposition of the SU(2)–bundle $E \to X$ over the embedded surface S as $E|_S = L \oplus L^*$, where L is a $U(1)$–bundle over S. This decomposition extends to N. We define two topological invariants,

$$k = c_2(E)[X] \quad \text{and} \quad l = -c_1(L)[S].$$

Here k is called the instanton number, and l is called the monopole number.

Next, we fix a connection A_0 over X of the form

$$A_0|_N = \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}.$$

Here b is a connection over L. This means that A^0 reduces to a $U(1)$–connection over S. We give the polar coordinates $(r, \theta) \in D^2$ on each fiber of N. Let η be a 1–form obtained by a pulled-back 1–form on ∂N which coincides $d\theta$ on each fiber, and ψ be a cutoff function supported on N. We define the singular base connection A^α by

$$A^\alpha = A_0 + i\psi \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \eta,$$

where $\alpha \in \left(0, \frac{1}{2}\right)$. Here α is called the holonomy parameter. Recall that η is defined only on $N \setminus S$, but extends by 0 to $X \setminus S$ after cutting off by ψ. A^α is a connection over $X \setminus S$. Let $\mathfrak{g}_E$ be the adjoint bundle of E. For $a \in \Omega^1(X \setminus S, \mathfrak{g}_E)$, $A^\alpha + a$ is called a singular connection.

Before defining the space of singular connections, we have to introduce functional spaces. We fix an orbifold metric on X, which can be written in the form

$$g^\nu = du^2 + dv^2 + dr^2 + \frac{r^2}{\nu^2}\, d\theta^2$$

on N, where (u, v) is a local coordinate of S. We say that this orbifold metric has cone angle $2\pi/\nu$. Then (X, g^ν) has a local structure $U/\mathbb{Z}_\nu$ near the singular locus S, where U is an open set in $\mathbb{R}^4$. The model connection A^α induces an SO(3)–adjoint connection on $\mathfrak{g}_E$. We define the covariant derivative $\check{\nabla}_{A^\alpha}$ on the bundle $\Lambda^m \otimes \mathfrak{g}_E$ using the adjoint connection of A^α and the Levi-Civita connection with respect to the metric g^ν. Let $F \to X$ be an orbifold vector bundle. The Sobolev space $\check{L}^p_{m,A^\alpha}(X \setminus S, F)$ is defined as the completion of the space of smooth sections of $F \to X$ by the norm

$$\|s\|^p_{\check{L}^p_{m,A^\alpha}} = \sum_{i=0}^{m} \int_{X \setminus S} |\check{\nabla}_{A^\alpha} s|^p \,\mathrm{dvol}_{g^\nu}.$$

If we use orbifold metrics, the "Fredholm package" works. Let $d^+_{A^\alpha} : \Omega^1(X \setminus S, \mathfrak{g}_E) \to \Omega^+(X \setminus S, \mathfrak{g}_E)$ be the linearized anti-self-dual operator defined by the metric g^ν, and $d^*_{A^\alpha} : \Omega^1(X \setminus S, \mathfrak{g}_E) \to \Omega^0(X \setminus S, \mathfrak{g}_E)$ be the formal adjoint of the covariant derivative for the metric g^ν. Consider the elliptic operator $D_{A^\alpha} = -d^*_{A^\alpha} \oplus d^+_{A^\alpha}$ acting on the Sobolev space

$$\check{L}^p_{m,A^\alpha}(X \setminus S, \Lambda^1 \otimes \mathfrak{g}_E) \to \check{L}^p_{m-1,A^\alpha}(X \setminus S, (\Lambda^0 \oplus \Lambda^+) \otimes \mathfrak{g}_E). \tag{2-1}$$

Proposition 2.1 *Let α be a rational holonomy parameter of the form $\alpha = p/q \in \left(0, \tfrac{1}{2}\right) \cap \mathbb{Q}$. Choose a cone angle $2\pi/\nu$ of orbifold metric so that $2\nu p/q \in \mathbb{Z}$. Then the operator D_{A^α} and its formal adjoint are Fredholm, and the Fredholm alternative holds.*

Let A^α_{ad} be the adjoint of the singular connection A^α and $\pi : U \to U/\mathbb{Z}_\nu$ be an orbifold chart with respect to the orbifold metric g^ν. If $\nu \in \mathbb{Z}_{>0}$ is chosen as in Proposition 2.1, the lift of the adjoint connection of $\pi^* A^\alpha$ has the asymptotically trivial holonomy along a small linking of the singular locus. Thus $\pi^* A^\alpha$ extends smoothly over U. This means that A^α_{ad} defines an orbifold connection. All analytical argument reduces to the orbifold setting. From now on, we always fix ν as in Proposition 2.1 for a given rational holonomy parameter.

Assume that $m > 2$. The space of singular connections with a holonomy parameter $\alpha \in \left(0, \tfrac{1}{2}\right)$ is given by

$$\mathcal{A}(X, S, \alpha) = \{A^\alpha + a \mid a \in \check{L}^2_{m,A^\alpha}(X \setminus S, \Lambda^1 \otimes \mathfrak{g}_E)\}.$$

This is an affine space as the nonsingular case. Notice that $\mathcal{A}(X, S, \alpha)$ is independent of the choice of the base connection A^α. We also introduce the group of gauge transformations,

$$\mathcal{G}(X, S) = \{g \in \mathrm{Aut}(E) \mid g \in \check{L}^2_{m+1,A^\alpha}(X \setminus S, \mathrm{End}(E))\}.$$

There is the smooth action of $\mathcal{G}(X, S)$ on $\mathcal{A}(X, S, \alpha)$, and we can take the quotient.

$$\mathcal{B}(X, S, \alpha) = \mathcal{A}(X, S, \alpha)/\mathcal{G}(X, S).$$

A singular connection with the 0–dimensional stabilizer for the action of $\mathcal{G}(X, S)$ is called an irreducible connection. A singular connection is called reducible if it is not irreducible. The quotient space $\mathcal{B}(X, S, \alpha)$ has a smooth Banach manifold structure except for orbits of reducible connections. The set of gauge equivalence classes of solutions for the anti-self-dual equation

$$M^\alpha(X, S) = \{[A] \in \mathcal{B}(X, S, \alpha) \mid F^+_A = 0\}$$

is called the moduli space of singular anti-self-dual connections. $M^\alpha(X,S)_d$ denotes the subset of $M^\alpha(X,S)$ with expected dimension d. For a generic orbifold metric with a fixed cone angle, the irreducible part of $M^\alpha(X,S)_d$ is a smooth manifold of dimension $d=\operatorname{ind}(d_A^*\oplus d_A^+)$, where $[A]\in M^\alpha(X,S)_d$. If $M^\alpha(X,S)_d$ consists of reducible connections, we modify the dimension of the moduli space so that $d=\operatorname{ind}(d_A^*\oplus d_A^+)+\dim H_A^0$, where H_A^i is an i^{th}cohomology group of the deformation complex. The index of the ASD-operator $d_A^*\oplus d_A^+$ is given by

$$\operatorname{ind}(d_A^*\oplus d_A^+)=8k+4l-3(1-b^1+b^+)-2(g(S)-1),$$

where $g(S)$ is the genus of the surface S. The index formula for the closed pair (X,S) does not depend on the holonomy parameter α. On the other hand, the energy integral $\kappa(A)=\|F_A\|_{\check{L}^2}$ for an ASD-connection A is given by

$$\kappa(A)=k+2\alpha l-\alpha^2 S\cdot S.$$

We always assume that an integer $\nu>0$ is chosen large enough for a fixed holonomy parameter $\alpha\in\mathbb{Q}\cap\left(0,\frac{1}{2}\right)$, under the condition $2\alpha\nu\in\mathbb{Z}$. Such choice of ν is related to the bubbling and compactification of moduli spaces. The details are described in [27; 28].

2.2 The Chern–Simons functional

We discuss singular connections over 3–manifolds. Let Y be an oriented integral homology 3–sphere and K be an oriented knot in Y. Let E be an SU(2)–bundle over Y. This is always topologically trivial. We fix a reduction of E to a line bundle over K as $E|_K=L\oplus L^*$, and fix orbifold metric g^ν along K as in Section 2.1. For a fixed $\alpha\in\mathbb{Q}\cap\left(0,\frac{1}{2}\right)$, we choose ν as in Proposition 2.1. We can similarly define the spaces of singular connections and gauge transformations:

$$\mathcal{A}(Y,K,\alpha)=\{A^\alpha+a\mid a\in\check{L}^2_{m,A^\alpha}(Y\setminus K,\mathfrak{g}_E)\},\quad \mathcal{G}(Y,K)=\{g\in\operatorname{Aut}(E)\mid g\in\check{L}^2_{m+1,A^\alpha}(Y\setminus K,\operatorname{End}(E))\}.$$

We define the quotient

$$\mathcal{B}(Y,K,\alpha)=\mathcal{A}(Y,K,\alpha)/\mathcal{G}(Y,K).$$

We use the notation $\mathcal{A}_m(Y,K,\alpha)$ if we wish to emphasize that the space of singular connection is defined by the completion of the Sobolev norm $\check{L}^2_m$.

We describe the topology of $\mathcal{G}(Y,K)$ and $\mathcal{B}(Y,K,\alpha)$. There are two other kinds of groups of gauge transformations,

$$\mathcal{G}_K=\{g\in\operatorname{Aut}(L|_K)\}\quad\text{and}\quad\mathcal{G}^K(Y,K)=\{g\in\operatorname{Aut}(E)\mid g|_K=\mathrm{id}\}.$$

Then there is the exact sequence

$$1\to\mathcal{G}^K(Y,K)\to\mathcal{G}(Y,K)\to\mathcal{G}_K\to 1.$$

There is the map $\mathcal{G}(Y,K)\to\mathbb{Z}\oplus\mathbb{Z}$ given by $d(g)=(k,l)$, where $k=\deg(g\colon Y\to\mathrm{SU}(2))$ and $l=\deg(g|_K\colon K\to U(1))$, and this map induces the isomorphism

$$\pi_0(\mathcal{G}(Y,K))\cong\mathbb{Z}\oplus\mathbb{Z}.$$

Using the homotopy exact sequence induced from the fibration $\mathcal{G}(Y,K) \to \mathcal{A}(Y,K,\alpha) \to \mathcal{B}(Y,K,\alpha)$, we also have the isomorphism

$$\pi_1(\mathcal{B}(Y,K,\alpha)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

We define an L^2–inner product on tangent spaces of $\mathcal{A}(Y,K,\alpha)$ as follows:

$$\langle a,b\rangle = \int_{Y\setminus K} -\operatorname{tr}(a \wedge *b).$$

The $*$–operator is given by the orbifold metric g^ν. The Chern–Simons functional $\mathrm{CS}\colon \mathcal{A}(Y,K,\alpha) \to \mathbb{R}$ is given by the formal gradient

$$\operatorname{grad}(\mathrm{CS})_A = \frac{1}{4\pi^2} * F_A$$

with respect to the above L^2–inner product on $T\mathcal{A}(Y,K,\alpha)$. This uniquely determines CS up to a constant. $A \in \mathcal{A}(Y,K,\alpha)$ is a critical point of CS if only if $F_A = 0$. The critical point set of CS is a space of flat connections on $Y \setminus K$ such that their holonomy along the meridian is conjugate to

$$\begin{bmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{bmatrix}.$$

Let Crit be the critical point set of the Chern–Simons functional $\mathrm{CS}\colon \mathcal{A}(Y,K,\alpha) \to \mathbb{R}$ and $\mathrm{Crit}^* = \mathrm{Crit} \cap \mathcal{A}^*(Y,K,\alpha)$. Let $\mathfrak{C}(Y,K,\alpha)$ and $\mathfrak{C}^*(Y,K,\alpha)$ be images of Crit and Crit^* by the natural projection $\mathcal{A}(Y,K,\alpha) \to \mathcal{B}(Y,K,\alpha)$. Then

$$\mathfrak{C}(Y,K,\alpha) = \mathcal{R}_\alpha(Y \setminus K, \mathrm{SU}(2)) \quad \text{and} \quad \mathfrak{C}^*(Y,K,\alpha) = \mathcal{R}^*_\alpha(Y \setminus K, \mathrm{SU}(2))$$

by the holonomy correspondence of flat connections and representations of the fundamental group.

We have to perturb the Chern–Simons functional to achieve transversality. This is done by introducing a cylinder function associated with a perturbation $\pi \in \mathcal{P}$

$$f_\pi\colon \mathcal{A}(Y,K,\alpha) \to \mathbb{R},$$

which we will construct in Section 2.4. Let Crit_π be the critical point set of $\mathrm{CS} + f_\pi$ and $\mathrm{Crit}^*_\pi = \mathrm{Crit}_\pi \cap \mathcal{A}^*(Y,K,\alpha)$. Their orbits of gauge transformations are denoted by $\mathfrak{C}_\pi(Y,K,\alpha)$ and $\mathfrak{C}^*_\pi(Y,K,\alpha)$.

We define (perturbed) topological energy $\mathcal{E}_\pi(\gamma)$ of a path $\gamma\colon [0,1] \to \mathcal{A}(Y,K,\alpha)$ as

$$\mathcal{E}_\pi(\gamma) = 2\{(\mathrm{CS} + f_\pi)(\gamma(1)) - (\mathrm{CS} + f_\pi)(\gamma(0))\}. \tag{2-2}$$

We also define the (perturbed) Hessian of $A \in \mathcal{A}(Y,K,\alpha)$ as

$$\mathrm{Hess}_{A,\pi}(a) = *d_A a + DV_\pi|_A(a),$$

where V_π is a gradient of f_π, and $DV_\pi|_A$ is its derivative at A.

For each $A \in \mathcal{A}(Y,K,\alpha)$, we can regard the Hessian as the operator,

$$\mathrm{Hess}_{A,\pi}\colon \check{L}^2_{m,A^\alpha}(Y \setminus K, \Lambda^1 \otimes \mathfrak{g}_E) \to \check{L}^2_{m-1,A^\alpha}(Y \setminus K, \Lambda^1 \otimes \mathfrak{g}_E).$$

Definition 2.2 $A \in \mathrm{Crit}^*_\pi$ is called nondegenerate if $\mathrm{Hess}_{A,\pi}|_{\mathrm{Ker}(d^*_A)}$ is invertible.

This means that the Hessian is nondegenerate to the vertical direction of the gauge orbit. For irreducible critical points of the unperturbed Chern–Simons functional, there is the following criterion for the nondegeneracy condition:

Proposition 2.3 [32, Lemma 3.13] *A critical point $A \in \mathrm{Crit}^*$ is nondegenerate if only if the kernel of the map*

$$H^1(Y \setminus K; \operatorname{ad}\rho) \to H^1(\mu_K; \operatorname{ad}\rho)$$

is zero, where this map is induced by the natural embedding $\mu_K \hookrightarrow Y \setminus K$ and $\rho\colon \pi_1(Y \setminus K) \to \mathrm{SU}(2)$ is the representation corresponding to the flat connection A.

We say that $[A] \in \mathfrak{C}_\pi$ is nondegenerate if one of its representatives $A \in \mathrm{Crit}_\pi$ (and hence all) are nondegenerate.

The nondegeneracy condition at the reducible critical point is given by a constraint on the holonomy parameter. Let θ_α be the gauge equivalence class of the reducible flat connection corresponding to the conjugacy class of an SU(2)–representation of $\pi_1(Y \setminus K)$ which factors through the abelianization $H_1(Y \setminus K, \mathbb{Z})$ and has a holonomy parameter α. Since Y is an integral homology 3–sphere, such θ_α uniquely exists. The following is obtained as a corollary of [32, Lemma 3.13]:

Proposition 2.4 [12, Lemma 15] *The unique flat reducible θ_α is isolated and nondegenerate if only if $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$.*

Let us fix the definition of the Chern–Simons functional. We fix a reducible flat connection $\tilde{\theta}_\alpha$ which represents θ_α and put the condition $\mathrm{CS}(\tilde{\theta}_\alpha) = 0$. Then the $\mathbb{R}$–valued functional CS is determined up to the choice of a representative of θ_α. From now on, we fix a representative $\tilde{\theta}_\alpha$ for each pair (Y, K).

2.3 The flip symmetry

The flip symmetry is an involution that acts on a family of configuration spaces $\bigcup_{\alpha\in(0,1/2)\cap\mathbb{Q}} \mathcal{B}(Y, K, \alpha)$. The flip symmetry changes holonomy conditions as $\alpha \mapsto \frac{1}{2} - \alpha$. The 4–dimensional version is introduced in [27], and the 3–dimensional version is similarly defined in [9]. We generalize the 3–dimensional version of the flip symmetry as follows. Let $\chi \in H^1(Y \setminus K, \mathbb{Z}_2) \cong \mathbb{Z}_2$ be a generator. Since $H^1(Y \setminus K, \mathbb{Z}_2) = \mathrm{Hom}(\pi_1(Y \setminus K), \mathbb{Z}_2)$, we can regard χ as a representation $\chi\colon \pi(Y \setminus K) \to \mathbb{Z}_2$. The representation χ satisfies $\chi(\mu_K) = -1$, and forms a flat line bundle L_χ over $Y \setminus K$ with a flat connection corresponding to χ. Since L_χ is a trivial line bundle and there is an isomorphism $E|_{Y\setminus K} \cong E|_{Y\setminus K} \otimes L_\chi$, we regard a connection $A \otimes \chi$ on $E|_{Y\setminus K} \otimes L_\chi$ as a connection on $E|_{Y\setminus K}$. Thus the action of $\chi \in H^1(Y \setminus K, \mathbb{Z}_2)$ on $\bigcup_{\alpha\in(0,1/2)\cap\mathbb{Q}} \mathcal{B}(Y, K, \alpha)$ is defined by

$$\chi[A] = [A \otimes \chi].$$

This action is called the flip symmetry, and gives the identification

$$\mathcal{B}(Y,K,\alpha)\cong\mathcal{B}\big(Y,K,\tfrac{1}{2}-\alpha\big).$$

In particular, it defines the involution on $\mathcal{B}\big(Y,K,\frac{1}{4}\big)$.

The flip symmetry can be restricted to the space $\bigcup_{\alpha\in(0,1/2)\cap\mathbb{Q}}\mathcal{R}_\alpha(Y\setminus K,\mathrm{SU}(2))$. In this case, the action of $\chi\in H^1(Y\setminus K,\mathbb{Z}_2)$ is simply described as $\chi[\rho]=[\rho\cdot\chi]$ where $\rho\cdot\chi\colon\pi_1(Y\setminus K)\to\mathrm{SU}(2)$ is the SU(2)–representation defined as $(\rho\cdot\chi)(g):=\rho(g)\chi(g)$ for $g\in\pi_1(Y\setminus K)$. If ρ satisfies

$$\rho(\mu_K)\sim\begin{bmatrix}e^{2\pi i\alpha}&0\\0&e^{-2\pi i\alpha}\end{bmatrix},$$

then

$$(\rho\cdot\chi)(\mu_K)\sim\begin{bmatrix}e^{2\pi i(1/2-\alpha)}&0\\0&e^{-2\pi i(1/2-\alpha)}\end{bmatrix}.$$

2.4 Holonomy perturbations

In this subsection, we review the construction and properties of the perturbation term of the Chern–Simons functional introduced by Floer [14] and Braam and Donaldson [1], and here we follow the notation of Kronheimer and Mrowka [32] and Daemi and Scaduto [9].

Let $q\colon S^1\times D^2\to Y\setminus K$ be a smooth immersion of a solid torus. Then $(s,z)\in S^1\times D^2$ denotes its coordinates, regarding S^1 as $\mathbb{R}/\mathbb{Z}$ and D^2 as the unit disk in $\mathbb{C}$. Let $G_E\to Y$ be the bundle of the group whose sections are gauge transformations of E. This is defined by $G_E=P\times_{\mathrm{SU}(2)}\mathrm{SU}(2)$, where P is the corresponding SU(2)–bundle, and SU(2) acts in the obvious way on P and by conjugation on the SU(2)–factor. $\mathrm{Hol}_q(A)\colon D^2\to q^*(G_E)$ is a section of $q^*(G_E)$ which assigns the holonomy $\mathrm{Hol}_{q(-,z)}(A)$ of connection $A\in\mathcal{A}(Y,K,\alpha)$ along the loop $q(-,z)\colon S^1\to Y\setminus K$ to each $z\in D^2$. Next, we repeat the above constructions for an r–tuple of smooth immersions of solid tori

$$\boldsymbol{q}=(q_1,\dots,q_r).$$

Assume that there is a positive number $\eta>0$ such that

$$q_i(s,z)=q_j(s,z)\quad\text{for all }(s,z)\in[-\eta,\eta]\times D^2. \tag{2-3}$$

Then there are identifications

$$q_i^*(G_E)\cong q_j^*(G_E)$$

over $[-\eta,\eta]\times D^2$, and $q^*(G_E^r)$ denotes the fiber product of $q_1^*(G_E),\dots,q_r^*(G_E)$ over $[-\eta,\eta]\times D^2$. Then we can construct a section $\mathrm{Hol}_{\boldsymbol{q}}(A)\colon D^2\to q^*(G_E^r)$ which assigns

$$(\mathrm{Hol}_{q_1(-,z)}(A),\dots,\mathrm{Hol}_{q_r(-,z)}(A))\in\mathrm{SU}(2)^r$$

for each $z\in D^2$. Next, we choose a smooth function on $\mathrm{SU}(2)^r$ which is invariant under the diagonal action of SU(2) on $\mathrm{SU}(2)^r$ by the adjoint action on each factor. This smooth function induces $h:q^*(G_E)\to\mathbb{R}$.

Definition 2.5 $\mathrm{Hol}_{\boldsymbol{q}}(A)$ and h are as above. Let μ be a 2–form on D^2 such that $\int_{D^2}\mu = 1$. A smooth function $f\colon \mathcal{A}(Y,K,\alpha)\to\mathbb{R}$ of the form

$$f(A)=\int_{D^2} h(\mathrm{Hol}_{\boldsymbol{q}}(A))\mu$$

is called a cylinder function.

Cylinder functions are determined by the choice of an r–tuple $\boldsymbol{q}$ and a function h. Note that the construction of cylinder functions is gauge invariant. Let $\mathcal{P}$ be the space of perturbations; see [32] for details. For each $\pi\in\mathcal{P}$, we can associate a cylinder function f_π. We call f_π the holonomy perturbation and $\mathrm{CS}+f_\pi$ the perturbed Chern–Simons functional.

Proposition 2.6 *There is a residual subset of the Banach space of perturbations $\mathcal{P}'\subset\mathcal{P}$ such that, for any sufficiently small $\pi\in\mathcal{P}'$, the perturbed Chern–Simons functional $\mathrm{CS}+f_\pi$ has the nondegenerate critical point set Crit_π^* and its image $\mathfrak{C}_\pi^*$ in $\mathcal{B}^*(Y,K,\alpha)$ is a finite point set. Moreover, the reducible critical point θ_α is unmoved under the perturbation and is nondegenerate if $\Delta_{(Y,K)}(e^{4\pi i\alpha})\neq 0$.*

Proof The finiteness property follows from a similar argument as in [32, Lemma 3.8]. The nondegeneracy condition follows from the fact that f_π is dense in $C^\infty(S)$ for any compact finite-dimensional submanifold $S\subset\mathcal{B}^*(Y,K,\alpha)$; see [10, Section 5] for details. The argument in [9, Subsection 2.4] is adapted to show that, for a suitable choice of an SU(2)–invariant smooth function h, the unique flat reducible θ_α is unmoved under small perturbations. By Proposition 2.4, the unique flat reducible θ_α is still isolated and nondegenerate for such perturbations under the condition $\Delta_{(Y,K)}(e^{4\pi i\alpha})\neq 0$. □

2.5 The moduli space over the cylinder

We discuss trajectories for the perturbed gradient flow. Let $(Z,S)=\mathbb{R}\times(Y,K)$ be a cylinder equipped with a product metric $g_Y^\nu+dt$. We introduce moduli spaces of instantons over the cylinder. $\mathbb{E}$ denotes the pullback of the SU(2)–bundle $E\to Y$ by the projection $\mathbb{R}\times Y\to Y$. Consider a connection A on $\mathbb{E}$ of the form $A=B(t)+C\,dt$, where $B(t)$ is a t–dependent singular connection on $Y\setminus K$. Let β_0 and β_1 be elements in $\mathfrak{C}_\pi^*$, and let B_0 and B_1 be their representatives in gauge equivariant classes. Consider a singular connection A_0 over the cylinder (Z,S) such that

$$A_0|_{(Y\setminus K)\times\{t\}}=B_1\quad\text{for } t\gg 0\qquad\text{and}\qquad A_0|_{(Y\setminus K)\times\{t\}}=B_0\quad\text{for } t\ll 0.$$

A_0 defines a path $\gamma\colon\mathbb{R}\to\mathcal{B}(Y,K,\alpha)$ by sending t to $[B(t)]$, and z denotes its relative homotopy class in $\pi_1(\mathcal{B}(Y,K,\alpha);\beta_0,\beta_1)$.

Then we define the space of singular connection indexed by z:

$$\mathcal{A}_z(Z,S;B_0,B_1)=\{A\mid A-A_0\in\check{L}^2_{m,A_0}(Z\setminus S,\Lambda^1\otimes\mathfrak{g}_{\mathbb{E}})\}.$$

We also define the group of gauge transformations:

$$\mathcal{G}_z(Z,S)=\{g\in \operatorname{Aut}(\mathbb{E})\mid \nabla_{A_0}^k g\in \check{L}^2(Z\setminus S,\operatorname{End}(\mathbb{E})),\ k=1,\dots,m+1\}.$$

The group $\mathcal{G}_z(Z,S)$ acts on $\mathcal{A}_z(Z,S)$. Taking the quotient gives the configuration space $\mathcal{B}_z(Z,S;\beta_0,\beta_1)$. We introduce the moduli space of (perturbed) instantons over the cylinder associated with the perturbed Chern–Simons functional $\mathrm{CS}+f_\pi$. This is the moduli space of solutions of the perturbed ASD-equation

$$M_z^\pi(\beta_0,\beta_1)=\{[A]\in \mathcal{B}_z(Z,S;\beta_0,\beta_1)\mid F_A^+ + \hat{V}_\pi(A)=0\}.$$

Here $\hat{V}_\pi$ is a term arising from the perturbation f_π. The perturbed version of the ASD-complex is given by

$$\Omega^0(Z\setminus S,\mathfrak{g}_{\mathbb{E}})\xrightarrow{d_A}\Omega^1(Z\setminus S,\mathfrak{g}_{\mathbb{E}})\xrightarrow{d_A^+ + D\hat{V}_\pi}\Omega^+(Z\setminus S,\mathfrak{g}_{\mathbb{E}}).$$

We consider the Fredholm operator

$$D_{A,\pi}\colon \check{L}^2_{m,A_0}(Z\setminus S,\Lambda^1\otimes\mathfrak{g}_{\mathbb{E}})\to \check{L}^2_{m-1,A_0}(Z\setminus S,(\Lambda^0\oplus\Lambda^+)\otimes\mathfrak{g}_{\mathbb{E}})$$

given by $D_{A,\pi}=-d_A^*\oplus(d_A^+ + D\hat{V}_\pi)$ and define the relative $\mathbb{Z}$–grading for $\beta_0,\beta_1\in\mathfrak{C}_\pi^*(Y,K,\alpha)$ as

$$\mathrm{gr}_z(\beta_0,\beta_1)=\mathrm{ind}(D_{A,\pi}),$$

where z is a path represented by A. Note that $\mathrm{ind}\, D_{A,\pi}$ is independent of the choice of perturbation π since the term $D\hat{V}_\pi$ is a compact operator. The following proposition gives the well-defined mod-4 grading on the critical point set:

Proposition 2.7 [32, Lemma 3.1] *Let $z\in\pi_1(\mathcal{B}(Y,K,\alpha))$ be a homotopy class represented by a path which connects B and $g^*(B)$, where $\beta=[B]$ and $d(g)=(k,l)$. For $\beta\in\mathfrak{C}^*$ and a homotopy class $z\in\pi_1(\mathcal{B}(Y,K,\alpha);\beta)$, we have*

$$\mathrm{gr}_z(\beta,\beta)=8k+4l.$$

The mod-4 value of the $\mathbb{Z}$–grading is independent of the choice of the homotopy class z, and hence we can write

$$\mathrm{gr}(\beta_0,\beta_1)\equiv \mathrm{gr}_z(\beta_0,\beta_1)\mod 4.$$

We also define the absolute $\mathbb{Z}$–grading by

$$\mathrm{gr}_z(\beta,\theta_\alpha)=\mathrm{ind}(D_{A,\pi}\colon \phi\check{L}^2_m\to\phi\check{L}^2_{m-1}),$$

where $\phi\check{L}^2_m$ is a weighted Sobolev space with a weight function ϕ which agrees with $e^{-\delta|t|}$ over two ends of the cylinder. Here $\delta>0$ is chosen to be small enough. Similarly, we can define the mod-4 grading

$$\mathrm{gr}(\beta)\equiv \mathrm{gr}_z(\beta,\theta_\alpha)\mod 4.$$

The moduli space $M_z^\pi(\beta_0,\beta_1)$ is called regular if the operator $D_{A,\pi}$ is surjective for all $[A]\in M_z^\pi(\beta_0,\beta_1)$. For a generic choice of perturbation, the moduli space $M_z^\pi(\beta_0,\beta_1)$ is a regular and smooth manifold of dimension $\mathrm{gr}_z(\beta_0, \beta_1)$. We explicitly write $M_z^\pi(\beta_0, \beta_1)_d$ if the moduli space $M_z^\pi(\beta_0, \beta_1)$ has dimension d, and write $\breve{M}_z^\pi(\beta_0, \beta_1)_{d-1} = M_z^\pi(\beta_0, \beta_1)_d/\mathbb{R}$. The argument in [10, Section 5] is adapted to our situation, and we have the following properties:

Proposition 2.8 *Let $\alpha \in \left(0, \frac{1}{2}\right)$ be a holonomy parameter with $\Delta_K(e^{4\pi i\alpha}) \neq 0$ and $\pi_0 \in \mathcal{P}'$ be a small perturbation such that $\mathfrak{C}_{\pi_0} = \{\theta_\alpha\} \sqcup \mathfrak{C}^*_{\pi_0}$ consists of finitely many nondegenerate points. Then there is a small perturbation $\pi \in \mathcal{P}'$ such that*

- (i) $f_\pi = f_{\pi_0}$ *in a neighborhood of* $\mathfrak{C}_{\pi_0}$,
- (ii) $\mathfrak{C}_\pi = \mathfrak{C}_{\pi_0}$,
- (iii) $M_z^\pi(\beta_1, \beta_2)$ *is regular for all homotopy classes z and critical points β_1 and β_2.*

Proof First, we fix a perturbation $\pi_0 \in \mathcal{P}$ as in Proposition 2.6. Then for each homotopy class z, we can find a perturbation $\pi_z \in \mathcal{P}$ which is supported away from critical points and the corresponding moduli space is regular. This essentially follows from the argument in [10, Section 5]. Since the subset $\mathcal{P}_z$ of regular perturbations as above forms an open dense subset in $\mathcal{P}$, we can find a desired perturbation π in the countable intersection $\bigcap_z \mathcal{P}_z$. □

From now on, we assume that the perturbation $\pi \in \mathcal{P}$ always satisfies the properties in Proposition 2.8 and we drop π from the notation $M_z^\pi(\beta_1, \beta_2)$.

2.6 Compactness

Consider a relative homotopy class $z \in \pi_1(\mathcal{B}(Y, K, \alpha), \beta_1, \beta_2)$. If $\beta_1 = \beta_2$ then we assume that z is a nontrivial homotopy class. Elements in $\breve{M}_z(\beta_1, \beta_2)$ are called unparametrized trajectories.

Definition 2.9 A collection $([A_1], \dots, [A_l]) \in \breve{M}_{z_1}(\beta_1, \beta_2) \times \cdots \times \breve{M}_{z_l}(\beta_{l-1}, \beta_l)$ of unparametrized trajectories is called an unparametrized broken trajectory from β_1 to β_l. If the composition of paths $z_1 \circ \cdots \circ z_l$ is contained in the homotopy class z, then $\breve{M}_z^+(\beta_1, \beta_l)$ denotes the space of unparametrized broken trajectories from β_1 to β_l.

The compactness property of moduli spaces is as follows; see also [32, Proposition 3.22].

Proposition 2.10 *Let $\beta_1, \beta_2 \in \mathfrak{C}_\pi$ and assume that $\dim M_z(\beta_1, \beta_2) < 4$. Then the space of unparametrized broken trajectories $\breve{M}_z^+(\beta_1, \beta_2)$ is compact.*

We can assign the energy $\mathcal{E}_\pi(z)$ to a homotopy class z. In singular gauge theory for general holonomy parameters, the counting $\#\bigcup_z \breve{M}_z(\beta_1, \beta_2)$ with $\mathrm{gr}_z(\beta_1, \beta_2) = 1$ can be infinite. Instead, we use the following finiteness result:

Proposition 2.11 [32, Proposition 3.23] *For a given constant $C > 0$, there are only finitely many critical points β_1 and β_2 and homotopy classes $z \in \pi_1(\mathcal{B}; \beta_1, \beta_2)$ such that the moduli space $M_z(\beta_1, \beta_2)$ is nonempty and $\mathcal{E}_\pi(z) < C$.*

Thus

$$\bigcup_{\substack{z \\ \mathcal{E}_\pi(z)<C}} \breve{M}_z(\beta_1,\beta_2)_0$$

is a finite point set for any $C > 0$.

The gluing formula of the index tells us

$$\mathrm{gr}_z(\beta_0,\theta_\alpha)+1+\mathrm{gr}_{z'}(\theta_\alpha,\beta_1)=\mathrm{gr}_{z'\circ z}(\beta_0,\beta_1) \tag{2-4}$$

since θ_α has a stabilizer S^1. From this relation, we conclude that any broken trajectories in $M_z^+(\beta_0,\beta_1)_d$ do not factor through θ_α if the dimension of $M_z(\beta_0,\beta_1)_d$ is less than 3.

2.7 Cobordisms

Let (W,S) be a pair of an oriented 4–manifold and an embedded oriented surface such that $\partial W = Y' \sqcup (-Y)$ and $\partial S = K \sqcup K'$. We call (W,S) the cobordism of pairs and write $(W,S)\colon (Y,K)\to(Y',K')$. Set

$$(W^+,S^+) := \mathbb{R}_{\le 0}\times(Y,K)\cup(W,S)\cup\mathbb{R}_{\ge 0}\times(Y',K').$$

We fix a metric on $W^+\setminus S^+$ with a cone angle $2\pi/\nu$ and cylindrical forms on each end. Let $\beta\in\mathcal{B}(Y,K,\alpha)$ and $\beta'\in\mathcal{B}(Y',K')$ be given connections and choose a singular $\mathrm{SU}(2)$–connection A_0 on (W^+,S^+) which has a limiting connection β or β'(up to gauge transformations) on each end of (W^+,S^+). Here z denotes the homotopy class of A. We define the space of connections and the group of gauge transformations as follows:

$$\mathcal{A}_z(W,S;\beta,\beta') := \{A \mid A-A_0\in \check{L}^2_{m-1,A_0}(W^+\setminus S^+,\mathfrak{g}_E\otimes\Lambda^1)\},$$
$$\mathcal{G}_z(W,S) := \{g\in \mathrm{Aut}(E)\mid \nabla^i_{A_0}g\in \check{L}^2(W^+\setminus S^+;\mathrm{End}(E)),\ i=1,\dots,m\}.$$

We also define the quotient

$$\mathcal{B}_z(W,S;\beta,\beta') := \mathcal{A}_z(W,S;\beta,\beta')/\mathcal{G}_z(W,S).$$

$\mathcal{B}(W,S;\beta,\beta')$ denotes the union of $\mathcal{B}_z(W,S;\beta,\beta')$ for all paths. The perturbed ASD equation on (W,S) has the form $F_A^+ + U_{\pi_W}=0$ where U_{π_W} is a t–dependent perturbation. More concretely this can be described as the following (the argument is based on [32]): Let $\pi,\pi_0\in\mathcal{P}_Y$ be two holonomy perturbations on $\mathbb{R}\times(Y,K)$. The perturbed ASD equation on $\mathbb{R}_{\le 0}\times(Y,K)$ has the form

$$F_A^+ + \psi(t)\widehat{V}_\pi + \psi_0(t)\widehat{V}_{\pi_0}=0,$$

where $\psi(t)$ is a smooth cutoff function such that $\psi(t)=1$ if $t<-1$ and $\psi(t)=0$ at $t=0$. Then ψ_0 is a smooth function supported on $(-1,0)\times Y$. We choose $\pi\in\mathcal{P}$ so that $\mathfrak{C}_\pi$ satisfies properties in Propositions 2.6 and 2.8. The perturbation term can be described similarly on another end. For generic choices of π_0 and $\pi_0'\in\mathcal{P}_{Y'}$, the irreducible part of the perturbed ASD-moduli space

$$M_z(W,S;\beta,\beta')\subset\mathcal{B}_z(W,S;\beta,\beta')$$

is a smooth manifold. Consider the ASD-operator

$$D_A = -d_A^* \oplus d_A^+ \colon \phi \check{L}^2_{m,A_0}(W \setminus S, \mathfrak{g}_E \otimes \Lambda^1) \to \phi \check{L}^2_{m-1,A_0}(W \setminus S, \mathfrak{g}_E \otimes (\Lambda^0 \oplus \Lambda^+)), \tag{2-5}$$

where ϕ is a weight function. If a limiting connection of A_0 is irreducible then we choose $\phi \equiv 1$ on that end of (W^+, S^+). If A_0 has a reducible limiting connection then we choose $\phi = e^{-\delta|t|}$ on that end, where $\delta > 0$ is small enough. $M(W, S; \beta, \beta')_d$ denotes the union of the moduli spaces $M_z(W, S; \beta, \beta')$ with ind $D_A = d$.

Definition 2.12 We define the topological energy of $A \in \mathcal{B}(W, S; \beta, \beta')$ as

$$\kappa(A) := \frac{1}{8\pi^2} \int_{W^+ \setminus S^+} \operatorname{Tr}(F_A \wedge F_A)$$

and the monopole number of A as

$$\nu(A) := \frac{i}{\pi} \int_{S^+} \Omega - 2\alpha S \cdot S,$$

where

$$F_A|_{S^+} = \begin{bmatrix} \Omega & 0 \\ 0 & -\Omega \end{bmatrix}.$$

For the cylinder $(W, S) = [0,1] \times (Y, K)$ and the trivial perturbation $\pi = 0$, the topological energy κ is related to the energy $\mathcal{E}$ of the Chern–Simons functional as $2\kappa(A) = \mathcal{E}(A)$. Consider an SU(2)–connection B on (Y, K), a connection A over the cylinder $\mathbb{R} \times (Y, K)$ which is asymptotic to B at $-\infty$, and a fixed reducible flat connection $\tilde{\theta}_\alpha$ such that $\operatorname{CS}(\tilde{\theta}_\alpha) = 0$ at ∞. Then $\operatorname{CS}(B) = \kappa(A)$ by construction.

Similarly we define an $\mathbb{R}$–valued function $\operatorname{hol}_K \colon \mathcal{A}(Y, K, \alpha) \to \mathbb{R}$ as follows:

Definition 2.13 Let A be an SU(2)–connection over the cylinder $[0,1] \times (Y, K)$ as above. We define $\operatorname{hol}_K(B) := -\nu(A)$.

If z is a path on (W, S) which is represented by a connection A, then we write $\kappa(z)$ for $\kappa(A)$ and $\nu(z)$ for $\nu(A)$ since these numbers are independent of the choice of A.

Let (X, Σ) be a pair of a 4–manifold and an embedded surface with boundary $\partial X = Y$ and $\partial\Sigma = K$ where K is an oriented knot in an oriented integral homology 3–sphere Y. We assume that $[\Sigma] = 0$. Let Θ_α be a singular flat reducible connection with a holonomy parameter $\alpha = n/m$ and whose lift to the m–fold cyclic branched covering $\widetilde{X}_m$ is the trivial connection. We write $H^i(X \setminus \Sigma; \Theta_\alpha)$ for the i^{th} cohomology of $X \setminus \Sigma$ with the local coefficient system twisted by Θ_α. Let $H^+(X \setminus \Sigma; \Theta_\alpha)$ and $H^-(X \setminus \Sigma; \Theta_\alpha)$ be the space of self-dual and anti-self-dual harmonic 2–forms on $X \setminus \Sigma$ twisted by Θ_α, respectively.

Lemma 2.14 *We define* $\chi(X \setminus \Sigma; \Theta_\alpha) = \sum_i (-1)^i \dim H^i(X \setminus \Sigma; \Theta_\alpha)$ *and*

$$\sigma(X \setminus \Sigma; \Theta_\alpha) = \dim H^+(X \setminus \Sigma; \Theta_\alpha) - \dim H^-(X \setminus \Sigma; \Theta_\alpha).$$

Then

$$\chi(X \setminus \Sigma; \Theta_\alpha) = \chi(X) - \chi(\Sigma) \quad \text{and} \quad \sigma(X \setminus \Sigma; \Theta_\alpha) = \sigma(X) + \sigma_\alpha(Y, K).$$

Proof Consider a rational holonomy parameter of the form $\alpha = n/m \in \mathbb{Q}$. We take an m–fold branched covering $\pi\colon \widetilde{X}_m \to X$ whose branched locus is Σ. The pullback of singular flat connection Θ_α extends as the trivial flat connection over $\widetilde{X}_m$. Let $\tau\colon \widetilde{X}_m \to \widetilde{X}_m$ be a generator of covering transformations. Then its induced action $\tilde{\tau}$ on the pulled-back bundle $\underline{\mathbb{C}}$ is multiplication by $e^{4\pi i n/m}$. The index of the twisted de Rham operator

$$d_{\Theta_\alpha} + d^*_{\Theta_\alpha}\colon \Omega^{\mathrm{even}}(X \setminus \Sigma; \Theta_\alpha) \to \Omega^{\mathrm{odd}}(X \setminus \Sigma; \Theta_\alpha)$$

coincides with the index of

$$d + d^*\colon \Omega^{\mathrm{even}}(\widetilde{X}_m; \underline{\mathbb{C}})^{\tilde{\tau}} \to \Omega^{\mathrm{odd}}(\widetilde{X}_m; \underline{\mathbb{C}})^{\tilde{\tau}}, \tag{2-6}$$

where $\Omega^*(\widetilde{X}_m; \underline{\mathbb{C}})^{\tilde{\tau}} = \{\omega \in \Omega^*(\widetilde{X}_m; \underline{\mathbb{C}}) \mid \omega(\tau(x)) = \tilde{\tau}(\omega(x))\}$. The index of (2-6) is given by $\chi(X) - \chi(\Sigma)$. This can be seen by taking cell complex $C_*(\widetilde{X}_m)$ of $\widetilde{X}_m$ in τ–equivariant way. Then there are decompositions of the underlying groups of the chain complex

$$C_*(\widetilde{X}_m) = C_*(\Sigma) \oplus C_*(\widetilde{X}_m \setminus \Sigma), \quad C_*(\widetilde{X}_m \setminus \Sigma) = \bigoplus_{i=1}^{n} C_i,$$

where each C_i is isomorphic to a copy of $C_*(X \setminus \Sigma)$. Since τ_* acts as the identity on $C_*(\Sigma)$ and in a cyclic way on $C_*(\widetilde{X}_m \setminus \Sigma) = \bigoplus_{i=1}^{n} C_i$, all eigenspaces of the action τ_* on $C_*(\widetilde{X}_m \setminus \Sigma)$ are isomorphic. On the other hand, there is the identity $\chi(\widetilde{X}_m) = m\chi(X) - (m-1)\chi(\Sigma)$. Thus the τ–invariant index of the de Rham operator is given by $\chi(X) - \chi(\Sigma)$.

Similarly, the index of the signature operator twisted by the local coefficient Θ_α coincides with the index of the signature operator over $\widetilde{X}_m$ which is restricted to $e^{4\pi i n/m}$–eigenspaces. This signature is equal to $\sigma(X) + \sigma_{n/m}(Y, K)$ by the formula in [41]. □

Proposition 2.15 *Let $(W, S)\colon (Y, K) \to (Y', K')$ be a cobordism of pairs and $[A]$ be an element of $\mathcal{B}(W, S; \theta_\alpha, \theta'_\alpha)$. Then the index of the ASD operator D_A is given by*

$$\operatorname{ind} D_A = 8\kappa(A) + 2(4\alpha - 1)\nu(A) - \tfrac{3}{2}(\sigma(W) + \chi(W)) + \chi(S) + 8\alpha^2 S \cdot S + \sigma_\alpha(Y, K) - \sigma_\alpha(Y', K') - 1.$$

Proof Let X be a compact 4–manifold with $\partial X = Y$ and $\Sigma \subset Y$ be a Seifert surface of the knot K. Pushing Σ into the interior of X, we obtain a pair (X, Σ) whose boundary is (Y, K). Moreover $[\Sigma] = 0$ in $H_2(X; \mathbb{Z})$. Similarly we can construct another pair (X', Σ') such that $(\partial X', \partial \Sigma') = (Y', K')$.

Set

$$(\overline{W}, \overline{S}) := (X, \Sigma) \cup_{(Y,K)} (W, S) \cup_{(Y',K')} (X', \Sigma').$$

Then $(\overline{W}, \overline{S})$ is a closed pair of a 4–manifold and an embedded surface. Let A_1 and A_2 be singular flat reducible connections over (X, Σ) and (X, Σ') which are extensions of θ_α and θ'_α, respectively. Let A be a connection which represents an element of $\mathcal{B}(W, S; \theta_\alpha, \theta'_\alpha)$. We consider the connection $A' = A_1 \#_{\theta_\alpha} A \#_{\theta'_\alpha} A_2$ over $(\overline{W}, \overline{S})$ obtained by the gluing.

Using the gluing formula for the index, we have

$$\operatorname{ind} D_{A'} = \operatorname{ind} D_{A_1} + \operatorname{ind} D_A + \operatorname{ind} D_{A_2} + 2,$$

where A' is a singular connection on the closed pair $(\overline{W}, \overline{S})$ obtained by gluing A_1, A_2 and A. Since A_1 is reducible, there is the decomposition $A_1 = \mathbf{1} \oplus B_\alpha$ with respect to the decomposition of the adjoint bundle $\underline{\mathbb{R}} \oplus L^{\otimes 2}$, where $\mathbf{1}$ denotes the trivial connection. The deformation complex for D_{A_1} decomposes into

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d^+} \Omega^+(X) \tag{2-7}$$

and

$$\Omega^0(X \setminus \Sigma; \mathrm{ad}\, B_\alpha) \xrightarrow{d_{B_\alpha}} \Omega^1(X \setminus \Sigma; \mathrm{ad}\, B_\alpha) \xrightarrow{d^+_{B_\alpha}} \Omega^+(X \setminus \Sigma; \mathrm{ad}\, B_\alpha). \tag{2-8}$$

The index of (2-7) is given by $-\frac{1}{2}(\sigma(X) + \chi(X)) - \frac{1}{2}$. On the other hand, the index of (2-8) is given by $-\sigma(X \setminus \Sigma; B_\alpha) - \chi(X \setminus \Sigma; B_\alpha)$. Using two formulae $\sigma(X \setminus \Sigma; B_\alpha) = \sigma(X) + \sigma_\alpha(Y, K)$ and $\chi(X \setminus \Sigma, B_\alpha) = \chi(X) - \chi(\Sigma)$ in Lemma 2.14, we obtain

$$\operatorname{ind} D_{A_1} = -\tfrac{3}{2}(\sigma(X) + \chi(X)) - \sigma_\alpha(Y, K) + \chi(\Sigma) - \tfrac{1}{2}.$$

Similarly, we have

$$\operatorname{ind} D_{A_2} = -\tfrac{3}{2}(\sigma(X') + \chi(X')) + \sigma_\alpha(Y', K') + \chi(\Sigma') - \tfrac{1}{2}$$

since $\sigma_\alpha(-Y', K') = -\sigma_\alpha(Y', K')$. The index formula for a closed pair in [27] gives

$$\operatorname{ind} D_{\bar{A}} = 8\kappa(\bar{A}) + 2(4\alpha - 1)\nu(\bar{A}) - \tfrac{3}{2}(\sigma(\overline{W}) + \chi(\overline{W})) + \chi(\bar{S}) + 8\alpha^2 S \cdot S + 2.$$

Hence we have the desired formula:

$$\operatorname{ind} D_A = 8\kappa(A) + 2(4\alpha - 1)\nu(A) - \tfrac{3}{2}(\sigma(W) + \chi(W)) + \chi(S) + 8\alpha^2 S \cdot S + \sigma_\alpha(Y, K) - \sigma_\alpha(Y', K') - 1. \quad \square$$

Remark 2.16 (i) The index formula in Proposition 2.15 recovers [9, Lemma 2.26] when $\alpha = \frac{1}{4}$.

(ii) For a cobordism of pairs $(W, S)\colon (Y, K) \to (Y', K')$, we define the integers

$$k(L) = -c_1(L)^2[W] \quad \text{and} \quad l(L) = -c_1(L)[S].$$

Then the Chern–Weil formula gives us another expression of the index formula in Proposition 2.15 as

$$\operatorname{ind} D_{A_L} = 8k(L) + 4l(L) - \tfrac{3}{2}(\sigma(W) + \chi(W)) + \chi(S) + \sigma_\alpha(Y, K) - \sigma_\alpha(Y', K') - 1.$$

Assume that the cobordism of pairs (W, S) satisfies $b^1(W) = b^+(W) = 0$. Then there exists a unique singular reducible instanton A_L corresponding to a decomposition $E = L \oplus L^*$.

Definition 2.17 We call A_L minimal if it minimizes $\operatorname{ind} D_{A_L}$ among all line bundles L.

Our definition of minimal reducible coincides with [8, Subsection 2.3] if $\alpha = \frac{1}{4}$.

Let us describe relations between CS and κ, and ν and hol_K over cobordisms. Consider a connection A over a cobordism $(W, S)\colon (Y, K) \to (Y', K')$ whose limiting connections are B on (Y, K) and B' on (Y', K'). Then the following statement holds:

Lemma 2.18 *Fix a reducible connection A_L over (W, S). Then there exist $k, l \in \mathbb{Z}$ such that*

$$\kappa(A) - \kappa(A_L) = \mathrm{CS}(B) - \mathrm{CS}(B') + k + 2\alpha l \quad \textit{and} \quad \nu(A) - \nu(A_L) = \mathrm{hol}_{K'}(B') - \mathrm{hol}_K(B) - 2l.$$

Proof Recall that $\mathbb{R}$–valued functions CS and hol are fixed by choosing reducible flat connections $\tilde{\theta}_\alpha$ and $\tilde{\theta}'_\alpha$ over pairs (Y, K) and (Y', K'). If we choose a reducible connection A_{L_0} so that it has two reducible limits $\tilde{\theta}_\alpha$ and $\tilde{\theta}'_\alpha$, then we have

$$\kappa(A) + \mathrm{CS}(B') = \mathrm{CS}(B) + \kappa(A_{L_0}) \quad \text{and} \quad \nu(A) - \mathrm{hol}_{K'}(B') = -\mathrm{hol}_K(B) + \nu(A_{L_0})$$

by construction. If we change A_L to other homotopy classes of reducible connections, terms $k + 2\alpha l$ and $-2l$ appear by gauge transformations. □

For a cobordism of pairs (W, S) and a fixed holonomy parameter α, we introduce real values $\kappa_0(W, S, \alpha)$ and $\nu_0(W, S, \alpha)$ as follows:

Definition 2.19 We define

$$\kappa_0(W, S, \alpha) := \min\{\kappa(A_L) \mid A_L \text{ minimal reducible}\},$$

$$\nu_0(W, S, \alpha) := \begin{cases} \nu(A_L), \text{ where } A_L \text{ is a minimal reducible with } \kappa_0 = \kappa(A_L) & \text{if } \alpha \neq \frac{1}{4}, \\ \min\{\nu(A_L) \mid A_L \text{ is a minimal reducible}\} & \text{if } \alpha = \frac{1}{4}. \end{cases}$$

Note that the homotopy class of the path $z \colon \theta_\alpha \to \theta'_\alpha$ represented by a minimal reducible A_L with $\kappa_0 = \kappa(A_L)$ is uniquely determined when $\alpha \neq \frac{1}{4}$. If $\alpha = \frac{1}{4}$ then homotopy classes of paths represented by minimal reducibles are not unique, but only finitely many exist. In particular, $\nu_0(W, S, \alpha)$ is well defined.

Remark 2.20 If the cobordism of pairs (W, S) has a flat minimal reducible with a holonomy parameter α, then $\kappa_0(W, S, \alpha) = \nu_0(W, S, \alpha) = 0$.

We write $\kappa_0 = \kappa_0(W, S, \alpha)$ and $\nu_0 = \nu_0(W, S, \alpha)$ for short.

Definition 2.21 Let $(W, S) \colon (Y, K) \to (Y', K')$ be a cobordism of pairs where K and K' are oriented knots in integral homology 3–spheres Y and Y'. Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. A cobordism of pairs (W, S) is called negative definite over $\mathscr{S}$ if

1. $b^1(W) = b^+(W) = 0$,
2. the index of the minimal reducibles is -1,
3. we have the nonzero element in $\mathscr{S}$

$$\eta^\alpha(W, S) := \sum_{A_L \text{ minimal}} (-1)^{c_1(L)^2} \lambda^{\kappa_0 - \kappa(A_L)} T^{\nu(A_L) - \nu_0}.$$

Remark 2.22 Our definition of the negative definite cobordism coincides with that of [8] when $\alpha = \frac{1}{4}$, since instantons have minimal energy if only if they have minimal index.

Let $(W_1, S_1) \colon (Y_1, K_1) \to (Y', K')$ and $(W_2, S_2) \colon (Y', K') \to (Y_2, K_2)$ be negative definite cobordisms. Note that their composition $(W_2 \circ W_1, S_2 \circ S_1) \colon (Y_1, K_1) \to (Y_2, K_2)$ is also a negative definite cobordism.

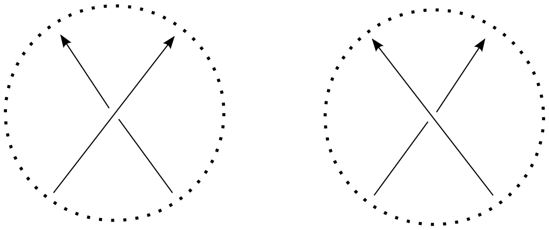

Figure 1: Positive (left) and negative (right) crossings of a knot.

A cylinder $[0,1]\times(Y,K)$ and a homology concordance $(Y,K)\to(Y',K')$ are examples of negative definite cobordisms. The following is also a basic example of negative definite cobordisms. Let K_+ be a knot in S^3 which has at least one positive crossing. Let K_- be a knot which is obtained by replacing one positive crossing in the knot K_+ by one negative crossing; see Figure 1.

Since S^3 is simply connected, K_+ and K_- are homotopic. Approximating homotopy from K_- to K_+ by a smooth map, we get a smoothly immersed surface $S\subset[0,1]\times S^3$ such that $S\cap\{0\}\times S^3=K_+$ and $S\cap\{1\}\times S^3=K_+$. Furthermore, we assume that S has a transverse self-intersection point. Let $S'\colon K_+\to K_-$ be an inverse cobordism of S. S has a positive self-intersection point in $[0,1]\times S^3$. Blowing up this self-intersection point, we obtain a new cobordism of pairs

$$(\overline{\mathbb{CP}}^2\,\#\,([0,1]\times S^3),\bar{S})\colon(S^3,K_-)\to(S^3,K_+). \tag{2-9}$$

$\bar{S}$ is an embedded surface in $\overline{\mathbb{CP}}^2\,\#\,([0,1]\times S^3)$ obtained by resolving the self-intersection of S, and it represents a homology class

$$2e\in H^2(\overline{\mathbb{CP}}^2;\mathbb{Z})\cong H^2(\overline{\mathbb{CP}}^2\,\#\,([0,1]\times S^3);\mathbb{Z}),$$

where e is an element represented by the exceptional curve. Similarly, we obtain a cobordism of pairs

$$(\overline{\mathbb{CP}}^2\,\#\,([0,1]\times S^3),\bar{S}')\colon(S^3,K_+)\to(S^3,K_-). \tag{2-10}$$

Cobordisms of pairs $(W,\bar{S})\colon(S^3,K_-)\to(S^3,K_+)$ and $(W',\bar{S}')\colon(S^3,K_+)\to(S^3,K_-)$ constructed as above are called *the cobordism of positive/negative crossing change*, respectively.

Proposition 2.23 *Fix a holonomy parameter $\alpha\in\left(0,\tfrac{1}{2}\right)\cap\mathbb{Q}$ with $\Delta_{K_+}(e^{4\pi i\alpha})\neq 0$ and $\Delta_{K_-}(e^{4\pi i\alpha})\neq 0$. Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. We assume that $\sigma_\alpha(K_+)=\sigma_\alpha(K_-)$. Then the cobordism of positive and negative crossing change are negative definite over $\mathscr{S}$.*

Proof Firstly, we show that (2-9) is a negative definite pair. Put $W=\overline{\mathbb{CP}}^2\,\#\,([0,1]\times S^3)$. Then it is clear that W satisfies Definition 2.21(1) since $H^1(W;\mathbb{Z})=0$ and $H^2(W;\mathbb{Z})=\mathbb{Z}$. Let A_m be a $U(1)$–reducible instanton corresponding to an element $m\in\mathbb{Z}=H^2(W;\mathbb{Z})$. Then

$$\bar{\kappa}(A_m)=-(c_1(L_m)+\alpha\bar{S})^2=(m+2\alpha)^2,$$

where L_m is a line bundle such that $c_1(L_m)[e]=-m$. We also have

$$\nu(A_m)=2c_1(L_m)[\bar{S}]=-4m.$$

The index computation yields that

$$\begin{aligned}\operatorname{ind}(D_{A_m}) &= 8(m+2\alpha)^2 + 2(4\alpha-1)(-4m) - 32\alpha^2 + \sigma_\alpha(K_+) - \sigma_\alpha(K_-) - 1\\ &= 8m(m+1) + \sigma_\alpha(K_-) - \sigma_\alpha(K_+) - 1.\end{aligned}$$

Thus $m = 0, -1$ minimize $\operatorname{ind} D_{A_m}$, and this means that A_0 and A_{-1} are minimal reducibles. Since $\sigma_\alpha(K_+) = \sigma_\alpha(K_-)$ by our assumption, the index for minimal instantons is -1 for the first case. Thus $(W, \bar{S})$ satisfies Definition 2.21(2). Since $\bar{\kappa}(A_m) = m^2 + 4\alpha m + 4\alpha^2$, we have

$$\eta^\alpha(W,\bar{S}) = \begin{cases} 1 - \lambda^{4\alpha-1}T^4 & \text{if } \alpha \le \frac{1}{4},\\ \lambda^{1-4\alpha}T^{-4} - 1 & \text{if } \alpha > \frac{1}{4}.\end{cases}$$

Since $1 - \lambda^{4\alpha-1}T^4$ is invertible when $\alpha \neq \frac{1}{4}$ and nonzero when $\alpha = \frac{1}{4}$ by assumption, $\eta^\alpha(W, \bar{S})$ is nonzero in $\mathscr{S}$. Hence $(W, \bar{S})$ satisfies Definition 2.21(3) and $(W, \bar{S})$ is a negative definite pair.

It is also obvious that $(W, \bar{S}')$ satisfies Definition 2.21(1). Since $\bar{S}'$ has the trivial homology class in $H_2(W;\mathbb{Z})$, minimal reducibles are only trivial with index -1. Hence $\eta^\alpha(W, \bar{S}') = 1 \neq 0 \in \mathscr{S}$. □

Next, we discuss the transversality of moduli spaces at reducibles. Following [26; 4], we introduce the perturbation supported on the interior of the cobordism. Let $\mathcal{I}$ be an infinite countable set of indexes and consider the following data:

- a collection of embedded 4–balls $\{B_i\}_{i\in\mathcal{I}}$ in $W^+ \setminus S^+$,
- a collection of submersions $q_i\colon S^1 \times B_i \to W^+ \setminus S^+$ such that $q_i(1,\cdot)$ is the identity,
- for any $x \in W \setminus S$, the set $\{q_{i,x} \mid i \in \mathcal{I},\ x \in B_i\}$ is a C^1–dense subset in the space of loops based at $x \in W \setminus S$.

For each $i \in \mathcal{I}$, consider a self-dual 2–form ω_i on B_i with $\operatorname{supp}(\omega_i) \subset B_i$. These self-dual 2–forms ω_i can be regarded as self-dual 2–forms on $W^+ \setminus S^+$. We define $V_{\omega_i}\colon \mathcal{A}_z(W, S; \beta, \beta') \to \Omega^+(W^+\setminus S^+; \mathfrak{su}(2))$ as

$$V_{\omega_i}(A) := \pi(\omega_i \otimes \operatorname{Hol}_{q_i}(A)),$$

where $\pi\colon \mathrm{SU}(2) \to \mathfrak{su}(2)$ is a map given by $g \mapsto g - \frac{1}{2}\operatorname{tr}(g)1$. The argument similar to [26] shows that there are constants $K_{n,i}$ and differentials of V_{ω_i} which satisfy the inequality

$$\|D^n V_{\omega_i} \mid (a_1, \dots, a_n)\|_{\check{L}^2_{m,A_0}} \le K_{n,i} \|\omega\|_{C^l} \prod_{i=1}^{n} \|a_i\|_{\check{L}^2_{m,A_0}},$$

where A_0 is a singular connection which represents the homotopy class z and $l \ge 3$. We choose a family of positive constants $\{C_i\}$ so that

$$C_i \ge \sup\{K_{n,i} \mid 0 \le n \le i\}.$$

Consider a family of self-dual 2–forms $\{\omega_i\}$ such that $\sum_{i\in\mathcal{I}} C_i \|\omega_i\|_{C^l}$ converges. For such a choice of $\{\omega_i\}$, $V_{\boldsymbol{\omega}} := \sum_{i\in\mathcal{I}} V_{\omega_i}\omega_i$ defines a smooth map

$$\mathcal{A}_z(W, S; \beta, \beta') \to \phi\check{L}^2_m(W^+ \setminus S^+, \Lambda^+ \otimes \mathfrak{su}(2))$$

between Banach manifolds.

We define $\mathcal{J} := \{(i,j) \in \mathcal{I}\times\mathcal{I} \mid i \neq j,\ B_{i,j} := B_i \cap B_j \neq \varnothing\}$ and $q_{i,j}\colon B_{i,j} \to W^+ \setminus S^+$ by

$$q_{i,j}|_{\{x\}\times S^1} := q_{i,x} * q_{j,x} * q_{i,x}^{-1} * q_{j,x}^{-1}$$

for each $(i,j) \in \mathcal{J}$. We choose a family of constants $\{C_{i,j}\}_{(i,j)\in\mathcal{J}}$ as before. Let $\omega_{i,j}$ be a self-dual 2–form on $B_{i,j}$. We introduce a Banach space $\mathcal{W}$ which consists of sequences of self-dual 2–forms $\omega = \{\omega_i\}_{i\in\mathcal{I}} \cup \{\omega_{i,j}\}_{(i,j)\in\mathcal{J}}$ with the following weighted l^1–norm:

$$\|\omega\|_{\mathcal{W}} := \sum_{i\in\mathcal{I}} C_i \|\omega_i\|_{C^l} + \sum_{(i,j)\in\mathcal{J}} C_{i,j} \|\omega_{i,j}\|_{C^l}.$$

For each $\omega \in \mathcal{W}$, we define a perturbation term

$$V_\omega(A) := \sum_{i\in\mathcal{I}} V_{\omega_i}(A) \otimes \omega_i + \sum_{(i,j)\in\mathcal{J}} V_{\omega_{i,j}}(A) \otimes \omega_{i,j}$$

which defines a smooth map $V_\omega\colon \mathcal{A}_z(W,S;\beta,\beta') \to \check{L}^2_{m,\epsilon}(W^+ \setminus S^+, \Lambda^+ \otimes \mathfrak{su}(2))$. We call

$$F_A^+ + U_{\pi_W}(A) + V_\omega(A) = 0$$

the secondary perturbed ASD-equation over the cobordism of pairs $(W,S)\colon (Y,K) \to (Y',K')$. Then $M^{\pi_W,\omega}(W,S;\beta,\beta')$ denotes the moduli space of solutions for the secondary perturbed ASD-equation.

Proposition 2.24 *Let (W,S) be a cobordism of pairs such that $b^1(W) = b^+(W) = 0$. Assume that the perturbation π_W is chosen so that the perturbed ASD-equation*

$$F_A^+ + U_{\pi_W}(A) = 0$$

cuts out the irreducible part of the moduli space transversely. Let $A^{\mathrm{ad}} = \mathbf{1} \oplus B$ be the adjoint connection of abelian reducible ASD connection $[A] \in M(W,S;\theta_\alpha,\theta'_\alpha)_{2d+1}$ with $\mathrm{ind}(d_B^* \oplus d_B^+) \geq 0$. Then for a small generic perturbation $\omega \in \mathcal{W}$, the secondary perturbed ASD-equation cuts out the irreducible part of the moduli space transversely. Moreover, $M^{\pi_W,\omega}(W,S;\theta_\alpha,\theta'_\alpha)_{2d+1}$ is regular at $[A]$ and has a neighborhood of $[A]$ which is homeomorphic to a cone on $\pm\mathbb{CP}^d$.

Proof For each connected component $M_z^{\pi_W,\omega}(W,S;\beta,\beta')$ of moduli spaces, the argument [4, Section 7] is adapted to our case and reducible points are regular for generic perturbations. Taking countable intersections of these subsets of regular perturbations in W, we can find a generic perturbation $\omega \in \mathcal{W}$ such that the statement holds. The claim about local structures around reducibles can be refined using the standard argument; see [11, Proposition 4.3.20], for example. □

Essentially the same argument is used in [8]. From now on, we assume that perturbations over the cobordism of pairs (W,S) are chosen so that they satisfy the statement of Proposition 2.24.

2.8 Orientation

We see the orientation of moduli spaces over the cylinder based on [32; 9]. Consider a reference connection A_0 on (W^+, S^+) as described in Section 2.7 and the ASD-operator (2-5). If the weight function ϕ has the form $e^{-\delta|t|}$ on one end, the functional space $\phi\check{L}^2_{m,A_0}$ consists of exponential decaying functions on that end. On the other hand, if the weight function ϕ has the form $e^{\delta|t|}$ on one end, the functional space $\phi\check{L}^2_{m,A_0}$ allows exponential growth functions. The index of the operator D_A depends on these choices of weighted functions. To distinguish these two situations, $\theta_{\alpha,\pm}$ denote reducible flat limits θ_α with weighted functions $e^{\pm\delta|t|}$. Let z be a path along (W,S) between two critical limits β and β' on (Y,K) and (Y',K'). The family index of $D_{\{A\}}$ defines a trivial line bundle $\det\operatorname{ind}(D_A)$ on each $\mathcal{B}_z(W,S,\beta,\beta')$. Let $\mathscr{O}_z[W,S;\beta_0,\beta_1]$ be the two-point set of the orientation of the determinant line bundle $\det\operatorname{ind}(D_{\{A\}})$. Then $\mathscr{O}_z[W,S;\beta,\beta']$ is the set of orientation of the moduli space $M^\alpha_z(W,S;\beta,\beta')$. There is a transitive and faithful $\mathbb{Z}_2$–action on $\mathscr{O}_z[W,S;\beta,\beta']$. For a composition of cobordisms $(W_2,S_2)\circ(W_1,S_1)$, there is a pairing

$$\Phi\colon \mathscr{O}_{z_1}[W_1,S_1;\beta,\beta']\otimes_{\mathbb{Z}_2}\mathscr{O}_{z_2}[W_2,S_2;\beta',\beta'']\to\mathscr{O}_{z_2\circ z_1}[W_2\circ W_1,S_2\circ S_1;\beta,\beta'']$$

which is induced from the gluing formula of the index. If we consider the gluing operation along the reducible connection θ_α, we choose $\beta'=\theta_{\alpha,+}$ at the first component and $\theta_{\alpha,-}$ at the second component. Since there is a natural isomorphism between $\mathscr{O}_z[W,S;\beta,\beta']$ and $\mathscr{O}_{z'}[W,S;\beta,\beta']$, we omit z from the above notation. We call an element of $\mathscr{O}[W,S;\theta_{\alpha+},\theta'_{\alpha-}]$ a homology orientation of (W,S). For a given knot in an integral homology 3–sphere (Y,K), we use the notation

$$\mathscr{O}[\beta]:=\mathscr{O}[Y\times I,K\times I;\beta,\theta_{\alpha-}]$$

if β is irreducible, and

$$\mathscr{O}[\theta_\alpha]:=\mathscr{O}[Y\times I,K\times I;\theta_{\alpha+},\theta_{\alpha-}].$$

There is an isomorphism

$$\mathscr{O}[W,S;\theta_{\alpha,+},\theta_{\alpha,-}]|_{[A_0]}\cong\Lambda^{\mathrm{top}}(H^1(W)\oplus H^+(W)),$$

and an element $o_W\in\mathscr{O}[W,S;\theta_{\alpha,+},\theta_{\alpha,-}]$ is called a homology orientation.

Now we describe how the orientation of the moduli space $M(W,S;\beta,\beta')$ is defined. Let $o_W\in\mathscr{O}[W,S;\theta_{\alpha,+},\theta_{\alpha,-}]$ be a given homology orientation for (W,S). We fix elements $o_\beta\in\mathscr{O}[\beta]$ and $o_{\beta'}\in\mathscr{O}[\beta']$. Then the orientation $o_{(W,S;\beta,\beta')}\in\mathscr{O}[W,S;\beta,\beta']$ is fixed so that

$$\Phi(o_\beta\otimes o_W)=\Phi(o_{(W,S;\beta,\beta')}\otimes o_{\beta'}).$$

The moduli space $\check{M}_z(\beta_0,\beta_1)$ is oriented in the following way. First, we fix orientations $o_{\beta_0}\in\mathscr{O}[\beta_0]$ and $o_{\beta_1}\in\Lambda[\beta_1]$. Then the orientation of $M(\beta_0,\beta_1)$ is determined as above. Note that there is an $\mathbb{R}$–action on $M_z(\beta_1,\beta_2)$. Let $\tau_s(t,y)=(t-s,y)$ be the transition on the cylinder $(Y,K)\times\mathbb{R}$. Then the $\mathbb{R}$–action on $M(\beta_0,\beta_1)$ is given by the pullback $[A]\mapsto[\tau^*A]$. Finally, we orient $\check{M}(\beta_1,\beta_2)$ so that $\mathbb{R}\times\check{M}(\beta_1,\beta_2)=M(\beta_1,\beta_2)$ is orientation preserving.

The boundary of moduli spaces is oriented so that the outward normal vector sits in the first place in the tangent space.

3 $\mathcal{S}$–complexes and Frøyshov type invariants

In this section, we extend the construction of $\mathcal{S}$–complexes $\widetilde{C}_*(Y,K)$ for (Y,K) in [9] to general holonomy parameters. We also introduce $\mathbb{Z}\times\mathbb{R}$–bigrading of $\mathcal{S}$–complexes with rational holonomy parameters for the specific choice of coefficient and its filtered subcomplex based on [35].

3.1 A review on $\mathcal{S}$–complexes and Frøyshov invariants

The $\mathcal{S}$–complex and Frøyshov type invariant introduced by [9; 8] are defined using purely algebraic objects:

Definition 3.1 Let R be an integral domain, and $\widetilde{C}_*$ be a finitely generated and graded free R–module. The triple $(\widetilde{C}_*,\tilde{d},\chi)$ is called an $\mathcal{S}$–complex if

1. $\tilde{d}\colon \widetilde{C}_*\to\widetilde{C}_*$ is a degree -1 homomorphism,
2. $\chi\colon \widetilde{C}_*\to\widetilde{C}_*$ is a degree $+1$ homomorphism,
3. $\tilde{d}$ and χ satisfy
   - $\tilde{d}^2=0$, $\chi^2=0$, and $\tilde{d}\chi+\chi\tilde{d}=0$,
   - $\operatorname{Ker}(\chi)/\operatorname{Im}(\chi)\cong R_{(0)}$, where $R_{(0)}$ is a copy of R in $\widetilde{C}_0$.

If (C_*,d) is a given chain complex with the coefficient ring R, we can form an $\mathcal{S}$–complex

$$\widetilde{C}_*=C_*\oplus C_{*-1}\oplus R,\quad \tilde{d}=\begin{bmatrix} d & 0 & 0\\ v & -d & \delta_2\\ \delta_1 & 0 & 0\end{bmatrix},\quad \chi=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}, \tag{3-1}$$

where $\delta_1\colon C_*\to R$, $\delta_2\colon R\to C_{*-1}$ and $v\colon C_*\to C_{*-2}$. Since there are conditions on $\tilde{d}$ and χ in Definition 3.1, the components in $\tilde{d}$ and χ have to satisfy the following relations:

$$\delta_1 d=0,\quad d\delta_2=0\quad\text{and}\quad dv-vd-\delta_2\delta_1=0. \tag{3-2}$$

Conversely, if the $\mathcal{S}$–complex $(\widetilde{C},\tilde{d},\chi)$ is given then there is a decomposition $\widetilde{C}_*=C_*\oplus C_{*-1}\oplus R$. The reader can find the details in [9, Section 4.1].

There is also the notion of an $\mathcal{S}$–morphism, which is a morphism of $\mathcal{S}$–complexes.

Definition 3.2 Let $(\widetilde{C}_*,\tilde{d},\chi)$ and $(\widetilde{C}'_*,\tilde{d}',\chi')$ be $\mathcal{S}$–complexes. Fix decompositions $\widetilde{C}_*=C_*\oplus C_{*-1}\oplus R$ and $\widetilde{C}'=C'_*\oplus C'_{*-1}\oplus R$. A chain map $\widetilde{m}\colon\widetilde{C}_*\to\widetilde{C}'_*$ is called an $\mathcal{S}$–morphism if it has the form

$$\widetilde{m}=\begin{bmatrix} m & 0 & 0\\ \mu & m & \Delta_2\\ \Delta_1 & 0 & \eta\end{bmatrix}, \tag{3-3}$$

where $\eta\neq 0\in R$.

The condition that $\widetilde{m}$ is a chain map is equivalent to the following relations:

$$md-dm=0,\quad \Delta_1 d+\eta\delta_1-\delta'_1 m=0,\quad d'\Delta_2-\delta'_2\eta+m\delta_2=0,$$
$$\mu d+mv-\Delta_2\delta_1-v'm+d'\mu-\delta'_2\Delta_1=0.$$

Definition 3.3 Let $\widetilde{m}, \widetilde{m}' \colon \widetilde{C}_* \to \widetilde{C}'_*$ be two $\mathcal{S}$–morphisms. An $\mathcal{S}$–chain homotopy of $\widetilde{m}$ and $\widetilde{m}'$ is a degree 1 map $\tilde{h} \colon \widetilde{C}_* \to \widetilde{C}'_*$ such that

$$\tilde{d}'\tilde{h} + \tilde{h}\tilde{d} = \widetilde{m} - \widetilde{m}', \chi'\tilde{h} + \tilde{h}\chi = 0.$$

Two $\mathcal{S}$–complexes $\widetilde{C}_*$ and $\widetilde{C}'_*$ are called $\mathcal{S}$–chain homotopy equivariant if there are $\mathcal{S}$–morphisms $\widetilde{m} \colon \widetilde{C}_* \to \widetilde{C}'_*$ and $\widetilde{m}' \colon \widetilde{C}'_* \to \widetilde{C}_*$ such that $\widetilde{m}\widetilde{m}'$ and $\widetilde{m}'\widetilde{m}$ are $\mathcal{S}$–chain homotopic to the identity.

Remark 3.4 Consider $\mathcal{S}$–morphisms $\widetilde{m} \colon \widetilde{C}_* \to \widetilde{C}'_*$ and $\widetilde{m}' \colon \widetilde{C}'_* \to \widetilde{C}_*$. If there are unit elements c and c' in the coefficient ring R, and two $\mathcal{S}$–chain homotopies

$$\widetilde{m}'\widetilde{m} \sim c\,\mathrm{id}_{\widetilde{C}_*} \quad \text{and} \quad \widetilde{m}\widetilde{m}' \sim c'\,\mathrm{id}_{\widetilde{C}'_*},$$

then the two $\mathcal{S}$–complexes $\widetilde{C}_*$ and $\widetilde{C}'_*$ are $\mathcal{S}$–chain homotopy equivalent since both $c^{-1}\widetilde{m}$ and $c'^{-1}\widetilde{m}$ are $\mathcal{S}$–chain homotopic to the composition $c^{-1}c'^{-1}\widetilde{m}\widetilde{m}'\widetilde{m}$.

The Frøyshov type invariant, defined from an $\mathcal{S}$–complex, assigns an integer $h(\widetilde{C}_*)$ to each $\mathcal{S}$–complex $\widetilde{C}_*$.

Definition 3.5 [9, Proposition 4.15]

- $h(\widetilde{C}_*) > 0$ if and only if there is an element $\beta \in C_*$ such that $d\beta = 0$ and $\delta_1\beta \neq 0$.
- If $h(\widetilde{C}_*) = k > 0$ then k is the largest integer such that there exists $\beta \in C_*$ satisfying
$$d\beta = 0, \quad \delta_1 v^{k-1}(\beta) \neq 0, \quad \delta_1 v^i \beta = 0 \quad \text{for } i \leq k-2.$$
- If $h(\widetilde{C}_*) = k \leq 0$ then there are elements $a_0, \dots, a_{-k} \in R$ and $\beta \in C_*$ such that
$$d\beta = \sum_{i=0}^{-k} v^i \delta_2(a_i).$$

The followings are basic properties of the Frøyshov type invariant:

Proposition 3.6 [9, Corollary 4.14] *If there is an $\mathcal{S}$–morphism $\widetilde{m} \colon \widetilde{C}_* \to \widetilde{C}'_*$ then $h(\widetilde{C}_*) \leq h(\widetilde{C}'_*)$.*

Given two $\mathcal{S}$–complexes $(\widetilde{C}_*, \tilde{d}, \chi)$ and $(\widetilde{C}'_*, \tilde{d}', \chi')$, the product $\mathcal{S}$–complex $(\widetilde{C}^{\otimes}_*, \tilde{d}^{\otimes}, \chi^{\otimes})$ is defined as

$$\widetilde{C}^{\otimes}_* = \widetilde{C}_* \otimes \widetilde{C}'_*, \quad \tilde{d}^{\otimes} = \tilde{d} \otimes 1 + \epsilon \otimes \tilde{d}' \quad \text{and} \quad \chi^{\otimes} = \chi \otimes 1 + \epsilon \otimes \chi',$$

where $\epsilon \colon \widetilde{C}'_* \to \widetilde{C}'_*$ is given by $\epsilon(\beta') = (-1)^{\deg(\beta')}\beta'$ on elements of homogeneous degree. Let $d^{\otimes}$, $v^{\otimes}$, $\delta_1^{\otimes}$ and $\delta_2^{\otimes}$ be components of $\tilde{d}^{\otimes}$ with respect to the splitting $\widetilde{C}^{\otimes} = C^{\otimes}_* \oplus C^{\otimes}_{*-1} \oplus R$. Using the decomposition $C^{\otimes}_* = (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'$, these maps are represented by

$$d^{\otimes} = \begin{bmatrix} d \otimes 1 + \epsilon \otimes d' & 0 & 0 & 0 \\ -\epsilon v \otimes 1 + \epsilon \otimes v' & d \otimes 1 - \epsilon \otimes d' & \epsilon \otimes \delta'_2 & -\delta'_2 \otimes 1 \\ \epsilon \otimes \delta'_1 & 0 & d & 0 \\ \delta'_1 \otimes 1 & 0 & 0 & d' \end{bmatrix}, \quad v^{\otimes} = \begin{bmatrix} v \otimes 1 & 0 & 0 & \delta_2 \otimes 1 \\ 0 & v \otimes 1 & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & \delta_1 \otimes 1 & 0 & v' \end{bmatrix},$$

$$\delta_1^{\otimes} = [0, 0, \delta_1, \delta'_1], \qquad \delta_2^{\otimes} = [0, 0, \delta_2, \delta'_2]^{\mathrm{T}}.$$

The Frøyshov type invariant behaves additively for the product of $\mathcal{S}$–complexes:

Proposition 3.7 [9, Corollary 4.28] $h(\widetilde{C}^{\otimes}_*) = h(\widetilde{C}_*) + h(\widetilde{C}'_*)$.

3.2 Floer homology groups with local coefficients

In this subsection, we construct the summand C_* in an $\mathcal{S}$–complex as a Floer chain group with local coefficients. Let (Y, K) be an oriented knot in an integral homology 3–sphere. We fix a holonomy parameter α so that $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$ to isolate the unique flat reducible connection θ_α. We assign an abelian group $\Delta_{[B]}$ for each elements $[B]$ in the configuration space $\mathcal{B}(Y, K, \alpha)$ and an isomorphism $\Delta_z \colon \Delta_{[B_0]} \to \Delta_{[B_1]}$ for each homotopy class $z \in \pi_1(\mathcal{B}(Y, K, \alpha), [B_0], [B_1])$. If this assignment is functorial, a Floer chain complex with the local coefficient Δ is defined as follows:

$$C^\alpha_*(Y, K, \Delta) = \bigoplus_{\beta \in \mathfrak{C}^*_\pi(Y,K,\alpha)} \Delta_\beta \mathscr{O}[\beta] \quad \text{and} \quad \langle d(\beta_0), \beta_1 \rangle = \sum_{z \colon \beta_0 \to \beta_1} \sum_{[\breve{A}] \in \breve{M}_z(\beta_0,\beta_1)} \epsilon([\breve{A}]) \otimes \Delta_z.$$

The $\mathbb{Z}/4$–grading of $C^\alpha_*(Y, K, \Delta)$ is defined by mod-4 grading for critical points. Consider a subring $\mathscr{R}_\alpha$ in the Novikov ring $\Lambda^{\mathbb{Z}[T^{-1}, T]\!]}$, which is introduced in Section 1.2.

Lemma 2.18 enables us to define a local coefficient system $\Delta = \Delta_{\mathscr{R}_\alpha}$ as follows:

$$\Delta_{\mathscr{R}_\alpha, [B]} := \mathscr{R}_\alpha \lambda^{\mathrm{CS}(B)} T^{\mathrm{hol}_K(B)} \quad \text{and} \quad \Delta_{\mathscr{R}_\alpha, z} := \#\breve{M}_z(\beta, \beta_1)_0 \lambda^{-\kappa(z)} T^{\nu(z)}.$$

Note that this definition is independent of choices of representatives of $[B]$ and θ_α. Write $C^\alpha_*(Y, K; \Delta_{\mathscr{R}_\alpha})$ for a chain complex with the local coefficient system over $\mathscr{R}_\alpha$. For any algebra $\mathscr{S}$ over $\mathscr{R}_\alpha$, we can extend the above construction to the coefficient $\mathscr{S}$.

Definition 3.8 Let (Y, K) be an oriented knot in an integral homology 3–sphere and $\mathscr{S}$ be an algebra over $\mathscr{R}_\alpha$. Fix $\alpha \in \left(0, \frac{1}{2}\right) \cap \mathbb{Q}$ so that $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$. The homology group of the $\mathbb{Z}/4$–graded chain complex $(C^\alpha_*(Y, K; \Delta_{\mathscr{S}}), d)$ is denoted by $I^\alpha_*(Y, K; \Delta_{\mathscr{S}})$. We call $I^\alpha_*(Y, K; \Delta_{\mathscr{S}})$ the irreducible singular instanton knot homology over the local coefficient $\mathscr{S}$ with the holonomy parameter α.

Let $(W, S) \colon (Y, K) \to (Y', K')$ be a negative definite cobordism over $\mathscr{S}$. We define an induced morphism $m = m_{(W,S)} \colon C^\alpha_*(Y, K; \Delta_{\mathscr{S}}) \to C^\alpha_*(Y', K'; \Delta_{\mathscr{S}})$ by

$$m(\beta) = \sum_{\beta' \in \mathfrak{C}^*(Y,K,\alpha)} \sum_{z \colon \beta \to \beta'} \#M_z(W, S; \beta, \beta')_0 \lambda^{\kappa_0 - \kappa(z)} T^{\nu(z) - \nu_0} \beta'.$$

Counting the boundary of 1–dimensional moduli space $M^+_z(W, S; \beta, \beta')_1$ for each homotopy class z, we obtain the relation

$$dm - md' = 0.$$

We remark that:

- For a composition of negative definite cobordisms $(W, S) := (W_0, S_0) \circ (W_1, S_1)$, there is a map ϕ such that
$$d\phi - \phi d = m_{(W_1,S_1)} \circ m_{(W_0,S_0)} - m_{(W,S)},$$
where metrics and perturbation data on (W, S) are given by the composition of those of (W_0, S_0) and (W_1, S_1).

- If $m_{(W,S)}$ and $m'_{(W,S)}$ are defined by different perturbations and metric data on an interior domain of (W,S), they are chain homotopic.
- If $(W,S)=[0,1]\times(Y,K)$ then $m_{(W,S)}$ is chain homotopic to the identity map.

Thus $C^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}})$ is an invariant of (Y,K) up to chain homotopy. We write $I^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}})$ for its homology group, and this is an invariant for (Y,K).

Remark 3.9 The above argument shows that the chain homotopy type of $C^{\alpha}_{*}(Y,K,\Delta_{\mathscr{S}})$ is independent of the choice of orbifold metric with the same cone angle $\nu\in\mathbb{Z}_{>0}$. Hence, more precisely, the module $I^{\alpha}_{*}(Y,K,\Delta_{\mathscr{S}})$ should be denoted by $I^{\alpha}_{*}(Y,K,\nu,\Delta_{\mathscr{S}})$. We implicitly assume that the cone angle ν is chosen as a large enough number so that gauge theory on the orbifold setup described in Section 2 works.

Next, we introduce the filtered construction for the Floer chain complex based on Nozaki, Sato and Taniguchi [35]. For the filtered construction, we have to introduce the lift of critical points. Let $\mathcal{G}_{(0,0)}$ be a normal subgroup of the gauge group $\mathcal{G}(Y,K)$ which is given by

$$\mathcal{G}_{(0,0)}:=\{g\mid k(g)=l(g)=0\}.$$

Consider the quotient of the space of singular connections

$$\widetilde{\mathcal{B}}(Y,K,\alpha):=\mathcal{A}(Y,K,\alpha)/\mathcal{G}_{(0,0)}.$$

The Chern–Simons functional descends on $\widetilde{\mathcal{B}}(Y,K,\alpha)$ as an $\mathbb{R}$–valued function, and we still use the same notation. The normal subgroup $\mathcal{G}_{(0,0)}$ is a connected component of the full gauge group $\mathcal{G}(Y,K)$ which corresponds to $(0,0)\in\mathbb{Z}\oplus\mathbb{Z}\cong\pi_0(\mathcal{G}(Y,K))$. Thus there is an action of $\pi_0(\mathcal{G}(Y,K))\cong\mathbb{Z}\oplus\mathbb{Z}$ on $\widetilde{\mathcal{B}}(Y,K,\alpha)$ as a covering transformation, and hence $\widetilde{\mathcal{B}}(Y,K,\alpha)$ is a covering space over $\mathcal{B}(Y,K,\alpha)$ with a fiber $\mathbb{Z}\oplus\mathbb{Z}$.

Definition 3.10 A lift of $[B]\in\mathcal{B}(Y,K,\alpha)$ to the covering space $\widetilde{\mathcal{B}}(Y,K,\alpha)$ is called a lift of $[B]$, and denoted by $\widetilde{[B]}$.

For a fixed lift $\widetilde{[B]}$ of $[B]\in\mathcal{B}(Y,K,\alpha)$, the fiber of the projection $\widetilde{\mathcal{B}}(Y,K,\alpha)\to\mathcal{B}(Y,K,\alpha)$ over a point $[B]$ can be described as

$$\mathscr{L}_{[B]}:=\{g^{*}(\widetilde{[B]})\in\widetilde{\mathcal{B}}(Y,K,\alpha)\mid g\in\pi_0(\mathcal{G}(Y,K))\}.$$

The fiber $\mathscr{L}_{[B]}$ can be seen as the set of lifts of the element $[B]$. There is another description of lifts: Let $\tilde{\theta}_{\alpha}$ be a lift of reducible flat connections θ_{α}. Then a lift $\tilde{\beta}$ of $\beta\in\mathfrak{C}^{*}_{\pi}$ is fixed by choosing a path $z\colon\beta\to\theta_{\alpha}$ of connections over the cylinder whose endpoint is $\tilde{\theta}_{\alpha}$.

We again choose the coefficient ring $\mathscr{R}_{\alpha}$ and fix a lift $\tilde{\beta}$ for each critical points $\beta\in\mathfrak{C}_{\pi}(Y,K,\alpha)$.

Then we modify the local coefficient system $\Delta_{\mathscr{R}_{\alpha}}$ so that

$$\Delta_{\mathscr{R}_{\alpha},\beta}=\mathscr{R}_{\alpha}\lambda^{\mathrm{CS}(\tilde{\beta})}T^{\mathrm{hol}_K(\tilde{\beta})}$$

with the same map $\Delta_{\mathscr{R}_\alpha, z}$. Once we fix an orientation of β, each summand $\Delta_\beta \mathscr{O}[\beta]$ in the chain complex is generated (over $\mathbb{Z}$) by the elements of the form $\lambda^k \xi_\alpha^l \tilde{\beta} = \lambda^{k+2\alpha l} T^{2l} \tilde{\beta}$ where $(k, l) \in \mathbb{Z} \oplus \mathbb{Z}$. The action of $\lambda^k \xi_\alpha^l$ corresponds to the action of the gauge transformation with $d(g) = (k, l)$. Such elements can be identified with the set of lifts $\mathscr{L}_\beta$ of the critical point β. Hence the chain complex $C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})$ can be seen as a $\mathbb{Z}$–module generated by the all lifts of $\mathfrak{C}_\pi(Y, K, \alpha)$ under the modification as above.

Once we fix lifts of generators, the chain complex $C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})$ admits a $(\mathbb{Z}\times\mathbb{R})$–bigrading as in [8], that is we can associate a pair of values which is defined as follows:

For a lift $\tilde{\beta}$ of the critical point $\beta \in \mathfrak{C}_\pi$, we define $\deg_{\mathbb{Z}}(\tilde{\beta}) := \mathrm{gr}_z(\beta)$ where z is a path corresponding to the lift $\tilde{\beta}$. Then we extend $\deg_{\mathbb{Z}}$ as

$$\deg_{\mathbb{Z}}(\lambda^i \xi_\alpha^j \tilde{\beta}) = 8i + 4j + \deg_{\mathbb{Z}}(\tilde{\beta}).$$

Next we define $\deg_{\mathbb{R}}$. For a lift $\tilde{\beta}$ of a critical point $\beta \in \mathfrak{C}_\pi$, we define $\deg_{\mathbb{R}}(\tilde{\beta}) := \mathrm{CS}(\tilde{\beta})$. This extends to elements of the form $\lambda^i \xi_\alpha^j \tilde{\beta}$ as

$$\deg_{\mathbb{R}}(\lambda^i \xi_\alpha^j \tilde{\beta}) = i + 2\alpha j + \deg_{\mathbb{R}}(\tilde{\beta}). \tag{3-4}$$

In general, an element $\gamma \in C_*^\alpha(Y, K, \alpha)$ has the form $\gamma = \sum_i a_i \gamma_i$ where $\gamma_i \in \bigcup_{\beta \in \mathfrak{C}_\pi} \mathscr{L}_\beta$. This is possibly an infinite sum. We define

$$\deg_{\mathbb{R}}(\gamma) = \max\{\deg_{\mathbb{R}}(\gamma_i) \mid a_i \neq 0\}$$

for $\gamma \neq 0$ and $\deg_{\mathbb{R}}(0) = -\infty$.

In summary, we have the following proposition:

Proposition 3.11 *Once we fix lifts of critical points of the Chern–Simons functional, the chain complex* $(C_*^\alpha(Y, K, \Delta_{\mathscr{R}_\alpha}), d)$ *admits the* $(\mathbb{Z}\times\mathbb{R})$*–bigrading.*

We write $C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]}$ for the chain complex $C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})$ with the $(\mathbb{Z}\times\mathbb{R})$–bigrading.

Let $\mathscr{C}^* \subset \mathbb{R}$ be a subset defined by $\mathscr{C}^* := \mathrm{CS}(\mathrm{Crit}^*)$. For $R \in \mathbb{R} \setminus \mathscr{C}^*$, we define a subset

$$C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R]} := \{\gamma \in C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]} \mid \deg_{\mathbb{R}}(\gamma) < R\}.$$

This defines a subcomplex of $C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]}$. For two numbers $R_0, R_1 \in (\mathbb{R} \setminus \mathscr{C}^*) \cup \{\pm\infty\}$ such that $R_0 \le R_1$, we define a quotient complex as follows:

$$C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]} := C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R_1]} / C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R_0]}.$$

Definition 3.12 For $R_0, R_1 \in \mathbb{R} \cup \{\pm\infty\}$ such that $R_0 \le R_1$ and $R_0, R_1 \notin \mathscr{C}^* \cup \mathscr{C}'^*$, we call $C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})$ a $[R_0, R_1]$–filtered chain complex.

Consider a negative definite cobordism $(W, S)\colon (Y, K) \to (Y', K')$ with $\kappa_0 = 0$. A cobordism map $m_{(W,S)}$ on $C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})$ induces a map

$$C_*^\alpha(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R]} \to C_*^\alpha(Y', K'; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R]}$$

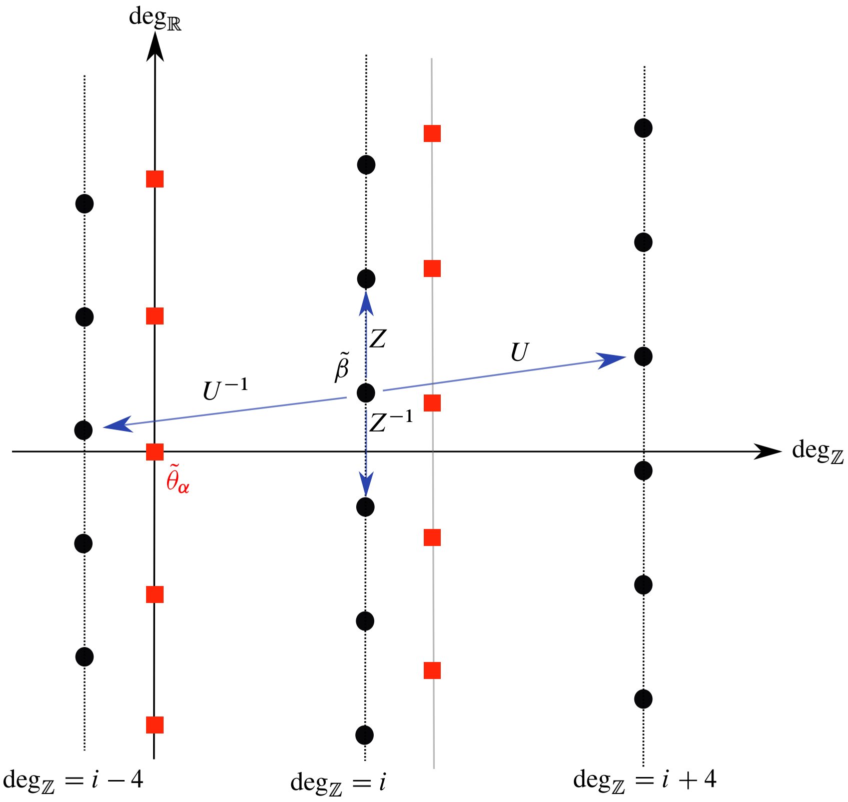

Figure 2: Dots represent lifts of the irreducible flat connection β and squares represent lifts of the reducible flat connection θ_α.

by the restriction, and hence this induces a map

$$m^{[R_0,R_1]}_{(W,S)}\colon C^\alpha_*(Y,K;\Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]} \to C^\alpha_*(Y',K';\Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]}. \tag{3-5}$$

As described before, the covering transformation on $\widetilde{\mathcal{B}}(Y,K,\alpha)$ is generated by multiplications of elements $\lambda^{\pm 1}$and $\xi_\alpha^{\pm 1}$. We also introduce other generators which fit the $(\mathbb{Z}\times\mathbb{R})$–bigrading on $C^\alpha_*(Y,K;\Delta_{\mathscr{R}_\alpha})$. Let us introduce two operators on $C^\alpha_*(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]}$,

$$Z^{\pm 1} := (\lambda^{1-4\alpha}T^{-4})^{\pm 1} \quad \text{and} \quad U^{\pm 1} := (\lambda^{2\alpha}T^2)^{\pm 1}. \tag{3-6}$$

These operators change the $(\mathbb{Z}\times\mathbb{R})$–bigrading as

$$\deg_{\mathbb{Z}}(Z^i\tilde{\beta}) = \deg_{\mathbb{Z}}(\tilde{\beta}) \quad \text{and} \quad \deg_{\mathbb{R}}(Z^i\tilde{\beta}) = \deg_{\mathbb{R}}(\tilde{\beta}) + (1-4\alpha)i \quad \text{for the operator } Z, \tag{3-7}$$

$$\deg_{\mathbb{Z}}(U^i\tilde{\beta}) = \deg_{\mathbb{Z}}(\tilde{\beta}) + 4i \quad \text{and} \quad \deg_{\mathbb{R}}(U^i\tilde{\beta}) = \deg_{\mathbb{R}}(\tilde{\beta}) + 2\alpha i \quad \text{for the operator } U. \tag{3-8}$$

See Figure 2 for the case $\alpha < \frac{1}{4}$. Since $\lambda = ZU^2$, actions of the two operations Z and U (and their inverses) on lifted critical points generate $C^\alpha_*(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]}$.

3.3 Maps δ_1, δ_2, Δ_1 and Δ_2

We introduce operators which are defined by counting instantons on a cylinder or a cobordism with the reducible limit. We remark that the sign convention of counting moduli spaces in this subsection is the same as that of [9]. Let (Y, K) and (Y', K') be two knots in integral homology 3–spheres. Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. In this subsection, we assume that the holonomy parameter α is chosen so that $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$ and $\Delta_{(Y',K')}(e^{4\pi i\alpha}) \neq 0$.

Definition 3.13 We define $\mathscr{S}$–linear chain maps $\delta_1\colon C^\alpha_*(Y, K; \Delta_{\mathscr{S}}) \to \mathscr{S}$ and $\delta_2\colon \mathscr{S} \to C^\alpha_{-2}(Y, K; \Delta_{\mathscr{S}})$ as follows.

For $\beta \in \mathfrak{C}^*_\pi(Y, K, \alpha)$,

$$\delta_1(\beta) := \sum_{z\colon \beta\to\theta_\alpha} \#\breve{M}_z(\beta, \theta_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)}$$

and

$$\delta_2(1) := \sum_{\substack{\beta\in\mathfrak{C}^*_\pi(Y,K,\alpha)\\ \mathrm{gr}(\beta)\equiv 2}} \sum_{z\colon \theta_\alpha\to\beta} \#\breve{M}_z(\theta_\alpha, \beta)_0 \lambda^{-\kappa(z)} T^{\nu(z)} \beta.$$

Since the compactified 1–dimensional moduli space $\breve{M}^+_z(\beta, \theta_\alpha)_1$ has oriented boundaries

$$\bigcup_{\substack{\gamma\in\mathfrak{C}^*_\pi\\ \mathrm{gr}(\gamma)\equiv 1}} \bigcup_{\substack{z_1,z_2\\ z_1\circ z_2=z}} \breve{M}_{z_1}(\beta, \gamma)_0 \times \breve{M}_{z_2}(\gamma, \theta_\alpha)_0,$$

it is straightforward to check that $d \circ \delta_1 = 0$. Similarly, $\delta_2 \circ d = 0$ holds.

Next, we define $\Delta_1\colon C^\alpha_*(Y, K; \Delta_{\mathscr{S}}) \to \mathscr{S}$ and $\Delta_2\colon \mathscr{S} \to C^\alpha_*(Y', K'; \Delta_{\mathscr{S}})$ for a cobordism of pairs $(W, S)\colon (Y, K) \to (Y', K')$:

Definition 3.14 We have

$$\Delta_1(\beta) := \sum_z \#M_z(W, S; \beta, \theta'_\alpha)_0 \lambda^{\kappa_0-\kappa(z)} T^{\nu(z)-\nu_0},$$

$$\Delta_2(1) := \sum_{\beta'\in\mathfrak{C}^*_\pi(Y',K',\alpha)} \sum_z \#M_z(W, S; \theta_\alpha, \beta')_0 \lambda^{\kappa_0-\kappa(z)} T^{\nu(z)-\nu_0} \beta'.$$

Proposition 3.15 *Let $m = m_{(W,S)}\colon C^\alpha_*(Y, K; \Delta_{\mathscr{S}}) \to C^\alpha_*(Y', K'; \Delta_{\mathscr{S}})$ be a cobordism map induced from the negative definite pair $(W, S)\colon (Y, K) \to (Y', K')$. Then the relations*

(i) $\Delta_1 \circ d + \eta\delta_1 - \delta'_1 \circ m = 0$,

(ii) $d' \circ \Delta_2 - \delta'_2\eta + m \circ \delta_2 = 0$,

hold, where $\eta = \eta^\alpha(W, S)$ and $'$ denotes corresponding maps for the pair (Y', K').

Proof The relation (i) is given by counting the ends of each component of the 1–dimensional moduli space $M_z(W,S;\beta,\theta'_\alpha)_1$ as in [9, Proposition 3.10]. The boundary components of $M_z(W,S;\beta,\theta'_\alpha)_1$ with the induced orientation are given by

(a) $$-\bigcup_{\beta_1\in\mathfrak{C}^*_\pi}\ \bigcup_{\substack{z_1,z_2\\ z_1\circ z_2=z}} \breve{M}_{z_1}(\beta,\beta_1)_0\times M_{z_2}(W,S;\beta_1,\theta'_\alpha)_0,$$

(b) $$\bigcup_{\gamma'\in\mathfrak{C}'^*_\pi}\ \bigcup_{\substack{z_1,z_2\\ z_1\circ z_2=z}} M_{z_1}(W,S;\beta,\beta')_0\times \breve{M}_{z_2}(\beta',\theta_\alpha)_0,$$

(c) $$-\bigcup_{\substack{z_1,z_2\\ z_1\circ z_2=z}} \breve{M}_{z_1}(\beta,\theta_\alpha)_0\times M_{z_2}(W,S;\theta_\alpha,\theta'_\alpha)_0.$$

Note that product orientations of $\breve{M}_{z_1}(\beta,\gamma)_0\times M_{z_2}(W,S;\gamma,\theta'_\alpha)_0$ and $\breve{M}_{z'}(\beta,\theta_\alpha)_0\times M(W,S;\theta_\alpha,\theta'_\alpha)_0$ are opposite to orientations induced as the boundaries of $M_z(W,S;\beta,\theta_\alpha)_1$. The signed counting of the boundary components of types (a) and (b) contribute to $-\Delta_1\circ d(\beta)$ and $\delta'_1\circ m(\beta)$, respectively. Since $M(W,S;\theta_\alpha,\theta'_\alpha)_0$ consists of minimal reducible elements, the counting of (c) gives $-\eta\delta_1(\beta)$. This proves (i). The relation (ii) can be similarly proved considering the ends of the 1–dimensional moduli space $M_z(W,S;\theta_\alpha,\beta')_1$. □

3.4 Maps v and μ

In this subsection, we introduce maps induced from the cobordism of pairs (W,S) with an embedded curve $\gamma\subset S$. Our assumptions for the choice of holonomy parameter α and the coefficient $\mathscr{S}$ are the same as the previous subsection. We remark that the sign convention of moduli spaces in this subsection is also the same as that of [9]. In particular, if $f\colon M\to N$ is a smooth map between oriented manifolds then $f^{-1}(y)$ for a regular value $y\in N$ is oriented so that

$$T_xM=N_xf^{-1}(y)\oplus T_xf^{-1}(y)$$

is orientation preserving, where $N_xf^{-1}(y)$ is a fiber of the normal bundle for $f^{-1}(y)$ and its orientation is induced from that of N. The mapping degree $\deg(f)$ is defined by using this orientation.

Assume that $\gamma\colon[0,1]\to S$ is a smoothly embedded loop. Fix a regular neighborhood $N_\gamma(\epsilon)$ of γ in W with radius $\epsilon>0$ and fix a basepoint $x_0\in\partial N_\gamma(\epsilon)$. We take a Seifert framing $\tilde{\gamma}_\epsilon\subset\partial N_\gamma(\epsilon)$ of γ so that it passes through the basepoint x_0. The bundle decomposition $E=L\oplus L^*$ over $S\subset W$ extends to $N_\gamma(\epsilon)$, and the holonomy of the adjoint connection of $[A]\in\mathcal{B}(W,S;\beta,\beta')$ yields $\mathrm{Hol}_{\tilde{\gamma}_\epsilon}(A^{\mathrm{ad}})\in S^1$. Put

$$h^\gamma_{\beta\beta'}(A):=\lim_{\epsilon\to 0}\mathrm{Hol}_{\tilde{\gamma}_\epsilon}(A^{\mathrm{ad}}).$$

The construction above gives a map

(3-9) $$h^\gamma_{\beta\beta'}\colon\mathcal{B}(W,S,\alpha;\beta,\beta')\to S^1.$$

Note that this map itself depends on the choice of the Seifert framing of γ and orientations of K and S. However, such dependence on auxiliary data can be ignored to define the following map:

Definition 3.16 Let β and β' be irreducible critical points of the (perturbed) Chern–Simons functional on (Y,K) and (Y',K'), respectively. We define a map $\mu=\mu_{(W,S,\gamma)}\colon C^{\alpha}_*(Y,K;\Delta_{\mathscr{S}})\to C^{\alpha}_*(Y',K';\Delta_{\mathscr{S}})$ by

$$\mu(\beta)=\sum_{\beta'\in\mathfrak{C}^*_\pi(Y,K,\alpha)}\ \sum_{z\colon\beta\to\beta'}\deg(h^{\gamma}_{\beta\beta'}|_{M_z(W,S;\beta,\beta')_1})\lambda^{\kappa_0-\kappa(z)}T^{\nu(z)-\nu_0}\beta'$$

for each $\beta\in\mathfrak{C}^*_\pi(Y,K,\alpha)$.

The map μ satisfies the following relation:

Proposition 3.17

$$d'\circ\mu-\mu\circ d=0.$$

Proof Consider the compactified 2–dimensional moduli space $M^+_z(W,S;\beta,\beta')_2$ which has oriented boundary of the types

$$-\bigcup_{\beta_1\in\mathfrak{C}^*_\pi(Y,K,\alpha)}\ \bigcup_{z'\circ z''=z}\breve{M}^+_{z'}(\beta,\beta_1)_{i-1}\times M^+_{z''}(W,S;\beta_1,\beta')_{2-i},$$
$$\bigcup_{\beta_1'\in\mathfrak{C}^*_\pi(Y',K',\alpha)}\ \bigcup_{z'\circ z''=z}M^+_{z'}(W,S;\beta,\beta_1')_{2-i}\times\breve{M}^+_{z''}(\beta_1',\beta')_{i-1},$$

where $i=1$ or 2. Count the boundary of the 1–dimensional submanifold $(h^{\gamma}_{\beta\beta'})^{-1}(s)\subset M^+(W,S;\beta,\beta')$ for a regular value $s\in S^1$. Since the closed loop γ is supported on a compact subset of S, $(h^{\gamma}_{\beta\beta'})^{-1}(s)$ intersects faces of the boundary of $M^+_z(W,S;\beta,\beta')$ with $i=1$. Thus

$$\#((h^{\gamma}_{\beta\beta'})^{-1}(s)\cap\partial M^+_z(W,S;\beta,\beta')_2)=d'\circ\mu-\mu\circ d=0. \qquad\square$$

We consider the case when $(W,S)=\mathbb{R}\times(Y,K)$ and $\gamma\subset S$ is a curve $\mathbb{R}\times\{y_0\}$ where y_0 is a fixed basepoint in K. Taking holonomy along γ, we obtain a map

$$h_{\beta_1\beta_2}\colon\mathcal{B}(Y,K,\alpha;\beta_1,\beta_2)\to S^1$$

similarly to (3-9), where β_i for $i=1,2$ are irreducible critical points of the Chern–Simons functional.

The holonomy map $h_{\beta_1\beta_2}$ is modified to extend broken trajectories as in [10]. Such modification of $h_{\beta_1\beta_2}$ near the broken trajectories gives the map

$$H_{\beta_1\beta_2}\colon\breve{M}(\beta_1,\beta_2)_d\to S^1$$

with the following properties:

(i) $H_{\beta_1\beta_2}=h_{\beta_1\beta_2}$ on the complement of a small neighborhood of $\partial\breve{M}^+(\beta_1,\beta_2)_d$.

(ii) $H_{\beta_1\beta_3}([A_1],[A_2])=H_{\beta_1\beta_2}([A_1])H_{\beta_2\beta_3}([A_2])$ on unparametrized broken trajectories

$$\breve{M}^+(\beta_1,\beta_2)_{i-1}\times\breve{M}^+(\beta_2,\beta_3)_{d-i},$$

where β_2 is irreducible.

(iii) $H_{\beta_1\beta_2}=1$ if $\dim\breve{M}(\beta_1,\beta_2)=0$.

Definition 3.18 We define the v–map $v\colon C^{\alpha}_*(Y,K;\Delta_{\mathscr{S}}) \to C^{\alpha}_*(Y,K;\Delta_{\mathscr{S}})$ by

$$v(\beta_1) = \sum_{\beta_2\in\mathfrak{C}^*_\pi}\ \sum_{z\colon\beta_1\to\beta_2} \deg(H_{\beta_1\beta_2}|_{\breve{M}_z(\beta_1,\beta_2)_1})\lambda^{-\kappa(z)}T^{\nu(z)}\beta_2.$$

The v–map does not commute with the differential of the chain complex. However, the following relation holds:

Proposition 3.19
$$dv - vd - \delta_2\delta_1 = 0.$$

Proof We consider the 1–dimensional moduli space

$$\breve{M}_{\gamma,z}(\beta_1,\beta_2)_1 := \breve{M}_z(\beta_1,\beta_2)_2 \cap (H_{\beta_1\beta_2})^{-1}(s)$$

for a generic $s\in S^1\setminus\{1\}$. As in the argument in the proof of [9, Proposition 3.16], the boundary of $\breve{M}_z(\beta_1,\beta_2)$ consists of unparametrized broken trajectories of the form

$$\mathfrak{a} = ([A_1],[A_2]),$$

and there are the following cases:

(I) $\mathfrak{a}\in\bigcup_{z'\circ z''=z}\breve{M}_{z'}(\beta_1,\beta_3)_0\times\breve{M}_{z''}(\beta_3,\beta_2)_1$ where $\beta_3\in\mathfrak{C}^*_\pi(Y,K,\alpha)$,

(II) $\mathfrak{a}\in\bigcup_{z'\circ z''=z}\breve{M}_{z'}(\beta_1,\beta_3)_1\times\breve{M}_{z''}(\beta_3,\beta_2)_0$ where $\beta_3\in\mathfrak{C}^*_\pi(Y,K,\alpha)$,

(III) $[A]\in\breve{M}_z(\beta_1,\beta_2)$ factors through the reducible critical point θ_α.

For (I), the corresponding oriented boundary components of $(H_{\beta_1\beta_2})^{-1}(s)\cap\breve{M}^+_z(\beta_1,\beta_2)_2$ are

$$\begin{aligned}(H_{\beta_1\beta_2})^{-1}(s)\cap -\Big(\bigcup_{\beta_3\in\mathfrak{C}^*_\pi(Y,K,\alpha)}\ \bigcup_{z'\circ z''=z}\breve{M}_{z'}(\beta_1,\beta_3)_0\times\breve{M}_{z''}(\beta_3,\beta_2)_1\Big)\\ = -\bigcup_{\beta_3\in\mathfrak{C}^*_\pi(Y,K,\alpha)}\ \bigcup_{z'\circ z''=z}\breve{M}_{z'}(\beta_1,\beta_3)_0\times(H_{\beta_3\beta_2})^{-1}(s)\cap\breve{M}_{z''}(\beta_3,\beta_2)_1,\end{aligned}$$

since $H_{\beta_1\beta_3}=1$. This contributes the term $-\langle vd(\beta_1),\beta_2\rangle$. For (II), the similar argument shows that this contributes to the term $\langle dv(\beta_1),\beta_2\rangle$. Case (III) requires gluing theory at the reducible. Let U be an open subset of $\breve{M}(\beta_1,\beta_2)$ which is given by

$$U = \{[A]\in\breve{M}(\beta_1,\beta_2)\mid \|A-\pi^*\theta_\alpha\|_{L^2_1((-1,1)\times(Y\setminus K))}<\epsilon\}.$$

U_z denotes the restriction of U to $M_z(\beta_1,\beta_2)$. There is the "ungluing" map

$$\breve{M}_z(\beta_1,\beta_2)\supset U_z \xrightarrow{\psi} (0,\infty)\times\bigcup_{z'\circ z''=z}\breve{M}_{z'}(\beta_1,\theta_\alpha)_0\times S^1\times\breve{M}_{z''}(\theta_\alpha,\beta_3)_0.$$

For $T>0$ large enough, consider a subset $U_{z,T}=\psi^{-1}(\{(t,[A_1],s,[A_2])\in U_z\mid t>T\})$ of U_z. Then

$$\psi(M_z(\beta_1,\beta_2)\cap U_{z,T}) = (T,\infty)\times\bigcup_{z'\circ z''=z}\breve{M}_{z'}(\beta_1,\theta_\alpha)\times\{s\}\times\breve{M}_{z''}(\theta_\alpha,\beta_2).$$

Thus corresponding boundaries of 1–manifold $H^{-1}_{\beta_1\beta_2}(s)\cap \breve{M}^+(\beta_1,\beta_2)_2$ with induced orientations are given by

$$-\bigcup_{z'\circ z''=z} \breve{M}_{z'}(\beta_1,\theta_\alpha)_0\times \breve{M}_{z''}(\theta_\alpha,\beta_2)_0.$$

The sign counting of this contributes to the term $-\langle \delta_2\delta_1(\beta_1),\beta_2\rangle$. Finally, we obtain the relation $\langle (dv-vd-\delta_2\delta_1)(\beta_1),\beta_2\rangle=0$. □

Next, we consider a negative definite pair $(W,S)\colon (Y,K)\to(Y',K')$ with an embedded curve $\gamma\colon[0,1]\to S$ such that $\gamma(0)=p\in K$ and $\gamma(1)=p'\in K'$. We identify γ with its image. We define

$$\gamma^+=(-\infty,0]\times\{p\}\cup\gamma\cup[0,\infty)\times\{p'\}\subset S^+.$$

Assume that $\beta\in\mathfrak{C}^*_\pi(Y,K,\alpha)$ and $\beta'\in\mathfrak{C}^*_\pi(Y',K',\alpha)$. For each $A\in\mathcal{A}(W,S;\beta,\beta')$, taking the holonomy of A^{ad} along the path γ^+, we obtain a map

$$h^\gamma_{\beta\beta'}\colon \mathcal{B}(W,S;\beta,\beta')\to S^1,$$

and its modification

$$H^\gamma_{\beta\beta'}\colon M^+(W,S;\beta,\beta')_d\to S^1$$

so that $H^\gamma_{\beta\beta'}=1$ on 0–dimensional unparametrized broken trajectories.

Definition 3.20 We define a map $\mu=\mu_{(W,S,\gamma)}\colon C^\alpha_*(Y,K;\Delta_{\mathscr{S}})\to C^\alpha_*(Y',K';\Delta_{\mathscr{S}})$ by

$$\mu(\beta)=\sum_{\beta'\in\mathfrak{C}^*_\pi(Y',K',\alpha)}\ \sum_{z\colon\beta\to\beta'}\deg(H^\gamma_{\beta\beta'}|_{M_z(W,S;\beta,\beta')_1})\lambda^{\kappa_0-\kappa(z)}T^{\nu(z)-\nu_0}\beta'.$$

Proposition 3.21 *Let $(W,S)\colon(Y,K)\to(Y',K')$ be a negative definite pair, and let m and μ denote its corresponding maps as above. Then*

$$d'\mu+\mu d+\Delta_2\delta_1-\delta_2'\Delta_1-v'm+mv=0,$$

where the prime denotes corresponding maps for the pair (Y',K').

Proof Consider a 2–dimensional moduli space $M^+_z(W,S;\beta,\beta')_2$ and its codimension 1 faces. Firstly, there are two types of ends of $M_z(W,S;\beta,\beta')_2$ in which $[A]\in M(W,S;\beta,\beta')_2$ is broken at irreducible critical points,

$$\breve{M}^+_{z'}(\beta,\beta_1)_{i-1}\times M^+_{z''}(W,S;\beta_1,\beta')_{2-i},\tag{I}$$

$$M^+_{z'}(W,S;\beta,\beta')_{2-i}\times \breve{M}^+_{z''}(\beta_1,\beta')_{1-i},\tag{II}$$

where $i=1,2$. Since

$$\begin{aligned}(H^\gamma_{\beta\beta'})^{-1}(s)\cap\bigcup_{\beta_1}\bigcup_{z'\circ z''=z}&\breve{M}_z(\beta,\beta_1)_0\times M^+_{z''}(W,S;\beta_1,\beta')_1\\ &=\bigcup_{\beta_1}\bigcup_{z'\circ z''}\breve{M}_{z'}(\beta,\beta_1)_0\times(H^\gamma_{\beta_1\beta'})^{-1}(s)\cap M_{z''}(W,S;\beta_1,\beta')_1,\end{aligned}$$

the signed counting of points in $\partial((H^{\gamma}_{\beta\beta'})^{-1}(s)\cap M^{+}(W,S;\beta,\beta')_2)$ which are contained in codimension 1 faces of type (I) with $i=1$ contributes to the term $-\langle \mu d(\beta),\beta'\rangle$. Next, we consider the case of type (I) with $i=2$. Since

$$\begin{aligned}(H^{\gamma}_{\beta\beta'})^{-1}(s)\cap\bigcup_{\beta_1}\bigcup_{z'\circ z''=z}\breve{M}_{z'}(\beta,\beta_1)_1\times M_z(W,S;\beta_1,\beta')_0\\ =\bigcup_{\beta_1}\bigcup_{z'\circ z''=z}(H_{\beta\beta_1})^{-1}(s)\cap\breve{M}_{z'}(\beta,\beta_1)_1\times M(W,S;\beta_1,\beta')_0,\end{aligned}$$

the signed counting of points in $\partial((H^{\gamma}_{\beta\beta'})^{-1}(s)\cap M^{+}(W,S;\beta,\beta')_2)$ which are contained in codimension 1 faces of type (I) with $i=2$ contributes to the term $-\langle mv(\beta),\beta'\rangle$. Similarly, a collection of codimension 1 faces of type (II) contributes to the term $-\langle d'\mu(\beta),\beta'\rangle$ if $i=1$ and $\langle v'm(\beta),\beta'\rangle$ if $i=2$. Finally, we consider the ends of $M_z(W,S;\beta,\beta')_2$ which break at reducibles. Such ends are described as in the poof of Proposition 3.19 and contribute to the term $-\langle(\Delta_2\delta_1-\delta_2\Delta_1(\beta),\beta'\rangle$. □

Corollary 3.22 *We have* $(\widetilde{C}^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}}),\tilde{d},\chi)$, *where*

$$\widetilde{C}^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}})=C^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}})\oplus C^{\alpha}_{*-1}(Y,K;\Delta_{\mathscr{S}})\oplus\mathscr{S},\quad \tilde{d}=\begin{bmatrix} d & 0 & 0\\ v & -d & \delta_1\\ \delta_2 & 0 & 0\end{bmatrix}\quad\text{and}\quad \chi=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}$$

form an $\mathcal{S}$*–complex. Moreover, if* $(W,S)\colon(Y,K)\to(Y',K')$ *is a given negative definite cobordism and* α *satisfies* $\Delta_{(Y,K)}(e^{4\pi i\alpha})\Delta_{(Y',K')}(e^{4\pi i\alpha})\neq 0$, *then*

$$\widetilde{m}_{(W,S)}=\begin{bmatrix} m & 0 & 0\\ \mu & m & \Delta_2\\ \Delta_1 & 0 & \eta\end{bmatrix}$$

defines an $\mathcal{S}$*–morphism* $\widetilde{m}_{(W,S)}\colon\widetilde{C}^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}})\to\widetilde{C}^{\alpha}_{*}(Y',K';\Delta_{\mathscr{S}})$.

Proof The arguments in Section 3.3 and Proposition 3.19 show that $(\widetilde{C}^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}}),\tilde{d},\chi)$ is an $\mathcal{S}$–complex. For a generic perturbation, moduli spaces over the negative definite pair (W,S) are regular at reducible points by Proposition 2.24, and hence the counting of reducibles $\eta=\eta^{\alpha}(W,S)$ is well defined. The arguments in Section 3.2 and Propositions 3.15, and 3.21 show that $\widetilde{m}_{(W,S)}$ is an $\mathcal{S}$–morphism. □

The $\mathcal{S}$–complex $\widetilde{C}^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}})$ itself depends on the choices of metric and perturbation. However, the standard argument (see [9, Theorem 3.33]) shows that its $\mathcal{S}$–chain homotopy class is a topological invariant of pairs (Y,K,p) with $\Delta_{(Y,K)}(e^{4\pi i\alpha})\neq 0$, where $K\subset Y$ is an oriented knot in an integer homology 3–sphere and $p\in K$ is a basepoint. The $\mathcal{S}$–chain homotopy type of an $\mathcal{S}$–complex itself depends on the choice of basepoint, however, there is a canonical isomorphism between two homology groups of $\mathcal{S}$–complexes which are defined by different choices of basepoints.

Definition 3.23 We call

$$h^{\alpha}_{\mathscr{S}}(Y,K):=h(\widetilde{C}^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}}))$$

the Frøyshov invariant for (Y,K) over $\mathscr{S}$ with a holonomy parameter α.

The $\mathcal{S}$–complex $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{S}})$ admits the following connected sum theorem:

Theorem 3.24 *Let (Y, K) and (Y', K') be two oriented knots in integral homology spheres and α be a holonomy parameter such that $\Delta_{(Y,K)}(e^{4\pi i\alpha})\Delta_{(Y',K')}(e^{4\pi i\alpha}) \neq 0$. Then*

$$\widetilde{C}^{\alpha}_{*}(Y \# Y', K \# K'; \Delta_{\mathscr{S}}) \simeq \widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{S}}) \otimes_{\mathscr{S}} \widetilde{C}^{\alpha}_{*}(Y', K'; \Delta_{\mathscr{S}}),$$

where $\simeq$ denotes an $\mathcal{S}$–chain homotopy equivalence.

The strategy of proof (found in the appendix) is essentially the same as [9, Section 6].

The following corollary gives the proof of Theorem1.6:

Corollary 3.25 *Let (Y, K) and (Y', K') be knots in integral homology 3–spheres and α be a holonomy parameter such that $\Delta_{(Y,K)}(e^{4\pi i\alpha})\Delta_{(Y',K')}(e^{4\pi i\alpha}) \neq 0$. Then*

$$h^{\alpha}_{\mathscr{S}}(Y \# Y', K \# K') = h^{\alpha}_{\mathscr{S}}(Y, K) + h^{\alpha}_{\mathscr{S}}(Y', K').$$

Moreover, if there are two negative definite cobordisms

$$(W, S)\colon (Y, K) \to (Y', K') \quad \text{and} \quad (W', S')\colon (Y', K') \to (Y, K),$$

then

$$h^{\alpha}_{\mathscr{S}}(Y, K) = h^{\alpha}_{\mathscr{S}}(Y', K').$$

Proof The first statement follows from Theorem 3.24 and Proposition 3.7. The second follows from Corollary 3.22 and Proposition 3.6. □

The filtered construction can be applied to an $\mathcal{S}$–complex for the coefficient $\mathscr{R}_\alpha$. A fixed lift $\tilde{\theta}_\alpha$ of a reducible flat connection can be identified with $1 \in \mathscr{R}_\alpha$, and $\mathscr{R}_\alpha$ itself can be identified with the set of all lifts of θ_α. We extend the $\mathbb{R}$–grading to $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})$. First, we define

$$\deg_{\mathbb{R}}(\delta) = \begin{cases} \max\{r \mid a_r \neq 0\} & \text{if } \delta \neq 0, \\ -\infty & \text{if } \delta = 0, \end{cases}$$

for $\delta = \sum_r a_r \lambda^r \in \mathscr{R}_\alpha$, $a_r \in \mathbb{Z}[T^{-1}, T]]$. Then for $(\beta, \gamma, \delta) \in C^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha}) \oplus C^{\alpha}_{*-1}(Y, K; \Delta_{\mathscr{R}_\alpha}) \oplus \mathscr{R}_\alpha$, we define

$$\widetilde{\deg}_{\mathbb{R}}(\beta, \gamma, \delta) := \max\{\deg_{\mathbb{R}}(\beta), \deg_{\mathbb{R}}(\gamma), \deg_{\mathbb{R}}(\delta)\}.$$

Obviously, we have the following proposition:

Proposition 3.26 *If we fix a lift of each critical point of the Chern–Simons functional, then the $\mathcal{S}$–complex $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})$ admits the $(\mathbb{Z}\times\mathbb{R})$–grading.*

Note that the $\mathbb{R}$–grading of $\mathcal{S}$–complexes extends to tensor products of $\mathcal{S}$–complexes in a natural way.

The filtered $\mathcal{S}$–complex $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]}$ for $R_0, R_1 \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \mathscr{C}^*$ with $R_0 < R_1$ can be defined as follows. Put $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R]} := \{(\beta, \gamma, \delta) \in \widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha}) \mid \widetilde{\deg}_{\mathbb{R}}(\beta, \gamma, \delta) < R\}$ and

$$\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]} := \widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R_1]} / \widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R_0]}$$

for $R_0 < R_1$.

3.5 Cobordism maps for immersed surfaces

Let $(W,S)\colon (Y,K)\to(Y',S')$ be a cobordism of pairs where S is possibly immersed. Blowing up all double points of S, we obtain a cobordism of pairs $(\overline{W},\overline{S})$ where $\overline{S}$ is an embedded surface.

Definition 3.27 We say (W,S) is negative definite if its blowup $(\overline{W},\overline{S})$ is negative definite. We define a cobordism map for a negative definite cobordism (W,S) where S is possibly immersed surface as

$$\widetilde{m}_{(W,S)} := \widetilde{m}_{(\overline{W},\overline{S})}.$$

We describe the relation between operations on immersed surface cobordisms and induced $\mathcal{S}$–morphisms:

Proposition 3.28 *Let $\mathscr{S}$ be an integral domain over the ring $\mathscr{R}_\alpha$. Assume that (W,S) is a negative definite pair over $\mathscr{S}$ where S is a possibly immersed surface. Let S^* be a surface obtained from S by a positive or negative twist move, or a finger move. Then $\widetilde{m}_{(W,S^*)}$ is $\mathcal{S}$–chain homotopic to $\widetilde{m}_{(W,S)}$ up to the multiplication of a unit element in $\mathscr{S}$.*

The definition of positive twist, negative twist and finger moves can be found in [17].

Proof Since the monotonicity condition cannot be assumed in our setting, we have to modify the argument in [25].

(i) **(positive twist move)** Consider the blowup at the positive self-intersection point $(\overline{W},\overline{S}^*)=(W,S)\,\#\,(\overline{\mathbb{CP}}^2,S_2)$, where S_2 is an embedded sphere whose homology class is $-2e\in H_2(\overline{\mathbb{CP}}^2;\mathbb{Z})$. Note that $\mathcal{R}_\alpha(S^3\setminus S^1,\mathrm{SU}(2))=\{\theta_\alpha\}$ for $\alpha\in\left(0,\tfrac{1}{2}\right)$. Assume that (W,S^*) has a metric g_T such that (S^3,S^1) has a neighborhood which is isometric to $[-T,T]\times(S^3,S^1)$, where $T>0$ is large enough. Let A_T be an instanton on $\{(W,S^*),g_T\}$ which is contained in the 0–dimensional moduli space. A_∞ denotes the limiting instanton of A_T with respect to $T\to\infty$, and A_1 and A_2 denote its restriction to components obtained by attaching cylindrical ends on (W,S) and $(\overline{\mathbb{CP}}^2,S_2)$, respectively. Then we have

$$\operatorname{ind}D_{A_1}+1+\operatorname{ind}D_{A_2}=\operatorname{ind}D_{A_\infty}\le 0.$$

The last inequality essentially follows from [27, Corollary 8.4] and our assumption. The index formula for the closed pair $(\overline{\mathbb{CP}}^2,S_2)$ shows that $\operatorname{ind}D_{A_2}\equiv -1 \bmod 4$, and we have $\operatorname{ind}D_{A_2}=-1$. By the perturbation, the instanton A_2 on $\overline{\mathbb{CP}}^2$ satisfies $H^1_{A_2}=H^2_{A_2}=0$, and the gluing along $\mathcal{R}_\alpha(S^3\setminus S^1)=\{\theta_\alpha\}$ is unobstructed. The moduli space $M(W,S^*;\beta,\beta')_0$ is diffeomorphic to

$$M(W,S;\beta,\beta')_0\times M^\alpha(\overline{\mathbb{CP}}^2,S_2)_0.$$

Note that there is a diffeomorphism $M^\alpha(\overline{\mathbb{CP}}^2\setminus D^4,S_2\setminus D^2;\theta_\alpha)_0\cong M^\alpha(\overline{\mathbb{CP}}^2,S_2)_0$ by the removable singularity theorem. Since $\operatorname{ind}D_{A_2}=-1$, A_2 is a minimal reducible. Moreover, minimal reducibles on $(\overline{\mathbb{CP}}^2,S_2)$ define elements in $M^\alpha(\overline{\mathbb{CP}}^2,S_2)_0$. Counting elements in the moduli space $M(W,S;\beta,\beta')_0$ defined by the limiting metric $\lim_{T\to\infty}g_T$ contributes the relation

$$\langle\widetilde{m}^{\infty}_{(\overline{W},\overline{S}^*)}\beta,\beta'\rangle=\eta^\alpha(\overline{\mathbb{CP}}^2,S_2)\langle\widetilde{m}_{(W,S)}\beta,\beta'\rangle.$$

Since

$$\eta^{\alpha}(\overline{\mathbb{CP}}^2, S_2) = \begin{cases} 1-\lambda^{4\alpha-1}T^4 & \text{if } \alpha \le \frac{1}{4}, \\ \lambda^{1-4\alpha}T^{-4}-1 & \text{if } \alpha > \frac{1}{4}, \end{cases}$$

$\eta^{\alpha}(\overline{\mathbb{CP}}^2, S_2)$ is a unit in $\mathscr{S}$. Considering the 1–parameter family of moduli spaces gives an $\mathcal{S}$–chain homotopy between $\widetilde{m}^{\infty}_{(\overline{W},\overline{S}^*)}$ and $\widetilde{m}_{(\overline{W},\overline{S}^*)}$. In particular, $\eta^{\alpha}(\overline{\mathbb{CP}}^2, S_2)$ has the top term 1, and hence the statement follows.

(ii) **(negative twist move)** In this case, we change S_2 in the above argument to an embedded sphere S_0 whose homology class is trivial. Thus we obtain $\eta^{\alpha}(\overline{\mathbb{CP}}^2, S_0) = 1$.

(iii) **(finger move)** Consider the decomposition $(W,S) = (W_1,S_1)\cup(W_2,S_2)$ where $W_2 = D^4$ and $S_2 = D^2\sqcup D^2$. Let $(\overline{W},\overline{S}^*) = (W_1,S_1)\cup(W_2',S_2')$ be the double blowup of (W,S^*). In this case, W_2 is a 4–manifold obtained by removing a disk from $\overline{\mathbb{CP}}^2\,\#\,\overline{\mathbb{CP}}^2$ and S_2 is two disjoint disks. Note that $\mathcal{R}_\alpha := \mathcal{R}_\alpha(S^3\backslash(S^1\sqcup S^1), \mathrm{SU}(2)) \cong [0,\pi]$ for fixed $\alpha\in\left(0,\frac{1}{2}\right)$, the interior of $\mathcal{R}_\alpha$ consists of irreducible flat connections and two endpoints are reducible. Moreover, the endpoint map $r_1\colon M(W_1,S_1;\beta,\beta')_0\to\mathcal{R}_\alpha$ has its image in the irreducible part of $\mathcal{R}_\alpha$. See [25, Lemma 3.2] for details.

We claim that the counting of the two moduli spaces $M(W_1,S_1;\beta,\beta')_0$ and $M(W,S;\beta,\beta')_0$ can be identified up to multiplication by a unit element in $\mathscr{S}$. Firstly, we define an $\mathcal{S}$–morphism $\widetilde{m}_{(W_1,S_1)}$ as

$$\langle m_{(W_1,S_1)}\beta,\beta'\rangle = \sum_{z} \#M_z(W_1,S_1;\beta,\beta')\lambda^{\kappa_0-\kappa(z)}T^{\nu(z)-\nu_0}\beta',$$

and similarly for other components in $\widetilde{m}_{(W_1,S_1)}$. Here $\kappa(z)$, κ_0, $\nu(z)$ and ν_0 are similarly defined as in Section 2.7. We have to modify the argument in [25] which is related to the unobstructed gluing along the pair $(S^3, S^1\sqcup S^1)$. For $\rho\in\mathcal{R}_\alpha$ which is in the image of r_1, we take its extension A_ρ to $(D^4, D^2\sqcup D^2)$. Consider the double $(S^4, S^2\sqcup S^2) = (D^4, D^2\sqcup D^2)\cup_{(S^3,S^1)}(D^4, D^2\sqcup D^2)$. Then $\operatorname{ind} D_{A_\rho\#A_\rho} = 2\operatorname{ind} D_{A_\rho}+1$ by the gluing formula. Consider the pair of connected sum $(S^4, S^2\sqcup S^2) = (S^4,S^2)\,\#_{(S^3,\varnothing)}\,(S^4,S^2)$. Then $\operatorname{ind} D_{A_\rho\#A_\rho} = 2\operatorname{ind} D_{A_\rho\#A_\rho}|_{(S^4,S^2)}+3$ and the left-hand side is equal to 1. Hence we have $\operatorname{ind} D_{A_\rho}=0$. Thus the relation

$$\operatorname{ind} D_{A_\rho} + \dim H^1_\rho = -\dim H^0_{A_\rho} + \dim H^1_{A_\rho} - \dim H^2_{A_\rho}$$

tells us that $H^2_{A_\rho}=0$ since $\dim H^0_{A_\rho}=0$ and $\dim H^1_{A_\rho}=1$. Here H^1_ρ is the cohomology with the local coefficient system associated with the flat connection ρ. Thus the Morse–Bott gluing of instantons over (W_1,S_1) and (W_2,S_2) is unobstructed. For a metric on (W,S) with a long neck along the cylinder $[0,1]\times(S^3, S^1\sqcup S^1)$, we have the diffeomorphism,

$$M(W,S;\beta,\beta')_0 \cong M(W_1,S_1;\beta,\beta')_0\,{}_{r}{\times}_{r'}\,M^{\alpha}(D^4, D^2\sqcup D^2)_1$$

where

$$r\colon M(W_1,S_1;\beta,\beta')_0\to\mathcal{R}_\alpha$$

and

$$r'\colon M^{\alpha}(D^4, D^2\sqcup D^2)_1\to\mathcal{R}_\alpha$$

are restriction maps. For simplicity, we consider the case $\alpha \leq \frac{1}{4}$. Since flat connections on $(S^3, S^1 \sqcup S^1)$ uniquely extend to $(D^4, D^2 \sqcup D^2)$, the induced cobordism map has the form

$$\widetilde{m}_{(W,S)} = \Big(1 + \sum_{k>0} c_k Z^{-k}\Big)\widetilde{m}_{(W_1,S_1)},$$

where $c_k \in \mathbb{Z}$ and $Z = \lambda^{1-4\alpha}T^{-4}$. Thus $\widetilde{m}_{(W,S)}$ and $\widetilde{m}_{(W_1,S_1)}$ differ by the multiplication of a unit element in $\mathscr{S}$.

Assume that the cobordism of pairs $(\overline{W}, \overline{S}^*)$ is equipped with a metric such that $(\overline{W}, \overline{S}^*)$ has a long neck along $(S^3, S^1 \sqcup S^1)$. Then the moduli space $M(\overline{W}, \overline{S}^*; \beta, \beta')_0$ decomposes into a union of fiber products

$$M(W_1, S_1, \beta, \beta')_d \, {}_{r_1}\!\times_{r_2} M^{\alpha}(W_2'; S_2')_{d'}$$

with $d + d' = 1$, where

$$r_1 \colon M(W_1, S_1, \beta, \beta')_d \to \mathcal{R}_\alpha \quad \text{and} \quad r_2 \colon M^{\alpha}(W_2', S_2')_{d'} \to \mathcal{R}_\alpha$$

are restriction maps. Since $d' \equiv 1 \bmod 4$ by the index formula, we have $d = 0$ and $d' = 1$. Thus there is the coefficient $c \in \mathscr{S}$ such that $\widetilde{m}_{(\overline{W},\overline{S}^*)} = c\widetilde{m}_{(W,S)}$. Consider the special case of a finger move which is the composition of one positive twist move and one negative twist move. In this case, the coefficient c turns out to be $1 - \lambda^{4\alpha-1}T^4$ for $\alpha \leq \frac{1}{4}$ and $\lambda^{1-4\alpha}T^{-4} - 1$ for $\alpha > \frac{1}{4}$ by the argument above. Finally, we conclude that there is a unit element $c \in \mathscr{S}$ such that $\widetilde{m}_{(W,S^*)}$ and $c\widetilde{m}_{(W,S)}$ are $\mathcal{S}$–chain homotopic. □

4 Nondegeneracy of the representation variety

In this section we will discuss conjugacy classes of representations

$$\rho \colon \pi_1(Y \setminus K) \to \mathrm{SU}(2),$$

with the condition

$$\rho(\mu_K) \sim \begin{bmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{bmatrix}.$$

We write $[\rho]$ for its conjugacy class to distinct elements in $\mathrm{Hom}(\pi_1(Y \setminus K), \mathrm{SU}(2))$ and $\mathcal{R}(Y \setminus K, \mathrm{SU}(2))$. Firstly, we introduce the method of taking cyclic branched coverings. Considering a knot in an integral homology 3–sphere (Y, K), we can take a cyclic branched covering $\widetilde{Y}_r(K)$ over Y branched along K. Let $N(K)$ be a tubular neighborhood of $K \subset Y$, and $V = Y \setminus \mathrm{int}(N(K))$ be its exterior. $\widetilde{V}$ denotes the r–fold unbranched covering over V with $\pi_1(\widetilde{V})$ being a kernel of $\pi_1(Y \setminus K) \to H_1(Y \setminus K, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$. $N(K)$ and $\widetilde{V}$ have a torus boundary, and let $h \colon \partial N(K) \to \partial \widetilde{V}$ be a gluing map which sends μ_K a meridian of K to its lift $\tilde{\mu}_K$. Then the r–fold cyclic branched covering over Y is defined by

$$\widetilde{Y}_r(K) = N(K) \cup_h \widetilde{V}.$$

Let $\tau\colon \widetilde{Y}_r \to \widetilde{Y}_r$ be a covering transformation. We define an induced action of $\pi_1(\widetilde{Y}_r, p)$ with a basepoint p missing the fixed point set of the τ–action on $\widetilde{Y}_r$. For this, we fix another basepoint q inside the fixed point set and a path connecting p and q. Such a choice of path defines a (noncanonical) isomorphism between $\pi_1(\widetilde{Y}_r, p)$ and $\pi_1(\widetilde{Y}_r, q)$. Since τ induces a natural action on $\pi_1(\widetilde{Y}_r, q)$, we define an induced action τ_* on $\pi_1(\widetilde{Y}_r, p)$ via the above isomorphism. The τ–action induces the action τ^* on $\operatorname{Hom}(\pi_1(\widetilde{Y}_r, p), \mathrm{SO}(3))$ by $\tau^*(\rho) = \rho \circ \tau_*$. This also defines an action on $\mathcal{R}^*(\widetilde{Y}_r, \mathrm{SO}(3))$ by $\tau^*[\rho] = [\tau^*(\rho)]$. We define the following subsets:

$$\mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)) = \{[\rho] \in \mathcal{R}^*(\widetilde{Y}_r, \mathrm{SO}(3)) \mid \tau^*[\rho] = [\rho]\},$$
$$\mathcal{R}^{*,\tau}(\widetilde{Y}, \mathrm{SU}(2)) = \{[\rho] \in \mathcal{R}^*(\widetilde{Y}, \mathrm{SU}(2)) \mid \mathrm{Ad}[\rho] \in \mathcal{R}^{*,\tau}(\widetilde{Y}, \mathrm{SO}(3))\}.$$

Since a different choice of basepoints of fundamental groups induces a canonical isomorphism on $\mathcal{R}^*(\widetilde{Y}_r, \mathrm{SO}(3))$, we may omit the choice of basepoints and a path between them from the notation here.

The aim of Section 4.1 is giving the construction of the lifting map

$$\Pi\colon \bigsqcup_{1 \le l \le r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \to \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2)),$$

which sends singular flat connections to nonsingular flat connections on a cyclic branched covering of the knot $K \subset Y$. We will see that the lifting map Π satisfies the following proposition.

Proposition 4.1 *Assume that the r–fold cyclic branched covering $\widetilde{Y}_r$ of a knot K in an integral homology 3–sphere Y is an integral homology 3–sphere. Then the lifting map Π gives a two-to-one correspondence*

$$\Pi\colon \bigsqcup_{1 \le l \le r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \to \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2)).$$

This is a generalization of the argument in [2].

Let $X(K)$ be the complement of a tubular neighborhood of the knot $K \subset S^3$. Its boundary $\partial X(K)$ is a torus. In Section 4.2, we will show that the restriction map $r\colon \mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \to \mathcal{R}(\partial X(T_{p,q}), \mathrm{SU}(2))$ is a smooth immersion of a 1–manifold without any perturbation of flat connections, using the setting of gauge theory by Herald [22] and computations of the group cohomology of π_1. In Section 4.3, we will give a proof of Theorem 1.9 using the results in Section 4.1.

4.1 The construction of the lifting map

We assign the second Stiefel–Whitney class $w \in H^2(Y, \mathbb{Z}_2)$ to $[\rho] \in \mathcal{R}(Y, \mathrm{SO}(3))$. We can construct a flat bundle $E = \widetilde{Y} \times_\rho \mathbb{R}^3$ from an SO(3)–representation ρ and define $w([\rho]) := w_2(E) \in H^2(Y, \mathbb{Z}_2)$, where $w_2(E)$ is the second Stiefel–Whitney class of E. If $w(\rho) = 0$ then the SO(3)–bundle E lifts to an SU(2)–bundle F. Let P and Q be the corresponding principal bundles of E and F, respectively. The natural map $p\colon Q \to P$ is a fiberwise double covering map. Let θ_ρ be a connection form on P which corresponds to the flat connection ρ. Then $p^*\theta_\rho$ defines a flat connection on Q. Thus each element of $\mathcal{R}(Y, \mathrm{SO}(3))$ lifts to $\mathcal{R}(Y, \mathrm{SU}(2))$ if its second Stiefel–Whitney class vanishes.

Proposition 4.2 *Let X be Y or $Y \setminus K$. Then there is an action of $H^1(X,\mathbb{Z}_2)$ on $\mathcal{R}(X,\mathrm{SU}(2))$ and the map $\mathrm{Ad}\colon \mathcal{R}(X,\mathrm{SU}(2)) \to \mathcal{R}^0(X,\mathrm{SO}(3))$ induces a bijection*

$$\mathcal{R}(X,\mathrm{SU}(2))/H^1(X,\mathbb{Z}_2) \cong \mathcal{R}^0(X,\mathrm{SO}(3)).$$

Here $\mathcal{R}^0(X,\mathrm{SO}(3))$ denotes the set of conjugacy classes of $\mathrm{SO}(3)$–representations whose second Stiefel–Whitney class vanishes.

Proof Let $\rho\colon \pi_1(X) \to \mathrm{SO}(3)$ be a representation whose second Stiefel–Whitney class vanishes and $\tilde{\rho}\colon \pi_1(X) \to \mathrm{SU}(2)$ be its $\mathrm{SU}(2)$–lift. Consider another lift $\tilde{\rho}'\colon \pi_1(X) \to \mathrm{SU}(2)$. Then there is a map $\chi\colon \pi_1(X) \to \{\pm 1\}$ such that $\tilde{\rho}'(g) = \chi(g)\tilde{\rho}(g)$ for any $g \in \pi_1(X)$. We can directly check that χ is a homomorphism and determine an element $\chi \in \mathrm{Hom}(\pi_1(X),\mathbb{Z}_2) = H^1(X,\mathbb{Z}_2)$. Conversely, two $\mathrm{SU}(2)$–representation $\sigma_1,\sigma_2\colon \pi(X) \to \mathrm{SU}(2)$ such that there exists $\chi \in \mathrm{Hom}(\pi_1(X),\mathbb{Z}_2)$ and satisfying $\sigma_1(g) = \chi(g)\sigma_2(g)$ for any $g \in \pi_1(X)$ induces the same $\mathrm{SO}(3)$–representation. We define an action of $H^1(X,\mathbb{Z}_2)$ on $\mathrm{Hom}(\pi_1(X),\mathrm{SU}(2))$ by $\sigma \mapsto \chi\cdot\sigma$, where $(\chi\cdot\sigma)(g) = \chi(g)\sigma(g)$ for $g \in \pi_1(X)$. The action of χ commutes with the conjugacy action and descends to $\mathcal{R}(X,\mathrm{SU}(2))$. □

Note that the action of $H^1(Y \setminus K,\mathbb{Z}_2)$ coincides with the flip symmetry. From Proposition 4.2, we get the following corollary:

Corollary 4.3 *For an integral homology 3–sphere Y, all elements in $\mathcal{R}(Y,\mathrm{SO}(3))$ have a unique lift in $\mathcal{R}(Y,\mathrm{SU}(2))$.*

Proof Since $H^2(Y,\mathbb{Z}_2) = 0$, the second Stiefel–Whitney class of $[\rho] \in \mathcal{R}(Y,\mathrm{SO}(3))$ vanishes, and ρ lifts to an $\mathrm{SU}(2)$–representation. By Proposition 4.2, this lift is unique since $H^1(Y,\mathbb{Z}_2) = 0$. □

If $[\rho] \in \mathcal{R}(Y \setminus K,\mathrm{SU}(2))$ satisfies

$$\rho(\mu_K) \sim \begin{bmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha}, \end{bmatrix}$$

then the induced $\mathrm{SO}(3)$–representation satisfies

$$\mathrm{Ad}\,\rho(\mu_K) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(4\pi\alpha) & -\sin(4\pi\alpha) \\ 0 & \sin(4\pi\alpha) & \cos(4\pi\alpha) \end{bmatrix}. \tag{4-1}$$

Let $\mathcal{R}_\alpha(Y \setminus K,\mathrm{SO}(3))$ be a subset of $\mathcal{R}(Y \setminus K,\mathrm{SO}(3))$ whose elements are represented by $\mathrm{SO}(3)$–representations of $\pi_1(Y \setminus K)$ such that their images of μ_K are conjugate to the right-hand side of (4-1).

Before proceeding with the argument, we introduce the orbifold fundamental group of $Y \setminus K$. (It appears in [2; 3], for example.)

Definition 4.4 The orbifold fundamental group of $Y \setminus K$ is $\pi_1^V(Y,K;r) := \pi_1(Y \setminus K)/\langle \mu_K^r \rangle$.

Proposition 4.5 *The orbifold fundamental group $\pi_1^V(Y,K;r)$ admits the split short exact sequence*

$$1 \to \pi_1(\widetilde{Y}_r) \to \pi_1^V(Y,K;r) \to \mathbb{Z}/r \to 1.$$

Proof Let $\widetilde{K} \subset \widetilde{Y}_r$ be the branched locus. Then there is the exact sequence

$$1 \to \pi_1(\widetilde{Y}_r \setminus \widetilde{K}) \to \pi_1(Y \setminus K) \to \mathbb{Z}/r \to 1.$$

since $\widetilde{Y}_r \setminus \widetilde{K} \to Y \setminus K$ is a regular covering. Applying the van Kampen theorem to $\widetilde{Y}_r \setminus \operatorname{int} N(\widetilde{K}) \cup N(\widetilde{K})$, we have $\pi_1(\widetilde{Y}_r) = \pi_1(\widetilde{Y}_r \setminus \widetilde{K})/\langle \mu_{\widetilde{K}} \rangle$. Since $\pi_1(\widetilde{Y}_r \setminus \widetilde{K}) \to \pi_1(Y \setminus K)$ maps $\mu_{\widetilde{K}}$ to μ_K^r, this induces $1 \to \pi_1(\widetilde{Y}_r) \to \pi_1^V(Y, K; r)$. Since $\pi_1(Y \setminus K) \to \mathbb{Z}/r$ maps μ_K^r to 1, this induces $\pi_1^V(Y, K; r) \to \mathbb{Z}/r$ which sends μ_K to a generator of $\mathbb{Z}/r$. The spitting $\mathbb{Z}/r \to \pi_1^V(Y, K; r)$ sends a generator of $\mathbb{Z}/r$ to μ_K. □

Lemma 4.6 *There is a natural one-to-one correspondence*

$$\mathcal{R}^*(\pi_1^V(Y, K; r), \mathrm{SO}(3)) \cong \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)).$$

Proof Let $\rho\colon \pi_1(Y \setminus K) \to \mathrm{SO}(3)$ be a representation with $[\rho] \in \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3))$. Then it factors through $\pi_1^V(Y, K; r)$. Conversely, any representation $\sigma\colon \pi_1^V(Y, K; r) \to \mathrm{SO}(3)$ satisfies

$$\sigma(\mu_K) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(4\pi\alpha) & -\sin(4\pi\alpha) \\ 0 & \sin(4\pi\alpha) & \cos(4\pi\alpha) \end{bmatrix},$$

where $\alpha = l/(2r)$ for some $0 < l < r$. Thus σ defines the desired representation of $\pi_1(Y \setminus K)$. □

Proposition 4.7 *There is a bijection*

$$\bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)) \cong \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)).$$

Proof Since there is the natural one-to-one correspondence in Lemma 4.6, we only have to construct

$$\mathcal{R}^*(\pi_1^V(Y, K; r), \mathrm{SO}(3)) \xrightarrow{\cong} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)).$$

This is induced from $\pi_1(\widetilde{Y}_r) \xrightarrow{i} \pi_1^V(Y, K; r)$ in the short exact sequence in Proposition 4.5. We claim that if $\rho\colon \pi_1^V(Y, K; r) \to \mathrm{SO}(3)$ is irreducible then $\rho \circ i$ is also irreducible. Since $\widetilde{Y}_r$ is an integral homology sphere, any reducible SO(3)–representation of $\pi_1(\widetilde{Y}_r)$ is the trivial representation. If $\rho \circ i$ is trivial, then ρ factors through $\pi_1^V(Y, K; r)/i(\pi_1(\widetilde{Y}_r)) \cong \mathbb{Z}/r$, and hence is reducible. This is a contradiction.

We will construct the inverse correspondence of the above. Let σ be an SO(3)–representation of $\pi_1(\widetilde{Y}_r)$ which represents an element in $\mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3))$. Since the conjugacy class of σ is fixed by the induced action of τ, there is a matrix $A \in \mathrm{SO}(3)$ such that

$$\tau^*\sigma(u) = A\sigma(u)A^{-1}$$

for any $u \in \pi_1(\widetilde{Y}_r)$. A is uniquely determined since σ is irreducible and has the trivial stabilizer $\{1\}$ in SO(3), and A is conjugate to the matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi l/r) & -\sin(2\pi l/r) \\ 0 & \sin(2\pi l/r) & \cos(2\pi l/r) \end{bmatrix}. \tag{4-2}$$

Since τ has order r, we get the relation

$$\sigma(u) = A^r \sigma(u) A^{-r}$$

for any $u \in \pi_1(\widetilde{Y}_r)$, and we get $A^r = 1$ using the irreducibility of σ. Thus we can assign a unique order-r element $A_\sigma \in \mathrm{SO}(3)$ for each σ. Finally, we assign a representation

$$\bar{\sigma}\colon \pi_1^V(Y, K; r) \cong \pi_1(\widetilde{Y}_r) \rtimes \mathbb{Z}/r \to \mathrm{SO}(3), \quad (u, t^k) \mapsto \sigma(u) \cdot A_\sigma^k, \tag{4-3}$$

to a given representation σ, where $t \in \mathbb{Z}/r$ is a generator. This satisfies $\bar{\sigma}(\mu_K) = A_\sigma$. The above construction gives the inverse of $\mathcal{R}^*(\pi_1^V(Y, K; r), \mathrm{SO}(3)) \ni [\rho] \mapsto [\rho \circ i] \in \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3))$. □

We write

$$\Pi'\colon \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)) \xrightarrow{\cong} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3))$$

for the bijection constructed above.

Definition 4.8 Let K be a knot in an integral homology 3–sphere Y, and assume that $\widetilde{Y}_r$ is also an integral homology 3–sphere. Then $\Pi\colon \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \to \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2))$ is given by the following composition:

$$\bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \xrightarrow{\mathrm{Ad}} \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)) \xrightarrow{\Pi'} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)) \xrightarrow{\mathrm{Ad}^{-1}} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2)).$$

We call $\Pi([\rho])$ a lift of $[\rho]$.

An SU(2)–representation that factors through Pin(2) subgroups is called a binary dihedral representation. An SO(3)–representation that factors through $O(2)$ subgroups is called a dihedral representation. Note that $O(2)$ is embedded in SO(3) as

$$\begin{bmatrix} A & 0 \\ 0 & \det A \end{bmatrix} \in \mathrm{SO}(3),$$

where $A \in O(2)$. The adjoint representation of a binary dihedral representation is a dihedral representation. In the proof of Proposition 4.1, which is an important property of the lift Π, we use the following lemma:

Lemma 4.9 [39] *The fixed point set of the $H^1(Y \setminus K, \mathbb{Z}_2)$–action on $\mathcal{R}(Y \setminus K, \mathrm{SU}(2))$ consists of conjugacy classes of binary dihedral representations.*

Proof Let $[\rho] \in \mathcal{R}(Y \setminus K, \mathrm{SU}(2))$ be a fixed point of the action of $H^1(Y \setminus K, \mathbb{Z}_2)$. We regard this as a representation $\rho\colon \pi_1(Y \setminus K) \to \mathrm{SU}(2)$ such that there exists $A \in \mathrm{SU}(2)$ and $(\chi \cdot \rho)(u) = A\rho(u)A^{-1}$ for any $u \in \pi_1(Y \setminus K)$. Here $\chi \in H^1(Y \setminus K)$ is a generator. Since χ has order 2, $\rho(u) = A^2 \rho(u) A^{-2}$. If ρ is reducible then its image is contained in a circle in SU(2) and is a binary dihedral representation.

Assume that ρ is irreducible and consider two cases, $A^2 = 1$ and $A^2 = -1$. We regard SU(2) as the unit sphere in the quaternions. Then $\mathrm{Pin}(2) = S^1 \cup jS^1$. If $A^2 = 1$ then $A = \pm 1$ and $-\rho(u) = \rho(u)$ for some $u \in \pi_1(Y \setminus K)$. This cannot happen in SU(2). If $A^2 = -1$ then we can assume that $A = i$ after a conjugation, and then $\rho(u) = \pm i\rho(u)i^{-1}$ for any $u \in \pi_1(Y \setminus K)$. If $\rho(u) = i\rho(u)i^{-1}$ then $\rho(u) \in S^1 = \{a + bi\}$. If $\rho(u) = -i\rho(u)i^{-1}$ then $\rho(u) \in jS^1 = \{cj + dk\}$. Thus the image of ρ is contained in $S^1 \cup jS^1$. □

Lemma 4.10 *Let $r \in 2\mathbb{Z}$. If $\rho\colon \pi_1^V(Y, K; r) \to \mathrm{SO}(3)$ is a dihedral representation, then its pullback $\pi_1(\widetilde{Y}_r) \to \mathrm{SO}(3)$ by the orbifold exact sequence in Proposition 4.5 is a reducible representation.*

Proof Since ρ factors through $O(2)$, we have a representation $\rho'\colon \pi_1^V(Y, K; r) \to O(2)$. Composing with $\det\colon O(2) \to \mathbb{Z}/2$, we have a representation $\det\circ\rho'\colon \pi_1^V(Y, K; r) \to \mathbb{Z}/2$. Since $\det\circ\rho'$ factors through the abelianization $\pi_1^V(Y, K; r) = \pi_1(Y \setminus K)/\langle {\mu_K}^r\rangle \xrightarrow{\mathrm{Ab}} \mathbb{Z}[\mu_K]/\langle {\mu_K}^r\rangle$, we have the diagram

$$\begin{tikzcd}
\pi_1(\widetilde{Y}_r) \arrow[r, "i"] & \pi_1^V(Y, K; r) \arrow[r, "\rho'"] \arrow[dr, "\mathrm{Ab}"] & O(2) \arrow[r, "\det"] & \mathbb{Z}/2 \\
& & \mathbb{Z}/r \arrow[ur] &
\end{tikzcd}$$

where $\pi_1(\widetilde{Y}_r) \xrightarrow{i} \pi_1^V(Y, K; r)$ is the inclusion map in the orbifold exact sequence. By construction, Ab coincides with the map $\pi_1^V(Y, K; r) \to \mathbb{Z}/r$ in the orbifold exact sequence. Thus $\mathrm{Ab}\circ i$ is the trivial representation, and hence $\det\circ\rho'\circ i$ is also the trivial representation. This implies that the image of $\rho'\circ i$ is contained in SO(2). Thus $\rho\circ i\colon \pi_1(\widetilde{Y}_r) \to \mathrm{SO}(3)$ factors through SO(2), and this means that $\rho\circ i$ is reducible. □

The following proposition gives the proof of Proposition 4.1:

Proposition 4.11 *Let $K \subset Y$ be a knot in an integral homology 3–sphere whose r–fold cyclic branched covering $\widetilde{Y}_r$ is also an integral homology 3–sphere. For each $[\rho] \in \mathcal{R}^{*,\tau}(\widetilde{Y}, \mathrm{SU}(2))$, $\Pi^{-1}([\rho])$ consists of two elements which correspond to each other by the flip symmetry.*

Proof Applying Proposition 4.2 to the 3–manifold $Y \setminus K$, we have a bijection

$$\mathcal{R}(Y \setminus K, \mathrm{SU}(2))/H^1(Y \setminus K, \mathbb{Z}_2) \cong \mathcal{R}(Y \setminus K, \mathrm{SO}(3)). \tag{4-4}$$

Note that $\mathcal{R}(Y \setminus K, \mathrm{SO}(3)) = \mathcal{R}^0(Y \setminus K, \mathrm{SO}(3))$ since $H^2(Y \setminus K, \mathbb{Z}_2) = 0$. We restrict this correspondence to elements with holonomy parameter $\alpha = l/(2r)$ for $l = 1, \dots, r-1$. Note that $H^1(Y \setminus K, \mathbb{Z}_2)$ acts on $\mathcal{R}^*_\alpha(Y \setminus K, \mathrm{SU}(2)) \cup \mathcal{R}^*_{1/2-\alpha}(Y \setminus K, \mathrm{SU}(2))$ since the flip symmetry changes the holonomy parameter as $\alpha \mapsto \frac{1}{2} - \alpha$, and the bijection (4-4) is restricted to

$$\left[\bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2))\right]\Big/ H^1(Y \setminus K, \mathbb{Z}_2) \cong \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)). \tag{4-5}$$

Fix $[\rho] \in \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2))$. The composition

$$\bigsqcup_{0<l<r} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)) \xrightarrow{\Pi'} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)) \xrightarrow{\mathrm{Ad}^{-1}} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2))$$

is bijective. Thus we have a unique element $[\rho'] \in \bigsqcup_{0<l<r} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3))$ which corresponds to $[\rho]$. $\Pi^{-1}([\rho])$ is the inverse image of $[\rho']$ by the map

$$\bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \xrightarrow{\mathrm{Ad}} \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)).$$

Finally, we prove that $\mathrm{Ad}^{-1}([\rho'])$ consists of two elements. Let $[\sigma] \in \mathrm{Ad}^{-1}([\rho'])$ be an element contained in $\mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2))$. If r is odd then $l/(2r) \neq \frac{1}{2} - l/(2r)$, and thus $[\rho] \neq \chi[\rho]$ in $\bigsqcup_{0<l<r} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2))$ since ρ and $\chi\rho$ have different holonomy parameters, where χ is a generator of $H^1(Y \setminus K, \mathbb{Z}_2) \cong \mathbb{Z}_2$. This means that the $H^1(Y \setminus K, \mathbb{Z}_2)$–action on $\bigcup_{0<l<r} \mathcal{R}_{l/(2r)}(Y \setminus K, \mathrm{SU}(2))$ is free and $\mathrm{Ad}^{-1}([\rho])$ consists of two elements. If r is even and $2l \neq r$, then $\mathrm{Ad}^{-1}([\rho])$ consists of two elements by the same reason. If $2l = r$ then $H^1(Y \setminus K, \mathbb{Z}_2)$ acts on $\mathcal{R}^*_{1/4}(Y \setminus K, \mathrm{SU}(2))$. The fixed points of the $H^1(Y \setminus K, \mathbb{Z}_2)$–action on $\mathcal{R}^*_{1/4}(Y \setminus K, \mathrm{SU}(2))$ are binary dihedral representations by Lemma 4.9. We show that $\Pi^{-1}([\rho])$ does not contain a binary dihedral representation. Let $\sigma' \colon \pi_1(Y \setminus K) \to \mathrm{SU}(2)$ be a binary dihedral representation with holonomy parameter $\alpha = \frac{1}{4}$. Then $\mathrm{Ad}\,\sigma'$ defines a dihedral representation $\pi_1^V(Y, K; r) \to \mathrm{SO}(3)$. Then the induced representation $\mathrm{Ad}\,\sigma' \circ i : \pi_1(\widetilde{Y}_r) \to \mathrm{SO}(3)$ is reducible by Lemma 4.10 and its SU(2)–lift is also reducible. This means that $\Pi^{-1}([\rho])$ does not contain any binary dihedral representation. Thus $H^1(Y \setminus K, \mathbb{Z}_2)$ acts freely on $\Pi^{-1}([\rho])$, and hence $\Pi^{-1}([\rho])$ consists of two elements which are related by the flip symmetry. □

4.2 Nondegeneracy results

The purpose of this subsection is to associate the nondegeneracy property of the critical point set $\mathfrak{C}$ of the singular Chern–Simons functional and the transversality of the moduli space of irreducible flat connections $\mathcal{R}^*(Y \setminus K, \mathrm{SU}(2))$. Let us recall the setting of the gauge theory used in [22; 23] to deal with the "pillowcase picture" of perturbed flat connections. In this subsection, Y denotes a (general) oriented closed 3–manifold and K is a knot in Y. Let E be an SU(2)–bundle over $X = Y \setminus N(K)$. We fix a Riemannian metric on X. We introduce the space of SU(2)–connections over X and $\partial X = T^2$ as follows:

$$\mathcal{A}_X = L^2_2(X, \mathfrak{su}(2) \otimes \Lambda^1), \quad \mathcal{A}_{T^2} = L^2_{3/2}(T^2, \mathfrak{su}(2) \otimes \Lambda^1).$$

Here we fix a trivialization of the SU(2)–bundle over X and ∂X, and identify the trivial connection to zero elements in each functional space. We also introduce spaces of gauge transformations:

$$\mathcal{G}_X = \{g \in \mathrm{Aut}(E) \mid g \in L^2_3\}, \quad \mathcal{G}_{T^2} = \{g \in \mathrm{Aut}(E|_{T^2}) \mid g \in L^2_{5/2}\}.$$

The action of gauge transformations on connections and $\mathfrak{su}(2)$–valued p–forms are given in obvious ways. $\mathcal{G}_X$ and $\mathcal{G}_{T^2}$ have Banach Lie group structures and act smoothly on $\mathcal{A}_X$ and $\mathcal{A}_{T^2}$, respectively. A connection whose stabilizer of gauge transformations is $\{\pm 1\}$ is called irreducible. $\mathcal{A}^*_X$ denotes the subset of irreducible connections.

We introduce the following spaces of p–forms with boundary conditions:

$$\Omega^p_\nu(X,\mathfrak{su}(2)) = \{\omega \in \Omega^p(X,\mathfrak{su}(2)) \mid *\omega|_{\partial X} = 0\}, \quad \Omega^p_\tau(X,\mathfrak{su}(2)) = \{\omega \in \Omega^p(X,\mathfrak{su}(2)) \mid \omega|_{\partial X} = 0\}.$$

We define the L^2–inner product on $\Omega^p(X,\mathfrak{su}(2))$ by the formula

$$\langle a,b\rangle = -\int_X \operatorname{tr}(a \wedge *b).$$

For each $A \in \mathcal{A}_X$, the slice of the action of $\mathcal{G}_X$ on $\mathcal{A}_X$ is given by

$$X_A = A + \operatorname{Ker} d_A^* \cap L^2_2\Omega^1_\nu(X,\mathfrak{su}(2)).$$

For each flat connection $A \in \mathcal{A}$, the space of harmonic p–forms is given by

$$\mathcal{H}^p(X;\operatorname{ad} A) = \{\omega \in \Omega^p_\nu(X,\mathfrak{su}(2)) \mid d_A\omega = 0,\ d_A^*\omega = 0\},$$
$$\mathcal{H}^p(X,\partial X;\operatorname{ad} A) = \{\omega \in \Omega^p_\tau(X,\mathfrak{su}(2)) \mid d_A\omega = 0,\ d_A^*\omega = 0\}.$$

The holonomy perturbation h defines a compact perturbation term $V_h\colon \mathcal{A}_X \to \Omega^1(X,\mathfrak{su}(2))$ and a perturbed flat connection can be defined as a solution of the equation

$$*F_A + V_h = 0. \tag{4-6}$$

$\mathcal{R}^{*,h}(X,\mathrm{SU}(2))$ denotes gauge equivalence classes of irreducible solutions for (4-6). Consider the restriction map $r\colon \mathcal{R}^{*,h}(X,\mathrm{SU}(2)) \to \mathcal{R}(T^2,\mathrm{SU}(2))$. For a generic perturbation h, $\mathcal{R}^{*,h}(X,\mathrm{SU}(2))$ is a smooth 1–manifold. Moreover, the restriction map r is a smooth immersion of $\mathcal{R}^{*,h}(X,\mathrm{SU}(2))$ to the smooth part of the pillowcase. The detailed argument is contained in [22]. Put

$$S_\alpha := \{\rho \in \mathcal{R}(T^2,\mathrm{SU}(2)) \mid \operatorname{tr}\rho(\mu_K) = 2\cos(2\pi i\alpha)\}.$$

This is a vertical slice in the pillowcase. Note that $\mathcal{R}_\alpha(X,\mathrm{SU}(2)) = r^{-1}(S_\alpha) \cap \mathcal{R}(X,\mathrm{SU}(2))$ and we define $\mathcal{R}^{*,h}_\alpha(X,\mathrm{SU}(2)) := r^{-1}(S_\alpha) \cap \mathcal{R}^{*,h}(X,\mathrm{SU}(2))$.

Proposition 4.12 *Let $K \subset Y$ be a knot in a closed 3–manifold, and α be an arbitrary holonomy parameter in $\left(0,\frac{1}{2}\right)$. Assume that $[\rho] \in \mathcal{R}^*_\alpha(Y \setminus K,\mathrm{SU}(2))$ is a nondegenerate critical point. We also assume that the image of $\mathcal{R}^*(Y \setminus K,\mathrm{SU}(2))$ by the restriction map r is contained in the smooth part of the pillowcase. Then $\mathcal{R}^*(Y \setminus K,\mathrm{SU}(2))$ is smooth near $[\rho]$. Moreover, the restriction map $r\colon \mathcal{R}^*(Y \setminus K,\mathrm{SU}(2)) \to \mathcal{R}(T^2,\mathrm{SU}(2))$ is an immersion to the smooth part of the pillowcase at $[\rho]$.*

For the proof of Proposition 4.12, we need gauge theory on 3–manifolds with the boundary described above.

Lemma 4.13 *Let B_0 be an abelian $\mathrm{SU}(2)$–flat connection on a torus T^2. Then the $\mathfrak{su}(2)$–valued harmonic form $h \in \mathcal{H}^1(T^2;\operatorname{ad} B_0)$ has the diagonal form*

$$h = \begin{bmatrix} ai & 0 \\ 0 & -ai \end{bmatrix},$$

where $a \in \Omega^1(T^2)$.

Proof Since B_0 is an abelian flat connection, it defines a splitting of the SU(2)–bundle E over T^2 into $E = L \oplus L^*$, where L is the trivial line bundle. Then the adjoint bundle of E has a splitting $\mathfrak{g}_E = \underline{\mathbb{R}} \oplus L^{\otimes 2}$. Let $d_{B_0} = d \oplus d_C$ be the covariant derivative induced on $\Omega^p(T^2, \mathfrak{g}_E) = \Omega^p(T^2, \underline{\mathbb{R}}) \oplus \Omega^p(T^2, L^{\otimes 2})$. Any section $\omega \in \Omega^p(T^2, \mathfrak{g}_E)$ has the form

$$\omega = \begin{bmatrix} ai & b \\ -\bar{b} & -ai \end{bmatrix},$$

where $a \in \Omega^p(T^2)$ and $b \in \Omega^p(T^2, L^{\otimes 2})$. The space of harmonic forms $\mathcal{H}^p(T^2; \operatorname{ad} B_0)$ splits into $\mathcal{H}^p(T^2; \underline{\mathbb{R}}) \oplus \mathcal{H}^p(T^2; L^{\otimes 2})$ with respect to the decomposition of $(\Omega^p(T^2, \mathfrak{g}_E), d_{B_0})$. Let us compute $\mathcal{H}^1(T^2; \operatorname{ad} B_0)$ using $H^1(\pi_1(T^2); \operatorname{ad}\rho)$, where ρ is an abelian SU(2)–representation corresponding to B_0. Let μ and λ be canonical generators of $\pi_1(T^2)$. Then the space of 1–cocycles consists of the element $\gamma \colon \pi_1(T^2) \to \mathfrak{su}(2) \cong \mathbb{R}^3$ such that

$$(1 - \operatorname{Ad}_{\rho(\mu)})\gamma(\lambda) = (1 - \operatorname{Ad}_{\rho(\lambda)})\gamma(\mu),$$

since μ and λ commute. Let $F \colon \mathbb{R}^3 \oplus \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map given by

$$F(x_1, x_2) = (1 - A_\mu)x_1 - (1 - A_\lambda)x_2,$$

where $A_\mu := \operatorname{Ad}_{\rho(\mu)}$ and $A_\lambda := \operatorname{Ad}_{\rho(\lambda)}$. Since A_μ and A_λ are SO(3)–linear transformations acting on $\mathbb{R}^3$, they have 1–dimensional axes of rotation $\mathbb{R}_\mu$ and $\mathbb{R}_\lambda$, respectively. Let $\mathbb{C}_\mu$ and $\mathbb{C}_\lambda$ be their orthogonal complement spaces. Then $\operatorname{Im}(1 - A_\mu) = \mathbb{C}_\mu$ and $\operatorname{Im}(1 - A_\lambda) = \mathbb{C}_\lambda$, and hence F is surjective. Thus the space of 1–cocycles is isomorphic to $\mathbb{R}^4$. On the other hand, the space of 1–coboundaries is spanned by $\operatorname{Im}(1 - \operatorname{Ad}_{\rho(g)})$ for all $g \in \pi_1(T^2)$, and this is 2–dimensional since ρ is reducible. Thus $\mathcal{H}^1(T^2; \operatorname{ad} B_0) \cong H^1(\pi_1(T^2); \operatorname{ad}\rho) \cong \mathbb{R}^2$. Therefore $H^1(T^2; L^{\otimes 2})$ vanishes since $\mathcal{H}^1(T^2; \underline{\mathbb{R}}) \cong H^1(T^2; \mathbb{R}) \cong \mathbb{R}^2$. This means that if $\omega \in \Omega^1(T^2, \mathfrak{g}_E)$ is a harmonic form then $b = 0$. Thus $h \in \mathcal{H}^1(T^2; \operatorname{ad} B_0)$ has only diagonal components. □

Since B_0 is a reducible connection with $U(1)$–stabilizer, $\mathcal{H}^0(T^2; \operatorname{ad} B_0) = \operatorname{Ker} d_{B_0} \cong \mathbb{R}$. We fix a generator $\gamma_0 \in \mathcal{H}^0(T^2; \operatorname{ad} B_0)$.

Lemma 4.14 *There is a $\mathcal{G}_{T^2}$–invariant neighborhood N_{B_0} of $B_0 \in \mathcal{A}_{T^2}$ and $\mathcal{G}_{T^2}$–invariant map $\eta \colon N_{B_0} \to \Omega^0(T^2, \mathfrak{su}(2))$ such that*

(1) $\eta(B_0) = \gamma_0$,

(2) $\int_{T^2} \operatorname{tr}(F_B \wedge \eta(B)) = 0$ *for all* $B \in N_{B_0}$.

Proof Take a small neighborhood of B_0 in the slice of the action of $\mathcal{G}_{T^2}$ on $\mathcal{A}_{T^2}$ as

$$X_{B_0,\epsilon} = \{B_0 + b \mid b \in L^2_{3/2}\Omega^1(T^2, \mathfrak{su}(2)), d^*_{B_0} b = 0, \|b\|_{L^2_{3/2}} < \epsilon\},$$

where $\epsilon > 0$ is small enough. Firstly, we define an $\Omega^0(T^2, \mathfrak{su}(2))$–valued map η on the slice $X_{B_0,\epsilon}$ and then extend it to a gauge-invariant neighborhood. For $B = B_0 + b \in X_{B_0,\epsilon}$, define

$$\eta(B) := \gamma_0.$$

Then

$$\int_{T^2}\operatorname{tr}(F_B\wedge\gamma_0)=\int_{T^2}\operatorname{tr}((d_{B_0}b+b\wedge b)\wedge\gamma_0)=\int_{T^2}\operatorname{tr}(d_{B_0}b\wedge\gamma_0)+\int_{T^2}\operatorname{tr}(b\wedge b\wedge\gamma_0).$$

Using Stokes' theorem and the condition $d_{B_0}\gamma_0=0$,

$$\int_{T^2}\operatorname{tr}(d_{B_0}b\wedge\gamma_0)=\int_{T^2}d\operatorname{tr}(b\wedge\gamma_0)=0.$$

Thus

$$\int_{T^2}\operatorname{tr}(F_B\wedge\eta(B))=\int_{T^2}\operatorname{tr}(b\wedge b\wedge\gamma_0). \tag{4-7}$$

Since $d^*_{B_0}b=0$, we have $h\in\mathcal{H}^1(T^2;\operatorname{ad}B_0)$ and $\omega\in\Omega^2(T^2,\mathfrak{su}(2))$ such that

$$b=h+d^*_{B_0}\omega. \tag{4-8}$$

Using (4-7) and (4-8),

$$\begin{aligned}
&\int_{T^2}\operatorname{tr}(F_B\wedge\eta(B))\\
&\quad=\int_{T^2}\operatorname{tr}[(h+d^*_{B_0}\omega)\wedge(h+d^*_{B_0}\omega)\wedge\gamma_0]\\
&\quad=\int_{T^2}\operatorname{tr}(h\wedge h\wedge\gamma_0)+\int_{T^2}\operatorname{tr}(d^*_{B_0}\omega\wedge h\wedge\gamma_0)+\int_{T^2}\operatorname{tr}(h\wedge d^*_{B_0}\omega\wedge\gamma_0)+\int_{T^2}\operatorname{tr}(d^*_{B_0}\omega\wedge d^*_{B_0}\omega\wedge\gamma_0).
\end{aligned}$$

Note that,

$$\begin{aligned}
\int_{T^2}\operatorname{tr}(*d_{B_0}*\omega\wedge h\wedge\gamma_0)&=\int_{T^2}\operatorname{tr}(d_{B_0}*\omega\wedge *h\wedge\gamma_0)\\
&=-\int_{T^2}\operatorname{tr}(*\omega\wedge d_{B_0}*h\wedge\gamma_0)+\int_{T^2}\operatorname{tr}(*\omega\wedge *h\wedge d_{B_0}\gamma_0)=0.
\end{aligned}$$

Here we use Stokes' theorem at the second equality. Similarly,

$$\begin{aligned}
\int_{T^2}\operatorname{tr}(h\wedge d^*_{B_0}\omega\wedge\gamma_0)&=-\int_{T^2}\operatorname{tr}(d_{B_0}*h\wedge *\omega\wedge\gamma_0)-\int_{T^2}\operatorname{tr}(*h\wedge *\omega\wedge d_{B_0}\gamma_0)=0,\\
\int_{T^2}\operatorname{tr}(d^*_{B_0}\omega\wedge d^*_{B_0}\omega\wedge\gamma_0)&=-\int_{T^2}\operatorname{tr}(d^2_{B_0}*\omega\wedge *\omega\wedge\gamma_0)-\int_{T^2}\operatorname{tr}(d_{B_0}*\omega\wedge *\omega\wedge d_{B_0}\gamma_0)=0.
\end{aligned}$$

Hence

$$\int_{T^2}\operatorname{tr}(F_B\wedge\eta(B))=\int_{T^2}\operatorname{tr}(h\wedge h\wedge\gamma_0).$$

Since $\gamma_0\in\mathcal{H}^0(T^2;\operatorname{ad}B_0)$ is an element of the Lie algebra of the stabilizer of B_0, $\operatorname{Stab}(B_0)=U(1)$ and it has the pointwise form

$$\gamma_0(x)=\begin{bmatrix} ri & 0\\ 0 & -ri\end{bmatrix}\in\mathfrak{su}(2),$$

where $r\in\mathbb{R}$. Similarly, $h\in\mathcal{H}^1(T^2;\operatorname{ad}B_0)$ has the form

$$h(x)=\begin{bmatrix} ai & 0\\ 0 & -ai\end{bmatrix}$$

by Lemma 4.13. By the pointwise computation of $\operatorname{tr}(h\wedge h\wedge\gamma_0)$, we obtain

$$\operatorname{tr}(h\wedge h\wedge\gamma_0)(x)=\operatorname{tr}\left(\begin{bmatrix} ai0\\ 0-ai\end{bmatrix}\wedge\begin{bmatrix} ai0\\ 0-ai\end{bmatrix}\wedge\begin{bmatrix} ri0\\ 0-ri\end{bmatrix}\right)=\operatorname{tr}\begin{bmatrix} -ra\wedge ai0\\ 0ra\wedge ai\end{bmatrix}=0.$$

Thus $\eta\colon X_{B_0,\epsilon} \to \Omega^0(T^2, \mathfrak{su}(2))$ satisfies $\int_{T^2} \operatorname{tr}(F_B \wedge \eta(B)) = 0$ for all $B \in X_{B_0,\epsilon}$. Define $N_{B_0} := \mathcal{G}_{T^2} \cdot X_{B_0,\epsilon}$ and extend η to N_{B_0} in a gauge-equivariant way (ie $\eta(g^*(B)) = g^{-1}\eta(B)g$). □

Let A_0 be a flat irreducible SU(2)–connection on X. We can assume that $A_0|_{T^2}$ is a noncentral flat connection on T^2 by the assumption of Proposition 4.12, and we write $A_0|_{T^2} = B_0$. Let U_{A_0} be a gauge invariant neighborhood of A_0 in $\mathcal{A}^*_X$. We define $\tilde{\eta}\colon U_{A_0} \to \Omega^0(X, \mathfrak{su}(2))$ as a smooth extension of η which satisfies

$$\tilde{\eta}(A)|_{\partial X} = \eta(A|_{\partial X}).$$

Here we assume that the extension $\tilde{\eta}$ satisfies $d_{A_0}\tilde{\eta}(A_0) \in \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0)$; this is possible by the following lemma:

Lemma 4.15 *For $\eta \in \mathcal{H}^0(T^2; \operatorname{ad} B_0)$ there is an extension $\tilde{\eta}$ on X such that $d_{A_0}\tilde{\eta} \in \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0)$.*

Proof For $\eta \in \mathcal{H}^0(T^2, \operatorname{ad} B_0)$, we take an arbitrary smooth extension $\tilde{\eta}$ to X. Then

$$d_{A_0}\tilde{\eta} \in \operatorname{Ker} d_{A_0}|_{\Omega^1_\tau(X,\mathfrak{su}(2))} = d_{A_0}\Omega^0_\tau(X, \mathfrak{su}(2)) \oplus \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0).$$

Let $d_{A_0}\tilde{\xi}$ be the $d_{A_0}\Omega^0_\tau(X, \mathfrak{su}(2))$–component of $d_{A_0}\tilde{\eta}$. Then $d_{A_0}(\tilde{\eta} - \tilde{\xi}) \in \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0)$ with $(\tilde{\eta} - \tilde{\xi})|_{\partial X} = \eta$. Hence we can choose an extension $\tilde{\eta}$ of η as $d_{A_0}\tilde{\eta} \in \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0)$. □

We define a map

$$\Phi\colon U_{A_0} \times L^2_2\Omega^0_\tau(X, \mathfrak{su}(2)) \times \mathbb{R} \to L^2_1(X, \mathfrak{su}(2) \otimes \Lambda^1)$$

by $\Phi(A, \zeta, t) = *F_A + d_A\zeta + t\, d_A\tilde{\eta}(A)$. The linearized operator of Φ at $(A_0, 0, 0)$ has the form

$$D\Phi_{(A_0,0,0)}(a, \zeta, t) = *d_{A_0}a + d_{A_0}\zeta + t\, d_{A_0}\tilde{\eta}(A_0).$$

Coker $D\Phi_{(A_0,0,0)}$ is $\mathcal{H}^1(X, \partial X; \operatorname{ad}_{A_0}) \cap (d_{A_0}\tilde{\eta}(A_0))^\perp$ by the Hodge decomposition.

Lemma 4.16 $\Phi(A, \zeta, t) = 0$ *if only if* $F_A = 0$, $\zeta = 0$ *and* $t = 0$.

Proof Assume that $\Phi(A, \zeta, t) = 0$. Then

$$\|F_A\|^2_{L^2} = -\int_X \operatorname{tr}(F_A \wedge *F_A) = \int_X \operatorname{tr}(F_A \wedge d_A\zeta) + t\int_X \operatorname{tr}(F_A \wedge d_A\tilde{\eta}(A)).$$

Using Stokes' theorem and the Bianchi identity,

$$\int_X \operatorname{tr}(F_A \wedge d_A\zeta) = \int_X d\operatorname{tr}(F_A \wedge \zeta) - \int_X \operatorname{tr}(d_A F_A \wedge \zeta) = \int_{T^2} \operatorname{tr}(F_{A|_{T^2}} \wedge \zeta|_{T^2}).$$

The last term vanishes by the boundary condition on ζ. Consider the remaining term

$$\int_X \operatorname{tr}(F_A \wedge d_A\tilde{\eta}(A)). \tag{4-9}$$

Using Stokes' theorem and the Bianchi identity, this is equal to

$$\int_{T^2} \operatorname{tr}(F_B \wedge \eta(B)),$$

where $B = A|_{T^2}$. By Lemma 4.14, this is equal to zero and we have $F_A = 0$. Thus $0 = *F_A = -d_A(\zeta + t\tilde{\eta}(A))$ by our assumption. Since A is an irreducible connection, d_A has trivial kernel and $\zeta = -t\tilde{\eta}(A)$. Restricting this to the boundary T^2, we have the relation $t\eta(B) = 0$. Since $\eta(B) = \gamma_0$ is a generator of $\mathcal{H}^0(T^2; \operatorname{ad} B_0)$, we have $\eta(B) \neq 0$. Hence $t = 0$ and $\zeta = 0$ follow.

Conversely, if we assume that $F_A = 0$, $\zeta = 0$ and $t = 0$, then clearly $\Phi(A, \zeta, t) = 0$. □

Lemma 4.16 means that the two equations $F_A = 0$ and $\Phi(A, \zeta, t) = 0$ have the same zero set near an irreducible flat connection A_0. Hence $\Phi = 0$ defines the space of flat connections near A_0.

Proof of Proposition 4.12 Natural embeddings $\mu_K \hookrightarrow \partial X \hookrightarrow X$ induce maps on cohomology groups with a local coefficient system,

(4-10)
$$H^1(X; \operatorname{ad}\rho) \xrightarrow{j} H^1(\partial X; \operatorname{ad}\rho) \to H^1(\mu_K; \operatorname{ad}\rho).$$

The nondegeneracy condition on $[\rho]$ is equivalent to the condition that the composition (4-10) is injective by Proposition 2.3. This implies that j is also injective. Thus the restriction map $\mathcal{R}^*(X, \mathrm{SU}(2)) \to \mathcal{R}(T^2, \mathrm{SU}(2))$ to the pillowcase is an immersion at $[\rho]$ if we show that $\mathcal{R}^*(X, \mathrm{SU}(2))$ has a smooth manifold structure near $[\rho]$. Next, we show that $\mathcal{R}^*(X, \mathrm{SU}(2))$ is a smooth manifold near $[\rho]$. Consider the long exact sequence of cohomology with local coefficient associated to the pair $(X, \partial X)$,

$$\cdots \to H^0(\partial X; \operatorname{ad}\rho) \xrightarrow{\partial} H^1(X, \partial X; \operatorname{ad}\rho) \to H^1(X; \operatorname{ad}\rho) \xrightarrow{j} H^1(\partial X; \operatorname{ad}\rho) \to \cdots.$$

The cokernel of the connecting homomorphism ∂ is zero since j is injective. Using the harmonic representative of the cohomology with local coefficient, the connecting homomorphism ∂ is given by $\gamma_0 \mapsto d_{A_0}\tilde{\eta}(A_0)$, where A_0 is an SU(2)–flat connection corresponding to ρ. Thus $\operatorname{Coker}\partial = \mathcal{H}^1(X, \partial X; \operatorname{ad}_{A_0}) \cap (d_{A_0}\tilde{\eta}(A_0))^{\perp} = 0$. This means that the equation $\Phi(A, \zeta, t) = 0$ has a surjective linearization map at $(A_0, 0, 0)$. Thus there is a neighborhood V_{A_0} of $A_0 \in \Phi^{-1}(0)$ which has a smooth structure by the implicit function theorem. Since A_0 is irreducible, the quotient singularity by gauge transformations $\mathcal{G}_X$ does not occur. Thus $\mathcal{R}^*(X, \mathrm{SU}(2))$ has a smooth manifold structure near $[\rho]$. □

By Proposition 2.3, the following shows that the singular Chern–Simons functional for a (p, q)–torus knot has nondegenerate irreducible critical points without perturbations:

Proposition 4.17 *For any* $[\rho] \in \mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$, *the natural map*

(4-11)
$$H^1(S^3 \setminus T_{p,q}; \operatorname{ad}\rho) \to H^1(\mu_{T_{p,q}}; \operatorname{ad}\rho)$$

is injective.

Proof Firstly, we compute $H^1(S^3 \setminus T_{p,q}; \operatorname{ad}\rho)$ using the group cohomology of $\pi_1(Y \setminus K)$. Since the fundamental group $\pi_1(S^3 \setminus T_{p,q})$ has a presentation $\langle x, y \mid x^p = y^q \rangle$, 1–cocycles $\gamma : \pi_1(S^3 \setminus T_{p,q}) \to \mathfrak{su}(2)$ satisfy the relation

$$(I + A_x + \cdots + A_x^{p-1})\gamma(x) = (I + A_y + \cdots + A_y^{q-1})\gamma(y)$$

where $A_x := \mathrm{Ad}_{\rho(x)}$ and $A_y := \mathrm{Ad}_{\rho(y)}$. Since A_x and A_y are SO(3)–linear transformation acting on $\mathbb{R}^3$, they have 1–dimensional axis of rotation $\mathbb{R}_x$ and $\mathbb{R}_y$, respectively. Let $\mathbb{C}_x$ and $\mathbb{C}_y$ denote their orthogonal complement spaces. Then $\mathrm{Im}(I - A_x) = \mathbb{C}_x$ and $\mathrm{Im}(I - A_y) = \mathbb{C}_y$. Note that $\rho(x)$ and $\rho(y)$ are contained in different great circles in $\mathrm{SU}(2) \cong S^3$ since ρ is an irreducible SU(2)–representation. Thus ρ satisfies $\rho(x)^p = \rho(y)^q = \pm 1$ and hence $A_x^p = A_y^q = I$. Thus $\mathrm{Ker}(I + A_x + \cdots + A_x^{p-1}) = \mathbb{C}_x$ and $\mathrm{Ker}(I + A_y + \cdots + A_y^{q-1}) = \mathbb{C}_y$. Since ρ is irreducible, $\mathbb{R}_x$ and $\mathbb{R}_y$ are independent in $\mathbb{R}^3$. Consider a linear map $L\colon \mathbb{R}^3 \oplus \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L(x_1, x_2) = (I + A_x + \cdots + A_x^{p-1})x_1 - (I + A_y + \cdots + A_y^{q-1})x_2.$$

This has rank 2, and the space of 1–cocycles has dimension 4. On the other hand, the space of 1–coboundaries is a subspace of $\mathbb{R}^3$ spanned by $\mathrm{Im}(I - \mathrm{Ad}_{\rho(g)})$ for all $g \in \pi_1(S^3 \setminus T_{p,q})$, and this coincides with $\mathbb{R}^3$ itself. Therefore $H^1(S^3 \setminus T_{p,q}; \mathrm{ad}\,\rho) \cong \mathbb{R}^4/\mathbb{R}^3 \cong \mathbb{R}$.

Next we compute $H^1(\mu_{T_{p,q}}; \mathrm{ad}\,\rho)$. Here the space of 1–cocycles is isomorphic to $\mathbb{R}^3$ since its elements are determined by choosing $\gamma(\mu) \in \mathfrak{su}(2) \cong \mathbb{R}^3$. The space of 1–coboundaries is $\mathrm{Im}(I - \mathrm{Ad}_{\rho(\mu)}) \cong \mathbb{C}$. Thus $H^1(\mu; \mathrm{ad}\,\rho) \cong \mathbb{R}^3/\mathbb{C} \cong \mathbb{R}$.

Finally, we prove that the map (4-11) is surjective. If $\gamma : \pi_1(S^3 \setminus T_{p,q}) \to \mathfrak{su}(2)$ represents a nonzero element in $H^1(S^3 \setminus T_{p,q}; \mathrm{ad}\,\rho)$ then $\gamma(g) \notin \mathrm{Im}(I - \mathrm{Ad}_{\rho(g)})$ for any $g \in \pi_1(S^3 \setminus T_{p,q})$. Thus $\gamma(\mu) \notin \mathrm{Im}(I - \mathrm{Ad}_{\rho(\mu)})$ for the meridian $\mu \in \pi_1(S^3 \setminus T_{p,q})$, and this means that the image of $[\gamma] \in H^1(S^3 \setminus T_{p,q}; \mathrm{ad}\,\rho)$ in $H^1(\mu_{T_{p,q}}; \mathrm{ad}\,\rho)$ is a nonzero element. □

Consider a knot K in S^3. Note that the image of the restriction map $r : \mathcal{R}^*(S^3 \setminus K, \mathrm{SU}(2)) \to \mathcal{R}(T^2, \mathrm{SU}(2))$ is contained in the smooth part of the pillowcase. By Propositions 4.17 and 4.12 we get the following statement:

Corollary 4.18 *The natural restriction map* $\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \to \mathcal{R}(T^2, \mathrm{SU}(2))$ *to the smooth part of the pillowcase is an immersion of a smooth* 1*–manifold.*

In fact, it is known that the irreducible representation variety $\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ is a disjoint union of $\frac{1}{2}(p-1)(q-1)$ segments; see [24].

4.3 Levine–Tristram signature and representation variety

The following statement relates the size of the set of singular flat connections over $S^3 \setminus T_{p,q}$ and the set of flat connections over the cyclic branched covering.

Lemma 4.19 *Let* p, q *and* r *be relatively coprime positive integers and* $\Sigma(p,q,r)$ *be a Brieskorn homology sphere. Then*

$$2|\mathcal{R}^*(\Sigma(p,q,r), \mathrm{SU}(2))| = \sum_{l=1}^{r-1} |\mathcal{R}^*_{l/(2r)}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|.$$

Proof We apply Proposition 4.1 to the r–fold cyclic branched covering $\Sigma(p,q,r)$ of $T_{p,q} \subset S^3$. Note that the covering transformation τ induces the trivial action on $\mathcal{R}^*(\Sigma(p,q,r),\mathrm{SU}(2))$ by [2], and hence there is a two-to-one correspondence

$$\bigsqcup_{l} \mathcal{R}^*_{l/(2r)}(S^3 \setminus T_{p,q},\mathrm{SU}(2)) \overset{\cong}{\Rightarrow} \mathcal{R}^*(\Sigma(p,q,r),\mathrm{SU}(2)). \qquad \square$$

The nondegeneracy condition at irreducible critical points can be interpreted in the pillowcase as follows:

Lemma 4.20 *Let $\alpha \in \left(0,\frac{1}{2}\right)$ be a fixed holonomy parameter. Assume that $[\rho] \in \mathcal{R}^*_\alpha(S^3 \setminus K,\mathrm{SU}(2))$ is nondegenerate. Then S_α and the image of $\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2))$ by the restriction map*

$$r\colon \mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2)) \to \mathcal{R}(T^2,\mathrm{SU}(2))$$

intersect transversely at $r([\rho])$.

Proof Consider the natural map $p\colon \mathcal{R}(T^2,\mathrm{SU}(2)) \to \mathcal{R}(\mu_K,\mathrm{SU}(2))$ induced from the embedding $\mu_K \hookrightarrow T^2$. Let $[\sigma] \in \mathcal{R}(\mu_K,\mathrm{SU}(2))$ be an element such that $\mathrm{tr}(\sigma(\mu_K)) = 2\cos(2\alpha\pi)$. Then $p^{-1}([\sigma]) = S_\alpha \subset \mathcal{R}(T^2,\mathrm{SU}(2))$ by definition. Note that S_α is contained in the smooth part of the pillowcase, and $[\sigma]$ is also contained in the smooth part of $\mathcal{R}(\mu_K,\mathrm{SU}(2))$ since the quotient singularity by the conjugacy action of SU(2) does not happen when $\alpha \neq 0, \frac{1}{2}$. Thus the kernel of the map

$$dp_{[\sigma']}\colon T_{[\sigma']}\mathcal{R}(T^2,\mathrm{SU}(2)) = H^1(T^2;\mathrm{ad}\,\rho) \to T_{[\sigma]}\mathcal{R}(\mu_K,\mathrm{SU}(2)) = H^1(\mu_K;\mathrm{ad}\,\rho)$$

induced on their tangent spaces is $T_{[\sigma]}S_\alpha$, where $[\sigma'] = r([\rho])$. Note that $\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2))$ is smooth near $[\rho]$ by Proposition 4.12. The composition of the natural maps

$$T_{[\rho]}\mathcal{R}(S^3 \setminus K,\mathrm{SU}(2)) = H^1(S^3 \setminus K;\mathrm{ad}\,\rho) \xrightarrow{j} H^1(T^2;\mathrm{ad}\,\rho) \to H^1(\mu_K;\mathrm{ad}\,\rho) \tag{4-12}$$

is injective by our nondegeneracy assumption. Thus the image of $H^1(S^3 \setminus K;\mathrm{ad}\,\rho)$ in $H^1(T^2;\mathrm{ad}\,\rho)$ is independent of $\mathrm{Ker}(H^1(T^2;\mathrm{ad}\,\rho) \to H^1(\mu_K;\mathrm{ad}\,\rho))$. This means that $r(\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2)))$ and S_α intersect transversely at $r([\rho])$. $\square$

There is a relation between $\sigma_\alpha(K)$ and $\mathcal{R}^*_\alpha(S^3 \setminus K,\mathrm{SU}(2))$. We use the following inequality in the proof of Proposition 4.22:

Lemma 4.21 *Let K be a knot in S^3. Assume that $\mathcal{R}^*_\alpha(S^3 \setminus K,\mathrm{SU}(2))$ is nondegenerate. Then*

$$|\sigma_\alpha(K)| \le 2|\mathcal{R}^*_\alpha(S^3 \setminus K,\mathrm{SU}(2))|$$

for $\alpha \in \left[0,\frac{1}{2}\right]$ with $\Delta_K(e^{4\pi i\alpha}) \neq 0$.

Proof By Proposition 4.12, $\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2)) \to \mathcal{R}(T^2,\mathrm{SU}(2))$ is an immersion to the smooth part of the pillowcase. By Proposition 4.17 and Lemma 4.20, the immersed image of $\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2))$ intersects transversely to S_α. After taking a small perturbation, the image of $\mathcal{R}^{*,h}(S^3 \setminus K,\mathrm{SU}(2))$ intersects to S_α transversely and the number of intersection points do not change,

$$|\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2)) \cap r^{-1}(S_\alpha)| = |\mathcal{R}^{*,h}(S^3 \setminus K,\mathrm{SU}(2)) \cap r^{-1}(S_\alpha)|. \tag{4-13}$$

If $\Delta_K(e^{4\pi i\alpha}) \neq 0$ and the perturbation h is chosen so that it satisfies the conditions in [23, Lemma 5.1] then the signed counting $\#\mathcal{R}^{*,h}_\alpha(S^3 \setminus K, \mathrm{SU}(2)))$ can be defined, and

$$\#\mathcal{R}^{*,h}_\alpha(S^3 \setminus K, \mathrm{SU}(2)) = -\tfrac{1}{2}\sigma_\alpha(K)$$

holds by [23, Corollary 0.2]. On the other hand, the left side of (4-13) is just the size of the set $\mathcal{R}^*_\alpha(S^3 \setminus K, \mathrm{SU}(2))$ by definition. □

Since $K = T_{p,q}$ satisfies the assumption of Lemma 4.21, $|\sigma_\alpha(T_{p,q})| \leq 2|\mathcal{R}^*_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|$ holds for $\alpha \in \left[0, \frac{1}{2}\right]$ with $\Delta_{T_{p,q}}(e^{4\pi i\alpha}) \neq 0$.

Proposition 4.22 *Let p, q and r be positive and relatively coprime integers. The formula*

$$-\tfrac{1}{2}\sigma_{l/(2r)}(T_{p,q}) = |\mathcal{R}^*_{l/(2r)}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|$$

holds for $1 \leq l \leq r-1$ with $\Delta_{T_{p,q}}(e^{2\pi i l/r}) \neq 0$.

For the proof we use the similar argument as in the proof of [2, Theorem 3.4].

Proof Consider a 4–ball B^4 and a torus knot in its boundary $T_{p,q} \subset S^3 = \partial B^4$, and take a Seifert surface S for $T_{p,q}$ as $S \subset B^4$ and $S \cap \partial B^4 = \partial S$. The r–fold cyclic branched covering of B^4 branched along S is the Milnor fiber

$$M(p,q,r) = \{(z_1, z_2, z_3) \mid z_1^p + z_2^q + z_3^r = \epsilon\} \cap B^6 \subset \mathbb{C}^3,$$

where $\epsilon > 0$ is small enough. Furthermore, $\partial M(p,q,r) = \Sigma(p,q,r)$ is an r–fold cyclic branched covering of $\partial B^4 = S^3$, branched along $T_{p,q}$. There is the following formula (see [13, Corollary 2.9]):

$$-\tfrac{1}{4}\sigma(M(p,q,r)) = |\mathcal{R}^*(\Sigma(p,q,r), \mathrm{SU}(2))|.$$

Using the signature formula in [41], Lemma 4.19 and decomposition of $\sigma(M(p,q,r))$ into the equivariant signature $\sigma(M(p,q,r); \frac{i}{r})$, we have

$$-\frac{1}{2}\sum_{l=1}^{r-1}\sigma_{l/(2r)}(T_{p,q}) = \sum_{l=1}^{r-1}|\mathcal{R}^*_{l/(2r)}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|.$$

Note that $\sigma_{l/(2r)}(T_{p,q}) \leq 0$ since $T_{p,q}$ is a positive knot. If we assume that the inequality in Lemma 4.21 is strict for some l, then $-\frac{1}{4}\sigma(M(p,q,r)) < |\mathcal{R}^*(\Sigma(p,q,r))|$, and this is a contradiction. □

Proof of Theorem 1.9 When $\alpha = 0$ or $\frac{1}{2}$, $\sigma_{T_{p,q}} = 0$ and $\mathcal{R}^*_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ is empty. So we consider the case $\alpha \in \left(0, \frac{1}{2}\right)$ with $\Delta_{T_{p,q}}(e^{4\pi i\alpha}) \neq 0$. Since the image of $\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ in the pillowcase intersects S_α transversely, there is a small $\epsilon > 0$ such that for any $\alpha' \in (\alpha - \epsilon, \alpha + \epsilon)$ we have $\Delta_{T_{p,q}}(e^{4\pi i\alpha}) \neq 0$ and

$$|\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \cap r^{-1}(S_\alpha)| = |\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \cap r^{-1}(S_{\alpha'})|.$$

Thus $|\mathcal{R}^*_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2))| = |\mathcal{R}^*_{\alpha'}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|$. The Levine–Tristram signature is piecewise constant and jumps at the roots of the Alexander polynomial. Hence $\sigma_{T_{p,q}}(e^{4\pi i\alpha}) = \sigma_{T_{p,q}}(e^{4\pi i\alpha'})$ if $\epsilon > 0$ is small enough. We can find a positive integer r which is coprime to p and q, and a positive integer l such that $l/(2r) \in (\alpha - \epsilon, \alpha + \epsilon)$. Then

$$-\tfrac{1}{2}\sigma_{l/(2r)}(T_{p,q}) = |\mathcal{R}^*_{l/(2r)}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|$$

by Proposition 4.22. Thus we have

$$-\tfrac{1}{2}\sigma_\alpha(T_{p,q}) = |\mathcal{R}^*_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|. \qquad \square$$

5 Properties of instanton knot invariants and their applications

In this section, we give the proof of Theorem 1.1, our main theorem. The important consequence of Section 5.1 is that the Floer chain $C^\alpha_*(T_{p,q}; \Delta_{\mathscr{S}})$ is supported only on the odd graded part. The key argument is that the Frøyshov invariant of a knot $K \subset S^3$ for an appropriate choice of coefficient $\mathscr{S}$ reduces to the Levine–Tristram signature. This is a generalization of the corresponding result in [8] and the argument is parallel. Section 5.2 gives the proof of Theorem 1.1 using this specific property of the Floer chain complex $\widetilde{C}^\alpha_*(T_{p,q}; \Delta_{\mathscr{S}})$ and the Frøyshov knot invariant.

5.1 The Frøyshov knot invariant and the structure theorem

Let W be a compact oriented smooth 4–manifold with $b^1(W) = b^+(W) = 0$, whose boundary $\partial W = Y$ is an integral homology 3–sphere. Let $K \subset Y$ be an oriented knot and $S \subset W$ be an embedded oriented surface with $\partial S = K$. Throughout this subsection, we assume that $\mathscr{S}$ is an integral domain over $\mathscr{R}_\alpha$. We define

$$K(A) := \kappa(A) + \big(\alpha - \tfrac{1}{4}\big)\nu(A) + \alpha^2 S \cdot S \quad \text{and} \quad d^\alpha(W, S) := 4K(A_{\min}) - g(S) - \tfrac{1}{2}\sigma_\alpha(Y, K) - 1$$

for each holonomy parameter $\alpha \in \big(0, \frac{1}{2}\big) \cap \mathbb{Q}$. Here $A_{\min}$ is a minimal reducible, and note that $K(A_{\min})$ is independent of the choice of minimal reducibles. Moreover $d^\alpha(W, S)$ is an integer by the index theorem. The value of the Frøyshov knot invariant is evaluated by the following proposition:

Proposition 5.1 *Let (W, S) and (Y, K) be as above and $\alpha \in \big(0, \frac{1}{2}\big) \cap \mathbb{Q}$ satisfy $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$. If $d := d^\alpha(W, S) \geq 0$ then there is a cycle $c^\alpha(W, S) \in C^\alpha_{2d+1}(Y, K; \Delta_{\mathscr{S}})$ satisfying*

$$\delta_1 v^j(c^\alpha(W, S)) = \begin{cases} 0 & \text{if } 0 \leq j < d, \\ \eta^\alpha(W, S) & \text{if } j = d. \end{cases}$$

Proof We define $c^\alpha(W, S) \in C^\alpha_{2d+1}(Y, K; \Delta_{\mathscr{S}})$ by

$$\langle c^\alpha(W, S), \beta\rangle = \sum_{[A] \in M(W,S;\beta)_0} \epsilon(A)\lambda^{\kappa_0 - \kappa(A)} T^{\nu(A) - \nu_0},$$

where $M(W, S, \beta)_0$ is a zero-dimensional moduli space. Since $dc^\alpha(W, S)$ corresponds to the counting of the boundary of the 1–dimensional moduli space $M^\alpha(W, S; \beta')^+_1$, we have $dc^\alpha(W, S) = 0$. Let $M(W, S, \theta_\alpha)_{2d+1}$ be the moduli space of instantons A over (W, S) which are asymptotic to θ_α and satisfy $\kappa(A) = \kappa(A_{\min})$. If $d \geq 0$ then we can perturb the ASD equation so that each reducible connection in $M_z(W, S; \theta_\alpha)_{2d+1}$ has a neighborhood which is homeomorphic to the cone of $\mathbb{CP}^d$. Removing small $(2d+1)$–balls of each reducible point from $M_z(W, S; \theta_\alpha)_{2d+1}$, we get a $(2d+1)$–manifold M'_z whose boundary is $\bigsqcup(\pm\mathbb{CP}^d)$, where the sign ± is determined by the orientation of each reducible point. Cutting down M'_z by codimension 2 divisors $\{V_i\}_{1\leq i\leq d}$ associated to d points in S, $M'_z \cap V_1 \cap \cdots \cap V_d$ is a 1–manifold with boundary. Then $\sum_z \#\partial(M'_z \cap V_1 \cap \cdots \cap V_d)\lambda^{\kappa_0-\kappa(z)}T^{\nu(z)-\nu_0} = \eta^\alpha(W, S)$. On the other hand, $M'_z \cap V_1 \cap \cdots \cap V_d$ has ends arising from the sliding end of instantons. Define $\psi \in C_1(Y, K; \Delta_{\mathscr{S}})$ by

$$\langle \beta, \psi \rangle = \sum_z \sum_{[A]} \epsilon([A])\lambda^{\kappa_0-\kappa(A)}T^{\nu(A)-\nu_0},$$

where $[A]$ runs through all elements in $\#(M_z(W, S; \beta)_{2d} \cap V_1 \cap \cdots \cap V_d)$ for each z. Then ψ and $v^d c^\alpha(W, S)$ are homologous. Since $\delta_1\psi = \eta^\alpha(W, S)$, we have $\delta_1 v^d c^\alpha(W, S) = \eta^\alpha(W, S)$. If $j < d$, $M_z(W, S; \theta_\alpha)_{2j+1}$ does not contain reducible points and we have $\delta_1 v^j c^\alpha(W, S) = 0$. □

Before the proof of Theorem 1.8, we state Lemma 5.2 and Proposition 5.3 related to two-bridge torus knots.

Lemma 5.2 *For any* $\alpha \in \left(0, \frac{1}{2}\right)$ *there exists an integer* $k > 0$ *such that* $\sigma_\alpha(T_{2,2k+1}) = -2$ *and* $\Delta_{T_{2,2k+1}}(e^{4\pi i\alpha}) \neq 0$.

Proof Consider the case $\alpha \leq \frac{1}{4}$. By [34, Proposition 1], $\sigma_\alpha(T_{2,2k+1})$ is given by

$$\sigma_\alpha(T_{2,2k+1}) = n_1 - n_2,$$

where n_1 is the number of lattice points $\left\{(1, m) \mid \left(k + \frac{1}{2}\right)(1 + 4\alpha) < m < 2k + 1\right\}$ and n_2 is the number of lattice points $\left\{(1, m) \mid 0 < m < \left(k + \frac{1}{2}\right)(1 + 4\alpha)\right\}$. Thus $\sigma_\alpha(T_{2,2k+1}) = -2$ if only if $1/(8k + 4) < \alpha < 3/(8k + 4)$. Moreover, note that the interval $(1/(8k + 4), 3/(8k + 4))$ does not contain any root of $\Delta_{T_{2,2k+1}}(t)$. Thus, for any $\alpha \leq \frac{1}{4}$, we can find $k > 0$ such that $\sigma_\alpha(T_{2,2k+1}) = -2$ and $\Delta_{T_{2,2k+1}}(e^{4\pi i\alpha}) \neq 0$. For the case $\alpha > \frac{1}{4}$, it follows that $\sigma_\alpha(T_{2,2k+1}) = -2$ if only if $\frac{1}{2} - 3/(8k + 4) < \alpha < \frac{1}{2} - 1/(8k + 4)$ by the flip symmetry. □

Proposition 5.3 *For any* $\alpha \in \mathbb{Q} \cap \left(0, \frac{1}{2}\right)$, *there is an integer* $k > 0$ *such that* $\Delta_{T_{2,2k+1}}(e^{4\pi i\alpha}) \neq 0$ *and* $h^\alpha_{\mathscr{S}}(T_{2,2k+1}) = 1$.

Proof By Lemma 5.2, we can find an integer $k > 0$ such that $\sigma_\alpha(T_{2,2k+1}) = -2$ and $\Delta_{T_{2,2k+1}}(e^{4\pi i\alpha}) \neq 0$. Consider a cobordism of pairs (W_k, S_k) obtained by the composition

$$(W_k, S_k)\colon (S^3, U) \to (S^3, T_{2,3}) \to \cdots \to (S^3, T_{2,2k-1}) \to (S^3, T_{2,2k+1}),$$

where $(S^3, T_{2,2i-1}) \to (S^3, T_{2,2i+1})$ is obtained by the crossing change of the knot. Put $(\overline{W}_k, \overline{S}_k) := (D^4, D^2) \cup_{(S^3,U)} (W_k, S_k)$. Then it is easy to see that $b^1(\overline{W}_k) = b^+(\overline{W}_k) = 0$ and $d^\alpha(\overline{W}_k, \overline{S}_k) = 0$ by the similar argument as in Proposition 2.23. Applying Proposition 5.1 to the pair $(\overline{W}_k, \overline{S}_k)$, we obtain a cycle $c^\alpha(\overline{W}_k, \overline{S}_k) \in C^\alpha_1(T_{2,2k+1})$ such that $\delta_1 c^\alpha(\overline{W}_k, \overline{S}_k) \neq 0$. This implies that $h^\alpha_{\mathscr{S}}(T_{2,2k+1}) \neq 0$. Since $\operatorname{rank} C^\alpha_*(T_{2,2n+1}) = 1$, we have $h^\alpha_{\mathscr{S}}(T_{2,2k+1}) = 1$. □

Proof of Theorem 1.8 Consider a knot $K \subset S^3$ and a holonomy parameter $\alpha \in \mathbb{Q} \cap \bigl(0, \frac{1}{2}\bigr)$ with $\Delta_K(e^{4\pi i\alpha}) \neq 0$. Since $K \subset S^3$ is homotopic to any knot, it can be deformed into $lT_{2,2n+1}$ by positive and negative crossing changes, where $l = -\frac{1}{2}\sigma_\alpha(K)$. This operation defines a cobordism of pairs $([0,1]\times S^3, S) \colon (S^3, K) \to (S^3, lT_{2,2n+1})$ where S is an immersed surface with normal self-intersection points. Let $S' \colon lT_{2,2n+1} \to K$ be the inverse cobordism of S. Since $\sigma_\alpha(K) = \sigma_\alpha(lT_{2,2n+1})$, two cobordisms S and S' induce negative definite cobordisms. Let $\widetilde{m}_S \colon \widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}}) \to \widetilde{C}^\alpha_*(lT_{2,2n+1}; \Delta_{\mathscr{S}})$ and $\widetilde{m}_{S'} \colon \widetilde{C}^\alpha_*(lT_{2,2n+1}; \Delta_{\mathscr{S}}) \to \widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}})$ be induced cobordism maps on $\mathcal{S}$–complexes. Since two immersed cobordisms $S' \circ S$ and $S \circ S'$ can be deformed into product cobordisms by finitely many finger moves, their induced maps $\widetilde{m}_{S' \circ S}$ and $\widetilde{m}_{S \circ S'}$ are S–chain homotopic to the identity up to the multiplication of unit elements by Proposition 3.28. By the functoriality of $\mathcal{S}$–morphisms, $\widetilde{m}_{S'} \circ \widetilde{m}_S$ and $\widetilde{m}_S \circ \widetilde{m}_S$ are $\mathcal{S}$–chain homotopic to the identity up to the multiplication of unit elements. The proof is completed by Remark 3.4. □

The proof of Theorem 1.7 immediately follows from Theorem 1.8:

Proof of Theorem 1.7 Comparing Frøyshov invariants for both sides of

$$\widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}}) \simeq C^\alpha_*(lT_{2,2k+1}; \Delta_{\mathscr{S}}),$$

we obtain $h^\alpha_{\mathscr{S}}(K) = l h^\alpha_{\mathscr{S}}(T_{2,2k+1})$ where $l = -\frac{1}{2}\sigma(K)$. Since $h^\alpha_{\mathscr{S}}(T_{2,2k+1}) = 1$ by Proposition 5.3, we obtain the desired formula. □

Remark 5.4 Since $\mathcal{S}$–chain homotopy equivalence of two $\mathcal{S}$–complexes $\widetilde{C}_* \simeq \widetilde{C}'_*$ implies chain homotopy equivalence between C_* and C'_*, assume that $\sigma_\alpha(K) \leq 0$. Then Theorems 1.5 and 1.8 imply the $\mathcal{S}$–chain homotopy equivalence $\widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}}) \simeq \widetilde{C}^\alpha_*(T_{2,2n+1}; \Delta_{\mathscr{S}})^{\otimes l}$, and hence we have the Euler characteristic formula

$$\chi(C^\alpha_*(K; \Delta_{\mathscr{S}})) = l\chi(C^\alpha_*(T_{2,2n+1}; \Delta_{\mathscr{S}})).$$

If $\sigma_\alpha(K) > 0$ then there is an $\mathcal{S}$–chain homotopy equivalence $\widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}}) \simeq \widetilde{C}^\alpha_*(-T_{2,2n+1}; \Delta_{\mathscr{S}})^{\otimes -l}$ and we have

$$\chi(C^\alpha_*(K; \Delta_{\mathscr{S}})) = -l\chi(C^\alpha_*(-T_{2,2n+1}; \Delta_{\mathscr{S}})).$$

By Proposition 5.3, $\chi(C^\alpha_*(T_{2,2n+1}; \Delta_{\mathscr{S}})) = -1$. On the other hand, $\chi(C^\alpha_*(-T_{2,2n+1}; \Delta_{\mathscr{S}})) = 1$ since if we reverse the orientation of the 3–manifold, the $\mathbb{Z}/4$–grading of the chain complex changes so that $\mathrm{gr}_{-Y}(\beta) \equiv 3 - \mathrm{gr}_Y(\beta)$, which follows from (2-4). In any case,

$$\chi(C^\alpha_*(K; \Delta_{\mathscr{S}})) = \tfrac{1}{2}\sigma_\alpha(K).$$

Note that this formula for the Euler characteristic is independent of the choice of the coefficient $\mathscr{S}$.

Proof of Theorem 1.10 Consider an arbitrary knot $K \subset S^3$. For any holonomy parameter $\alpha \in \bigl(0, \frac{1}{2}\bigr) \cap \mathbb{Q}$ with $\Delta_K(e^{4\pi i\alpha}) \neq 0$, the Floer chain complex $C^\alpha_*(K; \Delta_{\mathscr{S}})$ is defined and the relation $h^\alpha_{\mathscr{S}}(K) = -\frac{1}{2}\sigma_\alpha(K)$ holds. By the definition of the Frøyshov knot invariant, we have lower bounds of Floer homology groups

$$\operatorname{rank} I^\alpha_1(K; \Delta_{\mathscr{S}}) \geq \bigl\lceil -\tfrac{1}{4}\sigma_\alpha(K) \bigr\rceil \quad \text{and} \quad \operatorname{rank} I^\alpha_3(K; \Delta_{\mathscr{S}}) \geq \bigl\lfloor -\tfrac{1}{4}\sigma_\alpha(K) \bigr\rfloor$$

for any knot $K \subset S^3$ with $\sigma_\alpha(K) \leq 0$. In particular, $K = T_{p,q}$ satisfies this condition. Using the equality $\operatorname{rank} I_*(T_{p,q}) = -\frac{1}{2}\sigma_\alpha(T_{p,q})$, we obtain

$$\operatorname{rank} I_1^\alpha(T_{p,q};\Delta_{\mathscr{S}}) = \left\lceil -\tfrac{1}{4}\sigma_\alpha(T_{p,q})\right\rceil, \operatorname{rank} I_3^\alpha(T_{p,q};\Delta_{\mathscr{S}}) = \left\lfloor -\tfrac{1}{4}\sigma_\alpha(T_{p,q})\right\rfloor.$$

Since $I_*^\alpha(T_{p,q})$ is supported only on the odd graded part, we obtain the statement. □

5.2 An application to knot concordance

In this subsection, we complete the proof of our main theorem (Theorem 1.1).

The operators $Z^{\pm 1}$ and $U^{\pm 1}$ extend to the $\mathcal{S}$–complex $\widetilde{C}_*$ in the obvious way. We also introduce the operator

$$\mathcal{W}_{i,j,k} := \delta_1 v^i U^k Z^j : \widetilde{C}_* \to \widetilde{C}_*.$$

If $\deg_{\mathbb{R}}(Z\gamma) > \deg_{\mathbb{R}}(\gamma)$, the operator Z does not act on the filtered chain complex $C_*^{[-\infty,R]}$, and $\mathcal{W}_{i,j}$ does not directly induce a map on $\widetilde{C}_*^{[-\infty,R]}$. For this reason, we introduce the map $\mathcal{V}_{i,j,k}^{[-\infty,R]}$ on the filtered chain complex by the composition

$$\widetilde{C}_*^{[-\infty,R]} \hookrightarrow \widetilde{C}_*^{[-\infty,\infty]} \xrightarrow{\mathcal{W}_{i,j,k}} \widetilde{C}_*^{[-\infty,\infty]}.$$

We also introduce the operator $\mathcal{W}_{i,j,k}^{[R',R]}$ on the quotient filtered $\mathcal{S}$–complex $\widetilde{C}_*^{[R',R]}$ by the composition

$$\widetilde{C}_*^{[R',R]} \hookrightarrow \widetilde{C}_*^{[-\infty,\infty]} \xrightarrow{\mathcal{W}_{i,j,k}} \widetilde{C}_*^{[-\infty,\infty]} \twoheadrightarrow \widetilde{C}_*^{[R',\infty]}.$$

Here, the last map is a natural quotient map.

Proposition 5.5 *Let $S : T_{p,q} \to T_{p,q}$ be a given self-concordance. Then there is a dense subset $\mathcal{I} \subset \left(0, \frac{1}{2}\right)$ such that all elements in $\mathcal{R}_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ extend to elements in $\mathcal{R}_\alpha((S^3 \times [0,1]) \setminus S, \mathrm{SU}(2))$ for any $\alpha \in \mathcal{I}$.*

Proof We choose a dense subset $\mathcal{I} \subset \left(0, \frac{1}{2}\right)$ such that Theorem 1.10 holds for $T_{p,q}$. Since all irreducible critical points of the Chern–Simons functional of $T_{p,q}$ are nondegenerate by Proposition 4.17, we can choose a perturbation π so that it is supported away from flat connections. In particular, we can assume that the chain complex $C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}})$ is generated by $\mathcal{R}_\alpha^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$. Since the assertion for $\alpha = \frac{1}{4}$ is proved in [8], we assume that $\alpha \neq \frac{1}{4}$. In particular, we consider the case $\alpha < \frac{1}{4}$ for a while. Since the unique flat reducible θ_α with the holonomy parameter α on $S^3 \setminus T_{p,q}$ always extends to the concordance complement, it is enough to consider the extension problem for irreducibles. We choose a field $\mathscr{S} := \mathscr{R}_\alpha \otimes \mathbb{Q}$. By Theorems 1.7 and 1.10 we have

$$h_{\mathscr{S}}^\alpha(T_{p,q}) = -\tfrac{1}{2}\sigma_\alpha(T_{p,q}) = d,$$

where $d := \operatorname{rank} C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}})$. This implies that there is a cycle $\beta_0 \in C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}})$ such that $\delta_1 v^k(\beta_0) = 0$ if $k < d-1$ and $\delta_1 v^{d-1}(\beta_0) \neq 0$. Put $\beta_i := v^i(\beta_0)$ for $0 \leq i \leq d-1$. The chain complex $C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}_\alpha})$ admits a $(\mathbb{Z}\times\mathbb{R})$–bigrading by fixing lifts $\tilde{\rho}_1, \dots, \tilde{\rho}_d$ of singular flat connections $\rho_1, \dots, \rho_d \in \mathcal{R}_\alpha^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$. In particular, we may assume that $\deg_{\mathbb{Z}}(\tilde{\rho}_i) = 1$ or 3 by Theorem 1.10.

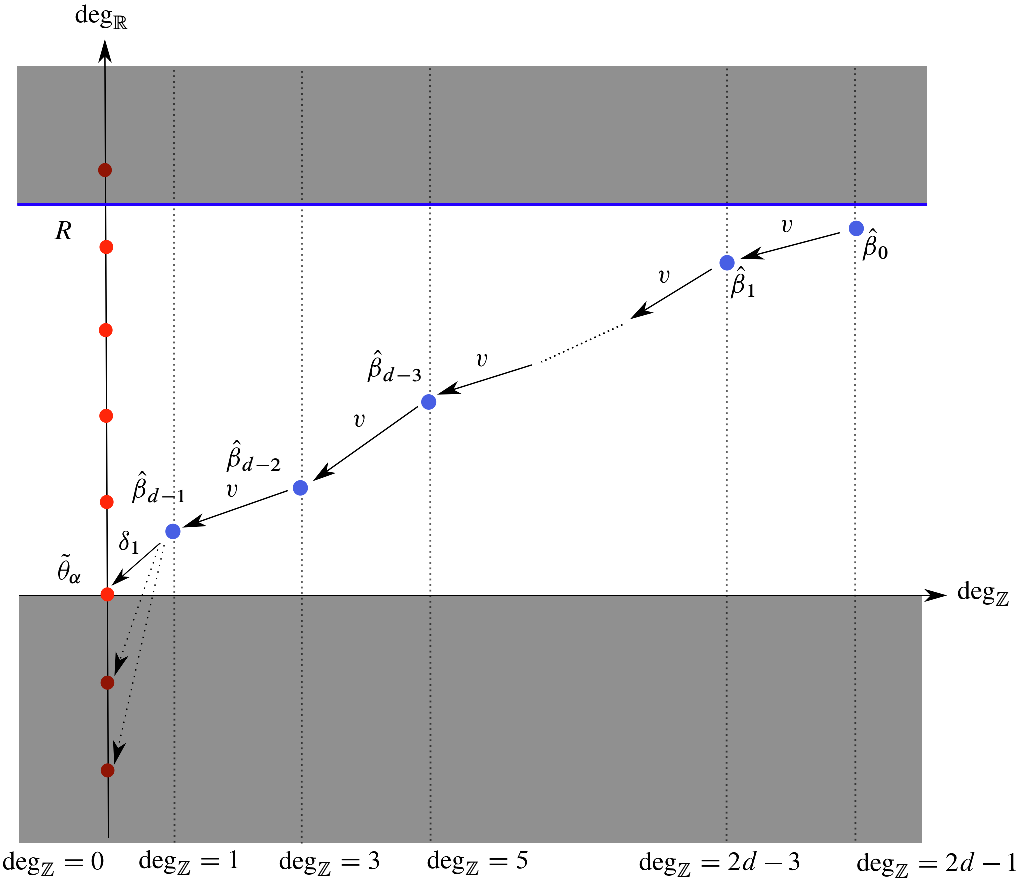

Figure 3: Elements $\hat\beta_0, \dots, \hat\beta_{d-1}$ and their $(\mathbb{Z}\times\mathbb{R})$–gradings.

Using properties of elements $\beta_0, \dots, \beta_{d-1}$, we fix elements $\hat\beta_0, \dots, \hat\beta_{d-1} \in C_*^\alpha(T_{p,q}; \Delta_{\mathscr{S}_\alpha})^{[-\infty,\infty]}$ in the following way. Firstly, there exists a cycle $\hat\beta_{d-1}$ such that $\deg_{\mathbb{Z}}(\hat\beta_{d-1}) = 1$ and satisfying

$$\delta_1(\hat\beta_{d-1}) = \sum_{k \le 0} c_k Z^k \tilde\theta_\alpha$$

with $c_0 \neq 0$ since $\delta_1(\beta_{d-1}) \neq 0$. Next, choose an element $\hat\beta_i$. Then $\hat\beta_{i-1}$ is defined as a cycle satisfying

$$v(\hat\beta_{i-1}) = \hat\beta_i.$$

Finally, we obtain cycles $\hat\beta_0, \dots, \hat\beta_{d-1}$ by induction.

Note that $\deg_{\mathbb{Z}}(\hat\beta_i) = 2(d-1) - 2i$ and $0 \le \deg_{\mathbb{R}}(\hat\beta_{i-1}) \le \cdots \le \deg_{\mathbb{R}}(\hat\beta_0)$. We fix $R > 0$ so that it satisfies $\deg_{\mathbb{R}}(\hat\beta_0) < R$ and $R \notin \mathscr{C}^*$; see Figure 3.

Let $-\epsilon < 0$ be a small negative number such that an interval $[-\epsilon, 0)$ does not contain any critical value of the Chern–Simons functional CS. Our aim is to show that the cobordism map m_S on the quotient filtered chain $\widetilde{C}_*^\alpha(T_{p,q}; \Delta_{\mathscr{S}})^{[-\epsilon,R]}$ is an isomorphism.

Since $\widetilde{m}_S$ preserves the $\mathbb{Z}$–grading, it is enough to show that $\widetilde{m}_S$ is an isomorphism on $C_1^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$ and $C_3^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$. We claim that $C_1^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$ and $C_3^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$ are generated by elements of the form

$$\{Z^jU^{2k+1-(d-1)}\hat{\beta}_{2k+1}\mid k, m_{2k+1}\le j\le n_{2k+1}\}\quad\text{or}\quad\{Z^jU^{2k-(d-1)}\hat{\beta}_{2k}\mid k, m_{2k}\le j\le n_{2k}\}$$

over $\mathbb{Q}$. To see this, consider the linear combination

$$\sum_{0\le i\le d-1}\ \sum_{m_j\le j\le n_i} c_{i,j}Z^jU^{i-(d-1)}\hat{\beta}_i=0, \tag{5-1}$$

where $c_{i,j}$ are rational coefficients. Then we consider applying operators $\mathcal{W}^{[-\epsilon,R]}_{i,j,k}$ to (5-1). Firstly, we apply the operator for $(i,j,k)=(0,-n_{d-1},0)$. Then we get $c_{0,n_{d-1}}\delta_1(\hat{\beta}_{d-1})=0$ with $\delta_1(\hat{\beta}_{d-1})$ nonzero. Since $\mathscr{S}$ is an integral domain, $c_{d-1,n_{d-1}}=0$. Next, we apply the operator $\mathcal{W}^{[-\epsilon,R]}_{(i,j,k)}$ for $(i,j,k)=(0,-n_{d-1}+1,0)$ to (5-1). Then we obtain $c_{d-1,n_{d-1}-1}$ using $c_{d-1,n_{d-1}}=0$. Inductively, we obtain

$$c_{d-1,n_{d-1}}=\cdots=c_{d-1,m_{d-1}}=0$$

by applying operators $\mathcal{W}^{[-\epsilon,R]}_{0,-n_{d-1},0},\dots,\mathcal{W}^{[-\epsilon,R]}_{0,-m_{d-1},0}$. We repeat a similar arguments using the operators $\{\mathcal{W}^{[-\epsilon,R]}_{1,j,1}\}_{m_{d-2}\le j\le n_{d-2}}$, and obtain

$$c_{d-2,n_{d-2}}=\cdots=c_{d-2,m_{d-2}}=0.$$

Inductively, we conclude that

$$c_{i,n_i}=\cdots=c_{i,m_i}=0$$

for all $0\le i\le d-1$. So $\{Z^jU^{i-(d-1)}\hat{\beta}_i\}$ for $0\le i\le d-1$ and $m_i\le j\le n_i$ are linearly independent. Put $\hat{\beta}_i':=m_S(\hat{\beta}_i)$. Since the induced cobordism map $\widetilde{m}_S$ on an $\mathcal{S}$–complex satisfies the relations in Proposition 3.15, the elements $\hat{\beta}_0,\dots,\hat{\beta}_{d-1}$ have the same properties, and the same technique shows that $\{Z^jU^{i-(d-1)}\hat{\beta}_i'\}$ for $0\le i\le d-1$ and $m_i\le j\le n_i$ are linearly independent. Moreover, $\deg_{\mathbb{R}}(\hat{\beta}_i)=\deg_{\mathbb{R}}(\hat{\beta}_i')$ by the construction of elements $\{\hat{\beta}_i\}$. We conclude that the map m_S is an isomorphism on $C_1^\alpha(T_{p,q};\Delta_{\mathscr{S}_\alpha})^{[-\epsilon,R]}$ and $C_3^\alpha(T_{p,q};\Delta_{\mathscr{S}_\alpha})^{[-\epsilon,R]}$.

Note that the chain complex $C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}})$ is generated by those irreducible singular flat connections. Then the degree 1 part $C_1^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$ of the quotient filtered chain complex is generated by elements of the forms $\{Z^j\tilde{\rho}_1\}_{m_1\le j\le n_1},\dots,\{Z^j\tilde{\rho}_l\}_{m_l\le j\le n_l}$ over $\mathbb{Q}$. We order these generators by values of the Chern–Simons functional. Then the cobordism map m_S can be represented by the form

$$\begin{bmatrix} L_1 & O & \cdots & & \\ & L_2 & O & \cdots & \\ & & \ddots & O & \cdots \\ & & & \ddots & O \\ & & & & L_k \end{bmatrix}, \tag{5-2}$$

where diagonal blocks L_i are components that correspond to the basis with the same value of the Chern–Simons functional. Note that components in L_i are defined by counting (perturbed) flat connections over the concordance complement. Since m_S is an isomorphism on the degree 1 part, the matrix (5-2) is invertible over $\mathbb{Q}$. Hence each diagonal block L_i is also invertible. In particular, they do not contain any zero-column. Since the regular condition on moduli space is an open condition with respect to choices of perturbation, all flat connections $\rho_1, \dots, \rho_l$ extend to flat connections over the concordance complement. The similar argument works for $\rho_{l+1}, \dots, \rho_d$, and thus all elements in $\mathcal{R}^*_\alpha(S^3 \setminus T_{p,q}; \Delta_{\mathscr{S}})$ extend to the concordance complement.

Finally, we consider the case $\alpha > \frac{1}{4}$. Here we only change the above argument at the following point: We apply the operator $\mathcal{W}^{[-\epsilon,R]}_{0,-m_{d-1}}$ on (5-1) the first time. Then we obtain $c_{d-1,m_{d-1}} = 0$. Next we apply the operator $\mathcal{W}^{[-\epsilon,R]}_{0,-m_{d-1}+1,0}$ and obtain $c_{d-1,m_{d-1}-1} = 0$. We inductively obtain $c_{d-1,n_{d-1}} = \cdots = c_{d-1,m_{d-1}} = 0$. The rest of the argument proceeds similarly, and finally all coefficients in (5-1) vanish. □

Proof of Theorem 1.1 Let $S\colon T_{p,q} \to K$ be a given concordance. Then we can construct a concordance $\overline{S} \circ S\colon T_{p,q} \to K \to T_{p,q}$ by the composition, where $\overline{S}$ is the opposite concordance of S. By Proposition 5.5 there exists a dense subset $\mathcal{I} \subset \left[0, \frac{1}{2}\right]$ such that there is a extension $\mathcal{R}_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \to \mathcal{R}_\alpha((S^3 \times [0,1]) \setminus \overline{S} \circ S, \mathrm{SU}(2))$ for any $\alpha \in \mathcal{I}$. Let $\alpha \in \left[0, \frac{1}{2}\right]$ be any holonomy parameter and consider the representation $\rho\colon \pi_1(S^3 \setminus T_{p,q}) \to \mathrm{SU}(2)$ with

$$\rho(\mu_{T_{p,q}}) \sim \begin{bmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{bmatrix}.$$

Then we can choose a sequence $\{\alpha_i\} \subset \mathcal{I}$ such that $\lim_{i\to\infty} \alpha_i = \alpha$ and SU(2) representations ρ_i of $\pi_1(S^3 \setminus T_{p,q})$ with

$$\rho_i(\mu_{T_{p,q}}) \sim \begin{bmatrix} e^{2\pi i\alpha_i} & 0 \\ 0 & e^{-2\pi i\alpha_i} \end{bmatrix},$$

since ρ_i extends to an SU(2) representation $\Phi_i : \pi_1((S^3 \times [0,1]) \setminus \overline{S} \circ S) \to \mathrm{SU}(2)$ and we can choose a convergent subsequence of $\{\Phi_i\}$ with the limiting representation $\Phi_\infty\colon \pi_1((S^3 \times [0,1]) \setminus \overline{S} \circ S) \to \mathrm{SU}(2)$. (Since SU(2) is compact, we can choose a convergent subsequence $\{\Phi_i(x_j)\}_i$ for each generator x_j of $\pi_1((S^3 \times [0,1]) \setminus \overline{S} \circ S)$, and $\lim_{i\to\infty} \Phi_i(x_j)$ defines a limiting representation Φ_∞.) By restriction, we get a representation $\pi_1((S^3 \times [0,1]) \setminus S) \to \mathrm{SU}(2)$ which is the extension of ρ. □

Appendix The connected sum theorem

In this section, we give the proof of the connected sum theorem. The connected sum theorem for nonsingular settings was proved in [20], and the singular setting with $\alpha = \frac{1}{4}$ was proved in [9]. We use an argument similar to [9] to prove our connected sum theorem (Theorem 3.24). Let us recall the settings which are introduced in [9, Section 6]. Let (Y, K) and (Y', K') be two given knots in integral homology 3–spheres. Fixing basepoints $p \in K$ and $p' \in K'$, we take a pair of the connected sum $(Y \# Y', K \# K')$ at these basepoints. We also fix a basepoint $p^\# \in K \# K'$. Construct a cobordism

$$(W, S)\colon (Y \sqcup Y', K \sqcup K') \to (Y \# Y', K \# K')$$

by attaching a pair of 1–handles $(D^1\times D^3, D^1\times D^1)$ to the product cobordism $[0,1]\times(Y\sqcup Y', K\sqcup K')$. Let

$$(W',S')\colon (Y\mathbin{\#}Y', K\mathbin{\#}K')\to(Y\sqcup Y', K\sqcup K')$$

be a cobordism of the opposite direction. We define three oriented piecewise smooth paths γ, γ' and $\gamma^\#$on $S\subset W$. Assume that these paths intersect the boundaries of the cobordism only at their edge points. The path γ starts from $p\in Y$ and ends at the basepoint $p^\#\in Y\mathbin{\#}Y'$. Similarly, γ' starts from $p'\in Y'$ and ends at $p^\#$, and $\gamma^\#$ starts from $p\in Y$ and ends at $p'\in Y'$. Let us define the paths σ, σ' and $\sigma^\#$ in S' as mirrors of γ, γ' and $\gamma^\#$, respectively. We use the notation β, β' and $\beta^\#$ (and their indexed versions) for critical points of the perturbed Chern–Simons functional on (Y,K), (Y',K') and $(Y\mathbin{\#}Y', K\mathbin{\#}K')$, respectively. Let $\theta_\alpha,\theta'_\alpha$ and $\theta^\#_\alpha$ denote unique flat reducibles on (Y,K), (Y',K') and $(Y\mathbin{\#}Y', K\mathbin{\#}K')$, respectively. We use the reduced notation for d–dimensional moduli spaces as follows:

$$M_z(\beta,\beta';\beta^\#)_d := M_z(W,S;\beta,\beta',\beta^\#)_d \quad\text{and}\quad M_z(\beta^\#;\beta,\beta')_d := M_z(W',S';\beta^\#,\beta,\beta')_d.$$

We drop z from the notation above if we consider all unions of z. We define maps

$$H^\gamma\colon \mathcal{B}(W,S;\beta,\beta',\beta^\#)\to S^1 \quad\text{and}\quad H^{\gamma'}\colon \mathcal{B}(W,S;\beta,\beta',\beta^\#)\to S^1$$

as in Section 3.4. The moduli spaces cut down by these maps are defined by

$$\begin{aligned}
M_{\gamma,z}(\beta,\beta';\beta^\#)_d &:= \{[A]\in M_z(\beta,\beta';\beta^\#)_{d+1}\mid H^\gamma([A])=s\},\\
M_{\gamma',z}(\beta,\beta';\beta^\#)_d &:= \{[A]\in M_z(\beta,\beta';\beta^\#)_{d+1}\mid H^{\gamma'}([A])=s'\},\\
M_{\gamma\gamma',z}(\beta,\beta';\beta^\#)_d &:= \{[A]\in M_z(\beta,\beta';\beta^\#)_{d+1}\mid H^\gamma([A])=s,\ H^{\gamma'}([A])=s'\},
\end{aligned}$$

where $s\in S^1$ is a generic point. The orientation of moduli spaces over (W,S) is defined in the following way. Let $o_W\in\mathscr{O}[W,S;\theta_{\alpha+},\theta'_{\alpha+},\theta^\#_{\alpha-}]$ be the canonical homology orientation of (W,S), and $o_\beta\in\mathscr{O}[\beta]$, $o_{\beta'}\in\mathscr{O}[\beta']$ and $o_{\beta^\#}\in\mathscr{O}[\beta^\#]$ be given orientations for generators. Then $o_{\beta,\beta';\beta^\#}\in\mathscr{O}[W,S;\beta,\beta';\beta^\#]$ is fixed so that the relation

$$\Phi(o_\beta\otimes o_{\beta'}\otimes o_W)=\Phi(o_{\beta,\beta';\beta^\#}\otimes o_{\beta^\#})$$

holds.

The argument of the proof consists of the following steps:

(I) A cobordism of pairs $(W,S)\colon (Y\sqcup Y', K\sqcup K')\to(Y\mathbin{\#}Y', K\mathbin{\#}K')$ induces an $\mathcal{S}$–morphism

$$\widetilde{m}_{(W,S)}\colon \widetilde{C}^\alpha_*(Y,K)\otimes\widetilde{C}^\alpha_*(Y',K')\to\widetilde{C}^\alpha_*(Y\mathbin{\#}Y', K\mathbin{\#}K').$$

(II) A cobordism of pairs $(W',S')\colon (Y\mathbin{\#}Y', K\mathbin{\#}K')\to(Y\sqcup Y', K\sqcup K')$ induces an $\mathcal{S}$–morphism

$$\widetilde{m}_{(W',S')}\colon \widetilde{C}^\alpha_*(Y\mathbin{\#}Y', K\mathbin{\#}K')\to\widetilde{C}^\alpha_*(Y,K)\otimes\widetilde{C}^\alpha_*(Y',K').$$

(III) Put $\widetilde{C}^\#:=\widetilde{C}^\alpha_*(Y\mathbin{\#}Y', K\mathbin{\#}K')$. The composition $\widetilde{m}_{(W,S)}\circ\widetilde{m}_{(W',S')}$ is $\mathcal{S}$–chain homotopic to $\mathrm{id}_{\widetilde{C}^\#}$ up to the multiplication of a unit element in $\mathscr{S}$.

(IV) The composition $\widetilde{m}_{(W',S')}\circ\widetilde{m}_{(W,S)}$ is $\mathcal{S}$–chain homotopic to $\mathrm{id}_{\widetilde{C}^\otimes}$.

A.1 Step I

We define a map $\widetilde{m}_{(W,S)}$ as follows. Using the decomposition of the Floer chain group

$$C^{\otimes} = (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'_*,$$

we define four maps:

$$\begin{aligned}
m = [m_1, m_2, m_3, m_4] &\colon (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'_* \to C^{\#}_*,\\
\mu = [\mu_1, \mu_2, \mu_3, \mu_4] &\colon (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'_* \to C^{\#}_*,\\
\Delta_1 = [\Delta_{1,1}, \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}] &\colon (C \otimes C')_0 \oplus (C \otimes C')_{-1} \oplus C_0 \oplus C'_0 \to \mathscr{S},\\
\Delta_2 &\colon \mathscr{S} \to C^{\#}_{-1}.
\end{aligned}$$

Each component of the above maps is defined as follows:

$$\langle m_1(\beta \otimes \beta'), \beta^{\#} \rangle = \sum_z \# M_{\gamma^{\#},z}(\beta, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle m_2(\beta \otimes \beta'), \beta^{\#} \rangle = \sum_z \# M_z(\beta, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle m_3(\beta), \beta^{\#} \rangle = \sum_z \# M_z(\beta, \theta'_\alpha; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \quad \langle m_4(\beta'), \beta^{\#} \rangle = \sum_z \# M_z(\theta_\alpha, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle \mu_1(\beta \otimes \beta'), \beta^{\#} \rangle = \sum_z \# M_{\gamma\gamma',z}(\beta, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle \mu_2(\beta \otimes \beta'), \beta^{\#} \rangle = \sum_z \# M_{\gamma,z}(\beta, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle \mu_3(\beta), \beta^{\#} \rangle = \sum_z \# M_{\gamma,z}(\beta, \theta'_\alpha; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle \mu_4(\beta'), \beta^{\#} \rangle = \sum_z \# M_{\gamma',z}(\theta_\alpha, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\Delta_{1,1}(\beta \otimes \beta') = \sum_z \# M_{\gamma^{\#},z}(\beta, \beta'; \theta^{\#}_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \quad \Delta_{1,2}(\beta \otimes \beta') = \sum_z \# M_z(\beta, \beta'; \theta^{\#}_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\Delta_{1,3}(\beta) = \sum_z \# M_z(\beta, \theta'_\alpha; \theta^{\#}_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \qquad \Delta_{1,4}(\beta') = \sum_z \# M_z(\theta'_\alpha, \beta'; \theta^{\#}_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle \Delta_2(1), \beta^{\#} \rangle = \sum_z \# M_z(\theta_\alpha, \theta'_\alpha; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)}.$$

As described in [9, Remarks 6.10 and 6.11], notice that

- $H^{-1}_{\beta\beta_1}(s) \cap H^{-1}_{\beta\beta'}(s') \cap M(\beta, \beta_1)_2 = \varnothing$ for distinct regular values $s, s' \in S^1$,
- $\# M_\gamma(\beta, \beta'; \beta^{\#}) - \# M_{\gamma'}(\beta, \beta'; \beta^{\#}) = \# M_{\gamma^{\#}}(\beta, \beta'; \beta^{\#})$.

Proposition A.1 *There are the following relations*:

$$d^{\#} \circ m = m \circ d^{\otimes}, \tag{A-1}$$

$$\delta^{\#}_1 \circ m = \Delta_1 \circ d^{\otimes} + \delta^{\otimes}_1, \tag{A-2}$$

$$m \circ \delta^{\otimes}_2 = \delta^{\#}_2 - d^{\#} \circ \Delta_2, \tag{A-3}$$

$$d^{\#} \circ \mu + \mu \circ d^{\otimes} = v^{\#} \circ m - m \circ v^{\otimes} + \delta^{\#}_2 \circ \Delta_1 - \Delta_2 \circ \delta^{\otimes}_1. \tag{A-4}$$

Proof The identity (A-1) decomposes into the following four relations:

$$d^\# m_1 = m_1(d\otimes 1) + m_1(\epsilon\otimes d') - m_2(\epsilon v\otimes 1) + m_3(\epsilon\otimes v') + m_3(\epsilon\otimes\delta_1') + m_4(\delta_1\otimes 1), \tag{A-5}$$

$$d^\# m_2 = m_2(d\otimes 1) - m_2(\epsilon\otimes d'), \tag{A-6}$$

$$d^\# m_3 = m_3(\epsilon\otimes\delta_2') + m_3 d, \tag{A-7}$$

$$d^\# m_4 = -m_2(\delta_2\otimes 1) + m_4 d'. \tag{A-8}$$

The identity (A-5) is obtained by counting the boundary of the compactified moduli space $M^+_{\gamma^\#,z}(\beta,\beta';\beta^\#)_1$ for each path z. In fact, the oriented boundary of $M^+_{\gamma^\#,z}(\beta,\beta';\beta^\#)_1$ consists of the following types of codimension 1 faces:

$$M_{\gamma^\#,z'}(\beta,\beta';\beta_1^\#)_0\times\breve{M}_{z''}(\beta_1^\#,\beta^\#)_0,\quad \breve{M}_{z'}(\beta,\beta_1)_0\times M_{\gamma^\#,z''}(\beta_1,\beta';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)}\breve{M}_{z'}(\beta',\beta_1')_0\times M_{\gamma,z''}(\beta,\beta_1';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)+1}(H^{-1}_{\beta\beta_1}(s)\cap M_{z'}(\beta,\beta_1)_1)\times M_{z''}(\beta_1,\beta';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)}(H^{-1}_{\beta'\beta_1'}(s)\cap M_{z'}(\beta',\beta_1')_1)\times M_{z''}(\beta,\beta_1';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)}\breve{M}_{z'}(\beta',\theta_\alpha')_0\times M_{z''}(\beta,\theta_\alpha';\beta^\#)_0,\quad \breve{M}_{z'}(\beta,\theta_\alpha)_0\times M_{z''}(\theta_\alpha,\beta';\beta^\#)_0.$$

The identities (A-6)–(A-8) are obtained by counting the compactified moduli spaces $M^+_z(\beta,\beta';\beta^\#)_1$, $M^+_z(\beta,\theta_\alpha';\beta^\#)_1$ and $M^+_z(\theta_\alpha,\beta';\beta^\#)_1$, respectively. We list up codimension 1 faces of each moduli space:

- codimension 1 faces of $\partial M^+_z(\beta,\beta';\beta^\#)_1$:
$$M_{z'}(\beta,\beta';\beta_1^\#)_0\times\breve{M}_{z''}(\beta_1^\#,\beta^\#)_0,\quad \breve{M}_{z'}(\beta,\beta_1)\times M_{z''}(\beta_1,\beta';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)}\breve{M}_{z'}(\beta',\beta_1')_0\times M_{z''}(\beta,\beta_1';\beta^\#)_0,$$
- codimension 1 faces of $\partial M^+_z(\beta,\theta_\alpha';\beta^\#)_1$:
$$M_{z'}(\beta,\theta_\alpha';\beta_1^\#)_0\times\breve{M}_{z''}(\beta_1^\#,\beta^\#)_0,\quad (-1)^{\mathrm{gr}(\beta)}\breve{M}_{z'}(\theta_\alpha',\beta')_0\times M_{z''}(\beta,\beta';\beta^\#)_0,$$
$$\breve{M}_{z'}(\beta,\beta_1)\times M_{z''}(\beta_1,\theta_\alpha';\beta^\#)_0,$$
- codimension 1 faces of $\partial M^+_z(\theta_\alpha,\beta';\beta^\#)_1$:
$$M_{z'}(\theta_\alpha,\beta';\beta_1^\#)_0\times\breve{M}_{z''}(\beta_1^\#,\beta^\#)_0,\quad \breve{M}_{z'}(\theta_\alpha,\beta)_0\times M_{z''}(\beta,\beta';\beta^\#)_0,$$
$$\breve{M}_{z'}(\beta',\beta_1')_0\times M_{z''}(\theta_\alpha,\beta_1';\beta^\#)_0.$$

The relation (A-2) decomposes into the following four identities:

$$\delta_1^\# m_1 = \Delta_{1,1}(d\otimes 1) + \Delta_{1,1}(\epsilon\otimes d') - \Delta_{1,2}(\epsilon v\otimes 1) + \Delta_{1,2}(\epsilon\otimes v') + \Delta_{1,3}(\epsilon\otimes\delta_1') + \Delta_{1,4}(\delta_1\otimes 1),$$
$$\delta_1^\# m_2 = \Delta_{1,2}(d\otimes 1) - \Delta_{1,2}(\epsilon\otimes d'),$$
$$\delta_1^\# m_3 = \Delta_{1,2}(\epsilon\otimes\delta_2') + \Delta_{1,3}d + \delta_1,$$
$$\delta_1^\# m_4 = -\Delta_{1,2}(\delta_2\otimes 1) + \Delta_{1,4}d' + \delta_1'.$$

Each relation is obtained by counting the boundaries of the compactified 1–dimensional moduli spaces $M^+_{\gamma^\#,z}(\beta,\beta';\theta_\alpha^\#)_1$, $M^+_z(\beta,\beta';\theta_\alpha^\#)_1$, $M^+_z(\theta_\alpha,\beta';\theta_\alpha^\#)_1$ and $M^+_z(\beta,\theta_\alpha';\theta_\alpha^\#)_1$ for each path z, and the argument is similar to the previous case. Note that $M(\theta_\alpha, \theta'_\alpha; \theta^\#_\alpha)_0$ consists of the unique reducible connection. The relation (A-3) reduces to

$$m_3\delta_2 + m_4\delta'_2 = \delta^\#_2 - d^\#\Delta_2,$$

and this reduces to the counting of the boundary of $M^+_z(\theta_\alpha, \theta'_\alpha; \beta^\#)_1$ whose codimension 1 faces are

$$\breve{M}_{z'}(\theta_\alpha, \beta)_0 \times M_{z''}(\beta, \theta'_\alpha; \beta^\#)_0, \quad \breve{M}_z(\theta'_\alpha, \beta')_0 \times M_{z''}(\beta, \beta'; \theta^\#_\alpha)_0,$$
$$\breve{M}_{z'}(\theta_\alpha, \theta'_\alpha; \theta^\#_\alpha)_0 \times \breve{M}_{z''}(\theta^\#_\alpha, \beta^\#)_0, \quad M_{z'}(\theta_\alpha, \theta'_\alpha; \beta^\#_1) \times \breve{M}_{z''}(\beta^\#_1, \beta^\#).$$

The relation (A-4) reduces to four identities:

$$\begin{aligned} d^\#\mu_1 + \mu_1(d \otimes 1) + \mu_1(\epsilon \otimes d') - \mu_2(\epsilon v \otimes 1) + \mu_2(\epsilon \otimes v') + \mu_3(\epsilon \otimes \delta'_1) + \mu_4(\delta_1 \otimes 1)& \\ = v^\# m_1 - m_1(v \otimes 1) + \delta^\#_2\Delta_{1,1},& \\ d^\#\mu_2 + (d \otimes 1) - \mu_2(\epsilon \otimes d') = v^\# m_2 - m_2(v \otimes 1) + m_4(\delta_1 \otimes 1),& \\ d^\#\mu_3 + \mu_2(\epsilon \otimes \delta'_2) + \mu_3 d = v^\# m_3 - m_3 v + \delta^\#_2\Delta_{1,3} - \Delta_2\delta_1,& \\ d^\#\mu_4 - \mu_2(\delta_2 \otimes 1) + \mu_4 d' = v^\# m_4 - m_1(\delta_1 \otimes 1) - m_4 v' + \delta^\#_2\Delta_{1,3} - \Delta_2\delta'_1.& \end{aligned}$$

These are obtained by counting the boundaries of $M^+_{\gamma\gamma',z}(\beta, \beta'; \beta^\#)_1$, $M^+_{\gamma,z}(\beta, \beta'; \beta^\#)_1$, $M^+_{\gamma,z}(\beta, \theta'_\alpha; \beta^\#)_1$ and $M^+_{\gamma'}(\theta_\alpha, \beta'; \beta^\#)_1$; see also [9, Remark 6.11]. □

A.2 Step II

We have

$$\begin{aligned} m' = [m'_1, m'_2, m'_3, m'_4]^{\mathrm{T}} &: C^\#_* \to (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'_*, \\ \mu' = [\mu'_1, \mu'_2, \mu'_3, \mu'_4]^{\mathrm{T}} &: C^\#_* \to (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'_*, \\ \Delta'_1 &: C^\#_1 \to \mathscr{S}. \\ \Delta'_2 = [\Delta'_{2,1}, \Delta'_{2,2}, \Delta'_{2,3}, \Delta'_{2,4}]^{\mathrm{T}} &: \mathscr{S} \to (C \otimes C')_{-1} \oplus (C \otimes C')_{-2} \oplus C_{-1} \oplus C'_{-1}, \end{aligned}$$

Each component of the above maps is defined as follows:

$$\langle m'_1(\beta^\#), \beta \otimes \beta' \rangle = \sum_z \#M_z(\beta^\#; \beta, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle m'_2(\beta^\#), \beta \otimes \beta' \rangle = \sum_z \#M_{\sigma,z}(\beta^\#; \beta, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle m'_3(\beta^\#), \beta \rangle = \sum_z \#M_z(\beta^\#; \beta, \theta'_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \quad \langle m'_4(\beta^\#), \beta' \rangle = \sum_z \#M_z(\beta^\#; \theta_\alpha, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu'_1(\beta \otimes \beta'), \beta^\# \rangle = \sum_z \#M_{\sigma,z}(\beta^\#; \beta, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu'_2(\beta \otimes \beta'), \beta^\# \rangle = \sum_z \#M_{\sigma\sigma',z}(\beta^\#; \beta, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu'_3(\beta), \beta^\# \rangle = \sum_z \#M_{\sigma,z}(\beta^\#; \beta, \theta'_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu'_4(\beta'), \beta^\# \rangle = \sum_z \#M_{\sigma',z}(\beta^\#; \theta_\alpha, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \Delta'_1(1), \beta^\# \rangle = \sum_z \#M_z(\beta^\#; \theta_\alpha, \theta'_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\Delta'_{2,1}(\beta\otimes\beta')=\sum_z \#M_z(\beta^{\#};\beta,\beta')_0\lambda^{-\kappa(z)}T^{\nu(z)},\quad \Delta'_{2,2}(\beta\otimes\beta')=\sum_z \#M_{\sigma,z}(\theta^{\#}_\alpha;\beta,\beta')_0\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\Delta'_{2,3}(\beta)=\sum_z \#M_z(\theta^{\#}_\alpha;\beta,\theta'_\alpha)_0\lambda^{-\kappa(z)}T^{\nu(z)},\qquad \Delta'_{2,4}(\beta')=\sum_z \#M_z(\theta^{\#}_\alpha;\theta'_\alpha,\beta')_0\lambda^{-\kappa(z)}T^{\nu(z)}.$$

Proposition A.2 *There are the following relations*:

$$d^{\otimes}\circ m'=m'\circ d^{\#},\tag{A-9}$$

$$\delta^{\otimes}_1\circ m'=\Delta'_1\circ d^{\#}+\delta^{\#}_1,\tag{A-10}$$

$$m'\circ\delta^{\#}_2=\delta^{\otimes}_2-d^{\otimes}\circ\Delta'_2,\tag{A-11}$$

$$d^{\otimes}\circ\mu'+\mu'\circ d^{\#}=v^{\otimes}\circ m'-m'\circ v^{\#}+\delta^{\otimes}_2\circ\Delta'_1-\Delta'_2\circ\delta^{\#}_1.\tag{A-12}$$

Proof The proof is similar to that of Proposition A.1. In this case, we consider the opposite cobordism (W',S'). □

A.3 Step III

Put $(W^o,S^o):=(W\circ W',S\circ S')$. We define compositions of paths $\rho^{\#}:=\gamma^{\#}\circ\sigma^{\#}$, $\rho:=\gamma\circ\sigma$ and $\rho':=\gamma'\circ\sigma'$; see Figure 5. We regard the configuration space of connections over (W^o,S^o) as the quotient of the space of SO(3)–adjoint connections by the determinant 1 gauge group $\mathcal{G}$. Then there is an exact sequence

$$\mathcal{G}\hookrightarrow\mathcal{G}^e\twoheadrightarrow H^1(W^o;\mathbb{Z}_2),$$

where $\mathcal{G}^e$ is an SO(3)–gauge transformation and the second map gives the obstruction to lifting an SO(3)–automorphism to an SU(2)–automorphism over the 1–skeleton. There is an action of $\mathcal{G}^e/\mathcal{G}\cong H^1(W^o,\mathbb{Z}_2)\cong\mathbb{Z}_2$ on the configuration space. In particular, there is an involution on the moduli space $M(W^o,S^o;\beta^{\#},\beta^{\#}_1)_d$ and we denote its quotient by $M(W^o,S^o;\beta^{\#},\beta^{\#}_1)^e_d$. We define

$$M_{\rho^{\#};z}(W^o,S^o;\beta^{\#},\beta^{\#}_1)^e_0:=\{[A]\in M_z(W^o,S^o;\beta^{\#},\beta^{\#}_1)^e_1\mid H^{\rho^{\#}}([A])=s\},$$
$$M_{\rho^{\#}\rho;z}(W^o,S^o;\beta^{\#},\beta^{\#}_1)^e_0:=\{[A]\in M_z(W^o,S^o;\beta^{\#},\beta^{\#}_1)^e_2\mid H^{\rho^{\#}}([A])=s,H^{\rho}([A])=s'\}.$$

The cardinality of these moduli spaces is half of that of the usual ones. Assume that (W^o,S^o) is equipped with a Riemannian metric with a long neck along the cylinder $[0,1]\times(Y\sqcup Y',K\sqcup K')$. Then we have a good gluing relation

$$\begin{aligned}M_{\rho^{\#}}(W^o,S^o;\beta^{\#},\beta^{\#}_1)^e_0=\bigsqcup_{\beta,\beta'}M_{\sigma^{\#}}(W',S';\beta^{\#},\beta,\beta')_0\times M(W,S;\beta,\beta';\beta^{\#}_1)_0&\\ \sqcup\bigsqcup_{\beta,\beta'}M(W',S';\beta^{\#},\beta,\beta')_0\times M_{\gamma^{\#}}(W,S;\beta,\beta';\beta^{\#}_1)_0&\\ \sqcup\bigsqcup_{\beta'\in\mathfrak{C}'^*}M(W',S';\beta^{\#},\theta_\alpha,\beta')_0\times M(W,S;\theta_\alpha,\beta';\beta^{\#}_1)_0&\\ \sqcup\bigsqcup_{\beta\in\mathfrak{C}^*}M(W',S';\beta^{\#},\beta,\theta_\alpha)_0\times M(W,S;\beta,\theta'_\alpha;\beta^{\#}_1)_0.&\end{aligned}$$

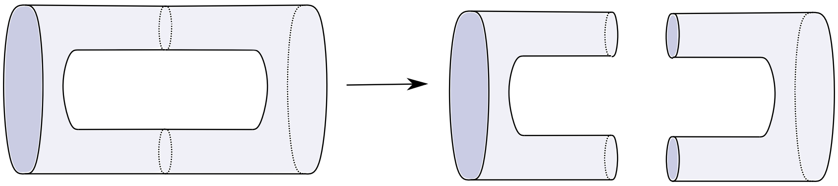

Figure 4: The family of metrics G^o.

Let $\widetilde{m}_{(W^o,S^o,\rho^\#)}\colon \widetilde{C}^\#_* \to \widetilde{C}^\#_*$ be an $\mathcal{S}$–morphism whose components m^o, μ^o, Δ^o_1 and Δ^o_2 are defined by

$$\begin{aligned}
\langle m^o(\beta^\#), \beta^\#_1\rangle &= \sum_z \# M_{\rho^\#,z}(W^o,S^o;\beta^\#,\beta^\#_1)^e_0 \lambda^{-\kappa(z)} T^{\nu(z)},\\
\langle \mu^o(\beta^\#), \beta^\#_1\rangle &= \sum_z \# M_{\rho^\#\rho,z}(W^o,S^o;\beta^\#,\beta^\#_1)^e_0 \lambda^{-\kappa(z)} T^{\nu(z)},\\
\Delta^o_1(\beta^\#) &= \sum_z \# M_{\rho^\#,z}(W^o,S^o;\beta^\#,\theta^\#_\alpha)^e_0 \lambda^{-\kappa(z)} T^{\nu(z)},\\
\Delta^o_2(1)\beta^\# &= \sum_z \# M_{\rho^\#,z}(W^o,S^o;\beta^\#,\theta^\#_\alpha)^e_0 \lambda^{-\kappa(z)} T^{\nu(z)}.
\end{aligned}$$

Proposition A.3 *We have that $\widetilde{m}_{(W,S)} \circ \widetilde{m}_{(W',S')}$ is $\mathcal{S}$–chain homotopic to $\widetilde{m}_{(W\circ W',S\circ S';\rho^\#)}$.*

Proof Let G^o be the 1–parameter family of metrics which stretch the cobordism (W^o,S^o) as in Figure 4. We modify the definition of the $\mathcal{S}$–chain homotopy in [9, Proposition 6.16] in the following way:

$$\begin{aligned}
\langle K^o(\beta^\#), \beta^\#_1\rangle &= \sum_z \#\Big\{[A] \in \bigcup_{g\in G^o} M^g_z(W^o,S^o;\beta^\#,\beta^\#_1)^e_0 \mid H^{\rho^\#}([A]) = s\Big\}\lambda^{-\kappa(z)} T^{\nu(z)},\\
\langle L^o(\beta^\#), \beta^\#_1\rangle &= \sum_z \#\Big\{[A] \in \bigcup_{g\in G^o} M^g_z(W^o,S^o;\beta^\#,\beta^\#_1)^e_0 \mid H^{\rho^\#}([A]) = s,\, H^{\rho}([A]) = t\Big\}\lambda^{-\kappa(z)} T^{\nu(z)},\\
\langle M^o_1(\beta^\#), 1\rangle &= \sum_z \#\Big\{[A] \in \bigcup_{g\in G^o} M^g_z(W^o,S^o;\beta^\#,\theta^\#)^e_0 \mid H^{\rho^\#}([A]) = s\Big\}\lambda^{-\kappa(z)} T^{\nu(z)},\\
\langle M^o_2(1), \beta^\#\rangle &= \sum_z \#\Big\{[A] \in \bigcup_{g\in G^o} M^g_z(W^o,S^o;\theta^\#_\alpha,\beta^\#)^e_0 \mid H^{\rho^\#}([A]) = s\Big\}\lambda^{-\kappa(z)} T^{\nu(z)}.
\end{aligned}$$

The rest of the argument is similar to [9, Proposition 6.16], and we can check that

$$H^o = \begin{bmatrix} K^o & 0 & 0\\ L^o & -K^o & M^o_2\\ M^o_1 & 0 & 0 \end{bmatrix}$$

gives an $\mathcal{S}$–chain homotopy from $\widetilde{m}_{(W^o,S^o)}$ to $\widetilde{m}_{(W\sqcup W',S\sqcup S')}$. □

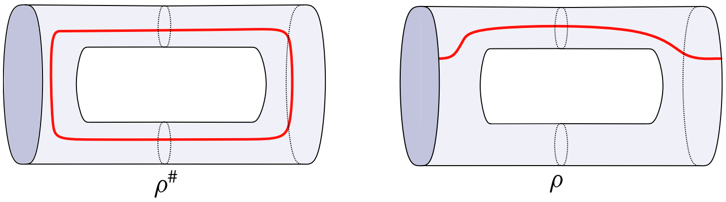

Figure 5: Paths on (W^o,S^o).

Proposition A.4 *We have that $\widetilde{m}_{(W\circ W', S\circ S';\rho^\#)}$ is $\mathcal{S}$–chain homotopic to $\mathrm{id}_{\widetilde{C}^\#}$, up to the multiplication of a unit element in $\mathscr{S}$.*

Proof As in the proof of [9, Proposition 6.17], we consider the decomposition,

$$(W^o, S^o) = (W^c, S^c) \cup (S^1\times D^3, S^1\times D^1)$$

along $(S^1\times S^2, S^1\times 2\mathrm{pt})$. We arrange the perturbation data on $(S^1\times D^3, S^1\times D^1)$ and the gluing region so that it is supported away from the moduli space of flat connections. Define the map $\widetilde{m}^+$ as $\widetilde{m}_{(W^o,S^o,\rho^\#)}$, but using the metric on (W^o, S^o) which is stretched along the gluing region $(S^1\times S^2, S^1\times 2\mathrm{pt})$. We write m^+, μ^+, Δ_1^+ and Δ_2^+ for corresponding components of $\widetilde{m}^+$.

Let μ_1 be a generator of the $S^2\setminus 2\mathrm{pt}$ factor and μ_2 be a generator of the S^1 factor in $\pi_1(S^1\times S^2\setminus S^1\times 2\mathrm{pt})$. The equivalence classes of the critical point set $\mathfrak{C}$ of the Chern–Simons functional on $(S^1\times S^2, S^1\times 2\mathrm{pt})$ can be identified with

$$\mathcal{R}_\alpha(S^1\times S^2\setminus S^1\times 2\mathrm{pt}) = \{\beta\in \mathrm{Hom}(\pi_1, \mathrm{SU}(2)) \mid \mathrm{tr}\,\beta(\mu_1) = 2\cos(2\pi\alpha)\}/\,\mathrm{SU}(2)$$

by the holonomy correspondence. The character variety $\mathcal{R}_\alpha(S^1\times S^2\setminus S^1\times 2\mathrm{pt})$ is identified with S^1 as follows. Let μ_1 be a generator of $\pi_1(S^1\times(S^2\setminus 2\mathrm{pt}))$ arising from the $S^2\setminus 2\mathrm{pt}$ factor, and μ_2 be another generator arising from the S^1 factor. Since $\mathrm{tr}\,\beta(\mu_1) = 2\cos(2\pi i\alpha)$, there is an element $g_\beta\in\mathrm{SU}(2)$ with $g_\beta\beta(\mu_1)g_\beta^{-1} = e^{2\pi i\alpha}\in S^1$. Since μ_1 and μ_2 commute and $\alpha\neq 0, \frac{1}{2}$, there is $\theta(\beta)\in[0,2\pi)$ and we have $g_\beta\beta(\mu_2)g_\beta^{-1} = e^{i\theta(\beta)}\in S^1$. The correspondence $\beta\mapsto e^{i\theta(\beta)}$ gives a bijection $\mathcal{R}_\alpha(S^1\times S^2\setminus S^1\times 2\mathrm{pt})\cong S^1$.

Let A be a singular flat connection which is the extension of $\rho\in\mathfrak{C}$ over $(S^1\times D^3, S^1\times D^1)$. Since all elements in $\mathfrak{C}$ have $U(1)$–stabilizer, $\dim H^0(S^1\times D^3\setminus S^1\times D^1;\mathrm{ad}\,A) = 1$. Also,

$$\dim H^1(S^1\times D^3\setminus S^1\times D^1;\mathrm{ad}\,A) = 1$$

by the computation of group cohomology of $\pi_1(S^1\times(D^3\setminus D^1))$. Thus the critical point set $\mathfrak{C} = \mathcal{R}_\alpha(S^1\times(S^2\setminus 2\mathrm{pt}))$ is Morse–Bott nondegenerate. Consider the closed pair $(S^1\times S^3, S^1\times S^1) = (S^1\times D^3, S^1\times D^1)\cup_{(S^1\times S^2, S^1\times 2\mathrm{pt})}(S^1\times D^3, S^1\times D^1)$. Then the gluing of the index formula is

$$2\,\mathrm{ind}\,D_A + \dim\mathfrak{C} + \dim\mathrm{Stab}(\rho) = \mathrm{ind}\,D_{A\#_\rho A}.$$

Since $\dim\mathfrak{C} = \dim\mathrm{Stab}(\rho) = 1$ and $\mathrm{ind}\,D_{A\#_\rho A} = 0$ by the index formula for a closed pair, $\mathrm{ind}\,D_A = -1$. This implies that

$$\dim H^2(S^1\times D^3\setminus S^1\times D^1;\mathrm{ad}\,A) = 0,$$

and hence the gluing theory is unobstructed at the flat connection. Morse–Bott gluing theory tells us that the moduli space $M_{\rho^\#}(W^o, S^o;\beta^\#,\beta_1^\#)_0$ has the structure of the union of fiber products as follows:

$$M(W^c, S^c;\beta^\#,\beta_1^\#)_d\times_{\mathfrak{C}} M_{\rho^\#}(S^1\times D^3, S^1\times D^1)^{\mathrm{irred}}_{d'}\quad\text{for } d+d'=1,$$
$$M(W^c, S^c;\beta^\#,\beta_1^\#)_1\times_{\mathfrak{C}} M_{\rho^\#}(S^1\times D^3, S^1\times D^1)^{\mathrm{red}}.$$

The first case is excluded for index reasons.

Consider the restriction map

$$r'\colon M_{\rho^\#}(S^1\times D^3, S^1\times D^1)^{\mathrm{red}}\to\mathfrak{C}.$$

By the holonomy condition $H^{\rho^\#}([A])=1$ on the moduli space $M(S^1\times D^3, S^1\times D^1)$, the image of r' consists of two points $\theta,\theta'\in\mathfrak{C}$. Hence, if the metric on (W^o, S^o) has a long neck along $(S^1\times S^2, S^2\times 2\mathrm{pt})$, the moduli space $M_{\rho^\#}(W^o, S^o;\beta^\#,\beta_1^\#)$ is two copies of

$$M(W^c, S^c;\beta^\#,\theta,\beta_1^\#)_0\times M(S^1\times D^3, S^1\times D^1;\theta)^{\mathrm{red}}.$$

In particular,

$$\begin{aligned}
&\sum_z \#M_{\rho^\#,z}(W^o,S^o;\beta^\#,\beta_1^\#)_0\lambda^{-\kappa(z)}T^{\nu(z)}\\
&\quad=2\sum_z\sum_{z'\circ z''=z}\#M_{z'}(S^1\times D^3,S^1\times D^1;\theta)^{\mathrm{red}}\lambda^{-\kappa(z')}T^{\nu(z')}\#M_{z''}(W^c,S^c;\beta^\#,\theta,\beta_1^\#)_0\lambda^{-\kappa(z'')}T^{\nu(z'')}\\
&\quad=2\Bigl(\sum_{k\le 0}c_kZ^k\Bigr)\sum_{z''}\#M_{z''}(W^c,S^c;\beta^\#,\theta,\beta_1^\#)_0\lambda^{-\kappa(z'')}T^{\nu(z'')}.
\end{aligned}$$

Since flat connections on $(S^1\times S^2, S^1\times 2\mathrm{pt})$ uniquely extend to $(S^1\times D^3, S^1\times D^1)$, we have $c_0=1$. Since $2\#M_{\rho^\#}(W^o,S^o;\beta^\#,\beta_1^\#)_0^e=\#M_{\rho^\#}(W^o,S^o;\beta^\#,\beta_1^\#)_0$, there is a unit element $C_1\in\mathscr{S}$ and we have

$$\langle m^+(\beta^\#),\beta_1^\#\rangle=C_1\sum_z\#M_z(W^c,S^c;\beta^\#,\theta,\beta_1^\#)_0\lambda^{-\kappa(z)}T^{\nu(z)}.$$

The same argument with $M_{\rho^\#\rho}(W^o,S^o;\beta^\#,\theta,\beta_1^\#)_0$, $M_{\rho^\#}(W^o,S^o;\beta^\#,\theta,\theta_\alpha^\#)_0$ and $M_{\rho^\#}(W^o,S^o;\theta_\alpha^\#,\theta,\beta_1^\#)_0$ instead of $M_{\rho^\#}(W^o,S^o;\beta^\#,\beta_1^\#)_0$ gives

$$\begin{aligned}
\langle\mu^+(\beta^\#),\beta_1^\#\rangle&=C_1\sum_z\#M_{\rho,z}(W^c,S^c;\beta^\#,\theta,\beta_1^\#)_0\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle\Delta_1^+(\beta^\#),1\rangle&=C_1\sum_z\#M_z(W^o,S^o;\beta^\#,\theta,\theta_\alpha^\#)_0\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle\Delta_2^+(1),\beta_1^\#\rangle&=C_1\sum_z\#M_z(W^o,S^o;\theta_\alpha^\#,\theta,\beta_1^\#)_0\lambda^{-\kappa(z)}T^{\nu(z)}.
\end{aligned}$$

Replacing the pair $(S^1\times D^3, S^1\times D^1)$ with $(D^2\times S^2, D^2\times 2\mathrm{pt})$, we obtain the product cobordism $[0,1]\times(Y\#Y', K\#K')$. By stretching the metric on $[0,1]\times(Y\#Y',K\#K')$ along the attaching domain, the moduli space $M([0,1]\times(Y\#Y',K\#K')\beta^\#,\beta_1^\#)_0$ has the structure of the union of fiber products

$$M(W^c,S^c;\beta^\#,\beta_1^\#)\times_{\mathfrak{C}}M(D^2\times S^2,D^2\times 2\mathrm{pt})^{\mathrm{red}}.\tag{A-13}$$

Let A' be an extended flat connection on $(D^2\times S^2, D^2\times 2\mathrm{pt})$ of the flat connection θ. Such A' uniquely exists. Moreover, it can be checked that the point Θ is unobstructed as follows. Consider the closed pair

$$(S^4,S^2):=(D^2\times S^2,D^2\times 2\mathrm{pt})\cup_{(S^1\times S^2,S^1\times 2\mathrm{pt})}(S^1\times D^3,S^1\times D^1)$$

and the glued reducible flat connection $A'\#_\theta A$ on (S^4,S^2). Then we have

$$\operatorname{ind}D_{A'\#_\theta A}=\operatorname{ind}D_{A'}+\dim\operatorname{Stab}(\theta)+\dim\mathfrak{C}+\operatorname{ind}D_A.$$

Since $b^1(X)=b^+(X)=0$ and $S\cong S^2$, the index formula for a closed pair shows that $\operatorname{ind}D_{A'\#A}=-1$. Moreover, $\operatorname{ind}D_A=-1$ by the previous argument. Thus $\operatorname{ind}D_{A'}=-2$.

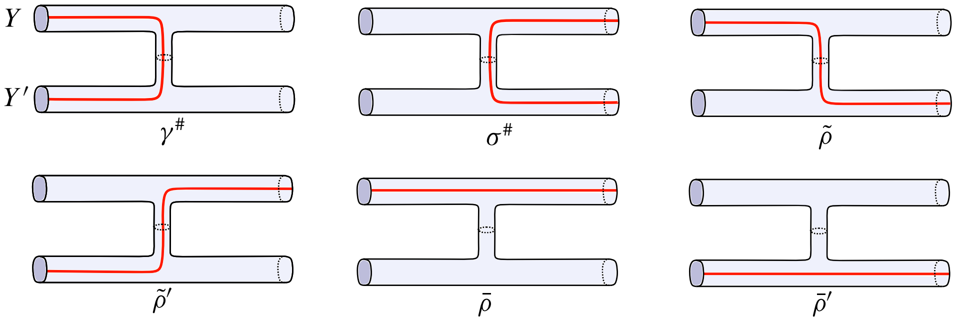

Figure 6: Paths on (W^I, S^I).

Since $\dim H^0(D^2 \times (S^2 \setminus 2\text{pt}); \operatorname{ad} A') = 1$ and $\dim H^1(D^2 \times (S^2 \setminus 2\text{pt}); \operatorname{ad} A') = 0$ by the computation of group cohomology, $\dim H^2(D^2 \times (S^2 \setminus 2\text{pt}); \operatorname{ad} A')$ vanishes.

Now, the fiber product structure (A-13) implies that there is a unit element $C_2 \in \mathscr{S}$ and

$$\langle m^+(\beta^\#), \beta_1^\# \rangle = C_2 \sum_z \# M_z([0,1] \times (Y \# Y', K \# K'); \beta^\#, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)}.$$

Similarly

$$\langle \mu^+(\beta^\#), \beta_1^\# \rangle = C_2 \sum_z \# M_{\rho,z}([0,1] \times (Y \# Y', K \# K'); \beta^\#, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\Delta_1^+(\beta^\#) = C_2 \sum_z \# M_z([0,1] \times (Y \# Y', K \# K'); \beta^\#, \theta_\alpha^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle \Delta_2^+(1), \beta^\# \rangle = C_2 \sum_z \# M_z([0,1] \times (Y \# Y', K \# K'); \theta_\alpha^\#, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)}.$$

Finally, there is a unit element $c \in \mathscr{S}$ and we have

$$\widetilde{m}^+ = c \widetilde{m}_{[0,1] \times (Y \# Y', K \# K')}.$$

The right-hand side is $\mathcal{S}$–chain homotopic to the identity since it is induced from the product cobordism. By construction, the unit element c has the top term 1, and hence $\widetilde{m}_{(W \circ W', S \circ S'; \rho^\#)}$ is $\mathcal{S}$–chain homotopic to the identity up to the multiplication of a unit element in $\mathscr{S}$. □

A.4 Step IV

Set $\bar{\rho} := \sigma \circ \gamma$, $\bar{\rho}' := \sigma' \circ \gamma'$, $\tilde{\rho} := \sigma' \circ \gamma$ and $\tilde{\rho}' := \sigma \circ \gamma'$; see Figure 6.

Proposition A.5 *We have that $\widetilde{m}_{(W', S')} \circ \widetilde{m}_{(W, S)}$ is $\mathcal{S}$–chain homotopic to $\widetilde{m}_{(W' \circ W, S' \circ S)}$.*

Proof Let G^I be a 1–parameter family of metrics stretching $(W^I, S^I) := (W' \circ W, S' \circ S)$ along $(Y \# Y', K \# K')$. Let $\widetilde{m}_{(W^I, S^I)}$ be the cobordism map for (W^I, S^I). We claim that there is an $\mathcal{S}$–chain homotopy H^I such that

$$\tilde{d}^\otimes H^I + H^I \tilde{d}^\otimes = \widetilde{m}_{(W', S')} \circ \widetilde{m}_{(W, S)} - \widetilde{m}_{(W^I, S^I)}.$$

Let us write each components of H^I as

$$H^I = \begin{bmatrix} K^I & 0 & 0 \\ L^I & -K^I & M_2^I \\ M_1^I & 0 & 0 \end{bmatrix},$$

where K^I and L^I are 4×4 matrices. Before defining each component of these matrices, we introduce the notation

$$\begin{aligned}
&\mathfrak{m}^I_z(\beta,\beta',\beta_1,\beta'_1) := \#\bigcup_{g\in G^I} M^g_z(W^I,S^I,\beta,\beta';\beta'_1,\beta'_1)_{-1},\\
&\mathfrak{m}^I_{\circ_1,\dots,\circ_d;z}(\beta,\beta',\beta_1,\beta'_1)\\
&\qquad\qquad := \#\Big\{[A]\in \bigcup_{g\in G^I} M^g_z(W^I,S^I,\beta,\beta';\beta'_1,\beta'_1)_{d-1} \mid H^{\circ_1}([A])=s_1,\dots,H^{\circ_d}([A])=s_d\Big\},
\end{aligned}$$

where $\circ_1,\dots,\circ_d$ are elements in the set of paths $\{\gamma^\#,\sigma^\#,\bar\rho,\bar\rho',\tilde\rho,\tilde\rho'\}$. Then each component of K^I, L^I, M^I_1 and M^I_2 is given as follows:

- components of K^I

$$\begin{aligned}
\langle K^I_{11}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{\gamma^\#;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{12}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{13}(\beta),\beta_1\otimes\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{z}(\beta,\theta'_\alpha;\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{14}(\beta'),\beta_1\otimes\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{21}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{\gamma^\#,\sigma^\#;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{22}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{\sigma^\#;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{23}(\beta),\beta_1\otimes\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{\sigma^\#;z}(\beta,\theta'_\alpha;\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{24}(\beta'),\beta_1\otimes\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{\sigma^\#;z}(\theta_\alpha,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{31}(\beta\otimes\beta'),\beta_1\rangle &= \sum_z \mathfrak{m}^I_{\gamma^\#;z}(\beta,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{32}(\beta\otimes\beta'),\beta_1\rangle &= \sum_z \mathfrak{m}^I_{z}(\beta,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{33}(\beta),\beta_1\rangle &= \sum_z \mathfrak{m}^I_{z}(\beta,\theta'_\alpha;\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{34}(\beta'),\beta_1\rangle &= \sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{41}(\beta\otimes\beta'),\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{\gamma^\#;z}(\beta,\beta';\theta_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{42}(\beta\otimes\beta'),\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{z}(\beta,\beta';\theta'_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{43}(\beta),\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{z}(\beta,\theta'_\alpha;\theta'_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},\\
\langle K^I_{44}(\beta'),\beta'_1\rangle &= \sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\beta';\theta'_\alpha,\beta_1)\lambda^{-\kappa(z)}T^{\nu(z)},
\end{aligned}$$

- components of L^I

$$\langle L^I_{11}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\gamma^\#,\bar\rho;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{12}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{13}(\beta),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho;z}(\beta,\theta'_\alpha;\beta_1,\beta'_1)_{\lambda}^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{14}(\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho';z}(\theta_\alpha,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{21}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#,\bar\rho;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{22}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#,\bar\rho;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{23}(\beta),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#,\bar\rho;z}(\beta,\theta'_\alpha;\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{24}(\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#,\bar\rho';z}(\theta_\alpha,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{31}(\beta\otimes\beta'),\beta_1\rangle=\sum_z \mathfrak{m}^I_{\gamma^\#,\bar\rho;z}(\beta,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{32}(\beta\otimes\beta'),\beta_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho;z}(\beta,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{33}(\beta),\beta_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho';z}(\beta,\theta'_\alpha;\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{34}(\beta'),\beta_1\rangle=\sum_z \mathfrak{m}^I_{\tilde\rho';z}(\theta_\alpha,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{41}(\beta\otimes\beta'),\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\gamma^\#,\bar\rho';z}(\beta,\beta';\theta_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{42}(\beta\otimes\beta'),\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\tilde\rho;z}(\beta,\beta';\theta'_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{43}(\beta),\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\tilde\rho;z}(\beta,\theta'_\alpha;\theta'_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{44}(\beta'),\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho';z}(\theta_\alpha,\beta';\theta'_\alpha,\beta_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

- components of M^I_1

$$M^I_{1,1}(\beta\otimes\beta')=\sum_z \mathfrak{m}^I_{\gamma^\#;z}(\beta,\beta';\theta_\alpha,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$M^I_{1,2}(\beta\otimes\beta')=\sum_z \mathfrak{m}^I_{z}(\beta,\beta';\theta_\alpha,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$M^I_{1,3}(\beta)=\sum_z \mathfrak{m}^I_{z}(\beta,\theta'_\alpha;\theta_\alpha,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},\quad M^I_{1,4}(\beta')=\sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\beta';\theta_\alpha,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$

- components of M^I_2

$$\langle M^I_{2,1}(1),\beta\otimes\beta'\rangle=\sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\theta'_\alpha;\beta,\beta')\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle M^I_{2,2}(1),\beta\otimes\beta'\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#;z}(\theta_\alpha,\theta'_\alpha;\beta,\beta')\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle M^I_{2,3}(1),\beta\rangle=\sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\theta'_\alpha;\beta,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},\quad \langle M^I_{2,4}(1),\beta'\rangle=\sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\theta'_\alpha;\theta_\alpha,\beta')\lambda^{-\kappa(z)}T^{\nu(z)}.$$

component of (A-14)	corresponding family of moduli spaces
(1, 1)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \beta_1')_0 \mid H^{\gamma^\#}([A]) = s\}$
(1, 2)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \beta_1')_0$
(1, 3)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \theta_\alpha'; \beta_1, \beta_1')_0$
(1, 4)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \theta_\alpha, \beta'; \beta_1, \beta_1')_0$
(2, 1)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \beta_1')_2 \mid H^{\gamma^\#}([A]) = s, H^{\sigma^\#}([A]) = t\}$
(2, 2)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \beta_1')_1 \mid H^{\sigma^\#}([A]) = s\}$
(2, 3)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \theta_\alpha'; \beta_1, \beta_1')_1 \mid H^{\sigma^\#}([A]) = s\}$
(2, 4)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \theta_\alpha, \beta'; \beta_1, \beta_1')_1 \mid H^{\sigma^\#}([A]) = s\}$
(3, 1)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \theta_\alpha')_1 \mid H^{\gamma^\#}([A]) = s\}$
(3, 2)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \theta_\alpha')_0$
(3, 3)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \theta_\alpha'; \beta_1, \theta_\alpha')_1 \mid H^{\bar{\rho}'}([A]) = t\}$
(3, 4)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \theta_\alpha, \beta'; \beta_1, \theta_\alpha')_0$
(4, 1)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \theta_\alpha, \beta_1')_1 \mid H^{\gamma^\#}([A]) = s\}$
(4, 2)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \theta_\alpha', \beta_1')_0$
(4, 3)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \theta_\alpha'; \theta_\alpha', \beta_1')_0$
(4, 4)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \theta_\alpha, \beta'; \theta_\alpha', \beta_1)_0$

Table 1

Then we can check that there are the following identities:

$$d^\otimes K^I + K^I d^\otimes = m'm - m^I, \tag{A-14}$$

$$v^\otimes K^I - d^\otimes L^I + \delta_2^\otimes M_1^I + L^I d^\otimes - K^I v^\otimes + M_2^I \delta_1^\otimes = \mu' m + m' \mu + \Delta_2' \Delta_1 - \mu^I, \tag{A-15}$$

$$\delta_1^\otimes K^I + M_1^I d^\otimes = \Delta_1' m + \Delta_1 - \Delta_1^I, \tag{A-16}$$

$$-d^\otimes M_2^I - K^I \delta_2^\otimes = m' \Delta_2 + \Delta_2' - \Delta_2^I. \tag{A-17}$$

The identities above are proved by counting oriented boundaries of corresponding moduli spaces. For example, such moduli spaces for identity (A-14) are given in Table 1. Other identities can be proved in similar ways. □

Proposition A.6 *We have that $\widetilde{m}_{(W' \circ W, S' \circ S)}$ is $\mathcal{S}$–chain homotopic to $\mathrm{id}_{\widetilde{C}^\otimes}$.*

Proof Consider a family of metrics G'^I on (W^I, S^I) which stretch the cobordism along (S^3, S^1) as in Figure 7. Let $\widetilde{m}^I$ be the map defined by a long stretched metric on (W^I, S^I). The family of metrics G'^I gives an $\mathcal{S}$–chain homotopy between $\widetilde{m}_{(W^I, S^I)}$ and $\widetilde{m}^I$. Let $(W^{c'}, S^{c'})$ be a disjoint union

$$(Y \times [0,1] \setminus D^4, K \times [0,1] \setminus D^2) \sqcup (Y' \times [0,1] \setminus D^4, K' \times [0,1] \setminus D^2).$$

We can also define $\widetilde{m}^I$ by counting instantons on $(W^{c'}, S^{c'})$. We will show that $\widetilde{m}^I$ is an isomorphism of $\mathcal{S}$–complexes. We obtain a pair of cylinders $[0,1] \times (Y \sqcup Y', K \sqcup K')$ by gluing back two pairs of disks

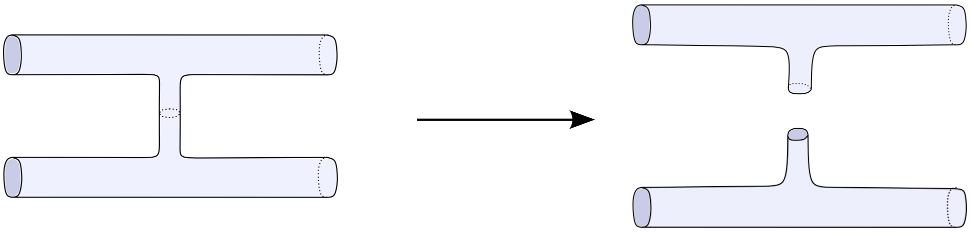

Figure 7: Family of metrics G'^I.

(D^4, D^2) to $(W^{c'}, S^{c'})$. Consider the character variety $\mathfrak{C}'$ with the holonomy parameter α on (S^3, S^1). For $0 < \alpha < \frac{1}{2}$, $\mathfrak{C}'$ is a one-point set which consists of the unique flat reducible θ_α, and the moduli space $M(D^4, D^2; \theta_\alpha)_0$ also consists of one element Θ_α which is the unique extension of θ_α to $D^4 \setminus D^2$. The computation of group cohomology of $\pi_1(S^3 \setminus S^1)$ shows that $\dim H^1(S^3 \setminus S^1; \operatorname{ad}\theta_\alpha) = 0$. Taking the double of (D^4, D^2), we have the relation of indices

$$2 \operatorname{ind} D_{\Theta_\alpha} + 1 = \operatorname{ind} D_{\Theta_\alpha \# \Theta_\alpha}.$$

Moreover, $\operatorname{ind} D_{\Theta_\alpha \# \Theta_\alpha} = -1$ by the index formula for the closed pair (S^4, S^2). Thus $\operatorname{ind} D_{\Theta_\alpha} = -1$ and $H^2(D^4 \setminus D^2; \operatorname{ad}\Theta_\alpha) = 0$. In particular, the gluing along $\mathfrak{C}'$ is unobstructed. The Morse–Bott gluing argument shows that

$$M([0,1] \times Y, [0,1] \times K; \beta, \beta_1)_d = M([0,1] \times Y \setminus D^4, [0,1] \times K \setminus D^2; \beta, \theta_\alpha, \beta')_d,$$

and similarly for the pair (Y', K'). Thus

$$\begin{aligned}\#M_z^{g^\infty}(W^I, S^I, \beta, \beta'; \beta_1, \beta_1') &= \#M_{z'}(W^I, S^I; \beta, \theta_\alpha, \beta_1) \#M_{z''}(W^I, S^I; \beta', \theta_\alpha, \beta_1') \\ &= \#M_{z'}(Y \times [0,1], K \times [0,1]; \beta, \beta_1) \#M_{z''}(Y' \times [0,1], K' \times [0,1]; \beta', \beta_1').\end{aligned}$$

Therefore $\widetilde{m}_{(W^I, S^I)}$ is $\mathcal{S}$–chain homotopic to the morphism $\widetilde{m}_{\text{prod}}$ which is induced from the product cobordism $(Y \sqcup Y', K \sqcup K') \times [0,1]$. The $\mathcal{S}$–morphism $\widetilde{m}_{\text{prod}}$ is an isomorphism of $\mathcal{S}$–complexes (see [9, Lemma 6.29]), and in fact $\mathcal{S}$–chain homotopic to the identity by the formal argument. □

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