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On the invariance of the Dowlin spectral sequence

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Given a link L, Dowlin constructed a filtered complex inducing a spectral sequence with E_2–page isomorphic to the Khovanov homology $\overline{\mathrm{Kh}}(L)$ and E_∞–page isomorphic to the knot Floer homology $\widehat{HFK}(m(L))$ of the mirror of the link. We prove that the E_k–page of this spectral sequence is also a link invariant, for $k \geq 3$.

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1 Introduction

Dowlin [2024] associated a filtered chain complex to a link L. The spectral sequence this filtered complex gives rise to has E_2–page isomorphic to the (reduced) Khovanov homology $\overline{\mathrm{Kh}}(L)$ and converges to the knot Floer homology $\widehat{HFK}(m(L))$ of the mirror of the link. The fact that the E_2– and E_∞–pages of the spectral sequence are link invariants, independent of the diagram used to construct the filtered complex, suggests that the same may be true of all the higher pages of the spectral sequence. This is the main result of this paper.

Theorem 1.1 *For $k \geq 2$, the E_k–page of Dowlin's spectral sequence does not depend on the diagram used to construct the filtered complex, and is thus a link invariant.*

This theorem provides a whole family of link invariants $\{E_k(L)\}_{k=2}^{\infty}$. The invariance of these higher pages of the Dowlin spectral sequence helps us further decipher the connection between Khovanov homology and knot Floer homology.

This result opens several research directions. The first is to find knots (or families of knots) which have the same Khovanov homology and knot Floer homology, but are distinguished by these higher page invariants. The ranks of Khovanov homology and knot Floer homology tend to coincide for knots with few crossings [Rasmussen 2005], so finding such examples may be computationally difficult.

A second direction is to consider implications in the study of transverse links. Plamenevskaya [2006] identified an invariant of transverse links $\psi(L) \in \mathrm{Kh}(L)$, which we can think of as residing in the E_2 page of the Dowlin spectral sequence. One could hope to define a countable family of transverse link invariants $\{\psi_k(L)\}_{k=2}^{\infty}$ by taking the image of ψ on each higher page E_k for $k \geq 2$, in the style of Baldwin [2011]. It might prove interesting to compare these invariants, especially the image of ψ on the E_∞ page $\widehat{HFK}(m(L))$ with known transverse link invariants [Baldwin et al. 2013].

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A third direction for future work would be the investigate potential relationships between the s invariant in Khovanov homology [Rasmussen 2010] and the τ invariant in knot Floer homology [Ozsváth and Szabó 2003]. For many knots, these invariants are related by the equation $s = 2\tau$; however, we also know of knots that break this rule [Hedden and Ording 2008]. Perhaps the spectral sequence could be used to explain this (lack of a) pattern.

Organization

We begin by reviewing the construction of $\mathrm{C}_2^-(L)$ in Section 2; as originally defined by Dowlin [2024], this filtered complex induces the spectral sequence from $\overline{\mathrm{Kh}}(L)$ to $\widehat{HFK}(m(L))$ for a given link L. In Section 3, we prove that the homotopy type of this complex is invariant under a diagrammatic change we call "relabeling vertices". We then discuss MOY moves, another set of operations on diagrams, and define maps associated to these moves in Section 4. With these in hand, we prove invariance of the higher pages of the spectral sequence in Section 5.

Conventions

There are a few homological algebra conventions that we need to establish.

- We call our complexes chain complexes, despite the fact that our differentials usually have degree 1 with respect to the homological grading.
- Our filtrations are *descending*, which is to say that $\mathscr{F}_i M \supseteq \mathscr{F}_j M$ when $i \leq j$.
- A *filtered quasi-isomorphism* $f : A \to B$ is a filtered chain map which induces a quasi-isomorphism between the associated graded complexes $\mathrm{gr}(f) \colon \mathrm{gr}(A) \to \mathrm{gr}(B)$. In other words, a filtered quasi-isomorphism induces a quasi-isomorphism between E_0–pages of spectral sequences, and equivalently induces isomorphisms between E_1–pages. If A and B are connected by a zigzag of filtered quasi-isomorphisms, then they have the same weak filtered homotopy type, a relationship which we denote by $A \simeq B$.
- Because the E_1–page of the filtered complex C_2^- is isomorphic to the Khovanov *complex*, and not the Khovanov *homology*, we need to work with invariance maps which are not filtered quasi-isomorphisms. Instead, they only induce quasi-isomorphisms on the E_1–pages, or equivalently induce isomorphisms on the E_2–pages. We call these maps E_1*–quasi-isomorphisms* (terminology from [Cirici et al. 2020]). As above, we write $A \simeq_1 B$ to denote that A and B are connected by a zigzag of E_1–quasi-isomorphisms.
- Since we work with two different notions of weak equivalence, we also need two different mapping cones for a filtered map $f \colon A \to B$, denoted by $\mathrm{cone}(f)$ and $\mathrm{cone}_1(f)$. Both of them have the same underlying unfiltered complex, but differ in the definition of the filtration. The former filtration is defined to be $\mathscr{F}_i(\mathrm{cone}(f)) = \mathscr{F}_i A \oplus \mathscr{F}_i B$, whereas the latter filtration is given by $\mathscr{F}_i(\mathrm{cone}_1(f)) = \mathscr{F}_i A \oplus \mathscr{F}_{i-1} B$.

Acknowledgements

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2 The spectral sequence

In this section, we review the construction of the spectral sequence from $\overline{\mathrm{Kh}}(L)$ to $\widehat{HFK}(L)$ for a link L, as originally defined by Dowlin [2024]. The spectral sequence arises from a filtered chain complex $\mathrm{C}_2^-(D)$ constructed from a *partially singular braid diagram* D associated to an unoriented link L. In Section 2.A, we define these diagrams, and in Section 2.B we associate a filtered chain complex to each such diagram. Finally, in Section 2.C, we discuss how to associate a partially singular braid diagram to an unoriented link L, and we characterize the set of moves connecting any two such partially singular braid diagrams.

2.A Partially singular braid diagrams

In this section, we define the types of diagrams we need to construct the spectral sequence.

We start by establishing some conventions regarding braid diagrams. If D is a closed braid diagram, we can consider it as a 4–valent graph embedded in $\mathbb{R}^2$ with vertices $V(D)$ the set of crossings, and edges $E(D)$ the set of arcs connecting these crossings. This agrees with the usual way of representing link diagrams as graphs. Given a graph G, recall that a *subdivision* H of G is a graph obtained by adding 2–valent vertices along edges of G.

Definition 2.1 A (*closed*) *partially singular braid diagram* is an oriented graph embedded in $\mathbb{R}^2$ which can be obtained as a subdivision of a closed braid diagram, equipped with the following extra information:

- a labeling of every 4–valent vertex as “positive”, “negative”, or “singular”,
- a further labeling of every singular vertex as either “fixed” or “free”, and
- exactly one distinguished edge, which is called the “decorated” edge.

An *open* partially singular braid diagram is defined identically to a closed one, except that it also has $2n$ 1–valent vertices (assuming n strands) corresponding to the endpoints of the strands. When drawing partially singular braid diagrams, we indicate fixed singular vertices by drawing a circle around them, as

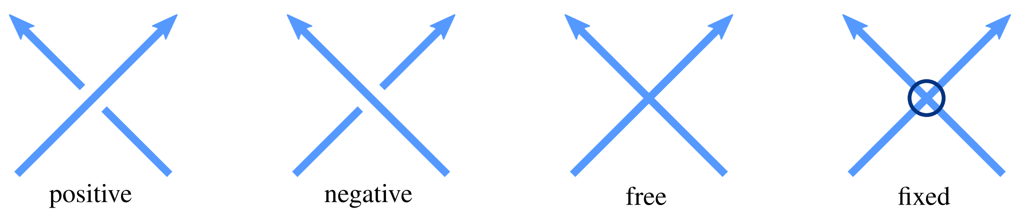


Figure 1: The different types of vertices in a partially singular braid diagram.

Figure 2: Other features that can occur in a braid diagram.

in Figure 1; 2–valent vertices are drawn simply as dots on the strands, and the decorated edge is denoted by two small lines, as in Figure 2.

Throughout, we assume the decorated edge is leftmost in the diagram. We also assume a fixed ordering of the vertices whenever we consider a partially singular braid diagram D. We let $\mathrm{Fixed}(D)$ denote the set of fixed singular vertices of D and $\mathrm{Free}(D)$ denote the set of free singular vertices of D.

A (*fully*) *singular braid diagram* is a partially singular braid diagram with no crossings. This type of diagram may arise from resolving a partially singular braid diagram D in the following sense. Let D be a partially singular braid diagram, with $c(D)$ the set of crossings of D; then a *resolution*, a function $I: c(D) \to \{0,1\}$, gives a fully singular braid diagram D_I by resolving each crossing according to Figure 3. In words, the 0–resolution of a positive crossing is a singular vertex, and the 1–resolution is the oriented smoothing with two subdivided edges. The 0– and 1–resolutions of a negative crossing are the 1– and 0–resolutions of a positive crossing, respectively. If a fully singular braid diagram S arises as a complete resolution of a partially singular braid diagram D, then $\mathrm{Fixed}(S) = \mathrm{Fixed}(D)$, and $\mathrm{Free}(S)$ contains all crossings in $\mathrm{Free}(D)$ as well as those which were singularized in the resolution.

2.B The filtered complex $\mathrm{C}_2^-(D)$

In this section, we recall Dowlin's construction of the filtered chain complex $\mathrm{C}_2^-(D)$ which gives rise to the spectral sequence connecting Khovanov homology to knot Floer homology. Throughout, let D be a

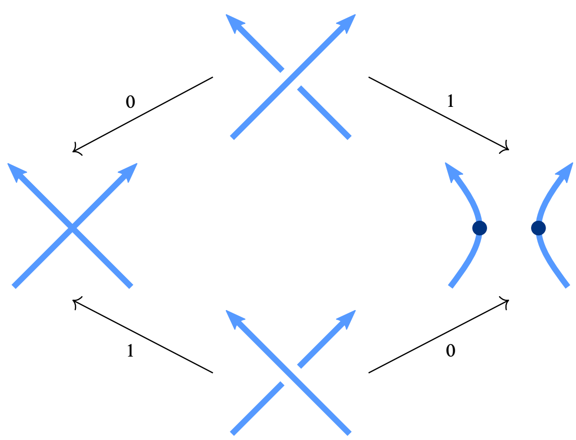


Figure 3: The 0– and 1–resolutions of positive and negative crossings.

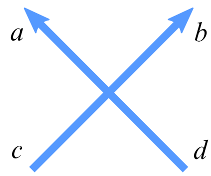


Figure 4: The local edge labels around a vertex.

partially singular braid diagram and I a resolution of D giving rise to the fully singular braid diagram D_I. We first construct $\mathrm{C}_2^-(D_I)$ for each resolution I, then combine these into a cube complex $\mathrm{C}_2^-(D)$ by adding "edge maps".

To begin, label each edge of D by a unique integer from 1 to $k = |E(D)|$, and let $R(D) = \mathbb{Q}[U_1, \dots, U_k]$ be the polynomial ring over $\mathbb{Q}$ generated by one variable for each edge. Note that, whenever crossings in D are resolved to get a diagram D', there is a natural bijection between edges in D and edges in D', so we can extend our edge labels to any resolution of D. To each vertex $v \in V(D)$ we associate two polynomials, $L(v)$ and $L^+(v)$. Label the adjacent edges to each vertex $v \in V(D)$ as in Figure 4; if we draw the vertex such that all edges are oriented upwards, then we label the edge in the top left by a, the remaining edges by b, c, and d as we traverse clockwise from the edge labeled a. Define $L(v) = U_a + U_b - U_c - U_d$ and $L^+(v) = U_a + U_b + U_c + U_d$.

One factor of $\mathrm{C}_2^-(D_I)$ does not depend on the specific resolution but only on D; we denote this factor by $\mathscr{L}_D^+$. Let

$$\mathscr{L}_D^+ := \bigotimes_{v \in \mathrm{Fixed}(D)} R(D) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(D).$$

It should be noted that $\mathscr{L}_D^+$ is not a chain complex, but rather a *matrix factorization* (or *curved complex*). A *matrix factorization* is a module M equipped with an endomorphism $\partial\colon M \to M$ such that $\partial^2 = \omega\,\mathrm{id}_M$ for some potentially nonzero scalar ω, which is called the *potential* of the matrix factorization. Despite the fact that ∂ does not square to zero, we may still refer to it as a *differential* on M; this is hopefully clear from context. In the case of $\mathscr{L}_D^+$, $\omega = \sum_{v \in \mathrm{Fixed}(D)} L(v)L^+(v)$, which is often nonzero in $R(D)$. Matrix factorizations are well-studied algebraic objects, but for our purposes we only need a few facts about them; these can be found in Section 3.

The other factor of $\mathrm{C}_2^-(D_I)$, which is not the same for every I and depends on the specific resolution, is the $R(D)$–module $Q(D_I) = R(D)/(L(D_I) + N(D_I))$. Here, $L(D_I)$ and $N(D_I)$ are two ideals of $R(D)$. The first of these is the *linear ideal* $L(D_I)$, defined as

$$L(D_I) := \sum_{v \in \mathrm{Free}(D_I)} (L(v)).$$

The second is the *nonlocal ideal* $N(D_I)$. Let Ω be a smoothly embedded disk in $\mathbb{R}^2$ that does not contain the decorated edge, and such that the boundary only intersects D transversely at edges. Let $\mathrm{In}(\Omega)$ (resp.

Out(Ω)) denote the set of edges that intersect the boundary of Ω and are oriented inward (resp. outward). We define $N(\Omega)$ to be the polynomial

$$N(\Omega) := \prod_{i \in \mathrm{Out}(\Omega)} U_i - \prod_{j \in \mathrm{In}(\Omega)} U_j.$$

The nonlocal ideal $N(D_I)$ is then generated by $N(\Omega)$ for all such embedded disks Ω:

$$N(D_I) := \sum_{\Omega} (N(\Omega)).$$

With the above definitions in hand, the complex $\mathrm{C}_2^-(D_I)$ is then defined as

$$\begin{aligned}\mathrm{C}_2^-(D_I) &:= Q(D_I) \otimes \mathscr{L}_D^+ \\ &:= R(D)/(L(D_I) + N(D_I)) \otimes \Big(\bigotimes_{v \in \mathrm{Fixed}(D)} R(D) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(D) \Big).\end{aligned}$$

It is shown in [Dowlin 2024, Lemma 2.4] that the potential ω of $\mathscr{L}_D^+$ is contained in $L(D_I) + N(D_I)$, and thus is zero in $Q(D_I)$. Thus, the endomorphism of $\mathrm{C}_2^-(D_I)$ induced by $\mathscr{L}_D^+$ squares to 0, so it is truly a differential; we denote it by d_0.

As a module, define

$$\mathrm{C}_2^-(D) := \bigoplus_{I \in \{0,1\}^{c(D)}} \mathrm{C}_2^-(D_I).$$

The differential on $\mathrm{C}_2^-(D)$ is defined as a sum $d_0 + d_1$, where d_0 is induced by the differential d_0 on the summands $\mathrm{C}_2^-(D_I)$, and d_1 is induced by edge maps that we have yet to define. In order to do so, we must first restrict the set of partially singular braid diagrams we are working with.

Definition 2.2 [Dowlin 2024, Definition 2.2] The set $\mathscr{D}^{\mathscr{R}}$ contains all partially singular braid diagrams D satisfying the following conditions for all $I \in \{0,1\}^{c(D)}$:

- D_I is connected, and
- the linear terms $L(v)$ for $v \in \mathrm{Free}(D_I)$ form a regular sequence[1] over $R(D)/N(D_I)$.

The latter condition is an algebraic restriction which is used in the proof of Theorem 3.1. It is equivalent to the existence of an ordering $v_1, \dots, v_k$ of the vertices in Free(D_I) such that $L(v_j)$ is not a zero divisor in $R(D)/(N(D_I) + (L(v_1), \dots, L(v_{j-1})))$ for each $1 \le j \le k$. Since $R(D)$ is a graded ring and the linear terms $L(v)$ are homogeneous of positive degree, if this condition is true for one ordering of Free(D_I), it is true for *every* ordering.

For the rest of the definition of $\mathrm{C}_2^-(D)$, we assume $D \in \mathscr{D}^{\mathscr{R}}$. Let I and J be two resolutions with $I \prec J$, ie I and J agree on all crossings except a single $c \in c(D)$, where $I(c) = 0$ and $J(c) = 1$. Let v be the vertex corresponding to c, and label the edges adjacent to v according to Figure 4.

[1] The $\mathscr{R}$ in $\mathscr{D}^{\mathscr{R}}$ likely stands for "regular".

The edge map $d_{I,J}\colon \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D_J)$ depends on whether c is a positive or negative crossing. If I and J differ at a positive crossing, let $\phi_+\colon Q(D_I) \to Q(D_J)$ be the unique $R(D)$–module map such that $\phi_+(1) = 1$, and define the edge map $d_{I,J}\colon \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D_J)$ to be $d_{I,J} = \phi_+ \otimes \mathrm{id}_{\mathscr{L}_D^+}$. Else, I and J differ at a negative crossing v. In this case, let $\phi_-\colon Q(D_I) \to Q(D_J)$ be the unique $R(D)$–module map such that $\phi_-(1) = U_b - U_c$, and define the edge map $d_{I,J}\colon \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D_J)$ to be $d_{I,J} = \phi_- \otimes \mathrm{id}_{\mathscr{L}_D^+}$. We may occasionally overload notation by referring to the edge map $d_{I,J}$ as $\phi_\pm$ when there is no risk of confusion.

Combining all of these maps together into a single map induces $d_1\colon \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D)$, given by

$$d_1 := \sum_{I \prec J} \epsilon(I, J) d_{I,J}.$$

Here, $\epsilon(I, J)$ is a *sign assignment*, which is a labeling of the edges of the cube of resolutions by $\{\pm 1\}$ satisfying the property that every square face has an odd number of -1–labeled edges. Such a sign assignment ensures that $(d_1)^2 = 0$, and any two choices of ϵ result in isomorphic complexes. As one example, we may let $\epsilon(I, J) = (-1)^k$, where k is the number of 1's that come before the place at which I and J differ, as in [Bar-Natan 2002]. We further abuse notation by referring to d_1, the signed sum of the edge maps $d_{I,J}$ for all $I \prec J$, itself as an edge map.

Consider $\mathrm{C}_2^-(D)$ as a chain complex with total differential $d_0 + d_1$. We filter $\mathrm{C}_2^-(D)$ by weight in the cube of resolutions, ie the filtration on $\mathrm{C}_2^-(D)$ is given by

$$\mathscr{F}_p \mathrm{C}_2^-(D) := \bigoplus_{w(I) \geq p} \mathrm{C}_2^-(D_I),$$

where $w(I) = \sum_{c \in c(D)} I(c)$ is the *weight* of I, ie the number of 1–resolved crossings of D_I. Note that d_0 preserves the weight, and d_1 increases it by 1, so the differential on $\mathrm{C}_2^-(D)$ is indeed filtered with respect to this decomposition.

Remark 2.3 We could have alternately defined $\mathrm{C}_2^-(D)$ by first defining $\mathrm{C}_2^-(S)$ for fully singular braid diagrams S, then defining $\mathrm{C}_2^-(D)$ to be the mapping cone

$$\mathrm{C}_2^-(D) := \mathrm{cone}_1(\phi \otimes \mathscr{L}_D^+) = (\mathrm{C}_2^-(D_0) \to \mathrm{C}_2^-(D_1)),$$

where D_0 and D_1 above are the 0– and 1–resolutions of a particular crossing, and $\phi\colon Q(D_0) \to Q(D_1)$ is the associated map of quotient modules. Iterating this construction produces a filtered complex that is isomorphic to the one that we defined previously.

2.C Diagrams associated to a link

Each partially singular braid diagram gives rise to an unoriented link by taking the *unoriented smoothing*.

Definition 2.4 Let D be a partially singular braid diagram. The *unoriented smoothing* $\mathrm{sm}(D)$ is the unoriented link obtained from D by smoothing each singular vertex in the way that does not respect the orientation.

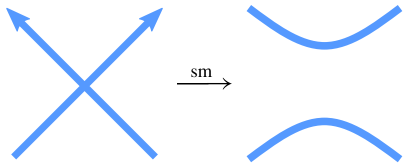


Figure 5: Unoriented smoothing of a crossing.

Figure 5 shows a local picture of smoothing a singular vertex, and Figure 6 gives an example of a partially singular braid diagram and the link obtained by taking the unoriented smoothing.

When $\mathrm{sm}(D)$ is an ℓ–component link, we can construct a "reduced" version of $\mathrm{C}_2^-(D)$. First, choose a set of edges $e_1, \dots, e_\ell \in E(D)$ such that each e_i is on a distinct component of $\mathrm{sm}(D)$. Then, let

$$\widehat{C}_2(D) := \mathrm{C}_2^-(D) \otimes \bigotimes_{e_i} (R(D) \xrightarrow{U_{e_i}} R(D)).$$

We define the differentials given by multiplication by U_{e_i} to have weight filtration degree 1. Therefore, we get a weight filtration on $\widehat{C}_2(D)$ induced by the above definition as a tensor product of filtered complexes. This is the filtered complex that is used to define the spectral sequence relating Khovanov homology and knot Floer homology.

Theorem 2.5 [Dowlin 2024, Theorem 1.6] *Let $D \in \mathfrak{D}^{\mathcal{R}}$ be a partially singular braid diagram with $\mathrm{sm}(D) = L$. The spectral sequence induced by the weight filtration on $\widehat{C}_2(D)$ has E_2–page isomorphic to $\overline{\mathrm{Kh}}(L)$ and converges to $\widehat{HFK}(L)$.*

Dowlin [2024] proves that every link can be realized as the unoriented smoothing of a diagram in $\mathfrak{D}^{\mathcal{R}}$ by first considering a braid whose plat closure is the desired link, then turning that braid into a partially singular braid diagram. We go about things similarly, but instead choose a different way of embedding braid closures into $\mathfrak{D}^{\mathcal{R}}$ that better fits our particular invariance proofs.

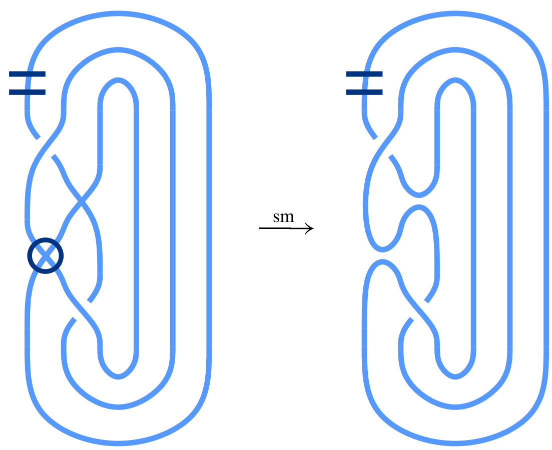


Figure 6: A diagram D and its unoriented smoothing $\mathrm{sm}(D)$.

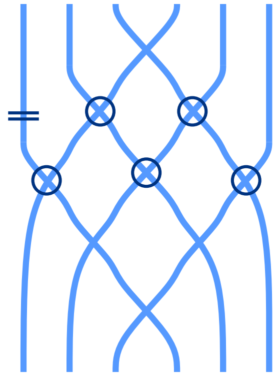


Figure 7: The partially singular braid diagram I_n in the case $n = 3$.

Proposition 2.6 *Let L be an unoriented link. There is a partially singular braid diagram $D \in \mathfrak{D}^{\mathcal{R}}$ such that $\mathrm{sm}(D) = L$.*

To prove Proposition 2.6, we make use of a special partially singular open braid diagram which we denote by I_n. This open diagram I_n consists of $2n$ upward oriented strands with $2n-1$ layers of singular vertices. The layers are symmetric, meaning layer i has singular vertices between the same strands as layer $2n-i$ for $1 \leq i < n$. The first layer has a singular vertex between the strands n and $n+1$. The second layer has two singular vertices; one between strands $n-1$ and n and one between strands $n+1$ and $n+2$. In general, the i^{th} layer has i consecutive singular vertices, beginning with one between strands $n+1-i$ and $n+2-i$ and ending with one between strands $n-1+i$ and $n+i$. We let $\mathrm{Fixed}(I_n)$ be the singular vertices in layers n and $n+1$, and let $\mathrm{Free}(I_n)$ be the rest of the singular vertices. See Figure 7 for I_n in the case $n = 3$.

Definition 2.7 Given a braid $\beta \in B_n$, let $I_n(\beta)$ denote the partially singular braid diagram D built by putting n downward-oriented strands to the right of β, and putting I_n above and taking the braid closure.

Proof of Proposition 2.6 Given an unoriented link L, let β be a braid with braid closure $\mathrm{cl}(\beta)$ isotopic to L, the existence of which is guaranteed by Alexander's theorem [1923]. The unoriented smoothing $\mathrm{sm}(I_n(\beta))$ is isotopic to the braid closure $\mathrm{cl}(\beta)$ of β itself, so $D = I_n(\beta)$ is a partially singular braid diagram with $\mathrm{sm}(D)$ isotopic to L. That $D \in \mathfrak{D}^{\mathcal{R}}$ is an application of [Dowlin 2024, Lemma 7.1]. More specifically, D contains a vertically mirrored copy of the open braid diagram S_{2n} defined in [Dowlin 2024], where it is proven that any such diagram is in $\mathfrak{D}^{\mathcal{R}}$. □

See Figure 8 for an example of the process of constructing a partially singular braid diagram with specified unoriented smoothing.

Let $\mathfrak{D}^{\mathcal{B}} = \{I_n(\beta) \mid \beta \in B_n, n \in \mathbb{Z}\}$ be the set of partially singular braid diagrams constructed as above.[2] Then we have the following classification theorem.

[2]Here, the $\mathcal{B}$ in $\mathfrak{D}^{\mathcal{B}}$ stands for "braid".

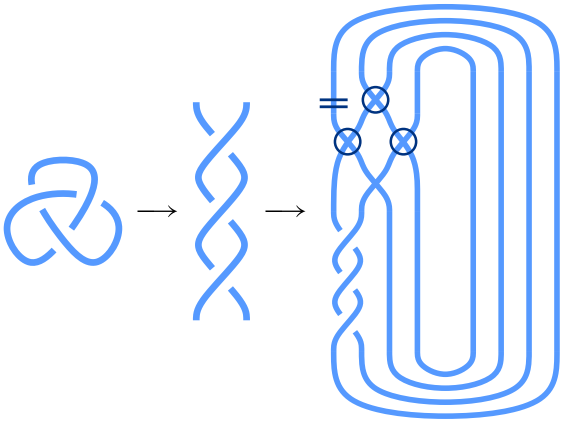


Figure 8: The process of constructing a partially singular braid diagram with smoothing isotopic to a given knot.

Theorem 2.8 *Two diagrams in* $\mathscr{D}^{\mathscr{B}}$ *have the same unoriented smoothing if and only if the underlying braids are connected by a finite sequence of Reidemeister II moves*, *Reidemeister III moves*, (*de*)*stabilizations*, *and conjugations.*

Proof This is just Markov's theorem [1936], repackaged. □

Since $\mathscr{D}^{\mathscr{B}} \subset \mathscr{D}^{\mathscr{R}}$, we can construct the complex $\mathrm{C}_2^-(D)$ for any $D \in \mathscr{D}^{\mathscr{B}}$. We overload notation by writing $\mathrm{C}_2^-(\beta)$ instead of $\mathrm{C}_2^-(I_n(\beta))$ for $\beta \in B_n$. We prove invariance of $\mathrm{C}_2^-(\beta)$ under the moves in Theorem 2.8 in Section 5 using maps defined in Section 4.

3 Vertex relabeling

Before we continue towards a proof of invariance, we detour to comment on a quirk of the construction of $\mathrm{C}_2^-(D)$. One natural question to ask is why $\mathrm{C}_2^-(D)$ treats fixed and free singular vertices differently. It turns out that, in order for $H_*(\mathrm{C}_2^-(D))$ to be isomorphic to $\widehat{HFK}(\mathrm{sm}(D))$, our diagram D needs to be in $\mathscr{D}^{\mathscr{R}}$, which means satisfying the regular sequence condition. This condition cannot be satisfied unless D contains sufficiently many fixed vertices in a sufficiently nice arrangement. On the other hand, we only know how to define the edge maps $d_{I,J}$ on free vertices, so we cannot make all of our vertices fixed either.

As a sort of compromise, we choose some of the vertices to be fixed and some to be free. We do not need to worry about which choice we have made when proving invariance under Reidemeister moves II and III in Section 5, since they only involve local pictures of diagrams which contain some crossings but no singular vertices. While not a local move, we define stabilization to be compatible with our vertex labeling as well. Conjugation, however, requires us to change which vertices are fixed and which are free; this is what motivates the following theorem.

While it is not immediately obvious, it turns out that the homotopy type of $\mathrm{C}_2^-(D)$ does not depend on the particular labeling of vertices as fixed or free in the following sense:

Theorem 3.1 *If $D, D' \in \mathfrak{D}^{\mathcal{R}}$ are identical partially singular braid diagrams up to relabeling of fixed and free vertices, then $\mathrm{C}_2^-(D) \simeq \mathrm{C}_2^-(D')$.*

To prove this, we need to introduce a slight variation on the technique of "excluding a variable" from [Rasmussen 2015, Lemma 3.8] or [Khovanov and Rozansky 2008a, Proposition 9]. Both sources are also good references for the relevant details on matrix factorizations, including the statement below on the effect of change of basis on matrix factorizations.

We include the necessary details on matrix factorizations below. Let R be a ring. For $a, b \in R$, let $\{a, b\}$ denote the matrix factorization

$$\{a,b\} := R \underset{a}{\overset{b}{\rightleftarrows}} R.$$

For $\vec{a}, \vec{b} \in R^n$, let

$$\{\vec{a},\vec{b}\} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$$

denote the matrix factorization

$$\{\vec{a},\vec{b}\} := \bigotimes_{i=1}^{n} \{a_i, b_i\} = \bigotimes_{i=1}^{n} R \underset{a_i}{\overset{b_i}{\rightleftarrows}} R.$$

We have already seen a matrix factorization of this form; if we let $\vec{a} = (L^+(v_1), \dots, L^+(v_n))$ and $\vec{b} = (L(v_1), \dots, L(v_n))$ for a partially singular braid diagram D with $\mathrm{Fixed}(D) = \{v_1, \dots, v_n\}$, then $\mathcal{L}_D^+ = \{\vec{a}, \vec{b}\}$. By definition, the potential ω associated to the matrix factorization $\{\vec{a}, \vec{b}\}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n.$$

Starting with a matrix factorization $\{\vec{a}, \vec{b}\}$, we can perform a change of basis operation to get an isomorphic one. Specifically, sending $\vec{e_i}$ to $\vec{e_i} + c\vec{e_j}$ for standard basis vectors $\vec{e_i}$ and $\vec{e_j}$ of R^n has the effect of replacing the matrix factorization by $\{\vec{a}', \vec{b}'\}$, where

$$\vec{a}'_k = \begin{cases} \vec{a}_k + c\vec{a}_j & \text{if } k = i, \\ a_k & \text{otherwise,} \end{cases}$$

and

$$\vec{b}'_k = \begin{cases} \vec{b}_k - c\vec{b}_i & \text{if } k = j, \\ b_k & \text{otherwise.} \end{cases}$$

For more details, see [Khovanov and Rozansky 2008a; Rasmussen 2015].

Let $C = \{\vec{a}, \vec{b}\}$ be any matrix factorization over R. We can decompose

$$C = C' \underset{a_1}{\overset{b_1}{\rightleftarrows}} C',$$

where $C' = \{\vec{a}', \vec{b}'\}$ is the factorization obtained by omitting the first components of $\vec{a}$ and $\vec{b}$. Define $\pi\colon C \to C' \otimes R/(b_1)$ by $\pi((c_1, c_2)) = c_2 \otimes 1$. Before proving Theorem 3.1, we first prove that if the potential of C is 0 and b_1 is a nonzero divisor in R, then π is a quasi-isomorphism.

Lemma 3.2 *If the potential of C is 0 and b_1 is a nonzero divisor in R, then π is a quasi-isomorphism of chain complexes.*

Proof It is clear that π is surjective; since b_1 is a nonzero divisor, multiplication by b_1 is injective, so we have the following short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C' & \xrightarrow{1} & C' & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow\downarrow & & {\scriptstyle a_1 b_1}\uparrow\downarrow{\scriptstyle 1} & & {\scriptstyle a_1}\uparrow\downarrow{\scriptstyle b_1} & & \uparrow\downarrow & & \uparrow\downarrow \\ 0 & \longrightarrow & C' & \xrightarrow{b_1} & C' & \longrightarrow & C' \otimes R/(b_1) & \longrightarrow & 0 \end{array}$$

Let C'' denote the first nonzero column in this sequence, the matrix factorization

$$C' \underset{a_1 b_1}{\overset{1}{\rightleftarrows}} C'.$$

By the corresponding long exact sequence in homology, it suffices to show that C'' is acyclic in order to prove that π is a quasi-isomorphism. We write C'' in matrix form, then apply our above remarks about change of basis:

$$\begin{pmatrix} a_1 b_1 & 1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \sim \begin{pmatrix} a_1 b_1 + a_2 b_2 & 1 \\ a_2 & 0 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \sim \begin{pmatrix} \omega & 1 \\ a_2 & 0 \\ \vdots & \vdots \\ a_n & 0 \end{pmatrix}.$$

Since we know that the potential $\omega = 0$, we see that

$$C'' = \{\vec{a}', \vec{0}\} \underset{0}{\overset{1}{\rightleftarrows}} \{\vec{a}', \vec{0}\},$$

and therefore is acyclic. □

With this lemma, we can now prove that $\mathrm{C}_2^-(D)$ is independent of vertex labeling for $D \in \mathscr{D}^{\mathscr{R}}$:

Proof of Theorem 3.1 Let $S \in \mathscr{D}^{\mathscr{R}}$ be a fully singular braid diagram, and let $w \in \mathrm{Fixed}(S)$ be some fixed vertex such that if w were instead free, the new diagram S' would still be in $\mathscr{D}^{\mathscr{R}}$. Note that $R(S') = R(S)$, and $Q(S') = Q(S)/(L(w))$. Since $\mathrm{C}_2^-(S) = Q(S) \otimes \mathscr{L}_S^+$, we may consider $\mathrm{C}_2^-(S)$ as the matrix factorization $\{\vec{a}, \vec{b}\}$ over $R = Q(S)$ with $\vec{a} = (L^+(v))_{v \in \mathrm{Fixed}(S)}$ and $\vec{b} = (L(v))_{v \in \mathrm{Fixed}(S)}$.

Assume without loss of generality that $b_1 = L(w)$. Since $S' \in \mathscr{D}^{\mathcal{R}}$, we know that the linear terms $L(v)$ for $v \in \mathrm{Free}(S')$ form a regular sequence over $R(S')/N(S') = R(S)/N(S)$, and in particular, $L(w)$ is a nonzero divisor in $Q(S)$, since $w \in \mathrm{Free}(S')$. We then get that

$$\begin{aligned}
\mathrm{C}_2^-(S) &= Q(S) \otimes_{R(S)} \mathscr{L}_S^+ \\
&\cong \{\vec{a}, \vec{b}\} \\
&\simeq Q(S)/(L(w)) \otimes_{Q(S)} (Q(S) \otimes_{R(S)} \{\vec{a}', \vec{b}'\}) && \text{(by Lemma 3.2)} \\
&\simeq \big(Q(S)/(L(w)) \otimes_{Q(S)} Q(S)\big) \otimes_{R(S)} \{\vec{a}', \vec{b}'\} && \text{(by associativity of } \otimes) \\
&\simeq Q(S)/(L(w)) \otimes_{R(S)} \{\vec{a}', \vec{b}'\} \\
&\cong Q(S') \otimes_{R(S)} \mathscr{L}_{S'}^+ \\
&\cong Q(S') \otimes_{R(S')} \mathscr{L}_{S'}^+ && \text{(since } R(S) = R(S')) \\
&= \mathrm{C}_2^-(S').
\end{aligned}$$

Therefore, we see that changing a fixed vertex to a free one in a fully singular diagram does not change the homotopy type of $\mathrm{C}_2^-(-)$ as long as both diagrams are in $\mathscr{D}^{\mathcal{R}}$.

Now, we need to extend this result. Let $D, D' \in \mathscr{D}^{\mathcal{R}}$ be partially singular braid diagrams that differ only on the labeling of a single vertex $w \in \mathrm{Fixed}(D) \cap \mathrm{Free}(D')$. We know that $\mathrm{C}_2^-(D_I) \simeq \mathrm{C}_2^-(D'_I)$ for all $I \in \{0,1\}^{c(D)}$. In particular, we have a map in one direction: $\pi : \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ is a filtered quasi-isomorphism inducing the above equivalence. Therefore, it suffices to show that π commutes with the edge map d_1, which is the sum of $d_{I,J}$. Since π is linear over $Q(S)$, we get that it is additionally $R(S)$–linear via the natural quotient map, and therefore commutes with scalar multiplication by elements of $R(S)$. Since the edge maps $d_{I,J}$ are defined via scalar multiplication by 1 or $U_b - U_c$, we see that π does in fact commute with the edge maps, and therefore extends to a filtered quasi-isomorphism $\pi : \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D')$ by Lemma A.4.

Given any two diagrams $D', D'' \in \mathscr{D}^{\mathcal{R}}$ that differ only by some number of vertex labels, we can construct a diagram $D \in \mathscr{D}^{\mathcal{R}}$ with $\mathrm{Fixed}(D) = \mathrm{Fixed}(D') \cup \mathrm{Fixed}(D'')$, and therefore get that

$$\mathrm{C}_2^-(D') \simeq \mathrm{C}_2^-(D) \simeq \mathrm{C}_2^-(D''),$$

thus proving the general case. □

4 MOY moves

Murakami, Ohtsuki, and Yamada [Murakami et al. 1998] studied local operations on singular diagrams (“MOY moves”). While originally formulated for oriented planar trivalent graphs, they are relevant to us because one can think of singular vertices in our braids and braid resolutions as pairs of trivalent vertices instead. Two of these moves, MOY I and MOY III, represent planar isotopy when applied to the unoriented smoothing of a diagram, and thus are useful to make up for the fact that we cannot isotope

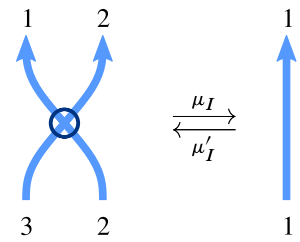


Figure 9: An MOY I move.

singularized crossings in the same ways that we can smoothed ones. The MOY II move corresponds to a cup/cap cobordism, but is rather more limited in its application. Nevertheless, these three moves suffice to define Reidemeister moves (and others) in Section 5. The maps that we choose to realize these moves are inspired by those used in [Khovanov and Rozansky 2008a; 2008b].

In this section, we construct filtered chain maps relating $\mathrm{C}_2^-(D)$ and $\mathrm{C}_2^-(D')$, where D and D' are partially singular braid diagrams connected by an MOY I, II, or III move.

4.A MOY I

Suppose D and D' are partially singular braid diagrams that differ by an MOY I move, as illustrated in Figure 9. In words, there is a fixed vertex v in D that meets the same edge e twice; without loss of generality, e is to the right of v. The diagram D' is then obtained from D by removing the edge e and relabeling v as a bivalent vertex.

Theorem 4.1 *There exist $R(D')$–linear filtered quasi-isomorphisms*

$$\mu_{\mathrm{I}}\colon \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D'), \quad \mu_{\mathrm{I}}'\colon \mathrm{C}_2^-(D') \to \mathrm{C}_2^-(D).$$

Under the identification $E_1(\mathrm{C}_2^-(-)) \cong \mathrm{CKh}^-(\mathrm{sm}(-))$, these maps induce the expected isomorphisms corresponding to planar isotopy.

First, suppose S and S' are fully singular braid diagrams that differ by an MOY I move, as illustrated in Figure 9. Specifically, S contains a fixed singular vertex v that meets the same edge twice. We would like to construct filtered chain maps $\mu_{\mathrm{I}}\colon \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ and $\mu_{\mathrm{I}}'\colon \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$. To start, let us characterize $\mathrm{C}_2^-(S)$ and $\mathrm{C}_2^-(S')$.

Without loss of generality, assume that the edge which is deleted by the MOY I move is to the right of the vertex. Label this edge with the variable U_2, label the top left edge U_1, and label the bottom left edge with U_3, again as in Figure 9.

Let R be the polynomial ring over all edges not shown in the local diagram; thus, $R(S') = R[U_1]$ and $R(S) = R(S')[U_2, U_3]$. We relate the associated quotient rings by the following proposition:

Proposition 4.2 *As $R(S')$ modules, $Q(S') \cong Q(S)/(U_1 + U_2)$.*

Proof We expand the right-hand side as a quotient of a free $R(S')$–module:

$$\begin{aligned}
Q(S)/(U_1+U_2) &\cong Q(S)\otimes_{R(S)} R(S)/(U_1+U_2)\\
&\cong R(S)/(L(S)+N(S))\otimes R(S)/(U_1+U_2)\\
&\cong R(S)/(L(S)+N(S)+(U_1+U_2))\\
&\cong R(S')[U_2,U_3]/(L(S)+N(S)+(U_1+U_2))\\
&\cong R(S')[U_2,U_3]/(L(S)+\widetilde{N}(S)+(U_2-U_3)+(U_1+U_2))\\
&\cong R(S')/(L(S)+\widetilde{N}(S)).
\end{aligned}$$

In the above, $\widetilde{N}(S)$ is the sum of the nonlocal relations other than U_1-U_3; this is exactly equal to $N(S')$, as any region intersecting these local diagrams can be made to avoid U_2 and any intersections with U_3 can be isotoped to intersect U_1 instead. Further, $L(S)=L(S')$. Thus,

$$Q(S)/(U_1+U_2)\cong R(S')/(L(S)+\widetilde{N}(S))=R(S')/(L(S')+N(S'))=Q(S'),$$

as desired. □

Proposition 4.3 *The chain complexes* $\mathrm{C}_2^-(S)$ *and* $\mathrm{C}_2^-(S')$ *are quasi-isomorphic as complexes over* $R(S')$.

Proof We can use Proposition 4.2 to expand $\mathrm{C}_2^-(S)$:

$$\begin{aligned}
\mathrm{C}_2^-(S) &= Q(S)\otimes_{R(S)}\mathscr{L}_S^+\\
&= Q(S)\otimes\left(R(S)\underset{U_1+2U_2+U_3}{\overset{U_1-U_3}{\rightleftarrows}} R(S)\otimes\widetilde{\mathscr{L}}_S^+\right)\\
&\cong Q(S)\otimes R(S)\underset{2U_1+2U_2}{\overset{0}{\rightleftarrows}} R(S)\otimes\widetilde{\mathscr{L}}_S^+ && \text{(using relation } U_1-U_3 \text{ in } N(S))\\
&\simeq Q(S)\otimes R(S)/(U_1+U_2)\otimes\widetilde{\mathscr{L}}_S^+ && \text{(replacing } 2U_1+2U_2 \text{ by the cokernel)}\\
&\cong \big(Q(S)\otimes R(S)/(U_1+U_2)\big)\otimes\widetilde{\mathscr{L}}_S^+\\
&\cong Q(S')\otimes_{R(S')}\mathscr{L}_{S'}^+ && \text{(by Proposition 4.2)}\\
&= \mathrm{C}_2^-(S').
\end{aligned}$$

In the above, let

$$\widetilde{\mathscr{L}}_S^+=\bigotimes_{w\in\mathrm{Fixed}(D)\setminus\{v\}} R(D)\underset{L^+(w)}{\overset{L(w)}{\rightleftarrows}} R(D),$$

and note $\widetilde{\mathscr{L}}_S^+=\mathscr{L}_{S'}^+$. Note that we may replace the mapping cone of $2U_1+2U_2$ by its cokernel in the fourth line only after checking that $2U_1+2U_2$ is not a zero divisor in $Q(S)$; by the logic in the proof of Proposition 4.2, we may choose a generating set of relations for $N(S)+L(S)$, none of which contain a term with a nonzero power of U_2. Therefore, $Q(S)$ is isomorphic to a free polynomial ring over U_2; since $2U_1+2U_2$ is a unit multiple (over $\mathbb{Q}$) of a monic polynomial in U_2, we therefore get that it is not a zero divisor in $Q(S)$. □

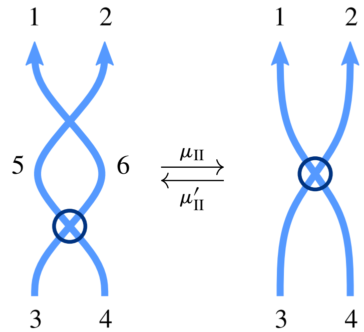


Figure 10: An MOY II move.

Let $\mu_{\mathrm{I}}\colon \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ be the quotient map implied by the above calculations. Explicitly on a simple tensor, $\mu_{\mathrm{I}}([r]\otimes(a,b)\otimes\tilde{s}) = [rb]\otimes\tilde{s}$. Let $\mu'_{\mathrm{I}}\colon \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$ be the splitting of μ_{I} given by inclusion into the first $R(S)$ summand in the equivalence of $R(S)/(U_1+U_2)$ and

$$R(S) \underset{2U_1+2U_2}{\overset{0}{\rightleftarrows}} R(S)$$

in the above proof. Explicitly on a simple tensor, $\mu'_{\mathrm{I}}([r]\otimes\tilde{s}) = [r]\otimes(0,1)\otimes\tilde{s}$.

For partially singular braid diagrams D and D' related by an MOY I move, we extend both maps to the cube of resolutions by defining $\mu_{\mathrm{I}}\colon \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ and $\mu'_{\mathrm{I}}\colon \mathrm{C}_2^-(D'_I) \to \mathrm{C}_2^-(D_I)$ as above for each $I \in \{0,1\}^{c(D)}$.

Proof of Theorem 4.1 It is clear that μ_{I} and μ'_{I} are filtered maps, since they are defined componentwise on the cube of resolutions.

We need to check that μ_{I} and μ'_{I} are chain maps, ie that they commute with the edge map d_1. Let $I, J \in \{0,1\}^{c(D)}$ with $I \prec J$. If I and J differ at a positive crossing, then $d_{I,J}$ is given by

$$\phi_+ \otimes \mathcal{L}_D^+ = 1 \otimes \mathcal{L}_D^+.$$

Otherwise, $d_{I,J}$ is given by $\phi_- \otimes \mathcal{L}_D^+ = (U_b - U_c)\otimes \mathcal{L}_D^+$. Either way, the edge maps are given by multiplication by an element of $R(D')$. Since μ_{II} and μ'_{II} were defined to be $R(D')$–linear, we get that they commute with d_1. □

4.B MOY II

Suppose D and D' are partially singular braid diagrams with D' the result of applying an MOY II move to D and reducing the number of crossings, as shown in Figure 10. In words, D contains a free vertex v_1, a fixed vertex v_2, and two edges e_5 and e_6 from v_2 to v_1. The diagram D' is obtained from D by removing e_5 and e_6 and merging v_1 and v_2 into a single fixed vertex.

Theorem 4.4 *There exists a direct sum decomposition $\mathrm{C}_2^-(D) \cong \mathrm{C}_2^-(D') \oplus \mathrm{C}_2^-(D')$ as filtered chain complexes over $R(D')$. Define $\mu_{\mathrm{II}}\colon \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D')$ to be projection onto the second summand,*

and define $\mu'_{\mathrm{II}}\colon \mathrm{C}_2^-(D') \to \mathrm{C}_2^-(D)$ to be inclusion into the first summand. Under the identification $E_1(\mathrm{C}_2^-(-)) \cong \mathrm{CKh}^-(\mathrm{sm}(-))$, the maps μ_{II} and μ'_{II} induce the maps on Khovanov homology corresponding to the cobordisms which delete and introduce a circle, respectively.

To start, let S and S' be fully singular braid diagrams again with S' the result of applying an MOY II move to S reducing the number of crossings. Let R be the polynomial ring over all edges not shown in the local diagrams, so that $R(S') = R[U_1, U_2, U_3, U_4]$, and $R(S) = R(S')[U_5, U_6]$.

Proposition 4.5 *As an $R(S')$–module, $Q(S) \cong Q(S')\langle 1\rangle \oplus Q(S')\langle U_6\rangle$.*

Proof First, note that $Q(S) = Q(S')[U_5, U_6]/(U_5 + U_6 - U_1 - U_2, U_5U_6 - U_1U_2)$. We do not need to consider any other nonlocal relations, as any region Ω intersecting these diagrams can be isotoped away from U_5 and U_6 to give an equivalent or stronger relation. We want to prove that $\{1, U_6\}$ is a basis for $Q(S)$ over $Q(S')$. To see that $\{1, U_6\}$ is a generating set, it is enough to note that in $Q(S)$, $U_5 = U_1 + U_2 - U_6$, and that $(U_1 + U_2 - U_6)U_6 - U_1U_2 = 0$, so $U_6^2 = (U_1 + U_2)U_6 - U_3U_4$. Linear independence follows from the fact that $U_6^2 - (U_1 + U_2)U_6 + U_1U_2$ is a monic polynomial of degree 2 in U_6. □

Using this proposition, we can decompose

$$\begin{aligned}
\mathrm{C}_2^-(S) &\cong Q(S) \otimes_{R(S)} \mathscr{L}_S^+ \\
&\cong Q(S) \otimes_{R(S)} \Big(\bigotimes_{v \in \mathrm{Fixed}(S)} R(S) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(S) \Big) \\
&\cong Q(S) \otimes_{R(S)} \Big(\bigotimes_{v \in \mathrm{Fixed}(S')} R(S) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(S) \Big) \\
&\cong \bigotimes_{v \in \mathrm{Fixed}(S')} Q(S) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S) \\
&\cong \bigotimes_{v \in \mathrm{Fixed}(S')} Q(S')\langle 1\rangle \oplus Q(S')\langle U_6\rangle \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S')\langle 1\rangle \oplus Q(S')\langle U_6\rangle \\
&\cong \Big(\bigotimes_{v \in \mathrm{Fixed}(S')} Q(S')\langle 1\rangle \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S')\langle 1\rangle \Big) \oplus \Big(\bigotimes_{v \in \mathrm{Fixed}(S')} Q(S')\langle U_6\rangle \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S')\langle U_6\rangle \Big) \\
&\cong (Q(S')\langle 1\rangle \otimes_{R(S')} \mathscr{L}_{S'}^+) \oplus (Q(S')\langle U_6\rangle \otimes_{R(S')} \mathscr{L}_{S'}^+) \\
&\cong \mathrm{C}_2^-(S')\langle 1\rangle \oplus \mathrm{C}_2^-(S')\langle U_6\rangle.
\end{aligned}$$

Define $\mu_{\mathrm{II}}\colon \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ to be projection onto the second summand in the above decomposition, and define $\mu'_{\mathrm{II}}\colon \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$ to be inclusion into the first summand. For partially singular braid

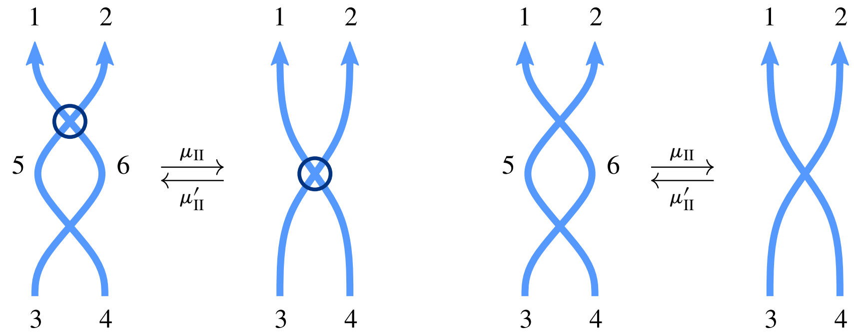


Figure 11: Variations of the MOY II move.

diagrams D and D' related by an MOY II move, we extend both maps to the cube of resolutions by defining $\mu_{\mathrm{II}}\colon \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ and $\mu'_{\mathrm{II}}\colon \mathrm{C}_2^-(D'_I) \to \mathrm{C}_2^-(D_I)$ as above for each $I \in \{0,1\}^{c(D)}$.

Proof of Theorem 4.4 It is clear that μ_{II} and μ'_{II} are filtered maps, since they are defined componentwise on the cube of resolutions. Next, we need to check that μ_{II} and μ'_{II} are chain maps, ie that they commute with the edge map d_1. Let $I, J \in \{0,1\}^{c(D)}$ with $I \prec J$. If I and J differ at a positive crossing, then $d_{I,J}$ is given by $\phi_+ \otimes \mathscr{L}_D^+ = 1 \otimes \mathscr{L}_D^+$. Otherwise, $d_{I,J}$ is given by $\phi_- \otimes \mathscr{L}_D^+ = (U_b - U_c) \otimes \mathscr{L}_D^+$. Either way, the edge maps are given by multiplication by an element of $R(D')$. Since μ_{II} and μ'_{II} were defined to be $R(D')$–linear, we get that they commute with d_1.

We used a direct sum decomposition of $\mathrm{C}_2^-(S)$ to define these maps on complete resolutions. We can see this direct sum decomposition on the cube of resolutions as well. Specifically, we have a split exact sequence:

$$0 \longrightarrow \mathrm{C}_2^-(D') \underset{1\otimes\mathscr{L}_D^+}{\overset{\mu'_{\mathrm{II}}}{\rightleftarrows}} \mathrm{C}_2^-(D) \underset{(U_6-U_1)\otimes\mathscr{L}_D^+}{\overset{\mu_{\mathrm{II}}}{\rightleftarrows}} \mathrm{C}_2^-(D') \longrightarrow 0$$

Finally, we want to show that these maps induce the correct morphisms on the Khovanov complex. The cobordism corresponding to the introduction of a circle is induced by multiplication by 1 [Bar-Natan 2005]. This should correspond to μ'_{II}, which we can see also induces multiplication by 1 on homology. The cobordism corresponding to the deletion of a circle should send $1 \mapsto 0$ and $X \mapsto 1$, where X is a variable associated to the shrinking circle. In our case, μ_{II} maps $1 \mapsto 0$ and $U_6 \mapsto 1$, inducing this same map on homology. □

One can repeat the same argument to show that we also have similar MOY II decompositions for the cases in Figure 11.

4.C MOY III

Suppose D and D' are fully singular braid diagrams with D' the result of applying an MOY III move to D and reducing the number of crossings, as shown in Figure 12. In words, D contains a fixed vertex v_1, free vertices v_2 and v_3, and edges $e_7\colon v_2 \to v_1$, $e_8\colon v_3 \to v_1$, and $e_9\colon v_3 \to v_2$. The diagram D' is obtained

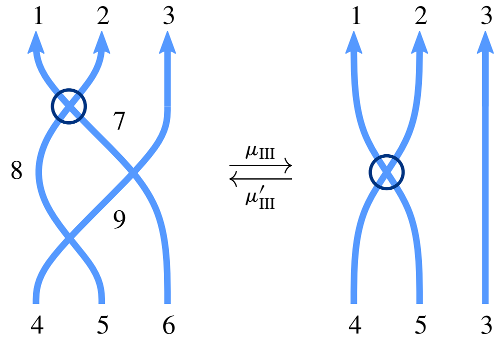


Figure 12: An MOY III move.

from D by removing the edges e_7, e_8, and e_9, merging v_1 and v_3 into a single fixed vertex, removing v_2, and merging e_6 into e_3.

Theorem 4.6 *There exist $R(D')$–linear filtered quasi-isomorphisms $\mu_{\mathrm{III}}\colon \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D')$ and $\mu'_{\mathrm{III}}\colon \mathrm{C}_2^-(D') \to \mathrm{C}_2^-(D)$. Furthermore, $\mathrm{C}_2^-(D')$ is isomorphic to a direct summand of $\mathrm{C}_2^-(D)$. Under the identification $E_1(\mathrm{C}_2^-(-)) \cong \mathrm{CKh}^-(\mathrm{sm}(-))$, these maps induce the expected isomorphisms corresponding to planar isotopy.*

Analogously to the MOY I and II cases, we again start by defining these maps on fully singular braid diagrams, then extending them to the cube of resolutions. Let S and S' be fully singular braid diagrams with S' the result of applying an MOY III move to S reducing the number of crossings as in Figure 12. Below, our goal is to prove that $\mathrm{C}_2^-(S) \cong \mathrm{C}_2^-(S') \oplus \Upsilon_L$, where Υ_L is some acyclic complex. Furthermore, the MOY III move has a nontrivial horizontal mirroring. We also prove that in the case where S and S' are connected by an MOY III move which is the mirror of Figure 12, we have $\mathrm{C}_2^-(S) \cong \mathrm{C}_2^-(S') \oplus \Upsilon_R$. While it is true that $\Upsilon_R = \Upsilon_L$, we neither need this fact nor prove it in this paper. Nevertheless, we may refer to the complex as Υ all the same.

We construct a map $\mu_{\mathrm{III}}\colon \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$ and another map $\mu'_{\mathrm{III}}\colon \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ which splits μ_{III}, thus proving that $\mathrm{C}_2^-(S')$ is a direct summand of $\mathrm{C}_2^-(S)$. Let S'' be the fully singular braid diagram

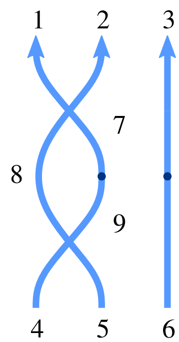


Figure 13: The fully singular diagram S'' used in the definitions of μ_{III} and μ'_{III}.

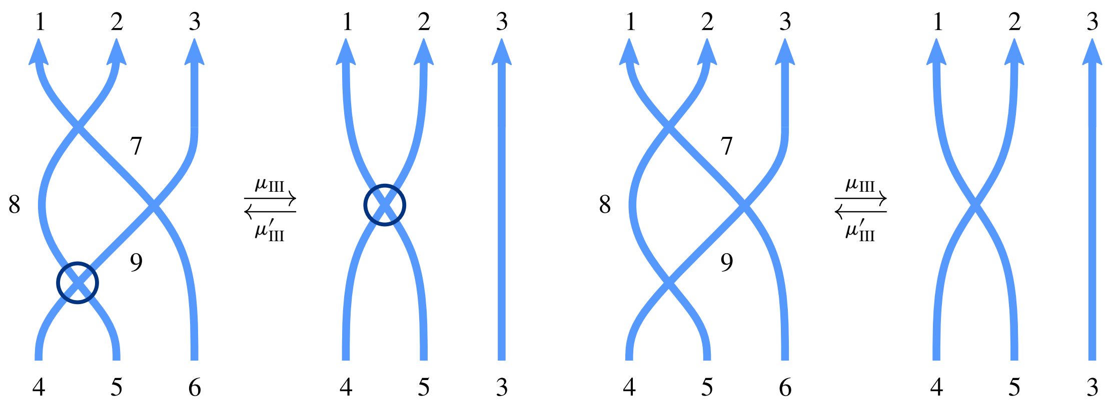


Figure 14: Variations of the MOY III move.

where the middle singular vertex v_2 is replaced by the oriented smoothing as in Figure 13, so that we may define the map $1 \otimes \mathscr{L}_S^+ : \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S'')$. We may then apply an MOY II move μ_{II} to the left two strands in S'' to get a map $\mu_{\mathrm{II}} : \mathrm{C}_2^-(S'') \to \mathrm{C}_2^-(S')$. Therefore, we define $\mu_{\mathrm{III}} = \mu_{\mathrm{II}} \circ (1 \otimes \mathscr{L}_S^+)$. We can also reverse the order of these operations to define $\mu'_{\mathrm{III}} = ((U_9 - U_3) \otimes \mathscr{L}_D^+) \circ \mu'_{\mathrm{II}}$. Note that the maps $1 \otimes \mathscr{L}_S^+$ and $(U_9 - U_3) \otimes \mathscr{L}_S^+$ are well defined since if v_2 were replaced by a positive or negative crossing, these would simply be multiples of the edge maps corresponding to resolutions of that crossing.

Proposition 4.7 *μ_{III} splits μ'_{III}, ie $\mu_{\mathrm{III}} \circ \mu'_{\mathrm{III}} = \mathrm{id}_{\mathrm{C}_2^-(S')}$.*

Proof We expand out the definitions of μ_{III} and μ'_{III} to get

$$\begin{aligned}
\mu_{\mathrm{III}} \circ \mu'_{\mathrm{III}} &= \mu_{\mathrm{II}} \circ (1 \otimes \mathscr{L}_S^+) \circ ((U_9 - U_3) \otimes \mathscr{L}_S^+) \circ \mu'_{\mathrm{II}} \\
&= \mu_{\mathrm{II}} \circ ((U_9 - U_3) \otimes \mathscr{L}_S^+) \circ \mu'_{\mathrm{II}} \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -U_3 & -U_4U_5 \\ 1 & U_4 + U_5 - U_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathscr{L}_S^+ \\
&= (1) \otimes \mathscr{L}_S^+ \\
&= \mathrm{id}_{\mathrm{C}_2^-(S')} .
\end{aligned}$$

□

Therefore, we get a direct sum decomposition $\mathrm{C}_2^-(S) \cong \Upsilon\langle 1\rangle \oplus \mathrm{C}_2^-(S')\langle U_9 - U_3\rangle$. For partially singular braid diagrams D and D' related by an MOY III move, we extend both maps to the cube of resolutions by defining $\mu_{\mathrm{III}} : \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ and $\mu'_{\mathrm{III}} : \mathrm{C}_2^-(D'_I) \to \mathrm{C}_2^-(D_I)$ as above for each $I \in \{0,1\}^{c(D)}$.

Proof of Theorem 4.6 It is clear that μ_{III} and μ'_{III} are filtered maps, since they are defined componentwise on the cube of resolutions. Furthermore, we can extend our proof of Proposition 4.7 to partially singular braid diagrams since both the edge maps and MOY II maps are defined on such diagrams, so $\mathrm{C}_2^-(D')$ really is a summand of $\mathrm{C}_2^-(D)$.

We also need to check that μ_{III} and μ'_{III} are chain maps, ie that they commute with the edge maps d_1. Let $I, J \in \{0,1\}^{c(D)}$ with $I \prec J$. If I and J differ at a positive crossing, then $d_{I,J}$ is given by $\phi_+ \otimes \mathscr{L}_D^+ = 1 \otimes \mathscr{L}_D^+$. Otherwise, $d_{I,J}$ is given by $\phi_- \otimes \mathscr{L}_D^+ = (U_b - U_c) \otimes \mathscr{L}_D^+$. Either way, the edge maps are given by multiplication by an element of $R(D')$. Since μ_{III} and μ'_{III} were defined to be $R(D')$–linear, we get that they commute with d_1. □

As before, a similar argument shows that we also have MOY III decompositions for the cases in Figure 14.

5 Invariance

In this section, we prove that $\mathrm{C}_2^-(\beta)$ is an invariant of the braid closure $\mathrm{cl}(\beta)$ by showing that it is invariant under each of the four moves of Theorem 2.8. The first two moves, Reidmeister II and III, apply to any partially singular braid diagram D, whereas the second two moves, stabilization and conjugation, are specific to diagrams of the form $D = I_n(\beta)$.

5.A Reidemeister II

We begin by proving invariance under Reidemeister II moves. There are two distinct such moves, but they are mirror images of each other, and their proofs are almost identical. We prove one of the cases in detail below.

Theorem 5.1 *If D and D' are two partially singular braid diagrams that differ by a Reidemeister II move, then $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$ over $R(D')$.*

Proof Let D and D' be the diagrams in Figure 15, with D' the result of eliminating two crossings from D' by means of a Reidemeister II move. We use Lemma A.1 to simplify $\mathrm{C}_2^-(D)$ and $\mathrm{C}_2^-(D')$ to see they have the same homotopy type. Label the edges of D with variables $U_1, \dots, U_6$ as in Figure 15, and order the crossings from top to bottom. Let $\phi_1 = \phi_+ = 1$ be the edge map corresponding to the top (positive)

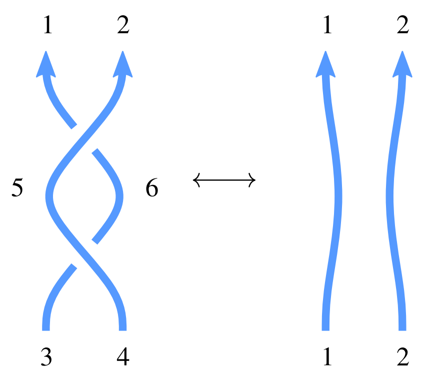


Figure 15: A Reidemeister II move.

vertex, and let $\phi_2 = \phi_- = U_6 - U_3$ be the edge map corresponding to the bottom (negative) vertex. Then, fixing a sign assignment without loss of generality, we expand the cube of resolutions for $\mathrm{C}_2^-(D)$ as:

$$\begin{array}{ccc} \mathrm{C}_2^-(D_{00}) & \xrightarrow{\phi_1} & \mathrm{C}_2^-(D_{10}) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle -\phi_2} \\ \mathrm{C}_2^-(D_{01}) & \xrightarrow{\phi_1} & \mathrm{C}_2^-(D_{11}) \end{array}$$

Note that $\mathrm{C}_2^-(D_{10})$ is isomorphic to $\mathrm{C}_2^-(D')$ via the removal of bivalent vertices, so our goal is to show that $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D_{10})$ as a filtered chain complex over $R(D')$. We work over the larger ring $R(D')[U_3, U_4]$, but do not enforce linearity with respect to U_5 or U_6. First, note that $\mathrm{C}_2^-(D_{00}) \cong \mathrm{C}_2^-(D_{11})$. We see that we can apply the MOY II decomposition from Section 4.B to write $\mathrm{C}_2^-(D_{01}) = \mathrm{C}_2^-(D_{11})\langle 1\rangle \oplus \mathrm{C}_2^-(D_{00})\langle U_6\rangle$. We compute the maps induced by ϕ_1 and ϕ_2 on these decompositions to get an isomorphic cube of resolutions:

$$\begin{array}{ccc} \mathrm{C}_2^-(D_{00}) & \xrightarrow{\quad 1 \quad} & \mathrm{C}_2^-(D_{10}) \\ \downarrow{\scriptstyle \begin{pmatrix} -U_3 \\ 1 \end{pmatrix}} & & \downarrow{\scriptstyle U_3 - U_2} \\ \mathrm{C}_2^-(D_{11})\langle 1\rangle \oplus \mathrm{C}_2^-(D_{00})\langle U_6\rangle & \xrightarrow{(1\ U_2)} & \mathrm{C}_2^-(D_{11}) \end{array}$$

This is the first of several times we use Lemma A.1 to simplify a cube of resolutions in this paper. This key lemma allows us to effectively cancel out isomorphisms of direct summands in cubes. In this case, it yields the E_1–quasi-isomorphic complex:

$$\begin{array}{ccc} 0 & \longrightarrow & \mathrm{C}_2^-(D_{10}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

We conclude by noting that $\mathrm{C}_2^-(D_{10}) \cong \mathrm{C}_2^-(D')$ as chain complexes over $R(D')[U_3, U_4]$. This proves invariance under one type of Reidemeister II move; the proof of the mirror-image move is analogous. □

5.B Reidemeister III

We aim to prove invariance under the Reidemeister III move shown in Figure 16, which corresponds to sliding a strand over a positive crossing. In terms of the braid group, it represents the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. All other variations of the Reidemeister III move follow from this one plus the Reidemeister II invariance result from Theorem 5.1.

Theorem 5.2 *If D and D' are two partially singular braid diagrams that differ by a Reidemeister III move, then $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$.*

Proof Let D be the diagram on the left and D' the diagram on the right in Figure 16. We aim to use Lemma A.1 to simplify $\mathrm{C}_2^-(D)$ and $\mathrm{C}_2^-(D')$ to see they have the same homotopy type.

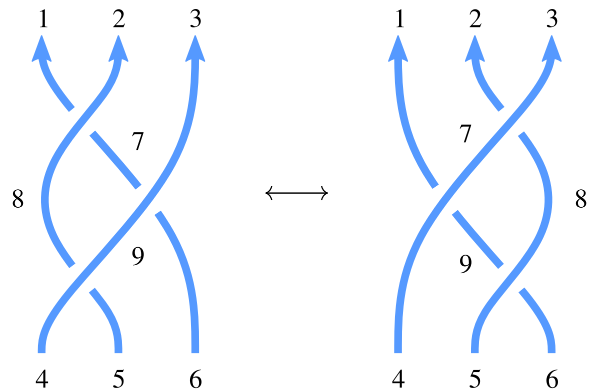
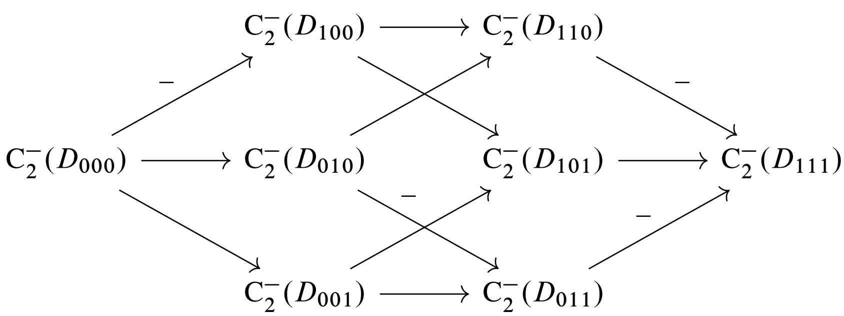


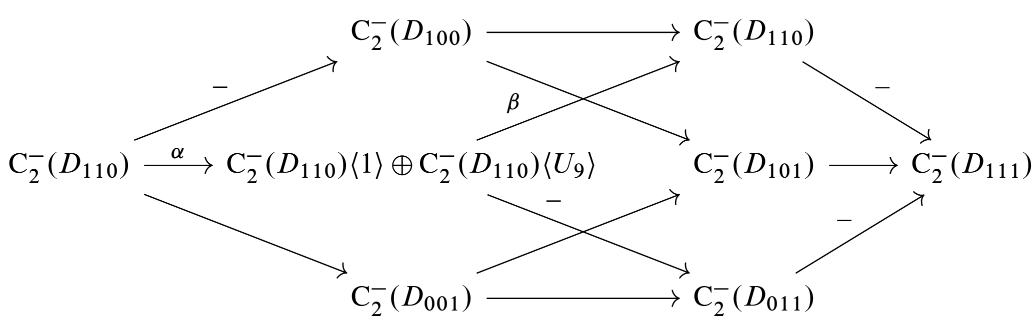
Figure 16: A Reidemeister III move.

To start, label the edges of D with variables $U_1, \dots, U_9$ as in Figure 16, and order the crossings from top to bottom. We expand the cube of resolutions for $\mathrm{C}_2^-(D)$ as:



Since our local picture of D consists of only positive crossings, all edge maps in this cube are given by $\phi_+ = 1$ up to a sign assignment, which we take to be the one in the above cube of resolutions without loss of generality.

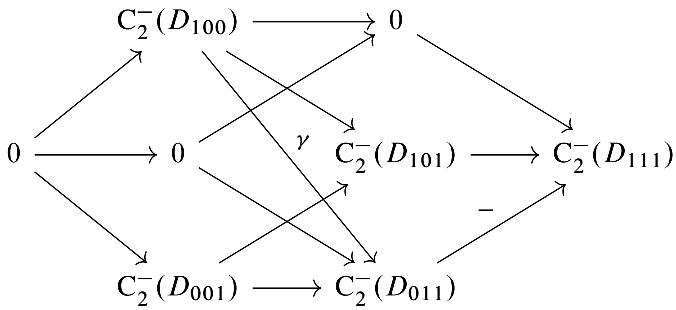
By Theorem 4.6, we note $\mathrm{C}_2^-(D_{000}) \cong \mathrm{C}_2^-(D_{110}) \oplus \Upsilon$, where Υ is acyclic. By a slight generalization of Lemma A.3, we get an E_1–quasi-isomorphic cube after replacing $\mathrm{C}_2^-(D_{000})$ by $\mathrm{C}_2^-(D_{110})$. Furthermore, Theorem 4.4 gives us that $\mathrm{C}_2^-(D_{010}) \cong \mathrm{C}_2^-(D_{110})\langle 1\rangle \oplus \mathrm{C}_2^-(D_{110})\langle U_9\rangle$. Therefore, the above cube is E_1–quasi-isomorphic to:



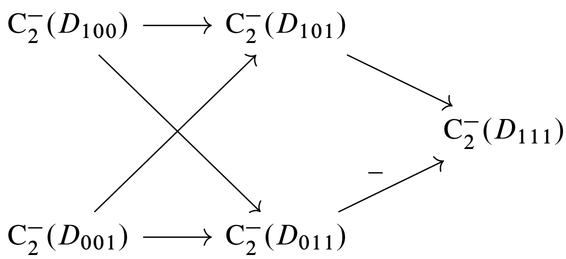
We compute the induced maps in the above cube to be

$$\alpha = \begin{pmatrix} -U_3 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & U_2 \end{pmatrix}.$$

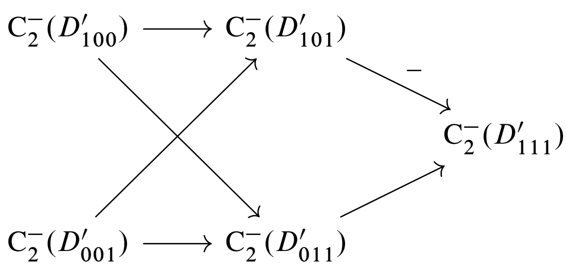
By Lemma A.1, we can cancel the isomorphisms of direct summands in the above cube to obtain the E_1–quasi-isomorphic complex:



Removing the trivial complexes in the above cube, and noting that the map γ induced by cancellation is given by multiplication by 1, we get the complex:



Now, recalling that we have a second diagram D' to work with, we may go through the same steps to simplify $C_2^-(D')$ to get the complex:



We conclude by noting that the reduced complexes for $C_2^-(D)$ and $C_2^-(D')$ are isomorphic via the map that reflects the complexes about a horizontal axis, ie swaps the 100– and 001–resolutions, swaps the 101– and 011–resolutions, and fixes the 111–resolution. This map is a chain map since all the edge maps are ± 1, and therefore $C_2^-(D) \simeq_1 C_2^-(D')$. □

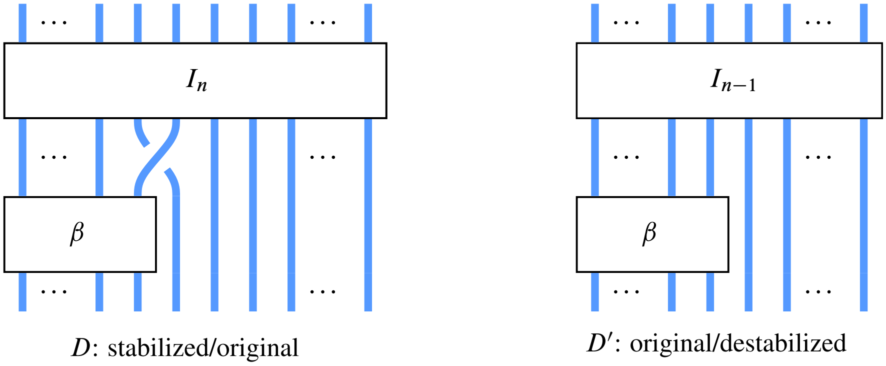


Figure 17: Diagrams related by a positive stabilization. Depending on context, we either consider the diagram D and its *destabilization* D', or we consider the diagram D' and its *stabilization* D.

5.C Stabilization

Let $\beta \in B_{n-1}$ be an element of the braid group for $n \geq 2$, and consider β as an element of B_n via the natural inclusion $B_{n-1} \hookrightarrow B_n$ adjoining a straight strand to the right of β. Let $\sigma_{n-1} \in B_n$ be the generator which introduces a positive crossing between strands $n-1$ and n. The *positive stabilization* of β is the braid $\sigma_{n-1}\beta \in B_n$. Analogously, the *negative stabilization* of β is the braid $\sigma_{n-1}^{-1}\beta \in B_n$. For a braid $\beta = \sigma_{n-1}^{\pm 1}\beta' \in B_n$ in the image of one of these operations, we say that $\beta' \in B_{n-1}$ is the *destabilization* of β.

Theorem 5.3 *The E_1–homotopy type of the filtered complex $\mathrm{C}_2^-(\beta)$ is invariant under positive and negative* (de)*stabilization*, *ie* $\mathrm{C}_2^-(\sigma_{n-1}\beta) \simeq_1 \mathrm{C}_2^-(\sigma_{n-1}^{-1}\beta) \simeq_1 \mathrm{C}_2^-(\beta)$.

We note that β, $\sigma_{n-1}\beta$, and $\sigma_{n-1}^{-1}\beta$ all have isotopic braid closures. Before we prove stabilization invariance, we need to relate diagrams containing the open braid diagrams I_n and I_{n-1}, as the (de)stabilization operations alter the number of strands of our partially singular braid diagrams. Therefore, we first note that we can see I_{n-1} as a subdiagram of I_n by ignoring the rightmost vertices in every row. Equivalently, we can build I_n inductively from I_{n-1} as in Figure 18.

Consider $I_n(\sigma_{n-1}\beta)$, as shown in Figure 19. Let R be the polynomial ring over all edges not labeled in Figure 19, and label the rest of the edges accordingly, so that $R(I_n(\sigma_{n-1}\beta)) = R[U_1, U_2, U_3, U_4, U_5]$.

Let $\phi = \phi_+ = 1$ be the edge map corresponding to the positive crossing of σ_{n-1}. We write the one-dimensional cube of resolutions corresponding to resolving the crossing

$$\mathrm{C}_2^-(I_n(\sigma_{n-1}\beta)_0) \xrightarrow{\phi} \mathrm{C}_2^-(I_n(\sigma_{n-1}\beta)_1).$$

The diagrams corresponding to these resolutions are illustrated in Figure 20.

Our goal now is to use MOY moves to modify both resolutions so that they can be represented using a common diagram, tracking the effect on the complexes. By an abuse of notation, we denote this common

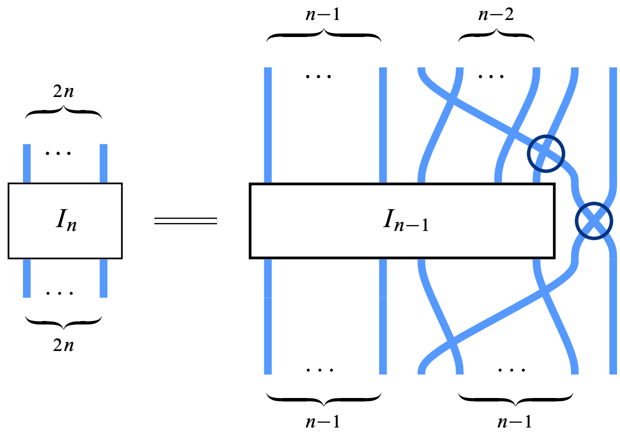


Figure 18: Constructing I_n from I_{n-1} by adding two more strands and marking two new vertices as fixed.

diagram $I'_n(\beta)$, which is gotten analogously to $I_n(\beta)$: we place a straight strand to the right of β, place n straight strands to the right of that, top this diagram with I'_n, and take the braid closure. It remains to define I'_n. We let I'_n be I_n, without the singular vertex between strands n and $n+1$ in the first layer. As for I_n, we can also define I'_n by building on I_{n-1}, as in Figure 21.

With this definition in mind, we now see that $I_n(\sigma_{n-1}\beta)_0$ is one MOY III move away from $I'_n(\beta)$, and $I_n(\sigma_{n-1}\beta)_1$ is one MOY II move away from $I'_n(\beta)$. On the one-dimensional cube of resolutions, then, we get

$$\mathrm{C}_2^-(I'_n(\beta))\langle U_3 - U_5\rangle \oplus \Upsilon \xrightarrow{\phi} \mathrm{C}_2^-(I'_n(\beta))\langle 1\rangle \oplus \mathrm{C}_2^-(I'_n(\beta))\langle U_4\rangle.$$

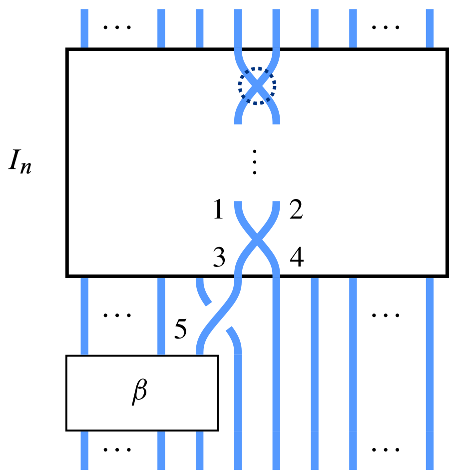


Figure 19: Relevant edge labels near σ_{n-1}. Note that the top vertex is fixed only when $n=2$, and is otherwise free for $n \geq 3$. For this reason, we use a dashed circle to indicate that the top vertex may be fixed or free.

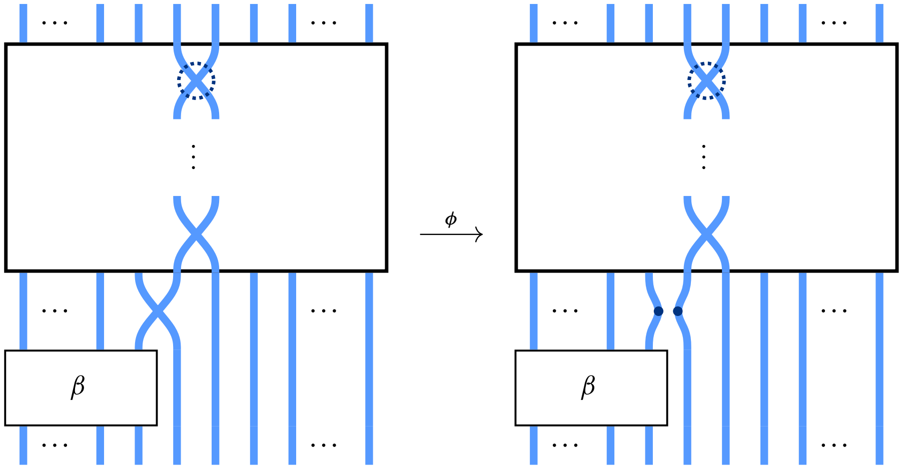


Figure 20: The mapping cone decomposition induced by σ_{n-1}.

Since Υ is acyclic, we can ignore it by Lemma A.2. We compute the map induced by ϕ on the summands as

$$\mathrm{C}_2^-(I_n'(\beta))\langle U_3-U_5\rangle \xrightarrow{\binom{U_1+U_2-U_5}{-1}} \mathrm{C}_2^-(I_n'(\beta))\langle 1\rangle \oplus \mathrm{C}_2^-(I_n'(\beta))\langle U_4\rangle.$$

Since the -1 entry represents an isomorphism of $\mathrm{C}_2^-(I_n'(\beta))$ summands, we can cancel it by Lemma A.1. This proves the following lemma.

Lemma 5.4 *The complexes $\mathrm{C}_2^-(\sigma_{n-1}\beta)$ and $\mathrm{C}_2^-(I_n'(\beta))$ have the same E_1–homotopy type.*

Therefore, to prove conjugation invariance, it remains to prove the following proposition.

Proposition 5.5 *The complexes $\mathrm{C}_2^-(I_n'(\beta))$ and $\mathrm{C}_2^-(\beta)$ have the same E_1–homotopy type.*

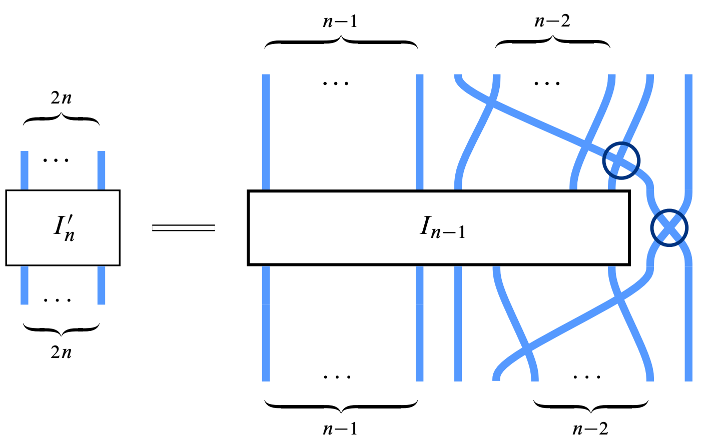


Figure 21: Building I_n' from I_{n-1}.

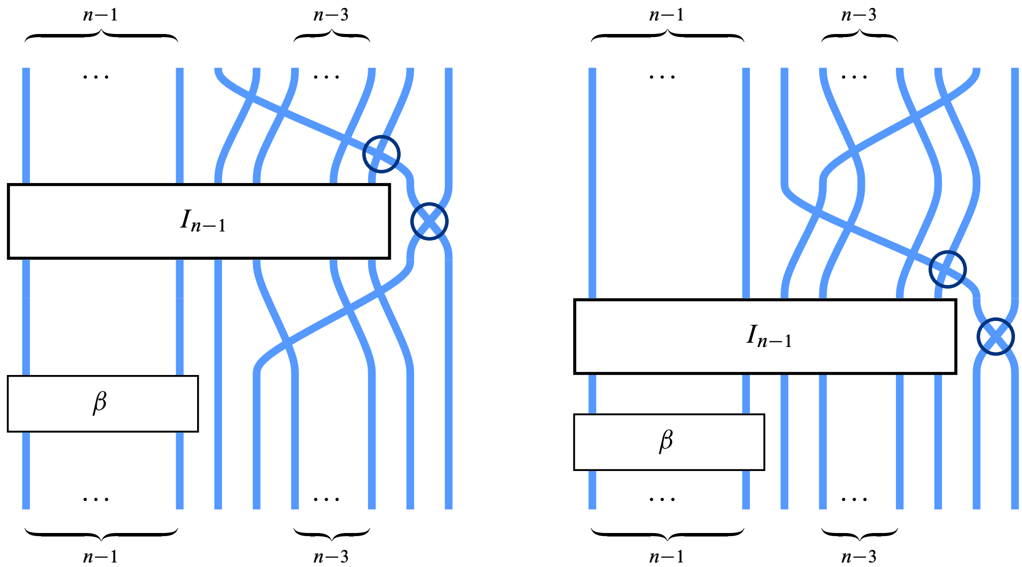


Figure 22: Shifting vertices in $I'_n(\beta)$.

Proof To begin, we note that $I'_n(\beta)$ and $I_{n-1}(\beta)$ are really braid closures, so for ease of understanding the upcoming MOY moves, we replace our usual depiction of $I'_n(\beta)$ with a shifted version, as in Figure 22.

In this shifted version, we identify the local picture on the left in Figure 23, consisting of a pair of intersecting strands and $n-2$ other strands which intersect both. We can apply $n-2$ MOY III moves to simplify this part of the diagram to the local picture on the right in Figure 23, consisting of $n-2$ straight strands and one pair of intersecting strands. By Theorem 4.6, each of these preserves the E_1–homotopy type of the complex. The global picture at this stage can be seen on the left in Figure 24. To arrive at the diagram for $I_{n-1}(\beta)$, we apply two MOY I moves to the two remaining fixed vertices outside of I_{n-1}. By Theorem 4.1, each of these preserves the E_1–homotopy type of the complex. This leads to the diagram on the right in Figure 24, which is exactly the diagram for $I_{n-1}(\beta)$. □

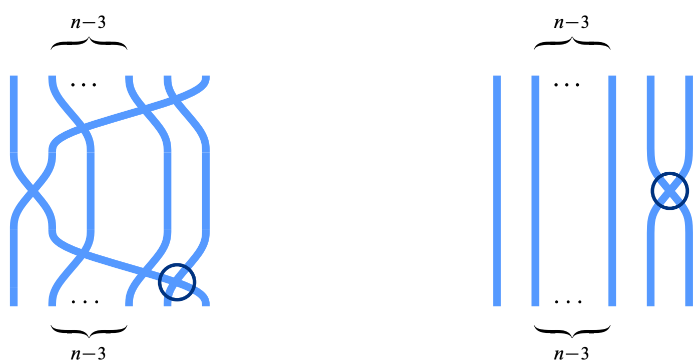


Figure 23: Local pictures of diagrams related by a sequence of $n-2$ MOY III moves.

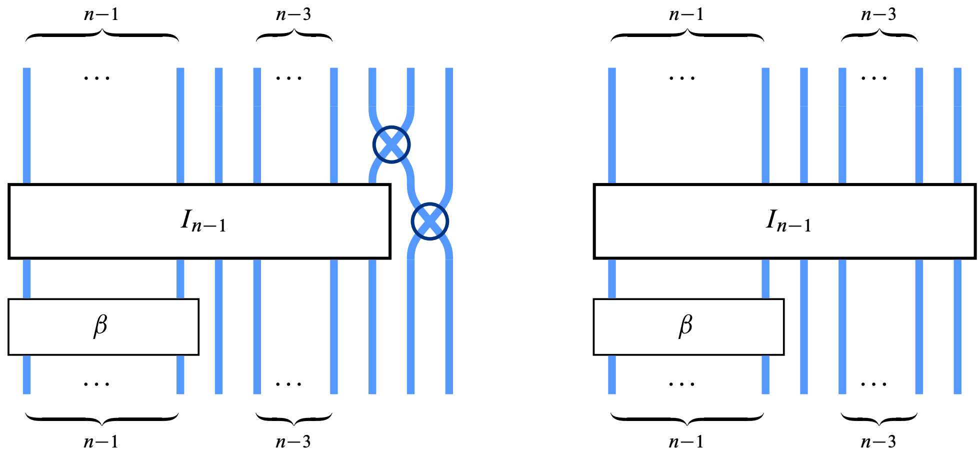


Figure 24: The last step in simplifying $I'_n(\beta)$ in the proof of Proposition 5.5.

Proof of Theorem 5.3 We have shown that $\mathrm{C}_2^-(\sigma_{n-1}\beta) \simeq \mathrm{C}_2^-(I'_n(\beta)) \simeq \mathrm{C}_2^-(\beta)$. We can simplify $\mathrm{C}_2^-(\sigma_{n-1}^{-1}\beta)$ to $\mathrm{C}_2^-(I'_n(\beta))$ as well. Using the same edge labels and notation as before, we write the cube of resolutions for $\mathrm{C}_2^-(\beta)$ as

$$\mathrm{C}_2^-(I_n(\beta)_0) \xrightarrow{\phi} \mathrm{C}_2^-(I_n(\beta)_1)$$

where this time $\phi = \phi_- = U_3 - U_5$. Applying an MOY II move to $I_n(\beta)_0$ and an MOY III move to $I_n(\beta)_1$ to write our complexes in terms of $\mathrm{C}_2^-(I'_n(\beta))$ gives us the complex

$$\mathrm{C}_2^-(I'_n(\beta))\langle 1\rangle \oplus \mathrm{C}_2^-(I'_n(\beta))\langle U_4\rangle \xrightarrow{\phi} \mathrm{C}_2^-(I'_n(\beta))\langle U_3 - U_5\rangle \oplus \Upsilon.$$

Again, excluding Υ and computing the map induced by ϕ, we get

$$\mathrm{C}_2^-(I'_n(\beta))\langle 1\rangle \oplus \mathrm{C}_2^-(I'_n(\beta))\langle U_4\rangle \xrightarrow{(1\ U_4)} \mathrm{C}_2^-(I'_n(\beta))\langle U_3 - U_5\rangle.$$

As before, we may cancel the 1 in the above matrix to see that $\mathrm{C}_2^-(\sigma_{n-1}^{-1}\beta) \simeq_1 \mathrm{C}_2^-(I'_n(\beta))$ as well. The rest of the proof follows from Proposition 5.5. Since we have covered both the positive and negative cases, this suffices to show invariance under stabilization. □

5.D Conjugation

Conjugation invariance is the following statement:

Theorem 5.6 *For any $\alpha, \beta \in B_n$, we have that $\mathrm{C}_2^-(\alpha^{-1}\beta\alpha) \simeq_1 \mathrm{C}_2^-(\beta)$.*

To begin, we prove a lemma relating complexes associated to diagrams that locally look like the pictures in Figure 25.

Lemma 5.7 *Let A and A' be partially singular braid diagrams that are identical outside of a specific region, where they look like the diagrams in Figure 25, ie A has two opposite crossings whereas A' has oriented smoothings. Then $\mathrm{C}_2^-(A) \simeq_1 \mathrm{C}_2^-(A')$.*

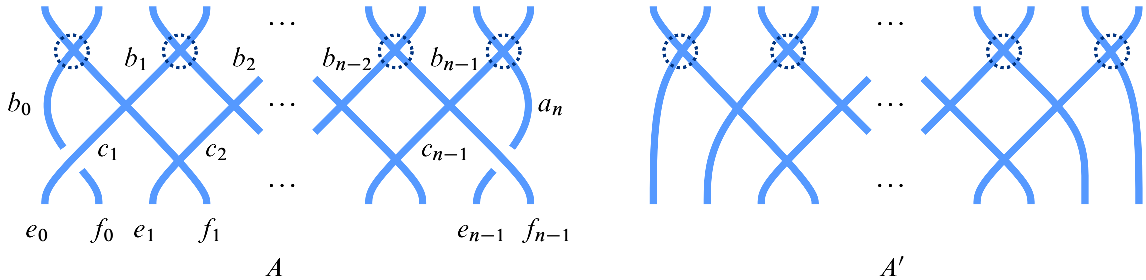


Figure 25: Local pictures of diagrams with equivalent $C_2^-(-)$. As before, the dashed circles indicate that the vertices may be fixed or free.

Proof Let $\phi_1 = \phi_+ = 1$ be the edge map corresponding to the left (positive) crossing, and let $\phi_2 = \phi_- = a_n - e_{n-1}$ be the edge map corresponding to the right (negative) crossing. We expand the cube of resolutions for $C_2^-(A)$ as follows:

$$\begin{array}{ccc} C_2^-(A_{00}) & \xrightarrow{\phi_1} & C_2^-(A_{10}) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_2} \\ C_2^-(A_{01}) & \xrightarrow{\phi_1} & C_2^-(A_{11}) \end{array}$$

The diagrams for these four partial resolutions look like Figure 26.

We can use MOY III moves to simplify three of the four corners of this cube. For A_{00}, we can start with a MOY III move on the left, simplifying the diagram. Each MOY III move we apply allows us to perform another, until we have done $n-1$ such moves moving left-to-right. We denote the resulting diagram A''_{00}; it is shown in Figure 27. By Theorem 4.6, A_{00} and A'_{00} are E_1–quasi-isomorphic.

Similarly, we can simplify A_{11} to A'_{11} by performing $n-1$ MOY III moves right-to-left, and we can simplify A_{01} to A'_{01} by performing $n-1$ MOY III moves left-to-right. In each case, Theorem 4.6 ensures

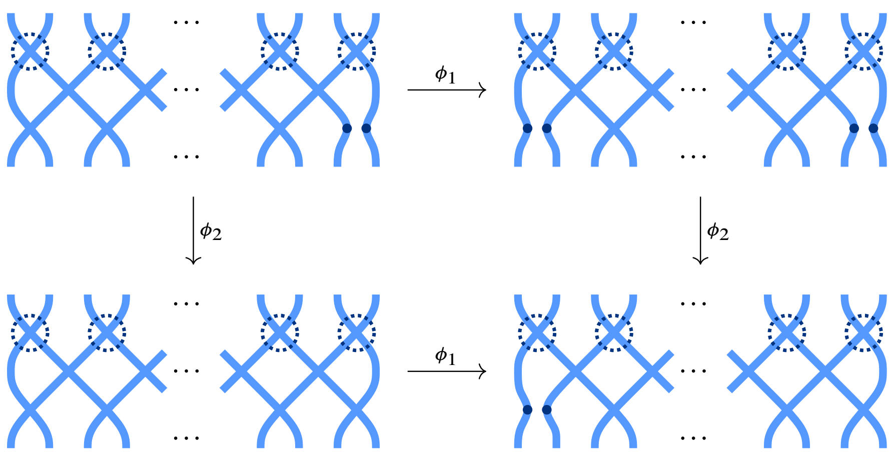


Figure 26: The cube of resolutions for diagram A in Figure 25.

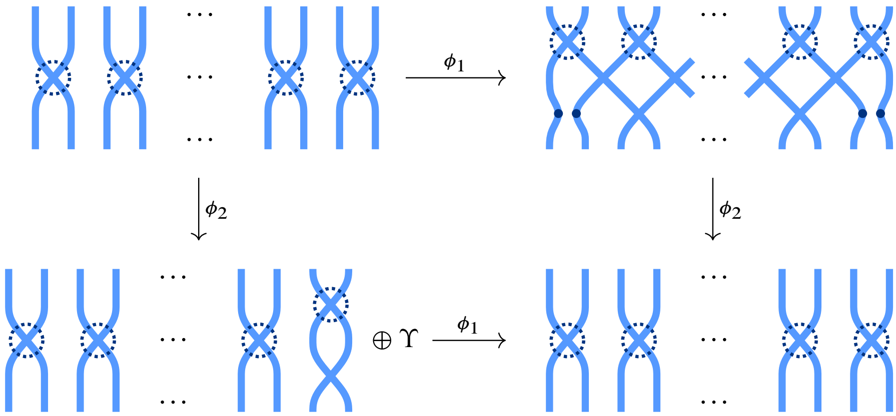


Figure 27: The reduced cube of resolutions.

we are preserving the E_1–quasi-isomorphism type. The resulting diagrammatic cube of resolutions is shown in Figure 27. Thus we obtain the complex

$$\begin{array}{ccc} C_2^-(A''_{00})\langle x\rangle & \xrightarrow{\phi_1} & C_2^-(A_{10}) \\ \big\downarrow{\scriptstyle \phi_2} & & \big\downarrow{\scriptstyle \phi_2} \\ C_2^-(A''_{01})\langle x\rangle \oplus \Upsilon & \xrightarrow{\phi_1} & C_2^-(A''_{11})\langle x\rangle \end{array}$$

This cube ignores the Υ summands in A_{00} and A_{11} by Lemma A.3, but retains the $\Upsilon := \Upsilon_1 \oplus \cdots \oplus \Upsilon_{n-1}$ summand in A_{01}. Further, as every vertex in the middle row is free, we may choose

$$x = (b_1 - c_1)\cdots(b_{n-1} - c_{n-1})$$

to be the generator for all three complexes modulo the linear ideal L.

We further decompose A''_{01} via an MOY II move on the right into two copies of A''_{00}, generated by x and $a_n x$. Additionally, we see that A''_{00} and A''_{11} are isomorphic. We can compute the maps induced by ϕ_1 and ϕ_2 and write our complex as

$$\begin{array}{ccc} C_2^-(A''_{00})\langle x\rangle & \xrightarrow{\phi_1} & C_2^-(A_{10}) \\ \Big\downarrow{\scriptstyle \begin{pmatrix} -e_{n-1} \\ 1 \\ * \end{pmatrix}} & & \Big\downarrow{\scriptstyle \phi_2} \\ C_2^-(A''_{00})\langle x\rangle \oplus C_2^-(A''_{00})\langle a_n x\rangle \oplus \Upsilon & \xrightarrow{\begin{pmatrix} 1 & f_{n-1} & * \end{pmatrix}} & C_2^-(A''_{00})\langle x\rangle \end{array}$$

We may cancel the 1's in the above matrices to reduce the complex by Lemma A.1 to obtain

$$\begin{array}{ccc} 0 & \longrightarrow & C_2^-(A_{10}) \\ \big\downarrow & & \big\downarrow \\ \Upsilon & \longrightarrow & 0 \end{array}$$

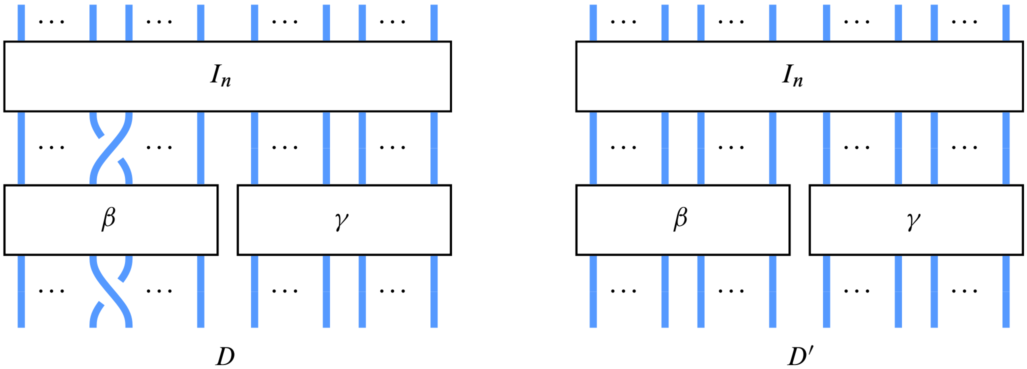


Figure 28: When $\gamma = 1$, the diagram D is the result of conjugating the braid β in D' by a generator of the braid group.

Since Υ is a direct sum of E_1–acyclic complexes, we see that the E_2–page of the above complex is isomorphic to that of $\mathrm{C}_2^-(A_{10})$, which is isomorphic to $\mathrm{C}_2^-(A')$, thereby proving Lemma 5.7 in the case of a positive crossing on the left and a negative one on the right.

The opposite case is analogous; applying the same moves (mirrored horizontally) results in the complex

$$\begin{array}{ccc} \mathrm{C}_2^-(A''_{00})\langle x\rangle & \xrightarrow{\begin{pmatrix} f_0 \\ -1 \\ * \end{pmatrix}} & \mathrm{C}_2^-(A''_{00})\langle x\rangle \oplus \mathrm{C}_2^-(A''_{00})\langle b_0 x\rangle \oplus \Upsilon \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle (1\ \ e_0\ \ *)} \\ \mathrm{C}_2^-(A_{01}) & \xrightarrow{\phi_1} & \mathrm{C}_2^-(A''_{00})\langle x\rangle \end{array}$$

which we can simplify to get that the E_2–page is the same as that of $\mathrm{C}_2^-(A_{01})$ and therefore $\mathrm{C}_2^-(A')$. □

With this lemma in hand, we are now prepared to prove conjugation invariance.

Proof of Theorem 5.6 It suffices to prove this in the case that $\alpha = \sigma_i^{\pm 1}$ is any generator of the braid group (or its inverse). Therefore, let $\sigma_i \in B_n$ be the generator which introduces a positive crossing between strands i and $i+1$.

Graphically, we would like to show that $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$, where D and D' are the partially singular braids depicted in Figure 28, when $\gamma = 1$. In order to prove this, we instead show that

$$\mathrm{C}_2^-(D'') \simeq_1 \mathrm{C}_2^-(D') \simeq_1 \mathrm{C}_2^-(D''')$$

for a generic $\gamma \in B_n$, where D'' and D''' are the diagrams in Figure 29.

Since we are considering the case $\alpha = \sigma_i$, note that in D'', the positive crossing occurs between strands i and $i+1$. If we decompose I_n, we see that for any i, we locally get a picture like Figure 25, where the top row of vertices is fixed if $i = 1$, and free if $i > 1$. Therefore, we may apply Lemma 5.7 directly to see

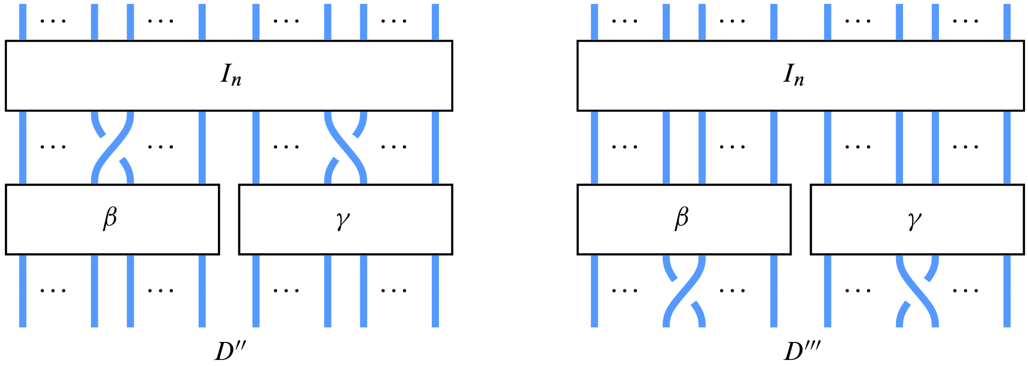


Figure 29: Alternate diagrams for proving conjugation invariance.

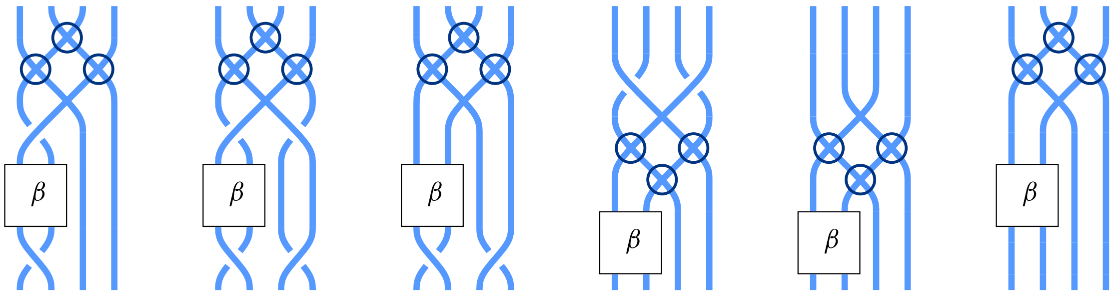


Figure 30: The steps to prove conjugation invariance for $n = 2$ and $\alpha = \sigma_1$.

that $\mathrm{C}_2^-(D'') \simeq_1 \mathrm{C}_2^-(D')$. Additionally, note that in D''', the negative crossing occurs between strands i and $i+1$. If we decompose I_n, we see that for any i, we locally get a picture like Figure 25, except that the bottom row of vertices is fixed if $i = 1$, and free if $i > 1$. In the latter case, this is not an issue and we may proceed as before to use Lemma 5.7 to prove that $\mathrm{C}_2^-(D''') \simeq_1 \mathrm{C}_2^-(D')$. If $i = 1$, then we first use Theorem 3.1 to relabel the top row of vertices as free and the bottom row as fixed; this diagram is still in $\mathfrak{D}^{\mathfrak{R}}$ as it contains the open braid S_{2n} from [Dowlin 2024] as a subdiagram, so we may proceed with the rest of the proof as usual.

Therefore, we can prove the desired equivalence $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$ when $\gamma = 1$ by first performing a Reidemeister II move to add two crossings to the right side of D, then using the equivalences $\mathrm{C}_2^-(D'') \simeq_1 \mathrm{C}_2^-(D')$ and $\mathrm{C}_2^-(D''') \simeq_1 \mathrm{C}_2^-(D')$ to simplify the diagram to D'. This proves Theorem 5.6 in the case of $\alpha = \sigma_i$. We illustrate these steps for the case $n = 2$ and $\alpha = \sigma_1$ in Figure 30. The proof for $\alpha = \sigma_i^{-1}$ is analogous, from which the proof for general $\alpha \in B_n$ follows. □

Appendix Homological algebra

In this section we review a few lemmas in homological algebra to aid in our calculations. We note that all four lemmas are true for filtered complexes, replacing maps with filtered maps and quasi-isomorphisms with filtered quasi-isomorphisms. Additionally, if we instead assume that filtered maps have filtration degree 1, then these lemmas still hold, replacing $\operatorname{cone}(f)$ with $\operatorname{cone}_1(f)$ and filtered quasi-isomorphism with E_1–quasi-isomorphism.

Lemma A.1 [Bar-Natan 2007, Lemma 4.2] *If $\varphi\colon A \to B$ is an isomorphism of complexes, then the double complexes*

$$C \xrightarrow{\binom{\alpha}{\beta}} A \oplus D \xrightarrow{\begin{pmatrix} \varphi & \delta \\ \gamma & \epsilon \end{pmatrix}} B \oplus E \xrightarrow{(\mu\ \nu)} F$$

and

$$C \xrightarrow{\beta} D \xrightarrow{\epsilon - \gamma\varphi^{-1}\delta} E \xrightarrow{\nu} F$$

are quasi-isomorphic.

This lemma is proved for the very general case of additive categories in [Bar-Natan 2007], and is well known in the specific case of free modules over a ring as the "cancellation lemma" or "reduction algorithm". We require it to prove invariance in Section 5, as it greatly simplifies calculations involving cubes of resolutions.

Lemma A.2 *Let $f: A \to B$ be a map of complexes, and suppose that $A \cong A' \oplus A''$ and $B \cong B' \oplus B''$, where A'' and B'' are acyclic. Let $\iota\colon A' \hookrightarrow A$ and $\pi: B \twoheadrightarrow B'$ be the associated inclusion and projection maps, respectively. Then $\operatorname{cone}(f) \simeq \operatorname{cone}(\pi \circ f \circ \iota)$.*

Proof First, we note that $\operatorname{cone}(\iota)$ is acyclic. One way to see this is via a cancellation argument: we have that $\operatorname{cone}(\iota) \cong (A' \to A' \oplus A'')$, which is quasi-isomorphic to A'' by Lemma A.1. Similarly, we get that $\operatorname{cone}(\pi)$, being quasi-isomorphic to B'', is acyclic as well.

For any two maps $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ of complexes, we have a long exact sequence relating the homology groups of $\operatorname{cone}(\alpha)$, $\operatorname{cone}(\beta)$, and $\operatorname{cone}(\beta \circ \alpha)$ (for example, via the octahedral axiom for triangulated categories applied to the derived category of R–modules). Therefore, we get that $\operatorname{cone}(f) \simeq \operatorname{cone}(f \circ \iota) \simeq \operatorname{cone}(\pi \circ f \circ \iota)$. □

Lemma A.3 *Let A, B, C, and D be complexes, and suppose that $A \cong A' \oplus A''$ and $D \cong D' \oplus D''$, where A'' and D'' are acyclic. Let $\iota\colon A' \hookrightarrow A$ and $\pi: D \twoheadrightarrow D'$ be the associated inclusion and projection maps, respectively. Then the following two cube of resolutions complexes have the same homotopy type:*

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ \big\downarrow{\scriptstyle g_1} & & \big\downarrow{\scriptstyle g_2} \\ C & \xrightarrow{f_2} & D \end{array} \qquad \begin{array}{ccc} A' & \xrightarrow{f_1 \circ \iota} & B \\ \big\downarrow{\scriptstyle g_1 \circ \iota} & & \big\downarrow{\scriptstyle \pi \circ g_2} \\ C & \xrightarrow{\pi \circ f_2} & D' \end{array}$$

Proof We know that the inclusion $\operatorname{cone}(g_1 \circ \iota) \hookrightarrow \operatorname{cone}(g_1)$ and the projection $\operatorname{cone}(g_2) \twoheadrightarrow \operatorname{cone}(\pi \circ g_2)$ are quasi-isomorphisms by the proof of Lemma A.2:

$$\begin{array}{ccc} A' & \xrightarrow{\iota} & A \\ \big\downarrow{\scriptstyle g_1 \circ \iota} & & \big\downarrow{\scriptstyle g_1} \\ C & \xrightarrow{\mathrm{id}_C} & C \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\mathrm{id}_B} & B \\ \big\downarrow{\scriptstyle g_2} & & \big\downarrow{\scriptstyle \pi \circ g_2} \\ D & \xrightarrow{\pi} & D' \end{array}$$

We can also view the maps $f_1 : A \to B$ and $f_2 : C \to D$ as components of a map $f : \mathrm{cone}(g_1) \to \mathrm{cone}(g_2)$. Therefore, we can compose f with the inclusion and projection to get a single map

$$f' : \mathrm{cone}(g_1 \circ \iota) \to \mathrm{cone}(\pi \circ g_2).$$

By the same long exact sequence logic as before, the cone of this map has the same homotopy type as f, ie

$$\mathrm{cone}(f') = \mathrm{cone}(\mathrm{cone}(g_1 \circ \iota) \to \mathrm{cone}(\pi \circ g_2)) \simeq \mathrm{cone}(\mathrm{cone}(g_1) \to \mathrm{cone}(g_2)) = \mathrm{cone}(f).$$

We conclude by noting that the complex on the left in Lemma A.3 is $\mathrm{cone}(f)$, and the complex on the right is $\mathrm{cone}(f')$. □

While the above lemma is phrased only for squares, it can be iterated to reduce summands of higher-dimensional cubes as well.

Since our complexes in this paper are often constructed as mapping cones, it helps to know when a quasi-isomorphism is induced by maps on the components of the cone.

Lemma A.4 *Suppose that we have the following commutative diagram of chain maps*:

$$\begin{array}{ccc} A_0 & \xrightarrow{g} & A_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ B_0 & \xrightarrow{g'} & B_1 \end{array}$$

Let $A = \mathrm{cone}(g)$ *and* $B = \mathrm{cone}(g')$, *so that we get a map* $f : A \to B$ *with components* f_0 *and* f_1. *If* f_0 *and* f_1 *are quasi-isomorphisms*, *then so is* f.

Proof By properties of the mapping cone, f induces a map of short exact sequences (with grading shifts suppressed):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A & \longrightarrow & A_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B & \longrightarrow & B_0 & \longrightarrow & 0 \end{array}$$

We can look at the induced map of long exact sequences in homology

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_*(A_1) & \longrightarrow & H_*(A) & \longrightarrow & H_*(A_0) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle H_*(f_1)} & & \downarrow{\scriptstyle H_*(f)} & & \downarrow{\scriptstyle H_*(f_0)} & & \\ \cdots & \longrightarrow & H_*(B_1) & \longrightarrow & H_*(B) & \longrightarrow & H_*(B_0) & \longrightarrow & \cdots \end{array}$$

to conclude that $H_*(f)$ must be an isomorphism as well, so f is a quasi-isomorphism. □

To prove the filtered generalizations of Lemma A.4, one replaces $H_*(-)$ with $E_1(-)$ or $E_2(-)$ (see [Weibel 1994, Exercise 5.4.4]).

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