AGT

Algebraic & Geometric Topology

Volume 24 (2024)

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Nikitas Nikandros
Constanze Roitzheim

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Algebraic & Geometric Topology **24**:9 (2024) 5161–5210
DOI: 10.2140/agt.2024.24.5161
Published: 27 December 2024

Monoidal properties of Franke's exotic equivalence

NIKITAS NIKANDROS
CONSTANZE ROITZHEIM

Franke's reconstruction functor $\mathcal{R}$ is known to provide examples of triangulated equivalences between homotopy categories of stable model categories, which are exotic in the sense that the underlying model categories are not Quillen equivalent. We show that, while not being a tensor-triangulated functor in general, $\mathcal{R}$ is compatible with monoidal products.

55P42; 18N55

1 Introduction

For several decades, Franke's exotic equivalence has been fascinating to homotopy theorists, as it is a rare example of a machinery that provides an equivalence up to homotopy between two model categories which are not Quillen equivalent. In practice, the known situations where Franke's construction can be applied to obtain the equivalence

$$\mathcal{R}\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{M})$$

link an algebraic model category ($\mathrm{D}^{([1],1)}(\mathcal{A})$ is the derived category of a flavour of chain complexes in a suitable abelian category $\mathcal{A}$) with a stable model category $\mathcal{M}$ which is not necessarily algebraic. Key examples include

- $\mathcal{A}$ the category of $\pi_*(R)$–modules for a ring spectrum R and $\mathcal{M}$ the category of modules over R, together with some extra assumption on the projective dimension of $\pi_*(R)$ as well as $\pi_*(R)$ being concentrated in degrees that are multiples of some $N > 1$,
- $\mathcal{A}$ the category of $E(1)_*E(1)$–comodules and $\mathcal{M}$ the category of K–local spectra at an odd prime.

In this paper, we will always assume that $\mathcal{R}$ exists and is an equivalence.

Both the algebraic side $\mathrm{D}^{([1],1)}(\mathcal{A})$ and the topological side $\mathrm{Ho}(\mathcal{M})$ are equipped with monoidal structures derived from the monoidal model category structures on $\mathrm{C}^{([1],1)}(\mathcal{A})$ and $\mathcal{M}$, so it is only natural to consider whether $\mathcal{R}$ is compatible with these. But as $\mathcal{R}$ is not derived from a Quillen functor $\mathrm{C}^{([1],1)}(\mathcal{A}) \to \mathcal{M}$, this problem requires a different approach working closely with the construction of $\mathcal{R}$ itself.

The example of K–local spectra at $p = 3$ tells us that we cannot expect $\mathcal{R}$ to be a monoidal functor in general: the preimage of the mod-3 Moore spectrum is a chain complex that is a monoid, whereas the mod-3 Moore spectrum has no associative multiplication [Ganter 2007, Remark 1.4.2]. However, we obtain the following, which is the main result of this article.

Theorem 1.0.1 *Let $(\mathcal{M}, \wedge)$ be a simplicial stable monoidal model category and let $(\mathcal{A}, \otimes)$ be a hereditary abelian monoidal category with enough projectives such that Franke's reconstruction functor $\mathcal{R}$ exists and is an equivalence. Then*

$$\mathcal{R}\colon (\mathrm{D}^{([1],1)}(\mathcal{A}), \otimes^{\mathbb{L}}) \to (\mathrm{Ho}(\mathcal{M}), \wedge^{\mathbb{L}})$$

commutes with the respective monoidal products up to a natural isomorphism

$$\mathcal{R}(M_* \otimes^{\mathbb{L}} N_*) \cong \mathcal{R}(M_*) \wedge^{\mathbb{L}} \mathcal{R}(N_*).$$

The reconstruction functor $\mathcal{R}$ rebuilds $\mathcal{M}$ from algebraic data in the following way. Firstly, part of the assumptions on $\mathcal{A}$ is that it splits into shifted copies of a smaller abelian category $\mathcal{B}$. This is then used to split an object of $\mathrm{C}^{([1],1)}(\mathcal{A})$ into pieces, which are placed in certain crown-shaped diagram C_N. Using this piecewise data, one then constructs a C_N–shaped diagram in $\mathcal{M}$. Finally, the homotopy colimit over C_N is applied to get to $\mathrm{Ho}(\mathcal{M})$. Specifically, $\mathcal{R}$ is the composite

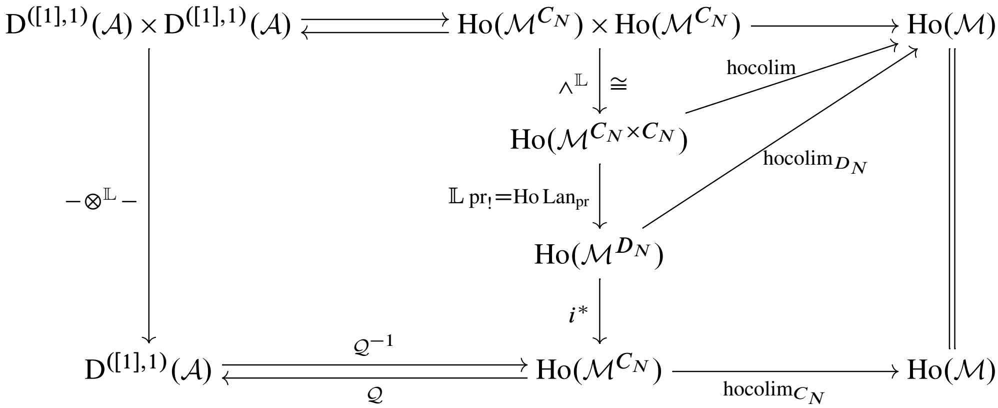
$$\mathcal{R}\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \xrightarrow{\mathcal{Q}^{-1}} \mathcal{L} \subseteq \mathrm{Ho}(\mathcal{M}^{C_N}) \xrightarrow{\mathrm{hocolim}_{C_N}} \mathrm{Ho}(\mathcal{M}).$$

We therefore take the diagram

$$\begin{array}{ccc} \mathrm{D}^{([1],1)}(\mathcal{A}) \times \mathrm{D}^{([1],1)}(\mathcal{A}) & \xrightarrow{\mathcal{R}\wedge^{\mathbb{L}}\mathcal{R}} & \mathrm{Ho}(\mathcal{M}) \times \mathrm{Ho}(\mathcal{M}) \\ {\scriptstyle \otimes^{\mathbb{L}}}\downarrow & & \downarrow{\scriptstyle \wedge^{\mathbb{L}}} \\ \mathrm{D}^{([1],1)}(\mathcal{A}) & \xrightarrow{\mathcal{R}} & \mathrm{Ho}(\mathcal{M}) \end{array}$$

which we would like to show to be commutative and refine it in the way below in order to deal with the different components of $\mathcal{R}$ separately:

(1.0.2)



Here, D_N denotes a suitable modification of the crown-shaped diagram C_N together with an inclusion $i\colon C_N \to D_N$. (All the ingredients will of course be defined in detail where appropriate.) The outline of our proof roughly follows the key points of [Ganter 2007]; however, we choose to work in a setting of model categories, which makes our exposition more explicit and straightforward. Overall, we have relied on contemporary methods and we refer to modern literature. Our techniques put Ganter's theorem in firm rigorous footing and in better context with other existing literature, as well as hopefully making it more adaptable to future generalisations.

This paper is organised as follows.

In Section 2 we recall the background and tools that we need for our main result and proof, namely simplicial replacements, homotopy Kan extensions, monoidal structures on diagram model categories, a specific mapping cone construction, calculating homotopy colimits using the homology of a category with coefficients in a functor, as well as a recap of the construction of Franke's functor.

In Section 3 we will begin by setting up one of our main results, which involves working out the middle vertical part of the diagram (1.0.2). The key ingredient is given by a spectral sequence argument calculating the vertices of the functor $\mathbb{L}\,\mathrm{pr}_!(X \wedge Y)$. We will then feed this into the definition of Franke's functor $\mathcal{Q}$ in order to obtain the necessarily formulas for monoidality on the left hand side of (1.0.2), dealing with underlying graded modules of the twisted chain complexes and the differentials separately.

Section 4 now wraps up the right hand side of the diagram (1.0.2) which mostly involves standard properties of homotopy colimits. We can finally assemble these results into the proof of the main theorem and finish with some examples.

Acknowledgements

This paper is based on the PhD thesis of Nikandros under the supervision of Roitzheim. We would like to acknowledge EPSRC grant EP/R513246/1 for funding this project. Furthermore, we would like to thank Nora Ganter, Irakli Patchkoria and Neil Strickland for helpful comments and support.

2 Preliminaries

In this section we will introduce some of the terminology that we need for our result. We assume that the reader is familiar with the basic background regarding simplicial sets, homological algebra and model categories.

The category of simplicial sets is denoted by $s\mathrm{Set}$. For $n \geq 0$, Δ^n denotes the standard n–simplex. For an arbitrary category $\mathcal{C}$, the notation $s\mathcal{C}$ stands for the simplicial objects in $\mathcal{C}$, ie $s\mathcal{C} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$. We have $I = \Delta^1$ and $I_+ = \Delta^1 \cup *$ and $S^0 = \Delta^0 \cup *$. Similarly, S^1 stands for the simplicial circle $I/(0 \sim 1)$, that is, $\Delta^1/\partial\Delta^1$.

We will let $\mathcal{A}$ be a graded ($\mathbb{Z}$–graded) abelian category, which means that $\mathcal{A}$ possesses a shift functor [1] which is an equivalence of categories, and $[n]$ denotes the n–fold iteration of [1]. The *graded global homological dimension* of $\mathcal{A}$, $\mathrm{gl.dim}\,\mathcal{A}$, is the supremum of the projective dimensions of objects in $\mathcal{A}$. An abelian category $\mathcal{A}$ is called *hereditary* if $\mathrm{gl.dim}\,\mathcal{A} = 1$. There are other, equivalent descriptions of hereditary abelian categories but this one suits our purposes best.

2.1 Model categories

We will now set up our background on model categories. We write any cofibrant replacement functor $Q\colon \mathcal{M} \to \mathcal{M}$ that comes with a natural weak equivalence $q\colon Q \to 1_{\mathcal{M}}$.

Convention 2.1.1 We let $\mathrm{Ho}(\mathcal{M})$ denote the category $\mathcal{M}_{\mathrm{cof}}[\mathcal{W}^{-1}]$, where $\mathcal{M}_{\mathrm{cof}}$ denotes the full subcategory of cofibrant objects of $\mathcal{M}$, and we denote the set of morphisms in $\mathrm{Ho}(\mathcal{M})$ by $[X, Y]$.

Convention 2.1.1 allows us to provide a very simple description of the *left derived functor* $\mathbb{L}F$ of a left Quillen functor $F\colon \mathcal{M} \to \mathcal{N}$. Indeed, the functor

$$F|_{\mathcal{M}_{\mathrm{cof}}}\colon \mathcal{M}_{\mathrm{cof}} \to \mathcal{N}_{\mathrm{cof}}$$

preserves weak equivalences and, therefore, it induces a functor between the localization. This functor is precisely $\mathbb{L}F$ with our convention.

Finally, an important class of model categories is the class of *simplicial model categories*. These are model categories which are enriched, tensored and cotensored over $s\mathrm{Set}$ and which satisfy the pushout-product axiom (SM7). If a simplicial model category is pointed, ie the terminal object is isomorphic to the initial one, then $\mathcal{M}$ is enriched over the category $s\mathrm{Set}_*$ of pointed simplicial sets. In particular, we have functors

$$- \otimes - \colon s\mathrm{Set}_* \times \mathcal{M} \to \mathcal{M}, \quad \mathrm{Map}_{\mathcal{M}}(-,-)\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to s\mathrm{Set}_*,$$

and the adjunction

$$\mathrm{Hom}_{\mathcal{M}}(K \wedge X, Y) \cong \mathrm{Hom}_{s\mathrm{Set}}(K, \mathrm{Map}_{\mathcal{M}}(X, Y)),$$

see [Barnes and Roitzheim 2020, Definition 6.1.28; Riehl 2014, Section 11.4].

2.1.1 Diagram categories We will use model structures on diagram categories throughout the paper. Below we introduce the definition of a direct category which is a generalization of the concept of a poset; see [Hovey 1999, Definition 5.1.1] for further details.

Definition 2.1.2 Let ω denote the poset category of the ordered set $\{0, 1, 2, \dots\}$. A small category J is called *direct* if there is a functor $f\colon J \to \omega$ that sends nonidentity morphisms to nonidentity morphisms. We refer to $f(j)$ as the *degree* of the object j. Dually, J is an *inverse* category if there is a functor $J^{\mathrm{op}} \to \omega$ that sends nonidentity morphisms to nonidentity morphisms

Any finite poset J is a direct category, and dually J^{op} is an inverse category. We provide some examples that will be useful later on.

Definition 2.1.3 Suppose $\mathcal{M}$ is a small category with small colimits, J a small category, z an object in J and J_z the category of all nonidentity morphisms with codomain z. The *latching space functor* $L_z\colon \mathcal{M}^J \to \mathcal{M}$ is the composition

$$\mathcal{M}^J \to \mathcal{M}^{J_z} \xrightarrow{\mathrm{colim}} \mathcal{M},$$

where the first arrow is the restriction functor. Equivalently the latching space of a diagram X is given by

$$L_z X = \operatorname{colim}(J_z \hookrightarrow J \xrightarrow{X} \mathcal{M}),$$

where $J_z \hookrightarrow J$ is the inclusion.

Note that we have a natural transformation $L_z X \to X_z$ for any fixed object $z \in J$.

We can now describe the *projective model structure* on $\mathcal{M}^J$; see [Hovey 1999, Theorem 5.1.3].

Proposition 2.1.4 *Given a model category $\mathcal{M}$ and a direct category J, there is a model structure on $\mathcal{M}^J$ in which a morphism $f: X \to Y$ is a weak equivalence (resp. fibration) if and only if the map $f_z: X_z \to Y_z$ is a weak equivalence (resp. fibration) for all $z \in J$. Furthermore, $f: X \to Y$ is an (acyclic) cofibration if and only if the induced map*

$$X_z \coprod_{L_z X} L_z Y \to Y_z$$

is an (acyclic) cofibration for all $z \in J$.

We will now give the finite posets J that are going to play a central role throughout this paper.

Example 2.1.5 By [1] we denote the poset $0 \le 1$. We are aware that early in this section we also denoted the shift functor on graded objects. Both are standard notation, and from our use of the poset $0 \le 1$ there is vanishingly little danger of confusing those two.

Example 2.1.6 Consider the poset

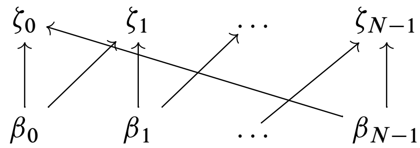
$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \big\downarrow & & \\ (0,1) & & \end{array}$$

denoted by $\ulcorner$. Let $\iota: [1] \to \ulcorner$ be the map of posets which sends 0 to $(0,0)$ and 1 to $(1,0)$. In other words, ι includes the interval [1] to the top horizontal line. Furthermore, consider the product of the interval posets $[1] \times [1]$. It is the poset

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \big\downarrow & & \big\downarrow \\ (0,1) & \longrightarrow & (1,1) \end{array}$$

and we let $i_\ulcorner : \ulcorner \to [1] \times [1]$ be the inclusion.

Example 2.1.7 Let $N \geq 2$ be a natural number. The poset C_N consists of elements $\{\beta_i, \zeta_i \mid i \in \mathbb{Z}/N\mathbb{Z}\}$ such that $\beta_i < \zeta_i$ and $\beta_i < \zeta_{i+1}$ for $i \in \mathbb{Z}/N\mathbb{Z}$, ie



Then $X \in \mathcal{M}^{C_N}$ is cofibrant if and only if the canonical map $L_z X \to X_z$ is a cofibration in $\mathcal{M}$, ie if and only if the X_{β_i}, X_{ζ_i} are cofibrant and the induced morphism

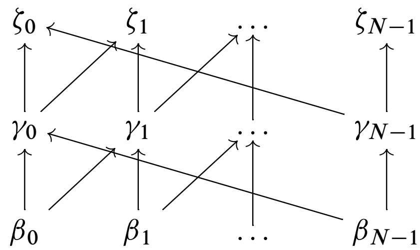
$$X_{\beta_{i-1}} \vee X_{\beta_i} \to X_{\zeta_i}$$

is a cofibration, where $\vee$ is the coproduct in $\mathcal{M}$. We will refer to an object $X \in \mathcal{M}^{C_N}$ as a *crowned diagram* due to the crown shape of the diagram C_N.

Example 2.1.8 Let D_N be the poset consisting of elements $\{\beta_n, \gamma_n, \zeta_n \mid n \in \mathbb{Z}/N\mathbb{Z}\}$ such that

$$\beta_n \leq \gamma_n \leq \zeta_n \quad \text{and} \quad \beta_n \leq \gamma_{n+1} \quad \text{and} \quad \gamma_n \leq \zeta_{n+1},$$

ie



Remark 2.1.9 In what follows, when we have a direct category I and a model category $\mathcal{M}$, the category of diagrams $\mathcal{M}^I$ will always have the model structure defined in Proposition 2.1.4 without further mention. If not, we will explicitly say so.

It follows that for any model category $\mathcal{M}$ and direct category J, there is a Quillen adjunction

$$\operatorname{colim}\colon \mathcal{M}^J \rightleftarrows \mathcal{M} : \operatorname{const}.$$

(Note that when we write an adjunction, the top arrow will always denote the left adjoint.)

Definition 2.1.10 The left derived functor of $\operatorname{colim}\colon \mathcal{M}^J \to \mathcal{M}$ is called the *homotopy colimit* and is denoted by

$$\operatorname{hocolim}\colon \operatorname{Ho}(\mathcal{M}^J) \to \operatorname{Ho}(\mathcal{M}).$$

If $J = \ulcorner$, then the homotopy colimit is called *homotopy pushout*. A particular example of homotopy pushout is the *homotopy cofiber* which is the homotopy pushout of a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ * & & \end{array}$$

and we write

$$\mathrm{hocofib}(f) := \mathrm{hocolim}(* \leftarrow X \xrightarrow{f} Y). \tag{2.1.11}$$

In general, for notational convenience sometimes a homotopy pushout is denoted by

$$\mathrm{hocolim}(Z \leftarrow X \rightarrow Y) := Z \coprod_X^h Y.$$

2.1.2 Homotopy colimits in simplicial model categories In Definition 2.1.10 we recalled the definition of the homotopy colimit as a derived functor. Here, we will present an alternative construction via simplicial techniques. After introducing some definitions we briefly explain how this method provides a good theory of homotopy colimits; see also [Riehl 2014, Chapters 4, 5; Shulman 2006, Section 7].

Let $\mathcal{M}$ be a model category and consider the category of simplicial objects $s\mathcal{M} = \mathcal{M}^{\Delta^{\mathrm{op}}}$. We consider $s\mathcal{M}$ as a simplicial category with tensors defined objectwise, ie for $K \in s\mathrm{Set}$ and $X \in s\mathcal{M}$ we have

$$(K \otimes X)_n = K \otimes X_n.$$

Now, let $\mathcal{M}$ be a simplicial model category. Given a simplicial object $X \in s\mathcal{M}$ we can construct an object in $\mathcal{M}$ via *geometric realization*, see [Hirschhorn 2003, Definition 18.6.2].

Definition 2.1.12 (geometric realization) Let $X \in \mathcal{M}^{\Delta^{\mathrm{op}}}$. *The geometric realization* of X, denoted as $|X|$, is defined as the coequalizer

$$\mathrm{coeq}\Big(\coprod_{\sigma\colon [n]\to[k]\in\Delta} \Delta^k \otimes X_n \rightrightarrows \coprod_{[n]\in\Delta} \Delta^n \otimes X_n \Big).$$

This is an example of a functor tensor product (coend). In this case, the geometric realization is the functor tensor product of $X\colon \Delta^{\mathrm{op}} \to \mathcal{M}$ and the functor $\Delta^\bullet\colon \Delta \to s\mathrm{Set}$, $[n] \mapsto \Delta^n$. In other words, the realization $|X|$ is the object

$$\Delta^\bullet \otimes_{\Delta^{\mathrm{op}}} X = \int^n \Delta^n \otimes X_n.$$

The following theorem is the cornerstone of our exposition of homotopy colimits using geometric realizations; see [Goerss and Jardine 1999, VII 3.6; Hirschhorn 2003, 18.4.11; Riehl 2014, Corollary 14.3.10]. For details for the Reedy model structure on $s\mathcal{M}$, see [Goerss and Jardine 1999, Definition 2.1].

Theorem 2.1.13 *If $\mathcal{M}$ is a simplicial model category, then*

$$|-|\colon \mathcal{M}^{\Delta^{op}} \to \mathcal{M}$$

is a left Quillen functor with respect to the Reedy model structure. In particular, $|-|$ sends Reedy cofibrant simplicial objects to cofibrant objects and preserves objectwise weak equivalences between them.

At this level of generality, this is the strongest result possible. It is not true that geometric realization preserves all objectwise weak equivalences. However, the above will suffice for our purposes. We can now start to work our way to the homotopy colimit of a diagram $X \in \mathcal{M}^J$ in a simplicial model category $\mathcal{M}$.

Our first definition towards this goal is the *simplicial replacement functor*. That is to say, given any diagram $F\colon I \to \mathcal{M}$ we can replace it with simplicial object in $\mathcal{M}$ with good properties.

Definition 2.1.14 (simplicial replacement) Let I be a small category and consider a diagram $X \in \mathcal{M}^I$. The *simplicial replacement* of X is the simplicial object in $\mathcal{M}$, denoted $\operatorname{srep} X$ given in simplicial degree $[n]$ by

$$(\operatorname{srep} X)_n = \coprod_{(i_0 \to i_1 \to \cdots \to i_n) \in N(I)_n} X_{i_0}.$$

The coproduct is indexed over the set of n–chains

$$\sigma = [i_0 \to i_1 \to \cdots \to i_n]$$

over the nerve of I. If $0 \le k < n$, then

$$d_k\colon (\operatorname{srep} X)_n \to (\operatorname{srep} X)_{n-1}$$

maps the term X_{i_n} indexed on σ to the term X_{i_n} indexed on

$$\sigma(k) = [i_0 \to i_1 \to i_{k-1} \to i_{k+1} \to \cdots \to i_n]$$

via the identity, while for $k = n$, the map d_n sends the term X_{i_n} to $X_{i_{n-1}}$ indexed on

$$\sigma(n) = [i_0 \to i_1 \to \cdots \to i_{n-1}]$$

via the induced map $X(i_n \to i_{n-1})$. The degeneracy maps

$$s_j\colon (\operatorname{srep} X)_n \to (\operatorname{srep} X)_{n+1}, \quad 0 \le j \le n$$

are easier to define. Each s_j sends the summand X_{i_n} corresponding to the summand

$$[i_0 \to i_1 \to \cdots \to i_n]$$

to the identical summand X_{i_n} corresponding to the chain in which one has inserted the identity map $i_j \to i_j$.

In other words, the simplicial replacement is the following simplicial object,

$$\coprod_{i_0} X_{i_0} \leftleftarrows \coprod_{i_0 \to i_1} X_{i_0} \leftleftarrows\!\!\leftarrow \coprod_{i_0 \to i_1 \to i_2} X_{i_0} \cdots,$$

where degeneracy maps are omitted. Note that this is can also be found in literature as the *simplicial bar construction* or *Bousfield–Kan construction* denoted by $B(*, I, X)$.

Remark 2.1.15 The colimit of a diagram $X \in \mathcal{M}^I$, if it exists, agrees with the colimit of $\mathrm{srep}(X) \in s\mathcal{M}$. Indeed, consider the colimit of the diagram $\mathrm{srep}(X)$ as the coequilizer

$$\coprod_i X_i \leftleftarrows \coprod_{j \leftarrow i} X_i,$$

but this is precisely the colimit of X. Therefore in this case, $\mathrm{srep}(X)$ has the augmentation

$$\mathrm{srep}(F) \to \operatorname*{colim}_I F,$$

where we regard the object $\mathrm{colim}_I F$ as a constant simplicial object.

We therefore reach the following result.

Lemma 2.1.16 *Given a diagram $X \in \mathcal{M}^I$ and its simplicial replacement $\mathrm{srep}(X) \in \mathcal{M}^{\Delta^{\mathrm{op}}}$, there is a canonical isomorphism*

$$\operatorname*{colim}_I X \cong \operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathrm{srep}(X)).$$

The proof can be found in [Riehl 2014, Lemma 4.4.2]. The following lemma will also be of importance, see [Riehl 2014, Lemma 5.1.2; Shulman 2006, Lemma 8.7].

Lemma 2.1.17 *Let I be a small category and let $\mathcal{M}$ be a simplicial model category. If $F \in \mathcal{M}^I$ is objectwise cofibrant, then $\mathrm{srep}(F) \in s\mathcal{M}$ is Reedy cofibrant.*

The above Lemma 2.1.16 and Theorem 2.1.13 essentially mean that geometric realization of objectwise cofibrant diagrams is a good model for calculating homotopy colimits. For details see [Riehl 2014, Theorem 6.6.1].

2.2 Homotopy Kan extensions

In this subsection we will introduce *homotopy Kan extensions*, the homotopy invariant version of ordinary Kan extensions, see eg [Hirschhorn 2003, Section 11.9].

Now, let $\mathcal{M}$ be a model category. Furthermore, let I, J be direct categories and $f: I \to J$ a functor. The pullback functor

$$f^*: \mathcal{M}^J \to \mathcal{M}^I$$

preserves weak equivalences, so it defines a functor between homotopy categories, which we denote by the same letter. Recall the functor $\mathrm{Lan}_f = f_!$, left adjoint to f^*. We have the following proposition.

Proposition 2.2.1 *Let $\mathcal{M}$ be a model category and let $f : I \to J$ be a map of direct categories. Then the adjunction*

$$f_! : \mathcal{M}^I \rightleftarrows \mathcal{M}^J : f^*$$

is a Quillen adjunction.

Proof This follows from the definition of the projective model structure; see Proposition 2.1.4. The functor f^* is a right adjoint by construction. It preserves weak equivalences and projective fibrations, which means that f^* is also a right Quillen functor. □

Thus, the derived functors of the adjoint pair $(f_!, f^*)$ define an adjoint pair on the level of homotopy categories

$$\mathbb{L}\mathrm{Lan}_f := \mathbb{L} f_! : \mathrm{Ho}(\mathcal{M}^I) \rightleftarrows \mathrm{Ho}(\mathcal{M}^J) : \mathbb{R} f^*.$$

A useful fact about homotopy Kan extensions is that they does not change the homotopy colimit of a diagram, which is similar to the properties of ordinary Kan extensions.

Corollary 2.2.2 *Let $\mathcal{M}$ be a model category, $f : I \to J$ a map of direct categories and let $X \in \mathcal{M}^I$. Then there is a canonical isomorphism in* $\mathrm{Ho}(\mathcal{M})$

$$\operatorname*{hocolim}_J \mathbb{L} f_! X \cong \operatorname*{hocolim}_I X.$$

Proof This follows from the fact that for every pair of left Quillen functors F and G there is a natural isomorphism

$$\mathbb{L}F \circ \mathbb{L}G \to \mathbb{L}(F \circ G),$$

see [Hovey 1999, Theorem 1.37], together with the natural isomorphism

$$\operatorname*{colim}_J \mathrm{Lan}_f X \cong \operatorname*{colim}_I X.$$ □

To conclude this section, we will shortly discuss how one calculates the values and edges of a homotopy Kan extension. Recall the notion of a *slice category* for given posets C and D and a functor $f : C \to D$, namely

$$f/d = \{c \in C \mid f(c) \le d\} \tag{2.2.3}$$

for $d \in D$. The following is [Cisinski 2009, Proposition 1.14], which tells us that homotopy Kan extensions can be computed pointwise.

Proposition 2.2.4 *Let $f : I \to J$ be a map of posets and let X be any functor $I \to \mathcal{M}$. For any object $j \in J$ there is a canonical isomorphism in* $\mathrm{Ho}(\mathcal{M})$

$$(\mathbb{L} f_! F)_j \cong \mathrm{hocolim}(f/j \xrightarrow{\pi} I \xrightarrow{X} \mathcal{M}).$$

2.3 Monoidal model categories

Let us now turn to some results concerning monoidal model categories, see eg [Hovey 1999, Definition 4.2.6], [Barnes and Roitzheim 2011, Definition 6.1.9] or [Riehl 2014, Definition 11.4.6] for definitions.

Remark 2.3.1 Let $(\mathcal{C}, \wedge)$ be a closed symmetric monoidal category and let $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ be maps in $\mathcal{C}$. The pushout-product map is the universal arrow

$$f \mathbin{\square} g\colon X_0 \wedge Y_1 \coprod_{X_0 \wedge Y_0} X_1 \otimes Y_0 \to X_1 \wedge Y_1.$$

Another way to see the pushout-product map is as a left Kan extension. Again, consider a cocomplete, (closed) monoidal category $(\mathcal{C}, \wedge)$. Let $[1] = \{0 \leq 1\}$. Furthermore, consider the following map of posets.

$$\mathrm{pr}\colon [1] \times [1] \to [1], \qquad \begin{aligned} (0,0), (1,0), (0,1) &\mapsto 0, \\ (1,1) &\mapsto 1. \end{aligned}$$

Now let f and g be morphisms in $\mathcal{C}$. We can consider them as objects in the arrow category $f, g \in \mathcal{C}^{[1]}$. The functors $f\colon [1] \to \mathcal{C}$ and $g\colon [1] \to \mathcal{C}$ give rise to their objectwise tensor product $f \wedge g$, see Definition 2.3.2. That is, the functor

$$f \wedge g\colon [1] \times [1] \to \mathcal{C}$$

is the following commutative diagram:

$$\begin{array}{ccc} X_0 \wedge Y_0 & \longrightarrow & X_1 \wedge Y_0 \\ \big\downarrow & & \big\downarrow \\ X_0 \wedge Y_1 & \longrightarrow & X_1 \wedge Y_0 \end{array}$$

Note that the slice category $\mathrm{pr}/0$ is the poset $\ulcorner$ and the slice $\mathrm{pr}/1$ is the whole square. It follows that the map

$$\operatorname*{colim}_{\ulcorner}(f \wedge g) \to \operatorname*{colim}_{[1]\times[1]}(f \wedge g)$$

induced by the inclusion $\ulcorner \hookrightarrow [1] \times [1]$ is exactly the map

$$f \mathbin{\square} g\colon X_0 \wedge Y_1 \coprod_{X_0 \wedge Y_0} X_1 \wedge Y_1 \to X_1 \wedge Y_1.$$

So indeed, $(\mathrm{Lan}_{\mathrm{pr}}(f \wedge g)) = \mathrm{pr}_!(f \wedge g) = f \mathbin{\square} g$.

2.3.1 Smash products for diagram categories A monoidal category $(\mathcal{M}, \wedge)$ gives rise to more monoidal categories by considering diagrams from small categories into $\mathcal{M}$. In our next example we discuss how this is related to model category theory.

Definition 2.3.2 Let $(\mathcal{M}, \wedge)$ be a monoidal category and let I and J be direct categories. We define the external product, which is the bifunctor

$$-\wedge - : \mathcal{M}^I \times \mathcal{M}^J \to \mathcal{M}^{I\times J}$$

sending (X, Y) to the diagram

$$X \wedge Y : I \times J \to \mathcal{M}, \quad (i, j) \mapsto X_i \wedge Y_j.$$

The external product is part of a two-variable adjunction. Since we do not use the extra structure will not define the other two functors in the two-variable adjunction. We have the following proposition.

Proposition 2.3.3 *Let $(\mathcal{M}, \wedge)$ be a monoidal model category. Then, the bifunctor*

$$-\wedge - : \mathcal{M}^I \times \mathcal{M}^J \to \mathcal{M}^{I\times J}$$

is a Quillen bifunctor, that is to say, it has a total left derived functor

$$-\wedge^{\mathbb{L}} - : \mathrm{Ho}(\mathcal{M}^I) \times \mathrm{Ho}(\mathcal{M}^J) \to \mathrm{Ho}(\mathcal{M}^{I\times J}).$$

Proof Suppose that the *injective* model structures $\mathcal{M}^I_{\mathrm{inj}}$, $\mathcal{M}^J_{\mathrm{inj}}$ and $\mathcal{M}^{I\times J}_{\mathrm{inj}}$ exist, eg if $\mathcal{M}$ is a combinatorial model category. Since in the injective model structures the cofibrations are the objectwise cofibrations, the above proposition follows directly. The universal property of $-\wedge^{\mathbb{L}} -$ implies that up to canonical isomorphism both constructions give the same result. □

We have the following corollary.

Corollary 2.3.4 *In the context of Proposition 2.3.3, there is a functor isomorphism*

$$\operatorname*{hocolim}_{I\times J}(X \wedge^{\mathbb{L}} Y) \cong (\operatorname*{hocolim}_{I} X) \wedge^{\mathbb{L}} (\operatorname*{hocolim}_{J} Y).$$

Proof From Proposition 2.3.3, it follows that the external product preserves diagram cofibrant objects and preserves trivial diagram cofibrations between diagram cofibrant objects. The result now follows from the strict formula

$$\operatorname*{colim}_{I\times J}(X \wedge Y) \cong (\operatorname*{colim}_{I} X) \wedge (\operatorname*{colim}_{J} Y)$$

as all the objects involved are cofibrant. □

As a consequence of Proposition 2.3.3, we also obtain the following.

Example 2.3.5 Let $(\mathcal{M}, \wedge)$ be a monoidal model category and let J be a direct category. Consider the diagram category $\mathcal{M}^J$ with the model structure of Proposition 2.1.4. The category $\mathcal{M}^J$ inherits a monoidal structure

$$\mathcal{M}^J \times \mathcal{M}^J \to \mathcal{M}^J, \quad (X, Y) \mapsto X \wedge Y,$$

where $X \wedge Y$ is the diagram $j \mapsto X_j \wedge Y_j$. By a proof analogous to that of Proposition 2.3.3, $(\mathcal{M}^J, \wedge)$ is a monoidal model category.

Corollary 2.3.6 *Let $(\mathcal{M}, \wedge)$ be a pointed symmetric monoidal model category, and let $f\colon X \to Y$ and $g\colon U \to V$ be morphisms in $\mathcal{M}$. There is a canonical isomorphism*

$$\mathrm{hocofib}(f) \wedge^{\mathbb{L}} \mathrm{hocofib}(g) \cong \mathrm{hocofib}(f \,\square^{\mathbb{L}}\, g).$$

We will provide a proof since it is important to our exposition. A different proof can be found in [Hovey 2014, Proposition 4.1].

Proof We may assume that X, Y, U, V are cofibrant in $\mathcal{M}$. By definition,

$$\mathrm{hocofib}(f) \wedge^{\mathbb{L}} \mathrm{hocofib}(g) = \mathrm{hocolim}(* \leftarrow X \xrightarrow{f} Y) \wedge^{\mathbb{L}} \mathrm{hocolim}(* \leftarrow U \xrightarrow{g} V).$$

By Corollary 2.3.4, this is isomorphic to

$$\mathrm{hocolim}\begin{pmatrix} * & \longleftarrow & X \wedge V & \longrightarrow & Y \wedge V \\ \uparrow & & \uparrow & & \uparrow \\ * & \longleftarrow & X \wedge U & \longrightarrow & Y \wedge U \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & * & \longrightarrow & * \end{pmatrix}. \tag{2.3.7}$$

We denote the above underlying $\ulcorner\times\ulcorner$–diagram by $\mathcal{Z}$. We define the following map of posets

$$\begin{aligned} \mathrm{pr}\colon \ulcorner\times\ulcorner \to \ulcorner, \qquad\qquad ((1,0),(1,0)) &\mapsto (1,0), \\ ((0,0),(0,0)), ((0,0),(1,0)), ((1,0),(0,0)) &\mapsto (0,0), \\ \text{else} &\mapsto (0,1), \end{aligned}$$

and consider the homotopy left Kan extension

$$\mathbb{L}\mathrm{pr}_!\colon \mathrm{Ho}(\mathcal{M}^{\ulcorner\times\ulcorner}) \to \mathrm{Ho}(\mathcal{M}^{\ulcorner}). \tag{2.3.8}$$

Applying the formula Proposition 2.2.4 to the diagram $\mathcal{Z}$ we obtain $(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(1,0)} = Y \wedge V$. Next, for the object $(0,0)$ the slice category $\mathrm{pr}/(0,0)$ is just the poset $\ulcorner$ and we have

$$(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(0,0)} = \mathrm{hocolim}\begin{pmatrix} X \wedge U & \xrightarrow{f\wedge 1} & Y \wedge U \\ \downarrow{\scriptstyle 1\wedge g} & & \\ X \wedge V & & \end{pmatrix}$$

and finally, $(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(0,1)} \cong *$. Note that

$$(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(0,0)} \to (\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(1,0)} = f \,\square^{\mathbb{L}}\, g.$$

Hence, the homotopy left Kan extension (2.3.8) of the underlying diagram (2.3.7) is the following $\ulcorner$–diagram:

$$\begin{array}{ccc} X \wedge V \coprod_{X\wedge U} Y \wedge U & \longrightarrow & Y \wedge V \\ \downarrow & & \\ * & & \end{array}$$

It follows directly that the homotopy colimit of this diagram is

$$\mathrm{hocofib}(f \,\square^{\mathbb{L}}\, g). \qquad \square$$

2.3.2 Stable model categories and triangulated categories Recall that the homotopy category $\mathrm{Ho}(\mathcal{M})$ of a pointed model category $\mathcal{M}$ supports a *suspension* functor

$$\Sigma\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$$

given by

$$\Sigma X := \mathrm{hocolim}(* \leftarrow X \rightarrow *),$$

with a right adjoint functor

$$\Omega\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$$

given by

$$\Omega X = \mathrm{holim}(* \rightarrow X \leftarrow *).$$

Definition 2.3.9 A *stable model category* is a pointed model category for which the functors Σ and Ω are inverse equivalences.

Example 2.3.10 The prototypical example of a stable model category is the category of spectra, Sp. There are of course many variants of spectra, but as our result does not depend on a choice of suitable, monoidal model category, we will not need to specify this further.

Example 2.3.11 Let $\mathcal{A}$ be a graded abelian category with enough projectives, and let $\mathrm{C}^{([1],1)}(\mathcal{A})$ denote the category of *twisted* $([1],1)$–chain complexes or *differential* objects. An object of $\mathrm{C}^{([1],1)}(\mathcal{A})$ is a pair (M_*, d) with $M_* \in \mathcal{A}$ together with a morphism (the differential)

$$d\colon M_* \to M_*[1],$$

such that $d[1] \circ d = 0$. The category $\mathrm{C}^{([1],1)}(\mathcal{A})$ admits a stable model structure, the *projective model structure*, where the weak equivalences are the homology isomorphisms and the fibrations are the surjections. In particular, the cofibrant objects are the projective objects of $\mathcal{A}$. We let $\mathrm{D}^{([1],1)}(\mathcal{A})$ denote the homotopy category of $\mathrm{C}^{([1],1)}(\mathcal{A})$. For an object $(M_*, d) \in \mathrm{C}^{([1],1)}(\mathcal{A})$ we define the homology $H(M) = \ker d/\operatorname{im} d$, and so we have the homology functor

$$H_*\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathcal{A}.$$

In the following we will let $(\mathcal{A}, \otimes, \mathbf{1})$ be an abelian symmetric monoidal category with enough projectives. In this case $(\mathrm{C}^{([1],1)}(\mathcal{A}), \otimes)$ is a monoidal stable model category. Finally, we mention the homology functor $H_*\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathcal{A}$ is a lax symmetric monoidal functor via the Künneth morphism.

We note that our methods throughout this paper also work in a setting where $\mathcal{A}$ does not have enough projectives. In the case of $\mathcal{A} = E(1)_*E(1)$-comod, $\mathrm{C}^{([1],1)}(\mathcal{A})$ can be equipped with a model structure where the cofibrant twisted chain complexes are degreewise projective as $E(1)_*$–modules. This *relative projective model structure* is also monoidal; see [Barnes and Roitzheim 2011, Section 5].

If $\mathcal{M}$ is a pointed simplicial model category, then the suspension functor

$$\Sigma\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$$

admits a simple description. Indeed, by the simplicial model category axioms, the functor

$$S^1 \wedge -\colon \mathcal{M} \to \mathcal{M}$$

defined using the tensor with simplicial sets is a left Quillen functor. Then, Σ can be defined as the left derived functor of $S^1 \wedge -$, ie

$$\Sigma X := S^1 \wedge^{\mathbb{L}} X = S^1 \wedge QX;$$

see [Hovey 1999, 6.1.1]. Note that if $\mathcal{M}$ is stable, then the homotopy category $\mathrm{Ho}(\mathcal{M})$ is a triangulated category with Σ a shift functor; see [Barnes and Roitzheim 2011, Theorem 4.2.1; Hovey 1999, 7.1.6].

In a simplicial model category $\mathcal{M}$ we can choose a particular *model* for the homotopy cofiber (2.1.11) of a morphism, which will help with computations. It is called the *mapping cone* construction.

Definition 2.3.12 Suppose $\mathcal{M}$ is a simplicial stable model category and $f\colon X \to Y$ a morphism in $\mathcal{M}_{\mathrm{cof}}$. Let $\mathrm{cone}(f)$ be the pushout of f along the canonical morphism

$$\mathrm{incl} \otimes 1\colon S^0 \otimes X \to (I,0) \otimes X = CX,$$

that is, $\mathrm{cone}(f)$ comes with the pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{incl}\otimes 1}\downarrow & & \downarrow \\ CX & \longrightarrow & \mathrm{cone}\, f \end{array}$$

Here $CX = (I,0) \otimes X$ denotes the *cone* of X. The natural map

$$\pi\colon (I,0) \otimes X \to S^1 \otimes X$$

and the trivial map

$$*\colon Y \to S^1 \otimes X$$

induce, using the universal property of pushout, a map $\partial\colon \mathrm{cone}(f) \to S^1 \otimes X$.

The fact that the mapping cone construction represents the homotopy cofiber and further details can be found in [Barnes and Roitzheim 2020, Section 4.3].

Definition 2.3.13 Let $\mathcal{M}$ be a simplicial stable model category and $f\colon X \to Y$ a morphism in $\mathcal{M}_{\mathrm{cof}}$. The *elementary triangle* associated to f is the triangle

$$X \xrightarrow{f} Y \xrightarrow{\iota} \mathrm{cone}(f) \xrightarrow{\partial} S^1 \otimes X.$$

A triangle (f, g, h)

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

in $\mathrm{Ho}(\mathcal{M})$ is called *distinguished* if it is isomorphic to an elementary one.

2.4 Homology of a category with coefficients in a functor

In this subsection we will introduce one our main tools, namely homology of a category with coefficients in a functor. It is a particular case of functor homology that assigns the groups $\mathrm{Tor}_*^I(F, G)$ to functors $F\colon I \to \mathcal{A}$ and $G\colon I^{\mathrm{op}} \to \mathcal{A}$ with $\mathcal{A}$ an abelian category. Since we do not need such generality, we will introduce it in a more down-to-earth way using simplicial techniques that dates back to Quillen. Traditional references include [Oberst 1967; 1968], more contemporary references include [Gálvez-Carrillo et al. 2013; Richter 2020, Chapters 15, 16].

Before we define the homology of a category with coefficients in a functor we will define the associated complex of a simplicial object in an abelian category.

Definition 2.4.1 Let $D \in s\mathcal{A}$ be a simplicial object in $\mathcal{A}$. We define the *associated complex* $(C_\bullet(U), \partial) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ by

$$C_n(D) = D_n, \quad \partial_n = \sum_{i=0}^{n} (-1)^n d_i \colon C_n(D) \to C_{n-1}(D).$$

Note that the simplicial identities imply $\partial^2 = 0$, so $C_\bullet(D)$ is indeed a chain complex. Moreover, this evidently defines a functor $C\colon s\mathcal{A} \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$. In other words, the associated complex to a simplicial object $D \in s\mathcal{A}$ is the following chain complex:

$$D_0 \xleftarrow{d_0 - d_1} D_1 \xleftarrow{d_0 - d_1 + d_2} D_2 \leftarrow \cdots. \tag{2.4.2}$$

Definition 2.4.3 Let I be a small category and consider a diagram $D\colon I \to \mathcal{A}$. The *homology of the category I with coefficients in the functor D* is defined as the homology of the complex $C_\bullet(D)$, ie the homology of the associated complex of the simplicial replacement $\mathrm{srep}(D) \in s\mathcal{A}$.

So, unwinding the definition, we start by first taking the simplicial replacement $\mathrm{srep}(D)\colon \Delta^{\mathrm{op}} \to \mathcal{A}$ of D, see Definition 2.1.14, that is, the diagram

$$\bigoplus_{i_0} D_{i_0} \leftleftarrows \bigoplus_{i_0 \to i_1} D_{i_0} \,\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\, \bigoplus_{i_0 \to i_1 \to i_2} D_{i_0} \cdots .$$

Then, we consider the associated chain complex (2.4.2) of $C_\bullet(D)$. Then we defined $H_p(I; D)$ to be the p^{th} homology group of the chain complex $C_\bullet(D)$.

Now we will investigate how these constructions help us calculate homotopy colimits. First, recall the following.

Definition 2.4.4 We call a functor $F_*\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$ *homological* if it satisfies the following conditions:

(i) F_* is a graded functor, that is to say, it commutes with suspensions, so there are natural equivalences

$$F_*(\Sigma X) \cong F_*(X)[1] := F_{*-1}(X)$$

which are part of the structure.

(ii) F_* is additive, ie it commutes with arbitrary coproducts.

(iii) F_* converts distinguished triangles into long exact sequences.

(iv) Furthermore, if $(\mathcal{M}, \wedge)$ is a monoidal model category and $(\mathcal{A}, \otimes)$ is a monoidal abelian category, we require that F_* is lax symmetric monoidal, that is, there is a natural Künneth morphism

$$\kappa_{X,Y}\colon F_*X \otimes F_*Y \to F_*(X \wedge^{\mathbb{L}} Y).$$

Now let $\mathcal{M}$ be a simplicial stable model category, let I be a direct category and let $X \in \mathrm{Ho}(\mathcal{M}^I)$. Further, let

$$F_*\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$$

be a homological functor into an (graded) abelian category. Then there is a spectral sequence

$$E^2_{pq} = H_p(I; F_q X) \Rightarrow F_{p+q}(\operatorname*{hocolim}_J X); \tag{2.4.5}$$

see [Richter 2020, 16.3.1]. The construction of the spectral sequence (2.4.5) arises from the skeletal filtration of a simplicial object. This spectral sequence will play a central role in our calculations for the monoidal properties of $\mathcal{Q}$ in Section 3.

2.5 Franke's realization functor

In this subsection we will recall the construction of Franke's equivalence

$$\mathcal{R}\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{M}).$$

For a detailed exposition we refer to [Patchkoria 2012, Section 3.3; Roitzheim 2008]. Recall that C_N is the crown-shaped poset from Example 2.1.7, and that the category $\mathrm{D}^{([1],1)}(\mathcal{A})$ above is the derived category of twisted chain complexes from Example 2.3.11, where $\mathcal{A}$ is a graded symmetric monoidal hereditary abelian category with enough projectives, $\mathcal{M}$ is a simplicial stable model category, and $F : \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$ is a homological functor. Also, we assume $\mathcal{A}$ splits into shifted copies of another abelian category $\mathcal{B}$,

$$\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{B}[i]$$

for $N > 1$. Under these assumptions, $\mathcal{R}$ exists and is an equivalence.

For an object $X \in \mathcal{M}^{C_N}$ we have the structure morphisms of X,

$$l_i\colon X_{\beta_i} \to X_{\zeta_i}, \quad k_i\colon X_{\beta_{i-1}} \to X_{\zeta_i}, \qquad i \in \mathbb{Z}/N\mathbb{Z}.$$

Furthermore, let

$$Z^{(i)}(X) = F_*(X_{\zeta_i}), \quad B^{(i)}(X) = F_*(X_{\beta_i}), \quad C^{(i)}(X) = F_*(\mathrm{cone}(k_i)),$$
$$\lambda^{(i)} := F_* l_i\colon B^{(i)}(X) \to Z^{(i)}(X), \quad i \in \mathbb{Z}/N\mathbb{Z},$$

where $\mathrm{cone}(k_i)$ denotes the cone construction from Definition 2.3.12. We will now list some additional assumptions that we need in order to assemble the $C^{(i)}$ into a chain complex C_*.

Definition 2.5.1 Consider the full subcategory $\mathcal{L}$ of $\mathrm{Ho}(\mathcal{M}^{C_N})$ consisting of those diagrams $X \in \mathrm{Ho}(\mathcal{M}^{C_N})$ which satisfy the following conditions:

- (i) The objects X_{β_i} and X_{ζ_i} are cofibrant in $\mathcal{M}$ for any $i \in \mathbb{Z}/N\mathbb{Z}$.
- (ii) The objects $F_*(X_{\beta_i})$ and $F_*(X_{\zeta_i})$ are contained in $\mathcal{B}[i]$ for any $i \in \mathbb{Z}/N\mathbb{Z}$.
- (iii) The map $\lambda^{(i)}\colon F_*(X_{\beta_i}) \to F_*(X_{\zeta_i})$ is a monomorphism for any $i \in \mathbb{Z}/N\mathbb{Z}$.

Next we construct a functor

$$\mathcal{Q}\colon \mathcal{L} \to \mathrm{C}^{([1],1)}(\mathcal{A}).$$

Let X be an object of $\mathcal{L}$. As the functor

$$F_*\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$$

is homological, the distinguished triangles

$$X_{\beta_{i-1}} \xrightarrow{k_i} X_{\zeta_i} \to \mathrm{cone}(k_i) \to \Sigma X_{\beta_{i-1}}$$

induce long exact sequences

$$\cdots \to B^{(i-1)}(X) \to Z^{(i)}(X) \xrightarrow{\iota^{(i)}} C^{(i)}(X) \xrightarrow{\rho^{(i)}} B^{(i-1)}(X)[1] \to Z^{(i)}(X)[1] \to \cdots.$$

Note that $B^{(i-1)}(X) \in \mathcal{B}[i-1]$ and $Z^{(i)}(X) \in \mathcal{B}[i]$ for all $i \in \mathbb{Z}/N\mathbb{Z}$, since $X \in \mathcal{L}$. Therefore, the morphisms $B^{(i-1)}(X) \to Z^{(i)}(X)$ and $B^{(i-1)}(X)[1] \to Z^{(i)}(X)[1]$ are zero. As a consequence, for any $i \in \mathbb{Z}/N\mathbb{Z}$ we actually obtain short exact sequence in $\mathcal{A}$,

$$0 \to Z^{(i)}(X) \xrightarrow{\iota^{(i)}} C^{(i)}(X) \xrightarrow{\rho^{(i)}} B^{(i-1)}(X)[1] \to 0. \tag{2.5.2}$$

Now consider the following objects in $\mathcal{A}$.

$$C_*(X) = C^{(0)}(X) \oplus C^{(1)}(X) \oplus \cdots \oplus C^{(N-1)}(X),$$
$$Z_*(X) = Z^{(0)}(X) \oplus Z^{(1)}(X) \oplus \cdots \oplus Z^{(N-1)}(X),$$
$$B_*(X) = B^{(0)}(X) \oplus B^{(1)}(X) \oplus \cdots \oplus B^{(N-1)}(X).$$

The morphisms $\lambda^{(i)}, \iota^{(i)}, \rho^{(i)}, i \in \mathbb{Z}/N\mathbb{Z}$, induce morphisms between the direct sums

$$\begin{aligned} \lambda\colon B_*(X) &\to Z_*(X), & \lambda &= \lambda^{(0)} \oplus \lambda^{(1)} \oplus \cdots \oplus \lambda^{(N-1)}, \\ \iota\colon Z_*(X) &\to C_*(X), & \iota &= \iota^{(0)} \oplus \iota^{(1)} \oplus \cdots \oplus \iota^{(N-1)}, \\ \rho\colon C_*(X) &\to B_*(X)[1], & \rho &= \rho^{(0)} \oplus \rho^{(1)} \oplus \cdots \oplus \rho^{(N-1)}. \end{aligned}$$

After summing up, we get a short exact sequence of objects in $\mathcal{A}$

$$0 \to Z_*(X) \xrightarrow{\iota} C_*(X) \xrightarrow{\rho} B_*(X)[1] \to 0. \tag{2.5.3}$$

Splicing this short exact sequence with its shifted copy gives an object in $\mathrm{C}^{([1],1)}(\mathcal{A})$. More precisely, define

$$d = \iota[1]\lambda[1]\rho\colon C_*(X) \to C_*(X)[1].$$

We have $d^2 = 0$ by construction and therefore we get a ([1], 1)–twisted complex. We have now arrived at the definition

$$\mathcal{Q}\colon \mathcal{L} \to \mathrm{C}^{([1],1)}(\mathcal{A}), \quad \mathcal{Q}(X) = \Big(\bigoplus_{i\in\mathbb{Z}/N\mathbb{Z}} F_*(\mathrm{cone}(k_i)), d\Big) = (C_*(X), d).$$

It can be shown that $\mathcal{Q}$ is in fact an equivalence of categories. The composite

$$\mathrm{C}^{([1],1)}(\mathcal{A}) \xrightarrow{\mathcal{Q}^{-1}} \mathcal{L} \xrightarrow{\mathrm{hocolim}} \mathrm{Ho}(\mathcal{M}). \tag{2.5.4}$$

factors over $\mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{M})$, which is Franke's realization functor $\mathcal{R}$. It follows from the construction of $\mathcal{R}$ that it commutes with suspensions and that $F_* \circ \mathcal{R} \cong H_*$.

3 Monoidal properties of $\mathcal{Q}$

In this section, we will examine properties of the bifunctor

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N})$$

via Theorem 3.1.5, which is one of the main ingredients of the diagram (1.0.2).

3.1 Preliminaries on crowned diagrams

Recall the poset C_N from Example 2.1.7 (the crown shape with two rows) and the poset D_N from Example 2.1.8 (the crown shape with three rows). We will be interested in two functors between these two categories. The first functor is the *projection functor*

$$\begin{aligned} \mathrm{pr}\colon C_N \times C_N \to D_N, \quad &(\beta_i, \beta_j) \mapsto \beta_{i+j}, \quad (\zeta_i, \zeta_j) \mapsto \zeta_{i+j}, \\ &(\zeta_i, \beta_j) \mapsto \gamma_{i+j}, \quad (\beta_i, \zeta_j) \mapsto \gamma_{i+j}. \end{aligned} \tag{3.1.1}$$

Note, that we really should be writing $\beta_{i\,(\mathrm{mod}\,N)}$ and $\gamma_{i+j\,(\mathrm{mod} N)}$ etc, but we commit a small abuse of notation and avoid this. The other functor that we will be interested in is the functor

$$i: C_N \to D_N, \quad \zeta_n \mapsto \zeta_n, \quad \beta_n \mapsto \gamma_n, \tag{3.1.2}$$

which is the inclusion of the crown shape C_N into the bottom two rows of D_N. Since weak equivalences in the diagram categories are given objectwise, the functor $i^*\colon \mathcal{M}^{D_N} \to \mathcal{M}^{C_N}$ preserves weak equivalences. Thus, it defines a functor on the homotopy categories, which we denote by the same letter, that is,

$$i^*\colon \mathrm{Ho}(\mathcal{M}^{D_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N}).$$

Next, recall the external smash product for diagrams $X \in \mathcal{M}^I$ and $Y \in \mathcal{M}^J$ for I and J direct categories from Definition 2.3.2. By choosing $I = J = C_N$, it follows formally that we have the bifunctor

$$- \wedge -\colon \mathcal{M}^{C_N} \times \mathcal{M}^{C_N} \to \mathcal{M}^{C_N \times C_N}. \tag{3.1.3}$$

By Proposition 2.3.3, the external product has a total left derived functor

$$- \wedge^{\mathbb{L}} -\colon \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N \times C_N}). \tag{3.1.4}$$

Given diagrams $X, Y \in \mathrm{Ho}(\mathcal{M}^{C_N})$, we can define the homotopy left Kan extension of the external smash product $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{C_N \times C_N})$ along the projection functor $\mathrm{pr}\colon C_N \times C_N \to D_N$, that is,

$$E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N}).$$

Now that we have all the necessary ingredients we can finally state the following theorem.

Theorem 3.1.5 *The bifunctor*

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N})$$

satisfies the following. Let $X, Y \in \mathcal{L}$ such that $F_*(X_{\alpha_n}), F_*(Y_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$. Then, $i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathcal{L}$, that is to say, we have a bifunctor

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}.$$

Furthermore, there is a natural isomorphism

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y).$$

The theorem has two parts. First, we show that $i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)$ is in fact a bifunctor

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}.$$

The second part is that for any two crowned diagrams $X, Y \in \mathcal{L}$ satisfying the stated hypotheses, there is a natural isomorphism

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y).$$

The two parts combined yield that the following diagram commutes (up to natural isomorphism):

$$\begin{array}{ccc}
\mathrm{C}^{([1],1)}(\mathcal{A}) \times \mathrm{C}^{([1],1)}(\mathcal{A}) & \xleftarrow{\mathcal{Q} \times \mathcal{Q}} & \mathcal{L} \times \mathcal{L} \\
{\scriptstyle \otimes}\downarrow & & \downarrow{\scriptstyle i^*\mathbb{L}\mathrm{pr}_!} \\
\mathrm{C}^{([1],1)}(\mathcal{A}) & \xleftarrow[\mathcal{Q}]{} & \mathcal{L}
\end{array}$$

The first part of Theorem 3.1.5 is the content of Section 3.3 and Proposition 3.3.1. The natural isomorphism

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y)$$

is the content of Sections 3.4 and 3.5 and Proposition 3.6.6.

3.2 Slice categories of the projection functor

Again, the values of $\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)$ are given by the formula in Proposition 2.2.4. That is, the values of E at the objects of D_N are given by

$$E_{\gamma_n} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_n}(X \wedge^{\mathbb{L}} Y), \tag{3.2.1}$$

$$E_{\zeta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y), \tag{3.2.2}$$

$$E_{\beta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\beta_n}(X \wedge^{\mathbb{L}} Y). \tag{3.2.3}$$

The structure morphisms of the diagram E, $\hat{l}_n \colon E_{\gamma_n} \to E_{\zeta_n}$ and $\hat{k}_n \colon E_{\gamma_{n+1}} \to E_{\zeta_n}$, are the edges of the homotopy Kan extension and are given by the natural maps

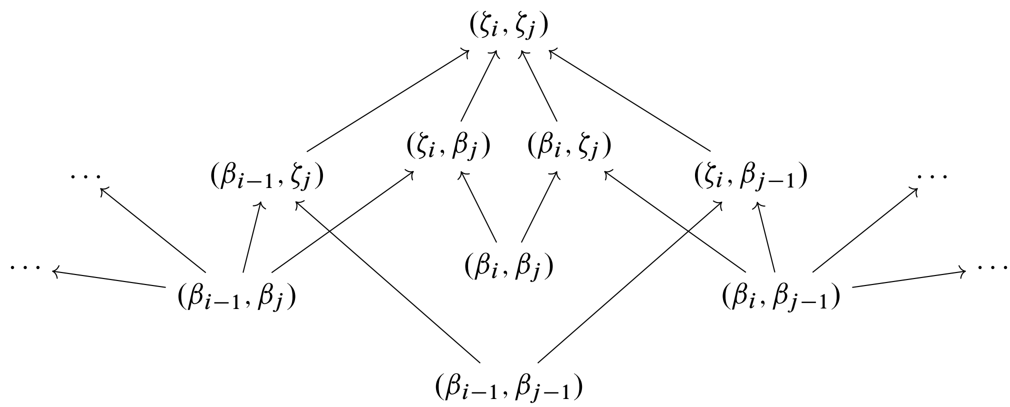
$$E_{\gamma_n} \cong \operatorname*{hocolim}_{\mathrm{pr}/\gamma_n}(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) \cong E_{\zeta_n}, \tag{3.2.4}$$

$$E_{\gamma_{n+1}} \cong \operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n+1}}(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) \cong E_{\zeta_n}, \tag{3.2.5}$$

induced by the maps of posets ϕ and ψ, respectively, see (3.2.8).

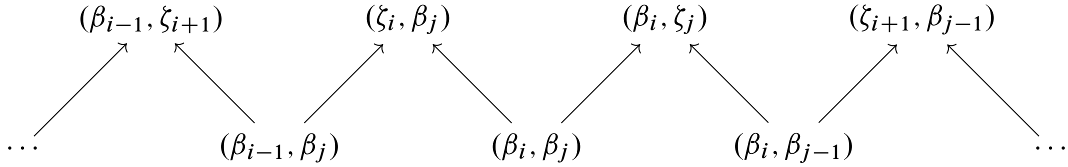
Since we are interested in the homotopy Kan extension of the functor $\mathrm{pr} \colon C_N \times C_N \to D_N$, we need to identify all the slice categories involved, ie pr/ζ_n, pr/γ_n and pr/β_n. We have the following three cases.

(i) The first case is pr/ζ_n. For $n \in \mathbb{Z}/N\mathbb{Z}$ and the object ζ_n we have the slice category pr/ζ_n



where $i + j \equiv n \pmod N$. Note that all the nonidentity morphisms are of the form $(1, l_i)$ or $(l_i, 1)$ and similarly $(1, k_i)$ or $(k_i, 1)$ for any $i \in \mathbb{Z}/N\mathbb{Z}$. The poset pr/ζ_n follows the same pattern to the left and to the right.

(ii) Next we have the case pr/γ_n. Let $n \in \mathbb{Z}/N\mathbb{Z}$ and consider now the slice category pr/γ_n which looks as follows,



where $i + j \equiv n \pmod N$. Similarly to the above all the nonidentity morphisms are of the form $(1, l_i)$ or $(l_i, 1)$ and $(1, k_i)$ or $(k_i, 1)$ for any $i \in \mathbb{Z}/N\mathbb{Z}$.

(iii) Next is the case pr/β_n. Let again $n \in \mathbb{Z}/N\mathbb{Z}$ but now we consider the slice category pr/β_n. Notice that it is

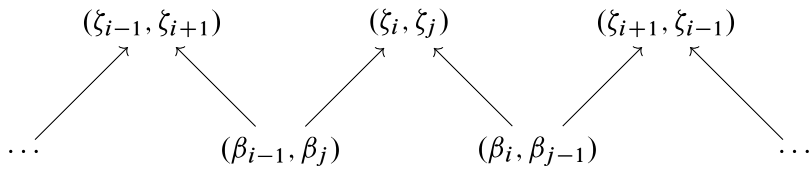
$$\cdots \qquad (\beta_{i-1}, \beta_{j+1}) \qquad (\beta_i, \beta_j) \qquad (\beta_{i+1}, \beta_{j-1}) \qquad \cdots$$

in which $i + j \equiv n \pmod N$. In other words, it is a discrete category. This means that

$$E_{\beta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\beta_n}(X \wedge^{\mathbb{L}} Y) \cong \bigoplus_{i+j=n} X_{\beta_i} \wedge^{\mathbb{L}} Y_{\beta_j}.$$

This is the only case that we can be explicit about the values of the homotopy left Kan extension $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)$.

(iv) Our last example is a particular subposet of pr/ζ_n and it is not strictly speaking a slice of any value. However it will be very useful for us is the following. Consider the following subposet $J_n \subseteq \mathrm{pr}/\zeta_n$ defined as follows



where $i + j \equiv n \pmod N$. In this poset, the nonidentity morphisms are of the form (k_i, l_i) or (l_i, k_i), unlike the examples above where one arrow was always the identity arrow.

Remark 3.2.6 Now let $\theta\colon J_n \to \mathrm{pr}/\zeta_n$ denote the inclusion of the subposet defined in (iv) into the poset in (i). We will define a map of posets

$$L\colon \mathrm{pr}/\zeta_n \to J_n,$$

where it suffices to define it for the part of the poset visible in (i) as the rest can be defined analogously. The map L is given by

$$L\colon \mathrm{pr}/\zeta_n \to J_n, \qquad \begin{aligned} (\beta_{i+1}, \beta_j) &\mapsto (\beta_{i+1}, \beta_j), \\ (\beta_i, \beta_{j+1}) &\mapsto (\beta_i, \beta_{j+1}), \\ \text{else} &\mapsto (\zeta_i, \zeta_j). \end{aligned}$$

We note that L left adjoint to θ- this can quickly be verified straight from the definition as the morphism sets in either poset are either empty or consist of exactly one element. As a consequence, since the inclusion map $\theta\colon J_n \to \mathrm{pr}/\zeta_n$ is a right adjoint, it is homotopy final, ie for any $F \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$ we have

$$\operatorname*{hocolim}_{J_n} \theta^*(F) \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} F.$$

In other words, the value E_{ζ_n} in (3.2.1) can be calculated as

$$E_{\zeta_n} \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y). \tag{3.2.7}$$

We discuss homotopy finality in more detail in Section 4.1; see Definition 4.1.3.

Given any of subposet of $C_N \times C_N$, eg pr/γ_n from example (ii), we can define the restriction of the external smash product $X \wedge Y \in \mathcal{M}^{C_N \times C_N}$ to pr/γ_n by taking the pullback along the inclusion $\nu\colon \mathrm{pr}/\gamma_n \to C_N \times C_N$, that is,

$$\nu^*\colon \mathcal{M}^{C_N \times C_N} \to \mathcal{M}^{\mathrm{pr}/\gamma_n}.$$

Notice that ν^* preserves weak equivalences so it induces a functor on homotopy categories

$$\nu^*\colon \mathrm{Ho}(\mathcal{M}^{C_N \times C_N}) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n}).$$

Moreover, we have maps between the subposets of $C_N \times C_N$. The morphisms $\gamma_n \to \zeta_n$ and $\gamma_{n+1} \to \zeta_n$ induce maps of posets

$$\psi\colon \mathrm{pr}/\gamma_n \to \mathrm{pr}/\zeta_n, \quad \phi\colon \mathrm{pr}/\gamma_{n-1} \to \mathrm{pr}/\zeta_n, \tag{3.2.8}$$

which in turn also induce pullback functors on the homotopy categories, that is,

$$\phi^*\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n}) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_{n-1}}), \quad \text{and} \quad \psi^*\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n}) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n}).$$

We conclude this section with a convention.

Convention 3.2.9 Because of the above, we will commit an abuse of notation and instead of writing, for example,

$$\phi^*(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n})$$

we will simply write

$$X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n}),$$

with the understanding that this diagram was given by a composition of restriction functors

$$\mathrm{Ho}(\mathcal{M}^{C_N \times C_N}) \xrightarrow{\pi^*} \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n}) \xrightarrow{\phi^*} \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n})$$

unless we need the extra notation for clarification.

Remark 3.2.10 Consider a diagram $F \in \mathrm{Ho}(\mathcal{M}^{C_N \times C_N})$. By Convention 2.1.1, we can assume that F is a projective cofibrant object, so in particular, it is objectwise cofibrant. The external smash product

$$- \wedge - : \mathcal{M}^{C_N} \times \mathcal{M}^{C_N} \to \mathcal{M}^{C_N \times C_N}$$

as defined in (3.1.3) is a Quillen bifunctor, so in particular it preserves cofibrant objects. This implies that $X \wedge^{\mathbb{L}} Y$ is cofibrant in $\mathcal{M}^{C_N \times C_N}$, so in particular it is objectwise cofibrant. Now, for any subposet

$$\iota\colon J \hookrightarrow C_N \times C_N,$$

eg any of the slice categories of the projection functor pr (3.1.1), we have the pullback functor

$$\iota^*\colon \mathcal{M}^{C_N \times C_N} \to \mathcal{M}^J.$$

This functor is not necessarily a left Quillen functor with respect the projective model structures; see Proposition 2.1.4. However, the diagram $\iota^*(X \wedge^{\mathbb{L}} Y)$, is objectwise cofibrant, which means that the geometric realization of the simplicial replacement still models the homotopy colimit of the diagram $\iota^*(X \wedge^{\mathbb{L}} Y)$. In particular, the skeletal filtration of all the restrictions is always Reedy cofibrant; see Lemma 2.1.17.

3.3 Spectral sequence calculations

The main result of this subsection is that given crowned diagrams $X, Y \in \mathcal{L}$ that for satisfying a simple condition, the diagram $i^*E = i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)$ is also in the subcategory $\mathcal{L}$, ie the objects X_{β_i} and X_{ζ_i} are cofibrant in $\mathcal{M}$, the objects $F_*(X_{\beta_i})$ and $F_*(X_{\zeta_i})$ are in $\mathcal{B}[i]$, and the map

$$\lambda^{(i)}\colon F_*(X_{\beta_i}) \to F_*(X_{\zeta_i})$$

is a monomorphism for any $i \in \mathbb{Z}/N\mathbb{Z}$; see Definition 2.5.1. Essentially, this condition is that for the given homological functor $F_*\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$, either the crowned diagram X or Y is objectwise projective.

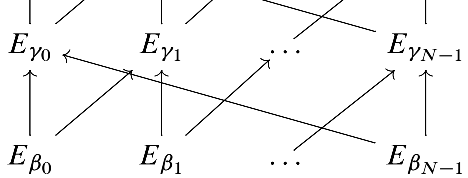
Proposition 3.3.1 *Let $X, Y \in \mathcal{L}$ such that $F_*(X_{\alpha_n}), F_*(Y_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$. Consider the homotopy left Kan extension $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N})$ of*

$$X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{C_N \times C_N})$$

along

$$\mathrm{pr}\colon C_N \times C_N \to D_N$$

with the values and morphisms given in (3.2.1)–(3.2.3) *and* (3.2.4), (3.2.5), *respectively.*



Then, for any $n \in \mathbb{Z}/N\mathbb{Z}$ *and any* $\alpha \in \{\beta, \zeta\}$ *we have* $F_*(E_{\alpha_n}) \in \mathcal{B}[n]$ *and the morphisms*

$$F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$$

induced by $E_{\gamma_n} \to E_{\zeta_n}$ *are monomorphisms.*

Corollary 3.3.2 *Let* X, Y *be crowned diagrams satisfying the hypothesis of Proposition 3.3.1. The top two rows of the diagram* $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)$ *form an object in* $\mathcal{L}$, *that is, the diagram* $i^*E \in \mathcal{L}$.

By our assumption, for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$ the objects $F_*(X_{\alpha_n})$ and $F_*(Y_{\alpha_n})$ are projective in $\mathcal{A}$. This ensures that there are natural Künneth isomorphisms

$$F_*(X_{\alpha_n} \wedge^{\mathbb{L}} Y_{\alpha_n}) \cong F_*(X_{\alpha_n}) \otimes F_*(Y_{\alpha_n}). \tag{3.3.3}$$

Since the values E_{ζ_n}, E_{γ_n} and E_{β_n} are computed via homotopy colimits, we will use (2.4.5), the spectral sequences converging to the homology of the homotopy colimit.

Lemma 3.3.4 *There are spectral sequences*

$$E^2_{pq} = H_p(\mathrm{pr}/\gamma_n; F_q(X \wedge^{\mathbb{L}} Y)) \Rightarrow F_{p+q}\Big(\operatorname*{hocolim}_{\mathrm{pr}/\gamma_n}(X \wedge^{\mathbb{L}} Y)\Big) \cong F_{p+q}(E_{\gamma_n}) \tag{3.3.5}$$

and

$$E'^2_{pq} = H_p(\mathrm{pr}/\zeta_n; F_q(X \wedge^{\mathbb{L}} Y)) \Rightarrow F_{p+q}\Big(\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y)\Big) \cong F_{p+q}(E_{\zeta_n}) \tag{3.3.6}$$

and natural morphisms of spectral sequences $f : \{E^2_{pq}\} \to \{E'^2_{pq}\}$ *induced by the map in* (3.2.4).

We will now begin the proof of Proposition 3.3.1.

Proof Our claim is that $F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$ is a monomorphism, where

$$E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N})$$

as before and $F_* : \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$ is our homological functor. To obtain information on $F_*(E_{\gamma_n})$ and $F_*(E_{\zeta_n})$, we will start by working out the spectral sequence (3.3.5), which we explained is a special case of the spectral sequence (2.4.5). The proof of the proposition is divided into three parts:

- calculating the E_2–term $H_p(\mathrm{pr}/\gamma_n; F_q(X \wedge^{\mathbb{L}} Y))$,
- calculating the E_2–term $H_p(\mathrm{pr}/\zeta_n; F_q(X \wedge^{\mathbb{L}} Y))$,
- showing that the induced map of spectral sequences gives the desired isomorphism.

Step 1 $H_p(\mathrm{pr}/\gamma_n; F_q(X \wedge^{\mathbb{L}} Y))$

We will use the simplicial replacement techniques explained in Section 2.4. Recall the poset pr/γ_n which is

$$\begin{array}{ccccccccccccccc} & & (\beta_{i-1},\zeta_{i+1}) & & & & (\zeta_i,\beta_j) & & & & (\beta_i,\zeta_j) & & & & (\zeta_{i+1},\beta_{j-1}) & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \cdots & & & & (\beta_{i-1},\beta_j) & & & & (\beta_i,\beta_j) & & & & (\beta_i,\beta_{j-1}) & & & & \cdots \end{array}$$

Here, $i+j=n \pmod N$, and so the functor $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n})$ looks as follows:

$$\begin{array}{ccccccccccccccc} & & X_{\beta_{i-1}} \wedge Y_{\zeta_{i+1}} & & & & X_{\zeta_i} \wedge Y_{\beta_j} & & & & X_{\beta_i} \wedge Y_{\zeta_j} & & & & X_{\zeta_{i+1}} \wedge Y_{\beta_{j-1}} & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \cdots & & & & X_{\beta_{i-1}} \wedge Y_{\beta_j} & & & & X_{\beta_i} \wedge Y_{\beta_j} & & & & X_{\beta_i} \wedge Y_{\beta_{j-1}} & & & & \cdots \end{array}$$

Our goal is to compute

$$H_p(\mathrm{pr}/\gamma_n; F_q(X \wedge^{\mathbb{L}} Y)) \quad \text{for all } p \geq 0 \text{ and all } q \in \mathbb{Z},$$

which are the E^2–terms of the spectral sequence (3.3.5). In order to do so, we apply the homological functor $F_n(-)$ to the previous diagram to get the diagram $F_n(X \wedge^{\mathbb{L}} Y) \in \mathcal{A}^{\mathrm{pr}/\gamma_n}$ which, by (3.3.3), is

(3.3.7)
$$\begin{array}{ccccccccccccccc} & & B^{(i-1)} \otimes \widetilde{Z}^{(j+1)} & & & & Z^{(i)} \otimes \widetilde{B}^{(j)} & & & & B^{(i)} \otimes \widetilde{Z}^{(j)} & & & & B^{(i+1)} \otimes \widetilde{Z}^{(j-1)} & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \cdots & & & & 0 & & & & B^{(i)} \otimes \widetilde{B}^{(j)} & & & & 0 & & & & \cdots \end{array}$$

We will write

(3.3.8) $$f_{ij} = \lambda_i \otimes 1 \colon B^{(i)} \otimes \widetilde{B}^{(j)} \to Z^{(i)} \otimes \widetilde{B}^{(j)},$$

(3.3.9) $$g_{ij} = 1 \otimes \tilde{\lambda}_j \colon B^{(i)} \otimes \widetilde{B}^{(j)} \to B^{(i)} \otimes \widetilde{Z}^{(j)},$$

to distinguish, for labelling purposes, the two different morphisms in the simplicial replacement below. Note that since B^i and $\widetilde{B}^j$ and projective in $\mathcal{A}$, by our convention they are automatically flat, hence the morphisms (3.3.8) and (3.3.9) are monomorphisms.

Next, we consider the simplicial replacement of the diagram $F_n(X \wedge^{\mathbb{L}} Y) \in \mathcal{A}^{\mathrm{pr}/\gamma_n}$, that is

$$\mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y)) \in \mathcal{A}^{\Delta^{\mathrm{op}}}.$$

Following Definition 2.1.14 we have that

$$\mathrm{srep}(F(X \wedge^{L} Y))_0 = \bigoplus_{i+j=n} \big((B^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (Z^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (B^{(i)} \otimes \widetilde{Z}^{(j)})\big),$$

$$\mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y))_1 = \bigoplus_{i+j=n} \big((B^{(i)} \otimes \widetilde{B}^{(j)})_{f_{ij}} \oplus (B^{(i)} \otimes \widetilde{B}^{(j)})_{g_{ij}}\big),$$

with face maps given by "source" and "target" respectively. Because of the shape of the poset pr/γ_n, for all $m \geq 2$ the simplices $\mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y))_m$ consist solely of degenerate simplices.

Now we consider the associated complex $C_*(F_n(X \wedge^{\mathbb{L}} Y))$ of this simplicial complex, see Definition 2.4.1. We briefly explain the differential of the complex $C_*(E(1)_{-n}(X \wedge^{\mathbb{L}} Y))$, namely the map

$$d = d_0 - d_1 \colon C_1(F_n(X \wedge^{\mathbb{L}} Y)) \to C_0(F_n(X \wedge^{\mathbb{L}} Y)).$$

Notice from (3.3.7), we can consider the simpler case where the diagram is

$$\begin{array}{ccc} Z^{(i)} \otimes \widetilde{B}^{(j)} & & B^{(i)} \otimes \widetilde{Z}^{(j)} \\ & \nwarrow^{f_{ij}} \quad \nearrow^{g_{ij}} & \\ & B^{(i)} \otimes \widetilde{B}^{(j)} & \end{array}$$

Then, the differential of the associated complex of the simplicial replacement of this diagram is

$$d_{ij} = d_0 - d_1 \colon (B^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (B^{(i)} \otimes \widetilde{B}^{(j)}) \to (B^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (Z^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (B^{(i)} \otimes \widetilde{Z}^{(j)}),$$
$$(x, y) \mapsto (x + y, -f_{ij}(x), -g_{ij}(y)).$$

The 0^{th} homology of the complex is just the pushout

$$B^{(i)} \otimes \widetilde{Z}^{(j)} \coprod_{B^{(i)} \otimes \widetilde{B}^{(j)}} Z^{(i)} \otimes \widetilde{B}^{(i)}.$$

The first homology is the kernel of the differential d_{ij}. Since the maps f_{ij} and g_{ij} are injective, this forces $d_{ij}(x, y) = 0$ if and only if $x = y = 0$, which implies that the first homology is trivial. It follows from the diagram (3.3.7) that the differential d on the complex $C_*(F_n(X \wedge^{\mathbb{L}} Y))$ is the direct sum of the differentials d_{ij} for $i + j = n$. Now that we know the differential of the complex $C_*(F_n(X \wedge^{\mathbb{L}} Y))$ we will compute its homology. It follows that $H_0(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y))$ is the colimit of the diagram $F_n(X \wedge^{\mathbb{L}} Y)$. By inspecting the diagram $F_n(X \wedge^{\mathbb{L}} Y)$ above we can see the colimit of the diagram is a direct sum (coproduct) of pushouts, that is,

$$H_0(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y)) = \operatorname*{colim}_{\mathrm{pr}/\gamma_n} F_n(X \wedge^{\mathbb{L}} Y) = \bigoplus_{i+j=n} \Bigl(Z^{(i)} \otimes \widetilde{B}^{(j)} \coprod_{B^{(i)} \otimes \widetilde{B}^{(j)}} B^{(i)} \otimes \widetilde{Z}^{(j)} \Bigr).$$

Similar to the simpler case, the first homology

$$H_1(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y))$$

is the kernel of the differential

$$d_0 - d_1 \colon \mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y))_1 \to \mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y))_0.$$

Since it is a direct sum of the simpler differentials d_{ij} as above, it follows that

$$H_1(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y)) = 0.$$

Of course, all the higher homologies

$$H_q(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y))$$

vanish for all $q \geq 2$.

Next, we apply the homological functor $F_{n-1}(-)$ to the diagram $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n})$ and we have the diagram $F_{n-1}(X \wedge^{\mathbb{L}} Y) \in \mathcal{A}^{\mathrm{pr}/\gamma_n}$ which is

$$\begin{array}{ccccccccccccc} & & 0 & & & & 0 & & & & 0 & & & & 0 & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \cdots & & & & B^{(i-1)} \otimes \widetilde{B}^{(j)} & & & & 0 & & & & B^{i} \otimes \widetilde{B}^{(j-1)} & & & & \cdots \end{array}$$

Clearly,

$$H_0(\mathrm{pr}/\gamma_n; F_{n-1}(X \wedge^{\mathbb{L}} Y)) = 0,$$

and

$$H_1(\mathrm{pr}/\gamma_n; F_{n-1}(X \wedge^{\mathbb{L}} Y)) = \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)}.$$

It follows that for all $p \geq 0$ and all $m \neq -n, -n-1 \mod N$, the terms $H_p(\mathrm{pr}/\gamma_n; F_m(X \wedge^{\mathbb{L}} Y))$ all vanish. This completes the computation of the E^2 terms of the spectral sequence. It is concentrated in degrees $(0, m)$ and $(1, m-1)$ with $m \equiv n \mod N$. Therefore the spectral sequence collapses and we have a short exact sequence

$$0 \to \bigoplus_{i+j=n} \Bigl(Z^{(i)} \otimes \widetilde{B}^{(j)} \bigoplus_{B^{(i)} \otimes \widetilde{B}^{(j)}} B^{(i)} \otimes \widetilde{Z}^{(j)} \Bigr) \to F_n(E_{\gamma_n}) \to \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)} \to 0.$$

This concludes the calculation of the spectral sequence (3.3.5).

Step 2 $H_p(\mathrm{pr}/\zeta_n; F_q(X \wedge^{\mathbb{L}} Y))$

We will now repeat the previous strategy and apply it to the spectral sequence (3.3.6). Recall the poset J_n from (iv), which is the following subposet of pr/ζ_n:

$$\begin{array}{ccccccccccc} & & (\zeta_{i-1}, \zeta_{i+1}) & & & & (\zeta_i, \zeta_j) & & & & (\zeta_{i+1}, \zeta_{i-1}) & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \cdots & & & & (\beta_{i-1}, \beta_j) & & & & (b_i, b_{j-1}) & & & & \cdots \end{array}$$

By Remark 3.2.6, the inclusion functor $\theta \colon J_n \to \mathrm{pr}/\zeta_n$ has a left adjoint L, and we have

$$E_{\zeta_n} \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y);$$

see (3.2.7). So, instead of the spectral sequence (3.3.6) we can compute the following spectral sequence

$$H_p\bigl(J_n; F_q(\theta^*(X \wedge^{\mathbb{L}} Y))\bigr) \Rightarrow F_{p+q}\Bigl(\operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y)\Bigr)$$

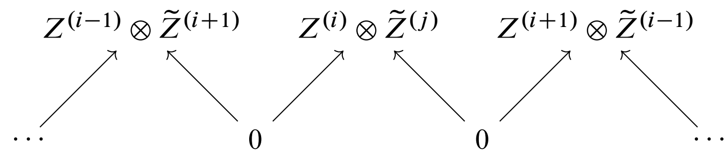
since both converge to the same target, ie the F_*–homology of E_{ζ_n},

$$F_*\Bigl(\operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y)\Bigr) \cong F_*\Bigl(\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y)\Bigr) \cong F_*(E_{\zeta_n}).$$

In fact this, can be made stronger. The adjoint pair $L\colon \mathrm{pr}/\zeta_n \rightleftarrows J_n \colon \theta$ induces a natural isomorphism

$$H_*(\mathrm{pr}/\zeta_n; F_q(X \wedge^{\mathbb{L}} Y)) \cong H_*(J_n, \theta^* F_q(X \wedge^{\mathbb{L}} Y)).$$

From the diagram J_n we again only need to consider $F_n(-)$ and $F_{-n-1}(-)$. Firstly, we apply $F_n(-)$ to the diagram $\theta^*(X \wedge^{\mathbb{L}} Y)$ and we get $F_n(\theta^*(X \wedge^{\mathbb{L}} Y)) \in \mathcal{A}^{J_n}$ as



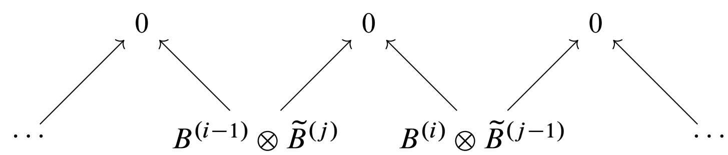
From this we get that

$$H_0\bigl(J_n; F_n(\theta^*(X \wedge^{\mathbb{L}} Y))\bigr) = \bigoplus_{i+j=n} Z^{(i)} \otimes \widetilde{Z}^{(j)}$$

and

$$H_p\bigl(J_n; F_n(\theta^*(X \wedge^{\mathbb{L}} Y))\bigr) = 0, \quad p \geq 1.$$

Next, we will apply the functor $F_{n-1}(-)$ to obtain the diagram $F_{n-1}(\theta^*(X \wedge^{\mathbb{L}} Y)) \in \mathcal{A}^{J_n}$ depicted by



From the above we get that

$$H_1\bigl(J_n; F_{n-1}(\theta^*(X \wedge^{\mathbb{L}} Y))\bigr) = \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)}$$

and

$$H_p\bigl(J_n; F_{n-1}(\theta^*(X \wedge^{\mathbb{L}} Y))\bigr) = 0 \quad \text{for } p = 0 \text{ and } p \geq 2.$$

This completes the computation of the E^2–term of the final spectral sequence. It is concentrated in degrees $(0, m)$ and $(1, m-1)$ with $m \equiv n \mod N$. Therefore, the spectral sequence collapses and we have a short exact sequence

$$0 \to \bigoplus_{i+j=n} Z^{(i)} \otimes \widetilde{Z}^{(j)} \to F_n(E_{\zeta_n}) \to \bigoplus_{i+j=n-1} B^{(i)} \otimes \widetilde{B}^{(j)} \to 0.$$

Step 3 the monomorphism $F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$

Now that we have calculated both spectral sequences we can continue with the proof that $F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$ is a monomorphism. The map of posets $\psi\colon \mathrm{pr}/\gamma_n \to \mathrm{pr}/\zeta_n$ induces morphisms on homologies of categories with coefficients $F_n(-)$ and $F_{n-1}(-)$ respectively, ie

$$H_*(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y)) \to H_*(\mathrm{pr}/\zeta_n; F_n(X \wedge^{\mathbb{L}} Y)) \cong H_*\bigl(J_n; F_n(\theta^*(X \wedge^{\mathbb{L}} Y))\bigr),$$
$$H_*(\mathrm{pr}/\gamma_n; F_{n-1}(X \wedge^{\mathbb{L}} Y)) \to H_*(\mathrm{pr}/\zeta_n; F_{n-1}(X \wedge^{\mathbb{L}} Y)) \cong H_*\bigl(J_n; F_{n-1}(\theta^*(X \wedge^{\mathbb{L}} Y))\bigr).$$

Hence we have a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{i+j=n} \big(Z^{(i)} \otimes \widetilde{B}^{(j)} \coprod_{B^{(i)}\otimes \widetilde{B}^{(j)}} B^{(i)} \otimes \widetilde{Z}^{(j)}\big) & \longrightarrow & F_n(E_{\gamma_n}) & \longrightarrow & \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow \cong & \\
0 \longrightarrow & \bigoplus_{i+j=n} Z^{(i)} \otimes \widetilde{Z}^{(j)} & \longrightarrow & F_n(E_{\zeta_n}) & \longrightarrow & \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)} & \longrightarrow 0
\end{array}$$

By naturality, the left vertical map is the direct sum of the pushout-product maps

$$\lambda_i \,\square\, \widetilde{\lambda}_j \colon \Big(Z^{(i)} \otimes \widetilde{B}^{(j)} \coprod_{B^{(i)}\otimes \widetilde{B}^{(j)}} B^{(i)} \otimes \widetilde{Z}^{(j)} \Big) \to Z^{(i)} \otimes \widetilde{Z}^{(j)}.$$

By Lemma 3.7.2, the map $\lambda_i \,\square\, \widetilde{\lambda}_j$ is injective which means that so is the left vertical map. The five lemma now implies that the morphism

$$F_n(E_{\gamma_n}) \to F_n(E_{\zeta_n})$$

is an injection. In particular, $F_*(E_{\gamma_n})$ and $F_*(E_{\zeta_n})$ are concentrated in the correct degrees and the induced morphisms $F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$ are injections. This concludes the proof of the proposition. □

Corollary 3.3.2 now follows: the diagram E is indeed in the subcategory $\mathcal{L} \subseteq \mathrm{Ho}(\mathcal{M}^{C_N})$ as the vertices are in the correct degree shifts of $\mathcal{B}$, and F applied to the edges $E_{\gamma_n} \to E_{\zeta_n}$ is a monomorphism, which is precisely how $\mathcal{L}$ was defined.

3.4 Cones

In the previous section we proved that for any two crowned diagrams $X, Y \in \mathcal{L}$ which are objectwise projective, $i^*E = i^*\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathcal{L}$. In this subsection we will prove that applying the functor $\mathcal{Q}$ to the object i^*E is a good model for the tensor product $\mathcal{Q}(X) \otimes \mathcal{Q}(Y)$. This will follow as a corollary from the following proposition.

Proposition 3.4.1 *Consider $E \in \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N})$ and let i^*E be the pullback of E along $i \colon C_N \to D_N$. For every $n \in \mathbb{Z}/N\mathbb{Z}$ we have a canonical isomorphism*

$$\mathrm{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}) \cong \bigvee_{i+j=n} \mathrm{cone}(k_i) \wedge^{\mathbb{L}} \mathrm{cone}(\tilde{k}_j),$$

where $k_i \colon X_{\beta_{i-1}} \to X_{\zeta_i}$ is a structure morphism of $X \in \mathcal{L}$.

Proof This proof has three main parts. Firstly, we will work out the morphism $i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}$ by calculating the relevant values of $E_{\beta_{n-1}}$ and E_{ζ_n} using their description as homotopy colimits over slice categories; see Section 3.2. We will arrive at the conclusion that the left-hand side is actually $\mathrm{hocolim}_{\mathrm{pr}/\zeta_n}(\mathrm{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y}))$, where $\varepsilon_{X \wedge^{\mathbb{L}} Y}$ is the counit of a certain adjunction. We will then explicitly determine the map of diagrams $\varepsilon_{X \wedge^{\mathbb{L}} Y}$ in Step 2 and calculate its cone in Step 3.

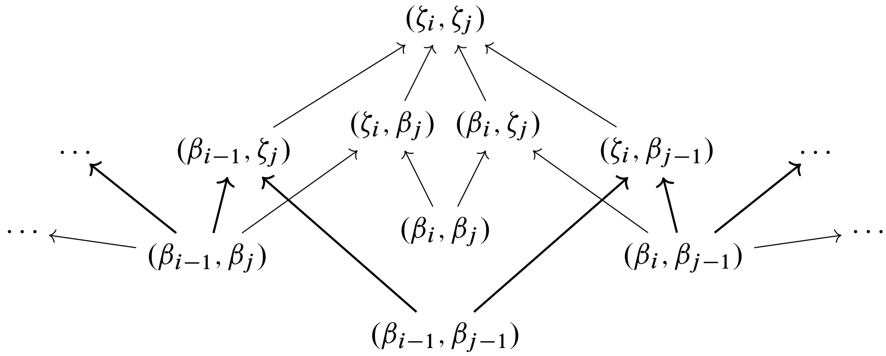
Step 1 unravelling $i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}$

Recall the slice categories of the map $\mathrm{pr}\colon C_N \times C_N \to D_N$ over ζ_n, pr/ζ_n from example (i), and pr/γ_n from example (ii). By definition of i, we have that

$$(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}) = E_{\gamma_{n-1}} \to E_{\zeta_n}.$$

Let us begin by recalling the diagram $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$. The thick arrows show the image of the map of posets $\phi\colon \mathrm{pr}/\gamma_{n-1} \to \mathrm{pr}/\zeta_n$:

(3.4.2)



Recall from (3.2.2) that

$$E_{\zeta_n} = \mathrm{hocolim}(\mathrm{pr}/\zeta_n \xrightarrow{\pi} C_N \times C_N \xrightarrow{X \wedge^{\mathbb{L}} Y} \mathcal{M}),$$

and we committed an abuse of notation by writing

$$E_{\zeta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} \pi^*(X \wedge^{\mathbb{L}} Y).$$

Also, recall from (3.2.5) that the morphism $E_{\gamma_{n-1}} \to E_{\zeta_n}$ is the canonical morphism

$$E_{\gamma_{n-1}} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n-1}} \phi^*(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) = E_{\zeta_n}$$

induced by the map of posets $\phi\colon \mathrm{pr}/\gamma_{n-1} \to \mathrm{pr}/\zeta_n$. The pullback functor

$$\phi^*\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n}) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_{n-1}})$$

has a left adjoint defined by the homotopy left Kan extension $\mathbb{L}\phi_!$, that is,

$$\mathbb{L}\phi_!\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_{n-1}}) \rightleftarrows \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})\colon \phi^*.$$

The counit of the derived adjunction $\varepsilon\colon \mathbb{L}\phi_!\phi^* \to \mathrm{Id}$ provides the canonical natural transformation

(3.4.3) $$\varepsilon_{X\wedge^{\mathbb{L}}Y}\colon \mathbb{L}\phi_!\phi^*(X \wedge^{\mathbb{L}} Y) \to X \wedge^{\mathbb{L}} Y.$$

Lastly, since $\mathbb{L}\phi_!$ is a homotopy left Kan extension, there is a canonical isomorphism

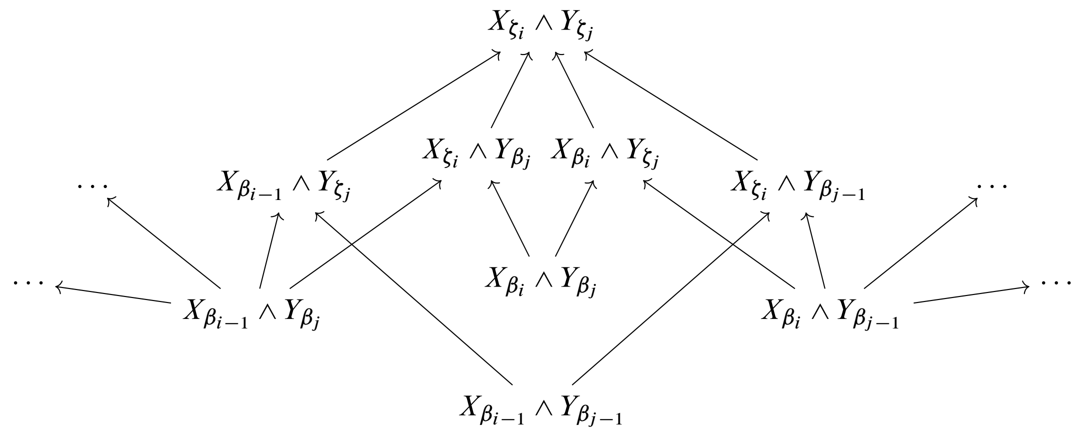
$$\operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n-1}} \phi^*(X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} \mathbb{L}\phi_!\phi^*(X \wedge^{\mathbb{L}} Y).$$

Putting all this together means that the left-hand side of Proposition 3.4.1 is

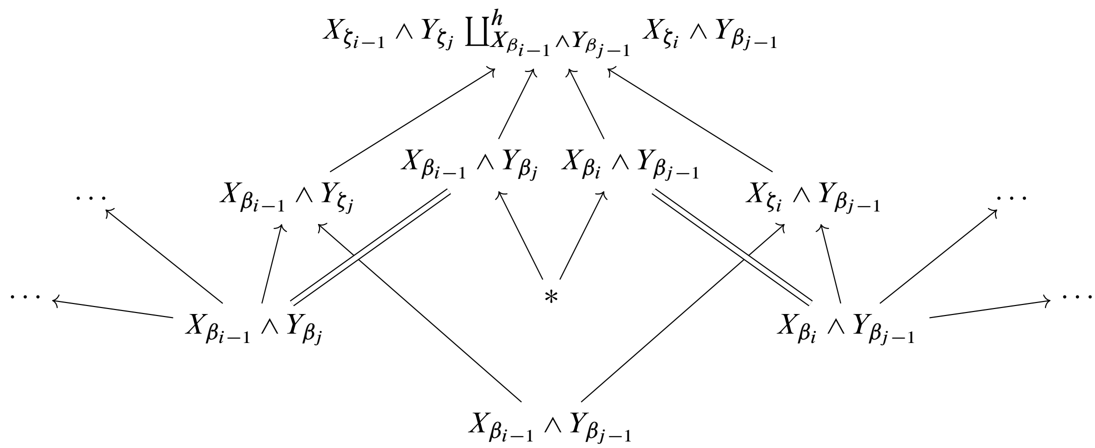
$$\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(\mathrm{cone}(\varepsilon_{X\wedge^{\mathbb{L}}Y})).$$

Step 2 working out $\varepsilon_{X\wedge^{\mathbb{L}}Y}$

The underlying diagram $X\wedge^{\mathbb{L}}Y\in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$ is

(3.4.4)


Furthermore, the homotopy left Kan extension $\mathbb{L}\phi_!\phi^*(X\wedge^{\mathbb{L}}Y)\in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$ is

(3.4.5)


We briefly explain how we calculated the left homotopy Kan extension $\mathbb{L}\phi_!(X\wedge^{\mathbb{L}}Y)$. From the formula of Proposition 2.2.4 for calculating homotopy Kan extensions, we can calculate the homotopy left Kan extension $\mathbb{L}\phi_!\phi^*$ at an object $(\alpha_s,\alpha_t)\in \mathrm{pr}/\zeta_n$ as

$$\mathbb{L}\phi_!(X\wedge^{\mathbb{L}}Y)_{(\alpha_s,\alpha_t)}\cong \mathrm{hocolim}\big(\phi/(\alpha_s,\alpha_t)\xrightarrow{\pi}\mathrm{pr}/\gamma_{n-1}\xrightarrow{\phi^*(X\wedge^{\mathbb{L}}Y)}\mathcal{M}\big).$$

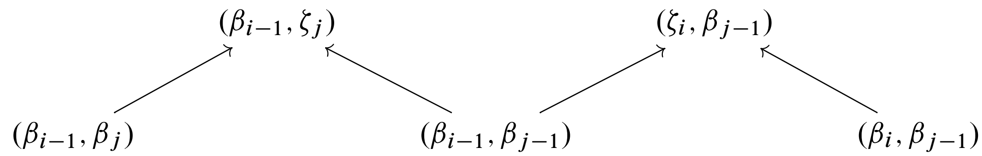
For the object (ζ_i,β_j), the slice $\phi/(\zeta_i,\beta_j)$ consists only of the object the object (β_{j-1},β_j), which implies that

$$(\mathbb{L}\phi_!)_{(\zeta_i,\beta_j)}=X_{\beta_{i-1}}\wedge Y_{\beta_j}.$$

For the object (β_i, ζ_j), the argument is the same as above. For (β_i, β_j), the slice category $\phi/(\beta_i, \beta_j)$ is empty, which means that

$$(\mathbb{L}\phi_!)_{(\beta_i,\beta_j)} \cong *.$$

For the object (ζ_i, ζ_j), the slice category $\phi/(\zeta_i, \zeta_j)$ is the poset



But the subposet

$$(\beta_{i-1}, \zeta_j) \longleftarrow (\beta_{i-1}, \beta_{j-1}) \longrightarrow (\zeta_i, \beta_{j-1})$$

is homotopy final, which means that

$$(\mathbb{L}\phi_!)_{(\zeta_i,\zeta_j)} \cong (\mathbb{L}\phi_!)_{(\zeta_i,\zeta_j)} \cong \operatorname{hocolim}\left(\begin{array}{ccc} X_{\beta_{i-1}} \wedge Y_{\beta_{j-1}} & \xrightarrow{k_i \wedge 1} & X_{\zeta_i} \wedge Y_{\beta_{j-1}} \\ \big\downarrow {\scriptstyle 1 \wedge \tilde{k}_j} & & \\ X_{\beta_{i-1}} \wedge Y_{\zeta_j} & & \end{array}\right).$$

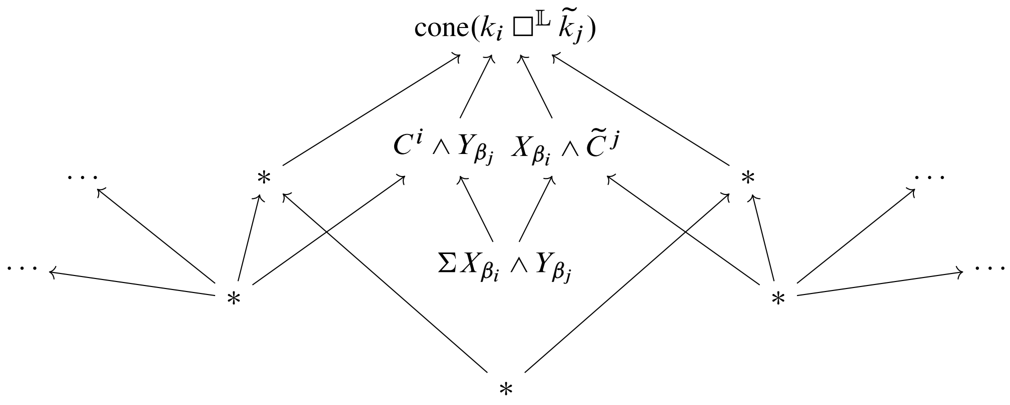
Step 3 calculating the cone in the left-hand side

Next, we calculate the cone of the natural transformation $\varepsilon_{X \wedge^{\mathbb{L}} Y}$ (3.4.3) of diagrams in $\mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$. We have the diagram $\operatorname{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y}) \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$, which is

$$\operatorname{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y})\colon \mathrm{pr}/\zeta_n \to \mathcal{M}, \quad (\alpha_s, \alpha_t) \mapsto \operatorname{cone}\big(\phi_!(X \wedge^{\mathbb{L}} Y)_{(\alpha_s,\alpha_t)} \to (X \wedge^{\mathbb{L}} Y)_{(\alpha_s,\alpha_t)}\big).$$

In other words, we are taking objectwise cones of the canonical map from the diagram (3.4.5) to the diagram (3.4.4). This means that $\operatorname{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y})$ is

(3.4.6)



Here, we have denoted

$$C^i := \operatorname{cone}(k_i) = \operatorname{cone}(X_{i-1} \to X_{\zeta_i}), \quad \tilde{C}^j := \operatorname{cone}(\tilde{k}_j) = \operatorname{cone}(Y_{j-1} \to Y_{\zeta_i}).$$

Next, we determine the homotopy colimit of the diagram $\mathrm{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y})$. One way is to observe that the homotopy colimit of the above diagram is isomorphic in $\mathrm{Ho}(\mathcal{M})$ to the homotopy colimit of (finite) coproduct of squares

$$\begin{array}{ccc} \Sigma X_{\beta_i} \wedge Y_{\beta_j} & \longrightarrow & \mathrm{cone}(k_i) \wedge Y_{\beta_j} \\ \downarrow & & \downarrow \\ X_{\beta_i} \wedge \mathrm{cone}(\widetilde{k}_j) & \longrightarrow & \mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \widetilde{k}_j) \end{array} \tag{3.4.7}$$

where we can consider the above as an object in $\mathrm{Ho}(\mathcal{M}^{[1]\times[1]})$. Formally this is obtained by taking the visually obvious map of posets $f\colon [1]\times[1] \to \mathrm{pr}/\zeta_n$, ie

$$(0,0) \mapsto (\beta_i, \beta_j), \quad (0,1) \mapsto (\zeta_i, \beta_j), \quad (1,0) \mapsto (\beta_i, \zeta_j), \quad (1,1) \mapsto (\zeta_i, \zeta_j)$$

and considering the pullback

$$f^*\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n}) \to \mathrm{Ho}(\mathcal{M}^{[1]\times[1]}).$$

The bottom right corner of the poset $[1]\times[1]$ is its final object, which implies that the homotopy colimit of the diagram (3.4.7) is naturally isomorphic to $\mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \widetilde{k}_j)$. Hence the homotopy colimit over pr/ζ_n is, up to natural isomorphism, the coproduct $\bigvee_{i+j=n} \mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \mathrm{cone}(\widetilde{k}_j))$. Another way of seeing this is by pulling back the above diagram to $\theta_n\colon J_n \to \mathrm{pr}/\zeta_n$, and we get the diagram

$$\begin{array}{ccccccccccc} & & \mathrm{cone}(k_{i-1} \,\square^{\mathbb{L}}\, k_{j+1}) & & & & \mathrm{cone}(k_i \,\square^{\mathbb{L}}\, k_j) & & & & \mathrm{cone}(k_{i+1} \,\square^{\mathbb{L}}\, k_{j-1}) & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \cdots & & & & * & & & & * & & & & \cdots \end{array}$$

All in all, we have that the homotopy colimit of the diagram (3.4.6) is

$$\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(\mathrm{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y})) \cong \bigvee_{i+j=n} \mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \widetilde{k}_j). \tag{3.4.8}$$

Finally, by Corollary 2.3.6, we have the canonical isomorphism

$$\mathrm{cone}(k_i \,\square^{\mathbb{L}}\, \widetilde{k}_j) \cong \mathrm{cone}(k_i) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_j)$$

for each pair $i, j \in \mathbb{Z}/N\mathbb{Z}$. The coproduct of these isomorphisms together with (3.4.8) gives us that

$$\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(\mathrm{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y})) \cong \bigvee_{i+j=n} \mathrm{cone}(k_i) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_j).$$

Let us now gather all this information to prove Proposition 3.4.1. Calculating the homotopy cofiber (cone) of the morphisms $i^* E_{\beta_{n-1}} \to i^* E_{\zeta_n}$ is the same thing as calculating the homotopy cofiber $E_{\gamma_{n-1}} \to E_{\zeta_n}$.

We have the following natural isomorphisms:

$$\begin{aligned}
\operatorname{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}) &= \operatorname{cone}(E_{\gamma_{n-1}} \to E_{\zeta_n}) \\
&= \operatorname{cone}\Big(\operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n-1}} \phi^*(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y)\Big) \\
&\cong \operatorname{cone}\Big(\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} \phi_!\phi^*(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y)\Big) \\
&\cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}\big(\operatorname{cone}\big(\phi_!\phi^*(X \wedge^{\mathbb{L}} Y) \to (X \wedge^{\mathbb{L}} Y)\big)\big) \\
&\cong \bigvee_{i+j=n} \operatorname{cone}(k_i \,\square^{\mathbb{L}}\, \tilde{k}_j) \\
&\cong \bigvee_{i+j=n} \operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone}(\tilde{k}_j). \qquad \square
\end{aligned}$$

Corollary 3.4.9 *Let X, Y, E as before and assume furthermore that $F_*(X_{\alpha_n}), F_*(Y_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\zeta, \beta\}$. Then there is a canonical isomorphism*

$$C^{(n)}(i^*E) = F_*(\operatorname{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n})) \cong \bigoplus_{i+j=n} C^{(i)}(X) \otimes C^{(j)}(Y).$$

Proof By our assumption, for any $\alpha \in \{\zeta, \beta\}$ and any $n \in \mathbb{Z}/N\mathbb{Z}$, the object $F_*X_{\alpha_n}$ is projective. Therefore, by definition, $Z^{(n)}(X)$ and $B^{(n-1)}(X)$ are projective. The short exact sequence (2.5.2) now implies that for any $i \in \mathbb{Z}/N\mathbb{Z}$ the graded object $C^{(i)}(X)$ is projective. It follows by our assumptions that

$$F_*(\operatorname{cone} k_s \wedge^{\mathbb{L}} \operatorname{cone} \tilde{k}_t) \cong F_*(\operatorname{cone} k_s) \otimes F_*(\operatorname{cone} k_t).$$

By Proposition 3.4.1 we have

$$\operatorname{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}) \cong \bigvee_{i+j=n} \operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone}(\tilde{k}_j),$$

and applying the functor $F_*(-)$ we have

$$\begin{aligned}
F_*(\operatorname{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n})) &\cong F_*\Big(\bigvee_{i+j=n} \operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone}(\tilde{k}_j)\Big) \\
&\cong \bigoplus_{i+j=n} F_*(\operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone} \tilde{k}_j) \\
&\cong \bigoplus_{i+j=n} F_*(\operatorname{cone} k_i) \otimes F_*(\operatorname{cone} \tilde{k}_j).
\end{aligned}$$

Shifting the above by $[n] = [i + j]$ we have

$$C^{(n)}(i^*E) \cong \bigoplus_{i+j=n} C^{(i)}(X) \otimes C^{(j)}(Y). \qquad \square$$

3.5 Differentials

In the previous subsection we proved that $C_*(i^*E) \cong C_*(X) \otimes C_*(Y)$ as objects in $\mathcal{A}$, so the diagram i^*E is a good candidate for the tensor product

$$C_*(X) \otimes C_*(Y).$$

The final step in order to show that indeed

$$\mathcal{Q}(i^*E) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y)$$

as objects in $\mathrm{C}^{([1],1)}(\mathcal{A})$ is to prove that the differential $d\colon C_*(i^*E) \to C_*(i^*E)[1]$ coincides with the differential of the tensor product $C_*(X) \otimes C_*(Y)$. That is, we have to show that

$$(C_*(i^*E), d) \cong (C_*(X) \otimes C_*(Y), d_\otimes),$$

where $d_\otimes$ is the differential of the tensor product of the dg-objects $(C_*(X), d)$ and $(C_*(Y), d)$.

3.5.1 Reduction to the case of disks We will reduce the proof to a much simpler case. Let $L_* \in \mathrm{C}^{([1],1)}(\mathcal{A})$ and choose $s \in \mathbb{Z}/N\mathbb{Z}$. Without loss of generality we will assume that L_* is degreewise projective. Consider the map of differential graded objects,

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & L_s & = & L_s & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \| & & \downarrow{\scriptstyle d^s} & & \downarrow & & \\ \dots & \xrightarrow[d^{s+2}]{} & L_{s+1} & \xrightarrow[d^{s+1}]{} & L_s & \xrightarrow[d^s]{} & L_{s-1} & \xrightarrow[d^{s-1}]{} & L_{s-2} & \xrightarrow[d^{s-2}]{} & \dots \end{array} \tag{3.5.1}$$

where we view the top differential graded object as an object in $\mathcal{B}[s-1] \oplus \mathcal{B}[s]$, meaning that it is concentrated in degrees $s-1$ and s modulo N. We denote this by $D^s(L_s)$, and we denote the above map of differential graded objects by $f_{L,s}\colon D^s(L_s) \to L_*$. Under the equivalence of categories $\mathcal{Q}\colon \mathcal{L} \to \mathrm{C}^{([1],1)}(\mathcal{A})$ there are crowned diagrams X and X' and a morphism $F\colon X \to X'$ such that the morphism $f_{L,s}$ is realized as $\mathcal{Q}(F)$. This means that there are isomorphisms

$$\mathcal{Q}(X) \cong D^s(L_s), \quad \mathcal{Q}(X') \cong L_*$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Q}(X) & \xrightarrow{\mathcal{Q}(F)} & \mathcal{Q}(X') \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ D^s(L_s) & \xrightarrow[f_{L,s}]{} & L_* \end{array}$$

Now let M_* be another differential graded object, which we also assume to be degreewise projective, and let $t \in \mathbb{Z}/N\mathbb{Z}$. Similarly to (3.5.1) we have the morphism

$$g_{M,t}\colon D^t(M_t) \to M_*.$$

Again, under the equivalence $\mathcal{Q}$ there are crowned diagrams Y and Y' and a morphism $G\colon Y \to Y'$ such that

$$\mathcal{Q}(Y) \cong D^t(M_t), \quad \mathcal{Q}(Y') \cong M_*$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Q}(Y) & \xrightarrow{\mathcal{Q}(G)} & \mathcal{Q}(Y') \\ \downarrow \cong & & \downarrow \cong \\ D^t(M_t) & \xrightarrow[g_{M,t}]{} & M_* \end{array}$$

We have the morphism of dg-objects $f_{L,s} \otimes g_{M,t}\colon D^s(L_s) \otimes D^t(M_t) \to L_* \otimes M_*$, which is

(3.5.2)
$$\begin{array}{ccccccc} \cdots \longrightarrow & L_s \otimes M_t & \longrightarrow & (L_s \otimes M_t) \oplus (L_s \otimes M_t) & \longrightarrow \cdots \\ & \downarrow & & \downarrow (d^s \otimes \mathrm{id}, \mathrm{id} \otimes \tilde{d}^t) & \\ \cdots \longrightarrow & \bigoplus_{i+j=n} L_i \otimes M_j & \longrightarrow & \bigoplus_{i+j=n+1} L_i \otimes M_j & \longrightarrow \cdots \end{array}$$

where the left vertical morphism is the inclusion of the $(s,t)^{\text{th}}$ summand and the right vertical map is the universal map out of the coproduct.

Now assume that

$$\mathcal{Q}(i^*\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y),$$

that is, we prove our claim for the case of $X \cong \mathcal{Q}^{-1}(D^s(L_s))$ and $Y \cong \mathcal{Q}^{-1}(D^t(M_t))$. The commutativity of the square (3.5.2) implies that the bottom vertical maps must also coincide with the tensor product $L_* \otimes M_*$, ie

$$Q(i^*\mathrm{pr}_!(X' \wedge^{\mathbb{L}} Y')) \cong L_* \otimes M_*$$

and the following diagram commutes degreewise:

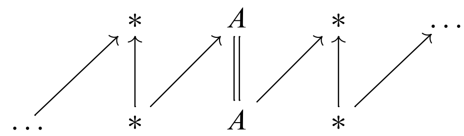
$$\begin{array}{ccc} Q(i^*\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) & \longrightarrow & Q(i^*\mathrm{pr}_!(X' \wedge^{\mathbb{L}} Y')) \\ \downarrow & & \downarrow \\ D^s(L_s) \otimes D^t(M_t) & \longrightarrow & L_* \otimes M_* \end{array}$$

The horizontal maps are indeed maps of dg-objects, so if we can show that the left hand vertical map is too, then the claim follows for the general L_* and M_*. The proof of the former will occupy the next subsection.

3.6 Differentials for disks

To prove the claim for disks, we discuss a crowned diagram that corresponds to the disks. By [Patchkoria 2012, Proposition 3.2.1], there is an object $A \in \mathrm{Ho}(\mathcal{M})$, such that $F_*A \in \mathcal{B}[s-1] = L_s$, which is due to the fact that our assumptions force the corresponding Adams spectral sequence to collapse. Consider the crowned diagram

$X =$



where the nontrivial entries are at the $(s-1)$–spot, ie

$$X_{\beta_{s-1}} = X_{\zeta_{s-1}} = A.$$

The diagram X is in $\mathcal{L}$ since

$$B_*(X) = B^{(s-1)}(X) = F_* X_{\beta_{s-1}} = F_* A \quad \text{and} \quad Z_*(X) = Z^{(s-1)}(X) = F_* X_{\zeta_{s-1}} = F_* A.$$

Next, we calculate $(C_*(X), d) \in \mathrm{C}^{([1],1)}(\mathcal{A})$. The only nontrivial cones are $\operatorname{cone}(k_{s-1})$ and $\operatorname{cone}(k_s)$. This means that

$$\begin{aligned} C^{(s)}(X) &= F_* \operatorname{cone}(k_s) = F_* \operatorname{cone}(A \to *) = F_*(\Sigma A) \cong (F_* A)[1], \\ C^{(s-1)}(X) &= F_* \operatorname{cone}(k_{s-1}) = F_* \operatorname{cone}(* \to A) = F_* A, \\ C_*(X) &= C^{(s-1)}(X) \oplus C^{(s)}(X). \end{aligned}$$

We obtain that $\lambda\colon B_*(X) \to Z_*(X)$ is the identity map, $\iota\colon Z_*(X) \to C_*(X)$ is inclusion to the first factor and $\rho\colon C_*(X) \to B_*(X)$ is the projection to the second factor. It follows that $d\colon C_*(X) \to C_*(X)[1]$ is the identity. Similarly, $D^t(L_t)$ is mapped to a crowned diagram Y in which

$$Y_{\beta_{t-1}} = Y_{\zeta_{t-1}} = \widetilde{A},$$

where the only nontrivial morphism is the identity.

We now have the ingredients to deal with the following proposition.

Proposition 3.6.1 *Let X and Y be the objects of $\mathcal{L}$ of the form $\mathcal{Q}^{-1}(D^s(L_s))$ and $\mathcal{Q}^{-1}(D^t(M_t))$. Then*

$$\mathcal{Q}(i^* \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong (C_*(X) \otimes C_*(Y), d_\otimes),$$

where $(C_*(X) \otimes C_*(Y), d_\otimes)$ is the tensor product of $C_*(X)$ and $C_*(Y)$ in $\mathrm{C}^{([1],1)}(\mathcal{A})$.

Proof We note that the tensor product $D^s(L_s) \otimes D^t(M_t)$ is concentrated in degrees $s+t$, $s+t-1$, and $s+t-2$ modulo N. As we already know that our chain complexes agree degreewise, these are the only degrees where we have to calculate our differential. As usual, we write $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)$.

We will work out the differential in the chain complex $\mathcal{Q}(i^* \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y))$, beginning with

$$\mathcal{Q}(i^* E)^{s+t} = C^{(s+t)}(i^* E) \to C^{(s+t-1)}(i^* E) = \mathcal{Q}(i^* E)^{s+t-1},$$

and we will discuss the other degree

$$C^{(s+t-1)}(i^* E) \to C^{(s+t-2)}(i^* E)$$

afterwards. Our proof is divided into the following steps.

We start by going through the definition $\mathcal{Q}$ applied to i^*E using the descriptions given in Section 2.5, where we will arrive at the exact triangle

$$\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}} \to \Sigma E_{\zeta_{s+t-1}} \to \Sigma\mathrm{cone}(\widehat{k}_{s+t-1}).$$

The next steps separately determine $E_{\zeta_{s+t-1}}$ followed by the maps

$$\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}}, \quad E_{\gamma_{s+t-1}} \to E_{\zeta_{s+t-1}} \quad \text{and} \quad E_{\zeta_{s+t-1}} \to \mathrm{cone}(\widehat{k}_{s+t-1}).$$

Putting those together, we obtain the desired differential.

Step 1 recalling the construction of $\mathcal{Q}(i^*E)^{s+t} \to \mathcal{Q}(i^*E)^{s+t-1}$

By Proposition 3.3.1 and Proposition 3.4.1 we can construct a diagram $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N})$ such that

$$i^*E \in \mathcal{L} \quad \text{and} \quad \mathrm{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}) \cong \bigvee_{i+j=n} \mathrm{cone}(k_i) \wedge \mathrm{cone}(\widetilde{k}_j).$$

For notational convenience we will write

$$\widehat{k}_n : i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n} \quad \text{and} \quad \hat{l}_n : i^*E_{\beta_n} \to E_{\zeta_n}$$

for the structure maps of the crowned diagram i^*E. We briefly recall the construction of the differential

$$d : C_*(i^*E) \to C_*(i^*E)[1], \quad d = \iota[1]\lambda[1]\rho C_*(i^*E) \to C_*(i^*E)[1].$$

Degreewise, the differential on $C^{(n)}(i^*E) \to C^{(n-1)}(i^*E)$ is given by applying $F_*(-)$ to the sequence of maps

$$\mathrm{cone}(\widehat{k}_n) \to \Sigma E_{\gamma_{n-1}} \to \Sigma E_{\zeta_{n-1}} \to \Sigma\mathrm{cone}(\widehat{k}_{n-1}). \tag{3.6.2}$$

Therefore, we have to show that for $n = s + t$ the sequence of maps (3.6.2) after applying $F_*(-)$ gives us the differential of the tensor product of disks. By Proposition 3.4.1, we have

$$\mathrm{cone}(\widehat{k}_{s+t}) \cong \mathrm{cone}(k_s) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t),$$
$$\mathrm{cone}(\widehat{k}_{s+t-1}) \cong (\mathrm{cone}(k_{s-1}) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t)) \vee (\mathrm{cone}(k_s) \wedge^{L} \mathrm{cone}(\widetilde{k}_{t-1})).$$

Recall that $A = X_{\beta_{s-1}} = X_{\zeta_{s-1}}$ and $\widetilde{A} = Y_{\beta_{s-1}} = Y_{\zeta_{s-1}}$ as before. Directly from the structure morphisms of the crowned diagrams X and Y we have

$$\mathrm{cone}(\widehat{k}_{s+t}) \cong (\Sigma A) \wedge (\Sigma \widetilde{A}), \quad \mathrm{cone}(\widehat{k}_{s+t-1}) \cong (A \wedge \Sigma\widetilde{A}) \vee (\Sigma A \wedge \widetilde{A}).$$

To analyse the sequence of maps (3.6.2) it remains to calculate $E_{\gamma_{s+t-1}}, E_{\zeta_{s+t-1}}, E_{\zeta_{s+t}}$ and the maps $E_{\gamma_{s+t-1}} \to E_{\zeta_{s+t-1}}$. The maps

$$\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}}, \tag{3.6.3}$$

$$\Sigma E_{\zeta_{s+t-1}} \to \Sigma\mathrm{cone}(\widehat{k}_{s+t-1}) \tag{3.6.4}$$

are the canonical maps that are given by construction of distinguished triangles in a simplicial stable model category. The map (3.6.3) is the canonical map

$$\operatorname{cone}(\widehat{k}_{s+t}) \to S^1 \wedge E_{\gamma_{s+t-1}};$$

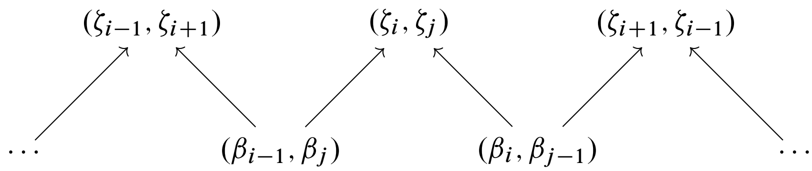
see Definition 2.3.13. Similarly, the map (3.6.4) is the suspension of the canonical map

$$E_{\zeta_{s+t-1}} \to \operatorname{cone}(\widehat{k}_{s+t-1});$$

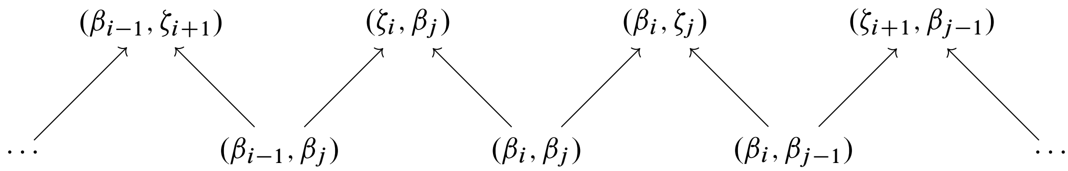
see Definition 2.3.13.

Step 2 calculating E_{ζ_n}

To compute the above, let us recall from (iv) the poset J_n with inclusion $\theta\colon J_n \hookrightarrow \mathrm{pr}/\zeta_n$ and left adjoint $L\colon \mathrm{pr}/\zeta_n \to J_n$. For $i+j \equiv n$ modulo N the poset J_n looks as follows:



Also, recall from (ii) the slice category pr/γ_n, which for $i+j \equiv n$ is



By definition of homotopy left Kan extensions, we have

$$E_{\zeta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} X \wedge^{\mathbb{L}} Y \cong \operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y), \quad E_{\gamma_n} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_n} (X \wedge^{\mathbb{L}} Y).$$

The maps

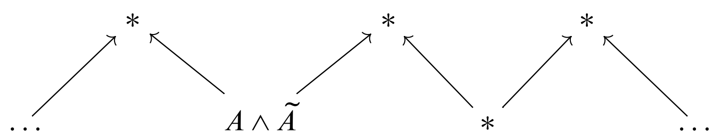
$$E_{\gamma_{n-1}} \to E_{\zeta_n} \quad \text{and} \quad E_{\gamma_{n-1}} \to E_{\zeta_{n-1}}$$

are the maps of homotopy colimits induced by the respective map of posets

$$\psi\colon \mathrm{pr}/\gamma_{n-1} \to \mathrm{pr}/\zeta_n \quad \text{and} \quad \phi\colon \mathrm{pr}/\gamma_{n-1} \to \mathrm{pr}/\zeta_{n-1}.$$

We start with calculating $E_{\zeta_{s+t-1}}$. The underlying diagram $\theta^*(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{J_{s+t-1}})$ is

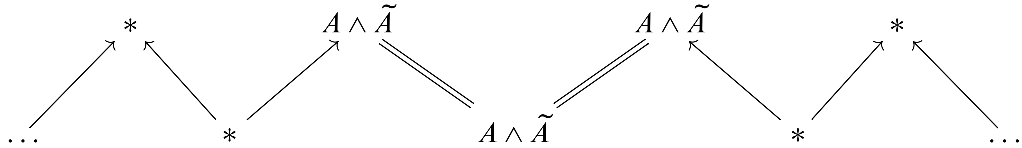
(3.6.5)



where the only nontrivial entry is at $(\beta_{s-1}, \beta_{t-1})$. From the diagram above we get

$$E_{\zeta_{s+t-1}} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_{s+t-1}} (X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{J_{s+t-1}} \theta^*(X \wedge^{\mathbb{L}} Y) \cong \Sigma A \wedge \widetilde{A}.$$

We do the same for $E_{\gamma_{s+t}}, E_{\gamma_{s+t-1}}$ and $E_{\gamma_{s+t-2}}$. The value $E_{\gamma_{s+t-2}}$ is the homotopy colimit of the diagram $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_{s+t-2}})$, which is

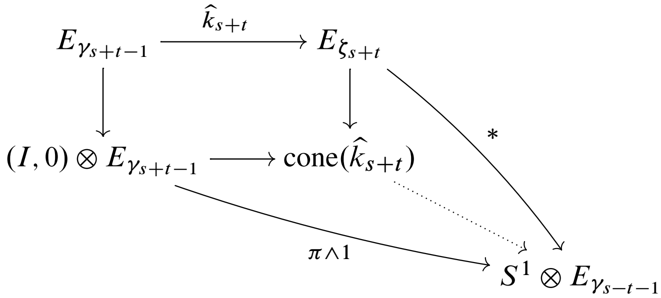


with nontrivial entries at the places $(\beta_{s-1}, \beta_{t-1})$ on the bottom, $(\zeta_{s-1}, \beta_{t-1})$ on the left and $(\beta_{s-1}, \zeta_{t-1})$ on the right. Thus, $E_{\gamma_{s+t-2}} \cong A \wedge \widetilde{A}$. Similarly, we have

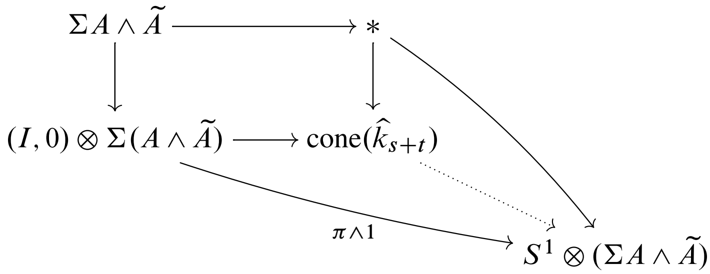
$$E_{\gamma_{s+t}} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_{s+t}} (X \wedge^{\mathbb{L}} Y) \cong *, \quad E_{\gamma_{s+t-1}} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_{s+t-1}} (X \wedge^{\mathbb{L}} Y) \cong \Sigma A \wedge \widetilde{A}.$$

Step 3 calculating $\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}}$

We move on to calculate the map $\mathrm{cone}(\widehat{k}_{s+t}) \to S^1 \otimes E_{\gamma_{s+t-1}}$. From Definition 2.3.13 we have the pushout square



which, based on our computations, is



Recall from Proposition 3.4.1, (3.4.8), and Corollary 2.3.6 that there is a series of canonical isomorphisms

$$\mathrm{cone}(\widehat{k}_{s+t}) \cong \mathrm{cone}(k_s \,\square^{\mathbb{L}}\, \widetilde{k}_t) \cong \mathrm{cone}(k_s) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t).$$

In our particular case, in which $k_s \colon A \to A$ and $\widetilde{k} \colon \widetilde{A} \to *$, this is

$$\mathrm{cone}(\widehat{k}_{s+t}) \cong \mathrm{cone}(k_s \,\square^{\mathbb{L}}\, k_t) \cong \Sigma^2 A \wedge \widetilde{A} \cong \Sigma A \wedge \Sigma \widetilde{A} \cong \mathrm{cone}(k_s) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t).$$

This implies that the universal map out of the pushout is the identity map. Thus, the map

$$\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}}$$

is the map $\Sigma A \wedge \Sigma\widetilde{A} \to \Sigma^2(A \wedge \widetilde{A})$, which is the composition of the canonical map (commutation of colimits) and the identity map.

Step 4 calculating $\hat{l}_{s+t-1}\colon E_{\gamma_{s+t-1}} \to E_{\zeta_{s+t-1}}$

From the posets above we can see directly that the map

$$\hat{l}_{s+t-1}\colon E_{\gamma_{s+t-1}} \to E_{\zeta_{s+t-1}}$$

is the identity map induced by

$$\psi\colon \mathrm{pr}/\gamma_{s+t-1} \to \mathrm{pr}/\zeta_{s+t-1}.$$

Therefore the map

$$\Sigma\hat{l}_{s+t-1}\colon E_{\gamma_{s+t-1}} \to \Sigma E_{\zeta_{s+t-1}}$$

is the identity map

$$1\colon \Sigma^2(A \wedge \widetilde{A}) \to \Sigma^2(A \wedge \widetilde{A}).$$

Step 5 calculating $E_{\zeta_{s+t-1}} \to \mathrm{cone}(\widehat{k}_{s+t-1})$

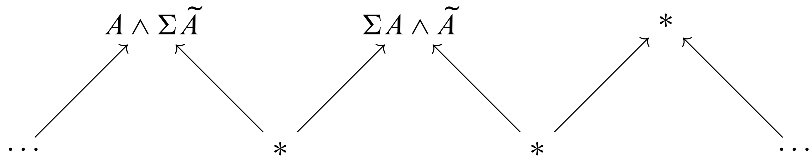
Lastly it remains to figure out the map

$$E_{\zeta_{s+t-1}} \to \mathrm{cone}(\widehat{k}_{s+t-1}).$$

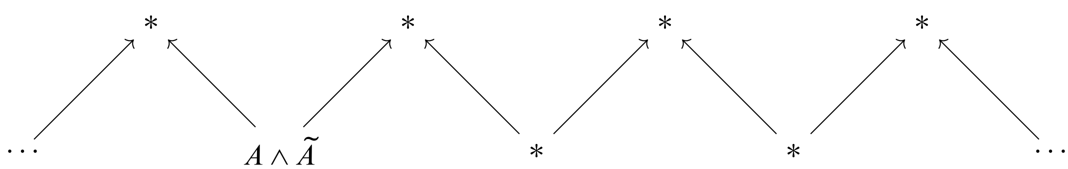
Recall from the proof of Proposition 3.4.1 that $\mathrm{cone}(\widehat{k}_{s+t-1})$ can be written as a homotopy colimit,

$$\mathrm{cone}(\widehat{k}_{s+t-1}) \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_{s+t-1}}(\mathrm{cone}(\varepsilon_{X\wedge^{\mathbb{L}}Y})),$$

where $\phi\colon \mathrm{pr}/\gamma_{s+t-2} \to \mathrm{pr}/\zeta_{s+t+1}$, and ε is the counit of the derived adjunction $(\mathbb{L}\phi_!, \phi^*)$. Pulling back the diagram $\mathrm{cone}(\varepsilon_{X\wedge^{\mathbb{L}}Y})$ to J_{s+t-1} along the inclusion $\theta\colon J_{s+t-1} \to \mathrm{pr}/\zeta_{s+t-1}$, we obtain the diagram



with nontrivial entries at (ζ_{s-1}, ζ_t) and (ζ_s, ζ_{t-1}) respectively. Recall the following diagram from (3.6.5) $\theta^*(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{J_{s+t-1}})$,



with the only nontrivial entry at $(\beta_{s-1}, \beta_{t-1})$, left top being (ζ_{s-1}, ζ_t) and right top being (ζ_s, ζ_{t-1}). Because of the shape of the underlying posets and the map, we can safely ignore the trivial entries, so the map $E_{\gamma_{s+t-1}} \to \mathrm{cone}(\widehat{k}_{s+t-1})$ can be taken as the map of homotopy pushouts

$$\mathrm{hocolim}(* \leftarrow A \wedge \widetilde{A} \to *) \to \mathrm{hocolim}(A \wedge \Sigma\widetilde{A} \leftarrow * \to \Sigma A \wedge \widetilde{A}),$$

induced by the following map of posets:

$$\begin{array}{ccccc} * & \longleftarrow & A \wedge \widetilde{A} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ A \wedge \Sigma \widetilde{A} & \longleftarrow & * & \longrightarrow & \Sigma A \wedge \widetilde{A} \end{array}$$

Now consider the above map of diagrams and the following map at the bottom:

$$\begin{array}{ccccc} * & \longleftarrow & A \wedge \widetilde{A} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ A \wedge \Sigma \widetilde{A} & \longleftarrow & * & \longrightarrow & \Sigma A \wedge \widetilde{A} \\ {\scriptstyle \tau}\downarrow & & \downarrow & & \| \\ \Sigma A \wedge \widetilde{A} & \longleftarrow & * & \longrightarrow & \Sigma A \wedge \widetilde{A} \end{array}$$

Here, τ is the map

$$A \wedge \Sigma \widetilde{A} = A \wedge (S^1 \wedge \widetilde{A}) \cong (A \wedge S^1) \wedge \widetilde{A} \xrightarrow{\tau} (S^1 \wedge A) \wedge \widetilde{A} \cong \Sigma A \wedge \widetilde{A}$$

and the first map is the associativity isomorphism. By Lemma 3.7.1 the induced map of homotopy colimits is, up to weak equivalence, the diagonal map

$$\text{diag}\colon \Sigma A \wedge \widetilde{A} \to (\Sigma A \wedge \widetilde{A}) \vee (\Sigma A \wedge \widetilde{A}).$$

Hence, the map (3.6.2) is up to weak equivalence the diagonal map but with a sign introduced by the twist map as above. This and Corollary 3.4.9 imply that indeed the differential

$$d\colon C^{(s+t)}(i^* E) \to C^{(s+t-1)}(i^* E)$$

coincides with the differential of the tensor product of

$$((D^s L^s) \otimes (D^t \widetilde{L}^t))^{s+t} \to ((D^s L^s) \otimes (D^t \widetilde{L}^t))^{s+t-1}.$$

Step 6 $\mathcal{Q}(i^* E)^{s+t-1} \to \mathcal{Q}(i^* E)^{s+t-2}$

We do not need to do any extra work to determine the other differential, namely to check that the differential

$$C^{(s+t-1)}(i^* E) \to C^{(s+t-2)}(i^* E)$$

coincides with the differential

$$((D^s L^s) \otimes (D^t \widetilde{L}^t))^{s+t-1} \to ((D^s L^s) \otimes (D^t \widetilde{L}^t))^{s+t-2},$$

since by construction $(C_*(i^* E), d)$ is a differential graded object and that means that by necessity $d[1] \circ d = 0$ on $C_*(i^* E)$. This concludes the proof. □

To conclude this section, by combining Corollary 3.4.9 and Proposition 3.6.1 we have proved the following proposition.

Proposition 3.6.6 *Let $X, Y \in \mathcal{L}$ such that $F_*(X_{\alpha_n}), F_*(Y_{\alpha_n}) \in \mathcal{A}_{\text{proj}}$ for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$. There is a natural isomorphism*

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y).$$ □

3.7 Technical lemmas

In this subsection we prove two technical lemmas that are used in the previous proofs. The first lemma is about the canonical map from the suspension of an object to the wedge product of suspensions in a stable, simplicial model category $\mathcal{M}$. The second lemma is about pushout-products of injective morphisms in a hereditary abelian category $\mathcal{A}$.

Lemma 3.7.1 *Let $\mathcal{M}$ be a stable simplicial model category and let $X \in \mathcal{M}$. Consider the following map of homotopy pushouts*

$$\mathrm{hocolim}(* \leftarrow X \rightarrow *) \rightarrow \mathrm{hocolim}(\Sigma X \leftarrow * \rightarrow \Sigma X).$$

Then the above map is, up to isomorphism in $\mathrm{Ho}(\mathcal{M})$, the diagonal map

$$\mathrm{diag}\colon \Sigma X \rightarrow \Sigma X \vee \Sigma X.$$

Proof Let $CX = (I, 0) \otimes X$ be the cone of X and let $i\colon X \rightarrow CX$ be the canonical inclusion, which is a cofibration. We choose a model for ΣX as the homotopy pushout

$$\Sigma X \cong \mathrm{hocolim}(CX \leftarrow X \rightarrow CX).$$

In fact, we can take this to be the ordinary pushout $\mathrm{colim}(CX \longleftarrow X \rightarrow CX)$ since $i\colon X \rightarrow CX$ is a cofibration. From this model we get directly that the induced map on pushouts

$$\begin{array}{ccccc} CX & \xleftarrow{i} & X & \xrightarrow{i} & CX \\ {\scriptstyle \pi\otimes 1}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi\otimes 1} \\ \Sigma X & \longleftarrow & * & \longrightarrow & \Sigma X \end{array}$$

where $\pi\colon I \rightarrow S^1$ is the projection is indeed the diagonal map $\mathrm{diag}\colon \Sigma X \rightarrow \Sigma X \vee \Sigma X$. Hence, the induced map of homotopy pushouts is the diagonal map up to natural isomorphism. □

Lemma 3.7.2 *Let $\mathcal{A}$ be a hereditary abelian category. Let $X, Y, U, V \in \mathcal{A}_{\text{proj}}$ and let $f\colon X \rightarrow Y$ and $g\colon U \rightarrow V$ be injective maps. Then the pushout-product map $f \mathbin{\square} g$ is injective.*

Proof Since $g\colon U \rightarrow V$ is monomorphism we have the short exact sequence

$$0 \rightarrow U \xrightarrow{g} V \xrightarrow{j} \mathrm{coker}\, g \rightarrow 0.$$

Notice that the dimension of the abelian category $E(1)_*$–modules is 1, which implies that $\operatorname{coker} g$ is a projective module since it is a submodule of V. Since X is flat, $X \otimes -$ is an exact functor, which means that the sequence

$$0 \to X \otimes U \xrightarrow{1\otimes g} X \otimes V \xrightarrow{1 \otimes j} X \otimes \operatorname{coker} g \to 0$$

is short exact. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X \otimes U & \xrightarrow{1\otimes g} & X \otimes V & \xrightarrow{1\otimes j} & X \otimes \operatorname{coker} g & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f\otimes 1} & & \downarrow & & \| & & \\
0 & \longrightarrow & Y \otimes U & \longrightarrow & P & \longrightarrow & X \otimes \operatorname{coker} g & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f \square g} & & \downarrow{\scriptstyle f\otimes 1} & & \\
0 & \longrightarrow & Y \otimes U & \xrightarrow[1\otimes g]{} & Y \otimes V & \xrightarrow[1\otimes j]{} & Y \otimes \operatorname{coker} g & \longrightarrow & 0
\end{array}$$

where P is the pushout of $1 \otimes g$ and $g \otimes 1$. Since the top left square is cocartesian, the canonical map $\operatorname{coker}(1 \otimes g) \xrightarrow{\cong} \operatorname{coker}(Y \otimes U \to P)$ is an isomorphism, so the middle row is also exact. Now note that the morphism $f \otimes 1\colon X \otimes \operatorname{coker} g \to Y \otimes \operatorname{coker} g$ is injective since $\operatorname{coker} g$ is projective. Applying the snake lemma gives us that $f \square g$ is a monomorphism. □

4 Main result

4.1 Homotopy colimit calculations

In this section we discuss how the functor $i^*\mathbb{L}\mathrm{pr}_!$ interacts with the homotopy colimits of the various diagram categories, giving us the right hand side of the main diagram (1.0.2). The main result of the section is the following.

Theorem 4.1.1 *For any pair of diagrams $(X, Y) \in \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N})$, the homotopy colimit of the diagram $i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{C_N})$ is naturally isomorphic to the smash product of the homotopy colimits of X and Y, that is,*

$$\operatorname*{hocolim}_{C_N}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \operatorname*{hocolim}_{C_N} X \wedge^{\mathbb{L}} \operatorname*{hocolim}_{C_N} Y.$$

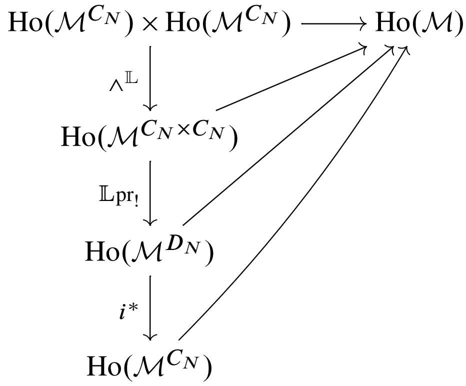
Recall that the functor

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N})$$

is the composition

$$\mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \xrightarrow{\wedge^{\mathbb{L}}} \mathrm{Ho}(\mathcal{M}^{C_N \times C_N}) \xrightarrow{\mathbb{L}\mathrm{pr}_!} \mathrm{Ho}(\mathcal{M}^{D_N}) \xrightarrow{i^*} \mathrm{Ho}(\mathcal{M}^{C_N}).$$

In order to prove Theorem 4.1.1 we will break it apart into smaller pieces. Consider the following diagram:



The top horizontal functor is the smash product of homotopy colimits of crowned diagrams, that is, $\mathrm{hocolim}_{C_N} X \wedge^{\mathbb{L}} \mathrm{hocolim}_{C_N} Y$. The three other functors are the homotopy colimit functors,

$$\operatorname*{hocolim}_{C_N \times C_N} \colon \mathrm{Ho}(\mathcal{M}^{C_N \times C_N}) \to \mathrm{Ho}(\mathcal{M}),$$
$$\operatorname*{hocolim}_{D_N} \colon \mathrm{Ho}(\mathcal{M}^{D_N}) \to \mathrm{Ho}(\mathcal{M}),$$
$$\operatorname*{hocolim}_{C_N} \colon \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}).$$

Theorem 4.1.1 asserts that the outer triangle above commutes up to isomorphism. This will follow once we show that all the small triangles commute up to isomorphism.

Lemma 4.1.2 *The top triangle and the middle triangle commute. That is,*

$$\operatorname*{hocolim}_{C_N} X \wedge^{\mathbb{L}} \operatorname*{hocolim}_{C_N} Y \cong \operatorname*{hocolim}_{C_N \times C_N}(X \wedge^{\mathbb{L}} Y)$$

and

$$\operatorname*{hocolim}_{C_N \times C_N}(X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{D_N} \mathrm{pr}_!(X \wedge^{\mathbb{L}} Y).$$

Proof The first assertion follows from Corollary 2.3.4 as a direct application for $C = D = C_N$. The second assertion follows from the fact that the homotopy colimit of a homotopy left Kan extension of a diagram is isomorphic to the homotopy colimit of the diagram itself [Richter 2020, Proposition 4.3.2]. □

We will prove Theorem 4.1.1 by proving that the functor $i \colon C_N \to D_N$ satisfies the following definition; see [Riehl 2014, Definition 8.5.1].

Definition 4.1.3 A functor between small categories $K \colon C \to D$ is *homotopy final* (or *homotopy terminal*) if for every object $d \in D$, the simplicial set $N(d/K)$ is contractible.

A convenient way to check whether a poset is contractible is given by Quillen [1978, Section 1.5]: a poset C is *conically contractible* if there is an object $c_0 \in C$ and a map of posets $f\colon C \to C$ such that $c \le f(c) \ge c_0$ for every $c \in C$. In this case one can show that the identity 1_C, the map f, and the constant map with value c_0 from C to itself are homotopic (that is to say, their realizations are homotopic), and hence C is contractible. So, given a diagram $E \in \mathrm{Ho}(\mathcal{M}^{D_N})$, to check that the canonical morphism

$$\phi_i\colon \operatorname*{hocolim}_{C_N} i^* E \to \operatorname*{hocolim}_{D_N} E$$

is an isomorphism it suffices to check that the slice categories α_n/i of the functor $i\colon C_N \to D_N$ are contractible for any $\alpha \in \{\zeta, \gamma, \beta\}$ and any $n \in \mathbb{Z}/N\mathbb{Z}$.

We will now apply this to our functor $i\colon C_N \to D_N$, which is the inclusion of the two-row crowned diagram into the three-row crowned diagram (3.1.2).

Lemma 4.1.4 *The functor $i\colon C_N \to D_N$ is homotopy final.*

Proof We will prove the above proposition by applying Quillen's criterion of conical contractible posets. First, we identity the slice categories ζ_n/i, γ_n/i and β_n/i and then we will check that they are indeed conically contractible. We start with ζ_n/i. By definition,

$$\zeta_n/i = \{\alpha_n \in C_N \mid i(\alpha_n) \ge \zeta_n\} = \{\zeta_n\}.$$

Since this poset contains only one element it is obviously contractible. The next slice categories are of the form γ_n/i. By definition,

$$\gamma_n/i = \{\alpha_n \in C_N \mid i(\alpha_n) \ge \gamma_n\},$$

that is, γ_n/i is the poset

$$\begin{array}{ccc} \zeta_n & & \zeta_{n+1} \\ \uparrow & \nearrow & \\ \beta_n & & \end{array}$$

We choose β_n and $1\colon \gamma_n/i \to \gamma_n/i$. Directly from above we can see that γ_n/i is conically contractible. The last case is the slices β_n/i. By definition,

$$\beta_n/i = \{\alpha_n \in C_N \mid i(\alpha_n) \ge \beta_n\},$$

which is the poset

$$\begin{array}{ccc} \zeta_n & & \zeta_{n+1} \\ \uparrow & \nearrow & \uparrow \\ \beta_n & & \beta_{n+1} \end{array}$$

We choose β_n and the map of $\beta_n/i \to \beta_n/i$ as

$$\zeta_n \mapsto \zeta_n, \quad \zeta_{n+1} \mapsto \zeta_{n+1}, \quad \beta_n \mapsto \beta_n, \quad \beta_{n+1} \mapsto \zeta_{n+1}.$$

With these choices, we can see that the poset β_n/i is conically contractible. □

Finally, we obtain the commutativity of the bottom triangle of our big diagram, which also concludes the proof of Theorem 4.1.1.

Corollary 4.1.5 *The bottom triangle of* (1.0.2) *commutes*, *that is*,

$$\operatorname*{hocolim}_{C_N} i^* E \cong \operatorname*{hocolim}_{D_N} E. \qquad \square$$

4.2 Proof of main theorem

Finally, we are in a position to assemble all our work into our main theorem.

Theorem 4.2.1 *Let $\mathcal{A}$ be a hereditary abelian category*, *and $\mathcal{M}$ be a monoidal stable model category such that Franke's functor*

$$\mathcal{R}\colon (\mathrm{D}^{([1],1)}(\mathcal{A}), \otimes^{\mathbb{L}}) \to (\mathrm{Ho}(\mathcal{M}), \wedge^{\mathbb{L}})$$

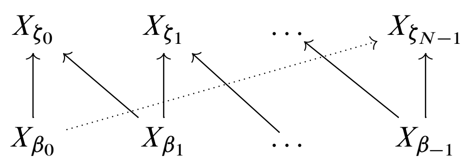
exists and is an equivalence. Then $\mathcal{R}$ preserves the monoidal products up to a natural isomorphism, *that is*,

$$\mathcal{R}(M_* \otimes^{\mathbb{L}} M_*) \cong \mathcal{R}(M_*) \wedge^{\mathbb{L}} \mathcal{R}(M_*).$$

Proof We assemble our proof along the lines of the diagram (1.0.2). Let M_* and N_* be objects in $\mathrm{D}^{([1],1)}(\mathcal{A})$. By Convention 2.1.1, both objects are cofibrant. Since M_* is cofibrant, the functor

$$M_* \otimes - \colon \mathrm{C}^{([1],1)}(\mathcal{A}) \to \mathrm{C}^{([1],1)}(\mathcal{A})$$

is left Quillen, see [Hovey 1999, Remark 4.2.3], which means it preserves cofibrant objects. Since both objects are cofibrant, the tensor product $M_* \otimes N_*$ represents the derived tensor product in $(\mathrm{D}^{([1],1)}(\mathcal{A}), \otimes^{\mathbb{L}})$ and in particular it also cofibrant. Recall from Example 2.3.11 that the cofibrant objects in $\mathrm{C}^{([1],1)}(\mathcal{A})$ are the projective objects in $\mathcal{A}$. This means, in particular, that M_*, N_* and $M_* \otimes N_*$ all belong to $\mathcal{A}_{\mathrm{proj}}$. We recall some notation from Section 3. Given a crowned diagram $X \in \mathcal{M}^{C_N}$ as



we set

$$Z^{(n)}(X) = F_*(X_{\zeta_n}), \quad B^{(n)}(X) = F_*(X_{\beta_n}), \quad C^{(n)}(X) = F_*(\mathrm{cone}(X_{\beta_{n-1}} \to X_{\zeta_n})).$$

Given $(M_*, d) \in \mathrm{C}^{([1],1)}(\mathcal{A})$, one can construct a crowned diagram X in $\mathcal{L}$ such that

$$(C_*(X), d) \cong (M_*, d), \quad Z_*(X) \cong \ker d, \quad B_*(X) \cong \operatorname{im} d.$$

By the discussion above, $M_* \in \mathcal{A}_{\mathrm{proj}}$. By assumption, $\mathcal{A}$ is a hereditary abelian category, in other words, $\mathrm{gl.dim}\,\mathcal{A} = 1$. This implies that $\ker d, \operatorname{im} d \in \mathcal{A}_{\mathrm{proj}}$ since they are submodules of M_*.

Hence, for the crowned diagram $X \cong \mathcal{Q}^{-1}(M_*)$ we have $F_*(X_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for every $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\zeta, \beta\}$. Similarly, for the dg-object (N_*, d) we get a crowned diagram $Y \cong \mathcal{Q}^{-1}(N_*)$ such that $F_*(Y_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for every $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\zeta, \beta\}$.

Now, by Theorem 3.1.5,

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y) = M_* \otimes N_*$$

and by Theorem 4.1.1,

$$\operatorname*{hocolim}_{C_N}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \operatorname*{hocolim}_{C_N} X \wedge^{\mathbb{L}} \operatorname*{hocolim}_{C_N} Y.$$

Finally, we recall that Franke's realization functor (2.5.4) is defined by

$$\mathcal{R} = \operatorname*{hocolim}_{C_N} \circ\, \mathcal{Q}^{-1},$$

which concludes the proof. □

The assumptions of Theorem 4.2.1 are satisfied in the following instances.

Example 4.2.2 From [Patchkoria 2012, Corollary 5.2.1] we know that

$$\mathcal{R}\colon D(\pi_* R) \to D(R) = \mathrm{Ho}(R\text{-mod})$$

is an equivalence for a ring spectrum R with $\pi_*(R)$ concentrated in degrees that are multiples of some $N > 1$ and global dimension of $\pi_*(R)$ equal to 1. This satisfies the assumption of our Theorem 4.2.1 and applies to $R = KU$, $R = KU_{(p)}$, $R = E(1)$ (complex K–theory), and $R = k(n)$ (connective Morava K–theory).

Example 4.2.3 By [Franke 1996; Roitzheim 2008] we know that

$$\mathcal{R}\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathrm{Ho}(L_1\mathcal{S})$$

is an equivalence. Here, $\mathcal{A}$ is the category of $E(1)_*E(1)$–comodules, and $L_1\mathcal{S}$ is a suitable category of spectra equipped with the K–local model structure at an odd prime. Note that as mentioned in Example 2.3.11, that while $\mathcal{A}$ does not have enough projectives, all our proofs also work when working with comodules whose underlying $E(1)_*$–module is projective; see also the first author's thesis [Nikandros 2022].

References

[Barnes and Roitzheim 2011] **D Barnes**, **C Roitzheim**, *Monoidality of Franke's exotic model*, Adv. Math. 228 (2011) 3223–3248 MR Zbl

[Barnes and Roitzheim 2020] **D Barnes**, **C Roitzheim**, *Foundations of stable homotopy theory*, Cambridge Stud. Adv. Math. 185, Cambridge Univ. Press (2020) MR Zbl

[Cisinski 2009] **D-C Cisinski**, *Locally constant functors*, Math. Proc. Cambridge Philos. Soc. 147 (2009) 593–614 MR Zbl

[Franke 1996] **J Franke**, *Uniqueness theorems for certain triangulated categories possessing an Adams spectral sequence*, preprint (1996)

[Gálvez-Carrillo et al. 2013] **I Gálvez-Carrillo**, **F Neumann**, **A Tonks**, *Thomason cohomology of categories*, J. Pure Appl. Algebra 217 (2013) 2163–2179 MR Zbl

[Ganter 2007] **N Ganter**, *Smash product of $\mathscr{E}(1)$–local spectra at an odd prime*, Cah. Topol. Géom. Différ. Catég. 48 (2007) 3–54 MR Zbl

[Goerss and Jardine 1999] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Progr. Math. 174, Birkhäuser, Basel (1999) MR Zbl

[Hirschhorn 2003] **P S Hirschhorn**, *Model categories and their localizations*, Math. Surv. Monogr. 99, Amer. Math. Soc., Providence, RI (2003) MR Zbl

[Hovey 1999] **M Hovey**, *Model categories*, Math. Surv. Monogr. 63, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[Hovey 2014] **M Hovey**, *Smith ideals of structured ring spectra*, preprint (2014) arXiv 1401.2850

[Nikandros 2022] **N Nikandros**, *Franke's realization functor and monoidal products*, PhD thesis, University of Kent (2022) Available at `https://www.proquest.com/docview/2892043197`

[Oberst 1967] **U Oberst**, *Basiserweiterung in der Homologie kleiner Kategorien*, Math. Z. 100 (1967) 36–58 MR Zbl

[Oberst 1968] **U Oberst**, *Homology of categories and exactness of direct limits*, Math. Z. 107 (1968) 87–115 MR Zbl

[Patchkoria 2012] **I Patchkoria**, *On the algebraic classification of module spectra*, Algebr. Geom. Topol. 12 (2012) 2329–2388 MR Zbl

[Quillen 1978] **D Quillen**, *Homotopy properties of the poset of nontrivial p–subgroups of a group*, Adv. Math. 28 (1978) 101–128 MR Zbl

[Richter 2020] **B Richter**, *From categories to homotopy theory*, Cambridge Stud. Adv. Math. 188, Cambridge Univ. Press (2020) MR Zbl

[Riehl 2014] **E Riehl**, *Categorical homotopy theory*, New Math. Monogr. 24, Cambridge Univ. Press (2014) MR Zbl

[Roitzheim 2008] **C Roitzheim**, *On the algebraic classification of K–local spectra*, Homology Homotopy Appl. 10 (2008) 389–412 MR Zbl

[Shulman 2006] **M Shulman**, *Homotopy limits and colimits and enriched homotopy theory*, preprint (2006) arXiv math/0610194

School of Mathematics, Statistics and Actuarial Science, University of Kent
Canterbury, United Kingdom

School of Mathematics, Statistics and Actuarial Science, University of Kent
Canterbury, United Kingdom

`nnikandros@gmail.com, c.roitzheim@kent.ac.uk`

Received: 31 January 2023 Revised: 28 November 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY
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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 9 (pages 4731–5219) 2024