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Cartesian fibrations of $(\infty, 2)$–categories

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We introduce four variance flavors of (co)cartesian fibrations of ∞–bicategories with ∞–bicategorical fibers, in the framework of scaled simplicial sets. Given a map $p\colon \mathcal{E} \to \mathcal{B}$ of ∞–bicategories, we define p–(co)cartesian arrows and inner/outer triangles by means of lifting properties against p, leading to a notion of 2–inner/outer (co)cartesian fibrations as those maps with enough (co)cartesian lifts for arrows and enough inner/outer lifts for triangles, together with a compatibility property with respect to whiskerings in the outer case. By doing so, we also recover in particular the case of ∞–bicategories fibered in ∞–categories studied in previous work. We also prove that equivalences of such fibrations can be tested fiberwise. As a motivating example, we show that the domain projection $\mathrm{d}\colon \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is a prototypical example of a 2–outer cartesian fibration, where $\mathrm{Fun}^{\mathrm{gr}}(X, Y)$ denotes the ∞–bicategory of functors, lax natural transformations and modifications. We then define 2–inner and 2–outer flavors of (co)cartesian fibrations of categories enriched in ∞–categories, and we show that a fibration $p\colon \mathcal{E} \to \mathcal{B}$ of such enriched categories is a (co)cartesian 2–inner/outer fibration if and only if the corresponding map $\mathrm{N}^{\mathrm{sc}}(p)\colon \mathrm{N}^{\mathrm{sc}}\mathcal{E} \to \mathrm{N}^{\mathrm{sc}}\mathcal{B}$ is a fibration of this type between ∞–bicategories.

18N65, 18N50, 55U35

Introduction

This paper is part of an ongoing series of works on the theory of $(\infty, 2)$–categories. We will generally refer to these as ∞–bicategories, and identify them with scaled simplicial sets satisfying suitable extension properties. Our goal here is to set up the fundamentals of a theory of (co)cartesian fibrations $\mathcal{E} \to \mathcal{B}$ of ∞–bicategories, whose fibers encode a family of ∞–bicategories $\mathcal{E}_b$ depending functorially on $b \in \mathcal{B}$.

When $\mathcal{E}$ and $\mathcal{B}$ are ∞–categories the corresponding notion of (co)cartesian fibration was set up by Lurie in [12]. Generalizing classical work of Grothendieck, Lurie showed that such fibrations over a base ∞–category $\mathcal{B}$ are in complete correspondence with functors $\mathcal{B} \to \mathcal{C}at_\infty$ in the cocartesian case, and contravariant such functors in the cartesian case. This Grothendieck–Lurie correspondence plays a key role in higher category theory, as it permits the handling of highly coherent pieces of structure, such as functors, in a relatively accessible manner.

When coming to consider the ∞–bicategorical counterpart, an immediate difference presents itself: here we have not just two different variances, but four, depending on whether or not the functorial dependence on 2–morphisms is covariant or contravariant. This additional axis of symmetry is already visible when the base is an ∞–bicategory and the fibers $\mathcal{E}_b$ are ∞–categories, since $\mathcal{C}at_\infty$ itself has a nontrivial ∞–bicategorical structure. A thorough treatment of this case was taken up in our previous work [6], where we have used the term inner (co)cartesian fibrations to indicate those cases where the dependence on 2–morphisms matches the one on 1–morphisms, and outer (co)cartesian fibration for those where these two dependencies have opposite variance. The division into these two types is also visible on the level of mapping ∞–categories: when both variances match (the inner case) the induced functor on mapping ∞–bicategories is a right fibration, and when they don't (the outer case) the induced functor on mapping ∞–categories is a left fibration. One of the main results of [6] is an ∞–bicategorical Grothendieck–Lurie correspondence for all the four variances. The proof crucially relies on the work of Lurie [13] in the inner cocartesian case, and the straightening–unstraightening Quillen equivalence he proved in that setting.

Our goal in the present paper is to extend these notions of fibrations to the setting where both the base $\mathcal{B}$ and the fibers $\mathcal{E}_b$ are ∞–bicategories. This requires defining, in addition to a notion of (co)cartesian 1–morphisms, a suitable notion of (co)cartesian 2–morphism. Working in the setting of scaled simplicial sets, such 2–morphisms are determined by triangles, though this correspondence is not perfect (the same 2–morphism can be encoded by different triangles corresponding to different factorizations of the target 1–morphism), and one needs to be a bit more careful about how to define this. To match with the notation for fibrations, we use the terms *inner* and *outer* to describe triangles which roughly correspond to cartesian and cocartesian 2–morphisms. For somewhat technical reasons one needs to separate those further into two types, which we call left and right inner/outer triangles. This distinction turns out to behave slightly differently in the inner and outer cases, an issue due to the built-in asymmetry of triangles, which encode a 2–morphism together with a factorization of its target 1–morphism, but not of its domain.

In the future continuation of the present work we plan to establish a complete Grothendieck–Lurie correspondence for such fibrations over a fixed base $\mathcal{B}$, showing that they encode the four possible variances of functors $\mathcal{B} \to \mathrm{BiCat}_\infty$. Though we will not arrive to this goal yet here, we do lay the foundations and prove the basic properties we expect to need, based on our current work in progress in that direction. In particular, we prove that equivalences between such fibrations are detected on fibers, and construct a certain universal example in the form of the domain projection $\mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{B}) \to \mathcal{B}$, where $\mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{B})$ is the ∞–bicategory of arrows in $\mathcal{B}$ and lax squares between them. We then establish a comparison between the notions of inner/outer (co)cartesian fibrations and an analogous one in the setting of categories enriched in marked simplicial sets. Such a comparison played an important role in [6] by implying that the $(\mathbb{Z}/2\times\mathbb{Z}/2)$–symmetry of the theory of ∞–bicategories switched between the four types of fibrations. This allowed one to reduce the Grothendieck–Lurie correspondence in [6] to the inner cocartesian case, and we expect a similar role to be played in a future generalization of that correspondence to fibrations.

After finishing the work on the present paper, we became aware of independent work of Abellán García and Stern [1], which investigates the outer variant of these fibrations using a model category structure on marked biscaled simplicial sets. By contrast, our approach here consistently covers all four variances, and also interacts well with the framework we set up in [6], a property we expect would be useful in a future proof of a complete ∞–bicategorical Grothendieck–Lurie correspondence for inner/outer (co)cartesian fibrations. Finally, let us also note that a Grothendieck–Lurie type correspondence for outer cartesian fibrations is sketched by Gaitsgory and Rozenblyum in [9, Appendix], though the argument relies in various parts on unproven statements. The extensive treatment of derived algebraic geometry developed in the body of their volumes [9; 10] based on that appendix makes for a powerful motivation for obtaining the ∞–bicategorical Grothendieck–Lurie correspondence rigorously. We also plan to pursue the study of inner/outer *locally* (co)cartesian fibrations, which encode lax/oplax 2–functors from an ∞–bicategory $\mathcal{B}$ to BiCat_∞, as we defined in [7, Section 3]. This will enable one to compare that last definition with that of [9; 10], which should also allow for the comparison of the two notions of Gray products, thus establishing many key unproven statements involving Gray products made in the appendix of loc. cit. The results of the present work are fundamental preliminaries for all these applications.

Organization of the paper

We begin with a preliminary section where we give pointers to the necessary definitions and results concerning marked and scaled simplicial sets, and recall the framework of inner/outer (co)cartesian fibrations set up in [6], which we rename 1–inner/outer (co)cartesian fibration in order to better distinguish between them and the type of fibrations introduced in the present paper. Next, we introduce the main concepts of this paper, namely those of left/right inner/outer triangles and the corresponding notions of fibrations. We establish fundamental results concerning closure properties of inner/outer triangles, as well as stability properties and a fiberwise criterion to test equivalences of fibrations. Finally, we prove Theorem 3.0.7, which concerns the domain projection on an ∞–bicategory and provides a prototypical example of a 2–outer cartesian fibration.

In Section 4 we briefly recall the theory of (co)cartesian fibrations for ordinary 2–categories, as developed by Buckley in [5], as a motivation for its enhancement to the setting of categories enriched in marked simplicial sets. We then provide an extension of the results of [6, Section 3], by showing that enriched 2–inner/outer (co)cartesian fibrations identify via the coherent nerve functor with the respective notion of fibrations between ∞–bicategories; see Theorem 4.2.4 for the precise statement.

Vistas and applications

A theory of fibrations provides the backbone to define symmetric monoidal $(\infty,2)$–categories, as a suitable class of inner cocartesian fibrations over Fin_*. We expect symmetric monoidal ∞–bicategories to play a fundamental role as their 1–dimensional counterpart, providing extra expressive power thanks to the 2–dimensional structure. For instance, they can be used to encode the relevant dualities of Ind-coherent sheaves of (derived) schemes of finite type, as in [9, Chapter 9].

Furthermore, inner/outer (locally) (co)cartesian fibrations are used as a tool in [9, Chapters 11 and 12] to establish an adjoint theorem that plays a crucial role in extending the quasicoherent sheaves functor from derived affine schemes to derived prestacks; see [9, Chapter 11, Sections 3.1.1 and 3.2.1].

Finally, the theory of fibrations can be used to define a simplicial version of the notion of (relative) $(\infty,2)$–operad; see Batanin [2; 3]. Possible applications in this direction come from symplectic geometry, and in particular from the functoriality of the Fukaya category; see for instance Bottman and Carmeli [4].

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1 Preliminaries

In this section we establish notation and recall some preliminary definitions and results that will be used.

1.1 Marked simplicial sets and enriched categories

Recall that a marked simplicial set is a pair (X, E_X) where X is a simplicial set and E_X is a collection of edges in X, called the *marked* edges, containing all degenerate edges. A map of marked simplicial sets $f\colon (X, E_X) \to (Y, E_Y)$ is a map of simplicial sets $f\colon X \to Y$ satisfying $f(E_X) \subseteq E_Y$. When denoting an explicit marked simplicial set we will often omit the reference to the degenerate edges. For example, we will write $(\Delta^n, \Delta^{\{0,1\}})$ for the marked simplicial set whose underlying simplicial set is the n–simplex and whose marked edges are all the degenerate edges together with the edge $\Delta^{\{0,1\}}$. The category of marked simplicial sets will be denoted by $\mathcal{S}\mathrm{et}_\Delta^+$. It is locally presentable and cartesian closed. For more background on marked simplicial sets we refer the reader to the comprehensive treatment in [12].

We will denote by $\mathcal{S}\mathrm{et}_\Delta^+$–$\mathcal{C}at$ the category of categories in enriched in $\mathcal{S}\mathrm{et}_\Delta^+$ with respect to the cartesian product on $\mathcal{S}\mathrm{et}_\Delta^+$, to which we will refer as *marked simplicial categories*. For a marked simplicial category $\mathcal{C}$ and two objects $x, y \in \mathcal{C}$, we will denote by $\mathcal{C}(x, y) \in \mathcal{S}\mathrm{et}_\Delta^+$ the associated mapping marked simplicial set. By an arrow $e\colon x \to y$ in a marked simplicial category $\mathcal{C}$ we will simply mean a vertex $e \in \mathcal{C}(x, y)_0$ in the corresponding mapping object.

We will generally consider $\mathcal{S}\mathrm{et}_\Delta^+$–$\mathcal{C}at$ together with its associated *Dwyer–Kan model structure*; see [12, Section A.3.2]. In this model structure the weak equivalences are the Dwyer–Kan equivalences, that is, the maps which are essentially surjective on homotopy categories and induce marked categorical equivalences on mapping objects. The fibrant objects are the enriched categories $\mathcal{C}$ whose mapping objects $\mathcal{C}(x, y)$ are all fibrant, that is, ∞–categories marked by their equivalences. The model category $\mathcal{S}\mathrm{et}_\Delta^+$–$\mathcal{C}at$ is then a presentation of the theory of $(\infty, 2)$–categories, and is Quillen equivalent to other known models; see Section 1.2 below.

We say that $\mathcal{E} \in \mathcal{S}\mathrm{et}_\Delta^+$–$\mathcal{C}at$ is a $\mathcal{C}at_\infty$*–category* if it is fibrant in the Dwyer–Kan model structure, ie if it is enriched over ∞–categories, with marking given by equivalences. A *fibration* of $\mathcal{C}at_\infty$–categories is a fibration between fibrant objects in the Dwyer–Kan model structure on $\mathcal{S}\mathrm{et}_\Delta^+$–$\mathcal{C}at$.

1.2 Scaled simplicial sets and ∞–bicategories

Definition 1.2.1 [13] A *scaled simplicial set* is a pair (X, T_X) where X is a simplicial set and T_X is a subset of the set of triangles of X, called *thin triangles*, containing the degenerate ones. A map of scaled simplicial sets $f\colon (X, T_X) \to (Y, T_Y)$ is a map of simplicial sets $f\colon X \to Y$ satisfying $f(T_X) \subseteq T_Y$.

We will denote by $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ the category of scaled simplicial sets. It is locally presentable and cartesian closed. When denoting an explicit scaled simplicial set we will often omit the reference to the degenerate edges. For example, we will write $(\Delta^n, \Delta^{\{0,1,n\}})$ for the scaled simplicial set whose underlying simplicial set is the n–simplex and whose thin triangles are all the degenerate triangles together with the triangle $\Delta^{\{0,1,n\}}$.

Definition 1.2.2 The set of *generating scaled anodyne maps* $\boldsymbol{S}$ is the set of maps of scaled simplicial sets consisting of

(i) the inner horns inclusions
$$(\Lambda^n_i, \{\Delta^{\{i-1,i,i+1\}}\}_{|\Lambda^n_i}) \to (\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\}) \quad \text{for } n \geq 2 \text{ and } 0 < i < n;$$

(ii) the map
$$(\Delta^4, T) \to (\Delta^4, T \cup \{\Delta^{\{0,3,4\}}, \Delta^{\{0,1,4\}}\}),$$
where
$$T \overset{\mathrm{def}}{=} \{\Delta^{\{0,2,4\}}, \Delta^{\{1,2,3\}}, \Delta^{\{0,1,3\}}, \Delta^{\{1,3,4\}}, \Delta^{\{0,1,2\}}\};$$

(iii) the set of maps
$$\Big(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\Big) \to \Big(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\Big) \quad \text{for } n \geq 3.$$

A general map of scaled simplicial set is said to be *scaled anodyne* if it belongs to the weakly saturated closure of $\boldsymbol{S}$.

Definition 1.2.3 An ∞*–bicategory* is a scaled simplicial set $\mathcal{C}$ which admits extensions along all maps in $\boldsymbol{S}$.

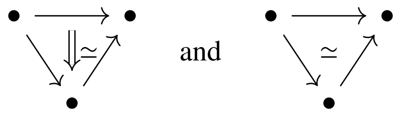
To avoid confusion we point out that scaled simplicial sets as in Definition 1.2.3 are referred to in [13] as *weak ∞–bicategories*, while the term ∞–bicategory was used to designate the stronger property of being fibrant in the *bicategorical model structure* on $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ constructed in loc. cit., whose cofibrations are the monomorphisms, and which serves as a model for the theory of $(\infty,2)$–categories. It is related to the model of marked simplicial categories mentioned above via a Quillen equivalence

$$\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \underset{\mathrm{N}^{\mathrm{sc}}}{\overset{\mathfrak{C}^{\mathrm{sc}}}{\rightleftarrows}} \mathcal{S}\mathrm{et}_\Delta^{+}\text{–}\mathcal{C}at,$$

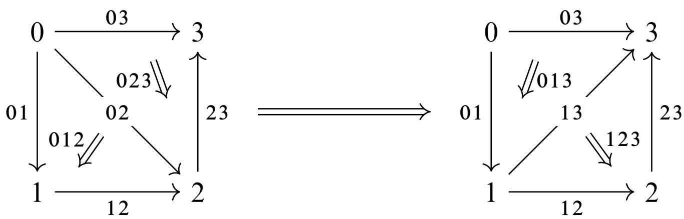
in which the right functor N^{sc} is called as the *scaled coherent nerve*. Nonetheless, as we have shown in [8], the weak and strong notions of ∞–bicategory in fact coincide. In particular, the fibrant objects in the bicategorical model structure can be characterized by the extension property of Definition 1.2.3, and the notion of weak ∞–bicategory will not be further mentioned in the present paper.

Notation 1.2.4 We will refer to the weak equivalences in the bicategorical model structure as *bicategorical weak equivalences*. Since all the objects in the bicategorical model structure are cofibrant, the left Quillen equivalence $\mathfrak{C}^{\mathrm{sc}}$ preserves and detects weak equivalences.

Notation 1.2.5 In a drawing, every 2–simplex filled by a 2–cell with the equivalence symbol, or simply filled by an equivalence symbol such as in the triangles

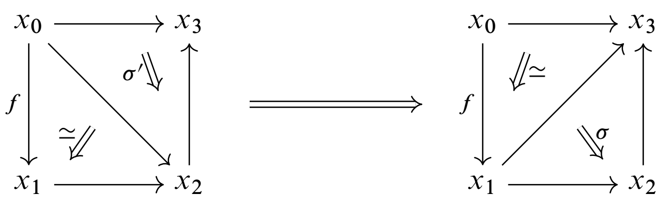


is a thin 2–simplex. As for 3–simplices, we will often draw them as planarized tetrahedra



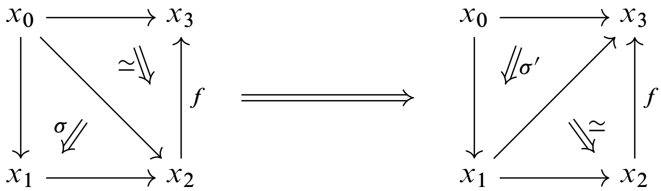
where an additional equivalence symbol can appear in some of the triangles to indicate their thinness.

Definition 1.2.6 Given a 3–simplex $\rho\colon \Delta^3 \to X$ of the form



we will say that ρ exhibits σ' as the *left whiskering* of σ by f.

Similarly, a 3–simplex $\rho\colon \Delta^3 \to X$ of the form



will be said to exhibit σ' as the *right whiskering* of σ by f.

Notation 1.2.7 Let X be a simplicial set. We will denote by $X_\flat = (X, \deg_2(X))$ the scaled simplicial set where the thin triangles of X are the degenerate 2–simplices, and by $X_\sharp = (X, X_2)$ the scaled simplicial set where all the triangles of X are thin. The assignments

$$X \mapsto X_\flat \quad \text{and} \quad X \mapsto X_\sharp$$

are left and right adjoint, respectively, to the forgetful functor $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \to \mathcal{S}\mathrm{et}_\Delta$.

Definition 1.2.8 Given a scaled simplicial set X, we define its *core* to be the simplicial set X^{th} spanned by those n–simplices of X whose 2–dimensional faces are all thin triangles. The assignment $X \mapsto X^{\mathrm{th}}$ is then right adjoint to the functor $(-)_\sharp\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$.

Warning 1.2.9 In [14, Tag 01XA], Lurie uses the term *pith* in place of core, and denotes it by $\mathrm{Pith}(\mathcal{C})$.

Remark 1.2.10 If $\mathcal{C}$ is an ∞–bicategory then its core $\mathcal{C}^{\mathrm{th}}$ is an ∞–category.

Definition 1.2.11 Let $\mathcal{C}$ be an ∞–bicategory. We will say that an edge in $\mathcal{C}$ is *invertible* if it is invertible when considered in the ∞–category $\mathcal{C}^{\mathrm{th}}$, that is, if its image in the homotopy category of $\mathcal{C}^{\mathrm{th}}$ is an isomorphism. We will sometimes refer to invertible edges in $\mathcal{C}$ as *equivalences*. We will denote by $\mathcal{C}^{\simeq} \subseteq \mathcal{C}^{\mathrm{th}}$ the subsimplicial set spanned by the invertible edges. Then $\mathcal{C}^{\simeq}$ is an ∞–groupoid (that is, a Kan complex), which we call the *core groupoid* of $\mathcal{C}$. It can be considered as the ∞–groupoid obtained from $\mathcal{C}$ by discarding all noninvertible 1–cells and 2–cells. If X is an arbitrary scaled simplicial set then we will say that an edge in X is *invertible* if its image in $\mathcal{C}$ is invertible for any bicategorical equivalence $X \to \mathcal{C}$ such that $\mathcal{C}$ is an ∞–bicategory. This does not depend on the choice of the ∞–bicategory replacement $\mathcal{C}$.

Notation 1.2.12 Let $\mathcal{C}$ be an ∞–bicategory and let $x, y \in \mathcal{C}$ be two vertices. In [13, Section 4.2], Lurie gives an explicit model for the mapping ∞-category from x to y in $\mathcal{C}$ that we now recall. Let $\mathrm{Hom}_{\mathcal{C}}(x,y)$ be the marked simplicial set whose n–simplices are given by maps $f\colon \Delta^n \times \Delta^1 \longrightarrow \mathcal{C}$ such that $f_{|\Delta^n \times \{0\}}$ is constant on x, $f_{|\Delta^n \times \{1\}}$ is constant on y, and the triangle $f_{|\Delta^{\{(i,0),(i,1),(j,1)\}}}$ is thin for every $0 \le i \le j \le n$. An edge $f\colon \Delta^1 \times \Delta^1 \longrightarrow \mathcal{C}$ of $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is marked exactly when the triangle $f_{|\Delta^{\{(0,0),(1,0),(1,1)\}}}$ is thin. The assumption that $\mathcal{C}$ is an ∞–bicategory implies that the marked simplicial set $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is *fibrant* in the marked categorical model structure, that is, it is an ∞–category whose marked edges are exactly the equivalences.

Definition 1.2.13 We will denote by $\mathcal{C}at_\infty$ the scaled coherent nerve of the (fibrant) marked simplicial subcategory $(\mathcal{S}\mathrm{et}_\Delta^+)^\circ \subseteq \mathcal{S}\mathrm{et}_\Delta^+$ spanned by the fibrant marked simplicial sets. We will refer to $\mathcal{C}at_\infty$ as the *∞–bicategory of ∞–categories.*

Definition 1.2.14 We define BiCat_∞ to be the scaled coherent nerve of the (large) marked simplicial category BiCat_Δ whose objects are the ∞–bicategories and whose mapping marked simplicial set, for $\mathcal{C}, \mathcal{D} \in \mathrm{BiCat}_\Delta$, is given by $\mathrm{BiCat}_\Delta(\mathcal{C},\mathcal{D}) := \mathrm{Fun}^{\mathrm{th}}(\mathcal{C},\mathcal{D})^\natural$. Here $\mathrm{Fun}^{\mathrm{th}}(\mathcal{C},\mathcal{D})$ is the core ∞–category of the internal hom scaled simplicial set $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, which happens to be an ∞–bicategory whenever $\mathcal{D}$ is (see [13, Proposition 3.1.8 and Lemma 4.2.6]), and by $(-)^\natural$ we mean the associated marked simplicial set in which the marked arrows are the equivalences. We will refer to BiCat_∞ as the *∞–bicategory of ∞–bicategories.*

Since the scaled coherent nerve functor N^{sc} is a right Quillen equivalence, it determines an equivalence

$$(\mathcal{S}\mathrm{et}_\Delta^+\text{–}\mathcal{C}at)_\infty \xrightarrow{\simeq} \mathrm{BiCat}_\infty^{\mathrm{th}} \tag{1}$$

between the ∞–category associated to the model category $\mathcal{S}\mathrm{et}_\Delta^+\text{–}\mathcal{C}at$ and the core ∞–category of BiCat_∞. In the model $\mathcal{S}\mathrm{et}_\Delta^+\text{–}\mathcal{C}at$ the $(\mathbb{Z}/2)^2$–action on the theory of $(\infty,2)$–categories can be realized by an action of $(\mathbb{Z}/2)^2$ on $\mathcal{S}\mathrm{et}_\Delta^+\text{–}\mathcal{C}at$ via model category isomorphisms: the operation $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$, which inverts only the direction of 1–morphisms, is realized by setting $\mathcal{C}^{\mathrm{op}}(x,y) = \mathcal{C}(y,x)$, while the operation $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{co}}$ of inverting only the direction of 2–morphisms is realized by setting $\mathcal{C}^{\mathrm{co}}(x,y) = \mathcal{C}(x,y)^{\mathrm{op}}$, where the right-hand side denotes the operation of taking opposites in marked simplicial sets. Through the equivalence (1) these two involutions induce a $(\mathbb{Z}/2)^2$–action on the core ∞–category $\mathrm{BiCat}_\infty^{\mathrm{th}}$, which we then denote by the same notation. In particular, we have involutions

$$(-)^{\mathrm{op}}\colon \mathrm{BiCat}_\infty^{\mathrm{th}} \to \mathrm{BiCat}_\infty^{\mathrm{th}} \quad \text{and} \quad (-)^{\mathrm{co}}\colon \mathrm{BiCat}_\infty^{\mathrm{th}} \to \mathrm{BiCat}_\infty^{\mathrm{th}},$$

the first inverting the direction of 1–morphisms and the second the direction of 2–morphisms.

Remark 1.2.15 The $(\mathbb{Z}/2)^2$–action on $\mathrm{BiCat}_\infty^{\mathrm{th}}$ does not extend to an action of $(\mathbb{Z}/2)^2$ on the ∞–bicategory BiCat_∞. Instead, it extends to a *twisted* action, that is, $(-)^{\mathrm{co}}$ and $(-)^{\mathrm{op}}$ extend to equivalences of the form

$$(-)^{\mathrm{co}}\colon \mathrm{BiCat}_\infty \xrightarrow{\simeq} \mathrm{BiCat}_\infty \quad \text{and} \quad (-)^{\mathrm{op}}\colon \mathrm{BiCat}_\infty \xrightarrow{\simeq} \mathrm{BiCat}_\infty^{\mathrm{co}}.$$

1.3 Join and slice

In [8, Section 2.2] and [6, Section 2.1] we used join and slice constructions in the setting of marked scaled simplicial sets, that is, simplicial sets X endowed both with a distinguished collection $E_X \subseteq X_1$ of marked edges and a distinguished collection $T_X \subseteq X_2$ of thin triangles. The category of marked scaled simplicial sets will be denoted by $\mathcal{S}\mathrm{et}_\Delta^{+,\mathrm{sc}}$. Though we will need only a limited amount of the generality used in [6], let us recall the construction and terminology used there for the sake of consistency. Given two marked scaled simplicial sets $(X, E_X, T_X), (Y, E_Y, T_Y)$, their join is the *scaled* simplicial set $(X * Y, T_{X*Y})$ whose underlying simplicial set $X * Y$ is the usual join of simplicial sets and the collection of thin triangles is

$$T_{X*Y} := [T_X \times Y_0] \cup [E_X \times E_Y] \cup [X_0 \times T_Y] \subseteq [X_2 \times Y_0] \cup [X_1 \times Y_1] \cup [X_0 \times Y_2] = (X * Y)_2.$$

For a fixed marked scaled simplicial set (Y, E_Y, T_Y), the functor $(X, E_X, T_X) \mapsto (X * Y, E_{X*Y})$ is a colimit-preserving functor from marked scaled simplicial sets to scaled simplicial sets under (Y, T_Y), and admits a right adjoint, that is, an associated slice construction. Given a marked scaled simplicial set (K, E_K, T_K) and a map of scaled simplicial set $p: (K, T_K) \to (Z, T_Z)$, we will denote by $(Z, T_Z)_{/p}$, or simply $Z_{/p}$ for brevity, the valued of this right adjoint at p. In particular, $Z_{/p}$ is the marked scaled simplicial set characterized by the property that maps of marked scaled simplicial sets $(X, E_X, T_X) \to Z_{/p}$ correspond to maps of scaled simplicial sets $(X * K, T_{X*K}) \to (Z, T_Z)$ under (K, E_K, T_K). We then write $\overline{Z}_{/p}$ for the *underlying scaled simplicial set* of $Z_{/p}$, obtained by forgetting the marking. To avoid confusion, let us emphasize that the thin triangles in $\overline{Z}_{/p}$ do depend on the marking E_K of K, and not only on the map of scaled simplicial sets p. Similarly, we denote by $Z_{p/}$ the marked scaled simplicial set representing the functor

$$(X, E_X, T_X) \mapsto \mathrm{Map}_{(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}})_{(K,T_K)/}}((K * X, T_{K*X}), T_{X*K}, (Z, T_Z)),$$

and by $\overline{Z}_{p/}$ its underlying scaled simplicial set.

We will mostly be interested in the case where the target Z is an ∞–bicategory $\mathcal{C}$, and K is either Δ^0 or ${}^\sharp\Delta^1$, the latter being the 1–simplex endowed with the maximal marking and the (unique) trivial scaling. In the latter case we will sometimes make use of [6, Notation 2.3.4], which we now recall for the convenience of the reader.

Notation 1.3.1 Given a scaled simplicial set (Z, T_Z) and an edge $e: x \to y$ in Z, we will denote by $Z_{/e^\sharp} \in \mathcal{S}\mathrm{et}_\Delta^{+,\mathrm{sc}}$ the result of the slice construction above applied to the marked scaled simplicial set ${}^\sharp\Delta^1$ and the map of scaled simplicial sets $\Delta^1_\flat \to (Z, T_Z)$ determined by e. Explicitly, the set of n–simplices in $Z_{/e^\sharp}$ is given by

$$(Z_{/e^\sharp})_n \overset{\text{def}}{=} \{\alpha: {}^\flat\Delta^n * {}^\sharp\Delta^1 \to Z \mid \alpha_{|\Delta^{\{n+1,n+2\}}} = e\},$$

where ${}^\flat\Delta^n$ denotes the n–simplex with minimal marking and minimal scaling. The marked edges of $Z_{/e^\sharp}$ are those which factor through ${}^\sharp\Delta^1 * {}^\sharp\Delta^1$, and the thin triangles are those which factor through $(\Delta^2)^\flat_\sharp * {}^\sharp\Delta^1$, where $(\Delta^2)^\flat_\sharp$ is the 2–simplex with minimal marking and maximal scaling, that is, its unique nondegenerate triangle is thin. As above, we will write $\overline{Z}_{/e^\sharp}$ for the underlying scaled simplicial set of $Z_{/e^\sharp}$.

When $K = \Delta^0$ the map p corresponds to a vertex $y \in \mathcal{C}$, and we will denote the associated marked scaled slice by $\mathcal{C}_{/y}$. The fiber $(\mathcal{C}_{/y})_x$ of the projection $\mathcal{C}_{/y} \to \mathcal{C}$ over a vertex x of $\mathcal{C}$ is then a marked scaled simplicial set all of whose triangles are thin. Its underlying marked simplicial set, denoted by $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)$ in [7, Section 2.3], is fibrant in the marked categorical model structure, so that one can treat it as an ∞–category (marked by its equivalences). As such, it is a model for the mapping ∞–category of $\mathcal{C}$; see [8, Proposition 2.23]. We then write $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{C}}(x,y)$ for the underlying simplicial set of $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)$.

1.4 1–Inner/outer (co)cartesian fibrations

The theory of inner and outer (co)cartesian fibrations of ∞–bicategories was developed in [6] as an analogue of the usual notion of (co)cartesian fibrations of ∞–categories. As in the latter case, such a fibration encodes the data of a family of ∞–categories functorially parametrized by the base $\mathcal{B}$, only that in the ∞–bicategorical setting there are four different variance flavors for this functorial dependence one can consider. Specifically, inner (resp. outer) cocartesian fibrations encode a covariant dependence on the 1–morphisms of $\mathcal{B}$ and a covariant (resp. contravariant) dependence on the level of 2–morphisms. Similarly, inner (resp. outer) cartesian fibrations encode a contravariant dependence on the 1–morphisms of $\mathcal{B}$ and a contravariant (resp. covariant) dependence on the level of 2–morphisms.

Below we recall the main definitions. We refer the reader to loc. cit. for a comprehensive treatment.

Definition 1.4.1 We will say that a map of scaled simplicial sets $X \to Y$ is a *weak fibration* if it has the right lifting property with respect to the following types of maps:

(i) All scaled inner horn inclusions of the form

$$(\Lambda^n_i, \{\Delta^{\{i-1,i,i+1\}}\}_{|\Lambda^n_i}) \subseteq (\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\}) \quad \text{for } n \geq 2 \text{ and } 0 < i < n.$$

(ii) The scaled horn inclusions of the form

$$\Big(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}_{|\Lambda^n_0}\Big) \subseteq \Big(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\Big) \quad \text{for } n \geq 2.$$

(iii) The scaled horn inclusions of the form

$$\Big(\Lambda^n_n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0, \{\Delta^{\{0,n-1,n\}}\}_{|\Lambda^n_n}\Big) \subseteq \Big(\Delta^n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0, \{\Delta^{\{0,n-1,n\}}\}\Big) \quad \text{for } n \geq 2.$$

Remark 1.4.2 The maps appearing in Definition 1.4.1 are all trivial cofibrations with respect to the bicategorical model structure. This means that any bicategorical fibration $\mathcal{E} \to \mathcal{B}$ is in particular a weak fibration. For example, if $\mathcal{E}$ is an ∞–bicategory then the terminal map $\mathcal{E} \to \Delta^0$ is a weak fibration.

Definition 1.4.3 Given a weak fibration $f : X \to Y$, we will say that f is

- a 1–*inner fibration* if it detects thin triangles and the underlying map of simplicial sets is an inner fibration, that is, satisfies the right lifting property with respect to inner horn inclusions;

- a 1–*outer fibration* if it detects thin triangles and the underlying map of simplicial sets satisfies the right lifting property with respect to the inclusions

$$\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \subseteq \Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0 \quad \text{and} \quad \Lambda^n_n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0 \subseteq \Delta^n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0 \quad \text{for } n \geq 2.$$

Note that the collection of 1–inner fibrations is closed under the $(-)^{\mathrm{op}}$ duality, and the same holds for the collection of 1–outer fibrations.

Remark 1.4.4 In [8] and [6] we used the terms inner and outer fibration for what we called above 1–inner and 1–outer fibrations, respectively. The reason for the terminology update is the desire to more clearly distinguish between these notions and those of 2–inner and 2–outer fibrations introduced in the present paper. In principle, 1–inner and 1–outer fibrations between ∞–bicategories are the functors which induce right and left fibrations, respectively, on the level of mapping ∞–categories. The notions of 2–inner and 2–outer fibration between ∞–bicategories correspond in turn to functors which induce cartesian and cocartesian fibrations on the level of mapping ∞–categories, together with the condition that composition of arrows preserves cocartesian edges; see Section 4.2.

Remark 1.4.5 In [14, Tag 01WF], Lurie uses the term *interior fibration* to encode what we just defined as 1–outer fibrations. Our choice in [6] (which already appeared in [8]) is motivated by the intent of highlighting that *special outer horns* admit fillers against such maps.

Definition 1.4.6 Let $p\colon X \to Y$ be a weak fibration. We will say that an edge $e\colon \Delta^1 \to X$ is *p–cartesian* if the dotted lift exists in any diagram of the form

$$\begin{array}{ccc} (\Lambda^n_n, \{\Delta^{\{0,n-1,n\}}\}_{|\Lambda^n_n}) & \xrightarrow{\sigma} & (X, T_X) \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ (\Delta^n, \{\Delta^{\{0,n-1,n\}}\}) & \longrightarrow & (Y, T_Y) \end{array}$$

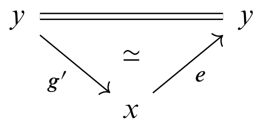
with $n \geq 2$ and $\sigma_{|\Delta^{\{n-1,n\}}} = e$. We will say that e is *p–cocartesian* if $e^{\mathrm{op}}\colon \Delta^1 \to X^{\mathrm{op}}$ is p^{op}–cartesian.

As in [6, Definition 2.3.1], we will also say that the edge $e\colon \Delta^1 \to X$ is *strongly* f*–(co)cartesian* if it is a (co)cartesian edge with respect to the underlying map of simplicial sets.

Remark 1.4.7 If $p\colon \mathcal{E} \to \mathcal{B}$ is weak fibration between ∞–bicategories, then any equivalence in $\mathcal{E}$ is both p–cartesian and p–cocartesian, see [6, Corollary 2.3.10]. On the other hand, if $e\colon x \to y$ is either a p–cartesian or p–cocartesian edge in $\mathcal{E}$ such that pe is an equivalence in $\mathcal{B}$ then e is an equivalence. To see this, let $g\colon y \to x$ be an inverse to pe in $\mathcal{B}$, equipped with thin triangles of the following forms:

$$\begin{array}{ccc} px & \xlongequal{\quad} & px \\ & {\scriptstyle pe}\searrow \;\simeq\; \nearrow{\scriptstyle g} & \\ & y & \end{array} \qquad \begin{array}{ccc} py & \xlongequal{\quad} & py \\ & {\scriptstyle g}\searrow \;\simeq\; \nearrow{\scriptstyle pe} & \\ & px & \end{array}$$

If e is p–cartesian then we can lift the right-hand side triangle to a triangle in $\mathcal{E}$ of the form



producing in particular a right inverse g' to e. But then g' is also p–cartesian by [6, Lemma 2.3.8], and since $pg' = g$ is invertible in $\mathcal{B}$ we deduce from the same argument that g' also has a right inverse in $\mathcal{E}$. It then follows from standard arguments (which can be applied on the level of the core ∞–category $\mathcal{C}^{\mathrm{th}}$) that e and g' are homotopy inverses, and in particular e is an equivalence. If e is assumed instead to be p–cocartesian then one proceeds in the same manner by lifting the left-hand side triangle to $\mathcal{E}$.

Remark 1.4.8 If $p\colon \mathcal{E} \to \mathcal{B}$ is a weak fibration and $e, e'\colon \Delta^1 \to \mathcal{E}$ are two arrows which are equivalent in $\mathrm{Fun}(\Delta^1, \mathcal{E})$, then e is p–cartesian if and only if e' is. To see this, note that in this case one has both an equivalence going from e to e' and an equivalence going from e' to e, and so it will suffice to show that if we have an equivalence $e \Rightarrow e'$ and e' is p–cartesian, then e is p–cartesian. Indeed, such an equivalence is given by a map $H\colon \Delta^1_\flat \times \Delta^1_\flat \to \mathcal{E}$ such that $H|_{\Delta^1_\flat \times \Delta^{\{0\}}} = e$ and $H|_{\Delta^1_\flat \times \Delta^{\{1\}}} = e'$. Since p–cartesian edges are closed under composition [6, Lemma 2.3.9] and every equivalence is p–cartesian (Remark 1.4.7), we get that H sends the diagonal edge $\Delta^{\{(0,0),(1,1)\}} \subseteq \Delta^1 \times \Delta^1$ to a p–cartesian edge. Then, from the partial two-out-of-three property for p–cartesian edges of [6, Lemma 2.3.8], we get that e is p–cartesian.

Passing to opposites, we also obtain from this argument that e is p–cocartesian if and only if e' is. In addition, if p is a 1–outer fibration then by [6, Proposition 2.3.7] the collection of strongly p–(co)cartesian arrows coincides with that of p–(co)cartesian arrows, and hence e is strongly p–(co)cartesian if and only if e' is so — alternatively, when p is a 1–outer fibration the statements of [6, Lemmas 2.3.8 and 2.3.9] also apply to strongly p–(co)cartesian arrows, so that the above argument can simply be carried out verbatim.

Definition 1.4.9 Let $f\colon X \to Y$ be a weak fibration of scaled simplicial sets. We will say that f is a *cartesian fibration* if for every $x \in X$ and an edge $e\colon y \to f(x)$ in Y there exists a f–cartesian edge $\widetilde{e}\colon \widetilde{y} \to x$ such that $f(\widetilde{e}) = e$. Dually, we will say that $f\colon X \to Y$ is a *cocartesian fibration* if $f^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is a cartesian fibration.

Definition 1.4.10 Let $f\colon X \to Y$ be a weak fibration of scaled simplicial sets. We will say that f is a 1–inner (resp. 1–outer) cartesian fibration if it is both a 1–inner (resp. 1–outer) fibration and a cartesian fibration. Dually, we will say that f is a 1–inner (resp. 1–outer) cocartesian fibration if p^{op} is a 1–inner (resp. 1–outer) cartesian fibration.

Remark 1.4.11 The classes of weak fibrations, 1–inner/outer fibrations and 1–inner/outer (co)cartesian fibrations are all closed under base change.

Example 1.4.12 If $\mathcal{C}$ is an ∞–bicategory, then the projection $\overline{\mathcal{C}}_{/x} \to \mathcal{C}$ is an example of a 1–outer cartesian fibration, where the cartesian edges are exactly those whose corresponding triangle in $\mathcal{C}$ is thin (equivalently, those corresponding to marked edges in $\mathcal{C}_{/x}$); see [6, Corollary 2.4.7]. Similarly, $\overline{\mathcal{C}}_{x/} \to \mathcal{C}$ is a 1–outer cocartesian fibration, with the cocartesian edges again corresponding to thin triangles. More generally, for any map of scaled simplicial sets $p\colon (K, T_K) \to \mathcal{C}$, the associated slice projections $\overline{\mathcal{C}}_{/p} \to \overline{\mathcal{C}}$ and $\overline{\mathcal{C}}_{p/} \to \mathcal{C}$ are 1–outer cartesian and cocartesian fibrations, respectively.

Proposition 1.4.13 *Any* (*co*)*cartesian fibration between* ∞*–bicategories is a fibration in the bicategorical model structure on* $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$.

Proof We prove the cartesian case, from which the cocartesian case can be deduced by passing to opposites. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration between ∞–bicategories. To show that p is a bicategorical fibration we need to produce the dotted lift in any square of the form

(2)
$$\begin{array}{ccc} K & \xrightarrow{f} & \mathcal{E} \\ \downarrow & \nearrow & \downarrow \\ L & \xrightarrow{g} & \mathcal{B} \end{array}$$

such that $K \to L$ is a bicategorical trivial cofibration of scaled simplicial sets. Since $\mathcal{E}$ is an ∞–bicategory it is in particular fibrant in the bicategorical model structure (see discussion in Section 1.2), and hence we can extend f to a map $h\colon L \to \mathcal{E}$. This is not yet a solution to the above lifting problem since the composite ph might be different from g. The two maps ph and g agree however on K by construction. Since the bicategorical model structure is cartesian closed and $\mathcal{B}$ is fibrant, we may solve the lifting problem

$$\begin{array}{ccc} [K \times \Delta^1_\flat] \coprod_{K \times \partial\Delta^1} [L \times \partial\Delta^1] & \longrightarrow & \mathcal{B} \\ \downarrow & \nearrow & \\ L \times \Delta^1_\flat & & \end{array}$$

yielding a natural transformation $H\colon L \times \Delta^1_\flat \to \mathcal{B}$ from g and ph which is constant on K. Since $K \to L$ is a trivial cofibration and $\mathcal{B}$ is fibrant the induced functor $\mathrm{Fun}(L, \mathcal{B}) \to \mathrm{Fun}(K, \mathcal{B})$ is an equivalence of ∞–bicategories, and since the arrow in $\mathrm{Fun}(L, \mathcal{B})$ associated to H maps to an identity arrow in $\mathrm{Fun}(K, \mathcal{B})$ we deduce that it must be invertible in $\mathrm{Fun}(L, \mathcal{B})$. In particular, the restriction of H to $\{l\} \times \Delta^1_\flat$ is an invertible arrow of $\mathcal{B}$ for every vertex $l \in L$. Now since p is a cartesian fibration we can choose for each $l \in L_0 \setminus K_0$ a p–cartesian lift $e_l\colon \{l\} \times \Delta^1_\flat \to \mathcal{E}$ of $H|_{\{l\} \times \Delta^1_\flat}$. Then each e_l is a p–cartesian lift of an equivalence, and is hence itself an equivalence, see Remark 1.4.7. Let $L' \subseteq L$ be the scaled simplicial subset whose underlying simplicial set is that of L and whose thin triangles are only those thin triangles which are contained in K. Applying [8, Proposition 2.38] we may solve the lifting problem

$$\begin{array}{ccc} [K \times \Delta^1_\flat] \coprod_{K \times \Delta^{\{1\}}} [L' \times \Delta^{\{1\}}] & \longrightarrow & \mathcal{E} \\ \downarrow & \overset{G}{\nearrow} & \downarrow \\ L' \times \Delta^1_\flat & \xrightarrow{H} & \mathcal{B} \end{array}$$

to yield a natural transformation $G\colon L'\times\Delta^1_\flat\to\mathcal{E}$ such that $G|_{\{l\}\times\Delta^1_\flat}=e_l$ for every $l\in L_0\setminus K_0$, where we point out that the assumption made in [8, Proposition 2.38] that p detects thin triangles is not needed since L' does not contain any thin triangles that are not in K, see [8, Remark 2.40]. In particular, G is a levelwise invertible natural transformation. We wish to show that $G|_{L'\times\Delta^{\{0\}}}$ extends to $L\times\Delta^{\{0\}}$, thus providing a solution to the original lifting problem (2). Indeed, if $\sigma\colon\Delta^2_\flat\to L'$ is a triangle that is thin in L then the composed map

$$H_\sigma\colon\Delta^2_\flat\times\Delta^1_\flat\xrightarrow{\sigma\times\mathrm{id}}L'\times\Delta^1_\flat\xrightarrow{G}\mathcal{E}$$

is a levelwise invertible natural transformation between two triangles, one of which is thin (since $G|_{L'\times\Delta^{\{1\}}}$ extends to $L\times\Delta^1_\flat$ by construction), and hence the other one is thin as well by [8, Corollary 3.5]. □

Over a fixed base $\mathcal{B}$, the collection of 1–inner cocartesian fibrations, and cocartesian edges preserving functors between them, can be organized into an ∞–bicategory $\mathrm{coCar}^{\mathrm{inn}}(\mathcal{B})$. This ∞–bicategory can be presented by a suitable model structure on the category of marked simplicial sets over the underlying simplicial set of $\mathcal{B}$, developed in [13, Section 3.2] using the machinery of categorical patterns. Lurie then constructs in loc. cit. a straightening-unstraightening Quillen equivalence between this model structure and the projective model structure on $\mathrm{Fun}(\mathfrak{C}^{\mathrm{sc}}(\mathcal{B}),\mathcal{S}\mathrm{et}^+_\Delta)$. In [6] we used this to establish the following ∞–bicategorical form of the Grothendieck–Lurie correspondence, for both cartesian and cocartesian, inner and outer flavors of fibrations:

Theorem 1.4.14 [6, Corollary 3.3.3] *For an ∞–bicategory $\mathcal{B}\in\mathrm{BiCat}_\infty$ there are natural equivalences of ∞–bicategories*

$$\mathrm{coCar}^{\mathrm{inn}}(\mathcal{B})\simeq\mathrm{Fun}(\mathcal{B},\mathcal{C}at_\infty),\qquad \mathrm{coCar}^{\mathrm{out}}(\mathcal{B})\simeq\mathrm{Fun}(\mathcal{B}^{\mathrm{co}},\mathcal{C}at_\infty),$$
$$\mathrm{Car}^{\mathrm{inn}}(\mathcal{B})\simeq\mathrm{Fun}(\mathcal{B}^{\mathrm{coop}},\mathcal{C}at_\infty),\qquad \mathrm{Car}^{\mathrm{out}}(\mathcal{B})\simeq\mathrm{Fun}(\mathcal{B}^{\mathrm{op}},\mathcal{C}at_\infty).$$

2 2–Inner/outer cartesian fibrations

In this section we will define the principal notion of this paper, namely that of 2–inner/outer (co)cartesian fibrations, and study their basic properties.

2.1 Inner and outer triangles

Definition 2.1.1 Let $p\colon\mathcal{E}\to\mathcal{B}$ be a weak fibration of ∞–bicategories and $\sigma\colon\Delta^2\to\mathcal{E}$ a triangle.

- We will say that σ is *left p–inner* if the corresponding arrow in $\overline{\mathcal{E}}_{/\sigma(2)}$ is strongly cartesian with respect to the projection $\overline{\mathcal{E}}_{/\sigma(2)}\to\mathcal{E}\times_{\mathcal{B}}\overline{\mathcal{B}}_{/p\sigma(2)}$.
- We will say that σ is *right p–inner* if the corresponding arrow in $\overline{\mathcal{E}}_{\sigma(0)/}$ is strongly cocartesian with respect to the projection $\overline{\mathcal{E}}_{\sigma(0)/}\to\mathcal{E}\times_{\mathcal{B}}\overline{\mathcal{B}}_{p\sigma(0)/}$.
- We will say that σ is *left p–outer* if the corresponding arrow in $\overline{\mathcal{E}}_{/\sigma(2)}$ is strongly cocartesian with respect to the projection $\overline{\mathcal{E}}_{/\sigma(2)}\to\mathcal{E}\times_{\mathcal{B}}\overline{\mathcal{B}}_{/p\sigma(2)}$.

- We will say that σ is *right p–outer* if the corresponding arrow in $\overline{\mathcal{E}}_{\sigma(0)/}$ is strongly cartesian with respect to the projection $\overline{\mathcal{E}}_{\sigma(0)/} \to \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{p\sigma(0)/}$.

Remark 2.1.2 Unwinding the definitions, we see that σ is left p–inner if and only if, for $n \geq 3$, every commutative square of the form

$$\begin{array}{ccc} & \Delta^{\{n-2,n-1,n\}} & \\ \swarrow & & \searrow^{\sigma} \\ \Lambda^n_{n-1} & \longrightarrow & \mathcal{E} \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & \mathcal{B} \end{array} \tag{3}$$

admits a diagonal filler as displayed by the dotted arrow, and right p–inner if the same holds for diagrams as above where Λ^n_{n-1} is replaced by Λ^n_1 and $\Delta^{\{n-2,n-1,n\}}$ by $\Delta^{\{0,1,2\}}$. On the other hand, σ is left p–outer if and only if every commutative square of the form

$$\begin{array}{ccc} & \Delta^{\{0,1,n\}} & \\ \swarrow & & \searrow^{\sigma} \\ \Lambda^n_{0} & \longrightarrow & \mathcal{E} \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & \mathcal{B} \end{array} \tag{4}$$

admits a diagonal filler as displayed by the dotted arrow, and right p–outer if and only if the same holds for diagrams as above where Λ^n_0 is replaced by Λ^n_n and $\Delta^{\{0,1,n\}}$ by $\Delta^{\{0,n-1,n\}}$.

Remark 2.1.3 It follows from Remark 2.1.2 that any thin triangle in $\mathcal{E}$ is both left and right p–inner. On the other hand, any thin triangle whose left leg is p–cocartesian is left p–outer and any thin triangle whose right leg p–cartesian is right p–outer.

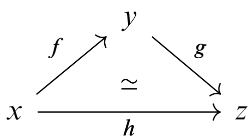
Remark 2.1.3 admits a type of a converse statement:

Proposition 2.1.4 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a weak fibration of ∞–bicategories. Suppose $\sigma\colon \Delta^2 \to \mathcal{E}$ is a triangle such that $p(\sigma)$ is thin in $\mathcal{B}$. If σ is either left or right p–inner then σ is thin in $\mathcal{E}$. The same holds if we assume that σ is left p–outer and left-degenerate or that σ is right p–outer and right-degenerate.*

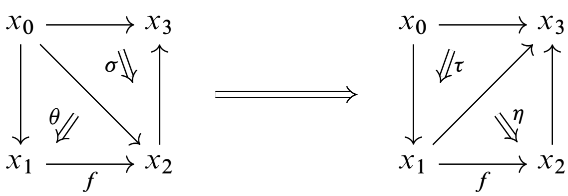
Proof Write $x = \sigma(0), z = \sigma(2)$. Suppose first that σ is left p–inner. The condition that $p(\sigma)$ is thin means that the arrow determined by $p(\sigma)$ in $\overline{\mathcal{B}}_{/pz}$ is cartesian with respect to the projection $\overline{\mathcal{B}}_{/pz} \to \mathcal{B}$; see Example 1.4.12. By base change it then follows that the arrow determined by σ in $\mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/pz}$ is cartesian with respect to the projection $\mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/pz} \to \mathcal{E}$. At the same time, the arrow determined by σ in $\overline{\mathcal{E}}_{/z}$ is cartesian with respect to the projection $\overline{\mathcal{E}}_{/z} \to \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/pz}$ (by the definition of being left p–inner) and so we conclude that the arrow determined by σ in $\overline{\mathcal{E}}_{/z}$ is cartesian with respect to the composed projection $\overline{\mathcal{E}}_{/z} \to \mathcal{E}$. By Example 1.4.12 we then get that σ is thin. The dual argument using $\overline{\mathcal{E}}_{x/}$ and $\overline{\mathcal{B}}_{px/}$ applies to the case where σ is right p–inner.

Now suppose that σ is left p–outer and left-degenerate. Then $p\sigma$ is left-degenerate and since $p\sigma$ is assumed thin it follows that the arrow in $\overline{\mathcal{B}}_{/pz}$ determined by $p(\sigma)$ is invertible (indeed, by Example 1.4.12 it is a cocartesian arrow with respect to the 1–outer fibration $\overline{\mathcal{B}}_{/pz} \to \overline{\mathcal{B}}$ lying over an equivalence). Since σ is left-degenerate we then have that the arrow determined by σ in $\mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/pz}$ is invertible as well. Since σ is left p–outer it now follows that the arrow in $\overline{\mathcal{E}}_{/z}$ determined by σ is a cocartesian lift of an equivalence along the 1–outer fibration $\overline{\mathcal{E}}_{/z} \to \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/pz}$, and is hence itself an invertible arrow $\overline{\mathcal{E}}_{/z}$. As such, this arrow is in particular cocartesian with respect to the projection $\overline{\mathcal{E}}_{z/} \to \mathcal{E}$ (Remark 1.4.7), and so we conclude that σ is thin by Example 1.4.12. The dual argument using $\overline{\mathcal{E}}_{x/}$ and $\overline{\mathcal{B}}_{px/}$ applies to the case where σ is right p–outer and right-degenerate. □

Remark 2.1.5 By [6, Lemma 2.3.9 and Lemma 2.3.8] the collection of strongly (co)cartesian arrows in a given 1–outer fibration is closed under composition and has a partial two-out-of-three property. More precisely, if $\mathcal{C} \to \mathcal{D}$ is a 1–outer fibration of ∞–bicategories and



is a thin triangle in $\mathcal{C}$ such that g is strongly cartesian then f is strongly cartesian if and only if h is strongly cartesian. Dually, if f is strongly cocartesian then g is strongly cocartesian if and only if h is strongly cocartesian. Now for any weak fibration $p\colon \mathcal{E} \to \mathcal{B}$ of ∞–bicategories, both $\overline{\mathcal{E}}_{x/} \to \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{x/}$ and $\overline{\mathcal{E}}_{/x} \to \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/x}$ are 1–outer fibrations for every $x \in \mathcal{E}$ by Remark 1.4.11 and Example 1.4.12. The above partial two-out-of-three property for strongly (co)cartesian edges then translates to a certain partial two-out-of-three property for inner/outer triangles. More precisely, suppose given a 3–simplex $\rho\colon \Delta^3 \to \mathcal{E}$ of the form



If θ is thin then we may consider ρ as encoding a thin triangle in $\overline{\mathcal{E}}_{/x_3}$ exhibiting the edge associated to σ as the composite of those associated to τ and η, whereas if η is thin then we may consider ρ as encoding thin a triangle in $\overline{\mathcal{E}}_{x_0/}$ exhibiting the edge associated to τ as the composite of those associated to θ and σ. We hence conclude the following:

(i) If θ is thin and η is left p–inner then τ is left p–inner if and only if σ is.

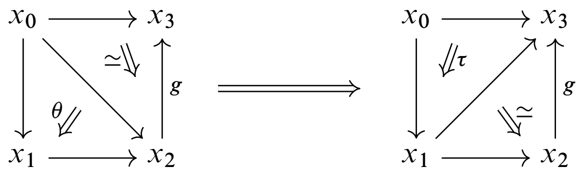
(ii) If η is thin and θ is right p–inner then σ is right p–inner if and only if τ is.

(iii) If θ is thin and τ is left p–outer then η is left p–outer if and only if σ is.

(iv) If η is thin and σ is right p–outer then θ is right p–outer if and only if τ is.

Combining this with Remark 2.1.3, we conclude that the collection of left p–inner triangles in $\mathcal{E}$ is closed under left whiskering with 1–morphisms and the collection of right p–inner triangles is closed under right whiskering with 1–morphisms. On the other hand, the collection of left p–outer triangles is only closed under left whiskering with p–cocartesian 1–morphisms and the collection of right p–outer triangles is only closed under right whiskering with p–cartesian arrows.

Combining Remark 2.1.5(4) with Remark 2.1.3 we get that if



is a 3–simplex such that g is p–cartesian then θ is right p–outer if and only if τ is. The "only if" direction of this implication also holds for right p–inner triangles by Remark 2.1.5(2) and Remark 2.1.3. The following lemma shows that on the other hand, the "if" direction of this implication actually holds for left p–inner and left p–outer triangles:

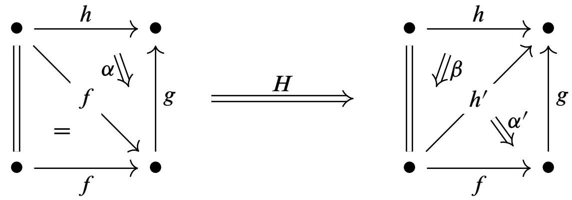
Lemma 2.1.6 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a weak fibration. Given a 3–simplex $\rho\colon \Delta^3 \to \mathcal{E}$ as above with g being p–cartesian, if τ is left p–inner or left p–outer then so is θ.*

Proof Consider the commutative diagram

$$\begin{array}{ccccc}
\overline{\mathcal{E}}_{/x_2} & \xleftarrow{\simeq} & \overline{\mathcal{E}}_{/g^\sharp} & \longrightarrow & \overline{\mathcal{E}}_{/x_3} \\
\downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_{2,3}} & & \downarrow{\scriptstyle q_3} \\
\overline{\mathcal{E}} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/px_2} & \xleftarrow{\simeq} & \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/pg^\sharp} & \longrightarrow & \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/px_3} \\
 & & \downarrow & & \downarrow \\
 & & \overline{\mathcal{B}}_{/pg^\sharp} & \longrightarrow & \overline{\mathcal{B}}_{/px_3}
\end{array}$$

in which the left-pointing horizontal arrows are trivial fibrations by [6, Lemma 2.4.6]. Now since the two faces of ρ leaning on g are thin we have that the 3–simplex ρ determines an arrow e in $\overline{\mathcal{E}}_{/g^\sharp}$, whose image in $\overline{\mathcal{E}}_{/x_2}$ is the arrow associated to θ. We conclude that θ is left p–inner (resp. left p–outer) if and only if e is $q_{2,3}$–cartesian (resp. $q_{2,3}$–cocartesian). Now the image of e in $\overline{\mathcal{E}}_{/x_3}$ is the arrow associated to τ, and since τ is assumed to be left p–inner (resp. left p–outer) its associated arrow is q_3–cartesian (resp. q_3–cocartesian) in $\overline{\mathcal{E}}_{/x_3}$. At the same time, the assumption that g is p–cartesian implies that the vertical external square on the right is homotopy cartesian by [6, Lemma 2.3.6], and since the bottom right square is also cartesian we get from the pasting lemma that the top right square is homotopy cartesian. We hence conclude that e is $q_{2,3}$–cartesian (resp. $q_{2,3}$–cocartesian), as desired. □

Remark 2.1.7 Inner and outer lifts in a given weak fibration are unique up to equivalence (once they exist). For example, suppose we have two left p–inner triangles α, α' in $\mathcal{E}$ whose restrictions to Λ^2_1 and whose images under p coincide. By Remark 2.1.2 (applied for σ' and $n = 3$) we can find a 3–simplex H of the form



where β lives over a degenerate triangle in $\mathcal{B}$. By Remark 2.1.5 we then have that β is left p–inner as well, and hence thin by Proposition 2.1.4. We may consider H as exhibiting an equivalence between α and α': its leftmost leg is invertible, the two faces leaning on this leg are thin, and the remaining two faces are α and α'. In a very similar manner, if α and α' are assumed instead to be right p–outer and such that their restrictions to Λ^2_2 and images under p coincide, then we construct the same type of 3–simplex H, only that this time we will take its $\Delta^{\{0,1,3\}}$–face to be degenerate and extend H from its right outer horn using the right p–outerness of α. Replacing p with $p^{\mathrm{op}} \colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ we get the analogous statements for the uniqueness of *right* inner and *left* outer lifts.

Remark 2.1.8 Let $p \colon \mathcal{E} \to \mathcal{B}$ be a weak fibration of ∞–bicategories. For given $x, z \in \mathcal{E}$, if we base change the weak fibration $\overline{\mathcal{E}}_{/z} \to \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/pz}$ along the map $\{x\} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/pz} \to \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/pz}$ then we get the map of maximally scaled simplicial sets, whose underlying map of simplicial sets is

$$p^{\triangleright}_{x,z} \colon \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}}(x, z) \to \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}}(px, pz),$$

which is a model for the induced map on mapping ∞–categories by [8, Section 2.3]; see in particular [8, Proposition 2.23]. Since base change maps detect (co)cartesian edges, it follows that for a triangle $\sigma \colon \Delta^2 \to \mathcal{E}$ such that $\sigma|_{\Delta^{\{0,1\}}}$ is degenerate, we have:

- If σ is left p–inner then the corresponding edge of $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}}(x, z)$ is $p^{\triangleright}_{x,z}$–cartesian.
- If σ is left p–outer then the corresponding edge of $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}}(x, z)$ is $p^{\triangleright}_{x,z}$–cocartesian.

2.2 2–Inner and 2–outer fibrations

Definition 2.2.1 Let $p \colon \mathcal{E} \to \mathcal{B}$ be a weak fibration and $\sigma \colon \Delta^2 \to \mathcal{B}$ a triangle.

- We say that σ has a *sufficient supply of left* (*resp. right*) *p–inner lifts* if for every $\rho \colon \Lambda^2_1 \to \mathcal{E}$ lifting $\sigma|_{\Lambda^2_1}$ there exists a left (resp. right) p–inner triangle $\tau \colon \Delta^2 \to \mathcal{E}$ such that $p\tau = \sigma$ and $\tau|_{\Lambda^2_1} = \rho$.
- We say that σ has a *sufficiently supply of left p–outer lifts* if for every $\rho \colon \Lambda^2_0 \to \mathcal{E}$ lifting $\sigma|_{\Lambda^2_0}$ and such that $\rho|_{\Delta^{\{0,1\}}}$ is p–cocartesian, there exists a left p–outer triangle $\tau \colon \Delta^2 \to \mathcal{E}$ such that $p\tau = \sigma$ and $\tau|_{\Lambda^2_0} = \rho$.

- We say that σ has a *sufficient supply of right* p*–outer lifts* if for every $\rho\colon \Lambda^2_2 \to \mathcal{E}$ lifting $\sigma|_{\Lambda^2_0}$ and such that $\rho|_{\Delta^{\{1,2\}}}$ is p–cartesian, there exists a right p–outer triangle $\tau\colon \Delta^2 \to \mathcal{E}$ such that $p\tau = \sigma$ and $\tau|_{\Lambda^2_2} = \rho$.

Definition 2.2.2 A weak fibration of scaled simplicial sets $p\colon \mathcal{E} \to \mathcal{B}$ is said to be a 2*–inner fibration* if every triangle in $\mathcal{B}$ admits both a sufficient supply of left p–inner lifts and a sufficient supply of right p–inner lifts.

We say that p is a 2–inner cartesian fibration if it is both a 2–inner fibration and a cartesian fibration (in the sense of Definition 1.4.9). We say that p is a 2*–inner cocartesian fibration* if $p^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a 2–inner cartesian fibration.

Definition 2.2.3 A weak fibration of scaled simplicial sets $p\colon \mathcal{E} \to \mathcal{B}$ is said to be a 2*–outer fibration* if the following conditions are satisfied:

(i) Every triangle in $\mathcal{B}$ admits both a sufficient supply of left p–outer lifts and a sufficient supply of right p–outer lifts.

(ii) The collection of left p–outer triangles in $\mathcal{E}$ is closed under right whiskering and the collection of right p–outer triangles is closed under left whiskering.

We say that p is a 2*–outer cartesian fibration* if it is both a 2–outer fibration and a cartesian fibration (in the sense of Definition 1.4.9). Dually, we say that p is a 2*–outer cocartesian fibration* if $p^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a 2–outer cartesian fibration.

Remark 2.2.4 For the previously introduced classes of fibrations, the class of 2–inner/outer (co)cartesian fibrations is readily seen to be closed under base change.

Definition 2.2.5 Given 2–inner/outer (co)cartesian fibrations $q\colon \mathcal{D} \to \mathcal{A}$ and $p\colon \mathcal{E} \to \mathcal{B}$, a *morphism of* 2*–inner/outer* (*co*)*cartesian fibrations* from q to p is a commutative square

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{g} & \mathcal{E} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{A} & \xrightarrow{f} & \mathcal{B} \end{array}$$

such that g sends q-(co)cartesian arrows to p–(co)cartesian arrows and left/right p–inner/outer triangles to left/right q–inner/outer triangles.

Proposition 2.2.6 *Let* $p\colon \mathcal{E} \to \mathcal{B}$ *be a weak fibration of* ∞*–bicategories. Then*:

(i) *If* p *is a* 2*–inner fibration then the induced map* $p_{x,y}\colon \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}}(x,y) \to \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}}(px,py)$ *is a cartesian fibration of* ∞*–categories for every* $x, y \in \mathcal{E}$.

(ii) *If* p *is a* 2*–outer fibration then the induced map* $p_{x,y}\colon \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}}(x,y) \to \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}}(px,py)$ *is a cocartesian fibration of* ∞*–categories for every* $x, y \in \mathcal{E}$.

Proof The condition that p is a weak fibration implies that the induced map $\overline{\mathcal{E}}_{/y} \to \mathcal{E} \times_{\mathcal{B}} \overline{\mathcal{B}}_{/py}$ is a 1–outer fibration (see Remark 1.4.11 and Example 1.4.12), and hence its base change $(\overline{\mathcal{E}}_{/y})_x \to (\overline{\mathcal{B}}_{/py})_{px}$ is a 1–outer fibration as well, and in particular a weak fibration. This last map is between ∞–bicategories in which every triangle is thin, and hence its underlying map of simplicial sets

$$p_{x,y} \colon \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}}(x,y) \to \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}}(px,py)$$

is an inner fibration between ∞–categories. We also note that arrows in $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}}(px,py)$ correspond to triangles $\sigma \colon \Delta^2 \to \mathcal{B}$ such that $\sigma|_{\Delta^{\{0,1\}}}$ is degenerate on px and $\sigma|_{\Delta^{\{2\}}} = py$. More precisely, these are arrows from $\sigma|_{\Delta^{\{0,2\}}}$ to $\sigma|_{\Delta^{\{1,2\}}}$, considered as vertices in $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}}(px,py)$.

Now if p is a 2–inner fibration, then for any choice of an edge $g \colon x \to y$ in $\mathcal{E}$ lifting $\sigma|_{\Delta^{\{1,2\}}}$, there exists a left p–inner triangle τ such that $\tau|_{\Delta^{\{0,1\}}}$ is degenerate, $\tau|_{\Delta^{\{1,2\}}} = g$ and $p\tau = \sigma$. By Remark 2.1.8 the triangle τ determines a $p_{x,y}$–cartesian lift with target g of the arrow in $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}}(px,py)$ determined by σ. Since g was arbitrary we conclude that $p_{x,y}$ is a cartesian fibration. Similarly, if p is a 2–outer fibration we have that for any choice of an edge $g \colon x \to y$ in $\mathcal{E}$ lifting $\sigma|_{\Delta^{\{0,2\}}}$, there exists a left p–outer triangle τ such that $\tau|_{\Delta^{\{0,1\}}}$ is degenerate, $\tau|_{\Delta^{\{0,2\}}} = g$ and $p\tau = \sigma$. By Remark 2.1.8, the triangle τ determined a $p_{x,y}$–cocartesian lift with domain g of the arrow in $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}}(px,py)$ determined by σ, and so $p_{x,y}$ is a now a cocartesian fibration. □

2.3 Congruent triangles

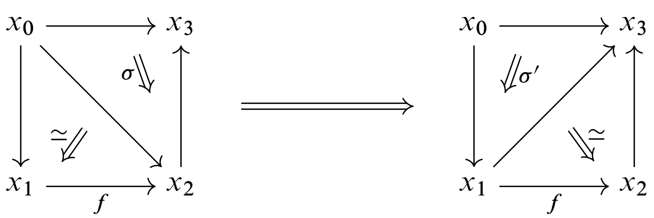
Our goal in the present subsection is to establish some preliminary results showing that for most questions about 2–outer (co)cartesian fibrations between ∞–bicategories, one may restrict attention to left/right outer triangles which are left/right-degenerate in the following sense:

Definition 2.3.1 We say that a triangle $\sigma \colon \Delta^2 \to X$ is *left degenerate* if the edge $\sigma|_{\Delta^{\{0,1\}}}$ of X is degenerate. The triangle σ is said to be *right degenerate* if the edge $\sigma|_{\Delta^{\{1,2\}}}$ is degenerate.

Warning 2.3.2 To avoid confusion, let us emphasize that left (or right) degenerate triangles in the sense of Definition 2.3.1 are not necessarily themselves degenerate. This terminology is also used in [1].

We will make use of the following construction.

Definition 2.3.3 Let X be a scaled simplicial set. Given a 3–simplex $\rho \colon \Delta^3 \to X$ of the form



we say that ρ exhibits σ as *left congruent* to σ', and σ' as *right congruent* to σ.

Lemma 2.3.4 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a weak fibration and let σ and σ' be two triangles in $\mathcal{E}$ such that σ is left congruent to σ' via a 3–simplex ρ as in Definition 2.3.3. Then the following holds:*

(i) *The triangle σ is left/right p–inner if and only if σ' is.*

(ii) *If f is p–cocartesian and σ' is left p–outer, then σ is left p–outer.*

(iii) *If f is p–cartesian and σ is right p–outer, then σ' is right p–outer.*

(iv) *If f is an equivalence, then σ is left/right p–outer if and only if σ' is.*

Proof The first three statements follows directly from the partial two-out-of-three properties elaborated in Remark 2.1.5 together with the fact that thin triangles are always left and right p–inner and that thin triangles with p–cocartesian left leg are left p–outer, while thin triangles with p–cartesian right leg are right p–outer; see Remark 2.1.3. To prove the last claim, we note that if f is an equivalence then the 3–simplex ρ determines in particular an equivalence between the arrows associated to σ and σ' in $\overline{\mathcal{E}}_{x_0/}$, where $x_0 = \sigma(0) = \sigma'(0)$, and hence each of these arrows is strongly cartesian with respect to the projection $\overline{\mathcal{E}}_{x_0/} \to \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{px_0/}$ if and only if the other is so; see Remark 1.4.8. Similarly, ρ determines an equivalence between the arrows associated to σ and σ' in $\overline{\mathcal{E}}_{/x_3}$, where $x_3 = \sigma(2) = \sigma'(2)$, and hence each of these arrows is strongly cocartesian with respect to the projection $\overline{\mathcal{E}}_{/x_3} \to \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{/px_3}$ if and only if the other is so. □

Lemma 2.3.5 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a weak fibration between ∞–bicategories and let σ and σ' be two triangles in $\mathcal{B}$ such that σ is left congruent to σ'. Then the following holds:*

(i) *If the left leg of σ' admits a sufficient supply of p–cocartesian lifts and σ' admits a sufficient supply of left p–outer lifts, then σ admits a sufficient supply of left p–outer lifts.*

(ii) *If the right leg of σ admits a sufficient supply of p–cartesian lifts and σ admits a sufficient supply of right p–outer lifts, then σ' admits a sufficient supply of right p–outer lifts.*

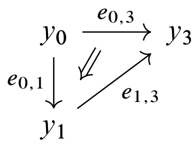
Proof We prove the first claim, the second claim then follows by applying the first claim to $\mathcal{E}^{\mathrm{op}}$ and switching the roles of σ and σ'. Let $\rho\colon \Delta^3 \to \mathcal{E}$ be a 3–simplex as above exhibiting σ as left congruent to σ'. In particular, $\sigma = \rho|_{\Delta^{\{0,2,3\}}}$. We need to show that for every pair of arrows

$$\begin{array}{ccc} y_0 & \xrightarrow{e_{0,3}} & y_3 \\ & \searrow^{e_{0,2}} & \\ & & y_2 \end{array}$$

of $\mathcal{E}$ lifting $\sigma|_{\Delta^{\{0,2\}}}$ and $\sigma|_{\Delta^{\{0,3\}}}$, respectively, with $e_{0,2}$ being p–cocartesian, there exists a left p–outer triangle lifting σ:

$$\begin{array}{ccc} y_0 & \xrightarrow{e_{0,3}} & y_3 \\ & \searrow^{e_{0,2}} \ \Downarrow & \uparrow{\scriptstyle e_{2,3}} \\ & & x_2 \end{array}$$

Now by assumption, the left leg of σ' admits a p–cocartesian lift $e_{0,1}\colon y_0 \to y_1$. Since σ' is assumed to have a sufficient supply of left p–outer lifts, we may now find a left p–outer lift τ' of σ', depicted as



At the same time, since $\rho|_{\Delta^{\{0,1,2\}}}$ is thin and $e_{0,1}$ is p–cocartesian the pair $e_{0,1}, e_{0,2}$ extends to a thin triangle $\theta\colon \Delta^{\{0,1,2\}} \to \mathcal{E}$ such that $p\theta = \rho|_{\Delta^{\{0,1,2\}}}$. Set $e_{1,2} = \theta|_{\Delta^{\{1,2\}}}$. By the two-out-of-three property for p–cocartesian edges — eg the dual of [6, Lemma 2.3.8] — we get that $e_{1,2}$ is p–cocartesian. Since $\rho|_{\Delta^{\{1,2,3\}}}$ is thin we may now lift it to a triangle $\eta\colon \Delta^{\{1,2,3\}} \to \mathcal{E}$ extending $e_{1,2}$ and $e_{1,3}$. The triangles τ', θ and η now glue to give a map $\alpha\colon \Lambda^3_1 \to \mathcal{E}$ lifting $\rho|_{\Lambda^3_1}$, where α sends $\Delta^{\{0,1,2\}}$ and $\Delta^{\{1,2,3\}}$ to thin triangles by construction. Since p is a weak fibration we may extend α to a map $\beta\colon \Delta^3 \to \mathcal{E}$ lifting ρ, so that $\tau := \beta|_{\Delta^{\{0,2,3\}}}$ gives in particular a triangle lifting σ and extending $e_{0,2}, e_{0,3}$. The 3–simplex β now exhibits τ as left congruent to τ' and hence τ is left p–outer by Lemma 2.3.4(2). □

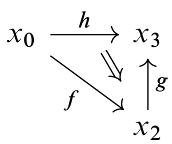
Remark 2.3.6 In the proof of Lemma 2.3.5(1) we have complete freedom in choosing the p–cocartesian lift $e_{0,1}$ of $\sigma'|_{\Delta^{\{0,1\}}}$. We may consequently slightly weaken the assumption on σ': it suffices to assume that for every choice of lift of $e_{1,3}$ of $\sigma'|_{\Delta^{\{1,2\}}}$ there exists some left p–outer lift τ' of σ' such that $\tau|_{\Delta^{\{1,2\}}} = e_{1,3}$ and $\tau|_{\Delta^{\{0,1\}}}$ is p–cartesian (as opposed to assuming this for any choice of p–cocartesian lift of $\tau|_{\Delta^{\{0,1\}}}$). For example, if σ' is left degenerate then it suffices to assume that it has a sufficient supply of *left degenerate* left p–outer lifts (that is, for each choice of a lift of $\sigma'|_{\Delta^{\{1,2\}}}$). A similar statement holds for the right p–outer case of Lemma 2.3.5(2).

Lemma 2.3.7 *Let $\mathcal{B}$ be an ∞–bicategory. Then the following holds*:

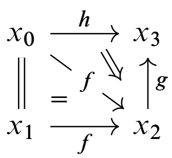
(i) *Any triangle σ in $\mathcal{B}$ is left congruent to a left degenerate triangle σ'.*

(ii) *Any triangle σ' in $\mathcal{B}$ is right congruent to a right degenerate triangle σ.*

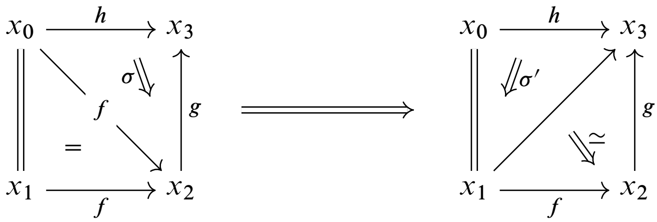
Proof We prove the first claim. The second claim then follows by applying the first statement to $\mathcal{B}^{\mathrm{op}}$ and switching the roles of σ and σ'. Let us depict σ as



Let $K \subseteq \Delta^3$ be the simplicial subset spanned by the faces $\Delta^{\{0,2,3\}}$ and $\Delta^{\{0,1,2\}}$ and let $\rho\colon K \to \mathcal{B}$ be the map which sends $\Delta^{\{0,2,3\}}$ to σ and $\Delta^{\{0,1,2\}}$ to the degenerate triangle whose left leg is degenerate and whose other two legs are f. We may then visualize ρ as



Now since $\mathcal{B}$ is an ∞–bicategory we may extend ρ to a map $\rho' \colon \Lambda^3_2 \to \mathcal{B}$ which sends the triangle $\Delta^{\{1,2,3\}}$ to a thin triangle, and then proceed to extend ρ' to a full 3–simplex $\Delta^3 \to \mathcal{B}$, which we can depict as



This 3–simplex then exhibits σ as left congruent to σ', and σ' is left degenerate, as desired. □

Corollary 2.3.8 *Let $p \colon \mathcal{E} \to \mathcal{B}$ be a weak fibration. If every left degenerate triangle in $\mathcal{B}$ has a sufficient supply of left p–outer lifts, then every triangle in $\mathcal{B}$ has a sufficient supply of left p–outer lifts. Similarly, if every right degenerate triangle in $\mathcal{B}$ has a sufficient supply of right p–outer lifts then every triangle in $\mathcal{B}$ has a sufficient supply of right p–outer lifts.*

Proof Combine Lemma 2.3.7 with Lemma 2.3.5 using the fact that any degenerate arrow in $\mathcal{B}$ has a sufficient supply of (co)cartesian lifts. □

Using Remark 2.3.6 we may also obtain the following strengthening of Corollary 2.3.8:

Corollary 2.3.9 *Let $p \colon \mathcal{E} \to \mathcal{B}$ be a weak fibration. If every left degenerate triangle in $\mathcal{B}$ has a sufficient supply of left degenerate left p–outer lifts, then every triangle in $\mathcal{B}$ has a sufficient supply of left p–outer lifts. Similarly, if every right degenerate triangle in $\mathcal{B}$ has a sufficient supply of right-degenerate right p–outer lifts then every triangle in $\mathcal{B}$ has a sufficient supply of right p–outer lifts.*

2.4 Homotopy invariance of fibrations

Our goal in this section is to prove the following homotopy invariance property for (co)cartesian fibrations.

Proposition 2.4.1 *Let*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\simeq} & \mathcal{E} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{A} & \xrightarrow{\simeq} & \mathcal{B} \end{array}$$

be a commutative diagram of ∞–bicategories whose vertical maps are both bicategorical fibrations and whose horizontal maps are bicategorical equivalences. Then p is a 2–inner/outer (co)cartesian fibration if and only if q is. In addition, an edge in $\mathcal{D}$ is q–(co)cartesian if and only if its image in $\mathcal{E}$ is p–(co)cartesian, and similarly a triangle in $\mathcal{D}$ is left/right q–inner/outer if and only if its image in $\mathcal{E}$ is so with respect to p.

The proof of Proposition 2.4.1 will require a couple of lemmas, and will be given at the end of the section.

Lemma 2.4.2 *Let*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\simeq} & \mathcal{E} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{A} & \xrightarrow{\simeq} & \mathcal{B} \end{array}$$

be a commutative diagram of ∞–bicategories whose vertical maps are both bicategorical fibrations and whose horizontal maps are bicategorical equivalences. Then an arrow in $\mathcal{D}$ is q–(co)cartesian if and only if its image in $\mathcal{E}$ is p–(co)cartesian.

Proof By [6, Proposition 2.3.7] we may replace the property of being cartesian with that of being weakly (co)cartesian. The desired claim now follows from the characterization [6, Proposition 2.3.3] of weakly (co)cartesian arrows in terms of mapping spaces. □

In any weak fibration $p\colon \mathcal{E} \to \mathcal{B}$ between ∞–bicategories, the collection of p–(co)cartesian arrows is closed under equivalences in the arrow category; see [6, Remark 2.3.12]. We now show that a similar property holds for inner/outer triangles:

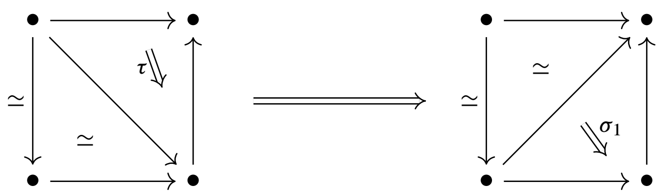
Lemma 2.4.3 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a weak fibration between ∞–bicategories, and let $H\colon \Delta^1_\flat \times \Delta^2_\flat \to \mathcal{E}$ be levelwise invertible natural transformation between triangles. Then $\sigma_0 := H|_{\Delta^{\{0\}}\times\Delta^2}$ is left/right p–inner/outer if and only if $\sigma_1 := H|_{\Delta^{\{1\}}\times\Delta^2}$ is so.*

Proof To fix ideas we prove the left inner and left outer cases, the right inner and right outer cases then follow by replacing p with opposite. In addition, it will suffice to prove that σ_0 is left p–inner/outer as soon as σ_1 is such, since we can get the other direction by replacing H by the inverse equivalence. We hence assume that σ_1 is left p–inner/outer.

For $i = 0, 1, 2$, let $\rho_i\colon \Delta^3 \to \Delta^1 \times \Delta^2$ be the 3–simplex given by

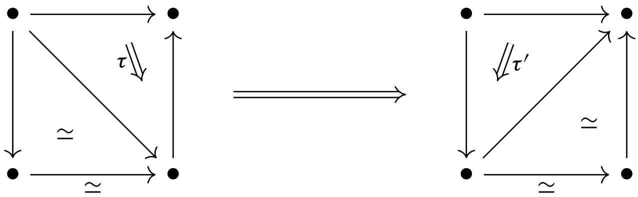
$$\rho_i(j) = \begin{cases} (0, j) & \text{if } j \le i, \\ (1, j-1) & \text{if } j > i. \end{cases}$$

We may then write $H\rho_0$ as

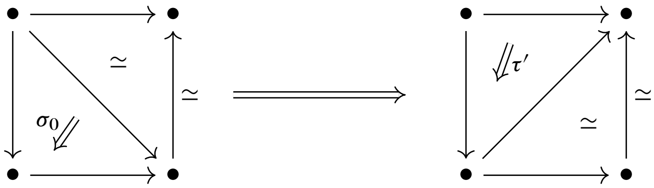


for some triangle τ. Since σ_1 is left p–inner/outer we get from the two-out-of-three properties of Remark 2.1.5 that τ is left p–inner/outer (in the outer case, we also point out that the triangle $H\rho_0|_{\Delta^{\{0,1,3\}}}$

is left p–outer by Remarks 2.1.3 and 1.4.7, since it is thin and its left leg is an equivalence). We now write $H\rho_1$ as



for some triangle τ'. Since τ is left p–inner/outer we now get from Points (1) and (4) of Lemma 2.3.4 that τ' is left p–inner/outer. Finally, we may write $H\rho_2$ as

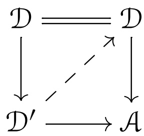


and so by Lemma 2.1.6 we conclude that σ_0 is left p–inner/outer as well, as desired. □

Proof of Proposition 2.4.1 We first note that since both vertical arrows are fibrant and cofibrant in the arrow category (with respect to the projective model structure), the existence of a levelwise equivalence from q to p implies the existence of a levelwise equivalence from p to q. It will hence suffice to show that if p is a 2–inner/outer (co)cartesian fibration then so is q. Now since weak equivalences between fibrant objects are preserved under base change along fibrations it follows from the two-out-of-three property that the map $\mathcal{D} \to \mathcal{E} \times_{\mathcal{B}} \mathcal{A}$ is an equivalence of ∞–bicategories. The map $\mathcal{E}' := \mathcal{E} \times_{\mathcal{B}} \mathcal{A} \to \mathcal{A}$ is then again a 2–inner/outer (co)cartesian fibration (Remark 2.2.4). We now factor the weak equivalence $\mathcal{D} \xrightarrow{\simeq} \mathcal{E}'$ as a composite

$$\mathcal{D} \xrightarrow{\simeq} \mathcal{D}' \xrightarrow{\simeq} \mathcal{E}',$$

where the first map is a trivial cofibration and the second a trivial fibration. Then $\mathcal{D}' \to \mathcal{E}' \to \mathcal{A}$ is a composite of a 2–inner/outer (co)cartesian fibration and a trivial fibration of scaled simplicial sets, and is hence itself again a 2–inner/outer (co)cartesian fibration. On the other hand, since $\mathcal{D} \to \mathcal{D}'$ is a trivial cofibration and $\mathcal{D} \to \mathcal{A}$ is in particular a bicategorical fibration we may solve the lifting problem



so that we obtain a retract diagram of arrows

$$\begin{array}{ccccc} \mathcal{D} & \xrightarrow{i} & \mathcal{D}' & \xrightarrow{r} & \mathcal{D} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ \mathcal{A} & = & \mathcal{A} & = & \mathcal{A} \end{array}$$

Now by Lemma 2.4.2 the map $\mathcal{D}' \to \mathcal{D}$ preserves (co)cartesian arrows over $\mathcal{A}$, and hence the fact that $\mathcal{D}'$ has a sufficient supply of q'–(co)cartesian edges implies that $\mathcal{D}$ has a sufficient supply of q–(co)cartesian edges. It is thus left to show that $\mathcal{D}$ also has a sufficient supply of left/right q–inner/outer triangles. Solving a lifting problem of the form

$$\begin{array}{ccc} [\Delta^1 \times \mathcal{D}] \coprod_{\partial\Delta^1 \times \mathcal{D}'} [\partial\Delta^1 \times \mathcal{D}] & \longrightarrow & \mathcal{D}' \\ \downarrow & \overset{H}{\nearrow} & \downarrow q' \\ \mathcal{D}' \times \Delta^1 & \longrightarrow & \mathcal{A} \end{array}$$

we obtain a natural transformation H over $\mathcal{A}$ from $i \circ r : \mathcal{D}' \to \mathcal{D}'$ to the identity $\mathrm{id}_{\mathcal{D}'}$, whose restriction to $\mathcal{D}$ is constant on i. Since i is a trivial cofibration, it is in particular essentially surjective, and so the natural transformation H is levelwise invertible. We now prove the inner case. Suppose we are given a triangle $\sigma\colon \Delta^2 \to \mathcal{A}$ and a lift $\rho\colon \Lambda^2_1 \to \mathcal{D}$ of $\sigma|_{\Lambda^2_1}$. Since q' is a left/right inner fibration we may extend $i\rho$ to a left/right q'–inner triangle $\tau : \Delta^2 \to \mathcal{D}'$ lifting σ. Evaluating the levelwise invertible natural transformation H at τ we obtain an equivalence $ir\tau \Rightarrow \tau$ covering the identity transformation on τ and restricting to the identity transformation from $i\rho$ to itself on Λ^2_1. By Lemma 2.4.3 we deduce that $ir\tau$ is also q'–inner. Since i admits a retraction and by the explicit description of left/right inner triangles in terms of lifting properties as in Remark 2.1.2, we then deduce that $r\tau$ itself is a left/right q–inner extension of ρ lifting σ. In the outer case, the argument for the existence of a sufficient supply of left/right q–outer triangles is completely analogous, but one also needs to show the closure of q–outer triangles under whiskering as required in Definition 2.2.3(2). But this again follows from the corresponding property for q' and the fact that $i : \mathcal{D} \to \mathcal{D}'$ detects left/right outer triangles, as it admits a retraction over $\mathcal{A}$. □

2.5 Equivalences of cartesian fibrations

This section is devoted to the proof of a fiberwise criterion to test equivalences of fibrations.

Theorem 2.5.1 *Consider a morphism of 2–inner/outer (co)cartesian fibrations (in the sense of Definition 2.2.5) given by a commutative diagram of the form*

(5)
$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{r} & \mathcal{E}' \\ p\downarrow & & \downarrow q \\ \mathcal{B} & \xrightarrow{f} & \mathcal{B}' \end{array}$$

where p and q are 2–inner/outer cartesian fibrations between ∞–bicategories. Suppose that f is an equivalence of ∞–bicategories. Then r is an equivalence if and only if the induced map $r_b : \mathcal{E}_b \to \mathcal{E}'_{f(b)}$ on the level of fibers is an equivalence of ∞–bicategories for all $b \in \mathcal{B}$.

Before proving this, we need a preliminary result, generalizing [12, Proposition 2.4.4.2]. Here we make use of the results from [6] involving the slice ∞–bicategories associated with marked scaled simplicial sets; see Section 1.3 and Notation 1.3.1.

Lemma 2.5.2 *Let $p\colon \mathcal{C}\to\mathcal{D}$ be a 2–inner/outer cartesian fibration of ∞–bicategories. Let $x,y\in\mathcal{C}$ be two objects, $\overline{e}\colon px\to py$ an arrow between their images in $\mathcal{D}$, and $e\colon x'\to y$ a p–cartesian lift of $\overline{e}$. Then the (homotopy) fiber of the map*

$$\phi\colon \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{C}}(x,y)\to\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{D}}(px,py)$$

at $\overline{e}$ is naturally equivalent to the mapping ∞–category $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{C}_{px}}(x,x')$, where $\mathcal{C}_{px}$ denotes the (homotopy) fiber of p over px.

Proof To begin with we note that the map ϕ is a cartesian (or cocartesian in the outer case) fibration of ∞–categories by Proposition 2.2.6 and hence the homotopy fiber in question is equivalent to the corresponding strict fiber. We now consider the following diagram

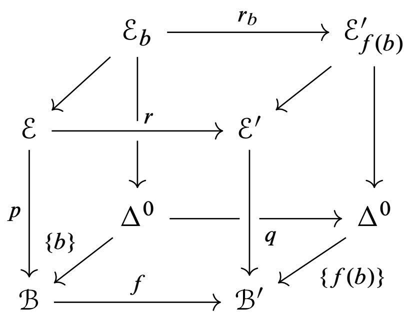
$$\overline{\mathcal{C}}_{/x'}\times_{\overline{\mathcal{D}}_{/px'}}\overline{\mathcal{D}}_{/\overline{e}^\sharp}\xleftarrow{\ \simeq\ }\overline{\mathcal{C}}_{/e^\sharp}\xrightarrow{\ \simeq\ }\overline{\mathcal{C}}_{/y}\times_{\overline{\mathcal{D}}_{/py}}\overline{\mathcal{D}}_{/\overline{e}^\sharp}$$

where the left diagonal map is a trivial fibration by [6, Lemma 2.4.6] and the right diagonal map is a trivial fibration by [6, Lemma 2.3.6] and the assumption that e is p–cartesian (and is hence in particular weakly p–cartesian, in the sense of loc. cit.). Taking fibers over $(x,s_1(\overline{e}))\in\mathcal{C}\times_{\mathcal{D}}\mathcal{D}_{/\overline{e}^\sharp}$ we thus get a zig-zag of equivalences

$$\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{C}_{p(x)}}(x,x')_\sharp\xleftarrow{\ \simeq\ }\overline{\mathcal{C}}_{/e^\sharp}\times_{\mathcal{C}\times_{\mathcal{D}}\mathcal{D}_{/\overline{e}^\sharp}}\{(x,s_1(\overline{e}))\}\xrightarrow{\ \simeq\ }\phi^{-1}(\overline{e})_\sharp$$

relating the fiber of ϕ over $\overline{e}$ to $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{C}_{p(x)}}(x,x')$, as desired. □

Proof of Theorem 2.5.1 We prove the 2–inner cartesian case. The proof for the remaining three variance flavors proceeds in exactly the same manner. Let us begin by proving the "only if" direction of the statement. Every object in the square (5) is fibrant, and the vertical maps are fibrations by Proposition 1.4.13, therefore every pullback along these maps is automatically a homotopy pullback. Consider the following commutative cube:



The front face and the side faces are homotopy pullbacks by assumption, so that the back one must also be such. Since the bottom horizontal map is an equivalence, the map r_b must be an equivalence as well.

Assume now that r is a fiberwise equivalence, and let us prove it is essentially surjective on objects and fully faithful. To see that r is essentially surjective, factor it as a composite $\mathcal{E} \to \mathcal{E}' \times_{\mathcal{B}'} \mathcal{B} \to \mathcal{E}'$, where the first map is essentially surjective since $r_b \colon \mathcal{E}_b \to \mathcal{E}'_{fb}$ is essentially surjective for every $b \in \mathcal{B}$, and the second is essentially surjective because $f \colon \mathcal{B} \to \mathcal{B}'$ is so. Concerning full-faithfulness, we consider the following commutative square of ∞–categories:

$$\begin{array}{ccc} \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}}(x,y) & \xrightarrow{\ r\ } & \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}'}(rx,ry) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}}(px,py) & \xrightarrow{\ f\ } & \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{B}'}(qrx,qry) \end{array} \tag{6}$$

Here, we have used the fact that $qr = fp$, and we denoted the induced action between the ∞–categories of morphisms with the same letter as that of the map between the relevant ∞–bicategories. Now, the vertical maps are cartesian fibrations of ∞–categories by Proposition 2.2.6, and the bottom horizontal map is an equivalence since f is fully faithful. Therefore, the top horizontal map is an equivalence if and only if the square (6) is a homotopy pullback, which happens precisely if the maps induced between the (strict) fibers of the vertical maps are equivalences. Thanks to Lemma 2.5.2, the fiber of the left-hand side map over $\overline{e} \colon p(x) \to p(y)$ coincides with $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}_{px}}(x,x')$ for some p–cartesian 1–simplex $e \colon x' \to y$ that lifts $\overline{e}$. Since r is assumed to preserve cartesian edges we have that $re \colon rx' \to ry$ is a q–cartesian lift of $f\overline{e}$, and so the fiber of q over $f(\overline{e})$ is given by $\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}'_{qrx}}(rx,rx')$. We can now finish the proof by observing that the induced map

$$\overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}_{px}}(x,x') \to \overline{\mathrm{Hom}}^{\triangleright}_{\mathcal{E}'_{qrx}}(rx,rx')$$

is an equivalence of ∞–categories, since $\mathcal{E}_{px} \to \mathcal{E}'_{fpx} \simeq \mathcal{E}'_{qrx}$ is assumed to be an equivalence of ∞–bicategories. □

3 The domain projection

In this section we analyze a key example of a 2–outer cartesian fibration: the domain projection $\mathrm{d} \colon \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ induced by the inclusion $\{0\} \hookrightarrow \Delta^1$, where $\mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$ is the ∞–bicategory whose objects are the arrows in $\mathcal{C}$ and whose morphisms are the lax-commutative squares. It is characterized by the property that for every scaled simplicial set (K, T_K), maps $(K, T_K) \to \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$ correspond to maps of scaled simplicial sets

$$\Delta^1_\flat \otimes (K, T_K) \to \mathcal{C},$$

where $\otimes$ denotes the *Gray product* of scaled simplicial sets [7, Definition 2.1]. Explicitly, $\Delta^1_\flat \otimes (K, T_K)$ is the scaled simplicial set whose underlying simplicial set is the cartesian product $\Delta^1 \times K$, and where a triangle $\sigma\colon \Delta^2 \to \Delta^1 \times K$ is thin if and only if its image in K belongs to T_K and either $\sigma|_{\Delta^{\{0,1\}}}$ maps to a degenerate edge in K or $\sigma|_{\Delta^{\{1,2\}}}$ maps to a degenerate edge in Δ^1. Here, we have used the fact that all the triangles in $\Delta^1_\flat$ are thin, otherwise the additional condition of projecting to a thin triangle in the first factor would have been necessary.

Remark 3.0.1 By the main result of [7] the functor $- \otimes -$ is a left Quillen bifunctor. It then follows that for any ∞–bicategory $\mathcal{C}$, the domain projection

$$\mathrm{d}\colon \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

is a bicategorical fibration of ∞–bicategories, and in particular a weak fibration; see Remark 1.4.2.

Our argument to show that the domain fibration is a 2–outer cartesian fibrations is based on the following extension lemma.

Lemma 3.0.2 *Let $\mathcal{C}$ be an ∞–bicategory, and for $n \geq 2$ suppose we are given an extension problem of the form*

$$\begin{array}{ccc} \Big([\Delta^1 \times \Lambda^n_n] \coprod_{\partial\Delta^1 \times \Lambda^n_n} [\partial\Delta^1 \times \Delta^n], T'\Big) & \xrightarrow{\rho} & \mathcal{C} \\ \downarrow & \nearrow & \\ (\Delta^1 \times \Delta^n, T) & & \end{array}$$

where

$$T = \{\Delta^{\{(0,n-1),(0,n),(1,n)\}}, \Delta^{\{(0,0),(1,n-1),(1,n)\}}\}$$

and T' is its restriction to the top left corner. If ρ sends $\Delta^{\{1\}} \times \Delta^{\{n-1,n\}}$ to an invertible edge in $\mathcal{C}$, then the dotted extension exists.

Proof We define a sequence of scaled maps

$$\tau_1, \dots, \tau_{n-1}\colon (\Delta^n, \Delta^{\{0,n-1,n\}}) \to (\Delta^1 \times \Delta^n, T)$$

in the following manner:

$$\tau_i(j) = \begin{cases} (0, j) & \text{if } j < i, \\ (1, j) & \text{if } j \geq i. \end{cases}$$

Set $K_0 := \big([\Delta^1 \times \Lambda^n_n] \coprod_{\partial\Delta^1 \times \Lambda^n_n} [\partial\Delta^1 \times \Delta^n], T'\big)$ and $K_i := K_{i-1} \cup \tau_i$ for $1 \leq i \leq n-1$, so that for every $1 \leq i \leq n-1$ we have pushout diagrams of the form

$$\begin{array}{ccc} (\Lambda^n_n, \{\Delta^{\{0,n-1,n\}}\}) & \longrightarrow & K_{i-1} \\ \downarrow & & \downarrow \\ (\Delta^n, \{\Delta^{\{0,n-1,n\}}\}) & \xrightarrow{\tau_i} & K_i \end{array}$$

These all have the property that the edge $\Delta^{\{n-1,n\}}$ is mapped to an equivalence in $\mathcal{C}$. All these maps admit lifts against $\mathcal{C}$ by [6, Corollary 2.3.10], and so we can extend the map ρ to a map $\rho'\colon K_{n-1} \to \mathcal{C}$. Consider now the $(n+1)$–simplices $\sigma_0, \dots, \sigma_n\colon \Delta^{n+1} \to \Delta^1 \times \Delta^n$ defined as

$$\sigma_i(j) = \begin{cases} (0, j) & \text{if } j \le i, \\ (1, j-1) & \text{if } j > i. \end{cases}$$

Observe that $\sigma_0, \dots, \sigma_{n-1}$ can be promoted to maps

$$(\Delta^{n+1}, \{\Delta^{\{0,n,n+1\}}\}) \to (\Delta^1 \times \Delta^n, T)$$

whereas σ_n can be promoted to a map

$$(\Delta^{n+1}, \{\Delta^{\{n-1,n,n+1\}}\}) \to (\Delta^1 \times \Delta^n, T)$$

We now set $K_{n+k} = K_{n-1+k} \cup \sigma_{n-k}$ for $0 \le k \le n$, and observe that $K_{2n} = (\Delta^1 \times \Delta^n, T)$. We then have a pushout diagram of the form

$$\begin{array}{ccc} (\Lambda^{n+1}_n, \{\Delta^{\{n-1,n,n+1\}}\}|_{\Lambda^{n+1}_n}) & \longrightarrow & K_{n-1} \\ \downarrow & & \downarrow \\ (\Delta^{n+1}, \{\Delta^{\{n-1,n,n+1\}}\}) & \xrightarrow{\sigma_n} & K_n \end{array}$$

and, for every $k > 0$, there are pushout diagrams of the form

$$\begin{array}{ccc} (\Lambda^{n+1}_{n+1}, \{\Delta^{\{0,n,n+1\}}\}) & \longrightarrow & K_{n-1+k} \\ \downarrow & & \downarrow \\ (\Delta^{n+1}, \{\Delta^{\{0,n,n+1\}}\}) & \xrightarrow{\sigma_{n-k}} & K_{n+k} \end{array}$$

where the corresponding image of the edge $\Delta^{\{n,n+1\}}$ in $\mathcal{C}$ is an equivalence. As before, this is enough to prove the existence of an extension to $\mathcal{C}$. □

Remark 3.0.3 Applying Lemma 3.0.2 to $\mathcal{C}^{\mathrm{op}}$ one obtains the following dual form of its statement, which we spell out for the convenience of the reader: suppose given an extension problem of the form

$$\begin{array}{ccc} \Big([\Delta^1 \times \Lambda^n_0] \coprod_{\partial\Delta^1 \times \Lambda^n_0} [\partial\Delta^1 \times \Delta^n], T'\Big) & \xrightarrow{\rho} & \mathcal{C} \\ \downarrow & \nearrow & \\ (\Delta^1 \times \Delta^n, T) & & \end{array}$$

where

$$T = \{\Delta^{\{(0,0),(1,0),(1,1)\}}, \Delta^{\{(0,0),(0,1),(1,n)\}}\}$$

and T' is its restriction to the top left corner. If ρ sends $\Delta^{\{1\}} \times \Delta^{\{0,1\}}$ to an invertible edge in $\mathcal{C}$ then the dotted lift exists.

We now consider the question of cartesian 1–simplices for the domain projection.

Lemma 3.0.4 *Let* $\mathcal{C}$ *be an* ∞*–bicategory and consider the domain projection* $\mathrm{d}\colon \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ *as above. Then a* 1*–simplex* $\alpha\colon \Delta^1 \to \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$ *is* d*–cartesian if its transpose* $\widehat{\alpha}\colon \Delta^1 \otimes \Delta^1 \to \mathcal{C}$ *corresponds to a commutative square* (*ie both its nondegenerate triangles are thin*) *with the side* $\widehat{\alpha}(\{1\}\times\Delta^1)$ *being an equivalence in* $\mathcal{C}$*. Pictorially,* $\widehat{\alpha}$ *looks like*

$$\begin{array}{ccc} a & \longrightarrow & x \\ {\scriptstyle g}\big\downarrow & \searrow^{k} \; \overset{\simeq}{\nearrow\!\!\!\nearrow} \;\; \overset{\simeq}{\swarrow\!\!\!\swarrow} & \big\downarrow{\scriptstyle \simeq} \\ b & \underset{l}{\longrightarrow} & y \end{array}$$

Proof Consider a 1–simplex $\alpha\colon \Delta^1 \to \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$, as described above. Given $n \geq 2$ and a solid square of the form

$$\begin{array}{ccc} & \Delta^{\{n-1,n\}} & \\ \swarrow & & \searrow^{\alpha} \\ (\Lambda^n_n, \Lambda^n_n \cap \Delta^{\{0,n-1,n\}}) & \longrightarrow & \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \\ \big\downarrow & \nearrow^{\cdots} & \big\downarrow{\scriptstyle \mathrm{d}} \\ (\Delta^n, \Delta^{\{0,n-1,n\}}) & \longrightarrow & \mathcal{C} \end{array}$$

we have to exhibit a filler as indicated by the dotted arrow. Note that this lifting problem corresponds to one of the form

$$\begin{array}{ccc} \Big(\Delta^1 \times \Lambda^n_n \coprod_{\Delta^{\{0\}}\times\Lambda^n_n} \Delta^{\{0\}} \times \Delta^n, T'\Big) & \xrightarrow{f} & \mathcal{C} \\ \big\downarrow & \nearrow^{\cdots} & \\ (\Delta^1 \times \Delta^n, T) & & \end{array} \tag{7}$$

in which $\Delta^1 \times \Delta^{\{n-1,n\}}$ is mapped to $\widehat{\alpha}$ under the adjunction $\Delta^1 \otimes - \dashv \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, -)$. Here T is the union of the triangles which are thin in $\Delta^1_\flat \otimes (\Delta^n, \{\Delta^{\{0,n-1,n\}}\})$, together with $\Delta^{\{(0,n-1),(0,n),(1,n)\}}$, and $T' \subseteq T$ is the subset of those triangles which are contained in the domain of the vertical arrow in (7). Since the edge $\Delta^{\{1\}} \times \Delta^{\{n-1,n\}}$ maps to an invertible edge of $\mathcal{C}$ by our assumption on α and T contains $\Delta^{\{1\}} \times \Delta^{\{0,n-1,n\}}$ we can extend the map f in (7) to a map

$$f'\colon \Big([\Delta^1 \times \Lambda^n_n] \coprod_{\partial\Delta^1\times\Lambda^n_n} [\partial\Delta^1 \times \Delta^n], T''\Big) \to \mathcal{C},$$

where T'' is the intersection of T with the triangles in $[\Delta^1 \times \Lambda^n_n] \coprod_{\partial\Delta^1\times\Lambda^n_n} [\partial\Delta^1 \times \Delta^n]$. Let $S = T'' \cup \{\Delta^{\{(0,n-1),(0,n),(1,n)\}}, \Delta^{\{(0,0),(1,n-1),(1,n)\}}\}$. Then S is contained in T and by Lemma 3.0.2 we may extend f' to $(\Delta^1 \times \Delta^n, S)$. When $n \geq 3$ we have that $T = T'' = S$ and so the proof is complete. In the case $n = 2$, one needs to additionally verify that the resulting extension sends $\Delta^{\{(0,0),(1,0),(1,1)\}}$ to a thin triangle. Indeed, this follows from [8, Proposition 3.4] since S contains $\Delta^{\{(0,0),(1,1),(1,2)\}}$, $\Delta^{\{(0,0),(1,0),(1,2)\}}$ and $\Delta^{\{(1,0),(1,1),(1,2)\}}$, and $\Delta^{\{(1,1),(1,2)\}}$ maps to an invertible edge in $\mathcal{C}$ by assumption. □

Lemma 3.0.5 *The domain projection*

$$\mathrm{d}\colon \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

of an ∞*–bicategory* $\mathcal{C}$ *has enough cartesian edges.*

Proof A lifting problem of the form

$$\begin{array}{ccc} \Delta^{\{1\}} & \xrightarrow{h} & \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \\ \downarrow & & \downarrow d \\ \Delta^1 & \xrightarrow{f} & \mathcal{C} \end{array}$$

corresponds to the following data in $\mathcal{C}$:

$$\begin{array}{ccc} x & & \\ f\downarrow & & \\ y & \xrightarrow{h} & z \end{array}$$

This can be extended to a 1–simplex in $\mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$, depicted as

$$\begin{array}{ccc} x & \xrightarrow{hf} & z \\ & \searrow^{hf} \quad = & \\ f\downarrow & \simeq & \Vert \\ y & \xrightarrow[h]{} & z \end{array}$$

where the bottom triangle is given by an extension along $\Lambda^2_1 \to \Delta^2_\sharp$ and the upper triangle is degenerate. Such a 1–simplex is then d–cartesian by Lemma 3.0.4. □

Lemma 3.0.6 *Let* $\alpha\colon \Delta^2 \to \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$ *be a triangle with transpose* $\widehat{\alpha}\colon \Delta^1 \otimes \Delta^2 \to \mathcal{C}$.

(i) *If* $\widehat{\alpha}|_{\Delta^{\{1\}} \times \Delta^2}$ *is thin and* $\alpha|_{\Delta^{\{1,2\}}}$ *satisfies the assumption of* Lemma 3.0.4 (*so that it is* d*–cartesian by that lemma*), *then* α *is right* d*–outer.*

(ii) *If* $\widehat{\alpha}|_{\Delta^{\{1\}} \times \Delta^2}$ *is thin and* $\alpha|_{\Delta^{\{0,1\}}}$ *is invertible, then* α *is left* d*–outer.*

Proof We first prove (i). By Remark 2.1.2 we need to consider a lifting problem of the form

$$\begin{array}{ccc} & \Delta^{\{0,n-1,n\}}_\flat & \\ \swarrow & & \searrow^{\alpha} \\ (\Lambda^n_n)_\flat & \longrightarrow & \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \\ \downarrow & \nearrow^{\cdots} & \downarrow \mathrm{d} \\ \Delta^n_\flat & \longrightarrow & \mathcal{C} \end{array}$$

for $n \geq 3$, where we have to exhibit a diagonal filler as indicated by the dotted arrow.

We obtain an equivalent lifting problem of the form

$$\begin{array}{ccc} \left(\Delta^1\times\Lambda^n_n \coprod_{\Delta^{\{0\}}\times\Lambda^n_n} \Delta^{\{0\}}\times\Delta^n, T\right) & \xrightarrow{f} & \mathcal{C} \\ \downarrow & \nearrow & \\ (\Delta^1\times\Delta^n, T) & & \end{array}$$

where T is the set of thin triangles in $\Delta^1_\flat\otimes\Delta^n_\flat$ together with $\Delta^{\{(0,n-1),(0,n),(1,n)\}}$ and $\Delta^{\{1\}}\times\Delta^{\{0,n-1,n\}}$ (all of whom are contained in the top left corner, since $n\geq 3$). Our assumption on α implies that the edge $\Delta^{\{1\}}\times\Delta^{\{n-1,n\}}$ is invertible in $\mathcal{C}$ and so by [6, Corollary 2.3.10] we can extend the map f to a map

$$f'\colon\left(\Delta^1\times\Lambda^n_n \coprod_{\partial\Delta^1\times\Lambda^n_n} \partial\Delta^1\times\Delta^n, T\right)\to\mathcal{C}.$$

Now since T contains $\Delta^{\{(0,0),(1,0),(1,n-1)\}}$, $\Delta^{\{(0,0),(1,0),(1,n)\}}$ and $\Delta^{\{1\}}\times\Delta^{\{0,n-1,n\}}$ and $\mathcal{C}$ is an ∞–bicategory, the map f' must also send $\Delta^{\{(0,0),(1,n-1),(1,n)\}}$ to a thin triangle; see [13, Remark 3.1.4]. We may consequently apply Lemma 3.0.2 in order to extend f' to all of $(\Delta^1\times\Delta^n, T)$, as desired.

We now prove (ii). By Remark 2.1.2 we now need to consider a lifting problem of the form

$$\begin{array}{ccccc} & & \Delta^{\{0,1,n\}}_\flat & & \\ & \swarrow & & \searrow^{\alpha} & \\ (\Lambda^n_0)_\flat & & \longrightarrow & & \mathrm{Fun}^{\mathrm{gr}}(\Delta^1,\mathcal{C}) \\ \downarrow & & \nearrow & & \downarrow \mathrm{d} \\ \Delta^n_\flat & & \longrightarrow & & \mathcal{C} \end{array}$$

with $n\geq 3$. We obtain an equivalent lifting problem of the form

$$\begin{array}{ccc} \left(\Delta^1\times\Lambda^n_0 \coprod_{\Delta^{\{0\}}\times\Lambda^n_0} \Delta^{\{0\}}\times\Delta^n, T\right) & \xrightarrow{f} & \mathcal{C} \\ \downarrow & \nearrow & \\ (\Delta^1\times\Delta^n, T) & & \end{array}$$

where T is the set of thin triangles in $\Delta^1_\flat\otimes\Delta^n_\flat$ together with $\Delta^{\{(0,0),(0,1),(1,1)\}}$ and $\Delta^{\{1\}}\times\Delta^{\{0,1,n\}}$. Since the image of the edge $\Delta^{\{1\}}\times\Delta^{\{0,1\}}$ is invertible in $\mathcal{C}$ we can extend the map f to a map

$$f'\colon\left([\Delta^1\times\Lambda^n_0] \coprod_{\partial\Delta^1\times\Lambda^n_0} [\partial\Delta^1\times\Delta^n], T\right)\to\mathcal{C}.$$

Now since T contains $\Delta^{\{(0,0),(1,0),(1,1)\}}$, $\Delta^{\{(0,0),(1,0),(1,n)\}}$ and $\Delta^{\{1\}}\times\Delta^{\{0,1,n\}}$ and $\mathcal{C}$ is an ∞–bicategory, the map f' must also send $\Delta^{\{(0,0),(1,1),(1,n)\}}$ to a thin triangle. Since f' also sends $\Delta^{\{(0,0),(0,1),(1,1)\}}$ and $\Delta^{\{(0,1),(1,1),(1,n)\}}$ to thin triangles, the same holds for $\Delta^{\{(0,0),(0,1),(1,n)\}}$. We may consequently apply the dual form of Lemma 3.0.2 (see Remark 3.0.3), in order to extend f' to all of $(\Delta^1\times\Delta^n, T)$, as desired. □

Finally, we are ready to prove the main result of this section.

Theorem 3.0.7 *Given an ∞–bicategory $\mathcal{C}$, the domain projection*

$$\mathrm{d}\colon \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

is a 2–outer cartesian fibration, whose p–cartesian 1–simplices are those described in Lemma 3.0.4. In addition, the right p–outer triangles whose right legs are cartesian are those described in Lemma 3.0.6(1), and the left p–outer triangles whose left leg is invertible are those described in Lemma 3.0.6(2).

Proof By Remark 3.0.1, the map d is a weak fibration and by Lemma 3.0.4 it has a sufficient supply of d–cartesian lifts for 1–morphisms. By the essential uniqueness of d–cartesian lifts we deduce that all d–cartesian arrows are of the form described in Lemma 3.0.4.

We now show that the triangles in $\mathcal{C}$ have a sufficient supply of right p–outer lifts. This translates into a lifting problem of the form

$$\begin{array}{ccc} \Lambda^2_2 & \xrightarrow{\rho} & \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \\ \downarrow & \nearrow & \downarrow \mathrm{d} \\ \Delta^2 & \xrightarrow{\gamma} & \mathcal{C} \end{array}$$

with $\rho|_{\Delta^{\{1,2\}}}$ a d–cartesian edge, and so (as argued just above) of the form described in Lemma 3.0.4. This means in particular that $\rho|_{\Delta^{\{1,2\}}}$ corresponds to a map $\Delta^1_\flat \otimes \Delta^{\{1,2\}}_\flat \to \mathcal{C}$ which sends both triangles to thin triangles and the edge $\Delta^{\{1\}} \times \Delta^{\{1,2\}}$ to an invertible edge in $\mathcal{C}$. Solving this lifting problem in a way that produces a triangle of $\mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$ of the form described in Lemma 3.0.6(1) then corresponds to solving a lifting problem of the form

$$\begin{array}{ccc} \Bigl(\Delta^1 \times \Lambda^2_2 \coprod_{\Delta^{\{0\}} \times \Lambda^2_2} \Delta^{\{0\}} \times \Delta^2, T'\Bigr) & \xrightarrow{\widehat{\rho} \cup \gamma} & \mathcal{C} \\ \downarrow & \nearrow & \\ (\Delta^1 \times \Delta^2, T) & & \end{array} \tag{8}$$

where T' is obtained by intersection from T, which in turn contains all triangles which are thin in $\Delta^1_\flat \otimes \Delta^2_\flat$ as well as $\Delta^{\{1\}} \times \Delta^2$ and $\Delta^{\{(0,1),(0,2),(1,2)\}}$, and $\widehat{\rho}$ sends $\Delta^{\{1\}} \times \Delta^{\{1,2\}}$ to an invertible edge in $\mathcal{C}$. Since $\mathcal{C}$ is an ∞–bicategory we may then extend $\widehat{\rho} \cup \gamma$ to a map

$$g\colon \Bigl(\Delta^1 \times \Lambda^2_2 \coprod_{\partial\Delta^1 \times \Lambda^2_2} \partial\Delta^1 \times \Delta^2, T''\Bigr) \to \mathcal{C},$$

where T'' is obtained from T by intersection. Applying Lemma 3.0.2 we may extend g to a map $g'\colon (\Delta^1 \times \Delta^2, T'' \cup \{\Delta^{\{(0,0),(1,1),(1,2)\}}\}) \to \mathcal{C}$. We then observe that there is exactly one triangle in T which is not in $T'' \cup \{\Delta^{\{(0,0),(1,1),(1,2)\}}\}$, and that is the triangle $\Delta^{\{(0,0),(1,0),(1,1)\}}$. But this triangle is sent by g to a thin triangle in $\mathcal{C}$ by [8, Proposition 3.4], since T contains $\Delta^{\{(0,0),(1,1),(1,2)\}}$, $\Delta^{\{(0,0),(1,0),(1,2)\}}$ and $\Delta^{\{(1,0),(1,1),(1,2)\}}$, and g sends $\Delta^{\{1\}} \times \Delta^{\{1,2\}}$ to an invertible edge.

We now show that the triangles in $\mathcal{C}$ have a sufficient supply of left p–outer lifts. Invoking Corollary 2.3.8, it will suffice to test this only for triangles in $\mathcal{B}$ whose left leg is degenerate. This then translates into a lifting problem of the form

$$\begin{array}{ccc} \Lambda^2_0 & \xrightarrow{\rho} & \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \\ \downarrow & \nearrow & \downarrow{\scriptstyle d} \\ \Delta^2 & \xrightarrow{\gamma} & \mathcal{C} \end{array}$$

with $\rho|_{\Delta^{\{0,1\}}}$ a d–cocartesian lift of an identity, and hence an invertible edge of $\mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$. This means in particular that $\rho|_{\Delta^{\{0,1\}}}$ corresponds to a map $\Delta^1_\flat \otimes \Delta^{\{0,1\}}_\flat \to \mathcal{C}$ which sends both triangles to thin triangles and the edges $\Delta^{\{0\}} \times \Delta^{\{0,1\}}$ and $\Delta^{\{1\}} \times \Delta^{\{0,1\}}$ to invertible edges in $\mathcal{C}$. Solving this lifting problem in a way that produces a triangle of $\mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$ of the form described in Lemma 3.0.6(2) then corresponds to solving a lifting problem of the form

(9)
$$\begin{array}{ccc} \Big(\Delta^1 \times \Lambda^2_0 \coprod_{\Delta^{\{0\}} \times \Lambda^2_0} \Delta^{\{0\}} \times \Delta^2, T'\Big) & \xrightarrow{\hat{\rho} \cup \gamma} & \mathcal{C} \\ \downarrow & \nearrow & \\ (\Delta^1 \times \Delta^2, T) & & \end{array}$$

where T' is obtained by intersection from T, which in turn contains all triangles which are thin in $\Delta^1_\flat \otimes \Delta^2_\flat$ as well as the triangles $\Delta^{\{1\}} \times \Delta^2$ and $\Delta^{\{(0,0),(0,1),(1,1)\}}$, and $\hat{\rho}$ sends $\Delta^{\{\varepsilon\}} \times \Delta^{\{0,1\}}$ to an invertible edge in $\mathcal{C}$ for $\varepsilon = 0, 1$. Since $\mathcal{C}$ is an ∞–bicategory we may then extend $\hat{\rho} \cup \gamma$ to a map

$$g \colon \Big(\Delta^1 \times \Lambda^2_0 \coprod_{\partial\Delta^1 \times \Lambda^2_0} \partial\Delta^1 \times \Delta^2, T''\Big) \to \mathcal{C},$$

where T'' is obtained from T by intersection. Applying the dual of Lemma 3.0.2 (Remark 3.0.3) we may extend g to a map $g' \colon (\Delta^1 \times \Delta^2, T'' \cup \{\Delta^{\{(0,0),(0,1),(1,2)\}}\}) \to \mathcal{C}$. We then observe that there are exactly two triangles in T which are not in T'', namely, $\Delta^{\{(0,0),(1,1),(1,2)\}}$ and $\Delta^{\{(0,1),(1,1),(1,2)\}}$. To finish the proof we will show that these two triangles are sent by g to thin triangles in $\mathcal{C}$. For the first one, we note that since g sends $\Delta^{\{(0,0),(1,0),(1,1)\}}$, $\Delta^{\{(0,0),(1,0),(1,2)\}}$ and $\Delta^{\{1\}} \times \Delta^2$ to thin triangles, then it must also send $\Delta^{\{(0,0),(1,1),(1,2)\}}$ to a thin triangle. Then, since g also sends $\Delta^{\{(0,0),(0,1),(1,1)\}}$ and $\Delta^{\{(0,0),(0,1),(1,2)\}}$ to thin triangles, and the edge $\Delta^{\{(0,0),(0,1)\}}$ to an equivalence, then it must send $\Delta^{\{(0,1),(1,1),(1,2)\}}$ to a thin triangle as well; see [8, Proposition 3.4].

Having provided sufficiently many left and right d–outer lifts of the form appearing in Lemma 3.0.6, the uniqueness of d–outer lifts, as expressed for example in Remark 2.1.7, shows that all right d–outer triangles whose right leg is cartesian are of the form described in 3.0.6(1), and all left d–outer triangles whose left leg is invertible are of the form described in 3.0.6(2). To show that the collection of right (resp. left) d–outer triangles is closed under right (resp. left) whiskering, it will hence suffice to show that the collection of triangles $\alpha \colon \Delta^2 \to \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$ whose adjoint $\hat{\alpha} \colon \Delta^1 \otimes \Delta^2 \to \mathcal{C}$ sends $\Delta^{\{1\}} \times \Delta^2$

to a thin triangle, is closed under both left and right whiskering. Indeed, this is exactly the preimage in $\mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C})$ under the *codomain projection* of the collection of thin triangles in $\mathcal{C}$, and the collection of thin triangles is closed under whiskering from both sides. □

Remark 3.0.8 Passing to opposites, Theorem 3.0.7 implies that the codomain projection

$$\mathrm{cod}\colon \mathrm{Fun}^{\mathrm{opgr}}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

is 2–outer cocartesian fibration, whose cocartesian arrows and outer triangles admit similar descriptions. Note that we have switched not only from domain to codomain but also from $\mathrm{Fun}^{\mathrm{gr}}$ to $\mathrm{Fun}^{\mathrm{opgr}}$. If we only switch from domain to codomain then the resulting projection

$$\mathrm{cod}\colon \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

is a 2–inner cocartesian fibration. This claim does not formally follow from the outer statement, but its proof can be obtained using a completely analogous argument, replacing the key extension result of Lemma 3.0.2 with one involving the pushout product of $\partial\Delta^1 \to \Delta^1$ and a suitable *inner* horn inclusion. Similarly, the projection

$$\mathrm{d}\colon \mathrm{Fun}^{\mathrm{opgr}}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

is a 2–inner cartesian fibration.

4 Enriched cartesian fibrations

In this section we define the notion of cartesian fibration in the context of marked simplicial categories. The definition is motivated by the one given for 2–categories by Buckley in [5], which we recall in Section 4.1.

4.1 Recollection: fibrations of 2–categories

Fibrations of 2–categories were initially introduced by Hermida [11]. A suitably modified definition was later given by Buckley [5], who also proved an (un)straightening-type result. In what follows we give a concise summary of the main results of loc. cit., to be considered as a motivation for the discussion of simplicial categories in Section 4.2.

Definition 4.1.1 Let $p\colon \mathcal{E} \to \mathcal{B}$ be a 2–functor between 2–categories.

- A 1–cell $f : x \to y$ in $\mathcal{E}$ is *p–cartesian* if the following square is a pullback of categories for every $a \in \mathcal{E}$:

$$\begin{array}{ccc} \mathcal{E}(a,x) & \xrightarrow{f\circ -} & \mathcal{E}(a,y) \\ \downarrow{\scriptstyle p_{a,x}} & & \downarrow{\scriptstyle p_{a,y}} \\ \mathcal{B}(pa,px) & \xrightarrow{p(f)\circ -} & \mathcal{B}(pa,py) \end{array}$$

- A 2–cell $\alpha\colon f \Rightarrow g : x \to y$ in $\mathcal{E}$ is *p–cartesian* if it is cartesian with respect to the induced functor $p_{x,y}\colon \mathcal{E}(x,y) \to \mathcal{B}(px,py)$.

The notion of cartesian fibration for 2–categories amounts to the existence of enough cartesian lifts, as in the 1–dimensional case, but it also requires an additional property: cartesian 2–cells must be closed under horizontal composition.

Definition 4.1.2 A 2–functor between 2–categories $p\colon \mathcal{E} \to \mathcal{B}$ is called a 2–*fibration* if it satisfies the following properties:

- (i) For every object $e \in \mathcal{E}$ and every 1–cell $f\colon b \to p(e)$ there exists a p–cartesian 1–cell $h\colon a \to e$ in $\mathcal{E}$ such that $p(h) = f$.
- (ii) For every pair of objects x, y in $\mathcal{E}$, the map $p_{x,y}\colon \mathcal{E}(x,y) \to \mathcal{B}(px, py)$ is a cartesian fibration of categories.
- (iii) Cartesian 2–cells are closed under horizontal composition, ie for every triple of objects (x, y, z) in $\mathcal{E}$, the functor $\circ_{x,y,z}\colon \mathcal{E}(y,z) \times \mathcal{E}(x,y) \to \mathcal{E}(x,z)$ sends $p_{y,z} \times p_{x,y}$–cartesian 1–cells to $p_{x,z}$–cartesian ones.

Replacing cartesian lifts for 1–cells by cocartesian lifts and similarly for 2–cells one may obtain four different variants of fibration, corresponding to the four possible types of variance for pseudofunctors $\mathcal{B} \to 2\text{–}\mathcal{C}at$.

Remark 4.1.3 Condition (iii) of Definition 4.1.2 is equivalent to requiring that given 1–cells in $\mathcal{E}$ of the form $f : w \to x$ and $g : y \to z$, the whiskering functors

$$- \circ f : E(x,y) \to \mathcal{E}(w,y) \quad \text{and} \quad g \circ - : \mathcal{E}(x,y) \to \mathcal{E}(x,z)$$

preserve cartesian 2–cells. This follows from the fact that horizontal composition can be obtained from vertical composition and whiskering composition.

The following result appears as Theorem 2.2.11 in [5].

Theorem 4.1.4 *There exists an equivalence of* 3*–categories between* $2\mathbf{Fib}_s(\mathcal{B})$ *and* $[\mathcal{B}^{\mathrm{op}}_{\mathrm{co}}, 2\text{–}\mathcal{C}at]$, *the former being the* 3*–category of fibrations equipped with a choice of cartesian lifts compatible with composition, while the latter is the* 3*–category of* (*strict*) 2*–functors into* $2\text{–}\mathcal{C}at$, *natural transformations and modifications.*

In the same paper, the author proves several weakenings of this statement, by looking at fibrations without a choice of lifts (which correspond to *pseudofunctors*) and fibrations of bicategories.

4.2 Cartesian fibrations of enriched categories

We now consider cartesian fibrations in the setting of $\mathcal{C}at_\infty$–categories. All definitions and statements can be dualized to the case of cocartesian fibrations by replacing all $\mathcal{C}at_\infty$–categories by their opposites.

Definition 4.2.1 Let $p\colon \mathcal{E} \to \mathcal{B}$ be a map of marked simplicial categories. A morphism $h \in \mathcal{E}(e', e)_0$ is said to be *p–cartesian* if the induced square

$$\begin{array}{ccc} \mathcal{E}(a, e') & \xrightarrow{h\circ -} & \mathcal{E}(a, e) \\ {\scriptstyle p_{a,e'}}\downarrow & & \downarrow{\scriptstyle p_{a,e}} \\ \mathcal{B}(pa, pe') & \xrightarrow{p(h)\circ -} & \mathcal{B}(pa, pe) \end{array}$$

is a homotopy pullback for the model structure on marked simplicial sets.

Definition 4.2.2 Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration of $\mathcal{C}at_\infty$–categories. We say that p is an *enriched cartesian fibration* if for any $e \in \mathcal{E}$ and any morphism $f \in \mathcal{B}(a, p(e))_0$ there exists a p–cartesian morphism $h \in \mathcal{E}(e', e)_0$, for some $e' \in \mathcal{E}$, such that $p(h) = f$.

Definition 4.2.3 Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration of $\mathcal{C}at_\infty$–categories. We say that p is an *enriched* 2*–inner* (resp. 2*–outer*) *fibration* if it satisfies the following properties:

(i) For every pair of objects (x, y) in $\mathcal{E}$, the map $p_{x,y}\colon \mathcal{E}(x, y) \to \mathcal{B}(px, py)$ is a cartesian (resp. cocartesian) fibration on the level of underlying simplicial sets.

(ii) For every pair of 0–simplices $u\colon y \to z$ and $v\colon w \to x$ in $\mathcal{E}$, the commutative squares

$$\begin{array}{ccc} \mathcal{E}(x, y) & \xrightarrow{u\circ -} & \mathcal{E}(x, z) \\ {\scriptstyle p_{x,y}}\downarrow & & \downarrow{\scriptstyle p_{x,z}} \\ \mathcal{B}(px, py) & \xrightarrow{p(u)\circ -} & \mathcal{B}(px, pz) \end{array} \qquad \begin{array}{ccc} \mathcal{E}(x, y) & \xrightarrow{-\circ v} & \mathcal{E}(w, y) \\ \downarrow{\scriptstyle p_{x,y}} & & \downarrow{\scriptstyle p_{w,y}} \\ \mathcal{B}(px, py) & \xrightarrow{-\circ p(v)} & \mathcal{B}(pw, py) \end{array}$$

are morphisms of cartesian (resp. cocartesian) fibrations, ie the top horizontal maps in both squares preserve cartesian (resp. cocartesian) edges.

We will say that p is *an enriched* 2*–inner* (resp. 2*–outer*) *cartesian fibration* if it is an enriched 2–inner (resp. 2–outer) fibration and an enriched cartesian fibration.

Our goal in the present section is to prove the following:

Theorem 4.2.4 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration of $\mathcal{C}at_\infty$–categories. Then $p\colon \mathcal{E} \to \mathcal{B}$ is an enriched* 2*–inner* (*resp.* 2*–outer*) *cartesian fibration in the sense of Definition 4.2.2 if and only if*

$$\mathrm{N}^{\mathrm{sc}}(p)\colon \mathrm{N}^{\mathrm{sc}}\mathcal{E} \to \mathrm{N}^{\mathrm{sc}}\mathcal{B}$$

is a 2*–inner* (*resp.* 2*–outer*) *cartesian fibration of ∞–bicategories.*

The remainder of this section is devoted to the proof of Theorem 4.2.4.

Definition 4.2.5 We will denote by $\square^n = (\Delta^1)^n$ the *simplicial n–cube* and by $\partial\square^n$ its *boundary*, so that the inclusion $\partial\square^n \subseteq \square^n$ can be identified with the pushout-product of $\partial\Delta^1 \hookrightarrow \Delta^1$ with itself n times. For $i = 1, \dots, n$ we denote by $\sqcap^{n,i}_{\varepsilon} \hookrightarrow \square^n$ the iterated pushout-product

$$[\partial\Delta^1 \hookrightarrow \Delta^1]\square \cdots \square[\Delta^{\{\varepsilon\}} \hookrightarrow \Delta^1]\square \cdots \square[\partial\Delta^1 \hookrightarrow \Delta^1],$$

where $\varepsilon \in \{0, 1\}$ and $[\Delta^{\{\varepsilon\}} \hookrightarrow \Delta^1]$ appears in the i^{th} factor.

Lurie introduces in [12, Definition 3.1.1.1] the class of *cartesian anodyne* morphisms (called *marked anodyne* in loc. cit.), which is the smallest weakly saturated class generated by a certain set of monomorphisms of marked simplicial sets listed in [12, Definition 3.1.1.1], which contain, in particular, the collection of all (minimally marked) inner horn inclusions, as well as the marked outer horn inclusion $(\Lambda^n_n, \{\Delta^{\{n-1,n\}}\}) \hookrightarrow (\Delta^n, \{\Delta^{\{n-1,n\}}\})$. Dually, we shall call *cocartesian anodyne* maps the smallest weakly saturated class generated by the opposites of those maps (or simply those maps whose opposites are cartesian anodyne). By [12, Proposition 3.1.2.3], (co)cartesian anodyne maps are closed under pushout-product with monomorphisms.

Lemma 4.2.6 *For* $\varepsilon \in \{0, 1\}$, $n \geq 1$ *and* $1 \leq i \leq n$ *let* $E^i_{\varepsilon} \subseteq (\square^n)_1$ *be the set of all degenerate edges together with the edge* $(\varepsilon, \dots, \varepsilon) \times \Delta^1 \times (\varepsilon, \dots, \varepsilon)$, *where* Δ^1 *sits in the* i^{th} *place. Then the inclusion of marked simplicial sets* $(\sqcap^{n,i}_1, E^i_1) \hookrightarrow (\square^n, E^i_1)$ *is cartesian anodyne, and the inclusion of marked simplicial sets* $(\sqcap^{n,i}_0, E^i_0) \hookrightarrow (\square^n, E^i_0)$ *is cocartesian anodyne.*

Proof We note that the $\varepsilon = 0$ and $\varepsilon = 1$ statements imply each other by passing to opposites. We hence just prove the cartesian case. Ignoring the order of factors, we may identify the map $\sqcap^{n,i}_1 \hookrightarrow \square^n$ with the box product of $\partial\square^{n-1} \hookrightarrow \square^{n-1}$ and $\Delta^{\{1\}} \hookrightarrow \Delta^1$. The nondegenerate m–simplices of $\square^{n-1}$ which are not in $\partial\square^{n-1}$ correspond to maps of posets $[m] \to [1]^{n-1}$ whose projection to each factor $[1]$ is surjective. It then follows that the initial vertex of such an m–simplex must be $(0, \dots, 0)$, and the final vertex must be $(1, \dots, 1)$. Adding these nondegenerate simplices one by one in an order that respects dimensions (that is, first all the 1–dimensional ones, then all the 2–dimensional ones, etc, up to dimension $n-1$) results in a factorization of the map $(\sqcap^{n,i}_1, E^i_1) \hookrightarrow (\square^n, E^i_1)$ into a finite composite of pushouts of maps of the form

$$\Big(\Delta^{\{1\}} \times \Delta^m \coprod_{\Delta^{\{1\}} \times \partial\Delta^m} \Delta^1 \times \partial\Delta^m, \{\Delta^1 \times \Delta^{\{m\}}\}\Big) \to (\Delta^1 \times \Delta^m, \{\Delta^1 \times \Delta^{\{m\}}\})$$

for $m \geq 1$. It will hence suffice to show that each of these maps is cartesian anodyne. For $\ell = 0, \dots, m$, let

$$\tau_\ell \colon \Delta^{m+1} \to \Delta^1 \times \Delta^m$$

be the map given on vertices by the formula

$$\tau_\ell(j) = \begin{cases} (0, j) & \text{if } j \leq \ell, \\ (1, j-1) & \text{if } j > \ell, \end{cases}$$

and for $k = 0, \dots, m+1$ let $Z_k \subseteq (\Delta^1 \times \Delta^m, \{\Delta^1 \times \Delta^{\{m\}}\})$ be the marked simplicial subset obtained as the union of $[\Delta^1 \times \partial\Delta^m] \coprod_{\Delta^{\{1\}} \times \partial\Delta^m} [\Delta^{\{1\}} \times \Delta^m]$ and the simplices τ_ℓ for $0 \le \ell < k$. Set

$$Z_0 \overset{\text{def}}{=} \Big([\Delta^1 \times \partial\Delta^m] \coprod_{\Delta^{\{1\}} \times \partial\Delta^m} [\Delta^{\{1\}} \times \Delta^m], \{\Delta^1 \times \Delta^{\{m\}}\} \Big).$$

We then have an ascending filtration of marked simplicial sets

$$Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_{m+1} = (\Delta^1 \times \Delta^m, \{\Delta^1 \times \Delta^{\{m\}}\}).$$

For each $k = 0, \dots, m-1$ we then find a pushout square of marked simplicial sets

$$\begin{array}{ccc} (\Lambda^{m+1}_{k+1})^\flat & \longrightarrow & Z_k \\ \downarrow & & \downarrow \\ (\Delta^{m+1})^\flat & \longrightarrow & Z_{k+1} \end{array}$$

so that $Z_k \to Z_{k+1}$ is inner anodyne, and in particular cartesian anodyne. Finally, in the last step $k = m$ we find a pushout square of the form

$$\begin{array}{ccc} (\Lambda^{m+1}_{m+1}, \{\Delta^{\{m,m+1\}}\}) & \longrightarrow & Z_m \\ \downarrow & & \downarrow \\ (\Delta^{m+1}, \{\Delta^{\{m,m+1\}}\}) & \longrightarrow & Z_{m+1} \end{array}$$

so that $Z_m \to Z_{m+1}$ is cartesian anodyne, as desired. □

We recall the comparison of the notion of (co)cartesian 1–cells between the enriched and scaled models:

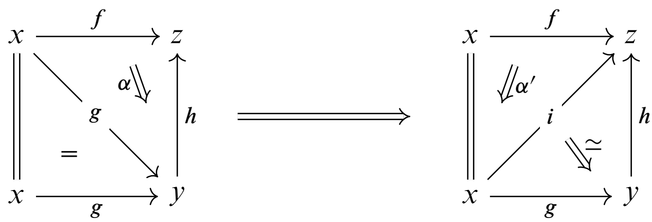
Proposition 4.2.7 [6, Proposition 3.1.3] *Given a fibration $p \colon \mathcal{E} \to \mathcal{B}$ of $\mathcal{C}at_\infty$–categories, an arrow in $\mathcal{E}$ is p–cartesian if and only if the corresponding 1–simplex of $\mathrm{N}^{\mathrm{sc}}\mathcal{E}$ is $\mathrm{N}^{\mathrm{sc}}(p)$–cartesian.*

Since the morphisms in a given $\mathcal{C}at_\infty$–category are in bijection with the edges of its scaled coherent nerve we readily obtain:

Corollary 4.2.8 *A fibration $p \colon \mathcal{E} \to \mathcal{B}$ of $\mathcal{C}at_\infty$–categories is cartesian if and only if $\mathrm{N}^{\mathrm{sc}}(p)$ is a cartesian fibration of ∞–bicategories* (*in the sense of Definition 1.4.9*).

We now consider the analogous question on the level of triangles.

Remark 4.2.9 Let $\mathcal{E}$ be a $\mathcal{C}at_\infty$–category and consider two 2–simplices α and α' of $\mathrm{N}^{\mathrm{sc}}\mathcal{E}$. If a 3–simplex ρ of $\mathrm{N}^{\mathrm{sc}}\mathcal{E}$ exhibits α as left-congruent to α' (cf Definition 2.3.3) in the form



then the 2–simplex ρ^* of $\mathcal{E}(x,z)$ corresponding to ρ goes from the 1–simplex α^* of $\mathcal{E}(x,z)$ corresponding to α to the composition of the 1–simplex $(\alpha')^*$ corresponding to α' followed by an equivalence. Hence, in the arrow ∞–category of $\mathcal{E}(x,z)$ the 3–simplex ρ induces an equivalence between α^* and the composition of $(\alpha')^*$ with an arrow of $\mathcal{E}(x,z)$ which is an equivalence.

Lemma 4.2.10 *Let $p\colon \mathcal{E}\to\mathcal{B}$ be a fibration of $\mathcal{C}at_\infty$–categories. Then a triangle $\alpha\colon \Delta^2\to \mathrm{N}^{\mathrm{sc}}\mathcal{E}$ of the form*

$$\begin{array}{ccc} x & \xrightarrow{\;h\;} & z \\ & \searrow^{f} \;\Downarrow\alpha\; \nearrow_{g} & \\ & y & \end{array}$$

is left $\mathrm{N}^{\mathrm{sc}}(p)$–inner if and only if the corresponding 1–simplex $\alpha^*\colon h\to gf$ in $\mathcal{E}(x,z)$ is $p_{x,z}$–cartesian and maps to a $p_{x',z}$–cartesian arrow in $\mathcal{E}(x',z)$ after precomposing with any arrow $x'\to x$. Similarly, α is right $\mathrm{N}^{\mathrm{sc}}(p)$–inner if and only if the corresponding 1–simplex $\alpha^*\colon h\to gf$ in $\mathcal{E}(x,z)$ is $p_{x,z}$–cartesian and maps to a $p_{x,z'}$–cartesian arrow in $\mathcal{E}(x,z')$ after postcomposing with any arrow $z\to z'$.

Proof We prove the left inner case. The proof for right inner triangles is completely analogous. Suppose first that α^* is $p_{x,z}$–cartesian in $\mathcal{E}(x,z)$ and maps to a $p_{x',z}$–cartesian arrow in $\mathcal{E}(x',z)$ after precomposing with any arrow $x'\to x$. By Remark 2.1.2 we have to provide a solution for every lifting problem of the form

$$\begin{array}{ccc} & \Delta^{\{n-2,n-1,n\}} & \\ \swarrow & & \searrow^{\alpha} \\ \Lambda^n_{n-1} & \longrightarrow & \mathrm{N}^{\mathrm{sc}}\mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{N}^{\mathrm{sc}}(p)} \\ \Delta^n & \longrightarrow & \mathrm{N}^{\mathrm{sc}}\mathcal{B} \end{array}$$

for $n\geq 3$. Transposing along the adjunction $\mathfrak{C}^{\mathrm{sc}}\dashv \mathrm{N}^{\mathrm{sc}}$, this corresponds to a lifting problem of the form

$$\begin{array}{ccc} & \mathfrak{C}^{\mathrm{sc}}\Delta^{\{n-2,n-1,n\}} & \\ \swarrow & & \searrow^{\alpha^*} \\ \mathfrak{C}^{\mathrm{sc}}\Lambda^n_{n-1} & \xrightarrow{\;f\;} & \mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathfrak{C}^{\mathrm{sc}}\Delta^n & \xrightarrow{\;g\;} & \mathcal{B} \end{array} \tag{10}$$

where we have committed a small abuse of language denoting by α^* also the transpose map $\mathfrak{C}^{\mathrm{sc}}\Delta^2\to\mathcal{E}$ induced by α. As a straightforward calculation shows, the lifting problem in (10) corresponds, at the level of marked simplicial sets, to the lifting problem

$$\begin{array}{ccc} (\sqcap_1^{n-1,n-1})^\flat & \longrightarrow & \mathcal{E}(x',z) \\ \downarrow & & \downarrow{\scriptstyle p_{x',z}} \\ (\square^{n-1})^\flat & \longrightarrow & \mathcal{B}(px',pz) \end{array}$$

where $x' := f(0)$. Moreover, the edge $(1,\dots,1)\times\Delta^1$ is mapped by the top horizontal map to the whiskering of the 1–simplex α^* by a sequence of 0–simplices connecting x' and x, and it is therefore $p_{x',z}$–cartesian in $\mathcal{E}(x',z)$. The desired lift hence exists by Lemma 4.2.6.

We now prove the "only if" direction, and so we assume that α is left p–inner. By Lemma 2.3.7 we may find a left degenerate triangle α' such that α is left congruent to α', so that α' is also left p–inner by Lemma 2.3.4. Then α' has the same first and last vertex as α and hence determines an edge $(\alpha')^*$ in $\mathcal{E}(x,z)$. Furthermore, as pointed out in the previous remark the 3–simplex exhibiting α' as left congruent to α also determines an equivalence in the arrow ∞–category of $\mathcal{E}(x,z)$ between the edge α^* and the edge given by $(\alpha')^*$ followed by an equivalence. Since every equivalence is $p_{x,z}$–cartesian and moreover the property of being $p_{x,z}$–cartesian is invariant under equivalences we may replace α by α', so that we may simply assume that α is left degenerate. We now consider the commutative diagram of marked simplicial sets

$$\begin{array}{ccccc} \mathrm{Hom}^{\triangleright}_{\mathrm{N}^{\mathrm{sc}}\mathcal{E}}(x,z) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{N}^{\mathrm{sc}}\mathcal{E}}(x,z) & \xleftarrow{\simeq} & \mathcal{E}(x,z) \\ \downarrow{\scriptstyle p^{\triangleright}_{x,y}} & & \downarrow & & \downarrow{\scriptstyle p_{x,y}} \\ \mathrm{Hom}^{\triangleright}_{\mathrm{N}^{\mathrm{sc}}\mathcal{B}}(x,z) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{N}^{\mathrm{sc}}\mathcal{B}}(x,z) & \xleftarrow{\simeq} & \mathcal{B}(x,z) \end{array} \tag{11}$$

in which the horizontal maps are marked categorical equivalences and the vertical maps are fibrations between fibrant objects in the marked categorical model structure. Here, the horizontal equivalences in the right column are given by the canonical isomorphism $\mathrm{Hom}_{\mathrm{N}^{\mathrm{sc}}\mathcal{E}}(x,z)\cong \mathrm{Un}^{\mathrm{sc}}\,\mathcal{E}(x,z)$, see [13, Remark 4.2.1], and the horizontal equivalences in the left column are established in [8, Proposition 2.33]. The triangle α determines arrows in both the top left and top right marked simplicial sets, and the images of these two arrows coincides in $\mathrm{Hom}_{\mathrm{N}^{\mathrm{sc}}\mathcal{E}}(x,z)$ by direct inspection. We hence obtain that the arrow α^* is $p_{x,z}$–cartesian in $\mathcal{E}(x,z)$ if and only if the arrow determined by α in $\mathrm{Hom}^{\triangleright}_{\mathrm{N}^{\mathrm{sc}}\mathcal{E}}(x,z)$ is $p^{\triangleright}_{x,z}$–cartesian. The latter, and hence the former, indeed holds when α is left p–inner by Remark 2.1.8. The desired implication is now a consequence of the closure of left p–inner triangles under left whiskering; see Remark 2.1.5. □

We now come to the outer counterpart of Lemma 4.2.10.

Lemma 4.2.11 *Let $p\colon\mathcal{E}\to\mathcal{B}$ be a fibration of $\mathcal{C}at_\infty$–categories. Let α^* be a triangle in $\mathrm{N}^{\mathrm{sc}}\mathcal{E}$ of the form*

$$\begin{array}{ccc} x & \xrightarrow{h} & z \\ & {\scriptstyle f}\searrow \;\Downarrow\alpha^*\; \nearrow{\scriptstyle g} & \\ & y & \end{array}$$

Then the following holds:

(i) *If the associated 1–simplex $\alpha^*\colon h\to gf$ is $p_{x,z}$–cocartesian in $\mathcal{E}(x,z)$ and $f\colon x\to y$ is $\mathrm{N}^{\mathrm{sc}}(p)$–cocartesian, then α is left $\mathrm{N}^{\mathrm{sc}}(p)$–outer.*

(ii) *If the associated 1–simplex $\alpha^*\colon h\to gf$ is $p_{x,z}$–cocartesian in $\mathcal{E}(x,z)$ and $g\colon y\to z$ is $\mathrm{N}^{\mathrm{sc}}(p)$–cartesian, then α is right $\mathrm{N}^{\mathrm{sc}}(p)$–outer.*

(iii) *If α is left $\mathrm{N}^{\mathrm{sc}}(p)$–outer and f is a degenerate edge, then α^* is $p_{x,z}$–cocartesian.*

(iv) *If α is right $\mathrm{N}^{\mathrm{sc}}(p)$–outer and g is a degenerate edge, then α^* is $p_{x,z}$–cocartesian.*

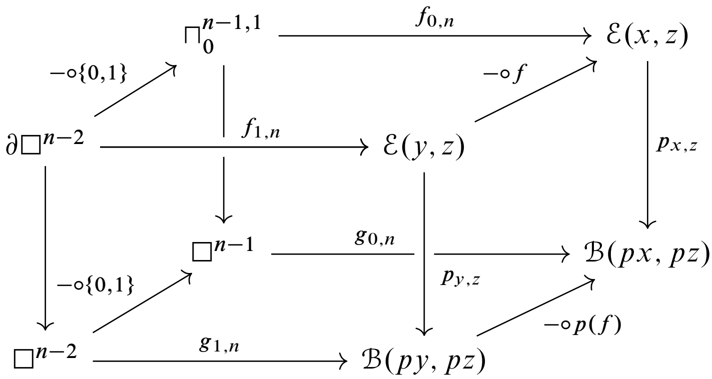
Proof Statements (iii) and (iv) follow from Remark 2.1.8 by using the commutative diagram (11) as in the proof of Lemma 4.2.10, and statements (i) and (ii) imply each other by passing to opposites. We now prove (i). Suppose that f is $\mathrm{N}^{\mathrm{sc}}(p)$–cocartesian and that α^* is $p_{x,z}$–cocartesian in $\mathcal{E}(x,z)$, and let us prove that α is left $\mathrm{N}^{\mathrm{sc}}(p)$–outer. By Remark 2.1.2 we have to provide a solution for every lifting problem of the form

(12)
$$\begin{array}{ccc} & \Delta^{\{0,1,n\}} & \\ \swarrow & & \searrow^{\alpha^*} \\ \Lambda^n_0 & \longrightarrow & \mathrm{N}^{\mathrm{sc}}\mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{N}^{\mathrm{sc}}(p)} \\ \Delta^n & \longrightarrow & \mathrm{N}^{\mathrm{sc}}\mathcal{B} \end{array}$$

Transposing along the adjunction $\mathfrak{C}^{\mathrm{sc}} \dashv \mathrm{N}^{\mathrm{sc}}$ we obtain an equivalent lifting problem of the form

$$\begin{array}{ccc} & \mathfrak{C}^{\mathrm{sc}}\Delta^{\{0,1,n\}} & \\ \swarrow & & \searrow^{\alpha^*} \\ \mathfrak{C}^{\mathrm{sc}}\Lambda^n_0 & \xrightarrow{\ f\ } & \mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathfrak{C}^{\mathrm{sc}}\Delta^n & \xrightarrow{\ g\ } & \mathcal{B} \end{array}$$

A simple combinatorial analysis shows that this amounts to compatibly solving the lifting problems determined by the back and front faces of the cube

(13)


where we have identified

$$\mathfrak{C}^{\mathrm{sc}}\Delta^n(0,n) = \square^{n-1}, \quad \mathfrak{C}^{\mathrm{sc}}\Lambda^n_0(0,n) = \sqcap^{n-1,1}_0, \quad \mathfrak{C}^{\mathrm{sc}}\Delta^n(1,n) = \square^{n-2}, \quad \mathfrak{C}^{\mathrm{sc}}\Lambda^n_0(1,n) = \partial\square^{n-2}.$$

Equivalently, we may consider this as a lifting problem in the arrow category of marked simplicial sets, involving the morphism between arrows encoded by the left square against the morphism between arrows encoded by the right square. We may endow this arrow category with the projective model structure, so that cofibrations are Reedy cofibrations and fibrations are levelwise. We then find that in the lifting problem encoded by the above cube, the left arrow constitutes a cofibration between cofibrant objects, while the right arrow is a fibration between fibrant objects (by our assumption that p is a fibration of $\mathcal{C}at_\infty$–categories).

Now, it is a general fact concerning model categories that the lifting property in a given square involving a cofibration between cofibrant objects against a fibration between fibrant objects is a *homotopy invariant property*, that is, it does not change if one replaces the given square by a levelwise weakly equivalent one of the same nature. In particular, in proving the existence of a lift we may as well replace the cube (13) with a levelwise weakly equivalent one, as long as we make sure it also has the property that its left square is Reedy cofibrant and its right square is levelwise fibrant with vertical legs fibrations. We now choose to make such a modification by simply replacing the corner $\mathcal{E}(y,z)$ with the fiber product

$$\mathcal{X} := \mathcal{E}(x,z) \times_{\mathcal{B}(px,pz)} \mathcal{B}(py,pz),$$

which is also a homotopy fiber product since the vertical legs are fibrations between fibrant objects. The map $\mathcal{E}(y,z) \to \mathcal{X}$ is an equivalence: indeed, by Proposition 4.2.7, $f \in \mathcal{E}(x,y)_0$ is a cocartesian arrow, and hence the right square in (13) is homotopy cartesian. We conclude that the new cube is levelwise equivalent to the old one, while clearly still keeping the same property of having its left square Reedy cofibrant and its right square levelwise fibrant with vertical maps fibrations. At the same time, by its construction, the data of a lift in the modified cube is the same as a lift in its back square

$$\begin{array}{ccc} \sqcap_0^{n-1,1} & \longrightarrow & \mathcal{E}(x,z) \\ \downarrow & & \downarrow{\scriptstyle p_{x,z}} \\ \square^{n-1} & \longrightarrow & \mathcal{B}(px,pz) \end{array}$$

where the edge corresponding to $\Delta^1 \times (0,\dots,0)$ in $\sqcap_0^{n-1,1}$ is sent to a $p_{x,z}$–cocartesian edge in $\mathcal{E}(x,z)$ by assumption. A solution therefore exists by Lemma 4.2.6, thus concluding the proof of the proposition. □

Proposition 4.2.12 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration of $\mathcal{C}at_\infty$–categories. Then p is an enriched 2–inner (resp. 2–outer) fibration if and only if*

$$\mathrm{N}^{\mathrm{sc}}(p)\colon \mathrm{N}^{\mathrm{sc}}\mathcal{E} \to \mathrm{N}^{\mathrm{sc}}\mathcal{B}$$

is a 2–inner (resp. 2–outer) fibration of ∞–bicategories.

Proof We first note that since p is a fibration between fibrant objects the same holds for $\mathrm{N}^{\mathrm{sc}}(p)$, and so the latter is always a weak fibration. We now recall that the vertices in $\mathrm{N}^{\mathrm{sc}}(\mathcal{E})$ correspond exactly to the objects of $\mathcal{E}$, and the edges $f\colon x \to y$ in $\mathrm{N}^{\mathrm{sc}}(\mathcal{E})$ correspond exactly to a pair of objects $x, y \in \mathcal{E}$ and a vertex $f \in \mathcal{E}(x,y)$. In addition, triangles

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ x & & \xrightarrow{\quad h \quad} & & z \end{array}$$

correspond bijectively to triples of objects $x, y, z \in \mathcal{E}$, triples of vertices $f \in \mathcal{E}(x,y)$, $g \in \mathcal{E}(y,z)$, $h \in \mathcal{E}(x,z)$ and an edge $\tau\colon h \Rightarrow g \circ f$ in $\mathcal{E}(x,z)$.

Considering inner lifts of triangles, it follows directly from Lemma 4.2.10 that the triangles in $\mathrm{N}^{\mathrm{sc}}(\mathcal{B})$ admits a sufficient supply of left (resp. right) $\mathrm{N}^{\mathrm{sc}}(p)$–inner lifts if and only if each $p_{x,y}\colon \mathcal{E}(x,y) \to \mathcal{B}(x,y)$ has a sufficient supply of cartesian lifts, and these cartesian lifts are closed under precomposition (resp. postcomposition) in $\mathcal{E}$. We may then conclude that $\mathrm{N}^{\mathrm{sc}}(p)$ is an 2–inner fibration if and only if $\mathcal{E}$ is a 2–inner fibration of $\mathcal{C}at_{\infty}$–categories.

We now consider the outer case. By Corollary 2.3.9 we may restrict attention to triangles in $\mathcal{B}$ with one leg degenerate and lifts whose same leg is degenerate in $\mathcal{E}$. We then deduce from Lemma 4.2.11 that the triangles in $\mathrm{N}^{\mathrm{sc}}(\mathcal{B})$ admits a sufficient supply of left (resp. right) p–outer lifts if and only if each $p_{x,y}\colon \mathcal{E}(x,y) \to \mathcal{B}(x,y)$ has a sufficient supply of cocartesian lifts. Since the definition of 2–outer fibration contains explicitly the closure under left/right whiskering (which corresponds, up to equivalence, to pre/post composition), we can conclude that $\mathrm{N}^{\mathrm{sc}}(p)$ is a 2–outer fibration if and only if $\mathcal{E}$ is an enriched 2–outer fibration of $\mathcal{C}at_{\infty}$–categories, as desired. □

We can now deduce our main result of interest:

Proof of Theorem 4.2.4 Our goal is to compare enriched 2–inner cartesian fibrations of $\mathcal{C}at_{\infty}$–categories with 2–inner cartesian fibrations of ∞–bicategories. The cartesian fibrational part is given by Corollary 4.2.8 while the 2–inner fibrational part is dealt with by Proposition 4.2.12. Combining the two results the theorem is thereby proven. □

We finish this section by collecting a few corollaries which can be easily deduced from the comparison of Theorem 4.2.4.

Corollary 4.2.13 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a 2–inner (co)cartesian fibration. Then a triangle $\sigma\colon \Delta^2 \to \mathcal{E}$ is left p–inner if and only if it is right p–inner.*

Proof Since N^{sc} is a right Quillen equivalence there exists a fibration $q\colon \mathcal{C} \to \mathcal{D}$ of $\mathcal{C}at_{\infty}$–categories which fits in a commutative diagram of the form

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ \simeq\ } & \mathrm{N}^{\mathrm{sc}}\mathcal{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{N}^{\mathrm{sc}}q} \\ \mathcal{B} & \xrightarrow{\ \simeq\ } & \mathrm{N}^{\mathrm{sc}}\mathcal{D} \end{array} \tag{14}$$

where the horizontal maps are equivalences of ∞–bicategories. Now the right vertical arrow is a bicategorical fibration, being the image under a right Quillen functor of a Dwyer–Kan fibration, while the left vertical arrow is a bicategorical fibration by Proposition 1.4.13. Applying Proposition 2.4.1 we now obtain that the right vertical map is a 2–inner/outer (co)cartesian fibration and that the top horizontal map preserves and detects left/right inner triangles. It will hence suffice to prove the left and right $\mathrm{N}^{\mathrm{sc}}(p)$–inner triangles coincide in $\mathrm{N}^{\mathrm{sc}}(\mathcal{C})$. By Lemma 4.2.10 this amounts to showing that for every $x, y \in \mathcal{C}$ and morphism $e \in \mathcal{C}(x,y)_1$ in the mapping ∞–category, the condition that e is cartesian with respect to $q_{x,y}\colon \mathcal{C}(x,y) \to \mathcal{D}(qx,qy)$ and remains cartesian after postcomposing with any morphism $y \to z$ is equivalent to the condition that e is $q_{x,y}$–cartesian and remains cartesian after precomposing with any morphism $w \to x$. Indeed, by Theorem 4.2.4 we have that $q\colon \mathcal{C} \to \mathcal{D}$ is an enriched 2–inner cartesian fibration of $\mathcal{C}at_\infty$–categories, and so both conditions are equivalent to e simply being $q_{x,y}$–cartesian. □

Corollary 4.2.14 *Let $\mathcal{E}$ and $\mathcal{B}$ two ∞–bicategories.*

(i) *A given 2–inner (co)cartesian fibration $p\colon \mathcal{E} \to \mathcal{B}$ is equivalent to a 1–inner (co)cartesian fibration if and only if every triangle is left and right inner.*

(ii) *A given 2–outer (co)cartesian fibration $p\colon \mathcal{E} \to \mathcal{B}$ is equivalent to a 1–outer (co)cartesian fibration if and only if every triangle whose left leg is p–cocartesian is left outer and every triangle whose right leg is p–cartesian is right outer.*

Proof From the explicit description of Remark 2.1.2 one immediately finds that if p is a 1–inner (co)cartesian fibration then every triangle is both left and right p–inner. Similarly, if p is a 1–outer cartesian fibration then any (co)cartesian arrow is automatically strongly (co)cartesian by [6, Proposition 2.3.7] (and its dual), and so every triangle whose left leg is p–cocartesian is left p–outer and any triangle whose right leg is p–cartesian is right p–outer.

To prove the "if" direction, we now invoke the fact that N^{sc} is a right Quillen equivalence to deduce the existence of fibration $q\colon \mathcal{C} \to \mathcal{D}$ of $\mathcal{C}at_\infty$–categories which fits in a commutative diagram of the form

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\simeq} & \mathrm{N}^{\mathrm{sc}}\mathcal{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{N}^{\mathrm{sc}}q} \\ \mathcal{B} & \xrightarrow{\simeq} & \mathrm{N}^{\mathrm{sc}}\mathcal{D} \end{array} \tag{15}$$

whose horizontal maps are equivalences of ∞–bicategories. Now the right vertical arrow is a bicategorical fibration, being the image under a right Quillen functor of a Dwyer-Kan fibration, while the left vertical arrow is a bicategorical fibration by Proposition 1.4.13. Applying Proposition 2.4.1 we now obtain that the right vertical map is a 2–inner/outer (co)cartesian fibration, and hence by Theorem 4.2.4 the functor $q\colon \mathcal{C} \to \mathcal{D}$ is an enriched 2–inner/outer (co)cartesian fibration of $\mathcal{C}at_\infty$–categories. For every $x, y \in \mathcal{E}$ with images $x', y' \in \mathrm{N}^{\mathrm{sc}}\mathcal{C}$ (which we can identify with objects of $\mathcal{C}$) we may then consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}^{\triangleright}_{\mathcal{E}}(x,y) & \xrightarrow{\simeq} & \mathrm{Hom}^{\triangleright}_{\mathrm{N}^{\mathrm{sc}}\mathcal{C}}(x',y') & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{N}^{\mathrm{sc}}\mathcal{C}}(x',y') & \xleftarrow{\simeq} & \mathcal{C}(x',y') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}^{\triangleright}_{\mathcal{B}}(px,py) & \xrightarrow{\simeq} & \mathrm{Hom}^{\triangleright}_{\mathrm{N}^{\mathrm{sc}}\mathcal{D}}(qx',qy') & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{N}^{\mathrm{sc}}\mathcal{D}}(qx',qy') & \xleftarrow{\simeq} & \mathcal{D}(qx',qy') \end{array}, \tag{16}$$

in which all vertical arrows are cartesian fibrations in the inner case and cocartesian fibrations in the outer case. Our assumption implies in particular that every left degenerate triangle in $\mathcal{E}$ is left p–inner/outer, which by Remark 2.1.8 implies that the leftmost vertical map in (16) is a left fibration in the outer case and a right fibration in the inner case. The same consequently holds for all vertical maps in (16), which implies that $q: \mathcal{C} \to \mathcal{D}$ is an enriched 1–inner/outer (co)cartesian fibration of $\mathcal{C}at_\infty$–categories. By [6, Proposition 3.1.3] this means that $\mathrm{N}^{\mathrm{sc}}q: \mathrm{N}^{\mathrm{sc}}\mathcal{C} \to \mathrm{N}^{\mathrm{sc}}\mathcal{D}$ is a 1–inner/outer (co)cartesian fibration. Since the square (15) is an equivalence between two bicategorical fibrations between fibrant objects, these two fibrations satisfy the same right lifting properties. Applying this to the right lifting properties of Definition 1.4.3, we conclude that if the right one is a 1–inner fibration then so is the left, and if the right one is a 1–outer fibration then so is the left. We may consequently deduce that $p: \mathcal{E} \to \mathcal{B}$ is a 1–inner/outer (co)cartesian fibration, as desired. □

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On the profinite distinguishability of hyperbolic Dehn fillings of finite-volume 3–manifolds

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We use the Culler–Shalen machine and tools from model theory to study the profinite rigidity of residually finite groups, especially 3–manifold groups. We borrow a transfer principle from model theory to apply to $\mathbb{C}$–character varieties in order to study cofinite collections of $\mathbb{F}_p$–character varieties and prove that under certain finiteness conditions weaker than non-Hakenness, they all have the same (finite) cardinality. We prove that residually finite groups satisfying a niceness property are almost relatively profinitely distinguishable within a geometrically relevant class, and we finish up by applying that result to knot complements in S^3 in particular.

20F65, 57M07; 03C07, 03C52

1 Introduction

One of the foundational lines of research in modern geometry has been the study of 3–manifold topology; some relevant recent progress has been made primarily through teasing out differences between different manifolds through appeal to their fundamental groups; see for example Wise [34], Agol [1], Wilton and Zalesskii [32] and Przytycki and Wise [22].

It is well known that fundamental groups of finite-volume hyperbolic 3–manifolds are finitely presented, and additionally that they are residually finite; see Maltsev [18]. Thus it arises as a natural line of inquiry to try to distinguish the fundamental groups of finite-volume 3–manifolds by looking at their finite quotients. This leads us to ideas of profinite equivalence and distinguishability; compare Noskov, Remeslennikov and Romankov [21], Reid [23] and Wilton and Zalesskii [33], among many others. In particular, Wilton and Zalesskii [32, Theorem 8.4] show that the profinite completion of a geometric 3–manifold group determines its geometry, and further, in [33], they show that the profinite completion of a 3–manifold group also determines the JSJ decomposition of the manifold.

The study of finite-volume hyperbolic 3–manifolds is itself a central area within 3–manifold topology: among 3–manifolds, the possession of a hyperbolic structure is the "generic" case, as we see by Thurston's hyperbolic Dehn surgery theorem [30, Theorem 2.6], and which is explored in more formal and precise detail by Maher in [17]. We recall further that the study of Dehn fillings is fundamental to 3–manifold topology, and that the volumes of the Dehn fillings $M_{p/q}$ — notation which is defined later on in Definition 2.6 — of a hyperbolic 3–manifold M are strictly less than and converge to vol M,[1] by [10, Thurston's theorem]. Additionally, Mostow's rigidity theorem says that geometric invariants of a complete and finite-volume hyperbolic 3–manifold M, such as volume, are topological invariants, and thus give invariants of their fundamental groups. Accordingly, we have even further reason to suspect that the study of finite-volume hyperbolic 3–manifolds might be fruitful. Mostow rigidity implies that the fundamental groups $\pi_1(M_{p/q})$ are different, and it becomes a natural question whether the profinite completions of these groups are also distinguishable. This is a question that we answer in the affirmative in cofinitely many cases.

It is an open question as to whether all finite-volume hyperbolic 3–manifold groups are absolutely profinitely rigid, and furthermore, it is also currently unknown whether there exists some pair of nonisometric hyperbolic manifolds that are not profinitely distinguishable. In particular, we have the following theorem, proved in Section 4, with the precise definition of the notation $|\chi^I_{\mathbb{C}}(\Gamma)|$ we use for character varieties in the theorem below found in Definition 2.12:

Theorem A *Let Γ be any finitely generated residually finite group with $|\chi^I_{\mathbb{C}}(\Gamma)| < \infty$, and let M be an oriented finite-volume hyperbolic 3–manifold with a single cusp. Then $\Lambda = \pi_1(M_{m/n})$ has $\widehat{\Gamma} \ncong \widehat{\Lambda}$ for all but finitely many choices of orbifold surgery coefficient m/n with hyperbolic Dehn filling $M_{m/n}$.*

Our journey takes us through representation theory, as well: it turns out to be easier to look at the $\mathrm{SL}(2,k)$–representations of 3–manifold groups over careful choices of field k rather than at the groups themselves, and in turn at the character variety of a given representation rather than at the representation itself. By a result of Culler and Shalen [8, Proposition 1.5.2], which here is Proposition 2.13, a point in the character variety of a 3–manifold group picks out an irreducible representation up to conjugacy, and by a result of [8] — Theorem 2.14 here — the character variety of a non-Haken 3–manifold group is in fact finite.

The "special sauce" here is our use of model theory as described by Marker [19], employing a Lefschetzian transfer principle rather than algebraic geometry in order to transport statements between the zero- and positive-characteristic cases of algebraically closed fields. We carefully construct model-theoretic predicates which we use to represent matrices, representations, and even character varieties, permitting us to bring otherwise totally unfamiliar compactness results from logic to bear on questions more at home in geometric group theory. In particular, we follow Culler and Shalen in observing that the conjugacy classes of representations of $\mathrm{Hom}_I(\Gamma,\mathrm{SL}(2,k))$ correspond to the points of $V_\Gamma(k)$, and then noting that the defining equations for $V_\Gamma(k)$ arise from the defining relations for Γ, of which we may assume that there are only finitely many; this gives us a robust, mostly bidirectional link between definable sets and affine algebraic varieties.

Most importantly, this framework ensures that we can pass back and forth between $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\overline{\mathbb{F}}_p)$–representations as needed, which is crucial, as both have desirable properties: $\mathrm{SL}(2,\overline{\mathbb{F}}_p)$ is

[1] In fact, the order type of the set of all volumes of Dehn fillings of all 3–manifolds is ω^ω!

locally finite, which we find useful in controlling profinite extensions of maps in both Lemmas 2.19 and 4.6, but by contrast, representations into $\mathrm{SL}(2,\mathbb{C})$ are both much better understood and more directly connected to more concrete geometric applications. In particular, any finite-volume hyperbolic manifold corresponds naturally to a finite-covolume lattice within $(\mathrm{P})\mathrm{SL}(2,\mathbb{C})$, and this gives us a canonical representation. This model-theoretic approach allows us to give a much cleaner and more elementary proof of the following result than existing ones, and to prove a result like the one that follows it:

Theorem 1.1 *The equality $|\chi^I_{\mathbb{C}}(\Gamma)| = n$ holds if and only if $|\chi^I_p(\Gamma)| = n$ for cofinitely many p as well.*

Corollary 1.1.1 *Let M be a one-cusped, finite-volume, hyperbolic 3–manifold. Suppose $M_{m/n}$ is a hyperbolic Dehn filling of M with (orbifold) surgery coefficients m/n and with finite character variety (for instance, a non-Haken such filling), and let $\Gamma = \pi_1(M_{m/n})$. Let M_* be the set of all Dehn fillings $M_{m'/n'}$, and let Λ_* be the set of all fundamental groups of those manifolds. Then Γ is profinitely almost distinguishable within Λ_*.*

Acknowledgements We owe a debt of gratitude to Bridson, McReynolds, Reid and Spitler [5], who have used representation theory to think about profinite distinguishability of specific 3–manifold groups in a different way. Additionally, while this paper was still in thesis form, Liu's [14] appeared on the arXiv, and proves a more general version of Corollary 1.1.1 using sophisticated methods more closely hewing to orthodox geometric group theory, and which as a consequence concerns itself purely with 3–manifold groups; by contrast, our borrowing from model theory has the major advantage of being a more elementary approach, which also permits the study of more general objects.

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2 Preliminaries

2.1 Hyperbolic geometry and Dehn surgery

For a very readable treatment of foundational concepts in 3–manifold topology, see Hatcher [12].

Definition 2.1 Let M be a 3–manifold, S a compact surface properly embedded in M. Suppose there exists some disk $D \subseteq M$ such that $D \cap S = \partial D$, with the intersection being transverse. If ∂D bounds no disk in S, then we call D a *nontrivial compressing disk*, and S is a *compressible surface*. Otherwise, if S neither has a nontrivial compressing disk nor is an embedded 2–sphere, then S is an *incompressible surface*.

Definition 2.2 Let M be a connected 3–manifold. We call M *prime* if there exist no 3–manifolds $N_1, N_2 \not\cong S^3$ such that $N_1 \# N_2 \cong M$. We call M *irreducible* if every $S^2 \subseteq M$ bounds a 3–ball.

Lemma 2.3 [20, Lemma 1] *Let M be a prime 3–manifold. Then either M is irreducible or $M \cong S^2 \times S^1$.*

Definition 2.4 Let M be a compact, orientable, prime 3–manifold. Then we say that M is *Haken* if it has at least one properly embedded two-sided incompressible surface; if it has none, we call it *non-Haken*.

Definition 2.5 Let M be a connected Riemannian manifold. We say that M is a *hyperbolic 3–manifold* if it is complete and everywhere locally isometric to the hyperbolic 3–space $\mathbb{H}^3$.

We may note here that many knot complements $S^3 \backslash K$ have hyperbolic structure; a notable family of exceptions is the set of torus knots. For example, the figure-eight knot has a complement with hyperbolic structure; on the other hand, the trefoil knot does not. Additionally, we note (after Benedetti and Petronio [4, Chapter D]) that the boundary components of finite-volume hyperbolic 3–manifolds always comprise zero or more tori.

Definition 2.6 Let M be a 3–manifold such that ∂M consists of a single torus $T \cong S^1 \times S^1$, with $H_1(\partial T)$ generated by choices of longitude l and meridian m. For $p/q \in \mathbb{Q} \cup \infty$, with p coprime to q, the *Dehn filling of M along T with slope p/q* is given by $M \cup T'_{p/q}$ for T' a solid torus $T' \cong B^2 \times S^1$, where the union is a gluing along T such that the meridian of T' maps to a corresponding curve in ∂T homotopic to $q \cdot [m] + p \cdot [l]$, and where $T'_\infty = \varnothing$. We denote the resulting manifold by $M_{p/q}$.

Remark If $\gcd(p,q) \neq 1$, then we instead have the *orbifold Dehn surgery of M along T with slope p/q*. The result will be similar to the manifold case above, except that the nature of $T'_{p/q}$ is much more complex, in general depending primarily on $\gcd(p,q)$.

For a more complete description and characterization of manifold Dehn fillings as a special case of orbifold Dehn surgeries, we recommend referring to Cooper and Futer [7, Section 2.4] or to Thurston's notes [29].

Theorem 2.7 (one-cusp case of Thurston's hyperbolic Dehn surgery theorem [30, Theorem 2.6]) *Let M be a hyperbolic 3–manifold with a single cusp, and let $M(p/q)$ be the manifold obtained through applying a hyperbolic Dehn filling with surgery coefficient p/q to that cusp. Then if p/q differs from finitely many exceptional slopes, $M_{p/q}$ is also hyperbolic.*

Proposition 2.8 (adapted from the orbifold version of Thurston's hyperbolic Dehn surgery theorem [9, Theorem 5.3]) *Let M be a compact 3–manifold whose interior admits a complete hyperbolic structure such that ∂M consists of a single torus. Then there is a neighborhood U of ∞ in S^2 such that for all $p/q \in U$, the manifold $M_{p/q}$ also admits a hyperbolic structure.*

We note that the groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$ show up frequently in this paper. This is due to the fact that $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$: for an orientable finite volume hyperbolic 3–manifold M, $\Gamma = \pi_1(M)$, we can define M as $\mathbb{H}^3/\Gamma$, where Γ is a subgroup of $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$, so that since $\widetilde{M} = \mathbb{H}^3$, $\mathrm{PSL}(2,\mathbb{C})$ itself acts by deck transformations on M. What is more, the inclusion representation $\rho\colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ lifts to the discrete faithful representation $\widehat{\rho}\colon \Gamma \to \mathrm{SL}(2,\mathbb{C})$, while other hyperbolic Dehn surgeries yield representations of Γ that are discrete but no longer faithful; for more on this, refer to MacLachlan and Reid in [16] on pages 111–112. In particular:

Proposition 2.9 *Let M be an orientable finite volume hyperbolic 3–manifold with a single cusp, with $\Gamma = \pi_1(M)$. Let $\Gamma_{p/q} = \pi_1(M_{p/q})$. Then the discrete faithful representation of $\Gamma_{p/q}$ into $\mathrm{SL}(2,\mathbb{C})$ is also a nonfaithful but still discrete $\mathrm{SL}(2,\mathbb{C})$–representation of Γ.*

2.2 Representation theory

Definition 2.10 Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$. The *trace field* of Γ, $\mathrm{TF}(\Gamma)$, is the field generated by all traces of all elements of Γ, which can also be written equivalently as $\mathbb{Q}(\operatorname{tr}\Gamma)$. The *degree* of the trace field is the degree of the extension of $\mathrm{TF}(\Gamma)$ over $\mathbb{Q}$.

We'll write $\mathbb{Q}(\operatorname{tr}\Gamma)$ whenever we want to emphasize the trace field's nature as a number field, and $\mathrm{TF}(\Gamma)$ otherwise.

Theorem 2.11 [16, Theorem 3.1.2] *Let M be a finite-volume orientable hyperbolic 3–manifold, so that $\Gamma = \pi_1(M)$ is Kleinian. Then $\mathbb{Q}(\operatorname{tr}\Gamma)$ is a finite-degree extension of $\mathbb{Q}$.*

Next, we define notation to be used for the sake of clarity and concision for the rest of the paper.

Definition 2.12 Let G be a group, and k a field. Let $\mathrm{Hom}_{\mathrm{Irr}}(G, \mathrm{SL}(2,k))$ be the set of irreducible representations of G in $\mathrm{SL}(2,k)$. We define

$$\chi_k^I(\Gamma) := \mathrm{Hom}_{\mathrm{Irr}}(\Gamma, \mathrm{SL}(2,k))/{\sim},$$

where $\sim$ denotes the conjugacy relation. In particular,

$$\chi_{\mathbb{C}}^I(\Gamma) := \mathrm{Hom}_{\mathrm{Irr}}(\Gamma, \mathrm{SL}(2,\mathbb{C}))/{\sim} \quad \text{and} \quad \chi_{\overline{\mathbb{F}}_p}^I(\Gamma) := \mathrm{Hom}_{\mathrm{Irr}}(\Gamma, \mathrm{SL}(2,\overline{\mathbb{F}}_p))/{\sim}.$$

This last we write as $\chi_p^I(\Gamma)$ for the sake of further concision.

Proposition 2.13 [8, Proposition 1.5.2] *Let Π be a finitely generated group, and let ρ and σ be representations of Π into $\mathrm{SL}(2,\mathbb{C})$ with corresponding characters χ_ρ and χ_σ. Let $\chi_\rho = \chi_\sigma$, and assume that ρ is an irreducible representation. Then ρ and σ are conjugate.*

Proposition 2.13 gives us a bijection between the set of irreducible characters, and the set of irreducible representations up to conjugacy. Since traces (and thus characters) are invariant under conjugation, trace is in fact a complete invariant of conjugacy classes of irreducible representations. It thus suffices to consider the trace field of any choice of character within each conjugacy class. Accordingly, we call $\chi^I_{\mathbb{C}}(\Gamma)$ the $\mathbb{C}$–character variety of Γ. A natural way to think about the image of the character is therefore to look at its trace field, and the degree of the trace field over $\mathbb{Q}$ is one of several natural measures of complexity. Keeping this framing in mind, we make use of the following corollary:

Corollary 2.13.1 [8, Corollary 1.4.5] *$\chi^I_{\mathbb{C}}(\Gamma)$ is a closed algebraic set.*

We remark that this corollary illuminates an important part of our approach: not only are we fine with reducible varieties, but we explicitly anticipate that our character varieties might be reducible, and that its irreducible components give us important information about the structure of $\mathrm{Hom}_{\mathrm{Irr}}(G, \mathrm{SL}(2, \mathbb{C}))$. In particular, the number of irreducible components (in fact, the number of points) is the key invariant.

The Culler–Shalen machine associates locally separating incompressible surfaces of a manifold to positive-dimensional components of its character variety; consequently whenever a manifold has a positive-dimensional character variety, we know it to be Haken. This gives us the following corollary, which is an important finiteness result:

Theorem 2.14 [25, Lemma 2.2] *Let M be a non-Haken 3–manifold of finite volume, with $\Gamma = \pi_1(M)$. Then $|\chi^I_{\mathbb{C}}(\Gamma)|$ is finite and Γ has finitely many representations into $\mathrm{SL}(2, \mathbb{C})$ up to conjugacy.*

2.3 Profinite groups

We now introduce the other aspect of geometric group theory that we make use of in this paper. For a more complete and formal treatment of the elementary characterizing properties of profinite completions of groups, we recommend reading through [23].

Keeping in mind the fact that any group can be made into a topological group by endowing it with the discrete topology, we remind the reader of the following definition:

Definition 2.15 A *profinite* group is a topological group isomorphic to some inverse limit of an inverse system of finite groups with the discrete topology. Equivalently, profinite groups are compact, Hausdorff, totally disconnected topological groups, as per Ribes and Zalesskii in [26, Theorem 1.1.12].

The *profinite completion* of a group G, which we denote by $\widehat{G}$, is that profinite group whose choice of finite groups is the set of all G/N, where N ranges over the normal subgroups of G of finite index, and the homomorphisms are given by the partial ordering of reverse containment of normal subgroups.

We care about the profinite completion of a residually finite group G primarily because it packages together all data on maps from G to its quotient groups of finite order.

We may think of residual finiteness as the capacity for at least one of the finite-index (normal) subgroups of G to tell an arbitrary $g \in G \backslash \{1\}$ apart from the identity, and we may think of the profinite completion of a group $\widehat{G}$ to be the packaging together of all of this finite-index information.

Lemma 2.16 [26, adapted from Lemma 1.1.7] *Let G be a group, with $\widehat{G}$ its profinite completion. Let $\iota\colon G \to \widehat{G}$ given by $g \mapsto ([g]_i)_{i\in I}$ be the canonical map sending g to the I–indexed tuple of equivalence classes of g under quotienting by the normal subgroups $\{N_i\}_{i\in I}$. Then ι has dense image.*

The following proposition is well-known: see for example Reid [24, Section 2.2] as well as Ribes and Zalesskii [26, page 78] .

Proposition 2.17 *Given a group G, denote by $\mathcal{N}(G)$ the set of all finite-index normal subgroups of G. Then the following are equivalent:*

- *G is residually finite.*
- $\displaystyle\bigcap_{N\in\mathcal{N}(G)} N = \{1\}.$
- *The natural homomorphism $\iota\colon G \to \widehat{G}$ is injective.*

Lemma 2.18 [26, Lemma 3.2.1, page 79] *This canonical map ι satisfies the universal property that for any profinite group H and group map $f\colon G \to H$, there exists a unique $g\colon \widehat{G} \to H$ such that $g \circ \iota = f$.*

Lemma 2.19 *The group $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$ is locally finite. Furthermore, for G a finitely generated group, $\rho\colon G \to \mathrm{SL}(2, \overline{\mathbb{F}}_p)$ a representation, $\operatorname{im}\rho$ is also finite, and ρ extends uniquely to a map $\widehat{\rho}\colon \widehat{G} \to \mathrm{SL}(2, \overline{\mathbb{F}}_p)$.*

Proof It suffices to show that every finitely generated subgroup H of $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$ has $|H|$ finite. To see this, we note that some generator of H must have an entry h such that the minimal $\mathbb{F}_{p^k}$ it belongs to is maximal among all entries of all generators of H, and that neither addition nor matrix multiplication can increase that k for any element of H; finally, every group of the form $\mathrm{SL}(2, \mathbb{F}_{p^k})$ is finite. □

Definition 2.20 We say that two groups G, H are *profinitely equivalent* if $\widehat{G} \cong \widehat{H}$.

Definition 2.21 Let G be a residually finite group, and S a set of groups. We say that a residually finite group G is *almost profinitely distinguishable* within S if there are at most finitely many residually finite groups $H \in S$ such that $\widehat{G} \cong \widehat{H}$, we have $G \cong H$.

3 Model theory and its uses

What could model theory be doing in a paper on geometric group theory and representation theory? Well, we use it as a means to prove the following theorem, whose proof can be found in Section 4.1:

Theorem 1.1 *The equality $|\chi^I_{\mathbb{C}}(\Gamma)| = n$ holds if and only if $|\chi^I_p(\Gamma)| = n$ for cofinitely many p as well.*

Where model theory comes in is its ability to permit us to pass back and forth between working over $\mathbb{C}$ and over cofinite collections of $\overline{\mathbb{F}}_p$ — this is an example of a transfer principle. For early examples of this, see Lefschetz [13], Weil [31] or Seidenberg [28]; for their context in a more general logical framework, see Ax [2] in his proof of the Ax–Grothendieck theorem, Barwise and Eklof [3], or Robinson and Tarski [27]; and for the earliest modern form of the principle as used here, see Chapter 2 of Cherlin [6]. Subsequently, the aim of Section 4.1 is to apply techniques as found in Marker [19, Section 1] to weld together *algebraic varieties over fields* and the *definable sets they coincide with*, and then use these techniques to interpret character varieties $\chi^I_{\mathbb{C}}(\Gamma)$ in terms of the first-order theory of $\mathbb{C}$, also called ACF_0.[2] We remark more formally on this in a remark in Section 4.1 but for now, the motivating slogan to keep in mind is this: "The relations of our groups are equally validly affine algebraic conditions, and conjugacy classes of representations correspond to points of the resulting variety".

In any case, if for whatever reason you want to blackbox this, you can simply use Theorem 1.1. Otherwise, however, we first need a few closely related definitions and a pair of results from model theory, which we detail later on in Section 4.1; for a background reference not only for the model-theoretic techniques we use here but also introductory model theory as a whole, refer to [19]. First, though, a few of the important basics of the language and notation of model theory:

Definition 3.1 The *arity* of a function or predicate is the number of arguments (zero or more) that that function or predicate accepts. For example, addition is a function of arity 2; we describe it as *being* 2*–ary* or as *having arity* 2. A nullary (0–ary) function is a constant, and a nullary predicate is either truth ($\top$) or falsehood ($\bot$).

The *alphabet* of first-order logic consists of the quantifiers $\forall, \exists$, the logical symbols $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$, disambiguating punctuation like parentheses and brackets, infinitely many variables (which we may notate as we choose, as long as it is clear that they are variables), the equals symbol $=$, any number of predicates of any arity, and any number of functions of any arity.

A *term*, which represents an object, is always either a variable or a function of any arity, and its arguments can themselves can be either variables or functions, to finite depth.

A *formula*, which represents a statement, is always finitely long, and consists of one or more predicates of any arity (including binary equality), possibly modified or joined by logical symbols, with zero or more of its variables quantified, or *bound*. An unbound variable is said to be *free*. A *sentence in first-order logic*, or $\mathscr{L}_1$–sentence, is a formula with no free variables.

A *signature* of a first-order theory T is the set of zero or more functions or relations, each of any arity, that we choose to represent important constants, functions, operations, and relations of our structure of interest.

A *first-order theory* is a set of axioms, which are sentences whose symbols come from the ordinary alphabet of first-order logic along with symbols from the theory's signature. A sentence S is said to *hold*

[2]This is not, strictly speaking, true, as we're about to see. However, $\mathbb{C}$ is, up to isomorphism, the unique algebraically closed field of characteristic 0, and transcendence degree over $\mathbb{Q}$ (and thus also cardinality) given by $\mathfrak{c}$.

in a first-order theory T, or equivalently that T *models* S, if S can be proven from the axioms of T. We use the symbol $\models$ to represent that, that is, $T \models S$.

A first-order theory T is said to be *consistent* if there exists no statement s for which $T \models s$ and also $T \models \sim s$, that is, no false statement can be proven from its axioms.

A first-order theory T is said to be *semantically complete*, or just *complete*, if for every statement s, $(T \models s) \to (T \vdash s)$, that is, every statement true within T is also provable in T. In fact, by Gödel's completeness theorem, every first-order theory is complete.

Definition 3.2 The *first-order theory of fields* has signature given by the constants $0, 1$ and the binary functions $+, \times$. It has the axioms that addition makes the set into an abelian group, multiplication is associative, commutative and distributive with identity 1, $\neg 0 = 1$, and removing 0 makes the set into an abelian group under multiplication.

Definition 3.3 The *first-order theory of algebraically closed fields*, ACF, extends the first-order theory of fields by appending countably many axioms, one for each natural number, each of the form that every nontrivial polynomial of degree n has at least one root.

Definition 3.4 The *first-order theory of algebraically closed fields of characteristic p*, ACF_p, extends ACF by appending the additional axiom that

$$\overbrace{1+1+\cdots+1}^{p \text{ copies of } 1} = 0.$$

Definition 3.5 The *first-order theory of algebraically closed fields of characteristic* 0, ACF_0, extends ACF by appending countably many axioms, one for each prime p, each of the form

$$\neg \overbrace{1+1+\cdots+1}^{p \text{ copies of } 1} = 0.$$

Remark ACF models all and only those statements true of (or in) every algebraically closed field. Because all of the ACF_p and ACF_0 both extend the axioms defining ACF, anything that ACF models, every ACF_p and ACF_0 also model, though the reverse need not be true — for instance, trivially

$$\mathrm{ACF}_5 \models 1+1+1+1+1 = 0,$$

but

$$\mathrm{ACF}, \mathrm{ACF}_0 \not\models 1+1+1+1+1 = 0.$$

Similarly, if ACF models a statement, then that statement also holds of any specific algebraically closed field, but the reverse need not be true.

Lemma 3.6 [19, Corollary 1.2] *Let* ACF_0 *be the first-order theory of algebraically closed fields of characteristic* 0, *and for rational prime* p, *let* ACF_p *be the first-order theory of algebraically complete fields of characteristic* p. *Let* Σ *be an* $\mathscr{L}_1$*–sentence. Then the following are equivalent*:

(i) $\mathrm{ACF}_0 \models \Sigma$.

(ii) $\mathrm{ACF}_p \models \Sigma$ *for cofinitely many choices of* p.

(iii) $\mathrm{ACF}_p \models \Sigma$ *for infinitely many choices of* p.

(iv) $\mathbb{C} \models \Sigma$.

Lemma 3.7 *Let* $S(k)$ *be a first-order statement in the theory of a field* k. *Then* $S(\overline{\mathbb{F}}_p)$ *holds for infinitely* (*in fact*, *cofinitely*) *many choices of* p *if and only if* $S(\mathbb{C})$ *holds.*

Proof Assume that $S(\mathbb{C})$ holds. By Lemma 3.6, this means that $\mathrm{ACF}_0 \models S$, and thus also that $\mathrm{ACF}_p \models S$ for cofinitely many p, that is, $S(\overline{\mathbb{F}}_p)$ holds for those p. On the other hand, assume that $S(\mathbb{C})$ does not hold. Then by Lemma 3.6 again we must have $\mathrm{ACF}_0 \models \neg S$, and thus also that $\mathrm{ACF}_p \models \neg S$ for cofinitely many p, that is, $S(\overline{\mathbb{F}}_p)$ does not hold for those p. □

4 Main result

4.1 More model theory

With the preliminaries well in hand, we can begin to discuss the specific way we apply tools from model theory to the study of profinite rigidity.

Whenever we want to leave the decision of which field we're working over until later, we will just write ACF. Fixing k to be some arbitrary algebraically closed field,[3] we start by looking at how we can use ACF to talk about matrices in $\mathrm{SL}(2,k)$. Let x_1, x_2, x_3, x_4 be variables in ACF. Then we define the predicate

$$M(x_1, x_2, x_3, x_4)$$

to be

$$x_1 \cdot x_4 - x_2 \cdot x_3 = 1.$$

The attentive reader may notice that this is exactly the defining relation for the determinant of a matrix in $\mathrm{M}(2,k)$ to be 1 in terms of its elements. More subtly, and perhaps more powerfully, one may note a tactic that will be used throughout this section: namely, that we will make our predicates complex and full of equations, so that they can do the heavy lifting that a mere abstract tuple cannot do. More simply, though, we bundle 4–tuples of variables that satisfy M and notate them as matrices $A \in \mathrm{SL}(2,\mathbb{C})$ unless we really do need access to the entries.

[3]This still works for k an arbitrary ring instead, but in that case, many of the properties below might be much weaker or have otherwise misleading names.

To write that a given matrix is the identity is actually even easier. We define the predicate

$$\mathrm{Id}(x_1, x_2, x_3, x_4)$$

to be

$$x_1 = 1 \wedge x_2 = 0 \wedge x_3 = 0 \wedge x_4 = 1.$$

We can also just write $\mathrm{Id}(A)$, when we have a 4–tuple as mentioned above.

Before we can look at how to extend our method for talking about matrices of $\mathrm{SL}(2,k)$ in ACF to a method for talking about representations $\Gamma \to \mathrm{SL}(2,k)$, we are going to need to be able to write predicates that verify that each relation of Γ is satisfied. Consider how, for some finitely presented group

$$\Gamma = \langle L | R \rangle,$$

one might describe a new representation $\rho\colon \Gamma \to \mathrm{SL}(2,k)$. It suffices to specify what the map does to a given choice of generators of Γ, $(l_i) \mapsto \rho(l_i)$. It is worth noting that this gives us a natural map $\mathrm{Hom}(\Gamma, \mathrm{SL}(2,k)) \to k^{4l}$ determined by which element of $\mathrm{SL}(2,k)$ that particular $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(2,k))$ sends the ordered l–tuple of generators of Γ to, interpreted by reading off the matrix entries. This is certainly injective — different elements of $\mathrm{Hom}(\Gamma, \mathrm{SL}(2,k))$ send at least one generator of Γ to different matrices. What [8] gives us is the conceptualization of the image as also some vanishing set $V_\Gamma(k)$, and then additionally, using the next few results (when we have proven them), that we can also can recognize which points of k^{4l} are in $V_\Gamma(k)$ and are thus true representations by understanding $V_\Gamma(k)$ as the vanishing set of the polynomial relations in R, along with the polynomials ensuring that the generators map to elements of $\mathrm{SL}(2,k)$. But to make use of all this, we have to tackle the challenge of how to communicate all of the machinery used here in ACF first. Bringing this to ACF, let $(A_i) := A_1, \dots, A_l$ be 4–tuples such that

$$\bigwedge_{i=1}^{l} M(A_i).$$

That is, the vector in k^{4l} can be thought of more helpfully as a l–tuple of 4–tuples, each of which we conceptualize as a matrix. As such, we write the l–tuple $(A_i)_{i=1}^{l}$ as $\vec{A}$. A couple of lemmata on the relation between the entries of matrices and those of their products and inverses mean we can use ACF to talk about the satisfaction of relations:

Lemma 4.1 *Let $A, B \in \mathrm{SL}(2,k)$. Then the entries of AB and A^{-1} are polynomial in the entries of A and B.*

Corollary 4.1.1 *Let $F\langle L\rangle$ be the set of freely reduced words on some finite set of letters and their inverses, and let $l = |L|$ and $r \in F\langle L\rangle$. Let $f_r : k^{4l} \to k^4$ be the map treating successive 4–tuples of the argument as matrix elements of a generating set, interpreting concatenation as matrix multiplication, and inverses of letters as inverses of generators, to take r to its image under this choice of assignment. Then for all $\vec{z} \in k^{4l}$, $f_r(\vec{z})$ is polynomial in the z_i.*

In particular, we use this in the case where the z_i represent elements of $\mathrm{SL}(2,\mathbb{C})$.

Now that we have ensured that nothing goes horribly wrong when we talk about matrix inversions, matrix products, and free reduction in $\mathrm{SL}(2,k)$, we can now figure out how to use ACF to write that a relationship $r \in R$ is satisfied by some l–tuple of matrices. For $r \in R$, we define the predicate $\mathrm{SAT}_r(\vec{A})$ to be

$$\mathrm{Id}(f_r(\vec{A})).$$

That is, the f_r map reads off the word r and takes the l–tuple it spells out from the A_i to the identity, as is required by the fact that the $r \in R$ are the relations of Γ.

Perhaps the most important immediate idea of this section is that we can use these previous predicates to write a statement in first-order logic that says whether or not a given $4l$–tuple is a representation of Γ into $\mathrm{SL}(2,k)$. In particular, we define $\mathrm{REP}_\Gamma(\vec{A})$ to be

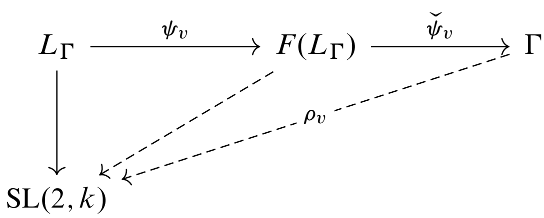
$$\Big(\bigwedge_{i=1}^{l} M(A_i)\Big) \wedge \Big(\bigwedge_{r\in R} \mathrm{SAT}_r(\vec{A})\Big),$$

where the Γ subscript reminds us that we started by fixing Γ and a presentation for Γ once and for all, and where the assumption of finite presentation ensures that the sentence is of finite length. Half-translating towards prose, this says that a $4l$–tuple A corresponds to a representation of Γ if all l of the 4–tuples are matrices of $\mathrm{SL}(2,k)$, and that all of the relations $r \in R$ of Γ are satisfied.

Proposition 4.2 [8, Corollary to Proposition 1.4.1] *Let Γ be a finitely presented group, with finite presentation $\Gamma = \langle L|R\rangle$, and let $V_\Gamma(k) = \{\vec{A} \mid \mathrm{REP}_\Gamma(\vec{A})\} = \{\vec{A} \mid V(f_r(\vec{A}) - I_2), M(A_i)_{i=1}^{l}\} \subseteq k^{4l}$. Then the map $\Phi_\Gamma\colon V_\Gamma \to \mathrm{Hom}(\Gamma, \mathrm{SL}(2,k))$ taking points in V_Γ to the representations that they determine is well-defined and is a bijection.*

Remark The Hilbert basis theorem permits us to assume finite generation without loss of generality, rather than the stronger condition of finite presentation, and, in proofs, to pass to the case of finite presentation rather than finite generation.

Proof We start by noting that the first part of REP_Γ uses the M predicate to check that the $4l$–tuple genuinely is an l–tuple of matrices in $\mathrm{SL}(2,k)$. Thinking once more of $\vec{A}$ as a $4l$–tuple v, this defines a map $\psi_v\colon L \to \mathrm{SL}(2,k), a_i \mapsto A_i$. Then by the universal property of free groups, this extends to a map $\check{\psi}_v\colon F(L) \to \mathrm{SL}(2,k)$ under multiplication and inverses:



Subsequently, the second part of REP_Γ uses each $\mathrm{SAT}(r)$ to check that each relation of Γ is satisfied, and by the universal property of groups defined by presentations, this uniquely determines a representation $\rho_v\colon \Gamma \to \mathrm{SL}(2,k)$. But then that means that $\Phi_\Gamma(v) = \rho_v$. □

Remark After [8, Section 1], we call $V_\Gamma(k) \subseteq k^{4l}$ the *definable k–points of the* SL$(2,k)$*–representation variety of* Γ, or (trading precision for readability by abusing notation once again) simply the SL$(2,k)$–*representation variety of* Γ. We justify this deliberate confusion as Culler and Shalen do: by observing that the conjugacy classes of representations of $\mathrm{Hom}_I(\Gamma, \mathrm{SL}(2,k))$ correspond to the points of $V_\Gamma(k)$ in the natural[4] way, and subsequently recalling that the defining equations for $V_\Gamma(k)$ arise from the defining relations for Γ, of which (by the Hilbert basis theorem) we may assume without loss of generality that there are only finitely many.

Now that we have established that we can talk about whether a given $4l$–tuple corresponds to a representation of Γ, we can talk about whether that representation is irreducible. As it turns out, though, it is much easier to start with reducibility. We recall that a representation into SL$(2,k)$ is *reducible* if the action by all of the images of the generators on k^2 fix some line through the origin: more formally, for some generator-dependent $\lambda \in k$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ ranges over the images of all generators of Γ. We can therefore define the predicate $\mathrm{RED}(\vec{A})$ to be

$$\mathrm{REP}_\Gamma(\vec{A}) \wedge \exists a\, \exists b \bigwedge_{i=1}^{l} \exists \lambda_i : ((a,b) \neq (0,0)) \wedge A_i \cdot \langle a,b \rangle = \lambda_i \langle a,b \rangle,$$

where we treat $\langle a,b \rangle$ as a column vector, and matrix and scalar multiplication are accordingly appropriately defined.

Proposition 4.3 *For all* $4l$*–tuples* (A_i), $\mathrm{ACF} \models \mathrm{RED}(\vec{A})$ *if and only if* $\Phi(\vec{A})$ *is a reducible representation.*

We can then talk about irreducibility, defining the predicate $\mathrm{IRREP}(\vec{A})$ to be

$$\mathrm{REP}_G(\vec{A}) \wedge \neg\, \mathrm{RED}(\vec{A}).$$

With a little more work, we can also talk about whether two representations are conjugate. We recall that two representations $\rho, \sigma : \Gamma \to \mathrm{SL}(2,k)$ are conjugate if there exists some matrix M such that for all i, $M\rho(x_i)M^{-1} = \sigma(x_i)$, where x_i is the i^{th} generator of G under some fixed choice of ordering. We can represent this in model theory by defining the predicate $\mathrm{CONJ}(\vec{A}, \vec{B})$ to be

$$\mathrm{REP}_\Gamma(\vec{A}) \wedge \mathrm{REP}_\Gamma(\vec{B}) \wedge \left(\exists C : M(C) \wedge \bigwedge_{j=1}^{l} (C A_j C^{-1} = B_j) \right).$$

Remark If $\mathrm{ACF} \models \mathrm{CONJ}(\vec{A}, \vec{B})$, then $\mathrm{ACF} \models \mathrm{REP}_\Gamma(\vec{A})$ and $\mathrm{ACF} \models \mathrm{REP}_\Gamma(\vec{B})$, so that $\Phi(\vec{A})$ and $\Phi(\vec{B})$ are both defined.

[4]By taking each point in $V_\Gamma(k)$ to correspond to the representation whose images under the representation are exactly the successive 4–tuples of coordinates of that point.

Proposition 4.4 *For all pairs $\vec{A}, \vec{B}$ of $4l$–tuples,* $\mathrm{ACF} \models \mathrm{CONJ}(\vec{A}, \vec{B})$ *if and only if* $\Phi(\vec{A}) \sim \Phi(\vec{B})$, *that is, there exists some* $C \in \mathrm{SL}(2,k)$ *with* $\tau_C \circ \Phi(\vec{A}) = \Phi(\vec{B})$, *where* τ_C *is the inner automorphism of* $\mathrm{SL}(2,k)$ *that* C *defines.*

The whole point of this section, of course, was to be able to talk about the number of irreducible representations of a given finitely generated group G into $\mathrm{SL}(2,k)$, up to conjugacy. However, this certainly is not a proper sentence in first-order logic. We might think of the "half-translated" version of $\Sigma_{\Gamma,n}$ as the following:

> There exist n irreducible representations up to conjugacy of Γ into $\mathrm{SL}(2,k)$, they are not conjugate to each other, and any other irreducible representation of G into $\mathrm{SL}(2,k)$ must be conjugate to one of the n representations previously mentioned.

We now have all the tools we need to write the previously mentioned sentence $\Sigma_{\Gamma,n}$; this will be almost exactly a predicate-by-predicate, symbol-by-symbol calque of what was written above as the half-translation, making use of the predicates we have constructed here. We write the first-order sentence $\Sigma_{\Gamma,n}$ as

$$\bigwedge_{j=1}^{n} \exists \vec{A}^{(j)} : \mathrm{IRREP}(\vec{A}^{(j)}) \wedge \Bigg(\bigwedge_{\substack{j,j'=1 \\ j \neq j'}}^{n} \neg\, \mathrm{CONJ}(\vec{A}^{(j)}, \vec{A}^{(j')}) \Bigg) \wedge \forall \vec{B} : \Bigg(\mathrm{IRREP}(\vec{B}) \Rightarrow \bigvee_{j=1}^{n} \mathrm{CONJ}(\vec{B}, \vec{A}^{(j)}) \Bigg).$$

Theorem 4.5 *Let Γ be a finitely generated group. Then the above sentence, $\Sigma_{\Gamma,n}$, is a sentence in* ACF *saying that for all algebraically closed fields k, $\Sigma_{\Gamma,n}(k)$ is true if and only if $|\chi_k^I(\Gamma)| = n$.*

Proof Assume first that Γ is finitely presented.

($\Leftarrow$) Suppose that $|\chi_k^I(\Gamma)| = n$. Then by completeness of first-order logic, to verify that $\mathrm{ACF} \models \Sigma_{\Gamma,n}$, it suffices to carefully step through the sentence itself to verify its meaning. We cut the sentence $\Sigma_{\Gamma,n}$ into three parts along the two conjunctions. The first clause asserts that there exists some family of n tuples of appropriate length $\{\vec{A}^{(i)}\}_{j=1}^{n}$, each of which corresponds to an irreducible representation in itself. The second clause asserts that any distinct pair of those representations is nonconjugate. The final clause asserts that for all tuples $\vec{B}$ of the same length as the $\vec{A}^{(j)}$, if B corresponds to an irreducible representation, then that representation must be conjugate to the representation corresponding to one of the $\vec{A}^{(j)}$. Taken in sum, the sentence asserts that $|\chi_k^I(\Gamma)| = n$, and since we know it to be the case, ACF models it.

($\Rightarrow$) Suppose that $\mathrm{ACF} \models \Sigma_{\Gamma,n}$. Since from the previous part we know that $\Sigma_{\Gamma,n}$ asserts that $|\chi_k^I(\Gamma)| = n$ within ACF, by consistency of first-order logic we know that $|\chi_k^I(\Gamma)| = n$.

Now if Γ is merely finitely generated and not finitely presented, we must invoke the Hilbert basis theorem: using it, we have that there exists a finitely presented $\widetilde{\Gamma}$, $\eta\colon \widetilde{\Gamma} \to \Gamma$ such that the induced map $\eta^*\colon \chi_k^I(\Gamma) \to \chi_k^I(\widetilde{\Gamma})$ is a bijection. This $\widetilde{\Gamma}$ is generated by any generating set for Γ, and its relations are the finitely many relations that correspond to the finitely many equations that the Hilbert basis theorem gives us. Then the sentence $\Sigma_{\widetilde{\Gamma},n}$ for $\widetilde{\Gamma}$ also works for Γ. □

Theorem 1.1 *The equality $|\chi^I_{\mathbb{C}}(\Gamma)| = n$ holds if and only if $|\chi^I_p(\Gamma)| = n$ for cofinitely many p as well.*

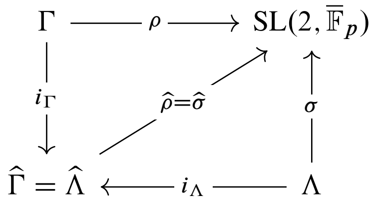
Proof This follows immediately from Lemma 3.6 and Theorem 4.5. We recall that we can use $\Sigma_{\Gamma,n}$ in ACF_0 and ACF_p to encode the finiteness property we care about. Using this, we can now apply transfer principles: since $|\chi^I_{\mathbb{C}}(\Gamma)| = n$, $|\chi^I_p(\Gamma)| = n$ for infinitely many p, and thus by Lemma 3.6, for cofinitely many p. □

4.2 Constraints on profinite completions

Now that we have established that we can relate (and thus constrain) information about representations into $\mathrm{SL}(2,\mathbb{C})$ and into $\mathrm{SL}(2,\overline{\mathbb{F}}_p)$, we can start to get a sense of what this means for the profinite distinguishability of groups.

Lemma 4.6 *Let Γ and Λ be two finitely generated groups such that $\widehat{\Gamma} \cong \widehat{\Lambda}$. Suppose that $|\chi^I_p(\Gamma)| = n$ for cofinitely many p. Then $|\chi^I_p(\Gamma)| = |\chi^I_p(\Lambda)| = n$ for those p.*

Proof Consider the commutative diagram



We note that by the universal property of profinite completions, any representation $\rho\colon \Gamma \to \mathrm{SL}(2,\overline{\mathbb{F}}_p)$ extends profinitely to a representation $\widehat{\rho}\colon \widehat{\Gamma} \to \mathrm{SL}(2,\overline{\mathbb{F}}_p)$ by the local finiteness of $\mathrm{SL}(2,\overline{\mathbb{F}}_p)$, as in Lemma 2.19. By the commutativity of the diagram, any representation from $\chi^I_p(\Gamma)$ must factor as a composition of the canonical injection into $\widehat{\Gamma}$ and a representation from $\chi^I_p(\widehat{\Gamma})$, so that $|\chi^I_p(\Gamma)| = |\chi^I_p(\widehat{\Gamma})|$. Looking at the other half of the diagram, we note that since $\widehat{\Gamma} = \widehat{\Lambda}$, the same argument applies in reverse: compositions of the canonical injection of Λ into $\widehat{\Lambda}$ with representations from $\chi^I_p(\widehat{\Lambda})$ must yield all of $\chi^I_p(\Lambda)$, so that $|\chi^I_p(\Lambda)| = |\chi^I_p(\widehat{\Lambda})|$. □

Theorem 4.7 *Let Γ and Λ be two finitely generated groups with $\widehat{\Gamma} \cong \widehat{\Lambda}$ and $|\chi^I_{\mathbb{C}}(\Gamma)| = n$ for some $n \in \mathbb{N}$. Then $|\chi^I_{\mathbb{C}}(\Lambda)| = |\chi^I_{\mathbb{C}}(\Gamma)| = n$.*

Proof Since $|\chi^I_{\mathbb{C}}(\Gamma)| = n$, by Theorem 1.1 we know that $|\chi^I_p(\Gamma)| = n$ for cofinitely many p. By profinite equivalence and Lemma 4.6, we know that $|\chi^I_p(\Lambda)| = |\chi^I_p(\Gamma)| = n$. Finally, by another application of Theorem 1.1, $|\chi^I_{\mathbb{C}}(\Lambda)| = n$. □

Theorem 4.8 *Let $\Gamma = \pi_1(M)$, where M is a compact hyperbolic 3–manifold. If $\deg(\mathrm{TF}(\Gamma)) \geq d$ for some $d \in \mathbb{N}$, then $|\chi^I_{\mathbb{C}}(\Gamma)| \geq d$.*

Proof Let Γ be the fundamental group of a finite-volume hyperbolic 3–manifold. Let $k = \mathbb{Q}(\operatorname{tr}\Gamma)$ be its trace field, so that $A_0\Gamma = \left\{\sum_{i<n} a_i\gamma_i \mid n \in \mathbb{N}, a_i \in \mathbb{Q}(\operatorname{tr}\Gamma), \gamma_i \in \Gamma\right\}$ is the quaternion algebra over Γ that k generates. Let $\theta\colon \Gamma \hookrightarrow A_0\Gamma$ be the natural inclusion, let $\phi\colon A_0\Gamma \to \mathrm{SL}(1, A_0\Gamma)$ be any irreducible representation, and let $\{\sigma_i\}\colon k \hookrightarrow \mathbb{C}$ be a family of distinct embeddings of the trace field. Then we may extend the scalars of ϕ by tensoring over the different images of k, taking $A_0\Gamma \otimes_{\sigma_i(k)} \mathbb{C}$ and the corresponding $\widehat{\phi}_i\colon A_0\Gamma \to \mathrm{SL}(1, A_0\Gamma \otimes_{\sigma_i(k)} \mathbb{C})$; we know that $A_0\Gamma \otimes_{\sigma_i(k)} \mathbb{C}$ also extends a quaternion algebra over $\mathbb{C}$ and is thus split, so that $\mathrm{SL}(1, A_0\Gamma \otimes_{\sigma_i(k)} \mathbb{C})$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})$.

Then the result will follow if $\rho_i = \widehat{\phi}_i \circ \theta$ is a representation $\rho_i\colon \Gamma \to \mathrm{SL}(2,\mathbb{C})$, and if for $i \neq j$, ρ_i and ρ_j are nonconjugate.

To see this, recall that k is a number field by [16, Theorem 3.1.2], and that k, being the trace field of Γ, is generated by traces. Denote its degree over $\mathbb{Q}$ as $d = \deg(k)$, so that the $\{\sigma_i\}_{i=1}^d\colon k \hookrightarrow \mathbb{C}$ are its d embeddings. Then since the maps $\{\sigma_i\}\colon k \to \mathbb{C}$ are all different, and are all embeddings, they cannot agree on every $\gamma \in \Gamma$: there must exist some $\gamma \in \Gamma$ such that $\widehat{\sigma_i} \circ \theta(\gamma) \neq \widehat{\sigma_j} \circ \theta(\gamma)$.

But then given that $\rho_i = \widehat{\sigma_i} \circ \theta$ and $\rho_j = \widehat{\sigma_j} \circ \theta$, for $\operatorname{tr}\colon \mathrm{SL}(2,\mathbb{C}) \to \mathbb{C}$ the trace map, $\operatorname{tr} \circ \rho_i(\gamma) \neq \operatorname{tr} \circ \rho_j(\gamma)$. Thus γ represents a witnessing element of Γ on which the representations ρ_i and ρ_j have different traces, which by Proposition 2.13 tells us that the two representations cannot be conjugate. □

Having shown that profinite equivalence means that the number of representations up to conjugacy into $\mathrm{SL}(2,k)$ (if finite) are the same between the profinitely equivalent groups, the goal is now to attack the main theorem.

5 Main theorem

We need one last result from Long and Reid in [15].[5]

Theorem 5.1 [15, Theorem 3.2] *Let M be an orientable hyperbolic 3–manifold of finite volume and with a single cusp, and $d \in \mathbb{N}$. Then there are only finitely many surgery coefficients m/n such that $\operatorname{tr}(\rho(\pi_1(M_{m/n})) \in k$, where $\deg(k/\mathbb{Q}) \leq d$ as an extension.*

Theorem A *Let Γ be any finitely generated residually finite group with $|\chi^I_{\mathbb{C}}(\Gamma)| < \infty$, and let M be an oriented finite-volume hyperbolic 3–manifold with a single cusp. Then for all but finitely many choices of orbifold surgery coefficient m/n with hyperbolic Dehn filling $M_{m/n}$, $\Lambda = \pi_1(M_{m/n})$ has $\widehat{\Gamma} \not\cong \widehat{\Lambda}$.*

[5]We may remark that p, q need not be coprime; that is, orbifold surgeries are covered by the quoted result.

Proof It suffices to show that $\widehat{\Gamma} \cong \widehat{\Lambda}$ only for finitely many Λ; we thus assume that $\widehat{\Gamma} \cong \widehat{\Lambda}$. By assumption, $|\chi^I_{\mathbb{C}}(\Gamma)| < \infty$; let $|\chi^I_{\mathbb{C}}(\Gamma)| = d$. However, by Theorem 4.7, we know that since $\widehat{\Gamma} \cong \widehat{\Lambda}$, $|\chi^I_{\mathbb{C}}(\Gamma)| = |\chi^I_{\mathbb{C}}(\Lambda)|$. Now, by Theorem 5.1, we know that there are at most finitely many choices of surgery coefficient resulting in a manifold with degree of trace field of fundamental group with at most a given degree $d+1$, and by Theorem 4.8, we know that if $\deg(\mathrm{TF}(\Lambda)) > d$, then $|\chi^I_{\mathbb{C}}(\Lambda)| > d$ as well, so that it is exactly these finitely many choices of surgery coefficient where it is even possible for us to have $|\chi^I_{\mathbb{C}}(\Lambda)| = d$. Finally, we recall that by Proposition 2.13, irreducible representations with the same trace are always conjugate, so we know that it suffices to check that the characters of the two groups differ. □

The above result thus lends itself to the following more geometrically focused corollary:

Corollary 1.1.1 *Let M be a one-cusped, finite-volume, hyperbolic 3–manifold. Suppose $M_{m/n}$ is a hyperbolic Dehn filling of M with (orbifold) surgery coefficients m/n and with finite character variety (for instance, a non-Haken such filling), and let $\Gamma = \pi_1(M_{m/n})$. Let M_* be the set of all Dehn fillings $M_{m'/n'}$, and let Λ_* be the set of all fundamental groups of those manifolds. Then Γ is profinitely almost distinguishable within Λ_*.*

Proof We may start by passing without loss of generality to the case where $M_{m'/n'}$ has hyperbolic structure, thanks to [32, Theorems A and 8.4]. By assumption, $|\chi^I_{\mathbb{C}}(\Gamma)| < \infty$, so Theorem A applies. □

Remark Liu in [14] proves a more general version of this case using completely different methods.

We can extend this to the question that actually motivated this entire line of inquiry, with a little help from [11].

Definition 5.2 A knot K is *small* if its complement $S^3 \backslash K$ contains no closed incompressible surface.

Theorem 5.3 [11, unnumbered theorem from pages 373–374] *Let K be a small knot. Then all but finitely many of its Dehn fillings $M_{m/n} = \{S^3 \backslash K\}_{m/n}$ are non-Haken.*

This allows us to narrow in on the trailhead for this line of thought: that we can use all of this to say something interesting about hyperbolic Dehn fillings of small knots.

Corollary 5.3.1 *Let K be a small knot such that $S^3 \backslash K = M$ is a one-cusped, finite-volume, hyperbolic 3–manifold. Let $\Gamma = \pi_1(M_{m/n})$ be the fundamental group of a non-Haken hyperbolic Dehn filling of M with (orbifold) surgery coefficients m/n. Let M_* be the set of all Dehn fillings $M_{m'/n'}$, and let Λ_* be the set of all fundamental groups of those manifolds. Then Γ is profinitely almost distinguishable within Λ_*.*

Proof This follows from Theorems 2.14 and 5.3 as a special case of Theorem A. □

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Index-bounded relative symplectic cohomology

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We study the relative symplectic cohomology with the help of an index-bounded contact form. For a Liouville domain with an index-bounded boundary, we construct a spectral sequence which starts from its classical symplectic cohomology and converges to the relative symplectic cohomology of it inside a Calabi–Yau manifold.

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1 Introduction

Given a closed symplectic manifold (M,ω) and a compact subset D of M, the relative symplectic cohomology $\mathrm{SH}_M(D)$ is a Floer-theoretic invariant, which captures both dynamical and topological information of the pair (M,D). Its construction by Varolgunes [2018] is based on the Hamiltonian Floer theory of M with more algebraic ingredients. Roughly speaking, one considers a family of increasing Hamiltonian functions that go to zero on D while going to positive infinity on $M-D$. Then $\mathrm{SH}_M(D)$ is defined as the homology of a completed telescope of Floer complexes, given by this family of Hamiltonian functions.

The idea of using Hamiltonian functions "localized at a subset" may date back to work of Cieliebak, Floer and Hofer [Cieliebak et al. 1995] and Viterbo [1990]. Recently there have been several new versions of Hamiltonian Floer theories related to this idea and aimed at various local-to-global problems. Besides [Varolgunes 2018], let us mention an incomplete list: [Groman 2023; McLean 2020; Venkatesh 2018]. A priori, the definitions in these papers are different, depending on whether M is open or closed, taking completion with the action filtration or the Novikov filtration, and the orders of taking different limits. It would be interesting to compare these theories to attack particular problems. But in this article we mainly focus on the version of $\mathrm{SH}_M(D)$ by Varolgunes.

Along its definition in [Varolgunes 2018], several good properties of this invariant $\mathrm{SH}_M(D)$ have been established, including the Hamiltonian isotopy invariance, a Künneth formula, a displaceability criterion and the Mayer–Vietoris property. These properties indicate that this invariant would play an important role in symplectic topology and mirror symmetry. One motivation of it comes from mirror symmetry suggested in [Seidel 2012] and the family Floer program. On the other hand, some symplectic topological applications have already appeared in [Tonkonog and Varolgunes 2023; Dickstein et al. 2024]. Also see its relation with quantum cohomology by Borman, Sheridan and Varolgunes [Borman et al. 2022].

Here we study a computational side of this invariant, and provide some applications to symplectic topology. The main goal is to construct a filtration on the underlying complex of this cohomology and look at the induced spectral sequence. Now we set up notation and state our results. Let (M,ω) be a closed symplectic manifold. We say M is *symplectic Calabi–Yau* if $c_1(TM)=0$. A *convex domain* (D,θ) in M is a compact codimension–0 symplectic submanifold of M, with a boundary ∂D and a 1–form θ locally defined near ∂D, such that the restriction of θ to ∂D is a contact form and the local Liouville vector field points outward. A *Liouville domain* (D,θ) in M is a convex domain in M such that the 1–form θ is defined on all of D and the restriction of ω on D is $d\theta$. We will focus on a special family of convex domains, whose restrictions of θ to ∂D are *index-bounded* contact forms; see Definition 2.2.

For a Liouville domain in M we will equip it with an auxiliary form $\tilde{\omega}$ which represents a class in $H^2(M,D;\mathbb{R})$, see Lemma 3.1. If $[\tilde{\omega}]$ has integral values on $H_2(M,D;\mathbb{Z})$, then we say it is integral. This auxiliary form $\tilde{\omega}$ will be used to characterize how far a Floer solution travels outside D.

Let Λ_0 be the Novikov ring and Λ_E be the truncated Novikov ring; see Section 2 for the notation. Our main result is the following.

Theorem 1.1 *Let (M,ω) be a closed symplectic Calabi–Yau manifold and D be a Liouville domain in M with an index-bounded boundary. Suppose that $[\tilde{\omega}]$ is integral. Given any positive number E, there is a truncated invariant $\mathrm{SH}_M(D;\Lambda_E)$ such that*:

(1) *There is a spectral sequence that starts from the classical symplectic cohomology $\mathrm{SH}(D;\Lambda_E)$ with coefficient Λ_E and converges to $\mathrm{SH}_M(D;\Lambda_E)$.*

(2) *If the class $[\tilde{\omega}]$ vanishes on $H_2(M,D)$, then the above spectral sequence degenerates at the first page, which shows that $\mathrm{SH}_M(D;\Lambda_E)\cong \mathrm{SH}(D;\Lambda_E)$.*

(3) *For an increasing sequence $E_1<E_2<\cdots$ that goes to positive infinity, the inverse limit of the truncated invariant recovers the relative symplectic cohomology*
$$\big(\varprojlim_{E_i} \mathrm{SH}^k_M(D;\Lambda_{E_i})\big)\otimes_{\Lambda_0}\Lambda \cong \mathrm{SH}^k_M(D)\otimes_{\Lambda_0}\Lambda.$$

Remark 1.2 In the definition of an index-bounded contact form, we assume that it is nondegenerate. We will give a perturbative method in Section 5 that works for Morse–Bott nondegenerate contact forms.

The proof of the above theorem draws much inspiration from [McLean 2020] and its usage of the index-bounded condition. Now we sketch the proof. The Hamiltonian functions we are using to compute $\mathrm{SH}_M(D)$ are approximately zero on D and positive infinity outside D, as a direct limit. The nonconstant periodic orbits of our interest lie around the boundary of D; see Figure 1. We call the integral of the Hamiltonian function over a periodic orbit the *level of the orbit*. Then the levels of these orbits can go to either zero or positive infinity. We first define a valuation on the free module generated by these orbits. However, due to the completion procedure, there will be elements with a negative infinite valuation which comes from limits of orbits going to infinite high level. Then we use the index-bounded condition to ignore these high limits of orbits. As a consequence, the original underlying complex of this relative invariant is quasi-isomorphic to a new complex without the high limits. And the same valuation on the new complex gives an exhaustive filtration, which will induce a convergent spectral sequence.

An important class of examples fitting into the above theorem comes from simply connected Lagrangian submanifolds in Calabi–Yau manifolds. Let L be a simply connected Lagrangian submanifold in a Calabi–Yau manifold M. Take D as a Weinstein neighborhood of L, which is isomorphic to a disk bundle $D_r T^*L$ of the cotangent bundle of L, with respect to some Riemannian metric g on L. There is a correspondence between the geodesics of g and the Reeb orbits on the cosphere bundle of T^*L. Hence the index-bounded condition for the contact form on the cosphere bundle will be satisfied if the metric g satisfies some relations between the length of closed geodesics and their Morse indices. For many simply connected manifolds, the existence of such a nice Riemannian metric is known. (In Section 5 we show it is true when g has a positive Ricci curvature.) Then we obtain a spectral sequence starting from $\mathrm{SH}(D;\Lambda_E)$ and converging to $\mathrm{SH}_M(D;\Lambda_E)$. Note that $\mathrm{SH}_M(D;\Lambda)\otimes_{\Lambda_0}\Lambda$ detects the displaceability of D inside M (Theorem 2.4), and it does not depend on r in the index-bounded case [Tonkonog and Varolgunes 2023, Proposition 1.13]. Hence we can let $r\to 0$ to detect the displaceability of L itself inside M. On the other hand, the usual invariant to detect the displaceability of L, the self-Lagrangian Floer cohomology $\mathrm{HF}(L)$ may not be defined due to possible holomorphic disks on L with Maslov index 0. Moreover, by using the Mayer–Vietoris property, we can also study the complement of Lagrangian submanifolds. We present one sample application of Theorem 1.1.

Proposition 1.3 *Let (M,ω) be a symplectic Calabi–Yau manifold with dimension greater than* 4 *and ω represents an integral class in $H^2(M)$. For a simply connected Lagrangian S in M and a Weinstein neighborhood U of S, we have that $M-U$ is not stably displaceable in M.*

Proof (See a more detailed proof in Proposition 5.9.) By using our spectral sequence we can show that $\mathrm{SH}^{2n}_M(U)\otimes_{\Lambda_0}\Lambda=0$, where $2n$ is the dimension of M. Hence the result follows from the stable displacement criterion (Theorem 2.4) and the Mayer–Vietoris property of the relative symplectic cohomology. □

Remark 1.4 This proposition can be regarded as an analogue of a result of Ishikawa [2016, Theorem 1.1]: if U is a round ball in a Calabi–Yau manifold M then $M-U$ is not stably displaceable; see also

[Tonkonog and Varolgunes 2023, Corollary 1.15]. Ishikawa's proof uses computations of spectral invariants of certain distance functions, which shows that $M-U$ is always a super-heavy set.

The outline of this article is as follows. In Section 2 we review backgrounds about Hamiltonian Floer theories. In Sections 3 and 4 we construct the filtration and show its properties to prove Theorem 1.1. In Section 5 we discuss some extensions of the theorems as well as applications.

Remark 1.5 We focus on the case that M is Calabi–Yau and D is index-bounded. More local-to-global results of $\mathrm{SH}_M(D)$ in other interesting cases can be found in [Borman et al. 2022; Groman and Varolgunes 2023].

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2 Background

Now we review the construction of symplectic cohomology theories. First we specify the ring and field that will be used. The Novikov ring Λ_0 and its field Λ of fractions are defined by

$$\Lambda_0=\left\{\sum_{i=0}^{\infty}a_iT^{\lambda_i}\;\middle|\;a_i\in\mathbb{C},\ \lambda_i\in\mathbb{R}_{\geq 0},\ \lambda_i<\lambda_{i+1},\ \lim_{i\to\infty}\lambda_i=+\infty\right\},$$

$$\Lambda=\left\{\sum_{i=0}^{\infty}a_iT^{\lambda_i}\;\middle|\;a_i\in\mathbb{C},\ \lambda_i\in\mathbb{R},\ \lambda_i<\lambda_{i+1},\ \lim_{i\to\infty}\lambda_i=+\infty\right\},$$

where T is a formal variable. The maximal ideal of Λ_0 is defined by

$$\Lambda_+=\left\{\sum_{i=0}^{\infty}a_iT^{\lambda_i}\;\middle|\;a_i\in\mathbb{C},\ \lambda_i\in\mathbb{R}_{>0},\ \lambda_i<\lambda_{i+1},\ \lim_{i\to\infty}\lambda_i=+\infty\right\}.$$

There is a valuation $v\colon\Lambda\to\mathbb{R}\cup\{+\infty\}$ by setting

$$v\left(\sum_{i=0}^{\infty}a_iT^{\lambda_i}\right):=\min_i\{\lambda_i\mid a_i\neq 0\}\quad\text{and}\quad v(0):=+\infty,$$

which makes Λ_0 a complete valuation ring. When we say the completion of a Λ_0–module we mean the completion with respect to this valuation. We write

$$\Lambda_{\geq r}:=v^{-1}([r,+\infty])\quad\text{and}\quad\Lambda_{>r}:=v^{-1}((r,+\infty])\quad\text{for all } r\in(-\infty,+\infty),$$

which are ideals of Λ_0. So Λ_0 is a short notation for $\Lambda_{\geq 0}$ and Λ_+ for $\Lambda_{>0}$. Later when we fix an energy bound $E>0$, we just write $\Lambda_E:=\Lambda_0/\Lambda_{\geq E}$.

2.1 Hamiltonian Floer theory on closed manifolds

Now we set up the background on Hamiltonian Floer theory. We work with a closed symplectic Calabi–Yau manifold (M,ω). Hence foundational details can be found in [Hofer and Salamon 1995; Salamon 1999].

A smooth function $H: M \to \mathbb{R}$ determines a smooth vector field X_H such that $dH(\cdot)=\omega(X_H,\cdot)$. We say H is a Hamiltonian function and X_H is the associated Hamiltonian vector field. Let $\mathcal{L}M := C^\infty(S^1,M)$ be the space of free loops in M, where we always view $S^1=\mathbb{R}/\mathbb{Z}$ and write t as the coordinate of S^1. We call t the time variable.

Moreover, we can consider a family of functions $H_t\colon M\times S^1\to\mathbb{R}$ parametrized by S^1. Then we have time-dependent Hamiltonian vector fields X_{H_t}. Integrating it we obtain a family of Hamiltonian symplectic diffeomorphisms $\phi^t\colon M\to M$. A loop $\gamma\in\mathcal{L}M$ is called a time–1 Hamiltonian orbit if $\gamma'(t)=X_{H_t}$. In this article, we only consider the component $\mathcal{L}_0M$ which contains contractible loops. Hence from now on, all Hamiltonian orbits are assumed to be contractible. We write

$$\mathcal{CP}_{H_t}=\{\gamma\in\mathcal{L}_0M \mid \gamma'(t)=X_{H_t}\}$$

as the set of contractible 1–periodic orbits of H_t. An orbit is nondegenerate if the Poincaré return map

$$d\phi^1\colon T_{\gamma(0)}M\to T_{\gamma(0)}M$$

does not have eigenvalue 1. And we say a Hamiltonian H_t is nondegenerate if all of its 1–periodic orbits are nondegenerate.

Next we assign an index $\mathrm{CZ}(\gamma)$ to each orbit γ, the Conley–Zehnder index; see [Salamon 1999, Lecture 2]. By the Calabi–Yau condition, this index does not depend on choices of cappings. We grade our orbits by setting

$$\mu_{H_t}(\gamma):=n+\mathrm{CZ}(\gamma), \tag{2-1}$$

where $2n$ is the real dimension of M. (Our grading is different from that in [Salamon 1999], since we use cohomology instead of homology.) When we say a degree–k or an index–k orbit we mean an orbit γ with $\mu_{H_t}(\gamma)=k$. We remark that for a constant orbit of a C^2–small Morse function, its degree defined above equals its Morse index.

Therefore $\mathcal{CP}_{H_t}$ becomes a graded set

$$\mathcal{CP}_{H_t}=\bigoplus_{k\in\mathbb{Z}}\mathcal{CP}^k_{H_t},$$

where $\mathcal{CP}^k_{H_t}$ is the set of orbits with index k.

Then for a nondegenerate Hamiltonian H_t, we define

$$\mathrm{CF}^k(H_t;\Lambda_0)=\left\{\sum_{i=1} c_i\gamma_i \;\middle|\; c_i\in\Lambda_0,\ \gamma_i\in\mathcal{CP}^k_{H_t}\right\}, \tag{2-2}$$

which is the free Λ_0–module generated by index–k orbits. Similarly we can define

$$\mathrm{CF}^k(H_t;\Lambda)=\left\{\sum_{i=1}c_i\gamma_i \;\middle|\; c_i\in\Lambda,\ \gamma_i\in\mathcal{CP}^k_{H_t}\right\}, \tag{2-3}$$

which is the Λ–vector space generated by index–k orbits. We don't grade the formal variable T.

By using a family of compatible almost-complex structures J_t, we can study the solutions of the Floer equation

$$\partial_s u+J_t(\partial_t u-X_{H_t})=0 \tag{2-4}$$

for $u:\mathbb{R}\times S^1\to M$. Here s is the $\mathbb{R}$–coordinate and t is the S^1–coordinate on the domain. For two orbits γ_-,γ_+ and a homotopy class $A\in\pi_2(M;\gamma_-\cup\gamma_+)$, consider the solution space

$$\begin{aligned}\mathcal{M}(\gamma_-,\gamma_+;A)=\big\{u:\mathbb{R}\times S^1\to M\mid \partial_s u+J_t(\partial_t u-X_{H_t})=0,\ u(-\infty,t)=\gamma_-,\\ u(+\infty,t)=\gamma_+,\ [u]=A\in\pi_2(M;\gamma_-\cup\gamma_+)\big\}/\sim.\end{aligned} \tag{2-5}$$

There is an $\mathbb{R}$–action on this space by translating the s–coordinate of a solution u, and $\sim$ is the quotient of this $\mathbb{R}$–action. The L^2–energy of a solution u is

$$0\le E(u)=\int|\partial_s u|^2=\int u^*\omega+\int_{\gamma_+}H_t-\int_{\gamma_-}H_t.$$

For generic pairs (H_t,J_t), the solution space is an l–dimensional manifold where

$$\mu_{H_t}(\gamma_+)-\mu_{H_t}(\gamma_-)=l+1.$$

We call a pair (H_t,J_t) satisfying the above condition a regular pair. By the Gromov–Floer compactness theorem, when γ_-,γ_+,A are fixed, the above solution spaces admit compactifications by adding broken Floer trajectories and J–holomorphic sphere bubbles. The bubbles can be ruled out by using the Calabi–Yau condition when the moduli space is 0– or 1–dimensional; see [Hofer and Salamon 1995]. There are also coherent orientations on these moduli spaces. In particular, when the moduli space is 0–dimensional, we can count the signed number of elements, which we denote by $n(\gamma_-,\gamma_+;A)$.

Then we define an operator

$$d:\mathrm{CF}^k(H_t)\to\mathrm{CF}^{k+1}(H_t),$$

with either Λ_0– or Λ–coefficients, by setting

$$d(\gamma_-):=\sum_{\gamma_+}\sum_{[u]=A}n(\gamma_-,\gamma_+;A)\cdot\gamma_+\cdot T^{\int u^*\omega+\int_{\gamma_+}H_t-\int_{\gamma_-}H_t}. \tag{2-6}$$

The right-hand side is summed over all γ_+ with $\mu_{H_t}(\gamma_+)-\mu_{H_t}(\gamma_-)=1$ and all classes A are in $\pi_2(M;\gamma_-\cup\gamma_-)$. It may not be a finite sum, but it converges as an element in $\mathrm{CF}^k(H_t)$, by the Gromov compactness theorem. Then we extend this operator Λ_0 or Λ–linearly to $\mathrm{CF}^k(H_t)$.

By the analysis of codimension–1 boundaries of $\mathcal{M}(\gamma_-,\gamma_+;A)$ with

$$\mu_{H_t}(\gamma_+)-\mu_{H_t}(\gamma_-)=2,$$

a big theorem in Hamiltonian Floer theory shows that $d^2=0$. Then we write the resulting cohomology groups as $\mathrm{HF}^k(H_t,J_t;\Lambda_0)$ and $\mathrm{HF}^k(H_t,J_t;\Lambda)$.

Another theorem shows that $\mathrm{HF}^k(H_t,J_t;\Lambda)$ is independent of the choices of generic pairs (H_t,J_t). Hence we can call it the Hamiltonian Floer cohomology of M. This invariance result is proved by considering continuation maps between different choices of (H_t,J_t). We sketch it here since we will use it later to define symplectic cohomology; see [Salamon 1999, Section 3.4] for a full proof.

For simplicity we only vary H_t. The case for J_t can be handled in the same way. Let H_t^α and H_t^β be two nondegenerate Hamiltonians. Assume that both (H_t^α,J_t) and (H_t^β,J_t) are regular for a fixed J_t. Then we choose a homotopy $H_{s,t}^{\alpha\beta}$ of Hamiltonians to connect H_t^α and H_t^β. That is,

(2-7)
$$H_{s,t}^{\alpha\beta}:\mathbb{R}\times S^1\times M\to\mathbb{R},\qquad H_{s,t}^{\alpha\beta}=\begin{cases}H_t^\alpha, & s\le -1,\\ H_t^\beta, & s\ge 1.\end{cases}$$

Then we consider the s–dependent Floer equation

(2-8)
$$\partial_s u+J_t(\partial_t u-X_{H_{s,t}^{\alpha\beta}})=0$$

and the moduli space

$$\mathcal{M}(\gamma_-^\alpha,\gamma_+^\beta;A)=\{u:\mathbb{R}\times S^1\to M\mid \partial_s u+J_t(\partial_t u-X_{H_{s,t}^{\alpha\beta}})=0,\\ u(-\infty,t)=\gamma_-^\alpha,u(+\infty,t)=\gamma_+^\beta,[u]=A\in\pi_2(M;\gamma_-^\alpha\cup\gamma_+^\beta)\}.$$

Note that now the equation is s–dependent; hence there is no $\mathbb{R}$–action. For a generic path $H_{s,t}^{\alpha\beta}$, the above moduli space is a manifold of dimension $\mu_{H_t^\beta}(\gamma_+)-\mu_{H_t^\alpha}(\gamma_-)$. And it admits a similar compactification by adding broken trajectories. When $\mu_{H_t^\beta}(\gamma_+)=\mu_{H_t^\alpha}(\gamma_-)$, we define an operator

(2-9)
$$f^{\alpha\beta}:\mathrm{CF}^k(H_t^\alpha)\to\mathrm{CF}^k(H_t^\beta)$$

by setting

$$f^{\alpha\beta}(\gamma_-^\alpha):=\sum_{\gamma_+^\beta}\sum_{[u]=A}n^{\alpha\beta}(\gamma_-^\alpha,\gamma_+^\beta;A)\cdot\gamma_+^\beta\cdot T^{\int u^*\omega+\int\partial_s(H_{s,t}^{\alpha\beta}(u(s,t)))},$$

where $n^{\alpha\beta}(\gamma_-^\alpha,\gamma_+^\beta;A)$ is a signed count of elements of the above moduli space. Note that the weight

(2-10)
$$\int u^*\omega+\int\partial_s(H_{s,t}^{\alpha\beta}(u(s,t)))=\int|\partial_s u|^2+\int\frac{\partial H_{s,t}^{\alpha\beta}}{\partial s}(u(s,t))$$

is not necessarily nonnegative; hence we need to use Λ–coefficients now. We call this weight the *topological energy* of a Floer solution. (It is nonnegative if the family of Hamiltonian functions satisfies that $\int\partial_s H_{s,t}^{\alpha\beta}\ge 0$.) Then one can show that $f^{\alpha\beta}$ is a chain map and moreover $f^{\alpha\beta}\circ f^{\beta\alpha}$ is chain homotopy equivalent to the identity map.

We add one more lemma here which will be used frequently in later sections.

Lemma 2.1 *Let M be a Calabi–Yau manifold and H_t be a nondegenerate Hamiltonian function on M. For a constant $\Delta > 0$, suppose that we have a homotopy $\{H_t^s\}_{s\in[0,1]}$ such that $H_t^s = H_t + s\Delta$. Then the continuation map between $\mathrm{CF}(H_t^0)$ and $\mathrm{CF}(H_t^1)$ is a multiplication by T^Δ. In particular, when $\Delta = 0$, this shows that the continuation map of a constant homotopy is the identity map.*

Proof Since our Hamiltonian functions are just a translation of a fixed one, the Floer equation of the continuation map (2-8) does not depend on s. Therefore any solution of it still carries an $\mathbb{R}$–action. By using the Calabi–Yau condition, we can pick a regular family J_t for H_t. For two different orbits γ and γ' with index k, any Floer solution of the continuation map equation connecting γ and γ' carries an $\mathbb{R}$–action. So the corresponding moduli space is at least 1–dimensional, contradicting that both γ and γ' have index k.

Then the continuation maps only exist when $\gamma = \gamma'$ and it will be a constant map. One can directly check that the constant map is regular and has contribution 1. Hence the continuation map between $\mathrm{CF}(H_t^0)$ and $\mathrm{CF}(H_t^1)$ is an identity matrix, weighted by the change of the Hamiltonian function which is T^Δ. □

2.2 Liouville domain and contact cylinder

Let (C,α) be a contact manifold with a contact form α. The *Reeb vector field* of α is the unique vector field R_α on C such that

$$d\alpha(R_\alpha,\cdot) = 0, \quad \alpha(R_\alpha) = 1.$$

Then a *Reeb orbit of length* $\lambda > 0$ is a map

$$\gamma(t)\colon \mathbb{R}/\lambda\mathbb{Z} \to C, \quad \frac{d}{dt}\gamma(t) = R_\alpha.$$

We write $\Gamma_{\alpha,\lambda} \subset C$ for the set formed by Reeb orbits of length λ and $\Gamma_\alpha := \bigcup_{\lambda>0}\Gamma_{\alpha,\lambda}$. We say a Reeb orbit is nondegenerate if the Poincaré return map of the Reeb flow does not have eigenvalue 1. And we say a contact form α is nondegenerate if all of its orbits are nondegenerate. We say α is Morse–Bott nondegenerate if, for all $\lambda > 0$, the set $\Gamma_{\alpha,\lambda}$ is a closed submanifold in C, the rank of $d\alpha\,|_{\Gamma_{\alpha,\lambda}}$ is locally constant, and $T_p\Gamma_{\alpha,\lambda} = \ker(T_p\phi_\lambda - id)$ for all $p \in \Gamma_{\alpha,\lambda}$, where ϕ_t is the Reeb flow. For a Morse–Bott nondegenerate contact form, one can define a Conley–Zehnder index of its Reeb orbits; see [Cieliebak and Mohnke 2018, Section 2.1] for more details on Reeb orbits and their indices. Then we have the following definition.

Definition 2.2 [Tonkonog and Varolgunes 2023, Definition 1.12] Suppose (M,ω) is Calabi–Yau. A contact hypersurface (C,α) in M is called index-bounded if:

(1) α is a nondegenerate contact form.

(2) All of its Reeb orbits are contractible inside M.

(3) For any integer k, the lengths of the Reeb orbits of Conley–Zehnder index k are bounded from above.

Similarly, we call a Liouville domain (D,θ) in M index-bounded if its boundary $(\partial D, \alpha = \theta\,|_{\partial D})$ is index-bounded.

Let (D,θ) be a Liouville domain in (M,ω). For small $\epsilon > 0$ there is an embedding

$$\overline{C} = [1-\epsilon, 1+\epsilon] \times \partial D \to M$$

such that ∂D is the image of $\{1\} \times \partial D$. We do not distinguish $\overline{C}$ from its image. Then $\overline{C}$ is called a contact cylinder associated to the Liouville domain D if $\omega\,|_{\overline{C}} = d(r\alpha)$, where r is the coordinate on $[1-\epsilon, 1+\epsilon]$. And we will write $D_{1+\epsilon} = D \cup \overline{C}$ as a compact neighborhood of D. The 1–form θ on D smoothly extends to a 1–form θ on $D_{1+\epsilon}$ such that $\omega\,|_{D_{1+\epsilon}} = d\theta$.

2.3 Relative symplectic cohomology of a Liouville domain

The relative symplectic cohomology is the homology of a suitable limit of complexes derived from Hamiltonian Floer theory of M. The whole construction [Varolgunes 2018] of relative symplectic cohomology defines a module $\mathrm{SH}_M(D)$ over the universal Novikov ring Λ_0 for any compact subset D of a closed symplectic manifold M. Now we briefly review its definition when D is a convex domain in a symplectic Calabi–Yau manifold M.

The following data is called an *acceleration data* for D:

(1) $H_{1,t} \le H_{2,t} \le \cdots$ a monotone sequence of nondegenerate Hamiltonian functions such that $H_{i,t}(x) \to 0$ on D and $H_{i,t}(x) \to +\infty$ on $M - D$.

(2) Monotone homotopies of Hamiltonians $\{H_{s,t}\}_{s\in[i,i+1]}$ for all i, which means that $H_{s,t}(x) \ge H_{s',t}(x)$ if $s \ge s'$ and $H_{s,t} = H_{i,t}$ if $s = i$.

(3) A family of almost-complex structures $\{J_{s,t}\}_{(s,t)\in[1,+\infty)\times S^1}$ such that, for each i, $(H_{i,t}, J_{i,t})$ is a regular pair, and, for each i, $(H_{s,t}, J_{s,t})_{s\in[i,i+1]}$ is a regular homotopy.

From an acceleration data, we obtain a sequence of chain complexes over Λ_0

$$\mathcal{CF}^k = \mathrm{CF}^k(H_{1,t}) \to \mathrm{CF}^k(H_{2,t}) \to \cdots$$

which are connected by continuation maps. Here each $\mathrm{CF}^k(H_{i,t})$ is the degree–k Floer complex of the Hamiltonian $H_{i,t}$. Since $H_{i,t} \le H_{i+1,t}$ are connected by a monotone family of Hamiltonians, the weight (2-10) in the continuation map is nonnegative.

Then the relative symplectic cohomology module $\mathrm{SH}^k_M(D;\Lambda_0)$ is defined as the cohomology

$$H(\widehat{\mathrm{tel}}(\mathcal{CF}^k);\Lambda_0) \tag{2-11}$$

of the completion $\widehat{\mathrm{tel}}(\mathcal{CF}^k)$ of the telescope

$$\mathrm{tel}(\mathcal{CF}^k) = \bigoplus_{n\in\mathbb{Z}_+} (C_n[1] \oplus C_n).$$

Here we write $C_n = \mathrm{CF}^k(H_{n,t})$. Another algebraic way to define it is that

$$(2\text{-}12)\qquad \begin{aligned} \mathrm{SC}^k_M(D;\Lambda_0) &:= \varprojlim_{r\geq 0} \varinjlim_{n} C_n \otimes_{\Lambda_0} \Lambda_0/\Lambda_{\geq r}, \\ \mathrm{SH}^k_M(D;\Lambda_0) &:= H(\mathrm{SC}^k_M(D);\Lambda_0). \end{aligned}$$

For the equivalence of these two definitions, see [Varolgunes 2018, Section 2]. And it is also shown that the definition of $\mathrm{SH}^k_M(D;\Lambda_0)$ is independent of various choices.

Proposition 2.3 [Varolgunes 2018, Proposition 3.3.4] (1) *Let H_s and H'_s be two different acceleration data. Then $H(\mathrm{SC}_M(D, H_s)) \cong H(\mathrm{SC}_M(D, H'_s))$ canonically. Therefore we simply denote this invariant by $\mathrm{SH}_M(D)$.*

(2) *Let $\phi\colon M \to M$ be a symplectomorphism. There exists a canonical isomorphism $\mathrm{SH}_M(D) = \mathrm{SH}_M(\phi(D))$ by relabeling an acceleration data by the map ϕ.*

(3) *For $D \subset D'$, there exist canonical restriction maps $\mathrm{SH}_M(D') \to \mathrm{SH}_M(D)$. This satisfies the presheaf property.*

Hence we can write this invariant as $\mathrm{SH}_M(D) := \mathrm{SH}_M(D;\Lambda_0)$ and its torsion-free part $\mathrm{SH}_M(D;\Lambda_0)\otimes_{\Lambda_0}\Lambda$. This invariant has many good properties. Notably it satisfies the Mayer–Vietoris exact sequence in some settings. Another property we will keep using here is the *stably displaceability condition*.

Theorem 2.4 [Varolgunes 2018, Theorem 4.0.1 and Remark 4.3.1] *If the compact subset $D \subset M$ is stably displaceable then $\mathrm{SH}_M(D;\Lambda_0) \otimes_{\Lambda_0} \Lambda = 0$.*

In practice when D is a convex domain, with a nondegenerate contact form on its boundary, we will use a particular class of acceleration data to compute the relative symplectic cohomology. For small $\epsilon > 0$ we fix a contact cylinder $\overline{C}$ associated to D and write $D_{1+\epsilon} = D \cup \overline{C}$ as a compact neighborhood of D. We introduce the notion of S–shaped Hamiltonian functions; see Figure 1.

Definition 2.5 A time-independent Hamiltonian function $H\colon M \to \mathbb{R}$ is called an S–shaped Hamiltonian if:

(1) H is cylindrical on the region $\left[1-\frac{\epsilon}{4}, 1+\frac{3\epsilon}{4}\right] \times \partial D \subset \overline{C}$. That is, $H(x) = H(x')$ if $r(x) = r(x')$, where r is the cylindrical coordinate. So we can write $H(x) = h(r(x))$ for some $h\colon \left[1-\frac{\epsilon}{4}, 1+\frac{3\epsilon}{4}\right] \to \mathbb{R}$ on the cylinder region.

(2) $h'(r)$ is concave and $h(r) = \lambda r + m$ on $\left[1+\frac{\epsilon}{4}, 1+\frac{\epsilon}{2}\right] \times \partial D \subset \overline{C}$ for some constants $\lambda > 0$ and m.

(3) The linear slope λ is not in the action spectrum of the contact form.

(4) H is a C^2–small Morse function on $D_{1-\epsilon/4}$, and it is a Morse function on $M - D_{1+3\epsilon/4}$ with small derivatives such that it only has constant orbits outside $\left[1-\frac{\epsilon}{4}, 1+\frac{3\epsilon}{4}\right] \times \partial D$.

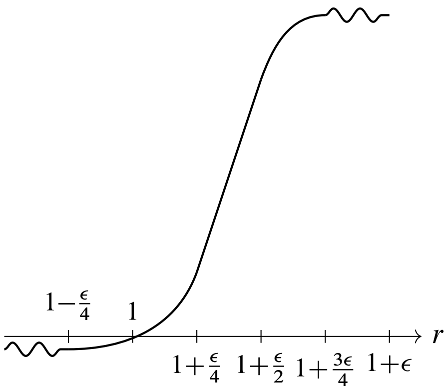

Figure 1: Hamiltonian functions in the cylindrical coordinate.

By this special shape of our Hamiltonian functions, we have further description of their 1–periodic orbits. First in the region where the Hamiltonians have small derivatives, there are only constant orbits. Hence all nonconstant orbits lie in the contact cylinder $\overline{C}$. By a direct computation we have that

$$X_{H_{n,t}}(r_0, c_0) = -\partial_r H_{n,t}(r_0) \cdot R_\alpha(c_0) \quad \text{for all } (r_0, c_0) \in \overline{C} = [1-\epsilon, 1+\epsilon] \times \partial D.$$

That is, the Hamiltonian vector field is proportional to the Reeb vector field with the negative slope as the ratio. Hence any τ–periodic Reeb orbit gives rise to a 1–periodic Hamiltonian orbit in $r_0 \times \partial D$ if and only if

$$\tau = \left| \int_0^1 \partial_r H_{n,t}(r_0)\, dt \right|.$$

Since the Hamiltonian is linear in the middle of the cylindrical region, with a slope which is not in the action spectrum of the contact form, there are no orbits in this region. Hence all the nonconstant 1–periodic Hamiltonian orbits can be separated into two groups. One group is located in the region $\left[1-\frac{\epsilon}{4}, 1+\frac{\epsilon}{4}\right] \times \partial D$ and we call them *lower* orbits. The other group is located in the region $\left[1+\frac{\epsilon}{2}, 1+\frac{3\epsilon}{4}\right] \times \partial D$ and we call them *upper* orbits.

Previously, an S–shaped Hamiltonian function is time-independent. So the nonconstant orbits appear in S^1–families. Now we use small time-dependent perturbations to make them nondegenerate by using the technique in [Cieliebak et al. 1996].

Proposition 2.6 [Cieliebak et al. 1996, Lemma 2.1 and Proposition 2.2] *Let H be a time-independent S–shaped Hamiltonian function and let γ be a nonconstant 1–periodic orbit of H such that γ is transversally nondegenerate. Pick U to be a neighborhood of γ which does not contain other 1–periodic orbits. Then there exists a time-dependent function H_t such that*:

(1) *The support of $H - H_t$ is in U.*

(2) *There are exactly two 1–periodic orbits $\gamma_\pm$ of H_t in U.*

(3) *$\int \gamma^*\theta = \int \gamma_+^*\theta = \int \gamma_-^*\theta$, where θ is the Liouville 1–form in the cylindrical region.*

(4) *The difference between the Conley–Zehnder indices of γ and $\gamma_\pm$ is bounded by 1.*

Note that the orbit γ is both a Hamiltonian orbit of H and a Reeb orbit of the contact form. One can compute its index in two ways. The following lemma relates these two indices.

Lemma 2.7 [McLean 2020, Lemma 5.25] *Let $\overline{C} = [1-\epsilon, 1+\epsilon] \times C$ be an index-bounded contact cylinder with cylindrical coordinate r and associated contact form α and let $\pi : \overline{C} \to C$ be the natural projection map. Let $f : [1-\epsilon, 1+\epsilon] \to \mathbb{R}$ be a smooth function, $R_{\lambda,[-m,m]}$ be the set of Reeb orbits of length λ and index in $[-m,m]$ and $O_{\lambda,[-m,m]}$ be the set of 1–periodic orbits of $f(r)$ contained in $\{r = (f')^{-1}(\lambda)\}$ of index in $[-m,m]$ which are null homologous in M. Then the map*

$$O_{\lambda,[-m,m]} \to R_{\lambda,[-m-1/2,m+1/2]}$$

sending $\gamma : \mathbb{R}/\mathbb{Z} \to \overline{C}$ to $\pi \circ \gamma \circ b_\lambda$ is well-defined, where

$$b_\lambda : [0,\lambda] \to [0,1], \qquad b_\lambda(t) := t/\lambda \quad \text{for all } t \in [0,\lambda].$$

Hence we can use the index-bounded condition, which was previously defined for Reeb orbits, in the setting of Hamiltonian orbits. Let D be a convex domain with an index-bounded boundary in a Calabi–Yau manifold. We start with a time-independent S–shaped Hamiltonian function and perturb it. Before perturbation, a nonconstant orbit γ satisfies the index-length relation in Definition 2.2 since it comes from a Reeb orbit. After perturbation, by the above proposition and lemma, the index-length relation still holds for new orbits $\gamma_\pm$.

Now we say a time-dependent function H_t is a time-dependent S–shaped Hamiltonian function if it is a perturbation of a time-independent S–shaped Hamiltonian function as in Proposition 2.6. Note that we only perturb the region where nonconstant orbits lie, so we can still talk about upper and lower orbits after the perturbation.

Definition 2.8 Let D be a convex domain in a Calabi–Yau manifold M. A time-dependent S–shaped Hamiltonian function H_t is called index-bounded if, for any integer k, there exists a constant $\mu_k > 0$ such that $\int \gamma^*\theta < \mu_k$ for all degree–k 1–periodic orbits of H_t.

The above discussion says that if D is a convex domain with an index-bounded boundary, then we can always find time-dependent nondegenerate S–shaped Hamiltonian functions which are index-bounded. In practice we will use families of time-dependent nondegenerate S–shaped Hamiltonian functions to compute $\mathrm{SH}_M(D)$.

2.4 Hamiltonian Floer theory on manifolds with convex boundary

Now we review the construction of Hamiltonian Floer theory on convex manifolds, and fix our notation along the way. Symplectic cohomology was first introduced by Cieliebak, Floer and Hofer [Cieliebak et al. 1995] and Viterbo [1999] in the exact setting, and by Ritter [2010] in the nonexact setting.

Let $(M, \omega = d\theta)$ be a Liouville domain, and let $\alpha := \theta\,|_{\partial M}$ be the contact form. Then we can attach a cylindrical end to M to get an open symplectic manifold

$$\widehat{M} := M \cup (\partial M \times [1, +\infty)).$$

Let r be the coordinate on $[1,+\infty)$. We equip the manifold $\widehat{M}$ with a smooth symplectic form $\hat{\omega}$, where $\hat{\omega}=\omega$ on M and $\hat{\omega}=d(r\alpha)$ on $\partial M\times[1,+\infty)$. And $(\widehat{M},\hat{\omega})$ will be called the *completion* of M. In the following we assume that α on ∂M is nondegenerate.

Now we define *admissible* Hamiltonian functions we will use. A Hamiltonian function $H_t\colon S^1\times\widehat{M}\to\mathbb{R}$ is called admissible if:

(1) It is a negative time-independent Morse function on M.

(2) All its contractible 1–periodic orbits in $\widehat{M}$ are nondegenerate.

(3) It is a linear function just depending on r with a positive slope on $\partial M\times[R_0,+\infty)$ for some $R_0>1$.

(4) The slope of the linear part is not an element in $\mathrm{Spec}(\alpha)$.

For an admissible Hamiltonian function, there are only finitely many 1–periodic orbits. Next with an admissible Hamiltonian H_t, we consider the degree–k Floer complex $\mathrm{CF}^k(H_t)$ as in (2-2) and (2-3). Then for suitably chosen almost-complex structures, we can use moduli spaces of Floer solutions to define differentials and continuation maps as in the closed case.

For a monotone family $H_{i,t}$ such that $H_{i,t}\le H_{i+1,t}$ and the linear slope of $H_{i,t}$ goes to positive infinity, we have a sequence of complexes

$$\mathcal{CF}^k=\mathrm{CF}^k(H_{1,t})\to\mathrm{CF}^k(H_{2,t})\to\cdots$$

connected by continuation maps. Here all Floer differentials and continuation maps are weighted by the topological energy; see (2-10). Since the symplectic form on our Liouville domain is exact, the images of an orbit under Floer differentials and continuation maps are finite sums of other orbits. Hence we can both define the classical symplectic cohomology over $\mathbb{C}$ or over Λ_0. The former theory can be defined by the latter one by setting $T=1$.

The classical symplectic cohomology of M over Λ_0 is defined as

$$\mathrm{SH}^k(M;\Lambda_0):=H(\mathrm{tel}(\mathcal{CF}^k)).$$

An essential difference between this definition and that of the relative symplectic cohomology is that the classical one does not complete $\mathrm{tel}(\mathcal{CF}^k)$ before taking homology.

The classical symplectic cohomology of M over $\mathbb{C}$ is defined as

$$\mathrm{SH}^k(M;\mathbb{C}):=H(\mathrm{tel}(\mathcal{CF}^k)\,|_{T=1}).$$

In other words, all differentials and continuation maps are defined without weights. Since $(M,d\theta)$ is an exact symplectic manifold, this reduction to $T=1$ is well-defined.

2.5 Lower semicontinuous Hamiltonian functions

Defining the classical $\mathrm{SH}^k(M;\mathbb{C})$ via taking a direct limit is equivalent to the definition via a single Hamiltonian function that is quadratic at infinity; see [Seidel 2008, Section 3]. Similarly, we will see that the relative symplectic cohomology $\mathrm{SH}_M(D)$ is related to a kind of Hamiltonian Floer theory for a lower semicontinuous Hamiltonian function F_0^D, where $F_0^D\mid_D=0$ and $F_0^D\mid_{M-D}=+\infty$. A general study of Hamiltonian Floer theory of lower semicontinuous functions can be found in [Groman 2023; McLean 2020]. Now we will discuss a special case of it for our purpose.

Fix a closed symplectic Calabi–Yau manifold M and let $F\colon M\times S^1\to\mathbb{R}$ be a lower semicontinuous function. Pick a monotone sequence $\{H_{n,t}\}$ of nondegenerate Hamiltonian functions such that

$$H_{1,t}\le H_{2,t}\le\cdots\le H_{n,t}\le\cdots\to F.$$

The Hamiltonian Floer cohomology $\mathrm{HF}(F)$ is defined as

$$H(\widehat{\mathrm{tel}}(\mathrm{CF}(H_{1,t})\to\mathrm{CF}(H_{2,t})\to\cdots)). \tag{2-13}$$

Also we have an equivalent definition in terms of (2-12).

From its definition, we can see that the lower semicontinuous Hamiltonian Floer cohomology $\mathrm{HF}(F)$ is a generalization of the relative symplectic cohomology. For a domain D in M, we have

$$\mathrm{SH}_M(D)\cong\mathrm{HF}(F_0^D).$$

This indicator-type function F_0^D is a very degenerate one, since it is identically zero on D. In practice, it is often more handy to replace its part on D by a fixed nondegenerate Hamiltonian function f. And our main result of this subsection is the following isomorphism, which shows that the choice f does not matter if we work over the Novikov field.

Proposition 2.9 *Let D be a domain in M. Let $F\colon M\times S^1\to\mathbb{R}$ be a lower semicontinuous function such that F is a smooth negative nondegenerate Hamiltonian function f on $D\times S^1$ and $F=+\infty$ otherwise. Then we have*

$$\mathrm{SH}_M(D)\otimes_{\Lambda_0}\Lambda\cong\mathrm{HF}(F)\otimes_{\Lambda_0}\Lambda.$$

Proof Pick a cofinal family $\{H_{n,t}\}$ of nondegenerate Hamiltonian functions such that

$$H_{1,t}\le H_{2,t}\le\cdots\le H_{n,t}\le\cdots\to F_0^D$$

and a cofinal family $\{K_{n,t}\}$ of nondegenerate Hamiltonian functions such that

$$K_{1,t}\le K_{2,t}\le\cdots\le K_{n,t}\le\cdots\to F.$$

By the nondegeneracy of f in $D\times S^1$, we can choose $K_{n,t}$ such that $K_{n,t}(x)=f(x,t)$ for all $n\in\mathbb{N}$, $(x,t)\in D\times S^1$. Moreover, since F is negative on D we can choose above families such that $H_{n,t}\ge K_{n,t}$ for all n. Moreover, we can assume that

$$\max_{(x,t)\in M\times S^1}(H_{n,t}-K_{n,t})=\max_{(x,t)\in D\times S^1}(H_{n,t}-K_{n,t})\quad\text{for all } n\in\mathbb{N}.$$

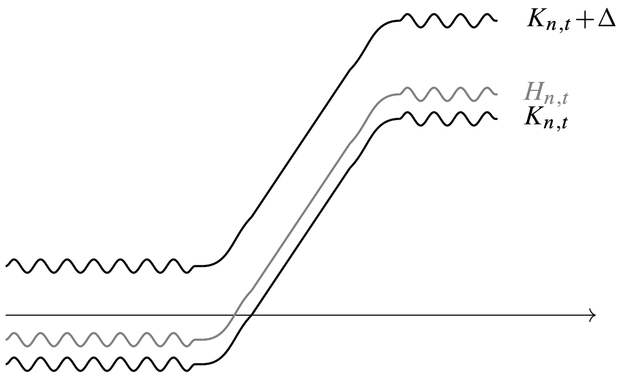

Figure 2: A sandwich of Hamiltonian functions.

Consider the continuation map

$$\mathrm{CF}(K_{i,t}) \to \mathrm{CF}(H_{i,t})$$

over Λ_0 for each i. These continuation maps induce a Λ_0–module map

$$\mathrm{HF}(F) \to \mathrm{HF}(F_0^D) \cong \mathrm{SH}_M(D).$$

Next we set $\Delta := -\min_{(x,t)\in D\times S^1} K_{1,t}$, and set $F+\Delta$ to be the lower semicontinuous function such that it is $f+\Delta$ on $D\times S^1$ and positive infinity otherwise. Note that the family $\{K_{n,t}+\Delta\}$ is a cofinal family for the function $F+\Delta$, so on $D\times S^1$ we have that $K_{n,t}+\Delta \geq K_{1,t}+\Delta \geq 0 \geq H_{n,t}$ for all n. On the other hand, since $H_{n,t}-K_{n,t}$ takes maximum on $D\times S^1$, we have $K_{n,t}+\Delta \geq H_{n,t}$ globally on $M\times S^1$, see Figure 2. This gives the second Λ_0–module map in

$$\mathrm{HF}(F) \to \mathrm{HF}(F_0^D) \to \mathrm{HF}(F+\Delta) \to \mathrm{HF}(F_0^D+\Delta').$$

Similarly we can find some other constant Δ' to define the third map. (For example, we can take $\Delta'=\Delta$. What we need is that $H_{n,t}+\Delta' \geq K_{n,t}+\Delta$ for all n.) Since the data to define $\mathrm{HF}(F+\Delta)$ is just a translation of the data to define $\mathrm{HF}(F)$, the composition of the first two maps is the multiplication by T^{Δ}, by Lemma 2.1. By the same reason, the composition of the last two maps is the multiplication by $T^{\Delta'}$. After tensoring with the Novikov field, these compositions become isomorphisms, which shows that the three Λ_0–modules $\mathrm{HF}(F)$, $\mathrm{HF}(F_0^D)$ and $\mathrm{HF}(F+\Delta)$ are all isomorphic over Λ. □

Therefore if we only care about the relative symplectic cohomology over the Novikov field, we can relax the condition of the acceleration data of Hamiltonian functions we used for computations. That is, the Hamiltonian functions converge to a fixed negative Morse function on D, instead of converging to zero.

3 A filtration on the completed telescope

Let (M,ω) be a closed Calabi–Yau symplectic manifold and let $D\subset M$ be a Liouville domain. Recall that the relative symplectic cohomology $\mathrm{SH}_M(D)$ is the homology of the completion $\widehat{\mathrm{tel}}(\mathcal{CF})$ of the telescope

$$\mathrm{tel}(\mathcal{CF})=\bigoplus_{i\in\mathbb{N}}(C_i[1]\oplus C_i).$$

Now we will define a filtration on $\widehat{\mathrm{tel}}(\mathcal{CF})$ which is compatible with the differentials.

3.1 Auxiliary symplectic forms

First we define some auxiliary symplectic forms with respect to our Liouville domain D. The boundary ∂D is a contact hypersurface in M and for small $\epsilon>0$ we write $D_{1+\epsilon}=D\cup([1,1+\epsilon]\times\partial D)$ as a compact neighborhood of D. That is, we fix a choice of a contact cylinder associated to D. On $D_{1+\epsilon}$ there is a 1–form θ such that $d\theta=\omega$. Then we can interpolate ω from $D_{1+3\epsilon/4}$ to $M-D_{1+\epsilon}$.

Lemma 3.1 *There exists a global* 2*–form* $\tilde{\omega}$ *and a* 1*–form* $\tilde{\theta}$ *on* M *such that*:

(1) $\omega=\tilde{\omega}+d\tilde{\theta}$ *on* M.

(2) *The support of* $\tilde{\theta}$ *is in* $D_{1+\epsilon}$ *and* $\tilde{\theta}=\theta$ *in* $D_{1+3\epsilon/4}$.

(3) *The support of* $\tilde{\omega}$ *is in* $M-D_{1+3\epsilon/4}$ *and* $\tilde{\omega}=\omega$ *in* $M-D_{1+\epsilon}$.

Proof Let $\rho(r)\colon[1,2]\to\mathbb{R}$ be a smooth increasing function such that

$$\rho(r)=\begin{cases}0, & 1\le r\le 1+\frac{3}{4}\epsilon,\\ 1, & 1+\epsilon\le r.\end{cases}$$

Next we define

$$\tilde{\omega}\,|_x=\begin{cases}0, & x\in D_{1+3\epsilon/4},\\ d(\rho(r)\theta), & x\in D_{1+\epsilon}-D_{1+3\epsilon/4},\\ \omega, & x\in M-D_{1+\epsilon},\end{cases}$$

$$\tilde{\theta}\,|_x=\begin{cases}\theta, & x\in D_{1+3\epsilon/4},\\ (1-\rho(r))\theta, & x\in D_{1+\epsilon}-D_{1+3\epsilon/4},\\ 0, & x\in M-D_{1+\epsilon}.\end{cases}$$

Then we can check that $\tilde{\omega}$ and $\tilde{\theta}$ satisfy the conditions we need; see Figure 3. □

By the definition of $\tilde{\omega}$, it represents a cohomology class $[\tilde{\omega}]\in H^2(M,D)$. The exact sequence of de Rham cohomology

$$\cdots\to H^1(D)\to H^2(M,D)\xrightarrow{j}H^2(M)\to H^2(D)\to\cdots$$

says that $j([\tilde{\omega}])=[\omega]$. So we also call $[\tilde{\omega}]$ a lift of $[\omega]$. In the case that $[\tilde{\omega}]$ takes integral values on $H_2(M,D;\mathbb{Z})$, we say $[\tilde{\omega}]$ is an integral lift of $[\omega]$. From now on, our Liouville domain D is always equipped with such an auxiliary form $\tilde{\omega}$ and we assume it represents an integral class.

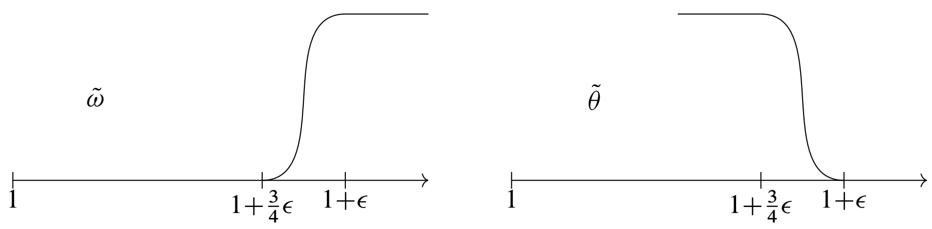

Figure 3: Cut-off functions for $\tilde{\omega}$ and $\tilde{\theta}$.

Example 3.2 If D is simply connected, then $H^1(D)=0$ and we have a unique lift $[\tilde{\omega}]$ of $[\omega]$. Moreover, the map $H_2(M)\to H_2(M,D)$ is surjective. Hence if $[\omega]$ represents an integral class in $H^2(M)$ then $[\tilde{\omega}]$ is automatically integral. A similar conclusion holds when the map $\pi_1(D)\to\pi_1(M)$ is injective.

For a Hamiltonian vector field which is small on $M-D_{1+3\epsilon/4}$, the Floer equation becomes very close to the genuine Cauchy–Riemann equation on that region. This implies the positivity of the ω–energy of solutions on that region. To prove this first we recall the following computation for a solution to the Floer equation. Let H be a Hamiltonian function and J be a compatible almost-complex structure. Let $g(\cdot,\cdot):=\omega(\cdot,J\,\cdot)$ be the induced Riemannian metric. For a solution u with finite energy we have

$$\omega([u])=\int u^*\omega=\int\omega(\partial_s u,\partial_t u)=\int|\partial_s u|_g^2+\int\omega(\partial_s u,X_H).$$

For the second term we have

$$\begin{aligned}\int\omega(\partial_s u,X_H)&=-\int dH(\partial_s u)\\&=-\int_{-\infty}^{+\infty}\frac{\partial}{\partial s}\int_0^1 H(u)\,dt\,ds=-\left(\int_0^1 H(u(+\infty,t))\,dt-\int_0^1 H(u(-\infty,t))\,dt\right)\\&\geq-\|H\|,\end{aligned}$$

where $\|H\|=\int_0^1(\max_{x\in M}H(x,t)-\min_{x\in M}H(x,t))\,dt$ is the Hofer norm of H. When we restrict our function to some region of M, the relative Hofer norm is defined in a similar way by taking the max and min on that region.

Lemma 3.3 *Let (M,ω) be a closed symplectic manifold and D be a Liouville domain in M. Let $\tilde{\omega}$ be an auxiliary form constructed above. We assume that $[\tilde{\omega}]$ is an integral lift of $[\omega]$. Then for any nondegenerate Hamiltonian function H with $\|H\|_{M-D_{1+3\epsilon/4}}<1$, for any finite-energy solution $u\colon S^1\times\mathbb{R}\to M$ of the Floer equation*

$$\partial_s u+J(\partial_t u-X_H)=0,$$

the integral $\int u^*\tilde{\omega}\geq 0$. Here J is a cylindrical almost-complex structure compatible with D.

Proof We prove this lemma by computation. By the definitions of $\tilde{\omega}$ and a cylindrical almost-complex structure J, we know that $\tilde{g}(\cdot,\cdot) := \tilde{\omega}(\cdot, J\,\cdot)$ defines a Riemannian metric on the region where $\tilde{\omega} \neq 0$. Then by the above computation we have that

$$\int u^*\tilde{\omega} = \int \tilde{\omega}(\partial_s u, \partial_t u) \geq \int |\partial_s u|^2_{\tilde{g}} - \|H\|_{M-D_{1+3\epsilon/4}} \geq -\|H\|_{M-D_{1+3\epsilon/4}}.$$

Since $\tilde{\omega}$ is supported in $M - D_{1+3\epsilon/4}$, we only need to consider the relative Hofer norm $\|H\|_{M-D_{1+3\epsilon/4}}$. In particular, if the image of u does not intersect $M - D_{1+3\epsilon/4}$, then $\int u^*\tilde{\omega} = 0$.

Since the nonconstant orbits of H are in $D_{1+3\epsilon/4}$, the cylinder u represents a relative homology class in $H_2(M, D)$. By our assumption that the number $\int u^*\tilde{\omega}$ is an integer, when $\|H\|_{M-D_{1+3\epsilon/4}} < 1$, the number $\int u^*\tilde{\omega}$ is nonnegative. □

Note that Lemma 3.3 also works for a family of Hamiltonian $\{H_{t,s}\}$ under small perturbations. That is, if $\|H_{t,s} - H_{t,s'}\|_{M-D_{1+3\epsilon/4}} < 1$ for any s, s' then we still have positivity of solutions of the parametrized Floer equation.

3.2 A filtration which is not exhaustive

The computation in the last subsection tells that when the Hamiltonian functions have small relative Hofer norms outside the Liouville domain, the outside part of the corresponding Floer solutions carry nonnegative $\tilde{\omega}$–energy. Now we use this fact to define a filtration on the telescope, which is the underlying complex of the relative symplectic cohomology.

First we consider the case of a single S–shaped Hamiltonian H such that $\|H\| < 1$ on $M - D_{1+3\epsilon/4}$. We define a valuation of a single element

$$a \cdot \gamma \in \mathrm{CF}^k(H) = \bigoplus_{\gamma \in \mathcal{CP}^k(H)} \Lambda_0 \cdot \gamma$$

by setting

(3-1) $$\sigma(a \cdot \gamma) = v(a) - \int_\gamma H - \int_\gamma \tilde{\theta},$$

where $v(a)$ is the valuation on Λ_0. If γ is a constant orbit, the integral $\int_\gamma \tilde{\theta}$ is zero. For a general sum $x = \sum_i a_i \gamma_i$ we define the valuation as

(3-2) $$\sigma(x) := \inf_i \{\sigma(a_i \gamma_i)\} = \inf_i \left\{ v(a_i) - \int_{\gamma_i} H_i - \int_{\gamma_i} \tilde{\theta} \right\}.$$

Lemma 3.4 *For the valuation σ we have*

$$\sigma(d(a \cdot \gamma)) \geq \sigma(a \cdot \gamma).$$

Hence it induces a filtration

(3-3) $$\mathcal{F}^\lambda \mathrm{CF}^k(H) = \{x \in \mathrm{CF}^k(H) \mid \sigma(x) \geq \lambda\}$$

on the complex $\mathrm{CF}^k(H)$.

Proof We prove this lemma by a direct computation. First from our definition

$$d(a\cdot\gamma)=a\cdot\sum_{\gamma',A}n(\gamma,\gamma';A)\cdot\gamma'\cdot T^{\omega(A)-\int_\gamma H+\int_{\gamma'}H},$$

where A is the homotopy class represented by a solution u connecting γ and γ'. Hence we have

$$\begin{aligned}\sigma(d(a\cdot\gamma))&=\sigma\Big(a\cdot\sum_{\gamma',A}n(\gamma,\gamma';A)\cdot\gamma'\cdot T^{\omega(A)-\int_\gamma H\int_{\gamma'}H}\Big)\\&=\inf_{\gamma',A}\Big\{v(a)+\omega(A)-\int_\gamma H+\int_{\gamma'}H-\int_{\gamma'}H-\int_{\gamma'}\tilde\theta\Big\}\\&=\inf_{\gamma',A}\Big\{v(a)+\omega(A)-\int_\gamma H-\int_{\gamma'}\tilde\theta\Big\}\\&=\inf_{\gamma',A}\Big\{v(a)+\tilde\omega(A)+d\tilde\theta(A)-\int_\gamma H-\int_{\gamma'}\tilde\theta\Big\}\\&=\inf_{\gamma',A}\Big\{v(a)+\tilde\omega(A)+\int_{\gamma'}\tilde\theta-\int_\gamma\tilde\theta-\int_\gamma H-\int_{\gamma'}\tilde\theta\Big\}\\&=\inf_{\gamma',A}\Big\{v(a)+\tilde\omega(A)-\int_\gamma\tilde\theta-\int_\gamma H\Big\}\\&=\inf_{\gamma',A}\{\sigma(a\cdot\gamma)+\tilde\omega(A)\}\geq\sigma(a\cdot\gamma).\end{aligned}\tag{3-4}$$

Here we use that $\omega=\tilde\omega+d\tilde\theta$ and the Stokes formula to compute $d\tilde\theta(A)$. The last inequality uses Lemma 3.3.

Similarly we can check that $\sigma(dx)\geq\sigma(x)$ for a sum $x=\sum_i a_i\gamma_i$. Hence this valuation σ induces a decreasing filtration on $\mathrm{CF}(H)$. □

The above computation also works for a 1–parameter family of Hamiltonian functions if the variation of these functions is sufficiently small outside the Liouville domain. More precisely, we pick a monotone 1–parameter family of Hamiltonian functions $H_{n,t}$ which form an acceleration data to compute the relative symplectic cohomology of D such that:

(1) The relative Hofer norm of $H_{n,t}$ on $M-D_{1+3\epsilon/4}$ is less than 1 for each n.

(2) The relative Hofer norm of $H_{n,t}-H_{n+1,t}$ on $M-D_{1+3\epsilon/4}$ is less than 1 for each n.

Then each of the continuation maps also satisfies the above lemma. Therefore the telescope given by

$$\mathcal{CF}:=\mathrm{CF}(H_{1,t})\to\mathrm{CF}(H_{2,t})\to\cdots\to\mathrm{CF}(H_{n,t})\to\cdots$$

satisfies that its differentials and continuation maps are compatible with σ. We write $\widehat{\mathrm{tel}}(\mathcal{CF})$ as the completion of the telescope of this 1–ray. For a general element $x=\sum_i a_i\gamma_i\in\widehat{\mathrm{tel}}(\mathcal{CF})$ we define $\sigma:\widehat{\mathrm{tel}}(\mathcal{CF})\to\mathbb{R}\cup\{-\infty\}$ by

$$\sigma(x):=\inf_i\{\sigma(a_i\gamma_i)\}=\inf_i\Big\{v(a_i)-\int_{\gamma_i}H_{i,t}-\int_{\gamma_i}\tilde\theta\Big\}.\tag{3-5}$$

This valuation gives a filtration on $\widehat{\mathrm{tel}}(\mathcal{CF})$. The differentials of the completed telescope are sums of the differentials and continuation maps in the Floer complex. Hence the differential of $\widehat{\mathrm{tel}}(\mathcal{CF})$ is compatible with this filtration, which makes $\widehat{\mathrm{tel}}(\mathcal{CF})$ a filtered differential graded module. The homology of $\widehat{\mathrm{tel}}(\mathcal{CF})$ is the relative symplectic cohomology.

Note that for a fixed Hamiltonian H this valuation takes values strictly in $\mathbb{R}$ since H is bounded and there are finitely many periodic orbits. But for a general element in the completion of the telescope the valuation can be negative infinity. Therefore the induced filtration is *not exhaustive*. That is,

$$\bigcup_{\lambda} \mathcal{F}^{\lambda}\, \widehat{\mathrm{tel}}(\mathcal{CF}) \neq \widehat{\mathrm{tel}}(\mathcal{CF}),$$

which is one main reason that the induced spectral sequence sometimes does not converge. We also remark that this filtration is *weakly convergent* since

$$\bigcap_{\lambda} \mathcal{F}^{\lambda}\, \widehat{\mathrm{tel}}(\mathcal{CF}) = \{0\}.$$

Now we recall two foundational theorems on spectral sequences from [McCleary 2001], one on the existence and the other on the convergence.

Definition 3.5 Let R be a commutative ring with unit. An R–module A is called a filtered differential graded module if:

(1) A is a direct sum of submodules, $A = \bigoplus_{n=0}^{\infty} A^n$.

(2) There is an R–linear map $d\colon A \to A$ satisfying $d \circ d = 0$.

(3) A has a filtration F and the differential d respects the filtration, that is, $d\colon F^pA \to F^pA$.

Theorem 3.6 [McCleary 2001, Theorem 2.6] *Each filtered differential graded module (A, d, F) determines a spectral sequence, $\{E_r^{*,*}, d_r\}$, $r = 1, 2, \dots$, with d_r of bidegree $(r, 1-r)$ and*

$$E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A).$$

Theorem 3.7 [McCleary 2001, Theorem 3.2] *Let (A, d, F) be a filtered differential graded module such that the filtration is exhaustive and weakly convergent. Then the associated spectral sequence with $E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A)$ converges to $H(A, d)$, that is,*

$$E_{\infty}^{p,q} \cong F^pH^{p+q}(A, d)/F^{p+1}H^{p+q}(A, d).$$

4 The induced spectral sequence

Previously we constructed a filtration on $\widehat{\mathrm{tel}}(\mathcal{CF})$ by using a special family of Hamiltonian functions that have small variations outside D. However, the induced spectral sequence does not always converge since the filtration is not exhaustive. Now we study the particular case that D is a Liouville domain with an index-bounded boundary in a symplectic Calabi–Yau manifold M.

First, we give an outline of the proof in this section. For a fixed degree k, we will study $\mathrm{SH}^k_M(D)$. In practice we will study the Hamiltonian Floer cohomology of a lower semicontinuous function such that it is a fixed nondegenerate Hamiltonian function on $D \times S^1$ and it is positive infinity outside. It is isomorphic to $\mathrm{SH}^k_M(D)$ over the Novikov field. Abusing the notation, we still write it as $\mathrm{SH}^k_M(D)$. Let $\widehat{\mathrm{tel}}(\mathcal{CF})$ be the completed telescope, constructed by a special family of Hamiltonian functions as in the previous section. Pick a sequence

$$0 < E_1 < E_2 < \cdots < E_l < \cdots$$

of positive numbers going to infinity. We have a sequence of telescopes $\mathrm{tel}(\mathcal{CF}) \otimes_{\Lambda_0} \Lambda_{E_l}$, which form an inverse system by using projection maps. The homology

$$\mathrm{SH}^k_M(D; \Lambda_{E_l}) := H^k(\mathrm{tel}(\mathcal{CF}) \otimes_{\Lambda_0} \Lambda_{E_l})$$

is called the truncated symplectic cohomology. Since projections are chain maps, we have an inverse system in the homology level

$$\cdots \leftarrow \mathrm{SH}^k_M(D; \Lambda_{E_{l-1}}) \leftarrow \mathrm{SH}^k_M(D; \Lambda_{E_l}) \leftarrow \cdots .$$

The inverse limit

$$\varprojlim_l \mathrm{SH}^k_M(D; \Lambda_{E_l})$$

is called the *reduced symplectic cohomology* in [Groman and Varolgunes 2023]. The relation between this reduced symplectic cohomology and the relative symplectic cohomology can be studied by verifying certain Mittag-Leffler condition of the above inverse systems.

The goal of this section is the following:

(1) For each E_l, we will construct two chain models which both compute $\mathrm{SH}^k_M(D; \Lambda_{E_l})$. They are defined by using particular Hamiltonians, which depend on E_l. The first chain model is a telescope, on which the filtration defined in the previous section is exhaustive. Hence we have a convergent spectral sequence for this chain model to compute $\mathrm{SH}^k_M(D; \Lambda_{E_l})$. This will establish (1), (2) in Theorem 1.1.

(2) The second chain model is a direct limit. It helps us to show $\mathrm{SH}^k_M(D; \Lambda_{E_l})$ is finitely generated, and the number of generators is independent of E_l, thanks to the index-bounded condition. By using this, we can verify a finite homological torsion criterion in [Groman and Varolgunes 2023], which shows that the reduced symplectic cohomology is isomorphic to the relative symplectic cohomology. This proves (3) in Theorem 1.1.

4.1 Ignoring upper orbits and convergence

For a monotone family of S–shaped Hamiltonian functions, the orbits form two groups: upper orbits and lower orbits. And the Floer differentials/continuation maps have four components: upper-to-upper, upper-to-lower, lower-to-upper and lower-to-lower. The next lemma says that, under the index-bounded condition, any lower-to-upper Floer trajectory has a big topological energy.

Lemma 4.1 *Given any energy bound $E>0$ and an integer k, there exists an S–shaped Hamiltonian function such that any lower-to-upper Floer differential with degree–k input has topological energy greater than E.*

Proof First, we choose an S–shaped Hamiltonian function H_t with $\|H\|_{M-D_{1+3\epsilon/4}}$ less than 1. For a Floer differential with degree–k input γ and degree–$(k+1)$ output γ', the energy weight is

$$\int u^*\omega-\int_\gamma H_t+\int_{\gamma'} H_t=-\int_\gamma \tilde{\theta}+\int_{\gamma'}\tilde{\theta}+\int u^*\tilde{\omega}-\int_\gamma H_t+\int_{\gamma'} H_t.$$

By the index-bounded condition, the difference $-\int_\gamma \tilde{\theta}+\int_{\gamma'}\tilde{\theta}$ is a bounded number which only depends on k. The computation in Lemma 3.3 says that $\int u^*\tilde{\omega}$ is nonnegative. Therefore if we choose an S–shaped Hamiltonian function such that its upper level is high enough, then the above energy weight is larger than E. □

Note that this estimate is uniform for all degree k orbits; hence we can make a subcomplex which only contains lower orbits, which gives the following lemma.

Lemma 4.2 *For any integer k and an energy bound $E>0$, let H be a Hamiltonian function which satisfies Lemma 4.1 for all three integers $k-1, k, k+1$ for E. Let $\mathrm{CF}^{k,L}(H)$ be the subspace of $\mathrm{CF}^k(H)$ which only contains lower orbits. Let d be the restriction of the Floer differential to $\mathrm{CF}^{k,L}(H)$. Then*

$$0\to \mathrm{CF}^{k-1,L}(H)\xrightarrow{d}\mathrm{CF}^{k,L}(H)\xrightarrow{d}\mathrm{CF}^{k+1,L}(H)\to 0$$

satisfies that $d\circ d=0$ over Λ_E.

Proof Take $\gamma\in \mathrm{CF}^{k-1,L}(H)$. The usual argument to show $d\circ d(\gamma)=0$ is to look at broken Floer trajectories. In our case, if it breaks along an upper orbit, then the energy weight is greater than E by Lemma 4.1, which is automatically zero over Λ_E. Hence we can ignore the upper orbits' contributions. □

The above two lemmas tell us for any fixed energy bound E and degree k, we can use only lower orbits to form a homology theory. By the same argument, lower-to-upper continuation maps have big topological energy for particular Hamiltonians.

Lemma 4.3 *Given any energy bound $E>0$ and an integer k, there exists a family of nondegenerate S–shaped Hamiltonian functions $\{H_n\}_{n\in\mathbb{N}}$ such that:*

1. $H_1\le H_2\le\cdots\le H_n\le H_{n+1}\le\cdots$.
2. $H_1=H_2=\cdots H_n=\cdots$ *on* $S^1\times D$.
3. H_1 *satisfies the above two lemmas.*
4. *Any lower degree–k orbits are inside D.*
5. *Any continuation map from a lower degree–k orbit to an upper degree–k orbit has topological energy greater than E.*

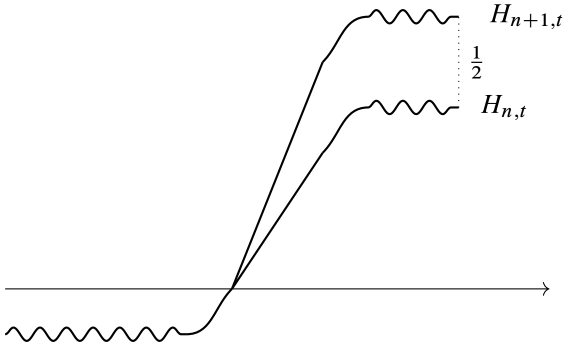

Figure 4: Hamiltonian functions with fixed lower parts and small variations outside.

Proof Take H_1 as a function which satisfies the above two lemmas. Then we construct H_2 in the following way:

(1) $H_1 = H_2$ on D.

(2) $H_1 + \frac{1}{2} = H_2$ on $M - D_{1+3\epsilon/4}$.

(3) Near ∂D, the function H_2 is obtained from a cylindrical function by adding time-dependent perturbations. We assume the perturbation is small such that $H_1 \leq H_2$ globally; see Figure 4.

Then we repeat this process inductively to get H_{n+1} from H_n. Hence the items (1)–(4) are satisfied, and we can connect H_n with H_{n+1} by a monotone homotopy. If we have a continuation map from a lower degree–k orbit γ of H_n to an upper degree–k orbit γ' of H_{n+1}, then it is weighted by an energy

$$\int u^*\omega - \int_\gamma H_n + \int_{\gamma'} H_{n+1} + \int (\partial_s H_s) = \int u^*\tilde{\omega} - \int_\gamma \tilde{\theta} + \int_{\gamma'} \tilde{\theta} - \int_\gamma H_n + \int_{\gamma'} H_{n+1} + \int (\partial_s H_s).$$

By the construction, we have that $\|H_n - H_{n+1}\|_{M-D_{1+3\epsilon/4}} < 1$; hence the first term is nonnegative. The monotonicity of $\partial_s H_s$ shows the last term is nonnegative and the index-bounded condition shows the second and third terms are bounded. So the whole energy is larger than E because the difference between the lower levels of H_n and the upper level of H_{n+1} are big enough. □

Therefore by using the above family of Hamiltonian functions, we are computing the Hamiltonian Floer cohomology of a semi-lower-continuous function, which is a nondegenerate Hamiltonian function on D and is positive infinity outside D. By Proposition 2.9, the resulting invariant is isomorphic to the relative symplectic cohomology over the Novikov field.

Now we consider two direct systems

$$\mathcal{CF}^k = \mathrm{CF}^k(H_{1,t}) \to \mathrm{CF}^k(H_{2,t}) \to \cdots,$$
$$\mathcal{CF}^{k,L} = \mathrm{CF}^{k,L}(H_{1,t}) \to \mathrm{CF}^{k,L}(H_{2,t}) \to \cdots$$

over Λ_E, induced by Hamiltonian functions defined in the above lemma. For each n, we have an inclusion of a subcomplex $\mathrm{CF}^{k,L}(H_{n,t}) \to \mathrm{CF}^k(H_{n,t})$ over Λ_E. We recall the following algebraic property of a telescope of subcomplexes. Suppose that we have a commutative diagram of chain complexes

$$\begin{array}{ccccccccc} \mathcal{C}' := & C_1' & \longrightarrow & C_2' & \longrightarrow & C_3' & \longrightarrow & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{C} := & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & \cdots \end{array}$$

where horizontal maps are chain maps (which we call continuation maps) and vertical maps are inclusion maps of subcomplexes. Then we have an induced map between telescopes $\mathrm{tel}(\mathcal{C}') \to \mathrm{tel}(\mathcal{C})$.

Lemma 4.4 [Borman et al. 2022, Lemma A.1] *Suppose that for every $\gamma \in C_n$ there exists $N(\gamma) > 0$ such that under continuation maps γ lands in $C'_{n+N(\gamma)}$. Then $\mathrm{tel}(\mathcal{C}') \to \mathrm{tel}(\mathcal{C})$ is a quasi-isomorphism.*

This lemma can be applied to our lower Floer complexes.

Proposition 4.5 *For any integer k and energy bound $E > 0$, there exists a family of Hamiltonian functions such that*

$$\mathrm{tel}(\mathcal{CF}^{k,L}) \to \mathrm{tel}(\mathcal{CF}^{k})$$

is a quasi-isomorphism over Λ_E.

Proof For any integer k and energy bound $E > 0$, we pick Hamiltonian functions as above to get a commutative diagram between $\mathrm{CF}^{k,L}(H_{n,t})$ and $\mathrm{CF}^{k}(H_{n,t})$ such that horizontal maps are continuations and vertical maps are inclusions. Next we check the condition in Lemma 4.4.

Pick $\gamma \in \mathrm{CF}^k(H_{n,t})$. If it is a lower orbit, then its images under continuation maps are always lower, by Lemma 4.3(5). So we assume γ is an upper orbit, and we will show after several continuation maps, it becomes lower or zero. Let $T^{E_1}\gamma_1$ be the image of γ under the continuation map $\mathrm{CF}^k(H_{n,t}) \to \mathrm{CF}^k(H_{n+1,t})$. Then

$$E_1 = -\int_\gamma \tilde{\theta} + \int_{\gamma_1} \tilde{\theta} + \int u_1^*\tilde{\omega} - \int_\gamma H_{n,t} + \int_{\gamma_1} H_{n+1,t}.$$

If γ_1 is a lower orbit, then we are done. Otherwise we consider $T^{E_2}\gamma_2$ as the image of $T^{E_1}\gamma_1$ under the continuation map $\mathrm{CF}^k(H_{n+1,t}) \to \mathrm{CF}^k(H_{n+2,t})$. We have

$$\begin{aligned} E_2 &= E_1 + \left(-\int_{\gamma_1} \tilde{\theta} + \int_{\gamma_2} \tilde{\theta} + \int u_2^*\tilde{\omega} - \int_{\gamma_1} H_{n+1,t} + \int_{\gamma_2} H_{n+2,t}\right) \\ &= -\int_\gamma \tilde{\theta} + \int_{\gamma_2} \tilde{\theta} + \int u_1^*\tilde{\omega} + \int u_2^*\tilde{\omega} - \int_\gamma H_{n,t} + \int_{\gamma_2} H_{n+2,t}. \end{aligned}$$

By the index-bounded condition, the first two terms are bounded. Moreover, our Hamiltonian functions have small variations outside; the $\tilde{\omega}$–energy terms are nonnegative. If γ_2 is still an upper orbit, we will consider its image under the third continuation map. Therefore after N compositions of continuation maps, either γ is sent to a lower orbit, or $E_N > E$, since $-\int_\gamma H_{n,t} + \int_{\gamma_N} H_{n+N,t}$ becomes arbitrarily large. This completes the proof by applying Lemma 4.4. □

We call the process to get this quasi-isomorphism *ignoring upper orbits*. And we define the truncated symplectic cohomology as

$$\mathrm{SH}^k_M(D;\Lambda_E) := H(\mathrm{tel}(\mathcal{CF}^{k,L});\Lambda_E) = H(\mathrm{tel}(\mathcal{CF}^{k});\Lambda_E).$$

Therefore Proposition 4.5 gives two ways to compute the truncated symplectic cohomology: by using $\mathrm{tel}(\mathcal{CF}^{k,L})$ or $\mathrm{tel}(\mathcal{CF}^{k})$. The former has the advantage that all generators of the underlying complex are local and the differentials are global. Moreover, the filtration on $\mathrm{tel}(\mathcal{CF}^{k})$ constructed in Section 3 becomes exhaustive when restricted to $\mathrm{tel}(\mathcal{CF}^{k,L})$. Hence we will get a convergent spectral sequence to compute the truncated symplectic cohomology. This will be discussed in next subsection. Now we study how this truncated symplectic cohomology is related to the usual relative symplectic cohomology.

First, note that a telescope is quasi-isomorphic to a direct limit. Hence we also have

$$\mathrm{SH}^k_M(D;\Lambda_E) = H(\mathrm{tel}(\mathcal{CF}^{k,L});\Lambda_E) = H\big(\varinjlim_n \mathcal{CF}^{k,L};\Lambda_E\big).$$

The following lemma shows that the direct limit has a simpler description in our index-bounded case.

Lemma 4.6 *For a fixed degree k, the direct limit $\varinjlim_n \mathcal{CF}^{k,L}$ which contains lower orbits is a finite-dimensional free Λ_E–module.*

Proof By the index-bounded condition and nondegeneracy of the contact form, for each fixed Hamiltonian, $\mathrm{CF}^{k,L}(H_{n,t})$ is a finite-dimensional free Λ_E–module, generated by degree–k orbits. On the other hand, the lower parts of the Hamiltonian functions $H_{n,t}$ are fixed. Hence we have a canonical identification between $\mathrm{CF}^{k,L}(H_{n,t})$ for different n. Next we study the continuation maps between $\mathrm{CF}^{k,L}(H_{n,t})$. After identifying the generators of $\mathrm{CF}^{k,L}(H_{n,t})$ for different n, the continuation maps can be written as $l\times l$ matrices $\{a^n_{ij}\}$ with entries $a^n_{ij}\in\Lambda_E$, where l is the dimension of $\mathrm{CF}^{k,L}(H_{n,t})$.

Let u be a Floer cylinder contributing to the continuation maps, and assume u is contained in the region where the Hamiltonian function is fixed. Then by regularity it can only be the identity map; see Lemma 2.1. There may be other Floer cylinders that travel outside D and contribute to the continuation maps, which have nontrivial topological energy. Hence the continuation maps, viewed as a matrix, have the following properties:

(1) The entries on the diagonal are $a^n_{jj} = 1 + b^n_{jj}$, with $v(b^n_{jj}) > 0$.

(2) The off-diagonal entries have strictly positive valuations.

Note that the determinant of the matrix $\{a^n_{ij}\}$ has a constant term 1, which says that the matrix is invertible. Hence each continuation map is an isomorphism of Λ_0–modules. So the direct limit $\varinjlim_n \mathcal{CF}^{k,L}$ is isomorphic to $\mathrm{CF}^{k,L}(H_{n,t})$, which is a finite-dimensional free Λ_E–module. □

To effectively use the truncated symplectic cohomology, there are two options: to show it is an invariant with good properties, or to relate it with the original relative symplectic cohomology $\mathrm{SH}^k_M(D)$.

For the first option, we expect the following is true:

Proposition 4.7 *The truncated symplectic cohomology* $\mathrm{SH}_M^k(D;\Lambda_E)$ *is a finite-dimensional* Λ_E*–module. Let* $\lambda > 0$ *be the smallest number such that*

$$T^{\lambda} \cdot \mathrm{SH}_M^k(D;\Lambda_E) = 0.$$

Then the displacement energy of D *is not less than* λ*.*

Proof The finite-dimensionality follows from the above lemma. We expect the proof of the energy relation is similar to [Varolgunes 2018, Remark 4.2.8] in the original relative symplectic cohomology setting. The full proof will be pursued in the future. □

The number λ is an analogue of the torsion threshold of the Lagrangian Floer cohomology (see [Fukaya et al. 2009, Theorem J]), which is related to the displacement energy of a Lagrangian submanifold. Since the energy bound E can be as large as one needs, this proposition is useful for most displacement problems.

For the second option, we have the following:

Proposition 4.8 *The inverse limit of the truncated symplectic cohomology recovers the original relative symplectic cohomology. That is,*

$$\big(\varprojlim_{E} \mathrm{SH}_M^k(D;\Lambda_E)\big) \otimes_{\Lambda_0} \Lambda \cong \mathrm{SH}_M^k(D) \otimes_{\Lambda_0} \Lambda.$$

Here the inverse limit is taken as E *goes to infinity.*

To prove the proposition, first we recall some results in homological algebra.

Definition 4.9 An inverse system

$$\mathcal{C} = C_1 \leftarrow C_2 \leftarrow \cdots$$

is said to satisfy the *Mittag-Leffler condition* if, for each $n \in \mathbb{N}$, there exists $i \geq n$ such that, for all $j \geq i$, we have

$$\mathrm{Im}(C_j \to C_n) \cong \mathrm{Im}(C_i \to C_n).$$

The Mittag-Leffler condition shows the vanishing of the $\varprojlim^1$ of an inverse system.

Proposition 4.10 [Weibel 1994, Proposition 3.5.7] *If an inverse system* $\mathcal{C}$ *satisfies the Mittag-Leffler condition, then* $\varprojlim^1(\mathcal{C}) = 0$.

Proposition 4.11 [Weibel 1994, Proposition 3.5.8] *For an inverse system*

$$\mathcal{C} = C_1 \leftarrow C_2 \leftarrow \cdots$$

of complexes, which satisfies the degreewise Mittag-Leffler condition, we have a short exact sequence

$$0 \to \varprojlim^1 H^*(C_n) \to H^*(\varprojlim C_n) \to \varprojlim H^*(C_n) \to 0.$$

Now we begin the proof of Proposition 4.8.

Proof Fix a sequence of positive numbers

$$0 < E_1 < E_2 < \cdots < E_l < \cdots$$

going to infinity. Let $\{G_n\}$ be a fixed acceleration data we used in Proposition 4.5 and Lemma 4.6. Then we have

$$\mathrm{SH}^k_M(D) \otimes_{\Lambda_0} \Lambda = H\big(\varprojlim_{E_l} \mathrm{tel}(\mathrm{CF}^k(G_n) \otimes_{\Lambda_0} \Lambda_{E_l})\big) \otimes_{\Lambda_0} \Lambda.$$

The complexes $\mathrm{tel}(\mathrm{CF}^k(G_n) \otimes_{\Lambda_0} \Lambda_{E_l})$ form an inverse system over E_l. Note that the Hamiltonian functions are independent of E_l, so the maps in this system are given by projection. Since projections are surjective, this inverse system satisfies the degreewise Mittag-Leffler condition. Hence we have a short exact sequence

$$0 \to \varprojlim{}^1 H(C_l) \to H(\varprojlim C_l) \to \varprojlim H(C_l) \to 0,$$

where $C_l := \mathrm{tel}(\mathrm{CF}^k(G_n) \otimes_{\Lambda_0} \Lambda_{E_l})$. In this short sequence, the middle term is what we need to compute and the right term is the inverse limit of truncated symplectic cohomology. Hence it suffices to show the vanishing of the left term.

In Lemma 4.6, we have showed that $H\big(\varinjlim_n \mathcal{CF}^{k,L}; \Lambda_{E_l}\big)$ is finite-dimensional by using acceleration datum depending on E_l. On the other hand, the quasi-isomorphisms before Lemma 4.6 shows that

$$H(C_l) = H\big(\varinjlim_n \mathcal{CF}^{k,L}; \Lambda_{E_l}\big).$$

So $H(C_l)$ is finite-dimensional for any E_l. Moreover, for different E_l the defining Hamiltonians for $\mathcal{CF}^{k,L}$ have the same fixed lower part. Hence the dimensions of $H(C_l)$ have a uniform upper bound, independent of l, given by the number of lower degree–k orbits of Hamiltonian functions with a fixed lower part. Therefore $H(C_l)$ over l is an inverse system of finite-dimensional modules with a uniform upper bound on ranks. In the following we will show it satisfies the degreewise Mittag-Leffler condition, which completes the proof. □

Now we prove that $H(C_l)$ over l is an inverse system which satisfies the degreewise Mittag-Leffler condition, by using the finite torsion criterion in [Groman and Varolgunes 2023].

Let V be a Λ_0–module. For any element $v \in V$ we define

$$\tau(v) := \inf\{\lambda \geq 0 \mid T^\lambda v = 0\}$$

and we define the *maximal torsion* of V as

$$\tau(V) := \sup_{v \in V, \tau(v) < +\infty} \tau(v);$$

see [Groman and Varolgunes 2023, Definition 6.15]. The following is a combination of Lemma 6.19 and Proposition 6.12 in [loc. cit.]. The invariant $\mathrm{SH}^k_{M,\lambda}(D)$ in [loc. cit.] is our truncated invariant $\mathrm{SH}^k_M(D; \Lambda_E))$ with $E = \lambda$.

Proposition 4.12 *If* $\mathrm{SH}^k_M(D)$ *has finite maximal torsion, then* $H(C_l)$ *over* l *is an inverse system which satisfies the degreewise Mittag-Leffler condition.*

Next we verify that $\mathrm{SH}^k_M(D)$ has a finite maximal torsion. Suppose $\mathrm{SH}^k_M(D)$ has an infinite maximal torsion. Then we have a sequence of elements x_n, y_n in the completed telescope such that

$$d(x_n)=0, \quad d(y_n)=T^{\tau_n}x, \quad 0<\tau_1<\tau_2\cdots<\tau_n<\cdots\to+\infty.$$

This shows that as l goes to infinity, the number of x_n's with valuations less than E_l goes to infinity. Hence the rank of $H(C_l)$ also goes to infinity, which contradicts that their ranks are uniformly bounded from above.

4.2 The first page of the spectral sequence

We showed that the truncated symplectic cohomology recovers the relative symplectic cohomology. Now we study how to compute the truncated symplectic cohomology.

For any integer k and energy bound $E>0$, we have three chain models

$$\mathrm{tel}(\mathcal{CF}^k), \quad \mathrm{tel}(\mathcal{CF}^{k,L}), \quad \varinjlim_n \mathcal{CF}^{k,L}$$

given by a particular family of Hamiltonian functions. They are all quasi-isomorphic; hence they all compute the truncated symplectic cohomology over Λ_E. Now we equip the second chain model with the filtration defined in (3-5). Recall that for a general element $x=\sum_i a_i\gamma_i\in\mathrm{tel}(\mathcal{CF}^{k,L})$ we define $\sigma\colon\mathrm{tel}(\mathcal{CF}^{k,L})\to\mathbb{R}\cup\{-\infty\}$ by

$$\sigma(x):=\inf_i\left\{v(a_i)-\int_{\gamma_i}H_{i,t}-\int_{\gamma_i}\tilde{\theta}\right\}. \tag{4-1}$$

And for any $p\in\mathbb{R}$ we define

$$F^p\,\mathrm{tel}(\mathcal{CF}^{k,L}):=\{x\in\mathrm{tel}(\mathcal{CF}^{k,L})\mid\sigma(x)\geq p\}. \tag{4-2}$$

By the computations in Section 3, we know that the differentials in the telescope are compatible with this filtration, which makes $\mathrm{tel}(\mathcal{CF}^{k,L})$ a filtered differential graded module. Moreover, since all generators in $\mathrm{tel}(\mathcal{CF}^{k,L})$ are lower orbits, the Hamiltonian terms in $\sigma(x)$ are uniformly bounded. By the index-bounded condition, the integrals of $\tilde{\theta}$ in $\sigma(x)$ are also uniformly bounded. Hence $\sigma(x)>-\infty$ for any x. This shows that the filtration is exhaustive. (Actually this filtration is bounded from below by some number.) Therefore Theorems 3.6 and 3.7 give us a spectral sequence which converges to the truncated symplectic cohomology. This proves the convergence part in Theorem 1.1(1). In the following we will compute the first page of this spectral sequence.

First we observe that by using the special family of S–shaped Hamiltonians, all Floer cylinders which are not contained in the Liouville domain have positive $\tilde{\omega}$–energy, given that the asymptotic boundaries of the cylinders are lower orbits.

This will be proved via Abouzaid and Seidel's integrated maximum principle. We follow [Borman et al. 2022, Section 3.4] and recall the setup now. Let (K, ω) be a symplectic manifold with a concave boundary (Y, α). That is, there is a symplectic embedding $(Y \times [c, c+\epsilon), d(r\alpha))$ onto a neighborhood of Y in K, where r is the Liouville coordinate. We will consider maps $u\colon (\Sigma, \partial\Sigma) \to (K, Y)$ solving the Floer equation with respect to a certain class of almost-complex structures and Hamiltonian perturbations. Here $(\Sigma, \partial\Sigma)$ is a general Riemann surface with boundary.

To define our Floer equation, we choose a family of compatible almost-complex structures J_z parametrized by $z \in \Sigma$, and a Hamiltonian-valued 1–form $\kappa \in \Omega^1(\Sigma; C^\infty(K))$. Note that we may interpret κ as a 1–form on $\Sigma \times K$. The de Rham differential has a decomposition

$$d := d_{\Sigma\times K} = d_\Sigma + d_K.$$

Then the nondegeneracy of ω gives us a Hamiltonian-vector-field-valued 1–form $X_\kappa \in \Omega^1(\Sigma; C^\infty(TW))$. The Floer equation in consideration is

$$(du - X_\kappa)^{0,1} = 0.$$

Proposition 4.13 [Borman et al. 2022, Proposition 3.9] *Suppose that*:

1. (1) *J_z is of contact type along Y for all $z \in \partial\Sigma$, that is, $dr \circ J_z = -r\alpha$.*
2. (2) *There exist 1–forms $\beta_1, \beta_2 \in \Omega^1(\Sigma)$ such that $\kappa = \beta_1 \cdot r + \beta_2$ in a neighborhood of Y.*
3. (3) $d_\Sigma\kappa - \{\kappa, \kappa\} - d\beta_2 \geq 0.$[1]

Then any smooth map $u\colon (\Sigma, \partial\Sigma \neq \varnothing) \to (K, Y)$ solving $(du - X_\kappa)^{0,1} = 0$, will satisfy that

$$\int_\Sigma u^*\omega - \int_{\partial\Sigma} u^*(c\alpha) \geq 0,$$

with equality if and only if u is contained in Y.

Next we apply the above proposition to our setting. We make two more assumptions on the S–shaped Hamiltonian functions which are used to define the telescope:

1. (1) Near the boundary of $D_{1+3\epsilon/4}$, the function H_1 is a function of the radial coordinate with small positive slope.
2. (2) All H_n's and all homotopies connecting them are given by translations in the s–coordinate outside $D_{1+3\epsilon/4}$. That is, we choose a smooth nondecreasing function $\phi(s)\colon \mathbb{R} \to \mathbb{R}$ such that

   $$\phi(s) = 0 \quad \text{for all } s \leq 0, \qquad \phi(s) = \tfrac{1}{2} \quad \text{for all } s \geq 1.$$

   Then define $H_n = H_1 + \frac{n}{2}$ and define the homotopy between H_n and H_{n+1} to be $H_s = H_n + \phi(s)$ outside $D_{1+3\epsilon/4}$.

[1] Here $\{\kappa, \kappa\}$ lives in $\Omega^2(\Sigma; C^\infty(K))$ and is defined by $\{\kappa, \kappa\}(v, w) := \{\kappa(v), \kappa(w)\}$ by the Poisson bracket.

Proposition 4.14 *Let D be a Liouville domain in M. Assume that $[\tilde{\omega}]$ is integral; see the paragraph after Lemma 3.1. And assume our acceleration data satisfies the above conditions. Let γ_-, γ_+ be two 1–periodic orbits of H_-, H_+ in the telescope which are lower orbits. Let u be a Floer solution connecting γ_-, γ_+. If $\mathrm{Im}(u)\cap(M-D_{1+3\epsilon/4})\neq\varnothing$, then $\tilde{\omega}(u)>\hbar$, where $\hbar$ is a positive number independent of $H_\pm$.*

Proof Suppose that $\mathrm{Im}(u)\cap(M-D_{1+3\epsilon/4})\neq\varnothing$. We pick a number ϵ' which is slightly greater than ϵ and $\partial D_{1+3\epsilon'/4}$ intersects $\mathrm{Im}(u)$ transversally. The intersection is a disjoint union of circles in $\mathbb{R}\times S^1$, which is nonempty when ϵ' is close to ϵ.

Then we set $M-D_{1+3\epsilon'/4}$ to be the concave manifold K in Proposition 4.13, and $c=1+\frac{3\epsilon'}{4}$. By the transversal intersection we get a new map $u'\colon(\Sigma,\partial\Sigma\neq\varnothing)\to(K,Y)$ solving the Floer equation. We will check the hypotheses in Proposition 4.13. First in our definition of the telescope we did use contact-type almost-complex structures in the cylindrical region. Hence (1) is satisfied.

In our case the 1–form satisfies $\kappa=H_s\,dt$. Near Y it is given by linear Hamiltonian functions, in terms of the r–coordinate: $\kappa=(ar+b+\phi(s))\,dt$. Hence (2) is satisfied, with $\beta_1=a\,dt,\ \beta_2=(b+\phi(s))\,dt$.

Next we verify (3). Since there is no ds term in κ, we have that $\{\kappa,\kappa\}(\partial_s,\partial_t)=0$. Moreover, we can compute that

$$\begin{aligned} d_\Sigma\kappa-d\beta_2&=\partial_sH_s\,ds\,dt-d\beta_2\\ &=\phi'(s)\,ds\,dt-\partial_s(b+\phi(s))\,ds\,dt\\ &=\phi'(s)\,ds\,dt-\phi'(s)\,ds\,dt=0. \end{aligned}$$

Therefore all hypotheses in Proposition 4.13 are satisfied and we get

$$0<\int_\Sigma u'^*\omega-\int_{\partial\Sigma}u'^*(c\alpha)=\int_\Sigma u'^*\omega-\int_{\partial\Sigma}u'^*\tilde{\theta}.$$

By our integrality assumption, the right-hand side of the equation is an integer. On the other hand, since the support of $\tilde{\omega}$ is outside $D_{1+3\epsilon/4}$, the integral

$$\int u^*\tilde{\omega}=\int u^*\omega-\int u^*\,d\tilde{\theta}$$

can be approximated by

$$\int_\Sigma u'^*\omega-\int_{\partial\Sigma}u'^*\tilde{\theta}=\int_\Sigma u'^*\omega-\int_{\partial\Sigma}u'^*(c\alpha)>0$$

as ϵ' tends to ϵ. This shows that if $\mathrm{Im}(u)\cap(M-D_{1+3\epsilon/4})\neq\varnothing$, then $\tilde{\omega}(u)\geq 1$. Finally, for example, we can set $\hbar=\frac{1}{2}$. □

Remark 4.15 The above proposition is an analogue of [Borman et al. 2022, Proposition 5.10]. However, our situation is easier since we can assume the Hamiltonian functions are translations of a fixed one outside $D_{1+3\epsilon/4}$.

Now we can use this positive number $\hbar$ to construct a $\mathbb{Z}$–valued filtration out of the $\mathbb{R}$–valued filtration (4-2). And for any $l\in\mathbb{Z}$ we define

(4-3) $$F^l\,\mathrm{tel}(\mathcal{CF}^{k,L}):=\{x\in\mathrm{tel}(\mathcal{CF}^{k,L})\mid\sigma(x)\geq l\hbar\}.$$

Then this new filtration induces a convergent spectral sequence. Its first page is calculated by using all differentials of which the change of the $\tilde{\omega}$–energy are less than $\hbar$. On the other hand, Floer solutions that are not contained in D are weighted by a positive $\tilde{\omega}$–energy greater than $\hbar$. Hence the first page of this spectral sequence is calculated by all differentials and continuation maps that are in D. Since in D our Hamiltonian function is a fixed function, the continuation maps are identity maps and the differentials give the classical symplectic cohomology $\mathrm{SH}^k(D;\Lambda_E)$. In particular, when $[\tilde{\omega}]=0\in H^2(M,D;\mathbb{Z})$, there will not be outside contributions; hence the spectral sequence degenerates at its first page. Now we have completed the proofs of (1) and (3) in Theorem 1.1.

5 Examples and extensions

We now discuss some applications of our spectral sequence, and how to construct perturbations in the Morse–Bott case.

Example 5.1 Let B be a round ball symplectically embedded in a Calabi–Yau manifold M^{2n} with an integral symplectic form. Then the boundary ∂B carries the standard contact structure on an odd-dimensional sphere, which is Morse–Bott index-bounded. After perturbing, the nondegenerate Reeb orbits on the sphere all have positive Conley–Zehnder indices. Our Hamiltonian flow is the reverse of the Reeb flow in the cylindrical region. Hence the degrees of nonconstant Hamiltonian orbits are all less than $2n$. Moreover, we can choose the fixed lower part of our Hamiltonians so that they do not have degree–$2n$ constant orbits. So the only degree–$2n$ constant orbits are upper constant orbits. Then we apply the ignoring-upper-orbits process to get that $\mathrm{SH}^{2n}_M(B)\otimes_{\Lambda_0}\Lambda=0$. Finally, the Mayer–Vietoris sequence shows that any neighborhood of $M-B$ has nonvanishing relative symplectic cohomology; hence it is not stably displaceable. This fact is already known in [Ishikawa 2016, Theorem 1.1] and [Tonkonog and Varolgunes 2023, Corollary 1.15] by using different methods (with stronger conclusions). We put our argument here to motivate Proposition 5.9.

5.1 Simply connected Lagrangians in Calabi–Yau manifolds

Let (L,g) be a Riemannian manifold and let T^*L be its cotangent bundle with the standard symplectic form. The unit disk bundle D_1T^*L, with respect to the metric g, is a Liouville domain with the unit sphere bundle ST^*L being its contact boundary. A closed geodesic $q(t)$ in L lifts to a Reeb orbit $\gamma(t)=(q(t),q'(t))$ in ST^*L. Pick a trivialization Φ of the contact distribution along γ. There is a Conley–Zehnder index $\mathrm{CZ}_\Phi(\gamma)$. On the other hand, the trivialization Φ also gives a trivialization of the symplectic vector bundle TT^*L along q. Hence there is a Maslov index $\mu_\Phi(q)$ of q, viewed as a loop in L. The relation between these two indices is the following lemma.

Lemma 5.2 [Cieliebak and Mohnke 2018, Lemma 2.1] *In the above notation, we have*

$$\mathrm{CZ}_\Phi(\gamma)+\mu_\Phi(q)=\mathrm{ind}(q),$$

where $\mathrm{ind}(q)$ *is the Morse index of* q *as a geodesic.*

Now we consider a Lagrangian submanifold L in a Calabi–Yau manifold M. Let D be a Weinstein neighborhood of L, which is isomorphic to D_1T^*L for some metric g on L. Then ∂D is a contact hypersurface in M. In particular, when L is simply connected we can use trivializations induced by disk cappings to compute these indices. The Maslov index is always zero and we have $\mathrm{CZ}_\Phi(\gamma) = \mathrm{ind}(q) \geq 0$.

Similar to the index-bounded condition, we can consider the following relation between the Morse index and the length of a closed geodesic.

Definition 5.3 A Riemannian metric g is called *index-bounded* if, for every $m > 0$, there exists $\mu_m > 0$ such that

$$\{\mathrm{length}(q) \mid -m < \mathrm{ind}(q) < m\} \subset (0, \mu_m)$$

for all closed geodesics. And the metric is called (Morse–Bott) nondegenerate if the length functional is (Morse–Bott) nondegenerate.

Hence if L admits an index-bounded Riemannian metric g then any Lagrangian embedding of L into a Calabi–Yau manifold admits an index-bounded neighborhood. This is true for Riemannian manifolds with a positive Ricci curvature.

Lemma 5.4 [Milnor 1963, Theorems 19.4 and 19.6] *Let (L, g) be a closed Riemannian manifold of dimension n whose Ricci curvature satisfies $\mathrm{Ric}_g \geq (n-1)C$ for some positive real number C. Then any closed geodesic on L with length λ has Morse index greater than $\lambda\sqrt{C}/\pi - 1$. In particular, g is index-bounded.*

Proof Let γ be a closed geodesic on L with length λ. For the constant C there exists an integer l such that

$$l\pi/\sqrt{C} < \lambda \leq (l+1)\pi/\sqrt{C}.$$

We cut γ into $l+1$ segments such that each of the first l segments has length slightly greater than $\pi/\sqrt{C}$. Then by the proof of [Milnor 1963, Theorem 19.6], any geodesic segment with length greater than $\pi/\sqrt{C}$ is unstable. Hence each of these l segments has index at least 1, and the index of γ is at least l.

Now let γ be a closed geodesic with index k and length λ, we know that $k > \lambda\sqrt{C}/\pi - 1$, which shows that $\lambda < (k+1)\pi/\sqrt{C}$. Hence g is index-bounded. □

Remark 5.5 The positivity of the Ricci curvature of a metric g is preserved under C^∞–small perturbations. Hence we obtain a nondegenerate index-bounded metric g_ϵ after perturbation.

Let (L, g_ϵ) be a closed Riemannian manifold with a positive Ricci curvature. Then g_ϵ is a nondegenerate index-bounded metric. Suppose there is a Lagrangian embedding $L \to M$ into a Calabi–Yau manifold M. Another consequence of the Bonnet–Myers theorem [Milnor 1963, Theorem 19.6] is that L has a finite fundamental group. Hence L is a Lagrangian submanifold of M with a vanishing Maslov class. Now let U be a Weinstein neighborhood of L induced by the metric g_ϵ. The above lemma tells us that U is a Liouville domain with a nondegenerate index-bounded boundary in M. In conclusion, our spectral sequence works for computing $\mathrm{SH}_M(U)$, given the integrality condition on $\tilde{\omega}$.

If we have two Riemannian manifolds both with positive Ricci curvature, then the product manifold also admits a metric with positive Ricci curvature. Hence we can also study Lagrangian submanifolds of product type.

Now we present the example of spheres, to demonstrate the above method. Let S^n be a sphere with dimension $n \geq 3$, and let g_R be the round metric on S^n. It is known that g_R is a Morse–Bott nondegenerate index-bounded metric. Next let g_ϵ be a C^∞–small generic perturbation of g_R such that it is a nondegenerate index-bounded metric.

The above discussion tells us that for any Lagrangian sphere $S = S^n$ with $n \geq 3$ in a Calabi–Yau manifold M with an integral symplectic form, a Weinstein neighborhood U of S induced by the metric g_ϵ has a nondegenerate index-bounded contact boundary. Hence our spectral sequence works for computing $\mathrm{SH}_M(U)$.

Lemma 5.6 *For (M, S, U) as above, we have $\mathrm{SH}_M(U) \otimes_{\Lambda_0} \Lambda \neq 0$.*

Proof For a given energy bound $E > 0$, there is a convergent spectral sequence which starts from the symplectic cohomology $\mathrm{SH}(T^*S^n; \Lambda_E)$ and converges to $\mathrm{SH}_M(U; \Lambda_E)$. In our degree notation (2-1), the usual symplectic cohomology $\mathrm{SH}(T^*S^n; \mathbb{C})$ is nonzero and 1–dimensional in degrees

$$\{n\} \cup \{i(1-n)+n, i(1-n)+n-1 \mid i \in \mathbb{Z}_+\}.$$

(Note that our Hamiltonian flow is the reverse of the Reeb flow in the cylindrical region.) The nonzero element in degree n cannot be killed in the spectral sequence when $n \geq 3$, since the differential only changes the degree by 1.

Hence for any energy bound $E > 0$, the truncated invariant satisfies that

$$\mathrm{SH}^n_M(U; \Lambda_E) \cong \Lambda_E.$$

Then by taking the inverse limit over E, we have that $\mathrm{SH}_M(U) \otimes_{\Lambda_0} \Lambda \neq 0$. □

So any Lagrangian sphere with dimension $n \geq 3$ in a Calabi–Yau manifold is stably nondisplaceable. This is known by using the Lagrangian Floer cohomology of S; see [Fukaya et al. 2009, Theorem L]. But by using the Mayer–Vietoris property, we can get more from the above lemma. Note that $\mathrm{SH}^2_M(U)$ is zero, and hence it cannot be isomorphic to the quantum cohomology of M. Pick a Weinstein neighborhood V of S, also induced by the same g_ϵ but with a smaller radius in the fiber direction, compared with U. Then $(M - V) \cup U = M$ and the boundaries of U, V do not intersect. The Mayer–Vietoris property says that $\mathrm{SH}^2_M(M - V) \otimes_{\Lambda_0} \Lambda \neq 0$.

Lemma 5.7 *For (M, S, U) as above, we have that $M - U$ is stably nondisplaceable.*

Proof Suppose that $M - U$ is stably displaceable. Then a neighborhood K of $M - U$ is also stably displaceable. We can choose V as above such that $M - V \subset K$, contradicting $\mathrm{SH}^2_M(M - V) \otimes_{\Lambda_0} \Lambda \neq 0$. □

This result is new and it contrasts the case where the ambient space is not Calabi–Yau: a Weinstein neighborhood U of a Lagrangian sphere can be compactified to be a quadric hypersurface $Q^n \subset \mathbb{C}P^{n+1}$. Note that Q^n is a monotone symplectic manifold, and the complement of U in Q^n is a small neighborhood of a divisor Q^{n-1}, which is stably displaceable.

Example 5.8 Let $L = S^2 \times S^3$ be a Lagrangian submanifold in a Calabi–Yau manifold M. Then there exists a Weinstein neighborhood of L which is a Liouville domain with a nondegenerate index-bounded boundary. So it's possible to use our spectral sequence to compute $\mathrm{SH}_M(L)$, which could determine the displaceability of L. However, due to the S^2–factor, we don't have an immediate nonvanishing result, compared with the case of spheres with dimension larger than 2. Hence $\mathrm{SH}_M(L)$ may depend on the ambient space.

On the other hand, the Lagrangian Floer cohomology of L may have obstructions to be defined. The obstruction lies in $H^2(L; \mathbb{Q})$; see [Fukaya et al. 2009, Theorem L].

Another application of the geodesic-Reeb orbit correspondence is a generalization of Lemma 5.7. If we only care about the complement of the Lagrangian, then no index-bounded Riemannian metric is needed.

Proposition 5.9 *Let (M^{2n}, ω) be a symplectic Calabi–Yau manifold with $n > 2$ and ω represents an integral class in $H^2(M)$. For a simply connected Lagrangian S in M and a Weinstein neighborhood U of S, we have that $M - U$ is not stably displaceable in M.*

Proof When $n = 3$, the only simply connected 3–manifold is the 3–sphere which has been discussed. So in the following we assume $n > 3$.

Let g be a nondegenerate Riemannian metric on S. By the discussion after Lemma 5.2, the Reeb orbits on the boundary of $U = D_1 T^* S$ all have nonnegative Conley–Zehnder indices. Then we pick a family of S–shaped Hamiltonian functions to be our acceleration data such that all the lower constant orbits have degrees less than $n + 1$. This can be achieved since $U = D_1 T^* S$. Note that the Hamiltonian flow is in the opposite direction of the Reeb flow, in the cylindrical region. From a Reeb orbit to its corresponding Hamiltonian orbit, the index is changed by a sign, plus a error term bounded by 1; see Lemma 2.7. Hence the Conley–Zehnder indices of nonconstant Hamiltonian orbits are all less than 2. Their degrees, defined as $\mathrm{CZ}(\gamma) + n$, are all less than $n + 3$ after time-dependent perturbations. So these Hamiltonian functions are index-bounded in degree $2n$, since all degree–$2n$ generators are upper constant orbits. Then the ignoring upper orbits process says that $\mathrm{SH}^{2n}_M(D; \Lambda_E) = 0$ for any $E > 0$. Finally we apply the Mayer–Vietoris argument in Lemma 5.7 to complete the proof. □

The disk cotangent bundle can be regarded as a Liouville domain with a smooth Lagrangian skeleton. On the other hand, certain Brieskorn manifolds are Liouville domains of which the Lagrangian skeletons are chains of spheres modeled by trees. In [Kwon and van Koert 2016], the Reeb orbits of many Brieskorn manifolds have been studied explicitly. Hence one can use the calculation of indices therein to get more applications, like the rigidity of symplectic embeddings of Brieskorn manifolds into Calabi–Yau manifolds.

5.2 Perturbation of Morse–Bott orbits

Let D be a Liouville domain with a contact boundary (C,α) in a closed Calabi–Yau manifold M such that α is Morse–Bott index-bounded. Then a time-independent S–shaped Hamiltonian function H has nondegenerate constant orbits, and nonconstant 1–periodic orbits that are Morse–Bott degenerate, given by the Reeb orbits of α. Now we will perturb H to get a nondegenerate S–shaped Hamiltonian function H_t such that it is index-bounded. We remark that we are perturbing the Hamiltonian function instead of perturbing the contact form, since the index-bounded condition may be destroyed by the latter perturbation.

Let Y be the set of l–periodic Reeb orbits of α. By the Morse–Bott condition, Y is a closed smooth submanifold of C. It may have several connected components, and we will construct our perturbation componentwisely. For simplicity, we assume that $l=1$. The general case is similar. Our perturbation is a modification of the case of a time-independent Hamiltonian function with transversally nondegenerate orbits, where $Y=S^1$.

There is an S^1–action on Y induced by the Reeb flow ϕ^t. For a Morse function $g\colon Y\to\mathbb{R}$, we twist it by the S^1–action to get a time-dependent function on Y:

$$g_t(y):=g(\phi^{1-t}(y)),\quad t\in[0,1],\ y\in Y.$$

Next let N be the normal bundle of Y in M. We extend g_t to be a function $\tilde{g}_t$ on N which is supported near the zero section. We also require that $\tilde{g}_t$ does not depend on the fiber direction in a small neighborhood of the zero section.

Now for a time-independent S–shaped Hamiltonian function $H\colon M\to\mathbb{R}$, it has degenerate 1–periodic orbits in the cylindrical region, which form the submanifold Y. Define $G_t^\epsilon\colon S^1\times M\to\mathbb{R}$ as

$$G_t^\epsilon(m)\colon H(m)+\epsilon\tilde{g}_t(m),\quad m\in M.$$

Our main result of this subsection is:

Proposition 5.10 *For small $\epsilon>0$, the 1–periodic orbits of G_t^ϵ in a small neighborhood of Y are in one-to-one correspondence with critical points of g. Let γ be a 1–periodic orbit of H on Y and γ^ϵ be a 1–periodic orbit of G_t^ϵ near Y. We have that:*

(1) $\int_\gamma\theta=\int_{\gamma^\epsilon}\theta$, *where θ is the Liouville 1–form on D.*
(2) $|\mathrm{CZ}(\gamma)-\mathrm{CZ}(\gamma^\epsilon)|\le\dim_{\mathbb{R}}Y$.

Proof This proposition, which is known to experts, is a modification of [Cieliebak et al. 1996, Lemma 2.1 and Proposition 2.2].

Let J be a compatible almost-complex structure on M which is cylindrical near C. By the construction, the Hamiltonian vector field of G_t^ϵ is

$$X_{G_t^\epsilon}(m) = X_H(m) + \epsilon J \nabla \tilde{g}_t(m).$$

Here the gradient is computed with respect to the metric $\omega(\cdot\,, J\,\cdot)$. Let $p \in Y$ be a critical point of g. Then $\phi^t(p)$ is a 1–periodic orbit of H on Y. We can also check that it is also a 1–periodic orbit of G_t^ϵ. Hence each critical point of g gives a 1–periodic orbit of G_t^ϵ. Next we will show there is no other 1–periodic orbit.

Let U be a neighborhood of Y in M which does not contain other 1–periodic orbits of H disjoint from Y. Then, for any open set $V \subset U$ with $Y \subset V$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ the 1–periodic orbits of G_t^ϵ in U are also in V. This is due to the compactness result in [Cieliebak et al. 1996, Lemma 2.2]. Hence when ϵ is small, any 1–periodic orbit of G_t^ϵ is close to a 1–periodic orbit of H on Y, particularly in the $W^{1,2}$–topology.

Next consider a nonlinear operator

$$A\colon W^{1,2}(S^1, N) \to L^2(S^1, TN)$$

given by

$$A(x(t)) := -J(x'(t) - X_H(x(t))).$$

By the Morse–Bott nondegeneracy, the linearization of A is nondegenerate in the normal direction of Y. More precisely, there exists a constant $c > 0$ such that for any 1–periodic orbit $x_0(t)$ of H on Y and a vector field $y(t)$ along $x_0(t)$ with $y(t) \notin TY$ for some t, we have

$$\|\mathrm{DA}(x_0) \cdot y(t)\| \geq c\|y(t)\|.$$

Now define another operator

$$f\colon W^{1,2}(S^1, N) \to L^2(S^1, TN)$$

given by

$$f(x(t)) := \tilde{g}_t'(x(t)).$$

Note that the kernel of the operator $A + \epsilon f$ is the set of all 1–periodic orbits of G_t^ϵ in N. Since any 1–periodic orbit of G_t^ϵ is close to a 1–periodic orbit $x_0(t)$ of H on Y, we can write it as $x_0 + y(t)$ with a vector field $y(t)$ along $x_0(t)$.

Then we use the Taylor expansion to calculate that

$$(A + \epsilon f)(x_0 + y) = A(x_0) + \mathrm{DA}(x_0) \cdot y + \epsilon f(x_0) + \epsilon Df(x_0) \cdot y + O(\|y\|^2).$$

Note that $A(x_0) = 0$ and $\|\mathrm{DA}(x_0) \cdot y\| \geq c\|y\|$. So we have that

$$\begin{aligned}\|(A + \epsilon f)(x_0 + y)\| &\geq c\|y\| + \epsilon\|f(x_0)\| - \epsilon c'\|y\| + O(\|y\|^2)\\ &\geq c''\|y\| + \epsilon\|f(x_0)\|\end{aligned}$$

when ϵ and $\|y\|$ are sufficiently small. Hence $(A+\epsilon f)(x_0+y)=0$ if and only if $y=0$ and $f(x_0)=0$, which are the orbits given by critical points of g. Geometrically these perturbed orbits are the same orbits which start at critical points of g. So the integrations of the Liouville 1–form do not change.

The proof of (2) will be a direct computation to relate the Conley–Zehnder index with the Morse index, similar to that in [Cieliebak et al. 1996]. □

Therefore, given a Liouville domain D with a contact boundary (C,α) in M such that α is Morse–Bott index-bounded, we can create nondegenerate index-bounded S–shaped Hamiltonian functions associated with D. Then we can use them to construct the spectral sequence as we did in the Morse index-bounded case.

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Heegaard Floer homology, knotifications of links, and plane curves with noncuspidal singularities

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We describe a formula for the H_1–action on the knot Floer homology of knotifications of links in S^3. Using our results about knotifications, we are able to study complex curves with noncuspidal singularities, which were inaccessible using previous Heegaard Floer techniques. We focus on the case of a transverse double point, and give examples of complex curves of genus g which cannot be topologically deformed into a genus $g-1$ surface with a single double point.

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1 Introduction

1.1 General context

Let C be a complex curve in $\mathbb{C}P^2$. The curve C is called *rational* if C is irreducible and there exists a continuous degree one map from S^2 to C. The curve C is called *cuspidal* if all its singularities have one branch (ie their links have one component).

Fernandez de Bobadilla, Luengo, Melle-Hernandez and Némethi [Fernández de Bobadilla et al. 2006] indicated a connection between Seiberg–Witten invariants and rational cuspidal curves. As a consequence of these connections, they stated a conjecture binding coefficients of Alexander polynomials of singular points of a rational cuspidal curve. A variant of this conjecture was proved in [Borodzik and Livingston 2014]; the proof used the relation of semigroups of singular points with V_s–invariants of knots together with the Ozsváth–Szabó d–invariant inequality.

The methods of [Borodzik and Livingston 2014] were later generalized by Bodnár, Celoria and Golla [Bodnár et al. 2016] and Borodzik, Hedden and Livingston [Borodzik et al. 2017] to the case of nonrational cuspidal curves. Their result does not generalize immediately to the case where C has noncuspidal singularities. In this case, the boundary of a suitably defined tubular neighborhood of C can be presented as a surgery on a connected sum of links of cuspidal singularities and *knotifications* of links of noncuspidal singularities of C.

Knotification is an operation described by Ozsváth and Szabó [2008a], which transforms an n–component link L in S^3 into a knot $\hat{L} \subset \#^{n-1} S^2 \times S^1$. The knot Floer homology $\mathit{HFK}^-(\hat{L})$ admits an action of the exterior algebra over $\mathbb{Z}$ on $n-1$ generators, which is identified with $\Lambda^* H_1(\#^{n-1} S^2 \times S^1)$. To apply the strategy of [Bodnár et al. 2016; Borodzik et al. 2017; Borodzik and Livingston 2014] to noncuspidal singularities, one must compute explicitly the action of $\Lambda^* H_1(\#^{n-1} S^2 \times S^1)$ on the knot Floer complex of the knotification. Performing explicit computations is often challenging, since computing the action of $\Lambda^* H_1(\#^{n-1} S^2 \times S^1)$ involves counting pseudoholomorphic curves in a symmetric product $\mathrm{Sym}^d(\Sigma)$ of a surface Σ in a Heegaard decomposition of $\#^{n-1} S^2 \times S^1$, which is used to compute the knot Floer complex. In this paper, we prove a general result which relates the homology action on the knotified link to counts of pseudoholomorphic curves on a Heegaard diagram for the original link in S^3. In many cases, this is more practical, since it allows us to compute pseudoholomorphic curves in a symmetric product of lower index d. For the links we consider, we are able to reduce the computations to $\mathrm{Sym}^1(S^2)$, which is completely combinatorial.

1.2 Main results

Given an n–component link $L \subset S^3$, we use Heegaard Floer TQFT to recover the knot Floer complex of the knotification $\hat{L}$ of L together with the action of $\Lambda^* H_1(\#^{n-1} S^2 \times S^1)$ on it. This result builds on recent developments in the Heegaard Floer TQFT due to the third author as well as many others; see [Hendricks et al. 2018; Juhász 2016; Zemke 2015; 2017; 2019c; 2019b]. Our main result concerning knotifications is Proposition 2.10, which describes the action of $\Lambda^* H_1(\#^{n-1} S^2 \times S^1)$ on the knot Floer homology of a knotification in terms of a link diagram for L.

Using this general result, we compute the knot Floer complexes of the knotifications of the $(2, 2n)$–torus link and of its mirror, as well as the action of $H_1(S^2 \times S^1)$. In particular, we are able to compute the invariants V_s^{bot} and V_s^{top} of these knots. To the best of our knowledge, these computations have not appeared in the literature before. For the reader's convenience, we present the precise result for the knotification of the torus link $T_{2,2n}$. For more details about its mirror, see Proposition 2.41.

Proposition 2.40 *Let $\widehat{T}_{2,2n}$ be the knotification of the torus link $T_{2,2n}$. The pair*

$$(\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,2n}), A_\gamma)$$

has a model where $\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,2n})$ is equal to $\mathcal{S}^n\{\frac{1}{2}, \frac{1}{2}\} \oplus \mathcal{S}^{n-1}\{-\frac{1}{2}, -\frac{1}{2}\}$ and A_γ maps $\mathcal{S}^n$ to $\mathcal{S}^{n-1}$ on the chain level. Here we recall that $\{i, j\}$ denotes a shift in the $(\mathrm{gr}_w, \mathrm{gr}_z)$–grading by (i, j), and $\mathcal{S}^n$ and $\mathcal{S}^{n-1}$ are the chain complexes in Definition 2.28.

Our main application is concerned with general curves in $\mathbb{C}P^2$. To generalize the results of [Bodnár et al. 2016; Borodzik et al. 2017] to the setting of complex curves $C \subset \mathbb{C}P^2$ with noncuspidal singularities, we take a precisely defined "tubular" neighborhood N of C. The boundary $Y = \partial N$ can be described as a surgery on a link L in $\#^\rho\, S^2 \times S^1$, where L is a suitable connected sum of knotifications of links of singularities and Borromean knots, and ρ can be expressed in terms of the topology of C. As in [Bodnár et al. 2016; Borodzik et al. 2017], the manifold Y bounds a four-manifold $X = \mathbb{C}P^2 \setminus N$ with trivial intersection form. Using Ozsváth and Szabó's d–invariant inequality in the version proved by Levine and Ruberman [2014], we obtain restrictions on $V_s^{\mathrm{top}}(L)$ and $V_s^{\mathrm{bot}}(L)$.

The main case we focus on is curves C with some finite number of cuspidal singularities as well as singularities whose links are $(2, 2n)$–torus links. We obtain the following result:

Theorem 6.4 *Let C be a reduced curve of degree d and genus g. Suppose that C has cuspidal singular points $p_1, \dots, p_\nu$ whose semigroup counting functions are $R_1, \dots, R_\nu$, respectively. Assume that, apart from these ν points, the curve C has, for each $n \geq 1$, $m_n \geq 0$ singular points whose links are $(2, 2n)$–torus links and no other singularities. Define*

$$\eta_+ = \sum_{n=1}^{\infty} m_n \quad \text{and} \quad \kappa_+ = \sum_{n=1}^{\infty} n m_n.$$

For any $k = 1, \dots, d-2$,

$$\max_{0 \leq j \leq g} \min_{0 \leq i \leq \kappa_+ - \eta_+} (R(kd + 1 - \eta_+ - 2i - 2j) + i + j) \leq \tfrac{1}{2}(k+1)(k+2) + g,$$

$$\min_{0 \leq j \leq g + \kappa_+} (R(kd + 1 - 2j) + j) \geq \tfrac{1}{2}(k+1)(k+2).$$

Here R denotes the infimal convolution of the functions $R_1, \dots, R_\nu$.

Although complex curves cannot have singularities whose links are (nonalgebraic) $(2, -2n)$–torus links, our techniques also obstruct smooth (nonalgebraic) surfaces with these singularities. See Theorem 6.8.

The technical statement in Theorem 6.4 is best understood by comparing the obstruction in the case of a single transverse double point to the genus $g = 1$ obstruction from [Bodnár et al. 2016; Borodzik et al. 2017]. We do this in Proposition 6.14, which we now summarize. Let C be a degree d curve, and define the quantity $\upsilon_k = \frac{1}{2}(k+1)(k+2)$ for $k = 1, \dots, d-2$. Write R for the semigroup counting function.

If C has genus 1, then the genus bound from [Bodnár et al. 2016; Borodzik et al. 2017] implies that, for each $k \in \{1, \dots, d-2\}$,

$$R(kd-1) \in \{\upsilon_k - 1, \upsilon_k\} \quad \text{and} \quad R(kd+1) \in \{\upsilon_k, \upsilon_k + 1\}. \tag{1.1}$$

In this case, the only constraint on $R(kd)$ is that it lies between $R(kd-1)$ and $R(kd+1)$, and hence $R(kd) \in \{\upsilon_k - 1, \upsilon_k, \upsilon_k + 1\}$.

On the other hand, our bounds from Theorems 6.4 and 6.8 give a slightly stronger obstruction than the bound for genus 1 curves in (1.1), based on the value of $R(kd)$. Since double points may be smoothed topologically, (1.1) must also hold for genus 0 curves C with a single double point. If C is a genus 0 curve with a single positive double point, then our bound implies

$$R(kd) \le \upsilon_k.$$

If instead C is a smooth curve with a negative double point, then we prove that $R(kd) \ge \upsilon_k$.

We compare our obstruction with known examples, focusing on the question of deforming a genus 1 surface into a surface with one double point. In Section 6.5 we provide concrete obstructions. For existing curves (ie curves that we can construct), there are obstructions to trading genus for negative double points; see Example 6.15.

We also compare our obstruction to the obstruction for genus 1 curves from [Borodzik et al. 2017]. In [loc. cit., Theorem 9.1], there is a list of genus one curves with a singularity whose link is the (p,q)–torus knot with p and q coprime. The curves in the list pass the obstruction provided in [loc. cit.], but it is not known whether these complex curves exist. We apply our bound to this list of potential examples. There is a remarkable case of a degree 27 curve with a $(10,73)$ singularity, where the genus cannot be traded for either a positive or a negative double point; see Table 1. While the curve passes all known criteria, we do not have a recipe to construct it.

1.3 Further applications and perspectives

There has been recent interest in the question of "trading genus for double points". To be more precise, given a surface of genus g, one can ask whether it is possible to deform it to a genus $g-1$ surface with an extra positive or negative double point. In the context of the surfaces in a four-ball with fixed boundaries, this question is related to studying the difference between the clasp number and the smooth four-ball genus; see [Daemi and Scaduto 2024; Feller and Park 2022; Juhász and Zemke 2020; Kronheimer and Mrowka 2021; Owens and Strle 2016]. We deal with a variation of this question, which concerns trading genus of a closed surface in $\mathbb{C}P^2$ for double points, while preserving the remaining singularities.

In Section 6.6, we consider another infinite family of higher genus curves constructed by Bodnár, Celoria and Golla. We show that the genus cannot be traded for a negative double point for any member of the family.

As a perspective and a possibility for future research, we indicate that the methods can be used to study line arrangements in $\mathbb{C}P^2$. The only missing ingredient is the computation of the Heegaard Floer chain complex of the (d,d)–torus link for $d > 2$, and understanding the H_1–action on the knotification of these links.

Organization

Section 2 reviews Heegaard Floer theory. After recalling various known definitions and results, we show how to obtain the knot Floer chain complex of the knotification of links, as well as the H_1/Tors–action. A detailed construction of the Heegaard Floer chain complex of the Hopf link is presented in Section 2.5. The generalization to knotifications of arbitrary $(2,2n)$–torus links is given in Section 2.6. We conclude Section 2 with Section 2.7, where we recall the computations of the Heegaard Floer chain complex of the Borromean knot $\mathcal{B}_0$.

Section 3 is devoted to a detailed study of correction terms. We recall the Levine–Ruberman versions of d–invariants and recall definitions of V_s–invariants.

Section 4 contains some important computations that happen behind a scene. We recall the computation of the Heegaard Floer chain complex of L–space knots, and in particular of algebraic knots, in Section 4.2. We show how to recover the V_s–invariant of a product of positive and negative staircases. A precise statement is given in Proposition 4.18. We show that the assumptions in the second item of that proposition are necessary in Section 4.4.

Next we consider tensor products of knot Floer chain complexes in manifolds with $b_1 > 0$. It turns out that most of the knots that we encounter share a property, which greatly facilitates our computations, namely they have *split towers*; see Definition 4.29.

Section 5 constructs a tubular neighborhood N of a singular curve and presents the boundary Y of this neighborhood as a surgery on a link L in $\#^\rho\, S^2 \times S^1$, where ρ is the first Betti number of C. We then compute homological invariants of Y, N and $\mathbb{C}P^2 \setminus N$. In particular, we study which Spin^c structures on Y extend over $\mathbb{C}P^2 \setminus N$. These computations are slight generalizations of calculations of [Bodnár et al. 2016; Borodzik et al. 2017; Borodzik and Livingston 2014].

Section 6 contains the proofs of Theorems 6.4 and 6.8. The main technical result is Proposition 6.3, which computes the d–invariants of Y in terms of the semigroup counting functions of knots of cuspidal singularities. We also compare Theorems 6.4 and 6.8 with bounds for cuspidal curves of higher genus in Section 6.4. Sections 6.5 and 6.6 provide explicit examples of curves for which our obstruction can be applied.

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2 Review of Heegaard Floer theory

2.1 Heegaard Floer complexes with multiple basepoints

Definition 2.1 A *multipointed Heegaard diagram* for a 3–manifold Y is a quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ where:

- Σ is a genus g surface, which splits Y into two genus g handlebodies, U_α and U_β, and $\boldsymbol{w} = (w_1, \dots, w_n)$ is a nonempty set of basepoints in Σ.
- $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{g+n-1})$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{g+n-1})$ are collections of simple closed curves on Σ, where $n = |\boldsymbol{w}|$. Each curve in $\boldsymbol{\alpha}$ bounds a compressing disk in U_α, and each curve in $\boldsymbol{\beta}$ bounds a compressing disk in U_β. Furthermore, the curves in $\boldsymbol{\alpha}$ are pairwise disjoint, and similarly for $\boldsymbol{\beta}$.
- The curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are transverse.
- The curves in $\boldsymbol{\alpha}$ are linearly independent in $H_1(\Sigma \setminus \boldsymbol{w})$, and similarly for $\boldsymbol{\beta}$.

Let $\mathbb{T}_\alpha, \mathbb{T}_\beta \subset \operatorname{Sym}^{g+n-1}(\Sigma)$ be two half-dimensional tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+n-1} \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+n-1}.$$

Ozsváth and Szabó [2004b, Section 2.6] describe a map

$$\mathfrak{s}_{\boldsymbol{w}} \colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \operatorname{Spin}^c(Y).$$

Given a Heegaard diagram of Y with a Spin^c structure $\mathfrak{s}$, we define a Floer chain complex $\mathrm{CF}^-(Y, \boldsymbol{w}, \mathfrak{s})$ over $\mathbb{F}[U_1, \dots, U_n]$, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. The chain complex is generated over $\mathbb{F}[U_1, \dots, U_n]$ by intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ satisfying $\mathfrak{s}_{\boldsymbol{w}}(\boldsymbol{x}) = \mathfrak{s}$.

For any $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the differential is defined by

$$\partial \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathbb{R}) U_1^{n_{w_1}(\phi)} \cdots U_n^{n_{w_n}(\phi)} \boldsymbol{y}. \tag{2.2}$$

Here, $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ denotes the set of homotopy classes of maps of a complex unit disk $\mathbb{D}$ to $\operatorname{Sym}^{g+n-1}(\Sigma)$ such that point $-i$ is mapped to $\boldsymbol{x}$, the point i is mapped to $\boldsymbol{y}$, $\partial\mathbb{D} \cap \{\operatorname{Re}(z) < 0\}$ is mapped to $\mathbb{T}_\beta$ and $\partial\mathbb{D} \cap \{\operatorname{Re}(z) > 0\}$ is mapped to $\mathbb{T}_\alpha$. The quantity $\mu(\phi)$ is the Maslov index of the disk. The space $\mathcal{M}(\phi)$ is the moduli space of J_s–holomorphic disks representing ϕ (for some 1–parameter family of almost complex structures J_s on $\mathrm{Sym}^{g+n-1}(\Sigma)$). The condition that $\mu(\phi)=1$ implies that $\mathcal{M}(\phi)/\mathbb{R}$ is generically a finite set of points. The integers $n_{w_i}(\phi)$ are intersection numbers of $\{w_i\}\times \mathrm{Sym}^{g+n-2}(\Sigma)\subset \mathrm{Sym}^{g+n-1}(\Sigma)$ with the image of ϕ.

The homology group $\mathrm{HF}^-(Y,\boldsymbol{w},\mathfrak{s})$ of $\mathrm{CF}^-(Y,\boldsymbol{w},\mathfrak{s})$ has the structure of an $\mathbb{F}[U_1,\dots,U_n]$–module.

If $c_1(\mathfrak{s})$ is torsion, then $\mathrm{CF}^-(Y,\boldsymbol{w},\mathfrak{s})$ admits an absolute $\mathbb{Q}$–valued grading, which we denote by $\mathrm{gr}_{\boldsymbol{w}}$. The differential decreases the grading by 1, so the grading descends to $\mathrm{HF}^-(Y,\boldsymbol{w},\mathfrak{s})$. Multiplication by any of the U_i decreases the grading by -2.

Formally inverting the variables $U_1,\dots,U_n$ in $\mathrm{CF}^-(Y,\boldsymbol{w},\mathfrak{s})$ gives a chain complex $\mathrm{CF}^\infty(Y,\boldsymbol{w},\mathfrak{s})$ over $\mathbb{F}[U_1,U_1^{-1},\dots,U_n,U_n^{-1}]$. The associated homology group is denoted by $\mathrm{HF}^\infty(Y,\boldsymbol{w},\mathfrak{s})$.

2.2 The link Floer complex

For links in S^3, Ozsváth and Szabó [2008a] introduced the link Floer homology, which generalizes the knot Floer homology defined separately in [Rasmussen 2003; Ozsváth and Szabó 2004a]. We presently recall their construction.

Definition 2.3 An *oriented multipointed link* $\mathbb{L}=(L,\boldsymbol{w},\boldsymbol{z})$ in a closed 3–manifold Y is an oriented link L with two disjoint collections of basepoints $\boldsymbol{w}=\{w_1,\dots,w_n\}$ and $\boldsymbol{z}=\{z_1,\dots,z_n\}$ such that, as one traverses L, the basepoints alternate between $\boldsymbol{w}$ and $\boldsymbol{z}$. Furthermore, each component of L has a positive (necessarily even) number of basepoints, and each component of Y contains at least one component of L.

Analogously to Definition 2.1, we have the following:

Definition 2.4 A *multipointed Heegaard link diagram* for $\mathbb{L}=(L,\boldsymbol{w},\boldsymbol{z})$ in Y is a tuple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{z})$ satisfying the following:

- $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{w})$ and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z})$ are embedded Heegaard diagrams for $(Y,\boldsymbol{w})$ and $(Y,\boldsymbol{z})$, respectively, in the sense of Definition 2.1.
- $L\cap\Sigma=\boldsymbol{w}\cup\boldsymbol{z}$, and furthermore L intersects Σ positively at $\boldsymbol{z}$ and negatively at $\boldsymbol{w}$.
- $L\cap U_\alpha$ (resp. $L\cap U_\beta$) is a boundary-parallel tangle in U_α (resp. U_β).

Given a multipointed Heegaard link diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{z})$ for $(Y,\mathbb{L})$, the *link Floer chain complex* is defined as follows. Let

$$\mathscr{R}^-=\mathbb{F}[\mathscr{U},\mathscr{V}],\quad \mathscr{R}^\infty=\mathbb{F}[\mathscr{U},\mathscr{U}^{-1},\mathscr{V},\mathscr{V}^{-1}].$$

Let $\mathfrak{s}$ be a Spin^c structure on Y. We define the chain complex $\mathcal{CFL}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z}, \mathfrak{s})$ to be the free $\mathscr{R}^-$–module generated by $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_{\boldsymbol{w}}(\boldsymbol{x}) = \mathfrak{s}$. The differential is given by

$$\partial \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \mathscr{U}^{n_{w_1}(\phi)+\cdots+n_{w_n}(\phi)} \mathscr{V}^{n_{z_1}(\phi)+\cdots+n_{z_n}(\phi)} \cdot \boldsymbol{y}, \tag{2.5}$$

extended $\mathscr{R}^-$–equivariantly. The differential ∂ squares to 0.

There is a larger version of the link Floer complex, which we call the *full link Floer complex*, denoted by $\mathcal{CFL}^-_{\mathrm{full}}(Y, \mathbb{L}, \mathfrak{s})$. As a module, $\mathcal{CFL}^-_{\mathrm{full}}(Y, \mathbb{L}, \mathfrak{s})$ is freely generated over the ring $\mathbb{F}[\mathscr{U}_1, \dots, \mathscr{U}_n, \mathscr{V}_1, \dots, \mathscr{V}_n]$ by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The differential is similar to (2.5), except we use the weight $n_{w_i}(\phi)$ for the variable $\mathscr{U}_i$, and the weight of $n_{z_i}(\phi)$ for the variable $\mathscr{V}_i$. In general, $\mathcal{CFL}^-_{\mathrm{full}}(Y, \mathbb{L}, \mathfrak{s})$ is a *curved chain complex*, ie $\partial^2 = \omega_{\mathbb{L}} \cdot \mathrm{id}$ for some $\omega_{\mathbb{L}} \in \mathbb{F}[\mathscr{U}_1, \dots, \mathscr{U}_n, \mathscr{V}_1, \dots, \mathscr{V}_n]$; see [Zemke 2017, Lemma 2.1].

2.3 Homological actions

Ozsváth and Szabó [2004b, Section 4.2.5] describe an action of $\Lambda^*(H_1(Y)/\mathrm{Tors})$ on the homology group $\mathrm{HF}^-(Y, \boldsymbol{w}, \mathfrak{s})$. For a multipointed 3–manifold $(Y, \boldsymbol{w})$, there is an analogous action of the relative homology group $H_1(Y, \boldsymbol{w})$ on $\mathrm{CF}^-(Y, \boldsymbol{w}, \mathfrak{s})$ [Zemke 2015]. In this section, we recall the construction and describe some analogs on link Floer homology.

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ is a multipointed Heegaard diagram, and λ is a path which connects two distinct basepoints $w_1, w_2 \in \boldsymbol{w}$, then there is a *relative homology action* A_λ, which is an endomorphism of $\mathrm{CF}^-(Y, \boldsymbol{w}, \mathfrak{s})$ and satisfies

$$A_\lambda \partial + \partial A_\lambda = U_1 + U_2. \tag{2.6}$$

See [Zemke 2015, Lemma 5.1].

The map A_λ is defined via the formula

$$A_\lambda(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi)=1}} a(\lambda, \phi) \#(\mathcal{M}(\phi)/\mathbb{R}) U_1^{n_{w_1}(\phi)} \cdots U_n^{n_{w_n}(\phi)} \cdot \boldsymbol{y}. \tag{2.7}$$

Here $a(\lambda, \phi) \in \mathbb{F}$ is a quantity determined as follows. Homotope the path λ so that it is an immersed curve in Σ, transverse to the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ curves. We write $D(\phi)$ for the *domain* of the class ϕ, which is a 2–chain on Σ with boundary in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$. We write $\partial D(\phi) = \partial_\alpha(\phi) + \partial_\beta(\phi)$. Then we set $a(\lambda, \phi) = \#(\partial_\alpha(\phi) \cap \lambda)$. Compare [Zemke 2015, Section 5.1]. Up to chain homotopy, the map A_λ only depends on the relative homology class of λ in Y, relative to its boundary. In particular, the map A_λ does not depend on the choice of representative on the surface Σ. See [Ni 2014, Lemma 2.4] for a proof in a related context, or [Zemke 2015, Lemma 5.6] for a similar proof in the present context.

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ is a multipointed Heegaard link diagram, and λ connects two basepoints w_1 and w_2, there is an analogous map A_λ on the link Floer homology. In contrast to (2.6), we have

$$A_\lambda \partial + \partial A_\lambda = \mathscr{U}_1 \mathscr{V}_1 + \mathscr{U}_2 \mathscr{V}_2, \tag{2.8}$$

where $\mathscr{V}_1$ denotes the variable for the basepoint z_1 which immediately follows w_1 with respect to the ordering of basepoints on the link, and similarly $\mathscr{V}_2$ is the variable for the basepoint z_2 which immediately follows w_2. The proof follows the same strategy as [Zemke 2015, Lemma 5.1]: One counts the ends of index 2 families of holomorphic disks. There are two types of ends: pairs of index 1 holomorphic disks as well as index 2 boundary degenerations. Pairs of index 1 holomorphic disks contribute to the left-hand side of (2.8), while the count of boundary degenerations, weighted by $a(\lambda, \phi)$, constitutes the right-hand side.

If $z_i \in \boldsymbol{z}$, then there is an endomorphism of $\mathcal{CFL}^-_{\text{full}}(Y, \mathbb{L}, \mathfrak{s})$ defined by

$$\Psi_{z_i}(\boldsymbol{x}) = \mathscr{V}_i^{-1} \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi)=1}} n_{z_i}(\phi) \, \#(\mathcal{M}(\phi)/\mathbb{R}) \mathscr{U}_1^{n_{w_1}(\phi)} \cdots \mathscr{U}_n^{n_{w_n}(\phi)} \mathscr{V}_1^{n_{z_1}(\phi)} \cdots \mathscr{V}_n^{n_{z_n}(\phi)} \cdot \boldsymbol{y}.$$

We call Ψ_{z_i} the *basepoint action* of z_i. Note that, since the contribution of each disk class ϕ is multiplied by $n_{z_i}(\phi)$ in the sum, the additional factor of $\mathscr{V}_i^{-1}$ never results in negative powers of $\mathscr{V}_i$, and hence the formula induces a well-defined endomorphism of $\mathcal{CFL}^-_{\text{full}}(Y, \mathbb{L}, \mathfrak{s})$.

Given $w_i \in \boldsymbol{w}$, there is an analogous endomorphism Φ_{w_i}. The map Ψ_{z_i} satisfies

$$\Psi_{z_i} \partial + \partial \Psi_{z_i} = \mathscr{U}_j + \mathscr{U}_{j+1},$$

where w_j and w_{j+1} are the $\boldsymbol{w}$ basepoints adjacent to z_i on the link. In particular, if we identify all of the $\mathscr{U}_i$ variables to a single $\mathscr{U}$, then Ψ_{z_i} is a chain map. See [Sarkar 2011, Lemma 4.1] or [Zemke 2017, Lemma 3.1]. Similarly, if z_i is on a link component which has only one other basepoint, then Ψ_{z_i} is also a chain map.

2.4 Heegaard Floer homology of a knotification

Definition 2.9 (knotification) Let $\mathcal{L} = L_1 \cup \cdots \cup L_n$ be a null-homologous link in a 3–manifold Y.

(1) A *partial knotification* of $\mathcal{L}$ with respect to components L_i and L_j is a $(n-1)$–component null-homologous link $\mathcal{L}_{ij}$ in $Y \# S^2 \times S^1$ obtained by connecting L_i and L_j with an oriented band going across the $S^2 \times S^1$ summand.

(2) A *knotification* of $\mathcal{L}$ is a knot $\widehat{\mathcal{L}}$ in $Y \# \#^{n-1} S^2 \times S^1$ obtained by consecutive partial knotifications.

The isotopy type $\widehat{\mathcal{L}}$ does not depend on the feet of the bands [Ozsváth and Szabó 2004a, Proposition 2.1].

Suppose $\mathbb{L} = (\mathcal{L}, \boldsymbol{w}, \boldsymbol{z})$ is an n–component link in $\#^m S^2 \times S^1$, equipped with $2n$ basepoints, and $\mathbb{L}'$ is a multipointed link in $\#^{m+1} S^2 \times S^1$, obtained by knotifying the components L_{n-1} and L_n of $\mathcal{L}$.

Furthermore, we assume that the basepoints on the link components $L_1, \dots, L_{n-2}$ are unchanged in $\mathbb{L}'$, and on L'_{n-1} we have only the two basepoints w_n and z_{n-1}. There are two natural maps

$$F : \mathcal{CFL}^-(\#^m S^2 \times S^1, \mathbb{L}) \to \mathcal{CFL}^-(\#^{m+1} S^2 \times S^1, \mathbb{L}'),$$
$$G : \mathcal{CFL}^-(\#^{m+1} S^2 \times S^1, \mathbb{L}') \to \mathcal{CFL}^-(\#^m S^2 \times S^1, \mathbb{L}).$$

The map F is the link cobordism map for a 4–dimensional 1–handle, followed by a saddle which crosses over the 1–handle. The decoration on the saddle consists of an arc, which connects the two link components of $\mathbb{L}$. Outside of the saddle region, the decoration consists of "vertical" arcs which connect $\mathbb{L}$ to $\mathbb{L}'$. See the left-hand side of [Zemke 2019a, Figure 5.1]. The map G is the map for the link cobordism obtained by reversing the orientation and turning around the above cobordism for F.

The following is a key lemma which we use to compute the H_1–action for knotified links:

Proposition 2.10 *Suppose $\mathbb{L}$, $\mathbb{L}'$, F and G are as above. Let λ be an arc in $\#^m S^2 \times S^1$ which connects the $\boldsymbol{w}$ basepoints of L_{n-1} and L_n. Let γ be the unique element of $H_1(\#^{m+1} S^2 \times S^1)$ obtained by joining the ends of λ across the 1–handle used in knotification. We have the following*:

(a) *F and G are homogeneously graded chain homotopy inverses.*

(b) *The map F satisfies*

$$F(A_\lambda + \mathscr{U}\Phi_{w_n}) \simeq F(A_\lambda + \mathscr{V}\Psi_{z_n}) \simeq A_\gamma F.$$

Proof To simplify the notation, we will describe the case when $\mathcal{L}$ is a link in S^3 with two components L_1 and L_2. We begin with claim (a). The proof is formally identical to the proof of [Zemke 2019a, Proposition 5.1] and follows from two 4–dimensional surgery relations [Zemke 2019a, Propositions 5.2 and 5.4].

We now move onto claim (b). We first show that

$$F(A_\lambda + \mathscr{V}\Psi_{z_2}) \simeq A_\gamma F. \tag{2.11}$$

By definition, we may take

$$F = S^-_{w_2,z_1} F^{\boldsymbol{w}}_B F_1, \tag{2.12}$$

where F_1 is the 1–handle map, $S^-_{w_2,z_1}$ is a quasidestabilization map, and $F^{\boldsymbol{w}}_B$ is a type-$\boldsymbol{w}$ saddle map; see [Zemke 2019c] for precise definitions of the relevant maps. Here B denotes the band (ie saddle) which crossed over the 1–handle used in the knotification operation.

We now have

$$F_1(A_\lambda + \mathscr{V}\Psi_{z_2}) = (A_\lambda + \mathscr{V}\Psi_{z_2})F_1 \tag{2.13}$$

by the same argument as Ozsváth and Szabó's proof that the 1–handle is a chain map [Ozsváth and Szabó 2006, Section 4.3]. Analogously, the computation of the quasistabilized differential in [Zemke 2017, Proposition 5.3] implies that

$$A_\gamma S^-_{w_2,z_1} = S^-_{w_2,z_1} A_\gamma.$$

Hence, it is sufficient to show that

$$F^{\boldsymbol{w}}_B(A_\lambda + \mathscr{V}\Phi_{z_1}) = A_\gamma F^{\boldsymbol{w}}_B.$$

We recall the definition of the map $F^{\boldsymbol{w}}_B$. We pick a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$ subordinate to the band [Zemke 2019c, Defintion 6.2]. The diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$ contains two canonical intersection points, $\Theta^{\boldsymbol{w}}_{\beta,\beta'}$ and $\Theta^{\boldsymbol{z}}_{\beta,\beta'}$, where $\Theta^{\boldsymbol{o}}_{\beta,\beta'}$ is the top degree generator with respect to the $\mathrm{gr}_{\boldsymbol{o}}$–grading for $\boldsymbol{o} \in \{\boldsymbol{w}, \boldsymbol{z}\}$. By definition,

$$F^{\boldsymbol{w}}_B(\boldsymbol{x}) = F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}).$$

Counting the ends of Maslov index 1 families of holomorphic triangles, weighted by $a(\lambda, \psi)$, we obtain the relation

$$F_{\alpha,\beta,\beta'}(A_\lambda(\boldsymbol{x}), \Theta^{\boldsymbol{z}}_{\beta,\beta'}) + A_\lambda(F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'})) = F^A_\lambda(\partial\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}) + F^A_\lambda(\boldsymbol{x}, \partial\Theta^{\boldsymbol{z}}_{\beta,\beta'}) + \partial F^A_\lambda(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'});$$

see [Zemke 2015, Lemma 5.2]. Here F^A_λ counts index 0 holomorphic triangles with an extra factor of $a(\lambda, \psi)$. Note that one might expect an extra term involving $F_{\alpha,\beta,\beta'}(\boldsymbol{x}, A_\lambda(\Theta^{\boldsymbol{z}}_{\beta,\beta'}))$; however, this term vanishes since A_λ weights disks based on their changes across the $\boldsymbol{\alpha}$ curves and $\Theta^{\boldsymbol{z}}_{\beta,\beta'} \in \mathbb{T}_\beta \cap \mathbb{T}_{\beta'}$. Since $\partial\Theta^{\boldsymbol{z}}_{\beta,\beta'} = 0$, we obtain that

(2.14) $$F^{\boldsymbol{w}}_B \circ A_\lambda + A_\lambda \circ F^{\boldsymbol{w}}_B \simeq 0.$$

Similarly, counting the ends of index 1 families of holomorphic triangles, weighted by $n_{z_2}(\psi)$, we obtain

$$\begin{aligned}F_{\alpha,\beta,\beta'}(\mathscr{V}\Psi_{z_2}(\boldsymbol{x}), \Theta^{\boldsymbol{z}}_{\beta,\beta'}) + F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})) + \mathscr{V}\Psi_{z_2}(F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}))\\ = F'(\partial\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}) + F'(\boldsymbol{x}, \partial\Theta^{\boldsymbol{z}}_{\beta,\beta'}) + \partial F'(\boldsymbol{x}, \Theta^{\boldsymbol{z}}_{\beta,\beta'}),\end{aligned}$$

where F' counts index 0 triangles weighted by a factor of $n_{z_1}(\psi)$. The above equation implies that

(2.15) $$F^{\boldsymbol{w}}_B \circ \mathscr{V}\Psi_{z_2} + \mathscr{V}\Psi_{z_2} \circ F^{\boldsymbol{w}}_B \simeq F_{\alpha,\beta,\beta'}(-, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})).$$

We claim now that the map $F_{\alpha,\beta,\beta'}(-, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'}))$ is null-homotopic. To establish this, it is sufficient to show that

(2.16) $$[\mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})] = 0,$$

where the brackets denote the induced element of homology. Indeed, assuming the existence of an $\eta \in \mathcal{CFL}^-(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$ such that $\partial\eta = \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})$, associativity of holomorphic triangles implies that

$$F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})) = \partial F_{\alpha,\beta,\beta'}(\boldsymbol{x}, \eta) + F_{\alpha,\beta,\beta'}(\partial\boldsymbol{x}, \eta),$$

so

(2.17) $$F_{\alpha,\beta,\beta'}(-, \mathscr{V}\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'})) \simeq 0.$$

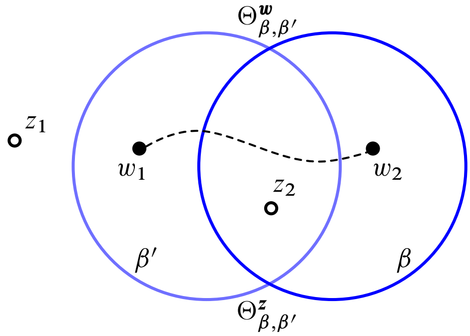


Figure 1: An unknot with four basepoints. The dashed arc is λ.

We will now demonstrate (2.16). We observe that the map Ψ_{z_2} commutes with the homotopy equivalences associated to changing Heegaard diagrams by [Zemke 2017, Lemma 3.2]. Furthermore, the homology class $[\Theta^{\boldsymbol{z}}_{\beta,\beta'}]$ is also preserved by these homotopy equivalences by [Zemke 2019c, Lemma 3.7], since it is the unique generator in its grading. In particular, we may verify (2.16) for any convenient choice of Heegaard diagram for an unknot with four basepoints. We perform the computation using the genus 0 Heegaard diagram shown in Figure 1. On this diagram, $\Psi_{z_2}(\Theta^{\boldsymbol{z}}_{\beta,\beta'}) = 0$.

Combining (2.14) and (2.15) with (2.17), we obtain

$$F^{\boldsymbol{w}}_B(A_\lambda + \mathscr{V}\Psi_{z_2}) \simeq (A_\lambda + \mathscr{V}\Psi_{z_2})F^{\boldsymbol{w}}_B. \tag{2.18}$$

Next, consider a path λ' from w_1 to w_2, which is a subarc of $\mathbb{L}'$. We choose λ' so that it is oriented from w_1 to w_2. There are two such subarcs of $\mathbb{L}'$, and we pick the one so that the portion of λ' nearest to w_1 is in the beta-handlebody (equivalently, we pick the one which goes over the band B before arriving at a $\boldsymbol{z}$ basepoint). Without loss of generality, we may assume that λ' crosses over z_2. See Figure 2. We define

$$\gamma := \lambda * \lambda',$$

where $*$ denotes concatenation.

On the Heegaard diagram, we may choose λ' to cross only the alpha curves between w_1 and z_2, and only the beta curves between z_2 and w_2. Clearly,

$$a(\lambda', \phi) = n_{w_2}(\phi) - n_{z_2}(\phi).$$

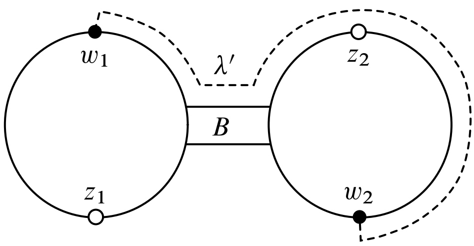


Figure 2: The configuration of the band B, the basepoints and the arc $\lambda' \subset \mathbb{L}'$.

Hence, $A_{\lambda'} = \mathscr{U}\Phi_{w_2} + \mathscr{V}\Psi_{z_2}$, or, equivalently,

(2.19)
$$\mathscr{V}\Psi_{z_2} = A_{\lambda'} + \mathscr{U}\Phi_{w_2}.$$

Combining (2.18) and (2.19), we obtain

(2.20)
$$\begin{aligned} F(A_\lambda + \mathscr{V}\Psi_{z_2}) &\simeq S^-_{w_2,z_1}(A_\lambda + A_{\lambda'} + \mathscr{U}\Phi_{w_2})F^{\boldsymbol{w}}_B F_1 \\ &\simeq S^-_{w_2,z_1}(A_\gamma + \mathscr{U}\Phi_{w_2})F^{\boldsymbol{w}}_B F_1 \\ &\simeq A_\gamma S^-_{w_2,z_1} F^{\boldsymbol{w}}_B F_1. \end{aligned}$$

The second line of (2.20) follows from the relation $A_\gamma \simeq A_\lambda + A_{\lambda'}$. The final line follows from (2.13), as well as the relation that $S^-_{w_2,z_1}\Phi_{w_2} \simeq S^-_{w_2,z_1} S^+_{w_2,z_1} S^-_{w_2,z_1} \simeq 0$ by [Zemke 2019c, Lemmas 4.11 and 4.13], completing the proof of (2.11).

Finally, to see that
$$F(A_\lambda + \mathscr{U}\Phi_{w_2}) \simeq A_\gamma F,$$
it is sufficient to show that $\mathscr{V}\Psi_{z_2} \simeq \mathscr{U}\Phi_{w_2}$ on $\mathcal{CFL}^-(\mathbb{L})$. To see this, we note that on a diagram for $\mathbb{L}$, we can consider a shadow of the link component L_2. The arc $L_2 \setminus \{w_2, z_2\}$ contains two subarcs, one of which intersects only the alpha curves, and one of which intersects only the beta curves. Hence $a(L_2, \phi) = n_{w_2}(\phi) - n_{z_2}(\phi)$ for any class of disks ϕ. On the other hand, this implies that the homology action associated to $0 = [L_2] \in H_1(S^3)$ satisfies
$$0 \simeq A_{L_2} = \mathscr{U}\Phi_{w_2} + \mathscr{V}\Psi_{z_2}. \qquad \square$$

The homology action on full knotifications may be computed by iterating the above result, via the following lemma:

Lemma 2.21 *Let $\mathbb{L}$, $\mathbb{L}'$, F and G be as in Proposition 2.10.*

1. *Suppose that $\gamma \in H_1(\#^m\, S^2 \times S^1)$. Write γ also for the induced element of $H_1(\#^{m+1}\, S^2 \times S^1)$. Then A_γ commutes with F and G up to chain homotopy.*
2. *If λ is an arc in $\#^m\, S^2 \times S^1$ which connects two components of $L_1, \dots, L_{n-2}$, then the relative homology map A_λ commutes with F and G up to chain homotopy.*
3. *If w and z are basepoints on one of the link components $L_1, \dots, L_{n-2}$, then Φ_w and Ψ_z commute with F and G up to chain homotopy.*

The proof of Lemma 2.21 is similar to the proof of Proposition 2.10 (though strictly easier), and hence we omit it. We refer the reader to [Zemke 2015, Section 5; 2019c, Section 4] for related results.

2.5 The Hopf link

Our next goal is to describe the $\mathcal{CFL}^-$–complexes for the $(2, 2n)$–torus links, denoted by $T_{2,2n}$, their mirrors and their knotifications. As the calculations are rather involved, we begin by describing the Floer chain complex for the link $T_{2,2}$ (ie. the positive Hopf link), leaving the general case to Section 2.6. While

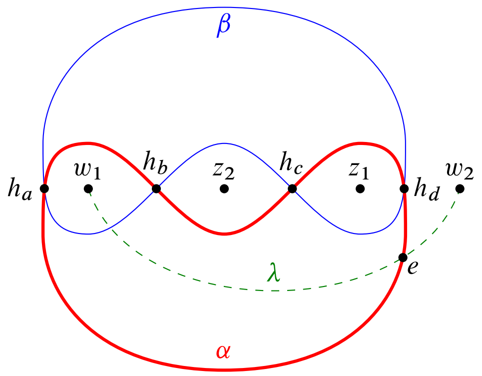


Figure 3: A genus 0 Heegaard diagram for the Hopf link. The thick (red) curve is the α curve, the thin (blue) curve is the β curve. The dotted curve is used to compute the action of $H_1(S^2 \times S^1; \mathbb{Z})$ on the knotification of the Hopf link.

the complex $\mathcal{CFL}^-(T_{2,2})$ is well known (it can be computed explicitly using a very simple diagram), to the best of our knowledge, the calculation of the action of $H_1(S^2 \times S^1)$ on the knot Floer chain complex of the knotification of $T_{2,2}$ is new.

As our main focus will eventually be the knotification of $T_{2,2}$, we restrict our attention to the link Floer complex over the ring $\mathscr{R}^- = \mathbb{F}[\mathscr{U}, \mathscr{V}]$, as opposed to the version with a variable for each basepoint.

Consider the diagram for the Hopf link, as in Figure 3. The complex $\mathcal{CFL}^-(T_{2,2})$ is generated over $\mathscr{R}^-$ by four elements, h_a, h_b, h_c and h_d, which correspond to the intersections of the α and β curves in Figure 3. The gradings are

$$(2.22)\qquad \begin{aligned} &(\mathrm{gr}_w(h_a), \mathrm{gr}_z(h_a)) = \left(\tfrac{1}{2}, -\tfrac{3}{2}\right), \quad (\mathrm{gr}_w(h_b), \mathrm{gr}_z(h_b)) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right), \\ &(\mathrm{gr}_w(h_c), \mathrm{gr}_z(h_c)) = \left(-\tfrac{3}{2}, \tfrac{1}{2}\right), \quad (\mathrm{gr}_w(h_d), \mathrm{gr}_z(h_d)) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right). \end{aligned}$$

The differential in the complex is computed by counting holomorphic disks of Maslov index 1. Counting bigons shows that

$$(2.23)\qquad \partial h_a = \partial h_c = 0, \quad \partial h_b = \partial h_d = \mathscr{U} h_a + \mathscr{V} h_c.$$

The homology of $\mathcal{CFL}^\infty(T_{2,2})$ is a direct sum of two copies of $\mathscr{R}^\infty$. One copy is spanned by $[h_b + h_d]$; the other copy is spanned by h_a or h_c.

We now describe the homology action A_γ on $\mathcal{CFK}^-(\widehat{T}_{2,2})$, where $\widehat{T}_{2,2}$ denotes the knotification of $T_{2,2}$, and γ is a generator of $H_1(S^2 \times S^1)$. We will use Proposition 2.10. The formula therein involves the relative homology action A_λ on $\mathcal{CFL}^-(T_{2,2})$, which we compute now. In our present case, the arc λ has only one intersection with an alpha curve, which occurs at a point labeled e in Figure 3. The map A_λ counts holomorphic disks of Maslov index 1, with weights corresponding to changes along the alpha boundary of a disk; see (2.7). Counting bigons with these weights, we obtain

$$(2.24)\qquad A_\lambda(h_a) = \mathscr{V}(h_b + h_d), \quad A_\lambda(h_b) = 0, \quad A_\lambda(h_c) = \mathscr{U}(h_b + h_d), \quad A_\lambda(h_d) = \mathscr{U} h_a.$$

We recall that, in Section 2.4, we defined a knotification map

$$F : \mathcal{CFL}^-(T_{2,2}) \to \mathcal{CFK}^-(\hat{T}_{2,2}),$$

which is a homotopy equivalence. In Proposition 2.10, we showed that

$$F(A_\lambda + \mathscr{U}\Phi_{w_2}) \simeq A_\gamma F.$$

Hence, as a model for the pair $(\mathcal{CFK}^-(\hat{T}_{2,2}), A_\gamma)$, we may use $(\mathcal{CFL}^-(T_{2,2}), A_\lambda + \mathscr{U}\Phi_{w_2})$. Hereafter, by a model for a chain complex (possibly with extra structure) defined up to chain homotopy equivalence, we mean a concrete chain complex in the class of an appropriate (usually bifiltered) chain homotopy equivalence. Abusing notation slightly, we will write A_γ for the endomorphism of $\mathcal{CFL}^-(T_{2,2})$ given by $A_\gamma := A_\lambda + \mathscr{U}\Phi_{w_2}$. One easily computes

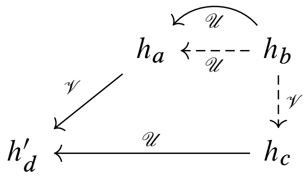
$$\Phi_{w_2}(h_d) = h_a,$$

and Φ_{w_2} vanishes on the other generators. Hence,

(2.25) $$A_\gamma(h_a) = \mathscr{V}(h_b + h_d), \quad A_\gamma(h_b) = \mathscr{U}h_a, \quad A_\gamma(h_c) = \mathscr{U}(h_b + h_d), \quad A_\gamma(h_d) = \mathscr{U}h_a.$$

With a change of basis $h'_d = h_b + h_d$, we obtain the following presentation of $(\mathcal{CFK}^-(\hat{T}_{2,2}), A_\gamma)$:

(2.26)



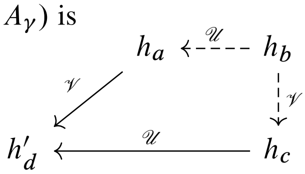
In (2.26), the dashed arrows denote differentials, and the solid arrows denote the action of A_γ.

We may obtain a simpler model of the homology action by replacing A_γ with $A_\gamma + [\partial, F]$, where F is the $\mathscr{R}^-$–equivariant map which satisfies

$$F(h_a) = h_a \quad \text{and} \quad F(h_b) = F(h_c) = F(h_d) = 0.$$

The resulting model for $(\mathcal{CFK}^-(\hat{T}_{2,2}), A_\gamma)$ is

(2.27)



2.6 The torus link $T_{2,2n}$

Before we start our computation of the Floer chain complex of the $(2, 2n)$–torus link and its knotification, we introduce a family of complexes $\mathcal{S}_n$ for $n \in \mathbb{Z}$, which play a prominent role in the present paper.

Definition 2.28 Let $n \geq 1$. We write $\mathcal{S}^n$ for the complex generated by elements $x_0, y_1, \ldots, y_{2n-1}, x_{2n}$ with differential $\partial(x_{2i}) = 0$ and

$$\partial(y_{2i+1}) = \mathscr{U}x_{2i} + \mathscr{V}x_{2i+2}.$$

The bigradings are given by $(\mathrm{gr}_w(x_j), \mathrm{gr}_z(x_j)) = (-j, j-2n)$ if j is even. The same formula holds for y_j if j is odd.

The complex $\mathcal{S}^{-n}$ is defined as the dual complex to $\mathcal{S}^n$. More specifically, it is generated by elements $\underline{x}_0, \underline{y}_1, \dots, \underline{y}_{2n-1}, \underline{x}_{2n}$ with differential $\partial(\underline{y}_{2i+1}) = 0$, $\partial(\underline{x}_{2i}) = \mathscr{V}\underline{y}_{2i-1} + \mathscr{U}\underline{y}_{2i+1}$, and the convention that $\underline{y}_{-1} = \underline{y}_{2n+1} = 0$. For j even, the grading of $\underline{x}_j$ is $(j, 2n-j)$, and an analogous formula holds for the grading of $\underline{y}_j$ if j is odd.

Remark 2.29 The complex $\mathcal{S}^n$ is the $\mathcal{CFK}^-$–complex of the positive torus knot $T_{2,2n+1}$, while $\mathcal{S}^{-n}$ is the complex for the negative torus knot $T_{2,-(2n+1)}$. Hence, we also call $\mathcal{S}^n$ a *staircase complex*. For details of staircase complexes, see Section 4.1.

Recall that $T_{2,2n} \subset S^3$ denotes a 2–component $(2,2n)$–torus link. In this subsection, we study the Floer chain complex $\mathcal{CFL}^-(T_{2,2n})$ as an $\mathscr{R}^-$–module. This gives the Floer chain complex $\mathcal{CFK}^-(S^2\times S^1, \widehat{T}_{2,2n})$, where $\widehat{T}_{2,2n}$ is the knotification of $T_{2,2n}$.

The Heegaard diagram of the link $T_{2,2n}$ in S^3 is shown in Figure 4 and the Floer chain complex is in Figure 5. The Heegaard diagram displayed therein is obtained from a doubly pointed open book whose page is a disk and whose monodromy is γ^n, where γ denotes a Dehn twist parallel to the boundary.

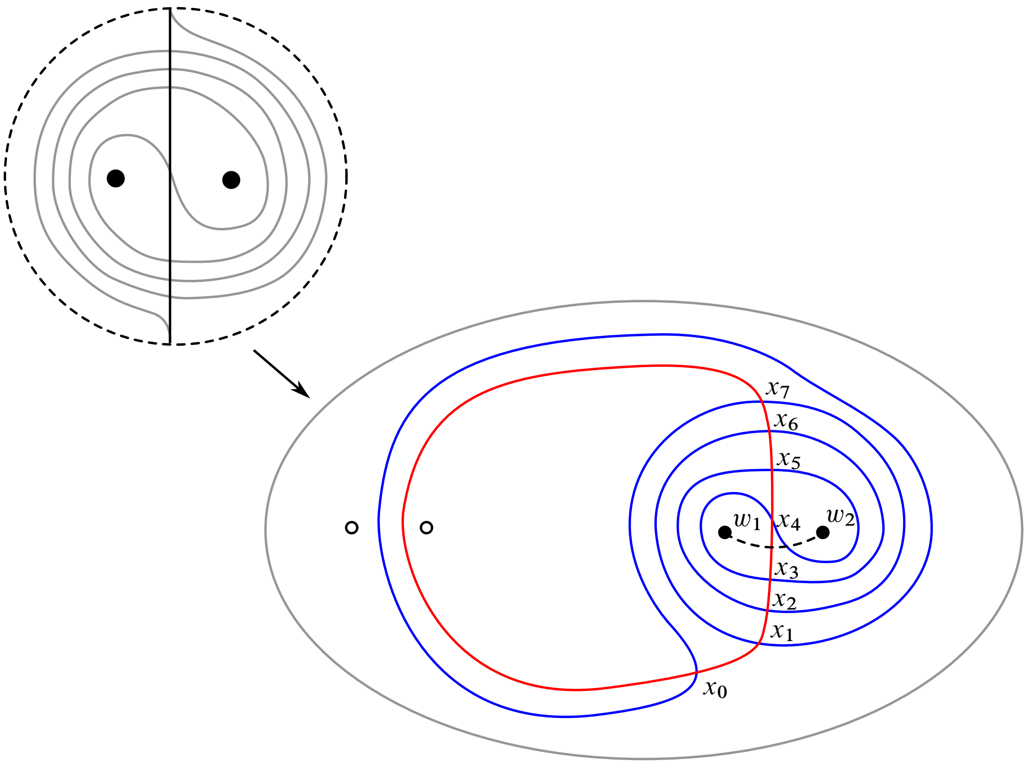


Figure 4: A Heegaard diagram for $T_{2,4}$ from a doubly pointed open book. The dashed line is an arc λ connecting w_1 and w_2.

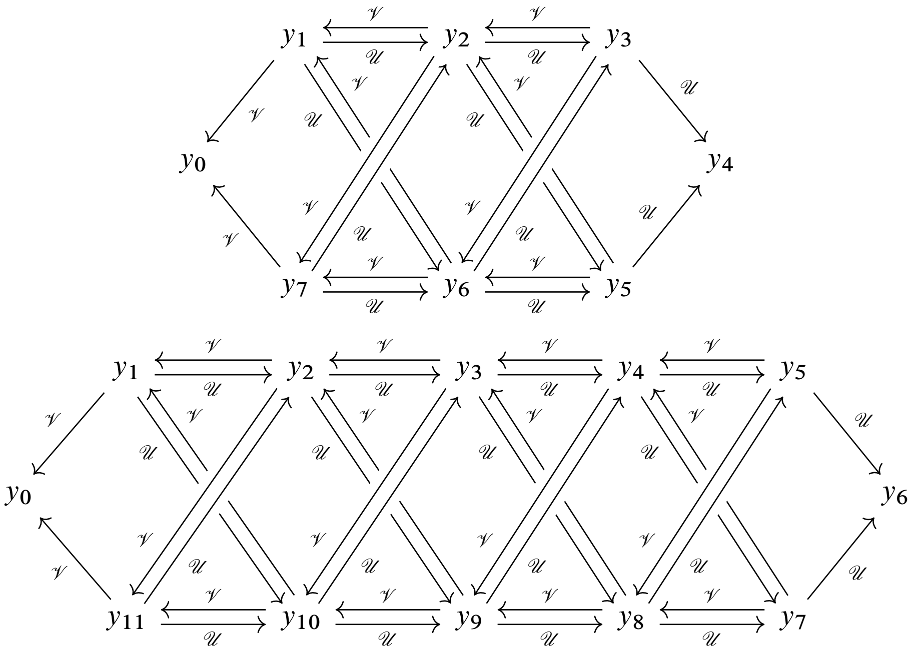


Figure 5: The chain complexes for $T_{2,4}$ (top) and $T_{2,6}$ (bottom).

It is easy to see that there are $4n$ generators $y_0, \dots, y_{4n-1}$ of the complex $\mathcal{CFL}^-(T_{2,2n})$. By counting bigons, one obtains formulas for the differential

$$\begin{aligned} \partial y_i &= \partial y_{4n-i} = \mathscr{V}(y_{i-1} + y_{4n-i+1}) + \mathscr{U}(y_{i+1} + y_{4n-i-1}) \quad \text{if } 2 \le i \le 2n-2, \\ \partial y_1 &= \partial y_{4n-1} = \mathscr{V} y_0 + \mathscr{U}(y_2 + y_{4n-2}), \\ \partial y_{2n-1} &= \partial y_{2n+1} = \mathscr{U} y_{2n} + \mathscr{V}(y_{2n-2} + y_{2n+2}), \\ \partial y_0 &= \partial y_{2n} = 0. \end{aligned} \tag{2.30}$$

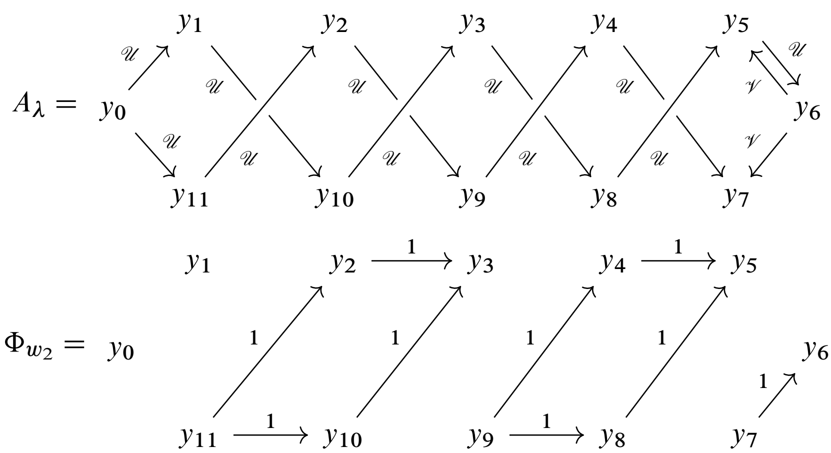


Figure 6: Figure 5 continued. The map A_λ on the complex for $T_{2,6}$ (top) and the map Φ_{w_2} (bottom).

It is convenient to do the following bigraded change of basis to the complex $\mathcal{CFL}^-(T_{2,2n})$. Namely we consider the basis $y_1, \dots, y_{2n-1}, x_0, \dots, x_{2n}$, where

$$x_i = y_i + y_{4n-i} \quad \text{if } 1 \le i \le 2n-1, \qquad x_0 = y_0, \qquad x_{2n} = y_{2n}. \tag{2.31}$$

With this change of basis, the differential takes the form

$$\partial y_i = \mathscr{V} x_{i-1} + \mathscr{U} x_{i+1} \quad \text{if } 1 \le i \le 2n-1, \qquad \partial x_i = 0. \tag{2.32}$$

The gradings of the generators in $\mathcal{CFL}^-(T_{2,2n})$ are summarized in the following lemma:

Lemma 2.33 *If* $1 \le i \le 2n-1$, *then*

$$(\mathrm{gr}_w(y_i), \mathrm{gr}_z(y_i)) = (\mathrm{gr}_w(x_i), \mathrm{gr}_z(x_i)) = \left(\tfrac{1}{2} - 2n + i, \tfrac{1}{2} - i\right).$$

If $i = 0$ *or* $i = 2n$, *then the same formula holds for* x_i.

Proof Recall that ∂ has $(\mathrm{gr}_w, \mathrm{gr}_z)$–bigrading $(-1,-1)$, and that $\mathscr{U}$ and $\mathscr{V}$ have bigradings $(-2,0)$ and $(0,-2)$, respectively. Using the description in Figure 6, it is easy to check that the formula holds up to an overall additive constant. That is, the formula holds for the relative gr_w– and gr_z–gradings. Hence, it is sufficient to show the absolute gr_w–grading is correct for one of the generators, and similarly for the gr_z–grading. To check the absolute gradings, we note that, if we set $\mathscr{V} = 1$ and $\mathscr{U} = 0$, then we recover the Heegaard Floer complex for $\widehat{\mathrm{CF}}(S^3, w_1, w_2)$, which is homotopy equivalent to $\mathbb{F}_{1/2} \oplus \mathbb{F}_{-1/2}$ as a gr_w–graded chain complex. In this case, the complex has generators x_{2n-1} and x_{2n}, which pins down their gr_w–grading. A similar argument computes the gr_z–gradings. □

We now compute the homology action A_γ on the complex of the knotification of $T_{2,2n}$. In order to use Proposition 2.10, we need to compute A_λ and Φ_{w_2}. For a choice of arc on the Heegaard surface as in Figure 4, by counting bigons we obtain that A_λ has the form

$$\begin{aligned} A_\lambda(y_0) &= \mathscr{U}(y_1 + y_{4n-1}), & A_\lambda(y_i) &= \mathscr{U} y_{i+1} && \text{if } 0 < i < 2n, \\ A_\lambda(y_{2n}) &= \mathscr{V}(y_{2n-1} + y_{2n+1}), & A_\lambda(y_i) &= \mathscr{U} y_{4n-i+1} && \text{if } 2n+1 < i < 4n. \end{aligned} \tag{2.34}$$

By (2.31), we have

$$\begin{aligned} A_\lambda(x_0) &= \mathscr{U} x_1, & A_\lambda(x_i) &= \mathscr{U} x_{i+1} \quad \text{if } 0 < i < 2n-1 \\ A_\lambda(x_{2n}) &= \mathscr{V} x_{2n-1}, & A_\lambda(x_{2n-1}) &= 0. \end{aligned} \tag{2.35}$$

Next, we need to understand the map Φ_{w_2}. Counting bigons on diagrams like those shown in Figure 4 implies that Φ_{w_2} takes the form

$$\begin{aligned} \Phi_{w_2}(y_{2i}) &= y_{2i+1} && \text{if } 0 < i < n, & \Phi_{w_2}(y_{2i}) &= y_{4n-2i+1} && \text{if } n < i < 2n, \\ \Phi_{w_2}(y_{2i+1}) &= y_{2i} + y_{4n-2i} && \text{if } n < i < 2n, & \Phi_{w_2}(y_{2n+1}) &= y_{2n}, \end{aligned} \tag{2.36}$$

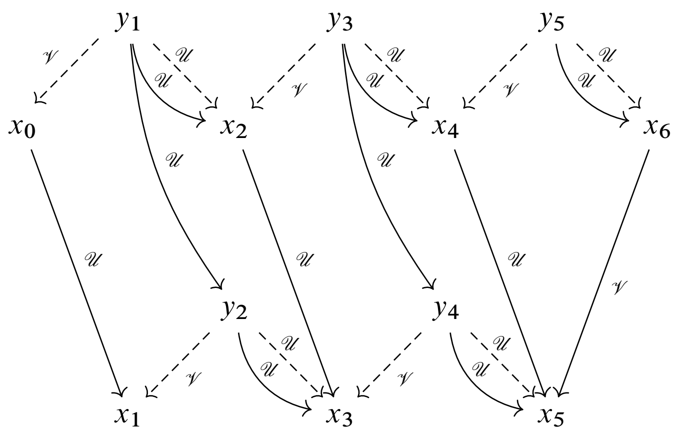
and Φ_{w_2} vanishes on all other generators.

Finally, we combine Proposition 2.10 with (2.35) and (2.36) to obtain the following formula for $A_\gamma \simeq A_\lambda + \mathscr{U}\Phi_{w_2}$ on the knotified complex, which we write in terms of the basis from (2.31):

(2.37)
$$\begin{aligned} A_\gamma(y_{2i+1}) &= \mathscr{U}x_{2i+2} + \mathscr{U}y_{2i+2} && \text{if } 0 \le i < n-1, \\ A_\gamma(y_{2i}) &= \mathscr{U}x_{2i+1} && \text{if } 0 < i < n-1, \\ A_\gamma(x_{2i}) &= \mathscr{U}x_{2i+1} && \text{if } 0 \le i < n, \\ A_\gamma(x_{2n}) &= \mathscr{V}x_{2n-1}, \end{aligned}$$

and A_γ vanishes on all other generators. The example of $T_{2,6}$ is shown below:

(2.38) $A_\gamma =$



The dashed lines denote the differential and the solid lines denote the A_γ–action. It is convenient to modify the map A_γ by a further chain homotopy, so that it takes one staircase summand to the other, with no self-arrows, as follows. Define a function $\delta\colon \mathbb{N} \to \mathbb{F}$ by

$$\delta(n) = \tfrac{1}{2}n(n-1) \mod 2.$$

Conceptually, it is easier to think of $\delta(n)$ as the sequence $0, 0, 1, 1, 0, 0, 1, 1, \ldots$. We define a homotopy F as follows. On the first staircase summand, we define F via

$$F(x_{2i}) = \delta(2i)\cdot x_{2i} \quad \text{if } 0 \le i \le n, \qquad F(y_{2i+1}) = \delta(2i+1)\cdot y_{2i+1} \quad \text{if } 0 \le i < n.$$

On the second staircase summand, we define F via

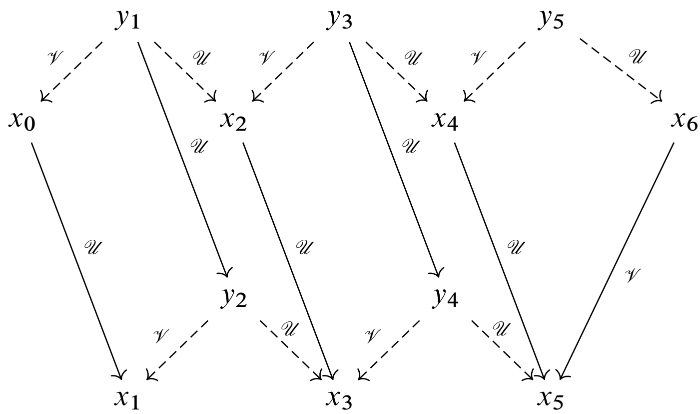
$$F(x_{2i+1}) = \delta(2i)\cdot x_{2i+1} \quad \text{if } 0 \le i < n, \qquad F(y_{2i}) = \delta(2i-1)\cdot y_{2i} \quad \text{if } 0 < i < n.$$

Writing A'_γ for $A_\gamma + [\partial, F]$, we compute that

$$\begin{aligned} A'_\gamma(x_{2i}) &= \mathscr{U}x_{2i+1} && \text{if } 0 \le i < n, \\ A'_\gamma(y_{2i+1}) &= \mathscr{U}y_{2i+2} && \text{if } 0 \le i < n-1, \\ A_{\gamma'}(x_{2n}) &= \mathscr{V}x_{2n-1}. \end{aligned}$$

Continuing our running example of $T_{2,6}$, (2.38) becomes

(2.39) $$A_\gamma + [\partial, F] =$$



We summarize the above computation as follows:

Proposition 2.40 *The pair $(\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,2n}), A_\gamma)$ has a model where $\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,2n})$ is equal to $\mathcal{S}^n\{\frac{1}{2}, \frac{1}{2}\} \oplus \mathcal{S}^{n-1}\{-\frac{1}{2}, -\frac{1}{2}\}$ and A_γ maps $\mathcal{S}^n$ to $\mathcal{S}^{n-1}$ on the chain level. Here, we recall that $\{i, j\}$ denotes a shift in the $(\mathrm{gr}_w, \mathrm{gr}_z)$–grading by (i, j), and $\mathcal{S}^n$ and $\mathcal{S}^{n-1}$ are the chain complexes in Definition 2.28.*

We now consider the mirror of the $(2, 2n)$–torus link, which we denote by $T_{2,-2n}$. We denote its knotification by $\widehat{T}_{2,-2n}$. On the level of Floer complexes, taking the mirror amounts to replacing the link Floer complex by the dual complex over the ring $\mathscr{R}^-$. In practice, this amounts to reversing all the arrows in the differential and multiplying the $(\mathrm{gr}_w, \mathrm{gr}_z)$–bigrading by an overall factor of -1. The homology action on the mirror is also the dual. We summarize this as follows:

Proposition 2.41 *The pair $(\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,-2n}), A_\gamma)$ has a model where $\mathcal{CFK}^-(S^2 \times S^1, \widehat{T}_{2,-2n})$ is equal to $\mathcal{S}^{-n}\{-\frac{1}{2}, -\frac{1}{2}\} \oplus \mathcal{S}^{-(n-1)}\{\frac{1}{2}, \frac{1}{2}\}$ and A_γ maps $\mathcal{S}^{-(n-1)}$ to $\mathcal{S}^{-n}$ on the chain level.*

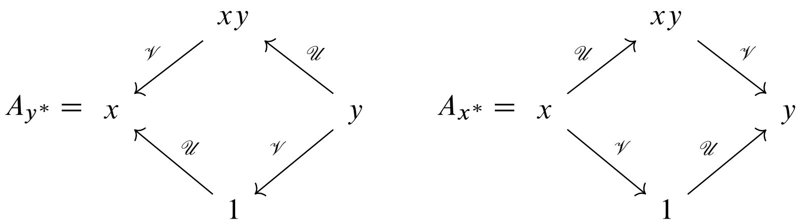
2.7 The Borromean knot $\mathcal{B}_0$

Let $\mathcal{B}_0 \subset \#^2 S^2 \times S^1$ be the Borromean knot, that is, the knot obtained from the Borromean rings by a zero-framed surgery on two of its components. The Heegaard Floer chain complex of $\mathcal{B}_0$ is described in [Ozsváth and Szabó 2004a, Proposition 9.2]. We adapt the calculation of [Borodzik et al. 2017, Section 5; Bodnár et al. 2016, Section 4] to the present context.

The chain complex $\mathcal{CFK}^-(\mathcal{B}_0)$ is homotopy equivalent to $\mathbb{F}^4 \otimes_{\mathbb{F}} \mathscr{R}^-$, with vanishing differential. We write 1, x, y and xy for the generators of $\mathbb{F}^4$, which we can think of as being generators of $H^*(\mathbb{T}^2)$. The bigradings are

$$(2.42) \qquad \begin{aligned} (\mathrm{gr}_w(1), \mathrm{gr}_z(1)) &= (1, -1), \\ (\mathrm{gr}_w(x), \mathrm{gr}_z(x)) = (\mathrm{gr}_w(y), \mathrm{gr}_z(y)) &= (0, 0), \\ (\mathrm{gr}_w(xy), \mathrm{gr}_z(xy)) &= (-1, 1). \end{aligned}$$

Up to an overall grading-preserving isomorphism, the $H_1(\#^2 S^2 \times S^1)$–module structure is uniquely determined by the formal properties of the action. In detail, if we write x^* and y^* for the two generators of $H_1(\#^2 S^2 \times S^1)$, then the module structure takes the form (up to overall isomorphism)



For the explicit description of the top and bottom towers of $\mathcal{CFK}^-(\mathcal{B}_0)$, see [Borodzik et al. 2017, Section 5].

3 Correction terms

3.1 Generalized correction terms of Levine and Ruberman

Suppose Y is an oriented closed three-dimensional manifold. The module $\mathrm{HF}^\infty(Y)$ is *standard* if, for each torsion Spin^c structure $\mathfrak{s}$,

$$\mathrm{HF}^\infty(Y,\mathfrak{s}) \cong \Lambda^* H^1(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}[U,U^{-1}]$$

as $\Lambda^*(H_1(Y;\mathbb{Z})/\mathrm{Tors}) \otimes_{\mathbb{Z}} \mathbb{F}[U]$–modules. Any manifold Y for which the triple cup product vanishes is standard; see [Lidman 2013] (and also [Levine and Ruberman 2014, Theorem 3.2]). In particular, a connected sum of finitely many copies of $S^1 \times S^2$ has standard HF^∞. Hence, a large surgery on a null-homologous knot in $\# S^1 \times S^2$ has standard HF^∞; see [Ozsváth and Szabó 2003]. This means that essentially all 3–manifolds we are going to consider have standard HF^∞.

There is an action (up to homotopy) of $\Lambda^*(H_1(Y)/\mathrm{Tors})$ on $\mathrm{CF}^-(Y,\mathfrak{s})$. Expanding on work of Ozsváth and Szabó [2003], Levine and Ruberman [2014] associate a d–invariant to any primitive subspace G of $H_1(Y)/\mathrm{Tors}$ (recall that a *primitive subspace* is a free submodule whose quotient is free) and any Spin^c structure $\mathfrak{s}$ on Y whose first Chern class is torsion as long as $\mathrm{HF}^\infty(Y)$ is standard. We denote this invariant by $d(Y,\mathfrak{s},G)$. For our purposes, the two most important instances are the invariants

$$d_{\mathrm{bot}}(Y,\mathfrak{s}) := d(Y,\mathfrak{s},H_1(Y)/\mathrm{Tors}), \quad d_{\mathrm{top}}(Y,\mathfrak{s}) := d(Y,\mathfrak{s},\{0\}),$$

which correspond approximately to the kernel and cokernel, respectively, of the $H_1(Y)/\mathrm{Tors}$–action.

The key property of these invariants is the following inequality, generalizing the Ozsváth–Szabó inequality:

Theorem 3.1 [Levine and Ruberman 2014, Theorem 4.7] *Suppose X is a connected four-manifold such that $b_2^+(X) = 0$ and $\partial X = Y$. Suppose $\mathfrak{s}$ is a Spin^c structure on Y that extends to a Spin^c structure $\mathfrak{t}$ on X. Then*

$$d(Y,\mathfrak{s},G) \geq \tfrac{1}{4}(c_1^2(\mathfrak{t}) + b_2^-(X)) + \tfrac{1}{2}b_1(Y) - \mathrm{rk}\, G$$

if G contains the kernel of the inclusion map from $H_1(Y)/\mathrm{Tors}$ to $H_1(X)/\mathrm{Tors}$.

3.2 V–invariants

The aim of this section is to gather several definitions of V_s–invariants. In the context of Heegaard Floer theory, all these definitions lead to the same invariants.

The first definition recalls the classical V_s–invariant for knots. The assumptions on C_* in Definition 3.2 are modeled on a knot Floer complex CFK^-.

Definition 3.2 (V_s–invariants for complexes over $\mathbb{F}[U, U^{-1}]$) Suppose C_* is a filtered chain complex of free $\mathbb{F}[U]$–modules (with multiplication by U decreasing the filtration level by 1 and the grading by 2) such that the homology of the localized complex $U^{-1}C_*$ is equal to $\mathbb{F}[U, U^{-1}]$. For $s \in \mathbb{Z}$, the invariant $V_s(C_*)$ is such that $-2V_s(C_*)$ is the maximal grading of an element $x \in C_*$ at filtration level at most s such that the class of $U^k x$ is nonzero in $H_*(C_*)$ for all $k \geq 0$.

Next we define the V_s–invariants of a bigraded $\mathscr{R}^-$–module, where $\mathscr{R}^- = \mathbb{F}[\mathscr{U}, \mathscr{V}]$. The definition is essentially taken from [Zemke 2019b, equation (10.3)]. Suppose C_* is a bigraded chain complex over $\mathscr{R}^-$ such that multiplication by $\mathscr{U}$ changes the grading by $(-2, 0)$, multiplication by $\mathscr{V}$ changes the grading by $(0, -2)$, and the differential changes the grading by $(-1, -1)$. Let $(\mathrm{gr}_w, \mathrm{gr}_z)$ denote the bigrading. It is not hard to see that the differential and multiplication by $\mathscr{U}\mathscr{V}$ preserves the difference $\mathrm{gr}_w - \mathrm{gr}_z$.

Definition 3.3 (V_s–invariants over $\mathscr{R}^-$) Suppose C_* is a chain complex over $\mathscr{R}^-$ such that

$$(\mathscr{U}, \mathscr{V})^{-1} \cdot H_*(C_*) \cong \mathscr{R}^\infty = \mathbb{F}[\mathscr{U}, \mathscr{V}, \mathscr{U}^{-1}, \mathscr{V}^{-1}] \tag{3.4}$$

as bigraded groups. (Here $(\mathscr{U}, \mathscr{V})^{-1}\cdot$ denotes localization at the nonzero monomials of $\mathscr{R}^-$.) We write $\mathscr{A}_s(C_*)$ for the subcomplex of C_* which has $\mathrm{gr}_w - \mathrm{gr}_z = 2s$. We can view $\mathscr{A}_s(C_*)$ as a complex over $\mathbb{F}[U]$, where U acts by $\mathscr{U}\mathscr{V}$. We define $d(\mathscr{A}_s(C_*))$ for the maximal gr_w–grading of a homogeneously graded, $\mathbb{F}[U]$–nontorsion element of $H_*(\mathscr{A}_s(C_*))$. We define

$$V_s(C_*) = -\tfrac{1}{2} d(\mathscr{A}_s(C_*)).$$

Remark 3.5 Suppose M is a graded module over $\mathscr{R}^-$ such that $(\mathscr{U}^{-1}, \mathscr{V}^{-1}) \cdot M \cong \mathscr{R}^\infty$ as bigraded groups. We define $V_s(M)$ to be $V_s(C_*)$, with C_* the chain complex with the same underlying module structure as M but trivial differential.

Remark 3.6 If C_* is the chain complex $\mathcal{CFL}^-(S^3, K)$ for a knot $K \subset S^3$, $V_s(C_*)$ is the classical V–function of the knot K. In this case, we also denote it by $V_s(K)$ if the context is clear. See [Zemke 2019b, Section 1.5] for translating between the chain complex $\mathcal{CFL}^-(S^3, K)$ and $\mathrm{CFK}^-(S^3, K)$.

Suppose C_* is as in Definition 3.3. Let $a, b \in \mathbb{Z}$. The chain complex $C_*\{a, b\}$ is defined as the chain complex equal to C_*, but with grading shifted by (a, b). That is, if $x \in C_*$ has bigrading (c, d), then $x \in C_*\{a, b\}$ has bigrading $(a + c, b + d)$.

Lemma 3.7 *Suppose C_* is a bigraded chain complex over $\mathscr{R}^-$ and let $D_* = C_*\{a,b\}$ be the chain complex with shifted grading. Then $V_{s+(a-b)/2}(D_*) = V_s(C_*) - \frac{1}{2}a$.*

Proof We use the fact that $\mathscr{A}_s(C_*) = \mathscr{A}_{s+(a-b)/2}(D_*)$. □

In our computations, we will need to show that V_s–invariants of locally equivalent complexes are the same. We recall the relevant definition:

Definition 3.8 Two chain complexes C_* and D_* are *locally equivalent* if there exist grading-preserving, $\mathscr{R}^-$–equivariant chain maps $f\colon C_* \to D_*$, $g\colon D_* \to C_*$ such that both f and g induce the identity map on $(\mathscr{U},\mathscr{V})^{-1} \cdot C_* \cong (\mathscr{U},\mathscr{V})^{-1} \cdot D_*$.

As an example, we quote the following result of Hedden, Kim and Livingston (note that ν^+–equivalence is equivalent to local equivalence; see [Hom 2017, Proposition 3.11]):

Proposition 3.9 [Hedden et al. 2016, Theorem B.1] *The tensor product $\mathcal{S}^k \otimes \mathcal{S}^\ell$ is locally equivalent to $\mathcal{S}^{k+\ell}$ for any integers k and l.*

For the following result, see [Zemke 2019a, Section 2], [Hom 2017] or [Kim and Park 2018, Section 3]:

Proposition 3.10 (a) *If C_* is locally equivalent to D_*, then $V_s(C_*) = V_s(D_*)$ for all s.*

(b) *If C_* is locally equivalent to D_* and E_* is locally equivalent to F_*, then $C_* \otimes E_*$ is locally equivalent to $D_* \otimes F_*$.*

We now extend Definition 3.3 to the case of chain complexes with a group action. Suppose C_* is a bigraded chain complex over $\mathscr{R}^-$ and H is a free abelian group such that the ring $\Lambda^* H$ acts on $H_*(C_*)$, and the action of H has degree $(-1,-1)$. Let $\mathrm{Tors} \subset H_*(C_*)$ denote the $\mathscr{R}^-$–torsion submodule. Define

$$\mathcal{H}^{\mathrm{top}} = \operatorname{coker}\bigl(H \otimes (H_*(C_*)/\mathrm{Tors}) \to (H_*(C_*)/\mathrm{Tors})\bigr),$$
$$\mathcal{H}^{\mathrm{bot}} = \bigcap_{\gamma \in H} \ker\bigl(\gamma\colon (H_*(C_*)/\mathrm{Tors}) \to (H_*(C_*)/\mathrm{Tors})\bigr).$$

By analogy with (3.4), we require that

$$(\mathscr{U},\mathscr{V})^{-1} \cdot \mathcal{H}^{\mathrm{top}} \cong \mathscr{R}^\infty \cong (\mathscr{U},\mathscr{V})^{-1} \cdot \mathcal{H}^{\mathrm{bot}}$$

as relatively bigraded $\mathscr{R}^-$–modules. Let $\mathcal{H}_s^{\mathrm{top}}$ (resp. $\mathcal{H}_s^{\mathrm{bot}}$) denote the $\mathbb{F}[U]$–submodule generated by homogeneously graded elements $x \in \mathcal{H}^{\mathrm{top}}$ (resp. $x \in \mathcal{H}^{\mathrm{bot}}$) such that $\mathrm{gr}_w(x) - \mathrm{gr}_z(x) = 2s$ (recall U acts by $\mathscr{U}\mathscr{V}$). We define $d_s^{\mathrm{top}}(C_*)$ to be the maximal gr_w–grading of a homogeneously graded, $\mathbb{F}[U]$–nontorsion element of $\mathcal{H}_s^{\mathrm{top}}$, and we define $d_s^{\mathrm{bot}}(C_*)$ analogously.

Definition 3.11 We set

$$V_s^{\mathrm{top}}(C_*) := -\tfrac{1}{2} d_s^{\mathrm{top}}(C_*) \quad \text{and} \quad V_s^{\mathrm{bot}}(C_*) = -\tfrac{1}{2} d_s^{\mathrm{bot}}(C_*).$$

Remark 3.12 If K is a null-homologous knot in a closed, oriented connected 3–manifold Y with standard $\mathrm{HF}^\infty(Y)$, for simplicity we write $\mathscr{A}_s(K)$ for $\mathscr{A}_s(\mathcal{CFL}^-(Y,K))$, and $V_s^{\mathrm{top}}(K)=-\frac{1}{2}d_s^{\mathrm{top}}(K)$ and $V_s^{\mathrm{bot}}(K)=-\frac{1}{2}d_s^{\mathrm{bot}}(K)$ for $V_s^{\mathrm{top}}(\mathcal{CFL}^-(Y,K))$ and $V_s^{\mathrm{bot}}(\mathcal{CFL}^-(Y,K))$, respectively.

3.3 Large surgery formula

To set up the notation, we recall the large surgery formula [Ozsváth and Szabó 2004b, Section 4] and relate the d–invariants of the surgery on a knot to its V_s–invariants. We first recall the description of Spin^c structures on a surgery.

Definition 3.13 Suppose Y is a closed 3–manifold and $K\subset Y$ is a null-homologous knot. Let $\mathfrak{s}\in\mathrm{Spin}^c(Y)$ and $q\in\mathbb{Z}_{>0}$. For any $m\in\left[-\frac{1}{2}q,\frac{1}{2}q\right)\cap\mathbb{Z}$ we denote by $\mathfrak{s}_m$ the unique Spin^c structure on $Y_q(K)$ such that $\mathfrak{s}_m$ extends to a Spin^c structure $\mathfrak{t}_m$ on W uniquely characterized by the properties that $\mathfrak{t}_m|_Y=\mathfrak{s}$ and $\langle c_1(\mathfrak{t}_m),F\rangle+q=2m$, where W is the trace of the surgery on K and F is the generator of $H_2(W)$ obtained by gluing a Seifert surface for K with the core of the two-handle.

With this notation, we state Ozsváth and Szabó's large surgery theorem [2004b, Theorem 4.1]:

Theorem 3.14 *Suppose $K\subset Y$ is a null-homologous knot in a closed 3–manifold. Suppose $q>2g_3(K)$ is an integer. For a Spin^c structure $\mathfrak{s}_m$ on Y as in Definition 3.13, there exists a quasi-isomorphism between $\mathrm{CF}^-(Y_q(K),\mathfrak{s}_m)$ and $\mathscr{A}_m$, where $\mathscr{A}_m$ is the $\mathbb{F}[U]$–subcomplex of $\mathcal{CFL}^-(Y,K,\mathfrak{s})$ of elements x with grading $\mathrm{gr}_w(x)-\mathrm{gr}_z(x)=2m$. If $\mathfrak{s}$ is torsion, then the quasi-isomorphism shifts the grading (Maslov grading on $\mathrm{CF}^-(Y_q(K),\mathfrak{s}_m)$ and gr_w–grading on $\mathscr{A}_m$) by $((q-2m)^2-q)/4q$.*

From this theorem we obtain the following well-known equalities:

Theorem 3.15 *Suppose $K\subset Y$ is as in Theorem 3.14 and $q>2g_3(K)$.*

(a) *If Y is a rational homology sphere, then $d(Y_q(K),\mathfrak{s}_m)=((q-2m)^2-q)/4q-2V_m(K)$;*

(b) *If $b_1(Y)>0$ and $\mathrm{HF}^\infty(Y)$ is standard, then $d^{\mathrm{top}}(Y_q(K),\mathfrak{s}_m)=((q-2m)^2-q)/4q-2V_m^{\mathrm{top}}(K)$ and $d^{\mathrm{bot}}(Y_q(K),\mathfrak{s}_m)=((q-2m)^2-q)/4q-2V_m^{\mathrm{bot}}(K)$.*

4 Staircase complexes and their tensor products

In this section, we introduce staircase complexes. Next we compute the correction terms of certain tensor products of staircase complexes.

4.1 Staircase complexes

A *positive staircase complex* $\mathcal{P}$ is a bigraded chain complex over $\mathscr{R}^-$ with generators $x_0,y_1,x_2,\dots,y_{2n-1},x_{2n}$ for some $n>0$ with differential given by $\partial y_{2i+1}=\mathscr{U}^{\alpha_i}\cdot x_{2i}+\mathscr{V}^{\beta_i}\cdot x_{2i+2}$, $\partial x_{2j}=0$, extended equivariantly over $\mathscr{R}^-$, for some positive integers α_i and β_i. We assume that ∂, $\mathscr{U}$ and $\mathscr{V}$ have bigradings $(-1,-1)$, $(-2,0)$ and $(0,-2)$, respectively. We assume that $\alpha_i = \beta_{n-i-1}$. Furthermore, we assume the gradings are normalized so that $H_*(\mathcal{P}/(\mathscr{U}-1)) \cong \mathbb{F}[\mathscr{V}]$ has generator with gr_z–grading 0, and $H_*(\mathcal{P}/(\mathscr{V}-1)) \cong \mathbb{F}[\mathscr{U}]$ has generator with gr_w–grading 0. A *negative staircase complex* is the dual complex of a positive staircase complex.

Example 4.1 The complex $\mathcal{S}^n$ of Definition 2.28 is a positive staircase complex for all $n > 0$. It is a negative staircase complex if $n < 0$.

Lemma 4.2 *Suppose that $\mathcal{P} = (P_1 \to P_0)$ is a positive staircase complex, viewed as a complex of free $\mathscr{R}^-$–modules, where P_1 is spanned by y_i and P_0 is spanned by x_i.*

1. *$H_*(\mathcal{P})$ is torsion-free as an $\mathscr{R}^-$–module.*
2. *There is a $(\mathrm{gr}_w, \mathrm{gr}_z)$–grading-preserving chain map*
$$F : \mathcal{P} \to \mathscr{R}^-$$
which sends $\mathscr{R}^-$–nontorsion cycles to $\mathscr{R}^-$–nontorsion cycles. Furthermore, F may be taken to map each generator of P_0 to a nonzero monomial in $\mathscr{R}^-$, and vanish on P_1.

Proof For the first claim, using the grading properties of $\mathcal{P}$ it is sufficient to show that $\mathscr{U}^i\mathscr{V}^j \cdot [x] \neq 0$ if $[x] \neq 0 \in H_*(\mathcal{P})$ when x is a homogeneously graded cycle in $\mathcal{P}$. Since the map from P_1 to P_0 is injective, there are no cycles with a nonzero summand in P_1. Hence, it is sufficient to see that, if $x \in P_0$ and $\mathscr{U}^i\mathscr{V}^j \cdot x \in \mathrm{im}(P_1)$, then $x \in \mathrm{im}(P_1)$. To see this, suppose that $y \in P_1$ is homogeneously graded and not a multiple of $\mathscr{U}$ or $\mathscr{V}$. We may write y as an $\mathscr{R}^-$ linear combination of $y_1, \dots, y_{2n-1}$. Let m (resp. M) be the minimal (resp. maximal) index which is supported by y. Hence, we may write $y = a_m y_m + \cdots + a_M y_M$ for $a_m, \dots, a_M \in \mathscr{R}^-$. We observe that

$$\mathrm{gr}_w(y_i) \geq \mathrm{gr}_w(y_{i+2}) \quad \text{and} \quad \mathrm{gr}_z(y_i) \leq \mathrm{gr}_z(y_{i+2}) \tag{4.3}$$

for all i. Since y is homogeneously graded, it follows that a_m is not a multiple of $\mathscr{V}$: if it were, then all other a_i would need to be a multiple of $\mathscr{V}$ for y to be homogeneously graded, which contradicts our assumption. Similarly, a_M is not a multiple of $\mathscr{U}$. We write $a_m = \mathscr{U}^{j_m}$ and $a_M = \mathscr{V}^{j_M}$ for some $j_m, j_M \in \mathbb{N}$. Then $\mathscr{U}^{j_m+\alpha_{(m-1)/2}} x_{m-1}$ and $\mathscr{V}^{j_M+\beta_{(M+1)/2}} x_{M+1}$ are summands of $\partial(y)$, and hence it is not a multiple of any element of $\mathscr{R}^-$.

For the second claim, if $x_i \in P_0$ is a generator, we define $F(x_i)$ to be the unique nonzero element of $\mathscr{R}^-$ in the same homogeneous grading as x. It follows from our normalization of the gradings of $H_*(\mathcal{P}/(\mathscr{U}-1)) \cong \mathbb{F}[\mathscr{V}]$ and $H_*(\mathcal{P}/(\mathscr{V}-1)) \cong \mathbb{F}[\mathscr{U}]$ as well as (4.3) that each generator of $\mathcal{P}$ has $(\mathrm{gr}_w, \mathrm{gr}_z)$–bigrading in $\mathbb{Z}^{\leq 0} \times \mathbb{Z}^{\leq 0}$, so this map is well defined. We leave it to the reader to verify that this map is a chain map and sends $\mathscr{R}^-$–nontorsion cycles to $\mathscr{R}^-$–nontorsion cycles. □

Definition 4.4 We call a complex $\mathcal{P}$ a *positive multistaircase* if it is the tensor product of a nonzero number of positive staircase complexes. We call $\mathcal{N}$ a *negative multistaircase* if it is the tensor product of a nonzero number of negative staircases.

The dual of a positive multistaircase is a negative multistaircase, and vice versa.

By construction, a positive staircase $\mathcal{P}$ has a $\mathbb{Z}$–filtration with two levels, and we write $\mathcal{P} = (P_1 \to P_0)$. Hence, a positive multistaircase with n factors has a $\mathbb{Z}$–filtration with $n+1$ nontrivial levels, for which we write

$$\mathcal{P} = (P_n \to P_{n-1} \to \cdots \to P_1 \to P_0). \tag{4.5}$$

If $\mathcal{P} = (P_n \to \cdots \to P_0)$ is a positive multistaircase, we say that $\mathcal{P}$ is an *exact* multistaircase if the following sequence is exact:

$$0 \to P_n \to \cdots \to P_0.$$

In particular, an exact multistaircase is a free resolution of its homology.

Remark 4.6 In general, the sequence in (4.5) will not be exact. As a concrete example, consider $\mathcal{C} = \mathcal{CFK}^-(T_{2,3})$ and the tensor product $\mathcal{P} = \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$. Write $\mathcal{P} = (P_3 \to P_2 \to P_1 \to P_0)$. Following our conventions, write x_0, y_1 and x_2 for the generators of the left-most factor of $\mathcal{C}$, where $\partial(y_1) = \mathscr{U} x_0 + \mathscr{V} x_2$. One easily computes that

$$y_1|x_2|x_0 + x_2|y_1|x_0 + x_2|x_0|y_1 + x_0|x_2|y_1 + y_1|x_0|x_2 + x_0|y_1|x_2 \in P_1$$

is a cycle. In the above, bars denote tensor products. It is not a boundary, since the differential has image in $\mathrm{im}(\mathscr{U}) + \mathrm{im}(\mathscr{V})$.

Lemma 4.7 (1) *Every positive staircase is exact.*

(2) *The tensor product of two positive staircases is exact.*

Proof Exactness of a positive staircase $\mathcal{P} = (P_1 \to P_0)$ amounts to the claim that the map $P_1 \to P_0$ is injective, which is easy to verify.

Next suppose $\mathcal{P} = (P_1 \to P_0)$ and $\mathcal{D} = (D_1 \to D_0)$ are staircases. We claim that their tensor product is also exact. Let $\mathcal{E} = (E_2 \to E_1 \to E_0)$ denote this tensor product. Clearly the map $E_2 \to E_1$ is injective, so it is sufficient to show that $H_1(\mathcal{E}) = 0$. The homology $H_*(\mathcal{E})$ decomposes as the direct sum $H_2(\mathcal{E}) \oplus H_1(\mathcal{E}) \oplus H_0(\mathcal{E})$. Since every $\mathscr{R}^-$–nontorsion element contains a nonzero summand of $H_0(\mathcal{E})$, it follows that $H_1(\mathcal{E})$ consists only of $\mathscr{R}^-$–torsion elements. Since $\mathcal{E}$ is bigraded, each element $[x] \in H_1(\mathcal{E})$ satisfies $\mathscr{U}^i \mathscr{V}^j \cdot [x] = 0$ for some i and j. In particular, if $x \in E_1$ is a cycle, then $\mathscr{U}^i \mathscr{V}^j \cdot x \in \mathrm{im}(E_2 \to E_1)$ for some i, j. In order to show that $H_1(\mathcal{E}) = 0$ it is sufficient to show that, if $\mathscr{U}^i \mathscr{V}^j \cdot x \in \mathrm{im}(E_2 \to E_1)$, then $x \in \mathrm{im}(E_2 \to E_1)$. We argue as follows. Note first that the map from E_2 to E_1 is the sum of the maps $P_1 \otimes D_1 \to P_1 \otimes D_0$ and $P_1 \otimes D_1 \to P_0 \otimes D_1$. Suppose that $\mathscr{U}^i \mathscr{V}^j \cdot x \in \operatorname{im}(E_2 \to E_1)$. Write $\mathscr{U}^i \mathscr{V}^j \cdot x = \partial(y)$. We may assume that x and y are homogeneously graded. Write $x = x_{0,1} + x_{1,0}$, where $x_{1,0} \in P_1 \otimes D_0$ and $x_{0,1} \in P_0 \otimes D_1$. Then $\mathscr{U}^i \mathscr{V}^j \cdot x_{0,1} \in \operatorname{im}(P_1 \to P_0) \otimes D_1$. Since $\mathcal{P}$ is exact and D_1 is free, we conclude that $x_{0,1} \in \operatorname{im}(P_1 \to P_0) \otimes D_1$. Hence there is some $y' \in P_1 \otimes D_1$ such that the map from $P_1 \otimes D_1$ to $P_0 \otimes D_1$ maps y' to $x_{0,1}$. Since the map from $P_1 \otimes D_1$ to $P_0 \otimes D_1$ is injective, we conclude that $\mathscr{U}^i \mathscr{V}^j y' = y$, so $\partial(y') = x_{0,1} + x_{1,0}$ and $x_{0,1} + x_{1,0} \in \operatorname{im}(E_2 \to E_1)$. □

4.2 The staircase complexes for L–space knots

A knot $K \subset S^3$ is called an *L–space knot* if there is a positive integer q such that $S^3_q(K)$ is an L–space, ie $\mathrm{HF}^-(S^3_q(K), \mathfrak{s}) \cong \mathbb{F}[U]$ for each $\mathfrak{s} \in \operatorname{Spin}^c(S^3_q(K))$. All algebraic knots are L–space knots; see [Hedden 2009, Theorem 1.10].

There is a simple description of Floer chain complexes of L–space knots, due to Ozsváth and Szabó [2005, Theorem 1.2]. (Note that, therein, only $\widehat{\mathrm{HFK}}(K)$ is described, but the algorithm actually produces a description of $\mathrm{CFK}^\infty(K)$.) We describe their algorithm presently. Let K be an L–space knot. Ozsváth and Szabó prove that the Alexander polynomial of K, which we denote by $\Delta_K(t)$, has the form

(4.8) $$\Delta_K(t) = t^{a_0} - t^{a_1} + \cdots + t^{a_{2r}},$$

where $0 = a_0 < a_1 < \cdots < a_{2r}$; that is, we use the normalization of Δ starting at degree 0. Define the gap function

$$\beta_i := a_i - a_{i-1}$$

for $1 \leq i \leq 2r$.

We now describe the complex $\mathcal{CFK}^-(K)$ over the ring $\mathscr{R}^-$. The complex $\mathcal{CFK}^-(K)$ is freely generated over $\mathscr{R}^-$ by elements

$$x_0, y_1, x_2, \ldots, y_{2r-1}, x_{2r}.$$

The differential takes the form

(4.9) $$\partial(x_{2i}) = 0 \quad \text{and} \quad \partial(y_{2i+1}) = \mathscr{U}^{\beta_{2i+1}} x_{2i} + \mathscr{V}^{\beta_{2i+2}} x_{2i+2}.$$

The $(\mathrm{gr}_w, \mathrm{gr}_z)$–bigradings are determined by the normalization that $\mathrm{gr}_w(\boldsymbol{x}_0) = 0$ and $\mathrm{gr}_z(\boldsymbol{x}_{2r}) = 0$. Recall that the variable $\mathscr{U}$ has bigrading $(-2, 0)$ and the variable $\mathscr{V}$ has bigrading $(0, -2)$.

The gradings can be expressed in the following way. Write

$$\Delta_K = 1 + (t-1)(t^{m_1} + \cdots + t^{m_s})$$

for some positive integers $m_1 < \cdots < m_s$. Note that the integers β_i compute the number of consecutive integers or consecutive gaps (depending on i) of the sequence $m_1, \ldots, m_s$; see [Borodzik and Livingston 2014, Lemma 4.2]. Define $S_K = \mathbb{Z}_{\geq 0} \setminus \{m_1, \ldots, m_s\}$, and

(4.10) $$R_K(t) = \# S_K \cap [0, t) \quad \text{if } t \in \mathbb{Z}.$$

With this notation, the gradings of the generator x_{2i} are $\mathrm{gr}_w(x_{2i}) = -2R_K(a_{2i})$ and $\mathrm{gr}_z(x_{2i}) = 2R_K(a_{2i}) - 2g_3(K)$; compare [Borodzik and Livingston 2014, Section 4]. Note that, with our normalization, $2g_3(K) = a_{2r} = m_s + 1$. If the context is clear, we sometimes write R instead of R_K to simplify the notation.

Example 4.11 If K is the $(2, 2n+1)$–torus knot, then the above procedure produces the complex $\mathcal{S}^n$ of Definition 2.28.

Remark 4.12 If K is an algebraic knot, the set S_K turns out to be a semigroup (note that, if K is only an L–space knot, S_K need not be a semigroup). In fact, this is the semigroup of that singular point. The function R_K is the *semigroup counting function*. See [Wall 2004, Section 4] for details on semigroups.

The next corollary is a compilation of [Borodzik and Livingston 2014, Proposition 5.6 and Lemma 6.2]:

Corollary 4.13 *The V_s–invariants of an L–space knot satisfy $V_{-g_3(K)+j}(K) = R_K(j) - j + g_3(K)$.*

The Künneth formula for the knot Floer chain complex allows us to compute the V_j–invariants of a connected sum of L–space knots. The following result is given in [Borodzik and Livingston 2014, formula (6.3)]:

Proposition 4.14 *Let $K_1, \dots, K_n$ be L–space knots. Set $K = K_1 \# \cdots \# K_n$ and let $g = g_3(K)$. Then*

$$V_j(K) + j = R_K(g + j),$$

where $R_K = R_{K_1} \diamond \cdots \diamond R_{K_n}$ is the infimal convolution of $R_{K_1}, \dots, R_{K_n}$.

We recall that, if $I, J : \mathbb{Z} \to \mathbb{Z}$ are two functions bounded from above, their *infimal convolution* is given by $I \diamond J(m) = \min_{i+j=m} I(i) + J(j)$.

4.3 V_s–invariants of tensor products of staircases

In this subsection, we compute the V_s–invariants of certain tensor products of staircases. We wish to understand the V_s–invariants of tensor products of staircases where some factors are positive and some negative. Of course, we may group factors and write such a complex as a tensor product of $\mathcal{N} \otimes \mathcal{P}$, where $\mathcal{N}$ is a negative multistaircase and $\mathcal{P}$ is a positive multistaircase. Clearly,

$$\mathcal{N} \otimes \mathcal{P} \cong \mathrm{Hom}_{\mathscr{R}^-}(\mathcal{N}^\vee, \mathcal{P}),$$

where $\mathrm{Hom}_{\mathscr{R}^-}(N^\vee, \mathcal{P})$ denotes the chain complex of $\mathscr{R}^-$–module homomorphisms from $\mathcal{N}^\vee$ to $\mathcal{P}$. In particular, to understand the V_s–invariants of arbitrary tensor products of positive and negative staircases, it is sufficient to understand the morphism complex between two positive multistaircases.

It is also helpful to note that, if $\mathcal{N}$ and $\mathcal{P}$ are multistaircases (of either sign), then a cycle $\phi \in \operatorname{Hom}_{\mathscr{R}^-}(\mathcal{N}^\vee, \mathcal{P})$ is $\mathscr{R}^-$–nontorsion as a morphism if and only if ϕ maps $\mathscr{R}^-$–nontorsion cycles to $\mathscr{R}^-$–nontorsion cycles.

The following result is by now classical (see [Borodzik and Livingston 2014, Proposition 5.1]):

Proposition 4.15 *Let $\mathcal{P} = (P_n \to \cdots \to P_0)$ be a positive multistaircase and let $s \in \mathbb{Z}$. Then*

$$V_s(\mathcal{P}) = \min_{x \in \mathcal{G}(P_0)} \max(\alpha(x), \beta(x) - s),$$

where $\alpha(x) = -\frac{1}{2}\operatorname{gr}_w(x)$, $\beta(x) = -\frac{1}{2}\operatorname{gr}_z(x)$ and $\mathcal{G}(P_0)$ denotes the set of homogeneously graded generators of P_0.

Proof Lemma 4.2 implies that a homogeneously graded element $x \in \mathcal{P}$ is an $\mathscr{R}^-$–nontorsion cycle if and only if its summand in P_0 may be written as an $\mathscr{R}^-$–linear combination of an odd number of distinct elements in the generating set $\mathcal{G}(P_0)$ with nonzero, homogeneously graded coefficients in $\mathscr{R}^-$. In particular, the individual elements of $\mathcal{G}(P_0)$ determine the correction terms V_s. The expression $-2\max(\alpha(x), \beta(x) - s)$ is the maximal gr_w–grading of an element of the form $\mathscr{U}^m \mathscr{V}^n x$ such that $m, n \geq 0$ and $x \in \mathscr{A}_s$. Taking the minimum over all $x \in \mathcal{G}(P_0)$ gives the result. □

We now pass to studying V_s–invariants of products of positive and negative multistaircases. We begin with the following statement, where we write $H_0(\mathcal{P})$ for $P_0/\operatorname{im} P_1$ for a multistaircase:

Proposition 4.16 *Suppose that $\mathcal{P} = (P_m \to \cdots \to P_0)$ and $\mathcal{Q} = (Q_n \to \cdots \to Q_0)$ are two positive multistaircases.*

1. *In general, $V_s(\operatorname{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q})) \geq V_s\big(\operatorname{Hom}_{\mathscr{R}^-}(H_*(\mathcal{P}), H_*(\mathcal{Q}))\big) = V_s\big(\operatorname{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q}))\big)$.*
2. *If $\mathcal{Q}$ is exact, then $V_s(\operatorname{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q})) = V_s\big(\operatorname{Hom}_{\mathscr{R}^-}(H_*(\mathcal{P}), H_*(\mathcal{Q}))\big)$.*

Proof There is a grading-preserving map of $\mathscr{R}^-$–modules

$$H_* \operatorname{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q}) \to \operatorname{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q})),$$

which sends $\mathscr{R}^-$–nontorsion elements to $\mathscr{R}^-$–nontorsion elements. Then the inequality of part (1) follows since the map sends $\mathscr{R}^-$–nontorsion elements in $\mathscr{A}_s(\operatorname{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q}))$ to $\mathscr{R}^-$–nontorsion elements in $\mathscr{A}_s\big(\operatorname{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q}))\big)$. The equality in part (1) follows since $H_*(\mathcal{P})$ decomposes as a direct sum

$$\bigoplus_{s=0}^{n} (\ker(P_i \to P_{i-1})/\operatorname{im}(P_{i+1} \to P_i)),$$

and $H_0(\mathcal{P}) = P_0/\operatorname{im} P_1$ is the only summand which contains $\mathscr{R}^-$–nontorsion elements.

We now consider the second claim. Suppose that $\mathcal{Q}$ is exact. We will show

$$V_s\big(\mathrm{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q}))\big) \geq V_s(\mathrm{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q})). \tag{4.17}$$

Suppose $\phi\colon H_0(\mathcal{P}) \to H_0(\mathcal{Q})$ is an $\mathscr{R}^-$–module homomorphism which maps $\mathscr{R}^-$–nontorsion elements to $\mathscr{R}^-$–nontorsion elements. It suffices to extend ϕ to obtain a commutative diagram

$$\begin{array}{ccccccccc} P_m \longrightarrow \cdots \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \twoheadrightarrow & H_0(\mathcal{P}) \\ & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi} \\ \cdots \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \twoheadrightarrow & H_0(\mathcal{Q}) \end{array}$$

since this extension gives an $\mathscr{R}^-$–nontorsion element in $\mathscr{A}_s(\mathrm{Hom}_{\mathscr{R}^-}(\mathcal{P}, \mathcal{Q}))$ corresponding to any $\mathscr{R}^-$–nontorsion element in $\mathscr{A}_s\big(\mathrm{Hom}_{\mathscr{R}^-}(H_0(\mathcal{P}), H_0(\mathcal{Q}))\big)$. The construction of the maps ϕ_i follows from the same procedure as in [Weibel 1994, Theorem 2.2.6] and the discussion below it. We briefly summarize the construction. The map ϕ_0 may be chosen since P_0 is free, and hence projective, and $Q_0 \to H_0(\mathcal{Q})$ is surjective. Having constructed ϕ_0, we next construct ϕ_1. Using exactness of $\mathcal{Q}$, we may factor $\phi_0 \circ (P_1 \to P_0)$ into $\mathrm{im}(Q_1 \to Q_0)$. Using the fact that P_1 is projective and $Q_1 \to \mathrm{im}(Q_1 \to Q_0)$ is surjective, we obtain a map ϕ_1. We repeat this process until we exhaust $\mathcal{P}$. This gives (4.17). □

Proposition 4.18 *Suppose $\mathcal{N} = (N_0 \to \cdots \to N_n)$ is a negative multistaircase, and $\mathcal{P} = (P_m \to \cdots \to P_0)$ is a positive multistaircase. Write $\mathcal{G}(P_i)$ for the generators of P_i, and similarly for $\mathcal{G}(N_i)$.*

(1) *In general,*

$$V_s(\mathcal{N} \otimes \mathcal{P}) \geq -\tfrac{1}{2} \min_{x \in \mathcal{G}(N_0)} \max_{y \in \mathcal{G}(P_0)} \min(\mathrm{gr}_w(x) + \mathrm{gr}_w(y), \mathrm{gr}_z(x) + \mathrm{gr}_z(y) + 2s). \tag{4.19}$$

(2) *If $\mathcal{P} = (P_1 \to P_0)$ is a positive staircase, then (4.19) is an equality.*

Proof We dualize, and consider the isomorphism $\mathcal{N} \otimes \mathcal{P} \cong \mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P})$. For the first claim, suppose $\phi \in \mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P})$ is an $\mathscr{R}^-$–nontorsion cycle which is of homogeneous grading $(d, d-2s)$, where $d = d\big(\mathscr{A}_s(\mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P}))\big)$. Note $\phi \in \mathscr{A}_s(\mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P}))$. For each $x^\vee \in \mathcal{G}(N_0^\vee)$, $\phi(x^\vee)$ is an $\mathscr{R}^-$–nontorsion cycle, and hence must contain a summand of the form $f \cdot y$ for some nonzero monomial $f \in \mathscr{R}^-$ and $y \in \mathcal{G}(P_0)$. By the definition of the grading of a morphism, we have

$$\mathrm{gr}_w(y) - \mathrm{gr}_w(x^\vee) + \mathrm{gr}_w(f) = d \quad \text{and} \quad \mathrm{gr}_z(y) - \mathrm{gr}_z(x^\vee) + \mathrm{gr}_z(f) = d - 2s.$$

Since $\mathrm{gr}_w(f) \leq 0$ and $\mathrm{gr}_z(f) \leq 0$, and $(\mathrm{gr}_w(x^\vee), \mathrm{gr}_z(x^\vee)) = (-\mathrm{gr}_w(x), -\mathrm{gr}_z(x))$, for each x,

$$d\big(\mathscr{A}_s(\mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P}))\big) \leq \max_{y \in \mathcal{G}(P_0)} \min(\mathrm{gr}_w(x) + \mathrm{gr}_w(y), \mathrm{gr}_z(x) + \mathrm{gr}_z(y) + 2s).$$

Taking the minimum over $x \in \mathcal{G}(N_0)$ gives the statement.

We now consider the second claim. Suppose that $\mathcal{P} = (P_1 \to P_0)$ is a positive staircase. Using Lemma 4.7 and Proposition 4.16, we know that

$$V_s(\mathcal{N} \otimes \mathcal{P}) = V_s\big(\mathrm{Hom}_{\mathscr{R}^-}(H_0(\mathcal{N}^\vee), H_0(\mathcal{P}))\big).$$

Fix $s \geq 0$. Let δ_s denote the right-hand side of (4.19) without the factor of $-\frac{1}{2}$. For each $x^\vee$ in $\mathcal{G}(N_0^\vee)$, we pick a $y_x \in \mathcal{G}(P_0)$ so that

$$\mathrm{gr}_w(y_x) - \mathrm{gr}_w(x^\vee) \geq d \quad \text{and} \quad \mathrm{gr}_z(y_x) - \mathrm{gr}_z(x^\vee) \geq d - 2s.$$

We set $\phi_0 \colon N_0^\vee \to P_0$ to be the map which takes $x^\vee$ to $f_x \cdot y_x$, where $f_x \in \mathscr{R}^-$ is the unique monomial such that ϕ_0 has bigrading $(d, d-2s)$. By composition, we obtain a map $\phi' \colon N_0^\vee \to H_0(\mathcal{P})$.

Claim *The map ϕ' vanishes on* $\mathrm{im}(N_1^\vee)$.

Given the claim, we quickly conclude the proof. In fact, we obtain a map ϕ from $H_0(\mathcal{N})$ to $H_0(\mathcal{P})$. Hence, we may use the second part of Proposition 4.16 to conclude that

$$d\big(\mathscr{A}_s(\mathrm{Hom}(\mathcal{N}^\vee, \mathcal{P}))\big) \geq \delta_s,$$

which completes the proof modulo the claim.

It remains to prove the claim. Let $y_1 \in N_1^\vee$. We consider the element $v = \partial(y_1) \in N_0^\vee$. We can write v as a sum $\sum_{x^\vee \in \mathcal{G}(N_0^\vee)} f_x \cdot x^\vee$, where each f_x is a monomial. Tensoring the maps from the second part of Lemma 4.2, we obtain a chain map from $\mathcal{N}^\vee$ to $\mathscr{R}^-$, which is nonzero only on $N_0^\vee$ and, furthermore, maps each generator of $N_0^\vee$ to a monomial. Using the fact that this map is a chain map, we see that the number of $x^\vee \in \mathcal{G}(N_0^\vee)$ where f_x is nonzero is even. It follows immediately that $\phi_0(v)$ is an $\mathscr{R}^-$–torsion cycle. By Lemma 4.2, $H_*(\mathcal{P})$ is torsion-free, so $[\phi_0(v)] = 0 \in H_*(\mathcal{P}) = P_0/\mathrm{im}(P_1)$. This proves the claim and completes the proof of Proposition 4.18. □

4.4 A counterexample

We give an example indicating that the second statement of Proposition 4.18 need not hold if $\mathcal{P}$ is a product of more than one positive staircase, even if $\mathcal{P}$ is exact.

Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be the staircases of torus knots $T_{6,7}$ and $T_{4,5}$, respectively. As described in Section 4.2, the generators of $\mathcal{P}^1$ are at bigradings $(-30,0)$, $(-30,-2)$, $(-20,-2)$, $(-20,-6)$, $(-12,-6)$, $(-12,-12)$, $(-6,-12)$, $(-6,-20)$, $(-2,-20)$, $(-2,-30)$ and $(0,-30)$. We denote these generators by $x_0, y_1, \ldots, x_{10}$. We have $\partial x_{2i} = 0$ and $\partial y_{2i+1} = \mathscr{U}^{\alpha_i} x_{2i+2} + \mathscr{V}^{\beta_i} x_{2i}$, where α_i and β_i are nonnegative integers determined by the condition that ∂ preserve the grading. In particular, the generators with odd index generate $\mathcal{P}_1^1$, while the generators with even index span $\mathcal{P}_0^1$.

Likewise, there are generators $x_0', y_1', \ldots, x_6'$ for $\mathcal{P}^2$ with bigradings $(-12,0)$, $(-12,-2)$, $(-6,-2)$, $(-6,-6)$, $(-2,-6)$, $(-2,-12)$ and $(0,-12)$.

Lemma 4.20 *Let $\mathcal{P} = \mathcal{P}^1 \otimes \mathcal{P}^2$. The only elements x in $\mathcal{P}$ such that $\mathrm{gr}_w(x) = \mathrm{gr}_z(x) > -18$ are linear combinations of $\mathscr{U}^i \mathscr{V}^j x_4 \otimes x_4'$ with $(i,j) = (0,1), (1,2)$ and $\mathscr{U}^{i'} \mathscr{V}^{j'} x_6 \otimes x_2'$ with $(i',j') = (1,0), (2,1)$.*

Proof This is by direct inspection. □

Now let $\mathcal{N}$ be the negative staircase complex of the mirror of the trefoil. It is generated by elements c_0, c_1 and c_2 at bigradings $(2,0)$, $(2,2)$ and $(0,2)$, respectively. The differential is given by $\partial c_0 = \mathscr{V} c_1$, $\partial c_2 = \mathscr{U} c_1$ and $\partial c_1 = 0$. That is, $c_0, c_2 \in \mathcal{N}_0$ and $c_1 \in \mathcal{N}_{-1}$.

Lemma 4.21 *There is no cycle $z \in \mathscr{A}_0(\mathcal{N} \otimes \mathcal{P})$ such that $\mathrm{gr}_w(z) \geq -12$ and $z \neq 0$.*

Proof Any such cycle would be a linear combination of elements of type $\mathscr{U}^i \mathscr{V}^j \cdot x_k \otimes x'_\ell \otimes c_m$. By Lemma 4.20, unless $(k,\ell) = (4,4)$ or $(6,2)$, the gr_w–grading of such a combination is at most -14. Hence, if $z \in \mathscr{A}_0(\mathcal{N} \otimes \mathcal{P})$ and $z \neq 0$ has $\mathrm{gr}_w(z) \geq -12$, then z has to be a linear combination of

$$x_4 \otimes_4 \otimes c_0 \quad \text{and} \quad x_6 \otimes x'_2 \otimes c_2.$$

But then z is not a cycle. □

Corollary 4.22 *We have $V_0(\mathcal{N} \otimes \mathcal{P}) \geq 7$.*

The following result shows that the right-hand side of (4.19) is strictly smaller than 7:

Lemma 4.23 *The expression*

$$-\tfrac{1}{2} \min_{x \in G(\mathcal{N}_0)} \max_{y \in G(\mathcal{P}_0)} \min(\mathrm{gr}_w(x) + \mathrm{gr}_w(y), \mathrm{gr}_z(x) + \mathrm{gr}_z(y))$$

is equal to 6.

Proof For $x = c_0$, the expression

$$\max_{y \in G(\mathcal{P}_0)} \min(\mathrm{gr}_w(x) + \mathrm{gr}_w(y), \mathrm{gr}_z(x) + \mathrm{gr}_z(y)) \tag{4.24}$$

is equal to -12 with the equality attained at $y = x_4 \otimes x'_4$. For $x = c_2$, (4.24) attains its maximal value -12 for $y = x_6 \otimes x'_2$. □

4.5 More on the V_s–invariants of tensor products of staircases

In this subsection, we highlight some special cases of Propositions 4.15 and 4.18 which will be useful for our purposes.

Corollary 4.25 *Suppose $\mathcal{P}$ is a positive multistaircase and, for $i \in \{1, \dots, r\}$, let $\mathcal{S}^{n_i}$ denote the staircase complex of Definition 2.28 with $\sum n_i \geq 0$. Then*

$$V_s(\mathcal{P} \otimes \mathcal{S}^{n_1} \otimes \cdots \otimes \mathcal{S}^{n_r}) = \min_{0 \leq j \leq \sum n_i} (V_{s+2j-\sum n_i}(\mathcal{P}) + j).$$

Proof By Proposition 3.9, we know that $\mathcal{S}^{n_1} \otimes \cdots \otimes \mathcal{S}^{n_r}$ is locally equivalent to $\mathcal{S}^n$, where $n = \sum n_i$, so, by Proposition 3.10, it suffices to prove the result when $i = 1$. Write $a_1, \dots, a_m$ for the generators of C_0, and write $x_0, x_2, \dots, x_{2n}$ for the generators of $\mathcal{S}^n_0$. Then $a_i \otimes x_{2j}$ forms a basis of homogeneously graded elements of $(\mathcal{P} \otimes \mathcal{S}^n)_0$. By Proposition 4.16,

$$V_s(\mathcal{P} \otimes \mathcal{S}^n) = \min_{\substack{1\le i\le m\\ 0\le j\le n}} \max(\alpha(a_i)+\alpha(x_{2j}), \beta(a_i)+\beta(x_{2j})-s).$$

We note that $\alpha(x_{2j}) = j$ and $\beta(x_{2j}) = n-j$, so we conclude that

$$\begin{aligned} V_s(\mathcal{P} \otimes \mathcal{S}^n) &= \min_{\substack{1\le i\le m\\ 0\le j\le n}} \max(\alpha(a_i)+j, \beta(a_i)+n-j-s) \\ &= \min_{0\le j\le n} \min_{1\le i\le m} \bigl(\max(\alpha(a_i), \beta(a_i)+n-2j-s)+j\bigr) \\ &= \min_{0\le j\le n} (V_{s+2j-n}(\mathcal{P})+j). \end{aligned}$$ □

We have the following corollary of Proposition 4.18:

Corollary 4.26 *Suppose $\mathcal{P}$ is a positive staircase and, for $i \in \{1, \dots, r\}$, let $\mathcal{S}^{n_i}$ denote the staircase complexes of Definition 2.28. Assume $\sum n_i < 0$. Then*

$$V_s(\mathcal{P} \otimes \mathcal{S}^{n_1} \otimes \cdots \otimes \mathcal{S}^{n_r}) = \max_{0\le j\le n} (V_{s-2j+n}(\mathcal{P}) - j),$$

where $n = -\sum n_i$.

Remark 4.27 In contrast to Corollary 4.25, where $\mathcal{P}$ was allowed to be a positive *multi*staircase (ie a tensor product of positive staircases), in Corollary 4.26 we require that $\mathcal{P}$ be a positive *staircase*.

Proof of Corollary 4.26 As in the proof of Corollary 4.25, $\mathcal{S}^{n_1} \otimes \cdots \otimes \mathcal{S}^{n_r}$ is locally equivalent to $\mathcal{S}^{-n}$ for some $n > 0$, so it is sufficient to consider the case when $i = 1$. Write $a_1, \dots, a_q$ for the generators of C_0, and $\tilde{x}_0, \tilde{x}_2, \dots, \tilde{x}_{2n}$ for the generators of the 0–level of $\mathcal{S}^{-n}$. According to Proposition 4.18,

$$\begin{aligned} V_s(\mathcal{P} \otimes \mathcal{S}^{-n}) &= \max_{0\le i\le n} \min_{1\le j\le q} \max(\alpha(a_j)+\alpha(\tilde{x}_{2i}), \beta(a_j)+\beta(\tilde{x}_{2i})-s) \\ &= \max_{0\le i\le n} \min_{1\le j\le q} \max(\alpha(a_j)-i, \beta(a_j)-n+i-s) \\ &= \max_{0\le i\le n} \min_{1\le j\le q} \bigl(\max(\alpha(a_j), \beta(a_j)-n+2i-s)-i\bigr) \\ &= \max_{0\le i\le n} (V_{s-2i+n}(\mathcal{P})-i). \end{aligned} \tag{4.28}$$ □

4.6 Knots with split towers

We now introduce the notion of a knot complex with *split towers*. The correction terms of a knot complex with split towers have a relatively simple form. An important example of a knot with split towers is connected sums of knotifications of positive and negative $(2, 2n)$–torus links.

Definition 4.29 (split towers) Let K be a knot in $Y = \#^m S^2 \times S^1$, and let $\mathcal{C}$ be a chain complex which is free and finitely generated over $\mathscr{R}^-$ and is homotopy equivalent to $\mathcal{CFK}^-(Y, K, \mathfrak{s}_0)$, where $\mathfrak{s}_0$ is the trivial Spin^c structure on Y. We say that $\mathcal{C}$ has *split towers* if there exists a basis $\gamma_1, \dots, \gamma_m$ of $H_1(\#^m S^2 \times S^1; \mathbb{Z})$ and subcomplexes $\mathcal{C}^I_* \subset \mathcal{C}$, indexed over subsets $I \subset \{\gamma_1, \dots, \gamma_m\}$, such that the following are satisfied:

(a) $\mathcal{C} = \bigoplus_{I \subset \{\gamma_1, \dots, \gamma_m\}} \mathcal{C}^I$.

(b) If $\gamma_i \notin I$, then A_{γ_i} takes $H_*(\mathcal{C}^I)$ to $H_*(\mathcal{C}^{I \cup \{\gamma_i\}})$, and becomes an isomorphism after inverting $\mathscr{U}$ and $\mathscr{V}$. If $\gamma_i \in I$, then A_{γ_i} vanishes on $H_*(\mathcal{C}^I)$, after inverting $\mathscr{U}$ and $\mathscr{V}$.

Abusing notation slightly, we say a knot K has split towers if there is a representative of $\mathcal{CFK}^-(Y, K)$ which has split towers. Note that, in many of our examples, the homology action actually respects the splitting on the chain level, ie A_{γ_i} maps $\mathcal{C}^I$ to $\mathcal{C}^{I \cup \{\gamma_i\}}$ if $\gamma_i \notin I$, and A_{γ_i} vanishes on $\mathcal{C}^I$ if $\gamma_i \in I$.

Example 4.30

- Any knot K in S^3 has split towers (trivially).
- The knotification of the $(2, 2n)$–torus link has split towers. See Proposition 2.40.
- The Borromean knot does not have split towers.

Lemma 4.31 *If K and K' have split towers, then $K \# K'$ has split towers.*

Proof This is a direct consequence of the Künneth formula. □

Proposition 4.32 *Suppose K is a knot in $\#^m S^2 \times S^1$ with split towers. Write*

$$\mathcal{C}^{\text{top}} = \mathcal{C}^{\varnothing} \quad \text{and} \quad \mathcal{C}^{\text{bot}} = \mathcal{C}^{\gamma_1, \dots, \gamma_m}.$$

Then

$$V_s^{\text{top}}(K) = V_s(\mathcal{C}^{\text{top}}) \quad \text{and} \quad V_s^{\text{bot}}(K) = V_s(\mathcal{C}^{\text{bot}}).$$

Suppose, additionally, that $n > 0$ and $\mathcal{B}_0$ is the Borromean knot. Then

$$V_s^{\text{top}}(K \# \#^n \mathcal{B}_0) = -\tfrac{1}{2}n + \min_{0 \le j \le n} (V_{s+2j-n}(\mathcal{C}^{\text{top}}) + j),$$
$$V_s^{\text{bot}}(K \# \#^n \mathcal{B}_0) = -\tfrac{1}{2}n + \max_{0 \le j \le n} (V_{s+2j-n}(\mathcal{C}^{\text{bot}}) + j).$$

Proof We consider first the proof that $V_s^{\text{top}}(K) = V_s(\mathcal{C}^{\text{top}})$. It is sufficient to show that

$$d_s^{\text{top}}(K) = d(\mathcal{C}_s^{\text{top}}), \tag{4.33}$$

where $\mathcal{C}_s^{\text{top}}$ denotes the subcomplex of $\mathcal{C}^{\text{top}}$ in Alexander grading s, and these d–invariants are defined in Definitions 3.3 and 3.11. By definition, $d_s^{\text{top}}(K)$ is the maximal grading of a homogeneously graded element of $H_*(\mathscr{A}_s(K))$ which maps to an element of $U^{-1} H_*(\mathscr{A}_s(K))$ having nontrivial image in $\mathcal{H}^{\text{top}}$.

Since K has split towers, by Definition 4.29 the cokernel $\mathcal{H}^{\mathrm{top}}$ is spanned by $U^{-1}H_*(\mathcal{C}_s^{\mathrm{top}})$, and $H_*(\mathcal{C}_s^I)$ has trivial image for $I \neq \varnothing$, equation (4.33) follows.

The claim about d^{bot} is similar. In this case, $d_s^{\mathrm{bot}}(K)$ is defined as the maximal grading of a homogeneous element in $H_*(\mathscr{A}_s(K))/\mathrm{Tors}$ which is in the image of $\mathcal{H}^{\mathrm{bot}}$. This is clearly $d(\mathcal{C}_s^{\mathrm{bot}})$.

We pass now to the second part of the proof. An analogous argument appeared in [Bodnár et al. 2016; Borodzik et al. 2017]; we recall it for completeness. The complex $\mathcal{CFK}^-(\mathcal{B}_0)$ is described in Section 2.7. Since $\mathcal{CFK}^-(\mathcal{B}_0)$ has vanishing differential, we obtain

$$H_*(\mathcal{CFK}^-(K) \otimes \mathcal{CFK}^-(\mathcal{B}_0)^{\otimes n}) \cong \mathcal{HFK}^-(K) \otimes_{\mathbb{F}} \mathbb{B}^{\otimes n},$$

where $\mathbb{B}$ is the 4–dimensional vector space spanned by 1, x, y and xy, whose bigradings are shown in (2.42).

We first consider the claim about V_s^{bot}. Using the H_1–action on $\mathcal{CFK}^-(\mathcal{B}_0)$ described in Section 2.7, one easily obtains the following: a cycle $x \in \mathscr{A}_s\big(K \# \#^n \mathcal{B}_0\big)$ is of homogeneous gr_w–grading d, is $\mathbb{F}[U]$–nontorsion, and maps to the kernel of the H_1–action in $U^{-1}H_*(\mathscr{A}_s(K \# B^{\# n}))$ if and only if it has the form

$$\sum_{\{a_1,\dots,a_n\}\in\{-1,1\}^n} x_{a_1,\dots,a_n} \otimes \epsilon_{a_1} \otimes \cdots \otimes \epsilon_{a_n},$$

where $\epsilon_{-1} = 1 \in \mathbb{B}$ and $\epsilon_1 = xy \in \mathbb{B}$ with $\mathrm{gr}_w = 1$ and -1, respectively. Moreover, each

$$x_{a_1,\dots,a_n} \in \mathcal{C}^{\mathrm{bot}}_{s+\sum a_i}(K)$$

is an $\mathbb{F}[U]$–nontorsion cycle of homogeneous gr_w–grading $d + \sum a_i$. Noting that $\sum a_i$ can be any integer of the form $n - 2j$ for $0 \le j \le n$, we obtain that

$$d^{\mathrm{bot}}\big(\mathscr{A}_s\big(K \# \#^n \mathcal{B}_0\big)\big) = \min_{0 \le j \le n} (d(\mathcal{C}^{\mathrm{bot}}_{s+n-2j}) - n + 2j).$$

Multiplying by $-\frac{1}{2}$ and switching j to $n - j$ yields the statement.

The proof for d^{top} is analogous. The cokernel of the H_1–action on $U^{-1}H_*\big(\mathscr{A}_s\big(K \# \#^n \mathcal{B}_0\big)\big)$ is spanned by any element of the form $x \otimes \epsilon_{a_1} \otimes \cdots \otimes \epsilon_{a_n}$ where ϵ_{a_i} are as above and $x \in \mathcal{C}^{\mathrm{top}}_{s+\sum a_i}(K)$ is a homogeneously graded, $\mathbb{F}[U]$–nontorsion element. Furthermore, any homogeneous element generating $U^{-1}H_*\big(\mathscr{A}_s\big(K \# \#^n \mathcal{B}_0\big)\big)$ is a sum of an odd number of such elements. The same argument as before shows that

$$d^{\mathrm{top}}\big(\mathscr{A}_s\big(K \# \#^n \mathcal{B}_0\big)\big) = \max_{0 \le j \le n} (d(\mathcal{C}^{\mathrm{top}}_{s+n-2j}) - n + 2j).$$

Multiplying by $-\frac{1}{2}$ and switching j to $n - j$ yields the statement. □

5 Topology of complex curves and their neighborhoods

In this section we give a precise definition of the notion of a tubular neighborhood of a possibly singular curve in $\mathbb{C}P^2$. We describe the boundary of this neighborhood in terms of the surgery on a link. We perform several helpful algebrotopological computations.

As the main focus of our article is on algebraic curves, we present the construction using the language of complex geometry. In Section 5.4 we will show how to generalize our results to the smooth category.

5.1 "Tubular" neighborhood of a complex curve

Let $C \subset \mathbb{C}P^2$ be a reduced complex curve of degree d. We do not insist that C be irreducible. We write $C_1, \dots, C_e$ for the irreducible components of C and let $d_1, \dots, d_e$ (resp. $g_1, \dots, g_e$) denote their degrees (resp. genera). Hereafter, by the *genus* $g(C)$ of a complex curve, we mean the genus of its normalization, that is, the geometric genus. From the topological perspective, the geometric genus of a singular curve is the sum of genera of connected components of the smooth locus of the curve, regarded as an open surface. We set $g = g_1 + \dots + g_e$.

We denote by $p_1, \dots, p_u$ the singular points of C. For each such singular point p_i, we denote by r_i the number of branches. Here, recall that a branch of C at p_i is a connected component of $B_i \cap (C \setminus \{p_i\})$ for a sufficiently small ball $B_i \subset \mathbb{C}^2$ centered at p_i. We write $\mathcal{L}_i$ for the link of singularity at p_i, whose components are $L_{i1}, \dots, L_{ir_i}$. We choose once and for all pairwise disjoint closed balls $B_1, \dots, B_u$ with centers $p_1, \dots, p_u$, respectively, and such that $C \cap \partial B_i$ is the link $\mathcal{L}_i$ and $C \cap B_i$ is homeomorphic to a cone over $\mathcal{L}_i$.

As the curve C is not smoothly embedded at its singular points, the notion of a tubular neighborhood of C requires some clarification. The following is an extension of the construction of [Borodzik and Livingston 2014].

Take a tubular neighborhood N_0 in $\mathbb{C}P^2 \setminus (B_1 \cup \dots \cup B_u)$ of the smooth part $C_0 := C \setminus (B_1 \cup \dots \cup B_u)$. Note that all components $C_1, \dots, C_e$ intersect each other; hence, C is connected. On the other hand, the balls $B_1, \dots, B_u$ contain all the intersection points between various curves $C_1, \dots, C_e$. Hence, C_0 has e connected components, which are $C_i \setminus (B_1 \cup \dots \cup B_u)$ for $i = 1, \dots, e$. We define N to be the union of N_0 and $B_1, \dots, B_u$. With $g = g_1 + \dots + g_e$, set

$$\rho = 2g - e + 1 + \sum_{i=1}^{u} (r_i - 1) = b_1(C) = \dim H_1(C; \mathbb{Q}). \tag{5.1}$$

To see that $\dim H_1(C; \mathbb{Q}) = \rho$, we consider the normalization C' of C. It is a surface of genus g with e connected components. So $\chi(C') = 2e - 2g$. Next, C arises from C' by gluing r_i–tuples of points (corresponding to singular points of C) for $i = 1, \dots, u$. Hence $\chi(C) = 2e - 2g - \sum (r_i - 1)$. Now C is connected, and $\dim H_2(C; \mathbb{Q}) = e$. From this, we recover the formula for $\dim H_1(C; \mathbb{Q})$.

Observe that C_0 arises from the normalization C' by removing $\sum r_i$ disks. The first disk for each connected component of C' kills an element in H_2, and all of the subsequent disks create a basis element in H_1. That is to say, $\dim H_1(C_0;\mathbb{Q}) = 2g+\sum r_i - e = \rho+u-1$. By duality, $\dim H_1(C_0,\partial C_0;\mathbb{Q}) = \rho+u-1$.

We now provide a surgery-theoretical description of N and its boundary Y. We first define a 3–manifold Z containing a link $\mathcal{L}$, as follows. We begin with the disjoint union $\mathcal{L}_0 := \mathcal{L}_1 \sqcup \cdots \sqcup \mathcal{L}_u$ in $Z_0 := S^3 \sqcup \cdots \sqcup S^3$. Next, we pick a collection of pairwise disjoint and properly embedded arcs $\lambda_1, \dots, \lambda_{\rho+u-1}$ on C_0 which form a basis of $H_1(C_0, \partial C_0)$. Such a collection of arcs cuts C_0 into a union of e disks, one for every connected component of C_0. We let $Z = \#^\rho\, S^2 \times S^1$ be the boundary of the 4–manifold Γ obtained by attaching $\rho+u-1$ 4–dimensional 1–handles to $\partial(B_1 \cup \cdots \cup B_u) = Z_0$, each containing a 2–dimensional band (corresponding to a λ_i), which we attach to $\mathcal{L}_0$. We let $\mathcal{L} \subset Z$ be the resulting link. By construction, $\mathcal{L}$ is a link inside of the connected sum of ρ copies of $S^1 \times S^2$. Furthermore, each component of $\mathcal{L}$ is null-homologous. The number of components of $\mathcal{L}$ is the number of disks $C_0 \setminus (\lambda_1 \cup \cdots \cup \lambda_{\rho+u-1})$. That is, $\mathcal{L}$ has e components, denoted henceforth by $L_1, \dots, L_e$, corresponding to connected components of C_0, ie to irreducible components of the complex curve C.

We have the following (compare [Borodzik et al. 2017, Theorem 3.1; Bodnár et al. 2016, Lemma 3.1]):

Proposition 5.2 *The 3–manifold $Y = \partial N$ is the surgery on $\mathcal{L} \subset Z$ with surgery coefficients $(d_1^2, \dots, d_e^2)$. The 4–manifold N is obtained by attaching e 2–handles to the boundary connected sum of ρ copies of $D^3 \times S^1$.*

Proof The fact that N is obtained by attaching e 2–handles to Γ along $\mathcal{L}$ follows from the fact that the complement $C_0 \setminus (\lambda_1, \dots, \lambda_{\rho+u-1})$ is a collection of disks $C_1', \dots, C_e'$ (we know that this complement has e components). The thickening of C_i' is a 2–handle in N. Upon renumbering, we may and will assume that C_i' is a subset of C_i and $\partial C_i' = L_i$, the component of $\mathcal{L}$. In particular, we know that N is the effect of a surgery on $\mathcal{L}$. It remains to determine the framing.

In order to do this, we recall that, if a 2–handle A is attached to B^4 along a knot $K \subset S^3 = \partial B^4$, the framing of the 2–handle is determined as a self-intersection number of the surface F obtained by capping the core C of the 2–handle with a Seifert surface for K. We note that the self-intersection number does not depend on the choice of the Seifert surface. Moreover, instead of a Seifert surface, we can take any smooth compact surface in B^4 whose boundary is K.

The same procedure applies for surgeries on null-homologous knots in $\#^\rho\, S^2 \times S^1$. In the present context, when we calculate the surgery coefficient at L_i, the role of the surface F is played by the union of C_i' and a surface in $\Gamma = \#^\rho\, B^3 \times S^1$ bounding L_i. A possible choice for F is then a smoothing of C_i, which essentially replaces $C_i \cap \Gamma$ by a smooth compact surface in Γ with boundary L_i. That is to say, the self-intersection number of F is exactly the self-intersection number of C_i, which is d_i^2. □

Remark 5.3 If $e = 1$, $\mathcal{L}$ is a knot. This knot can be obtained as a connected sum of $\widehat{\mathcal{L}}_1, \dots, \widehat{\mathcal{L}}_u$ and g copies of the Borromean knot. Here the hat denotes knotification.

5.2 Algebraic topology

In this section, we describe some basic algebrotopological facts about the tubular neighborhood N, and its boundary Y. Our description of Spin^c structures is similar to that in [Manolescu and Ozsváth 2010, Section 11.1].

Recall that, if N is a manifold obtained by gluing e handles along a null-homologous link to a four-manifold Γ with $H_2(\Gamma;\mathbb{Z})=0$, we can speak not only of a framing of handles, but of a framing matrix. An argument using the Mayer–Vietoris sequence reveals that $H_2(N;\mathbb{Z})=\mathbb{Z}^e$ is generated by the cores of the handles capped by Seifert surfaces of the components of the link. The *framing matrix*, denoted by Ξ, is the matrix of the intersection form $H_2(N;\mathbb{Z})\times H_2(N;\mathbb{Z})\to\mathbb{Z}$. In particular, the diagonal entries are surgery coefficients. The off-diagonal terms are linking numbers of the corresponding links (these are well defined as long as the components are null-homologous).

In the present situation, by Proposition 5.2, the surgery coefficients are $(d_1^2,\dots,d_e^2)$. The same argument shows that the off-diagonal terms are given by the intersection number of C_i with C_j. That is, the framing matrix has the form

$$\Xi=\{d_id_j\}_{i,j=1}^e.$$

Note that this construction in particular reveals that $\mathrm{lk}(L_i,L_j)=d_id_j$. We let $W_\Lambda(\mathcal{L})$ denote the 2–handle cobordism from Z to Y. Recall that N is the union of the 1–handlebody Γ and $W_\Lambda(\mathcal{L})$.

There is a map

$$\mathcal{F}\colon H^2(W_\Lambda(\mathcal{L}))\to\mathbb{Z}^e\oplus H^2(Z), \tag{5.4}$$

given by

$$\mathcal{F}(c)=(\langle c,[\widehat{F}_1]\rangle,\dots,\langle c,[\widehat{F}_e]\rangle,c|_Z).$$

Here $\widehat{F}_i$ is the surface obtained by capping a Seifert surface for L_i in Z with the core of the 2–handle. An easy argument involving the Mayer–Vietoris sequence on the handle attachment regions in Z shows that $\mathcal{F}$ is an isomorphism.

Dually, we may view $W_\Lambda(\mathcal{L})$ as being obtained by attaching 2–handles to a link $\mathcal{L}^*$ in Y. We consider the Mayer–Vietoris sequence obtained by viewing W_Λ as the union of $[0,1]\times Y$ and e 2–handles. A portion of this exact sequence reads

$$H^1(\mathcal{L}^*)\to H^2(W_\Lambda(Y))\to H^2(Y)\to 0.$$

In particular, $H^2(Y)$ is the quotient of $H^2(W_\Lambda(Y))$ by the image of $H^1(\mathcal{L}^*)$. Furthermore, from the definition of the coboundary map in the Mayer–Vietoris exact sequence, an element of $H^1(\mathcal{L}^*)$ acts by the Poincaré duals of the cores of the 2–handles attached along $\mathcal{L}$. Using the isomorphism $\mathcal{F}$ from (5.4), we thus obtain

$$H^2(Y)\cong(\mathbb{Z}^e/\mathrm{im}(\Xi))\oplus H^2(Z). \tag{5.5}$$

There are analogous descriptions for Spin^c structures on Y and $W_\Lambda(\mathcal{L})$, as follows. Consider the map

(5.6) $$\mathcal{T}_W : \mathrm{Spin}^c(W_\Lambda(\mathcal{L})) \hookrightarrow \mathbb{Q}^e \times \mathrm{Spin}^c(Z),$$

given by

$$\mathcal{T}_W(\mathfrak{s}) = \bigl(\tfrac{1}{2}(\langle c_1(\mathfrak{s}), [\widehat{F}_1]\rangle - [\widehat{F}]\cdot[\widehat{F}_1]), \dots, \tfrac{1}{2}(\langle c_1(\mathfrak{s}), [\widehat{F}_e]\rangle - [\widehat{F}]\cdot[\widehat{F}_e]), \mathfrak{s}|_Z\bigr),$$

where $[\widehat{F}]$ is the sum of the $[\widehat{F}_i]$. Similar to the argument for cohomology, an easy application of Mayer–Vietoris shows that $\mathcal{T}_W$ is an isomorphism onto its image. Since $c_1(\mathfrak{s})$ is a characteristic vector, $\langle c_1(\mathfrak{s}), [\widehat{F}_i]\rangle - [\widehat{F}_i]^2$ is even as well. Using this, it is not hard to identify the image of $\mathcal{T}_W$ as $\mathbb{H}(\mathcal{L}) \times \mathrm{Spin}^c(Z)$, where $\mathbb{H}(\mathcal{L})$ is the affine lattice in $\mathbb{Q}^e$ generated by tuples $(a_1, \dots, a_e)$ where

$$a_i - \tfrac{1}{2}\,\mathrm{lk}(\mathcal{L}_i, \mathcal{L} \setminus \mathcal{L}_i) \in \mathbb{Z} \quad \text{for all } i.$$

The linking number is computed as

(5.7) $$\mathrm{lk}(\mathcal{L}_i, \mathcal{L} \setminus \mathcal{L}_i) = d_i(d_1 + d_2 + \cdots + d_e) - d_i^2.$$

A similar argument as for cohomology implies $\mathrm{Spin}^c(Y)$ is isomorphic to the quotient of $\mathrm{Spin}^c(W_\Lambda(\mathcal{L}))$ by the action of the Poincaré duals of the cores of the 2–handles attached to $\mathcal{L}$. This translates into the isomorphism

(5.8) $$\mathcal{T}_Y : \mathrm{Spin}^c(Y) \cong (\mathbb{H}(\mathcal{L})/\mathrm{im}(\Xi)) \times \mathrm{Spin}^c(Z).$$

With respect to the isomorphisms $\mathcal{F}$ and $\mathcal{T}_W$, the Chern class map takes the simple form

$$c_1(s_1, \dots, s_e, \mathfrak{t}) = (2s_1 + [\widehat{F}]\cdot[\widehat{F}_1], \dots, 2s_e + [\widehat{F}]\cdot[\widehat{F}_e], c_1(\mathfrak{t})).$$

Since $Z = \#^\rho\, S^2 \times S^1$ bounds the 1–handlebody $\Gamma \subset N$, we know that $\delta(H^1(Z)) = \{0\} \subset H^2(N)$. Hence, a Mayer–Vietoris argument identifies $\mathrm{Spin}^c(N)$ with the set of Spin^c structures on $W_\Lambda(\mathcal{L})$ which extend over Γ, or equivalently the ones which have torsion restriction to Z. Hence,

$$\mathrm{Spin}^c(N) \cong \mathbb{H}(\mathcal{L}).$$

The following is helpful for understanding $H^2(Y)$:

Lemma 5.9 *Suppose $\Xi = \{a_{ij}\}_{i,j=1}^e$ is a matrix such that $a_{ij} = d_i d_j$, for some nonzero integers d_i. Then $\mathbb{Z}^e/\mathrm{im}(\Xi) \cong \mathbb{Z}^{e-1} \oplus \mathbb{Z}/\theta^2$, where $\theta = \gcd(d_1, \dots, d_e)$.*

Proof Recall that

$$\Xi = \begin{pmatrix} d_1d_1 & d_1d_2 & \cdots & d_1d_e \\ d_2d_1 & d_2d_2 & \cdots & d_2d_e \\ \vdots & \vdots & \ddots & \vdots \\ d_ed_1 & d_ed_2 & \cdots & d_ed_e \end{pmatrix}.$$

It is clear that $\operatorname{im}(\Xi)$ is the span of $\theta(d_1,\dots,d_e)^T$, by considering the image of the standard basis in $\mathbb{R}^n$. By module theory over a principal ideal domain, $\mathbb{Z}^e/\operatorname{im}(\Xi)\cong\mathbb{Z}^{e-1}\oplus\operatorname{Tors}(\mathbb{Z}^e/\operatorname{im}(\Xi))$. By definition, $\operatorname{Tors}(\mathbb{Z}^e/\operatorname{im}(\Xi))$ is generated by the set of vectors v in $\mathbb{Z}^e$ such that $n[v]=m[\theta(d_1,\dots,d_e)^T]$ for some integers n and m. Clearly, $\operatorname{Tors}(\mathbb{Z}^e/\operatorname{im}(\Xi))$ is generated by the vector $(d_1/\theta,\dots,d_e/\theta)^T$, which has order θ^2. □

Combining Lemma 5.9 with (5.5), we conclude that

(5.10) $$b_1(Y)=e-1+b_1(Z)=e-1+\rho.$$

If $j\in 2\mathbb{Z}+1$, let $\mathfrak{c}_j$ denote the Spin^c structure on $\mathbb{C}P^2$ which satisfies

(5.11) $$\langle c_1(\mathfrak{c}_j),E\rangle=j,$$

where E is a complex line. In terms of the isomorphism in (5.8),

(5.12) $$\mathcal{T}_Y(\mathfrak{c}_j|_Y)=\bigl(\tfrac12(jd_1-d_1(d_1+\cdots+d_e)),\dots,\tfrac12(jd_e-d_e(d_1+\cdots+d_e)),0\bigr).$$

We now let X denote the complement of the interior of N in $\mathbb{C}P^2$.

Lemma 5.13 (1) *X has trivial intersection form.*

(2) *Suppose $\mathfrak{s}$ is a torsion* Spin^c *structure on Y. Then $\mathfrak{s}$ extends over X if and only if it extends over $\mathbb{C}P^2$.*

Proof The proof follows arguments identical to those in [Borodzik et al. 2017, Sections 3 and 4]; therefore, we provide only a sketch. Claim (1) follows from the fact that the inclusion map $H_2(X)\to H_2(\mathbb{C}P^2)$ vanishes, since all elements of $H_2(X)$ are disjoint from C.

Claim (2) is proven as follows. A Spin^c structure on Y always extends over $W_\Lambda(\mathcal{L})$. Furthermore, the isomorphisms in (5.6) and (5.8) are clearly compatible with the natural restriction maps from $\operatorname{Spin}^c(W_\Lambda(\mathcal{L}))$ to $\operatorname{Spin}^c(Y)$ and $\operatorname{Spin}^c(Z)$. A Spin^c structure on $W_\Lambda(\mathcal{L})$ extends over N if and only if it restricts to the torsion Spin^c structure on Z. Hence, a Spin^c structure on Y extends over N if and only if the Spin^c factor on $\operatorname{Spin}^c(Z)$ in (5.8) is torsion. In particular, any torsion Spin^c structure on Y extends over N. Since a Spin^c structure on Y extends over $\mathbb{C}P^2$ if and only if it extends over both X and N, the claim follows. □

5.3 d–invariant inequalities for the neighborhood of C

We are now in position to prove an inequality for the d–invariants of boundaries of neighborhoods of complex curves in $\mathbb{C}P^2$ as in Section 5.1. With the notation from that subsection, we have the following result:

Proposition 5.14 *For any* Spin^c *structure $\mathfrak{s}$ on Y that extends over X and whose first Chern class is torsion,*

$$d_{\mathrm{bot}}(Y,\mathfrak{s})\ge-\tfrac12(\rho+e-1),\quad d_{\mathrm{top}}(Y,\mathfrak{s})\le\tfrac12(\rho+e-1).$$

Proof By (5.10), we know that $b_1(Y) = \rho + e - 1$. The intersection form on X is trivial by Lemma 5.13. From Theorem 3.1, we obtain

$$d_{\text{bot}}(Y, \mathfrak{s}) = d(Y, \mathfrak{s}, H_1(Y)/\text{Tors}) \geq -\tfrac{1}{2}(\rho + e - 1),$$

since the terms involving c_1^2 and $b_2^-(X)$ vanish.

Since the intersection form on X vanishes, we may reverse the orientation of X and Y and appeal to the same argument to get that

$$d_{\text{bot}}(-Y, \mathfrak{s}) = d(-Y, \mathfrak{s}, H_1(Y)/\text{Tors}) \geq -\tfrac{1}{2}(\rho + e - 1). \tag{5.15}$$

It follows from [Levine and Ruberman 2014, Proposition 4.2] and the fact that $d^*(Y, \mathfrak{s}, H_1(Y)/\text{Tors}) = d_{\text{top}}(Y, \mathfrak{s})$ (see [loc. cit., page 6]) that

$$d_{\text{bot}}(-Y, \mathfrak{s}) = -d_{\text{top}}(Y, \mathfrak{s}).$$

Combining this with (5.15), we conclude that

$$d_{\text{top}}(Y, \mathfrak{s}) \leq \tfrac{1}{2}(\rho + e - 1).$$ □

5.4 Singular curves in smooth category

The methods we use in this article work in a smooth category. The term "smooth surface with singularities" might be misleading; therefore, we make precise our terminology. The definition we give is quite general.

Definition 5.16 A *singular curve in the smooth category* $C \subset \mathbb{C}P^2$ is a closed subset of $\mathbb{C}P^2$ such that there exist finitely pairwise disjoint closed balls $B_1, \dots, B_u$ in $\mathbb{C}P^2$ such that, with $C_0 = \overline{C \setminus (B_1 \cup \dots \cup B_u)}$,

- C is connected;
- the subset C_0 is a smoothly embedded surface whose boundary belongs to $B_1 \cup \dots \cup B_u$;
- the intersection $B_i \cap C$ is a link (we call it $\mathcal{L}_i$).

The definition means that we do not have to control any possible pathological behavior of C inside balls. We let $C_{01}, \dots, C_{0e}$ be the connected components of C_0. The quantity e plays the same role as the number of irreducible components of an algebraic curve.

Choose $j = 1, \dots, e$. For any $i = 1, \dots, u$ such that $\mathcal{L}_{ij} := B_i \cap C_{0j} \neq \varnothing$, let S_{ij} be a minimal genus surface in B_{ij} whose boundary is $\mathcal{L}_{ij}$. Let $\widetilde{C}_j$ be a closed surface obtained by removing $B_i \cap C_{0j}$, gluing S_{ij} and possibly smoothing corners. The surface $\widetilde{C}_j$ is called a *smooth model* of C_{0j}.

Note that $\widetilde{C}_j$ determines a class in $H_2(\mathbb{C}P^2; \mathbb{Z})$. If S_{ij} and S'_{ij} are two choices of minimal genus surfaces for $\mathcal{L}_{ij}$, then $S_{ij} \cup -S'_{ij}$ is homologically trivial (as a surface in the ball B_{ij}). Hence, the class of $\widetilde{C}_j$ does not depend on the particular choice of S_{ij}. We let d_j be the integer such that $[\widetilde{C}_j] = d_j \cdot 1 \in H_2(\mathbb{C}P^2; \mathbb{Z})$, where we write 1 for the class of a line. We call d_j the *smooth degree* of C_j.

Definition 5.17 A singular curve is the smooth category is called *adjunctive* if, for all $j = 1, \dots, e$, we have $g(\widetilde{C}_j) = \frac{1}{2}(d_j - 1)(d_j - 2)$.

Definition 5.18 Let C be an adjunctive singular curve in the smooth category.

- C is of *algebraic type* if all links $\mathcal{L}_i$ are algebraic links.
- C is of *weakly algebraic type* if all links $\mathcal{L}_i$ are either algebraic links or their mirrors.

Remark 5.19 The distinction between the requirements that $\mathcal{L}_i$ be an algebraic link or an L–space link is motivated by applications in algebraic geometry. In our paper, we never use the fact that the links $\mathcal{L}_i$ are algebraic links, instead of merely L–space links. We note that there are some nontrivial differences between L–space knots and algebraic knots. For example, the set S_K defined in Section 4.2 is not necessarily a semigroup if K is merely an L–space knot. We recall that S_K is used to define the function R_K, which is referred to as the *semigroup counting function*. In our theory, we never need S_K to be a semigroup, so the mathematical part of the theory goes through.

We now define the analogs of ρ, Y and N from Section 5.1 in the case of a singular curve in the smooth category. First set g_j to be the genus of C_{0j} (not of $\widetilde{C}_j$). Set $g=g_1+\cdots+g_e$ and $\rho=2g-e+1+\sum(r_i-1)$, where r_i is the number of components of $\mathcal{L}_i$.

We now repeat the procedure from Section 5.1, omitting the proofs if they are the same as in that subsection. We pick $\lambda_1,\dots,\lambda_{\rho+u-1}$ to be arcs on C_0 which form a basis of $H_1(C_0,\partial C_0;\mathbb{Z})$. We let Γ be the 4–manifold obtained by attaching $\rho+u-1$ 4–dimensional 1–handles to $\partial(B_1\cup\cdots\cup B_u)$ as in Section 5.1. We set $Z=\partial\Gamma$; then $Z=\#^{\rho}\, S^2\times S^1$. Finally, $\mathcal{L}=C\cap Z$. This is an e–component link. The set $C\setminus\Gamma$ is a disjoint union of e disks $C'_{01},\dots,C'_{0e}$. Reindexing these disks if necessary, we may and will assume that C'_{0i} is a subset of C_{0i}. Let N be the handlebody Γ with attached 2–handles whose cores are $C'_{01},\dots,C'_{0e}$. The manifold $Y=\partial N$ is the surgery on $\mathcal{L}$ with framings equal to $d_1^2,\dots,d_e^2$.

With these definitions, the results of Sections 5.2 and 5.3 hold for singular curves in smooth category.

6 Nonrational noncuspidal complex curves

6.1 General estimates

We now pass to the main applications of our paper. Suppose $C\subset\mathbb{C}P^2$ is a degree d curve. We mostly focus on the case where C is complex curve, but also consider the case where C is only a smooth surface, embedded away from a finite set of singular points, as in Definition 5.16. We further assume that the singularities of C are restricted to the following:

- There are ν cuspidal (unibranched) singular points $p_1,\dots,p_\nu$. We write $K_1,\dots,K_\nu$ for their links, and set $K=K_1\#\cdots\# K_\nu$.
- There are m_n singular points whose link is $T_{2,2n}$.
- There are $\underline{m}_n$ singular points whose link is $-T_{2,2n}$.

Define

$$\kappa_+ = \sum_n n m_n, \quad \kappa_- = \sum_n n \underline{m}_n, \quad \eta_+ = \sum_n m_n, \quad \eta_- = \sum_n \underline{m}_n.$$

Additionally, we assume that the curve is adjunctive (see Definition 5.17); that is, its genus g is given by

$$g = g(C) = \tfrac{1}{2}(d-1)(d-2) - g_3(K) - (\kappa_+ + \kappa_-). \tag{6.1}$$

For algebraic curves, $\kappa_- = 0$ and (6.1) is the adjunction formula. If C is a singular curve in the smooth category of algebraic type (ie $\kappa_- = 0$; see Definition 5.18), the adjunction inequality implies that $g(C)$ is greater than or equal to the right-hand side of (6.1). If C is of weak algebraic type (see Definition 5.18), the relation between $g(C)$ and the right-hand side of (6.1) can be more involved, so the condition (6.1) is a significant restriction on $g(C)$.

We define

$$K_+ = K \# \#^n m_n \widehat{T}_{2,2n}, \quad K_- = \#^n \underline{m}_n \widehat{T}_{2,-2n}, \quad \widetilde{K} = K_+ \# K_-, \quad \widehat{K} = \widetilde{K} \# \#^g \mathcal{B}_0, \tag{6.2}$$

where $\widehat{T}_{2,2n}$ denotes the knotification of the torus link $T_{2,2n}$ and $\widehat{T}_{2,-2n}$ denotes the knotification of its mirror.

Since the knots $K_1, \dots, K_\nu$ are algebraic knots and so, in particular, L–space knots, their knot Floer complexes are staircase complexes, which we denote by $\mathcal{C}(K_i)$. In particular,

$$\mathcal{CFK}^-(K) = \mathcal{C}(K_1) \otimes \cdots \otimes \mathcal{C}(K_\nu)$$

is a positive multistaircase complex. Note that, by Proposition 2.40 and Example 4.30, the knots K_+, K_- and $\widetilde{K}$ have split towers. The following relations follow from Proposition 2.40, the Künneth theorem for connected sums, and Proposition 3.9, where we write $\cong$ for homotopy equivalence of chain complexes and $\simeq_{\mathrm{loc}}$ for local equivalence, and the brackets denote an overall grading shift:

$$\begin{aligned}
\mathcal{C}^{\mathrm{top}}(K_+) &\cong \mathcal{C}^{\mathrm{top}}(K) \otimes \bigotimes_n (\mathcal{S}^n)^{\otimes m_n} \{\tfrac{1}{2}\eta_+, \tfrac{1}{2}\eta_+\}, \\
\mathcal{C}^{\mathrm{bot}}(K_+) &\cong \mathcal{C}^{\mathrm{bot}}(K) \otimes \bigotimes_n (\mathcal{S}^{n-1})^{\otimes m_n} \{-\tfrac{1}{2}\eta_+, -\tfrac{1}{2}\eta_+\}, \\
\mathcal{C}^{\mathrm{top}}(K_-) &\cong \bigotimes_n (\mathcal{S}^{-(n-1)})^{\otimes \underline{m}_n} \{\tfrac{1}{2}\eta_-, \tfrac{1}{2}\eta_-\}, \\
\mathcal{C}^{\mathrm{bot}}(K_-) &\cong \bigotimes_n (\mathcal{S}^{-n})^{\otimes \underline{m}_n} \{-\tfrac{1}{2}\eta_-, -\tfrac{1}{2}\eta_-\}, \\
\mathcal{C}^{\mathrm{top}}(\widetilde{K}) &\cong \mathcal{C}^{\mathrm{top}}(K_+) \otimes \mathcal{C}^{\mathrm{top}}(K_-) \simeq_{\mathrm{loc}} \mathcal{C}(K) \otimes \mathcal{S}^{\kappa_+ - (\kappa_- - \eta_-)} \{\tfrac{1}{2}(\eta_+ + \eta_-), \tfrac{1}{2}(\eta_+ + \eta_-)\}, \\
\mathcal{C}^{\mathrm{bot}}(\widetilde{K}) &\cong \mathcal{C}^{\mathrm{bot}}(K_+) \otimes \mathcal{C}^{\mathrm{bot}}(K_-) \simeq_{\mathrm{loc}} \mathcal{C}(K) \otimes \mathcal{S}^{\kappa_+ - \eta_+ - \kappa_-} \{\tfrac{1}{2}(\eta_+ + \eta_-), \tfrac{1}{2}(\eta_+ + \eta_-)\}.
\end{aligned}$$

We set

$$\delta_1 := \kappa_+ - (\kappa_- - \eta_-), \quad \delta_2 := (\kappa_+ - \eta_+) - \kappa_-.$$

Whether the staircases in $\mathcal{C}^{\mathrm{top}}(\widetilde{K})$ and $\mathcal{C}^{\mathrm{bot}}(\widetilde{K})$ are positive or negative depends on the signs of δ_1 and δ_2. The following proposition is the main tool towards Theorems 6.4 and 6.8:

Proposition 6.3 *Suppose K, $\widetilde{K}$ and $\widehat{K}$ are as above and let $R = R_K$ be the infimal convolution of the semigroup counting functions for knots $K_1, \dots, K_\nu$.*

(a) *If $\delta_1 \geq 0$, then*
$$V_s^{\mathrm{top}}(\widetilde{K}) = -\tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\leq j\leq\delta_1}(V_{s+2j-\delta_1}(K) + j),$$
$$\begin{aligned} V_s^{\mathrm{top}}(\widehat{K}) &= -\tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\leq j\leq\delta_1+g}(V_{s+2j-\delta_1-g}(K) + j) \\ &= -\tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\leq j\leq\delta_1+g}\big(R(g_3(K)+s+2j-\delta_1-g) - (s+j-\delta_1-g)\big). \end{aligned}$$

(b) *If $\delta_2 \geq 0$, then*
$$V_s^{\mathrm{bot}}(\widetilde{K}) = \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\leq j\leq\delta_2}(V_{s+2j-\delta_2}(K) + j),$$
$$\begin{aligned} V_s^{\mathrm{bot}}(\widehat{K}) &= \tfrac{1}{4}(\eta_+ + \eta_-) - \tfrac{1}{2}g + \max_{0\leq i\leq g}\min_{0\leq j\leq\delta_2}(V_{s+2j+2i-g-\delta_2}(K) + i + j) \\ &= -\tfrac{1}{2}g + \tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0\leq i\leq g}\min_{0\leq j\leq\delta_2}\big(R(g_3(K)+s+2j+2i-g-\delta_2) - (s+i+j-g-\delta_2)\big). \end{aligned}$$

(c) *If $\delta_1 < 0$ and $\mathcal{C}(K)$ is a positive staircase (not just a positive multistaircase), then*
$$V_s^{\mathrm{top}}(\widetilde{K}) = -\tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0\leq j\leq-\delta_1}(V_{s-2j-\delta_1}(K) - j),$$
$$\begin{aligned} V_s^{\mathrm{top}}(\widehat{K}) &= \tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\leq i\leq g}\max_{0\leq j\leq-\delta_1}(V_{s-2j-2i+g-\delta_1}(K) - i - j) \\ &= \tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\leq i\leq g}\max_{0\leq j\leq-\delta_1}\big(R(g_3(K)+s-2j-2i+g-\delta_1) - (s-i-j+g-\delta_1)\big). \end{aligned}$$

(d) *If $\delta_2 < 0$ and $\mathcal{C}(K)$ is a positive staircase, then*
$$V_s^{\mathrm{bot}}(\widetilde{K}) = \tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0\leq j\leq-\delta_2}(V_{s-2j-\delta_2}(K) - j),$$
$$\begin{aligned} V_s^{\mathrm{bot}}(\widehat{K}) &= \tfrac{1}{2}g + \tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0\leq j\leq g-\delta_2}(V_{s-2j+g-\delta_2}(K) - j) \\ &= \tfrac{1}{2}g + \tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0\leq j\leq g-\delta_2}\big(R(g_3(K)+s-2j+g-\delta_2) - (s-j+g-\delta_2)\big). \end{aligned}$$

Proof The proof is similar in all cases and consists of gathering Corollaries 4.25 and 4.26, Propositions 4.32 and 4.14, and Lemma 3.7. For the reader's convenience, we present details of the computations of V^{top} for cases (a) and (c).

If $\delta_1 \geq 0$, then by Corollary 4.25 and Lemma 3.7,
$$V_s^{\mathrm{top}}(\widetilde{K}) = -\tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\leq j\leq\delta_1}(V_{s+2j-\delta_1}(K) + j).$$

Combining this with Proposition 4.32, we obtain
$$V_s^{\mathrm{top}}(\widehat{K}) = -\tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\leq j\leq\delta_1+g}(V_{s+2j-\delta_1-g}(K) + j).$$

By Proposition 4.14,

$$V_s^{\mathrm{top}}(\hat{K}) = -\tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\le i\le g}\ \max_{0\le j\le -\delta_1} \bigl(R(g_3(K)+s-2j-2i+g-\delta_1)-(s-i-j+g-\delta_1)\bigr).$$

This proves item (a). If $\delta_1 < 0$ and $\mathcal{C}(K)$ is a positive staircase, then, by Corollary 4.26,

$$V_s^{\mathrm{top}}(\tilde{K}) = -\tfrac{1}{4}(\eta_+ + \eta_-) + \max_{0\le j\le -\delta_1} (V_{s-2j-\delta_1}(K) - j).$$

Combining Propositions 4.32 and 4.14, we have

$$\begin{aligned} V_s^{\mathrm{top}}(\hat{K}) &= \tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\le i\le g}\ \max_{0\le j\le -\delta_1} (V_{s-2j-2i+g-\delta_1}(K) - i - j) \\ &= \tfrac{1}{2}g - \tfrac{1}{4}(\eta_+ + \eta_-) + \min_{0\le i\le g}\ \max_{0\le j\le -\delta_1} \bigl(R(g_3(K)+s-2j-2i+g-\delta_1)-(s-i-j+g-\delta_1)\bigr). \end{aligned}$$

This proves item (c). □

Proposition 6.3 allows us to express the d–invariants of the boundary $Y = \partial N$ of the tubular neighborhood of C in terms of the R_K–functions of singular points. In our applications, we will focus on two cases:

(1) **Algebraic case** We assume that C has only algebraic singularities; that is, $\underline{m}_n = 0$ for all $n > 0$. This corresponds to items (a) and (b) of Proposition 6.3.

(2) **Single knot case** We assume that $\nu = 1$, so K is a positive staircase and $m_n = 0$ for all $n > 0$. We will use items (c) and (d) of Proposition 6.3.

The first case is considered in Section 6.2. The second is addressed in Section 6.3.

6.2 Curves with no negative double points

For the reader's convenience, we repeat the statement from the introduction of the next result.

Theorem 6.4 *Let C be a reduced curve with degree d and genus g. Suppose that C has cuspidal singular points $p_1, \dots, p_\nu$ whose semigroup counting functions are $R_1, \dots, R_\nu$, respectively. Assume that, apart from these ν points, the curve C has, for each $n \ge 1$, $m_n \ge 0$ singular points whose links are $T_{2,2n}$ (A_{2n-1} singular points) and no other singularities. Define*

$$\eta_+ = \sum_n m_n \quad \text{and} \quad \kappa_+ = \sum_n n m_n.$$

For any $k = 1, \dots, d-2$,

$$\text{(6.5)}\qquad \begin{aligned} \max_{0\le j\le g}\ \min_{0\le i\le \kappa_+-\eta_+} (R(kd+1-\eta_+-2i-2j)+i+j) &\le \tfrac{1}{2}(k+1)(k+2)+g, \\ \min_{0\le j\le g+\kappa_+} (R(kd+1-2j)+j) &\ge \tfrac{1}{2}(k+1)(k+2). \end{aligned}$$

Here R denotes the infimal convolution of the functions $R_1, \dots, R_\nu$.

Proof Let Y be the boundary of a tubular neighborhood of C. Then Y is the result of a d^2–surgery on $\hat{K} \subset \#^\rho S^2 \times S^1$ obtained as in Section 6.2, where we readily compute from (5.1) that $\rho = 2g + \eta_+$. Note that, by (6.1), the genus $g_3(K)$ is less than or equal to $\frac{1}{2}(d-1)(d-2) < \frac{1}{2}d^2$. Hence, the surgery coefficient is greater than twice the genus of K. In particular, the large surgery formula can be applied [Ozsváth and Szabó 2008b, Theorem 4.10].

Let $\mathfrak{s}_j$ for $j \in \left[-\frac{1}{2}d^2, \frac{1}{2}d^2\right) \cap \mathbb{Z}$ denote the Spin^c structures on Y as in Definition 3.13. By Lemma 5.13, $\mathfrak{s}_j$ extends to $\mathbb{C}P^2 \setminus N$ if and only if $\mathfrak{s}_j$ is a restriction of $\mathfrak{c}_h$ for some h, where $\mathfrak{c}_h$ is as in (5.11). By (5.12), we infer that this holds if and only if $j = md$ with $m \in \mathbb{Z}$ if d is odd and $m \in \frac{1}{2} + \mathbb{Z}$ if d is even. Compare with [Borodzik and Livingston 2014, Lemma 3.1].

By Proposition 5.14, for any $md \in \left[-\frac{1}{2}d^2, \frac{1}{2}d^2\right)$ such that $m + \frac{1}{2}(d-1)$ is an integer,

$$d_{\mathrm{bot}}(Y, \mathfrak{s}_{md}) \geq -\tfrac{1}{2}\eta_+ - g, \quad d_{\mathrm{top}}(Y, \mathfrak{s}_{md}) \leq \tfrac{1}{2}\eta_+ + g. \tag{6.6}$$

By Theorem 3.15, (6.6) translates to the inequalities

$$\begin{aligned} V_{md}^{\mathrm{top}}(\widehat{K}) &\geq \tfrac{1}{8}(d-2m+1)(d-2m-1) - \tfrac{1}{4}\eta_+ - \tfrac{1}{2}g, \\ V_{md}^{\mathrm{bot}}(\widehat{K}) &\leq \tfrac{1}{8}(d-2m+1)(d-2m-1) + \tfrac{1}{4}\eta_+ + \tfrac{1}{2}g. \end{aligned} \tag{6.7}$$

We compute V_{md}^{top} and V_{md}^{bot} from Proposition 6.3. Using $g_3(K) = \frac{1}{2}(d-1)(d-2) - g - \kappa_+$, we rewrite the equations of Proposition 6.3(a)–(b) as

$$V_{md}^{\mathrm{top}}(\widehat{K}) = -\tfrac{1}{2}g - \tfrac{1}{4}\eta_+ + \min_{0 \leq j \leq \kappa_+ + g} \left(R\left(\tfrac{1}{2}(d-1)(d-2) + md + 2j - 2\kappa_+ - 2g\right) - (md + j - \kappa_+ - g)\right),$$

$$\begin{aligned} V_{md}^{\mathrm{bot}}(\widehat{K}) = -\tfrac{1}{2}g + \tfrac{1}{4}\eta_+ + \max_{0 \leq i \leq g} \min_{0 \leq j \leq \kappa_+ - \eta_+} \Big(&R\left(\tfrac{1}{2}(d-1)(d-2) + md + 2j + 2i - 2g - 2\kappa_+ + \eta_+\right) \\ &- (md + i + j - g - \kappa_+ + \eta_+)\Big). \end{aligned}$$

Comparing this with (6.7), we obtain

$$\min_{0 \leq j \leq \kappa_+ + g} R\left(\tfrac{1}{2}(d-1)(d-2) + md + 2j - 2\kappa_+ - 2g\right) - (md + j - \kappa_+ - g) \geq \tfrac{1}{8}(d-2m+1)(d-2m-1)$$

and

$$\begin{aligned} \max_{0 \leq i \leq g} \min_{0 \leq j \leq \kappa_+ - \eta_+} R\left(\tfrac{1}{2}(d-1)(d-2) + md + 2i + 2j - 2(\kappa_+ - \eta_+) - \eta_+ - 2g\right) - (md + j - \kappa_+ + \eta_+ - 2g) \\ \leq \tfrac{1}{8}(d-2m+1)(d-2m-1) + g. \end{aligned}$$

With a change $j \mapsto \kappa_+ + g - j$ in the first inequality and $i \mapsto g - i$ and $j \mapsto \kappa_+ - \eta_+ - j$ in the second, we obtain

$$\min_{0 \leq j \leq \kappa_+ + g} R\left(\tfrac{1}{2}(d-1)(d-2) + md - 2j\right) - md + j \geq \tfrac{1}{8}(d-2m+1)(d-2m-1),$$

$$\max_{0 \leq i \leq g} \min_{0 \leq j \leq \kappa_+ - \eta_+} R\left(\tfrac{1}{2}(d-1)(d-2) + md - 2i - 2j - \eta_+\right) - md + j \leq \tfrac{1}{8}(d-2m+1)(d-2m-1) + g.$$

With $m = k - \frac{1}{2}(d-3)$, after straightforward calculations we obtain

$$\min_{0 \leq j \leq g + \kappa_+} (R(kd + 1 - 2j) + j) \geq \tfrac{1}{2}(k+1)(k+2),$$

$$\max_{0 \leq j \leq g} \min_{0 \leq i \leq \kappa_+ - \eta_+} (R(kd + 1 - \eta_+ - 2i - 2j) + i + j) \leq \tfrac{1}{2}(k+1)(k+2) + g. \qquad \square$$

6.3 Negative double points

We now specialize to the case where C is a surface which has a single algebraic singularity and $\underline{m}_n \geq 0$ singular points whose links are $(2,-2n)$–torus links (which are not algebraic).

Theorem 6.8 *Suppose C is a genus g degree d singular curve in the smooth category as in Section 5.4 with a cuspidal singular point p, $\underline{m}_n$ singularities whose link is $-T_{2,2n}$ for each $n \geq 1$, and no other singular points. Suppose further that C is adjunctive.*

Then, for any $k = 1, \dots, d-2$,

$$\max_{0 \leq j \leq g+\kappa_-} (R(kd+1-2j)+j) \leq \tfrac{1}{2}(k+1)(k+2)+g+\kappa_-,$$
$$\min_{0 \leq i \leq g} \max_{0 \leq j \leq \kappa_- - \eta_-} (R(kd+1-2i-2j-\eta_-)+i+j) \geq \tfrac{1}{2}(k+1)(k+2)+\kappa_- - \eta_-,$$

where R is the semigroup counting function for the singular point p and $\eta_- = \sum \underline{m}_n$ and $\kappa_- = \sum \underline{m}_n n$.

Remark 6.9 With the assumptions on the singularities of C, the condition that C be adjunctive (spelled out in Definition 5.17) is equivalent to saying that the genus of C is given by (6.1).

Proof The beginning of the proof is exactly the same as in the proof of Theorem 6.4. The boundary Y of the tubular neighborhood of C is a result of a surgery with coefficient d^2 on the knot $\widehat{K}$ in $\#^{2g+\eta_-} S^2 \times S^1$. In particular, (6.7) holds with η_- replacing η_+:

$$\begin{aligned} V_{md}^{\text{top}}(\widehat{K}) &\geq \tfrac{1}{8}(d-2m+1)(d-2m-1) - \tfrac{1}{4}\eta_- - \tfrac{1}{2}g, \\ V_{md}^{\text{bot}}(\widehat{K}) &\leq \tfrac{1}{8}(d-2m+1)(d-2m-1) + \tfrac{1}{4}\eta_- + \tfrac{1}{2}g. \end{aligned} \tag{6.10}$$

With $g_3(K) = \frac{1}{2}(d-1)(d-2) - g - \kappa_-$, the equations of Proposition 6.3(c)–(d) take the form

$$V_{md}^{\text{top}}(\widehat{K}) = \tfrac{1}{2}g - \tfrac{1}{4}\eta_- + \min_{0 \leq i \leq g} \max_{0 \leq j \leq \kappa_- - \eta_-} \Big(R\big(\tfrac{1}{2}(d-1)(d-2)+md-2j-2i-\eta_-\big) - (md-i-j+g+\kappa_- - \eta_-)\Big),$$

$$V_{md}^{\text{bot}}(\widehat{K}) = \tfrac{1}{2}g + \tfrac{1}{4}\eta_- + \max_{0 \leq j \leq g+\kappa_-} \big(R\big(\tfrac{1}{2}(d-1)(d-2)+md-2j\big) - (md-j+g+\kappa_-)\big).$$

Comparing this with (6.10), after changes analogous to in Section 6.2, we arrive at

$$\max_{0 \leq j \leq g+\kappa_-} (R(kd+1-2j)+j) \leq \tfrac{1}{2}(k+1)(k+2)+g+\kappa_-,$$
$$\min_{0 \leq i \leq g} \max_{0 \leq j \leq \kappa_- - \eta_-} (R(kd+1-2i-2j-\eta_-)+i+j) \geq \tfrac{1}{2}(k+1)(k+2)+\kappa_- - \eta_-. \qquad \square$$

6.4 Special cases of Theorems 6.4 and 6.8

The bounds in Theorems 6.4 and 6.8 are fairly general, but clarity is the price. To illustrate these bounds, we provide several special cases.

Corollary 6.11 (a) *Suppose C is a genus g, degree d curve with singular points $p_1, \dots, p_\nu$ and η_+ positive double points. Assume also that C has no other critical points. Then, for $k = 1, \dots, d-2$,*

$$\max_{0 \le j \le g} (R(kd + 1 - \eta_+ - 2j) + j) \le \tfrac{1}{2}(k+1)(k+2) + g,$$
$$\min_{0 \le j \le g+\eta_+} (R(kd + 1 - 2j) + j) \ge \tfrac{1}{2}(k+1)(k+2),$$

where R denotes the infimal convolution of the functions $R_{K_1}, \dots, R_{K_\nu}$.

(b) *Suppose C is a genus g, degree d curve with a singular point p and η_- negative double points. Assume that C has genus as in (6.1). Then, for $k = 1, \dots, d-2$,*

$$\max_{0 \le j \le g+\eta_-} (R(kd + 1 - 2j) + j) \le \tfrac{1}{2}(k+1)(k+2) + g + \eta_-,$$
$$\min_{0 \le j \le g} (R(kd + 1 - \eta_- - 2j) + j) \ge \tfrac{1}{2}(k+1)(k+2),$$

where R is the semigroup counting function for the singular point p.

Proof Items (a) and (b) follow from Theorems 6.4 and 6.8, respectively, noting that $\kappa_+ = \eta_+$ and $\kappa_- = \eta_-$. □

Specifying further $\eta_+ = 0$ in Corollary 6.11(a) recovers the following result of [Bodnár et al. 2016; Borodzik et al. 2017]:

Corollary 6.12 *Suppose C is a cuspidal curve of genus g and degree d. Let R be the convolution of semigroup counting functions of the singular points of C. Then*

$$\max_{0 \le j \le g} (R(kd + 1 - 2j) + j) \le \tfrac{1}{2}(k+1)(k+2) + g,$$
$$\min_{0 \le j \le g} (R(kd + 1 - 2j) + j) \ge \tfrac{1}{2}(k+1)(k+2). \tag{6.13}$$

We now compare the cases $g = 0$ and $\eta_+ = 1$, $g = 0$ and $\eta_- = 1$, and $g = 1$ and $\eta_+ = \eta_- = 0$.

Proposition 6.14 *Let C be a degree d curve with one cuspidal singular point, whose semigroup counting function is denoted by R. Assume C has at most one ordinary double point ($\eta_+ + \eta_- \le 1$) and no other singularities. For all $k = 1, \dots, d-2$, set $\upsilon_k = \frac{1}{2}(k+1)(k+2)$.*

(a) *If $g = 1$ and $\eta_+ = \eta_- = 0$, then $R(kd-1) \in \{\upsilon_k - 1, \upsilon_k\}$ and $R(kd+1) \in \{\upsilon_k, \upsilon_k + 1\}$.*

(b) *If $g = 0$ and $\eta_+ = 1$, then $R(kd-1) \in \{\upsilon_k - 1, \upsilon_k\}$ and $R(kd+1) \in \{\upsilon_k, \upsilon_k + 1\}$, but also*

$$R(kd) \le \upsilon_k.$$

(c) *If $g = 0$ and $\eta_- = 1$, then $R(kd-1) \in \{\upsilon_k - 1, \upsilon_k\}$ and $R(kd+1) \in \{\upsilon_k, \upsilon_k + 1\}$, but also*

$$R(kd) \ge \upsilon_k.$$

Proof Item (a) is an immediate consequence of (6.13).

For item (b), note that Corollary 6.11(a) implies that $R(kd) \leq \upsilon_k$ and $R(kd+1) \geq \upsilon_k$, $R(kd-1) \geq \upsilon_k - 1$. Since $R(j+1) - R(j) \in \{0, 1\}$ for all j, the statement follows trivially.

The proof of item (c) is analogous. Corollary 6.11(c) implies that $R(kd+1) \leq \upsilon_k + 1$, $R(kd-1) \leq \upsilon_k$ and $R(kd) \geq \upsilon_k$. Again, the statement follows trivially. □

Proposition 6.14 can be interpreted as follows. Suppose C is a genus one curve with a single cuspidal singular point. Then the semigroup counting function R satisfies the constraints of item (a) of Proposition 6.14. If, for some $k = 1, \dots, d-2$, we have $R(kd) = \upsilon_k + 1$, then the function R does not satisfy the constraints of item (b). That is, C cannot be deformed to a curve with genus 0 and the same (topological type of) cuspidal singularity. That is, we cannot "trade genus for a positive double point".

If, for some k, we have $R(kd) = \upsilon_k - 1$, then the same argument shows that we cannot "trade genus for a negative double point".

6.5 Unicuspidal curves of genus 1

We will now check, for concrete examples, whether the genus can be traded for double points.

Example 6.15 Let $\phi_0 = 0$, $\phi_1 = 1$, $\phi_n = \phi_{n-1} + \phi_{n-2}$ be the Fibonacci sequence. Borodzik et al. [2017, Proposition 9.12], based on a construction of Orevkov [2002], constructed a family of genus 1 cuspidal curves C_n of degree ϕ_{4n} with a single singularity whose link is the $(\phi_{4n-2}, \phi_{4n+2})$–torus knot for $n = 2, 3, \dots$.

By Proposition 6.14(c), we deduce that the genus cannot be traded for negative double points. Indeed, a classical identity on Fibonacci numbers, $\phi_{k-2}+\phi_{k+2} = 3\phi_k$, shows that the semigroup generated by ϕ_{4n-2} and ϕ_{4n+2} has precisely nine elements in the interval $[0, 3\phi_{4n})$: $0, \phi_{4n-2}, \dots, 7\phi_{4n-2}$ and ϕ_{4n+2}. In fact, $7\phi_{4n-2} < 3\phi_{4n} < 8\phi_{4n-2}$ (we leave the proof of this to the reader) and $\phi_{4n+2} + \phi_{4n-2} = 3\phi_{4n}$. In particular, $R(3\phi_{4n}) = 9 < 10 = \upsilon_3 = \frac{1}{2}(3+1)(3+2)$.

Borodzik et al. [2017, Theorem 9.1] gave a complete list of candidates for curves of genus 1 with one singularity whose link is a torus link $T_{p,q}$. The list contains one infinite family (Orevkov curves) and a finite list of special cases. We apply our obstructions to these curves and obtain the following result:

Proposition 6.16 *Suppose C is a genus one, degree d curve, having a single singularity, whose link is a (p,q)–torus knot. Then either C is the Orevkov curve (of Example 6.15), or the values of (p,q) and d are one of*

(a) $d = 4$ *and* $(p,q) = (2,5)$;

(b) $d = 5$ *and* $(p,q) = (2,11)$;

case	(d,p,q)	positive	negative	existence
(a)	$(4,2,5)$	passes	passes	exists
(b)	$(5,2,11)$	passes	passes	exists
(c)	$(6,3,10)$	passes	$k=1$	
(d)	$(15,6,37)$	passes	$k=2$	
(e)	$(24,9,64)$	passes	$k=3$	
(f)	$(27,10,73)$	$k=12$	$k=8$	
(g)	$(33,12,91)$	$k=7$	$k=8$	
(h)	$(3p,p,9p+1)$	passes	fails if $p\geq 3$	

Table 1: Curves of Proposition 6.16 and the criteria of Proposition 6.14. "Positive" refers to item (b) of the proposition, "negative" refers to item (c). If the curve does not pass the criteria, we indicate the minimal k for which $R(kd)>\upsilon_k$ (case (b)) or $R(kd)<\upsilon_k$ (case (c)).

(c) *$d=6$ and $(p,q)=(3,10)$;*

(d) *$d=15$ and $(p,q)=(6,37)$;*

(e) *$d=24$ and $(p,q)=(9,64)$;*

(f) *$d=27$ and $(p,q)=(10,73)$;*

(g) *$d=33$ and $(p,q)=(12,91)$;*

(h) *$d=3p$ and $(p,q)=(p,9p+1)$ for $p=2,\dots,11$.*

By definition, all cases satisfy the statement of Proposition 6.14(a). We applied the criteria of Proposition 6.14(b)–(c). The results are in Table 1. We indicate that some of the examples predicted by Proposition 6.16 have not been either effectively constructed or obstructed by other means.

6.6 Generalized Orevkov curves

Bodnár et al. [2016] constructed a family of curves generalizing Orevkov's construction. Their work can be regarded as a generalization of the construction of [Borodzik et al. 2017, Proposition 9.12]. To begin with, fix $k\geq 2$. The *Lucas sequence* is the sequence L_i^k defined recursively via $L_0^k=k-1$, $L_1^k=1$, $L_{i+1}^k=L_i^k+L_{i-1}^k$. Here i is allowed to take all integer values.

Theorem 6.17 (BCG family; see [Bodnár et al. 2016, Theorem 1.7]) *For any $i\geq 2$, there exists a genus $\frac{1}{2}k(k-1)$ curve of degree L_{4i-1}^k with precisely one singularity whose link is the (L_{4i-3}^k,L_{4i+1}^k)–torus knot.*

For any $j\geq 1$, there exists a genus $\frac{1}{2}k(k-1)$ curve of degree $-L_{-4j-1}^k$ with singularity whose link is the $(-L_{-4j+1}^k,-L_{-4j-3}^k)$–torus knot.

Now we apply Corollary 6.11.

Proposition 6.18 *None of the curves of the BCG family can be transformed into a curve with genus one less and one negative double point.*

Proof We follow the same strategy as in Example 6.15. We begin with the first family. Suppose $i \geq 2$. Let S be the semigroup associated with the (L^k_{4i-3}, L^k_{4i+1})–torus knot, and let R be the counting function for it. The recursive formula for Lucas numbers implies that $L^k_s + L^k_{s+4} = 3L^k_{s+2}$ for all s. Moreover,

(6.19) $$L^k_{s+4} = L^k_{s+3} + L^k_{s+2} = 2L^k_{s+2} + L^k_{s+1} = 3L^k_{s+1} + 2L^k_s = 5L^k_s + 3L^k_{s-1} < 8L^k_s$$

as long as $s \geq 0$. In particular, $3L^k_{s+1} < 9L^k_s$. Therefore, all possible elements in $S \cap [0, 3L^k_{4j-1}]$ are $0, \dots, 8L^k_{4j-3}$ and L^k_{4j+1}. Hence, $R(3L^k_{4j-1}) \leq 9$, violating the second inequality in Corollary 6.11(b).

As for the second family, write $\widetilde{L}^k_i = (-1)^{i+1} L^k_{-i}$ for $i > 0$ and note that $\widetilde{L}^k_{i+1} = \widetilde{L}^k_i + \widetilde{L}^k_{i-1}$. Moreover, for $i > 0$, $\widetilde{L}^k_i$ is an increasing sequence of positive numbers. We have $\widetilde{L}^k_{s+4} + \widetilde{L}^k_s = 3\widetilde{L}^k_{s+2}$ and, for s odd, $\widetilde{L}^k_{s+4} < 8\widetilde{L}^k_s$ by the same argument as in (6.19). We conclude as in the first case. □

It is unknown whether it is possible to trade genus for *positive* double points in any curves in the BCG family.

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Classifying spaces of infinity-sheaves

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We prove that the set of concordance classes of sections of an ∞–sheaf on a manifold is representable, extending a theorem of Madsen and Weiss for sheaves of sets. This is reminiscent of an h–principle in which the role of isotopy is played by concordance. As an application, we offer an answer to the question: what does the classifying space of a Segal space classify?

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1 Introduction

Let F be an ∞–sheaf (alias homotopy sheaf; see Definition 2.15) on Man, the site of finite-dimensional smooth manifolds without boundary and smooth maps. For a manifold M, an element of $F(M \times \mathbb{R})$ is called a *concordance*. Two elements σ_0 and σ_1 in $F(M)$ are said to be *concordant* (and we write $\sigma_0 \sim_c \sigma_1$) if there exists a concordance whose restriction to $M \times \{k\}$ is σ_k for $k = 0, 1$.

Concordance is an equivalence relation, and a familiar one in many situations. Here are three examples. When $F = C^\infty(-, N)$, maps are concordant if and only if they are smoothly homotopic. For the sheaf of closed differential n–forms, two sections (ie closed n–forms) are concordant if and only if they are cohomologous. For the stack of vector bundles, a pair of vector bundles are concordant if and only if they are isomorphic. In these three cases, concordance classes have a well-known description in terms of homotopy classes of maps into a space, namely the space underlying N, the Eilenberg–Mac Lane space $K(\mathbb{R}, n)$ and the space $BO(n)$, respectively. In this paper we generalize these classical representability results: concordance classes of sections of *any* ∞–sheaf F is represented by a space BF, which we call the *classifying space* of F.

We now assemble the ingredients to state our results precisely. We denote by $\mathbb{A}^n$ the smooth extended simplex, that is, the subspace of $\mathbb{R}^{n+1}$ whose coordinates sum to one. By varying n, this defines a cosimplicial object in Man. Define a presheaf

$$\boldsymbol{B}F(M) := |[k] \mapsto F(M \times \mathbb{A}^k)|$$

with values in spaces (ie simplicial sets), where $|-|$ denotes the homotopy colimit of the simplicial space. The construction $\boldsymbol{B}$ is a form of localization; it is the universal way to render the maps $F(M) \to F(M \times \mathbb{A}^1)$ invertible for all M and F. It is a familiar construction in the motivic literature, for example in the work of Morel and Voevodsky [1999] (who call it Sing), but it has also appeared in the context of geometric topology in [Waldhausen 1985; Weiss and Williams 1995; Madsen and Weiss 2007]. The link between $\boldsymbol{B}F$ and the concordance relation $\sim_c$ is the bijection $\pi_0 \boldsymbol{B}F(M) \cong \pi_0 F(M)/{\sim_c}$.

Define the *classifying space* BF as a Kan complex replacement of $\boldsymbol{B}F(*)$ and denote by $\operatorname{Sing} M$ the usual singular simplicial set of M. Our main result is:

Theorem 1.1 *Let F be an ∞–sheaf on* Man. *There is an evaluation map*

$$(\boldsymbol{B}F)(M) \to \boldsymbol{R}\mathrm{map}(\operatorname{Sing} M, BF),$$

which is a natural weak equivalence of spaces for every manifold M.

It is not difficult to show — essentially by a variant of Brown's representability theorem; see Section 2 — that Theorem 1.1 is equivalent to the following:

Theorem 1.2 *If F is an ∞–sheaf, then $\boldsymbol{B}F$ is an ∞–sheaf.*

These statements may be regarded as analogues of the h–principle, where the usual relation of *isotopy* is replaced by that of *concordance*. Here we have in mind the strand of the h–principle that gives conditions (eg microflexibility) which guarantee that an isotopy-invariant functor (eg a sheaf) is an ∞–sheaf. The relation of concordance is more severe than that of isotopy, and this explains why the hypotheses are less restrictive than those of typical h–principles, eg there are no dimension restrictions, open versus closed manifolds, etc.

Just as with the h–principle, the key step in our proof involves verifying certain fibration properties. As such, a significant part of the paper is a study of weak lifting properties for maps of simplicial spaces.

We introduce the notion of *weak Kan fibration* of simplicial spaces and simplicial sets. A crucial result shows that weak Kan fibrations are realization fibrations (see Definition 3.14 and Theorem 3.17); this implies that geometric realization is stable under homotopy pullback along weak Kan fibrations.

We emphasize that these results — and hence Theorem 1.2 — do not follow from formal considerations. There are simple counterexamples in the category of schemes, as in the $\mathbb{A}^1$–homotopy theory of [Morel and Voevodsky 1999, Section 3, Example 2.7]. The ∞–sheaf property is a homotopy limit condition whereas $\boldsymbol{B}$ involves geometric realization, a homotopy colimit. Commuting these is a subtle issue. This is where we use the weak Kan property to prove Theorem 1.2. The verification that $\boldsymbol{B}F$ is weak Kan and certain restriction maps are weak Kan fibrations follows from a geometric argument about smooth maps (Lemma 4.21). We also rely on the existence of partitions of unity.

Our main results improve on prior work of others, although the techniques we use differ. The π_0–statement of Theorem 1.1 is due to Madsen and Weiss [2007, Appendix A] when F is a sheaf of *sets* (or of discrete categories). Although our theorem does extend the result of Madsen and Weiss from π_0 to π_n in the case of sheaves of sets, the main objective of our work is to extend it from sheaves of sets to ∞–sheaves of spaces. The argument of Madsen and Weiss shares some features with ours (in that certain locally constancy conditions along simplices of a triangulation are enforced), but does not extend to ∞–sheaves. Moreover, unlike theirs, our arguments apply in the topological or PL category too: Theorem 1.1 remains true if we consider topological or PL manifolds instead of smooth manifolds. In fact, our arguments simplify significantly in those cases (see Section 4 for explanations).

Bunke, Nikolaus and Völkl [2016, Section 7] have proved a version of Theorem 1.1 for ∞–sheaves on compact manifolds with values in *spectra*. From the point of view of Theorem 1.2, this case is essentially formal since, in a stable setting, homotopy pullback squares are homotopy pushout squares, and so (for finite covers) the problem of commuting homotopy pullbacks with geometric realization disappears. It has also been pointed out to us by John Francis that Ayala, Francis and Rozenblyum [2019, 2.3.16 and 2.4.5] gave related results that are proved in the context of stratified spaces. As we understand it, these results are both more general (they apply to *stratified* spaces) and less general than ours and those of Madsen and Weiss (they apply to a certain class of *isotopy* sheaves on groupoids). This restricted class of sheaves is, from the point of view of Theorem 1.1, too severe as it excludes many ∞–sheaves, even set-valued ones.

Applications of Theorem 1.1 abound. This stems from the fact that we not only prove abstract representability, but also give a formula for the representing space. This formula can be identified with classical constructions. Two illustrative examples, connecting back to the beginning of this introduction, are the classical de Rham theorem and the classification of vector bundles (with or without connection). These are obtained by applying the main theorem to the sheaf of differential n–forms and the stack of vector bundles (sheaves of sets and stacks are examples of ∞–sheaves). In Section 6, we discuss a further application: a classification of $\boldsymbol{C}$–bundles, where $\boldsymbol{C}$ is a Segal space. More recently, Pavlov [2024, Theorem 13.8] proved a generalization of Theorem 1.1 for presheaves with values in model categories Quillen equivalent to model categories of algebras over simplicial algebraic theories (such as chain complexes, connective spectra, and various flavors of connective ring spectra).

We mention here another consequence of Theorem 1.1. Let D denote the full subcategory of Man spanned by $\mathbb{R}^n$ with $n \geq 0$. The ∞–categories $\mathsf{Sh}(\mathsf{Man})$ and $\mathsf{Sh}(\mathsf{D})$ of ∞–sheaves on Man and D, respectively (with respect to the usual open covers by codimension zero embeddings), are examples of ∞–toposes, as are the slice ∞–categories $\mathsf{Sh}(\mathsf{Man})/F$ for an ∞–sheaf F on Man, and $\mathsf{Sh}(\mathsf{D})/F$ for an ∞–sheaf F on D.

Proposition 1.3 *The functor B from $\mathsf{Sh}(\mathsf{Man})$ to spaces is homotopy left adjoint to the functor which sends a space to the constant ∞–sheaf on that space. Moreover, $\boldsymbol{B}$ is homotopy left adjoint to the inclusion of the ∞–category of constant ∞–sheaves on Man into $\mathsf{Sh}(\mathsf{Man})$. These two statements also hold if Man is replaced by D, in which case we can also formulate the adjunction using ∞–presheaves instead of ∞–sheaves. In particular, the inclusion $\Delta \to \mathsf{D}$ is a homotopy initial functor (hence also an initial functor), ie the functor B computes the homotopy colimit over D^{op}.*

Proof By a constant ∞–sheaf, we mean the homotopy sheafification of a constant presheaf. Every constant presheaf on D is an ∞–sheaf, and, for a space K, the canonical functor from the constant ∞–sheaf to the mapping space ∞–sheaf, $\mathrm{const}_K \to \mathrm{map}(-, K)$, is an objectwise weak equivalence. On the other hand, a constant presheaf on Man is in general not an ∞–sheaf. However, since every open cover can be refined by a good open cover, the homotopy sheafification of a presheaf on Man is determined by its restriction to D. Therefore, $\mathrm{const}_K \to \mathrm{map}(-, K)$ is also a weak equivalence in $\mathsf{Sh}(\mathsf{Man})$.

If F is a representable presheaf, represented by a manifold M, then $BF \simeq M$. From this it follows that

$$\mathrm{map}(F, \mathrm{const}_K) \simeq \mathrm{map}(BF, K)$$

for F a representable, and, extending by colimits, the same is true for any presheaf F, and then for any ∞–sheaf (the mapping space on the left is computed in the ∞–category of ∞–sheaves or, equivalently, since const_K is an ∞–sheaf, in the ∞–category of presheaves).

As for the second statement, we have by Theorem 1.1 that $\boldsymbol{B}F$ is a constant ∞–sheaf on Man for any ∞–sheaf F. Then, by Yoneda, $\mathrm{map}(\boldsymbol{B}F, \mathrm{const}_K) \simeq \mathrm{map}(BF, K)$, which, by the first part, implies the statement.

The argument for D remains valid for ∞–presheaves because the constant ∞–presheaf is an ∞–sheaf in this case. This also implies the claim about homotopy initiality of $\Delta \to \mathsf{D}$. □

In other words, BF is the shape of $\mathsf{Sh}(\mathsf{Man})/F$ as in [Lurie 2009, Chapter 7], or, equivalently, the fundamental ∞–groupoid of F in the sense of [Schreiber 2013, Section 3.4], and $\boldsymbol{B}F$ is the shape modality of F in the sense of [Schreiber 2013, Definition 3.4.4]. For a different proof, see also [Pavlov

2024, Proposition 12.10]. The statement about the homotopy adjunction continues to hold if ∞–sheaves on Man are replaced by ∞–presheaves on Man, and $\boldsymbol{B}$ precomposed with the associated ∞–sheaf functor is weakly equivalent to $\boldsymbol{B}$, ie the shape of the associated ∞–sheaf of an ∞–presheaf of F can be computed as the shape of F [Pavlov 2024, Proposition 13.9].

The following formal consequence of Theorem 1.1 proved to be useful in applications. Sati and Schreiber [2021, Theorem 3.3.53] proposed the name "smooth Oka principle" for Proposition 1.4, in analogy to the Oka principle in complex geometry, and also gave an alternative derivation of Proposition 1.4 from Theorem 1.1.

Proposition 1.4 *Let F be an ∞–sheaf on* Man. *There is an evaluation map*

$$\boldsymbol{B}\operatorname{Hom}(M,F)\to\boldsymbol{R}\operatorname{Hom}(\boldsymbol{B}M,\boldsymbol{B}F)$$

which is a natural weak equivalence of ∞–sheaves for every manifold M. Here Hom *denotes the internal hom of ∞–sheaves, whereas $\boldsymbol{R}$*Hom *is the derived internal hom* ($\operatorname{Hom}(M,-)$ *is automatically derived*).

Proof The left side $\boldsymbol{B}\operatorname{Hom}(M,F)$ is a concordance-invariant ∞–sheaf because $\boldsymbol{B}$ lands in concordance-invariant presheaves by Corollary 2.11 and ∞–sheaves by Theorem 1.2. The right side $\operatorname{Hom}(\boldsymbol{B}M,\boldsymbol{B}F)$ is a concordance-invariant ∞–sheaf because the left Bousfield localization that produces concordance-invariant ∞–sheaves is a cartesian localization (since open covers are closed under products with a fixed manifold) and derived internal homs in cartesian left Bousfield localizations preserve local objects. Therefore, the map under consideration is a map between concordance-invariant ∞–sheaves, so it is a weak equivalence if and only if its evaluation at the point is a weak equivalence. Evaluating at the point gives the map

$$(\boldsymbol{B}F)(M)\to\boldsymbol{R}\operatorname{map}(\boldsymbol{B}M,\boldsymbol{B}F)\simeq\boldsymbol{R}\operatorname{map}(BM,BF)=\boldsymbol{R}\operatorname{map}(\operatorname{Sing}M,BF),$$

which is a weak equivalence by Theorem 1.1. □

Another application of this work is a construction of classifying spaces of field theories. This has recently been done in [Grady and Pavlov 2020]; see in particular Theorem 8.2.9 there. Stolz and Teichner [2011] have conjectured that concordance classes of particular classes of field theories determine cohomology theories. By Brown representability, this conjecture requires concordance classes of field theories to define a representable functor. In brief, they define field theories as functors out of a category of bordisms equipped with a smooth map to a fixed manifold M. When the relevant bordism category is fully extended, field theories are an ∞–sheaf evaluated on M. The main result of this paper then shows that concordance classes of fully extended field theories are representable. Furthermore, we identify a formula for the classifying space of field theories.

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2 The concordance resolution is concordance-invariant

Notation 2.1 Throughout, *space* will mean simplicial set. The category of such is denoted by S. A *simplicial space* is a simplicial object in spaces, and the category of such is denoted by sS. Of course, this is the same as a bisimplicial set, though the terminology emphasizes that there is a preferred simplicial direction. A simplicial set is often viewed as a simplicial *discrete* space, by regarding a set as a discrete (or constant) simplicial set. We denote the diagonal of a bisimplicial set X by $|X|$; this is our preferred model for homotopy colimits of simplicial spaces.

Definition 2.2 We write Δ^n for the representable simplicial *set* and $\Delta[n]$ for the corresponding simplicial discrete space. Similarly, we write $\partial\Delta^n$ and Λ^n_k for the simplicial *set* boundary and k^{th} horn, respectively, and $\partial\Delta[n]$ and $\Lambda_k[n]$ for the corresponding simplicial spaces.

Definition 2.3 We denote by Man the (discrete) category of smooth manifolds (of any dimension) and smooth maps, equipped with the Grothendieck topology of open covers.

Definition 2.4 A presheaf $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$ is *concordance-invariant* if, for all manifolds M, the map $F(M) \to F(M\times\mathbb{R})$ induced by the projection $M\times\mathbb{R}\to M$ is a weak equivalence.

Definition 2.5 Set

$$\mathbb{A}\colon \Delta \to \mathsf{Man}, \quad \mathbb{A}^n = \Big\{x\in\mathbb{R}^{n+1} \;\Big|\; \sum_i x_i = 1\Big\}.$$

Given a presheaf $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$, denote by $\boldsymbol{B}F$ the presheaf

$$\boldsymbol{B}F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}, \quad \boldsymbol{B}F(M) := |[k]\mapsto F(M\times\mathbb{A}^k)|,$$

where $|-|$ denotes the diagonal of a bisimplicial set.

In this section we show that $\boldsymbol{B}F$ is always concordance-invariant. Furthermore, if $\boldsymbol{B}F$ is an ∞-*sheaf*, then it is representable. The arguments are largely formal, so to make this structure more transparent we begin the discussion for an arbitrary category enriched over S and later specialize to the category of manifolds. These results are mostly a repackaging of [Morel and Voevodsky 1999]; see also [Herrmann and Strunk 2011].

Notation 2.6 Let $\boldsymbol{D}_0$ be a discrete category with products and $\mathbb{A}^\bullet$ a cosimplicial object in $\boldsymbol{D}_0$. This data defines a category enriched in spaces, denoted by $\boldsymbol{D}$, by declaring the set of n–simplices of the mapping space $\mathrm{map}_{\boldsymbol{D}}(X, Y)$ to be $\hom_{\boldsymbol{D}_0}(X \times \mathbb{A}^n, Y)$. The example that will be of interest to us here is $\boldsymbol{D}_0 = \mathsf{Man}$.

In this setting it makes sense to talk about concordance-invariant presheaves on $\boldsymbol{D}_0$.

Definition 2.7 Given a category $\boldsymbol{D}_0$ as in Notation 2.6, a presheaf $F\colon \boldsymbol{D}_0^{\mathrm{op}} \to \mathsf{S}$ is *concordance-invariant* if the map induced by the projection

$$F(X) \to F(X \times \mathbb{A}^1)$$

is a weak equivalence of spaces for all $X \in \boldsymbol{D}_0$.

A functor on $\boldsymbol{D}_0$ that can be enriched, ie lifted to a functor on $\boldsymbol{D}$, necessarily sends smooth homotopies to simplicial homotopies and smooth homotopy equivalences to simplicial homotopy equivalences. As such, it is automatically concordance-invariant. In Proposition 2.13 below, we will prove the converse.

Definition 2.8 Given a category $\boldsymbol{D}_0$ as in Notation 2.6, the *concordance resolution* of a functor $F\colon (\boldsymbol{D}_0)^{\mathrm{op}} \to \mathsf{S}$ is the functor

$$F(- \times \mathbb{A}^\bullet)\colon (\boldsymbol{D}_0)^{\mathrm{op}} \to \mathsf{sS}, \quad X \mapsto F(X \times \mathbb{A}^\bullet).$$

We denote the homotopy colimit of $F(- \times \mathbb{A}^\bullet)$ by

$$\boldsymbol{B}F(X) := \operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} F(X \times \mathbb{A}^n) = |F(X \times \mathbb{A}^\bullet)|.$$

Here we use the diagonal of a bisimplicial set as a model for the homotopy colimit over Δ^{op}.

For the case $\boldsymbol{D}_0 = \mathsf{Man}$ this definition of $\boldsymbol{B}F$ coincides with Definition 2.5.

Proposition 2.9 *Given a category $\boldsymbol{D}_0$ as in Notation 2.6, for any presheaf $F\colon \boldsymbol{D}_0^{\mathrm{op}} \to \mathsf{S}$, the functor $F(- \times \mathbb{A}^\bullet)$ lifts to an enriched functor $\boldsymbol{D}^{\mathrm{op}} \to \mathsf{sS}$.*

Proof We will define a simplicial map

$$\mathrm{map}(X, Y) \to \mathrm{map}(F(Y \times \mathbb{A}^\bullet), F(X \times \mathbb{A}^\bullet)).$$

Let $g\colon X \times \mathbb{A}^n \to Y$ be a morphism in $\boldsymbol{D}_0$. Given a morphism $\alpha\colon [k] \to [n]$ in Δ, consider the composition

$$F(Y \times \mathbb{A}^k) \xrightarrow{F(g \times \mathrm{id}_{\mathbb{A}^k})} F(X \times \mathbb{A}^n \times \mathbb{A}^k) \xrightarrow{F(\mathrm{id}_X \times \mathbb{A}^\alpha \times \mathrm{id}_{\mathbb{A}^k})} F(X \times \mathbb{A}^k \times \mathbb{A}^k) \xrightarrow{F(\mathrm{id}_X \times d)} F(X \times \mathbb{A}^k),$$

where $d\colon \mathbb{A}^k \to \mathbb{A}^k \times \mathbb{A}^k$ is the diagonal map. This is functorial in g and α and so defines a map between hom sets

$$\hom(X \times \mathbb{A}^n, Y) \to \hom(F(Y \times \mathbb{A}^\bullet) \times \Delta[n], F(X \times \mathbb{A}^\bullet)) \tag{2.10}$$

for each $n \geq 0$. Therefore, $F(- \times \mathbb{A}^\bullet)$ is enriched over spaces. □

Corollary 2.11 *Given a category $\boldsymbol{D}_0$ as in Notation 2.6, for any presheaf $F\colon \boldsymbol{D}_0^{\mathrm{op}} \to \mathsf{S}$, the presheaf $\boldsymbol{B}F$ is enriched over spaces and is concordance-invariant.*

Proof To see that $\boldsymbol{B}F$ is enriched, and hence concordance-invariant, postcompose (2.10) with the homotopy colimit functor (alias geometric realization or diagonal) and use the fact that it commutes with products. □

Remark 2.12 The functor $\boldsymbol{B}$ is homotopy left adjoint to the discretization functor

$$i^*\colon \mathsf{PSh}(\boldsymbol{D}) \to \mathsf{PSh}(\boldsymbol{D}_0),$$

given by the restriction along the inclusion $i\colon \boldsymbol{D}_0 \to \boldsymbol{D}$. This follows from the fact that $\boldsymbol{B}$ is a left adjoint functor whose value on a representable presheaf on $X \in \boldsymbol{D}_0$ is the representable presheaf on $i(X) \in \boldsymbol{D}$.

The following proposition implies that the category of enriched presheaves on $\boldsymbol{D}$ and the category of concordance-invariant presheaves on $\boldsymbol{D}_0$ have equivalent homotopy theories:

Proposition 2.13 *Given a category $\boldsymbol{D}_0$ as in Notation 2.6, a presheaf $F\colon \boldsymbol{D}_0^{\mathrm{op}} \to \mathsf{S}$ is concordance-invariant if and only if the map $F(X) \to i^*\boldsymbol{B}F(X)$ is a weak equivalence for all X.*

Proof If F is concordance-invariant then the simplicial object $F(X \times \mathbb{A}^\bullet)$ is homotopically constant with value $F(X)$. For the converse, consider the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & i^*\boldsymbol{B}F(X) \\ \downarrow & & \downarrow \\ F(X \times \mathbb{A}^1) & \longrightarrow & i^*\boldsymbol{B}F(X \times \mathbb{A}^1) \end{array}$$

The horizontal maps are weak equivalences by assumption. The vertical map on the right is a weak equivalence since $\boldsymbol{B}F$ is concordance-invariant. Thus, by the two-out-of-three property, the left map is a weak equivalence. □

Corollary 2.14 *The restriction functor*

$$i^*\colon \mathsf{PSh}(\boldsymbol{D}) \to \mathsf{PSh}(\boldsymbol{D}_0)$$

is a right Quillen equivalence, where $\mathsf{PSh}(\boldsymbol{D})$ *is equipped with the projective model structure and* $\mathsf{PSh}(\boldsymbol{D}_0)$ *is equipped with the* $\mathbb{A}^1$*–invariant projective model structure, ie the left Bousfield localization of the projective model structure with respect to the map* $\mathbb{A}^1 \to \mathbb{A}^0$*.*

2a Concordance-invariant ∞–sheaves on manifolds are representable

Definition 2.15 A presheaf $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$ is an ∞*–sheaf* if, for every manifold M and open cover $\{U_i \to M\}_{i\in I}$, the canonical map from $F(M)$ to the homotopy limit (over Δ) of the cosimplicial space

$$\prod_{i_0\in I} F(U_{i_0}) \substack{\leftarrow\\[-1em]\leftarrow\\[-1em]\leftarrow} \prod_{i_0,i_1\in I} F(U_{i_0}\cap U_{i_1}) \substack{\leftarrow\\[-1em]\leftarrow} \cdots$$

is a weak equivalence of spaces.

Any ∞–sheaf F satisfies $F(\varnothing)\simeq *$. This is implied by the descent condition for the empty cover of the empty manifold.

Remark 2.16 A set-valued sheaf is an ∞–sheaf of sets, and conversely. Indeed, the (homotopy) limit of a cosimplicial discrete space is, by initiality, computed by the limit of its truncation to its 1–coskeleton. A *stack is* a groupoid-valued ∞–sheaf [Hollander 2008]. Common alternative terminologies for ∞–sheaves include ∞–stacks and homotopy sheaves.

The following proposition is due to [Morel and Voevodsky 1999; Dugger 2001]:

Proposition 2.17 *Given a presheaf* $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$*, the presheaf* $\boldsymbol{B}F$ *is an* ∞*–sheaf if and only if the evaluation map*

$$\boldsymbol{B}F(M) \to \mathrm{map}(\mathrm{map}(*, M), BF)$$

is a weak equivalence for every M*, where the evaluation map is the adjoint to the simplicial map*

$$\mathrm{map}(*, M) \to \mathrm{map}(\boldsymbol{B}F(M), \boldsymbol{B}F(*)) \to \mathrm{map}(\boldsymbol{B}F(M), BF)$$

gotten by the enrichment afforded by Corollary 2.11 and the map $\boldsymbol{B}F(*) \to BF$ *being the Kan complex replacement of* $\boldsymbol{B}F(*)$*. Here* $\mathrm{map}(*, M)$ *is weakly equivalent to* $\mathrm{Sing}\, M$*, the singular simplicial set of* M*.*

Proof Take a good open cover $\{U_i\}_{i\in I}$ of M and let $\boldsymbol{U}_\bullet\colon \Delta^{\mathrm{op}} \to \mathsf{Man}$ denote its Čech nerve. There is a commutative square

$$\begin{array}{ccc}
\boldsymbol{B}F(M) & \longrightarrow & \boldsymbol{R}\mathrm{map}(\mathrm{Sing}\, M, \boldsymbol{B}F(*)) \\
\downarrow & & \downarrow \\
\mathrm{holim}_{n\in\Delta}\, \boldsymbol{B}F(\boldsymbol{U}_n) & \longrightarrow & \mathrm{holim}_{n\in\Delta}\, \boldsymbol{R}\mathrm{map}(\mathrm{Sing}\, \boldsymbol{U}_n, \boldsymbol{B}F(*))
\end{array}$$

where Sing denotes the singular simplicial set functor. The right-hand vertical arrow is an equivalence since $\operatorname{hocolim}_{n\in\Delta} U_n \simeq M$. The lower horizontal arrow is an equivalence since $\boldsymbol{B}F(V) \simeq \boldsymbol{B}F(*)$ for V contractible (by concordance-invariance of $\boldsymbol{B}F$). The statement now follows by the two-out-of-three property. □

2b Homotopy groups of *BF*

In this section, we explain how to compute the homotopy groups of BF. For a basepoint b in the d–dimensional sphere S^d and $x \in F(*)$, let $\boldsymbol{B}F(S^d)_x$ denote the homotopy fiber of

$$\boldsymbol{B}F(b)\colon \boldsymbol{B}F(S^d) \to \boldsymbol{B}F(*)$$

over the image of x in $\boldsymbol{B}F(*)$.

Proposition 2.18 *Let $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$ be a presheaf satisfying the ∞–sheaf property with respect to finite covers, and let $x \in F(*)$. The map*

$$\pi_0 \boldsymbol{B}F(S^d)_x \to \pi_d(\boldsymbol{B}F(*), x) = \pi_d(BF, x)$$

is an isomorphism.

Proof Under the assumption on F, the conclusion of Theorem 1.1 holds for any *compact* manifold M. This will be shown in Corollary 5.10. Therefore, the top map in the commutative square

$$\begin{array}{ccc} \boldsymbol{B}F(S^d) & \longrightarrow & \boldsymbol{R}\mathrm{map}(S^d, \boldsymbol{B}F(*)) \\ {\scriptstyle \boldsymbol{B}F(b)}\downarrow & & \downarrow{\scriptstyle \boldsymbol{R}\mathrm{map}(b, \boldsymbol{B}F(*))} \\ \boldsymbol{B}F(*) & \longrightarrow & \boldsymbol{R}\mathrm{map}(*, \boldsymbol{B}F(*)) \end{array}$$

is a weak equivalence. The bottom map is a weak equivalence by construction. Thus the induced map of vertical homotopy fibers over a point $x \in F(*)$ is a weak equivalence. Taking π_0 of the map between homotopy fibers, we obtain the result. □

Remark 2.19 Elements in $\pi_0 \boldsymbol{B}F(S^d)_x$ are concordance classes of sections of F over S^d which restrict to x on $b \in S^d$. We postpone the explanation to Lemma 4.13.

Remark 2.20 In the special case when F is a concrete sheaf of sets, ie a diffeological space in the sense of [Souriau 1980], Proposition 2.18 resolves in the affirmative a conjecture of [Christensen and Wu 2014, Section 1] on the isomorphism of smooth homotopy groups of a diffeological space with the simplicial homotopy groups of its smooth singular complex. Christensen and Wu [2014, Theorem 4.11] proved this conjecture for projectively fibrant diffeological spaces, whereas Proposition 2.18 proves it for arbitrary simplicial presheaves satisfying the ∞–sheaf property for finite covers, a much bigger class that includes all sheaves of sets, in particular all diffeological spaces.

3 Weak Kan fibrations

As a warmup to the ideas in this section, we will prove that the concordance relation $\sim_c$ is an equivalence relation when F is an ∞–sheaf. This generalizes the standard fact that smooth homotopy is an equivalence relation, but the core of the argument is identical: gluing a pair of smooth maps along an open submanifold yields a smooth map.

Lemma 3.1 *If $F: \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$ is an ∞–sheaf, then $\sim_c$ is an equivalence relation on $F(M)_0$, the set of 0–simplices in $F(M)$.*

Proof Reflexivity and symmetry are obvious. To establish transitivity, suppose σ_0, σ_1 and σ_2 are such that $\sigma_0 \sim_c \sigma_1$ and $\sigma_1 \sim_c \sigma_2$. Let i_k denote the inclusion of $M \times \{k\}$ into $M \times \mathbb{A}^1$ and pick sections σ_{01} and σ_{12} over $M \times \mathbb{A}^1$ such that $i_0^*\sigma_{01} = \sigma_0$, $i_1^*\sigma_{01} = i_0^*\sigma_{12} = \sigma_1$ and $i_1^*\sigma_{12} = \sigma_2$. Take a smooth map $r: \mathbb{A}^1 \to \mathbb{A}^1$ which fixes 0 and 1 and maps the complement of a small neighborhood of $\frac{1}{2}$ to $\{0, 1\}$. The sections $r^*\sigma_{01}$ and $r^*\sigma_{12}$ over $M \times \mathbb{A}^1$ are such that the restriction of $r^*\sigma_{01}$ to an *open* neighborhood of $[1, \infty)$ agrees with the restriction of $r^*\sigma_{12}$ to an *open* neighborhood of $(-\infty, 0]$. So, using the sheaf property and reparametrizing, we may glue these sections to obtain a section σ_{012} over $M \times \mathbb{A}^1$ with $i_0^*\sigma_{012} = \sigma_0$ and $i_1^*\sigma_{012} = \sigma_2$, ie $\sigma_0 \sim_c \sigma_2$. □

The fact that $\sim_c$ is an equivalence relation is a shadow of an important property possessed by the concordance resolution: it is a 0–*weak Kan complex* (Definition 3.9). Informally, the *weak* nature can be seen in the proof of transitivity at the point where sections σ_{01} and σ_{12} are replaced by $r^*\sigma_{01}$ and $r^*\sigma_{12}$. This step is essential since sections cannot be glued along closed sets. The failure of gluing along closed sets also means that concordance resolution does not satisfy the usual Kan condition as it does not have the right lifting property with respect to $\Lambda_1^2 \to \Delta^2$. Similar features of smooth geometry allow us to show that certain restriction maps for the concordance resolution have analogous weak fibrancy properties. The key definition formalizing this property is that of a *weak Kan fibration*.

3a Kan fibrations and weak Kan fibrations of simplicial spaces

In this section, we define and investigate weak Kan fibrations of simplicial spaces (or sets). These generalize Kan fibrations and are related to (and inspired by) Dold fibrations [1963] of topological spaces. We refer the reader to the appendix for background on simplicial spaces.

The following definition is discussed by Lurie [2011; 2018, Definition A.5.2.1]. Our definition is essentially the same, except that we formulate it for Reedy fibrant simplicial spaces, to avoid mentioning Reedy fibrant replacements.

Definition 3.2 Let $f: X \to Y$ be a Reedy fibration between Reedy fibrant simplicial spaces. We say that f is a *Kan fibration* if it has the right lifting property with respect to all horn inclusions (Definition 2.2).

That is, for every solid square

$$\begin{array}{ccc} \Lambda_k[n] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta[n] & \longrightarrow & Y \end{array}$$

there is a lift as pictured, where $n \geq 1$ and $0 \leq k \leq n$. Similarly, we say that f is a *trivial Kan fibration* if it has the right lifting property with respect to $\partial\Delta[n] \to \Delta[n]$ for all $n \geq 0$.

Unfortunately, Definition 3.2 is not applicable in the situations of interest to us. In particular, none of the maps in the crucial Propositions 4.1, 4.2 and 4.3 satisfy Definition 3.2, so a result like [Lurie 2018, Theorem A.5.4.1] is not applicable. An explicit counterexample is provided by Example 4.10. Therefore, we relax Definition 3.2 to Definition 3.8. First, we define an appropriate generalization of the right lifting property.

Definition 3.3 Let $f: X \to Y$ be a Reedy fibration between Reedy fibrant simplicial spaces. We say that f has the *weak right lifting property* (weak RLP) with respect to a map $i: A \hookrightarrow B$ (and i has the weak LLP with respect to f) if for every commutative square

$$\begin{array}{ccc} A & \xrightarrow{\beta} & X \\ {\scriptstyle i}\downarrow & \overset{\widetilde{\alpha}}{\nearrow} & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\alpha} & Y \end{array}$$

there is a lift $\widetilde{\alpha}$ as pictured, making the lower triangle commute strictly and the upper triangle commute *up to a specified vertical homotopy*. Such a homotopy consists of a map of simplicial spaces

$$H: A \times \Delta[1] \to X$$

subject to the requirement that $H_0 = \beta$, $H_1 = \widetilde{\alpha} \circ i$ and $f \circ H = \alpha \circ i \circ \pi$, where π denotes the projection of $A \times \Delta[1]$ onto A.

Remark 3.4 It will be useful to have some reformulations of Definition 3.3. Under the Reedy fibrancy hypothesis above, the map $f: X \to Y$ has the weak RLP with respect to $i: A \to B$ if and only if for every commutative square in the background of

$$\begin{array}{ccccc} A & & \longrightarrow & & X \\ & \searrow & & \nearrow & \\ {\scriptstyle i}\downarrow & & M(i) & & \downarrow{\scriptstyle f} \\ & & \downarrow & & \\ B & & \longrightarrow & & Y \\ & \searrow & & \nearrow & \\ & & B & & \end{array}$$

there exists a map from the *mapping cylinder* $M(i) = B \sqcup_i (A \times \Delta[1])$ to X making the diagram commute strictly. Here the map in the foreground $M(i) \to B$ collapses $A \times \Delta[1]$ to A; we denote it by $\pi(i)$.

To put it differently, consider the category $\mathsf{sS}^{[1]} = \mathrm{Fun}(0 \to 1, \mathsf{sS})$ whose objects are maps of simplicial spaces and morphisms are commutative squares. Then the requirement above is that the map induced by precomposition

$$\mathrm{map}\begin{pmatrix} M(i) & X \\ \downarrow & , \downarrow \\ B & Y \end{pmatrix} \to \mathrm{map}\begin{pmatrix} A & X \\ \downarrow , & \downarrow \\ B & Y \end{pmatrix}$$

(with the square on the left in the diagram above) is surjective on 0–simplices. Here map denotes the space of morphisms in $\mathsf{sS}^{[1]}$.

The Reedy model structure appears in the previous definitions as an artifact that guarantees homotopy-invariance with respect to degreewise weak homotopy equivalences of simplicial spaces. But it is possible (and worthwhile) to formulate a more homotopical definition of the weak RLP.

Definition 3.5 A map $f : X \to Y$ between arbitrary simplicial spaces satisfies the *weak right lifting property* (weak RLP) with respect to a map $i : A \to B$ if

$$\boldsymbol{R}\mathrm{map}(\pi(i), f) \to \boldsymbol{R}\mathrm{map}(i, f)$$

is surjective on π_0, where $\boldsymbol{R}$map refers to the derived mapping space computed in the category $\mathsf{sS}^{[1]}$ with objectwise weak equivalences.

Remark 3.6 We emphasize the homotopy-invariance properties of this definition: a map f has the weak RLP with respect to a map i if and only if it has the weak RLP with respect to any map (degreewise) weakly equivalent to i. Also, if a map f has the weak RLP with respect to i, then so does any map (degreewise) weakly equivalent to f.

Proposition 3.7 *A Reedy fibration f satisfies the weak RLP in the sense of Definition 3.3 if and only if it satisfies the weak RLP in the sense of Definition 3.5.*

Proof Equip sS with the Reedy (= injective) model structure and, relative to it, also equip $\mathsf{sS}^{[1]}$ with the injective model structure. In this model structure on $\mathsf{sS}^{[1]}$, all objects are cofibrant. Cofibrations are morphisms which are objectwise Reedy cofibrations of simplicial spaces, ie degreewise injections. Fibrant objects are Reedy fibrations between Reedy fibrant simplicial spaces.

The morphism $i \to \pi(i)$, ie the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & M(i) \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \end{array}$$

is a cofibration since the horizontal maps are degreewise injections. Also, the map f is a fibrant object in $\mathsf{sS}^{[1]}$. Therefore, the induced map

$$\mathrm{map}(\pi(i), f) \to \mathrm{map}(i, f)$$

is a fibration between Kan complexes, which is weakly equivalent to

$$\boldsymbol{R}\mathrm{map}(\pi(i), f) \to \boldsymbol{R}\mathrm{map}(i, f).$$

The result now follows, since a fibration of simplicial sets is surjective on π_0 if and only if it is surjective on 0–simplices. □

Definition 3.8 A map $X \to Y$ between simplicial spaces is a *weak Kan fibration* if it has the weak right lifting property with respect to the maps

$$h_i : \mathrm{Sd}^i(\Lambda_k[n]) \hookrightarrow \mathrm{Sd}^i(\Delta[n])$$

for each $i \geq 0$, $n \geq 1$ and $0 \leq k \leq n$. Here the functor $\mathrm{Sd}\colon \mathsf{sS} \to \mathsf{sS}$ is as in Definition A.12.

The definition also makes sense for maps $X \to Y$ of simplicial sets by regarding them as simplicial discrete spaces.

Note that every simplicial space is a weak Kan complex, in the sense that the map $X \to *$ is a weak Kan fibration. This may seem odd at first, but it makes sense in light of Section 3b, as every space is tautologically quasifibrant. A more interesting variation is:

Definition 3.9 Given $l \geq -1$, a *l-weak Kan complex* is a simplicial space X such that the terminal map $X \to *$ is a weak Kan fibration in which the vertical homotopies preserve the l–skeleton of the horn inclusions h_i.

So a (-1)–weak Kan complex is simply a weak Kan complex, ie a simplicial space, and a ∞–weak Kan complex is a Kan complex (in the usual sense, as in Definition 3.2). For a 0–weak Kan complex X, the image of $\pi_0 X_1 \to \pi_0 X_0 \times \pi_0 X_0$ is an equivalence relation (cf Lemma 3.1). We will not make use of the notion of l–weak Kan complex for $l > 0$, and for $l = 0$ we will use it in Lemma 5.13.

Example 3.10 A Kan fibration (Definition 3.2) is a weak Kan fibration. Of course, the usual definition of a Kan fibration does not mention subdivisions; this is because a map satisfying the strict RLP against all horn inclusions $\Lambda_k[n] \to \Delta[n]$ automatically satisfies the same property against all subdivisions of those, since these subdivided horn inclusions can be presented using cobase changes and the strict LLP is stable under cobase change, so the strict LLP for horn inclusions implies the strict LLP for subdivided horn inclusions. The same is *not* true for the weak RLP, so we need to take subdivisions seriously.

Likewise, if X is a l–weak Kan complex and $K \hookrightarrow L$ is an inclusion of simplicial sets, it does not follow automatically that $X^L \to X^K$ is a weak Kan fibration.

Lemma 3.11 *Kan's* Ex *functor* (see Definition A.12) *preserves weak Kan fibrations and so does* Ex^∞.

Proof The functor Ex is right adjoint to the subdivision functor, so we are investigating a square of the form

(3.12)
$$\begin{array}{ccc} \mathrm{Sd}^{i+1}\Lambda_k[n] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Sd}^{i+1}\Delta[n] & \longrightarrow & Y \end{array}$$

Since f is a weak Kan fibration, there is a lift as shown, and a homotopy $H\colon \mathrm{Sd}^{i+1}\Lambda_k[n]\times\Delta[1]\to X$. Since there is always a map from the subdivision of the product to the product of the subdivisions, we can precompose H with

$$\mathrm{Sd}(\mathrm{Sd}^i\Lambda_k[n]\times\Delta[1])\to\mathrm{Sd}^{i+1}\Lambda_k[n]\times\mathrm{Sd}\,\Delta[1]\to\mathrm{Sd}^{i+1}\Lambda_k[n]\times\Delta[1].$$

This gives the required homotopy for the upper triangle which is vertical over Y, proving the result. The case of Ex^∞ follows automatically since a map from a finite-dimensional simplicial space to $\mathrm{Ex}^\infty X$ factors through some finite $\mathrm{Ex}^i X$. □

Remark 3.13 Weak Kan fibrations are stable under various operations. They are stable under *homotopy* base change (with respect to degreewise weak equivalences of simplicial spaces). In other words, weak Kan fibrations that are moreover Reedy fibrations between Reedy fibrant objects are stable under pullback.

Weak Kan fibrations are also stable under fiberwise homotopy retracts (that is, if $g\colon W\to Y$ is a homotopy retract over Y of a weak Kan fibration $f\colon X\to Y$ then g is a weak Kan fibration). In particular, weak Kan fibrations are stable under fiberwise homotopy equivalences. Moreover, if we allow subdivisions of the vertical homotopies in the definition of the weak lifting property, ie if we replace $\Delta[1]$ by $\mathrm{Sd}^i\Delta[1]$ for $i\geq 0$ in Definition 3.3, then a composition of two weak Kan fibrations is also a weak Kan fibration. (Although, the resulting notion would presumably be weaker than the one we are using.) Since these properties will not be used in what follows, and the proofs are not particularly difficult, we omit further explanations.

3b Weak Kan fibrations are realization fibrations

Definition 3.14 [Rezk 2014] A map $f\colon X\to Y$ of simplicial spaces is a *realization fibration* if for every $Z\to Y$ the induced map

(3.15)
$$|X\times^h_Y Z|\to|X|\times^h_{|Y|}|Z|$$

is a weak equivalence of spaces. The vertical bars refer to the diagonal simplicial set, which models the homotopy colimit over Δ^{op}.

Remark 3.16 Realization fibrations are related to quasifibrations in the sense of [Dold and Thom 1958]. For example, if a map $f\colon X \to Y$ between simplicial *sets* is a realization fibration, then the map (3.15) with Z a point is identified with the inclusion of the fiber of f into the homotopy fiber. That is, f is a quasifibration. On the other hand, not all quasifibrations are realization fibrations: realization fibrations are stable under homotopy pullback, whereas quasifibrations need not be.

We now turn to the main technical result of the section.

Theorem 3.17 *A weak Kan fibration is a realization fibration.*

As we already emphasized, the subdivisions of simplices and horns in the definition of a weak Kan fibration are important for a number of reasons. Lemma 3.11 is one such reason, that will be exploited later on. Below is another.

Example 3.18 This is an example of a map which has the weak right lifting property against the (nonsubdivided) map (h_0) but is not a realization fibration. Suppose X is the union of three nondegenerate 1–simplices as in the picture

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & \longrightarrow & \bullet \end{array}$$

and $Y = \Delta[1]$. Let $f\colon X \to Y$ be the projection in the vertical direction. This is not a Kan fibration: there are 2–horns $\Lambda^k[2]$ in X that cannot be filled. On the other hand, f has the *weak* right lifting property with respect to the map (h_0). But f is not a quasifibration, since at one point the fiber is disconnected while the homotopy fiber is not. Hence f cannot be a realization fibration. This is not a contradiction: f is not a weak Kan fibration as it does not have the weak right lifting property for (h_2), the second subdivision of the horn inclusion.

Proposition 3.19 *Suppose $f\colon X \to Y$ is a Reedy fibration between Reedy fibrant simplicial spaces. Then f is a weak Kan fibration of simplicial spaces if and only if $|f|\colon |X| \to |Y|$ is a weak Kan fibration of simplicial sets. The vertical bars denote the diagonal functor.*

Proof Let us denote by κ the map $\mathrm{sd}^i\,\Lambda^n_k \to \mathrm{sd}^i\,\Delta^n$ for $i \geq 0$. Write δ for the functor which sends a simplicial set K to the corresponding simplicial discrete space $[n] \mapsto K_n$. Recall the commutative square of spaces $\kappa \to \pi(\kappa)$ where $\pi(\kappa)$ is the projection of mapping cylinder $M(\kappa)$ onto $\mathrm{sd}^i\,\Delta^n$. By Definition 3.8, the map f is a weak Kan fibration if and only if

$$\boldsymbol{R}\mathrm{map}\big(\delta(\pi(\kappa)), f\big) \to \boldsymbol{R}\mathrm{map}(\delta(\kappa), f)$$

is surjective on π_0.

The realization (ie diagonal) functor has a left adjoint $d_!\colon \mathsf{S} \to \mathsf{sS}$, and there is a natural transformation $d_! \to \delta$ which is a weak equivalence (Lemma A.8). Therefore, f is a weak Kan fibration if and only if

$$\boldsymbol{R}\mathrm{map}(d_!\pi(\kappa), f) \to \boldsymbol{R}\mathrm{map}(d_!\kappa, f)$$

is surjective on π_0. Regarding $\mathsf{sS}^{[1]}$ with the injective model structure, the map $d_!\kappa \to d_!\pi(\kappa)$ is a cofibration between cofibrant objects (since $d_!$ sends monomorphisms to monomorphisms), and the target f is fibrant by hypothesis. As such, the above holds if and only if the Kan fibration between Kan simplicial sets

$$\mathrm{map}(d_!\pi(\kappa), f) \to \mathrm{map}(d_!(\kappa), f)$$

is surjective. By adjunction, this holds if and only if $|f|$ has the weak RLP with respect to κ. (Note that $|f|$, being a map between simplicial discrete spaces, is automatically a Reedy fibration between Reedy fibrant objects, and so a fibrant object in $\mathsf{sS}^{[1]}$.) □

Remark 3.20 The same proof, with κ of the form $\Lambda^n_k \to \Delta^n$ and with $\pi(\kappa)$ replaced by the identity $\Delta^n \to \Delta^n$, shows that a Reedy fibration between Reedy fibrant simplicial spaces f is a (trivial) Kan fibration if and only if $|f|$ is a (trivial) Kan fibration.

In order to prove that weak Kan fibrations are realizations fibrations, we will use the following criterion:

Theorem 3.21 [Rezk 2014] *A map $f: X \to Y$ of simplicial spaces is a realization fibration if and only if, for all maps $\Delta[m] \to Y$ and $\Delta[0] \to \Delta[m]$, the induced map on pullbacks*

$$X \times^h_Y \Delta[0] \to X \times^h_Y \Delta[m]$$

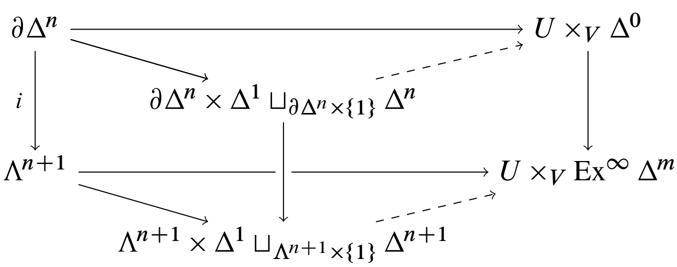
is a weak equivalence after realization.

Proof of Theorem 3.17 Let $f: X \to Y$ be a weak Kan fibration. We will verify that f satisfies the condition in Theorem 3.21. We may assume, without loss of generality, that f is a Reedy fibration between Reedy fibrant simplicial spaces. Then the homotopy pullbacks above become pullbacks. Using the fact that realization and Ex^∞ commute with finite limits, our task is then to show that

$$\mathrm{Ex}^\infty |X| \times_{\mathrm{Ex}^\infty |Y|} \mathrm{Ex}^\infty \Delta^0 \to \mathrm{Ex}^\infty |X| \times_{\mathrm{Ex}^\infty |Y|} \mathrm{Ex}^\infty \Delta^m$$

is a weak equivalence of Kan complexes. Since f is a weak Kan fibration, the same can be said of $|f|$ by Proposition 3.19 and of $\mathrm{Ex}^\infty |f|$ by Lemma 3.11. To simplify the notation, let us write $g: U \to V$ for $\mathrm{Ex}^\infty |f|: \mathrm{Ex}^\infty |X| \to \mathrm{Ex}^\infty |Y|$.

In view of Proposition A.1 (and Corollary A.2 and Example A.5), we will show that for every solid diagram



there are dashed maps as pictured. Let us write $A \to B$ for the middle vertical arrow. Consider the map

$$\Lambda^{n+1} \sqcup_{\partial\Delta^n \times \{0\}} A \to \operatorname{Ex}^\infty \Delta^m \tag{3.22}$$

determined by the lower horizontal map in the diagram above and by the map $A \to \Delta^0 \to \operatorname{Ex}^\infty \Delta^m$. Since the terminal map $\operatorname{Ex}^\infty \Delta^m \to \Delta^0$ is a trivial Kan fibration, the map (3.22) extends along the inclusion

$$\Lambda^{n+1} \sqcup_{\partial\Delta^n \times \{0\}} A \hookrightarrow B.$$

Next, we want to define a map $B \to U$ which is compatible with the composition

$$B \to \operatorname{Ex}^\infty \Delta^m \to V$$

of the map we have just constructed. This will give us the lower dashed map in the diagram above. Consider the diagram

$$\begin{array}{ccc} \Lambda^{n+1} & \longrightarrow & U \\ \big\downarrow & & \big\downarrow g \\ \Delta^{n+1} & \longrightarrow & V \end{array}$$

where the lower map is the composition $\Delta^{n+1} \to B \to \operatorname{Ex}^\infty \Delta^m \to V$. Since g is a weak Kan fibration, we obtain a lift and vertical homotopy, ie the required map $B \to U$. Therefore, we have defined a map $B \to U \times_V \operatorname{Ex}^\infty \Delta^m$, whose restriction to A factors through $U \times_V \Delta^0$. □

4 Weak Kan fibrancy of the concordance resolution

Let $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$ be an ∞–sheaf. In this section and the next, we apply the general theory developed in the previous sections to prove Theorem 1.2. The goal of this section is to prove the following three propositions:

Proposition 4.1 *For any ∞–sheaf $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$, the simplicial space $[n] \to F(\mathbb{A}^n)$ is a 0–weak Kan complex.*

Proposition 4.2 *For any ∞–sheaf $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$, the map of simplicial spaces*

$$F(\mathbb{A}^\bullet \times \mathbb{A}^1) \to F(\mathbb{A}^\bullet \times \partial\mathbb{A}^1)$$

is a weak Kan fibration. Here $\partial\mathbb{A}^1 = \mathbb{A}^0 \sqcup \mathbb{A}^0$ is the disjoint union of two points.

Proposition 4.3 *Let $* \hookrightarrow S^n$ be an inclusion of a basepoint into the smooth n–dimensional sphere. For any ∞–sheaf $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$, the induced map of simplicial spaces*

$$F(\mathbb{A}^\bullet \times S^n) \to F(\mathbb{A}^\bullet)$$

is a weak Kan fibration.

We use these to prove Theorem 1.2 in Section 5, and the reader may wish to jump directly to that section to see these propositions in action. In fact, in that section and in the remainder of the paper, we will only need Propositions 4.2 and 4.3. We include Proposition 4.1 because its proof anticipates much of the proof of the other two propositions.

The proofs of these propositions are based on the following simple observation. If one were to try to prove that $F(\mathbb{A}^\bullet)$ is a Kan complex in the sense of Definition 3.2, an obvious approach would be to construct a deformation retraction from the simplex $\mathbb{A}^n$ to its horn. This is possible in the topological and PL settings, but not possible smoothly, and a rigorous proof of this is provided in Example 4.10. However, the basic idea can be salvaged if one only asks for $F(\mathbb{A}^\bullet)$ to be 0-*weak* Kan, which roughly translates into asking for a retraction up to a suitable homotopy. This parallels the proof that concordance is an equivalence relation (Lemma 3.1): we modify smooth maps between manifolds (via homotopies) to achieve certain constancy properties. A relative version of this line of reasoning applies to the maps in Propositions 4.2 and 4.3.

4a The sheaf associated to a simplicial set

Definition 4.4 Denote by

$$\|-\|_{\mathrm{pre}}\colon \mathsf{sS} \to \mathsf{PSh}(\mathsf{Man}, \mathsf{S})$$

the simplicial left adjoint functor that sends $\Delta[n]$ to the representable presheaf of $\mathbb{A}^n$, $\hom(-, \mathbb{A}^n)$. The corresponding simplicial right adjoint functor is

$$\mathsf{PSh}(\mathsf{Man}, \mathsf{S}) \to \mathsf{sS}, \quad F \mapsto (n \mapsto F(\mathbb{A}^n)).$$

For a simplicial space K, the presheaf $\|K\|_{\mathrm{pre}}\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$ is given by the coend

$$K_n \otimes_{[n]\in\Delta} \hom(-, \mathbb{A}^n).$$

If K is a simplicial set (ie a simplicial discrete space), then $\|K\|_{\mathrm{pre}}$ is a presheaf of sets.

Remark 4.5 The adjunction of Definition 4.4 is Quillen if both categories are equipped with the projective model structure or with the injective model structure. Therefore, there is a weak equivalence

$$\boldsymbol{R}\mathrm{map}(\|K\|_{\mathrm{pre}}, F) \to \boldsymbol{R}\mathrm{map}(K, F(\mathbb{A}^\bullet))$$

natural in the simplicial space K and the presheaf F.

Remark 4.6 Note that $\|K\|_{\mathrm{pre}}$ is usually not a sheaf of sets. For a simple example illustrating this, take $\|\Lambda^2_1\|$ which is the pushout

$$\hom(-, \mathbb{A}^1) \sqcup_{\hom(-,\mathbb{A}^0)} \hom(-, \mathbb{A}^1)$$

and pick an open cover of $\mathbb{R}^1$ by two open sets and compatible sections over each that do not lift to a section over $\mathbb{R}^1$.

Definition 4.7 Given a simplicial set K, denote by $\|K\|$ the associated sheaf of sets of $\|K\|_{\mathrm{pre}}$.

Recall that, for presheaves of *sets*, the notions of a sheaf and ∞–sheaf coincide (Remark 2.16). For such presheaves, sheafification and ∞–sheafification also agree. A reference is Dugger, Hollander and Isaksen [2004, Proposition A.2]. (One can also prove this directly by comparing the usual sheafification formula, given by the so-called plus construction, to its ∞–counterpart applied to a presheaf of sets.) If we tried to define $\|K\|$ for simplicial *spaces* K, we would have had to use ∞–sheafification from the start.

Remark 4.8 Since sheafification is left adjoint to the inclusion of sheaves into presheaves, for F an ∞–sheaf the weak equivalence above lifts to a weak equivalence

$$\boldsymbol{R}\mathrm{map}(\|K\|, F) \xrightarrow{\simeq} \boldsymbol{R}\mathrm{map}(K, F(\mathbb{A}^\bullet))$$

natural in the simplicial space K and the ∞–sheaf F.

Remark 4.9 To be more concrete, suppose K is the simplicial set associated to a simplicial complex with vertex set V and take X to the union of affine subspaces spanned by the simplices of K. Then $\|K\|(M)$ is the set of smooth maps $M \to \mathbb{R}^V$ that land in X.

Example 4.10 Consider the sheaf of sets $F = \|\Lambda_1[2]\|$. The simplicial object $n \mapsto F(\mathbb{A}^n)$ is not a Kan complex in the sense of Definition 3.2. Indeed, consider the horn $\Lambda_1[2] \to (n \mapsto F(\mathbb{A}^n))$ given by the adjoint map of the identity map on F. This horn does not admit a filling by $\Delta[2]$. Indeed, such a filling has to be a section $s \in F(\mathbb{A}^2)$ that restricts to the identity map on $\|\Lambda_1[2]\|$. Since $\|\Lambda_1[2]\|$ is the sheafification of the sheaf of sets $\|\Lambda_1[2]\|_{\mathrm{pre}}$, in some neighborhood of the vertex $1 \in \mathbb{A}^2$, the section s must factor through one of the two 1–dimensional faces of $\Lambda_1[2]$. However, possessing such a factorization means that s cannot restrict to the identity map on any neighborhood of the vertex 1 in $\|\Lambda_1[2]\|$. Thus, $n \mapsto F(\mathbb{A}^n)$ is not a Kan complex in the sense of Definition 3.2.

Proposition 4.11 *Let F be an injectively fibrant object of* $\mathsf{PSh}(\mathsf{Man}, \mathsf{S})$. *Then the maps in Propositions 4.2 and 4.3 are Reedy fibrations between Reedy fibrant simplicial spaces.*

Proof We show that the map in Proposition 4.2 is a Reedy fibration. The argument for the one in Proposition 4.3 is similar. Let $A \to B$ be a trivial Reedy cofibration of simplicial spaces, ie a map of simplicial spaces which is a degreewise monomorphism and a degreewise weak equivalence. Using the adjunction of Definition 4.4, denote by Q the pushout of presheaves

(4.12)
$$\begin{array}{ccc} \|A\|_{\mathrm{pre}} \times \partial\mathbb{A}^1 & \longrightarrow & \|B\|_{\mathrm{pre}} \times \partial\mathbb{A}^1 \\ \downarrow & & \downarrow \\ \|A\|_{\mathrm{pre}} \times \mathbb{A}^1 & \longrightarrow & Q \end{array}$$

By adjunction, $F(\mathbb{A}^\bullet \times \mathbb{A}^1) \to F(\mathbb{A}^\bullet \times \partial\mathbb{A}^1)$ has the right lifting property with respect to $A \to B$ if and only if every solid diagram of presheaves

$$\begin{array}{ccc} Q & \longrightarrow & F \\ \big\downarrow & \nearrow & \\ \|B\|_{\mathrm{pre}} \times \mathbb{A}^1 & & \end{array}$$

has a lift as pictured. The existence of this lift is part of the so-called pushout–product axiom, which holds for the category of presheaves on Man. But we'll provide a short argument here. The top horizontal map in square (4.12) is a trivial cofibration since $\|-\|_{\mathrm{pre}}$ is a left Quillen functor. Since trivial cofibrations are stable under cobase change, the lower horizontal map is also a trivial cofibration. But the composition

$$\|A\|_{\mathrm{pre}} \times \mathbb{A}^1 \to Q \to \|B\|_{\mathrm{pre}} \times \mathbb{A}^1$$

is also a trivial cofibration by hypothesis, and so, by two-out-of-three, the right-hand map is a weak equivalence. The right-hand map is also an injective cofibration, as can be checked directly. Since F is injectively fibrant, we conclude that the dashed map exists. $\square$

Lemma 4.13 *Let $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$ be a presheaf satisfying the ∞–sheaf property with respect to finite covers and let $x \in F(*)$. Let $F(S^d)_x$ denote the homotopy fiber over x of the map $F(S^d) \to F(*)$ induced by a choice of basepoint in the d–sphere S^d. There is a canonical isomorphism*

$$(\pi_0 F(S^d)_x)/{\sim} \to \pi_0 \boldsymbol{B}F(S^d)_x,$$

where $\sim$ is the equivalence relation of concordance and $\boldsymbol{B}F(S^d)_x$ is the homotopy fiber of $\boldsymbol{B}F(S^d) \to \boldsymbol{B}F(*)$ as in Proposition 2.18. Therefore, the quotient of $\pi_0 F(S^d)_x$ by the equivalence relation of concordance is isomorphic to $\pi_d(\boldsymbol{B}F(*), x)$.

Proof By Proposition 4.3, the map induced by a choice of basepoint in S^d,

$$F(\mathbb{A}^\bullet \times S^d) \to F(\mathbb{A}^\bullet),$$

is a weak Kan fibration. Applying the homotopy colimit functor to this map and then taking its homotopy fiber over $x \in F(\mathbb{A}^0)$ yields the space $\boldsymbol{B}F(S^d)_x$, by definition.

Let $Z_\bullet := \mathrm{hofiber}_x(F(\mathbb{A}^\bullet \times S^d) \to F(\mathbb{A}^\bullet))$. The canonical map

$$|Z_\bullet| \to \boldsymbol{B}F(S^d)_x$$

is a weak equivalence since weak Kan fibrations are realization fibrations (Theorem 3.17). Moreover, weak Kan fibrations are stable under base change, so $Z_\bullet$ is a 0–weak Kan complex and, as such, the relation on $\pi_0 Z_0$ determined by the faces $\pi_0 Z_1 \to \pi_0 Z_0 \times \pi_0 Z_0$ is an equivalence relation (cf Lemma 3.1). This equivalence relation is the concordance relation on $\pi_0 Z_0 = \pi_0 F(S^d)_x$.

The second statement follows from Proposition 2.18. $\square$

4b Closed simplices

Definition 4.14 Denote by $\mathbf{\Delta}^n\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{Sets}$ the subsheaf of the representable sheaf of sets $\hom(-, \mathbb{A}^n)$ consisting of sections $X \to \mathbb{A}^n$ whose image is contained in the closed simplex $\mathbb{R}^{n+1}_{\geq 0} \cap \mathbb{A}^n \subset \mathbb{A}^n$.

Definition 4.15 Given a simplicial set K, denote by $\overline{\|K\|}\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{Sets}$ the subsheaf of sets of $\|K\|$ consisting of smooth maps $U \to M$ that factor locally through some *closed* simplex of K in M. In formulas,

$$\overline{\|K\|} = L(\overline{\|K\|}_{\mathrm{pre}}) = L(K_n \otimes_{[n]\in\Delta} \mathbf{\Delta}^n),$$

where $\mathbf{\Delta}^n$ is as in Definition 4.14, $L(W)$ denotes the associated sheaf of a presheaf of sets W, and $\overline{\|K\|}_{\mathrm{pre}}$ denotes the left adjoint functor of $F \mapsto \mathrm{map}(\mathbf{\Delta}^\bullet, F)$ applied to the simplicial set K.

Concretely, if K is a triangulation of a smooth manifold N, a section of $\|K\|$ over a manifold M is a smooth map $M \to N$ which factors *locally* through some simplex of the triangulation.

Lemma 4.16 *Let K be a simplicial set for which faces of nondegenerate simplices are nondegenerate. Then the sheaf $\|K\|$ weak deformation retracts onto $\overline{\|K\|}$. That is, there is a map $h\colon \|K\| \times \mathbb{A}^1 \to \|K\|$ whose restriction to $\|K\| \times \{0\}$ is the identity, whose restriction to $\|K\| \times \{1\}$ factors through $\overline{\|K\|}$, and $h(\overline{\|K\|} \times \mathbb{A}^1) \subset \overline{\|K\|}$. Moreover, for any subcomplex L of K, h restricts to a weak deformation retraction of $\|L\|$ onto $\overline{\|L\|}$.*

Proof Consider a homotopy

$$\lambda^n\colon \mathbb{A}^n \times [0,1] \to \mathbb{A}^n$$

that in barycentric coordinates is constructed as follows. Fix $c\colon \mathbb{A}^1 \to [0,\infty)$ a smooth function with $c \equiv 0$ on $(-\infty, 0]$ and strictly increasing on $[0,\infty)$. Then we set

$$\lambda^n((x_0,\ldots,x_n),t) = (y_0/C_t,\ldots,y_n/C_t),$$

where $y_i = tc(x_i) + (1-t)x_i$ and $C_t = \sum_{i=0}^n y_i$. Extend λ^n to $\mathbb{A}^n \times \mathbb{A}^1$ by precomposing λ^n with $\mathrm{id} \times f\colon \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n \times [0,1]$, where f is a smooth function which takes value 0 on a neighborhood of $(-\infty,0]$ and 1 on a neighborhood of $[1,\infty)$. This gives a weak deformation retraction $h\colon \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n$ of $\mathbb{A}^n$ onto the subsheaf $\mathbf{\Delta}^n$ for each fixed n.

The map λ is functorial with respect to injections $[m] \to [n]$. And we may replace the category Δ in the coend defining $\|K\|_{\mathrm{pre}}$ with the subcategory Δ_{inj} of injective maps. To see this, we can express the coend over Δ (respectively Δ_{inj}) as a colimit over the category $\mathrm{simp}(K)$ of simplices of K (respectively the subcategory $\mathrm{ndsimp}(K)$ of nondegenerate simplices). The inclusion $\mathrm{ndsimp}(K) \to \mathrm{simp}(K)$ is terminal (alias final) under the assumption that each face of a nondegenerate simplex of K is nondegenerate. Hence, λ defines a weak deformation retraction $h\colon \|K\|_{\mathrm{pre}} \times \mathbb{A}^1 \to \|K\|_{\mathrm{pre}}$ of $\|K\|_{\mathrm{pre}}$ in $\overline{\|K\|}_{\mathrm{pre}}$.

Now, composing h with the sheafification we obtain a map $\|K\|_{\mathrm{pre}} \times \mathbb{A}^1 \to \|K\|$ which, by the universal property, factors through $\|K\| \times \mathbb{A}^1$. This factorization is the required weak deformation retraction of $\|K\|$ onto $\overline{\|K\|}$. □

Proposition 4.17 *For any manifold M and any presheaf $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$, the restriction map*

$$\boldsymbol{R}\mathrm{map}(M \times \mathbb{A}^\bullet, F) \to \boldsymbol{R}\mathrm{map}(M \times \boldsymbol{\Delta}^\bullet, F)$$

is a weak equivalence after realization.

Proof We can assume F to be injectively fibrant, so $\boldsymbol{R}\mathrm{map}$ can be replaced with map. By replacing F with $\mathrm{map}(M, F)$ we can assume $M = \mathbb{R}^0$. We will verify the conditions of Proposition A.9 and show that every square

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & \mathrm{Ex}^\infty F(\mathbb{A}^\bullet) \\ \big\downarrow & & \big\downarrow \\ \Delta[n] & \longrightarrow & \mathrm{Ex}^\infty \mathrm{map}(\boldsymbol{\Delta}^\bullet, F) \end{array}$$

admits a lift making the upper and lower triangles commute up to homotopy, and such that these two homotopies are compatible on $\partial\Delta[n]$. For a finite-dimensional simplicial compact space K, ie having a compact space of nondegenerate simplices, a map $K \to \mathrm{Ex}^\infty Y$ factors through some finite stage, and so it corresponds to a map $\mathrm{Sd}^i K \to Y$. Then, by adjunction, the square above amounts to a map $P \to F$, where

$$P := \|\mathrm{Sd}^i\, \partial\Delta[n]\| \sqcup_{\overline{\|\mathrm{Sd}^i\, \partial\Delta[n]\|}} \overline{\|\mathrm{Sd}^i\, \Delta[n]\|}.$$

By Lemma 4.16, there is a self-homotopy H of $\|\mathrm{Sd}^i\, \Delta[n]\|$ such that $H_0 = \mathrm{id}$, H_1 factors through

$$\overline{\|\mathrm{Sd}^i\, \Delta[n]\|} \to \|\mathrm{Sd}^i\, \Delta[n]\|$$

and H preserves $\|\mathrm{Sd}^i\, \partial\Delta[n]\|$, $\overline{\|\mathrm{Sd}^i\, \partial\Delta[n]\|}$ and $\overline{\|\mathrm{Sd}^i\, \Delta[n]\|}$. This defines a map

$$\|\mathrm{Sd}^i\, \Delta[n]\| \to P \to F$$

and a self-homotopy of P, giving the lift and homotopies that were needed. □

4c Smooth maps with prescribed constancy conditions

Let K be a subdivision of the standard n–simplex; that is, K is an ordered (locally finite) simplicial complex, $|K| = \Delta^n \subset \mathbb{A}^n$ and every simplex of K is contained (affinely) in a simplex of Δ^n. We have a map

$$j\colon \|K\| \to \mathbb{A}^n$$

that is linear on each simplex. Note, however, that j is *not* induced by a simplicial map. The map j is not an inclusion but its restriction to $\overline{\|K\|}$ is, and its image is $\boldsymbol{\Delta}^n$.

Proposition 4.18 *The inclusion of sheaves* $j\colon \overline{\|K\|} \hookrightarrow \mathbf{\Delta}^n$ *admits a weak deformation retraction. More precisely, there is a map* $r\colon \mathbf{\Delta}^n \to \overline{\|K\|}$ *and a smooth homotopy*

$$\{h_t\colon \mathbb{A}^n \to \mathbb{A}^n\}_{t\in[0,1]}$$

which restricts to a homotopy $\mathbf{\Delta}^n \times \mathbb{A}^1 \to \mathbf{\Delta}^n$ *between the identity and* jr, *and to a homotopy*

$$\overline{\|K\|} \times \mathbb{A}^1 \to \overline{\|K\|}$$

between the identity and rj.

This is a consequence of the lemma below:

Lemma 4.19 *Given a subdivision* K *of the* n*–simplex* Δ^n *as in Proposition 4.18, there exists a smooth homotopy* $\{h_t\colon \mathbb{A}^n \to \mathbb{A}^n\}_{t\in[0,1]}$ *such that*

(i) h_0 *is the identity,*

(ii) h_t *maps each closed simplex* $\mathbf{\Delta}^n \subset \overline{\|K\|}$ *to itself for all* t, *and*

(iii) *each closed simplex* $\mathbf{\Delta}^n \subset \overline{\|K\|} \subset \mathbb{A}^n$ *has a neighborhood in* $\mathbb{A}^n$ *which gets mapped to that same simplex by* h_1.

Proof We use the following terminology during this proof: for a simplex σ of K, a homotopy of maps $(f_t\colon \mathbb{A}^n \to \mathbb{A}^n)_{t\in[c,d]}$ satisfies property (iii)$_\sigma$ if σ has a neighborhood in $\mathbb{A}^n$ which gets mapped to σ by f_d.

Fix some k with $-1 \le k \le n$ and suppose per induction that we have already constructed a smooth homotopy $(h_t\colon \mathbb{A}^n \to \mathbb{A}^n)_{t\in[0,a]}$ for some $a<1$ satisfying conditions (i), (ii) and (iii)$_\sigma$ for every simplex σ of dimension at most k.

We want to extend this to a smooth homotopy $(h_t\colon \mathbb{A}^n \to \mathbb{A}^n)_{t\in[0,b]}$, where $b>a$, that satisfies conditions (i), (ii) and (iii)$_\sigma$ for every simplex σ of dimension at most $k+1$.

For a closed k–simplex τ, let W_τ be a neighborhood of τ in $\mathbb{A}^n$ which gets mapped to τ by h_a. This exists by the inductive assumption. We shall define a homotopy

$$(g_t\colon \mathbb{A}^n \to \mathbb{A}^n)_{t\in[0,b-a]},$$

where $g_0=\mathrm{id}$, g_t maps each simplex of $\overline{\|K\|}$ to itself for all t, and g_{b-a} maps an appropriate subset of the interior of each $(k+1)$–simplex in $\mathbb{A}^n$ to that same simplex. *Appropriate* means it should be large enough so that its union with the W_τ, over all boundary faces $\tau\subset\sigma$, contains σ, and small enough so that the various open subsets for different $(k+1)$–simplices are disjoint. Once the homotopy g_t is given, we can simply define $(h_t)_{t\in[0,b]}$ as the concatenation of $(h_t)_{t\in[0,a]}$ and $(g_{t-a}h_a)_{t\in[a,b]}$. (In order for the concatenation to be smooth, we may arrange so that the homotopy $(h_t)_{t\in[0,a]}$ is stationary for t close to a and the homotopy (g_t) is stationary for t close to 0.)

To describe g_t we first choose, for each $(k+1)$–simplex σ, a small tubular neighborhood $U(\sigma)$ of $\mathrm{int}(\sigma)$ in $\mathbb{A}^n$ such that, for each point $x \in \mathrm{int}(\sigma)$ and every closed simplex τ of $\|K\|$, the intersection $U(\sigma)_x \cap \tau$ is a linear cone in $\mathbb{A}^n$. That is, there exist linearly independent vectors $v_1, \dots, v_l$ such that points in $U(\sigma)_x \cap \tau$ are of the form $c_0 v_0 + c_1 v_1 + \cdots + c_l v_l$ with $c_i \geq 0$. By shrinking if necessary, we may also assume that $U(\sigma) \cap U(\sigma')$ is empty if σ and σ' are distinct $(k+1)$–simplices.

Pick an open subset $V(\sigma)$ of the interior of each $(k+1)$–simplex σ, with compact closure, whose union with $\bigcup_{\text{face } \tau} W_\tau \cap \sigma$ is the closed simplex σ. Then use the linear coordinates on the tubular neighborhood $U(\sigma)$ to obtain a map

$$\psi : U(\sigma) \to U(\sigma)$$

over $\mathrm{int}(\sigma)$ satisfying the following conditions: for x close to the boundary of σ, ψ_x is the identity; for x in $V(\sigma)$, $\psi_x(v) = 0$ for $v \in U(\sigma)_x$ and $|v|$ small and $\psi_x(v) = v$ for $|v|$ large. Extend by the identity to obtain a map $g_1^\sigma : \mathbb{A}^n \to \mathbb{A}^n$. Linearly interpolate between the identity and g_1^σ to get a homotopy (g_t^σ) and concatenate the (g_t^σ) for all σ, to obtain the homotopy (g_t). □

Proof of Proposition 4.18 The lemma gives us a smooth homotopy h on $\mathbb{A}^n$. Condition (ii) implies that h restricts to a homotopy $\{h_t : \overline{\|K\|} \to \overline{\|K\|}\}$. Condition (iii) gives the required factorization of h_1 as

$$\mathbf{\Delta}^n \xrightarrow{r} \overline{\|K\|} \xrightarrow{j} \mathbf{\Delta}^n$$

in the category of sheaves of sets, where the factorization of h_1 through $\overline{\|K\|}$ defines r. □

Remark 4.20 Lemma 4.19 admits a more general version which applies to arbitrary manifolds M equipped with a suitable triangulation, though we will not require that level of generality. This is claimed in [Madsen and Weiss 2007, Appendix A.1].

An inclusion of simplicial complexes $L \hookrightarrow K$ is called a *relative horn inclusion* if K is obtained from L by attaching a simplex along a horn in L (we assume that the horn is embedded in L). The following lemma will be crucial to the proof of Proposition 4.1:

Lemma 4.21 *Let B be a subdivision of Δ^n. Given a sequence of relative horn inclusions*

$$A = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_l, = B,$$

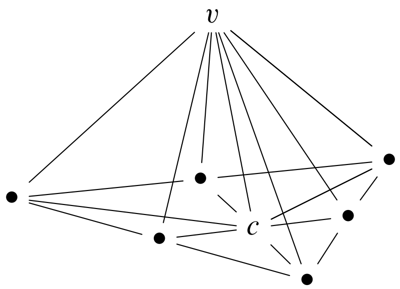
there exists a weak deformation retraction of $\overline{\|B\|}$ onto $\overline{\|A\|}$. That is, a homotopy $H : \overline{\|B\|} \times \mathbb{A}^1 \to \overline{\|B\|}$ such that H restricts to a homotopy $\overline{\|A\|} \times \mathbb{A}^1 \to \overline{\|A\|}$, $H_0 = \mathrm{id}$ and H_1 factors through $\overline{\|A\|}$.

We introduce some terminology in preparation for the proof of this lemma. Let $0 < k \leq l$. A homotopy

$$H^{(k)} : \overline{\|B\|} \times \mathbb{A}^1 \to \overline{\|B\|}$$

is said to have *property* (σ_k) if $H^{(k)}$ restricts to a homotopy $\overline{\|A_k\|} \times \mathbb{A}^1 \to \overline{\|A_k\|}$, $H_0 = \mathrm{id}$ and H_1 factors through $\overline{\|A_k\|}$.

Proof Before tackling the lemma in full generality, we prove it for the easy case of $A_0 \hookrightarrow A_1 = B$ for a single horn inclusion $\Lambda \hookrightarrow \Delta^n$. Choose a subdivision T of Δ^n which is the simplicial cone on a subdivision of Δ^{n-1}. For concreteness, we take T to be the simplicial cone of $\operatorname{sd}\Delta^{n-1}$, the first barycentric subdivision of Δ^{n-1}. (That is, T is the nerve of the category obtained by adjoining a terminal object v to the poset of nondegenerate simplices of the standard $(n-1)$–simplex.) We refer to T as the *cone-subdivision* of the n–simplex. Here is a picture for $n = 3$:



By Proposition 4.18, we have a weak deformation retraction

$$h\colon \mathbf{\Delta}^n \times \mathbb{A}^1 \to \mathbf{\Delta}^n$$

of $\mathbf{\Delta}^n$ onto $\overline{\|T\|}$. Let T' be the simplicial complex obtained from T by discarding the vertex $c \in T$ corresponding to the top simplex in Δ^{n-1}. (To obtain a simplicial complex, we must also discard all the simplices in T that have c as a face.) Then T' is a subdivision of the n–dimensional horn and h restricts to a weak deformation retraction of $\overline{\|\Lambda\|}$ onto $\overline{\|T'\|}$.

The inclusion $i : T' \hookrightarrow T$ admits a retraction $r : T \to T'$, essentially given by collapsing c onto v. This is a simplicial map; it is the application of the appropriate degeneracy map on each simplex of T. Moreover, we can construct a homotopy on each simplex of T between the identity and said degeneracy map. This can be done by linear interpolation, for example. Thus we obtain a deformation retraction

$$h' : \overline{\|T\|} \times \mathbb{A}^1 \to \overline{\|T\|}$$

of $\overline{\|T\|}$ in $\overline{\|T'\|}$. Clearly, the composition (concatenation) of h and h' gives a homotopy H satisfying the conditions of the lemma, ie having property (σ_0).

With this special case in hand we proceed to the general one, arguing by induction. Suppose we have constructed a homotopy $H^{(k)}$ having property (σ_k). We now construct a homotopy $H^{(k-1)}$ having property (σ_{k-1}) as follows. Firstly, take a subdivision K of A_k (and hence a subdivision of A_{k-1}) that restricts to the cone triangulation on the simplex attached to A_{k-1}. Lemma 4.19 gives us a homotopy on $\mathbf{\Delta}^n$ that restricts to a homotopy

$$f : \overline{\|A_k\|} \times \mathbb{A}^1 \to \overline{\|A_k\|}$$

with $f_0 = \mathrm{id}$ and which factors through $\overline{\|K\|}$ at time 1.

Now, by collapsing the cone subdivision of the attached simplex to the (subdivided) horn using the simplicial map from the case of a single horn inclusion, we obtain a homotopy

$$g\colon \overline{\|K\|}\times\mathbb{A}^1 \to \overline{\|K\|}\subset\overline{\|A_k\|}$$

with $g_0=\mathrm{id}$ and which factors through $\overline{\|A_{k-1}\|}$ at time 1. Compose (concatenate) the homotopies f and g to obtain a homotopy h on $\|A_k\|$. Then define $H^{(k-1)}$ to be composition of $H^{(k)}$ and h. For this composition to be smooth, we emphasize that it is important to first apply Lemma 4.19 to the whole of $\overline{\|A_k\|}$, not just the on the simplex that we are collapsing. This completes the induction. $\square$

Corollary 4.22 *The extended-simplices version of Lemma 4.21 holds. Namely, under the conditions of that lemma, $\|B\|$ weak deformation retracts onto $\|A\|$. That is, there exists a homotopy $G\colon\|B\|\times\mathbb{A}^1\to\|B\|$ such that G restricts to a homotopy $\|A\|\times\mathbb{A}^1\to\|A\|$, $G_0=\mathrm{id}$ and G_1 factors through $\|A\|$.*

Proof Starting from the homotopy $H\colon\overline{\|B\|}\times\mathbb{A}^1\to\overline{\|B\|}$ of Lemma 4.21, we obtain a homotopy

$$\widetilde{H}\colon \|B\|\times\mathbb{A}^1 \xrightarrow{\lambda} \overline{\|B\|}\times\mathbb{A}^1 \xrightarrow{H} \overline{\|B\|} \xrightarrow{i} \|B\|,$$

where i is the inclusion and λ is the map constructed in Lemma 4.16. It is clear that $\widetilde{H}$ restricts to a homotopy on $\|A\|$, $\widetilde{H}_0=\lambda i$ and $\widetilde{H}_1$ factors through $\|A\|$. Now define G as the concatenation of the homotopy on $\|B\|$ from Lemma 4.16 (between the identity and λi) with the homotopy $\widetilde{H}$. $\square$

4d Proof of Propositions 4.1, 4.2 and 4.3

For concreteness, we assume F to be injectively fibrant (by replacing it if necessary) so that Proposition 4.11 applies.

Proof of Proposition 4.1 Let $I\colon A\hookrightarrow B$ denote the map

$$h_i\colon \mathrm{Sd}^i(\Lambda_k[n])\hookrightarrow \mathrm{Sd}^i(\Delta[n])$$

with $i\geq 0$, $n\geq 1$ and $0\leq k\leq n$, as in Definition 3.8. We write $\iota=\|I\|_{\mathrm{pre}}$. By adjunction of Definition 4.4, it suffices to find weak liftings

$$\begin{array}{ccc} \|A\|_{\mathrm{pre}} & \xrightarrow{f} & F \\ \downarrow{\scriptstyle\iota} & \nearrow{\scriptstyle\tilde{\alpha}} & \\ \|B\|_{\mathrm{pre}} & & \end{array}$$

together with a homotopy $H\colon\|A\|_{\mathrm{pre}}\times\|\Delta^1\|_{\mathrm{pre}}\to F$ between $\tilde{\alpha}\iota$ and f. Indeed, this gives us the required homotopy $A\times\Delta[1]\to F(\mathbb{A}^\bullet)$ by precomposing H with $\|A\times\Delta^1\|_{\mathrm{pre}}\to\|A\|_{\mathrm{pre}}\times\|\Delta^1\|_{\mathrm{pre}}$ and applying the adjunction again. It also suffices to solve the above problem with $\|-\|_{\mathrm{pre}}$ replaced by $\|-\|$ everywhere. This is allowed since F is an ∞–sheaf and the map $\|-\|_{\mathrm{pre}}\to\|-\|$ is by definition a sheafification.

The strategy is to find a homotopy retraction of ι, ie a map $r\colon \|B\| \to \|A\|$ together with a homotopy $\|A\| \times \|\Delta^1\| \to \|A\|$ between $r\iota$ and the identity. In fact, we construct this homotopy on $\|A\|$ as the restriction of a homotopy on $\|B\|$. (We will need this stronger statement in the proof of Proposition 4.2.) This is achieved by a direct application of Corollary 4.22. More precisely, we choose a sequence of relative horn inclusions from $A = \mathrm{sd}^i\,\Lambda^n_k$ to $B = \mathrm{sd}^i\,\Delta^n$ for each $i \geq 0$, $n > 0$ and $0 \leq k \leq n$, and apply Corollary 4.22. (A proof that such a sequence exists can be found in [Moss 2020, Proposition 19].) That the resulting homotopy preserves the 0–simplices of each horn follows directly from its construction. □

Proof of Proposition 4.2 We keep the notation $I\colon A \hookrightarrow B$ for the map h_i as in Definition 3.8. As in the proof of Proposition 4.1, we use the adjunction of Definition 4.4 and the natural transformation $\|-\|_{\mathrm{pre}} \to \|-\|$ to reduce the problem to constructing certain maps of sheaves of sets.

Let P denote the pushout of sheaves of sets

$$\|A\| \times \mathbb{A}^1 \sqcup_{\|A\|\times\partial\mathbb{A}^1} \|B\| \times \partial\mathbb{A}^1.$$

To verify the weak RLP with respect to $\|I\|$, it suffices to prove that, for any map $\alpha\colon P \to F$, there is a dashed map $\tilde{\alpha}$ as in

$$\begin{array}{ccc} P & \xrightarrow{\ \alpha\ } & F \\ {\scriptstyle\iota}\big\downarrow & \overset{\tilde{\alpha}}{\nearrow} & \\ \|B\| \times \mathbb{A}^1 & & \end{array}$$

making the diagram commute up to a homotopy $H : P \times \mathbb{A}^1 \to F$ from α to $\tilde{\alpha}\iota$ which is *fixed* on $\|B\| \times \partial\mathbb{A}^1$ pointwise. Being fixed pointwise means that the restriction of H to $(\|B\| \times \partial\mathbb{A}^1) \times \mathbb{A}^1$ factors as the projection to $\|B\| \times \partial\mathbb{A}^1$ followed by α. The result now follows from the lemma below. □

Lemma 4.23 *Suppose $I\colon A \hookrightarrow B$ is the map h_i as in Definition 3.8 and P is the pushout of sheaves of sets $\|A\| \times \mathbb{A}^1 \sqcup_{\|A\|\times\partial\mathbb{A}^1} \|B\| \times \partial\mathbb{A}^1$. Then $\|B\| \times \mathbb{A}^1$ weak deformation retracts onto P, relative to $\|B\| \times \partial\mathbb{A}^1$.*

Proof We need to show that there is a homotopy $H\colon (\|B\| \times \mathbb{A}^1) \times \mathbb{A}^1 \to \|B\| \times \mathbb{A}^1$ such that

(1) H restricts to a homotopy $P \times \mathbb{A}^1 \to P$ which fixes $\|B\| \times \partial\mathbb{A}^1$ pointwise;

(2) H_0 is the identity and H_1 factors through P.

Choose a bump function $c\colon \mathbb{A}^1 \to [0,1] \subset \mathbb{A}^1$ with $c \equiv 0$ in an open neighborhood of $(-\infty, 0] \cup [1, \infty)$, and $c \equiv 1$ in a open neighborhood J of $\frac{1}{2}$, $c(t)$ increasing for $t \leq \frac{1}{2}$ and decreasing for $t \geq \frac{1}{2}$. Then choose a function $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ with $f(t) \equiv 0$ when $c \neq 1$ and $t \leq \frac{1}{2}$ and $f(t) \equiv 1$ when $c \neq 1$ and $t > \frac{1}{2}$.

Let $H_B\colon \|B\| \times \mathbb{A}^1 \to \|B\|$ be the weak deformation retraction constructed in the proof of Proposition 4.1. Define maps $R_1, R_2\colon (\|B\| \times \mathbb{A}^1) \times \mathbb{A}^1 \to \|B\| \times \mathbb{A}^1$ as

$$R_1(x,t,s) = \big(H_B(x, s\cdot c(t)), t\big) \quad \text{and} \quad R_2(x,t,s) = (x, s\cdot f(t) + (1-s)t)$$

for $x \in \|B\|$ and $t, s \in \mathbb{A}^1$. Then define H as

$$(x,t,s) \mapsto \begin{cases} R_1(x,t,2s) & \text{for } s \le \frac{1}{2}, \\ R_2(R_1(x,t,1), 2s-1) & \text{for } s > \frac{1}{2}. \end{cases}$$

Now R_1 and R_2 separately satisfy condition (1), so H does as well. As for property (2), we have that $H(x,t,1) = (x', t')$, where $x' = H_B(x, c(t))$ and $t' = f(t)$. If $t \in J$ then $c = 1$ and so $x' \in \|A\|$, and if $t \notin J$ then $f(t) \in \partial\mathbb{A}^1$. Therefore, H_1 factors through P. □

Proof of Proposition 4.3 The proof is of the same sort as that of Proposition 4.2, using the adjunction of Definition 4.4 and the natural transformation $\|-\|_{\mathrm{pre}} \to \|-\|$ to reduce the problem to constructing certain maps of sheaves of sets. Let $I : A \hookrightarrow B$ be the map h_i as in Definition 3.8. Consider the pushout

$$P := S^n \times \|A\| \sqcup_{*\times\|A\|} * \times \|B\|.$$

The same manipulations as before show that, to verify the weak RLP with respect to i, it suffices to prove that, for any map $\alpha\colon P \to F$, there is a map $\tilde{\alpha}$ as in

$$\begin{array}{ccc} P & \xrightarrow{\ \alpha\ } & F \\ {\scriptstyle \iota}\big\downarrow & \nearrow{\scriptstyle \tilde{\alpha}} & \\ S^n \times \|B\| & & \end{array}$$

making the diagram commute up to a homotopy $P \times \mathbb{A}^1 \to F$ from α to $\tilde{\alpha}\iota$ which is fixed on $* \times \|B\|$ pointwise. The result now follows from a lemma analogous to Lemma 4.23, below. □

Lemma 4.24 *Suppose $I : A \hookrightarrow B$ is the map h_i as in Definition 3.8 and P is the pushout of sheaves of sets $S^n \times \|A\| \sqcup_{*\times\|A\|} * \times \|B\|$. Then $\|B \times S^n\|$ weak deformation retracts onto P, relative to $\|B\| \times *$.*

Proof As before, we need to show that there is a homotopy $H\colon (\|B\| \times S^n) \times \mathbb{A}^1 \to \|B\| \times S^n$ such that

(1) H restricts to a homotopy $P \times \mathbb{A}^1 \to P$ which fixes $\|B\| \times *$ pointwise;

(2) H_0 is the identity and H_1 factors through P.

Let $* \in D_\epsilon \subset D_\delta \subset S^n$ be open disk neighborhoods of radii ϵ and δ with $\epsilon < \delta$. Choose a smooth function $c\colon S^n \to [0,1] \subset \mathbb{A}^1$ such that $c|_{S^n \setminus D_\epsilon} \equiv 0$ and $c(*) = 1$. For example, we can choose c to be a bump function with support in D_ϵ that is 1 at the basepoint.

Also choose a homotopy $h\colon \mathbb{A}^1 \times S^n \to S^n$ such that $h(1,-)|_{D_\epsilon}$ is constant to $* \in S^n$ and $h(t,-)|_{S^n\setminus D_\delta} = \mathrm{id}$. In words, this homotopy collapses a neighborhood of the basepoint to itself.

Let $H_B\colon \|B\|\times\mathbb{A}^1\to\|B\|$ be the map constructed in the proof of Proposition 4.1. We construct H as the composition of two homotopies. Define

$$R_1\colon S^n\times\|B\|\times\mathbb{A}^1\to S^n\times\|B\|,\quad R_1(z,x,t)=\bigl(z,H_B\bigl(x,t(1-c(z))\bigr)\bigr),$$

and

$$R_2\colon S^n\times\|B\|\times\mathbb{A}^1\to S^n\times\|B\|,\quad R_2(z,x,t)=(h(t,z),x).$$

The first homotopy collapses $S^n\times\|B\|$ onto $S^n\times\|A\|$ outside a neighborhood $N\subset D_\epsilon$ of the basepoint. The second collapses $D_\epsilon\times\|B\|$ to $*\times\|B\|$. The composition of these homotopies satisfies the claimed properties. □

5 The shape functor preserves the ∞–sheaf property

In this section we assemble the previous results to prove Theorem 1.2. Our approach uses the following characterization of ∞–sheaves:

Theorem 5.1 *A presheaf $F\colon \mathsf{Man}^{\mathrm{op}}\to\mathsf{S}$ is an ∞–sheaf if and only if $F(\varnothing)\simeq *$ and*

(1) *for all manifolds M and open covers of M with two elements $\{U,V\}$, the commutative square*

$$\begin{array}{ccc} F(M) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(U\cap V) \end{array}$$

is a homotopy pullback square; and

(2) *if M is a (possibly infinite) disjoint union of submanifolds U_i, then $F(M)\to\prod_i^h F(U_i)$ is a weak equivalence.*

This is probably well known and is similar to a special case of [Weiss 1999, Theorem 5.2; Boavida de Brito and Weiss 2013, Theorem 7.2]. For completeness, we provide a proof below. In preparation, we record the following:

Lemma 5.2 *A presheaf $F\colon \mathsf{Man}^{\mathrm{op}}\to\mathsf{S}$ is an ∞–sheaf if and only if, for every open cover $\{U_i\}_{i\in I}$, the canonical map*

$$F(M)\to\operatorname*{holim}_{S\subset I} F(U_S)$$

is a weak equivalence, where the homotopy limit ranges over all ***finite, nonempty subsets*** *$S\subset I$ and U_S is notation for $\bigcap_{i\in S}U_i$.*

Proof For the duration of this proof we will write $\underline{n}$ for the set $\{0,\dots,n\}$ to distinguish it from the total ordered set $[n] := \{0 \leq \cdots \leq n\}$.

We need to show that

(5.3)
$$\operatorname*{holim}_{[n]\in\Delta} \prod_{i_0,\dots,i_n\in I} F(U_{i_0}\cap\cdots\cap U_{i_n}) \simeq \operatorname*{holim}_{S\subset I} F(U_S).$$

First suppose that I is finite and pick a total order on I. We can then replace the homotopy limit on the left by replacing the product over sequences $i_0,\dots,i_n$ with the product over ordered sequences $i_0 \leq \cdots \leq i_n$. To see this, view the left-hand side of the display as a homotopy limit of a (covariant) functor from D to spaces, where D is the category whose objects are sequences $i_0,\dots,i_n$ and morphisms are induced by Δ and correspond to merging repeated elements or adding new ones. More precisely, an object is a map $i\colon \underline{n} \to I$ and a morphism from $t\colon \underline{m} \to I$ to $i\colon \underline{n} \to I$ is an order-preserving map $\theta\colon \underline{m} \to \underline{n}$ such that $t = \theta i$.

Let D' be the full subcategory of D consisting of *ordered* sequences $i_0 \leq \cdots \leq i_n$, ie functors $[n] \to I$. There is a canonical functor $o\colon D \to D'$ that orders each sequence. Namely, for a sequence $i_0,\dots,i_n$, ie a map $i\colon \underline{n} \to I$, precompose with the unique bijection $f\colon \underline{n} \to \underline{n}$ such that if is order-preserving and the restriction of f to $(if)^{-1}(j)$ is order-preserving for each $j \in I$. Given a morphism θ in D from $t\colon \underline{m} \to I$ to $i\colon \underline{n} \to I$, its image under o is the morphism from $o(t) = gt$ to $o(i) = if$ in D' given by $f^{-1}\theta g\colon [m] \to [n]$ (this is indeed order-preserving since g reverses the order of two elements if and only if f does).

We claim that this functor o, sending i to if, is homotopy initial. That is, for each ordered sequence $[\underline{i}] := (i_0 \leq \cdots \leq i_n)$, the comma category $o/[\underline{i}]$ is contractible. To prove this, we show that the identity map on the classifying space of $o/[\underline{i}]$ is null-homotopic by considering functors $A, \mathrm{const}\colon o/[\underline{i}] \to o/[\underline{i}]$. The functor A sends an object in $o/[\underline{i}]$, ie a morphism $o(a_0,\dots,a_k) \to [\underline{i}]$ in D', to the object $o(i_0,\dots,i_n,a_0,\dots,a_k) \to [\underline{i}]$ which merges all repetitions. The functor const sends all objects to the identity $o(i_0,\dots,i_n) \to [\underline{i}]$. For each object $o(a_0,\dots,a_k) \to [\underline{i}]$, there are morphisms in $o/[\underline{i}]$,

$$o(a_0,\dots,a_k) \to o(i_0,\dots,i_n,a_0,\dots,a_k) \leftarrow o(i_0,\dots,i_n),$$

induced by faces in Δ, ie adding new elements. These define natural transformations $\mathrm{id} \Rightarrow A \Leftarrow \mathrm{const}$, proving the claim.

Secondly, given an ordered sequence, we can forget its ordering and view it as a finite subset of I. This construction defines a functor η from D' to the category of nonempty subsets of I. Then, for each $S \subset I$, the comma category η/S is contractible since it is the category of simplices of the nerve of S (viewed as a poset with respect to the total ordering induced from the inclusion $S \subset I$). In other words, η is homotopy initial.

We have shown that, for a finite set I, the map induced by ηo gives a weak equivalence (5.3). It is clear that this is natural with respect to inclusions of finite sets $I' \subset I$. So, to prove (5.3) for a general indexing set I, we can reduce to the finite case by taking the homotopy limit of weak equivalences (5.3) over all finite subsets of I. In more detail, we have a commutative square

$$\begin{array}{ccc} \operatorname{holim}_{S\subset I} F(U_S) & \longrightarrow & \operatorname{holim}_{J\subset I} \operatorname{holim}_{S\subset J} F(U_S) \\ \downarrow & & \downarrow \\ \operatorname{holim}_{i:\underline{n}\to I} F(U_{i_0}\cap\cdots\cap U_{i_n}) & \longrightarrow & \operatorname{holim}_{J\subset I} \operatorname{holim}_{i:\underline{n}\to J} F(U_{i_0}\cap\cdots\cap U_{i_n}) \end{array}$$

where the first homotopy limits in the right column run over finite subsets $J\subset I$. We have shown that the right-hand map is a weak equivalence. The horizontal maps are also weak equivalences, which we can see by identifying the double homotopy limits with a single homotopy limit over a Grothendieck construction (Thomason's homotopy colimit theorem). We will explain this briefly for the lower map; the upper one is similar and almost identical to an argument in the proof of Theorem 5.1. In that case, the double homotopy limit is identified with the homotopy limit over the category whose objects are pairs $\underline{n}\to J\subset I$ with J finite, and morphisms are maps of such. The forgetful functor from this category to D, sending a pair $\underline{n}\to J\subset I$ to its composite $\underline{n}\to I$, is homotopy initial since the relevant comma categories have a terminal object. □

Proof of Theorem 5.1 If F is an ∞–sheaf then conditions (1) and (2) are immediate in view of Lemma 5.2.

To show the converse, suppose first that M is a compact manifold and take an open cover $\{U_i\}_{i\in I}$ of M. For every finite subcover $\{U_j\}_{j\in J}$ with $J\subset I$ of $\{U_i\}_{i\in I}$, the homotopy limit

$$\operatorname*{holim}_{S\subset J} F(U_S)$$

is indexed over a finite category (a cube) and so it is equivalent to an iterated homotopy pullback. Condition (1) applied inductively shows that this iterated homotopy pullback is $F(M)$. Then consider the square

$$\begin{array}{ccc} F(M) & \xrightarrow{\simeq} & \operatorname{holim}_{J\subset I} F(M) \\ \downarrow & & \downarrow\simeq \\ \operatorname{holim}_{S\subset I} F(U_S) & \xrightarrow{\simeq} & \operatorname{holim}_{J\subset I} \operatorname{holim}_{S\subset J} F(U_S) \end{array}$$

where the outer homotopy limits in the right column are indexed by finite refinements, ie finite subsets $J\subset I$ such that $\{U_j\}_{j\in J}$ is still a cover. The right-hand map is a weak equivalence by the observation just made. The poset of finite refinements is filtered, and hence contractible, and so the top horizontal map in the square is also a weak equivalence. As for the lower horizontal map, one can, by Thomason's homotopy colimit theorem, express the double homotopy limit as a homotopy limit over the category — call it P — whose objects are pairs $S\subset J\subset I$ and morphisms are inclusions of such. The forgetful map η from P to the poset of finite, nonempty subsets $S\subset I$ is homotopy initial since, for each $S\subset I$, the overcategory η/S is the filtered poset of all refinements J containing S.

To prove the noncompact case, we will assume for the moment that F satisfies the following condition, which implies condition (2):

(2′) For any manifold M and an open cover $\{V_i\}_{i\geq 0}$ by a nested sequence of open sets with $\overline{V_i} \subset V_{i+1}$, the canonical map
$$F(M) = F\Big(\bigcup_i V_i\Big) \to \operatorname*{holim}_i F(V_i)$$
is a weak homotopy equivalence.

Assuming F satisfies (1) and (2′), we will now prove that it satisfies the ∞–sheaf condition for any noncompact manifold. So let M be a noncompact manifold, and take an exhaustion of M by interiors of compact manifolds $V_0 \subset V_1 \subset \cdots$ with $M = \bigcup V_i$. Such an exhaustion can be obtained by picking a smooth proper map $f\colon M \to \mathbb{R}$ and setting V_i to be the interior of $f^{-1}((-\infty, i])$. Then, for an open cover $\{U_i \to M\}_{i\in I}$,

(5.4)
$$\operatorname*{holim}_{S\subset I} F(U_S) \simeq \operatorname*{holim}_{S\subset I} \operatorname*{holim}_{j\geq 0} F(V_j \cap U_S)$$

by (2′) applied to the covers $\{U_S \cap V_j \to U_S\}_j$ for each S. Now commute the homotopy limits and use that the cover $\{V_j \cap U_i \to V_j\}_i$ (for a fixed j) has a finite subcover to conclude, using (1), that (5.4) is weakly equivalent to
$$\operatorname*{holim}_{j\geq 0} F(V_j).$$
By invoking (2′) again, this homotopy limit is weakly equivalent to $F(M)$.

We are left to show that conditions (1) and (2) jointly imply condition (2′). Suppose $\{V_i\}$ is an open cover as in (2′). Let W_0 be the disjoint union of $V_{k+2} \setminus \overline{V}_k$, taken over k even, and let W_1 be the disjoint union of $V_{k+2} \setminus \overline{V}_k$, taken over k odd. Then W_0 and W_1 form an open cover of M so, by (1), we have an equivalence
$$F(M) \to \operatorname{holim}(F(W_0) \to F(W_0 \cap W_1) \leftarrow F(W_1)).$$
By (2), the target is equivalent to
$$\operatorname{holim}\Big(\operatorname*{holim}_i F(W_0 \cap V_i) \to \operatorname*{holim}_i F(W_0 \cap W_1 \cap V_i) \leftarrow \operatorname*{holim}_i F(W_1 \cap V_i)\Big),$$
which, by commuting homotopy limits and using (1), is equivalent to $\operatorname{holim}_i F(V_i)$. □

Remark 5.5 In a previous iteration of this paper, Theorem 5.1 had a stronger condition (2). The referee kindly pointed out to us that our proof implied the new (weaker) statement and suggested the simple argument of the last paragraph of the proof.

Remark 5.6 The same proof works for topological manifolds. The main observation for the noncompact case is that there exists a proper map $M \to \mathbb{R}$ for M a topological manifold (the requirement is that partitions of unity exist). For generalizations of these statements, see [Pavlov 2022].

We shall tackle properties (1) and (2) for $\boldsymbol{B}F$ separately below. We call them the *finite* and *noncompact* cases, respectively.

5a The finite case

Theorem 5.7 *Let $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$ be an ∞–sheaf and M a smooth manifold with U and V two open subsets of M such that $U \cup V = M$. Then the commutative square*

$$\begin{array}{ccc} \boldsymbol{B}F(M) & \longrightarrow & \boldsymbol{B}F(V) \\ \downarrow & & \downarrow \\ \boldsymbol{B}F(U) & \longrightarrow & \boldsymbol{B}F(U \cap V) \end{array}$$

is homotopy cartesian.

Proof The (homotopy) pullback

$$(\boldsymbol{B}F(U) \times \boldsymbol{B}F(V)) \times_{\mathrm{map}(\partial\Delta^1, \boldsymbol{B}F(U\cap V))} \mathrm{map}(\Delta^1, \boldsymbol{B}F(U \cap V))$$

is identified with

$$(\boldsymbol{B}F(U) \times \boldsymbol{B}F(V)) \times^h_{\boldsymbol{B}F(U\cap V\times\partial\mathbb{A}^1)} \boldsymbol{B}F(U \cap V \times \mathbb{A}^1) \tag{5.8}$$

since $\boldsymbol{B}F$ is concordance-invariant. By Proposition 4.2, we may commute the homotopy pullback with geometric realization, and thus (5.8) is identified with the geometric realization of the simplicial space

$$(F(U \times \mathbb{A}^\bullet) \times F(V \times \mathbb{A}^\bullet)) \times^h_{F(U\cap V\times\partial\mathbb{A}^1\times\mathbb{A}^\bullet)} F(U \cap V \times \mathbb{A}^1 \times \mathbb{A}^\bullet). \tag{5.9}$$

To prove that the map from $F(M \times \mathbb{A}^\bullet)$ to (5.9) is a weak equivalence after realization, we first refine the cover in a convenient way using a partition of unity subordinate to $\{U, V\}$. So let $f_U\colon M \to [0,1]$ and $f_V\colon M \to [0,1]$ with $f_U + f_V \equiv 1$, and $\mathrm{supp}(f_U) \subset U$ and $\mathrm{supp}(f_V) \subset V$. Take $U' = f_U^{-1}\bigl(\frac{2}{3}, 1\bigr]$ and $V' = f_V^{-1}\bigl(\frac{2}{3}, 1\bigr]$. Notice that $U' \cap V' = \varnothing$, and $\{U', V', U \cap V\}$ covers M. Let $c\colon \mathbb{A}^1 \to \mathbb{A}^1$ be a cutoff function with $c|_{(-\infty,1/3)} \equiv 0$ and $c|_{(2/3,\infty)} \equiv 1$, and define $f := c \circ f_V|_{U\cap V}$, so that $f\colon U \cap V \to \mathbb{A}^1$.

Rearrange (5.9) as an iterated homotopy pullback and consider the maps

$$\begin{array}{c} F(U \times \mathbb{A}^\bullet) \times^h_{F(U\cap V\times\mathbb{A}^\bullet)} F(U \cap V \times \mathbb{A}^1 \times \mathbb{A}^\bullet) \times^h_{F(U\cap V\times\mathbb{A}^\bullet)} F(V \times \mathbb{A}^\bullet) \\ \big\downarrow \mathrm{res} \\ F(U' \times \mathbb{A}^\bullet) \times^h_{F(U'\cap V\times\mathbb{A}^\bullet)} F(U \cap V \times \mathbb{A}^1 \times \mathbb{A}^\bullet) \times^h_{F(U\cap V'\times\mathbb{A}^\bullet)} F(V' \times \mathbb{A}^\bullet) \\ \mathrm{pr}^* \big\uparrow \big\downarrow f^* \\ F(U' \times \mathbb{A}^\bullet) \times^h_{F(U'\cap V\times\mathbb{A}^\bullet)} F(U \cap V \times \mathbb{A}^\bullet) \times^h_{F(U\cap V'\times\mathbb{A}^\bullet)} F(V' \times \mathbb{A}^\bullet) \end{array}$$

The restriction map from $F(M \times \mathbb{A}^\bullet)$ to the last space is a weak equivalence since $\{U', U \cap V, V'\}$ covers M and F is an ∞–sheaf. Similarly, the map res is a levelwise weak equivalence since $\{U', U \cap V\}$

covers U and $\{V', U\cap V\}$ covers V. The arrow pr^* is induced by the projection $\mathrm{pr}\colon (U\cap V)\times\mathbb{A}^1 \to U\cap V$. We obtain a map $U\cap V \to (U\cap V)\times\mathbb{A}^1$ from $f\colon U\cap V\to\mathbb{A}^1$, defined in the previous paragraph. By construction, $f|_{U'\cap V}=0$ and $f|_{U\cap V'}=1$, which is precisely the compatibility condition required to extend to a map on sections in the fibered product, which we denote by f^*. Notice that, since $\mathrm{pr}\circ(\mathrm{id}_{U\cap V}, f)=\mathrm{id}_{U\cap V}$, we have $(\mathrm{id}_{U\cap V}, f)^*\circ\mathrm{pr}^*=\mathrm{id}$. It remains to show that $\mathrm{pr}^*\circ(\mathrm{id}_{U\cap V}, f)^*$ is homotopic to the identity.

We consider the interpolation $h\colon ((U\cap V)\times\mathbb{A}^1)\times\mathbb{A}^1\to\mathbb{A}^1$ between $f\circ\mathrm{pr}$ and the projection map $q\colon (U\cap V)\times\mathbb{A}^1\to\mathbb{A}^1$, given by

$$h(t)=(1-t)\cdot q+t\cdot(f\circ\mathrm{pr})$$

and extend it to a smooth homotopy

$$H=(\mathrm{id}_{U\cap V}, h)\colon ((U\cap V)\times\mathbb{A}^1)\times\mathbb{A}^1\to((U\cap V)\times\mathbb{A}^1).$$

Since $F(-\times\mathbb{A}^\bullet)$ sends smooth homotopies to simplicial homotopies (Proposition 2.9) and the map H fixes $(U\cap V)\times\partial\mathbb{A}^1$ pointwise, the map H induces the required simplicial homotopy from $\mathrm{pr}^*\circ(\mathrm{id}_{U\cap V}, f)^*$ to id. □

Corollary 5.10 *Let F be a presheaf on* Man *which satisfies the ∞–sheaf condition with respect to finite covers. Then the evaluation map*

$$\boldsymbol{B}F(M)\to\mathrm{map}(\mathrm{Sing}\,M, BF)$$

is a natural weak equivalence of spaces for every compact manifold M.

Proof This follows from Theorem 5.7 and the proof of Proposition 2.17 applied to a finite good open cover of M. □

The beginning of the proof of Theorem 5.7 has the following obvious generalization, which is just a consequence of Proposition 4.2:

Definition 5.11 For a diagram $F\to G\leftarrow H$ of ∞–sheaves, define the *geometric* homotopy pullback to be the ∞–sheaf whose value at a manifold M is the homotopy pullback of the diagram

$$\begin{array}{ccc} & & G(M\times\mathbb{A}^1) \\ & & \big\downarrow{\scriptstyle \text{endpoints}} \\ F(M)\times H(M) & \longrightarrow & G(M)\times G(M) \end{array}$$

Then, by Proposition 4.2, the classifying space functor $\boldsymbol{B}$ sends geometric homotopy pullbacks of ∞–sheaves to homotopy pullbacks of spaces.

5b The noncompact case

Theorem 5.12 *Let $\{U_i\}_{i\geq 0}$ be a collection of manifolds (possibly noncompact). For an ∞–sheaf $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{S}$, the natural map*

$$\boldsymbol{B}F\Big(\coprod_i U_i\Big) \to {\prod_i}^h \boldsymbol{B}F(U_i)$$

is a weak equivalence.

We deduce this from the lemma below, by setting $F_i = F(U_i \times -)$:

Lemma 5.13 *Let $\{F_i\}_{i\in I}$ be a collection of ∞–sheaves indexed over a possibly infinite set I. Then the map*

(5.14)
$$\Big|{\prod_i}^h F_i(\mathbb{A}^\bullet)\Big| \to {\prod_i}^h |F_i(\mathbb{A}^\bullet)|$$

is a weak equivalence of spaces. In other words, the functor B preserves small homotopy products.

We use the symbol $\prod^h$ for the homotopy product, ie the derived functor of the product. This has a different meaning in simplicial spaces (with degreewise weak equivalences) and simplicial sets (with the usual weak equivalences). In the simplicial space case, it means: replace each factor by a degreewise fibrant simplicial space and then compute the product; in the simplicial set case, it means: replace each factor by a Kan complex and then compute the product. The homotopy product (of spaces or of simplicial spaces) agrees with the nonderived product when the indexing set is finite. In general, they do not agree when the set is infinite but Lemma 5.13 says they agree for the concordance resolution.

Proof of Lemma 5.13 The following elegant argument was suggested to us by a referee. Without loss of generality, we can assume F_i to be injectively fibrant, by performing an injective fibrant replacement if necessary. Thus, F_i is valued in Kan complexes and the homotopy product ${\prod_i}^h F_i$ can be computed as the ordinary product $\prod_i F_i$. It then suffices to show that the map (5.14) induces an isomorphism on homotopy groups for all degrees and basepoints.

We have

$$\pi_n\Big(\prod_i |F_i(\mathbb{A}^\bullet)|, x\Big) \cong \prod_i \pi_n(|F_i(\mathbb{A}^\bullet)|, x_i) \cong \prod_i \pi_0 F_i(S^n)_{x_i}/\sim,$$

where $F_i(S^n)_{x_i}$ is the fiber $F_i(S^n) \to F_i(*)$ over x_i and $\sim$ is the equivalence relation of concordance. The second isomorphism follows from Lemma 4.13. On the other hand, appealing again to Lemma 4.13 but now for the ∞–sheaf $\prod_i F_i$, we have that

$$\pi_n\Big(\Big|\prod_i F_i(\mathbb{A}^\bullet)\Big|, x\Big) \cong \pi_0\Big(\prod_i F_i(S^n)_{x_i}\Big)\Big/\sim \;\cong \Big(\prod_i \pi_0 F_i(S^n)_{x_i}\Big)\Big/\sim,$$

where in this case $\sim$ is the equivalence relation of componentwise concordance. The canonical map

$$\prod_i \pi_0 F_i(S^n)_{x_i}/\!\sim \;\to \Big(\prod_i \pi_0 F_i(S^n)_{x_i}\Big)\Big/\!\sim$$

is indeed a bijection since, in the category of Sets, infinite products commute with taking quotients by equivalence relations. □

6 What does the classifying space of an ∞–category classify?

In this section, we suggest an answer to the question in the title. This expands on earlier questions and earlier answers in [Moerdijk 1995; Weiss 2005]. Even earlier results on concordance classification of $\boldsymbol{C}$–bundles on manifolds (or paracompact spaces) for a topological category $\boldsymbol{C}$ (or even just a simplicial space) are due to [Segal 1978; Stasheff 1972]. Our point of view is particularly close to Segal's.

Throughout this section, we will take $\boldsymbol{C}$ to be a Segal space, although the discussion holds more generally for any simplicial space. For convenience, we assume that $\boldsymbol{C}$ is Reedy fibrant as a simplicial space; otherwise, the mapping spaces below need to be derived. (For definitions and more explanations, see [Rezk 2001].) For example, $\boldsymbol{C}$ could be the (Reedy fibrant replacement of the) nerve of a (topological) category. Informally, the following data should produce something deserving the name of a *$\boldsymbol{C}$–bundle on a manifold M*:

- an open cover $\mathcal{U} = \{U_i\}$ of M and a total order on its indexing set I,
- maps $\{\phi_i : U_i \to \boldsymbol{C}_0 = \mathbf{ob}(\boldsymbol{C})\}$,
- maps $\{\phi_{i<j} : U_i \cap U_j \to \boldsymbol{C}_1 = \mathbf{mor}(\boldsymbol{C})\}$,
- etc.

These data are then required to satisfy compatibility conditions; eg for a point $x \in U_i \cap U_j$, $\phi_{i<j}(x)$ is a morphism in $\boldsymbol{C}$ from $\phi_i(x)$ to $\phi_j(x)$. As everywhere else in this paper, *space* means *simplicial set*, so in the above a map from U_i is taken to mean a map of simplicial sets from the singular simplices of U_i to a given simplicial set.

We make the above informal description a $\boldsymbol{C}$–bundle precise, as follows:

Definition 6.1 A *$\boldsymbol{C}$–bundle* is an open cover $\mathcal{U} = \{U_i \to M\}_{i \in I}$ (we stress that here we do not require that I be totally ordered) with a simplicial space map $N\mathcal{U} \to \boldsymbol{C}$, where $N\mathcal{U}$ denotes the nerve of the following topological poset. The space of objects is

$$\coprod_{\varnothing \neq S \subset I} U_S,$$

where the coproduct runs over nonempty finite subsets S of I and $U_S := \bigcap_{s\in S} U_s$. Given objects (R,x) and (S,y) with $x \in U_R$ and $y \in U_S$, there is a morphism $(R,x) \to (S,y)$ if and only if $R \subset S$ (so that $U_S \subset U_R$) and $x = y$. Therefore, the space of morphisms is

$$\coprod_{\varnothing \neq R \subset S} U_S.$$

We view $N\mathcal{U}$ as a simplicial space. Since $N\mathcal{U}$ is Reedy cofibrant, the mapping space $\mathrm{map}(N\mathcal{U}, \boldsymbol{C})$ agrees with the derived mapping space $\boldsymbol{R}\mathrm{map}(N\mathcal{U}, \boldsymbol{C})$.

Remark 6.2 The informal description can be viewed as a special case of the definition by setting the images of certain morphisms — prescribed according to the ordering of I — to be identities. Conversely, given a $\boldsymbol{C}$–bundle, it is sometimes possible to construct a $\boldsymbol{C}$–bundle as in the informal description above by adding to the collection $\mathcal{U}$ all finite intersections of open sets in the original cover and choosing a total ordering on the resulting collection.

We now build a space of $\boldsymbol{C}$–bundles. First, a definition:

Definition 6.3 For a manifold M, we define a simplicially enriched category $\mathsf{Cov}(M)$ of open covers $\mathcal{U}$ of M and their refinements. Recall that a refinement $\mathcal{U} \to \mathcal{V}$ is a choice of function $\alpha\colon I \to J$ between the indexing sets of the covers such that $U_i \subset V_{\alpha(i)}$ for each $i \in I$. We define a k–simplex in the space of morphisms of $\mathsf{Cov}(M)$,

$$\mathrm{map}(\mathcal{U}, \mathcal{V}),$$

to be a $(k+1)$–tuple of refinements $\alpha_0, \dots, \alpha_k \colon \mathcal{U} \to \mathcal{V}$. The face and degeneracy maps are clear.

The space $\mathrm{map}(\mathcal{U}, \mathcal{V})$ may of course be empty. If it is nonempty, it is the nerve of a groupoid, and for every pair of objects α_0, α_1 there is by construction a unique morphism $\alpha_0 \to \alpha_1$. It follows that every k–sphere in $\mathrm{map}(\mathcal{U}, \mathcal{V})$ has an unique filler for every $k \geq 0$. Therefore, $\mathrm{map}(\mathcal{U}, \mathcal{V})$ is either empty or contractible. As such, $\mathsf{Cov}(M)$ is equivalent (as a simplicially enriched category) to the preorder of open covers $\mathcal{U}$ of M with order relation $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{U}$ refines $\mathcal{V}$.

The assignment $\mathcal{U} \mapsto N\mathcal{U}$ defines a simplicially enriched functor from $\mathsf{Cov}(M)$ to the category of simplicial spaces, since $\mathrm{map}(\mathcal{U}, \mathcal{V})$ is a subspace of the space $\mathrm{map}(N\mathcal{U}, N\mathcal{V})$ of simplicial space maps. Indeed, each refinement $\mathcal{U} \to \mathcal{V}$ defines a map of simplicial spaces $N\mathcal{U} \to N\mathcal{V}$. For each pair of refinements $\alpha_0, \alpha_1 \colon \mathcal{U} \to \mathcal{V}$, the relations $U_i \subset V_{\alpha_0(i)}$ and $U_i \subset V_{\alpha_1(i)}$ imply that $U_i \subset V_{\alpha_0(i)} \cap V_{\alpha_1(i)}$ and, as such, define a simplicial map $N\mathcal{U} \times \Delta[1] \to N\mathcal{V}$. More generally, a choice of refinements $\alpha_0, \dots, \alpha_k \colon \mathcal{U} \to \mathcal{V}$ implies the relation $U_i \subset V_{\alpha_0(i)} \cap \dots \cap V_{\alpha_k(i)}$ and so defines a map $N\mathcal{U} \times \Delta[k] \to N\mathcal{V}$.

Definition 6.4 The ∞–sheaf of $\boldsymbol{C}$–bundles is the functor which to a manifold M associates the space

$$\boldsymbol{C}(M) := \operatorname*{hocolim}_{\mathcal{U} \to M} \mathrm{map}(N\mathcal{U}, \boldsymbol{C})$$

given by the homotopy colimit of the enriched functor $\mathcal{U} \mapsto \mathrm{map}(N\mathcal{U}, \boldsymbol{C})$ on $\mathsf{Cov}(M)$. This functor is indeed enriched since, on morphisms, it is the restriction of the canonical map of spaces

$$\mathrm{map}(N\mathcal{U}, N\mathcal{V}) \to \mathrm{map}_{\mathsf{S}}(\mathrm{map}(N\mathcal{V}, \boldsymbol{C}), \mathrm{map}(N\mathcal{U}, \boldsymbol{C}))$$

to $\mathrm{map}_{\mathsf{Cov}(\boldsymbol{M})}(\mathcal{U}, \mathcal{V})$.

The formula in this definition applies even if M has corners. So we may view $\boldsymbol{C}$ as a functor on the larger category of manifolds with corners and smooth maps. In this setting, the subsheaf of sets $\boldsymbol{\Delta}^n \subset \mathbb{A}^n$ of Section 4b is representable.

Proposition 6.5 *There is a canonical weak equivalence of simplicial spaces* $\mathrm{Ex}^\infty \boldsymbol{C} \to \boldsymbol{C}(\boldsymbol{\Delta}^\bullet)$.

Proof Let Cov denote the category $\mathsf{Cov}(\boldsymbol{\Delta}^n)$ of open covers of $\boldsymbol{\Delta}^n$ and refinements. Let $\mathsf{Cov}_{\mathsf{sd}}$ be the full subcategory of Cov spanned by open covers by open stars of the vertices of some barycentric subdivision of $\boldsymbol{\Delta}^n \subset \mathbb{A}^n$. The set of objects of $\mathsf{Cov}_{\mathsf{sd}}$ is therefore identified with the nonnegative integers: for each $i \geq 0$, the corresponding open cover $\mathcal{U}(i)$ of $\boldsymbol{\Delta}^n$ is indexed by the set of vertices of the i^{th} barycentric subdivision of $\boldsymbol{\Delta}^n$. The simplicial space $N\mathcal{U}(i)$ is degreewise weakly equivalent to the simplicial discrete space $\mathrm{Sd}^{i+1}\Delta[n]$. To see this, note that, for a subset $S \subset \mathrm{Sd}^i \Delta[n]_0$, the space U_S is the open star of the unique nondegenerate simplex in $\mathrm{Sd}^i \Delta[n]$ with vertex set S, if that simplex exists, and otherwise is empty; and the 0–simplices of $\mathrm{Sd}^{i+1}\Delta[n]$ are by definition the nondegenerate simplices of $\mathrm{Sd}^i \Delta[n]$.

For each $i \geq 0$, there is a contractible choice of morphisms $\mathcal{U}(i+1) \to \mathcal{U}(i)$ in $\mathsf{Cov}_{\mathsf{sd}}$. Among these, we are interested in a specific morphism, namely the one whose underlying function between indexing sets $\mathrm{Sd}^{i+1}\Delta[n]_0 = \mathrm{Sd}(\mathrm{Sd}^i \Delta[n])_0 \to \mathrm{Sd}^i \Delta[n]_0$ is the last vertex map. The corresponding functor $\mathbb{N} \to \mathsf{Cov}_{\mathsf{sd}}$ that selects these morphisms is an equivalence of simplicial categories.

Write $j : \mathsf{Cov}_{\mathsf{sd}} \hookrightarrow \mathsf{Cov}$ for the inclusion. Clearly, every open cover of $\boldsymbol{\Delta}^n$ can be refined by one in $\mathsf{Cov}_{\mathsf{sd}}$. That is to say, for every open cover $\mathcal{V}$ of $\boldsymbol{\Delta}^n$, the comma category $j/\mathcal{V}$ is nonempty. The category $j/\mathcal{V}$ is equivalent to the discrete category (preorder) of open covers $\mathcal{U}(i)$ in $\mathsf{Cov}_{\mathsf{sd}}$ such that $\mathcal{U}(i) \leq \mathcal{V}$ with refinement relation $\leq$. Clearly, $\mathcal{U}(i) \leq \mathcal{U}(i')$ if and only if $i \geq i'$. From this description it is clear that $j/\mathcal{V}$ is contractible. This shows that j is homotopy final, ie that the homotopy colimit defining $\boldsymbol{C}(\boldsymbol{\Delta}^n)$ may be indexed by the smaller $\mathsf{Cov}_{\mathsf{sd}}$.

To summarize, we have weak equivalences

$$\operatorname*{hocolim}_{i>0} \mathrm{map}(\mathrm{Sd}^i \Delta[n], \boldsymbol{C}) \to \operatorname*{hocolim}_{\mathcal{U}\in\mathsf{Cov}_{\mathsf{sd}}} \mathrm{map}(N\mathcal{U}, \boldsymbol{C}) \xrightarrow{j^*} \boldsymbol{C}(\boldsymbol{\Delta}^n)$$

which are functorial in n, and so the result follows. □

Theorem 6.6 *For every smooth manifold M, the natural map*

$$\boldsymbol{BC}(M) \to \boldsymbol{R}\mathrm{map}(M, B\boldsymbol{C})$$

is a weak equivalence. Here $B\boldsymbol{C}$ denotes the classifying space of $\boldsymbol{C}$, ie the geometric realization of $\boldsymbol{C}$ viewed as a simplicial space, and $\boldsymbol{BC}$ is the functor $\boldsymbol{B}$ applied to the ∞–sheaf in Definition 6.4.

Proof This is immediate from Theorem 1.1, together with the identification of $|\boldsymbol{C}(\boldsymbol{\Delta}^\bullet)|$ with $|\mathrm{Ex}^\infty \boldsymbol{C}|$ from Proposition 6.5, and $|\mathrm{Ex}^\infty \boldsymbol{C}|$ with $B\boldsymbol{C} = |\boldsymbol{C}|$ from Proposition A.13. □

Remark 6.7 Clearly, if $\boldsymbol{C} \to \boldsymbol{D}$ is a map inducing a weak equivalence between classifying spaces $B\boldsymbol{C} \to B\boldsymbol{D}$, then $\boldsymbol{BC}(M) \to \boldsymbol{BD}(M)$ is a weak equivalence for every manifold M. This is the case, for example, if $\boldsymbol{C} \to \boldsymbol{D}$ is a Dwyer–Kan equivalence of Segal spaces.

Appendix Technical lemmas on simplicial sets and spaces

This appendix contains characterizations of weak equivalences in simplicial sets and simplicial spaces in the form that is needed in the paper. For simplicial sets, this is classical but we could not find the statements that we need in the literature (Examples A.5 and A.6). For simplicial spaces, this is less standard, although it may well be known.

Aa Special criteria for simplicial weak equivalences

Proposition A.1 [Dugger and Isaksen 2004, Proposition 4.1] *A map $f: X \to Y$ between Kan complexes is a weak equivalence if and only if for every $n \geq 0$ and every commutative square*

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

there is a lift as pictured making the upper triangle commute and the lower triangle commute up to a homotopy $H: \Delta^n \times \Delta^1 \to Y$ which is fixed on $\partial\Delta^n$.

In Dugger and Isaksen's terminology, a map f solving the lifting problem of this proposition is said to have the *relative homotopy lifting property* (RHLP) with respect to $\partial\Delta^n \to \Delta^n$.

It will be useful to think of these lifting properties in the following way. Let $\mathsf{S}^{[1]}$ denote the category whose objects are maps of simplicial sets and morphisms are commutative squares. Let τ denote the morphism in $\mathsf{S}^{[1]}$

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^n \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ \Delta^n & \longrightarrow & \Delta^n \times \Delta^1 \sqcup_{\partial\Delta^n \times \Delta^1} \partial\Delta^n \end{array}$$

(with source i and target j). Proposition A.1 then reads: a map f between Kan complexes is a weak equivalence if and only if $\tau^*: \mathrm{map}(j, f) \to \mathrm{map}(i, f)$ is a surjection.

Corollary A.2 *Let $\tau' \colon i' \to j'$ be a commutative square weakly equivalent to τ. (That is, τ' is related to τ by a zigzag of weak equivalences of squares.) A map $f : X \to Y$ of simplicial sets is a weak equivalence if and only if*

$$\boldsymbol{R}\mathrm{map}(j', f) \to \boldsymbol{R}\mathrm{map}(i', f)$$

is a surjection on π_0, where $\boldsymbol{R}$map refers to the homotopy function complex in $\mathsf{S}^{[1]}$ relative to objectwise weak equivalences.

Proof Since derived mapping spaces are invariant by weak equivalences by definition or construction, it suffices to prove that f is a weak equivalence if and only if

$$\boldsymbol{R}\mathrm{map}(j, f) \to \boldsymbol{R}\mathrm{map}(i, f) \tag{A.3}$$

is a surjection on π_0. To interpret the derived mapping spaces, let us equip $\mathsf{S}^{[1]}$ with the *projective* model structure. In this model structure, an object (ie map) is fibrant if source and target are Kan simplicial sets. Without loss of generality, we may assume that f is fibrant. Cofibrant objects are simplicial maps that are cofibrations (between cofibrant objects, which is no condition here). Cofibrations are commutative squares

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \longrightarrow & B' \end{array}$$

(with source i and target j) where the top map and the map

$$A' \sqcup_A B \to B'$$

are cofibrations of simplicial sets. It is not difficult to see that the morphism τ is then a cofibration between cofibrant objects. It follows that

$$\tau^* \colon \mathrm{map}(j, f) \to \mathrm{map}(i, f)$$

is identified with (A.3) and is a Kan fibration. Since a Kan fibration is surjective if and only if it is surjective on π_0, the result follows. □

Below are three examples which give rise to equivalent lifting problems:

Example A.4 Let τ' be the morphism in $\mathsf{S}^{[1]}$

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \partial\Delta^n \times \Delta^1 \sqcup_{\partial\Delta^n \times \{1\}} \Delta^n \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle j'} \\ \Delta^n & \longrightarrow & \Delta^n \times \Delta^1 \end{array}$$

Then τ' is weakly equivalent to τ and is a projective cofibration.

Example A.5 Let τ' be the morphism in $\mathsf{S}^{[1]}$

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \partial\Delta^n \times \Delta^1 \sqcup_{\partial\Delta^n \times \{1\}} \Delta^n \\ {\scriptstyle i'}\big\downarrow & & \big\downarrow{\scriptstyle j'} \\ \Lambda^{n+1} & \longrightarrow & \Lambda^{n+1} \times \Delta^1 \sqcup_{\Lambda^{n+1} \times \{1\}} \Delta^{n+1} \end{array}$$

Then τ' is weakly equivalent to τ and is a projective cofibration.

Example A.6 Let D be the simplicial set defined as the quotient Δ^2/d_0 where $d_0\colon \Delta^1 \to \Delta^2$ is the face that misses 0. The two remaining faces d_1, d_2 give two inclusions $\Delta^1 \to D$. Let τ' be the morphism in $\mathsf{S}^{[1]}$

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^n \sqcup_{\partial\Delta^n} \partial\Delta^n \times \Delta^1 \\ {\scriptstyle i'}\big\downarrow & & \big\downarrow{\scriptstyle j'} \\ \Delta^n & \longrightarrow & \Delta^n \times \Delta^1 \sqcup_{\partial\Delta^n \times \Delta^1} \partial\Delta^n \times D \end{array}$$

Then τ' is weakly equivalent to τ and is a projective cofibration.

So, in view of the previous result, a map $f : X \to Y$ between Kan complexes is a weak equivalence if and only if

$$(\tau')^*\colon \mathrm{map}(j', f) \to \mathrm{map}(i', f)$$

is surjective for $\tau'\colon i' \to j'$ as in the examples above.

Ab Criteria for realization weak equivalences

Definition A.7 A simplicial space is a contravariant functor from Δ to spaces (alias simplicial sets).

A simplicial space $[m] \mapsto X_m$ may be viewed as a bisimplicial set, ie a contravariant functor from $\Delta \times \Delta$ to Sets. However, the two Δ directions play different roles and it is important to distinguish them.

A map $X \to Y$ between simplicial spaces is a (degreewise) *weak equivalence* if, for each $m \geq 0$, the map $X_m \to Y_m$ is a weak equivalence of spaces. We write $\boldsymbol{R}\mathrm{map}(X, Y)$ for the homotopy function complex with respect to degreewise weak equivalences. This may be computed as $\mathrm{map}(X^c, Y^f)$ in a model structure on simplicial spaces with levelwise weak equivalences, for a cofibrant replacement $X^c \to X$ and a fibrant replacement $Y \to Y^f$. There are two canonical choices for such a model structure: the Reedy (= injective) model structure and the projective model structure.

The diagonal functor $d : \mathsf{sS} \to \mathsf{S}$ has a left adjoint $d_!$ which is the unique colimit-preserving functor with $d_!(\Delta^n) = \Delta^n \otimes \Delta[n]$. (For a simplicial set K and a simplicial space X, the tensor $K \otimes X$ is the simplicial space with n–simplices $K \times X_n$.)

There is another colimit-preserving functor $\delta\colon \mathsf{S} \to \mathsf{sS}$ defined by $\delta(\Delta^n) = \Delta[n]$, ie pullback along the projection onto the first factor $\Delta \times \Delta \to \Delta$. The projection $\Delta^n \otimes \Delta[n] \to \Delta^0 \otimes \Delta[n]$ induces a natural transformation $d_! \to \delta$.

Lemma A.8 *For each simplicial set X, the natural map $d_!(X) \to \delta(X)$ is a degreewise weak equivalence of simplicial spaces.*

Proof For representables, this is clear. A general simplicial set X is a filtered colimit of finite-dimensional simplicial sets X_i and filtered colimits of simplicial spaces are homotopy colimits, so it is enough to prove the statement for finite-dimensional simplicial sets. Suppose that we have proved the statement for all simplicial sets of dimension $< n$. We want to prove the statement for a simplicial set X of dimension n. Let $\mathrm{sk}_{n-1} X$ denote the $(n-1)^{\text{th}}$ skeleton of X, so that we have a pushout

$$\begin{array}{ccc} \coprod_{X_n} \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \coprod_{X_n} \Delta^n & \longrightarrow & X \end{array}$$

Since $d_!$ and δ are colimit-preserving, the result follows by induction and the case of representables. $\square$

Proposition A.9 *Let $f : X \to Y$ be a map between Reedy fibrant simplicial spaces which satisfy the Kan condition. Then $|f|\colon |X| \to |Y|$ is a weak equivalence if and only if every square*

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & Y \end{array}$$

has a lift as pictured making the lower triangle commute up to a given homotopy $\Delta[n] \times \Delta[1] \to Y$ and making the upper triangle commute up to a given homotopy $\partial\Delta[n] \times \Delta[1] \to X$. These two homotopies are required to be homotopic as maps $\partial\Delta[n] \times \Delta[1] \to Y$ and the homotopy should be constant on $\partial\Delta[n] \times \partial\Delta[1]$.

Proof Since X and Y are Kan complexes and d preserves Kan fibrations (Remark 3.20), $|X|$ and $|Y|$ are Kan complexes. In view of Proposition A.1, Example A.6 and the remarks that follow it, $|f|$ is a weak equivalence if and only if the map

$$\tau'^{*} : \mathrm{map}(i', |f|) \to \mathrm{map}(j', |f|)$$

is surjective (using the notation from Example A.6). By adjunction, this is equivalent to saying that

(A.10) $$(d_!\tau')^{*} : \mathrm{map}(d_! i', f) \to \mathrm{map}(d_! j', f)$$

is surjective. Since f is a map between Reedy fibrant simplicial spaces, it is a fibrant object in $\mathsf{sS}^{[1]}$ with the *projective* model structure on the category of functors $[1] \to \mathsf{sS}$, where sS is equipped with the Reedy model structure. Since $d_!$ preserves monomorphisms, $d_!\tau$ is a cofibration between cofibrant objects in that same model structure (see the proof of Corollary A.2). Therefore, the map (A.10) is a fibration and as such it is surjective if and only if it is surjective on π_0. These considerations also lead us to identify (A.10) with the map on derived mapping spaces

$$(d_!\tau')^*\colon \boldsymbol{R}\mathrm{map}(d_!i', f) \to \boldsymbol{R}\mathrm{map}(d_!j', f),$$

which by Lemma A.8 is identified with

$$\delta(\tau')^*\colon \boldsymbol{R}\mathrm{map}(\delta(i'), f) \to \boldsymbol{R}\mathrm{map}(\delta(j'), f).$$

Since $\delta(\tau')$ is also a cofibration between cofibrant objects, this map is identified with

$$\delta(\tau')^*\colon \mathrm{map}(\delta(i'), f) \to \mathrm{map}(\delta(j'), f).$$

The surjectivity of this last map is equivalent to the existence of the lift in the statement of the proposition. □

Corollary A.11 *Let $f\colon X \to Y$ be a map between Reedy fibrant simplicial spaces. Suppose that for every $j \ge 0$ and every square*

$$\begin{array}{ccc} \mathrm{Sd}^j\,\partial\Delta[n] & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \mathrm{Sd}^j\,\Delta[n] & \longrightarrow & Y \end{array}$$

there is a lift as pictured making the lower triangle commute up to a given homotopy $\mathrm{Sd}^j\,\Delta[n] \times \Delta[1] \to Y$ and making the upper triangle commute up to a given homotopy $\mathrm{Sd}^j\,\partial\Delta[n] \times \Delta[1] \to X$. These two homotopies are required to be homotopic as maps $\mathrm{Sd}^j\,\partial\Delta[n] \times \Delta[1] \to Y$ and the homotopy should be constant on $\mathrm{Sd}^j\,\partial\Delta[n] \times \partial\Delta[1]$. Then $|f|\colon |X| \to |Y|$ is a weak equivalence.

Proof Apply Proposition A.9, replacing X and Y by the simplicial spaces $\mathrm{Ex}^\infty X$ and $\mathrm{Ex}^\infty Y$, which satisfy the Kan condition by Proposition A.13. □

Ac Properties of subdivisions of simplicial spaces

Recall the simplicial subdivision $\mathrm{sd}\,\Delta^n$, ie the nerve of the poset of nonempty subsets of $[n] = \{0, \dots, n\}$.

Definition A.12 Denote by

$$\mathrm{Sd}\colon \mathsf{sS} \to \mathsf{sS}$$

the simplicial left adjoint functor that sends $\Delta[n]$ to $\mathrm{sd}\,\Delta^n$ viewed as a simplicial *discrete* space. Denote by

$$\mathrm{Ex}\colon \mathsf{sS} \to \mathsf{sS}$$

the simplicial right adjoint functor of Sd.

Every simplicial discrete space is Reedy cofibrant, so, replacing X by a Reedy fibrant simplicial space X^f, we may write the right derived functor of Ex evaluated at X as the (honest) mapping space $\operatorname{map}(\operatorname{Sd}\Delta[n], X^f)$.

There is a natural map $\gamma\colon \operatorname{Sd}\Delta[n] \to \Delta[n]$, sending a subset $\{i_0, \dots, i_k\} \subset [n]$ to i_k (the last vertex). The colimit

$$X \xrightarrow{\gamma^*} \operatorname{Ex} X \xrightarrow{\gamma^*} \operatorname{Ex}^2 X \to \cdots$$

is denoted by $\operatorname{Ex}^\infty X$. The map γ has a section $\Delta[n] \to \operatorname{Sd}\Delta[n]$ so if X is Reedy fibrant, all the maps in the tower are degreewise cofibrations and so the colimit computes the homotopy colimit.

We collect the important properties of the Ex^∞ endofunctor below. These parallel (or, rather, include) the well-known ones for simplicial *sets*.

Proposition A.13 *For a simplicial space X:*

(1) *$\operatorname{Ex}^\infty X$ is a Kan simplicial space.*

(2) *$X \to \operatorname{Ex}^\infty X$ is a weak equivalence after geometric realization.*

(3) *For each i, including $i = \infty$, Ex^i preserves (trivial) Kan fibrations, zero simplices and finite homotopy limits*

Proof By construction, the functor Ex^i for $0 \le i \le \infty$ sends weak equivalences of simplicial spaces to weak equivalences. If X is Reedy fibrant then

$$\operatorname{map}(\operatorname{Sd}\Delta[n], X) \to \operatorname{map}(\operatorname{Sd}\partial\Delta[n], X)$$

is a fibration (since $\operatorname{Sd}\partial\Delta[n] \to \operatorname{Sd}\Delta[n]$ is a degreewise monomorphism, hence a cofibration). Therefore, $\operatorname{Ex} X$ is Reedy fibrant. By standard compactness arguments, it follows that $\operatorname{Ex}^\infty X$ is also Reedy fibrant. Hence, in proving (1), (2) and (3), we may assume from the outset that X is Reedy fibrant.

The arguments to prove (1) and (3) are identical to the classical ones for simplicial sets, so we do not reproduce them here. As for (2), take a trivial Kan fibration $X' \to X$, where X' is a simplicial *set* (see [Lurie 2011, Proposition 7]), and consider the square

$$\begin{array}{ccc} X' & \longrightarrow & \operatorname{Ex} X' \\ \downarrow & & \downarrow \\ X & \longrightarrow & \operatorname{Ex} X \end{array}$$

Since the diagonal preserves trivial Kan fibrations, the vertical maps are weak equivalences after applying the diagonal (for the right-hand one, use part (3)). The top horizontal map is a weak equivalence; see eg [Goerss and Jardine 1999, Chapter III, Theorem 4.6]. We conclude that the diagonal of the lower map is a weak equivalence. □

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Chern character for infinity vector bundles

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Coherent sheaves on general complex manifolds do not necessarily have resolutions by finite complexes of vector bundles. However, D Toledo and Y L L Tong showed that one can resolve coherent sheaves by objects analogous to chain complexes of holomorphic vector bundles, whose cocycle relations are governed by a coherent infinite system of homotopies. In modern language, such objects are obtained by the ∞–sheafification of the simplicial presheaf of chain complexes of holomorphic vector bundles. We define a Chern character as a map of simplicial presheaves, whereby the connected components of its sheafification recover the Chern character of Toledo and Tong. As a consequence, our construction extends O'Brian, Toledo and Tong's definition of the Chern character to the settings of stacks and in particular the equivariant setting. Even in the classical setting of complex manifolds, the induced maps on higher homotopy groups provide new Chern–Simons, and higher Chern–Simons, invariants for coherent sheaves.

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1 Introduction

The celebrated Hirzebruch–Riemann–Roch theorem (HRR) [Hirzebruch 1954] is a generalization of the classical Riemann–Roch theorem for holomorphic line bundles on compact Riemann surfaces. In HRR, the setting of a line bundle on a Riemann surface is generalized to an arbitrary holomorphic bundle E on a smooth projective variety X over the complex numbers. The main tool in proving HRR is a resolution of the diagonal $X \to X \times X$, thought of as a coherent sheaf on $X \times X$, by a finite chain complex of vector bundles.

The Atiyah–Singer index theorem [1968], and the theory of elliptic and pseudoelliptic differential operators, can further be thought of as a far-reaching generalization of HRR and other high-powered theorems, such as the Gauss–Bonnet theorem, to a much vaster context. For example, using the Atiyah–Singer index theorem one can readily extend HRR to holomorphic bundles on even compact complex manifolds that are not necessarily algebraic (see for example [Freed 2021] for an exposition).

Unfortunately, such techniques, found in work by Atiyah, Bott and Patodi [1973] as well as by Gilkey [1973], use differential geometric methods that heavily rely on an auxiliary choice of a Hermitian metric on the manifold as well as the bundle. For example, one uses the metric to establish a heat flow and smooth out the diagonal de Rham current $X \to X \times X$ into a differential form (the heat kernel). However, generally, in complex geometry choosing a metric can be thought of as unnatural and out of context unless within the very specialized realm of Kähler geometry.

Casting this as a deficiency is not only a matter of taste but concerns applications of these ideas to settings where local automorphisms are involved, such as the equivariant as well as the "stacky" discussion. One would therefore desire an intrinsic complex geometric discussion, whereby one establishes HRR, and similar theorems, for general complex manifolds and holomorphic vector bundles outside metric geometry.

Toledo and Tong [1976; 1978a; 1978b; 1986] and O'Brian, Toledo and Tong [1981a; 1981b; 1981c] made several remarkable conceptual breakthroughs by providing local Čech cohomological proofs of HRR [1981b] and Grothendieck–Riemann–Roch (GRR) [1981a]. Through the modern lens, one may interpret their work as a hands-on theory of infinity stacks, which only much more recently has been made into a full-fledged mathematical theory. One of the key constructions by O'Brian, Toledo and Tong [1981c] is that of the Chern class for a coherent analytic sheaf on a complex manifold. While their construction is the one we focus on here, there is also another approach to calculating Chern classes for coherent analytic sheaves, as shown in [Green 1980; Toledo and Tong 1986] and later formalized by Timothy Hosgood [2020; 2023; 2024].

To get a taste for the type of math Toledo and Tong invented and utilized, consider the question of resolving the diagonal $X \to X \times X$, or more generally an arbitrary coherent sheaf, on a complex manifold, by a finite chain complex of vector bundles. One knows that when the complex manifold admits a positive line bundle such resolutions always exist (see [Griffiths and Harris 1978, page 705]). While in the algebraic setting the canonical line bundle provides such a line bundle, general complex manifolds may not support them. Toledo and Tong obviated such difficulties by resolving the problems in a homotopical setting in which strict identities are replaced with a coherent infinite system of homotopies. For instance, as a complex vector bundle is a bunch of transition functions satisfying the familiar cocycle conditions, they showed that, by requiring the cocycle condition to hold up to an infinite system of homotopies, not only could every coherent sheaf on a complex manifold be resolved by these more general objects, but also all of the necessary complex geometric arguments would remain valid.

Let us be more specific and start with a coherent sheaf on a complex manifold. Choose a good Stein cover for the manifold on which the coherent sheaf can be locally resolved by a chain complex of vector bundles; such a cover always exists. By restricting these resolutions to double intersections, we obtain two resolutions for the same coherent sheaf on that intersection which, by the uniqueness of resolutions over Stein manifolds, are then related by a quasi-isomorphism. On triple intersections, the three relevant quasi-isomorphisms may not fit to give you a chain complex of vector bundles, but the discrepancy can be killed by a homotopy. These assigned homotopies to triple intersections may not satisfy the required compatibilities on quadruple intersections but the discrepancy can be killed by a higher homotopy. Repeating this pattern ad infinitum gives rise to an infinite system of homotopies.

Historically, the use of coherent infinite systems of homotopy in a different context was known to some algebraic topologists almost 30 years prior but even there it was considered rather esoteric. Jim Stasheff [1963a; 1963b] showed how the based loop space of a pointed space was an A_∞ monoid. Nowadays these mathematical objects are inescapable and it is common knowledge among a large group of algebraic topologists that A_∞ algebras are just as good as differential associative algebras and have the same homotopy theories [Lefèvre-Hasegawa 2003]. Similarly, Toledo and Tong showed that these generalized objects are just as good as chain complexes of vector bundles as far as coherent cohomologies were concerned. While they did not make a formal claim about their corresponding homotopy theories, they showed how Ext and Tor of such generalized objects can be defined, calculated and, subsequently, be used to prove duality theorems à la Grothendieck and establish HRR and GRR.

Surprisingly, since their work very little has been done to formalize the homotopy theory of these objects. For example, in *Descente pour les n–champs*, André Hirschowitz and Carlos Simpson [1998] write:

> Dans les travaux de O'Brian, Toledo et Tong consacrés à une autre question issue de SGA 6, celle des formules de Riemann–Roch, on trouve des calculs de Čech qui sont certainement un exemple de situation de descente pour les complexes. Un meilleur cadre général pour ces calculs pourrait contribuer à notre compréhension des formules de Riemann–Roch.

This roughly translates to the following:

> In the work of O'Brian, Toledo and Tong devoted to another question arising from SGA 6 regarding the Riemann–Roch formulas, one can find Čech calculations that are an example of descent for complexes. A better general framework for these calculations could contribute to our understanding of the Riemann–Roch formulas.

Here we have taken the first step in providing a homotopy-theoretic framework for some of Toledo and Tong's mathematical objects. By simply finding the right homotopy-theoretic setting, their constructions extend far beyond what they had intended and point to new and exciting advances. For example, their construction of a Chern character for coherent sheaves in Hodge cohomology is easily generalized to the equivariant setting, or even to the setting of stacks. In addition, secondary and higher Chern characters are now an inseparable part of the discussion.

The inherent inclusion of these higher Chern characters points to the possibility of proving a version of GRR as a commutative diagram of spaces such that, after applying π_0 to the diagram, one would obtain a diagram of sets which is O'Brian, Toledo and Tong's GRR. Note that classical objects such as K–groups and cohomology groups are sets with additional algebraic structures.

In Section 2 we begin by defining the simplicial presheaves $\mathbf{IVB}$ and $\mathbf{\Omega}$, which will be the domain and codomain of our Chern map, respectively. For a fixed complex manifold $U \in \mathbb{C}\mathrm{Man}$, we first consider the dg-category $\mathrm{Perf}^{\nabla}(U)$ of finite chain complexes of holomorphic bundles with connection,[1] where there is no requirement that morphisms be compatible with connections. Taking the maximal Kan complex of the dg-nerve, we obtain a simplicial set $\mathbf{Perf}(U)$. Applying this construction objectwise over $\mathbb{C}\mathrm{Man}$ and noting that maps $f \in \mathbb{C}\mathrm{Man}^{\mathrm{op}}(U,V)$ induce maps of Kan complexes $\mathbf{Perf}(U) \xrightarrow{f^*} \mathbf{Perf}(V)$ via pullbacks, we obtain a simplicial presheaf $\mathbf{Perf}$ which is fibrant in the (global) projective model structure. Since the simplices $\mathbf{Perf}(U)_n = \mathrm{sSet}(\Delta^n, \mathbf{Perf}(U))$ lack the cyclic structure we will need later on to construct our trace map, we define a weakly equivalent (see Proposition 2.10) simplicial presheaf $\mathbf{IVB}(U)_n := \mathrm{sSet}(\widehat{\Delta}^n, \mathbf{Perf}(U))$ given by mapping the cyclic sets $\widehat{\Delta}^n$ into $\mathbf{Perf}(U)$. Here, $\widehat{\Delta}^n$ is the nerve of the category whose set of objects is $\mathbb{Z}/(n+1)\mathbb{Z}$ and all hom-sets have a single morphism (see Example 2.8). Next, we define $\mathbf{\Omega}$ in the same way we did in our previous paper [2022]; more precisely, $\mathbf{\Omega}(U)$ is the simplicial set whose k–simplices are decorations of all i–dimensional faces of the standard k–simplex with sequences of forms, all even for i even, and all odd for i odd, in such a way that the alternating sum of all forms sitting on the $(i-1)$–dimensional faces of any i–dimensional face add up to 0.

The Chern map $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ is then defined in Section 3 as follows. An n–simplex in $\mathbf{IVB}(U)_n$ consists of $n+1$ dg-bundles with connection $(\mathcal{E}_i, d_i, \nabla_i)$, and a set of maps $g = \{(g_{(i_0\dots i_k)}\colon \mathcal{E}_{i_k} \to \mathcal{E}_{i_0})\}_{(i_0,\dots,i_k)\in\widehat{\Delta}^n}$ satisfying the Maurer–Cartan condition (see Definition 3.3). First, in Definition 3.7, we define a trace map Tr_g similar to that of O'Brian, Toledo and Tong [1981c, Proposition 3.2], satisfying the condition

$$\mathrm{Tr}_g \circ (\hat{\hat{\delta}} + D + [g,-]) = \delta \circ \mathrm{Tr}_g. \tag{3-8}$$

Using this trace map, $\mathbf{Ch}$ is then defined (in Definition 3.13) by assigning to an n–simplex in $\mathbf{IVB}(U)_n$ as above decorations of the nondegenerate k–faces of Δ^n given by the elements in $\mathbf{\Omega}(U)_n$

$$\mathrm{Tr}_g(A^k)_\alpha \cdot \frac{u^k}{k!} = \mathrm{Tr}_g\big((\nabla(d+g))^k\big)_\alpha \cdot \frac{u^k}{k!} = \sum \pm \mathrm{tr}(g\cdot\nabla(d+g)\cdot\nabla(d+g)\cdots\nabla(d+g))_\alpha \cdot \frac{u^k}{k!} \tag{3-11}$$

[1] The use of Perf^{∇} is meant to allude to the study of perfect complexes.

for $k > 0$ and for $k = 0$ we assign the Euler characteristic. Our first main result is that this provides a map of (objectwise Kan) simplicial presheaves:

Theorem 3.14 *The Chern character* $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ *defined above is a map of simplicial presheaves.*

In Section 4 we construct what we call the Čech sheafification, $\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}} \to \mathbf{\Omega}^{\check{\dagger}}$ of the Chern map. Given a simplicial presheaf $\boldsymbol{F}$, the idea is that, for each open cover $(U_i \to X)_{i\in I}$, we can form the Čech nerve simplicial presheaf, $\check{N}U_\bullet$, and then compute the homotopy limit induced by the simplicial mapping space $\underline{\mathrm{sPre}}(\check{N}U_\bullet, \boldsymbol{F}) = \mathrm{holim}_i \prod_{\alpha_0,\dots,\alpha_i} \boldsymbol{F}(U_{\alpha_0\dots\alpha_i})$ by taking the totalization of the induced cosimplicial simplicial set $\boldsymbol{F}(\check{N}U_\bullet)$ defined in (4-1). The Čech sheafification $F^{\check{\dagger}}$ is then defined (Definition 4.1) by taking the colimit over all covers:

$$\boldsymbol{F}^{\check{\dagger}}(X) := \operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)). \tag{4-2}$$

As the construction is functorial in simplicial presheaves and preserves Kan complexes, we obtain a sheafified Chern map, $\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}} \to \mathbf{\Omega}^{\check{\dagger}}$, which is a map of Kan complexes. The rest of the section is devoted to showing how $\mathbf{Ch}^{\check{\dagger}}$ is related to the Chern character map of O'Brian, Toledo and Tong [1981c], which begins with Theorem 4.9, stating that the twisting cochains of [loc. cit.] include into the vertices of $\mathbf{IVB}^{\check{\dagger}}$. The full correspondence is given in Theorem 4.18, which shows that, if we restrict **IVB** to the simplicial presheaf **CohSh** considering only nonpositively graded chain complexes whose homology is concentrated in degree zero, then we fully recover the data from the Chern map in [loc. cit.] by the connected components of our sheafified Chern map:

Theorem 4.18 *For a given coherent sheaf, the formula for the Chern character* (4-15) *from* [loc. cit.] *is given by the terms in the formula* (4-14) *of the Chern character map*

$$\{\text{isomorphism classes of coherent sheaves}\} \simeq \pi_0(\mathbf{CohSh}^{\check{\dagger}}) \xrightarrow{\pi_0(\mathbf{Ch}^{\check{\dagger}})} \pi_0(\mathbf{\Omega}^{\check{\dagger}}) \simeq \bigoplus_{\substack{p,q\\ p+q \text{ even}}} H^p(\Omega^q) \tag{4-16}$$

applied to the corresponding twisting cochain interpreted (*by* Theorem 4.9) *as a* 0*–simplex in* $\mathbf{CohSh}^{\check{\dagger}}$.

Section 5 upgrades the results from the previous section to statements about (hyper)sheaves. Recall that a simplicial presheaf is a (hyper)sheaf if it is objectwise Kan and satisfies descent with respect to all hypercovers. By restricting our attention to simplicial presheaves of finite homotopy type taking values in Kan complexes, we prove in Proposition 5.2 that the aforementioned Čech sheafification construction computes the (hyper)sheafification. In particular, Proposition 5.12 states that, if we restrict to complex manifolds of bounded dimension, and restrict the homotopy type of **IVB**, then $\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}}_{\le n} \to \mathbf{\Omega}^{\check{\dagger}}$ is a map of hypersheaves. If instead we consider again **CohSh**, we see that its sheafification is a classifying stack for coherent sheaves, $\mathbb{R}\mathrm{Hom}(X, \mathbf{CohSh}) \simeq \mathbf{CohSh}^{\check{\dagger}}$:

Theorem 5.11 *The simplicial presheaf* **CohSh** *is a classifying prestack for coherent sheaves.*

Once again restricting to manifolds of bounded dimension, Theorem 5.13 states that our sheafified Chern map $\check{\mathbf{Ch}}^{\dagger}\colon \check{\mathbf{CohSh}}^{\dagger} \to \check{\mathbf{\Omega}}^{\dagger}$ is a map of (hyper)sheaves whose connected components yields the Chern map from [loc. cit.]. Finally, Remark 5.14 describes how our Chern character map generalizes to all stacks, with an eye towards future work in the equivariant setting.

Notation 1.1 The simplicial category is denoted by $\mathbf{\Delta}$. It has objects $[n] = \{0, \dots, n\}$ for $n \in \mathbb{N}_0$, and morphisms $\phi \in \mathbf{\Delta}([k],[n])$ that are nondecreasing maps $\phi\colon [k] \to [n]$, ie $\phi(i) \leq \phi(j)$ for $i \leq j$. The morphisms are generated by face maps $\delta_j : [n] \to [n+1]$ (the injection that skips the element j in $[n+1]$) and degeneracies $\sigma_j : [n] \to [n-1]$ (the surjection that maps j and $j+1$ to j).

Simplicial objects in a category $\mathcal{C}$ are functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, where the induced face and degeneracy morphisms are denoted by d_j and s_j, respectively. We denote the category of simplicial sets by $\mathrm{sSet} = \mathrm{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$. Cosimplicial objects in a category $\mathcal{C}$ are functors $\mathbf{\Delta} \to \mathcal{C}$.

Given an object X in a locally small category $\mathcal{C}$, we can consider its representable presheaf $yX := \mathcal{C}(-, X)$ given by the Yoneda embedding. Further, given a presheaf F on $\mathcal{C}$, we can consider its simplicially constant presheaf cF defined by $cF(Y)_n := F(Y)$. When context is clear we may drop the "y" or "c". For example, for an object X we might write X for the simplicial presheaf defined by $X(Y)_n := \mathcal{C}(Y, X)$.

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2 The simplicial presheaves IVB and $\mathbf{\Omega}$

We define two simplicial presheaves on the site of complex manifolds; first, $\mathbf{IVB}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ is the presheaf which will later give rise to infinity vector bundles (see Definition 4.4), and $\mathbf{\Omega} : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ is the presheaf of holomorphic forms. In the next section we will then define the Chern character map as a map of simplicial presheaves $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$.

Let $\mathbb{C}\mathrm{Man}$ be the category whose objects consist of complex manifolds, and morphisms are holomorphic maps. Furthermore, denote by dgCat the category of differential graded categories, ie categories $\mathcal{C}$ such that, for any two objects C_1 and C_2 of $\mathcal{C}$, the space of morphisms $\mathrm{Hom}(C_1, C_2)$ is a cochain complex, with the composition being a cochain map and the identity morphisms being closed.

Definition 2.1 Let $\mathrm{Perf}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{dgCat}$ be given by setting $\mathrm{Perf}(U)$ to be the dg-category whose objects $\mathcal{E} = (E_\bullet, d, \nabla) \in \mathrm{Perf}(U)$ are finite chain complexes of holomorphic vector bundles $E_\bullet \to U$ over U with differential $d\colon E_\bullet \to E_{\bullet+1}$, and with a holomorphic connection ∇ on $E_\bullet$. Morphisms $\mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ consist of graded morphisms of vector bundles $f : E_\bullet \to E'_\bullet$ which *need not have any special* *compatibility* with respect to the connections ∇ and ∇'. The dg structure on $\mathrm{Hom}(\mathcal{E},\mathcal{E}')$ is the induced one by the differential and gradings on $\mathcal{E}$ and $\mathcal{E}'$; in particular, the differential of an $f \in \mathrm{Hom}(\mathcal{E},\mathcal{E}')$ is defined to be $D(f) := f \circ d - (-1)^{|f|} d' \circ f$.

A holomorphic map $\varphi\colon U \to U'$ induces a functor $\mathrm{Perf}(\varphi)\colon \mathrm{Perf}(U') \to \mathrm{Perf}(U)$ by pulling back bundles via φ.

Since $\mathrm{Perf}(U)$ is a dg-category, we can apply the dg-nerve $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))$, which gives a simplicial set.

Note 2.2 Explicitly, we can describe the simplicial structure of the dg-nerve $\mathrm{dg}\mathcal{N}(\mathcal{C})$ of a dg-category $\mathcal{C}$ (for us, it will always be $\mathcal{C} = \mathrm{Perf}(U)$) as follows; see [Lurie 2017, 1.3.1.6; Faonte 2017, Definition 2.2.8]:

(1) A 0–simplex in $\mathrm{dg}\mathcal{N}(\mathcal{C})_0$ consists of an object $\mathcal{E}$ of $\mathcal{C}$.

(2) A 1–simplex in $\mathrm{dg}\mathcal{N}(\mathcal{C})_1$ consists of $(\mathcal{E}_1, \mathcal{E}_0, g_{01})$, ie two objects $\mathcal{E}_0$ and $\mathcal{E}_1$ of $\mathcal{C}$ and a morphism $g_{01}\colon \mathcal{E}_1 \to \mathcal{E}_0$ in $\mathcal{C}$ of degree 0, which is closed, ie $Dg_{01} = 0$, where we denoted the differential in $\mathrm{Hom}_{\mathcal{C}}$ by D. (In the case of $\mathcal{C} = \mathrm{Perf}(U)$, the internal differential D is given by the differentials d and d' on E and E', respectively, via $Df = f \circ d - (-1)^{|f|} d' \circ f$, so that $Dg_{01} = 0$ means that g_{01} is a chain map of dg-vector bundles.)

(3) A 2–simplex in $\mathrm{dg}\mathcal{N}(\mathcal{C})_2$ consists of $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, g_{01}, g_{12}, g_{02}, g_{012})$, ie three objects $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{C}$, three morphisms $g_{ij}\colon \mathcal{E}_j \to \mathcal{E}_j$ of degree 0, where $i, j \in \{0,1,2\}$ with $i < j$, and another morphism $g_{012}\colon \mathcal{E}_2 \to \mathcal{E}_0$ of degree -1 satisfying $Dg_{012} = g_{01} \circ g_{12} - g_{02}$.

(4) An n–simplex in $\mathrm{dg}\mathcal{N}(\mathcal{C})_n$ consists of $n+1$ dg-vector bundles $\mathcal{E}_0, \dots, \mathcal{E}_n$ and morphisms

$$g_{i_0 \dots i_k}\colon \mathcal{E}_{i_k} \to \mathcal{E}_{i_0}$$

of degree $1-k$ for each sequence $i_0, \dots, i_k \in \{0, \dots, n\}$ with $i_0 < \dots < i_k$ and $k \geq 1$ such that

$$D(g_{i_0 \dots i_k}) = \sum_{j=1}^{k-1} (-1)^{j-1} g_{i_0 \dots \hat{i}_j \dots i_k} + \sum_{j=1}^{k-1} (-1)^{k(j-1)+1} g_{i_0 \dots i_j} \circ g_{i_j \dots i_k}. \tag{2-1}$$

(5) For a morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$, there is an induced map $\phi^\sharp_{\mathrm{dg}\mathcal{N}}\colon \mathrm{dg}\mathcal{N}(\mathcal{C})_m \to \mathrm{dg}\mathcal{N}(\mathcal{C})_n$, given by mapping $(\mathcal{E}_i, g_{i_0 \dots i_k})_{\text{all indices}} \in \mathrm{dg}\mathcal{N}(\mathcal{C})_m$ to $(\mathcal{E}_{\phi(i)}, \tilde{g}_{i_0 \dots i_k})_{\text{all indices}} \in \mathrm{dg}\mathcal{N}(\mathcal{C})_n$, which is defined by either $\tilde{g}_{i_0 \dots i_k} = g_{\phi(i_0) \dots \phi(i_k)}$ if ϕ is injective on $\{i_0, \dots, i_k\}$, or $\tilde{g}_{i_0 i_1} = \mathrm{id}_{E_{\phi(i_0)}}$ if $\phi(i_0) = \phi(i_1)$, or $\tilde{g}_{i_0 \dots i_k} = 0$ in all other cases, ie when $k \geq 2$ and $\phi(i_p) = \phi(i_{p+1})$ for some $p = 0, \dots, k-1$.

In the later sections, we will use the dg-nerve of U as *local* building blocks of chain complexes of vector bundles on a complex manifold. To obtain a reasonable gluing, we will want the chain maps $g_{i_0 i_1}$ to be *homotopy equivalences*. This can be achieved in a natural way by taking the maximal Kan subcomplex $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ$ of $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))$; see [Joyal 2002, Corollary 1.5].

Definition 2.3 Let $\mathbf{Perf}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ be the simplicial presheaf given by $\mathbf{Perf}(U) := \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ$, ie the maximal Kan subcomplex of the dg-nerve of $\mathrm{Perf}(U)$.

We have the following characterization of the simplices of $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$ via [Joyal 2002, Theorem 2.2], for example:

Lemma 2.4 *An n–simplex in $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$ consists precisely of an n–simplex in $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))$ as described in Note 2.2(4) above, with the extra condition that all morphisms $g_{i_0 i_1}\colon \mathcal{E}_{i_1} \to \mathcal{E}_{i_0}$ are **homotopy equivalences**.*

Now, all chain maps $g_{i_0 i_1}$ on the edges of all simplices of $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$ are homotopy equivalences. In order to be able to define the Chern character below, we will need to find homotopy inverses of these together with compatible higher homotopies. This can be achieved as follows. First, using the Yoneda lemma for simplicial sets, we know that the n–simplices of a simplicial set $X_\bullet$ are precisely the simplicial set maps from $\Delta^n := \mathbf{\Delta}(-,[n])$ into $X_\bullet$, ie $X_n = X([n]) \cong \mathrm{Nat}(\mathbf{\Delta}(-,[n]), X) = \mathrm{sSet}(\Delta^n, X)$. Thus, we define $\mathbf{Perf}^\Delta\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by setting

$$\mathbf{Perf}^\Delta(U)_n := \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ_n = \mathrm{sSet}\big(\Delta^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ\big). \tag{2-2}$$

More generally, we define:

Definition 2.5 Let Q be a cosimplicial simplicial set, ie $Q\colon \mathbf{\Delta} \to \mathrm{sSet}$. In more detail, we denote by $Q^n = Q([n]) \in \mathrm{sSet}$ the image of $[n] \in \mathbf{\Delta}$ under Q, which is itself a simplicial set, $Q^n_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$, $Q^n_k := Q^n([k]) \in \mathrm{Set}$. Then, define $\mathbf{Perf}^Q\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by setting

$$\mathbf{Perf}^Q(U)_n := \mathrm{sSet}\big(Q^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ\big). \tag{2-3}$$

Since $\{Q^n\}_n$ is a cosimplicial object in sSet, this induces, for each $(f\colon [n] \to [m]) \in \mathbf{\Delta}$, a map $\mathbf{Perf}^Q(U)_m \to \mathbf{Perf}^Q(U)_n$, making $\mathbf{Perf}^Q(U)$ into a simplicial set.

For a holomorphic map $\varphi\colon U \to U'$, the induced map $\mathbf{Perf}^Q(U') \to \mathbf{Perf}^Q(U)$ is given by composition with the map $\mathrm{Perf}(U') \to \mathrm{Perf}(U)$ from Definition 2.1, ie by pulling back via φ.

We are mainly interested in the following Examples 2.6 and 2.8.

Example 2.6 Let $\Delta\colon \mathbf{\Delta} \to \mathrm{sSet}$ be given by $\Delta^n := \mathbf{\Delta}(-,[n])$ be the standard simplicial n–simplex given by morphisms of $\mathbf{\Delta}$ into $[n]$, ie its k–simplices $\phi \in \Delta^n_k = \mathbf{\Delta}([k],[n])$ are nondecreasing maps from $[k]$ to $[n]$, ie if we set $i_j := \phi(j)$, these are sequences of indices $(i_0 \le \cdots \le i_k)$ with $i_0, \dots, i_k \in \{0, \dots, n\}$. Face maps are $d_j\colon \Delta^n_k \to \Delta^n_{k-1}$ that remove the j^{th} index i_j, and degeneracies $s_j\colon \Delta^n_k \to \Delta^n_{k+1}$ that repeat the j^{th} index i_j.

By Yoneda, any simplicial set map $\Delta^n \to X$ is completely determined by the image of its nondegenerate n–simplex. Thus, by (2-2), $\mathbf{Perf}^\Delta(U)$ has n–simplices given as described precisely by Lemma 2.4, ie by Note 2.2 with homotopy equivalences on edges.

Before we give our second main example for $\mathbf{Perf}^Q$, we record a useful lemma about simplicial set maps into the dg-nerve $\mathbf{Perf}^Q(U)$.

Lemma 2.7 *Let $X_\bullet$ be a simplicial set, and let $\mathcal{C}$ be a dg-category (for us, $\mathcal{C} = \mathrm{Perf}(U)$). Then a simplicial set map $X \to \mathrm{dg}\,\mathcal{N}(\mathcal{C})$ is precisely given by the following data:*

(1) *For each 0–simplex $\alpha \in X_0$, we have an object $\mathcal{E}_\alpha$ of $\mathcal{C}$.*

(2) *For each nondegenerate k–simplex $\alpha \in X_k$ with $k \geq 1$, there is a morphism $g_\alpha\colon \mathcal{E}_{\alpha(k)} \to \mathcal{E}_{\alpha(0)}$ of degree $1-k$ satisfying the compatibility condition*

$$D(g_\alpha) = \sum_{j=1}^{k-1}(-1)^{j-1} g_{\alpha(0,\dots,\hat{j},\dots,k)} + \sum_{j=1}^{k-1}(-1)^{k(j-1)+1} g_{\alpha(0,\dots,j)} \circ g_{\alpha(j,\dots,k)}. \tag{2-4}$$

Here, for a disjoint union decomposition $\{0,\dots,k\} = \{i_0,\dots,i_p\} \sqcup \{j_0,\dots,j_q\}$ with $i_0 < i_1 < \dots < i_p$ and $j_0 < j_1 < \dots < j_q$, we denote by $\alpha(i_0,\dots,i_p) := d_{j_0} \circ \dots \circ d_{j_q}(\alpha) \in X_p$ the face of α corresponding to indices $\{i_0,\dots,i_p\} \subseteq \{0,\dots,k\}$.

In particular, a simplicial set map $X \to \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$ has the same data as given above with the extra condition that the maps g_α for $\alpha \in X_1$ are homotopy equivalences.

Proof Let $\mathcal{F}\colon X \to \mathrm{dg}\,\mathcal{N}(\mathcal{C})$ be a map of simplicial sets and, for $l \geq 0$, let $\alpha \in X_l$ be an l–simplex. Thus, $\mathcal{F}(\alpha) \in \mathrm{dg}\,\mathcal{N}(\mathcal{C})_l$, and, by Note 2.2, there are dg-vector spaces $\mathcal{E}^\alpha_0,\dots,\mathcal{E}^\alpha_l$, and for all $i_0,\dots,i_k \in \{0,\dots,l\}$, $k \geq 1$ with $i_0 < \dots < i_k$, there are maps $g^\alpha_{i_0\dots i_k}\colon \mathcal{E}^\alpha_{i_k} \to \mathcal{E}^\alpha_{i_0}$ satisfying (2-1). We claim that the data of the highest maps $g^\rho_{0\dots p}$ for all nondegenerate $\rho \in X_p$ is sufficient to recover all other maps $g^\alpha_{i_0\dots i_k}$: For $\alpha \in X_l$ and $i_0,\dots,i_k \in \{0,\dots,l\}$ with $i_0 < \dots < i_k$ with $k < l$, we use the commutative diagram for $\phi\colon [k] \to [l]$, $\phi(p) := i_p$,

$$\begin{array}{ccc} X_l & \xrightarrow{\mathcal{F}_l} & \mathrm{dg}\,\mathcal{N}(\mathcal{C})_l \\ {\scriptstyle \phi^\sharp_X}\big\downarrow & & \big\downarrow{\scriptstyle \phi^\sharp_{\mathrm{dg}\,\mathcal{N}}} \\ X_k & \xrightarrow{\mathcal{F}_k} & \mathrm{dg}\,\mathcal{N}(\mathcal{C})_k \end{array}$$

mapping the $i_0 < \dots < i_k$–component $g^\alpha_{i_0\dots i_k}$ of $\mathcal{F}_l(\alpha)$ under $\phi^\sharp_{\mathrm{dg}\,\mathcal{N}}$ to $\tilde{g}_{0\dots k} = g^\alpha_{i_0\dots i_k}$ (by Note 2.2(5), since ϕ is injective). Now, $\phi = \delta_{j_q} \circ \dots \circ \delta_{j_0}$ for $\{i_0,\dots,i_k\} \sqcup \{j_0,\dots,j_q\} = \{0,\dots,k\}$ with $j_0 < j_1 < \dots < j_q$, so that the left vertical map $\phi^\sharp_X$ maps $\phi^\sharp_X(\alpha) = d_{j_0} \circ \dots \circ d_{j_q}(\alpha) = \alpha(i_0,\dots,i_k)$. Then, $\mathcal{F}_k$ maps this to the $0 < \dots < k$–component $g^{\alpha(i_0,\dots,i_k)}_{0\dots k}$. By the commutativity of the diagram, we get that $g^\alpha_{i_0\dots i_k} = g^{\alpha(i_0,\dots,i_k)}_{0\dots k}$. This shows that the maps $g_\alpha := g^\alpha_{0\dots l}$ for all $\alpha \in X_l$ for $l \geq 1$ together with the implicit dg-vector spaces $\mathcal{E}_\alpha = \mathcal{F}_0(\alpha)$ for all 0–simplices $\alpha \in X_0$ give the complete data of the map of simplicial sets $\mathcal{F}\colon X \to \mathrm{dg}\,\mathcal{N}(\mathcal{C})$. Equation (2-1) for $g^\alpha_{0\dots l}$ using a fixed $\alpha \in X_l$ becomes precisely (2-4) via the identifications $g_\alpha = g^\alpha_{0\dots l}$ and $g^\alpha_{i_0\dots i_k} = g^{\alpha(i_0,\dots,i_k)}_{0\dots k}$.

Note moreover, by a similar argument, that degenerate simplices map to either the identity $g_{s_j(\alpha)} = \mathrm{id}_{E_\alpha}$ for $\alpha \in X_0$, or $g_{s_j(\alpha)} = 0$ for $\alpha \in X_l$ with $l \geq 1$.

Finally, $\mathcal{F}\colon X \to \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$ lands in $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$ precisely if all maps g_α given by $\mathcal{F}(\alpha)$ for $\alpha \in X_1$ are homotopy equivalences by Lemma 2.4. □

Example 2.8 Let $\widehat{\Delta}\colon \mathbf{\Delta} \to \mathrm{sSet}$ be given as follows. Let $\widehat{\Delta}^n \in \mathrm{sSet}$ be the nerve of the category $E\mathbb{Z}_{n+1}^{\mathrm{Cat}}$, whose objects are elements of $\mathbb{Z}_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$, and which has exactly one morphism between any two objects. More explicitly, $\widehat{\Delta}^n = E\mathbb{Z}_{n+1} = \mathcal{N}(E\mathbb{Z}_{n+1}^{\mathrm{Cat}})$ has k–simplices given by a sequence of k composable morphisms $[\![i_0]\!] \to [\![i_1]\!] \to \cdots \to [\![i_k]\!]$ where $[\![i_0]\!], \dots, [\![i_k]\!] \in \mathbb{Z}_{n+1}$, or, more concisely, the k–simplices $\widehat{\Delta}^n_k$ are sequences $(i_0, \dots, i_k)$ of indices $i_0, \dots, i_k \in \{0, \dots, n\}$, ie $\widehat{\Delta}^n_k \cong \{0, \dots, n\}^k$. Face maps $d_j : \widehat{\Delta}^n_k \to \widehat{\Delta}^n_{k-1}$ remove the j^{th} index i_j, and degeneracies $s_j : \widehat{\Delta}^n_k \to \widehat{\Delta}^n_{k+1}$ repeat the j^{th} index i_j. For example, for the simplicial set $\widehat{\Delta}^1$ a k–simplex consists of a sequence $(i_0, \dots, i_k)$ of 0's and 1's; a k–simplex is degenerate if and only if any two adjacent indices are equal, $i_j = i_{j+1}$; thus there are exactly two nondegenerate k–simplices: $(0, 1, 0, 1, \dots)$ and $(1, 0, 1, 0, \dots)$ for any k. The geometric realization of $\widehat{\Delta}^1$ is thus S^∞.

By Lemma 2.7, any simplicial set map $\widehat{\Delta}^n \to \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))$ is given by $n+1$ holomorphic dg-vector bundles with holomorphic connections $\mathcal{E}_0, \dots, \mathcal{E}_n$ together with maps $g_{i_0 \dots i_k} : E_{i_k} \to E_{i_0}$ for a nondegenerate k–simplex $\alpha = (i_0, \dots, i_k) \in \widehat{\Delta}^n_k = \{0, \dots, n\}^k$ without directly repeating indices, satisfying (2-4):

$$
\begin{aligned}
\text{(2-5)}\quad g_{i_0\dots i_k} \circ d + (-1)^k \cdot d \circ g_{i_0\dots i_k} &= D(g_{i_0\dots i_k}) \\
&= \sum_{j=1}^{k-1} (-1)^{j-1} g_{i_0\dots \hat{i}_j \dots i_k} + \sum_{j=1}^{k-1} (-1)^{k(j-1)+1} g_{i_0\dots i_j} \circ g_{i_j \dots i_k}.
\end{aligned}
$$

Note furthermore that, for a *degenerate* simplex $(i_0, \dots, i_k)$ of $\widehat{\Delta}^n$ where the two consecutive indices $i_j = i_{j+1}$ are equal, we also have a map $g_{jj} = \mathrm{id}_{E_j}$ or $g_{i_0 \dots jj \dots i_k} = 0$ when $k \geq 2$ satisfying (2-5).

For a morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$ we get an induced map of simplicial sets $\phi_\bullet\colon \widehat{\Delta}^n_\bullet \to \widehat{\Delta}^m_\bullet$ by mapping $\phi_k\colon \widehat{\Delta}^n_k \to \widehat{\Delta}^m_k$, $\phi_k(i_0, \dots, i_k) = (\phi(i_0), \dots, \phi(i_k))$. This gives the cosimplicial simplicial set $\widehat{\Delta}$. In particular, we can use Definition 2.5 to get the simplicial set $\mathbf{Perf}^{\widehat{\Delta}}(U)$, whose n–simplices are precisely $\mathbf{Perf}^{\widehat{\Delta}}(U)_n = \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ\big)$, ie simplicial set maps from $\widehat{\Delta}^n$ to $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$, which were described explicitly in the previous paragraph.

We note that, for the simplicial presheaf $\mathbf{Perf}^{\widehat{\Delta}}$, the "maximal Kan" condition follows automatically.

Lemma 2.9 *Simplicial set maps from $\widehat{\Delta}^n$ to $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))$ take values in its maximal Kan subsimplex, ie*

$$
\text{(2-6)}\qquad \mathbf{Perf}^{\widehat{\Delta}}(U)_n = \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ\big) = \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))\big).
$$

Proof Any edge $g_{i_0 i_1}$ is automatically a homotopy equivalence with chain homotopy inverse $g_{i_1 i_0}$, since we have the homotopies $g_{i_0 i_1 i_0} \circ d + d \circ g_{i_0 i_1 i_0} = g_{i_0 i_0} - g_{i_0 i_1} \circ g_{i_1 i_0} = \mathrm{id}_{E_{i_0}} - g_{i_0 i_1} \circ g_{i_1 i_0}$ and $g_{i_1 i_0 i_1} \circ d + d \circ g_{i_1 i_0 i_1} = g_{i_1 i_1} - g_{i_1 i_0} \circ g_{i_0 i_1} = \mathrm{id}_{E_{i_1}} - g_{i_1 i_0} \circ g_{i_0 i_1}$. The claim follows from Lemma 2.4. □

Note that there is a map of cosimplicial simplicial sets $\Delta \to \widehat{\Delta}$, given by $\Delta^n_k \to \widehat{\Delta}^n_k$, $\Delta^n_k = \mathbf{\Delta}([k], [n]) \ni \phi \mapsto (i_0, \dots, i_k) := (\phi(0), \dots, \phi(k)) \in \widehat{\Delta}^n_k$. We thus get an induced map of simplicial sets $\mathbf{Perf}^{\widehat{\Delta}}(U) \to \mathbf{Perf}^{\Delta}(U)$.

Proposition 2.10 *For an object $U \in \mathbb{C}\mathrm{Man}$, the map of simplicial sets $\mathbf{Perf}^{\widehat{\Delta}}(U) \to \mathbf{Perf}^{\Delta}(U)$ is a weak equivalence.*

Proof Since $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^{\circ}$ is (by definition) a Kan complex, and by Definition 2.5 both $\mathbf{Perf}^{\widehat{\Delta}}(U) := \mathrm{sSet}\big(\widehat{\Delta}^{\bullet}, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^{\circ}\big)$ and $\mathbf{Perf}^{\Delta}(U) = \mathrm{sSet}\big(\Delta^{\bullet}, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^{\circ}\big)$, the proposition follows from Proposition A.1. □

In the later sections we mainly use $\mathbf{Perf}^{Q}$ for $Q = \widehat{\Delta}$, and we therefore make the following definition:

Definition 2.11 Denote by $\mathbf{IVB} := \mathbf{Perf}^{\widehat{\Delta}} : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$, ie by (2-3),

$$\mathbf{IVB}(U)_n = \mathbf{Perf}^{\widehat{\Delta}}(U)_n = \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^{\circ}\big). \tag{2-7}$$

For a motivation of this notation, see Definition 4.4.

The reason why we want to consider the cosimplicial simplicial set $\widehat{\Delta}$ is that it has an important additional cyclic structure which Δ is lacking, as we will explain now.

Definition 2.12 Let $\mathbf{\Delta} C$ be the cyclic category; see [Loday 1992, 6.1.1]. More precisely, $\mathbf{\Delta} C$ has the same objects $[n] = \{0, \dots, n\}$ for $n \in \mathbb{N}_0$ as $\mathbf{\Delta}$, and has morphisms generated by face maps δ_j and degeneracy maps σ_j (as in $\mathbf{\Delta}$; see Notation 1.1), together with an additional cyclic operator $\tau_n : [n] \to [n]$; see [Loday 1992, 6.1.1] for more details. It is convenient to represent morphisms $\phi \in \mathbf{\Delta} C([k], [n])$ by set maps $\phi \colon [k] \to [n]$ such that there exists a nondecreasing function $\tilde{\phi} \colon \{0, \dots, k\} \to \mathbb{N}_0$ satisfying $\tilde{\phi}(k) \le \tilde{\phi}(0) + n$ and $\phi(j) \equiv \tilde{\phi}(j) \pmod{\mathbb{Z}_n}$.

Then a cyclic object in a category $\mathcal{C}$ is a functor $X : \mathbf{\Delta} C^{\mathrm{op}} \to \mathcal{C}$. Since $\mathbf{\Delta} C \cong \mathbf{\Delta} C^{\mathrm{op}}$ are isomorphic [Loday 1992, 6.1.11], cyclic objects in $\mathcal{C}$ are cocyclic objects in $\mathcal{C}$ and vice versa. We denote the category of cyclic sets $X : \mathbf{\Delta} C \to \mathrm{Set}$ by cSet. Note that there is functor $\mathbf{\Delta} \to \mathbf{\Delta} C$, which makes every cyclic object into a simplicial object by precomposition $(\mathbf{\Delta} C^{\mathrm{op}} \xrightarrow{X} \mathcal{C}) \mapsto (\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta} C^{\mathrm{op}} \xrightarrow{X} \mathcal{C})$, and similarly every cocyclic object is a cosimplicial object. In particular, every cosimplicial cyclic set is a cosimplicial simplicial set.

Remark 2.13 The canonical cyclic sets $\Delta C^n := \mathbf{\Delta} C(-, [n])$ assemble for various n to a cocyclic cyclic set $\Delta C^{\bullet} \colon \mathbf{\Delta} C \to \mathrm{cSet}$. In particular, this is also a cosimplicial cyclic set $\mathbf{\Delta} \to \mathbf{\Delta} C \xrightarrow{\Delta C^{\bullet}} \mathrm{cSet}$, so that we also have a third example of a simplicial presheaf $\mathbf{Perf}^{\Delta C}$ using our setup from Definition 2.5. By Lemma 2.7, an n–simplex in $\mathbf{Perf}^{\Delta C}(U)$ is given by maps $g_{i_0 \dots i_k}$ for any "cyclic set of indices" $i_0 = \phi(0), \dots, i_k = \phi(k)$ for some $\phi \in \mathbf{\Delta} C([k], [n])$ (for example, for $n = 9$ we would have maps such as $g_{457034} \colon E_4 \to E_4$). Unfortunately, the analog of Proposition 2.10 does not hold, ie $\mathbf{Perf}^{\Delta C}(U)$ and $\mathbf{Perf}^{\Delta}(U)$ are in general not weakly equivalent. (For example, the nondegenerate simplices of $\mathbf{\Delta} C^1$ as sequences of indices are (0), (1), $(0, 1)$, $(1, 0)$, $(0, 1, 0)$, $(1, 0, 1)$ but no higher ones due to cyclicity, so that the geometric realization of $\mathbf{\Delta} C^1$ is the 2–sphere S^2.)

Now, while Δ^n is not a cyclic set, $\widehat{\Delta}^n$ is a cyclic set, and we will need to use the additional cyclic structure of $\widehat{\Delta}$ below to define our Chern character map.

Lemma 2.14 *The simplicial set $\widehat{\Delta}^n$ as described in the first paragraph of Example 2.8, together with the operator $t_k : \widehat{\Delta}^n_k \to \widehat{\Delta}^n_k$ given by $t_k(i_0, \dots, i_{k-1}, i_k) = (i_k, i_0, \dots, i_{k-1})$, makes $\widehat{\Delta}^n$ into a cyclic set. This, in turn, makes $\widehat{\Delta}$ into a cosimplicial cyclic set.*

Proof One checks that t_k has the correct compatibility (see [Loday 1992, 6.1.2(b)–(c)]) with the face and degeneracy maps d_j and s_j. For a morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$, the induced map of simplicial sets $\phi_\bullet\colon \widehat{\Delta}^n_\bullet \to \widehat{\Delta}^m_\bullet$, $\phi_k : \widehat{\Delta}^n_k \to \widehat{\Delta}^m_k$, $\phi_k(i_0, \dots, i_k) = (\phi(i_0), \dots, \phi(i_k))$, respects not only the face and degeneracy maps, but also the t_k operator, ie $\widehat{\Delta}\colon \mathbf{\Delta} \to \mathrm{cSet}$ is a cosimplicial cyclic set. □

We have thus defined the simplicial presheaf $\mathbf{IVB} = \mathbf{Perf}^{\widehat{\Delta}}$, which will be the domain of our Chern character map for holomorphic dg-vector bundles over U with connection. As for the range of the Chern character map, we use the same presheaf $\mathbf{\Omega}$ that we used in our previous work [2022, Definition 2.3] (for the Chern character map of holomorphic vector bundles that were not differential graded). For completeness sake, we will briefly review the definition of $\mathbf{\Omega} : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$.

Definition 2.15 For an object $U \in \mathbb{C}\mathrm{Man}$, consider the (nonnegatively graded) cochain complex of holomorphic forms $\Omega^\bullet_{\mathrm{hol}}(U)$ on U with zero differential $d = 0$. Let u be a formal variable of degree $|u| = -2$, denote by $\Omega^\bullet_{\mathrm{hol}}(U)[u]$ polynomials in u, and by $\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\le 0}$ its quotient by its positive degree part $\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet>0}$. Applying the Dold–Kan functor to this chain complex gives a simplicial abelian group $\mathrm{DK}(\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\le 0})$, for which we consider its underlying simplicial set, denoted by an underline, ie $\mathbf{\Omega}(U) = \underline{\mathrm{DK}}(\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\le 0})$:

$$\mathbf{\Omega} : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \xrightarrow{\Omega^\bullet_{\mathrm{hol}}(-)[u]^{\bullet\le 0}} \mathrm{Ch}^{\le 0} \xrightarrow{\underline{\mathrm{DK}}} \mathrm{sSet}.$$

Since holomorphic forms pull back via a holomorphic map $\varphi : U \to U'$, this assignment defines a simplicial presheaf $\mathbf{\Omega} : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by $\mathbf{\Omega} := \underline{\mathrm{DK}}(\Omega^\bullet_{\mathrm{hol}}(\,\cdot\,)[u]^{\bullet\le 0}) : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$.

Note 2.16 If $C = (C^{\bullet\le 0}, d_C)$ is a nonpositively graded chain complex, then the Dold–Kan functor $\mathrm{DK}(C) \in \mathrm{Ab}^{\mathbf{\Delta}^{\mathrm{op}}}$, which is a simplicial abelian group, can be described as follows; see our previous work [2022, Appendix B]. For $n \ge 0$, we may define $\mathrm{DK}(C)_n$ to be the abelian group (under addition) of cochain maps from the normalized cells of the standard simplex Δ^n to C, ie we may set

$$\mathrm{DK}(C)_n := \mathrm{Chain}(N(\mathbb{Z}\Delta^n), C). \tag{2-8}$$

Thus, this means that an element of $\mathrm{DK}(C)_n$ is given by a labeling of the nondegenerate cells of the standard simplex Δ^n by elements of C in such a way that, for a k–cell α of Δ^n whose boundary $(k-1)$–cells are $d_j(\alpha)$, we have

$$d_C(\alpha) = \sum_{j=0}^{k} (-1)^j \cdot d_j(\alpha). \tag{2-9}$$

In the situation of Definition 2.15, the chain complex $C = \Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\le 0}$ has a zero internal differential, ie $d_C = 0$.

3 Chern character Ch: IVB → Ω

We now define a map of simplicial presheaves $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$, where $\mathbf{IVB} = \mathbf{Perf}^{\widehat{\Delta}}$ from Definition 2.11 and $\mathbf{\Omega}$ is from Definition 2.15. We start by defining cochains on a simplicial set X with values in a dg-category $\mathcal{C}$ (for us $\mathcal{C} = \mathrm{Perf}(U)$), and, in the case when X is a cyclic set, its trace map. The main example to keep in mind for the following definitions is the cyclic set $X = \widehat{\Delta}^n$.

Definition 3.1 A *labeling* of a simplicial set X by a dg-category $\mathcal{C}$ is a set map from the vertices of X to the objects of $\mathcal{C}$, $L\colon X_0 \to \mathrm{Obj}(\mathcal{C})$, ie a choice of an object $\mathcal{E}_\alpha := L(\alpha)$ of $\mathcal{C}$ for each $\alpha \in X_0$.

Definition 3.2 Let X be a simplicial set, let $\mathcal{C}$ be dg-category, and let $L\colon X_0 \to \mathrm{Obj}(\mathcal{C})$ be a labeling such that we have a choice of objects $\mathcal{E}_\alpha$ for each $\alpha \in X_0$. We define the *cochains on X with values in $\mathcal{C}$* to be

$$C_L^\bullet(X,\mathcal{C}) := \prod_{p\geq 1}\prod_{\alpha\in X_p} \mathrm{Hom}_{\mathcal{C}}^\bullet(\mathcal{E}_{\alpha(p)},\mathcal{E}_{\alpha(0)}), \tag{3-1}$$

where we used notation from Lemma 2.7 to denote the first and last vertices of $\alpha \in X_p$ by $\alpha(0)$ and $\alpha(p)$, respectively. In components, we will write $f \in C_L^\bullet(X,\mathcal{C})$ as $f = \{f_\alpha\}_{\alpha\in X}$, where, for $\alpha \in X_p$ and $p \geq 1$, we have $f_\alpha \in \mathrm{Hom}_{\mathcal{C}}^\bullet(\mathcal{E}_{\alpha(p)},\mathcal{E}_{\alpha(0)})$.

Note that $C_L^\bullet(X,\mathcal{C})$ is a dg-algebra:

(1) A cochain f of bidegree (p,q) assigns to a p–cell $\alpha \in X_p$ a degree q map $f_\alpha \in \mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p)},\mathcal{E}_{\alpha(0)})$, and is zero elsewhere; in this case the total degree of f is $|f| = p+q$.

(2) A differential $\hat{\delta}\colon C_L^p(X,\mathcal{C}) \to C_L^{p+1}(X,\mathcal{C})$ is induced by the face maps $d_i : X^{p+1} \to X^p$, so that if $\alpha \in X_{p+1}$ is a $(p+1)$-simplex of X, then the deleted Čech differential of f, denoted by $\hat{\delta} f$, is defined by

$$(\hat{\delta} f)_\alpha := \sum_{i=1}^{p} (-1)^i f_{d_i(\alpha)} = \sum_{i=1}^{p} (-1)^i f_{\alpha(0,\dots,\hat{\imath},\dots,p+1)}. \tag{3-2}$$

Note that d_0 and d_{p+1} are not used in the differential, which ensures the terms in the sum are all maps in $\mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p+1)},\mathcal{E}_{\alpha(0)})$.

(3) An internal differential $D\colon C_L^\bullet(X,\mathcal{C}) \to C_L^\bullet(X,\mathcal{C})$ is induced by the dg structure on $\mathcal{C}$, so that, if $\alpha \in X_p$ is a p–simplex and $f_\alpha \in \mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p)},\mathcal{E}_{\alpha(0)})$, then $(Df)_\alpha := (-1)^{p+q+1}\cdot D(f_\alpha) = (-1)^p\cdot(d\circ f_\alpha - (-1)^q \cdot f_\alpha \circ d)$ as a homomorphism in $\mathrm{Hom}^{q+1}(\mathcal{E}_{\alpha(p)},\mathcal{E}_{\alpha(0)})$.

(4) There is a product $f\cdot g$ on $C_L^\bullet(X,\mathcal{C})$, which, for $\alpha \in X_{p+r}$ is the extension of the maps $\mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p)},\mathcal{E}_{\alpha(0)}) \times \mathrm{Hom}_{\mathcal{C}}^s(\mathcal{E}_{\alpha(p+r)},\mathcal{E}_{\alpha(p)}) \to \mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p+r)},\mathcal{E}_{\alpha(0)})$,

$$(f_{\alpha(0,\dots,p)}, g_{\alpha(p,\dots,p+r)}) \mapsto (f\cdot g)_{\alpha(0,\dots,p+r)} := (-1)^{q\cdot r}\cdot f_{\alpha(0,\dots,p)} \circ g_{\alpha(p,\dots,p+r)}, \tag{3-3}$$

on the components of $C_L^\bullet(X,\mathcal{C})$ to all of $C_L^\bullet(X,\mathcal{C})$.

We note that, in particular, $Df = d\cdot f - (-1)^{|f|} f\cdot d =: [d, f]$. It is well known (and straightforward to check) that with these definitions the cochains on X with values in $\mathcal{C}$, $C_L^\bullet(X,\mathcal{C})$, becomes a dg-algebra.

Definition 3.3 Given a simplicial set X, a dg-category $\mathcal{C}$ and a labeling L, we say an element $g \in C_L^\bullet(X, \mathcal{C})$ is a *Maurer–Cartan element* if

$$\hat{\delta} g + Dg + g \cdot g = 0 \tag{3-4}$$

Definition 3.4 Let $X_\bullet$ be a simplicial set, and let $\mathcal{C}$ be a dg-category. Then, by Lemma 2.7, a simplicial set map $\mathcal{F}\colon X \to \mathrm{dg}\,\mathcal{N}(\mathcal{C})$ induces objects $\mathcal{E}_\alpha$ for each 0–simplex $\alpha \in X_0$, and maps $g_\alpha \colon \mathcal{E}_{\alpha(p)} \to \mathcal{E}_{\alpha(0)}$ for every $\alpha \in X_p$ with $p \geq 1$ (for degenerate simplices, we take $g_\alpha = \mathrm{id}_{\mathcal{E}_{\alpha(0)}}$ when $\alpha \in X_1$, and $g_\alpha = 0$ when $\alpha \in X_p$ for $p \geq 2$). Thus, we can define a labeling $L := \mathcal{F}_0 \colon X_0 \to \mathrm{dg}\,\mathcal{N}(\mathcal{C})_0 = \mathrm{Obj}(\mathcal{C})$ of X by $\mathcal{C}$ via $L(\alpha) := \mathcal{E}_\alpha$ for $\alpha \in X_0$. Moreover, the g_α for $\alpha \in X_p$ for $p \geq 1$, assemble to an element $g = \{g_\alpha\}_{\alpha \in X} \in C_L^\bullet(X, \mathcal{C})$.

Corollary 3.5 *The element $g \in C_L^\bullet(X, \mathcal{C})$ from Definition 3.4 is a Maurer–Cartan element, ie g satisfies (3-4). Moreover, g has components of bidegree $(p, 1-p)$ for $p \geq 1$, so that g is of total degree 1.*

Proof Each g_α for $\alpha \in X_p$ is of bidegree $(p, 1-p)$; see Lemma 2.7(2). For $\alpha \in X_{p+r}$ with $p, r \geq 1$, we have $g_{\alpha(0,\dots,p)} \cdot g_{\alpha(p,\dots,p+r)} = (-1)^{(1-p)(r-p)} g_{\alpha(0,\dots,p)} \circ g_{\alpha(p,\dots,p+r)}$, and since $(-1)^{(1-p)(r-p)} = (-1)^{(r+p)(p-1)}$, we see that (3-4) becomes exactly (2-4). □

Now consider the case $\mathcal{C} = \mathrm{Perf}(U)$. In this case, $C_L^\bullet(X, \mathcal{C})$ becomes a direct product of holomorphic sections, ie

$$C_L^\bullet(X, \mathrm{Perf}(U)) = \prod_{p \geq 1} \prod_{\alpha \in X_p} \Gamma_{\mathrm{hol}}(U, \mathrm{Hom}(E_{\alpha(p)}, E_{\alpha(0)})),$$

since morphisms $\mathrm{Hom}_{\mathcal{C}}(\mathcal{E}_1, \mathcal{E}_2)$, which are bundle maps, are in correspondence with holomorphic sections of the $\mathrm{Hom}(E_1, E_2)$–bundle. Since we want to include higher holomorphic forms as well, we will include this dg-algebra in a larger dg-algebra of all holomorphic forms $C_L^\bullet(X, \mathrm{Perf}(U)) \hookrightarrow C_L^\bullet(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, defined as follows.

Definition 3.6 Let X be a simplicial set and consider the dg-category $\mathrm{Perf}(U)$. Let $L \colon X_0 \to \mathrm{Obj}(\mathrm{Perf}(U))$ be a labeling as in Definition 3.2, ie $\mathcal{E}_\alpha = L(\alpha)$. We define the dg-algebra

$$C_L^\bullet(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U)) := \prod_{p \geq 0} \prod_{\alpha \in X_p} \Omega_{\mathrm{hol}}^\bullet(U, \mathrm{Hom}^\bullet(E_{\alpha(p)}, E_{\alpha(0)})), \tag{3-5}$$

where we again denoted the first and last vertices of $\alpha \in X_p$ by $\alpha(0)$ and $\alpha(p)$, respectively. In components, we will write $f \in C_L^\bullet(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$ as $f = \{f_\alpha\}_{\alpha \in X}$, where, for $\alpha \in X_p$, we have $f_\alpha \in \Omega_{\mathrm{hol}}^\bullet(U, \mathrm{Hom}^\bullet(E_{\alpha(p)}, E_{\alpha(0)}))$. Note that in (3-5) we included the 0–simplices ($p = 0$) when compared to (3-1).

The dg-algebra structure on $C_L^\bullet(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$ is defined as follows:

(1) $f \in C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$ has triple degree (k, p, q) if it assigns to a p–cell $\alpha \in X_p$ a holomorphic k–form with values in the appropriate Hom-bundle of degree q, $f_\alpha \in \Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^q(E_{\alpha(p)}, E_{\alpha(0)}))$, and vanishes elsewhere; in this case the total degree of f is $|f| = k + p + q$.

(2) A differential $\hat{\delta}\colon C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U)) \to C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$, the deleted Čech differential, is defined just as in Definition 3.2(2), ie for $f \in C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$,

$$(\hat{\delta} f)_\alpha := \sum_{i=1}^{p} (-1)^i f_{d_i(\alpha)} = \sum_{i=1}^{p} (-1)^i f_{\alpha(0,\dots,\hat{\imath},\dots,p+1)}. \tag{3-6}$$

(3) A differential $D\colon C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U)) \to C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$, the internal differential, is defined similarly to Definition 3.2(3), ie if $f_\alpha \in \Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^q(E_{\alpha(p)}, E_{\alpha(0)}))$, then $(Df)_\alpha \in \Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^{q+1}(E_{\alpha(p)}, E_{\alpha(0)}))$,

$$(Df)_\alpha := (-1)^p \cdot (d_{\alpha(0)} \circ f_\alpha - (-1)^{k+q} \cdot f_\alpha \circ d_{\alpha(p)}),$$

where d_i denotes the differential of E_i.

(4) There is a product $f \cdot g$ similar to Definition 3.2(4). More explicitly, consider the maps

$$\begin{aligned} \Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^q(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)})) \times \Omega^l_{\mathrm{hol}}(U, \mathrm{Hom}^s(\mathcal{E}_{\alpha(p+r)}, \mathcal{E}_{\alpha(p)})) \\ \to \Omega^{k+l}_{\mathrm{hol}}(U, \mathrm{Hom}^q(\mathcal{E}_{\alpha(p+r)}, \mathcal{E}_{\alpha(0)})), \\ (f_{\alpha(0,\dots,p)}, g_{\alpha(p,\dots,p+r)}) \mapsto (f \cdot g)_{\alpha(0,\dots,p+r)} := (-1)^{(k+q)\cdot r} \cdot f_{\alpha(0,\dots,p)} \circ g_{\alpha(p,\dots,p+r)}, \end{aligned} \tag{3-7}$$

where $\circ$ denotes wedging forms and composing Hom-spaces, and extend them from the components of $C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$ to the whole space.

We note that, again, $Df = d \cdot f - (-1)^{|f|} f \cdot d = [d, f]$. Just as in Definition 3.2, $C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$ becomes a dg-algebra, and the inclusion $C_L^\bullet(X, \mathrm{Perf}(U)) \hookrightarrow C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$ is a dg-algebra morphism. Note that this inclusion consists of two separate inclusions of holomorphic functions into holomorphic forms, $\Gamma_{\mathrm{hol}}(-) \hookrightarrow \Omega_{\mathrm{hol}}(-)$, as well as nonzero simplices into all simplices, $\prod_{p \geq 1}(-) \hookrightarrow \prod_{p \geq 0}(-)$. Note further, that $Df = d \cdot f - (-1)^{|f|} f \cdot d$, where $d = \{d_\alpha\}_{\alpha \in X} \in C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$ is given by the differentials $d_\alpha = d_{E_\alpha}$ for $\alpha \in X_0$ and $d_\alpha = 0$ for all other α.

Finally we remark that every Maurer–Cartan element in $C_L^\bullet(X, \mathrm{Perf}(U))$ is also a Maurer–Cartan element in the larger dg-algebra $C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$.

Now, for a vector bundle E, there is a trace map $\mathrm{tr}\colon \mathrm{Hom}(E, E) \to \mathbb{C}$. Following ideas of O'Brian, Toledo and Tong [1981c, page 238], we will define a trace map

$$\prod_{p \geq 0} \prod_{\alpha \in X_p} \Omega^\bullet_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E_{\alpha(p)}, E_{\alpha(0)})) \to \prod_{p \geq 0} \prod_{\alpha \in X_p} \Omega^\bullet_{\mathrm{hol}}(U, \mathbb{C}).$$

Note that the left-hand side is $C_L^\bullet(X, \Omega(U) \hat{\otimes} \mathrm{Perf}(U))$. We denote the right-hand side by $C^\bullet(X, \Omega_{\mathrm{hol}}(U))$. To fit this into our current setting, we need an additional *cyclic* structure on X.

Definition 3.7 Let X be a cyclic set. Let $\alpha \in X_p$ be a p–simplex, ie by our convention $\alpha = \alpha(0, \dots, p)$, then, using the additional operator $\tau_p \colon [p] \to [p]$, we denote the induced map $t_p \colon X_p \to X_p$ by $\alpha(p, 0, \dots, p-1) := t_p(\alpha)$.

Now let $L \colon X_0 \to \mathrm{Obj}(\mathrm{Perf}(U))$ be a labeling, and let g be a Maurer–Cartan element of $C_L^\bullet(X, \mathrm{Perf}(U))$. Then we define the *trace* map

$$\mathrm{Tr}_g \colon C_L^\bullet(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U)) \to C^\bullet(X, \Omega_{\mathrm{hol}}(U)),$$
$$(\mathrm{Tr}_g(f))_{\alpha \in X_s} := \sum_{0 \le k \le l \le s} (-1)^{(k+1)\cdot s + l - k} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,k)} \circ f_{\alpha(k,\dots,l)}).$$

Note that the trace on the right makes sense, since it is applied to $\mathrm{Hom}(E_{\alpha(l)}, E_{\alpha(l)})$.

The following proposition follows the arguments from [loc. cit., Proposition 3.2]:

Proposition 3.8 *Let X be a cyclic set with labeling L, and let g be a Maurer–Cartan element in $C_L^\bullet(X, \mathrm{Perf}(U))$. Then the trace map Tr_g satisfies*

$$\mathrm{Tr}_g \circ (\hat{\delta} + D + [g, -]) = \delta \circ \mathrm{Tr}_g, \tag{3-8}$$

where δ is the (full) Čech differential including first and last term, ie $(\delta f)_\alpha := \sum_{j=0}^{p+1} (-1)^j f_{\alpha(0,\dots,\hat{j},\dots,p+1)}$ for $\alpha \in X_{p+1}$.

Proof Let $f \in C_L^\bullet(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, and let $\alpha \in X_s$. Then

$$\big(\delta(\mathrm{Tr}_g(f))\big)_\alpha = \sum_{j=0}^{s} (-1)^j \cdot \mathrm{Tr}_g(f)_{\alpha(0,\dots,\hat{j},\dots,s)} = A + B + C$$

equals the sum of the three terms

$$A := \sum_{0 \le k \le l \le s} \sum_{j=k+1}^{l-1} (-1)^{j + (k+1)(s-1) + l - k - 1} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,k)} \circ f_{\alpha(k,\dots,\hat{j},\dots,l)}),$$

$$B := \sum_{0 \le k \le l \le s} \sum_{j=0}^{k-1} (-1)^{j + k(s-1) + l - k} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,\hat{j},\dots,k)} \circ f_{\alpha(k,\dots,l)}),$$

$$C := \sum_{0 \le k \le l \le s} \sum_{j=l+1}^{s} (-1)^{j + (k+1)(s-1) + l - k} \cdot \mathrm{tr}(g_{\alpha(l,\dots,\hat{j},\dots,s,0,\dots,k)} \circ f_{\alpha(k,\dots,l)}).$$

The first term A in the above sum is equal to

$$\begin{aligned} A &= \sum_{0 \le k \le l \le s} \sum_{j=k+1}^{l-1} (-1)^{j + (k+1)s + l} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,k)} \circ f_{\alpha(k,\dots,\hat{j},\dots,l)}) \\ &= \sum_{0 \le k \le l \le s} (-1)^{(k+1)s + l - k} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,k)} \circ (\hat{\delta} f)_{\alpha(k,\dots,l)}) = \big(\mathrm{Tr}_g(\hat{\delta}(f))\big)_\alpha. \end{aligned}$$

To evaluate $B+C$, note that

$$\text{(3-9)}\quad \sum_{0\le k\le l\le s}(-1)^{(k+1)s+1}\cdot \mathrm{tr}((\hat\delta(g))_{\alpha(l,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$= \sum_{0\le k\le l\le s}\sum_{j=l+1}^{s}(-1)^{(k+1)s+1+j-l}\cdot \mathrm{tr}(g_{\alpha(l,\dots,\hat\jmath,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$+\sum_{0\le k\le l\le s}\sum_{j=0}^{k-1}(-1)^{(k+1)s+1+s-l+1+j}\cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,\hat\jmath,\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$= C+B.$$

We claim that this is equal to $\big(\mathrm{Tr}_g(D(f)+[g,-](f))\big)_\alpha$, which we evaluate now. By Definition 3.6, we may write $D(f)=d\cdot f-(-1)^{|f|}f\cdot d=[d,f]$, where $|f|$ denotes the total degree of f. Thus, if we define $\tilde g:=d+g$, ie for $\alpha\in X_0$, $\tilde g_\alpha=d_\alpha$, and for $\alpha\in X_k$ with $k\ge 1$, $\tilde g_\alpha=g_\alpha$, then $D(f)+[g,-](f)=[d+g,f]=[\tilde g,f]$. With this, we write $(\mathrm{Tr}_g([\tilde g,f]))_\alpha=\big(\mathrm{Tr}_g(\tilde g\cdot f-(-1)^{|\tilde g|\cdot|f|}f\cdot\tilde g)\big)_\alpha=E+F$, which are given as follows. First,

$$E:=\mathrm{Tr}_g(\tilde g\cdot f)_\alpha=\sum_{0\le j\le l\le s}(-1)^{(j+1)s+l-j}\cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,j)}\circ(\tilde g\cdot f)_{\alpha(j,\dots,l)})$$
$$=\sum_{0\le j\le k\le l\le s}(-1)^{(j+1)s+l-j+(1-k+j)(l-j)}\cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,j)}\circ\tilde g_{\alpha(j,\dots,k)}\circ f_{\alpha(k,\dots,l)}),$$

where we used that the (de Rham, Čech, Hom)–triple degree of $\tilde g_{\alpha(j,\dots,k)}$ is $(0,k-j,1-k+j)$. For the second term, we get

$$F:=\mathrm{Tr}_g(-(-1)^{|\tilde g|\cdot|f|}f\cdot\tilde g)_\alpha$$
$$=\sum_{0\le k\le j\le s}(-1)^{|f|+1+(k+1)s+j-k}\cdot \mathrm{tr}(g_{\alpha(j,\dots,s,0,\dots,k)}\circ(f\cdot\tilde g)_{\alpha(k,\dots,j)})$$
$$=\sum_{0\le k\le l\le j\le s}(-1)^{|f|+1+(k+1)s+j-k+(|f|-l+k)(j-l)}\cdot \mathrm{tr}(g_{\alpha(j,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)}\circ\tilde g_{\alpha(l,\dots,j)})$$
$$=\sum_{0\le k\le l\le j\le s}(-1)^{|f|+1+(k+1)s+j-k+(|f|-l+k)(j-l)+(|f|+1-1-s+j-l)(1-j+l)}\cdot \mathrm{tr}(\tilde g_{\alpha(l,\dots,j)}\circ g_{\alpha(j,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)}),$$

where we used that $\mathrm{tr}(h\circ k)=(-1)^{a\cdot b}\cdot\mathrm{tr}(k\circ h)$ when the (Hom-degree) + (de Rham degree) = (total degree) − (Čech degree) of h and k is a and b, respectively, and that the Čech-degree of any $h_{\alpha(j,\dots,s,0,\dots,l)}$ is $1+s-j+l$. With this, we obtain

$$\text{(3-10)}\quad \sum_{0\le k\le l\le s}(-1)^{(k+1)s}\cdot \mathrm{tr}((\tilde g\cdot\tilde g)_{\alpha(l,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$=\sum_{0\le k\le l\le j\le s}(-1)^{(k+1)s+(1-j+l)(1+s-j+k)}\cdot \mathrm{tr}(\tilde g_{\alpha(l,\dots,j)}\circ g_{\alpha(j,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$+\sum_{0\le j\le k\le l\le s}(-1)^{(k+1)s+(l-s-j)(k-j)}\cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,j)}\circ\tilde g_{\alpha(j,\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$=F+E,$$

where we used that $\tilde{g} = d + g$ and $d \cdot d = 0$, and that the (de Rham, Čech, Hom)–triple degree of $g_{\alpha(l,\dots,s,0,\dots,j)}$ is $(0, 1+s-l+j, l-s-j)$. Comparing the left-hand sides of (3-9) and (3-10), and using that g is a Maurer–Cartan element, so that $\hat{\delta} g = -(Dg + g \cdot g) = -\tilde{g} \cdot \tilde{g}$, we obtain that

$$B + C = (3\text{-}9) = (3\text{-}10) = E + F = (\mathrm{Tr}_g([\tilde{g}, f]))_\alpha = \big(\mathrm{Tr}_g(D(f) + [g, -](f))\big)_\alpha. \qquad \square$$

Remark 3.9 The trace map of O'Brian, Toledo and Tong [1981c, Section 3] satisfies some additional properties which carry over to our trace map from Definition 3.7 . For example, following the algebraic proof from [loc. cit., Proposition 3.8], Tr_g vanishes on graded commutators: for a Maurer–Cartan element g and cocycles $u, v \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, $\mathrm{Tr}_g(u \cdot v)$ and $\mathrm{Tr}_g(v \cdot u)$ are cohomologous (up to sign) in $C^\bullet(X, \Omega_{\mathrm{hol}}(U))$.

We have one further structure on $C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$ coming from the holomorphic connections ∇ of the objects $\mathcal{E}$ of $\mathrm{Perf}(U)$. Note that there is an induced connection on the Hom-bundle $\mathrm{Hom}^\bullet(E, E')$ of two graded bundles E and E' with connections, which we also denote by $\nabla\colon \Omega^\bullet_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E, E')) \to \Omega^{\bullet+1}_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E, E'))$, and which is a graded derivation with respect to the wedge composition $\circ$ using the total degree of $\Omega^\bullet_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E, E'))$.

Definition 3.10 Define $\nabla\colon C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U)) \to C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$ to be given in components by the maps $(-1)^p \cdot \nabla\colon \Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^q(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)})) \to \Omega^{k+1}_{\mathrm{hol}}(U, \mathrm{Hom}^q(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)}))$. More explicitly, for $f \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, $f = \{f_\alpha\}_{\alpha \in X}$, we define $\nabla f = \{(\nabla f)_\alpha\}_{\alpha \in X}$ to be given by $(\nabla f)_\alpha := (-1)^p \cdot \nabla(f_\alpha)$ when $\alpha \in X_p$.

One can check that $\nabla \circ \hat{\delta} = -\hat{\delta} \circ \nabla$, and that $\nabla(f \cdot g) = \nabla(f) \cdot g + (-1)^{|f|} f \cdot \nabla(g)$, where $|f|$ is the total degree of the triple grading.

Definition 3.11 Let X be a cyclic set and let $\mathcal{F}\colon X \to \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))$ be a simplicial set map. By Definition 3.4, we get a labeling $L\colon X_0 \to \mathrm{Obj}(\mathrm{Perf}(U))$, and a Maurer–Cartan element

$$g \in C^\bullet_L(X, \mathrm{Perf}(U)) \hookrightarrow C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U)).$$

For a vertex $\alpha \in X_0$, denote by d_{E_α} the internal differential of the chain complex of vector bundles $\mathcal{E}_\alpha$, out of which we build the element $d = \{d_\alpha\}_{\alpha \in X} \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, given by $d_\alpha := d_{E_\alpha}$, and which has triple degree $(0, 0, 1)$. Then $d + g \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, and we call

$$A := \nabla(d + g) \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$$

the *Atiyah class*, which is concentrated in degrees $(1, k, 1-k)$ for $k \geq 0$.

Proposition 3.12 *We have* $(\hat{\delta} + D + [g, -])(A) = 0$, *and thus*

$$\delta(\mathrm{Tr}_g(A^k)) = 0 \quad \textit{for all } k \geq 0.$$

Proof We apply ∇ to the Maurer–Cartan equation (3-4), ie to $\hat{\delta}g + Dg + g\cdot g = 0$. Using $\nabla\hat{\delta}g = -\hat{\delta}\nabla g$, and $\nabla(g\cdot g) = \nabla g\cdot g - g\cdot\nabla g = -[g,\nabla g]$ together with

$$\nabla Dg = \nabla(d\cdot g + g\cdot d) = \nabla d\cdot g - d\cdot\nabla g + \nabla g\cdot d - g\cdot\nabla d = -D(\nabla g) - [g,\nabla d],$$

we obtain

$$0 = \nabla(\hat{\delta}g + Dg + g\cdot g) = -\hat{\delta}(\nabla g) - D(\nabla g) - [g,\nabla d] - [g,\nabla g] = -(\hat{\delta} + D + [g,-])(\nabla g + \nabla d).$$

In the last equality, we also used that $\hat{\delta}(\nabla d) = 0$ (since the deleted Čech differential vanishes on 0–simplices), and from $d^2 = 0$ it follows that $0 = \nabla(d\cdot d) = \nabla d\cdot d - d\cdot\nabla d = -D(\nabla d)$. This shows that, for $A = \nabla(d+g)$, we have $(\hat{\delta} + D + [g,-])(A) = 0$.

Since $(\hat{\delta} + D + [g,-])$ is a derivation on $C^\bullet_L(X, \Omega(U)\mathbin{\hat{\otimes}}\mathrm{Perf}(U))$, the k^{th} powers of a also satisfy $(\hat{\delta} + D + [g,-])(A^k) = 0$. Thus,

$$\delta(\mathrm{Tr}_g(A^k)) \overset{\text{(3-8)}}{=} \mathrm{Tr}_g((\hat{\delta} + D + [g,-])(A^k)) = 0. \qquad \square$$

We are now ready to define our Chern character map $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$, which is a map of simplicial presheaves, as shown in Theorem 3.14 below.

Definition 3.13 We define the *Chern character* as a map $\mathbf{Ch}\colon\mathbf{IVB}\to\mathbf{\Omega}$; that is, for a complex manifold U and $k \geq 0$, we define a map $\mathbf{Ch}(U)_n\colon \mathbf{IVB}(U)_n \to \mathbf{\Omega}(U)_n$.

For an n–simplex $\mathcal{F}\in\mathbf{IVB}(U)_n = \mathrm{sSet}\big(\hat{\Delta}^n, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))\big)$, we have (by Definition 2.5 and Example 2.8) the data of $n+1$ dg-vector bundles $\mathcal{E}_0,\dots,\mathcal{E}_n$, and maps $g_{i_0\dots i_k}\colon E_{i_k}\to E_{i_0}$, so $g = \{g_{(i_0\dots i_k)}\}_{(i_0,\dots,i_k)\in\hat{\Delta}^n}$ satisfies the Maurer–Cartan equation by Corollary 3.5. To this we associate $\mathbf{Ch}(U)_n(\mathcal{F})\in\mathbf{\Omega}(U)_n$, which is a labeling of the nondegenerate cells of Δ^n by elements in $\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\leq 0}$ (by Definition 2.15 and Note 2.16). Consider a nondegenerate k–cell of Δ^n given by the vertices $i_0,\dots,i_k$ of Δ^n with $i_0 < \dots < i_k$.

If $k = 0$, then we assign the Euler characteristic $\chi(E_{i_0})$ to this cell. If $k > 0$, then we use $\alpha = (i_0,\dots,i_k)\in\hat{\Delta}^n_k$ to assign the following expression to this cell:

$$\text{(3-11)}\quad \mathrm{Tr}_g(A^k)_\alpha\cdot\frac{u^k}{k!} = \mathrm{Tr}_g\big((\nabla(d+g))^k\big)_\alpha\cdot\frac{u^k}{k!} = \sum \pm\mathrm{tr}(g\cdot\nabla(d+g)\cdot\nabla(d+g)\cdots\nabla(d+g))_\alpha\cdot\frac{u^k}{k!}.$$

For example, here are the assignments for simplicial degrees 0, 1 and 2:

***n* = 0** A 0–simplex $\mathcal{F}\in\mathbf{IVB}(U)_0$ is just the data of one object $\mathcal{E} = (E\to U,\nabla)$ of $\mathrm{Perf}(U)$. Then $\mathbf{Ch}(U)_0(\mathcal{F})$ is the labeling of the Δ^0 by Euler characteristic of $\mathcal{E}$, denoted by $\chi(E)\in\Omega^0_{\mathrm{hol}}(U)[u]^{\bullet\leq 0}$.

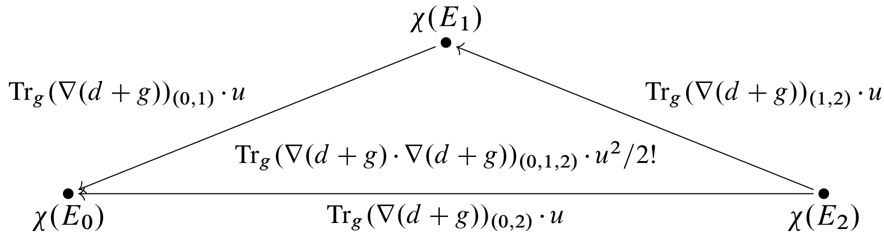
***n* = 1** A 1–simplex $\mathcal{F}\in\mathbf{IVB}(U)_1$ consists of bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ and sequences of morphisms $g_{0101\dots}$ and $g_{1010\dots}$. Then $\mathbf{Ch}(U)_1(\mathcal{F})$ is the labeling of Δ^1 given by $\chi(\mathcal{E}_i)$ on the vertices of Δ^1, and on the edge of Δ^1 we place the labeling $\mathrm{Tr}_g(\nabla(d+g))_{(0,1)}\cdot u\in\Omega^1_{\mathrm{hol}}(U)[u]^{\bullet\leq 0}$, where $(0,1)\in\hat{\Delta}^1$:

$$\underset{\bullet}{\chi(E_0)} \xleftarrow{\quad \mathrm{Tr}_g(\nabla(d+g))_{(0,1)}\cdot u \quad} \underset{\bullet}{\chi(E_1)}$$

Explicitly, the trace has terms (using $g_i = d_{E_i}$ for the internal differential of E_i)

$$\mathrm{Tr}_g(\nabla(d+g))_{(0,1)} = \mathrm{tr}(g_{101}\nabla g_1 - g_{010}\nabla g_0 + g_{10}\nabla g_{01})$$

***n* = 2** A 2–simplex $\mathcal{F} \in \mathbf{IVB}(U)_2$ consists of bundles $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ and sequences of morphisms $g_{i_0 i_1 \dots i_p}$ for $p \geq 1$ and $i_l \in \{0,1,2\}$ for any $0 \leq l \leq p$. Then, $\mathbf{Ch}(U)_2(\mathcal{F})$ is the labeling of Δ^2 given by $\chi(\mathcal{E}_i) \in \Omega^0_{\mathrm{hol}}(U)[u]^{\bullet \leq 0}$ on the vertices, $\mathrm{Tr}_g(\nabla(d+g))_{(i,j)} \cdot u \in \Omega^1_{\mathrm{hol}}(U)[u]^{\bullet \leq 0}$ on the edge of Δ^1 we place the labeling $\mathrm{Tr}_g(\nabla(d+g) \cdot \nabla(d+g))_{(0,1,2)} \cdot u^2/2! \in \Omega^2_{\mathrm{hol}}(U)[u]^{\bullet \leq 0}$ on the nondegenerate 2–cell, where $(0,1,2) \in \widehat{\Delta}^2$:



Explicitly, we have (again using $g_i = d_{E_i}$ for the internal differential of E_i)

$$\begin{aligned}\mathrm{Tr}_g(\nabla(d+g)\cdot\nabla(d+g))_{(0,1,2)} = {}& \mathrm{tr}(g_{20}\nabla g_0 \nabla g_{012} + g_{20}\nabla g_{01}\nabla g_{12} + g_{20}\nabla g_{012}\nabla g_2) \\ & - \mathrm{tr}(g_{201}\nabla g_1 \nabla g_{12} + g_{201}\nabla g_{12}\nabla g_2) \\ & - \mathrm{tr}(g_{120}\nabla g_0 \nabla g_{01} + g_{120}\nabla g_{01}\nabla g_1) \\ & + \mathrm{tr}(g_{2012}\nabla g_2 \nabla g_2 + g_{1201}\nabla g_1\nabla g_1 + g_{0120}\nabla g_0 \nabla g_0).\end{aligned}$$

Theorem 3.14 *The Chern character* $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ *defined above is a map of simplicial presheaves.*

Proof We use the notation from Definition 3.13. First, we note that $\mathbf{Ch}(U)_n(\mathcal{F})$ is a well-defined element of $\mathbf{\Omega}(U)_n$, ie we still need to show that the labeling satisfies (2-9). Since the internal differential vanishes for $\Omega^\bullet_{\mathrm{hol}}(\cdot)[u]^{\bullet \leq 0}$, this amounts to showing that, for each p–cell given by $\alpha = (i_0, \dots, i_p)$, the sum of the labelings on the boundary cells vanishes. This follows since

$$\sum_{j=0}^{k} (-1)^j \cdot d_j\left((\mathrm{Tr}_g(A^k))_\alpha \cdot \frac{u^k}{k!}\right) = \sum_{j=0}^{k} (-1)^j \cdot (\mathrm{Tr}_g(A^k))_{\alpha(0,\dots,\hat{j},\dots,k)} \cdot \frac{u^k}{k!} = \delta\big((\mathrm{Tr}_g(A^k))_\alpha\big) \cdot \frac{u^k}{k!} = 0,$$

using Proposition 3.12 for the last equality. Next, we show that $\mathbf{Ch}(U)\colon \mathbf{IVB}(U) \to \mathbf{\Omega}(U)$ is a map of simplicial sets, ie that it respects the face and degeneracy maps. If $\delta_j : [n] \to [n+1]$ is the j^{th} face map, then $d_j : \mathbf{IVB}(U)_{n+1} \to \mathbf{IVB}(U)_n$ is given by precomposition with $\widehat{\Delta}^n \to \widehat{\Delta}^{n+1}$, $\{0,\dots,n\}^k \ni (i_0,\dots,i_k) \mapsto (\delta_j(i_0),\dots,\delta_j(i_k)) \in \{0,\dots,n+1\}^k$. Thus, for $\mathcal{F} \in \mathbf{IVB}(U)_{n+1}$ with corresponding Maurer–Cartan element g, we have $\mathbf{Ch}(U)_n \circ d_j(\mathcal{F})|_{\alpha=(i_0<\dots<i_k)} = \mathrm{Tr}_g(A^k)_{(\delta_j(i_0)<\dots<\delta_j(i_k))} \cdot u^k/k!$. This is equal to taking $\mathbf{Ch}(U)_{n+1}(\mathcal{F}) \in \mathrm{DK}(C)_{n+1} = \mathrm{Chain}(N(\mathbb{Z}\Delta^{n+1}), C)$, where $C = \Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet \leq 0}$, after applying $d_j : \mathrm{DK}(C)_{n+1} \to \mathrm{DK}(C)_n$ to it, and looking at the labeling of the cell $i_0 < \dots < i_k$ of Δ^n. Similarly, if $\sigma_j : [n] \to [n-1]$ is the j^{th} degeneracy, and $s_j : \mathbf{IVB}(U)_{n-1} \to \mathbf{IVB}(U)_n$ is the induced map, then, for

$\mathcal{F} \in \mathbf{IVB}(U)_{n-1}$ with corresponding Maurer–Cartan element g, we get $\mathbf{Ch}(U)_n \circ s_j(\mathcal{F})|_{\alpha=(i_0<\cdots<i_k)} = \mathrm{Tr}_g(A^k)_{(\sigma_j(i_0)\le\cdots\le\sigma_j(i_k))} \cdot u^k/k!$. Now, if σ is injective on $\{i_0,\ldots,i_k\}$, then, by Note 2.2(5), this is equal to $\mathrm{Tr}_g(A^k)_{(\sigma_j(i_0)<\cdots<\sigma_j(i_k))} \cdot u^k/k!$, which is the labeling of $s_j \circ \mathbf{Ch}(U)_{n-1}(\mathcal{F})$ at $i_0<\cdots<i_k$. In the case where σ_j is not injective on $\{i_0,\ldots,i_k\}$, we get that $g_{\sigma_j(i_0)\ldots\sigma_j(i_k)}$ is either the identity or zero, so, in either case, $\nabla g_{\sigma_j(i_0)\ldots\sigma_j(i_k)} = 0$, and thus $\mathbf{Ch}(U)_n \circ s_j(\mathcal{F})|_{\alpha=(i_0<\cdots<i_k)} = 0$, which is equal to the degeneracy $s_j : \mathrm{DK}(C)_{n-1} \to \mathrm{DK}(C)_n$ applied to $\mathbf{Ch}(U)_{n-1}(\mathcal{F})$ at the cell $i_0<\cdots<i_k$.

Finally, we show that $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ is a map of simplicial presheaves, ie that under a holomorphic map $\varphi\colon U \to U'$, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{IVB}(U') & \xrightarrow{\mathbf{Ch}(U')} & \mathbf{\Omega}(U') \\ {\scriptstyle \mathbf{IVB}(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathbf{\Omega}(\varphi)} \\ \mathbf{IVB}(U) & \xrightarrow{\mathbf{Ch}(U)} & \mathbf{\Omega}(U) \end{array}$$

This follows, since both compositions are given by pullback via φ, ie for $\mathcal{F}' \in \mathbf{IVB}(U')$ with induced Maurer–Cartan element g' and induced differential d' on E'_α, we have

$$\begin{aligned} \mathbf{Ch}(U)_n \circ \mathbf{IVB}_n(\varphi)(\mathcal{F}')|_{\alpha=(i_0<\cdots<i_k)} &= \mathrm{Tr}_{\varphi^*g'}\Big(\big(((\varphi^*\nabla)(\varphi^*(d'+g'))\big)^k\Big)_\alpha \cdot \frac{u^k}{k!} \\ &= \varphi^*\big(\mathrm{Tr}_{g'}\big((\nabla(d'+g'))^k\big)_\alpha\big) \cdot \frac{u^k}{k!} \\ &= \mathbf{\Omega}(\varphi)_n \circ \mathbf{Ch}(U')_n(\mathcal{F}')|_{\alpha=(i_0<\cdots<i_k)}. \end{aligned}$$ □

4 A higher Chern character for coherent sheaves

In this section, we apply a construction, which we will call Čech sheafification, to the Chern character map $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ from Definition 3.13. More precisely, an endofunctor on simplicial presheaves $\boldsymbol{F} \mapsto \boldsymbol{F}^{\check{\dagger}}$ is defined as the colimit over all Čech covers of the totalization of the presheaf applied to the cover (see Definition 4.1), and then an explicit interpretation is offered for the induced map $\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}} \to \mathbf{\Omega}^{\check{\dagger}}$. Theorem 4.9 states that 0–simplices of $\mathbf{IVB}^{\check{\dagger}}$ are twisting cochains (up to equivalence) in the sense of O'Brian, Toledo and Tong [1981c], and Theorem 4.18 states that the induced Chern character $\mathbf{Ch}^{\check{\dagger}}$ recovers the Chern character from [loc. cit.].

To fix some notation, let $(U_i \to X)_{i\in I}$ be an open cover, which is a particular diagram in $\mathbb{C}\mathrm{Man}$. To this cover we associate the augmented simplicial presheaf $\check{N}U_\bullet \to X$ whose p–simplices are coproducts of representable presheaves given by $(p+1)$–fold intersections of the cover,

$$\check{N}U_p = \coprod_{i_0,\ldots,i_p\in I} yU_{i_0,\ldots,i_p},$$

where yU denotes the Yoneda functor applied to U, ie $yU : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$, $V \mapsto \mathbb{C}\mathrm{Man}(V,U)$, interpreted as a constant simplicial set. Given another simplicial presheaf $\boldsymbol{F}$ we abuse notation by writing $\boldsymbol{F}(\check{N}U_\bullet)$

for the cosimplicial simplicial set with

$$F(\check{N}U_\bullet)^l_p := \prod_{i_0,\ldots,i_l} F(U_{i_0,\ldots,i_l})_p. \tag{4-1}$$

Definition 4.1 Given a simplicial presheaf $F : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$, define its *Čech sheafification* on a test manifold $X \in \mathbb{C}\mathrm{Man}$ to be the simplicial set given by

$$F^{\check{\dagger}}(X) := \underset{(U_\bullet \to X) \in \check{S}}{\mathrm{colim}} \mathrm{Tot}(F(\check{N}U_\bullet)), \tag{4-2}$$

where $\check{S}$ is the category of all Čech covers, and Tot is the totalization, which is reviewed in Appendix B. (For further details about the totalization, see our previous paper [2022, Appendix D.1] and [Hirschhorn 2003, Definition 18.6.3]; specific examples of Tot are worked out in Note 4.5 below, as well as in our previous paper [2022, Proof of Proposition 3.16].)

While $F^{\check{\dagger}}$ may not be a hypersheaf in general, Section 5 discusses the sheaf property and there the above definition is justified.

Proposition 4.2 *If F is a simplicial presheaf which takes values in Kan complexes, then its Čech sheafification is a Kan complex.*

Proof By Proposition C.1, for an open cover $U_\bullet$ of X, $\mathrm{Tot}(\mathbf{IVB}(\check{N}U_\bullet))$ is a Kan complex. Now, since our colimit over Čech covers is directed once we pass to simplicial presheaves $\check{N}U_\bullet$, one can check by hand that the colimit in $\mathbf{IVB}^{\check{\dagger}}(X)$ sends a diagram of projectively fibrant objects to a projectively fibrant object (ie $\mathbf{IVB}^{\check{\dagger}}$ takes values in Kan complexes). □

Definition 4.3 The *Čech sheafified Chern character map* $\mathbf{Ch}^{\check{\dagger}} : \mathbf{IVB}^{\check{\dagger}} \to \mathbf{\Omega}^{\check{\dagger}}$ is the map obtained by applying Čech sheafifications to the Chern character map from Definition 3.13.

4.1 Čech sheafification of IVB as twisting cochains

In this subsection, the vertices of the simplicial presheaf, $\mathbf{IVB}^{\check{\dagger}}$, are examined and shown in Theorem 4.9 to be precisely the twisting cochains of O'Brian, Toledo and Tong [1981c] up to equivalence. We thus define:

Definition 4.4 An *infinity vector bundle* over a complex manifold X is a 0–simplex of $\mathbf{IVB}^{\check{\dagger}}(X)$.

The following note looks at the k–simplices of $\mathbf{IVB}^{\check{\dagger}}(X)$ in general, before focusing more specifically on the 0–simplices:

Note 4.5 Fix a complex manifold X. Definition 4.1 applied to $F = \mathbf{IVB}$ yields

$$\mathbf{IVB}^{\check{\dagger}}(X) = \underset{(U_\bullet \to X) \in \check{S}}{\mathrm{colim}} \mathrm{Tot}(\mathbf{IVB}(\check{N}\mathcal{U}_\bullet)). \tag{4-3}$$

Now fix a Čech cover, $U_\bullet \to X$, and denote by $K_\bullet^\bullet$ the cosimplicial simplicial set whose l–cosimplices are given by

$$K^l := \mathbf{IVB}(\check{N}\mathcal{U}_l) = \mathbf{Perf}^{\widehat{\Delta}}(\check{N}\mathcal{U}_l).$$

Following (B-2), a k–simplex in $\mathrm{Tot}(K)$ consists of a collection $\{x^{(k,l)}\}_{l\geq 0}$ with

$$\begin{aligned} x^{(k,l)} \in \mathrm{sSet}(\Delta^k \times \Delta^l, K^l) &= \mathrm{sSet}(\Delta^k \times \Delta^l, \mathbf{Perf}^{\widehat{\Delta}}(\check{N}\mathcal{U}_l)) \\ &\overset{(2\text{-}3)}{=} \mathrm{sSet}\big(\Delta^k \times \Delta^l, \mathrm{sSet}\big(\widehat{\Delta}, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(\check{N}\mathcal{U}_l))^\circ\big)\big) \\ &= \mathrm{sSet}\big(\operatorname*{colim}_{\Delta^p \to \Delta^k \times \Delta^l} \widehat{\Delta}^p, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(\check{N}\mathcal{U}_l))^\circ\big), \end{aligned}$$

where in the last equality the calculation from (B-8) is used. Thus, according to Appendix B, page 4985, these are given by p–cells

$$x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right]} \in \mathrm{dg}\mathcal{N}(\mathrm{Perf}(\check{N}\mathcal{U}_l))^\circ_p \tag{4-4}$$

for certain paths $\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right]$ in the $(k+1)\times(l+1)$ grid (ie for any path within the indices of a nondecreasing path). Given such a path $\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right]$, and a choice of a component $i_0,\dots,i_l \in \check{N}\mathcal{U}_l$ describing an $(l+1)$–fold intersection, the p–cell (4-4) decorates each index $\left[\begin{smallmatrix}\alpha_j\\ \beta_j\end{smallmatrix}\right]$ with a bundle-with-connection

$$E^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\alpha_j\\ \beta_j\end{smallmatrix}\right];i_0,\dots,i_l} \to U_{i_0,\dots,i_l},$$

and decorates subpaths $\left[\begin{smallmatrix}\tilde\alpha_0\\ \tilde\beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\tilde\alpha_q\\ \tilde\beta_q\end{smallmatrix}\right]$ of these indices with maps between them. To be precise, before taking into account any simplicial or coherence conditions, the p–cell (4-4) is itself (by Example 2.8 and Lemma 2.9) given by the data

$$x^{(k,l)} = \{x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right]}\}, \tag{4-5}$$

where we vary over the components $i_0,\dots,i_l$ of $\check{N}\mathcal{U}_l$ and

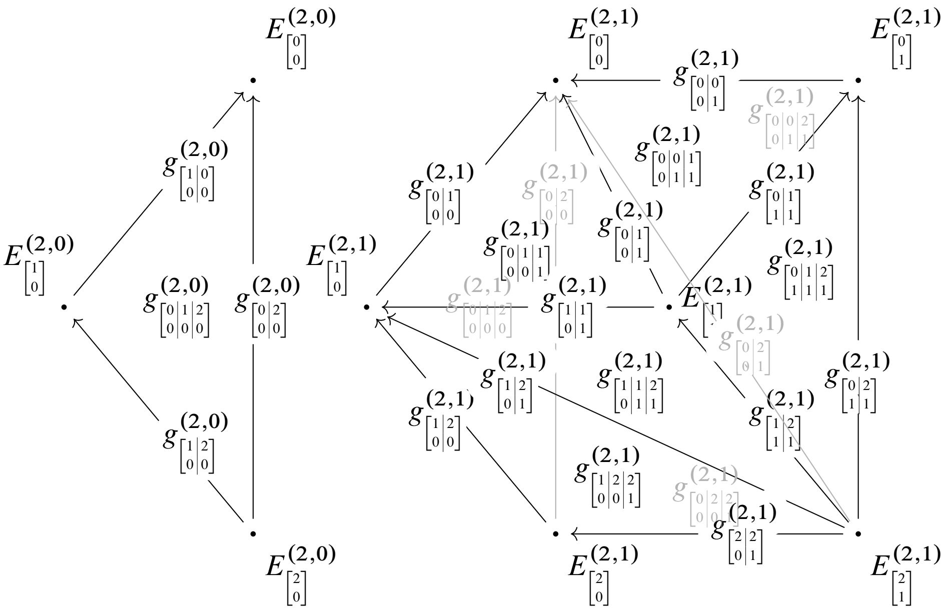
$$\begin{aligned} x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right]} = \Big(& E^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\alpha_j\\ \beta_j\end{smallmatrix}\right];i_0,\dots,i_l} \to U_{i_0,\dots,i_l}, \nabla^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\alpha_j\\ \beta_j\end{smallmatrix}\right];i_0,\dots,i_l}, \\ & g^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\tilde\alpha_0\\ \tilde\beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\tilde\alpha_q\\ \tilde\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} : E^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\tilde\alpha_q\\ \tilde\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} \to E^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\tilde\alpha_0\\ \tilde\beta_0\end{smallmatrix}\right];i_0,\dots,i_l} \Big); \end{aligned}$$

here the g's are associated to any subsequence $\left[\begin{smallmatrix}\tilde\alpha_0\\ \tilde\beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\tilde\alpha_q\\ \tilde\beta_q\end{smallmatrix}\right]$ for $q\geq 0$, of the indices from $\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right]$. Moreover, these g's satisfy the relations from (2-5). Since the simplices of $x^{(k,l)}$ fit together via the simplicial set relations, the above data (4-5) does not depend on the chosen p–cell determined by $\left[\begin{smallmatrix}\alpha_0\\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_p\\ \beta_p\end{smallmatrix}\right]$, and thus $x^{(k,l)}$ is given by the data

$$x^{(k,l)} = \Big(E^{(k,l)}_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right];i_0,\dots,i_l} \to U_{i_0,\dots,i_l}, \nabla^{(k,l)}_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right];i_0,\dots,i_l}, g^{(k,l)}_{\left[\begin{smallmatrix}\tilde\alpha_0\cdots\tilde\alpha_q\\ \tilde\beta_0\cdots\tilde\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} : E^{(k,l)}_{\left[\begin{smallmatrix}\tilde\alpha_q\\ \tilde\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} \to E^{(k,l)}_{\left[\begin{smallmatrix}\tilde\alpha_0\\ \tilde\beta_0\end{smallmatrix}\right];i_0,\dots,i_l} \Big). \tag{4-6}$$

For example, for $k=2$ and $l=0,1$, some of this data is visualized below, where both the nablas and the open set indices $i_0,\dots,i_l$ are suppressed for better readability:

on U_{i_0} for each fixed i_0 on U_{i_0,i_1} for each fixed i_0 and i_1



Now, by the compatibility relations (B-7) in $\mathrm{Tot}(K)$, the data given by the right-hand side of (4-6) is determined by the lowest l for which a given set of indices $\left[\begin{smallmatrix}\tilde\alpha_0 & | & \cdots & | & \tilde\alpha_q\\ \tilde\beta_0 & | & \cdots & | & \tilde\beta_q\end{smallmatrix}\right]$ can be obtained via a face map. For example,

$$E^{(k,l+1)}_{\left[\begin{smallmatrix}\alpha\\ \delta_j(\beta)\end{smallmatrix}\right];i_0,\dots,i_{l+1}} \overset{\text{(B-7)}}{=} (\text{component of } d^j(x^{(k,l)})) = E^{(k,l)}_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right];i_0,\dots,\hat{i}_j,\dots,i_{l+1}}|_{U_{i_0,\dots,i_{l+1}}},$$

where d^j acts by pulling back a bundle to a subset (by Definitions 2.5 and 2.1), ie by restricting the vector bundle to this subset. In particular,

$$E^{(k,l)}_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right];i_0,\dots,i_l} = E^{(k,0)}_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i_\beta}|_{U_{i_0,\dots,i_l}},$$

and similar statements apply to the g's.

Thus, the data of a k–simplex in $\mathrm{Tot}(K)$ is given by (suppressing the tildes)

(1) chain complexes of holomorphic vector bundles $E_{\alpha;i} := E^{(k,0)}_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i} \to U_i$ with differential $g_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i} = g^{(k,0)}_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i}$ for any index $\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right]$ on the $(k+1)\times(0+1)$ grid;

(2) connections $\nabla_{\alpha;i} := \nabla^{(k,0)}_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i}$ on $E_{\alpha;i}$;

(3) maps

$$g_{\left[\begin{smallmatrix}\alpha_0 & | & \cdots & | & \alpha_q\\ \beta_0 & | & \cdots & | & \beta_q\end{smallmatrix}\right];i_0,\dots,i_l} := g^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & | & \cdots & | & \alpha_q\\ \beta_0 & | & \cdots & | & \beta_q\end{smallmatrix}\right];i_0,\dots,i_l} : E_{\alpha_q;i_{\beta_q}}|_{U_{i_0,\dots,i_l}} \to E_{\alpha_0;i_{\beta_0}}|_{U_{i_0,\dots,i_l}}$$

for $l \geq 1$ and for any β's which include all the indices from 0 to l, ie for $\{\beta_0, \dots, \beta_q\} = \{0, \dots, l\}$; this is because if there was a $j \in \{0, \dots, l\}$ with $j \notin \{\beta_0, \dots, \beta_q\}$, then the map $g^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \alpha_q \\ \beta_0 & | \cdots | & \beta_q\end{smallmatrix}\right]; i_0, \dots, i_j, \dots, i_l}$ would, according to (B-7), just be the restriction

$$g^{(k,l-1)}_{\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \alpha_q \\ \gamma_0 & | \cdots | & \gamma_q\end{smallmatrix}\right]; i_0, \dots, \hat{i}_j, \dots, i_l}\Big|_{U_{i_0,\dots,i_j,\dots,i_l}},$$

where $\beta_i = \delta_j(\gamma_i)$ for all i, and so the data could be recovered from the map $g^{(k,l-1)}_{\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \alpha_q \\ \gamma_0 & | \cdots | & \gamma_q\end{smallmatrix}\right]; i_0, \dots, \hat{i}_j, \dots, i_l}$ via restriction.

Of course, as before, the sequence of indices $\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \alpha_q \\ \beta_0 & | \cdots | & \beta_q\end{smallmatrix}\right]$ has to come from a nondecreasing set of indices on a $(k+1) \times (l+1)$ grid (see Section B.3). Sometimes we simply write $g_{\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \alpha_q \\ \beta_0 & | \cdots | & \beta_q\end{smallmatrix}\right]}$ when the context of the open set $U_{i_0,\dots,i_l}$ is clear.

In particular note that:

- Using the fact that we land in the maximal Kan subcomplex $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ$ of $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))$, for $q = 1$, the maps on 1–cells $g_{\left[\begin{smallmatrix}\alpha_0 & | & \alpha_1 \\ \beta_0 & | & \beta_1\end{smallmatrix}\right]; i_0, i_1}$ are all quasi-isomorphisms.
- Finally, these maps satisfy the relations from (2-5) on $U_{i_0,\dots,i_l}$:

$$g_{\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \alpha_q \\ \beta_0 & | \cdots | & \beta_q\end{smallmatrix}\right]} \circ g_{\left[\begin{smallmatrix}\alpha_q \\ \beta_q\end{smallmatrix}\right]} + (-1)^q \cdot g_{\left[\begin{smallmatrix}\alpha_0 \\ \beta_0\end{smallmatrix}\right]} \circ g_{\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \alpha_q \\ \beta_0 & | \cdots | & \beta_q\end{smallmatrix}\right]} = \sum_{j=1}^{q-1} (-1)^{j-1} g_{\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \hat{\alpha}_j & | \cdots | & \alpha_q \\ \beta_0 & | \cdots | & \hat{\beta}_j & | \cdots | & \beta_q\end{smallmatrix}\right]} + \sum_{j=1}^{q-1} (-1)^{q(j-1)+1} g_{\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \alpha_j \\ \beta_0 & | \cdots | & \beta_j\end{smallmatrix}\right]} \circ g_{\left[\begin{smallmatrix}\alpha_j & | \cdots | & \alpha_q \\ \beta_j & | \cdots | & \beta_q\end{smallmatrix}\right]}. \tag{4-7}$$

The above note is applied below to the case of 0–simplices, in order to relate them to twisting cochains defined by O'Brian, Toledo and Tong [1981c, Definition 1.3], which we now briefly review.

Note 4.6 Let $(U_i \to X)_{i \in I}$ be a given cover, and let $E_i^\bullet \to U_i$ be graded holomorphic vector bundles over U_i. Then, according to [loc. cit.], a is a *twisting cochain* if $a = \sum_{j \geq 0} a^{j,1-j}$ with $a^{j,1-j} \in C^j(\mathcal{U}, \mathrm{Hom}^{1-j}(E, E))$, which is given by a collection of bundle morphisms on intersections of open sets, $a^{j,1-j} = \{a_{i_0,\dots,i_j} : E_{i_j}|_{U_{i_0,\dots,i_j}} \to E_{i_0}|_{U_{i_0,\dots,i_j}}\}_{i_0,\dots,i_j \in I}$, satisfying conditions [loc. cit., (1.5)] on each $U_{i_0,\dots,i_q}$

$$\sum_{j=1}^{q-1} (-1)^j a_{i_0,\dots,\hat{i}_j,\dots,i_q} + \sum_{j=0}^{q} (-1)^{(1-j)(q-j)} a_{i_0,\dots,i_j} \circ a_{i_j,\dots,i_q} = 0. \tag{4-8}$$

Note that, compared to the data of a k–simplex in $\check{\mathbf{IVB}}^\dagger(X)$ (see Note 4.5(1)–(3)), there is a priori no chosen connection. A version of $\check{\mathbf{IVB}}^\dagger(X)$ is also provided then without connection. Recall from (2-7) that $\mathbf{IVB}(U)_n = \mathrm{sSet}\big(\hat{\Delta}^n, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ\big)$.

Definition 4.7 Define $\widetilde{\mathrm{Perf}}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{dgCat}$ by setting $\widetilde{\mathrm{Perf}}(U)$ to be the dg-category of finite chain complexes of holomorphic vector bundles, just as in Definition 2.1, but with the difference that we do not choose any connection on $E_\bullet$. Analogously to **IVB** from Definition 2.11, define $\widetilde{\mathbf{IVB}}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by setting $\widetilde{\mathbf{IVB}}(U)_n := \mathrm{sSet}\big(\hat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\widetilde{\mathrm{Perf}}(U))^\circ\big)$.

For a Čech cover $(U_\bullet \to X)$, Note 4.5 can be repeated to obtain an explicit description of $\mathrm{Tot}(\widetilde{\mathbf{IVB}}(\check{N}\mathcal{U}_\bullet))$. Indeed the data of a k–simplex of $\mathrm{Tot}(\widetilde{\mathbf{IVB}}(\check{N}\mathcal{U}_\bullet))$ is given by the data of chain complexes of holomorphic vector bundles $E_{\alpha;i}$ as in (1) together with maps $g_{\left[\begin{smallmatrix}\alpha_0 \\ \beta_0\end{smallmatrix}\middle|\cdots\middle|\begin{smallmatrix}\alpha_q \\ \beta_q\end{smallmatrix}\right];i_0,\dots,i_l}$ as in (3), but *without* any connections as stated in (2).

The following lemma relates the above definition to the one with connections:

Lemma 4.8 *The dg-functor* Perf $\to \widetilde{\mathrm{Perf}}$ *that forgets the connection induces a map of simplicial presheaves* $\mathbf{IVB} \to \widetilde{\mathbf{IVB}}$*, which after applying the Čech sheafification (Definition 4.1) yields an isomorphism of simplicial sets* $\check{\mathbf{IVB}}^\dagger(X) \xrightarrow{\cong} \check{\widetilde{\mathbf{IVB}}}^\dagger(X)$.

Proof For a fixed cover $(U_\bullet \to X)$, the forgetful map $\mathrm{Tot}(\mathbf{IVB}(\check{N}\mathcal{U}_\bullet)) \to \mathrm{Tot}(\widetilde{\mathbf{IVB}}(\check{N}\mathcal{U}_\bullet))$ forgets the information of the connections as stated in (2) in Note 4.5. Taking colimit over covers, this descends to a well-defined map $\check{\mathbf{IVB}}^\dagger(X) \to \check{\widetilde{\mathbf{IVB}}}^\dagger(X)$ which is surjective, since every complex manifold has a (Stein) open cover such that, for every open set of the cover, there exists a connection on the corresponding bundles.

It remains to check injectivity. Assume that two k–simplices $x, x' \in \check{\mathbf{IVB}}^\dagger(X)_k$ are mapped, respectively, to $\tilde{x}, \tilde{x}' \in \check{\widetilde{\mathbf{IVB}}}^\dagger(X)_k$ by forgetting the connections, and that these are equal, ie $\tilde{x} = \tilde{x}'$. This means that there is a zigzag of refinements and extensions with respect to the colimit over covers which connects $\tilde{x}$ and $\tilde{x}'$ in $\check{\widetilde{\mathbf{IVB}}}^\dagger(X)_k$. Since every k–simplex in $\check{\widetilde{\mathbf{IVB}}}^\dagger(X)$ has a refinement which is in the image of $\check{\mathbf{IVB}}^\dagger(X)$ under the forgetful functor, (ie it has a choice of connections on the bundles for each open set,) it is enough to consider the case where $\tilde{x}$ and $\tilde{x}'$ are both refinements of $\tilde{y} \in \check{\widetilde{\mathbf{IVB}}}^\dagger(X)_k$, where $\tilde{y}$ may *not* be in the image of the forgetful functor. In order to prove injectivity, it is enough to show that there exists a $\tilde{z}$ for which both $\tilde{x}$ and $\tilde{x}'$ are refinements, and which is in the image of the forgetful functor, so that taking a preimage z of $\tilde{z}$ shows that x and x' are equal in $\check{\mathbf{IVB}}^\dagger(X)_k$. To this end, note that, if x and x' are represented on fixed covers $U_\bullet$ and $U'_\bullet$, respectively. Then we define $\tilde{z}$ represented on the cover $U_\bullet \sqcup U'_\bullet$ as follows. To define the bundle data (1) for $\tilde{z}$, if V is an open set in the cover $U_\bullet$ or $U'_\bullet$ pick the bundle for that open set from $\tilde{x}$ or $\tilde{x}'$, respectively, which we note to be equal to bundles from $\tilde{y}$ appropriately restricted. To define the maps g from (3) for $\tilde{z}$, if $V_1, \dots, V_l$ are open sets from $U_\bullet \sqcup U'_\bullet$, we have bundles over V_i coming from the data $\tilde{y}$, and so we take the maps of bundles as provided by $\tilde{y}$. Note that $\tilde{x}$ and $\tilde{x}'$ both extend $\tilde{z}$, and, moreover, $\tilde{z}$ is in the image of the forgetful functor by the extension z of x and x', since there are connections on each of the bundles coming from the data (2) provided by x and x'. □

With this definition, the main theorem of this section is stated below.

Theorem 4.9 *The equivalence classes of O'Brian, Toledo and Tong* [1981c] *of twisting cochains inject into the vertices of* $\check{\mathbf{IVB}}^{\dagger}(X)$.

Proof By Lemma 4.8, we may forget about the connections, and simply inject twisting cochains into vertices of $\widetilde{\mathbf{IVB}}^{\dagger}(X)$. By Note 4.6, a twisting cochain on a cover $(U_i \to X)_{i\in I}$ with holomorphic vector bundles $E_i^\bullet \to U_i$ is given by a collection $a = \{a_{i_0,\dots,i_j}\}_{i_0,\dots,i_j\in I, j\ge 0}$ satisfying (4-8). To this, we assign the data of a 0–simplex in $\check{\mathbf{IVB}}^{\dagger}(X)$ as stated in (1) and (3) from page 4962 as follows. First, the $E_{0;i} \to U_i$ from (1) are just the given E_i. As for the g's in (1) and (3), define

$$g^{(k,l)}_{\left[\begin{smallmatrix}0\\\beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\\cdots\end{smallmatrix}\middle|\begin{smallmatrix}0\\\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} := a_{i_{\beta_0},\dots,i_{\beta_q}}. \tag{4-9}$$

Note that the twisting cochain equations (4-8) imply (4-7). Moreover, the equivalence of twisting cochains is generated by refinements and extensions (see [loc. cit., page 232, above Proposition 1.10]), which identifies the corresponding infinity vector bundles (due to the colimit in (4-3)).

To check injectivity, we give a map in the opposite direction, which is a left-inverse to the above map. Explicitly, for a 0–simplex in $\widetilde{\mathbf{IVB}}^{\dagger}(X)$ represented by a cover $(U_i \to X)_i$ and bundles $E_i^\bullet$ with maps g as in (1) and (3), we define the twisting cochain

$$a_{i_0,\dots,i_j} := g^{(k,l)}_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\middle|\begin{smallmatrix}0\\1\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\\cdots\end{smallmatrix}\middle|\begin{smallmatrix}0\\j\end{smallmatrix}\right];i_0,\dots,i_j}, \tag{4-10}$$

which preserves the twisting cochain equations (4-8) due to (4-7). The colimit construction implies equivalence of twisting cochains. The composition of these two constructions, which maps twisting cochains to $\widetilde{\mathbf{IVB}}^{\dagger}(X)_0$ via (4-9) and then back to twisting cochains via (4-10), is the identity on twisting cochains.

As a final remark, we note that there are different (nonequivalent) choices for a left-inverse other than (4-10). In fact, equation (4-9) assigns the *same* homotopy $a_{j_0,\dots,j_q}$ to any

$$g_{\left[\begin{smallmatrix}\alpha_0\\\beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots\\\cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_q\\\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} \quad \text{with } i_{\beta_0} = j_0, \dots, i_{\beta_q} = j_q, \tag{4-11}$$

while in $\widetilde{\mathbf{IVB}}^{\dagger}(X)_0$ these maps (4-11) may generally be different. Therefore, any choice (consistent within the Maurer–Cartan equation (4-7)) may thus be used as a left-inverse for (4-9). □

To end this subsection, consider the restriction of the simplicial presheaf **IVB** to the one which only utilizes chain complexes of vector bundles whose homology is concentrated in degree zero. Below we show that the associated simplicial presheaf contains (after sheafification) all of the data of isomorphism classes of coherent sheaves in its vertices.

Note 4.10 For the reader's convenience, we review here a construction from [Toledo and Tong 1978a, Section 2]. Let $X \in \mathbb{C}\mathrm{Man}$ and $a_\bullet$ be a twisting cochain for a cover $(U_\bullet \to X)$ with holomorphic vector bundles $E_\bullet^\bullet$ (see O'Brian, Toledo and Tong [1981c] or Note 4.6 above). Consider the locally defined sheaf of $\mathcal{O}_X$–modules, $\mathcal{H}_i := H_\bullet(\Gamma(E_i), a_i)$, given by the homology of sections of $E_i^\bullet$ with differential a_i over U_i. Since each $a_{i,j}$ gives a quasi-isomorphism on the level of complexes, there is an induced isomorphism of sheaves on homology $a_{i,j}: U_{i,j}|_{\mathcal{H}_j} \xrightarrow{\sim} U_{i,j}|_{\mathcal{H}_i}$. Taking the colimit[2] of the $\mathcal{H}_i$ over the diagram induced by these a_{ij} produces a sheaf on X which we will call *the homology sheaf* and denote by $\mathcal{H}$. This construction further produces a map[3] of simplicial presheaves

$$\check{\mathbf{IVB}}^{\dagger} \xrightarrow{\mathcal{H}} \mathcal{N}(\mathrm{Sh}\mathcal{O}^\bullet), \tag{4-12}$$

where $\mathcal{N}$ denotes the nerve, and $\mathrm{Sh}\mathcal{O}^\bullet$ is the category of sheaves of graded $\mathcal{O}_X$–modules (without differential) with morphisms given by isomorphisms. The relevance of this construction to coherent sheaves is recorded in the following definition and proposition.

Definition 4.11 The simplicial presheaf $\mathbf{CohSh} \hookrightarrow \mathbf{IVB}$ is the subsimplicial presheaf defined by considering the full subpresheaf of dg-categories, $\mathrm{Perf}_{\mathrm{coh}} \hookrightarrow \mathrm{Perf}$ utilizing only chain complexes of bundles whose homology is concentrated in degree zero and then taking $\mathbf{CohSh}(X)_n := \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}_{\mathrm{coh}}(U))^\circ\big)$.

Lemma 4.12 *Given a manifold M and a coherent sheaf $\mathcal{F}$, there exists an open cover by relatively compact Stein open submanifolds on which $\mathcal{F}$ is locally resolved by a chain complex of vector bundles.*

Proof M admits a cover $\{U_i\}_{i\in I}$ by Stein open subsets. For each Stein submanifold U_i, it admits an open cover by relatively compact open sets $\{V_{i,j}\}_{i\in I, j\in J_i}$. Now, for each relatively compact open submanifold $V_{i,j}$, we cover it one final step further by open Stein sets $W_{i,j,k}$. As each $W_{i,j,k}$ is a subset of a relatively compact open Stein manifold U_i, then, by [Field 1982, Theorem 7.2.6], $\mathcal{F}$ admits a resolution by vector bundles on $W_{i,j,k}$. □

Proposition 4.13 *The set of isomorphism classes of coherent sheaves on X is in bijective correspondence with the connected components of $\check{\mathbf{CohSh}}^{\dagger}(X)$.*

Proof Recall the map $\mathcal{H}: \check{\mathbf{IVB}}^{\dagger}(X) \to \mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X^\bullet)$ from Note 4.10. But, since **CohSh** requires the local chain complex's homology to be concentrated in degree zero, the map's image lands in $\mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X) \hookrightarrow \mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X^\bullet)$, where $\mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X)$ is the nerve of the category of sheaves of $\mathcal{O}_X$–modules (concentrated in degree 0). Since the image of our map is precisely an $\mathcal{O}_X$ which satisfies the properties of a coherent sheaf, then the map factors through the nerve of the groupoid of coherent sheaves with isomorphisms, $\mathcal{H}: \check{\mathbf{CohSh}}^{\dagger}(X) \to \mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X) \hookrightarrow \mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X)$ which in turn is well defined as a map which sends connected components of $\check{\mathbf{CohSh}}^{\dagger}$ to connected components of $\mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X)$, ie precisely the isomorphism classes of $\mathrm{CohSh}\,\mathcal{O}_X$.

[2]Here we mean the concrete set-theoretic colimit given by a coproduct of $\mathcal{H}_i$ and then mod out by the equivalence generated by $a_{i,j}$ on $U_{i,j}$.

[3]Which, importantly, is *not* coming from a map of complexes or even graded modules.

To observe injectivity, we consider the image of two vertices $x, y \in \mathbf{CohSh}^{\check{\dagger}}(X)_0$, represented by cocycle data on some common refinement by a Stein cover, $(U_\bullet \to X)$, whose images $\mathcal{H}(x), \mathcal{H}(y) \in \mathcal{N}(\mathrm{CohSh}\, \mathcal{O}_X)$ are connected by an edge. In particular, this means that the global homology sheaves for x and y are isomorphic as $\mathcal{O}$–modules. In order to construct an edge $z \in \mathbf{CohSh}^{\check{\dagger}}(X)_1$ connecting x and y, we first need local quasi-isomorphisms connecting the local resolutions for the chain complexes of bundles x and y, respectively. These maps are given by recalling that these complexes over a Stein space are projective resolutions [Forstnerič 2011, Corollary 2.4.5] and so maps on homology induce chain maps between the complexes [Hilton and Stammbach 1971, Theorem 4.1]. So far, these quasi-isomorphisms produce the edge data for z on U_i, and the 1–skeleton of the edge data for z on higher intersections. To move up to the 2–skeleton, say on $U_{i,j}$, we see that we now have two quasi-isomorphisms between the complexes for x and y: one restricted from the quasi-isomorphism over U_i and the other from U_j. Again appealing to [Hilton and Stammbach 1971, Theorem 4.1] we now know these two quasi-isomorphisms are chain-homotopic and this provides all of the data for z on U_i's, $U_{i,j}$'s, and the 2–skeleton of the data on higher intersections. Now, by O'Brian, Toledo and Tong [1981c, Lemma 1.6] and the ensuing discussion there, one uses an inductive argument for how our higher homotopies of z would be constructed to satisfy the Maurer–Cartan equation and since their constructions include into ours (see our proof of Theorem 4.9), one indeed can construct an edge z connecting x and y to prove injectivity.

For surjectivity, applying Lemma 4.12 and then following [Toledo and Tong 1978a, Propsoition 2.4], for a coherent sheaf $\mathcal{F}$ there exists a Stein open cover $(U_i \hookrightarrow X)_{i\in I}$, so we can choose a twisting cochain class in $\mathbf{CohSh}^{\check{\dagger}}(X)_0$ by locally/projectively resolving the coherent sheaf by a complex of vector bundles, coherent on intersections $U_{i,j}$ up to quasi-isomorphisms, and further coherent on $U_{i_0,\dots,i_p}$ by higher homotopies which again exist by virtue of Lemma 1.6 of O'Brian, Toledo and Tong [1981c] and the discussion which follows it. It follows that the map $\mathcal{H}$ is surjective on connected components since in the proof of Theorem 4.9, we show how their constructions include into ours. □

4.2 Čech sheafification of the Chern map Ch

This section continues the study of the Čech sheafified Chern character map $\mathbf{Ch}^{\check{\dagger}} : \mathbf{IVB}^{\check{\dagger}} \to \mathbf{\Omega}^{\check{\dagger}}$ (where $\boldsymbol{F}^{\check{\dagger}}(X) = \mathrm{colim}_{(U_\bullet \to X)\in \check{S}} \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet))$ was defined in (4-2)). In Theorem 4.9 twisting cochains à la [loc. cit.] were already interpreted as 0–simplices of $\mathbf{IVB}^{\check{\dagger}}$. Next, in Note 4.16, $\mathbf{\Omega}^{\check{\dagger}}$ is explicitly described as well as the map $\mathbf{Ch}^{\check{\dagger}}$ for the case of 0–simplices. Comparing the formulas for the Čech sheafified Chern character map $\mathbf{Ch}^{\check{\dagger}}$ with the Chern character map from [loc. cit.] for a coherent sheaf (which is reviewed in 4.17), shows, that these are given by precisely the same formulas. This result is stated in Theorem 4.18.

The following note reviews $\mathrm{Tot}(\mathbf{\Omega}(\check{N}U_\bullet))$:

Note 4.14 Fix a Čech cover $(U_\bullet \to X)$. Then $\mathrm{Tot}(\mathbf{\Omega}(\check{N}U_\bullet))$ is the totalization of the cosimplicial simplicial set $\mathbf{\Omega}(\check{N}U_\bullet) = \underline{\mathrm{DK}}(\Omega^\bullet_{\mathrm{hol}}(\check{N}U)[u]^{\bullet\le 0})$. Recall from Note 2.16 that the n–simplices of Dold and Kan applied to the chain complex $\Omega^{\bullet}_{\mathrm{hol}}(V)[u]^{\bullet\le 0}$ for some open set V, are decorations of the standard n–simplex, ie they assign to each l–simplex, polynomials $a \in \Omega^{\bullet}_{\mathrm{hol}}(V)[u]^{\bullet\le 0}$ of total degree $-l$,

$$a = \begin{cases} \sum_{j=0}^{\infty} a^{2j} \cdot u^{l/2+j} & \text{when } l \text{ is even,} \\ \sum_{j=0}^{\infty} a^{2j+1} \cdot u^{l+1/2+j} & \text{when } l \text{ is odd,} \end{cases} \tag{4-13}$$

where $a^p \in \Omega^p_{\mathrm{hol}}(V)$. The condition (2-9) imposed for these decorations is that the alternating sum of the faces of a l–simplex agrees with applying the chain complex's differential to the data of the l–simplex:

$$0 = d_C(a) = \sum_{j=0}^{l} (-1)^j d_j(a),$$

where C is the complex $C = \Omega^{\bullet}_{\mathrm{hol}}(V)[u]^{\bullet\le 0}$ with zero differential $d_C = 0$, (see Definition 2.15).

Now, from Sections B.1 and B.2, 0–simplices of the totalization $\mathrm{Tot}(\mathbf{\Omega}(\check{N}\mathcal{U}_\bullet))_0$ consist of coherent decorations of the standard n–simplex by data coming from $\mathbf{\Omega}(\check{N}\mathcal{U}_n)$:

- on each U_i, a 0–simplex in $\underline{\mathrm{DK}}(\Omega^{\bullet}_{\mathrm{hol}}(U_i)[u]^{\bullet\le 0})$, ie a polynomial a_i as in (4-13) with $l = 0$: $a_i = \sum_{j=0}^{\infty} a_i^{2j} \cdot u^j$,
- on each U_{i_0,i_1}, a 1–simplex in $\underline{\mathrm{DK}}(\Omega^{\bullet}_{\mathrm{hol}}(U_{i_0,i_1})[u]^{\bullet\le 0})$, ie a polynomial a_{i_0,i_1} as in (4-13) with $l = 1$: $a_{i_0,i_1} = \sum_{j=0}^{\infty} a_{i_0,i_1}^{2j+1} \cdot u^{j+1}$,
- on each $U_{i_0,\dots,i_l}$, an l–simplex in $\underline{\mathrm{DK}}(\Omega^{\bullet}_{\mathrm{hol}}(U_{i_0,\dots,i_l})[u]^{\bullet\le 0})$, ie a polynomial $a_{i_0,\dots,i_l}$ as in (4-13).

These polynomials satisfy the conditions

$$0 = \sum_{j=0}^{l} (-1)^j d_j(a_{i_0,\dots,i_l}) = \sum_{j=0}^{l} U_{i_0,\dots,i_l}|_{a_{i_0,\dots,\hat{i}_j,\dots,i_l}},$$

where the last equality follows from (B-5) and Example B.1.

Recall from [Grothendieck 1966] that the *Hodge cohomology* $\bigoplus_{p,q} H^p(X, \Omega^q)$ is given by a sum over the p^{th} sheaf cohomology of the sheaf of holomorphic q forms (see also "Hodge theory" or "Hodge decomposition" [Frölicher 1955]). O'Brian, Toledo and Tong [1981c, Section 4] defined the Chern character as an element in $\bigoplus_k H^k(X, \Omega^k)$. Below we see how our $\mathbf{\Omega}^{\check{\dagger}}$ relates to the Hodge cohomology.

Proposition 4.15 *The set of connected components of $\mathbf{\Omega}^{\check{\dagger}}(X)$ forms a ring which is isomorphic to the even part of the Hodge cohomology ring,*

$$\pi_0(\mathbf{\Omega}^{\check{\dagger}}(X)) \simeq \bigoplus_{\substack{p,q \\ p+q \text{ even}}} H^p(X, \Omega^q).$$

Proof The proof follows first from a direct observation that the vertices of $\mathrm{Tot}(\mathbf{\Omega}(\check{N}U_\bullet))$ are precisely (since the differentials are all zero) a direct sum of Čech l–cocycles of holomorphic forms (even degree forms for l even and odd degree forms for l odd), and then from the observation that edges in $\mathrm{Tot}(\mathbf{\Omega}(\check{N}U_\bullet))$ correspond to Čech coboundaries. □

We next illustrate our sheafified Chern map $\check{\mathbf{Ch}}^{\dagger}$.

Note 4.16 Consider a Čech cover $(U_\bullet \to X)$, and a vertex in $\mathrm{Tot}(\mathbf{IVB}(\check{N}\mathcal{U}))_0$ as provided by Note 4.5, ie the data of holomorphic bundles $E_{0;i}$ with

- differentials $d = g_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right];i}$ from (1);
- connections $\nabla_{0;i}$ from (2); and
- maps $g_{\left[\begin{smallmatrix}0 & \cdots & 0\\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right];i_0,\dots,i_l}$ from (3).

Then our sheafified Chern character map $\check{\mathbf{IVB}}^{\dagger} \xrightarrow{\check{\mathbf{Ch}}^{\dagger}} \check{\mathbf{\Omega}}^{\dagger}$ simply applies the Chern character $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ from Definition 3.13 locally to the data in our vertex by allowing the indices from that definition to be given by the indices of the open cover. To clarify, the vertex above gets mapped to the following vertex in $\mathrm{Tot}(\mathbf{\Omega}(\check{N}\mathcal{U}))_0$:

- On each U_i, assign the Euler characteristic of $E_{0;i}$, denoted by $\chi(E_{0;i})\cdot u^0 \in \Omega^0_{\mathrm{hol}}(U_i)[u]^{\bullet\le 0}$.
- On each U_{i_0,i_1}, using $g = \{g_{\left[\begin{smallmatrix}0 & \cdots & 0\\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right];i_0,i_1}\}_{(\beta_0,\dots,\beta_q)\in\hat{\Delta}^1}$, assign the monomial
$$\mathrm{Tr}_g(\nabla(d+g))_{(0,1)}\cdot u \in \Omega^1_{\mathrm{hol}}(U_{i_0,i_1})[u]^{\bullet\le 0},$$
and restrict the Euler characteristic above on the vertices (see Definition 3.13):
$$\overset{U_{i_0,i_1}|_{\chi(E_{0;i_0})}}{\bullet} \xleftarrow{\mathrm{Tr}_g(\nabla(d+g))_{(0,1)}\cdot u} \overset{U_{i_0,i_1}|_{\chi(E_{0;i_1})}}{\bullet}$$
- For each $U_{i_0,i_1,\dots,i_l}$, using $g = \{g_{\left[\begin{smallmatrix}0 & \cdots & 0\\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right];i_0,i_1,\dots,i_l}\}_{(\beta_0,\dots,\beta_q)\in\hat{\Delta}^l}$, assign the monomial
$$\mathrm{Tr}_g\big((\nabla(d+g))^l\big)_{(0,1,\dots,l)}\cdot\frac{u^l}{l!} \in \Omega^l_{\mathrm{hol}}(U_{i_0,i_1,\dots,i_l})[u]^{\bullet\le 0} \tag{4-14}$$
to the top cell and to each face assign appropriate restrictions of the monomials defined for lower intersections.

The above formula is now compared to the one provided by O'Brian, Toledo and Tong for the Chern character map of a coherent sheaf.

Note 4.17 O'Brian, Toledo and Tong [1981c] construct characteristic classes for coherent sheaves via the following four steps:

(i) Given a coherent sheaf, a twisting cochain a is constructed using [loc. cit., below Lemma 1.6]. This construction is well defined with respect to equivalences of twisting cochains; see [loc. cit., Proposition 1.10].

(ii) Connection data is chosen for a so that we obtain a twisting cochain with holomorphic connection data; see [loc. cit., above Proposition 4.4].

(iii) The Atiyah class is represented by the class ∇a in [loc. cit., Proposition 4.4].

(iv) The Chern character is defined [loc. cit., above Proposition 4.5] using the trace map τ_a to be given by

$$\text{ch} := \sum_{k \geq 0} \text{ch}_k := \sum_{k \geq 0} \frac{1}{k!} \tau_a((\nabla a)^k). \tag{4-15}$$

Note that the trace map τ_a from [loc. cit., above Proposition 3.2] is defined in the same way as our trace map Tr_g in Definition 3.7.

Comparing the formulas (4-14) and (4-15) for the Chern character, these involve the same trace terms, and so we obtain the following theorem:

Theorem 4.18 *For a given coherent sheaf, the formula for the Chern character* (4-15) *from* [loc. cit.] *is given by the terms in the formula* (4-14) *of the Chern character map*

$$\{\text{isomorphism classes of coherent sheaves}\} \simeq \pi_0(\mathbf{CohSh}^{\check{\dagger}}) \xrightarrow{\pi_0(\mathbf{Ch}^{\check{\dagger}})} \pi_0(\mathbf{\Omega}^{\check{\dagger}}) \simeq \bigoplus_{\substack{p,q \\ p+q \text{ even}}} H^p(\Omega^q) \tag{4-16}$$

applied to the corresponding twisting cochain interpreted (*by Theorem 4.9*) *as a* 0*–simplex in* $\mathbf{CohSh}^{\check{\dagger}}$.

Proof A twisting cochain a defines the Maurer–Cartan element via (4-9). With this, the terms in the traces in (4-14) and (4-15) coincide. (We note that the additional factor u^l in (4-15) does not add any extra information, as the power l is precisely the "Čech degree" given by the number of intersections in $U_{i_0,\dots,i_l}$.). Finally, the left and right isomorphisms in (4-16) are given by Propositions 4.13 and 4.15, respectively. □

Note, in particular, that our sheafified $\mathbf{Ch}^{\check{\dagger}}$ provides not only a Chern character to resolutions of coherent sheaves but also provides invariants for morphisms and higher homotopies between these resolutions.

Remark 4.19 A version of the Chern–Simons invariant for the straight line path between connections is computed by $\pi_1(\mathbf{Ch}^{\check{\dagger}})$ as we outline here. In the case where $\mathbf{Vect} \hookrightarrow \mathbf{CohSh}$ is the full subcategory of vector bundles, a loop representing a class in $\pi_1(\mathbf{Vect}^{\check{\dagger}})$ is given by a vector bundle $E \to X$ and locally chosen connections $\{(E_i \to U_i, \nabla_i)\}$, along with a bundle automorphism $f \colon E \to E$. Then $\mathbf{Ch}^{\check{\dagger}}$ sends the vertex of this loop to the Chern character $\mathbf{Ch}^{\check{\dagger}}(\{(E_i, \nabla_i, g_{ij})\})$ and the edge induced by f is sent to an odd Čech–Hodge form, which we denote by $\mathbf{Ch}^{\check{\dagger}}(f)$, whose differential is the difference between $\mathbf{Ch}^{\check{\dagger}}(\{(E_i, \nabla_i, g_{ij})\})$ and $\mathbf{Ch}^{\check{\dagger}}(\{(E_i, f^*\nabla_i, g_{ij})\})$. Since $\mathbf{Ch}^{\check{\dagger}}(\{(E_i, \nabla_i, g_{ij})\}) = \mathbf{Ch}^{\check{\dagger}}(\{(E_i, f^*\nabla_i, g_{ij})\})$, f is sent to a closed odd form in the Čech–Hodge complex. Moreover, if two loops f and f' in $\mathbf{Vect}^{\check{\dagger}}$ are homotopic, then the difference between $\mathbf{Ch}^{\check{\dagger}}(f)$ and $\mathbf{Ch}^{\check{\dagger}}(f')$ is exact, and so $\pi_1(\mathbf{Ch}^{\check{\dagger}})$ indeed computes a higher invariant.

5 The induced map on classifying stacks

In this section we show that the previously considered Čech sheafified Chern character map (Definition 4.3) is a map of simplicial sheaves when we restrict $\mathbf{IVB}^{\check{\dagger}}$ (see Definition 4.1 and Note 4.5) to the subsimplicial sheaf $\mathbf{IVB}^{\check{\dagger}}_{\leq n}$ which considers complexes of vector bundles of a fixed length, n (see Definition 5.4). Moreover, each of these simplicial presheaves contains a subsimplicial presheaf which considers complexes, $\mathbf{CohSh}^{\check{\dagger}}$ and $\mathbf{CohSh}^{\check{\dagger}}_{\leq n}$ respectively (see Definition 4.11), whose homology is concentrated in degree zero, yielding the commutative diagram

$$\begin{array}{ccccccc}
\mathbf{CohSh}_{\leq n} & & & \hookrightarrow & & & \mathbf{CohSh} \\
\big\downarrow & \searrow & & & & \swarrow & \big\downarrow \\
 & & \mathbf{CohSh}^{\check{\dagger}}_{\leq n} & \hookrightarrow & \mathbf{CohSh}^{\check{\dagger}} & & \\
 & & \big\downarrow & & \big\downarrow & & \\
 & & \mathbf{IVB}^{\check{\dagger}}_{\leq n} & \hookrightarrow & \mathbf{IVB}^{\check{\dagger}} & & \\
\big\downarrow & \nearrow & & & & \nwarrow & \big\downarrow \\
\mathbf{IVB}_{\leq n} & & & \hookrightarrow & & & \mathbf{IVB}
\end{array}$$

(see Proposition 4.13 for a justification of our notation **CohSh**). As such we offer in Theorem 5.13 an upgrade on the statement of Theorem 4.18 to a statement about sheaves.

5.1 Sheaves in the local projective model structure

This section's main goal is to sort out which of the (maps of) presheaves in this paper are in fact (maps of) sheaves.

Given the Verdier site à la Dugger, Hollander and Isaksen [2004, Section 9] of complex manifolds and holomorphic maps, $\mathbb{C}\mathrm{Man}$, the category of simplicial presheaves $\mathrm{sPre}(\mathbb{C}\mathrm{Man})$ has multiples model structures. One particular choice is the (global) projective model structure whose weak equivalences are objectwise weak equivalences of simplicial sets and whose fibrations are objectwise fibrations of simplicial sets [Blander 2001, Theorem 1.5]. Further this model structure forms a (proper simplicial cellular) simplicial model category when we use the simplicial mapping space $\underline{\mathrm{sPre}}(X, Y)_n := \mathrm{sPre}(X \otimes \Delta^n, Y)$. After localizing this simplicial category over the class of maps induced by hypercovers, we further obtain the local projective (proper simplicial cellular) model structure $\mathrm{sPre}(\mathbb{C}\mathrm{Man})_{\mathrm{proj,loc}}$ [loc. cit., Theorem 1.6]. The relevant criteria in this structure for us is that an object in $\mathrm{sPre}(\mathbb{C}\mathrm{Man})_{\mathrm{proj,loc}}$ is fibrant if it is fibrant in the projective model structure and satisfies descent with respect to any hypercover thanks to Dugger, Hollander and Isaksen [2004]. Such an object is referred to below as a (hyper)sheaf.

In presenting a classifying stack (ie classifying hypersheaf) for coherent sheaves, one could produce a simplicial presheaf, $\boldsymbol{F} \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})$, and prove (at the very least) that for any manifold $X \in \mathbb{C}\mathrm{Man}$, the set of equivalence classes of coherent sheaves coincides with the connected components of the derived mapping space, $\mathbb{R}\mathrm{Hom}(X, \boldsymbol{F})$. Since we are working with the local projective simplicial model category of simplicial presheaves this mapping space can be computed by cofibrantly approximating X with $\widetilde{X} \to X$ (which in this case is the identity since X is representable and thus cofibrant), fibrantly approximating $\boldsymbol{F}$ by $\boldsymbol{F} \to \widehat{\boldsymbol{F}}$, and defining the right derived mapping space (ie the homotopy function complex from [Hirschhorn 2003, Section 17]) as the simplicial mapping space on the replacements:

$$\mathbb{R}\mathrm{Hom}(X, \boldsymbol{F}) := \underline{\mathrm{sPre}}(\widetilde{X}, \widehat{\boldsymbol{F}}) = \underline{\mathrm{sPre}}(X, \widehat{\boldsymbol{F}}). \tag{5-1}$$

Thus $\widehat{\boldsymbol{F}}$ would provide a more concrete description of this classifying stack and any map of simplicial presheaves $\boldsymbol{F} \to \boldsymbol{\Omega}$ provides cohomological invariants by inducing a map between fibrant replacements $\widehat{\boldsymbol{F}} \to \widehat{\boldsymbol{\Omega}}$; offering more explicit, cocycle-level cohomological invariants.

It is not immediate that our Čech sheafification computes the fibrant replacement. Below we first show that if $\boldsymbol{F}$ is already a hypersheaf then $\boldsymbol{F}^{\check{\dagger}}$ is again a hypersheaf, even though this result is not used in this paper.

Proposition 5.1 *If $\boldsymbol{F}$ is a hypersheaf, then $\boldsymbol{F}^{\check{\dagger}}$ is a hypersheaf and the natural map $\boldsymbol{F} \to \boldsymbol{F}^{\check{\dagger}}$ is an objectwise weak equivalence.*

Proof By construction, we have already shown in the proof of Proposition 5.2 that Čech sheafification preserves objectwise fibrancy without any assumptions on the homotopy type of $\boldsymbol{F}$. To see that there is an objectwise weak equivalence, we compute

$$\boldsymbol{F}^{\check{\dagger}}(X) = \operatorname*{colim}_{(\mathcal{W}\to X)\in S} \underline{\mathrm{sPre}}(\mathcal{W}, \boldsymbol{F}),$$

where S is a full subcategory of the overcategory $\mathbb{C}\mathrm{Man}/X$, whose objects are hypercovers $\mathcal{W} \to X$. Since $\boldsymbol{F}$ already satisfies descent, ie $\underline{\mathrm{sPre}}(\mathcal{W}, \boldsymbol{F}) \xleftarrow{\sim} \underline{\mathrm{sPre}}(X, \boldsymbol{F})$,

$$\boldsymbol{F}^{\check{\dagger}}(X) \xleftarrow{\sim} \operatorname*{colim}_{(\mathcal{W}\to X)\in S} \underline{\mathrm{sPre}}(X, \boldsymbol{F}) \xleftarrow{\sim} \underline{\mathrm{sPre}}(X, \boldsymbol{F}) = \boldsymbol{F}(X).$$

Now, to show that the Čech sheafification preserves hyperdescent, we choose a hypercover $\mathcal{U} \to X$ and argue that the natural map $\underline{\mathrm{sPre}}(X, \boldsymbol{F}^{\check{\dagger}}) \to \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}^{\check{\dagger}})$ is a weak equivalence of simplicial sets. On the one hand, we have

$$\underline{\mathrm{sPre}}(X, \boldsymbol{F}^{\check{\dagger}}) = \boldsymbol{F}^{\check{\dagger}}(X) \xleftarrow{\sim} \boldsymbol{F}(X),$$

while, on the other hand, we have

$$\begin{aligned}\underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}^{\check{\dagger}}) \xrightarrow{\sim} \underline{\mathrm{sPre}}\Bigl(\operatorname*{hocolim}_{i\in\Delta} \coprod_{i,\alpha_i} U_{i,\alpha_i}, \boldsymbol{F}^{\check{\dagger}}\Bigr) &= \operatorname*{holim}_{i\in\Delta} \prod_{i,\alpha_i} \underline{\mathrm{sPre}}(U_{i,\alpha_i}, \boldsymbol{F}^{\check{\dagger}}) \\ &= \operatorname*{holim}_{i\in\Delta} \prod_{i,\alpha_i} \boldsymbol{F}^{\check{\dagger}}(U_{i,\alpha_i}) \xleftarrow{\sim} \operatorname*{holim}_{i\in\Delta} \prod_{i,\alpha_i} \boldsymbol{F}(U_{i,\alpha_i}) = \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}) \xleftarrow{\sim} \boldsymbol{F}(X),\end{aligned}$$

where the last weak equivalence follows from $\boldsymbol{F}$ already satisfying descent. After repeated application of the two-out-of-three property for weak equivalences, we see that $\check{\boldsymbol{F}}^{\dagger}$ satisfies descent as well. □

Under a modest boundedness condition on a simplicial presheaf $\boldsymbol{F}$ which takes values in Kan complexes, its Čech sheafification (Definition 4.1) is a sheaf; this result is key to the rest of this paper.

Proposition 5.2 *Let $\boldsymbol{F} \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})$ be a projectively fibrant simplicial presheaf whose homotopy groups are all trivial above level n. Then $\check{\boldsymbol{F}}^{\dagger}$ is a fibrant approximation of $\boldsymbol{F}$ in the local projective model structure of simplicial presheaves on complex manifolds.*

Proof Given a projectively fibrant simplicial presheaf $\boldsymbol{F} \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})$ we can consider its fibrant replacement in the local projective model structure $\boldsymbol{F} \xrightarrow{\sim} \boldsymbol{F}' \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})_{\mathrm{loc}}$. By [Lurie 2017, Remark 6.2.2.12], we see that in general we can compute this fibrant replacement on a test manifold $X \in \mathbb{C}\mathrm{Man}$ with the *hypersheafification* of $\boldsymbol{F}$, written $\boldsymbol{F}^{\dagger}$, by taking a homotopy colimit of the simplicial mapping space $\underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F})$ over all hypercovers $(\mathcal{U} \to X)$. Below, as is standard, we identify the manifold X with its representable simplicial presheaf, ie with the functor $Y \mapsto \mathbb{C}\mathrm{Man}(Y, X)$, postcomposed by the functor which sends sets to simplicially constant simplicial sets. Thus, if S denotes the category of all hypercovers,

$$\boldsymbol{F}^{\dagger}(X) := \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}).$$

More formal references for this fact include [Anel and Subramaniam 2020, Example 3.4.9; Low 2015, Proposition 6.6]. We can now follow a series of steps to rewrite the above sheafification up to weak equivalence: Starting with

$$\boldsymbol{F}^{\dagger}(X) := \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}) = \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \underline{\mathrm{sPre}}\Big(\operatorname*{hocolim}_{i\in\boldsymbol{\Delta}} \mathcal{U}_i, \boldsymbol{F}\Big), \tag{5-2}$$

pulling the homotopy colimit out as a homotopy limit, and then using the fact that $\boldsymbol{F}$ is of bounded homotopy type so $\boldsymbol{F} \xrightarrow{\sim} \mathbf{cosk}_n \boldsymbol{F}$ with both of these projectively fibrant,

$$\boldsymbol{F}^{\dagger}(X) = \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathcal{U}_i, \boldsymbol{F}) \xrightarrow{\sim} \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathcal{U}_i, \mathbf{cosk}_n \boldsymbol{F}).$$

Now, using the skeleton–coskeleton adjunction and then that we can change the indexing set of hypercovers to also be n–skeletal,

$$\boldsymbol{F}^{\dagger}(X) \xrightarrow{\sim} \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathbf{sk}_n \mathcal{U}_i, \boldsymbol{F}) = \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S_{\le n}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathbf{sk}_n \mathcal{U}_i, \boldsymbol{F}).$$

Now, since Čech covers are cofinal in bounded hypercovers on a paracompact manifold [Schreiber 2013, Proposition 3.6.63], denoting by $\check{S}$ the category of Čech covers,

$$\begin{aligned}\operatorname*{hocolim}_{(\mathcal{U}\to X)\in S_{\le n}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathbf{sk}_n \mathcal{U}_i, \boldsymbol{F}) &\xleftarrow{\sim} \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in \check{S}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathbf{sk}_n \check{N}U_i, \boldsymbol{F}) \\ &= \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in \check{S}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\check{N}U_i, \mathbf{cosk}_n \boldsymbol{F}) \xleftarrow{\sim} \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in \check{S}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\check{N}U_i, \boldsymbol{F}).\end{aligned}$$

Next we apply a simplicial Yoneda lemma and then use the fact that Tot computes holim when the cosimplicial simplicial set is Reedy fibrant [Hirschhorn 2003, Theorem 18.7.4] to obtain

$$\operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in\check{S}} \operatorname*{holim}_{i\in\mathbf{\Delta}} \underline{\mathrm{sPre}}(\check{N}U_i, \boldsymbol{F}) = \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in\check{S}} \operatorname*{holim}_{i} \prod_{\alpha_0,\dots,\alpha_i} \boldsymbol{F}(U_{\alpha_0,\dots,\alpha_i}) \xrightarrow{\sim} \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in\check{S}} \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)),$$

and finally we use the fact that the colimit over Čech covers is a filtered colimit to compute hocolim with a colim to obtain

$$\operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in\check{S}} \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)) \xrightarrow{\sim} \operatorname*{colim}_{(\check{N}U_\bullet\to X)\in\check{S}} \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)) = \boldsymbol{F}^{\check{\dagger}}(X).$$

By Proposition 4.2, $\boldsymbol{F}^{\check{\dagger}}$ is already globally projectively fibrant (ie takes values in Kan complexes). Now it remains to show that $\boldsymbol{F}^{\check{\dagger}}$ satisfies hyperdescent. Given a hypercover, $\mathcal{U}\to X$, we use the commutative square

$$\begin{array}{ccc} \underline{\mathrm{sPre}}(X, \boldsymbol{F}^{\dagger}) = \boldsymbol{F}^{\dagger}(X) & \longrightarrow & \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}^{\dagger}) \\ \downarrow & & \downarrow \\ \underline{\mathrm{sPre}}(X, \boldsymbol{F}^{\check{\dagger}}) = \boldsymbol{F}^{\check{\dagger}}(X) & \longrightarrow & \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}^{\check{\dagger}}) \end{array} \tag{5-3}$$

where the equalities are given by Yoneda. Since $\boldsymbol{F}^{\dagger}$ satisfies descent, the top horizontal map is a weak equivalence by definition of descent. The left vertical map was proven to be an equivalence above. With $\mathcal{U}$ projectively cofibrant it follows that the simplicial mapping spaces preserve the weak equivalence $\boldsymbol{F}^{\dagger} \xrightarrow{\sim} \boldsymbol{F}^{\check{\dagger}}$ between projectively fibrant objects and so the right vertical map is a weak equivalence. Thus, by the two-out-of-three property afforded to our model category, we have shown that the bottom horizontal map is a weak equivalence. Since we have shown that $\boldsymbol{F}^{\check{\dagger}}$ is projectively fibrant, satisfies hyperdescent, and that $\boldsymbol{F} \xrightarrow{\sim} \boldsymbol{F}^{\check{\dagger}}$, then $\boldsymbol{F}^{\check{\dagger}}$ is a fibrant replacement of $\boldsymbol{F}$ in the local projective model structure. □

Lemma 5.3 *Let* $\mathrm{Ch}^{\le 0}(\mathcal{A})$ *be the* dg*–category of nonpositively graded chain complexes over some additive category* $\mathcal{A}$*, where the hom-complex* $\mathrm{Ch}^{\bullet}(E,E')$ *consists of chain maps and (higher) chain homotopies from* E *to* E'*, and let* $\mathcal{Q}\hookrightarrow\mathrm{Ch}^{\le 0}(\mathcal{A})$ *be a full subcategory which only considers complexes of height at most* m *for some fixed* $m\in\mathbb{N}$*. Then the simplicial set* $\mathrm{dg}\mathcal{N}(\mathcal{Q})\simeq \mathbf{cosk}_{m+1}\,\mathrm{dg}\mathcal{N}(\mathcal{Q})$ *is* $(m+1)$*–coskeletal.*

Proof For any two objects in $\mathcal{Q}$ and for an integer $k>m+1$, we have $\mathcal{Q}^k(E,E')=0$ due to the restricted height of all complexes in our dg-category. Thus the only way to decorate a k–simplex with $k>m+1$ is to have the boundary data all satisfy the condition $\hat{\delta}g+g\cdot g=0$ and then uniquely assign a 0–homotopy to the $(m+1)$–simplex. But recall that, whenever each decorated boundary simplex has a unique filler, this means the simplicial set is isomorphic to its coskeleton, so in our case we have $\mathrm{dg}\mathcal{N}(\mathcal{Q})\simeq\mathbf{cosk}_{m+1}\,\mathrm{dg}\mathcal{N}(\mathcal{Q})$, as required. □

Definition 5.4 Define $\mathrm{Perf}_{\leq n}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{dgCat}$ by setting $\mathrm{Perf}_{\leq n}(U)$ to be the dg-category of finite chain complexes of holomorphic vector bundles just as in Definition 2.1, but with the difference that we require the complexes to be trivial above level n. Analogously to **IVB** from Definition 2.11, we then define $\mathbf{IVB}_{\leq n}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by setting $\mathbf{IVB}_{\leq n}(U)_n := \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\mathcal{N}(\mathrm{Perf}_{\leq n}(U))^{\circ}\big)$.

Corollary 5.5 *The fibrant replacement of* $\mathbf{IVB}_{\leq n}$ *in the local projective model structure can be computed by its Čech sheafification*, $\mathbf{IVB}_{\leq n} \xrightarrow{\sim} \mathbf{IVB}^{\check{\dagger}}_{\leq n}$.

Proof By construction, $\mathbf{IVB}_{\leq n}$ is still (globally) projectively fibrant, while combining Lemma 5.3 and Proposition A.1 gives us that $\mathbf{IVB}_{\leq n}$ is (globally) a homotopy-$(n+1)$ type. □

Lemma 5.6 *Let* $Ch^{\leq 0}(\mathcal{A})$ *be the* dg*–category of nonpositively graded chain complexes over some additive category* $\mathcal{A}$, *where the hom-complex* $Ch^{\bullet}(E, E')$ *consists of chain maps and* (*higher*) *chain homotopies from* E *to* E', *and let* $\mathcal{Q} \hookrightarrow Ch^{\leq 0}(\mathcal{A})$ *be a full subcategory which only considers complexes with homology concentrated in degree zero. Then the* (*Kan replacement of the*) *simplicial set* $\mathrm{dg}\mathcal{N}(Q)$ *is a* 1*–type.*

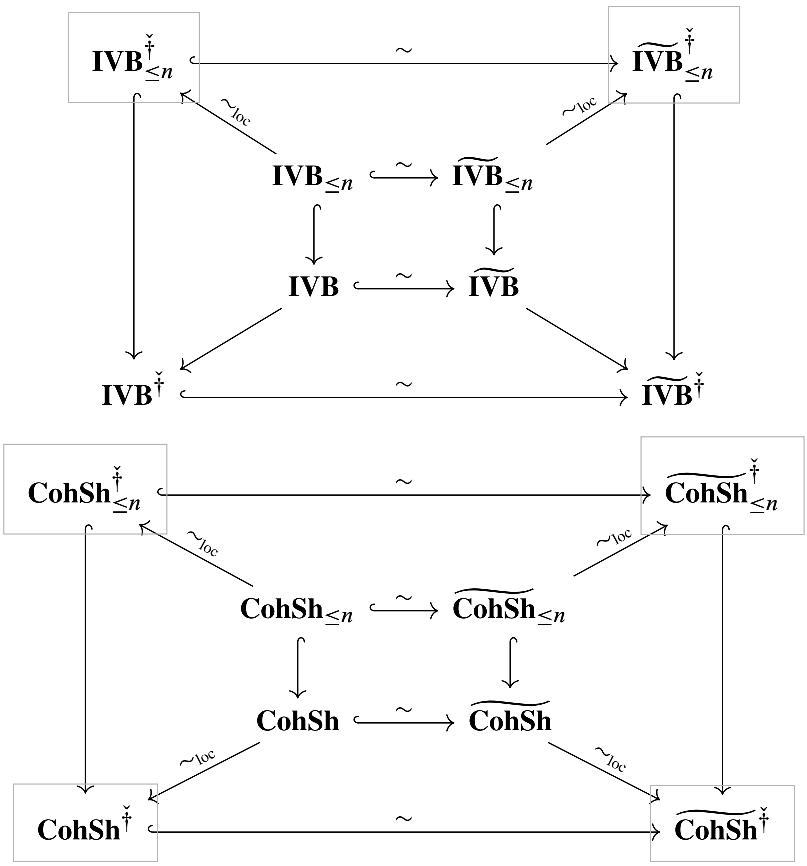
Proof If necessary, first replace $\mathrm{dg}\mathcal{N}(Q)$ with its maximal Kan subcomplex which only uses quasi-isomorphisms on edges. We will prove that $\pi_n(\mathrm{dg}\mathcal{N}(Q))$ is trivial for $n \geq 2$. A class in π_n consists of an n–simplex in $\mathrm{dg}\mathcal{N}(Q)$ whose entire boundary is in the image of a single vertex. Thus the vertices are given by the same chain complex, $E_0 = E, \dots, E_n = E$, the quasi-isomorphisms on the edges are the identity maps, and any homotopy decorating a $k < n$ face is the zero homotopy. By the definition of $\mathrm{dg}\mathcal{N}(Q)$, this data satisfies the condition $\hat{\delta}(g) + Dg + g \cdot g = 0$ using the notation of Definition 3.3. Since in this case $\hat{\delta}(g) + g \cdot g$ is an alternating sum of compositions of 0–homotopies and/or identity maps, one can show that the above condition reduces to $Dg = 0$. However, since E is a complex whose homology is concentrated in degree zero and $g \in Q^{1-n}(E, E)$ with $n \geq 2$, g is exact. From here we can fill this n–sphere with a higher homotopy and kill the class representing g in π_n. □

By a similar argument for Corollary 5.5 we can use the above lemma to see that **CohSh** is a 1–type and thus $\mathbf{CohSh}^{\check{\dagger}}$ is a sheaf, but without needing to further restrict the height of any chain complexes.

Corollary 5.7 *The simplicial presheaf* **CohSh** *is a* 1*–type and its fibrant replacement in the local projective model structure can be computed by its Čech sheafification*, $\mathbf{CohSh} \xrightarrow{\sim} \mathbf{CohSh}^{\check{\dagger}}$.

Remark 5.8 Now that under the right circumstances the Čech sheafification can act as a fibrant replacement functor, we can briefly present a different argument for Lemma 4.8 which makes use of equivalences being preserved under the various constructions we use to pass from the dgCat–valued presheaf Perf^{∇} to the simplicial presheaf $\mathbf{IVB}^{\check{\dagger}}$. The main idea used in the proof for Lemma 4.8 is that for a complex manifold X, and a point $x \in X$, there exists an (Stein) open subset $x \in U \subset X$ on which we have an equivalence of dg-categories, $\mathrm{Perf}^\nabla(U) \xrightarrow{\sim} \widetilde{\mathrm{Perf}}^\nabla(U)$, where the tilde again means we forget connection data. Since the dg-nerve construction preserves (weak) equivalences, we then obtain an equivalence of simplicial sets, $\mathbf{IVB}(U) \xrightarrow{\sim} \widetilde{\mathbf{IVB}}(U)$. We claim this then says that we have a weak equivalence for each stalk $\mathbf{IVB}_x \xrightarrow{\sim} \widetilde{\mathbf{IVB}}_x$ and thus a local weak equivalence of simplicial presheaves à la Jardine, $\mathbf{IVB} \xrightarrow{\sim} \widetilde{\mathbf{IVB}}$. The local weak equivalences for the local projective model structure happen to coincide with those of Jardine and thus we obtain a weak equivalence in the local projective model structure which is necessarily preserved under our (Čech) fibrant replacement functor if we restrict appropriately: $\mathbf{IVB}^{\check{\dagger}}_{\le n} \xrightarrow{\sim} \widetilde{\mathbf{IVB}}^{\check{\dagger}}_{\le n}$.

Remark 5.9 At this point, we'd like to take stock and summarize the relationships amongst some of the different constructions involving **IVB**. By the functoriality of our constructions, we obtain two commutative cubes of simplicial presheaves which actually fit together to form a commutative hypercube via the inclusion $\mathbf{CohSh} \hookrightarrow \mathbf{IVB}$:



where the hypersheaves are highlighted with boxes; we used $\sim$ to denote a global projective (ie objectwise) weak equivalence and $\sim_{\mathrm{loc}}$ to denote a local projective weak equivalence. Recall that the global weak equivalences are preserved in the local model structure.

Recall that in Proposition 4.13 we showed that $\check{\mathbf{CohSh}}^{\dagger}$ stands a chance of classifying coherent sheaves since the correspondence is bijective on connected components. We know, however, that $\mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X^{\bullet})$ is a 1–type and so, if we knew that $\check{\mathbf{CohSh}}^{\dagger}$ was also a 1–type, then it would only remain to prove the correspondence on π_1.

Lemma 5.10 *Given* $\boldsymbol{F} \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})$ *which is objectwise an* n*–type (ie* $\boldsymbol{F} \xrightarrow{\sim} \mathbf{cosk}_n\, F$ *for some* n*),* $\check{\boldsymbol{F}}^{\dagger}$ *is again an* n*–type.*

Proof We begin by noting that, if $\boldsymbol{F} \xrightarrow{\sim} \mathbf{cosk}_n\, \boldsymbol{F}$, then

$$\check{\boldsymbol{F}}^{\dagger}(X) = \operatorname*{colim}_{(U_\bullet\to X)\in\check{S}} \underline{\mathrm{sPre}}(\check{N}U_\bullet, \boldsymbol{F}) \xrightarrow{\sim} \operatorname*{colim}_{(U_\bullet\to X)\in\check{S}} \underline{\mathrm{sPre}}(\check{N}U_\bullet, \mathbf{cosk}_n\, \boldsymbol{F})$$
$$\xrightarrow{\sim} \operatorname*{colim}_{(U_\bullet\to X)\in\check{S}} \mathrm{Tot}(\mathbf{cosk}_n\, \boldsymbol{F}(\check{N}U_\bullet)) \xrightarrow{\sim} \operatorname*{colim}_{(U_\bullet\to X)\in\check{S}} \mathbf{cosk}_n\, \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)),$$

where we used that Tot computes the homotopy limit in this case and then we commuted the right adjoint $\mathbf{cosk}_n$ across this concrete limit, and now again using that Tot computes the holim,

$$\operatorname*{colim}_{(U_\bullet\to X)\in\check{S}} \mathbf{cosk}_n\, \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)) \xleftarrow{\sim} \operatorname*{colim}_{(U_\bullet\to X)\in\check{S}} \mathbf{cosk}_n\, \underline{\mathrm{sPre}}(\check{N}U_\bullet, \boldsymbol{F}).$$

While we would love to commute this coskeleton across the colimit, we must proceed differently. Recall that filtered colimits commute with finite limits, and, since each homotopy group can be written as a finite limit, we have, for $m > n$,

$$\pi_m(\check{\boldsymbol{F}}^{\dagger}(X)) \simeq \pi_m\Big(\operatorname*{colim}_{(U_\bullet\to X)\in\check{S}} \mathbf{cosk}_n\, \underline{\mathrm{sPre}}(\check{N}U_\bullet, \boldsymbol{F})\Big)$$
$$\simeq \operatorname*{colim}_{(U_\bullet\to X)\in\check{S}} \pi_m(\mathbf{cosk}_n\, \underline{\mathrm{sPre}}(\check{N}U_\bullet, \boldsymbol{F})) = \operatorname*{colim}_{(U_\bullet\to X)\in\check{S}} 0 = 0. \qquad \square$$

Theorem 5.11 *The simplicial presheaf* **CohSh** *is a classifying prestack for coherent sheaves.*

Proof Recall from [Hirschhorn 2003, Section 17] that the derived mapping space $\mathbb{R}\mathrm{Hom}(A, B)$ in a simplicial model category $\mathcal{C}$ can be computed by considering the simplicial mapping space $\underline{\mathcal{C}}(\widetilde{A}, B')$, where we use the cofibrant replacement $\widetilde{A} \xrightarrow{\sim}$ of A and the fibrant replacement $B \xrightarrow{\sim} B'$ of B. Then, since Corollary 5.7 tells us that **CohSh** is a 1–type whose (local projective) fibrant replacement is given by its Čech sheafification, we can compute the (local projective) derived mapping space from a manifold $X \in \mathbb{C}\mathrm{Man}$ (via its cofibrant representable presheaf) into **CohSh** as

$$\mathbb{R}\mathrm{Hom}(X, \mathbf{CohSh}) := \underline{\mathrm{sPre}}(\widetilde{X}, \mathbf{CohSh}') \simeq \underline{\mathrm{sPre}}(X, \check{\mathbf{CohSh}}^{\dagger}) = \check{\mathbf{CohSh}}^{\dagger}(X).$$

After combining Proposition 4.13 and Lemma 5.10, it remains to be shown that the map $\mathcal{H}\colon \check{\mathbf{CohSh}}^{\dagger}(X) \to \mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X)$ is an isomorphism of fundamental groups. The ideas used to prove this fact are analogous to those of Proposition 4.13 but we will summarize them here for ease of reading. Given a vertex $\mathcal{E} = (U_\bullet, E_\bullet, g_\bullet) \in \check{\mathbf{CohSh}}^{\dagger}(X)_0$ and the coherent sheaf $\mathcal{F} := \mathcal{H}(\mathcal{E}) \in \mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X)_0$, we want to prove that there is an isomorphisms of based homotopy groups, $\pi_1(\check{\mathbf{CohSh}}^{\dagger}(X), \mathcal{E}) \xrightarrow{\pi_1(\mathcal{H})} \pi_1(\mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X), \mathcal{F})$.

To prove injectivity, if two loops in $\mathbf{CohSh}^{\check{\dagger}}(X)_1$, $a_\bullet, b_\bullet\colon \mathcal{E}\to\mathcal{E}$ have connected images in $\mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X)$, then by definition of the nerve of a groupoid, we have a commutative square of isomorphisms in $\mathrm{CohSh}\,\mathcal{O}_X$ where all four corners are the coherent sheaf $\mathcal{F}$. Lifting this commutative square to a homotopy in $\mathbf{CohSh}^{\check{\dagger}}(X)_1$ once again uses the fact that chain maps which induce the same map on homology are homotopic [Hilton and Stammbach 1971, Theorem 4.1] (and then the discussion of O'Brian, Toledo and Tong [1981c, near Lemma 1.6]). To prove surjectivity, a loop $f\colon \mathcal{F}=\mathcal{H}(\mathcal{E})\to\mathcal{F}=\mathcal{H}(\mathcal{E})$ in $\mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X)_1$ is lifted to a loop in $\mathbf{CohSh}^{\check{\dagger}}(X)$ on $\mathcal{E}$ by using the fact that an isomorphism on homology lifts to a quasi-isomorphism of chain complexes [Hilton and Stammbach 1971, Theorem 4.1] (and then, again, the discussion of O'Brian, Toledo and Tong [1981c, near Lemma 1.6]). □

If we knew that $\mathbf{\Omega}$ somehow used complexes of bounded height, then our Čech sheafified Chern map from Definition 4.3 could be seen to restrict to a map of sheaves $\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}}_{\leq n}\to\mathbf{\Omega}^{\check{\dagger}}$ out of infinity vector bundles of bounded complex height. One way to resolve this is by restricting our site as recorded below:

Proposition 5.12 *On the site $\mathbb{C}\mathrm{Man}_{\leq n}$ of complex manifolds of dimension at most n, the Čech sheafification of the restricted Chern map,*

$$\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}}_{\leq n}\to\mathbf{\Omega}^{\check{\dagger}},$$

is a map of hypersheaves.

Proof By Corollary 5.5, $\mathbf{IVB}^{\check{\dagger}}_{\leq n}$ is already a sheaf. Now that we have restricted the site to $\mathbb{C}\mathrm{Man}_{\leq n}$, $\mathbf{\Omega}$ only makes use of chain complexes of length at most n and so it is coskeletal and, by Proposition 5.2, its sheafification is a hypersheaf. □

By different application of the same ideas above, we end with an upgrade on Theorem 4.18:

Theorem 5.13 *On the site $\mathbb{C}\mathrm{Man}_{\leq n}$ of complex manifolds of dimension at most n, the Čech sheafification of the Chern map restricted to coherent sheaves,*

$$\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{CohSh}^{\check{\dagger}}\to\mathbf{\Omega}^{\check{\dagger}},$$

is a map of hypersheaves which restricts on π_0 to the Chern character (4-15) from O'Brian, Toledo and Tong [1981c].

Proof By Theorem 5.11, $\mathbf{CohSh}^{\check{\dagger}}$ is already a sheaf. Now that we have restricted the site to $\mathbb{C}\mathrm{Man}_{\leq n}$, $\mathbf{\Omega}$ only makes use of chain complexes of length at most n and so it is coskeletal and, by Proposition 5.2, its sheafification is a hypersheaf. The fact that on π_0 it recovers the Chern map from O'Brian, Toledo and Tong [1981c] was already recorded in Theorem 4.18. □

Remark 5.14 For an arbitrary stack (ie hypersheaf) $\boldsymbol{F}$, recall as in (5-1) that the right derived mapping space

$$\mathbb{R}\mathrm{Hom}(\boldsymbol{F},\boldsymbol{G}) := \underline{\mathrm{sPre}}(\widetilde{\boldsymbol{F}},\widehat{\boldsymbol{G}})$$

for a simplicial model category can be computed by taking the simplicial mapping space between a cofibrant replacement of $\boldsymbol{F}$ and a fibrant replacement of $\boldsymbol{G}$. Letting $\boldsymbol{G}_1 = \mathbf{IVB}$ and $\boldsymbol{G}_2 = \boldsymbol{\Omega}$, Proposition 5.12 says that our presheafified Chern map $\mathbf{Ch}\colon \mathbf{IVB} \to \boldsymbol{\Omega}$ from Definition 3.13 induces a map of fibrant (ignoring the restrictions of sites and homotopy types for the moment) replacements $\check{\mathbf{Ch}}^{\dagger}\colon \check{\mathbf{IVB}}^{\dagger} \to \check{\boldsymbol{\Omega}}^{\dagger}$, and thus a map of right derived mapping spaces:

$$\mathbb{R}\mathrm{Hom}(\boldsymbol{F}, \mathbf{IVB}) = \underline{\mathrm{sPre}}(\tilde{\boldsymbol{F}}, \check{\mathbf{IVB}}^{\dagger}) \xrightarrow{\check{\mathbf{Ch}}^{\dagger}} \underline{\mathrm{sPre}}(\tilde{\boldsymbol{F}}, \check{\boldsymbol{\Omega}}^{\dagger}) =: \mathbb{R}\mathrm{Hom}(\boldsymbol{F}, \boldsymbol{\Omega}). \tag{5-4}$$

When $\boldsymbol{F} = X$ is the representable simplicial presheaf for a complex manifold, the above is explicitly calculated using Note 4.16. However, (5-4) suggests a reasonable definition for a *generalized Chern character map*. In a sequel to this paper, we will study this map for the case when a Lie group G acts on the complex manifold X and $\boldsymbol{F}_n = X \times G^{\times n}$ (see our previous paper [2022, Definition 5.1]), extending this paper to the equivariant setting.

Appendix A A weak equivalence $\mathrm{sSet}(\hat{\Delta}^\bullet, K) \to \mathrm{sSet}(\Delta^\bullet, K)$

In this appendix, we prove Proposition A.1:

Proposition A.1 *If K is a Kan complex, then there exists a weak equivalence $F^\sharp\colon \mathrm{sSet}(\hat{\Delta}^\bullet, K) \to \mathrm{sSet}(\Delta^\bullet, K)$.*

In order to define $F^\sharp$, we first establish some notation. Recall from Example 2.6 that Δ^n is the simplicial set whose k–simplices are nondecreasing sequences $(i_0 \le \cdots \le i_k)$ with $i_0, \dots, i_k \in \{0, \dots, n\}$, and recall from Example 2.8 that $\hat{\Delta}^n$ is the simplicial set whose k–simplices are any sequences $(i_0, \dots, i_k)$ with $i_0, \dots, i_k \in \{0, \dots, n\}$. Both Δ^n and $\hat{\Delta}^n$ have face maps d_j given by removing the j^{th} index i_j, and degeneracy maps s_j given by repeating the j^{th} index i_j. Furthermore both $\Delta^\bullet$ and $\hat{\Delta}^\bullet$ are cosimplicial simplicial sets, so that for $\phi\colon [n] \to [m]$ in $\boldsymbol{\Delta}$ we get an induced map of $\phi_\bullet\colon \tilde{\Delta}^n_\bullet \to \tilde{\Delta}^m_\bullet$ via $\phi_k\colon \tilde{\Delta}^n_k \to \tilde{\Delta}^m_k$, $\phi_k(i_0, \dots, i_k) = (\phi(i_0), \dots, \phi(i_k))$, where $\tilde{\Delta}^\bullet$ is either $\Delta^\bullet$ or $\hat{\Delta}^\bullet$. Thus, there is an induced map of cosimplicial simplicial sets $F^\bullet\colon \Delta^\bullet \to \hat{\Delta}^\bullet$, $(i_0 \le \cdots \le i_k) \mapsto (i_0, \dots, i_k)$. For any simplicial set X, both $X = \mathrm{sSet}(\Delta^\bullet, X)$ and $\hat{X} := \mathrm{sSet}(\hat{\Delta}^\bullet, X)$ are simplicial sets, and there is an induced map $F^\sharp\colon \hat{X} \to X$ by precomposition with F.

Our first step towards proving Proposition A.1 is to show that $\hat{K}$ is also a Kan complex:

Proposition A.2 *If K is a Kan complex, then $\hat{K}$ is a Kan complex.*

To begin with, here is a useful lemma:

Lemma A.3 *A map $c\colon \Delta^n \to \hat{K} = \mathrm{sSet}(\hat{\Delta}^\bullet, K)$ is determined by the element $\bar{c} = c(0 \le \cdots \le n)\colon \hat{\Delta}^n \to K$. Then $\delta_i(c) = c \circ \delta_i\colon \Delta^{n-1} \cong \delta_i(\Delta^{n-1}) \subset \Delta^n \xrightarrow{c} \hat{K}$ is determined by $\delta_i(\bar{c}) = c \circ \delta_i\colon \hat{\Delta}^{n-1} \cong \delta_i(\hat{\Delta}^{n-1}) \subset \hat{\Delta}^n \xrightarrow{c} K$.*

Proof Note that $\delta_i(\Delta^{n-1}) \subset \Delta^n$ are sequences that do not include i, which are generated by the $(n-1)$–simplex $(0 \le \cdots \le i-1 \le i+1 \le \cdots n) = d_i(0 \le \cdots \le n) \in \Delta^n_{n-1}$. Thus $\delta_i(c)$ is determined by the image of the simplex $d_i(0 \le \cdots \le n)$. Now $c(d_i(0 \le \cdots \le n)) = d_i(c(0 \le \cdots \le n)) = d_i(\bar{c}) = c \circ \delta_i$. □

Proof of Proposition A.2 Denote by $\Lambda^n_i := \bigcup_{j \neq i} \delta_j \Delta^{n-1}$ the i^{th} horn of Δ^n, which is a subsimplicial set of Δ^n. Similarly, denote by $\widehat{\Lambda}^n_i := \bigcup_{j \neq i} \delta_j \widehat{\Delta}^{n-1}$ the i^{th} horn of $\widehat{\Delta}^n$, which is a subsimplicial set of $\widehat{\Delta}^n$. As noted before, a simplicial set map $\Delta^n \to \widehat{K}$ is the same as an element $\widehat{K}_n$, ie a simplicial set map $\widehat{\Delta}^n \to K$. Similarly, a simplicial set map $\Lambda^n_i \to \widehat{K}$ is given by n maps $\delta_j \Delta^{n-1} \to \widehat{K}$, ie n maps $\widehat{\Delta}^{n-1} \to K$ (see Lemma A.3), which are compatible at their common boundary, ie whose induced common boundary maps $\widehat{\Delta}^{n-2} \to K$ coincide, and thus this is the same as a simplicial set map $\widehat{\Lambda}^n_i \to K$. Thus, the Kan condition for $\widehat{K}$ (left side of (A-1)) becomes equivalent to lifting a horn $\widehat{\Lambda}^n_i \to X$ to a map $\widehat{\Delta}^n \to X$ (right side of (A-1)):

(A-1)
$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \widehat{K} \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & * \end{array} \quad\Longleftrightarrow\quad \begin{array}{ccc} \widehat{\Lambda}^n_i & \longrightarrow & K \\ \big\downarrow & \nearrow & \big\downarrow \\ \widehat{\Delta}^n & \longrightarrow & * \end{array}$$

Since K is a Kan complex, we have such a lift if $\widehat{\Lambda}^n_i \to \widehat{\Delta}^n$ is an trivial cofibration, ie if this map is injective and a weak equivalence. Clearly, $\widehat{\Lambda}^n_i \to \widehat{\Delta}^n$ is injective, and the weak equivalence follows since both $\widehat{\Lambda}^n_i$ and $\widehat{\Delta}^n$ are contractible, ie they have zero homotopy groups. First, it is well known that EG for any group G is contractible, since it has an extra degeneracy $s_{-1}(g_0, \dots, g_k) = (e, g_0, \dots, g_k)$; see for example [Goerss and Jardine 1999, Lemma III.5.1 and Example III.5.2]. Thus, $\widehat{\Delta}^n = E\mathbb{Z}_{n+1}$ is contractible, and, from the explicit extra degeneracy, we can see that it preserves $\widehat{\Lambda}^n_0$. Thus, $\widehat{\Lambda}^n_0$ is contractible as well. Now, there is a $\mathbb{Z}_{n+1}$–action on $E\mathbb{Z}_{n+1}$, which, in particular, can be used to map $\widehat{\Lambda}^n_0$ isomorphically to any other $\widehat{\Lambda}^n_i$, showing that indeed all $\widehat{\Lambda}^n_i$ are contractible. (Or, alternatively, one obtains that the extra degeneracy $s_{-1}(i_0, \dots, i_k) = (i, i_0, \dots, i_k)$ of $\widehat{\Delta}^n$ preserves $\widehat{\Lambda}^n_i$.) □

In order to prove Proposition A.1, we need one more ingredient. Denote by $\widehat{\Theta}^n := \bigl(\bigcup_{\text{all } j} \delta_j \widehat{\Delta}^{n-1}\bigr) \cup \Delta^n$ the subsimplicial set of $\widehat{\Delta}^n$ generated by *all* $\widehat{\Delta}^{n-1}$ boundary components, together with $\Delta^n \cong F^n(\Delta^n) \subset \widehat{\Delta}^n$.

Lemma A.4 *$\widehat{\Theta}^n$ is contractible.*

Proof For a subset $A \subset \{0, \dots, n\}$, denote by $\widehat{\Upsilon}^n_A := \bigl(\bigcup_{j \in A} \delta_j \widehat{\Delta}^{n-1}\bigr) \cup \Delta^n$ the subsimplicial set $\widehat{\Upsilon}^n_A \subset \widehat{\Delta}^n$, given by Δ^n with "thickened" boundary components determined by A. In particular, $\widehat{\Upsilon}^n_{\{\}} = \Delta^n$ and $\widehat{\Upsilon}^n_{\{0,\dots,n\}} = \widehat{\Theta}^n$. (Note that $\widehat{\Upsilon}^n_A$ may be explicitly described to have p–simplices given by sequences $(i_0, \dots, i_p) \in \{0, \dots, n\}^p$ such that either $i_0 \le \cdots \le i_p$, or there exists an element $i \in A$ such that $i_0 \neq i, \dots, i_p \neq i$, or both.) We show that the $|\widehat{\Upsilon}^n_A|$ are contractible for all n and A. Since all $|\widehat{\Upsilon}^n_A|$ are CW–complexes, this is equivalent to showing that the $|\widehat{\Upsilon}^n_A|$ are connected and have zero homotopy groups.

We will repeatedly use the fact that, if X, Y, $X \cap Y$ and $X \cup Y$ are CW–complexes, and X, Y and $X \cap Y$ are contractible, then $X \cup Y$ is also contractible (which follows since $X \cup Y$ is certainly connected, has vanishing π_1 due to van Kampen, vanishing homology groups due to Mayer–Vietoris, and thus vanishing homotopy groups due to Hurewicz).

When $n = 1$, using that $\widehat{\Delta}^0 = \Delta^0$, we have for any $A \subset \{0,1\}$ that $\widehat{\Upsilon}^1_A = \Delta^1$, and $|\Delta^1|$ is contractible.

Now, for $n > 1$, assume by induction, that the $|\widehat{\Upsilon}^k_B|$ are contractible for all $k < n$ and all $B \subset \{0,\dots,k\}$. We perform a second induction on the number of elements of $A \subset \{0,\dots,n\}$. First, note that $\widehat{\Upsilon}^n_{\{\}} = \Delta^n$, and $|\Delta^n|$ is contractible. Thus, assume by induction that all $|\widehat{\Upsilon}^n_A|$ with $|A| < l$ are contractible. Now, let $A = \{i_1,\dots,i_l\} \subset \{0,\dots,n\}$ be an l–element set with, say, $i_1 < \dots < i_l$. Writing

$$\widehat{\Upsilon}^n_{\{i_1,\dots,i_l\}} = \widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}} \cup \delta_{i_l} \widehat{\Delta}^{n-1},$$

we know by induction that $|\widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}}|$ is contractible, and also $|\delta_{i_l} \widehat{\Delta}^{n-1}| \approx |\widehat{\Delta}^{n-1}|$ is contractible (which was reviewed in the proof of Proposition A.2). Furthermore, $\widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}} \cap \delta_{i_l} \widehat{\Delta}^{n-1} = \delta_{i_l} \widehat{\Upsilon}^{n-1}_{\{i_1,\dots,i_{l-1}\}} \cong \widehat{\Upsilon}^{n-1}_{\{i_1,\dots,i_{l-1}\}}$, and, by the first induction, $|\widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}}| \cap |\delta_{i_l} \widehat{\Delta}^{n-1}| = |\widehat{\Upsilon}^{n-1}_{\{i_1,\dots,i_{l-1}\}}|$ is contractible as well. Thus, by the above fact, $|\widehat{\Upsilon}^n_{\{i_1,\dots,i_l\}}| = |\widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}}| \cup |\delta_{i_l} \widehat{\Delta}^{n-1}|$ is also contractible. □

We are now ready to prove Proposition A.1.

Proof of Proposition A.1 Since both K and $\widehat{K}$ are Kan complexes, it suffices to show that $F^\sharp \colon \widehat{K} \to K$ induces isomorphisms on all simplicial homotopy groups (since these coincide with the homotopy groups of their geometric realizations; see [May 1967, Theorems 16.1 and 16.6]).

First, for $n = 0$, F induces a map $\pi_0(\widehat{K}) \to \pi_0(K)$ which is onto since $\widehat{\Delta}^0 = \Delta^0$ and thus $\widehat{K}_0 = K_0$. To see that the induced map $\pi_0(\widehat{K}) \to \pi_0(K)$ is one-to-one, assume $a, b \in K_0$ are equivalent $a \sim b$ in $\pi_0(K)$. Since K is a Kan complex, this means that (instead of a sequence of 1–simplices) there exists a single $c \in K_1$ such that $d_0(c) = a$ and $d_1(c) = b$. We need to check that $a \sim b$ in $\pi_0(\widehat{K})$, ie there exists a $\widehat{c} \in \widehat{K}_1$ with $d_0(\widehat{c}) = a$ and $d_1(\widehat{c}) = b$. Thus we need a simplicial set map $\Delta^1 \to \widehat{K}$, ie a map $\widehat{\Delta}^1 \to K$ making the following diagram commute:

$$\begin{array}{ccc} \widehat{\Theta}^1 = \Delta^1 \cup \delta_0 \widehat{\Delta}^0 \cup \delta_1 \widehat{\Delta}^0 & \xrightarrow{c \cup a \cup b} & K \\ \downarrow & \nearrow & \downarrow \\ \widehat{\Delta}^1 & \longrightarrow & * \end{array}$$

Note that the top arrow is well defined, and, since the left map is a trivial cofibration (ie injective and a weak equivalence) and K is a Kan complex, it follows that it lifts to a map $\widehat{\Delta}^1 \to K$, as needed.

Now, for $n \geq 1$, F induces a map $\pi_n(\widehat{K}, *) \to \pi_n(K, *)$ which is onto: if $c \in K_n$ with $d_i(c) = *$ for all i, represents an element of $\pi_n(K, *)$, then we want to produce a $\widehat{c} \in \widehat{K}_n$, ie $\widehat{c} \colon \widehat{\Delta}^n \to K$, with $d_i(\widehat{c}) = *$ for all i and which restricts to c under F. Thus, we need to find a lift making the following diagram commute:

$$\begin{array}{ccc} \widehat{\Theta}^n = \Delta^n \cup \delta_0 \widehat{\Delta}^{n-1} \cup \dots \cup \delta_n \widehat{\Delta}^{n-1} & \xrightarrow{c \cup * \cup \cdots \cup *} & K \\ \downarrow & \nearrow & \downarrow \\ \widehat{\Delta}^n & \longrightarrow & * \end{array}$$

Again, the top arrow is well defined, since c restricts trivially to its boundaries. Just as before, we can find a lift, because $\widehat{\Theta}^n \to \widehat{\Delta}^n$ is a trivial cofibration and K is a Kan complex. Finally, we need to check that F induces a map $\pi_n(\widehat{K}, *) \to \pi_n(K, *)$, which is one-to-one. Since this map is a map of groups, it suffices to check that the kernel is trivial. More explicitly, we need to show that if $\widehat{c} \in K_n$ with $d_i(\widehat{c}) = *$ for all i represents a class of $\pi_n(\widehat{K}, *)$, which maps to $c = \widehat{c} \circ F^n : \Delta^n \xrightarrow{F^n} \widehat{\Delta}^n \xrightarrow{\widehat{c}} K$ which is trivial in $\pi_n(K, *)$, then $\widehat{c}$ is trivial in $\pi_n(\widehat{K}, *)$. For c to be trivial in $\pi_n(K, *)$ means that there is an $(n+1)$–simplex $q \in K_{n+1}$ such that $d_0(q) = c$ and $d_i(q) = *$ for all $i \geq 1$. We thus have the setup for the diagram

$$\begin{array}{ccc} \widehat{\Theta}^{n+1} = \Delta^{n+1} \cup \delta_0 \widehat{\Delta}^n \cup \delta_1 \widehat{\Delta}^n \cup \dots \cup \delta_n \widehat{\Delta}^n & \xrightarrow{q \cup \widehat{c} \cup * \cup \cdots \cup *} & K \\ \downarrow & \nearrow & \downarrow \\ \widehat{\Delta}^{n+1} & \longrightarrow & * \end{array}$$

Since $\widehat{\Theta}^{n+1} \to \widehat{\Delta}^{n+1}$ is a trivial cofibration and K is a Kan complex, there exists a lift $\widehat{q} \in \widehat{K}_{n+1}$ with $d_0(\widehat{q}) = \widehat{c}$ and $d_i(\widehat{q}) = *$ for all $i \geq 1$. This shows that $\widehat{c}$ does indeed represent the trivial class in $\pi_n(\widehat{K}, *)$. □

Appendix B Explicit description of totalization

We now review the notion of totalization of a cosimplicial simplicial set.

B.1 Totalization

We recall from our previous work [2022, Definition D.1] and [Hirschhorn 2003, Definition 18.6.3]] the definition of totalization. Let $K^\bullet : \mathbf{\Delta} \to \mathrm{sSet}$ be a cosimplicial simplicial set, ie $K^l := K([l])$ is a simplicial set $K^l = K^l_\bullet$. Then, the totalization $\mathrm{Tot}(K^\bullet_\bullet)$ of K is defined as the simplicial set, which is the equalizer of the maps

(B-1) $$\mathrm{Tot}(K^\bullet_\bullet) \to \prod_{[l] \in \mathbf{Obj}(\mathbf{\Delta})} (K^l)^{\Delta^l} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{\rho : [n] \to [m]} (K^m)^{\Delta^n}$$

Here, by definition, $(K^p)^{\Delta^q}$ is the simplicial set whose n–simplices are simplicial set maps $((K^p)^{\Delta^q})_n = \mathrm{sSet}((\Delta^n \times \Delta^q)_\bullet, K^p_\bullet)$. Then a k–simplex in the totalization is given by some collection

(B-2) $$\{x^{(k,l)}\}_{l \geq 0}, \quad \text{where } x^{(k,l)} \in \mathrm{sSet}(\Delta^k \times \Delta^l, K^l),$$

satisfying the coherence condition that they are in the above equalizer. Explicitly, for a fixed $j = 0, \dots, l+1$ the map $\delta_j : [l] \to [l+1]$ which skips j induces the maps

$$x^{(k,l+1)} \in \mathrm{sSet}(\Delta^k \times \Delta^{l+1}, K^{l+1}) \xrightarrow{d_j} \mathrm{sSet}(\Delta^k \times \Delta^l, K^{l+1}), \tag{B-3}$$

$$x^{(k,l)} \in \quad \mathrm{sSet}(\Delta^k \times \Delta^l, K^l) \quad \xrightarrow{d^j} \mathrm{sSet}(\Delta^k \times \Delta^l, K^{l+1}). \tag{B-4}$$

Then, for $x^{(k,l+1)}$ and $x^{(k,l)}$ as above,

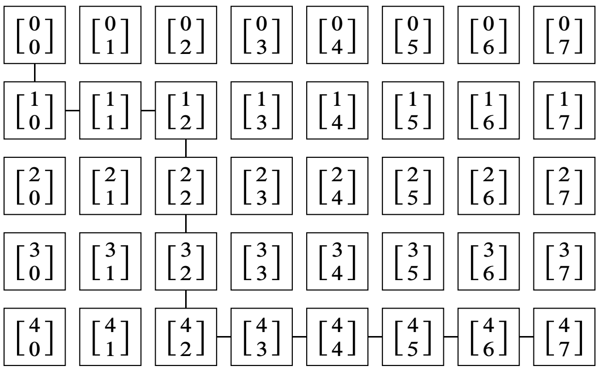
$$d_j(x^{(k,l+1)}) = d^j(x^{(k,l)}). \tag{B-5}$$

Thus, a k–simplex, $\{x^{(k,l)}\}_{l=0,1,\dots}$, in the totalization of a cosimplicial simplicial set, $\mathrm{Tot}(K^\bullet_\bullet)$ is given by maps $x^{(k,l)} \in \mathrm{sSet}((\Delta^k \times \Delta^l)_\bullet, K^l_\bullet)$ for each $l = 0, 1, \dots,$ which can be thought of as a coherent "decoration" of the simplicial sets $\Delta^k \times \Delta^l$, for $l = 0, 1, \dots,$ by simplices in $K^l_\bullet$.

B.2 Simplices of $\Delta^k \times \Delta^l$

We now recall that there is a nice book-keeping device for the simplices of $\Delta^k \times \Delta^l$. In fact, the p–simplices of $\Delta^k \times \Delta^l$ can be described by nondecreasing paths with $p+1$ vertices in a $(k+1) \times (l+1)$ grid; we also call this a p–path. For example, the maximally nondegenerate (4+7)–simplices of $\Delta^4 \times \Delta^7$ can be labeled by paths[4] through a $(4+1) \times (7+1)$ grid, necessarily starting from $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and ending at $\left[\begin{smallmatrix}4\\7\end{smallmatrix}\right]$. For example, the following path of labels, which we denote by $\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}0&1&1&1&2&3&4&4&4&4&4&4\\0&0&1&2&2&2&2&3&4&5&6&7\end{array}\right]$, labels an element of $(\Delta^4 \times \Delta^7)_{11}$:

(B-6)



We can apply $x^{(4,7)} \in \mathrm{sSet}(\Delta^4 \times \Delta^7, K^7)$ to this path, which will give an element

$$x^{(4,7)}_{\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}0&1&1&1&2&3&4&4&4&4&4&4\\0&0&1&2&2&2&2&3&4&5&6&7\end{array}\right]} \in K^7_{11}$$

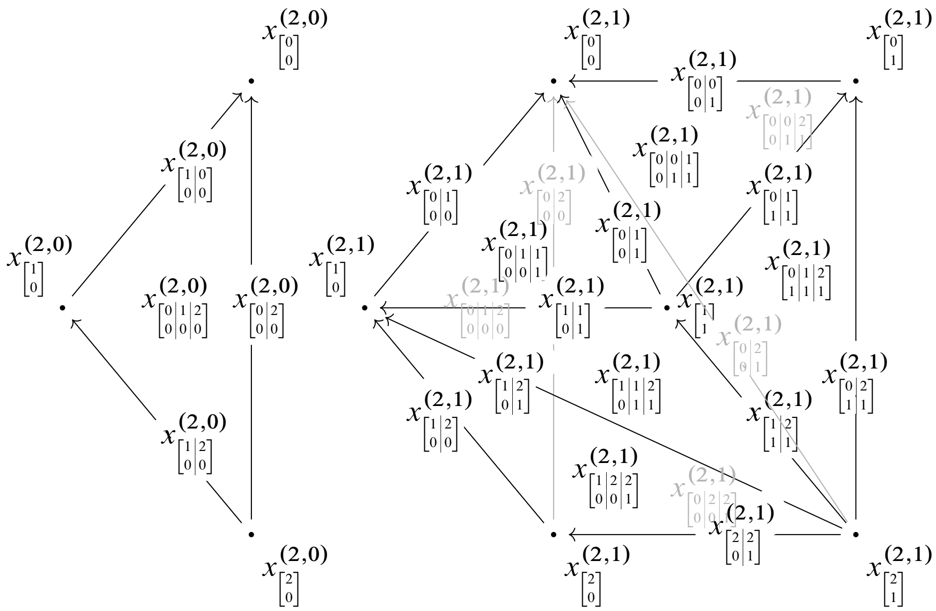
(note the simplicial degree 11 comes from the 11–path with 12 vertices). Note that, just as the simplices of the standard n–simplex have direction, these paths must be nondecreasing in both directions. Additionally, the faces of a p–simplex of $\Delta^k \times \Delta^l$ given by a path would consist of subsequences of that path, eg $\left[\begin{array}{c|c|c|c|c|c}1&1&2&4&4&4\\0&2&2&3&4&6\end{array}\right]$ describes a 5–simplex in $(\Delta^4 \times \Delta^7)_5$ which is a lower face of the above 11–simplex. Degenerate simplices are described by paths where at least one of the indices is repeated, eg $\left[\begin{array}{c|c|c|c|c}1&2&4&4&4\\2&2&3&3&6\end{array}\right]$.

[4]Informally, this path might be referred to as a "taxi-cab" path as it only moves in a rectangular fashion.

Using this notation, the coherence condition (B-5) can be stated more precisely as follows. Let $K_\bullet^\bullet$ be a cosimplicial simplicial set and let $\delta_j\colon [l] \to [l+1]$ be the map that skips j. We have the coface maps, $d^j\colon K_\bullet^l \to K_\bullet^{l+1}$, as well as the maps d_j in (B-3) given by precomposition with $\Delta_\bullet^l \to \Delta_\bullet^{l+1}$. Then we can explicitly describe the k–simplices of the totalization, $\mathrm{Tot}(K_\bullet^\bullet)_k$, as collections $\{x^{(k,l)} \in \mathrm{sSet}(\Delta^k \times \Delta^l, K_\bullet^l)\}_{l=0,1,\dots}$, which, applied to p–simplices of $\Delta^k \times \Delta^l$ labeled by the paths $\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]$ with $0 \le \alpha_0 \le \cdots \le \alpha_p \le k$ and $0 \le \beta_0 \le \cdots \le \beta_p \le l$ as described above, assign elements $x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]} \in K_p^l$, satisfying

$$x^{(k,l+1)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \delta_j(\beta_0) & \cdots & \delta_j(\beta_p)\end{smallmatrix}\right]} = d^j\big(x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]}\big) \in K_p^{l+1}. \tag{B-7}$$

For example, for $k=2$, we have the assignments, for $l=0,1$,



As an example, for $\delta_0\colon [0] \to [1]$, equation (B-7) yields $x^{(2,1)}_{\left[\begin{smallmatrix}0&1&2\\1&1&1\end{smallmatrix}\right]} = d^0\big(x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right]}\big)$, which relates the cells for different l's.

Note that, for a fixed k and l, the

$$x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]} \in K_p^l$$

are in fact determined by the maximal paths

$$x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_{k+l}\\ \beta_0 & \cdots & \beta_{k+l}\end{smallmatrix}\right]} \in K_{k+l}^l,$$

since each p–path is a subpath of a maximal path and so the p–cell is in the image of some face map $K_{k+l}^l \to K_p^l$ for some map $[p] \to [k+l]$.

Example B.1 For example, for a simplicial presheaf $\boldsymbol{F}:\mathbb{C}\mathrm{Man}^{\mathrm{op}}\to\mathrm{sSet}$, and an open cover $\mathcal{U}=\{U_i\}_{i\in\mathcal{I}}$ of $X\in\mathbb{C}\mathrm{Man}$, we take

$$K_p^l=\boldsymbol{F}_p(\check{N}\mathcal{U}_l)=\prod_{i_0,\dots,i_l\in\mathcal{I}}\boldsymbol{F}_p(U_{i_0,\dots,i_l}).$$

In this case a p–cell in $x\in K_p^l$ is given by $x=\{x_{i_0,\dots,i_l}\}$, where, for each $(l+1)$–fold intersection $U_{i_0,\dots,i_l}$, $x_{i_0,\dots,i_l}\in\boldsymbol{F}_p(U_{i_0,\dots,i_l})$ is a p–cell. Note that the map $d^j:K^l\to K^{l+1}$ in (B-4) and (B-7) is induced by the inclusions $\mathrm{incl}\colon U_{i_0,\dots,i_{l+1}}\hookrightarrow U_{i_0,\dots,\hat{i}_j,\dots,i_{l+1}}$ as $\boldsymbol{F}_p(\mathrm{incl})\colon\boldsymbol{F}_p(U_{i_0,\dots,\hat{i}_j,\dots,i_{l+1}})\to\boldsymbol{F}_p(U_{i_0,\dots,i_{l+1}})$. In particular, continuing the example from the figure above, $x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right]}$ and $x^{(2,1)}_{\left[\begin{smallmatrix}0&1&2\\1&1&1\end{smallmatrix}\right]}$ have components

$$x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right];i}\in\boldsymbol{F}_2(U_i),\qquad x^{(2,1)}_{\left[\begin{smallmatrix}0&1&2\\1&1&1\end{smallmatrix}\right];i_0i_1}\in\boldsymbol{F}_2(U_{i_0i_1}),$$

respectively and the compatibility of (B-7) now yields,

$$x^{(2,1)}_{\left[\begin{smallmatrix}0&1&2\\1&1&1\end{smallmatrix}\right];i_0i_1}=d^0\big(x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right]}\big)=U_{i_0i_1}|_{x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right];i_1}}.$$

B.3 Totalization for the case $K=\mathrm{sSet}(\widehat{\Delta},\widetilde{K})$

We are interested in the totalization of $K_\bullet^\bullet=\mathbf{Perf}^{\widehat{\Delta}}(\check{N}U)=\mathrm{sSet}(\widehat{\Delta},\mathbf{Perf}(\check{N}U))$. Thus, assume now that we have a cosimplicial simplicial set $K_\bullet^\bullet$, which is of the form $K_p^l:=\mathrm{sSet}(\widehat{\Delta}^p,\widetilde{K}^l)$ for some other cosimplicial simplicial set $\widetilde{K}_\bullet^\bullet$. By rewriting simplicial sets as colimits of their simplices, and using continuity of the hom-functor in the category sSet, we see that

$$\text{(B-8)}\qquad \begin{aligned}\mathrm{sSet}(\Delta^k\times\Delta^l,K^l)&=\mathrm{sSet}\big(\operatorname*{colim}_{\Delta^p\to\Delta^k\times\Delta^l}\Delta^p,K^l\big)=\lim_{\Delta^p\to\Delta^k\times\Delta^l}\mathrm{sSet}(\Delta^p,K^l)\\&=\lim_{\Delta^p\to\Delta^k\times\Delta^l}K_p^l=\lim_{\Delta^p\to\Delta^k\times\Delta^l}\mathrm{sSet}(\widehat{\Delta}^p,\widetilde{K}^l)\\&=\mathrm{sSet}\big(\operatorname*{colim}_{\Delta^p\to\Delta^k\times\Delta^l}\widehat{\Delta}^p,\widetilde{K}^l\big).\end{aligned}$$

We see from the above identification that decorations of simplicial sets $\Delta^k\times\Delta^l$ by simplices in $K_\bullet^l$ is equivalent to first gluing the simplicial sets $\widehat{\Delta}^n$ along the corresponding Δ^n sitting inside $\Delta^k\times\Delta^l$, and then decorating this colimit made of various $\widehat{\Delta}^n$ by simplices in $\widetilde{K}^l$. Using the description of $\widehat{\Delta}$ from Example 2.8, it now follows that the k–simplices of $\mathrm{Tot}(K_\bullet^\bullet)$ are in fact given by

$$x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0&\cdots&\alpha_p\\\beta_0&\cdots&\beta_p\end{smallmatrix}\right]}\in\widetilde{K}_p^l,$$

where this time the path described by $\left[\begin{smallmatrix}\alpha_0&\cdots&\alpha_p\\\beta_0&\cdots&\beta_p\end{smallmatrix}\right]$ is now permitted to move horizontally and vertically in each direction in the grid, ie possibly decreasing, but within the indices of a nondecreasing path. For example, in the $(2+1)\times(3+1)$ grid of vertices, take the 5–cell given by the map $\Delta^5\hookrightarrow\Delta^2\times\Delta^3$

whose nondecreasing path is $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\middle|\begin{smallmatrix}0\\1\end{smallmatrix}\middle|\begin{smallmatrix}0\\2\end{smallmatrix}\middle|\begin{smallmatrix}1\\2\end{smallmatrix}\middle|\begin{smallmatrix}1\\3\end{smallmatrix}\middle|\begin{smallmatrix}2\\3\end{smallmatrix}\right]$. Then, for the corresponding $\widehat{\Delta}^5$, there is a nondegenerate 9–simplex

$$x^{(2,3)}_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\middle|\begin{smallmatrix}0\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\3\end{smallmatrix}\middle|\begin{smallmatrix}0\\1\end{smallmatrix}\middle|\begin{smallmatrix}0\\2\end{smallmatrix}\middle|\begin{smallmatrix}1\\2\end{smallmatrix}\middle|\begin{smallmatrix}0\\1\end{smallmatrix}\middle|\begin{smallmatrix}2\\3\end{smallmatrix}\middle|\begin{smallmatrix}1\\3\end{smallmatrix}\middle|\begin{smallmatrix}2\\3\end{smallmatrix}\right]} \in \widetilde{K}^3_9, \tag{B-9}$$

which is both increasing and decreasing using the indices of the 5–path $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\middle|\begin{smallmatrix}0\\1\end{smallmatrix}\middle|\begin{smallmatrix}0\\2\end{smallmatrix}\middle|\begin{smallmatrix}1\\2\end{smallmatrix}\middle|\begin{smallmatrix}1\\3\end{smallmatrix}\middle|\begin{smallmatrix}2\\3\end{smallmatrix}\right]$ in $\Delta^2 \times \Delta^3$. Thus, in the totalization $\mathrm{Tot}(K)$, a 2–simplex $x = \{x^{(2,l)}\}$ needs to assign such an element in $\widetilde{K}^3_9$ to the 9–path from (B-9). However, note that there is no assignment to the path $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}0\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, because every map $\Delta^n \to \Delta^k \times \Delta^l$ is necessarily nondecreasing in both components and so one can never obtain both $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ in the same path. To summarize, a cell in $\mathrm{Tot}(K)$ has to assign elements in $\widetilde{K}$ exactly to any path which uses the indices of a nondecreasing path.

Finally, note that the coherence condition on these simplices of the totalization is the same as expressed in (B-7).

Appendix C Totalization and fibrant objects

The purpose of this appendix is to prove Proposition C.1.

Proposition C.1 *If $\boldsymbol{F}$ is a projectively fibrant simplicial presheaf* (*such as $\boldsymbol{F} = \mathbf{IVB}$*) *then* $\mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet))$ *is a Kan complex.*

We start with the following lemma:

Lemma C.2 *The totalization functor* (see Appendix B) $\mathrm{Tot}\colon (\mathrm{Set}^{\Delta^{\mathrm{op}}})^{\Delta} \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ *is a right adjoint.*

Proof We prove this directly by defining the left adjoint L. For any simplicial set $X^\bullet$, let $L(X^\bullet)$ be the cosimplicial simplicial set $n \mapsto X^\bullet \times \Delta^n$, where Δ^n is the standard n–simplex.

To show that these functors form an adjoint pair, let $X^\bullet$ be a simplicial set and $Y^\bullet_\bullet$ be a cosimplicial simplicial set. Since $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ is a simplicial model category (under the usual Quillen structure), $\mathrm{Set}^{\Delta^{\mathrm{op}}}(X \times \Delta^n, Y^n_\bullet)$ is in bijection with $\mathrm{Set}^{\Delta^{\mathrm{op}}}(X, (Y^n_\bullet)^{\Delta^n})$. Since $\mathrm{Tot}(Y^\bullet_\bullet) = (Y^\bullet_\bullet)^{\Delta}$, we have our bijection. □

Lemma C.3 *The functors* (L, Tot) *form a Quillen adjunction between the Reedy model structure* [Hirschhorn 2003, Section 15] *of cosimplicial simplicial sets and the usual Quillen model structure on simplicial sets.*

Proof It is enough to show that L preserves cofibrations and trivial cofibrations. Suppose $f\colon X^\bullet \to Y^\bullet$ is a cofibration of simplicial sets, ie a levelwise monomorphism. By [Hirschhorn 2003, Theorem 15.9.9], to show that $L(f)$ is a Reedy cofibration, it is enough to show that $L(f)$ is a monomorphism that takes the maximal augmentation of $L(X^\bullet)$ isomorphically onto the maximal augmentation of $L(Y^\bullet)$.

Since $L(f) = f \times \mathrm{Id}$ and f is a levelwise monomorphism, $L(f)$ is a monomorphism. The maximal augmentation of $L(X^\bullet)$ and $L(Y^\bullet)$ are empty. So L preserves cofibrations.

Suppose $f : X^\bullet \to Y^\bullet$ is a trivial cofibration. We need to show that $L(f) \colon L(X^\bullet) \to L(Y^\bullet)$ is a Reedy weak equivalence. Since $L(f) = f \times \mathrm{Id}$, then $L_n F \colon X^\bullet \times \Delta^n \to Y^\bullet \times \Delta^n$ is a weak equivalence. □

Lemma C.4 *Let X be a Reedy fibrant cosimplicial simplicial set. Then $\mathrm{Tot}(X)$ is a Kan complex.*

Proof Since Tot is a right adjoint, it preserves fibrations and terminal objects. So Tot preserves fibrant objects. □

Lemma C.5 *Let V be a manifold and $U_\bullet$ be an open cover of V. Let $\boldsymbol{F}$ be a simplicial presheaf that takes values in Kan complexes. Then $\boldsymbol{F}(\check{N}U_\bullet) \colon \Delta \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ (see (4-1)) is a Reedy fibrant cosimplicial simplicial set.*

Proof This proof uses some conventions from [Hirschhorn 2003, Section 15] for the Reedy model structure and is analogous to that of Block, Holstein and Wei [2017, Proposition 4.3]. We need to show that the matching map $\boldsymbol{F}(\check{N}U_n) \to M_n(\boldsymbol{F}(\check{N}U_\bullet))$ is a fibration for each n, where

$$\boldsymbol{F}(\check{N}U_n) := \underline{\mathrm{sPre}}\Big(\coprod_{i_0,\dots,i_n} yU_{i_0,\dots,i_n}, \boldsymbol{F}\Big) = \prod_{i_0,\dots,i_n} \boldsymbol{F}(U_{i_0,\dots,i_n}).$$

Write $\check{N}U_n$ as the coproduct

$$\check{N}U_n = \coprod_{\substack{i_0,\dots,i_n \\ i_j \neq i_{j+1}}} yU_{i_0,\dots,i_n} \amalg \Big(\coprod_{k=1}^{n} \coprod_{\substack{i_0,\dots,i_n \\ i_{j_1}=i_{j_1+1},\dots,i_{j_k}=i_{j_k+1}}} yU_{i_0,\dots,i_n} \Big)$$

and apply $\boldsymbol{F}$ to get

$$\prod_{\substack{i_0,\dots,i_n \\ i_j \neq i_{j+1}}} \boldsymbol{F}(U_{i_0,\dots,i_n}) \times \prod_{k=1}^{n} \Big(\prod_{\substack{i_0,\dots,i_n \\ i_{j_1}=i_{j_1+1},\dots,i_{j_k}=i_{j_k+1}}} \boldsymbol{F}(U_{i_0,\dots,i_n}) \Big).$$

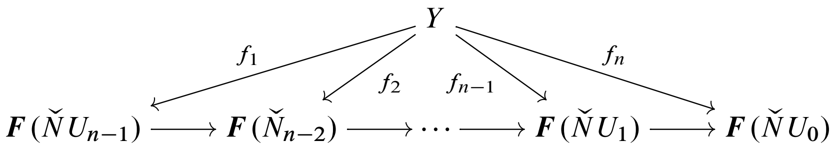
First note that the right side of this cartesian product is the matching object at n, $M_n \boldsymbol{F}(\check{N}U)$. This is seen directly by showing that this product is the terminal object in the category of cones under $\boldsymbol{F}(\check{N}U)$ restricted to the matching category $\partial([n] \downarrow \overleftarrow{\Delta})$ (see [Hirschhorn 2003, Definition 15.2.3.2]). The product

$$\begin{array}{c} \displaystyle\prod_{k=1}^{n} \Big(\prod_{\substack{i_0,\dots,i_n \\ i_{j_1}=i_{j_1+1},\dots,i_{j_k}=i_{j_k+1}}} \boldsymbol{F}(U_{i_0,\dots,i_n}) \Big) \\ \swarrow \quad \swarrow \qquad\qquad \searrow \quad \searrow \\ \boldsymbol{F}(\check{N}U_{n-1}) \longrightarrow \boldsymbol{F}(\check{N}U_{n-2}) \longrightarrow \cdots \longrightarrow \boldsymbol{F}(\check{N}U_1) \longrightarrow \boldsymbol{F}(\check{N}U_0) \end{array}$$

is a cone under $\boldsymbol{F}(\check{N}U)$, where $\boldsymbol{F}(\check{N}U_{n-j}) = \prod_{i_0,\dots,i_{n-j}} \boldsymbol{F}(U_{i_0,\dots,i_{n-j}})$ and the vertical maps are projections.

Now, suppose we have a cone under $\boldsymbol{F}(\check{N}U)$:

(C-1)



Then, to define the map Y into the product, send y to $(f_1(y), f_2(y), \dots, f_n(y))$.

Finally, we see that the matching map $\boldsymbol{F}(\check{N}U_n) \times M_n(\check{N}U) \to M_n(\check{N}U)$ is the projection onto the second factor. Since $\boldsymbol{F}(\check{N}U_n)$ is a Kan complex, the projection is a fibration. □

Applying Lemma C.4 to Lemma C.5 proves Proposition C.1.

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Derived character maps of group representations

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We define and study (derived) character maps of finite-dimensional representations of ∞–groups. As models for ∞–groups we take homotopy simplicial groups, ie the homotopy simplicial algebras over the algebraic theory of groups (in the sense of Badzioch (2002)). We introduce cyclic, symmetric and representation homology for "group algebras" $k[\Gamma]$ of such groups and construct canonical trace maps (natural transformations) relating these homology theories. We show that, in the case of one-dimensional representations, our trace maps are of topological origin: they are induced by natural maps of (iterated) loop spaces known in stable homotopy theory. Using this topological interpretation, we deduce some algebraic results on representation homology: in particular, we prove that the symmetric homology of group algebras and one-dimensional representation homology are naturally isomorphic, provided the base ring k is a field of characteristic zero. We also study the stable behavior of the derived character maps of n–dimensional representations as $n \to \infty$, in which case we show that these maps "converge" to become isomorphisms.

18A25, 18G15, 19D55, 55N35; 14A30, 55P42

1 Introduction

If Γ is a finite group and k is a field of characteristic zero, every finite-dimensional k–linear representation $\varrho\colon \Gamma \to \mathrm{GL}_n(k)$ is semisimple and determined (up to equivalence) by its character: the trace function $\langle g\rangle \mapsto \mathrm{Tr}_n[\varrho(g)]$ defined on the set $\langle\Gamma\rangle$ of conjugacy classes of elements of Γ; moreover, for each $n \geq 0$, there are finitely many equivalence classes of such representations. These well-familiar facts from representation theory of finite groups generalize to arbitrary groups by means of algebraic geometry. For any discrete group Γ, the set of all n–dimensional representations of Γ can be naturally given the structure of an affine algebraic variety (more precisely, an affine k–scheme) $\mathrm{Rep}_n(\Gamma)$ called the *representation variety* of Γ. The equivalence classes of n–dimensional representations of Γ are classified by the orbits of the general linear group GL_n that acts algebraically on $\mathrm{Rep}_n(\Gamma)$ by conjugation. The classes of semisimple representations correspond to the closed orbits[1] and are parametrized by the affine quotient scheme

$$\mathrm{Rep}_n(\Gamma) /\!\!/ \mathrm{GL}_n(k) := \mathrm{Spec}\, \mathcal{O}[\mathrm{Rep}_n(\Gamma)]^{\mathrm{GL}_n}$$

called the *character variety* of Γ. Now, the characters of representations assemble into a linear map

$$\mathrm{Tr}_n(\Gamma) \colon k\langle\Gamma\rangle \to \mathcal{O}[\mathrm{Rep}_n(\Gamma)]^{\mathrm{GL}_n} \tag{1-1}$$

defined on the k–vector space spanned by the conjugacy classes of elements of Γ. A well-known theorem of C Procesi [59] asserts that the characters of Γ, ie the images of the map (1-1), generate $\mathcal{O}[\mathrm{Rep}_n(\Gamma)]^{\mathrm{GL}_n}$ as a commutative k–algebra, and thus, by Nullstellensatz, detect the semisimple representations of Γ when k is algebraically closed. In general, the equivariant geometry of $\mathrm{Rep}_n(\Gamma)$ is closely related to representation theory of Γ, the geometric structure of GL_n–orbits in $\mathrm{Rep}_n(\Gamma)$ determining the algebraic structure of representations. Since the late 1980s this relation has been extensively studied and exploited in many areas of mathematics, most notably in geometric group theory and low-dimensional topology; see eg Lubotzky and Magid [50] and Sikora [66].

Derived algebraic geometry allows one to extend — and in some sense to complete — this beautiful connection between representation theory and geometry. For any affine algebraic group G defined over a commutative ring k (eg $G = \mathrm{GL}_n(k)$), the classical representation scheme $\mathrm{Rep}_G(\Gamma)$ parametrizing the representations of Γ in G admits a natural derived extension $\mathrm{DRep}_G(\Gamma)$ called the *derived G–representation scheme*[2] of Γ. This derived scheme is represented by a simplicial commutative k–algebra $\mathcal{O}[\mathrm{DRep}_G(\Gamma)]$ whose homotopy groups $\pi_i\mathcal{O}[\mathrm{DRep}_G(\Gamma)]$ are nonabelian homological invariants of Γ (or its classifying space $B\Gamma$). Following [13; 14], we set

$$\mathrm{HR}_*(\Gamma, G(k)) := \pi_*\mathcal{O}[\mathrm{DRep}_G(\Gamma)] \tag{1-2}$$

and call (1-2) the *representation homology of Γ with coefficients in G*. By definition, $\mathrm{HR}_*(\Gamma, G(k))$ is a graded commutative algebra, whose degree zero part is canonically isomorphic to the coordinate ring of $\mathrm{Rep}_G(\Gamma)$:

$$\mathrm{HR}_0(\Gamma, G(k)) \cong \mathcal{O}[\mathrm{Rep}_G(\Gamma)]. \tag{1-3}$$

[1] At least when Γ is finitely generated.

[2] The first construction of this kind, the derived moduli space $\mathbf{R}\mathrm{Loc}_G(X)$ of G–local systems over a pointed connected space X, was introduced by Kapranov [43]. In recent years, several other constructions and generalizations of $\mathbf{R}\mathrm{Loc}_G(X)$ have been studied in derived algebraic geometry; most notably, in the work of Toën and Vezzosi [69], but see also Pridham [58], Pantev, Toën, Vaquié and Vezzosi [55], Pantev and Toën [54] and Toën [68]. A brief review and comparison of these constructions can be found in Berest, Ramadoss and Yeung [12, Appendix].

Apart from groups, representation homology can be also defined for various kinds of algebras (eg associative and Lie algebras, see Berest, Ramadoss et al. [9; 11; 7; 8]) as well as for topological spaces [14; 13; 12]. What is surprising perhaps is that, in the case of discrete groups, the representation homology admits a simple interpretation in terms of classical (abelian) homological algebra: namely, as shown in [14], there is a natural isomorphism

$$\mathrm{HR}_*(\Gamma, G(k)) \cong \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \mathcal{O}(G)), \tag{1-4}$$

where $\mathfrak{G}$ is the (skeletal) category of f.g. free groups on which the group algebra $k[\Gamma]$ of the group Γ and the coordinate algebra $\mathcal{O}(G)$ of the algebraic group G are represented by monoidal functors: $\mathfrak{G} \to \mathrm{Mod}_k$ (contravariant and covariant, respectively). In the present paper, we will use formula (1-4) to *define* representation homology for *homotopy simplicial groups*, which are natural ("up to homotopy") generalizations of the usual ("strict") simplicial groups; see Badzioch [4]. In addition, we will also define cyclic and symmetric homology for such groups, extending the original approach of Connes [25] and Fiedorowicz and Loday [31].

Now, returning to the classical character map (1-1), we observe that its domain can be identified with the 0^{th} cyclic homology of the group algebra $k[\Gamma]$:

$$\mathrm{HC}_0(k[\Gamma]) \cong k\langle\Gamma\rangle. \tag{1-5}$$

With identifications (1-3) and (1-5), we can rewrite (1-1) in the form

$$\mathrm{Tr}_n(\Gamma)\colon \mathrm{HC}_0(k[\Gamma]) \to \mathrm{HR}_0(\Gamma, \mathrm{GL}_n(k))^{\mathrm{GL}_n}, \tag{1-6}$$

which suggests that there might exist a natural extension of this map to higher cyclic homology with values in representation homology of Γ:

$$\mathrm{Tr}_n(\Gamma)_*\colon \mathrm{HC}_*(k[\Gamma]) \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_n(k))^{\mathrm{GL}_n}. \tag{1-7}$$

The maps (1-7) do exist, and we call them the *derived character maps of n–dimensional representations of* Γ. Our goal is to define and study such maps for an arbitrary homotopy simplicial group Γ and an arbitrary affine algebraic group G; see Definition 3.15.

In the case of associative algebras, the derived character maps were originally constructed in [9], using nonabelian homological algebra. This construction was extended to Lie algebras in [8], where it was shown, among other things, that the derived character maps of Lie algebra representations are Koszul dual (in an appropriate sense) to the classical Loday–Quillen–Tsygan maps [49; 70]. The case of groups that we treat in this paper is special for several reasons. First, as mentioned above, the representation homology of groups admits a natural interpretation in terms of functor homology that is parallel to A Connes' well-known interpretation of cyclic homology. We will show that behind this "parallelism" there is actually a connection: a simple formula for the derived character maps (1-7) relating cyclic homology to representation homology via standard homological algebra; see Section 3.4.

Second, the cyclic homology of group algebras has a natural topological realization that goes back to the work of Goodwillie, Burghelea, Fiedorowicz and others (see Loday [47, Chapter 7]): specifically,

$$\mathrm{HC}_*(k[\Gamma]) \cong \mathrm{H}_*(ES^1 \times_{S^1} \mathcal{L}(B\Gamma); k), \tag{1-8}$$

where the right-hand side is the S^1–equivariant homology of the free loop space $\mathcal{L}(B\Gamma) := \mathrm{Map}(S^1, B\Gamma)$ of the classifying space of Γ. In fact, the isomorphism (1-8) is just one on a list of several classical isomorphisms relating the algebraic homology theories associated with so-called crossed simplicial groups (see Fiedorowicz and Loday [31]) to (stable) homotopy theory:

$$\begin{aligned} \mathrm{HH}_*(k[\Gamma]) &\cong \mathrm{H}_*(\mathcal{L}(B\Gamma); k), \\ \mathrm{HC}_*(k[\Gamma]) &\cong \mathrm{H}_*(ES^1 \times_{S^1} \mathcal{L}(B\Gamma); k), \\ \mathrm{HS}_*(k[\Gamma]) &\cong \mathrm{H}_*(\Omega\Omega^\infty\Sigma^\infty(B\Gamma); k), \\ \mathrm{HB}_*(k[\Gamma]) &\cong \mathrm{H}_*(\Omega^2\Sigma(B\Gamma); k), \\ \mathrm{HO}_*(k[\Gamma]) &\cong \mathrm{H}_*(E(\mathbb{Z}/2)_+ \wedge_{\mathbb{Z}/2} \Omega\Omega^\infty\Sigma^\infty(B\Gamma); k), \end{aligned} \tag{1-9}$$

where Ω, Σ and $\Omega^\infty\Sigma^\infty$ denote the based loop, the (reduced) suspension, and the stable homotopy functors, respectively. The first two of the above isomorphisms (for Hochschild and cyclic homology) are well known: they were originally established in Goodwillie [33] and Burghelea and Fiedorowicz [20], and their proofs appear in Loday's textbook [47], see also his [48] for a nice self-contained exposition. The last three (for the symmetric HS_*, braided HB_* and hyperoctahedral HO_* homologies) are less known: they were discovered by Fiedorowicz [30] in the early 1990s, but detailed proofs were published only recently; see Ault [2] and Graves [37].

The second (and perhaps, the main) goal of this paper is to extend the above list of isomorphisms by adding to it representation homology. To be precise, for any commutative ring k, let $\mathrm{HR}_*(k[\Gamma]) := \mathrm{HR}_*(\Gamma, k^\times)$ denote the one-dimensional representation homology of Γ. We prove (see Lemma 4.1 and Theorem 4.2):

Theorem 1.1 *For any homotopy simplicial group* Γ, *there is a natural isomorphism*

$$\mathrm{HR}_*(k[\Gamma]) \cong \mathrm{H}_*(\Omega\mathrm{SP}^\infty(B\Gamma); k), \tag{1-10}$$

where $\mathrm{SP}^\infty(B\Gamma)$ *denotes the Dold–Thom space of the classifying space of* Γ.

Apart from the Hochschild and cyclic theories, most interesting on the list (1-9) is the *symmetric homology* theory HS_* introduced by Fiedorowicz [30] and studied by Ault [2; 3]. Roughly speaking, HS_* is defined[3] in the same way as HC_*, with Connes' cyclic category ΔC replaced by the symmetric category ΔS, where the family of the symmetric groups $\{S_{n+1}^{\mathrm{op}}\}_{n\geq 0}$ is used instead of the cyclic groups $\{C_{n+1}\}_{n\geq 0}$. Now, the natural inclusions of groups $C_{n+1} \hookrightarrow S_{n+1}$ extend to a functor $\iota\colon \Delta C^{\mathrm{op}} \hookrightarrow \Delta S$, which, in turn,

[3] See Sections 3.3 and 4.2 for precise definitions of $\mathrm{HC}_*(k[\Gamma])$ and $\mathrm{HS}_*(k[\Gamma])$ in the context of homotopy simplicial groups.

induces a natural map $\mathrm{HC}_*(k[\Gamma]) \to \mathrm{HS}_*(k[\Gamma])$. It turns out that, with identifications (1-9), this last map is induced (on homology) by a map of topological spaces

$$\mathtt{CS}_{B\Gamma}\colon ES^1 \times_{S^1} \mathcal{L}(B\Gamma) \to \Omega\Omega^\infty\Sigma^\infty(B\Gamma). \tag{1-11}$$

The map (1-11) is actually defined as a natural transformation $\mathtt{CS}_X$ on the (homotopy) category of all pointed spaces; it was originally constructed by Carlsson and Cohen [21], and its relation to symmetric homology was noticed in [30]. We will refer to (1-11) as the *Carlsson–Cohen map* for $B\Gamma$.

We can now state our second observation that provides a topological interpretation of the derived character maps (1-7) for one-dimensional representations. To shorten notation we will write the maps (1-7) for $n = 1$ as

$$\mathrm{Tr}(\Gamma)_*\colon \mathrm{HC}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma]). \tag{1-12}$$

The next theorem encapsulates the main results of Section 4.3 (see Proposition 4.8 and Corollary 4.10), Section 5.2 (see Proposition 5.2) and Section 5.3 (see Proposition 5.3).

Theorem 1.2 *With isomorphisms (1-9) and (1-10), the derived character maps (1-12) are induced on homology by a natural map of topological spaces*

$$\mathtt{CR}_{B\Gamma}\colon ES^1 \times_{S^1} \mathcal{L}(B\Gamma) \to \Omega\mathrm{SP}^\infty(B\Gamma). \tag{1-13}$$

The map (1-13) factors (as a homotopy natural transformation) through the Carlsson–Cohen map (1-11):

$$ES^1 \times_{S^1} \mathcal{L}(B\Gamma) \xrightarrow{\mathtt{CS}_{B\Gamma}} \Omega\Omega^\infty\Sigma^\infty(B\Gamma) \xrightarrow{\mathtt{SR}_{B\Gamma}} \Omega\mathrm{SP}^\infty(B\Gamma), \tag{1-14}$$

where the induced map $\mathtt{SR}$ is the (looped once) canonical natural transformation $\Omega^\infty\Sigma^\infty \to \mathrm{SP}^\infty$ relating stable homotopy to (reduced) singular homology of pointed spaces.

Theorem 1.2 shows that, for any homotopy simplicial group Γ, the derived character map (1-12) factors through symmetric homology, and the induced map

$$\mathtt{SR}_{B\Gamma,*}\colon \mathrm{HS}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma]) \tag{1-15}$$

is determined by a map of spaces that is well known in topology. Using topological results, we then conclude (see Corollary 5.5 and Remark 5.6):

Corollary 1.3 *If k is a field of characteristic 0, the map (1-15) is an isomorphism, at least when $B\Gamma$ is a simply connected space.*

The results stated above are all concerned with derived characters of one-dimensional representations. For higher-dimensional representations ($n > 1$), the maps (1-7) are more complicated: in particular, they do not seem to factor through $\mathrm{HS}_*(k[\Gamma])$, and in general, the relation between symmetric homology and representation homology remains mysterious. However, when $n \to \infty$, things become more tractable. Assuming that k is a field of characteristic 0, we can naturally pass to the projective limit

$$\mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty(k))^{\mathrm{GL}_\infty} := \varprojlim_n \mathrm{HR}_*(\Gamma, \mathrm{GL}_n(k))^{\mathrm{GL}_n}$$

and construct the *stable* character maps

$$\mathrm{Tr}_\infty(\Gamma)_*\colon \overline{\mathrm{HC}}_*(k[\Gamma]) \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty(k))^{\mathrm{GL}_\infty}, \tag{1-16}$$

where $\overline{\mathrm{HC}}$ stands for the reduced cyclic homology. In this case, we have the following result, the proof of which is parallel to [11] and outlined in the last section of the paper; see Theorem 6.2.

Theorem 1.4 *Let Γ be a homotopy simplicial group such that $B\Gamma$ is a simply connected space of finite (rational) type. Then the stable character maps (1-16) induce an algebra isomorphism*

$$\Lambda\,\mathrm{Tr}_\infty(\Gamma)_*\colon \Lambda_k[\overline{\mathrm{HC}}_*(k[\Gamma])] \xrightarrow{\sim} \mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty}, \tag{1-17}$$

where $\Lambda_k[\overline{\mathrm{HC}}_*(k[\Gamma])]$ is the graded symmetric algebra generated by the reduced cyclic homology of $k[\Gamma]$.

We close this introduction by mentioning one application of stable character maps in derived Poisson geometry. If Γ is a simplicial group model of a simply connected closed manifold X of dimension d (so that $X \simeq B\Gamma$), then, by (1-9), we can identify $\overline{\mathrm{HC}}_*(k[\Gamma])$ with the reduced S^1–equivariant homology $\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}(X);k)$ of the free loop space of X. Thanks to the work of Chas and Sullivan, the latter is known to carry the so-called *string topology* Lie bracket, making the symmetric algebra $\Lambda_k[\overline{\mathrm{HC}}_*(k[\Gamma])] \cong \Lambda_k[\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}(X);k)]$ a graded Poisson algebra. On the other hand, the representation homology ring $\mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$ acquires a $(2-d)$–shifted graded Poisson structure from the Poincaré duality pairing on (the cohomology of) X. As an application of Theorem 1.4, we show that under the isomorphism (1-17), these two Poisson structures agree, ie the map (1-17) is an isomorphism of graded Poisson algebras; see Corollary 6.3.

The paper is organized as follows. In Section 2, we review basic facts from abstract homotopy theory concerning homotopy colimits. The new result proved in this section is Proposition 2.6, which we refer to as the "Shapiro Lemma for model categories". This proposition provides a key step for proofs of main theorems in Section 4 and may be of independent interest. In Section 3, after reviewing basic theory of homotopy simplicial groups (Section 3.1), we define representation homology (Section 3.2), and cyclic homology (Section 3.3) for such groups and construct the derived character maps relating the two (Section 3.4). In Section 4, we prove Theorem 1.1 (Section 4.1) and then, after defining symmetric homology for homotopy simplicial groups (Section 4.2), we prove part of Theorem 1.2 (see Proposition 4.8 and Corollary 4.10 in Section 4.3). The proof of Theorem 1.2 is completed in Section 6, where we study the maps (1-13) and (1-14) in topological terms, using Goodwillie homotopy calculus and classical operads; see Propositions 5.2 and 5.3. Finally, in Section 6, we describe the stabilization procedure for the derived character maps as $n \to \infty$ and sketch the proofs of Theorem 1.4 and Corollary 6.3. Each of the six sections begins with a short introduction that provides more details about its contents.

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2 Shapiro Lemma for model categories

In this section, we prove one general result in abstract homotopy theory concerning homotopy colimits that will provide a key step for our Theorem 1.1. We call this result (Proposition 2.6) the "Shapiro Lemma for model categories" as it appears to be a nonabelian generalization of the classical Shapiro Lemma in the context of model categories. We begin with a brief overview of the theory of homotopy colimits. The standard reference for this material is the last two chapters of Hirschhorn's book [39] but many results that we mention are classical and go back to Bousfield and Kan [19] and Quillen [61]. Our exposition is inspired by Cisinski's beautiful paper [24] that treats homotopy colimits axiomatically by analogy with derived direct image functors in algebraic geometry (unlike [24], however, we do not use the language of Grothendieck derivators). With the exception of Proposition 2.6, which (to the best of our knowledge) is new, all results in this section are known.

2.1 Notation and conventions

Throughout this section, $\mathcal{M}$ will denote a fixed model category which we assume to be cofibrantly generated and having all small limits and colimits. Unless stated otherwise, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ will denote small categories that we will use to index diagrams in $\mathcal{M}$. For a small category $\mathcal{A}$, the category of $\mathcal{A}$–diagrams in $\mathcal{M}$ (ie all functors $\mathcal{A} \to \mathcal{M}$) will be denoted by $\mathcal{M}^{\mathcal{A}}$. As usual, Cat will stand for the category of all small categories with morphisms being arbitrary functors.

2.2 Homotopy colimits

For any small category $\mathcal{A}$, the category $\mathcal{M}^{\mathcal{A}}$ has a projective (aka Bousfield–Kan) model structure inherited from $\mathcal{M}$: the weak equivalences and fibrations are defined in this model structure objectwise, while the cofibrations are determined by the lifting axiom of model categories (specifically, as morphisms having the left lifting property with respect to fibrations which are also weak equivalences in $\mathcal{M}^{\mathcal{A}}$). Since $\mathcal{M}$ is cofibrantly generated, such a model structure on $\mathcal{M}^{\mathcal{A}}$ always exists and is cofibrantly generated; see [39, Theorem 11.6.1].

Any functor $f: \mathcal{A} \to \mathcal{B}$ (a morphism in Cat) defines the *pullback functor* on the diagram categories $f^*: \mathcal{M}^{\mathcal{B}} \to \mathcal{M}^{\mathcal{A}}$, which is obtained by restricting diagrams $\mathcal{B} \to \mathcal{M}$ along f. This pullback functor preserves objectwise weak equivalences and fibrations and, since $\mathcal{M}$ has small colimits, admits a left adjoint

(2-1) $$f_!: \mathcal{M}^{\mathcal{A}} \rightleftarrows \mathcal{M}^{\mathcal{B}} : f^*$$

defined on a diagram $X: \mathcal{A} \to \mathcal{M}$ as the left Kan extension $f_!(X) := \mathrm{Lan}_f(X)$ of X along f. Thus, the functors (2-1) form a Quillen pair between the model categories $\mathcal{M}^{\mathcal{A}}$ and $\mathcal{M}^{\mathcal{B}}$. Then, by Quillen's adjunction theorem [39, Theorem 8.5.8], they admit total (left and right) derived functors

(2-2) $$\boldsymbol{L} f_!: \mathtt{Ho}(\mathcal{M}^{\mathcal{A}}) \rightleftarrows \mathtt{Ho}(\mathcal{M}^{\mathcal{B}}) : f^*$$

that form an adjunction between the homotopy categories of diagrams induced by (2-1).

The derived pushforward functor $\boldsymbol{L} f_!$ is called the *homotopy left Kan extension* along f. It is a generalization of the classical *homotopy colimit functor* $\operatorname{hocolim}_{\mathcal{A}}\colon \mathtt{Ho}(\mathcal{M}^{\mathcal{A}}) \to \mathtt{Ho}(\mathcal{M})$ that corresponds to the trivial map $\mathcal{A} \to *$, where $*$ denotes the one-point category (the terminal object in Cat). In this last case, we will use the classical notation writing $\operatorname{hocolim}_{\mathcal{A}}(X)$ instead of $\boldsymbol{L}(\mathcal{A} \to *)_!(X)$ for $X\colon \mathcal{A} \to \mathcal{M}$. We summarize the main properties of this construction in the following theorem.

Theorem 2.1 [24] *Let $\mathcal{M}$ be a model category with all small limits and colimits.*

(1) ***2–Functoriality*** *The pullback functors f^* fit together to give a strict, weakly product-preserving 2–functor*[4] $\mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ *that takes a small category $\mathcal{A} \in \mathrm{Cat}$ to the homotopy category $\mathtt{Ho}(\mathcal{M}^{\mathcal{A}})$. By adjunction, this implies, in particular, the existence of natural weak equivalences*

$$\boldsymbol{L}(fg)_! \simeq \boldsymbol{L} f_! \boldsymbol{L} g_! \tag{2-3}$$

for any composable morphisms f and g in Cat.

(2) ***Reflexivity*** *For any $\mathcal{A} \in \mathrm{Cat}$, the functor $i^*\colon \mathtt{Ho}(\mathcal{M}^{\mathcal{A}}) \to \mathtt{Ho}(\mathcal{M}^{\mathcal{A}^{\delta}})$ corresponding to the inclusion of the underlying discrete subcategory $\mathcal{A}^{\delta} \subset \mathcal{A}$ is conservative, ie reflects the weak equivalences in $\mathcal{M}^{\mathcal{A}^{\delta}}$.*

(3) ***Base change*** *For any $f\colon \mathcal{A} \to \mathcal{B}$ and any object $b \in \mathcal{B}$, the 2–commutativity of the fiber square*

$$\begin{array}{ccc} f \downarrow b & \xrightarrow{\ \pi\ } & \mathcal{A} \\ {\scriptstyle p}\big\downarrow & \Swarrow & \big\downarrow{\scriptstyle f} \\ * & \xrightarrow{\ b\ } & \mathcal{B} \end{array}$$

induces a change-of-base natural transformation that is a natural weak equivalence

$$\boldsymbol{L} p_! \pi^* \xrightarrow{\ \sim\ } b^* \boldsymbol{L} f_!.$$

For a diagram $X\colon \mathcal{A} \to \mathcal{M}$, this simply says that

$$\boldsymbol{L} f_! X(b) \simeq \operatorname{hocolim}_{f \downarrow b}(\pi^* X), \tag{2-4}$$

where $f \downarrow b$ is the comma category of the functor $f\colon \mathcal{A} \to \mathcal{B}$ over the object $b \in \mathcal{B}$.

Remark 2.2 In terminology of [24, Definition 1.6, pages 205–206], the properties (1)–(3) of Theorem 2.1 can be summarized by saying that the 2–functor $\mathtt{Ho}(\mathcal{M}^{-})\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ is a weak left derivator (*un dérivateur faible à gauche*) associated to the model category $\mathcal{M}$.

The properties of homotopy colimits listed in Theorem 2.1 are essentially formal. The next result — called the cofinality theorem — gives a deeper property of homotopy-theoretic nature that is very useful

[4] Here, $\mathrm{Cat}^{\mathrm{op}}$ stands for the opposite 2–category of small categories, while CAT denotes the "2–category" of all (not necessarily small) categories.

in computations. To state this result we recall that a functor $f:\mathcal{A}\to\mathcal{B}$ is *right homotopy cofinal* if its comma category $b\downarrow f$ under each object $b\in\mathcal{B}$ is (weakly) contractible, ie $B(b\downarrow f)\simeq\mathrm{pt}$. As an example, we point out that every right adjoint functor is right homotopy cofinal: indeed, if $f:\mathcal{A}\to\mathcal{B}$ admits a left adjoint, say $g:\mathcal{B}\to\mathcal{A}$, then each comma-category $b\downarrow f$ has an initial object (namely, (b,η_b), where $\eta_b: b\to fg(b)$ is the unit of the adjunction evaluated at $b\in\mathcal{B}$), hence $b\downarrow f$ is contractible for any $b\in\mathcal{B}$.

Theorem 2.3 (Cofinality) *If $f:\mathcal{A}\to\mathcal{B}$ is right homotopy cofinal, then the natural map*

$$\mathrm{hocolim}_{\mathcal{A}}(f^*X)\xrightarrow{\sim}\mathrm{hocolim}_{\mathcal{B}}(X)$$

is a weak equivalence for any diagram $X:\mathcal{B}\to\mathcal{M}$.

For the proof of Theorem 2.3 we refer to [39, Theorem 19.6.7]. As an application, we prove one simple lemma that we will need for our computations. Given a functor $f:\mathcal{A}\to\mathcal{B}$, we recall that its *fiber category* $f^{-1}(b)$ over an object $b\in\mathcal{B}$ is the subcategory of $\mathcal{A}$ consisting of all objects $a\in\mathcal{A}$ such that $f(a)=b$ and all morphisms $\varphi\in\mathrm{Hom}_{\mathcal{A}}(a,a')$ such that $f(\varphi)=\mathrm{Id}_b$. Note that the fiber inclusion functor $i: f^{-1}(b)\hookrightarrow\mathcal{A}$ factors through the comma-category $f\downarrow b$ over b:

(2-5)
$$\begin{array}{ccc} f^{-1}(b) & \overset{i}{\hookrightarrow} & \mathcal{A} \\ {\scriptstyle j}\downarrow & \nearrow{\scriptstyle \pi} & \\ f\downarrow b & & \end{array}$$

defining the "comparison" functor

(2-6)
$$j: f^{-1}(b)\to f\downarrow b,\quad a\mapsto\big(a,\ f(a)=b\xrightarrow{\mathrm{Id}}b\big).$$

Recall that a functor $f:\mathcal{A}\to\mathcal{B}$ is *precofibered* if (2-6) has a left adjoint for every object $b\in\mathcal{B}$; see [60, Section 1]).

Lemma 2.4 *If $f:\mathcal{A}\to\mathcal{B}$ is precofibered, then, for any diagram $X:\mathcal{A}\to\mathcal{M}$,*

$$(\boldsymbol{L}f_!X)(b)\simeq\mathrm{hocolim}_{f^{-1}(b)}(i^*X).$$

Proof By assumption, the inclusion functor $j: f^{-1}(b)\to f\downarrow b$ is right adjoint, hence right homotopy cofinal. By the base change formula (2-4) and Cofinality Theorem 2.3, we conclude

$$\begin{aligned}(\boldsymbol{L}f_!X)(b)\simeq\mathrm{hocolim}_{f\downarrow b}(\pi^*X)\simeq\mathrm{hocolim}_{f^{-1}(b)}(j^*\pi^*X)&=\mathrm{hocolim}_{f^{-1}(b)}((\pi j)^*X)\\ &=\mathrm{hocolim}_{f^{-1}(b)}(i^*X),\end{aligned}$$

where the last identification follows from (2-5). □

In practice, precofibered functors arise from the so-called Grothendieck construction; see [67]. Given a functor $F:\mathcal{C}\to\mathrm{Cat}$ (ie a strict diagram of small categories), its *Grothendieck construction* is defined to be the small category $\mathcal{C}\int F$ with $\mathrm{Ob}(\mathcal{C}\int F):=\{(c,x)\mid c\in\mathcal{C},\ x\in F(c)\}$ and morphism sets

(2-7)
$$\mathrm{Hom}_{\mathcal{C}\int F}((c,x),(c',x')):=\{(\varphi,f)\mid\varphi\in\mathrm{Hom}_{\mathcal{C}}(c,c'),\ f\in\mathrm{Hom}_{F(c')}(F(\varphi)x,x')\}.$$

The composition in $\mathcal{C} \int F$ is given by $(\varphi, f) \circ (\varphi', f') = (\varphi\varphi', f F(\varphi) f')$. The category $\mathcal{C} \int F$ comes equipped with a natural (forgetful) functor

$$p : \mathcal{C} \int F \to \mathcal{C}, \quad (c, x) \mapsto c,$$

which is precofibered (in fact, cofibered) over $\mathcal{C}$. Notice that $p^{-1}(c) = F(c)$ for any object $c \in \mathcal{C}$. Hence, by Lemma 2.4, for any functor $X : \mathcal{C} \int F \to \mathcal{M}$,

$$(\boldsymbol{L} p_! X)(c) \simeq \operatorname{hocolim}_{F(c)}[X(c)], \tag{2-8}$$

where $X(c) := i_c^* X$ is the restriction of X to $F(c)$ via the inclusion functor

$$i_c : F(c) \to \mathcal{C} \int F, \qquad x \mapsto (c, x), \quad (x \xrightarrow{f} x') \mapsto (\mathrm{Id}_c, f).$$

By 2–functoriality of homotopy Kan extensions (see (2-3)), equation (2-8) implies the weak equivalence

$$\operatorname{hocolim}_{\mathcal{C} \int F}(X) \simeq \operatorname{hocolim}_{c \in \mathcal{C}}\big(\operatorname{hocolim}_{F(c)} X(c)\big), \tag{2-9}$$

which is known as *Thomason's formula* for homotopy colimits over $\mathcal{C} \int F$; see [23, Theorem 26.8].

An important special case arises when we apply the Grothendieck construction to a set-valued functor $F : \mathcal{C} \to \mathrm{Set}$, regarding sets as discrete categories (ie by embedding $\mathrm{Set} \hookrightarrow \mathrm{Cat}$). In this case, the category $\mathcal{C} \int F$ is usually denoted by $\mathcal{C}_F$ and called the *category of elements of* F as its object set $\mathrm{Ob}(\mathcal{C}_F)$ can be identified with $\coprod_{c \in \mathcal{C}} F(c)$ (we will still write the objects of $\mathcal{C}_F$ as pairs (c, x), where $c \in \mathcal{C}$ and $x \in F(c)$). The Hom-sets in $\mathcal{C}_F$ are given by $\mathrm{Hom}_{\mathcal{C}_F}((c, x), (c', x')) = \{\varphi \in \mathrm{Hom}_{\mathcal{C}}(c, c') \mid F(\varphi)x = x'\}$; cf (2-7). If we take $\mathcal{M} = \mathrm{sSet}$ to be the category of simplicial sets (equipped with standard Quillen model structure) and apply Thomason's formula (2-9) to the trivial diagram $X : \mathcal{C}_F \to *$ in $\mathcal{M}$, then for any functor $F : \mathcal{C} \to \mathrm{Set}$, we get

$$\operatorname{hocolim}_{\mathcal{C}}(F) \cong N_*(\mathcal{C}_F), \tag{2-10}$$

where $N_*(\mathcal{C}_F)$ denotes the simplicial nerve of the category $\mathcal{C}_F$. Formula (2-10) is known as the *Bousfield–Kan construction* for homotopy colimits in $s\mathrm{Set}$; see [19].

2.3 Homotopy coends

Homotopy coends are special kinds of homotopy colimits defined for *bifunctors*, ie the diagrams of the form $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{M}$. There is a broader range of techniques for manipulating with such homotopy colimits, which makes them more accessible for computations. The homotopy coends are defined in terms of the so-called *twisted arrow category* $\mathcal{F}(\mathcal{C})$ introduced by Quillen [61]. It can be described as the category of elements of the bifunctor $\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ of the given category $\mathcal{C}$:

$$\mathcal{F}(\mathcal{C}) := (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \int \mathrm{Hom}. \tag{2-11}$$

We will actually be dealing with the opposite category $\mathcal{F}(\mathcal{C})^{\mathrm{op}}$ which can be explicitly described as follows: the objects of $\mathcal{F}(\mathcal{C})^{\mathrm{op}}$ are the morphisms $\{\varphi\colon c \to d\}$ in $\mathcal{C}$, and the Hom–sets are commutative squares

$$\begin{array}{ccc} d & \xleftarrow{\ \beta\ } & d' \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ c & \xrightarrow{\ \alpha\ } & c' \end{array} \tag{2-12}$$

ie $\mathrm{Hom}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}(\varphi, \varphi')$ consists of the pairs of morphisms (α, β) in $\mathcal{C}$ such that $\varphi = \beta\varphi'\alpha$, with compositions defined in the obvious way. Note that $\mathcal{F}(\mathcal{C})^{\mathrm{op}} \not\simeq \mathcal{F}(\mathcal{C}^{\mathrm{op}})$ in general. Now, there are two natural functors

$$s^{\mathrm{op}}\colon \mathcal{F}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}, \qquad (c \xrightarrow{\varphi} d) \mapsto c, \tag{2-13}$$

$$t^{\mathrm{op}}\colon \mathcal{F}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}, \quad (c \xrightarrow{\varphi} d) \mapsto d, \tag{2-14}$$

called the (opposite) source and target functors, respectively. We have:

Lemma 2.5 (Quillen) *The functors (2-13) and (2-14) are both right homotopy cofinal.*

Proof Since $\mathcal{F}(\mathcal{C})$ is defined by Grothendieck construction (2-11), the canonical (forgetful) functor

$$s \times t\colon \mathcal{F}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$$

is precofibered. It follows (cf [61, Example, page 94]) that both $s\colon \mathcal{F}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ and $t\colon \mathcal{F}(\mathcal{C}) \to \mathcal{C}$ are precofibered. Hence the inclusions $s^{-1}(c) \hookrightarrow s \downarrow c$ and $t^{-1}(d) \hookrightarrow t \downarrow d$ induce weak equivalences of classifying spaces

$$B(s^{-1}(c)) \simeq B(s \downarrow c), \quad B(t^{-1}(d)) \simeq B(t \downarrow d). \tag{2-15}$$

On the other hand, by inspection, $s^{-1}(c) = c \downarrow \mathcal{C}$ and $t^{-1}(d) = (\mathcal{C} \downarrow d)^{\mathrm{op}}$ are the slice and coslice categories respectively. Since both $c \downarrow \mathcal{C}$ and $(\mathcal{C} \downarrow d)^{\mathrm{op}}$ have initial objects, they are contractible for all $c, d \in \mathcal{C}$. To complete the proof it remains to note that $(c \downarrow s^{\mathrm{op}}) = (s \downarrow c)^{\mathrm{op}}$ and $(d \downarrow t^{\mathrm{op}}) = (t \downarrow d)^{\mathrm{op}}$, where s^{op} and t^{op} are the functors (2-13) and (2-14). Hence

$$B(c \downarrow s^{\mathrm{op}}) = B(s \downarrow c)^{\mathrm{op}} \simeq B(s \downarrow c) \simeq B(s^{-1}(c)) \simeq \mathrm{pt},$$

and similarly $B(d \downarrow t^{\mathrm{op}}) \simeq \mathrm{pt}$. This shows that s^{op} and t^{op} are right homotopy cofinal. □

In view of Lemma 2.5, for any diagrams $X\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{M}$ and $Y\colon \mathcal{C} \to \mathcal{M}$ Theorem 2.3 gives two natural weak equivalences

$$s^*\colon \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}(s^*Y) \xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{C}}(Y), \tag{2-16}$$

$$t^*\colon \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}(t^*X) \xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{C}^{\mathrm{op}}}(X). \tag{2-17}$$

These equivalences can be used to express arbitrary homotopy colimits over $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ as homotopy coends which we introduce next. Set

$$\pi^{\mathrm{op}} := t^{\mathrm{op}} \times s^{\mathrm{op}}\colon \mathcal{F}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C},$$

and for a bifunctor $D\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{M}$, define its *homotopy coend* by

$$\int_{\boldsymbol{L}}^{c\in\mathcal{C}} D(c,c) := \operatorname{hocolim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}(\pi^* D), \tag{2-18}$$

where $\pi^* D := D \circ \pi^{\mathrm{op}}\colon \mathcal{F}(\mathcal{C})^{\mathrm{op}} \to \mathcal{M}$. This is indeed the (left) derived functor of the classical coend functor, which is usually denoted by

$$\int^{c\in\mathcal{C}} D(c,c) := \operatorname{colim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}(\pi^* D). \tag{2-19}$$

The notation (2-18) is very convenient as it suggests the analogy with (definite) integrals in calculus. For example, for a bifunctor $D\colon (\mathcal{A}\times\mathcal{B})^{\mathrm{e}} \to \mathcal{M}$ defined on a product of two small categories $(\mathcal{A}\times\mathcal{B})^{\mathrm{e}} := \mathcal{A}^{\mathrm{op}} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \times \mathcal{B}$ there is a natural weak equivalence

$$\int_{\boldsymbol{L}}^{(a,b)\in\mathcal{A}\times\mathcal{B}} D(a,b;a,b) \simeq \int_{\boldsymbol{L}}^{a\in\mathcal{A}} \int_{\boldsymbol{L}}^{b\in\mathcal{B}} D(a,b;a,b),$$

which is analogous to the classical Fubini theorem in calculus (and thus called the Fubini theorem for homotopy coends). Another useful formula that we will need is

$$\int_{\boldsymbol{L}}^{c\in\mathcal{C}} \boldsymbol{L}F[D(c,c)] \simeq \boldsymbol{L}F\left[\int_{\boldsymbol{L}}^{c\in\mathcal{C}} D(c,c)\right], \tag{2-20}$$

where F is a left Quillen functor between model categories. This formula is a consequence of a more general (well-known) result that the derived functors of left Quillen functors preserve homotopy colimits (for a short proof, see eg [74, Proposition 3.15]).

We are now in a position to state the main result of this section.

Proposition 2.6 (Shapiro Lemma for model categories) *Let $\mathcal{M}$ be a model category, $\mathcal{C}$ a small category, and $F\colon \mathcal{C} \to \mathrm{Set}$ a set-valued functor on $\mathcal{C}$. For any contravariant diagram $X\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{M}$ such that $X(c)$ is cofibrant in $\mathcal{M}$ for all $c \in \mathcal{C}$, there is a natural weak equivalence*

$$\operatorname{hocolim}_{\mathcal{C}_F^{\mathrm{op}}}(p^* X) \simeq \int_{\boldsymbol{L}}^{c\in\mathcal{C}} X(c) \otimes F(c), \tag{2-21}$$

where $\mathcal{C}_F$ is the category of elements of F, and $\otimes$ denotes the natural (tensor) action[5] *of* Set *on $\mathcal{M}$.*

For the proof of Proposition 2.6, we need the following observation.

Lemma 2.7 *For any set-valued functor $F\colon \mathcal{C} \to \mathrm{Set}$, the functor $\mathcal{F}(p)^{\mathrm{op}}\colon \mathcal{F}(\mathcal{C}_F)^{\mathrm{op}} \to \mathcal{F}(\mathcal{C})^{\mathrm{op}}$ induced by the canonical projection $p\colon \mathcal{C}_F \to \mathcal{C}$ is precofibered.*

Proof The proof is by direct verification: we give some details in order to introduce notation and make a few observations that we will use later. We set $f := \mathcal{F}(p)^{\mathrm{op}}$ and describe first the fiber category $f^{-1}(\varphi)$

[5] That is, $\otimes$ is the bifunctor $\mathcal{M} \times \mathrm{Set} \to \mathcal{M}$ defined by $A \otimes S = \coprod_S A$, where $\coprod_S A$ is the coproduct of copies of A indexed by the elements of S.

for $(\varphi\colon c\to d)\in\mathcal{F}(\mathcal{C})^{\mathrm{op}}$. The objects of $f^{-1}(\varphi)$ are the morphisms in $\mathcal{C}_F$ of the form $(c,x)\xrightarrow{\varphi}(d,y)$ such that $y=F(\varphi)(x)$. We will write the object $(c,x)\xrightarrow{\varphi}(d,F(\varphi)(x))$ of $\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}$ as (φ,x). Thus,

$$\mathrm{Ob}(f^{-1}(\varphi))=\{(\varphi,x)\mid x\in F(c)\}.$$

Further, the morphisms $(\varphi,x)\to(\varphi,y)$ in $f^{-1}(\varphi)$ are precisely the morphisms in $\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}$, ie commutative diagrams of the form

$$\begin{array}{ccc} (d,F(\varphi)(x)) & \xleftarrow{\ \beta\ } & (d,F(\varphi)(y)) \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ (c,x) & \xrightarrow{\ \alpha\ } & (c,y) \end{array}$$

mapped to the identity by f. This last condition implies that $\alpha=\mathrm{Id}_c$ and $\beta=\mathrm{Id}_d$. Hence,

$$\mathrm{Hom}_{f^{-1}(\varphi)}\big((\varphi,x),(\varphi,y)\big)=\begin{cases}\{\mathrm{Id}\} & \text{if } x=y,\\ \varnothing & \text{otherwise.}\end{cases}$$

Hence, $f^{-1}(\varphi)\cong F(c)$, where the set $F(c)$ is viewed as a discrete category.

Next, for $(\varphi\colon c\to d)\in\mathcal{F}(\mathcal{C})^{\mathrm{op}}$, the objects of $f\downarrow\varphi$ are given by

$$\mathrm{Ob}(f\downarrow\varphi)=\{[\psi\colon k\to l,z,\alpha,\beta]\mid(\psi,z)\in\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}},(\alpha,\beta)\in\mathrm{Hom}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}(\psi,\varphi)\},$$

while the morphisms $[\psi,z,\alpha,\beta]\to[\psi',z',\alpha',\beta']$ in $f\downarrow\varphi$ are the commutative diagrams in $\mathcal{C}_F$ of the form

$$\begin{array}{ccc} (l,F(\psi)(z)) & \xleftarrow{\ \delta\ } & (l',F(\psi')(z')) \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\psi'} \\ (k,z) & \xrightarrow{\ \gamma\ } & (k',z') \end{array}$$

such that

$$\begin{array}{ccccc} l & & \xleftarrow{\ \delta\ } & & l' \\ & \nwarrow{\scriptstyle\beta} & & {\scriptstyle\beta'}\nearrow & \\ {\scriptstyle\psi}\uparrow & & d & & \uparrow{\scriptstyle\psi'} \\ & & \uparrow{\scriptstyle\varphi} & & \\ k & & \xrightarrow{\ \gamma\ } & & k' \\ & {\scriptstyle\alpha}\searrow & & \swarrow{\scriptstyle\alpha'} & \\ & & c & & \end{array}$$

commutes in $\mathcal{C}$. In particular, a morphism $[\psi,z,\alpha,\beta]\to[\varphi,x,\mathrm{Id}_c,\mathrm{Id}_d]$ in $f\downarrow\varphi$ is represented by

$$\begin{array}{ccc} (l,F(\psi)(z)) & \xleftarrow{\ \beta\ } & (d,F(\varphi)(x)) \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ (k,z) & \xrightarrow{\ \alpha\ } & (c,x) \end{array}$$

Such a diagram exists if and only if $x=F(\alpha)(z)$, in which case it is unique. Hence,

$$\mathrm{Hom}_{f\downarrow\varphi}([\psi,z,\alpha,\beta],[\varphi,x,\mathrm{Id}_c,\mathrm{Id}_d])=\begin{cases}\{(\alpha,\beta)\} & \text{if } x=F(\alpha)(z),\\ \varnothing & \text{otherwise,}\end{cases}$$

where (α,β) is viewed as a morphism $(\psi,z)\to(\varphi,x)$ in $\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}$ (rather than $\mathcal{F}(\mathcal{C})^{\mathrm{op}}$).

Now, consider the assignment

$$\Phi\colon f\downarrow\varphi\to f^{-1}(\varphi),\quad [\psi,z,\alpha,\beta]\mapsto(\varphi,F(\alpha)(z)).$$

If $(\gamma,\delta)\colon(\psi,z)\to(\psi',z')$ is a morphism in $f\downarrow\varphi$, then $z'=F(\gamma)(z)$ and $\alpha'\circ\gamma=\alpha$. Hence, letting Φ map (γ,δ) to the identity on $(\varphi,F(\alpha)(z))$ makes Φ a *functor*. We then note that

$$\mathrm{Hom}_{f^{-1}(\varphi)}(\Phi([\psi,z,\alpha,\beta]),(\varphi,x))=\mathrm{Hom}_{f^{-1}(\varphi)}((\varphi,F(\alpha)(z)),(\varphi,x))=\begin{cases}\{\mathrm{Id}\} & \text{if } x=F(\alpha)(z),\\ \varnothing & \text{otherwise.}\end{cases}$$

Hence, there is a natural bijection

$$\mathrm{Hom}_{f^{-1}(\varphi)}(\Phi([\psi,z,\alpha,\beta]),(\varphi,x))\cong\mathrm{Hom}_{f\downarrow\varphi}([\psi,z,\alpha,\beta],[\varphi,x,\mathrm{Id}_c,\mathrm{Id}_d]),$$

showing that Φ is left adjoint to the canonical inclusion

$$f^{-1}(\varphi)\hookrightarrow f\downarrow\varphi,\quad (\varphi,x)\mapsto[\varphi,x,\mathrm{Id}_c,\mathrm{Id}_d].$$

This shows that f is precofibered, as desired. □

Proof of Proposition 2.6 By formula (2-17) (applied to the category $\mathcal{C}_F$), there is a natural weak equivalence

$$t^*\colon \mathrm{hocolim}_{\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}}(t^*p^*X)\xrightarrow{\sim}\mathrm{hocolim}_{\mathcal{C}_F^{\mathrm{op}}}(p^*X),$$

where

$$t^*p^*X\colon\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}\xrightarrow{t^{\mathrm{op}}}\mathcal{C}_F^{\mathrm{op}}\xrightarrow{p^{\mathrm{op}}}\mathcal{C}^{\mathrm{op}}\xrightarrow{X}\mathcal{M}.$$

On the other hand, by definition (2-18),

$$\int_{\boldsymbol{L}}^{c\in\mathcal{C}}X(c)\otimes F(c)=\mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}[\pi^*(X\otimes F)],$$

where

$$\pi^*(X\otimes F)\colon\mathcal{F}(\mathcal{C})^{\mathrm{op}}\xrightarrow{\pi^{\mathrm{op}}}\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\xrightarrow{X\times F}\mathcal{M}\times\mathrm{Set}\xrightarrow{\otimes}\mathcal{M}.$$

To prove the desired proposition we thus need to show that

$$\mathrm{hocolim}_{\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}}(t^*p^*X)\simeq\mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}[\pi^*(X\otimes F)]. \tag{2-22}$$

By Theorem 2.1(1) (see (2-3)), it suffices to show that there is an weak equivalence of $\mathcal{F}(\mathcal{C})^{\mathrm{op}}$–diagrams

$$\boldsymbol{L}f_!(t^*p^*X)\simeq\pi^*(X\otimes F), \tag{2-23}$$

where $f\colon\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}\to\mathcal{F}(\mathcal{C})^{\mathrm{op}}$ is the functor induced by the canonical projection $p\colon\mathcal{C}_F\to\mathcal{C}$. Thanks to Lemma 2.7, we can use Lemma 2.4 to evaluate the homotopy Kan extension in (2-23) in terms of homotopy colimits over fiber categories. Specifically, for any $\varphi\in\mathcal{F}(\mathcal{C})^{\mathrm{op}}$, we have

$$\boldsymbol{L}f_!(t^*p^*X)(\varphi)\simeq\mathrm{hocolim}_{f^{-1}(\varphi)}(i^*t^*p^*X),$$

where $i\colon f^{-1}(\varphi)\hookrightarrow\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}$. In the proof of Lemma 2.7, we have described the fiber category $f^{-1}(\varphi)$: namely, $f^{-1}(\varphi)$ is isomorphic to the discrete category $F(c)$ for any $(\varphi\colon c\to d)\in\mathcal{F}(\mathcal{C})^{\mathrm{op}}$. Now, since $i^*t^*p^*X=i^*f^*t^*X=(fi)^*t^*X=t^*X(\varphi)=X(d)$ and since X is objectwise cofibrant in $\mathcal{M}$, we have

$$\mathrm{hocolim}_{f^{-1}(\varphi)}(i^*t^*p^*X)\simeq\coprod_{F(c)}^{\boldsymbol{L}}X(d)\simeq\coprod_{F(c)}X(d),$$

which is precisely the value of $\pi^*(X \otimes F)$ at φ. Thus, $\boldsymbol{L} f_!(t^* p^* X)\varphi \simeq \pi^*(X \otimes F))\varphi$ in $\mathcal{M}$ for all $\varphi \in \mathcal{F}(\mathcal{C})^{\mathrm{op}}$. By Theorem 2.1(2), this implies (2-23). Summing up, we have constructed the pullback–pushforward diagram

$$\begin{array}{ccc} & \operatorname{hocolim}_{\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}}(t^* p^* X) & \\ {\scriptstyle t^*}\swarrow & & \searrow{\scriptstyle \boldsymbol{L} f_!} \\ \operatorname{hocolim}_{\mathcal{C}_F^{\mathrm{op}}}(p^* X) & & \operatorname{hocolim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}[\pi^*(X \otimes F)] \end{array}$$

each arrow in which is a weak equivalence. This shows that the objects in both sides of (2-21) are weakly equivalent in $\mathcal{M}$, as claimed by the proposition. □

Remark 2.8 The proof of Proposition 2.6 shows that the additional assumption on the diagram X to be objectwise cofibrant in $\mathcal{M}$ is not needed if the coproducts in $\mathcal{M}$ preserve weak equivalences, eg if $\mathcal{M}$ is a cofibrant model category such as the category sSet of simplicial sets with Quillen model structure.

In the special case, if we take $\mathcal{M} = \mathrm{Ch}(\mathrm{Mod}_k)$ to be the category of chain complexes of k–modules equipped with standard projective model structure (see [40, Theorem 2.3.11]), Proposition 2.6 implies the following classical result in homological algebra.

Corollary 2.9 (Shapiro Lemma) *Let k be a commutative ring, $\mathcal{C}$ a small category, and $\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$ the (abelian) category of $\mathcal{C}^{\mathrm{op}}$–diagrams of k–modules. Then, for any functor $F : \mathcal{C} \to \mathrm{Set}$, and for any module $X \in \mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$ such that $X(c)$ is k–projective for all $c \in \mathcal{C}$, there is a natural isomorphism*

$$\mathrm{Tor}_*^{\mathcal{C}_F}(p^* X, k) \cong \mathrm{Tor}_*^{\mathcal{C}}(X, k[F]),$$

where $k[F] : \mathcal{C} \xrightarrow{F} \mathrm{Set} \xrightarrow{k[-]} \mathrm{Mod}_k$ is the k–linear functor generated by F.

The Shapiro Lemma appears in [47, Appendix C.12], where it is proven in the special case $X = k$ (the constant $\mathcal{C}^{\mathrm{op}}$–diagram valued at k); in the general form, the result of Corollary 2.9 is stated, for example, in [26].

As another immediate consequence of Proposition 2.6, we get a derived version of the classical "coend formula" for left Kan extensions; see [51, Theorem X.4.1].

Corollary 2.10 *Let $f : \mathcal{A} \to \mathcal{B}$ be a functor between small categories. Let $X : \mathcal{A} \to \mathcal{M}$ be an $\mathcal{A}$–diagram in a model category $\mathcal{M}$ such that $X(a)$ is cofibrant for all $a \in \mathcal{A}$. Then, for all objects $b \in \mathcal{B}$,*

$$\boldsymbol{L} f_!(X)(b) \simeq \int_{\boldsymbol{L}}^{a \in \mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(f(a), b) \otimes X(a). \tag{2-24}$$

Proof To apply Proposition 2.6 take $\mathcal{C} = \mathcal{A}^{\mathrm{op}}$ and $F = \mathrm{Hom}_{\mathcal{B}}(f(-), b) : \mathcal{C} \to \mathrm{Set}$. Then $\mathcal{C}_F^{\mathrm{op}} \cong f \downarrow b$ and the equivalence (2-24) is obtained as a combination of (2-4) and (2-21). □

The result of Corollary 2.10 must be well known to experts, although we could not find an exact reference in the literature.

3 Representation and cyclic homology of homotopy simplicial groups

In this section, we define representation homology of groups with coefficients in a commutative Hopf algebra $\mathcal{H}$, following the approach of [13; 14]. Taking $\mathcal{H} = \mathcal{O}(G)$, where G is an affine algebraic group, we then construct the derived character maps for G–representations of Γ. In the case when $G = \mathrm{GL}_n$, these maps specialize to the character maps (1-7) announced in the Introduction. Unlike in [13; 14], we will work here with *homotopy* simplicial groups (in the sense of Badzioch [4]), which are more general and flexible objects than the usual (strict) simplicial groups. In Section 3.1, we define the classifying spaces for such groups, and in Section 3.3, the cyclic bar construction and cyclic homology, both of which may be of independent interest. We begin by reviewing the main results of [4] specializing to the algebraic theory of groups.

3.1 Homotopy simplicial groups

Let $\mathfrak{G}$ be the small category whose objects $\langle n\rangle$ are the finitely generated free groups $\mathbb{F}_n = \mathbb{F}\langle x_1, x_2, \dots, x_n\rangle$, one for each $n \geq 0$ (with convention that $\langle 0\rangle$ is the trivial group), and the morphisms are arbitrary group homomorphisms. Every discrete group Γ defines a contravariant functor $\underline{\Gamma}\colon \mathfrak{G}^{\mathrm{op}} \to \mathrm{Set}$, $\langle n\rangle \mapsto \Gamma^n$, which is simply the restriction of the Yoneda functor $\mathrm{Hom}(–, \Gamma)\colon \mathrm{Gr}^{\mathrm{op}} \to \mathrm{Set}$ to $\mathfrak{G} \subset \mathrm{Gr}$. More generally, every simplicial group $\Gamma \in \mathrm{sGr}$ (ie a simplicial object in Gr) defines a functor

$$\underline{\Gamma}\colon \mathfrak{G}^{\mathrm{op}} \to \mathrm{sSet}, \quad \langle n\rangle \mapsto \Gamma^n, \tag{3-1}$$

where Γ^n denotes the product of n copies of the underlying simplicial set of Γ. The functors (3-1) can be characterized by the property of being product-preserving. To make it precise, observe that the category $\mathfrak{G}$ carries a (strict) monoidal structure $\coprod\colon \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ given by the coproduct (free product) of free groups: $\langle n\rangle \coprod \langle m\rangle = \langle n+m\rangle$. The opposite category $\mathfrak{G}^{\mathrm{op}}$ is thus equipped with the dual monoidal structure, which we simply denote by $\Pi\colon \mathfrak{G}^{\mathrm{op}} \times \mathfrak{G}^{\mathrm{op}} \to \mathfrak{G}^{\mathrm{op}}$. Every object $\langle n\rangle^{\mathrm{o}} \in \mathfrak{G}^{\mathrm{op}}$ comes equipped with n natural projections

$$p_{n,k}\colon \langle n\rangle^{\mathrm{o}} \to \langle 1\rangle^{\mathrm{o}} \quad \text{for } 1 \leqslant k \leqslant n \tag{3-2}$$

that correspond to the canonical inclusions $i_{n,k}\colon \langle 1\rangle \hookrightarrow \langle n\rangle$ given by $x_1 \mapsto x_k$ in $\mathfrak{G}$. We say that a functor $\mathcal{F}\colon \mathfrak{G}^{\mathrm{op}} \to \mathrm{sSet}$ is *product-preserving* if the maps induced by (3-2),

$$\mathcal{F}(p_n) := \prod_{k=1}^{n} \mathcal{F}(p_{n,k})\colon \mathcal{F}\langle n\rangle \to (\mathcal{F}\langle 1\rangle)^n, \tag{3-3}$$

are isomorphisms in sSet for all $n \geqslant 0$. It is easy to show that assigning to a simplicial group $\Gamma \in \mathrm{sGr}$ the functor (3-1) defines an equivalence of categories

$$\mathrm{sGr} \xrightarrow{\sim} \mathrm{sSet}_{\otimes}^{\mathfrak{G}^{\mathrm{op}}}, \tag{3-4}$$

where $\mathrm{sSet}_{\otimes}^{\mathfrak{G}^{\mathrm{op}}}$ denotes the full subcategory of product-preserving functors in the diagram category $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$. We will use (3-4) to identify $\mathrm{sGr} = \mathrm{sSet}_{\otimes}^{\mathfrak{G}^{\mathrm{op}}}$, thus regarding the simplicial groups as functors of the form (3-1). Now, the homotopy simplicial groups are obtained by replacing the assumption that the maps (3-3) are isomorphisms in sSet with that of being weak equivalences, which is a more natural condition from the point of view of homotopy theory. Precisely:

Definition 3.1 (Badzioch [4]) A *homotopy simplicial group* is a functor $\mathcal{F}\colon \mathfrak{G}^{\mathrm{op}} \to \mathrm{sSet}$ that is weakly product-preserving in the sense that the maps (3-3) are weak equivalences in sSet for all $n \geqslant 0$ (with the convention that $\mathcal{F}\langle 0\rangle \simeq \mathrm{pt}$).

The category of homotopy simplicial groups (ie the full subcategory of all weakly product-preserving functors in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$) does not carry any model structure as it is not closed under colimits. Instead, as suggested in [4], one can put a new model structure on the diagram category $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ in which the homotopy simplicial groups are exhibited as fibrant objects; cf [4, Proposition 5.5]. We call this model structure the *Badzioch model structure* and denote it by sGr^h. To be precise, sGr^h is defined by localizing (ie taking the left Bousfield localization of) the standard projective model structure on $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ with respect to the set of maps

$$S = \Big\{ i_n\colon \coprod_{k=1}^{n} \mathrm{Hom}_{\mathfrak{G}}(-,\langle 1\rangle) \to \mathrm{Hom}_{\mathfrak{G}}(-,\langle n\rangle) \Big\}_{n\geq 0}$$

induced by the natural inclusions $i_{n,k}\colon \langle 1\rangle \to \langle n\rangle$ in $\mathfrak{G}$. By definition, the underlying category of sGr^h is that of $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ but its class of weak equivalences is larger: in addition to all weak equivalences of $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ (which are objectwise equivalences of diagrams of simplicial sets), the weak equivalences of sGr^h include the set S and are thus called the S*–local* weak equivalences. There is a canonical localization functor $\mathcal{L}_S\colon \mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}} \to \mathrm{sGr}^h$ that takes a diagram $\Gamma \in \mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ to its functorial fibrant replacement in the model structure sGr^h. In this way, one can make any diagram in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ a homotopy simplicial group. On the other hand, the model category of (strict) simplicial groups sGr is related to sGr^h by a Quillen adjunction

$$K\colon \mathrm{sGr}^h \rightleftarrows \mathrm{sGr} : J \tag{3-5}$$

which is obtained by localizing (at S) the Quillen adjunction $K\colon \mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}} \rightleftarrows \mathrm{sGr} : J$ between sGr and the model category of all diagrams $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$. In particular, the right adjoint functor in (3-5) is given by the inclusion $J(\Gamma) = \underline{\Gamma}$ (see (3-1)), while the left adjoint — called the rigidification functor — is described explicitly in Lemma 3.5 below. Now, the main result of [4] reads:

Theorem 3.2 (Badzioch) *The adjunction (3-5) is a Quillen equivalence.*

Remark 3.3 Theorem 3.2 was proved in [4, Theorem 6.4] for an arbitrary one-sorted algebraic theory. It was extended to all multisorted theories in [17], and further to limit theories and to diagrams in model categories other than sSet in [63].

Next, recall that there is a classical adjunction, called the *Kan loop group construction* [42], that relates the model category sGr of (strict) simplicial groups to that of (reduced) simplicial sets:

$$\mathbb{G}\colon \mathrm{sSet}_0 \rightleftarrows \mathrm{sGr} : \overline{W}. \tag{3-6}$$

The left adjoint $\mathbb{G}$ is called the *Kan loop group functor*, and the right adjoint $\overline{W}$ is the *classifying complex functor* on simplicial groups. The properties of these functors are well known and discussed in detail, for example, in [32, Chapter V]; see also [14, Section 2.2]. Here, we mention only two important facts: first, the pair (3-6) is a Quillen equivalence, both $\mathbb{G}$ and $\overline{W}$ being homotopy invariant functors; see [32, V.6.4]. Second, for any reduced simplicial set X, there is a weak homotopy equivalence (see [32, V.5.11])

$$|\mathbb{G}(X)| \simeq \Omega|X|, \tag{3-7}$$

where $\Omega|X|$ is the (Moore) based loop space of the geometric realization of X. The equivalence (3-7) clarifies the topological meaning of the Kan loop group functor $\mathbb{G}$ (and justifies its name). Combining now Badzioch's theorem (Theorem 3.2) with Kan's construction, we get natural equivalences of homotopy categories

$$\mathtt{Ho}(\mathrm{sGr}^h) \xrightarrow{\boldsymbol{L}K} \mathtt{Ho}(\mathrm{sGr}) \xrightarrow{\overline{W}} \mathtt{Ho}(\mathrm{sSet}_0) \xrightarrow{|-|} \mathtt{Ho}(\mathrm{Top}_{0,*}) \tag{3-8}$$

induced by the above indicated functors. This leads us to the following definition.

Definition 3.4 For a homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$, we define its *classifying space* $B\Gamma$ by

$$B\Gamma := |\overline{W}\boldsymbol{L}K(\Gamma)| \tag{3-9}$$

where $\boldsymbol{L}K\colon \mathtt{Ho}(\mathrm{sGr}^h) \to \mathtt{Ho}(\mathrm{sGr})$ is the derived rigidification functor; see (3-11).

Note that if Γ is a (strict) simplicial group, ie $\Gamma = J(\Gamma)$, then $B\Gamma \cong |\overline{W}\Gamma|$, since $\boldsymbol{L}K \circ J \cong \mathrm{Id}$. Thus the above definition is a natural extension of Kan's definition of classifying spaces for simplicial groups (which is, in turn, an extension of the classical definition of $B\Gamma$ for ordinary discrete groups).

We conclude this section by giving a simple formula for the Badzioch rigidification functor that did not seem to appear explicitly in [4].

Lemma 3.5 *The functor* $K\colon \mathrm{sGr}^h \to \mathrm{sGr}$ *in* (3-5) *is given by the coend*

$$K(\Gamma) = \int^{\langle n\rangle\in\mathfrak{G}} \Gamma\langle n\rangle \otimes \mathbb{F}\langle n\rangle, \tag{3-10}$$

where $\mathbb{F}\colon \mathfrak{G} \hookrightarrow \mathrm{sGr}$ *given by* $\langle n\rangle \mapsto \mathbb{F}_n$ *is the natural inclusion functor, and* $\otimes\colon \mathrm{sSet}\times\mathrm{sGr} \to \mathrm{sGr}$ *is the standard simplicial tensor action on the category of simplicial groups.*

It follows from Lemma 3.5 that the derived functor $\boldsymbol{L}K$ can be written as the homotopy coend

$$\boldsymbol{L}K(\Gamma) = \int_{\boldsymbol{L}}^{\langle n\rangle\in\mathfrak{G}} \Gamma\langle n\rangle \otimes \mathbb{F}\langle n\rangle. \tag{3-11}$$

For the proof of Lemma 3.5 and formula (3-11) (in the general setting of [4]) we refer to our forthcoming paper [10].

3.2 Representation homology

Let k be a commutative ring. Recall that, for a small category $\mathcal{C}$, we denote by $\mathrm{Mod}_k(\mathcal{C})$ and $\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$ the categories of all covariant and contravariant functors from $\mathcal{C}$ to Mod_k, respectively. It is well known that these are abelian categories with sufficiently many projective and injective objects. Recall also (see eg [47, Appendix C.10]) that there is a natural biadditive functor

$$-\otimes_{\mathcal{C}} -\colon \mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}}) \times \mathrm{Mod}_k(\mathcal{C}) \to \mathrm{Mod}_k$$

called the *functor tensor product*. Explicitly, for $\mathcal{M}\colon \mathcal{C} \to \mathrm{Mod}_k$ and $\mathcal{N}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_k$, it is defined by

$$\mathcal{N} \otimes_{\mathcal{C}} \mathcal{M} := \Big[\bigoplus_{c\in\mathcal{C}} \mathcal{N}(c) \otimes_k \mathcal{M}(c) \Big] / R \tag{3-12}$$

where R is the k–submodule spanned by elements of the form $\mathcal{N}(\varphi)x \otimes y - x \otimes \mathcal{M}(\varphi)y$ for all $x \in \mathcal{N}(c')$, $y \in \mathcal{M}(c)$ and $\varphi \in \mathrm{Hom}_{\mathcal{C}}(c, c')$. The functor (3-12) is right exact (with respect to each argument), preserves sums, and is left balanced. Its classical (left) derived functors with respect to each argument are canonically isomorphic and their common value is denoted by $\mathrm{Tor}_*^{\mathcal{C}}(\mathcal{N}, \mathcal{M})$. More generally, we can extend the bifunctor (3-12) to chain complexes of $\mathcal{C}$–modules, ie the categories $\mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}}))$ and $\mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}))$, and define

$$\mathrm{Tor}_*^{\mathcal{C}}(\mathcal{N}, \mathcal{M}) := \mathrm{H}_*(\mathcal{N} \otimes_{\mathcal{C}}^{\boldsymbol{L}} \mathcal{M}) \tag{3-13}$$

for any $\mathcal{N} \in \mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}}))$ and $\mathcal{M} \in \mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}))$. Note that $\mathcal{N} \otimes_{\mathcal{C},k}^{\boldsymbol{L}} \mathcal{M}$ is an object in the (unbounded) derived category $\mathcal{D}(k) = \mathcal{D}(\mathrm{Mod}_k)$ of k–modules, and (3-13) is just the usual hyper-Tor functor on chain complexes. Next, observe that there is a natural functor

$$\mathrm{sSet}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{k[-]} \mathrm{sMod}_k(\mathcal{C}^{\mathrm{op}}) \xhookrightarrow{N} \mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})) \tag{3-14}$$

transforming the $\mathcal{C}^{\mathrm{op}}$–diagrams in sSet (simplicial presheaves on $\mathcal{C}$) to chain complexes over $\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$. Here N stands for the classical Dold–Kan normalization functor that identifies simplicial objects in $\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$ with nonnegatively graded chain complexes in $\mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}}))$. Abusing notation, we will write the functor (3-14) simply as $k[–]$.

We are now in a position to define representation homology of homotopy simplicial groups with coefficients in commutative Hopf algebras. We recall the well-known fact (see eg [62, Proposition 14.1.6]) that every such algebra $\mathcal{H}$ defines a covariant functor (a left $\mathfrak{G}$–module) by the rule

$$\underline{\mathcal{H}}\colon \mathfrak{G} \to \mathrm{Mod}_k, \quad \langle n \rangle \mapsto \mathcal{H}^{\otimes n}. \tag{3-15}$$

In particular, if G is an affine algebraic group (eg $G = \mathrm{GL}_n(k)$) with coordinate ring $\mathcal{H} = \mathcal{O}(G)$, then (3-15) can be written in the form $\langle n \rangle \mapsto \mathcal{O}[\mathrm{Rep}_G(\langle n \rangle)]$, which makes the functoriality clear.

Definition 3.6 The *representation homology* of a homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$ with coefficients in $\mathcal{H}$ is defined by

$$\mathrm{HR}_*(\Gamma, \mathcal{H}) := \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \underline{\mathcal{H}}),$$

where $k[\Gamma]$ and $\underline{\mathcal{H}}$ are viewed as chain complexes of $\mathfrak{G}$–modules and Tor stands for the hyper-Tor functor over $\mathfrak{G}$ defined by (3-13).

In the special case when G is an affine algebraic group over k and $\mathcal{H} = \mathcal{O}(G)$, we simply write $\mathrm{HR}_*(\Gamma, G)$ instead of $\mathrm{HR}_*(\Gamma, \mathcal{O}(G))$.

The next lemma shows that the above definition agrees with the Badzioch model structure on sGr^h.

Lemma 3.7 *If two homotopy simplicial groups* Γ *and* Γ' *are weakly equivalent in* sGr^h*, then*

$$\mathrm{HR}_*(\Gamma, \mathcal{H}) \cong \mathrm{HR}_*(\Gamma', \mathcal{H}) \tag{3-16}$$

for any commutative Hopf algebra $\mathcal{H}$*.*

Proof By [4, Proposition 5.6], if two homotopy simplicial groups Γ and Γ' are S–locally weakly equivalent, then their underlying diagrams are, in fact, weakly equivalent in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$. It therefore suffices to show that (3-16) holds for any objectwise weak equivalent diagrams $\Gamma, \Gamma' \colon \mathfrak{G}^{\mathrm{op}} \to \mathrm{sSet}$. To this end, observe that the linearization functor

$$k[\text{–}] \colon \mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}} \to \mathrm{sMod}_k(\mathfrak{G}^{\mathrm{op}}) \tag{3-17}$$

is left Quillen with respect to the projective model structures (its right adjoint is the forgetful functor). Since the weak equivalences in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ are defined objectwise and the model structure on sSet is cofibrant, being left Quillen, the functor (3-17) is actually homotopy invariant: ie it maps weakly equivalent objects in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ to weakly equivalent objects in $\mathrm{sMod}_k(\mathfrak{G}^{\mathrm{op}})$, which, in turn, are transformed by the normalization functor N to quasi-isomorphic complexes in $\mathrm{Ch}(\mathrm{Mod}_k(\mathfrak{G}^{\mathrm{op}}))$. Thus if $\Gamma \simeq \Gamma'$ in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$, then $k[\Gamma] \otimes^{\boldsymbol{L}}_{\mathfrak{G},k} \underline{\mathcal{H}} \simeq k[\Gamma'] \otimes^{\boldsymbol{L}}_{\mathfrak{G},k} \underline{\mathcal{H}}$ in $\mathcal{D}(k)$, which implies (3-16). □

Remark 3.8 Recall that the category sGr^h is obtained from $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ via a left Bousfield localization: its objects are arbitrary diagrams of simplicial sets $\Gamma \colon \mathfrak{G}^{\mathrm{op}} \to \mathrm{sSet}$ (not just homotopy simplicial groups). The result of Lemma 3.7 does *not* hold for arbitrary diagrams in sGr^h, since the functor (3-17) does not map all S–local weak equivalences to objectwise weak equivalences in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$. This last fact can be easily seen by evaluating (3-17) on representable simplicial presheaves on $\mathfrak{G}$.

An important consequence of Lemma 3.7 is that the representation homology of a homotopy simplicial group Γ depends only on the homotopy type of its classifying space $B\Gamma$ (Definition 3.4). In fact, we have

$$\mathrm{HR}_*(\Gamma, \mathcal{H}) \cong \mathrm{HR}_*(B\Gamma, \mathcal{H}), \tag{3-18}$$

where the HR on the right-hand side stands for representation homology of topological spaces as defined in [14], using a (nonabelian) derived representation functor. Indeed, by Badzioch [4, Theorem 3.1], every homotopy simplicial group Γ is weakly equivalent to a strict one, say Γ'; hence

$$B\Gamma \simeq B\Gamma' \simeq \overline{W}\Gamma'. \tag{3-19}$$

On the other hand, by [14, Theorem 4.2], $\mathrm{HR}_*(\Gamma', \mathcal{H}) \cong \mathrm{HR}_*(\overline{W}\Gamma', \mathcal{H})$, which together with (3-19) and the isomorphism (3-16) of Lemma 3.7 implies (3-18).

We conclude this section by briefly explaining how our approach (Definition 3.6) relates to derived algebraic geometry (DAG). For a model of DAG, we will take the simplicial presheaf model developed in [69]. Given a homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$ and an affine algebraic group (scheme) G over k with coordinate algebra $\mathcal{H} = \mathcal{O}(G)$, we introduce the *derived representation scheme of* Γ *in* G:

$$\mathrm{DRep}_G(\Gamma) := \mathbf{R}\mathrm{Spec}(k[\Gamma] \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G)). \tag{3-20}$$

Here, $\mathbf{R}$Spec denotes the Toën–Vezzosi derived Yoneda functor that assigns to a (homotopy) simplicial commutative algebra A — a derived ring in terminology of [69] — the simplicial presheaf (prestack)

$$\mathbf{R}\mathrm{Spec}(A)\colon \mathrm{dAff}_k^{\mathrm{op}} := \mathrm{sComm}_k \to \mathrm{sSet}, \quad B \mapsto \mathrm{Map}(QA, B),$$

where QA is a cofibrant replacement of A and Map is the simplicial mapping space (function complex) in sComm_k. The prestack $\mathbf{R}\mathrm{Spec}(A)$ satisfies the descent condition for étale hypercoverings and hence defines a derived stack (which is a derived affine scheme in the sense of [69]). On the other hand, for any pointed space (simplicial set) X, we can define the *pointed* mapping stack $\mathbf{Map}_*(X, BG)$ to be the homotopy fiber of the canonical map in the (homotopy) category of derived stacks:

$$\mathbf{Map}_*(X, BG) := \mathrm{hofib}[\mathbf{Map}(X, BG) \to BG], \tag{3-21}$$

where $\mathbf{Map}(X, BG)$ stands for the (unpointed) derived mapping stack defined in [69, 2.2.6.2]. This last mapping stack is a basic object of derived algebraic geometry that plays an important role in applications; see eg [55]. Now, its relation to representation homology is clarified by the following:

Proposition 3.9 [12] *There is a* (*weak*) *equivalence of derived stacks*

$$\mathrm{DRep}_G(\Gamma) \simeq \mathbf{Map}_*(B\Gamma, BG).$$

For a detailed proof of Proposition 3.9 and more explanations we refer to [12, Appendix A.1].

3.3 Cyclic homology

We now define cyclic homology for homotopy simplicial groups. To this end, we will associate to each $\Gamma \in \mathrm{sGr}^h$ a cyclic module $k[B^{\mathrm{cyc}}\Gamma]$ that generalizes the classical cyclic bar construction $C_*(k[\Gamma])$ when Γ is an ordinary discrete group. We begin by recalling basic definitions.

Let Δ denote the (co)simplicial category whose objects are finite ordered sets $[n]=\{0<1<\cdots<n\}$ and morphisms are (nonstrictly) order-preserving maps. The category Δ is generated by two families of maps,

$$d_n^i\colon [n-1]\to[n] \quad \text{for } 0\leqslant i\leqslant n \text{ and } n\geqslant 1,$$
$$s_n^j\colon [n+1]\to[n] \quad \text{for } 0\leqslant j\leqslant n \text{ and } n\geqslant 0,$$

called the (co)face and (co)degeneracy maps, respectively. These maps satisfy the standard (co)simplicial relations listed, for example, in [47, Appendix B.3]. Connes' *cyclic category* ΔC is a natural extension of Δ that has the same objects and is generated by the morphisms of Δ and the cyclic maps $\tau_n\colon[n]\to[n]$, $n\geqslant 0$, satisfying $\tau_n^{n+1}=\mathrm{Id}$; see [47, 6.11]. Formally, the category ΔC can be characterized by the two properties

(Cyc1) For each $n\geqslant 0$, $\mathrm{Aut}_{\Delta C}([n])\cong C_{n+1}$, where $C_{n+1}=\mathbb{Z}/(n+1)\mathbb{Z}$, and

(Cyc2) any morphism $f\colon[n]\to[m]$ in ΔC can be factored uniquely as $f=g\circ\varphi$, where $g\in\mathrm{Hom}_\Delta([n],[m])$ and $\varphi\in\mathrm{Aut}_{\Delta C}([n])$.

These show that it is a crossed simplicial category associated to the family of cyclic groups $\{C_{n+1}\}_{n\geqslant 0}$; see [47, 6.3.0]. Recall that a *cyclic set* (resp. a *cyclic module*) is defined to be a contravariant functor on ΔC, ie $\Delta C^{\mathrm{op}}\to\mathrm{Set}$ (resp. $\Delta C^{\mathrm{op}}\to\mathrm{Mod}_k$), while a *cocyclic set* (resp. a *cocyclic module*) is a covariant functor $\Delta C\to\mathrm{Set}$ (resp. $\Delta C\to\mathrm{Mod}_k$).

Now, if Γ is an ordinary discrete group, there is a natural functor

$$B_*^{\mathrm{cyc}}\Gamma\colon \Delta C^{\mathrm{op}}\to\mathrm{Set} \tag{3-22}$$

called the cyclic bar construction of Γ that has the property that $k[B_*^{\mathrm{cyc}}\Gamma]\cong C_*(k[\Gamma])$, where $C_*(k[\Gamma])$ is the standard cyclic module associated to $k[\Gamma]$ as an associative k–algebra. Explicitly, the functor (3-22) is defined (see [47, 7.3.10]) by

$$d_i(g_0,\dots,g_n)=\begin{cases}(g_0,\dots,g_{i-1},g_ig_{i+1},\dots,g_n) & \text{if } 0\leqslant i<n,\\ (g_ng_0,g_1,\dots,g_{n-1}) & \text{if } i=n,\end{cases}$$
$$s_j(g_0,\dots,g_n)=(g_0,\dots,g_j,1,g_{j+1},\dots,g_n),$$
$$t_n(g_0,\dots,g_n)=(g_n,g_0,g_1,\dots,g_{n-1}),$$

where $(g_0,\dots,g_n)\in\Gamma^{n+1}$. Clearly, $\Gamma\mapsto B_*^{\mathrm{cyc}}\Gamma$ gives a functor $B_*^{\mathrm{cyc}}\colon\mathrm{Gr}\to\mathrm{Set}^{\Delta C^{\mathrm{op}}}$. If we identify $\mathrm{Gr}=\mathrm{Set}_\otimes^{\mathfrak{G}^{\mathrm{op}}}$ as in (3-4), then it turns out that B_*^{cyc} coincides with the pullback functor for a certain natural map $\Psi_{\mathrm{cyc}}\colon\Delta C\to\mathfrak{G}$ in Cat. Specifically,

$$\Psi_{\mathrm{cyc}}\colon\Delta C\to\mathfrak{G} \tag{3-23}$$

is defined on objects by

$$\Psi_{\mathrm{cyc}}([n]):=\langle n+1\rangle=\mathbb{F}\langle x_0,\dots,x_n\rangle,$$

and on morphisms by the formulas

$$\Psi_{\mathrm{cyc}}(d_n^i)\colon \langle n\rangle \to \langle n+1\rangle,$$

$$(x_0, x_1, \dots, x_{n-1}) \mapsto \begin{cases} (x_0, \dots, x_{i-1}, x_i x_{i+1}, \dots, x_n) & \text{if } 0 \le i < n, \\ (x_n x_0, x_1, \dots, x_{n-1}) & \text{if } i = n, \end{cases} \tag{3-24}$$

$$\Psi_{\mathrm{cyc}}(s_n^j)\colon \langle n+2\rangle \to \langle n+1\rangle, \quad (x_0, \dots, x_{n+1}) \mapsto (x_0, \dots, x_j, 1, x_{j+1}, \dots, x_n),$$

$$\Psi_{\mathrm{cyc}}(\tau_n)\colon \langle n+1\rangle \to \langle n+1\rangle, \quad (x_0, x_1, \dots, x_n) \mapsto (x_n, x_0, x_1, \dots, x_{n-1}).$$

where $(x_0, x_1, \dots, x_n)$ is an ordered sequence of generators of the free group $\mathbb{F}\langle x_0, \dots, x_n\rangle$.

Lemma 3.10 *For any discrete group Γ there is a natural isomorphism of cyclic sets*

$$B^{\mathrm{cyc}}\Gamma \cong \Psi_{\mathrm{cyc}}^*(\underline{\Gamma}),$$

where $\underline{\Gamma}\colon \mathfrak{G}^{\mathrm{op}} \to \mathrm{Set}$ is the functor corresponding to Γ under the identification (3-1).

Proof Straightforward. □

Remark 3.11 The functor (3-23) was defined in [16] on a slightly larger — the so-called epicyclic — category $\Delta\Psi$, which is an extension of ΔC describing the Adams operations on cyclic modules.

Lemma 3.10 motivates the following definition.

Definition 3.12 For a homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$, we define its *cyclic bar construction* by

$$B^{\mathrm{cyc}}\Gamma := \Psi_{\mathrm{cyc}}^*(\Gamma)\colon \Delta C^{\mathrm{op}} \to \mathrm{sSet}, \tag{3-25}$$

and its *cyclic homology* by

$$\mathrm{HC}_*(k[\Gamma]) := \mathrm{Tor}_*^{\Delta C^{\mathrm{op}}}(k, k[B^{\mathrm{cyc}}\Gamma]) \cong \mathrm{Tor}_*^{\Delta C}(k[B^{\mathrm{cyc}}\Gamma], k). \tag{3-26}$$

The same argument as in (the proof of) Lemma 3.7 shows that $\mathrm{HC}_*(k[\Gamma])$ depends only on the homotopy type of Γ in the Badzioch model category sGr^h, and hence, on the homotopy type of its classifying space $B\Gamma$. In view of Lemma 3.10, the above definition of $\mathrm{HC}_*(k[\Gamma])$ for Γ an ordinary discrete group coincides with the classical (Connes') definition of cyclic homology of group algebras; see [47, 6.2.8].

3.4 Derived character maps

Next, we will construct a family of natural transformations relating the cyclic homology to representation homology of a homotopy simplicial group Γ. In the special case when $\mathcal{H} = \mathcal{O}(\mathrm{GL}_n)$, this family contains a distinguished element determined by the usual trace Tr_n, that gives the derived character map (1-7) announced in the introduction. With our current definitions of representation and cyclic homology the construction is actually very simple. It is based on two lemmas. The first one is a standard result of homological algebra that simply exhibits the naturality of derived tensor products (3-13).

Lemma 3.13 *Let $f:\mathcal{A}\to\mathcal{B}$ be a functor between small categories. For any complexes $\mathcal{N}\in\mathrm{Ch}(\mathrm{Mod}_k\mathcal{B}^{\mathrm{op}})$ and $\mathcal{M}\in\mathrm{Ch}(\mathrm{Mod}_k\mathcal{B})$, there is a natural map $f^*\mathcal{N}\otimes^{\boldsymbol{L}}_{\mathcal{A},k} f^*\mathcal{M}\to\mathcal{N}\otimes^{\boldsymbol{L}}_{\mathcal{B},k}\mathcal{M}$ in the derived category $\mathcal{D}(k)$ of k–modules that induces*

$$f^*:\mathrm{Tor}^{\mathcal{A}}_*(f^*\mathcal{N},f^*\mathcal{M})\to\mathrm{Tor}^{\mathcal{B}}_*(\mathcal{N},\mathcal{M}).$$

To apply this lemma in our situation we recall that every commutative Hopf k–algebra $\mathcal{H}$ defines the covariant functor $\underline{\mathcal{H}}:\mathfrak{G}\to\mathrm{Mod}_k$ by formula (3-15). Restricting this functor via the morphism (3-23) gives rise to a cocyclic k–module that we denote by

$$B_{\mathrm{cyc}}\mathcal{H}:=\Psi^*_{\mathrm{cyc}}(\underline{\mathcal{H}}):\Delta C\to\mathrm{Mod}_k.$$

On the other hand, by Definition 3.12, $\Psi^*_{\mathrm{cyc}}(k[\Gamma])=k[B^{\mathrm{cyc}}(\Gamma)]$ for any homotopy simplicial group Γ. Thus, by Lemma 3.13, the functor Ψ_{cyc} induces a canonical map

$$\Psi^*_{\mathrm{cyc}}:\mathrm{Tor}^{\Delta C}_*(k[B^{\mathrm{cyc}}\Gamma],B_{\mathrm{cyc}}\mathcal{H})\to\mathrm{Tor}^{\mathfrak{G}}_*(k[\Gamma],\underline{\mathcal{H}}). \tag{3-27}$$

The target of this map is precisely $\mathrm{HR}_*(\Gamma,\mathcal{H})$ (see Definition 3.6), while the domain differs from $\mathrm{HC}_*(k[\Gamma])$ in the second argument of Tor (cf Definition 3.12). To connect the two Tors we will use the following lemma, which we state in the language of affine algebraic groups.

Lemma 3.14 *Let G be an affine algebraic group defined over k, and let $\mathcal{O}(G)$ be its coordinate algebra. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Mod}_k(\Delta C)}(k,B_{\mathrm{cyc}}[\mathcal{O}(G)])\cong\mathcal{O}(G)^G, \tag{3-28}$$

where $\mathcal{O}(G)^G$ denotes the invariant subalgebra of $\mathcal{O}(G)$ under the adjoint G–action.

Proof For $m\geq 0$, denote by $\pi_m:[0]\to[m]$ the composition of maps $d^0_m d^0_{m-1}\cdots d^0_1$ in ΔC. It follows from (3-24) that $\Psi_{\mathrm{cyc}}(\pi_m):\langle 1\rangle\to\langle m+1\rangle$ is the homomorphism of groups taking the generator x of $\mathbb{F}\langle x\rangle$ to the product of generators $x_0x_1\ldots x_m$ in $\mathbb{F}\langle x_0,\ldots,x_m\rangle$. The corresponding map $[B_{\mathrm{cyc}}\mathcal{O}(G)](\pi_m):\mathcal{O}(G)\to\mathcal{O}(G)^{\otimes(m+1)}$ can thus be identified with the m–fold coproduct in $\mathcal{O}(G)$,

$$\Delta^{(m)}_G:\mathcal{O}(G)\to\mathcal{O}(G^{m+1}),\qquad P\mapsto[(g_0,g_1,\ldots,g_m)\mapsto P(g_0g_1\cdots g_m)]. \tag{3-29}$$

Now, it is easy to check that, for a fixed $P\in\mathcal{O}(G)^G$, the maps $\Delta^{(m)}_G(P):k\to\mathcal{O}(G^{m+1})$ taking $1\in k$ to $\Delta^{(m)}_G(P)$ assemble to a morphism of cocylic modules $\Delta_G(P):k\to B_{\mathrm{cyc}}[\mathcal{O}(G)]$, the commutativity with cyclic operators τ_m being a consequence of the G–invariance of P. We claim that the assignment $P\mapsto\Delta_G(P)$ defines a k–linear isomorphism

$$\Delta_G:\mathcal{O}(G)^G\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{Mod}_k(\Delta C)}(k,B_{\mathrm{cyc}}[\mathcal{O}(G)]). \tag{3-30}$$

The inverse of (3-30) can be constructed as follows. Let $\varphi\in\mathrm{Hom}_{\mathrm{Mod}_k(\Delta C)}(k,B_{\mathrm{cyc}}[\mathcal{O}(G)])$. Note that, for all $[m]\in\Delta C$, its components $\varphi_{[m]}:k\to\mathcal{O}(G)^{\otimes(m+1)}\cong\mathcal{O}(G^{m+1})$ are k–linear maps. Define $T\varphi:=\varphi_{[0]}(1)\in\mathcal{O}(G)$, where $1\in k$. Since φ is a natural transformation,

$$\varphi_{[m]}(1)=\{[B_{\mathrm{cyc}}\mathcal{O}(G)](\pi_m)\}(\varphi_{[0]}(1))=\{[B_{\mathrm{cyc}}\mathcal{O}(G)](\pi_m)\}(T\varphi)=\Delta^m(T\varphi),$$

where Δ^m is defined in (3-29). Similarly, applying $B_{\mathrm{cyc}}[\mathcal{O}(G)]$ to the cyclic operators τ_m in ΔC, we have

$$\varphi_{[m]}(1) = \{[B_{\mathrm{cyc}}\mathcal{O}(G)](\tau_m)\}(\varphi_{[m]}(1)),$$

from which it follows that $T\varphi(g_0 \cdots g_m) = T\varphi(g_m g_0 \cdots g_{m-1})$ for all $g_0, \dots, g_m \in G$. This is equivalent to the assertion that $T\varphi \in \mathcal{O}(G)^G$. Thus T defines a k–linear map

$$\mathrm{Hom}_{\mathrm{Mod}_k(\Delta C)}(k, B_{\mathrm{cyc}}[\mathcal{O}(G)]) \to \mathcal{O}(G)^G, \quad \varphi \mapsto T\varphi.$$

It is clear from its construction that the above map is the inverse of (3-30). □

We can now make the following definition.

Definition 3.15 Let $\Gamma \in \mathrm{sGr}^h$ be a homotopy simplicial group. For an affine algebraic group G and an $\mathrm{Ad}\, G$–invariant polynomial $P \in \mathcal{O}(G)^G$, we define the *derived G–character map of* Γ *associated to* P by

$$\chi_{G,P}(\Gamma)_* \colon \mathrm{HC}_*(k[\Gamma]) \xrightarrow{(\Delta_G P)_*} \mathrm{Tor}_*^{\Delta C}(k[B^{\mathrm{cyc}}\Gamma], B_{\mathrm{cyc}}[\mathcal{O}(G)]) \xrightarrow{\Psi^*_{\mathrm{cyc}}} \mathrm{HR}_*(\Gamma, G), \tag{3-31}$$

where $(\Delta_G P)_*$ is a linear map induced by the map of cocyclic modules $\Delta_G P \colon k \to B_{\mathrm{cyc}}[\mathcal{O}(G)]$ (see (3-29) and (3-30)), and Ψ^*_{cyc} is the map (3-27) defined for $\mathcal{H} = \mathcal{O}(G)$.

Explicitly, if we choose a projective resolution $Q \xrightarrow{\sim} k[\Gamma]$ of $k[\Gamma]$ in the (abelian) category $\mathrm{Mod}_k(\mathfrak{G}^{\mathrm{op}})$, applying the functor Ψ^*_{cyc} gives a projective resolution $\Psi^*_{\mathrm{cyc}} Q \xrightarrow{\sim} k[B^{\mathrm{cyc}}\Gamma]$ of the cyclic module $k[B^{\mathrm{cyc}}\Gamma]$ in $\mathrm{Mod}_k(\Delta C^{\mathrm{op}})$. The map (3-31) is then induced by a map of chain complexes

$$\chi_{G,P}(\Gamma)_* \colon (\Psi^*_{\mathrm{cyc}} Q) \otimes_{\Delta C} k \to Q \otimes_{\mathfrak{G}} \mathcal{O}(G) \tag{3-32}$$

which, in turn, is induced by the following map (see (3-12))

$$\bigoplus_{[m] \in \Delta C} Q\langle m+1 \rangle \to \bigoplus_{\langle n \rangle \in \mathfrak{G}} Q\langle n \rangle \otimes \mathcal{O}(G)^{\otimes n}, \quad v_{m+1} \mapsto v_{m+1} \otimes \Delta_G^{(m)}(P), \tag{3-33}$$

where $v_{m+1} \in Q\langle m+1 \rangle$ and $\Delta_G^{(m)}(P) \in \mathcal{O}(G)^{\otimes(m+1)}$ is defined by (3-29).

In the special case when $G = \mathrm{GL}_n(k)$ and $P = \mathrm{Tr}_n \in \mathcal{O}(\mathrm{GL}_n)$ is the usual trace function on $(n \times n)$–matrices, we denote the map (3-31) by

$$\mathrm{Tr}_n(\Gamma)_* \colon \mathrm{HC}_*(k[\Gamma]) \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_n(k)), \tag{3-34}$$

and call it the *derived character map of n–dimensional representations of* Γ. In the rest of the paper, we will study the maps $\mathrm{Tr}_n(\Gamma)_*$ in two extreme cases: $n = 1$ and $n = \infty$. In the first case, we will give a topological realization of $\mathrm{Tr}(\Gamma)_* := \mathrm{Tr}_1(\Gamma)_*$ by showing that this map is induced on homology by a natural map of topological spaces; in the second case, we will show that $\mathrm{Tr}_\infty(\Gamma)_* := \varprojlim \mathrm{Tr}_n(\Gamma)_*$ extends to an isomorphism between the graded symmetric algebra generated by $\overline{\mathrm{HC}}_*(k[\Gamma])$ and the GL_∞–invariant subalgebra of the stable representation homology $\mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty(k))$. We close this section with a general remark linking the above construction to earlier work.

Remark 3.16 If Γ is an ordinary discrete or (strict) simplicial group, then $k[\Gamma]$ is naturally a simplicial associative k–algebra. By (a monoidal version of) the classical Dold–Kan correspondence (see [64]), we can therefore view $k[\Gamma]$ as a differential-graded (DG) associative k–algebra. For such algebras (defined over a field k of characteristic 0), the derived character maps of n–dimensional representations were constructed in [9]. One can show that these maps agree with (3-34) in the case of group algebras, although the comparison is not entirely trivial as the methods used in [9] and the present paper are quite different. We will address this question in our forthcoming paper [10] in greater generality.

4 Topological realization of derived character maps

In this and the next sections, we will prove our main results (Theorems 1.1 and 1.2) stated in the introduction. Here we will construct the required spaces and maps simplicially: in terms of homotopy colimits of small diagrams of simplicial sets and associated natural maps. Then, in the next section, we will reproduce these maps in topological terms, using Goodwillie homotopy calculus and topological operads. The connection between the two approaches seems instructive and deserves a further investigation.

4.1 The space X_Γ

Recall that $\mathfrak{G}$ denotes the skeleton of the category of finitely generated free groups. There is a natural *abelianization functor*

(4-1) $$\underline{\mathbb{Z}}\colon \mathfrak{G} \to \mathrm{Set}, \quad \langle n\rangle \mapsto \mathbb{Z}^n,$$

that takes the free group $\langle n\rangle = \mathbb{F}_n$ to (the underlying set of) its abelianization $\langle n\rangle_{\mathrm{ab}} = \mathbb{Z}^n$. As in Section 2, we can form the category of elements of (4-1), using the Grothendieck construction

(4-2) $$\mathfrak{G}_{\mathbb{Z}} := \mathfrak{G} \textstyle\int \underline{\mathbb{Z}}.$$

The objects of $\mathfrak{G}_{\mathbb{Z}}$ are given explicitly by

$$\mathrm{Ob}(\mathfrak{G}_{\mathbb{Z}}) = \{(\langle n\rangle; k_1, \dots, k_n)) \mid \langle n\rangle \in \mathfrak{G}, (k_1, \dots, k_n) \in \mathbb{Z}^n\}$$

and the morphism sets are

$$\mathrm{Hom}_{\mathfrak{G}_{\mathbb{Z}}}((\langle n\rangle; k_1, \dots, k_n), (\langle m\rangle; l_1, \dots, l_m)) = \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(\langle n\rangle, \langle m\rangle) \mid \varphi_{\mathrm{ab}}(k_1, \dots, k_n) = (l_1, \dots, l_m)\}.$$

Note that the abelianized map $\varphi_{\mathrm{ab}}\colon \mathbb{Z}^n \to \mathbb{Z}^m$ above is represented by an integral $(m \times n)$–matrix, $\varphi_{\mathrm{ab}} \in \mathbb{M}_{m\times n}(\mathbb{Z})$, and its action on n–tuples of integers is simply given by matrix multiplication. The category (4-2) comes together with the canonical (forgetful) functor

(4-3) $$p\colon \mathfrak{G}_{\mathbb{Z}} \to \mathfrak{G}, \quad (\langle n\rangle; k_1, \dots, k_n) \mapsto \langle n\rangle.$$

Given a homotopy simplicial group $\Gamma\colon \mathfrak{G}^{\mathrm{op}} \to \mathrm{sSet}$, we now define

(4-4) $$X_\Gamma := |\mathrm{hocolim}_{\mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}}(p^*\Gamma)|,$$

where p^* is the pullback functor $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}} \to \mathrm{sSet}^{\mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}}$ associated to (4-3). The relation of the space (4-4) to representation homology becomes clear from the following observation.

Lemma 4.1 *For any* Γ *and any commutative ring* k, *there is a natural isomorphism*

$$\mathrm{H}_*(X_\Gamma, k) \cong \mathrm{HR}_*(\Gamma, k^\times), \tag{4-5}$$

where $k^\times = \mathrm{GL}_1(k)$ *denotes the multiplicative group of the ring* k.

Proof We have the sequence of natural isomorphisms

$$H_*(X_\Gamma, k) \cong \mathrm{Tor}_*^{\mathfrak{G}_\mathbb{Z}^{\mathrm{op}}}(k, k[p^*\Gamma]) \cong \mathrm{Tor}_*^{\mathfrak{G}_\mathbb{Z}}(k[p^*\Gamma], k) = \mathrm{Tor}_*^{\mathfrak{G}_\mathbb{Z}}(p^*k[\Gamma], k) \cong \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], k[\mathbb{Z}]),$$

where the first two are standard (see eg [47, Appendix C.10]) and the last one follows from the classical Shapiro Lemma (see Corollary 2.9). To complete the proof it remains to note that $k[\mathbb{Z}]$ can be identified with $\mathcal{O}[k^\times]$ as a commutative Hopf algebra. □

As in the introduction, we shorten notation for one-dimensional representation homology, writing

$$\mathrm{HR}_*(k[\Gamma]) := \mathrm{HR}_*(\Gamma, k^\times). \tag{4-6}$$

Our next goal is to identify the homotopy type of the space X_Γ in terms of the classifying space of Γ. The following theorem is one of the main results of the present paper.

Theorem 4.2 *For any homotopy simplicial group* Γ, *there is a weak equivalence in* Top_*:

$$X_\Gamma \simeq \Omega\mathrm{SP}^\infty(B\Gamma), \tag{4-7}$$

where $B\Gamma$ *is the classifying space of* Γ (Definition 3.4).

Before proving this theorem, we recall a few basic facts about the Dold–Thom space and related constructions; see eg [38, Chapter 4.K]. For any pointed connected CW complex X, the *Dold–Thom space* $\mathrm{SP}^\infty(X)$ is defined as the infinite symmetric product: namely,

$$\mathrm{SP}^\infty(X) = \varinjlim_n \mathrm{SP}^n(X), \tag{4-8}$$

where $\mathrm{SP}^n(X) := X^n/S_n$ with S_n acting on X^n the natural way (by permuting the factors). The maps $\mathrm{SP}^n(X) \to \mathrm{SP}^{n+1}(X)$ along which the inductive limit (4-8) is taken are induced by the natural inclusion

$$X^n \hookrightarrow X^{n+1}, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, *),$$

where $*$ stands for the basepoint of X. The Dold–Thom Theorem asserts that, for all $i \geq 1$, there are isomorphisms of abelian groups

$$\pi_i[\mathrm{SP}^\infty(X)] \cong H_i(X, \mathbb{Z})$$

that are natural in pointed connected CW complexes X. In fact, this classical theorem provides a topological realization for the Hurewicz homomorphisms, in the sense that the natural map of spaces

$$X = \mathrm{SP}^1(X) \hookrightarrow \mathrm{SP}^\infty(X) \tag{4-9}$$

induces the homomorphisms of groups: $\pi_i(X) \to \mathrm{H}_i(X, \mathbb{Z})$ for all $i \geqslant 1$.

Now, let $\mathcal{F}X$ denote the homotopy fiber of the inclusion map (4-9) so that we have a homotopy fibration sequence

$$\mathcal{F}X \to X \to \mathrm{SP}^{\infty}(X). \tag{4-10}$$

There is an alternative way to obtain this fibration sequence, using Kan's simplicial group model $\mathbb{G}(X)$ of the space[6] X. Namely (see eg [5, Section 7]), (4-10) arises from the short exact sequence of simplicial groups

$$1 \to \mathbb{G}_2(X) \to \mathbb{G}(X) \to \mathbb{A}(X) \to 1 \tag{4-11}$$

by applying the classifying space functor $B = |\overline{W}(\text{–})|$. Here $\mathbb{G}_2(X) := [\mathbb{G}(X), \mathbb{G}(X)]$ denotes the commutator subgroup of the Kan loop group $\mathbb{G}(X)$ and $\mathbb{A}(X)$ its abelianization:

$$\mathbb{A}(X) := (\mathbb{G}X)_{\mathrm{ab}} := \mathbb{G}(X)/\mathbb{G}_2(X) \tag{4-12}$$

Thus, we have $\mathrm{SP}^{\infty}(X) \simeq B\mathbb{A}(X) = |\overline{W}\mathbb{A}(X)|$, which, by Kan's theorem (see (3-7)), implies

$$\Omega\mathrm{SP}^{\infty}(X) \simeq \Omega|\overline{W}\mathbb{A}(X)| \simeq |\mathbb{G}\overline{W}\mathbb{A}(X)| \simeq |\mathbb{A}(X)|. \tag{4-13}$$

Note that for any reduced simplicial set X, $\mathbb{A}(X) \cong \widetilde{\mathbb{Z}}[X]$ is just the reduced free simplicial abelian group generated by X. After these preliminary remarks we can proceed with:

Proof of Theorem 4.2 As a first step we apply Proposition 2.6 to express the homotopy colimit (4-4) as a homotopy coend:

$$\mathrm{hocolim}_{\mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}}(p^*\Gamma) \simeq \int_{\boldsymbol{L}}^{\langle n\rangle \in \mathfrak{G}} \Gamma\langle n\rangle \times \underline{\mathbb{Z}}^n. \tag{4-14}$$

Next, observe that the bifunctor

$$\Gamma \times \underline{\mathbb{Z}} \colon \mathfrak{G}^{\mathrm{op}} \times \mathfrak{G} \to \mathrm{sSet}, \quad (\langle n\rangle, \langle m\rangle) \mapsto \Gamma\langle n\rangle \times \mathbb{Z}^m, \tag{4-15}$$

that appears in the homotopy coend (4-14) can be factored as

$$\mathfrak{G}^{\mathrm{op}} \times \mathfrak{G} \xrightarrow{\Gamma\otimes\mathbb{F}} \mathrm{sGr} \xrightarrow{(-)_{\mathrm{ab}}} \mathrm{sAb} \xrightarrow{\text{forget}} \mathrm{sSet}$$

where the first arrow is precisely the bifunctor $\Gamma \otimes \mathbb{F}$ that appears in formula (3-10) of Lemma 3.5, expressing the rigidification functor K. This last bifunctor takes an object $(\langle n\rangle, \langle m\rangle) \in \mathfrak{G}^{\mathrm{op}} \times \mathfrak{G}$ to the simplicial group $\coprod_{\Gamma\langle n\rangle} \mathbb{F}_m$, which is given, in each simplicial degree, by a free product of copies of the free group $\mathbb{F}_m$ indexed by the components of the simplicial set $\Gamma\langle n\rangle$. Hence $\Gamma \otimes \mathbb{F}$ is an objectwise cofibrant diagram in sGr, and therefore

$$\boldsymbol{L}(\Gamma \otimes \mathbb{F})_{\mathrm{ab}} \simeq (\Gamma \otimes \mathbb{F})_{\mathrm{ab}} \cong \Gamma \times \underline{\mathbb{Z}}, \tag{4-16}$$

[6] Abusing notation, we will use the same symbol X for a (pointed connected) space and its (reduced) simplicial set model.

where $\boldsymbol{L}(-)_{\mathrm{ab}}$ stands for the (left) derived functor of the abelianization functor $(-)_{\mathrm{ab}}\colon \mathrm{sGr}\to\mathrm{sAb}$. Since the abelianization functor is left Quillen, its derived functor commutes with homotopy coends; see (2-20). Hence, combining (3-11) with (4-16), we get

$$\text{(4-17)}\quad \boldsymbol{L}[\boldsymbol{L}K(\Gamma)]_{\mathrm{ab}} \simeq \int_{\boldsymbol{L}}^{\langle n\rangle\in\mathfrak{G}} \boldsymbol{L}(\Gamma\langle n\rangle\otimes\mathbb{F}\langle n\rangle)_{\mathrm{ab}} \simeq \int_{\boldsymbol{L}}^{\langle n\rangle\in\mathfrak{G}} (\Gamma\langle n\rangle\otimes\mathbb{F}\langle n\rangle)_{\mathrm{ab}} \simeq \int_{\boldsymbol{L}}^{\langle n\rangle\in\mathfrak{G}} \Gamma\langle n\rangle\times\underline{\mathbb{Z}}^{n}.$$

On the other hand, $\boldsymbol{L}[\boldsymbol{L}K(\Gamma)]_{\mathrm{ab}}\simeq[\mathbb{G}\overline{W}\boldsymbol{L}K(\Gamma)]_{\mathrm{ab}}=\mathbb{A}(\overline{W}\boldsymbol{L}K(\Gamma))$, hence, by (4-13), we have

$$\text{(4-18)}\qquad |\boldsymbol{L}[\boldsymbol{L}K(\Gamma)]_{\mathrm{ab}}|\simeq|\mathbb{A}(\overline{W}\boldsymbol{L}K(\Gamma))|\simeq\Omega\mathrm{SP}^{\infty}(B\Gamma).$$

Combining now (4-14), (4-17) and (4-18), we get the desired equivalence $X_\Gamma\simeq\Omega\mathrm{SP}^{\infty}(B\Gamma)$. □

Note that Theorem 4.2 combined with Lemma 4.1 implies Theorem 1.1 stated in the introduction.

4.2 Symmetric homology

In Section 3.3, we defined cyclic homology of homotopy simplicial groups by associating to each $\Gamma\in\mathrm{sGr}^h$ a cyclic bar construction $B^{\mathrm{cyc}}\Gamma\colon\Delta C^{\mathrm{op}}\to\mathrm{sSet}$; see Definition 3.12. In this section, we introduce an analogue of this construction for symmetric groups. Recall that the *symmetric crossed simplicial category* ΔS is defined to be an extension of Δ that has the same objects as Δ (and ΔC) with morphisms characterized by the two properties (cf [47, 6.1.4]):

(Sym1) For each $n\geqslant 0$, $\mathrm{Aut}_{\Delta S}([n])\cong S_{n+1}^{\mathrm{op}}$, where S_{n+1} is the $(n+1)^{\mathrm{th}}$ symmetric group.

(Sym2) Any morphism $f\colon[n]\to[m]$ in ΔS can be factored uniquely as the composite $f=g\circ\sigma$ with $g\in\mathrm{Hom}_\Delta([n],[m])$ and $\sigma\in\mathrm{Aut}_{\Delta S}([n])\cong S_{n+1}^{\mathrm{op}}$.

There is an inclusion functor (a morphism in Cat)

$$\text{(4-19)}\qquad \iota\colon\Delta C^{\mathrm{op}}\xrightarrow{\sim}\Delta C\hookrightarrow\Delta S,$$

where the first arrow is an isomorphism of categories (called Connes' duality) and the second one is induced by the natural inclusion of groups $C_{n+1}\hookrightarrow S_{n+1}$(cf [47, 6.1.11]). Explicitly, the functor (4-19) is given on objects by $\iota([n])=[n]$ and on generators by the following formulas

$$\text{(4-20)}\qquad \begin{aligned} \iota(d_i^n) &= \begin{cases} s_{n-1}^i & \text{if } 0\le i<n\\ s_{n-1}^0\circ(n,0,1,\dots,n-1) & \text{if } i=n,\end{cases}\\ \iota(s_j^n) &= d_{n+1}^{j+1},\\ \iota(t_n) &= (n,0,1,\dots,n-1),\end{aligned}$$

where $d_i^n\colon[n]\to[n-1]$, $s_j^n\colon[n]\to[n+1]$ and $t_n\colon[n]\to[n]$ denote the generators of ΔC^{op} dual (opposite) to the generators d_n^i, s_n^j and τ_n of ΔC, respectively.

Lemma 4.3 *The functor $\Psi_{\mathrm{cyc}}^{\mathrm{op}}\colon \Delta C^{\mathrm{op}} \to \mathfrak{G}^{\mathrm{op}}$ defined by (3-23) and (3-24) extends through ι, giving a commutative diagram of small categories*

(4-21)
$$\begin{array}{ccc} \Delta C^{\mathrm{op}} & \xrightarrow{\Psi_{\mathrm{cyc}}^{\mathrm{op}}} & \mathfrak{G}^{\mathrm{op}} \\ {\scriptstyle \iota}\big\downarrow & \nearrow_{\Psi_{\mathrm{sym}}} & \\ \Delta S & & \end{array}$$

Proof In order to construct the functor Ψ_{sym} it is convenient to use the following notation for morphisms in ΔS; cf [2, Section 1.1]. Any morphism $f\colon [n] \xrightarrow{\sigma} [n] \xrightarrow{g} [m]$ in ΔS can be written uniquely as a "tensor product" of $m+1$ noncommutative monomials $X_0, X_1, \ldots, X_m$ in $n+1$ formal variables $\{x_0, x_1, \ldots, x_n\}$:

(4-22)
$$f = X_0 \otimes X_1 \otimes \ldots \otimes X_m,$$

where each X_i is the product $x_{i_1} x_{i_2} \ldots x_{i_r}$ of $r = |f^{-1}(i)|$ variables whose indices i_k occur in the fiber $f^{-1}(i)$ and that are ordered in the same way as numbers in $\{\sigma(0), \ldots, \sigma(n)\}$, ie $\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_r)$. For example, if $f\colon [4] \to [3]$ is given by the composition $g \circ \sigma$ in ΔS, where $g \in \mathrm{Hom}_\Delta([4],[3])$ is defined by $g(0) = g(1) = 0$, $g(2) = g(3) = 1$ and $g(4) = 3$ and $\sigma \in \mathrm{Aut}_{\Delta S}([4]) = S_5^{\mathrm{op}}$ is the permutation

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 4 & 2 & 3 \end{pmatrix}$$

then f is represented by $x_1 x_0 \otimes x_3 x_4 \otimes 1 \otimes x_2$. The composition of morphisms $f_1 \circ f_2$ is defined by a natural substitution rule: for example, if $f_1\colon [3] \to [3]$ and $f_2\colon [4] \to [3]$ in ΔS are represented by

$$f_1 = 1 \otimes x_0 \otimes 1 \otimes x_3 x_2 x_1, \qquad f_2 = x_2 x_1 \otimes x_4 \otimes 1 \otimes x_0 x_3,$$

then $f_1 \circ f_2\colon [4] \to [3]$ can be computed as

$$\begin{aligned} f_1 \circ f_2 &= (1 \otimes X_0 \otimes 1 \otimes X_3 X_2 X_1) \circ (\overbrace{x_2 x_1}^{X_0} \otimes \overbrace{x_4}^{X_1} \otimes \overbrace{1}^{X_2} \otimes \overbrace{x_0 x_3}^{X_3}) \\ &= 1 \otimes x_2 x_1 \otimes 1 \otimes (x_0 x_3) \cdot 1 \cdot (x_4) = 1 \otimes x_2 x_1 \otimes 1 \otimes x_0 x_3 x_4. \end{aligned}$$

With this notation, we define the functor

(4-23)
$$\Psi_{\mathrm{sym}}\colon \Delta S \to \mathfrak{G}^{\mathrm{op}}$$

on objects by

$$\Psi_{\mathrm{sym}}([n]) = \langle n+1 \rangle,$$

and on morphisms by the following formula: if $f \in \mathrm{Hom}_{\Delta S}([n],[m])$ is represented by

$$f = (x_{i_1} \ldots x_{i_r}) \otimes \cdots \otimes (x_{k_1} \ldots x_{k_s}),$$

then

(4-24)
$$\Psi_{\mathrm{sym}}(f)\colon \langle m+1 \rangle \to \langle n+1 \rangle, \quad X_0 \mapsto x_{i_1} \cdots x_{i_r}, \ \ldots, \ X_m \mapsto x_{k_1} \cdots x_{k_s},$$

where

$$\langle m+1 \rangle = \mathbb{F}\langle X_0, \ldots, X_m \rangle \quad \text{and} \quad \langle n+1 \rangle = \mathbb{F}\langle x_0, \ldots, x_n \rangle.$$

Note that the maps (4-20) can be rewritten in this tensor notation as

$$\iota(d_i^n) = \begin{cases} x_0 \otimes \ldots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \ldots \otimes x_n & \text{if } 0 \le i < n, \\ x_n x_0 \otimes x_1 \otimes \ldots \otimes x_{n-1} & \text{if } i = n, \end{cases}$$

$$\iota(s_j^n) = x_0 \otimes \ldots \otimes x_j \otimes 1 \otimes x_{j+1} \otimes \ldots \otimes x_n,$$

$$\iota(\tau_n) = x_n \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_{n-1}.$$

The commutativity of (4-21) can now be checked by a trivial calculation that we leave to the reader. □

With the functor $\Psi_{\mathrm{sym}} \colon \Delta S \to \mathfrak{G}^{\mathrm{op}}$ in hand, we can now define a symmetric bar construction in the same way as we defined the cyclic bar construction in Definition 3.12.

Definition 4.4 For a homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$, its *symmetric bar construction* is the functor

$$B_{\mathrm{sym}}\Gamma := \Psi_{\mathrm{sym}}^{*}\Gamma \colon \Delta S \to \mathrm{sSet}, \tag{4-25}$$

and its *symmetric homology* is defined by

$$\mathrm{HS}_*(k[\Gamma]) := \mathrm{Tor}_*^{\Delta S}(k, k[B_{\mathrm{sym}}\Gamma]). \tag{4-26}$$

Remark 4.5 The same argument as (in the proof of) Lemma 3.7 shows that $\mathrm{HS}_*(k[\Gamma])$ depends only on the homotopy type of Γ in sGr^h and hence on the homotopy type of the space $B\Gamma$.

Remark 4.6 For Γ an ordinary discrete group, the definition (4-25) agrees with Fiedorowicz's original definition of the symmetric bar construction; see [30] and also [2]. In this case, formula (4-26) defines the symmetric homology of the group algebra $k[\Gamma]$. Note that, unlike $B^{\mathrm{cyc}}\Gamma$ (see (3-25)), the functor $B_{\mathrm{sym}}\Gamma \colon \Delta S \to \mathrm{sSet}$ is covariant on ΔS (which we emphasize by writing sym as a subscript).

Remark 4.7 To study symmetric homology it is often convenient to work with the *augmented* symmetric category ΔS_+, which is defined by adding to ΔS the initial object $[-1]$ and morphisms $[-1] \to [n]$, one for each $n \geqslant -1$; see [2]. It is easy to see that the map Ψ_{sym} defined in Lemma 4.3 extends to ΔS_+,

$$\Psi_{\mathrm{sym},+} \colon \Delta S_+ \to \mathfrak{G}^{\mathrm{op}}, \tag{4-27}$$

by letting $\Psi_{\mathrm{sym},+}([-1]) := \langle 0 \rangle$. Now, the category ΔS_+ is isomorphic to the category of so-called *finite associative sets*, $\mathcal{F}(\mathrm{as})$, introduced in [57]; see also [62, Section 15.4] for a detailed discussion. The latter is known to be a permutative category (PROP) that describes the associative unital algebras; see [56] and also [62]. Its opposite category $\mathcal{F}(\mathrm{as})^{\mathrm{op}}$ describes the coassociative counital coalgebras. If we identify $\Delta S_+ = \mathcal{F}(\mathrm{as})$, the restriction functor $\Psi_{\mathrm{sym},+}^{*} \colon \mathrm{Mod}_k(\mathfrak{G}) \to \mathrm{Mod}_k[\mathcal{F}(\mathrm{as})^{\mathrm{op}}]$ associated to the opposite of (4-27) takes commutative Hopf algebras viewed as functors (3-15) on $\mathfrak{G}$ to the underlying coassociative coalgebras viewed as functors on $\mathcal{F}(\mathrm{as})^{\mathrm{op}}$. In other words, the morphism $\Psi_{\mathrm{sym},+}^{\mathrm{op}}$ is isomorphic to a morphism of PROPs, $\mathcal{F}(\mathrm{as})^{\mathrm{op}} \to \mathfrak{G}$, that "forgets" the algebra structure on commutative Hopf algebras.

4.3 Symmetric homology vs representation homology

Recall that in Section 3.4, we constructed the derived character map $\mathrm{Tr}(\Gamma)_*$ relating the cyclic homology of Γ to its (one-dimensional) representation homology:

$$\mathrm{Tr}(\Gamma)_*\colon \mathrm{HC}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma]). \tag{4-28}$$

On the other hand, as a consequence of Lemma 4.3, we have a restriction map

$$\iota^*\colon \mathrm{HC}_*(k[\Gamma]) = \mathrm{Tor}_*^{\Delta C^{\mathrm{op}}}(k, k[B^{\mathrm{cyc}}\Gamma]) \to \mathrm{Tor}_*^{\Delta S}(k, k[B_{\mathrm{sym}}\Gamma]) = \mathrm{HS}_*(k[\Gamma]) \tag{4-29}$$

induced by the isomorphism of cyclic spaces

$$B^{\mathrm{cyc}}\Gamma \cong \iota^* B_{\mathrm{sym}}\Gamma. \tag{4-30}$$

The next proposition shows that the derived character map (4-28) factors through (4-29), thus relating representation homology to symmetric homology.

Proposition 4.8 *For any homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$, there is a natural map*

$$\widetilde{\Psi}^*_{\mathrm{sym}}\colon \mathrm{HS}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma]) \tag{4-31}$$

such that

$$\begin{array}{ccc} \mathrm{HC}_*(k[\Gamma]) & \xrightarrow{\mathrm{Tr}(\Gamma)_*} & \mathrm{HR}_*(k[\Gamma]) \\ & {\scriptstyle \iota^*}\searrow \quad \nearrow{\scriptstyle \widetilde{\Psi}^*_{\mathrm{sym}}} & \\ & \mathrm{HS}_*(k[\Gamma]) & \end{array} \tag{4-32}$$

Proof As our notation suggests, the map (4-31) is actually induced by a morphism $\widetilde{\Psi}_{\mathrm{sym}}$ in Cat. We construct $\widetilde{\Psi}_{\mathrm{sym}}$ by lifting the functor Ψ_{sym} of Lemma 4.3 to the (opposite) category of elements of the abelianization functor (4-1):

$$\begin{array}{ccccc} & & & \nearrow^{\widetilde{\Psi}_{\mathrm{sym}}} & \mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}} \\ & & & & \downarrow{\scriptstyle p^{\mathrm{op}}} \\ \Delta C^{\mathrm{op}} & \xrightarrow{\iota} & \Delta S & \xrightarrow{\Psi_{\mathrm{sym}}} & \mathfrak{G}^{\mathrm{op}} \end{array} \tag{4-33}$$

The existence of such a lifting is a consequence of the following observation. Consider the composition of functors

$$\Delta S^{\mathrm{op}} \xrightarrow{\Psi^{\mathrm{op}}_{\mathrm{sym}}} \mathfrak{G} \xrightarrow{(-)_{\mathrm{ab}}} \mathrm{Ab} \tag{4-34}$$

that takes an object $[n] \in \Delta S$ to the abelian group $\mathbb{Z}^{n+1}$. If we represent a morphism $f\colon [n] \to [m]$ in ΔS using the tensor notation (4-22), then $\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}}\colon \mathbb{Z}^{m+1} \to \mathbb{Z}^{n+1}$, the value of (4-34) on f, is represented by an integral $(n+1) \times (m+1)$–matrix whose rows are indexed by $0 \leqslant i \leqslant n$ and columns by $0 \leqslant j \leqslant m$, and the j^{th} column consists entirely of 0s and 1s, with the 1s occurring in positions indicated by the elements of $f^{-1}(j)$.

For example, if $f\colon [4]\to[3]$ in ΔS is represented by the product $x_1x_0\otimes x_3x_4\otimes 1\otimes x_2$, then $\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}}\colon \mathbb{Z}^4\to\mathbb{Z}^5$ is given by

$$\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}}=\begin{pmatrix}1&0&0&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\\0&1&0&0\end{pmatrix}.$$

Observe that for any morphism f in ΔS, the matrix $\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}}$ thus defined has exactly one nonzero entry in each row and that entry is 1. Hence $\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}}$ maps the vector $(1,1,\dots,1)^t\in\mathbb{Z}^{m+1}$ to the vector $(1,1,\dots,1)^t\in\mathbb{Z}^{n+1}$. This shows that there is a well-defined functor

$$\widetilde{\Psi}_{\mathrm{sym}}\colon \Delta S\to\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}},\quad [n]\mapsto(\langle n+1\rangle;1,1,\dots,1),\tag{4-35}$$

that makes the diagram (4-33) commutative. It follows from (4-33) that

$$k[B_{\mathrm{sym}}\Gamma]=\Psi^*_{\mathrm{sym}}(k[\Gamma])=\widetilde{\Psi}^*_{\mathrm{sym}}(k[p^*\Gamma]).$$

Hence, by Lemma 3.13, the functor (4-35) induces a natural map

$$\mathrm{HS}_*(k[\Gamma])=\mathrm{Tor}^{\Delta S}_*(k,k[B_{\mathrm{sym}}\Gamma])\xrightarrow{\widetilde{\Psi}^*_{\mathrm{sym}}}\mathrm{Tor}^{\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}}}_*(k,k[p^*\Gamma]).\tag{4-36}$$

We claim that if the target of the map (4-36) is identified with the representation homology of $k[\Gamma]$ via the Shapiro isomorphism (see Corollary 2.9), then the required factorization property (4-32) holds. To verify this we fix a projective resolution $Q\xrightarrow{\sim}k[\Gamma]$ of $k[\Gamma]$ in $\mathrm{Mod}_k(\mathfrak{G}^{\mathrm{op}})$. Then $p^*(Q)\xrightarrow{\sim}p^*k[\Gamma]=k[p^*\Gamma]$ gives a projective resolution of $k[p^*\Gamma]$ in $\mathrm{Mod}_k(\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}})$, and the Shapiro isomorphism

$$\mathrm{Tor}^{\mathfrak{G}_{\mathbb{Z}}}_*(k[p^*\Gamma],k)\xrightarrow{\sim}\mathrm{Tor}^{\mathfrak{G}}_*(k[\Gamma],p_!(k))$$

is induced by the composition

$$p^*(Q)\otimes_{\mathfrak{G}_{\mathbb{Z}}}k\xrightarrow{\mathrm{id}\otimes\varepsilon_k}p^*(Q)\otimes_{\mathfrak{G}_{\mathbb{Z}}}p^*p_!(k)\xrightarrow{p^*}Q\otimes_{\mathfrak{G}}p_!(k),$$

where the first map is given by the adjunction unit $\varepsilon\colon\mathrm{id}\to p^*p_!$ and the second is the restriction map via p. Explicitly, using the definition (3-12) of functor tensor products, we can represent the above composite map as

$$\bigoplus_{(\langle n\rangle;k_1,\dots,k_n)\in\mathfrak{G}_{\mathbb{Z}}}Q\langle n\rangle\to\bigoplus_{\langle n\rangle\in\mathfrak{G}}Q\langle n\rangle\otimes k[\mathbb{Z}^n],\quad (v_n)_{(\langle n\rangle;k_1,\dots,k_n)\in\mathfrak{G}_{\mathbb{Z}}}\mapsto(v_n\otimes(k_1,\dots,k_n))_{\langle n\rangle\in\mathfrak{G}},\tag{4-37}$$

where $v_n\in Q\langle n\rangle$ and the subscripts denote the indices of the corresponding components of direct sums. Now, using the same resolution Q, we can write explicitly the composition of maps (4-29) and (4-36):

$$\Psi^*_{\mathrm{cyc}}(Q)\otimes_{\Delta C}k\xrightarrow{\iota^*}\Psi^*_{\mathrm{sym}}(Q)\otimes_{\Delta S^{\mathrm{op}}}k\xrightarrow{\widetilde{\Psi}^*_{\mathrm{sym}}}p^*(Q)\otimes_{\mathfrak{G}_{\mathbb{Z}}}k.$$

At the level of chain complexes, this last composition is induced by the map

$$\text{(4-38)}\qquad \bigoplus_{[m]\in\Delta C} Q\langle m+1\rangle \to \bigoplus_{[m]\in\Delta S^{\mathrm{op}}} Q\langle m+1\rangle \to \bigoplus_{(\langle n\rangle;k_1,\dots k_n)\in\mathfrak{G}_{\mathbb{Z}}} Q\langle n\rangle,$$
$$(v_{m+1})_{[m]\in\Delta C} \mapsto (v_{m+1})_{[m]\in\Delta S^{\mathrm{op}}} \mapsto (v_{m+1})_{(\langle m+1\rangle;1,1,\dots,1)\in\mathfrak{G}_{\mathbb{Z}}}.$$

Combining (4-37) and (4-38), we see that the resulting map

$$\bigoplus_{[m]\in\Delta C} Q\langle m+1\rangle \to \bigoplus_{\langle n\rangle\in\mathfrak{G}} Q\langle n\rangle\otimes k[\mathbb{Z}^n], \quad (v_{m+1})_{[m]\in\Delta C}\mapsto (v_{m+1}\otimes(1,1,\dots,1))_{\langle m+1\rangle\in\mathfrak{G}},$$

coincides exactly with the map (3-33) representing the derived character $\chi_{\mathrm{GL}_1,\mathrm{Tr}_1}(\Gamma)_* = \mathrm{Tr}(\Gamma)_*$. This finishes the proof of the proposition. □

Remark 4.9 The proof of Proposition 4.8 shows that, apart from (4-35), any functor of the form

$$\text{(4-39)}\qquad \widetilde{\Psi}^{(m)}_{\mathrm{sym}}\colon \Delta S\to\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}},\quad [n]\mapsto(\langle n+1\rangle; m,m,\dots,m),$$

where $m\in\mathbb{Z}$ is a fixed integer, satisfies the lifting property (4-33). It is easy to see that there are no other solutions to this lifting problem. Among (4-39) the functor $\widetilde{\Psi}^{(0)}_{\mathrm{sym}}$ corresponding to $m=0$ is the only one that factors through $\mathfrak{G}^{\mathrm{op}}$: $\widetilde{\Psi}^{(0)}_{\mathrm{sym}} = s^{\mathrm{op}}\circ\Psi_{\mathrm{sym}}$, where $s\colon\mathfrak{G}\hookrightarrow\mathfrak{G}_{\mathbb{Z}}$ is the zero section of p.

Next, we observe that the linear maps factoring $\mathrm{Tr}(\Gamma)_*$ in (4-32) arise (on homology) from the natural maps of topological spaces induced by the functors (4-19) and (4-35) (cf Lemma 4.1):

$$\text{(4-40)}\qquad |\mathrm{hocolim}_{\Delta C^{\mathrm{op}}}(B^{\mathrm{cyc}}\Gamma)| \xrightarrow{\iota^*} |\mathrm{hocolim}_{\Delta S}(B_{\mathrm{sym}}\Gamma)| \xrightarrow{\widetilde{\Psi}^*_{\mathrm{sym}}} |\mathrm{hocolim}_{\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}}}(p^*\Gamma)|.$$

(Here, abusing notation, we denote these topological maps by the same symbols as the corresponding linear maps.) By Theorem 4.2, we know that

$$\text{(4-41)}\qquad |\mathrm{hocolim}_{\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}}}(p^*\Gamma)| \simeq \Omega\mathrm{SP}^\infty(B\Gamma).$$

On the other hand, by theorems of Goodwillie (see [47, Theorem 7.2.4]) and Fiedorowicz [30] (see [2, Section 5.3]),

$$\text{(4-42)}\qquad |\mathrm{hocolim}_{\Delta C^{\mathrm{op}}}(B^{\mathrm{cyc}}\Gamma)| \simeq ES^1\times_{S^1}\mathcal{L}(B\Gamma),$$
$$\text{(4-43)}\qquad |\mathrm{hocolim}_{\Delta S}(B_{\mathrm{sym}}\Gamma)| \simeq \Omega\Omega^\infty\Sigma^\infty(B\Gamma),$$

where $\mathcal{L}(B\Gamma) := \mathrm{Map}(S^1, B\Gamma)$ and $\Omega^\infty\Sigma^\infty(B\Gamma) := \mathrm{hocolim}_{n\to\infty}\Omega^n\Sigma^n(B\Gamma)$ denote the free loop space and the infinite loop space of $B\Gamma$, respectively.

Combining (4-40) with equivalences (4-41)–(4-43), we can thus refine the result of Proposition 4.8 as follows:

Corollary 4.10 *The derived character map*

$$\mathrm{Tr}(\Gamma)_* \colon \mathrm{HC}_*(k[\Gamma]) \xrightarrow{\iota^*} \mathrm{HS}_*(k[\Gamma]) \xrightarrow{\widetilde{\Psi}^*_{\mathrm{sym}}} \mathrm{HR}_*(k[\Gamma])$$

is induced on homology by a natural map of topological spaces in $\mathtt{Ho}(\mathrm{Top}_*)$*:*

$$ES^1 \times_{S^1} \mathcal{L}(B\Gamma) \xrightarrow{\mathtt{CS}_{B\Gamma}} \Omega\Omega^\infty\Sigma^\infty(B\Gamma) \xrightarrow{\mathtt{SR}_{B\Gamma}} \Omega\mathrm{SP}^\infty(B\Gamma). \tag{4-44}$$

In the next section, we will describe the maps $\mathtt{CS}$ and $\mathtt{SR}$ in topological terms in two ways: using the classical "little cubes" operads and the Goodwillie calculus of homotopy functors.

4.4 Generalization to monoids

All results of this section generalize naturally to (simplicial) monoids. We briefly outline this generalization as we will need it in Section 5.3. Instead of $\mathfrak{G}$, we start with the category $\mathfrak{M} \subset \mathrm{Mon}$ whose objects are finitely generated free monoids[7] $\langle n\rangle$, one for each $n \geq 0$. In this case, the abelianization functor reads

$$\underline{\mathbb{N}} \colon \mathfrak{M} \to \mathrm{Set}, \quad \langle n\rangle \mapsto \mathbb{N}^n,$$

where $\mathbb{N}$ is the set of natural numbers, ie the underlying set of the free abelian monoid of rank one. The associated category of elements $\mathfrak{M}_{\mathbb{N}} := \mathfrak{M} \textstyle\int \underline{\mathbb{N}}$ has an explicit description similar to that of $\mathfrak{G}_{\mathbb{Z}}$: its objects are $(\langle n\rangle; k_1, \dots, k_n))$, where $\langle n\rangle$ is the free monoid on n generators and $(k_1, \dots, k_n) \in \mathbb{N}^n$. Any simplicial monoid M gives a functor $M \colon \mathfrak{M}^{\mathrm{op}} \to \mathtt{sSet}$ that restricts to $\mathfrak{M}^{\mathrm{op}}_{\mathbb{N}}$ via the canonical projection $p \colon \mathfrak{M}_{\mathbb{N}} \to \mathfrak{M}$. The analogue (generalization) of Theorem 4.2 says:

Proposition 4.11 *For any simplicial monoid M, there is a weak equivalence in* Top_**:*

$$|\mathrm{hocolim}_{\mathfrak{M}^{\mathrm{op}}_{\mathbb{N}}}(p^*M)| \simeq \Omega\mathrm{SP}^\infty(BM), \tag{4-45}$$

where BM is the classifying space of M.

Proof The same argument as in the proof of Theorem 4.2 — based on Proposition 2.6 — shows

$$\mathrm{hocolim}_{\mathfrak{M}^{\mathrm{op}}_{\mathbb{N}}}(p^*M) \simeq \boldsymbol{L}(M)_{\mathrm{ab}},$$

where $\boldsymbol{L}(-)_{\mathrm{ab}}$ denotes the derived abelianization functor on simplicial monoids. To compute this last functor, instead of the Kan loop group, we will use the 2–sided (simplicial) bar resolution (5-22): $B_*(\underline{\mathcal{C}}_1, \underline{\mathcal{C}}_1, M) \xrightarrow{\sim} M$ in $\mathtt{sSet}_*$, where $\underline{\mathcal{C}}_1$ is the monad associated to the (simplicial analogue of) little 1–cube operad; see (5-24). Since $(\underline{\mathcal{C}}_1(X))_{\mathrm{ab}} = \underline{\mathcal{C}}_0(X)$, we have

$$|\boldsymbol{L}(M)_{\mathrm{ab}}| \simeq |B_*(\underline{\mathcal{C}}_1, \underline{\mathcal{C}}_1, M)_{\mathrm{ab}}| \simeq |B_*(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_1, M)| \simeq \Omega\mathrm{SP}^\infty(BM),$$

where the last equivalence is a result of Lemma 5.4 below; see (5-27). □

[7] Abusing notation, we will use the same symbols to denote the objects of $\mathfrak{M}$ and $\mathfrak{G}$.

The relation between monoids and groups is determined by the canonical (group completion) functor $l\colon \mathfrak{M} \to \mathfrak{G}$. This last functor extends naturally to a functor $\tilde{l}\colon \mathfrak{M}_{\mathbb{N}} \to \mathfrak{G}_{\mathbb{Z}}$, and the maps $\Psi_{\mathrm{sym}}\colon \Delta S \to \mathfrak{G}^{\mathrm{op}}$ and $\tilde{\Psi}_{\mathrm{sym}}\colon \Delta S \to \mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}$ defined by (4-23) and (4-35) factor through l and $\tilde{l}$ respectively, giving the commutative diagram

(4-46)
$$\begin{array}{ccccccc} & & & \overset{\tilde{\Psi}_{\mathrm{sym}}}{\nearrow} & \mathfrak{M}_{\mathbb{N}}^{\mathrm{op}} & \xrightarrow{\ \tilde{l}\ } & \mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}} \\ & & & & \big\downarrow p & & \big\downarrow p \\ \Delta C^{\mathrm{op}} & \xrightarrow{\ \iota\ } & \Delta S & \xrightarrow{\ \Psi_{\mathrm{sym}}\ } & \mathfrak{M}^{\mathrm{op}} & \xrightarrow{\ l\ } & \mathfrak{G}^{\mathrm{op}} \end{array}$$

As a consequence of Proposition 4.11, we get:

Corollary 4.12 *For any homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$, there is a weak equivalence*

$$|\mathrm{hocolim}_{\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}}(p^* l^* \Gamma)| \simeq \Omega \mathrm{SP}^{\infty}(B\Gamma).$$

Proof Apply Proposition 4.11 to the simplicial group $\boldsymbol{L}K(\Gamma)$ viewed as a simplicial monoid. □

Remark 4.13 Corollary 4.12 can be also deduced from Theorem 4.2 if we notice that the natural map

$$\mathrm{hocolim}_{\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}}(p^* l^* \Gamma) \xrightarrow{\sim} \mathrm{hocolim}_{\mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}}(p^* \Gamma)$$

is a weak equivalence for any Γ. This last fact follows from Theorem 2.3, the assumptions of which hold thanks to the known properties of the group completion functor; cf [14, Lemma 3.2].

5 Topological character maps via Goodwillie calculus and operads

In this section, we will describe the maps CS and SR explicitly in topological terms, using Goodwillie calculus and classical operads. The latter approach is based on ideas of Fiedorowicz [30] that were developed by Ault in [2]. The former is inspired by results of Biedermann and Dwyer that appeared in [18]. The interpretation in terms of Goodwillie derivatives leads to a natural nonlinear (polynomial) generalization of topological character maps that deserves a further study; see Section 5.4.

5.1 Goodwillie homotopy calculus

Goodwillie calculus provides a universal approximation ("Taylor decomposition") of arbitrary homotopy functors in terms of polynomial homotopy functors. This method, introduced by T Goodwillie in the series of papers [34; 35; 36], has been studied extensively in recent years and has found many interesting applications; see eg the survey papers [1] and [45].

Recall that by a homotopy functor we mean a functor on topological spaces that preserves weak homotopy equivalences. A homotopy functor $F\colon \mathrm{Top}_* \to \mathrm{Top}_*$ is called *n–excisive* (or *polynomial of degree $\le n$*) if it takes any strongly co-Cartesian $(n+1)$–dimensional cubical diagram in Top_* to a Cartesian diagram;

see [1, Definition 1.1.2]. For $n = 0$, this simply means that F is homotopically constant: ie $F(X) \simeq F(*)$ for any $X \in \mathrm{Top}_*$. For $n = 1$, this is the usual Mayer–Vietoris property: a functor F is 1–excisive if and only if it maps homotopy pushout squares to homotopy pullback squares in Top_*; see [1, Example 1.1.4]. For $n > 1$, F enjoys a higher-dimensional version of the Mayer–Vietoris property that reduces to the usual one inductively in n.

The main construction of Goodwillie calculus can be described as follows (cf [36, Theorem 1.8]).

Theorem 5.1 (Goodwillie) *For any homotopy functor $F\colon \mathrm{Top}_* \to \mathrm{Top}_*$ on pointed spaces, there exists a natural tower*

(5-1)
$$\begin{array}{ccc} & & \vdots \\ & & \big\downarrow{\scriptstyle p_3} \\ & & P_2F(X) \\ & \nearrow^{\delta_2} & \big\downarrow{\scriptstyle p_2} \\ & & P_1F(X) \\ & \nearrow^{\delta_1} & \big\downarrow{\scriptstyle p_1} \\ F(X) & \xrightarrow{\delta_0} & P_0F(X) \end{array}$$

of functors (fibrations) under F, satisfying the following properties: for all $n \geq 0$,

(1) *$P_nF\colon \mathrm{Top}_* \to \mathrm{Top}_*$ is an n–excisive functor, and*

(2) *$\delta_n\colon F \to P_nF$ is the universal weak natural transformation to an n–excisive functor.*

The last property needs an explanation. By a *weak* natural transformation $\delta\colon F \to P$ one means a pair ("zig-zag") of natural transformations $F \xrightarrow{\delta'} G \xleftarrow{\delta''} P$, where δ'' is a natural weak equivalence, ie $\delta''_X\colon G(X) \xleftarrow{\sim} P(X)$ is a weak homotopy equivalence for all spaces $X \in \mathrm{Top}_*$. Note that if F and P are homotopy functors, a weak natural transformation $\delta\colon F \to P$ induces a well-defined natural transformation between the corresponding functors on the homotopy category $\mathtt{Ho}(\mathrm{Top}_*)$. Property (2) of Theorem 5.1 then says that the weak natural transformation $\delta_n\colon F \to P_nF$ is homotopically initial among all natural transformations from F to n–excisive functors.

Given a homotopy functor $F\colon \mathrm{Top}_* \to \mathrm{Top}_*$, we define its n^{th} *layer* to be the homotopy fiber

$$D_nF(X) := \mathrm{hofib}\{P_nF(X) \xrightarrow{p_n} P_{n-1}F(X)\}, \tag{5-2}$$

where p_n is the canonical projection at the n^{th} stage of the Goodwillie tower (5-1). A remarkable fact discovered in [36] (see [1, Example 1.2.4]) is that all layers of a homotopy functor F are naturally infinite loop spaces. More precisely, for each $n \geqslant 0$, there is a spectrum ∂_nF equipped with a (naïve) action of the symmetric group S_n such that

$$D_nF(X) \simeq \Omega^\infty(\partial_nF \wedge (\Sigma^\infty X)^{\wedge n})_{hS_n}, \tag{5-3}$$

where $(\Sigma^\infty, \Omega^\infty)$ are the suspension spectrum and the infinite loop space functors, respectively. The spectrum $\partial_n F$ is called the n^{th} *Goodwillie derivative of* F (at the basepoint $*$).

5.2 The map CS

Recall that, by Corollary 4.10, the derived character map $\mathrm{Tr}(\Gamma)_*$ is induced by the composition of natural maps in $\mathtt{Ho}(\mathrm{Top}_*)$:

(5-4) $$ES^1 \times_{S^1} \mathcal{L}(X) \xrightarrow{\mathtt{CS}_X} \Omega\Omega^\infty\Sigma^\infty(X) \xrightarrow{\mathtt{SR}_X} \Omega\mathrm{SP}^\infty(X),$$

where $X = B\Gamma$. Since the classifying space functor on homotopy simplicial groups induces an equivalence $\mathtt{Ho}(\mathrm{sGr}^h) \cong \mathtt{Ho}(\mathrm{Top}_{0,*})$, the maps (5-4) are defined on (the homotopy types of) all pointed connected spaces. To analyze these maps we introduce the notation

$$\Theta(X) := ES^1 \times_{S^1} \mathcal{L}(X) = ES^1 \times_{S^1} \mathrm{Map}(S^1, X),$$

and define $\bar{\Theta}\colon \mathrm{Top}_* \to \mathrm{Top}_*$ by

(5-5) $$\bar{\Theta}(X) := \Theta(X)/\Theta(*) \cong ES^1 \times_{S^1} \mathcal{L}(X)/BS^1 \cong ES^1_+ \wedge_{S^1} \mathcal{L}(X).$$

Note that (5-5) is a *reduced* homotopy functor, so that $P_0\bar{\Theta}(X) \simeq \bar{\Theta}(*) = \{*\}$ and $P_1\bar{\Theta}(X) \cong D_1\bar{\Theta}(X)$ for any space $X \in \mathrm{Top}_*$; see (5-2).

The next proposition shows that the natural transformation CS in (5-4), relating cyclic to symmetric homology, essentially coincides with the first Goodwillie layer of the functor (5-5). We deduce this from results of Carlsson and Cohen [21] by elaborating on a remark of Fiedorowicz [30].

Proposition 5.2 *The map* CS *in* (5-4) *is represented by*

$$\begin{array}{ccccc} ES^1 \times_{S^1} \mathcal{L}(X) & & \xrightarrow{\quad \mathtt{CS}_X \quad} & & \Omega\Omega^\infty\Sigma^\infty(X) \\ \Vert & & & & \downarrow \wr \\ \Theta(X) & \xrightarrow{\text{can}} & \bar{\Theta}(X) & \xrightarrow{\delta_{1,X}} & D_1\bar{\Theta}(X) \end{array}$$

where the right vertical arrow is a natural weak equivalence and δ_1 *is the first layer of the functor* (5-5).

Proof As noticed in [30, Remark 1.4], the map $\mathtt{CS}_X$ factors in the homotopy category as

(5-6) $$ES^1 \times_{S^1} \mathcal{L}(X) \xrightarrow{\text{can}} ES^1_+ \wedge_{S^1} \mathcal{L}(X) \xrightarrow{f_X} \Omega\Omega^\infty\Sigma^\infty(X),$$

where f_X is a certain natural map constructed in [21]. We review the construction of f_X and compare it to a well-known general formula for the first Goodwillie layer of a reduced homotopy functor.

First, we recall a standard stabilization construction due to Waldhausen [72]. For a pointed space X, denote by $CX = X \wedge I$ and $\Sigma X = X \wedge S^1$ the reduced cone and the reduced suspension of X, respectively.

The latter can be obtained by gluing two copies of the former along a common base which is identified with X: this yields the natural pushout square in Top_*:

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow{\scriptstyle j_0} \\ CX & \xrightarrow{j_1} & \Sigma X \end{array} \tag{5-7}$$

Applying the given functor F to (5-7) and taking the homotopy pullback along the maps j_0 and j_1 induces a natural map

$$F(X) \to \operatorname{holim}[F(CX) \to F(\Sigma X) \leftarrow F(CX)]. \tag{5-8}$$

Since the functor F is homotopic and reduced, we have $F(CX) \simeq F(*) \simeq \{*\}$, which implies that the homotopy colimit in (5-8) is equivalent to $\Omega F(\Sigma X)$. Thus we get a natural map $s\colon F(X) \to \Omega F(\Sigma X)$. This last map can be iterated any number of times:

$$s_n\colon F(X) \to \Omega^n F(\Sigma^n X), \quad n \geq 0, \tag{5-9}$$

and eventually stabilized, defining the map

$$s_\infty\colon F(X) \to \varinjlim_n \Omega^n F(\Sigma^n X) = \Omega^\infty F \Sigma^\infty(X). \tag{5-10}$$

In particular, (5-10) exists for our functor $F = \bar{\Theta}$; see (5-5).

Next, for each $n \geqslant 0$, define $\Sigma^n X \to \bar{\Theta}\Sigma^n(\Sigma X)$ to be the composition of the natural maps

$$\begin{aligned} \Sigma^n X \xrightarrow{\varepsilon} \Omega\Sigma(\Sigma^n X) = \Omega(\Sigma^{n+1} X) \hookrightarrow \mathcal{L}(\Sigma^{n+1} X) \simeq ES^1 \times \mathcal{L}(\Sigma^{n+1} X) \\ \twoheadrightarrow ES^1_+ \wedge_{S^1} \mathcal{L}(\Sigma^{n+1} X) = \bar{\Theta}\Sigma^n(\Sigma X), \end{aligned}$$

where $\varepsilon\colon \mathrm{id} \to \Omega\Sigma$ is the adjunction unit of (Σ, Ω). Looping n times yields an inductive system of maps

$$i_n\colon \Omega^n \Sigma^n X \to \Omega^n \bar{\Theta}\Sigma^n(\Sigma X) \quad \text{for all } n \geq 0, \tag{5-11}$$

which, by [21, Lemma 4.1], induce in the limit a homotopy equivalence

$$i_\infty\colon \Omega^\infty \Sigma^\infty X \xrightarrow{\sim} \Omega^\infty \bar{\Theta}\Sigma^\infty(\Sigma X). \tag{5-12}$$

Finally, we note the canonical identifications

$$\begin{aligned} \Omega^\infty \bar{\Theta}\Sigma^\infty(X) := \varinjlim_n \Omega^n \bar{\Theta}\Sigma^n(X) = \varinjlim_n \Omega^{n+1} \bar{\Theta}\Sigma^{n+1}(X) = \varinjlim_n \Omega[\Omega^n \bar{\Theta}\Sigma^n(\Sigma X)] \\ \cong \Omega \varinjlim_n [\Omega^n \bar{\Theta}\Sigma^n(\Sigma X)] = \Omega\Omega^\infty \bar{\Theta}\Sigma^\infty(\Sigma X). \end{aligned} \tag{5-13}$$

The Carlsson–Cohen map f_X that appears in (5-6) can now be represented by the zig-zag of natural transformations

$$\bar{\Theta}(X) \xrightarrow{s_\infty} \Omega^\infty \bar{\Theta}\Sigma^\infty(X) \overset{(5\text{-}13)}{\cong} \Omega\Omega^\infty \bar{\Theta}\Sigma^\infty(\Sigma X) \xleftarrow{\Omega i_\infty} \Omega\Omega^\infty \Sigma^\infty(X), \tag{5-14}$$

where the leftmost arrow is the Waldhausen stabilization map (5-10) for $\bar{\Theta}$ and the rightmost arrow is a natural weak equivalence induced by (5-12). To complete the proof we note that $P_1(F) \simeq \Omega^\infty F \Sigma^\infty$ for any reduced homotopy functor F, and the universal natural transformation $\delta_1 : F \to P_1 F = D_1 F$ coincides (up to homotopy) with the stabilization map (5-10); see eg [45, Example 5.3]. □

5.3 The map SR

We now turn to the second map $\mathtt{SR}_X$ in (5-4) that relates symmetric homology to representation homology. In this section, we construct this map topologically by a method similar to that of Proposition 5.2; its relation to Goodwillie calculus will be discussed in Section 5.4. Our starting point is the well-known fact that the Dold–Thom functor $\mathrm{SP}^\infty : \mathrm{Top}_* \to \mathrm{Top}_*$ factors through the category of abelian topological monoids — in fact, $\mathrm{SP}^\infty(X)$ is the free abelian topological monoid generated by the space X; see eg [53]. This implies that SP^∞ is a linear (ie 1–excisive) functor. The latter can be seen directly as follows. Consider the natural maps (5-9) for the functor $F = \mathrm{SP}^\infty$ constructed in the proof of Proposition 5.2:

$$s_n : \mathrm{SP}^\infty(X) \to \Omega^n \mathrm{SP}^\infty \Sigma^n(X) \quad \text{for } n \geq 0. \tag{5-15}$$

The maps (5-15) are all weak equivalences, which follows immediately from the commutative diagrams

$$\begin{array}{ccc} \pi_i \mathrm{SP}^\infty(X) & \xrightarrow{\pi_i(s_n)} & \pi_i \Omega^n \mathrm{SP}^\infty \Sigma^n(X) \\ \downarrow \wr & & \downarrow \wr \\ \tilde{H}_i(X) & \xrightarrow{\sim} & \tilde{H}_{i+n}(\Sigma^n X) \end{array}$$

where the vertical arrows are isomorphisms by the Dold–Thom theorem. Thus, in the limit, we get

$$s_\infty : \mathrm{SP}^\infty(X) \xrightarrow{\sim} \Omega^\infty \mathrm{SP}^\infty \Sigma^\infty(X), \tag{5-16}$$

showing that $\mathrm{SP}^\infty \simeq P_1(\mathrm{SP}^\infty) \simeq D_1(\mathrm{SP}^\infty)$, whence the linearity of SP^∞.

On the other hand, for all $n \geq 0$, we have canonical maps $\Sigma^n X \to \mathrm{SP}^\infty(\Sigma^n X)$ inducing the Hurewicz homomorphisms; see (4-9). Applying loop functors to these maps yields an inductive system of maps

$$i_n : \Omega^n \Sigma^n(X) \to \Omega^n \mathrm{SP}^\infty \Sigma^n(X) \quad \text{for } n \geq 0 \tag{5-17}$$

which, in the limit, induces

$$i_\infty : \Omega^\infty \Sigma^\infty(X) \to \Omega^\infty \mathrm{SP}^\infty \Sigma^\infty(X). \tag{5-18}$$

Unlike the analogous map (5-12) for the functor $\bar{\Theta}$, (5-18) is not a weak equivalence in general. Nevertheless, looping it once and combining with (5-16), we get the pair of natural transformations

$$\Omega\Omega^\infty \Sigma^\infty(X) \xrightarrow{\Omega i_\infty} \Omega\Omega^\infty \mathrm{SP}^\infty \Sigma^\infty(X) \xleftarrow{\Omega s_\infty} \Omega \mathrm{SP}^\infty(X), \tag{5-19}$$

where the rightmost one is a natural weak equivalence. Our goal is to prove the following analogue of Proposition 5.2.

Proposition 5.3 *The map* SR *is represented by the weak natural transformation* (5-19). *Thus, in the homotopy category,* SR_X *is equivalent to the map*

$$(\Omega s_\infty)^{-1}(\Omega i_\infty)\colon \Omega\Omega^\infty\Sigma^\infty(X) \to \Omega\mathrm{SP}^\infty(X) \tag{5-20}$$

which is the (once looped) canonical natural transformation relating stable homotopy to (reduced) singular homology of pointed spaces.

To prove this proposition we will reinterpret the map (5-20) in terms of (topological) operads. The standard reference for the background material that we need is [52] (for a brief introduction, see also [62, Chapter 12]).

Recall that an *operad* $\mathcal{C}$ in Top_* is a collection of pointed spaces $\{\mathcal{C}(j)\}_{j\geq 0}$ with $\mathcal{C}(0) := \{*\}$ such that each $\mathcal{C}(j)$ carries a right S_j–action and there are composition laws

$$\mathcal{C}(k)\times\mathcal{C}(j_1)\times\cdots\times\mathcal{C}(j_k)\to\mathcal{C}(j_1+\cdots+j_k)$$

satisfying natural associativity and unitality conditions. If $\mathcal{C}$ is an operad, a $\mathcal{C}$*–space* is a pointed space X equipped with an action of $\mathcal{C}$, which is given by a sequence of S_j–equivariant maps $\theta_j\colon \mathcal{C}(j)\times X^j\to X$, with $\theta_0\colon \mathcal{C}(0)\hookrightarrow X$ being the basepoint inclusion, that satisfy associativity and unitality conditions compatible with those of $\mathcal{C}$. Every operad $\mathcal{C}$ determines a monad $\underline{\mathcal{C}}$ on Top_* (ie a monoid with respect to $\circ$ in the category of endofunctors $\mathrm{Top}_*\to\mathrm{Top}_*$) in such a way that the notion of a $\mathcal{C}$–space is equivalent to that of $\underline{\mathcal{C}}$–algebra. Explicitly, given an operad $\mathcal{C}$, the corresponding monad $\underline{\mathcal{C}}\colon \mathrm{Top}_*\to\mathrm{Top}_*$ is defined by

$$\underline{\mathcal{C}}(X) := \coprod_{j\geq 0}(\mathcal{C}(j)\times_{S_j} X^j)/\sim \tag{5-21}$$

where the equivalence relation is of the form

$$(c,x_1,\dots,x_{i-1},*,x_{i+1},\dots,x_j)\sim(\sigma_i(c),x_1,\dots,x_{i-1},x_{i+1},\dots,x_j)$$

for certain natural maps $\sigma_i\colon \mathcal{C}(j)\to\mathcal{C}(j-1)$; see [52, Construction 2.4]. A $\underline{\mathcal{C}}$*–algebra* is then defined to be a space $A\in\mathrm{Top}_*$ with an action map $\xi\colon \underline{\mathcal{C}}(A)\to A$ satisfying natural associativity and unitality conditions. Opposite to the notion of a $\underline{\mathcal{C}}$–algebra is that of a $\underline{\mathcal{C}}$*–functor*, which is a functor F on Top_* equipped a morphism $F\circ\underline{\mathcal{C}}\to F$ defining a right action of $\underline{\mathcal{C}}$ on F. Associated to a triple $(F,\underline{\mathcal{C}},A)$, there is a *two-sided bar construction* $B(F,\underline{\mathcal{C}},A)$ defined as the geometric realization of a simplicial space $B_*(F,\underline{\mathcal{C}},A)\in s\mathrm{Top}_*$ with components

$$B_n(F,\underline{\mathcal{C}},A) := F\underline{\mathcal{C}}^n(A)\quad\text{for } n\geq 0, \tag{5-22}$$

where the faces $d_i\colon B_n\to B_{n-1}$ and degeneracies $s_j\colon B_n\to B_{n+1}$ are determined by the structure maps of A and F; see [52, Construction 9.6].

Now, our main examples will be the so-called *little cubes operads* $\{\mathcal{C}_0,\mathcal{C}_1,\mathcal{C}_2,\dots\}$ originally introduced by Boardman and Vogt; see [52, Section 4]. The $\mathcal{C}_0$ and $\mathcal{C}_1$ are discrete operads[8] defined by $\mathcal{C}_0(j) := \{*\}$

[8]These operads are denoted in [52] by $\mathfrak{N}$ and $\mathfrak{M}$, respectively.

and $\mathcal{C}_1(j) := S_j$ for all $j \geq 0$, with the S_j–action being trivial in the former case, and induced by multiplication in S_j in the latter. A $\mathcal{C}_0$–space is just an abelian monoid in Top_*, and the monad associated to $\mathcal{C}_0$ is precisely the Dold–Thom functor

$$\underline{\mathcal{C}}_0(X) \cong \mathrm{SP}^{\infty}(X). \tag{5-23}$$

A $\mathcal{C}_1$–space is just a monoid in Top_* (ie an associative H–space with 1), and the monad associated to $\mathcal{C}_1$ yields the classical James functor

$$\underline{\mathcal{C}}_1(X) \cong J(X), \tag{5-24}$$

where $J(X) = (\coprod_{n\geq 0} X^n)/{\sim}$ is the free topological monoid generated by X. For $n \geq 2$, the operad $\mathcal{C}_n$ is not discrete: for $j \geq 1$, the space $\mathcal{C}_n(j)$ can be represented by the j–tuples of "little n–cubes" (ie linear embeddings $I^n \hookrightarrow I^n$ with parallel axes and disjoint interiors) with the natural (permutation) S_j–action. Thus, for $n \geq 2$, each $\mathcal{C}_n(j)$ is homotopy equivalent to $\mathrm{conf}_j(\mathbb{R}^n)$, the configuration space of unordered j–tuples of points in $\mathbb{R}^n$ equipped with the canonical free S_j–action. Natural inclusions of cubes $I^n \hookrightarrow I^{n+1}$ induce the embeddings of spaces $\mathcal{C}_n(j) \hookrightarrow \mathcal{C}_{n+1}(j)$, and hence the maps of operads $\mathcal{C}_n \hookrightarrow \mathcal{C}_{n+1}$ for all $n \geq 2$. This allows one to define the operad $\mathcal{C}_\infty := \varinjlim_n \mathcal{C}_n$. Since $\pi_i[\mathcal{C}_n(j)] \cong \pi_i[\mathrm{conf}_j(\mathbb{R}^n)] = 0$ for $i \leq n-2$, each component $\mathcal{C}_\infty(j)$ of $\mathcal{C}_\infty$ is contractible, and as the S_j–action on $\mathcal{C}_\infty(j)$ (induced from $\mathcal{C}_n(j)$) is free, $\mathcal{C}_\infty$ is an E_∞–operad. Finally, we recall May's approximation theorem [52, Theorem 2.7], that asserts that the natural map of monads $\alpha_n : \underline{\mathcal{C}}_n(X) \to \underline{\mathcal{C}}_n\Omega^n\Sigma^n(X) \to \Omega^n\Sigma^n(X)$ gives a homotopy equivalence

$$\underline{\mathcal{C}}_n(X) \simeq \Omega^n\Sigma^n(X) \quad \text{for all } n = 1, 2, \dots, \infty, \tag{5-25}$$

whenever X is connected.

We can now state the following result, which is probably well known to experts.

Lemma 5.4 (cf [30]) *For any topological monoid M, there are natural homotopy equivalences*

$$B(\underline{\mathcal{C}}_\infty, \underline{\mathcal{C}}_1, M) \simeq \Omega\Omega^\infty\Sigma^\infty(BM), \tag{5-26}$$

$$B(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_1, M) \simeq \Omega\mathrm{SP}^\infty(BM), \tag{5-27}$$

and the map (5-20) for $X = BM$ is equivalent to the map

$$B(\underline{\mathcal{C}}_\infty, \underline{\mathcal{C}}_1, M) \to B(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_1, M) \tag{5-28}$$

induced by the canonical (unique) morphism of operads $\mathcal{C}_\infty \to \mathcal{C}_0$.

Proof The equivalence (5-26) was originally proved by Fiedorowicz (see [30, Proposition 1.7] and also [2, Lemma 39]); the proof of (5-27) is similar. We describe these equivalences in both cases. First,

$$\begin{aligned} B(\underline{\mathcal{C}}_\infty, \underline{\mathcal{C}}_1, M) \simeq B(\Omega^\infty\Sigma^\infty, \underline{\mathcal{C}}_1, M) \simeq B(\Omega\Omega^\infty\Sigma^\infty\Sigma, \underline{\mathcal{C}}_1, M) &\simeq \Omega\Omega^\infty\Sigma^\infty B(\Sigma, \underline{\mathcal{C}}_1, M) \\ &\simeq \Omega\Omega^\infty\Sigma^\infty(BM), \end{aligned}$$

where the first equivalence is induced by (5-25), the second is obvious, the third is a formal property of the bar construction (see [52, Lemma 9.7]), and the last one follows from a theorem of Fiedorowicz (see [29, Corollary 9.7]) that yields $B(\Sigma,\underline{C}_1,M)\simeq BM$ for any topological monoid M. Similarly,

$$B(\underline{C}_0,\underline{C}_1,M)\cong B(\mathrm{SP}^\infty,\underline{C}_1,M)\simeq B(\Omega\mathrm{SP}^\infty\Sigma,\underline{C}_1,M)\simeq \Omega\mathrm{SP}^\infty B(\Sigma,\underline{C}_1,M)\simeq \Omega\mathrm{SP}^\infty(BM),$$

where the first identification follows from (5-23), the second is induced by the equivalence (5-15), which is a consequence of the Dold–Thom theorem, the third follows from [52, Lemma 9.7], and the last one is [29, Corollary 9.7]. The last statement of the lemma is now deduced by comparing the above equivalences with the construction of the map (5-20) given in the beginning of Section 5.3. ◻

Proof of Proposition 5.3 For any topological monoid M, consider the diagram of spaces

(5-29)
$$\begin{array}{ccccc}
|\mathrm{hocolim}_{\Delta S_+}(B_{\mathrm{sym}}M)| & \xrightarrow{f_\infty} & B(\underline{C}_\infty,\underline{C}_1,M) & \xrightarrow{(5\text{-}26)} & \Omega\Omega^\infty\Sigma^\infty(BM)\\
\downarrow{\scriptstyle \widetilde{\Psi}^*_{\mathrm{sym}}} & & \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle (5\text{-}20)}\\
|\mathrm{hocolim}_{\mathfrak{M}^{\mathrm{op}}_{\mathbb{N}}}(p^*M)| & \xrightarrow{f_0} & B(\underline{C}_0,\underline{C}_1,M) & \xrightarrow{(5\text{-}27)} & \Omega\mathrm{SP}^\infty(BM)
\end{array}$$

In this diagram all horizontal maps are natural weak equivalences: f_∞ is the equivalence constructed by Fiedorowicz in [30] (see [2, Theorem 38]), f_0 is the equivalence (4-45) of Proposition 4.11, and (5-26) and (5-27) are the equivalences described in Lemma 5.4. The map $\widetilde{\Psi}^*_{\mathrm{sym}}$ is induced by the functor $\widetilde{\Psi}_{\mathrm{sym}}$ defined in (4-46). To prove the proposition we need to show that the diagram (5-29) commutes. By Lemma 5.4, we already know that the rightmost square of (5-29) commutes; thus it suffices to prove the commutativity of the leftmost square. For this, we shall describe the maps f_∞ and f_0 explicitly.

The map f_∞ is explicitly constructed in the proof of [2, Lemma 36]. As in loc. cit. we let $\mathcal{N}\colon \mathrm{Top}_*\to\mathrm{Top}_*$ denote the functor defined as the coend

$$\mathcal{N}(X):=\int^{[n]\in\Delta S_+} N([n]\downarrow\Delta S_+)\times B_{\mathrm{sym}}J(X)[n].$$

By [2, Lemma 36], there is an equivalence of functors $\Theta\colon\mathcal{N}\simeq\underline{C}_\infty$, inducing an equivalence of bar constructions $B(\mathcal{N},\underline{C}_1,M)\simeq B(\underline{C}_\infty,\underline{C}_1,M)$. The identification $|\mathrm{hocolim}_{\Delta S_+}(B_{\mathrm{sym}}M)|\simeq B(\mathcal{N},\underline{C}_1,M)$ by [52, Lemma 9.7] then yields f_∞.

The map f_0 can be constructed in a similar way. Let $\mathcal{P}\colon\mathrm{Top}_*\to\mathrm{Top}_*$ denote the functor

$$\mathcal{P}(X):=\int^{(\langle n\rangle;k_1,\dots,k_n)\in\mathfrak{M}^{\mathrm{op}}_{\mathbb{N}}} N((\langle n\rangle;k_1,\dots,k_n)\downarrow\mathfrak{M}^{\mathrm{op}}_{\mathbb{N}})\times p^*J(X)(\langle n\rangle;k_1,\dots,k_n).$$

Identifying $J(X)(\langle n\rangle)=\mathrm{Hom}_{\mathrm{Mon}}(\langle n\rangle,J(X))$ and recalling that $\underline{C}_0(X)=\mathrm{SP}^\infty(X)$ is the abelianization of $J(X)$, we note that the map

$$\coprod N((\langle n\rangle;k_1,\dots,k_n)\downarrow\mathfrak{M}^{\mathrm{op}}_{\mathbb{N}})\times p^*J(X)(\langle n\rangle;k_1,\dots,k_n)\to\underline{C}_0(X)=\mathrm{SP}^\infty(X),$$
$$y\times\varphi\mapsto\varphi_{\mathrm{ab}}(k_1,\dots,k_n),$$

descends to the coend to yield a natural equivalence

$$\Lambda : \mathcal{P}(X) \simeq \mathcal{C}_0(X),$$

which, in turn, yields an equivalence of bar constructions

$$B(\mathcal{P}, \mathcal{C}_1, M) \simeq B(\mathcal{C}_0, \mathcal{C}_1, M).$$

Composing this with the identification $|\mathrm{hocolim}_{\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}}(p^* M)| \simeq B(\mathcal{P}, \mathcal{C}_1, M)$ by [52, Lemma 9.7] yields the map f_0.

It can be easily verified that the diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\Theta} & \mathcal{C}_\infty \\ {\scriptstyle \widetilde{\Psi}^*_{\mathrm{sym}}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ \mathcal{P} & \xrightarrow{\Lambda} & \mathcal{C}_0 \end{array}$$

commutes. It follows that the first square in the diagram (5-29) commutes. Finally, we note that in the case when $M = \Gamma$, a simplicial group, the map $\widetilde{\Psi}^*_{\mathrm{sym}}$ in the diagram (5-29) may be identified with the corresponding map in (4-40) by Corollary 4.12 (also see Remark 4.13). This completes the proof of the desired proposition. □

Corollary 5.5 *Let Γ be a (homotopy) simplicial group such that $X = B\Gamma$ has homotopy type of a simply connected CW complex which is of (locally) finite rational type. If k is a field of characteristic zero, then the map $\mathtt{SR}_X$ induces an isomorphism*

$$\mathrm{HS}_*(k[\Gamma]) \cong \mathrm{HR}_*(k[\Gamma]).$$

Proof As mentioned above, the natural map $i_\infty : \Omega^\infty \Sigma^\infty(X) \to \mathrm{SP}^\infty(X)$, defined by composing (5-18) with the inverse of (5-16) in $\mathtt{Ho}(\mathrm{Top}_*)$, is not an equivalence in general. However, it is known that for any connected CW complex X, this map induces an isomorphism of cohomology rings

$$i_\infty^* : H^*(\mathrm{SP}^\infty(X), k) \xrightarrow{\sim} H^*(\Omega^\infty \Sigma^\infty(X), k) \tag{5-30}$$

provided the coefficients are taken in a field k of characteristic zero; see eg [22, Section 7.3]. Now, under our assumption on X, both $\mathrm{SP}^\infty(X)$ and $\Omega^\infty\Sigma^\infty(X)$ are simply connected spaces of finite rational type. Hence, there is a natural (Cotor) spectral sequence with E_2–term $E_2^{*,*}(Z) = \mathrm{Ext}^*_{H^*(Z,k)}(k,k)$ that converges to $H_*(\Omega Z, k)$ for any simply connected space Z; see eg [22, Section 5.5, (5.13)]. By naturality, the map (5-30) induces an isomorphism $E_2^{*,*}(\Omega^\infty\Sigma^\infty X) \xrightarrow{\sim} E_2^{*,*}(\mathrm{SP}^\infty X)$ of such spectral sequences for $Z = \Omega^\infty\Sigma^\infty(X)$ and $Z = \mathrm{SP}^\infty(X)$. This last isomorphism is compatible with the map $\Omega i_\infty : H_*(\Omega\Omega^\infty\Sigma^\infty(X), k) \to H_*(\Omega \mathrm{SP}^\infty(X), k)$ which, by Proposition 5.3, coincides with $\mathtt{SR}_X : \mathrm{HS}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma])$ for $X = B\Gamma$. Thus, by the Comparison Theorem for spectral sequences (see [73, Theorem 5.2.12]), we conclude that $\mathtt{SR}_X$ is an isomorphism. □

Remark 5.6 We expect that the result of Corollary 5.5 holds for *any* homotopy simplicial group Γ, including the usual (discrete) groups, for which $B\Gamma$ is a $K(1,\Gamma)$–space, ie certainly not simply connected.

5.4 Polynomial extensions

There is a natural way to describe and generalize the map $\mathtt{SR}$ via Goodwillie calculus. As we have seen above, the Dold–Thom functor SP^∞ is 1–excisive, hence there is a canonical (up to homotopy) natural transformation $\beta_1 \colon P_1(\mathrm{id}) \to \mathrm{SP}^\infty$, where $P_1(\mathrm{id}) = D_1(\mathrm{id})$ is the first layer of the functor id. The latter is known to be the stable homotopy functor $P_1(\mathrm{id}) \simeq \Omega^\infty\Sigma^\infty$ and $\beta_1 \simeq i_\infty$. Thus $\mathtt{SR} \simeq \Omega\beta_1$. It turns out that the map β_1 can be extended naturally to higher layers — and in fact, to the entire Goodwillie tower of the functor id. This is based on results of the paper [18] that compares the Goodwillie tower of the identity with the lower central series of the Kan loop group.

Recall that, for any connected space X, we can identify $\mathrm{SP}^\infty(X) \simeq B[\mathbb{A}(X)]$, where $\mathbb{A}(X) := \mathbb{G}(X)_{\mathrm{ab}}$ is the abelianization of the Kan loop group $\mathbb{G}(X)$ of (a reduced simplicial set representing) X; see (4-12). Now, instead of just abelianization, consider the lower central series of $\mathbb{G}(X)$,

$$\cdots \to \mathbb{G}(X)/\mathbb{G}_{n+1}(X) \to \mathbb{G}(X)/\mathbb{G}_n(X) \to \cdots \to \mathbb{G}(X)/\mathbb{G}_2(X) = \mathbb{A}(X),$$

where $\mathbb{G}_n(X)$ are the simplicial subgroups of $\mathbb{G}(X)$ defined inductively by

$$\mathbb{G}_1(X) := \mathbb{G}(X) \qquad \text{and} \qquad \mathbb{G}_{n+1}(X) := [\mathbb{G}(X), \mathbb{G}_n(X)] \quad \text{for } n \geq 1.$$

It is shown in [18] that the functor $X \mapsto B[\mathbb{G}(X)/\mathbb{G}_{n+1}(X)]$ is n–excisive for each $n \geq 1$, and there exists a canonical (up to homotopy) morphism of towers

(5-31)
$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_n(\mathrm{id})(X) & \longrightarrow & P_{n-1}(\mathrm{id})(X) & \longrightarrow & \cdots & \longrightarrow & P_1(\mathrm{id})(X) \\
 & & \Big\downarrow{\scriptstyle \beta_n} & & \Big\downarrow{\scriptstyle \beta_{n-1}} & & \vdots & & \Big\downarrow{\scriptstyle \beta_1} \\
\cdots & \longrightarrow & B[\mathbb{G}(X)/\mathbb{G}_{n+1}(X)] & \longrightarrow & B[\mathbb{G}(X)/\mathbb{G}_n(X)] & \longrightarrow & \cdots & \longrightarrow & B[\mathbb{A}(X)]
\end{array}$$

where the rightmost vertical arrow is precisely the map $\beta_1 \colon P_1(\mathrm{id}) \to \mathrm{SP}^\infty$. This morphism induces natural maps on the layers of the Goodwillie tower

(5-32)
$$\beta_n \colon D_n(\mathrm{id})(X) \to B[\mathbb{G}_n(X)/\mathbb{G}_{n+1}(X)] \quad \text{for } n \geq 1,$$

that we can describe in explicit terms. First of all, by a theorem of B Johnson [41] (cf [1, Example 1.2.5]), all Goodwillie derivatives of the identity functor are known: for $n \geq 1$, the spectrum $\partial_n(\mathrm{id})$ is equivalent to a wedge of $(n-1)!$ copies of the $(1-n)$–sphere spectrum $\mathbb{S}^{1-n} = \Sigma^{1-n}(\mathbb{S}^0)$. Hence, by formula (5-3), we have

(5-33)
$$D_n(\mathrm{id})(X) \simeq \Omega^\infty\Big(\bigvee_{(n-1)!} \Sigma^{1-n}(\Sigma^\infty X)^{\wedge n}\Big)_{hS_n}.$$

On the other hand, the Kan simplicial group $\mathbb{G}(X)$ is (degreewise) free for any X. Hence, by classic PBW Theorem (see eg [65, I.4.3]), for all $n \geq 1$, there are natural isomorphisms of simplicial abelian groups

(5-34)
$$\mathbb{G}_n(X)/\mathbb{G}_{n+1}(X) \cong \mathrm{Lie}_n(\mathbb{A}X),$$

where Lie_n denotes (the simplicial extension of) the degree n graded component of the free graded Lie algebra functor $\mathrm{Lie}_*(A) = \bigoplus_{n\geq 1} \mathrm{Lie}_n(A)$ on abelian groups A. Thus, with identifications (5-33) and (5-34), the morphism of towers (5-31) (looped once) induces on layers natural maps

$$\mathtt{SR}_X^{(n)}\colon \Omega\Omega^\infty\Big(\bigvee_{(n-1)!}\Sigma^{1-n}(\Sigma^\infty X)^{\wedge n}\Big)_{hS_n} \to |\mathrm{Lie}_n(\mathbb{A}X)| \quad \text{for } n\geq 1. \tag{5-35}$$

These can be viewed as nonlinear (polynomial) extensions of our topological trace map $\mathtt{SR}$: in fact, for $n=1$, the map (5-35) coincides with (5-20) under identification (4-13):

$$\mathtt{SR}_X^{(1)}\colon \Omega\Omega^\infty\Sigma^\infty(X) \to |\mathbb{A}(X)| \simeq \Omega\mathrm{SP}^\infty(X),$$

while for $n=2$, it reads

$$\mathtt{SR}_X^{(2)}\colon \Omega\Omega^\infty\Sigma^{-1}(\Sigma^\infty X\wedge\Sigma^\infty X)_{h\mathbb{Z}_2} \to |\mathrm{Lie}_2(\mathbb{A}X)|$$

since the $\mathbb{Z}_2$–action on the spectrum $\partial_2(\mathrm{id}) \simeq \mathbb{S}^{-1}$ is known to be trivial; see [1, Example 1.2.5]. It would be interesting to see whether the maps (5-35) for $n\geq 2$ can be naturally represented by homotopy colimits (similar to $\mathtt{SR}_{B\Gamma} \simeq \widetilde{\Psi}^*_{\mathrm{sym}}$ for $n=1$, see (4-40)), and, in particular, whether the induced maps $\mathtt{SR}^{(n)}_{B\Gamma,*}$ can be described in terms of functor homology (extending the result of Corollary 4.10). The existence of such a description might lead to an interesting link between Goodwillie calculus and homological algebra of polynomial functors (as developed recently in [26; 28; 27; 71]).

6 Stable character maps and derived Poisson brackets

In this section, we study the behavior of the derived character maps (1-7) in the limit as $n\to\infty$. We show that, on simply connected spaces, these maps stabilize, inducing an isomorphism between the graded symmetric algebra generated by the S^1–equivariant homology of the free loop space of $X = B\Gamma$ and the invariant part of the representation homology in the projective limit $\varprojlim \mathrm{HR}_*(\Gamma, \mathrm{GL}_n)^{\mathrm{GL}_n}$. This result is a topological counterpart of a stabilization theorem proved for representation homology of algebras in [11]. In case when X represents a closed manifold, so that its S^1–equivariant homology carries the Chas–Sullivan bracket, we show that the stable character map is an isomorphism of Lie algebras, where the Lie bracket on representation homology is induced by a natural derived Poisson structure on the Quillen model of X.

6.1 Stabilization of derived character maps

For this section, let k be a field of characteristic 0. The (homotopy) group homomorphisms $\Gamma\to\{1\}$ and $\{1\}\to\Gamma$) induce morphisms of cyclic modules $k[B^{\mathrm{cyc}}\Gamma] \to k[B^{\mathrm{cyc}}\{1\}] = k$ and $k = k[B^{\mathrm{cyc}}\{1\}] \to k[B^{\mathrm{cyc}}\Gamma]$, respectively. In this way, the trivial cyclic module k is a direct summand of $k[B^{\mathrm{cyc}}\Gamma]$ yielding a direct sum decomposition

$$k[B^{\mathrm{cyc}}\Gamma] \cong k \oplus k[\overline{B^{\mathrm{cyc}}\Gamma}].$$

The *reduced* cyclic homology $\overline{\mathrm{HC}}_*(k[\Gamma])$ is defined by

$$\overline{\mathrm{HC}}_*(k[\Gamma]) := \mathrm{Tor}_*^{\Delta C}(k[\overline{B^{\mathrm{cyc}}\Gamma}], k),$$

so that

$$\mathrm{HC}_*(k[\Gamma]) \cong \mathrm{HC}_*(k) \oplus \overline{\mathrm{HC}}_*(k[\Gamma]).$$

On the other hand, the homomorphism of group schemes $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$ (given by padding with 1 on the bottom right corner) induces a morphism of commutative Hopf algebras $\mathcal{O}(\mathrm{GL}_{n+1}) \to \mathcal{O}(\mathrm{GL}_n)$, and hence, a morphism of left $\mathfrak{G}$–modules $\underline{\mathcal{O}}(\mathrm{GL}_{n+1}) \to \underline{\mathcal{O}}(\mathrm{GL}_n)$. This induces a morphism on representation homologies

(6-1) $$\mu_{n+1,n}\colon \mathrm{HR}_*(\Gamma, \mathrm{GL}_{n+1}) = \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \underline{\mathcal{O}}(\mathrm{GL}_{n+1})) \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_n) = \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \underline{\mathcal{O}}(\mathrm{GL}_n)).$$

It is not difficult to verify that (6-1) restricts to a morphism on the invariant part of the representation homologies

(6-2) $$\mu_{n+1,n}\colon \mathrm{HR}_*(\Gamma, \mathrm{GL}_{n+1})^{\mathrm{GL}_{n+1}} \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_n)^{\mathrm{GL}_n}.$$

Lemma 6.1 *The following diagram commutes for all n:*

$$\begin{array}{ccc} \overline{\mathrm{HC}}_*(k[\Gamma]) & \xrightarrow{\mathrm{Tr}_{n+1}(\Gamma)} & \mathrm{HR}_*(\Gamma, \mathrm{GL}_{n+1})^{\mathrm{GL}_{n+1}} \\ & \searrow^{\mathrm{Tr}_n(\Gamma)} & \downarrow{\scriptstyle \mu_{n+1,n}} \\ & & \mathrm{HR}_*(\Gamma, \mathrm{GL}_n)^{\mathrm{GL}_n} \end{array}$$

Proof Since any homotopy simplicial group is weakly equivalent to a cofibrant strict simplicial group, we may assume without loss of generality that Γ is a cofibrant strict simplicial group. Continuing to denote the map $\overline{k[\Gamma]} \otimes_{\Delta C} k \to k[\Gamma] \otimes_{\mathfrak{G}} \underline{\mathcal{O}}(\mathrm{GL}_n)$ induced by $\Delta_{\mathrm{GL}_n}\mathrm{tr}$ by $\mathrm{Tr}_n(\Gamma)$, we then need to verify that the following diagram commutes

(6-3) $$\begin{array}{ccc} \overline{k[\Gamma]} \otimes_{\Delta C} k & \xrightarrow{\mathrm{Tr}_{n+1}(\Gamma)} & k[\Gamma] \otimes_{\mathfrak{G}} \underline{\mathcal{O}}(\mathrm{GL}_{n+1}) \\ & \searrow^{\mathrm{Tr}_n(\Gamma)} & \downarrow{\scriptstyle \mu_{n+1,n}} \\ & & k[\Gamma] \otimes_{\mathfrak{G}} \underline{\mathcal{O}}(\mathrm{GL}_n) \end{array}$$

By (the proof of) [44, Theorem 4.1], $\mathrm{Tr}_n(\Gamma_m)$ is induced (in each simplicial degree m) by the composite map

$$\Gamma_m \xrightarrow{\rho_n} \mathrm{GL}_n(\mathcal{O}[\mathrm{Rep}_n(\Gamma_m)]) \hookrightarrow \mathbb{M}_n(\mathcal{O}[\mathrm{Rep}_n(\Gamma_m)]) \xrightarrow{\mathrm{Tr}} \mathcal{O}[\mathrm{Rep}_n(\Gamma_m)] \cong k[\Gamma_m] \otimes_{\mathfrak{G}} \underline{\mathcal{O}}(\mathrm{GL_n}),$$

where ρ_n denotes the universal n–dimensional representation. A similar argument shows that the diagram

$$\begin{array}{ccc} \Gamma_m & \xrightarrow{\rho_{n+1}} & \mathrm{GL}_{n+1}(\mathcal{O}[\mathrm{Rep}_{n+1}(\Gamma_m)]) \\ {\scriptstyle \rho_n}\downarrow & & \downarrow{\scriptstyle \mu_{n+1,n}} \\ \mathrm{GL}_n(\mathcal{O}[\mathrm{Rep}_n(\Gamma_m)]) & \hookrightarrow & \mathrm{GL}_{n+1}(\mathcal{O}[\mathrm{Rep}_n(\Gamma_m)]) \end{array}$$

commutes. Here, the lower horizontal arrow is given by padding by 1 on the bottom right. It follows that

$$\mathrm{Tr}_n(\Gamma_m)(\langle\gamma\rangle - 1) = \mu_{n+1,n} \circ \mathrm{Tr}_{n+1}(\Gamma_m)(\langle\gamma\rangle - 1)$$

for every conjugacy class $\langle\gamma\rangle$ in Γ_m. This shows commutativity of the diagram (6-3) in every simplicial degree, proving the desired lemma. □

By Lemma 6.1, the family of maps $\{\mathrm{Tr}_n(\Gamma)\}_{n\geq 1}$ yields a k–linear map

$$\mathrm{Tr}_\infty(\Gamma)\colon \overline{\mathrm{HC}}_*(k[\Gamma]) \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty} := \varprojlim_n \mathrm{HR}_*(\Gamma, \mathrm{GL}_n)^{\mathrm{GL}_n}, \tag{6-4}$$

where the inverse limit is taken along the maps (6-2). The map $\mathrm{Tr}_\infty(\Gamma)$, which we call the *stable character map*, induces a morphism of graded commutative k–algebras

$$\Lambda\,\mathrm{Tr}_\infty(\Gamma)\colon \Lambda_k[\overline{\mathrm{HC}}_*(k[\Gamma])] \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty}. \tag{6-5}$$

Next, recall that a simplicial group Γ is said to be a *simplicial group model* of a pointed, connected topological space X if Γ maps to X under (3-8), ie $|\overline{W}(\Gamma)|$ is weakly equivalent to X. In this case, it is well known that

$$\mathrm{HC}_*(k[\Gamma]) \cong \mathrm{H}_*^{S^1}(\mathcal{L}X; k), \tag{6-6}$$

where $\mathcal{L}X$ is the free loop space of X, and the representation homology $\mathrm{HR}_*(\Gamma, G)$, which is an invariant of (the homotopy type of) X by Lemma 3.7 is denoted by $\mathrm{HR}_*(X, G)$. The isomorphism (6-6) restricts to an isomorphism of graded k–modules

$$\overline{\mathrm{HC}}_*(k[\Gamma]) \cong \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k). \tag{6-7}$$

Here, $\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k)$ stands for the *reduced* S^1–equivariant homology of $\mathcal{L}X$, ie

$$\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}_X; k) := \mathrm{Ker}[\pi_*\colon \mathrm{H}_*^{S^1}(\mathcal{L}X) \to \mathrm{H}_*^{S^1}(\mathrm{pt})].$$

The map π_* is induced on S^1–equivariant homology by the map $\mathcal{L}X \to \mathrm{pt}$. The derived character map $\mathrm{Tr}_n(X) := \mathrm{Tr}_n(\Gamma)$ is thus a morphism of graded k–vector spaces

$$\mathrm{Tr}_n(X)\colon \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k) \to \mathrm{HR}_*(X, \mathrm{GL}_n)^{\mathrm{GL}_n}, \tag{6-8}$$

and the stable character map becomes

$$\mathrm{Tr}_\infty(X)\colon \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k) \to \mathrm{HR}_*(X, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty}. \tag{6-9}$$

The following theorem is the main result of this section.

Theorem 6.2 *Let X be a simply connected space of finite (rational) type. The stable character map (6-9) induces an isomorphism of graded commutative algebras*

$$\Lambda\,\mathrm{Tr}_\infty(X)\colon \Lambda_k[\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k)] \xrightarrow{\sim} \mathrm{HR}_*(X, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty}.$$

If, moreover, X is a simply connected manifold of dimension d then $\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;k)$ is equipped with the *Chas–Sullivan bracket* (also called the *string topology bracket*), a graded Lie bracket of (homological) degree $2-d$. This Lie bracket arises out of a derived Poisson structure (in the sense of [15, Section 3.1]) on an algebra weakly equivalent to $k[\Gamma]$. On the other hand, the representation homologies $\mathrm{HR}_*(X,\mathrm{GL}_n)^{\mathrm{GL}_n}$ are equipped with graded, ((2−d)–shifted) Poisson structures arising from the Poincaré duality pairing on the cohomology of X. Passing to the inverse limit, one obtains a graded ((2−d)–shifted) Poisson structure on $\mathrm{HR}_*(X,\mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$. As an application of Theorem 6.2, we obtain the following corollary which allows us to express the Chas–Sullivan bracket in terms of a graded Poisson bracket.

Corollary 6.3 *The map*

$$\Lambda\,\mathrm{Tr}_\infty(X)\colon \Lambda_k[\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;k)] \xrightarrow{\sim} \mathrm{HR}_*(X,\mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$$

is an isomorphism of graded $(2-d)$–shifted Poisson algebras.

6.2 Proofs of Theorem 6.2 and Corollary 6.3

The shortest way to prove Theorem 6.2 and Corollary 6.3 is to apply the results of the paper [11] that deals with stabilization of representation homology and derived character maps for (augmented) associative algebras. These results being applicable in our case follows from Remark 3.16. In what follows we outline key steps and necessary modifications of the arguments of [11], leaving details for interested readers.

Sketch of proof of Theorem 6.2 Let L_X denote a (cofibrant) Quillen model of X. Since X is of finite rational type, L_X may be chosen to be semifree, and finitely generated in each homological degree. By Remark 3.16, if suffices to prove the assertions of this theorem working with UL_X instead of $k[\Gamma]$. Further, since X is simply connected, the generators of L_X are in positive homological degree. Theorem 6.2 follows from (a minor modification of the proof of) [11, Theorem 7.8]. Indeed, since $R=UL_X$ is freely generated by finitely many generators in each homological degree, and since all its generators are in positive homological degree, the arguments of [11, Section 7.4] go through to show that for each $k>0$, the map

$$\tilde{\mu}_{n+1,n}\colon R_{n+1}^{\mathrm{GL},\leqslant k} \to R_n^{\mathrm{GL},\leqslant k} \tag{6-10}$$

is an isomorphism for n sufficiently large (ie for all $n>N(k)$, for some $N(k)$ which possibly depends on k). Here R_n^{GL} is the representation DG algebra as in [11, formula (2.10)], whose homology is isomorphic to $\mathrm{HR}_*(R,n)^{\mathrm{GL}_n} \cong \mathrm{HR}_*(X,\mathrm{GL}_n)^{\mathrm{GL}_n}$ and $R_n^{\mathrm{GL},\leqslant k}$ stands for the (brutal) truncation of R_n^{GL} to homological degrees $\leqslant k$. The map (6-10) is defined as in [11, Section 4] (where it is denoted by $\mu_{n+1,n}$). On homologies, (6-10) induces the map $\mu_{n+1,n}\colon \mathrm{HR}_*(X,\mathrm{GL}_{n+1})^{\mathrm{GL}_{n+1}} \to \mathrm{HR}_*(X,\mathrm{GL}_n)^{\mathrm{GL}_n}$. As in the proof of [11, Theorem 7.8] (see also [11, Proposition 7.5], which is the crux thereof), it then follows that the map

$$\Lambda\,\mathrm{Tr}_\infty(X)\colon \Lambda_k[\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;k)] \to \mathrm{H}_*[R_\infty^{\mathrm{GL}}]$$

is an isomorphism of graded commutative algebras, where $R^{\mathrm{GL}}_\infty = \varprojlim_n R^{\mathrm{GL}}_n$. The desired verification is thus complete once we check that $\mathrm{H}_*[R^{\mathrm{GL}}_\infty] \cong \varprojlim_n \mathrm{H}_*[R^{\mathrm{GL}}_n]$. By (6-10), the inverse system $\{R^{\mathrm{GL}}_n\}$ is Mittag-Leffler. Equation (6-10) further implies that for each k, the inverse system $\{\mathrm{H}_{k+1}(R^{\mathrm{GL}}_n)\}$ stabilizes, ie becomes constant for large n, and is thus Mittag-Leffler. It follows that $\lim^1_n \mathrm{H}_{k+1}(R^{\mathrm{GL}}_n) = 0$. That $\mathrm{H}_*[R^{\mathrm{GL}}_\infty] \cong \varprojlim_n \mathrm{H}_*[R^{\mathrm{GL}}_n]$, as desired, then follows from [73, Theorem 3.5.8]. This outlines the proof of Theorem 6.2. □

Sketch of proof of Corollary 6.3 Moreover (see [15, Section 4.2] for example), L_X may be chosen so that its universal enveloping algebra UL_X is equipped with a derived Poisson structure inducing the Chas–Sullivan bracket on its (reduced) cyclic homology (which is isomorphic to $\overline{\mathrm{H}}^{S^1}_*(\mathcal{L}X;k)$). More precisely, L_X may be chosen to be Koszul dual to the (graded linear dual of) the Lambrechts–Stanley model of X (see [46]), which is equipped with a cyclic pairing. Now, if Γ is a simplicial group model of X, then $k[\Gamma]$ is weakly equivalent to UL_X. By Remark 3.16, if suffices to prove the assertions of this theorem working with UL_X instead of $k[\Gamma]$. In this setting, it follows immediately from [15, Theorem 5.1] (also see [6, Theorems 2 and 3.1]) that the cyclic pairing on (the graded linear dual of) the Lambrechts–Stanley model of X yields a graded ((2$-d$)–shifted) Poisson structure on $\mathrm{HR}_*(X,\mathrm{GL}_n)^{\mathrm{GL}_n}$ such that the derived character map $\mathrm{Tr}_n\colon \overline{\mathrm{H}}^{S^1}_*(\mathcal{L}X;k) \to \mathrm{HR}_*(X,\mathrm{GL}_n)^{\mathrm{GL}_n}$ is a homomorphism of graded Lie algebras. Moreover, the maps $\mu_{n+1,n}\colon \mathrm{HR}_*(UL_X, n+1)^{\mathrm{GL}_{n+1}} \to \mathrm{HR}_*(UL_X,n)^{\mathrm{GL}_n}$ are easily seen to be homomorphisms of graded Poisson algebras in the setting of [15, Section 5]. Hence, $\mathrm{HR}_*(X,\mathrm{GL}_\infty)^{\mathrm{GL}} \cong \mathrm{HR}_*(UL_X,\infty)^{\mathrm{GL}}$ acquires the structure of a graded Poisson algebra. It follows that $\mathrm{Tr}_\infty(X)\colon \overline{\mathrm{H}}^{S^1}_*(\mathcal{L}X;k) \to \mathrm{HR}_*(X,\mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$ is a homomorphism of grade Lie algebras, which implies that $\Lambda\,\mathrm{Tr}_\infty(X)\colon \Lambda_k[\overline{\mathrm{H}}^{S^1}_*(\mathcal{L}X;k)] \to \mathrm{HR}_*(X,\mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$ is a homomorphism of graded Poisson algebras, where the Poisson structure in the left-hand side is obtained by extending the Chas–Sullivan bracket using the Leibniz rule. That it is an *isomorphism* of graded Poisson algebras then follows from Theorem 6.2. □

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Instanton knot invariants with rational holonomy parameters and an application for torus knot groups

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There are several knot invariants in the literature that are defined using singular instantons. Such invariants provide strong tools to study the knot group and give topological applications, for instance, the topology of knots in terms of representations of fundamental groups. In particular, it has been shown that any traceless representation of the torus knot group can be extended to any concordance from the torus knot to another knot. Daemi and Scaduto proposed a generalization that is related to a version of the slice-ribbon conjecture for torus knots. Our results provide further evidence towards the positive answer to this question. We use a generalization of Daemi and Scaduto's equivariant singular instanton Floer theory following Echeverria's earlier work. We also determine the irreducible singular instanton homology of torus knots for all but finitely many rational holonomy parameters as $\mathbb{Z}/4$–graded abelian groups.

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1 Introduction

1.1 Background

Floer homology is an infinite-dimensional analog of Morse homology. In the context of gauge theory, instanton Floer homology (see Floer [14]), Heegaard Floer homology (see Ozsváth and Szabó [37]) and monopole Floer homology (Kronheimer and Mrowka [30]) have provided strong topological invariants for low-dimensional manifolds. Knot invariants have also been developed in Floer theories. This list of knot invariants includes knot Floer homology introduced by Ozsváth and Szabó [36] and Rasmussen [40] in Heegaard Floer theory, and Kronheimer and Mrowka [31] in monopole Floer theory. In the field of instanton Floer theory, invariants of knots were constructed by Floer [15] and Braam and Donaldson [1] via framed surgery of knots. It is conjectured that their instanton knot invariants are related to knot invariants in Ozsváth and Szabó [36] and Rasmussen [40] by Kronheimer and Mrowka [31]. Collin and Steer [3] and Kronheimer and Mrowka [32] developed other type invariants for knots. While knot invariants in [15; 1] are related to invariants of 3–manifolds via surgery along knots, knot invariants in [3; 32] are related to 3–manifold invariants via branched covering.

The advantage of instanton invariants is that they are directly related to fundamental groups of the knot complement. For example, Kronheimer and Mrowka [29] show that the knot group $\pi_1(S^3 \setminus K)$ for a nontrivial knot $K \subset S^3$ admits nonabelian representation $\pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$. This is a refinement of the result by Papakyriakopoulos [38] which states that $K \subset S^3$ is unknot if only if $\pi_1(S^3 \setminus K)$ is infinitely cyclic. A concordance analog of the result of Kronheimer and Mrowka [29] was given by Daemi and Scaduto [8] using a version of instanton Floer theory. Daemi and Scaduto [8] also show the following statement which is specific to torus knots:

Theorem 1 [8, Theorem 8] *Let $S\colon T_{p,q} \to K$ be a given smooth concordance. Then any traceless* SU(2)*–representation of $\pi_1(S^3 \setminus T_{p,q})$ extends over the concordance complement.*

Here $T_{p,q}$ denotes the (p,q)–torus knot in S^3, where p and q are positive coprime integers. An SU(2)–traceless representation of $\pi_1(S^3 \setminus K)$ is an SU(2)–representation of $\pi_1(S^3 \setminus K)$ which sends a homotopy class of meridian μ_K of K to a traceless element in SU(2). The motivation of this theorem is related to a version of the slice-ribbon conjecture. A concordance $S\colon K \to K'$ is called ribbon concordance if the projection $S^3 \times [0,1] \supset S \to [0,1]$ is a Morse function without any local maximums. Consider a knot K which is concordant to the unknot U. The slice-ribbon conjecture proposed by Fox [16] states that there is a ribbon concordance from U to K under this assumption. A generalization of the slice-ribbon conjecture by Daemi and Scaduto [8] is:

Conjecture 2 [8, Question 2] *Let K be a knot which is concordant to the (p,q)–torus knot $T_{p,q}$. Then there is a ribbon concordance from $T_{p,q}$ to K.*

A necessary condition to show that a concordance $S\colon K \to K'$ is ribbon can be stated in terms of representations of knot groups. For a topological space X, we write $\mathcal{R}(X, \mathrm{SU}(2))$ for the SU(2)–character variety of X (ie the space of conjugacy classes of SU(2)–representations of $\pi_1(X)$).

Theorem 3 (Gordon [21, Lemma 3.1] and Daemi, Lidman, Vela-Vick and Wong [7, Proposition 2.1]) *Let $S\colon K \to K'$ be a ribbon concordance between two knots. Then the inclusion $i\colon S^3 \setminus K \to S^3 \times [0,1]$ induces a surjection $i^*\colon \mathcal{R}(S^3 \times [0,1] \setminus S, \mathrm{SU}(2)) \to \mathcal{R}(S^3 \setminus K, \mathrm{SU}(2))$.*

Hence Theorem 1 gives a piece of evidence towards Conjecture 2. The traceless condition on representations of $\pi_1(S^3 \setminus T_{p,q})$ arises from the specific type of knot invariants developed by Daemi and Scaduto [9]. In light of Theorem 3 and Conjecture 2, it is natural to ask the following question:

Question 4 *Can we drop the traceless condition in Theorem 1?*

We will affirmatively solve this question. To explain our strategy, let us describe the technical background of Daemi and Scaduto's work. It mainly consists of three ingredients: singular gauge theory, equivariant Floer theory and the Chern–Simons filtration.

Firstly, let us explain the notion of singular connections. Let $K \subset Y$ be a knot in a 3–manifold. Roughly speaking, an SU(2)–singular connection A is an SU(2)–connection defined over the knot complement with the holonomy condition

$$\lim_{r\to 0} \mathrm{Hol}_{\mu(r)}(A) \sim \begin{bmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{bmatrix}, \tag{1-1}$$

where $\mu(r)$ is a radius-r meridian of $K \subset Y$ and α is a fixed parameter in $\left(0, \frac{1}{2}\right)$. Here $\sim$ indicates that the two matrices are conjugate in SU(2). The parameter α is called the holonomy parameter of the singular connection A. In particular, a singular flat SU(2)–connection corresponds to an SU(2)–representation of $\pi_1(Y \setminus K)$ which sends the meridian μ_K of knot K to an element which is conjugate to the matrix in (1-1). Kronheimer and Mrowka developed a singular version of Yang–Mills gauge theory in [27; 28; 32]. These Floer homology theories constructed via singular connections are called singular instanton homology. Singular gauge theory has different features compared to nonsingular. In fact, singular Floer homology cannot be defined over the coefficient ring $\mathbb{Z}$ for a general holonomy parameter α. To be more precise, singular instanton Floer homology is defined over $\mathbb{Z}$ only for $\alpha = \frac{1}{4}$. This is called *the monotonicity condition*. Most of the works in singular instanton homology including [9; 8] impose the monotonicity condition. This is why the statement of Theorem 1 includes the traceless condition.

Next, we discuss the equivariant Floer theory. Frøyshov developed the homology cobordism invariant in [18; 19] based on the equivariant Floer theory for integral homology 3–spheres, which was introduced by Donaldson [10]. The equivariant Floer theory introduced by Daemi and Scaduto [9] produces invariants for a knot K in an integral homology 3–sphere Y, and this can be regarded as the counterpart of Frøyshov's work in singular gauge theory. Daemi and Scaduto's construction uses in a crucial way the $U(1)$–reducible singular flat connection θ which corresponds to the conjugacy class of the representation

$$\pi_1(Y \setminus K) \to H_1(Y \setminus K; \mathbb{Z}) \to \mathrm{SU}(2)$$

whose image of the meridian μ_K of $K \subset Y$ is trace-free. Here $\pi_1(Y \setminus K) \to H_1(Y \setminus K; \mathbb{Z})$ is the abelianization. In this situation, the construction which is similar to Floer's instanton homology [14] produces a chain complex $C_*(Y, K)$ for a knot in an integral homology 3–sphere. Its homology group $I_*(Y, K)$ can be interpreted as a categorification of the knot signature for the case $Y = S^3$. Daemi and Scaduto [9] also introduced chain complexes which have the form

$$\widetilde{C}_*(Y, K) := C_*(Y, K) \oplus C_{*-1}(Y, K) \oplus \mathbb{Z}.$$

Such objects are called $\mathcal{S}$–complexes. This can be interpreted as a version of S^1–equivariant Floer theory. Let $\mathcal{B}(Y, K)$ be the configuration space of singular connections over (Y, K) with a holonomy parameter $\alpha = \frac{1}{4}$. Then there is a configuration space $\mathcal{B}(Y, K)_0$ of framed connections. The Chern–Simons functional on $\mathcal{B}(Y,K)$ lifts to $\mathcal{B}(Y,K)_0$ in an equivariant way. An $\mathcal{S}$–complex $\widetilde{C}_*(Y,K)$ is related to the lifted S^1–equivariant Chern–Simons functional on $\mathcal{B}(Y,K)_0$.

Another feature of Daemi and Scaduto's construction is the Chern–Simons filtration of $\mathcal{S}$–complexes. While the usual instanton Floer theory is the analog of Morse theory on the configuration space, its filtered version can be seen the Morse theory on the universal covering of the configuration space. The Chern–Simons filtration gives more refined structures on $\mathcal{S}$–complexes. The counterpart idea in nonsingular instanton Floer theory was used by Daemi [5] and Nozaki, Sato and Taniguchi [35], which provided homology cobordism invariants.

Any of the above versions of singular instanton Floer theories can be extended to arbitrary holonomy parameters if the integer coefficient ring is replaced with a Novikov ring Λ by Echeverria's work [12]. To be more precise, the holonomy parameter should satisfy the technical condition $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$, where $\Delta_{(Y,K)}$ is the Alexander polynomial for $K \subset Y$. One of the flavors of Echeverria's Floer homology is a categorification of the Levine–Tristram signature when $Y = S^3$. For a knot K in an integral homology 3–sphere Y, the Levine–Tristram signature is given by

$$\sigma_\alpha(Y,K) := \operatorname{sign}[(1-e^{4\pi i\alpha})V + (1-e^{-4\pi i\alpha})V^T],$$

where V is a Seifert matrix form of $K \subset Y$. For the case $Y = S^3$, we omit Y from the notation.

Our strategy to drop the traceless condition from Theorem 1 is constructing a family of $\mathcal{S}$–complexes for general holonomy parameters.

1.2 Summary of results

First we state our main theorem, which gives the positive answer to Question 4:

Theorem 1.1 *For a given knot K and a smooth concordance $S: T_{p,q} \to K$, any* SU(2)*–representation of $\pi_1(S^3 \setminus T_{p,q})$ extends to an* SU(2)*–representation of $\pi_1((S^3 \times [0,1]) \setminus S)$.*

The proof of Theorem 1.1 requires the special property that all generators of singular instanton homology for torus knots have odd gradings. The outline of the proof is as follows. After extending the condition of Daemi and Scaduto [9], we define analogous knot Floer theory of [9] for all $\alpha \in \mathcal{I}$, where $\mathcal{I}$ is a dense subset of $\left[0, \frac{1}{2}\right]$. This means that all SU(2)–representations of $\pi_1(S^3 \setminus T_{p,q})$ with the holonomy parameter $\alpha \in \mathcal{I}$ extend to the concordance complement. The limiting argument shows that this extension property is true for all SU(2)–representations of $\pi_1(S^3 \setminus T_{p,q})$ with any holonomy parameter $\alpha \in \left[0, \frac{1}{2}\right]$.

As described above, singular instanton knot homology (see Echeverria [12]) and its equivariant counterparts are key tools for the proof, so we review the essential properties of these objects we use. We consider the Novikov ring $\Lambda^{\mathbb{Z}[T^{-1},T]\!]}$ which is given by

$$\Lambda^{\mathbb{Z}[T^{-1},T]\!]} := \left\{ \sum_{r\in\mathbb{R}} p_r \lambda^r \;\middle|\; p_r \in \mathbb{Z}[T^{-1},T]\!], \forall C > 0, \#\{p_r \neq 0\}_{r>C} < \infty \right\}.$$

Let α be a parameter in $\left(0, \frac{1}{2}\right)$. We introduce the subring $\mathscr{R}_\alpha$ of $\Lambda^{\mathbb{Z}[T^{-1},T]\!]}$

$$\mathscr{R}_\alpha := \begin{cases} \mathbb{Z}[\xi_\alpha^{\pm 1}][\![\lambda^{-1}, \lambda] & \text{if } \alpha \le \frac{1}{4}, \\ \mathbb{Z}[\lambda^{\pm 1}][\![\xi_\alpha^{-1}, \xi_\alpha] & \text{if } \alpha > \frac{1}{4}, \end{cases}$$

where $\xi_\alpha = \lambda^{2\alpha} T^2$. The geometric aspect of the subring $\mathscr{R}_\alpha$ is described in Section 3.2.

Theorem 1.2 *Let $\mathscr{S}$ be an algebra over $\mathscr{R}_\alpha$. Let $K \subset Y$ be an oriented knot in an integral homology 3–sphere. Choose a holonomy parameter $\alpha \in \mathbb{Q} \cap \left(0, \frac{1}{2}\right)$ so that $\Delta_{(Y,K)}(e^{4\pi i \alpha}) \neq 0$. Then we can associate a $\mathbb{Z}/4$–graded module $I_*^\alpha(Y, K; \Delta_{\mathscr{S}})$ over $\mathscr{S}$ to this parameter. Moreover, if $\mathscr{S}$ is an integral domain, we can associate a $\mathbb{Z}/4$–graded $\mathcal{S}$–complex $(\widetilde{C}_*^\alpha(Y, K; \Delta_{\mathscr{S}}), \tilde{d}, \chi)$ to a given triple (Y, K, α) with $\Delta_{(Y,K)}(e^{4\pi i \alpha}) \neq 0$, up to $\mathcal{S}$–chain homotopy equivalence.*

The precise definition of an $\mathcal{S}$–complex can be seen in Section 3.1. We call $I_*^\alpha(Y, K; \Delta_{\mathscr{S}})$ *the irreducible singular instanton knot homology over $\mathscr{S}$ with the holonomy parameter α*. For the case $Y = S^3$, we drop Y from the notation. The difference between the construction of our Floer homology $I_*^\alpha(Y, K; \Delta_{\mathscr{S}})$ and $I_*(Y, K, \alpha)$ introduced by [12] is the choice of local coefficients. The construction of $(C_*^\alpha(Y, K), d)$ and $(\widetilde{C}_*^\alpha(Y, K), \tilde{d}, \chi)$ depends on additional data (metric and perturbation), however their chain homotopy classes in the sense of $\mathcal{S}$–complexes are independent of such choices.

Remark 1.3 To be more precise, we need to specify the choice of positive integer $\nu \in \mathbb{Z}_{>0}$, called the cone angle, to define the invariant $I_*^\alpha(Y, K, \Delta_{\mathscr{S}})$. The details are included in Remark 3.9. As conjectured in [12], we expect that the invariant $I_*^\alpha(Y, K, \Delta_{\mathscr{S}})$ does not depend on the choice of cone angle $\nu \in \mathbb{Z}_{>0}$, and hence it is reasonable to drop ν from the notation. Similar remarks are applied to the dependence of invariants $\widetilde{C}_*^\alpha(Y, K, \Delta_{\mathscr{S}})$ and $h_{\mathscr{S}}^\alpha(Y, K)$ which appear later. For a given holonomy parameter α, we always assume that the cone angle ν is a large enough integer.

Remark 1.4 For the coefficient $\mathscr{S} = \mathscr{R}_\alpha$, we consider underlying groups of $C_*^\alpha(Y, K; \Delta_{\mathscr{S}})$ and $\widetilde{C}^\alpha(Y, K, \Delta_{\mathscr{S}})$ as $\mathbb{Z}$–modules. Then if we fix the choice of auxiliary data, there exists a functional giving the $(\mathbb{Z}\times\mathbb{R})$–bigraded structure on sets of generators of these underlying groups. Moreover, they have a filtered structure induced from the $\mathbb{R}$–grading. The precise descriptions of the $(\mathbb{Z}\times\mathbb{R})$–bigrading and the filtered structure are contained in Sections 3.2 and 3.4.

The following statement describes the behavior of $\mathcal{S}$–complexes under the connected sum:

Theorem 1.5 *Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. Let $K \subset Y$ and $K' \subset Y'$ be two oriented knots in integral homology 3–spheres. Fix a holonomy parameter $\alpha \in \mathbb{Q} \cap \left(0, \frac{1}{2}\right)$ such that*

$$\Delta_{(Y,K)}(e^{4\pi i \alpha}) \Delta_{(Y',K')}(e^{4\pi i \alpha}) \neq 0.$$

Then there is a chain homotopy equivalence of $\mathcal{S}$–complexes

$$\widetilde{C}_*^\alpha(Y \# Y', K \# K'; \Delta_{\mathscr{S}}) \simeq \widetilde{C}_*^\alpha(Y, K; \Delta_{\mathscr{S}}) \otimes_{\mathscr{S}} \widetilde{C}_*^\alpha(Y', K'; \Delta_{\mathscr{S}}).$$

The precise definition of *a chain homotopy equivalence of $\mathcal{S}$–complexes* can be seen in Definition 3.3. This is a generalization of the connected sum theorem by Daemi and Scaduto [9]. The method of the proof of [9, Theorem 6.1] cannot be directly adapted to prove Theorem 1.5 since we have to deal with the nonmonotonicity situation which arises for general holonomy parameters.

As described in [9], we can associate an integer-valued invariant which is called the Frøyshov type invariant to a given $\mathcal{S}$–complex. Our construction of $\mathcal{S}$–complexes provides an integer-valued invariant $h^{\alpha}_{\mathscr{S}}(Y,K)$ for a knot in a homology 3–sphere (Y,K). We call $h^{\alpha}_{\mathscr{S}}(Y,K)$ *the Frøyshov invariant for* (Y,K) *over* $\mathscr{S}$ *with the holonomy parameter* α. We drop Y from the notation when $Y = S^3$. Note that Echeverria [12] also introduced the Frøyshov type invariant denoted by $h(Y,K,\alpha)$, which is constructed from singular instanton Floer homology with a different local coefficient system from our setting. The invariant $h^{\alpha}_{\mathscr{S}}(Y,K)$ satisfies the following properties:

Theorem 1.6 *Let (Y,K) and (Y',K') be two pairs of integral homology 3–spheres and knots. Assume that $\alpha \in \mathbb{Q} \cap \left(0,\frac{1}{2}\right)$ satisfies $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$ and $\Delta_{(Y',K')}(e^{4\pi i\alpha}) \neq 0$. Then*

$$h^{\alpha}_{\mathscr{S}}(Y \# Y', K \# K') = h^{\alpha}_{\mathscr{S}}(Y,K) + h^{\alpha}_{\mathscr{S}}(Y',K').$$

Moreover, if (Y,K) and (Y',K') are homology concordant, then

$$h^{\alpha}_{\mathscr{S}}(Y,K) = h^{\alpha}_{\mathscr{S}}(Y',K').$$

Let us consider a knot in S^3. It has been shown that the Frøyshov type invariant in [9] reduces to knot signature (see Daemi and Scaduto [8, Theorem 7]). The invariant $h^{\alpha}_{\mathscr{S}}$ reduces to the Levine–Tristram signature as follows:

Theorem 1.7 *Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. For any knot $K \subset S^3$ and for a holonomy parameter $\alpha \in \left(0,\frac{1}{2}\right) \cap \mathbb{Q}$ with $\Delta_K(e^{4\pi i\alpha}) \neq 0$, the following equality holds:*

$$h^{\alpha}_{\mathscr{S}}(K) = -\tfrac{1}{2}\sigma_\alpha(K).$$

For a given knot $K \subset S^3$ and integer l, we define a knot $lK \subset S^3$ so that

$$lK := \begin{cases} \#_l K & \text{if } l > 0, \\ U \text{ (unknot)} & \text{if } l = 0, \\ \#_{-l}(-K) & \text{if } l < 0, \end{cases}$$

where $-K$ is the mirror of K with the reverse orientation. More strongly, $\mathcal{S}$–complexes have the following structure theorem:

Theorem 1.8 *Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. Then for a knot K in S^3 and for a holonomy parameter $\alpha \in \mathbb{Q} \cap \left(0,\frac{1}{2}\right)$ with $\Delta_K(e^{4\pi i\alpha}) \neq 0$, there is a two-bridge torus knot $T_{2,2n+1}$ such that $\Delta_{T_{2,2n+1}}(e^{4\pi i\alpha}) \neq 0$, $\sigma_\alpha(T_{2,2n+1}) = -2$ and the relation*

$$\widetilde{C}^{\alpha}_{*}(K;\Delta_{\mathscr{S}}) \simeq \widetilde{C}^{\alpha}_{*}(lT_{2,2n+1};\Delta_{\mathscr{S}})$$

holds, where $l = -\frac{1}{2}\sigma_\alpha(K)$.

For the proof of Theorem 1.8, it is essential to observe behaviors of morphisms of $\mathcal{S}$–complexes induced from cobordisms between pairs (Y, K) and (Y', K'). In [8], techniques demonstrated by Kronheimer [25] are used to describe behaviors of morphisms of $\mathcal{S}$–complexes for the case $\alpha = \frac{1}{4}$. However, such techniques do not directly adapt to prove Theorem 1.8 because of the lack of the monotonicity condition.

Theorems 1.5 and 1.8 imply the Euler characteristic formula

$$\chi(I^{\alpha}(K, \Delta_{\mathscr{S}})) = \tfrac{1}{2}\sigma_{\alpha}(K) \tag{1-2}$$

(see Section 5.1). Since our grading convention of generators coincides with that of Echeverria [12], the above argument also gives an alternative proof of the Euler characteristic formula in [12, Theorem 17] for the case $Y = S^3$.

Next, we focus on (p, q)–torus knot $T_{p,q}$ in the 3–sphere. We always assume that p and q are positive coprime integers. The following is a characteristic property of the torus knot and a key lemma for the proof of Theorem 1.1. Let $\mathcal{R}_{\alpha}(Y \setminus K, \mathrm{SU}(2))$ be the space of conjugacy classes of SU(2)–representations of $\pi_1(Y \setminus K)$ with the holonomy parameter α. Let $\mathcal{R}^*_{\alpha}(Y \setminus K, \mathrm{SU}(2))$ be its irreducible part.

Theorem 1.9 *For any $\alpha \in \left[0, \frac{1}{2}\right]$ with $\Delta_{T_{p,q}}(e^{4\pi i \alpha}) \neq 0$,*

$$|\mathcal{R}^*_{\alpha}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))| = -\tfrac{1}{2}\sigma_{\alpha}(T_{p,q}).$$

Here $|S|$ for a set S denotes the size of this set. In [23], Herald introduced the *signed count* of elements in the character variety $\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$ for a general knot K with a fixed holonomy parameter. One first perturbs $\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$ into a discrete set $\mathcal{R}^{*,h}_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$ and then associates a sign to each element of this set. The sum of these signs is Herald's signed count of $\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$, which we denote by $\#\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2))$. In general:

$$\#\mathcal{R}^*_{\alpha}(S^3 \setminus K, \mathrm{SU}(2)) = -\tfrac{1}{2}\sigma_{\alpha}(K).$$

See Herald [23, Corollary 0.2] and Lin [33] for the case $\alpha = \frac{1}{4}$. In the case $K = T_{p,q}$, the character variety $\mathcal{R}^*_{\alpha}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ is already discrete and one does not make any perturbation. Theorem 1.9 implies that all elements of $\mathcal{R}^*_{\alpha}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ have positive signs.

Theorem 1.9 implies that $C^{\alpha}_*(T_{p,q}; \Delta_{\mathscr{S}})$ is supported only on the odd graded components. In particular, its homology groups are isomorphic to chain complexes,

$$I^{\alpha}_*(T_{p,q}; \Delta_{\mathscr{S}}) \cong C^{\alpha}_*(T_{p,q}; \Delta_{\mathscr{S}}),$$

since all differentials of chain complexes are trivial. This can be interpreted as the counterpart of the computation of instanton homology of Brieskorn homology 3–spheres; see Fintushel and Stern [13].

Theorem 1.7 implies that $\operatorname{rank} C^{\alpha}_*(T_{p,q}; \Delta_{\mathscr{S}}) = h^{\alpha}_{\mathscr{S}}(T_{p,q})$, and by the definition of the invariant $h^{\alpha}_{\mathscr{S}}$:

Theorem 1.10 *Let $\mathscr{S}$ be an algebra over $\mathscr{R}_\alpha$. For $\alpha \in \left(0, \frac{1}{2}\right) \cap \mathbb{Q}$ with $\Delta_{T_{p,q}}(e^{4\pi i \alpha}) \neq 0$, there is an isomorphism*

$$I_*^\alpha(T_{p,q}; \Delta_{\mathscr{S}}) \cong \mathscr{S}_{(1)}^{\lceil -\sigma_\alpha(T_{p,q})/4 \rceil} \oplus \mathscr{S}_{(3)}^{\lfloor -\sigma_\alpha(T_{p,q})/4 \rfloor}$$

as a $\mathbb{Z}/4$–graded abelian group.

Theorem 1.10 describes the grading of generators of the Floer chain and it is independent of the choice of local coefficient system. A similar structure theorem holds for singular instanton knot homology introduced by Echeverria [12].

Theorem 1.8 implies that $\mathcal{S}$–complexes for knots are determined by the Levine–Tristram signature without the $(\mathbb{Z}\times\mathbb{R})$–grading structure. On the other hand, the $\mathbb{R}$–grading from the Chern–Simons filtration can be expected to have stronger information on the knot concordance. In upcoming work of Daemi, Sato, Scaduto, Taniguchi and the author [6], relying on the results here, we will introduce a generalization of the Γ–invariant of Daemi and Scaduto [9] for rational holonomy parameters, which can be regarded as a gauge-theoretic refinement of the Levine–Tristram signature. Our techniques are also used in the future work of Daemi and Scaduto to construct families of hyperbolic knots that are minimal with respect to the ribbon partial order; see Gordon [21, Conjecture 1.1].

1.3 Outline

In Section 2, we review the background of SU(2)–singular gauge theory for rational holonomy parameters. We also introduce the generalized definition of negative definite cobordism. In Section 3, we construct Floer chain groups and $\mathcal{S}$–complexes parametrized by holonomy parameters α, and introduce the Frøyshov type invariant. The argument is almost parallel to [9; 8], however, we need a careful choice of local coefficient system if we introduce the bigraded structure on the Floer chain complex. We also prove Theorem 1.6. In Section 4, we prove the Levine–Tristram signature formula for torus knots (Theorem 1.9). In the proof of Theorem 1.9, we use the correspondence of singular flat connections and nonsingular flat connections over the branched covering space. We also use the pillowcase picture of the SU(2)–character variety for the knot complement space. We prove Theorems 1.7, 1.8 and 1.10 in Section 5.1, and finally, we give the proof of our main theorem (Theorem 1.1) in Section 5.2. The bigraded structure of $\mathcal{S}$–complexes plays an important role in the proof of the main theorem. The appendix consists of the proof of the connected sum theorem (Theorem 1.5).

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2 Background on singular instantons

In this section, we review singular gauge theory, mainly developed by Kronheimer and Mrowka [32]. We also give a generalization of the setting of singular SU(2)–gauge theory adopted by Daemi and Scaduto [9; 8].

2.1 The space of singular connections

We review the construction of singular instantons over a closed pair of a 4–manifold and a surface. Let X be a closed and oriented smooth 4–manifold and S be a closed and oriented embedded surface in X. Let N be a tubular neighborhood of S in X. We identify N with a disk bundle over S and ∂N with a circle bundle over S. Let η be a connection 1–form on a circle bundle ∂N. This means that η is $U(1)$–invariant. We fix a decomposition of the SU(2)–bundle $E \to X$ over the embedded surface S as $E|_S = L \oplus L^*$, where L is a $U(1)$–bundle over S. This decomposition extends to N. We define two topological invariants,

$$k = c_2(E)[X] \quad \text{and} \quad l = -c_1(L)[S].$$

Here k is called the instanton number, and l is called the monopole number.

Next, we fix a connection A_0 over X of the form

$$A_0|_N = \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}.$$

Here b is a connection over L. This means that A^0 reduces to a $U(1)$–connection over S. We give the polar coordinates $(r, \theta) \in D^2$ on each fiber of N. Let η be a 1–form obtained by a pulled-back 1–form on ∂N which coincides $d\theta$ on each fiber, and ψ be a cutoff function supported on N. We define the singular base connection A^α by

$$A^\alpha = A_0 + i\psi \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \eta,$$

where $\alpha \in \left(0, \frac{1}{2}\right)$. Here α is called the holonomy parameter. Recall that η is defined only on $N \setminus S$, but extends by 0 to $X \setminus S$ after cutting off by ψ. A^α is a connection over $X \setminus S$. Let $\mathfrak{g}_E$ be the adjoint bundle of E. For $a \in \Omega^1(X \setminus S, \mathfrak{g}_E)$, $A^\alpha + a$ is called a singular connection.

Before defining the space of singular connections, we have to introduce functional spaces. We fix an orbifold metric on X, which can be written in the form

$$g^\nu = du^2 + dv^2 + dr^2 + \frac{r^2}{\nu^2} d\theta^2$$

on N, where (u, v) is a local coordinate of S. We say that this orbifold metric has cone angle $2\pi/\nu$. Then (X, g^ν) has a local structure $U/\mathbb{Z}_\nu$ near the singular locus S, where U is an open set in $\mathbb{R}^4$. The model connection A^α induces an SO(3)–adjoint connection on $\mathfrak{g}_E$. We define the covariant derivative $\check{\nabla}_{A^\alpha}$ on the bundle $\Lambda^m \otimes \mathfrak{g}_E$ using the adjoint connection of A^α and the Levi-Civita connection with respect to the metric g^ν. Let $F \to X$ be an orbifold vector bundle. The Sobolev space $\check{L}^p_{m,A^\alpha}(X \setminus S, F)$ is defined as the completion of the space of smooth sections of $F \to X$ by the norm

$$\|s\|^p_{\check{L}^p_{m,A^\alpha}} = \sum_{i=0}^{m} \int_{X\setminus S} |\check{\nabla}_{A^\alpha} s|^p \, \mathrm{dvol}_{g^\nu}.$$

If we use orbifold metrics, the "Fredholm package" works. Let $d^+_{A^\alpha} : \Omega^1(X \setminus S, \mathfrak{g}_E) \to \Omega^+(X \setminus S, \mathfrak{g}_E)$ be the linearized anti-self-dual operator defined by the metric g^ν, and $d^*_{A^\alpha} : \Omega^1(X \setminus S, \mathfrak{g}_E) \to \Omega^0(X \setminus S, \mathfrak{g}_E)$ be the formal adjoint of the covariant derivative for the metric g^ν. Consider the elliptic operator $D_{A^\alpha} = -d^*_{A^\alpha} \oplus d^+_{A^\alpha}$ acting on the Sobolev space

$$\check{L}^p_{m,A^\alpha}(X \setminus S, \Lambda^1 \otimes \mathfrak{g}_E) \to \check{L}^p_{m-1,A^\alpha}(X \setminus S, (\Lambda^0 \oplus \Lambda^+) \otimes \mathfrak{g}_E). \tag{2-1}$$

Proposition 2.1 *Let α be a rational holonomy parameter of the form $\alpha = p/q \in \left(0, \frac{1}{2}\right) \cap \mathbb{Q}$. Choose a cone angle $2\pi/\nu$ of orbifold metric so that $2\nu p/q \in \mathbb{Z}$. Then the operator D_{A^α} and its formal adjoint are Fredholm, and the Fredholm alternative holds.*

Let A^α_{ad} be the adjoint of the singular connection A^α and $\pi : U \to U/\mathbb{Z}_\nu$ be an orbifold chart with respect to the orbifold metric g^ν. If $\nu \in \mathbb{Z}_{>0}$ is chosen as in Proposition 2.1, the lift of the adjoint connection of $\pi^* A^\alpha$ has the asymptotically trivial holonomy along a small linking of the singular locus. Thus $\pi^* A^\alpha$ extends smoothly over U. This means that A^α_{ad} defines an orbifold connection. All analytical argument reduces to the orbifold setting. From now on, we always fix ν as in Proposition 2.1 for a given rational holonomy parameter.

Assume that $m > 2$. The space of singular connections with a holonomy parameter $\alpha \in \left(0, \frac{1}{2}\right)$ is given by

$$\mathcal{A}(X, S, \alpha) = \{A^\alpha + a \mid a \in \check{L}^2_{m,A^\alpha}(X \setminus S, \Lambda^1 \otimes \mathfrak{g}_E)\}.$$

This is an affine space as the nonsingular case. Notice that $\mathcal{A}(X, S, \alpha)$ is independent of the choice of the base connection A^α. We also introduce the group of gauge transformations,

$$\mathcal{G}(X, S) = \{g \in \mathrm{Aut}(E) \mid g \in \check{L}^2_{m+1,A^\alpha}(X \setminus S, \mathrm{End}(E))\}.$$

There is the smooth action of $\mathcal{G}(X, S)$ on $\mathcal{A}(X, S, \alpha)$, and we can take the quotient.

$$\mathcal{B}(X, S, \alpha) = \mathcal{A}(X, S, \alpha)/\mathcal{G}(X, S).$$

A singular connection with the 0–dimensional stabilizer for the action of $\mathcal{G}(X, S)$ is called an irreducible connection. A singular connection is called reducible if it is not irreducible. The quotient space $\mathcal{B}(X, S, \alpha)$ has a smooth Banach manifold structure except for orbits of reducible connections. The set of gauge equivalence classes of solutions for the anti-self-dual equation

$$M^\alpha(X, S) = \{[A] \in \mathcal{B}(X, S, \alpha) \mid F^+_A = 0\}$$

is called the moduli space of singular anti-self-dual connections. $M^\alpha(X,S)_d$ denotes the subset of $M^\alpha(X,S)$ with expected dimension d. For a generic orbifold metric with a fixed cone angle, the irreducible part of $M^\alpha(X,S)_d$ is a smooth manifold of dimension $d = \mathrm{ind}(d_A^* \oplus d_A^+)$, where $[A] \in M^\alpha(X,S)_d$. If $M^\alpha(X,S)_d$ consists of reducible connections, we modify the dimension of the moduli space so that $d = \mathrm{ind}(d_A^* \oplus d_A^+) + \dim H_A^0$, where H_A^i is an i^{th}cohomology group of the deformation complex. The index of the ASD-operator $d_A^* \oplus d_A^+$ is given by

$$\mathrm{ind}(d_A^* \oplus d_A^+) = 8k + 4l - 3(1 - b^1 + b^+) - 2(g(S) - 1),$$

where $g(S)$ is the genus of the surface S. The index formula for the closed pair (X,S) does not depend on the holonomy parameter α. On the other hand, the energy integral $\kappa(A) = \|F_A\|_{\check{L}^2}$ for an ASD-connection A is given by

$$\kappa(A) = k + 2\alpha l - \alpha^2 S \cdot S.$$

We always assume that an integer $\nu > 0$ is chosen large enough for a fixed holonomy parameter $\alpha \in \mathbb{Q} \cap \left(0, \frac{1}{2}\right)$, under the condition $2\alpha\nu \in \mathbb{Z}$. Such choice of ν is related to the bubbling and compactification of moduli spaces. The details are described in [27; 28].

2.2 The Chern–Simons functional

We discuss singular connections over 3–manifolds. Let Y be an oriented integral homology 3–sphere and K be an oriented knot in Y. Let E be an SU(2)–bundle over Y. This is always topologically trivial. We fix a reduction of E to a line bundle over K as $E|_K = L \oplus L^*$, and fix orbifold metric g^ν along K as in Section 2.1. For a fixed $\alpha \in \mathbb{Q} \cap \left(0, \frac{1}{2}\right)$, we choose ν as in Proposition 2.1. We can similarly define the spaces of singular connections and gauge transformations:

$$\mathcal{A}(Y,K,\alpha) = \{A^\alpha + a \mid a \in \check{L}^2_{m,A^\alpha}(Y \setminus K, \mathfrak{g}_E)\}, \quad \mathcal{G}(Y,K) = \{g \in \mathrm{Aut}(E) \mid g \in \check{L}^2_{m+1,A^\alpha}(Y \setminus K, \mathrm{End}(E))\}.$$

We define the quotient

$$\mathcal{B}(Y,K,\alpha) = \mathcal{A}(Y,K,\alpha)/\mathcal{G}(Y,K).$$

We use the notation $\mathcal{A}_m(Y,K,\alpha)$ if we wish to emphasize that the space of singular connection is defined by the completion of the Sobolev norm $\check{L}^2_m$.

We describe the topology of $\mathcal{G}(Y,K)$ and $\mathcal{B}(Y,K,\alpha)$. There are two other kinds of groups of gauge transformations,

$$\mathcal{G}_K = \{g \in \mathrm{Aut}(L|_K)\} \quad \text{and} \quad \mathcal{G}^K(Y,K) = \{g \in \mathrm{Aut}(E) \mid g|_K = \mathrm{id}\}.$$

Then there is the exact sequence

$$1 \to \mathcal{G}^K(Y,K) \to \mathcal{G}(Y,K) \to \mathcal{G}_K \to 1.$$

There is the map $\mathcal{G}(Y,K) \to \mathbb{Z} \oplus \mathbb{Z}$ given by $d(g) = (k,l)$, where $k = \deg(g\colon Y \to \mathrm{SU}(2))$ and $l = \deg(g|_K\colon K \to U(1))$, and this map induces the isomorphism

$$\pi_0(\mathcal{G}(Y,K)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Using the homotopy exact sequence induced from the fibration $\mathcal{G}(Y,K) \to \mathcal{A}(Y,K,\alpha) \to \mathcal{B}(Y,K,\alpha)$, we also have the isomorphism

$$\pi_1(\mathcal{B}(Y,K,\alpha)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

We define an L^2–inner product on tangent spaces of $\mathcal{A}(Y,K,\alpha)$ as follows:

$$\langle a,b\rangle = \int_{Y\setminus K} -\operatorname{tr}(a \wedge *b).$$

The $*$–operator is given by the orbifold metric g^ν. The Chern–Simons functional $\mathrm{CS}\colon \mathcal{A}(Y,K,\alpha) \to \mathbb{R}$ is given by the formal gradient

$$\operatorname{grad}(\mathrm{CS})_A = \frac{1}{4\pi^2} * F_A$$

with respect to the above L^2–inner product on $T\mathcal{A}(Y,K,\alpha)$. This uniquely determines CS up to a constant. $A \in \mathcal{A}(Y,K,\alpha)$ is a critical point of CS if only if $F_A = 0$. The critical point set of CS is a space of flat connections on $Y \setminus K$ such that their holonomy along the meridian is conjugate to

$$\begin{bmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{bmatrix}.$$

Let Crit be the critical point set of the Chern–Simons functional $\mathrm{CS}\colon \mathcal{A}(Y,K,\alpha) \to \mathbb{R}$ and $\mathrm{Crit}^* = \mathrm{Crit} \cap \mathcal{A}^*(Y,K,\alpha)$. Let $\mathfrak{C}(Y,K,\alpha)$ and $\mathfrak{C}^*(Y,K,\alpha)$ be images of Crit and Crit^* by the natural projection $\mathcal{A}(Y,K,\alpha) \to \mathcal{B}(Y,K,\alpha)$. Then

$$\mathfrak{C}(Y,K,\alpha) = \mathcal{R}_\alpha(Y \setminus K, \mathrm{SU}(2)) \quad \text{and} \quad \mathfrak{C}^*(Y,K,\alpha) = \mathcal{R}^*_\alpha(Y \setminus K, \mathrm{SU}(2))$$

by the holonomy correspondence of flat connections and representations of the fundamental group.

We have to perturb the Chern–Simons functional to achieve transversality. This is done by introducing a cylinder function associated with a perturbation $\pi \in \mathcal{P}$

$$f_\pi\colon \mathcal{A}(Y,K,\alpha) \to \mathbb{R},$$

which we will construct in Section 2.4. Let Crit_π be the critical point set of $\mathrm{CS} + f_\pi$ and $\mathrm{Crit}^*_\pi = \mathrm{Crit}_\pi \cap \mathcal{A}^*(Y,K,\alpha)$. Their orbits of gauge transformations are denoted by $\mathfrak{C}_\pi(Y,K,\alpha)$ and $\mathfrak{C}^*_\pi(Y,K,\alpha)$.

We define (perturbed) topological energy $\mathcal{E}_\pi(\gamma)$ of a path $\gamma\colon [0,1] \to \mathcal{A}(Y,K,\alpha)$ as

$$\mathcal{E}_\pi(\gamma) = 2\{(\mathrm{CS} + f_\pi)(\gamma(1)) - (\mathrm{CS} + f_\pi)(\gamma(0))\}. \tag{2-2}$$

We also define the (perturbed) Hessian of $A \in \mathcal{A}(Y,K,\alpha)$ as

$$\operatorname{Hess}_{A,\pi}(a) = *d_A a + DV_\pi|_A(a),$$

where V_π is a gradient of f_π, and $DV_\pi|_A$ is its derivative at A.

For each $A \in \mathcal{A}(Y,K,\alpha)$, we can regard the Hessian as the operator,

$$\operatorname{Hess}_{A,\pi}\colon \check{L}^2_{m,A^\alpha}(Y \setminus K, \Lambda^1 \otimes \mathfrak{g}_E) \to \check{L}^2_{m-1,A^\alpha}(Y \setminus K, \Lambda^1 \otimes \mathfrak{g}_E).$$

Definition 2.2 $A \in \mathrm{Crit}^*_\pi$ is called nondegenerate if $\operatorname{Hess}_{A,\pi}|_{\operatorname{Ker}(d^*_A)}$ is invertible.

This means that the Hessian is nondegenerate to the vertical direction of the gauge orbit. For irreducible critical points of the unperturbed Chern–Simons functional, there is the following criterion for the nondegeneracy condition:

Proposition 2.3 [32, Lemma 3.13] *A critical point $A \in \mathrm{Crit}^*$ is nondegenerate if only if the kernel of the map*

$$H^1(Y \setminus K; \mathrm{ad}\,\rho) \to H^1(\mu_K; \mathrm{ad}\,\rho)$$

is zero, where this map is induced by the natural embedding $\mu_K \hookrightarrow Y \setminus K$ and $\rho\colon \pi_1(Y \setminus K) \to \mathrm{SU}(2)$ is the representation corresponding to the flat connection A.

We say that $[A] \in \mathfrak{C}_\pi$ is nondegenerate if one of its representatives $A \in \mathrm{Crit}_\pi$ (and hence all) are nondegenerate.

The nondegeneracy condition at the reducible critical point is given by a constraint on the holonomy parameter. Let θ_α be the gauge equivalence class of the reducible flat connection corresponding to the conjugacy class of an SU(2)–representation of $\pi_1(Y \setminus K)$ which factors through the abelianization $H_1(Y \setminus K, \mathbb{Z})$ and has a holonomy parameter α. Since Y is an integral homology 3–sphere, such θ_α uniquely exists. The following is obtained as a corollary of [32, Lemma 3.13]:

Proposition 2.4 [12, Lemma 15] *The unique flat reducible θ_α is isolated and nondegenerate if only if $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$.*

Let us fix the definition of the Chern–Simons functional. We fix a reducible flat connection $\tilde{\theta}_\alpha$ which represents θ_α and put the condition $\mathrm{CS}(\tilde{\theta}_\alpha) = 0$. Then the $\mathbb{R}$–valued functional CS is determined up to the choice of a representative of θ_α. From now on, we fix a representative $\tilde{\theta}_\alpha$ for each pair (Y, K).

2.3 The flip symmetry

The flip symmetry is an involution that acts on a family of configuration spaces $\bigcup_{\alpha\in(0,1/2)\cap\mathbb{Q}} \mathcal{B}(Y, K, \alpha)$. The flip symmetry changes holonomy conditions as $\alpha \mapsto \frac{1}{2} - \alpha$. The 4–dimensional version is introduced in [27], and the 3–dimensional version is similarly defined in [9]. We generalize the 3–dimensional version of the flip symmetry as follows. Let $\chi \in H^1(Y \setminus K, \mathbb{Z}_2) \cong \mathbb{Z}_2$ be a generator. Since $H^1(Y \setminus K, \mathbb{Z}_2) = \mathrm{Hom}(\pi_1(Y \setminus K), \mathbb{Z}_2)$, we can regard χ as a representation $\chi\colon \pi(Y \setminus K) \to \mathbb{Z}_2$. The representation χ satisfies $\chi(\mu_K) = -1$, and forms a flat line bundle L_χ over $Y \setminus K$ with a flat connection corresponding to χ. Since L_χ is a trivial line bundle and there is an isomorphism $E|_{Y\setminus K} \cong E|_{Y\setminus K} \otimes L_\chi$, we regard a connection $A \otimes \chi$ on $E|_{Y\setminus K} \otimes L_\chi$ as a connection on $E|_{Y\setminus K}$. Thus the action of $\chi \in H^1(Y \setminus K, \mathbb{Z}_2)$ on $\bigcup_{\alpha\in(0,1/2)\cap\mathbb{Q}} \mathcal{B}(Y, K, \alpha)$ is defined by

$$\chi[A] = [A \otimes \chi].$$

This action is called the flip symmetry, and gives the identification

$$\mathcal{B}(Y,K,\alpha)\cong\mathcal{B}\big(Y,K,\tfrac{1}{2}-\alpha\big).$$

In particular, it defines the involution on $\mathcal{B}\big(Y,K,\frac{1}{4}\big)$.

The flip symmetry can be restricted to the space $\bigcup_{\alpha\in(0,1/2)\cap\mathbb{Q}}\mathcal{R}_\alpha(Y\setminus K,\mathrm{SU}(2))$. In this case, the action of $\chi\in H^1(Y\setminus K,\mathbb{Z}_2)$ is simply described as $\chi[\rho]=[\rho\cdot\chi]$ where $\rho\cdot\chi\colon\pi_1(Y\setminus K)\to\mathrm{SU}(2)$ is the SU(2)–representation defined as $(\rho\cdot\chi)(g):=\rho(g)\chi(g)$ for $g\in\pi_1(Y\setminus K)$. If ρ satisfies

$$\rho(\mu_K)\sim\begin{bmatrix}e^{2\pi i\alpha}&0\\0&e^{-2\pi i\alpha}\end{bmatrix},$$

then

$$(\rho\cdot\chi)(\mu_K)\sim\begin{bmatrix}e^{2\pi i(1/2-\alpha)}&0\\0&e^{-2\pi i(1/2-\alpha)}\end{bmatrix}.$$

2.4 Holonomy perturbations

In this subsection, we review the construction and properties of the perturbation term of the Chern–Simons functional introduced by Floer [14] and Braam and Donaldson [1], and here we follow the notation of Kronheimer and Mrowka [32] and Daemi and Scaduto [9].

Let $q\colon S^1\times D^2\to Y\setminus K$ be a smooth immersion of a solid torus. Then $(s,z)\in S^1\times D^2$ denotes its coordinates, regarding S^1 as $\mathbb{R}/\mathbb{Z}$ and D^2 as the unit disk in $\mathbb{C}$. Let $G_E\to Y$ be the bundle of the group whose sections are gauge transformations of E. This is defined by $G_E=P\times_{\mathrm{SU}(2)}\mathrm{SU}(2)$, where P is the corresponding SU(2)–bundle, and SU(2) acts in the obvious way on P and by conjugation on the SU(2)–factor. $\mathrm{Hol}_q(A)\colon D^2\to q^*(G_E)$ is a section of $q^*(G_E)$ which assigns the holonomy $\mathrm{Hol}_{q(-,z)}(A)$ of connection $A\in\mathcal{A}(Y,K,\alpha)$ along the loop $q(-,z)\colon S^1\to Y\setminus K$ to each $z\in D^2$. Next, we repeat the above constructions for an r–tuple of smooth immersions of solid tori

$$\boldsymbol{q}=(q_1,\dots,q_r).$$

Assume that there is a positive number $\eta>0$ such that

$$q_i(s,z)=q_j(s,z)\quad\text{for all }(s,z)\in[-\eta,\eta]\times D^2.\tag{2-3}$$

Then there are identifications

$$q_i^*(G_E)\cong q_j^*(G_E)$$

over $[-\eta,\eta]\times D^2$, and $q^*(G_E^r)$ denotes the fiber product of $q_1^*(G_E),\dots,q_r^*(G_E)$ over $[-\eta,\eta]\times D^2$. Then we can construct a section $\mathrm{Hol}_{\boldsymbol{q}}(A)\colon D^2\to q^*(G_E^r)$ which assigns

$$(\mathrm{Hol}_{q_1(-,z)}(A),\dots,\mathrm{Hol}_{q_r(-,z)}(A))\in\mathrm{SU}(2)^r$$

for each $z\in D^2$. Next, we choose a smooth function on $\mathrm{SU}(2)^r$ which is invariant under the diagonal action of SU(2) on $\mathrm{SU}(2)^r$ by the adjoint action on each factor. This smooth function induces $h:q^*(G_E)\to\mathbb{R}$.

Definition 2.5 $\mathrm{Hol}_{\boldsymbol{q}}(A)$ and h are as above. Let μ be a 2–form on D^2 such that $\int_{D^2}\mu = 1$. A smooth function $f\colon \mathcal{A}(Y, K, \alpha) \to \mathbb{R}$ of the form

$$f(A) = \int_{D^2} h(\mathrm{Hol}_{\boldsymbol{q}}(A))\mu$$

is called a cylinder function.

Cylinder functions are determined by the choice of an r–tuple $\boldsymbol{q}$ and a function h. Note that the construction of cylinder functions is gauge invariant. Let $\mathcal{P}$ be the space of perturbations; see [32] for details. For each $\pi \in \mathcal{P}$, we can associate a cylinder function f_π. We call f_π the holonomy perturbation and $\mathrm{CS} + f_\pi$ the perturbed Chern–Simons functional.

Proposition 2.6 *There is a residual subset of the Banach space of perturbations $\mathcal{P}' \subset \mathcal{P}$ such that, for any sufficiently small $\pi \in \mathcal{P}'$, the perturbed Chern–Simons functional $\mathrm{CS} + f_\pi$ has the nondegenerate critical point set Crit^*_π and its image $\mathfrak{C}^*_\pi$ in $\mathcal{B}^*(Y, K, \alpha)$ is a finite point set. Moreover, the reducible critical point θ_α is unmoved under the perturbation and is nondegenerate if $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$.*

Proof The finiteness property follows from a similar argument as in [32, Lemma 3.8]. The nondegeneracy condition follows from the fact that f_π is dense in $C^\infty(S)$ for any compact finite-dimensional submanifold $S \subset \mathcal{B}^*(Y, K, \alpha)$; see [10, Section 5] for details. The argument in [9, Subsection 2.4] is adapted to show that, for a suitable choice of an $\mathrm{SU}(2)$–invariant smooth function h, the unique flat reducible θ_α is unmoved under small perturbations. By Proposition 2.4, the unique flat reducible θ_α is still isolated and nondegenerate for such perturbations under the condition $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$. □

2.5 The moduli space over the cylinder

We discuss trajectories for the perturbed gradient flow. Let $(Z, S) = \mathbb{R} \times (Y, K)$ be a cylinder equipped with a product metric $g^\nu_Y + dt$. We introduce moduli spaces of instantons over the cylinder. $\mathbb{E}$ denotes the pullback of the $\mathrm{SU}(2)$–bundle $E \to Y$ by the projection $\mathbb{R} \times Y \to Y$. Consider a connection A on $\mathbb{E}$ of the form $A = B(t) + C\,dt$, where $B(t)$ is a t–dependent singular connection on $Y \setminus K$. Let β_0 and β_1 be elements in $\mathfrak{C}^*_\pi$, and let B_0 and B_1 be their representatives in gauge equivariant classes. Consider a singular connection A_0 over the cylinder (Z, S) such that

$$A_0|_{(Y\setminus K)\times\{t\}} = B_1 \quad \text{for } t \gg 0 \qquad \text{and} \qquad A_0|_{(Y\setminus K)\times\{t\}} = B_0 \quad \text{for } t \ll 0.$$

A_0 defines a path $\gamma\colon \mathbb{R} \to \mathcal{B}(Y, K, \alpha)$ by sending t to $[B(t)]$, and z denotes its relative homotopy class in $\pi_1(\mathcal{B}(Y, K, \alpha); \beta_0, \beta_1)$.

Then we define the space of singular connection indexed by z:

$$\mathcal{A}_z(Z, S; B_0, B_1) = \{A \mid A - A_0 \in \check{L}^2_{m,A_0}(Z \setminus S, \Lambda^1 \otimes \mathfrak{g}_{\mathbb{E}})\}.$$

We also define the group of gauge transformations:

$$\mathcal{G}_z(Z,S)=\{g\in \mathrm{Aut}(\mathbb{E})\mid \nabla^k_{A_0}g\in \check{L}^2(Z\setminus S,\mathrm{End}(\mathbb{E})),\ k=1,\dots,m+1\}.$$

The group $\mathcal{G}_z(Z,S)$ acts on $\mathcal{A}_z(Z,S)$. Taking the quotient gives the configuration space $\mathcal{B}_z(Z,S;\beta_0,\beta_1)$. We introduce the moduli space of (perturbed) instantons over the cylinder associated with the perturbed Chern–Simons functional $\mathrm{CS}+f_\pi$. This is the moduli space of solutions of the perturbed ASD-equation

$$M_z^\pi(\beta_0,\beta_1)=\{[A]\in \mathcal{B}_z(Z,S;\beta_0,\beta_1)\mid F_A^+ +\widehat{V}_\pi(A)=0\}.$$

Here $\widehat{V}_\pi$ is a term arising from the perturbation f_π. The perturbed version of the ASD-complex is given by

$$\Omega^0(Z\setminus S,\mathfrak{g}_\mathbb{E})\xrightarrow{d_A}\Omega^1(Z\setminus S,\mathfrak{g}_\mathbb{E})\xrightarrow{d_A^+ + D\widehat{V}_\pi}\Omega^+(Z\setminus S,\mathfrak{g}_\mathbb{E}).$$

We consider the Fredholm operator

$$D_{A,\pi}\colon \check{L}^2_{m,A_0}(Z\setminus S,\Lambda^1\otimes\mathfrak{g}_\mathbb{E})\to \check{L}^2_{m-1,A_0}(Z\setminus S,(\Lambda^0\oplus\Lambda^+)\otimes\mathfrak{g}_\mathbb{E})$$

given by $D_{A,\pi}=-d_A^*\oplus(d_A^+ + D\widehat{V}_\pi)$ and define the relative $\mathbb{Z}$–grading for $\beta_0,\beta_1\in\mathfrak{C}^*_\pi(Y,K,\alpha)$ as

$$\mathrm{gr}_z(\beta_0,\beta_1)=\mathrm{ind}(D_{A,\pi}),$$

where z is a path represented by A. Note that $\mathrm{ind}\, D_{A,\pi}$ is independent of the choice of perturbation π since the term $D\widehat{V}_\pi$ is a compact operator. The following proposition gives the well-defined mod-4 grading on the critical point set:

Proposition 2.7 [32, Lemma 3.1] *Let $z\in\pi_1(\mathcal{B}(Y,K,\alpha))$ be a homotopy class represented by a path which connects B and $g^*(B)$, where $\beta=[B]$ and $d(g)=(k,l)$. For $\beta\in\mathfrak{C}^*$ and a homotopy class $z\in\pi_1(\mathcal{B}(Y,K,\alpha);\beta)$, we have*

$$\mathrm{gr}_z(\beta,\beta)=8k+4l.$$

The mod-4 value of the $\mathbb{Z}$–grading is independent of the choice of the homotopy class z, and hence we can write

$$\mathrm{gr}(\beta_0,\beta_1)\equiv \mathrm{gr}_z(\beta_0,\beta_1)\mod 4.$$

We also define the absolute $\mathbb{Z}$–grading by

$$\mathrm{gr}_z(\beta,\theta_\alpha)=\mathrm{ind}(D_{A,\pi}\colon \phi\check{L}^2_m\to\phi\check{L}^2_{m-1}),$$

where $\phi\check{L}^2_m$ is a weighted Sobolev space with a weight function ϕ which agrees with $e^{-\delta|t|}$ over two ends of the cylinder. Here $\delta>0$ is chosen to be small enough. Similarly, we can define the mod-4 grading

$$\mathrm{gr}(\beta)\equiv\mathrm{gr}_z(\beta,\theta_\alpha)\mod 4.$$

The moduli space $M_z^\pi(\beta_0,\beta_1)$ is called regular if the operator $D_{A,\pi}$ is surjective for all $[A]\in M_z^\pi(\beta_0,\beta_1)$. For a generic choice of perturbation, the moduli space $M_z^\pi(\beta_0,\beta_1)$ is a regular and smooth manifold of dimension $\mathrm{gr}_z(\beta_0,\beta_1)$. We explicitly write $M_z^\pi(\beta_0,\beta_1)_d$ if the moduli space $M_z^\pi(\beta_0,\beta_1)$ has dimension d, and write $\breve{M}_z^\pi(\beta_0,\beta_1)_{d-1} = M_z^\pi(\beta_0,\beta_1)_d/\mathbb{R}$. The argument in [10, Section 5] is adapted to our situation, and we have the following properties:

Proposition 2.8 *Let $\alpha \in \left(0,\frac{1}{2}\right)$ be a holonomy parameter with $\Delta_K(e^{4\pi i\alpha}) \neq 0$ and $\pi_0 \in \mathcal{P}'$ be a small perturbation such that $\mathfrak{C}_{\pi_0} = \{\theta_\alpha\} \sqcup \mathfrak{C}^*_{\pi_0}$ consists of finitely many nondegenerate points. Then there is a small perturbation $\pi \in \mathcal{P}'$ such that*

(i) *$f_\pi = f_{\pi_0}$ in a neighborhood of $\mathfrak{C}_{\pi_0}$,*

(ii) *$\mathfrak{C}_\pi = \mathfrak{C}_{\pi_0}$,*

(iii) *$M_z^\pi(\beta_1,\beta_2)$ is regular for all homotopy classes z and critical points β_1 and β_2.*

Proof First, we fix a perturbation $\pi_0 \in \mathcal{P}$ as in Proposition 2.6. Then for each homotopy class z, we can find a perturbation $\pi_z \in \mathcal{P}$ which is supported away from critical points and the corresponding moduli space is regular. This essentially follows from the argument in [10, Section 5]. Since the subset $\mathcal{P}_z$ of regular perturbations as above forms an open dense subset in $\mathcal{P}$, we can find a desired perturbation π in the countable intersection $\bigcap_z \mathcal{P}_z$. □

From now on, we assume that the perturbation $\pi \in \mathcal{P}$ always satisfies the properties in Proposition 2.8 and we drop π from the notation $M_z^\pi(\beta_1,\beta_2)$.

2.6 Compactness

Consider a relative homotopy class $z \in \pi_1(\mathcal{B}(Y,K,\alpha),\beta_1,\beta_2)$. If $\beta_1 = \beta_2$ then we assume that z is a nontrivial homotopy class. Elements in $\breve{M}_z(\beta_1,\beta_2)$ are called unparametrized trajectories.

Definition 2.9 A collection $([A_1],\dots,[A_l]) \in \breve{M}_{z_1}(\beta_1,\beta_2)\times\cdots\times\breve{M}_{z_l}(\beta_{l-1},\beta_l)$ of unparametrized trajectories is called an unparametrized broken trajectory from β_1 to β_l. If the composition of paths $z_1\circ\cdots\circ z_l$ is contained in the homotopy class z, then $\breve{M}_z^+(\beta_1,\beta_l)$ denotes the space of unparametrized broken trajectories from β_1 to β_l.

The compactness property of moduli spaces is as follows; see also [32, Proposition 3.22].

Proposition 2.10 *Let $\beta_1,\beta_2 \in \mathfrak{C}_\pi$ and assume that $\dim M_z(\beta_1,\beta_2) < 4$. Then the space of unparametrized broken trajectories $\breve{M}_z^+(\beta_1,\beta_2)$ is compact.*

We can assign the energy $\mathcal{E}_\pi(z)$ to a homotopy class z. In singular gauge theory for general holonomy parameters, the counting $\#\bigcup_z \breve{M}_z(\beta_1,\beta_2)$ with $\mathrm{gr}_z(\beta_1,\beta_2) = 1$ can be infinite. Instead, we use the following finiteness result:

Proposition 2.11 [32, Proposition 3.23] *For a given constant $C > 0$, there are only finitely many critical points β_1 and β_2 and homotopy classes $z \in \pi_1(\mathcal{B};\beta_1,\beta_2)$ such that the moduli space $M_z(\beta_1,\beta_2)$ is nonempty and $\mathcal{E}_\pi(z) < C$.*

Thus

$$\bigcup_{\substack{z \\ \mathcal{E}_\pi(z)<C}} \breve{M}_z(\beta_1,\beta_2)_0$$

is a finite point set for any $C>0$.

The gluing formula of the index tells us

$$\mathrm{gr}_z(\beta_0,\theta_\alpha)+1+\mathrm{gr}_{z'}(\theta_\alpha,\beta_1)=\mathrm{gr}_{z'\circ z}(\beta_0,\beta_1) \tag{2-4}$$

since θ_α has a stabilizer S^1. From this relation, we conclude that any broken trajectories in $M_z^+(\beta_0,\beta_1)_d$ do not factor through θ_α if the dimension of $M_z(\beta_0,\beta_1)_d$ is less than 3.

2.7 Cobordisms

Let (W,S) be a pair of an oriented 4–manifold and an embedded oriented surface such that $\partial W=Y'\sqcup(-Y)$ and $\partial S=K\sqcup K'$. We call (W,S) the cobordism of pairs and write $(W,S)\colon(Y,K)\to(Y',K')$. Set

$$(W^+,S^+):=\mathbb{R}_{\le 0}\times(Y,K)\cup(W,S)\cup\mathbb{R}_{\ge 0}\times(Y',K').$$

We fix a metric on $W^+\setminus S^+$ with a cone angle $2\pi/\nu$ and cylindrical forms on each end. Let $\beta\in\mathcal{B}(Y,K,\alpha)$ and $\beta'\in\mathcal{B}(Y',K')$ be given connections and choose a singular SU(2)–connection A_0 on (W^+,S^+) which has a limiting connection β or β'(up to gauge transformations) on each end of (W^+,S^+). Here z denotes the homotopy class of A. We define the space of connections and the group of gauge transformations as follows:

$$\mathcal{A}_z(W,S;\beta,\beta'):=\{A\mid A-A_0\in\check{L}^2_{m-1,A_0}(W^+\setminus S^+,\mathfrak{g}_E\otimes\Lambda^1)\},$$
$$\mathcal{G}_z(W,S):=\{g\in\mathrm{Aut}(E)\mid\nabla^i_{A_0}g\in\check{L}^2(W^+\setminus S^+;\mathrm{End}(E)),\ i=1,\dots,m\}.$$

We also define the quotient

$$\mathcal{B}_z(W,S;\beta,\beta'):=\mathcal{A}_z(W,S;\beta,\beta')/\mathcal{G}_z(W,S).$$

$\mathcal{B}(W,S;\beta,\beta')$ denotes the union of $\mathcal{B}_z(W,S;\beta,\beta')$ for all paths. The perturbed ASD equation on (W,S) has the form $F_A^++U_{\pi_W}=0$ where U_{π_W} is a t–dependent perturbation. More concretely this can be described as the following (the argument is based on [32]): Let $\pi,\pi_0\in\mathcal{P}_Y$ be two holonomy perturbations on $\mathbb{R}\times(Y,K)$. The perturbed ASD equation on $\mathbb{R}_{\le 0}\times(Y,K)$ has the form

$$F_A^++\psi(t)\hat{V}_\pi+\psi_0(t)\hat{V}_{\pi_0}=0,$$

where $\psi(t)$ is a smooth cutoff function such that $\psi(t)=1$ if $t<-1$ and $\psi(t)=0$ at $t=0$. Then ψ_0 is a smooth function supported on $(-1,0)\times Y$. We choose $\pi\in\mathcal{P}$ so that $\mathfrak{C}_\pi$ satisfies properties in Propositions 2.6 and 2.8. The perturbation term can be described similarly on another end. For generic choices of π_0 and $\pi_0'\in\mathcal{P}_{Y'}$, the irreducible part of the perturbed ASD-moduli space

$$M_z(W,S;\beta,\beta')\subset\mathcal{B}_z(W,S;\beta,\beta')$$

is a smooth manifold. Consider the ASD-operator

(2-5) $$D_A = -d_A^* \oplus d_A^+ : \phi \check{L}^2_{m,A_0}(W \setminus S, \mathfrak{g}_E \otimes \Lambda^1) \to \phi \check{L}^2_{m-1,A_0}(W \setminus S, \mathfrak{g}_E \otimes (\Lambda^0 \oplus \Lambda^+)),$$

where ϕ is a weight function. If a limiting connection of A_0 is irreducible then we choose $\phi \equiv 1$ on that end of (W^+, S^+). If A_0 has a reducible limiting connection then we choose $\phi = e^{-\delta|t|}$ on that end, where $\delta > 0$ is small enough. $M(W, S; \beta, \beta')_d$ denotes the union of the moduli spaces $M_z(W, S; \beta, \beta')$ with ind $D_A = d$.

Definition 2.12 We define the topological energy of $A \in \mathcal{B}(W, S; \beta, \beta')$ as

$$\kappa(A) := \frac{1}{8\pi^2} \int_{W^+ \setminus S^+} \mathrm{Tr}(F_A \wedge F_A)$$

and the monopole number of A as

$$\nu(A) := \frac{i}{\pi} \int_{S^+} \Omega - 2\alpha S \cdot S,$$

where

$$F_A|_{S^+} = \begin{bmatrix} \Omega & 0 \\ 0 & -\Omega \end{bmatrix}.$$

For the cylinder $(W, S) = [0, 1] \times (Y, K)$ and the trivial perturbation $\pi = 0$, the topological energy κ is related to the energy $\mathcal{E}$ of the Chern–Simons functional as $2\kappa(A) = \mathcal{E}(A)$. Consider an SU(2)–connection B on (Y, K), a connection A over the cylinder $\mathbb{R} \times (Y, K)$ which is asymptotic to B at $-\infty$, and a fixed reducible flat connection $\tilde{\theta}_\alpha$ such that $\mathrm{CS}(\tilde{\theta}_\alpha) = 0$ at ∞. Then $\mathrm{CS}(B) = \kappa(A)$ by construction.

Similarly we define an $\mathbb{R}$–valued function $\mathrm{hol}_K : \mathcal{A}(Y, K, \alpha) \to \mathbb{R}$ as follows:

Definition 2.13 Let A be an SU(2)–connection over the cylinder $[0, 1] \times (Y, K)$ as above. We define $\mathrm{hol}_K(B) := -\nu(A)$.

If z is a path on (W, S) which is represented by a connection A, then we write $\kappa(z)$ for $\kappa(A)$ and $\nu(z)$ for $\nu(A)$ since these numbers are independent of the choice of A.

Let (X, Σ) be a pair of a 4–manifold and an embedded surface with boundary $\partial X = Y$ and $\partial \Sigma = K$ where K is an oriented knot in an oriented integral homology 3–sphere Y. We assume that $[\Sigma] = 0$. Let Θ_α be a singular flat reducible connection with a holonomy parameter $\alpha = n/m$ and whose lift to the m–fold cyclic branched covering $\widetilde{X}_m$ is the trivial connection. We write $H^i(X \setminus \Sigma; \Theta_\alpha)$ for the i^{th} cohomology of $X \setminus \Sigma$ with the local coefficient system twisted by Θ_α. Let $H^+(X \setminus \Sigma; \Theta_\alpha)$ and $H^-(X \setminus \Sigma; \Theta_\alpha)$ be the space of self-dual and anti-self-dual harmonic 2–forms on $X \setminus \Sigma$ twisted by Θ_α, respectively.

Lemma 2.14 *We define* $\chi(X \setminus \Sigma; \Theta_\alpha) = \sum_i (-1)^i \dim H^i(X \setminus \Sigma; \Theta_\alpha)$ *and*

$$\sigma(X \setminus \Sigma; \Theta_\alpha) = \dim H^+(X \setminus \Sigma; \Theta_\alpha) - \dim H^-(X \setminus \Sigma; \Theta_\alpha).$$

Then

$$\chi(X \setminus \Sigma; \Theta_\alpha) = \chi(X) - \chi(\Sigma) \quad \textit{and} \quad \sigma(X \setminus \Sigma; \Theta_\alpha) = \sigma(X) + \sigma_\alpha(Y, K).$$

Proof Consider a rational holonomy parameter of the form $\alpha = n/m \in \mathbb{Q}$. We take an m–fold branched covering $\pi\colon \widetilde{X}_m \to X$ whose branched locus is Σ. The pullback of singular flat connection Θ_α extends as the trivial flat connection over $\widetilde{X}_m$. Let $\tau\colon \widetilde{X}_m \to \widetilde{X}_m$ be a generator of covering transformations. Then its induced action $\tilde{\tau}$ on the pulled-back bundle $\underline{\mathbb{C}}$ is multiplication by $e^{4\pi i n/m}$. The index of the twisted de Rham operator

$$d_{\Theta_\alpha} + d^*_{\Theta_\alpha}\colon \Omega^{\mathrm{even}}(X \setminus \Sigma; \Theta_\alpha) \to \Omega^{\mathrm{odd}}(X \setminus \Sigma; \Theta_\alpha)$$

coincides with the index of

$$d + d^*\colon \Omega^{\mathrm{even}}(\widetilde{X}_m; \underline{\mathbb{C}})^{\tilde{\tau}} \to \Omega^{\mathrm{odd}}(\widetilde{X}_m; \underline{\mathbb{C}})^{\tilde{\tau}}, \tag{2-6}$$

where $\Omega^*(\widetilde{X}_m; \underline{\mathbb{C}})^{\tilde{\tau}} = \{\omega \in \Omega^*(\widetilde{X}_m; \underline{\mathbb{C}}) \mid \omega(\tau(x)) = \tilde{\tau}(\omega(x))\}$. The index of (2-6) is given by $\chi(X) - \chi(\Sigma)$. This can be seen by taking cell complex $C_*(\widetilde{X}_m)$ of $\widetilde{X}_m$ in τ–equivariant way. Then there are decompositions of the underlying groups of the chain complex

$$C_*(\widetilde{X}_m) = C_*(\Sigma) \oplus C_*(\widetilde{X}_m \setminus \Sigma), \quad C_*(\widetilde{X}_m \setminus \Sigma) = \bigoplus_{i=1}^{n} C_i,$$

where each C_i is isomorphic to a copy of $C_*(X \setminus \Sigma)$. Since τ_* acts as the identity on $C_*(\Sigma)$ and in a cyclic way on $C_*(\widetilde{X}_m \setminus \Sigma) = \bigoplus_{i=1}^{n} C_i$, all eigenspaces of the action τ_* on $C_*(\widetilde{X}_m \setminus \Sigma)$ are isomorphic. On the other hand, there is the identity $\chi(\widetilde{X}_m) = m\chi(X) - (m-1)\chi(\Sigma)$. Thus the τ–invariant index of the de Rham operator is given by $\chi(X) - \chi(\Sigma)$.

Similarly, the index of the signature operator twisted by the local coefficient Θ_α coincides with the index of the signature operator over $\widetilde{X}_m$ which is restricted to $e^{4\pi i n/m}$–eigenspaces. This signature is equal to $\sigma(X) + \sigma_{n/m}(Y, K)$ by the formula in [41]. □

Proposition 2.15 *Let $(W, S)\colon (Y, K) \to (Y', K')$ be a cobordism of pairs and $[A]$ be an element of $\mathcal{B}(W, S; \theta_\alpha, \theta'_\alpha)$. Then the index of the ASD operator D_A is given by*

$$\operatorname{ind} D_A = 8\kappa(A) + 2(4\alpha - 1)\nu(A) - \tfrac{3}{2}(\sigma(W) + \chi(W)) + \chi(S) + 8\alpha^2 S \cdot S + \sigma_\alpha(Y, K) - \sigma_\alpha(Y', K') - 1.$$

Proof Let X be a compact 4–manifold with $\partial X = Y$ and $\Sigma \subset Y$ be a Seifert surface of the knot K. Pushing Σ into the interior of X, we obtain a pair (X, Σ) whose boundary is (Y, K). Moreover $[\Sigma] = 0$ in $H_2(X; \mathbb{Z})$. Similarly we can construct another pair (X', Σ') such that $(\partial X', \partial\Sigma') = (Y', K')$.

Set

$$(\overline{W}, \overline{S}) := (X, \Sigma) \cup_{(Y,K)} (W, S) \cup_{(Y',K')} (X', \Sigma').$$

Then $(\overline{W}, \overline{S})$ is a closed pair of a 4–manifold and an embedded surface. Let A_1 and A_2 be singular flat reducible connections over (X, Σ) and (X, Σ') which are extensions of θ_α and θ'_α, respectively. Let A be a connection which represents an element of $\mathcal{B}(W, S; \theta_\alpha, \theta'_\alpha)$. We consider the connection $A' = A_1 \#_{\theta_\alpha} A \#_{\theta'_\alpha} A_2$ over $(\overline{W}, \overline{S})$ obtained by the gluing.

Using the gluing formula for the index, we have

$$\operatorname{ind} D_{A'} = \operatorname{ind} D_{A_1} + \operatorname{ind} D_A + \operatorname{ind} D_{A_2} + 2,$$

where A' is a singular connection on the closed pair $(\overline{W}, \overline{S})$ obtained by gluing A_1, A_2 and A. Since A_1 is reducible, there is the decomposition $A_1 = \mathbf{1} \oplus B_\alpha$ with respect to the decomposition of the adjoint bundle $\underline{\mathbb{R}} \oplus L^{\otimes 2}$, where $\mathbf{1}$ denotes the trivial connection. The deformation complex for D_{A_1} decomposes into

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d^+} \Omega^+(X) \tag{2-7}$$

and

$$\Omega^0(X \setminus \Sigma; \mathrm{ad}\, B_\alpha) \xrightarrow{d_{B_\alpha}} \Omega^1(X \setminus \Sigma; \mathrm{ad}\, B_\alpha) \xrightarrow{d^+_{B_\alpha}} \Omega^+(X \setminus \Sigma; \mathrm{ad}\, B_\alpha). \tag{2-8}$$

The index of (2-7) is given by $-\frac{1}{2}(\sigma(X) + \chi(X)) - \frac{1}{2}$. On the other hand, the index of (2-8) is given by $-\sigma(X \setminus \Sigma; B_\alpha) - \chi(X \setminus \Sigma; B_\alpha)$. Using two formulae $\sigma(X \setminus \Sigma; B_\alpha) = \sigma(X) + \sigma_\alpha(Y, K)$ and $\chi(X \setminus \Sigma, B_\alpha) = \chi(X) - \chi(\Sigma)$ in Lemma 2.14, we obtain

$$\mathrm{ind}\, D_{A_1} = -\tfrac{3}{2}(\sigma(X) + \chi(X)) - \sigma_\alpha(Y, K) + \chi(\Sigma) - \tfrac{1}{2}.$$

Similarly, we have

$$\mathrm{ind}\, D_{A_2} = -\tfrac{3}{2}(\sigma(X') + \chi(X')) + \sigma_\alpha(Y', K') + \chi(\Sigma') - \tfrac{1}{2}$$

since $\sigma_\alpha(-Y', K') = -\sigma_\alpha(Y', K')$. The index formula for a closed pair in [27] gives

$$\mathrm{ind}\, D_{\bar{A}} = 8\kappa(\bar{A}) + 2(4\alpha - 1)\nu(\bar{A}) - \tfrac{3}{2}(\sigma(\overline{W}) + \chi(\overline{W})) + \chi(\bar{S}) + 8\alpha^2 S \cdot S + 2.$$

Hence we have the desired formula:

$$\mathrm{ind}\, D_A = 8\kappa(A) + 2(4\alpha - 1)\nu(A) - \tfrac{3}{2}(\sigma(W) + \chi(W)) + \chi(S) + 8\alpha^2 S \cdot S + \sigma_\alpha(Y, K) - \sigma_\alpha(Y', K') - 1. \qquad \square$$

Remark 2.16 (i) The index formula in Proposition 2.15 recovers [9, Lemma 2.26] when $\alpha = \frac{1}{4}$.

(ii) For a cobordism of pairs $(W, S) \colon (Y, K) \to (Y', K')$, we define the integers

$$k(L) = -c_1(L)^2[W] \quad \text{and} \quad l(L) = -c_1(L)[S].$$

Then the Chern–Weil formula gives us another expression of the index formula in Proposition 2.15 as

$$\mathrm{ind}\, D_{A_L} = 8k(L) + 4l(L) - \tfrac{3}{2}(\sigma(W) + \chi(W)) + \chi(S) + \sigma_\alpha(Y, K) - \sigma_\alpha(Y', K') - 1.$$

Assume that the cobordism of pairs (W, S) satisfies $b^1(W) = b^+(W) = 0$. Then there exists a unique singular reducible instanton A_L corresponding to a decomposition $E = L \oplus L^*$.

Definition 2.17 We call A_L minimal if it minimizes $\mathrm{ind}\, D_{A_L}$ among all line bundles L.

Our definition of minimal reducible coincides with [8, Subsection 2.3] if $\alpha = \frac{1}{4}$.

Let us describe relations between CS and κ, and ν and hol_K over cobordisms. Consider a connection A over a cobordism $(W, S) \colon (Y, K) \to (Y', K')$ whose limiting connections are B on (Y, K) and B' on (Y', K'). Then the following statement holds:

Lemma 2.18 *Fix a reducible connection A_L over (W, S). Then there exist $k, l \in \mathbb{Z}$ such that*

$$\kappa(A) - \kappa(A_L) = \mathrm{CS}(B) - \mathrm{CS}(B') + k + 2\alpha l \quad \textit{and} \quad \nu(A) - \nu(A_L) = \mathrm{hol}_{K'}(B') - \mathrm{hol}_K(B) - 2l.$$

Proof Recall that $\mathbb{R}$–valued functions CS and hol are fixed by choosing reducible flat connections $\tilde{\theta}_\alpha$ and $\tilde{\theta}'_\alpha$ over pairs (Y, K) and (Y', K'). If we choose a reducible connection A_{L_0} so that it has two reducible limits $\tilde{\theta}_\alpha$ and $\tilde{\theta}'_\alpha$, then we have

$$\kappa(A) + \mathrm{CS}(B') = \mathrm{CS}(B) + \kappa(A_{L_0}) \quad \text{and} \quad \nu(A) - \mathrm{hol}_{K'}(B') = -\mathrm{hol}_K(B) + \nu(A_{L_0})$$

by construction. If we change A_L to other homotopy classes of reducible connections, terms $k + 2\alpha l$ and $-2l$ appear by gauge transformations. □

For a cobordism of pairs (W, S) and a fixed holonomy parameter α, we introduce real values $\kappa_0(W, S, \alpha)$ and $\nu_0(W, S, \alpha)$ as follows:

Definition 2.19 We define

$$\kappa_0(W, S, \alpha) := \min\{\kappa(A_L) \mid A_L \text{ minimal reducible}\},$$

$$\nu_0(W, S, \alpha) := \begin{cases} \nu(A_L), \text{ where } A_L \text{ is a minimal reducible with } \kappa_0 = \kappa(A_L) & \text{if } \alpha \neq \frac{1}{4}, \\ \min\{\nu(A_L) \mid A_L \text{ is a minimal reducible}\} & \text{if } \alpha = \frac{1}{4}. \end{cases}$$

Note that the homotopy class of the path $z\colon \theta_\alpha \to \theta'_\alpha$ represented by a minimal reducible A_L with $\kappa_0 = \kappa(A_L)$ is uniquely determined when $\alpha \neq \frac{1}{4}$. If $\alpha = \frac{1}{4}$ then homotopy classes of paths represented by minimal reducibles are not unique, but only finitely many exist. In particular, $\nu_0(W, S, \alpha)$ is well defined.

Remark 2.20 If the cobordism of pairs (W, S) has a flat minimal reducible with a holonomy parameter α, then $\kappa_0(W, S, \alpha) = \nu_0(W, S, \alpha) = 0$.

We write $\kappa_0 = \kappa_0(W, S, \alpha)$ and $\nu_0 = \nu_0(W, S, \alpha)$ for short.

Definition 2.21 Let $(W, S)\colon (Y, K) \to (Y', K')$ be a cobordism of pairs where K and K' are oriented knots in integral homology 3–spheres Y and Y'. Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. A cobordism of pairs (W, S) is called negative definite over $\mathscr{S}$ if

(1) $b^1(W) = b^+(W) = 0$,

(2) the index of the minimal reducibles is -1,

(3) we have the nonzero element in $\mathscr{S}$

$$\eta^\alpha(W, S) := \sum_{A_L \text{ minimal}} (-1)^{c_1(L)^2} \lambda^{\kappa_0 - \kappa(A_L)} T^{\nu(A_L) - \nu_0}.$$

Remark 2.22 Our definition of the negative definite cobordism coincides with that of [8] when $\alpha = \frac{1}{4}$, since instantons have minimal energy if only if they have minimal index.

Let $(W_1, S_1)\colon (Y_1, K_1) \to (Y', K')$ and $(W_2, S_2)\colon (Y', K') \to (Y_2, K_2)$ be negative definite cobordisms. Note that their composition $(W_2 \circ W_1, S_2 \circ S_1)\colon (Y_1, K_1) \to (Y_2, K_2)$ is also a negative definite cobordism.

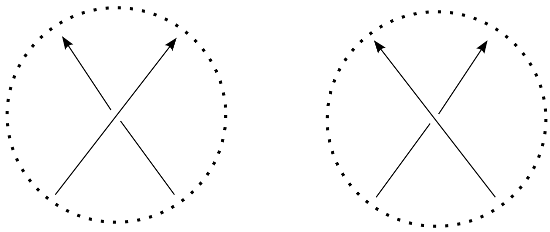

Figure 1: Positive (left) and negative (right) crossings of a knot.

A cylinder $[0,1]\times(Y,K)$ and a homology concordance $(Y,K)\to(Y',K')$ are examples of negative definite cobordisms. The following is also a basic example of negative definite cobordisms. Let K_+ be a knot in S^3 which has at least one positive crossing. Let K_- be a knot which is obtained by replacing one positive crossing in the knot K_+ by one negative crossing; see Figure 1.

Since S^3 is simply connected, K_+ and K_- are homotopic. Approximating homotopy from K_- to K_+ by a smooth map, we get a smoothly immersed surface $S\subset[0,1]\times S^3$ such that $S\cap\{0\}\times S^3=K_+$ and $S\cap\{1\}\times S^3=K_+$. Furthermore, we assume that S has a transverse self-intersection point. Let $S'\colon K_+\to K_-$ be an inverse cobordism of S. S has a positive self-intersection point in $[0,1]\times S^3$. Blowing up this self-intersection point, we obtain a new cobordism of pairs

$$(\overline{\mathbb{CP}}^2\#([0,1]\times S^3),\overline{S})\colon(S^3,K_-)\to(S^3,K_+). \tag{2-9}$$

$\overline{S}$ is an embedded surface in $\overline{\mathbb{CP}}^2\#([0,1]\times S^3)$ obtained by resolving the self-intersection of S, and it represents a homology class

$$2e\in H^2(\overline{\mathbb{CP}}^2;\mathbb{Z})\cong H^2(\overline{\mathbb{CP}}^2\#([0,1]\times S^3);\mathbb{Z}),$$

where e is an element represented by the exceptional curve. Similarly, we obtain a cobordism of pairs

$$(\overline{\mathbb{CP}}^2\#([0,1]\times S^3),\overline{S}')\colon(S^3,K_+)\to(S^3,K_-). \tag{2-10}$$

Cobordisms of pairs $(W,\overline{S})\colon(S^3,K_-)\to(S^3,K_+)$ and $(W',\overline{S}')\colon(S^3,K_+)\to(S^3,K_-)$ constructed as above are called *the cobordism of positive/negative crossing change*, respectively.

Proposition 2.23 *Fix a holonomy parameter $\alpha\in\left(0,\frac{1}{2}\right)\cap\mathbb{Q}$ with $\Delta_{K_+}(e^{4\pi i\alpha})\neq0$ and $\Delta_{K_-}(e^{4\pi i\alpha})\neq0$. Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. We assume that $\sigma_\alpha(K_+)=\sigma_\alpha(K_-)$. Then the cobordism of positive and negative crossing change are negative definite over $\mathscr{S}$.*

Proof Firstly, we show that (2-9) is a negative definite pair. Put $W=\overline{\mathbb{CP}}^2\#([0,1]\times S^3)$. Then it is clear that W satisfies Definition 2.21(1) since $H^1(W;\mathbb{Z})=0$ and $H^2(W;\mathbb{Z})=\mathbb{Z}$. Let A_m be a $U(1)$–reducible instanton corresponding to an element $m\in\mathbb{Z}=H^2(W;\mathbb{Z})$. Then

$$\bar\kappa(A_m)=-(c_1(L_m)+\alpha\overline{S})^2=(m+2\alpha)^2,$$

where L_m is a line bundle such that $c_1(L_m)[e]=-m$. We also have

$$\nu(A_m)=2c_1(L_m)[\overline{S}]=-4m.$$

The index computation yields that

$$\begin{aligned}\operatorname{ind}(D_{A_m}) &= 8(m+2\alpha)^2 + 2(4\alpha-1)(-4m) - 32\alpha^2 + \sigma_\alpha(K_+) - \sigma_\alpha(K_-) - 1 \\ &= 8m(m+1) + \sigma_\alpha(K_-) - \sigma_\alpha(K_+) - 1.\end{aligned}$$

Thus $m = 0, -1$ minimize $\operatorname{ind} D_{A_m}$, and this means that A_0 and A_{-1} are minimal reducibles. Since $\sigma_\alpha(K_+) = \sigma_\alpha(K_-)$ by our assumption, the index for minimal instantons is -1 for the first case. Thus $(W, \bar{S})$ satisfies Definition 2.21(2). Since $\bar{\kappa}(A_m) = m^2 + 4\alpha m + 4\alpha^2$, we have

$$\eta^\alpha(W, \bar{S}) = \begin{cases} 1 - \lambda^{4\alpha-1} T^4 & \text{if } \alpha \le \frac{1}{4}, \\ \lambda^{1-4\alpha} T^{-4} - 1 & \text{if } \alpha > \frac{1}{4}. \end{cases}$$

Since $1 - \lambda^{4\alpha-1}T^4$ is invertible when $\alpha \neq \frac{1}{4}$ and nonzero when $\alpha = \frac{1}{4}$ by assumption, $\eta^\alpha(W, \bar{S})$ is nonzero in $\mathscr{S}$. Hence $(W, \bar{S})$ satisfies Definition 2.21(3) and $(W, \bar{S})$ is a negative definite pair.

It is also obvious that $(W, \bar{S}')$ satisfies Definition 2.21(1). Since $\bar{S}'$ has the trivial homology class in $H_2(W; \mathbb{Z})$, minimal reducibles are only trivial with index -1. Hence $\eta^\alpha(W, \bar{S}') = 1 \neq 0 \in \mathscr{S}$. □

Next, we discuss the transversality of moduli spaces at reducibles. Following [26; 4], we introduce the perturbation supported on the interior of the cobordism. Let $\mathcal{I}$ be an infinite countable set of indexes and consider the following data:

- a collection of embedded 4–balls $\{B_i\}_{i \in \mathcal{I}}$ in $W^+ \setminus S^+$,
- a collection of submersions $q_i : S^1 \times B_i \to W^+ \setminus S^+$ such that $q_i(1, \cdot)$ is the identity,
- for any $x \in W \setminus S$, the set $\{q_{i,x} \mid i \in \mathcal{I}, x \in B_i\}$ is a C^1–dense subset in the space of loops based at $x \in W \setminus S$.

For each $i \in \mathcal{I}$, consider a self-dual 2–form ω_i on B_i with $\operatorname{supp}(\omega_i) \subset B_i$. These self-dual 2–forms ω_i can be regarded as self-dual 2–forms on $W^+ \setminus S^+$. We define $V_{\omega_i} : \mathcal{A}_z(W, S; \beta, \beta') \to \Omega^+(W^+ \setminus S^+; \mathfrak{su}(2))$ as

$$V_{\omega_i}(A) := \pi(\omega_i \otimes \operatorname{Hol}_{q_i}(A)),$$

where $\pi : \mathrm{SU}(2) \to \mathfrak{su}(2)$ is a map given by $g \mapsto g - \frac{1}{2}\operatorname{tr}(g)1$. The argument similar to [26] shows that there are constants $K_{n,i}$ and differentials of V_{ω_i} which satisfy the inequality

$$\|D^n V_{\omega_i} \mid (a_1, \dots, a_n)\|_{\check{L}^2_{m,A_0}} \le K_{n,i} \|\omega\|_{C^l} \prod_{i=1}^{n} \|a_i\|_{\check{L}^2_{m,A_0}},$$

where A_0 is a singular connection which represents the homotopy class z and $l \ge 3$. We choose a family of positive constants $\{C_i\}$ so that

$$C_i \ge \sup\{K_{n,i} \mid 0 \le n \le i\}.$$

Consider a family of self-dual 2–forms $\{\omega_i\}$ such that $\sum_{i \in \mathcal{I}} C_i \|\omega_i\|_{C^l}$ converges. For such a choice of $\{\omega_i\}$, $V_{\boldsymbol{\omega}} := \sum_{i \in \mathcal{I}} V_{\omega_i} \omega_i$ defines a smooth map

$$\mathcal{A}_z(W, S; \beta, \beta') \to \phi \check{L}^2_m(W^+ \setminus S^+, \Lambda^+ \otimes \mathfrak{su}(2))$$

between Banach manifolds.

We define $\mathcal{J} := \{(i,j) \in \mathcal{I} \times \mathcal{I} \mid i \neq j,\ B_{i,j} := B_i \cap B_j \neq \varnothing\}$ and $q_{i,j} \colon B_{i,j} \to W^+ \setminus S^+$ by

$$q_{i,j}|_{\{x\}\times S^1} := q_{i,x} * q_{j,x} * q_{i,x}^{-1} * q_{j,x}^{-1}$$

for each $(i,j) \in \mathcal{J}$. We choose a family of constants $\{C_{i,j}\}_{(i,j)\in\mathcal{J}}$ as before. Let $\omega_{i,j}$ be a self-dual 2–form on $B_{i,j}$. We introduce a Banach space $\mathcal{W}$ which consists of sequences of self-dual 2–forms $\omega = \{\omega_i\}_{i\in\mathcal{I}} \cup \{\omega_{i,j}\}_{(i,j)\in\mathcal{J}}$ with the following weighted l^1–norm:

$$\|\omega\|_{\mathcal{W}} := \sum_{i\in\mathcal{I}} C_i \|\omega_i\|_{C^l} + \sum_{(i,j)\in\mathcal{J}} C_{i,j} \|\omega_{i,j}\|_{C^l}.$$

For each $\omega \in \mathcal{W}$, we define a perturbation term

$$V_\omega(A) := \sum_{i\in\mathcal{I}} V_{\omega_i}(A) \otimes \omega_i + \sum_{(i,j)\in\mathcal{J}} V_{\omega_{i,j}}(A) \otimes \omega_{i,j}$$

which defines a smooth map $V_\omega \colon \mathcal{A}_z(W,S;\beta,\beta') \to \check{L}^2_{m,\epsilon}(W^+ \setminus S^+, \Lambda^+ \otimes \mathfrak{su}(2))$. We call

$$F_A^+ + U_{\pi_W}(A) + V_\omega(A) = 0$$

the secondary perturbed ASD-equation over the cobordism of pairs $(W,S)\colon (Y,K) \to (Y',K')$. Then $M^{\pi_W,\omega}(W,S;\beta,\beta')$ denotes the moduli space of solutions for the secondary perturbed ASD-equation.

Proposition 2.24 *Let (W,S) be a cobordism of pairs such that $b^1(W) = b^+(W) = 0$. Assume that the perturbation π_W is chosen so that the perturbed ASD-equation*

$$F_A^+ + U_{\pi_W}(A) = 0$$

cuts out the irreducible part of the moduli space transversely. Let $A^{\mathrm{ad}} = \mathbf{1} \oplus B$ be the adjoint connection of abelian reducible ASD connection $[A] \in M(W,S;\theta_\alpha,\theta'_\alpha)_{2d+1}$ with $\operatorname{ind}(d_B^* \oplus d_B^+) \geq 0$. Then for a small generic perturbation $\omega \in \mathcal{W}$, the secondary perturbed ASD-equation cuts out the irreducible part of the moduli space transversely. Moreover, $M^{\pi_W,\omega}(W,S;\theta_\alpha,\theta'_\alpha)_{2d+1}$ is regular at $[A]$ and has a neighborhood of $[A]$ which is homeomorphic to a cone on $\pm\mathbb{CP}^d$.

Proof For each connected component $M_z^{\pi_W,\omega}(W,S;\beta,\beta')$ of moduli spaces, the argument [4, Section 7] is adapted to our case and reducible points are regular for generic perturbations. Taking countable intersections of these subsets of regular perturbations in W, we can find a generic perturbation $\omega \in \mathcal{W}$ such that the statement holds. The claim about local structures around reducibles can be refined using the standard argument; see [11, Proposition 4.3.20], for example. □

Essentially the same argument is used in [8]. From now on, we assume that perturbations over the cobordism of pairs (W,S) are chosen so that they satisfy the statement of Proposition 2.24.

2.8 Orientation

We see the orientation of moduli spaces over the cylinder based on [32; 9]. Consider a reference connection A_0 on (W^+,S^+) as described in Section 2.7 and the ASD-operator (2-5). If the weight function ϕ has the form $e^{-\delta|t|}$ on one end, the functional space $\phi\check{L}^2_{m,A_0}$ consists of exponential decaying functions on that end. On the other hand, if the weight function ϕ has the form $e^{\delta|t|}$ on one end, the functional space $\phi\check{L}^2_{m,A_0}$ allows exponential growth functions. The index of the operator D_A depends on these choices of weighted functions. To distinguish these two situations, $\theta_{\alpha,\pm}$ denote reducible flat limits θ_α with weighted functions $e^{\pm\delta|t|}$. Let z be a path along (W,S) between two critical limits β and β' on (Y,K) and (Y',K'). The family index of $D_{\{A\}}$ defines a trivial line bundle $\det\operatorname{ind}(D_A)$ on each $\mathcal{B}_z(W,S,\beta,\beta')$. Let $\mathscr{O}_z[W,S;\beta_0,\beta_1]$ be the two-point set of the orientation of the determinant line bundle $\det\operatorname{ind}(D_{\{A\}})$. Then $\mathscr{O}_z[W,S;\beta,\beta']$ is the set of orientation of the moduli space $M^\alpha_z(W,S;\beta,\beta')$. There is a transitive and faithful $\mathbb{Z}_2$–action on $\mathscr{O}_z[W,S;\beta,\beta']$. For a composition of cobordisms $(W_2,S_2)\circ(W_1,S_1)$, there is a pairing

$$\Phi\colon \mathscr{O}_{z_1}[W_1,S_1;\beta,\beta']\otimes_{\mathbb{Z}_2}\mathscr{O}_{z_2}[W_2,S_2;\beta',\beta'']\to\mathscr{O}_{z_2\circ z_1}[W_2\circ W_1,S_2\circ S_1;\beta,\beta'']$$

which is induced from the gluing formula of the index. If we consider the gluing operation along the reducible connection θ_α, we choose $\beta'=\theta_{\alpha,+}$ at the first component and $\theta_{\alpha,-}$ at the second component. Since there is a natural isomorphism between $\mathscr{O}_z[W,S;\beta,\beta']$ and $\mathscr{O}_{z'}[W,S;\beta,\beta']$, we omit z from the above notation. We call an element of $\mathscr{O}[W,S;\theta_{\alpha+},\theta'_{\alpha-}]$ a homology orientation of (W,S). For a given knot in an integral homology 3–sphere (Y,K), we use the notation

$$\mathscr{O}[\beta]:=\mathscr{O}[Y\times I,K\times I;\beta,\theta_{\alpha-}]$$

if β is irreducible, and

$$\mathscr{O}[\theta_\alpha]:=\mathscr{O}[Y\times I,K\times I;\theta_{\alpha+},\theta_{\alpha-}].$$

There is an isomorphism

$$\mathscr{O}[W,S;\theta_{\alpha,+},\theta_{\alpha,-}]|_{[A_0]}\cong\Lambda^{\mathrm{top}}(H^1(W)\oplus H^+(W)),$$

and an element $o_W\in\mathscr{O}[W,S;\theta_{\alpha,+},\theta_{\alpha,-}]$ is called a homology orientation.

Now we describe how the orientation of the moduli space $M(W,S;\beta,\beta')$ is defined. Let $o_W\in\mathscr{O}[W,S;\theta_{\alpha,+},\theta_{\alpha,-}]$ be a given homology orientation for (W,S). We fix elements $o_\beta\in\mathscr{O}[\beta]$ and $o_{\beta'}\in\mathscr{O}[\beta']$. Then the orientation $o_{(W,S;\beta,\beta')}\in\mathscr{O}[W,S;\beta,\beta']$ is fixed so that

$$\Phi(o_\beta\otimes o_W)=\Phi(o_{(W,S;\beta,\beta')}\otimes o_{\beta'}).$$

The moduli space $\breve{M}_z(\beta_0,\beta_1)$ is oriented in the following way. First, we fix orientations $o_{\beta_0}\in\mathscr{O}[\beta_0]$ and $o_{\beta_1}\in\Lambda[\beta_1]$. Then the orientation of $M(\beta_0,\beta_1)$ is determined as above. Note that there is an $\mathbb{R}$–action on $M_z(\beta_1,\beta_2)$. Let $\tau_s(t,y)=(t-s,y)$ be the transition on the cylinder $(Y,K)\times\mathbb{R}$. Then the $\mathbb{R}$–action on $M(\beta_0,\beta_1)$ is given by the pullback $[A]\mapsto[\tau^*A]$. Finally, we orient $\breve{M}(\beta_1,\beta_2)$ so that $\mathbb{R}\times\breve{M}(\beta_1,\beta_2)=M(\beta_1,\beta_2)$ is orientation preserving.

The boundary of moduli spaces is oriented so that the outward normal vector sits in the first place in the tangent space.

3 $\mathcal{S}$–complexes and Frøyshov type invariants

In this section, we extend the construction of $\mathcal{S}$–complexes $\widetilde{C}_*(Y,K)$ for (Y,K) in [9] to general holonomy parameters. We also introduce $\mathbb{Z}\times\mathbb{R}$–bigrading of $\mathcal{S}$–complexes with rational holonomy parameters for the specific choice of coefficient and its filtered subcomplex based on [35].

3.1 A review on $\mathcal{S}$–complexes and Frøyshov invariants

The $\mathcal{S}$–complex and Frøyshov type invariant introduced by [9; 8] are defined using purely algebraic objects:

Definition 3.1 Let R be an integral domain, and $\widetilde{C}_*$ be a finitely generated and graded free R–module. The triple $(\widetilde{C}_*,\tilde{d},\chi)$ is called an $\mathcal{S}$–complex if

(1) $\tilde{d}\colon \widetilde{C}_*\to\widetilde{C}_*$ is a degree -1 homomorphism,

(2) $\chi\colon \widetilde{C}_*\to\widetilde{C}_*$ is a degree $+1$ homomorphism,

(3) $\tilde{d}$ and χ satisfy
 - $\tilde{d}^2=0$, $\chi^2=0$, and $\tilde{d}\chi+\chi\tilde{d}=0$,
 - $\operatorname{Ker}(\chi)/\operatorname{Im}(\chi)\cong R_{(0)}$, where $R_{(0)}$ is a copy of R in $\widetilde{C}_0$.

If (C_*,d) is a given chain complex with the coefficient ring R, we can form an $\mathcal{S}$–complex

$$\widetilde{C}_*=C_*\oplus C_{*-1}\oplus R,\quad \tilde{d}=\begin{bmatrix} d & 0 & 0\\ v & -d & \delta_2\\ \delta_1 & 0 & 0\end{bmatrix},\quad \chi=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}, \tag{3-1}$$

where $\delta_1\colon C_*\to R$, $\delta_2\colon R\to C_{*-1}$ and $v\colon C_*\to C_{*-2}$. Since there are conditions on $\tilde{d}$ and χ in Definition 3.1, the components in $\tilde{d}$ and χ have to satisfy the following relations:

$$\delta_1 d=0,\quad d\delta_2=0\quad\text{and}\quad dv-vd-\delta_2\delta_1=0. \tag{3-2}$$

Conversely, if the $\mathcal{S}$–complex $(\widetilde{C},\tilde{d},\chi)$ is given then there is a decomposition $\widetilde{C}_*=C_*\oplus C_{*-1}\oplus R$. The reader can find the details in [9, Section 4.1].

There is also the notion of an $\mathcal{S}$–morphism, which is a morphism of $\mathcal{S}$–complexes.

Definition 3.2 Let $(\widetilde{C}_*,\tilde{d},\chi)$ and $(\widetilde{C}'_*,\tilde{d}',\chi')$ be $\mathcal{S}$–complexes. Fix decompositions $\widetilde{C}_*=C_*\oplus C_{*-1}\oplus R$ and $\widetilde{C}'=C'_*\oplus C'_{*-1}\oplus R$. A chain map $\tilde{m}\colon\widetilde{C}_*\to\widetilde{C}'_*$ is called an $\mathcal{S}$–morphism if it has the form

$$\tilde{m}=\begin{bmatrix} m & 0 & 0\\ \mu & m & \Delta_2\\ \Delta_1 & 0 & \eta\end{bmatrix}, \tag{3-3}$$

where $\eta\neq 0\in R$.

The condition that $\tilde{m}$ is a chain map is equivalent to the following relations:

$$md-dm=0,\quad \Delta_1 d+\eta\delta_1-\delta'_1 m=0,\quad d'\Delta_2-\delta'_2\eta+m\delta_2=0,$$
$$\mu d+mv-\Delta_2\delta_1-v'm+d'\mu-\delta'_2\Delta_1=0.$$

Definition 3.3 Let $\widetilde{m}, \widetilde{m}' \colon \widetilde{C}_* \to \widetilde{C}'_*$ be two $\mathcal{S}$–morphisms. An $\mathcal{S}$–chain homotopy of $\widetilde{m}$ and $\widetilde{m}'$ is a degree 1 map $\tilde{h} \colon \widetilde{C}_* \to \widetilde{C}'_*$ such that

$$\tilde{d}'\tilde{h} + \tilde{h}\tilde{d} = \widetilde{m} - \widetilde{m}', \; \chi'\tilde{h} + \tilde{h}\chi = 0.$$

Two $\mathcal{S}$–complexes $\widetilde{C}_*$ and $\widetilde{C}'_*$ are called $\mathcal{S}$–chain homotopy equivariant if there are $\mathcal{S}$–morphisms $\widetilde{m} \colon \widetilde{C}_* \to \widetilde{C}'_*$ and $\widetilde{m}' \colon \widetilde{C}'_* \to \widetilde{C}_*$ such that $\widetilde{m}\widetilde{m}'$ and $\widetilde{m}'\widetilde{m}$ are $\mathcal{S}$–chain homotopic to the identity.

Remark 3.4 Consider $\mathcal{S}$–morphisms $\widetilde{m} \colon \widetilde{C}_* \to \widetilde{C}'_*$ and $\widetilde{m}' \colon \widetilde{C}'_* \to \widetilde{C}_*$. If there are unit elements c and c' in the coefficient ring R, and two $\mathcal{S}$–chain homotopies

$$\widetilde{m}'\widetilde{m} \sim c \,\mathrm{id}_{\widetilde{C}_*} \quad \text{and} \quad \widetilde{m}\widetilde{m}' \sim c' \,\mathrm{id}_{\widetilde{C}'_*},$$

then the two $\mathcal{S}$–complexes $\widetilde{C}_*$ and $\widetilde{C}'_*$ are $\mathcal{S}$–chain homotopy equivalent since both $c^{-1}\widetilde{m}$ and $c'^{-1}\widetilde{m}$ are $\mathcal{S}$–chain homotopic to the composition $c^{-1}c'^{-1}\widetilde{m}\widetilde{m}'\widetilde{m}$.

The Frøyshov type invariant, defined from an $\mathcal{S}$–complex, assigns an integer $h(\widetilde{C}_*)$ to each $\mathcal{S}$–complex $\widetilde{C}_*$.

Definition 3.5 [9, Proposition 4.15]

- $h(\widetilde{C}_*) > 0$ if and only if there is an element $\beta \in C_*$ such that $d\beta = 0$ and $\delta_1\beta \neq 0$.
- If $h(\widetilde{C}_*) = k > 0$ then k is the largest integer such that there exists $\beta \in C_*$ satisfying

$$d\beta = 0, \quad \delta_1 v^{k-1}(\beta) \neq 0, \quad \delta_1 v^i \beta = 0 \quad \text{for } i \leq k-2.$$

- If $h(\widetilde{C}_*) = k \leq 0$ then there are elements $a_0, \dots, a_{-k} \in R$ and $\beta \in C_*$ such that

$$d\beta = \sum_{i=0}^{-k} v^i \delta_2(a_i).$$

The followings are basic properties of the Frøyshov type invariant:

Proposition 3.6 [9, Corollary 4.14] *If there is an $\mathcal{S}$–morphism $\widetilde{m} \colon \widetilde{C}_* \to \widetilde{C}'_*$ then $h(\widetilde{C}_*) \leq h(\widetilde{C}'_*)$.*

Given two $\mathcal{S}$–complexes $(\widetilde{C}_*, \tilde{d}, \chi)$ and $(\widetilde{C}'_*, \tilde{d}', \chi')$, the product $\mathcal{S}$–complex $(\widetilde{C}^{\otimes}_*, \tilde{d}^{\otimes}, \chi^{\otimes})$ is defined as

$$\widetilde{C}^{\otimes}_* = \widetilde{C}_* \otimes \widetilde{C}'_*, \quad \tilde{d}^{\otimes} = \tilde{d} \otimes 1 + \epsilon \otimes \tilde{d}' \quad \text{and} \quad \chi^{\otimes} = \chi \otimes 1 + \epsilon \otimes \chi',$$

where $\epsilon \colon \widetilde{C}'_* \to \widetilde{C}'_*$ is given by $\epsilon(\beta') = (-1)^{\deg(\beta')}\beta'$ on elements of homogeneous degree. Let $d^{\otimes}$, $v^{\otimes}$, $\delta_1^{\otimes}$ and $\delta_2^{\otimes}$ be components of $\tilde{d}^{\otimes}$ with respect to the splitting $\widetilde{C}^{\otimes} = C^{\otimes}_* \oplus C^{\otimes}_{*-1} \oplus R$. Using the decomposition $C^{\otimes}_* = (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'$, these maps are represented by

$$d^{\otimes} = \begin{bmatrix} d \otimes 1 + \epsilon \otimes d' & 0 & 0 & 0 \\ -\epsilon v \otimes 1 + \epsilon \otimes v' & d \otimes 1 - \epsilon \otimes d' & \epsilon \otimes \delta'_2 & -\delta'_2 \otimes 1 \\ \epsilon \otimes \delta'_1 & 0 & d & 0 \\ \delta'_1 \otimes 1 & 0 & 0 & d' \end{bmatrix}, \quad v^{\otimes} = \begin{bmatrix} v \otimes 1 & 0 & 0 & \delta_2 \otimes 1 \\ 0 & v \otimes 1 & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & \delta_1 \otimes 1 & 0 & v' \end{bmatrix},$$

$$\delta_1^{\otimes} = [0, 0, \delta_1, \delta'_1], \qquad \delta_2^{\otimes} = [0, 0, \delta_2, \delta'_2]^{\mathrm{T}}.$$

The Frøyshov type invariant behaves additively for the product of $\mathcal{S}$–complexes:

Proposition 3.7 [9, Corollary 4.28] $h(\widetilde{C}^{\otimes}_*) = h(\widetilde{C}_*) + h(\widetilde{C}'_*)$.

3.2 Floer homology groups with local coefficients

In this subsection, we construct the summand C_* in an $\mathcal{S}$–complex as a Floer chain group with local coefficients. Let (Y, K) be an oriented knot in an integral homology 3–sphere. We fix a holonomy parameter α so that $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$ to isolate the unique flat reducible connection θ_α. We assign an abelian group $\Delta_{[B]}$ for each elements $[B]$ in the configuration space $\mathcal{B}(Y, K, \alpha)$ and an isomorphism $\Delta_z\colon \Delta_{[B_0]} \to \Delta_{[B_1]}$ for each homotopy class $z \in \pi_1(\mathcal{B}(Y, K, \alpha), [B_0], [B_1])$. If this assignment is functorial, a Floer chain complex with the local coefficient Δ is defined as follows:

$$C^\alpha_*(Y, K, \Delta) = \bigoplus_{\beta\in\mathfrak{C}^*_\pi(Y,K,\alpha)} \Delta_\beta \mathscr{O}[\beta] \quad \text{and} \quad \langle d(\beta_0), \beta_1\rangle = \sum_{z\colon \beta_0\to\beta_1} \sum_{[\breve{A}]\in\breve{M}_z(\beta_0,\beta_1)} \epsilon([\breve{A}]) \otimes \Delta_z.$$

The $\mathbb{Z}/4$–grading of $C^\alpha_*(Y, K, \Delta)$ is defined by mod-4 grading for critical points. Consider a subring $\mathscr{R}_\alpha$ in the Novikov ring $\Lambda^{\mathbb{Z}[T^{-1},T]\!]}$, which is introduced in Section 1.2.

Lemma 2.18 enables us to define a local coefficient system $\Delta = \Delta_{\mathscr{R}_\alpha}$ as follows:

$$\Delta_{\mathscr{R}_\alpha,[B]} := \mathscr{R}_\alpha \lambda^{\mathrm{CS}(B)} T^{\mathrm{hol}_K(B)} \quad \text{and} \quad \Delta_{\mathscr{R}_\alpha,z} := \#\breve{M}_z(\beta, \beta_1)_0 \lambda^{-\kappa(z)} T^{\nu(z)}.$$

Note that this definition is independent of choices of representatives of $[B]$ and θ_α. Write $C^\alpha_*(Y, K; \Delta_{\mathscr{R}_\alpha})$ for a chain complex with the local coefficient system over $\mathscr{R}_\alpha$. For any algebra $\mathscr{S}$ over $\mathscr{R}_\alpha$, we can extend the above construction to the coefficient $\mathscr{S}$.

Definition 3.8 Let (Y, K) be an oriented knot in an integral homology 3–sphere and $\mathscr{S}$ be an algebra over $\mathscr{R}_\alpha$. Fix $\alpha \in \left(0, \frac{1}{2}\right) \cap \mathbb{Q}$ so that $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$. The homology group of the $\mathbb{Z}/4$–graded chain complex $(C^\alpha_*(Y, K; \Delta_{\mathscr{S}}), d)$ is denoted by $I^\alpha_*(Y, K; \Delta_{\mathscr{S}})$. We call $I^\alpha_*(Y, K; \Delta_{\mathscr{S}})$ the irreducible singular instanton knot homology over the local coefficient $\mathscr{S}$ with the holonomy parameter α.

Let $(W, S)\colon (Y, K) \to (Y', K')$ be a negative definite cobordism over $\mathscr{S}$. We define an induced morphism $m = m_{(W,S)}\colon C^\alpha_*(Y, K; \Delta_{\mathscr{S}}) \to C^\alpha_*(Y', K'; \Delta_{\mathscr{S}})$ by

$$m(\beta) = \sum_{\beta'\in\mathfrak{C}^*(Y,K,\alpha)} \sum_{z\colon \beta\to\beta'} \#M_z(W, S; \beta, \beta')_0 \lambda^{\kappa_0-\kappa(z)} T^{\nu(z)-\nu_0} \beta'.$$

Counting the boundary of 1–dimensional moduli space $M^+_z(W, S; \beta, \beta')_1$ for each homotopy class z, we obtain the relation

$$dm - md' = 0.$$

We remark that:

- For a composition of negative definite cobordisms $(W, S) := (W_0, S_0) \circ (W_1, S_1)$, there is a map ϕ such that
$$d\phi - \phi d = m_{(W_1,S_1)} \circ m_{(W_0,S_0)} - m_{(W,S)},$$
where metrics and perturbation data on (W, S) are given by the composition of those of (W_0, S_0) and (W_1, S_1).

- If $m_{(W,S)}$ and $m'_{(W,S)}$ are defined by different perturbations and metric data on an interior domain of (W,S), they are chain homotopic.
- If $(W,S)=[0,1]\times(Y,K)$ then $m_{(W,S)}$ is chain homotopic to the identity map.

Thus $C^\alpha_*(Y,K;\Delta_{\mathscr{S}})$ is an invariant of (Y,K) up to chain homotopy. We write $I^\alpha_*(Y,K;\Delta_{\mathscr{S}})$ for its homology group, and this is an invariant for (Y,K).

Remark 3.9 The above argument shows that the chain homotopy type of $C^\alpha_*(Y,K,\Delta_{\mathscr{S}})$ is independent of the choice of orbifold metric with the same cone angle $\nu\in\mathbb{Z}_{>0}$. Hence, more precisely, the module $I^\alpha_*(Y,K,\Delta_{\mathscr{S}})$ should be denoted by $I^\alpha_*(Y,K,\nu,\Delta_{\mathscr{S}})$. We implicitly assume that the cone angle ν is chosen as a large enough number so that gauge theory on the orbifold setup described in Section 2 works.

Next, we introduce the filtered construction for the Floer chain complex based on Nozaki, Sato and Taniguchi [35]. For the filtered construction, we have to introduce the lift of critical points. Let $\mathcal{G}_{(0,0)}$ be a normal subgroup of the gauge group $\mathcal{G}(Y,K)$ which is given by

$$\mathcal{G}_{(0,0)}:=\{g\mid k(g)=l(g)=0\}.$$

Consider the quotient of the space of singular connections

$$\widetilde{\mathcal{B}}(Y,K,\alpha):=\mathcal{A}(Y,K,\alpha)/\mathcal{G}_{(0,0)}.$$

The Chern–Simons functional descends on $\widetilde{\mathcal{B}}(Y,K,\alpha)$ as an $\mathbb{R}$–valued function, and we still use the same notation. The normal subgroup $\mathcal{G}_{(0,0)}$ is a connected component of the full gauge group $\mathcal{G}(Y,K)$ which corresponds to $(0,0)\in\mathbb{Z}\oplus\mathbb{Z}\cong\pi_0(\mathcal{G}(Y,K))$. Thus there is an action of $\pi_0(\mathcal{G}(Y,K))\cong\mathbb{Z}\oplus\mathbb{Z}$ on $\widetilde{\mathcal{B}}(Y,K,\alpha)$ as a covering transformation, and hence $\widetilde{\mathcal{B}}(Y,K,\alpha)$ is a covering space over $\mathcal{B}(Y,K,\alpha)$ with a fiber $\mathbb{Z}\oplus\mathbb{Z}$.

Definition 3.10 A lift of $[B]\in\mathcal{B}(Y,K,\alpha)$ to the covering space $\widetilde{\mathcal{B}}(Y,K,\alpha)$ is called a lift of $[B]$, and denoted by $\widetilde{[B]}$.

For a fixed lift $\widetilde{[B]}$ of $[B]\in\mathcal{B}(Y,K,\alpha)$, the fiber of the projection $\widetilde{\mathcal{B}}(Y,K,\alpha)\to\mathcal{B}(Y,K,\alpha)$ over a point $[B]$ can be described as

$$\mathscr{L}_{[B]}:=\{g^*(\widetilde{[B]})\in\widetilde{\mathcal{B}}(Y,K,\alpha)\mid g\in\pi_0(\mathcal{G}(Y,K))\}.$$

The fiber $\mathscr{L}_{[B]}$ can be seen as the set of lifts of the element $[B]$. There is another description of lifts: Let $\tilde{\theta}_\alpha$ be a lift of reducible flat connections θ_α. Then a lift $\tilde{\beta}$ of $\beta\in\mathfrak{C}^*_\pi$ is fixed by choosing a path $z\colon\beta\to\theta_\alpha$ of connections over the cylinder whose endpoint is $\tilde{\theta}_\alpha$.

We again choose the coefficient ring $\mathscr{R}_\alpha$ and fix a lift $\tilde{\beta}$ for each critical points $\beta\in\mathfrak{C}_\pi(Y,K,\alpha)$.

Then we modify the local coefficient system $\Delta_{\mathscr{R}_\alpha}$ so that

$$\Delta_{\mathscr{R}_\alpha,\beta}=\mathscr{R}_\alpha\lambda^{\mathrm{CS}(\tilde{\beta})}T^{\mathrm{hol}_K(\tilde{\beta})}$$

with the same map $\Delta_{\mathscr{R}_\alpha,z}$. Once we fix an orientation of β, each summand $\Delta_\beta \mathscr{O}[\beta]$ in the chain complex is generated (over $\mathbb{Z}$) by the elements of the form $\lambda^k \xi_\alpha^l \tilde{\beta} = \lambda^{k+2\alpha l} T^{2l} \tilde{\beta}$ where $(k,l) \in \mathbb{Z} \oplus \mathbb{Z}$. The action of $\lambda^k \xi_\alpha^l$ corresponds to the action of the gauge transformation with $d(g) = (k,l)$. Such elements can be identified with the set of lifts $\mathscr{L}_\beta$ of the critical point β. Hence the chain complex $C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})$ can be seen as a $\mathbb{Z}$–module generated by the all lifts of $\mathfrak{C}_\pi(Y,K,\alpha)$ under the modification as above.

Once we fix lifts of generators, the chain complex $C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})$ admits a $(\mathbb{Z}\times\mathbb{R})$–bigrading as in [8], that is we can associate a pair of values which is defined as follows:

For a lift $\tilde{\beta}$ of the critical point $\beta \in \mathfrak{C}_\pi$, we define $\deg_\mathbb{Z}(\tilde{\beta}) := \mathrm{gr}_z(\beta)$ where z is a path corresponding to the lift $\tilde{\beta}$. Then we extend $\deg_\mathbb{Z}$ as

$$\deg_\mathbb{Z}(\lambda^i \xi_\alpha^j \tilde{\beta}) = 8i + 4j + \deg_\mathbb{Z}(\tilde{\beta}).$$

Next we define $\deg_\mathbb{R}$. For a lift $\tilde{\beta}$ of a critical point $\beta \in \mathfrak{C}_\pi$, we define $\deg_\mathbb{R}(\tilde{\beta}) := \mathrm{CS}(\tilde{\beta})$. This extends to elements of the form $\lambda^i \xi_\alpha^j \tilde{\beta}$ as

$$\deg_\mathbb{R}(\lambda^i \xi_\alpha^j \tilde{\beta}) = i + 2\alpha j + \deg_\mathbb{R}(\tilde{\beta}). \tag{3-4}$$

In general, an element $\gamma \in C_*^\alpha(Y,K,\alpha)$ has the form $\gamma = \sum_i a_i \gamma_i$ where $\gamma_i \in \bigcup_{\beta \in \mathfrak{C}_\pi} \mathscr{L}_\beta$. This is possibly an infinite sum. We define

$$\deg_\mathbb{R}(\gamma) = \max\{\deg_\mathbb{R}(\gamma_i) \mid a_i \neq 0\}$$

for $\gamma \neq 0$ and $\deg_\mathbb{R}(0) = -\infty$.

In summary, we have the following proposition:

Proposition 3.11 *Once we fix lifts of critical points of the Chern–Simons functional, the chain complex $(C_*^\alpha(Y,K,\Delta_{\mathscr{R}_\alpha}), d)$ admits the $(\mathbb{Z}\times\mathbb{R})$–bigrading.*

We write $C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]}$ for the chain complex $C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})$ with the $(\mathbb{Z}\times\mathbb{R})$–bigrading.

Let $\mathscr{C}^* \subset \mathbb{R}$ be a subset defined by $\mathscr{C}^* := \mathrm{CS}(\mathrm{Crit}^*)$. For $R \in \mathbb{R} \setminus \mathscr{C}^*$, we define a subset

$$C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,R]} := \{\gamma \in C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]} \mid \deg_\mathbb{R}(\gamma) < R\}.$$

This defines a subcomplex of $C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]}$. For two numbers $R_0, R_1 \in (\mathbb{R} \setminus \mathscr{C}^*) \cup \{\pm\infty\}$ such that $R_0 \leq R_1$, we define a quotient complex as follows:

$$C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]} := C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,R_1]} / C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,R_0]}.$$

Definition 3.12 For $R_0, R_1 \in \mathbb{R} \cup \{\pm\infty\}$ such that $R_0 \leq R_1$ and $R_0, R_1 \notin \mathscr{C}^* \cup \mathscr{C}'^*$, we call $C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})$ a $[R_0,R_1]$–filtered chain complex.

Consider a negative definite cobordism $(W,S)\colon (Y,K) \to (Y',K')$ with $\kappa_0 = 0$. A cobordism map $m_{(W,S)}$ on $C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})$ induces a map

$$C_*^\alpha(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,R]} \to C_*^\alpha(Y',K';\Delta_{\mathscr{R}_\alpha})^{[-\infty,R]}$$

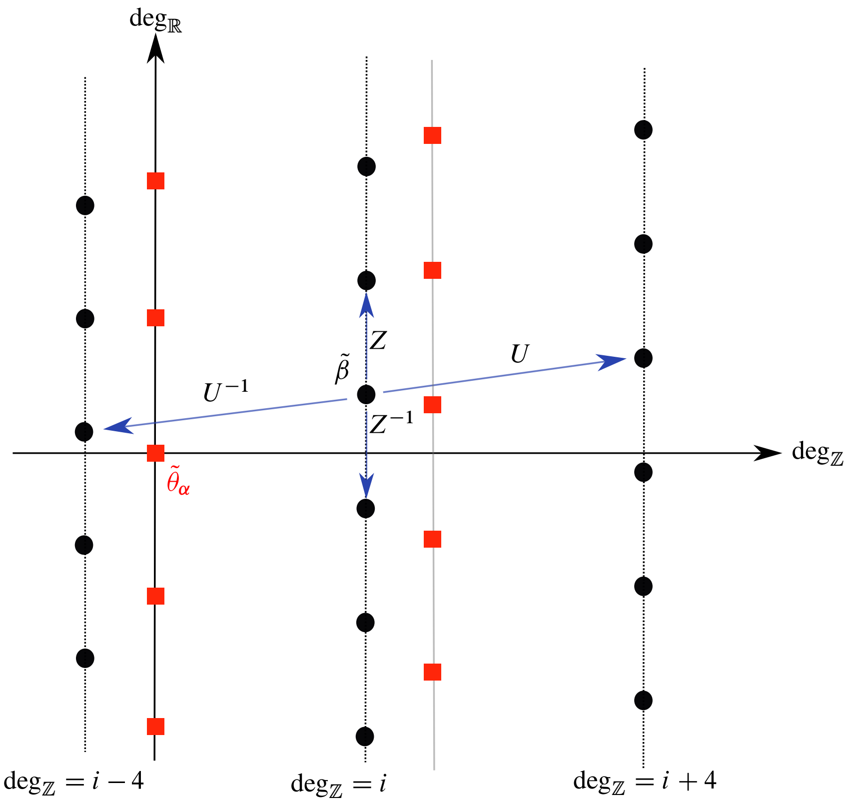

Figure 2: Dots represent lifts of the irreducible flat connection β and squares represent lifts of the reducible flat connection θ_α.

by the restriction, and hence this induces a map

$$m^{[R_0,R_1]}_{(W,S)}\colon C^\alpha_*(Y,K;\Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]} \to C^\alpha_*(Y',K';\Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]}. \tag{3-5}$$

As described before, the covering transformation on $\widetilde{\mathcal{B}}(Y,K,\alpha)$ is generated by multiplications of elements $\lambda^{\pm 1}$and $\xi_\alpha^{\pm 1}$. We also introduce other generators which fit the $(\mathbb{Z}\times\mathbb{R})$–bigrading on $C^\alpha_*(Y,K;\Delta_{\mathscr{R}_\alpha})$. Let us introduce two operators on $C^\alpha_*(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]}$,

$$Z^{\pm 1} := (\lambda^{1-4\alpha}T^{-4})^{\pm 1} \quad\text{and}\quad U^{\pm 1} := (\lambda^{2\alpha}T^2)^{\pm 1}. \tag{3-6}$$

These operators change the $(\mathbb{Z}\times\mathbb{R})$–bigrading as

$$\deg_{\mathbb{Z}}(Z^i\tilde\beta) = \deg_{\mathbb{Z}}(\tilde\beta) \quad\text{and}\quad \deg_{\mathbb{R}}(Z^i\tilde\beta) = \deg_{\mathbb{R}}(\tilde\beta) + (1-4\alpha)i \quad\text{for the operator } Z, \tag{3-7}$$

$$\deg_{\mathbb{Z}}(U^i\tilde\beta) = \deg_{\mathbb{Z}}(\tilde\beta) + 4i \quad\text{and}\quad \deg_{\mathbb{R}}(U^i\tilde\beta) = \deg_{\mathbb{R}}(\tilde\beta) + 2\alpha i \quad\text{for the operator } U. \tag{3-8}$$

See Figure 2 for the case $\alpha < \frac{1}{4}$. Since $\lambda = ZU^2$, actions of the two operations Z and U (and their inverses) on lifted critical points generate $C^\alpha_*(Y,K;\Delta_{\mathscr{R}_\alpha})^{[-\infty,\infty]}$.

3.3 Maps δ_1, δ_2, Δ_1 and Δ_2

We introduce operators which are defined by counting instantons on a cylinder or a cobordism with the reducible limit. We remark that the sign convention of counting moduli spaces in this subsection is the same as that of [9]. Let (Y, K) and (Y', K') be two knots in integral homology 3–spheres. Let $\mathscr{S}$ be an integral domain over $\mathscr{R}_\alpha$. In this subsection, we assume that the holonomy parameter α is chosen so that $\Delta_{(Y,K)}(e^{4\pi i\alpha}) \neq 0$ and $\Delta_{(Y',K')}(e^{4\pi i\alpha}) \neq 0$.

Definition 3.13 We define $\mathscr{S}$–linear chain maps $\delta_1 : C^\alpha_*(Y, K; \Delta_{\mathscr{S}}) \to \mathscr{S}$ and $\delta_2 : \mathscr{S} \to C^\alpha_{-2}(Y, K; \Delta_{\mathscr{S}})$ as follows.

For $\beta \in \mathfrak{C}^*_\pi(Y, K, \alpha)$,

$$\delta_1(\beta) := \sum_{z:\,\beta\to\theta_\alpha} \#\breve{M}_z(\beta, \theta_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)}$$

and

$$\delta_2(1) := \sum_{\substack{\beta\in\mathfrak{C}^*_\pi(Y,K,\alpha)\\ \mathrm{gr}(\beta)\equiv 2}} \sum_{z:\,\theta_\alpha\to\beta} \#\breve{M}_z(\theta_\alpha, \beta)_0 \lambda^{-\kappa(z)} T^{\nu(z)} \beta.$$

Since the compactified 1–dimensional moduli space $\breve{M}^+_z(\beta, \theta_\alpha)_1$ has oriented boundaries

$$\bigcup_{\substack{\gamma\in\mathfrak{C}^*_\pi\\ \mathrm{gr}(\gamma)\equiv 1}} \bigcup_{\substack{z_1,z_2\\ z_1\circ z_2=z}} \breve{M}_{z_1}(\beta, \gamma)_0 \times \breve{M}_{z_2}(\gamma, \theta_\alpha)_0,$$

it is straightforward to check that $d \circ \delta_1 = 0$. Similarly, $\delta_2 \circ d = 0$ holds.

Next, we define $\Delta_1 : C^\alpha_*(Y, K; \Delta_{\mathscr{S}}) \to \mathscr{S}$ and $\Delta_2 : \mathscr{S} \to C^\alpha_*(Y', K'; \Delta_{\mathscr{S}})$ for a cobordism of pairs $(W, S): (Y, K) \to (Y', K')$:

Definition 3.14 We have

$$\Delta_1(\beta) := \sum_z \#M_z(W, S; \beta, \theta'_\alpha)_0 \lambda^{\kappa_0-\kappa(z)} T^{\nu(z)-\nu_0},$$

$$\Delta_2(1) := \sum_{\beta'\in\mathfrak{C}^*_\pi(Y',K',\alpha)} \sum_z \#M_z(W, S; \theta_\alpha, \beta')_0 \lambda^{\kappa_0-\kappa(z)} T^{\nu(z)-\nu_0} \beta'.$$

Proposition 3.15 *Let $m = m_{(W,S)} : C^\alpha_*(Y, K; \Delta_{\mathscr{S}}) \to C^\alpha_*(Y', K'; \Delta_{\mathscr{S}})$ be a cobordism map induced from the negative definite pair $(W, S): (Y, K) \to (Y', K')$. Then the relations*

(i) $\Delta_1 \circ d + \eta\delta_1 - \delta'_1 \circ m = 0$,

(ii) $d' \circ \Delta_2 - \delta'_2\eta + m \circ \delta_2 = 0$,

hold, where $\eta = \eta^\alpha(W, S)$ and $'$ denotes corresponding maps for the pair (Y', K').

Proof The relation (i) is given by counting the ends of each component of the 1–dimensional moduli space $M_z(W,S;\beta,\theta'_\alpha)_1$ as in [9, Proposition 3.10]. The boundary components of $M_z(W,S;\beta,\theta'_\alpha)_1$ with the induced orientation are given by

(a) $$-\bigcup_{\beta_1\in\mathfrak{C}^*_\pi}\bigcup_{\substack{z_1,z_2\\ z_1\circ z_2=z}}\breve{M}_{z_1}(\beta,\beta_1)_0\times M_{z_2}(W,S;\beta_1,\theta'_\alpha)_0,$$

(b) $$\bigcup_{\gamma'\in\mathfrak{C}'^*_\pi}\bigcup_{\substack{z_1,z_2\\ z_1\circ z_2=z}}M_{z_1}(W,S;\beta,\beta')_0\times\breve{M}_{z_2}(\beta',\theta_\alpha)_0,$$

(c) $$-\bigcup_{\substack{z_1,z_2\\ z_1\circ z_2=z}}\breve{M}_{z_1}(\beta,\theta_\alpha)_0\times M_{z_2}(W,S;\theta_\alpha,\theta'_\alpha)_0.$$

Note that product orientations of $\breve{M}_{z_1}(\beta,\gamma)_0\times M_{z_2}(W,S;\gamma,\theta'_\alpha)_0$ and $\breve{M}_{z'}(\beta,\theta_\alpha)_0\times M(W,S;\theta_\alpha,\theta'_\alpha)_0$ are opposite to orientations induced as the boundaries of $M_z(W,S;\beta,\theta_\alpha)_1$. The signed counting of the boundary components of types (a) and (b) contribute to $-\Delta_1\circ d(\beta)$ and $\delta'_1\circ m(\beta)$, respectively. Since $M(W,S;\theta_\alpha,\theta'_\alpha)_0$ consists of minimal reducible elements, the counting of (c) gives $-\eta\delta_1(\beta)$. This proves (i). The relation (ii) can be similarly proved considering the ends of the 1–dimensional moduli space $M_z(W,S;\theta_\alpha,\beta')_1$. □

3.4 Maps v and μ

In this subsection, we introduce maps induced from the cobordism of pairs (W,S) with an embedded curve $\gamma\subset S$. Our assumptions for the choice of holonomy parameter α and the coefficient $\mathscr{S}$ are the same as the previous subsection. We remark that the sign convention of moduli spaces in this subsection is also the same as that of [9]. In particular, if $f\colon M\to N$ is a smooth map between oriented manifolds then $f^{-1}(y)$ for a regular value $y\in N$ is oriented so that

$$T_xM=N_xf^{-1}(y)\oplus T_xf^{-1}(y)$$

is orientation preserving, where $N_xf^{-1}(y)$ is a fiber of the normal bundle for $f^{-1}(y)$ and its orientation is induced from that of N. The mapping degree $\deg(f)$ is defined by using this orientation.

Assume that $\gamma\colon[0,1]\to S$ is a smoothly embedded loop. Fix a regular neighborhood $N_\gamma(\epsilon)$ of γ in W with radius $\epsilon>0$ and fix a basepoint $x_0\in\partial N_\gamma(\epsilon)$. We take a Seifert framing $\tilde{\gamma}_\epsilon\subset\partial N_\gamma(\epsilon)$ of γ so that it passes through the basepoint x_0. The bundle decomposition $E=L\oplus L^*$ over $S\subset W$ extends to $N_\gamma(\epsilon)$, and the holonomy of the adjoint connection of $[A]\in\mathcal{B}(W,S;\beta,\beta')$ yields $\mathrm{Hol}_{\tilde{\gamma}_\epsilon}(A^{\mathrm{ad}})\in S^1$. Put

$$h^\gamma_{\beta\beta'}(A):=\lim_{\epsilon\to0}\mathrm{Hol}_{\tilde{\gamma}_\epsilon}(A^{\mathrm{ad}}).$$

The construction above gives a map

(3-9) $$h^\gamma_{\beta\beta'}\colon\mathcal{B}(W,S,\alpha;\beta,\beta')\to S^1.$$

Note that this map itself depends on the choice of the Seifert framing of γ and orientations of K and S. However, such dependence on auxiliary data can be ignored to define the following map:

Definition 3.16 Let β and β' be irreducible critical points of the (perturbed) Chern–Simons functional on (Y, K) and (Y', K'), respectively. We define a map $\mu = \mu_{(W,S,\gamma)}\colon C^{\alpha}_{*}(Y, K; \Delta_{\mathscr{S}}) \to C^{\alpha}_{*}(Y', K'; \Delta_{\mathscr{S}})$ by

$$\mu(\beta) = \sum_{\beta' \in \mathfrak{C}^{*}_{\pi}(Y,K,\alpha)} \sum_{z\colon \beta \to \beta'} \deg(h^{\gamma}_{\beta\beta'}|_{M_z(W,S;\beta,\beta')_1}) \lambda^{\kappa_0 - \kappa(z)} T^{\nu(z)-\nu_0} \beta'$$

for each $\beta \in \mathfrak{C}^{*}_{\pi}(Y, K, \alpha)$.

The map μ satisfies the following relation:

Proposition 3.17
$$d' \circ \mu - \mu \circ d = 0.$$

Proof Consider the compactified 2–dimensional moduli space $M^{+}_{z}(W, S; \beta, \beta')_2$ which has oriented boundary of the types

$$-\bigcup_{\beta_1 \in \mathfrak{C}^{*}_{\pi}(Y,K,\alpha)} \bigcup_{z' \circ z'' = z} \breve{M}^{+}_{z'}(\beta, \beta_1)_{i-1} \times M^{+}_{z''}(W, S; \beta_1, \beta')_{2-i},$$
$$\bigcup_{\beta'_1 \in \mathfrak{C}^{*}_{\pi}(Y',K',\alpha)} \bigcup_{z' \circ z'' = z} M^{+}_{z'}(W, S; \beta, \beta'_1)_{2-i} \times \breve{M}^{+}_{z''}(\beta'_1, \beta')_{i-1},$$

where $i = 1$ or 2. Count the boundary of the 1–dimensional submanifold $(h^{\gamma}_{\beta\beta'})^{-1}(s) \subset M^{+}(W, S; \beta, \beta')$ for a regular value $s \in S^1$. Since the closed loop γ is supported on a compact subset of S, $(h^{\gamma}_{\beta\beta'})^{-1}(s)$ intersects faces of the boundary of $M^{+}_{z}(W, S; \beta, \beta')$ with $i = 1$. Thus

$$\#((h^{\gamma}_{\beta\beta'})^{-1}(s) \cap \partial M^{+}_{z}(W, S; \beta, \beta')_2) = d' \circ \mu - \mu \circ d = 0. \qquad \square$$

We consider the case when $(W, S) = \mathbb{R} \times (Y, K)$ and $\gamma \subset S$ is a curve $\mathbb{R} \times \{y_0\}$ where y_0 is a fixed basepoint in K. Taking holonomy along γ, we obtain a map

$$h_{\beta_1\beta_2}\colon \mathcal{B}(Y, K, \alpha; \beta_1, \beta_2) \to S^1$$

similarly to (3-9), where β_i for $i = 1, 2$ are irreducible critical points of the Chern–Simons functional.

The holonomy map $h_{\beta_1\beta_2}$ is modified to extend broken trajectories as in [10]. Such modification of $h_{\beta_1\beta_2}$ near the broken trajectories gives the map

$$H_{\beta_1\beta_2}\colon \breve{M}(\beta_1, \beta_2)_d \to S^1$$

with the following properties:

(i) $H_{\beta_1\beta_2} = h_{\beta_1\beta_2}$ on the complement of a small neighborhood of $\partial \breve{M}^{+}(\beta_1, \beta_2)_d$.

(ii) $H_{\beta_1\beta_3}([A_1], [A_2]) = H_{\beta_1\beta_2}([A_1]) H_{\beta_2\beta_3}([A_2])$ on unparametrized broken trajectories
$$\breve{M}^{+}(\beta_1, \beta_2)_{i-1} \times \breve{M}^{+}(\beta_2, \beta_3)_{d-i},$$
where β_2 is irreducible.

(iii) $H_{\beta_1\beta_2} = 1$ if $\dim \breve{M}(\beta_1, \beta_2) = 0$.

Definition 3.18 We define the v–map $v\colon C^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}}) \to C^{\alpha}_{*}(Y,K;\Delta_{\mathscr{S}})$ by

$$v(\beta_1) = \sum_{\beta_2 \in \mathfrak{C}^*_\pi} \sum_{z\colon \beta_1 \to \beta_2} \deg(H_{\beta_1\beta_2}|_{\breve{M}_z(\beta_1,\beta_2)_1}) \lambda^{-\kappa(z)} T^{\nu(z)} \beta_2.$$

The v–map does not commute with the differential of the chain complex. However, the following relation holds:

Proposition 3.19
$$dv - vd - \delta_2\delta_1 = 0.$$

Proof We consider the 1–dimensional moduli space

$$\breve{M}_{\gamma,z}(\beta_1,\beta_2)_1 := \breve{M}_z(\beta_1,\beta_2)_2 \cap (H_{\beta_1\beta_2})^{-1}(s)$$

for a generic $s \in S^1 \setminus \{1\}$. As in the argument in the proof of [9, Proposition 3.16], the boundary of $\breve{M}_z(\beta_1,\beta_2)$ consists of unparametrized broken trajectories of the form

$$\mathfrak{a} = ([A_1],[A_2]),$$

and there are the following cases:

(I) $\mathfrak{a} \in \bigcup_{z'\circ z''=z} \breve{M}_{z'}(\beta_1,\beta_3)_0 \times \breve{M}_{z''}(\beta_3,\beta_2)_1$ where $\beta_3 \in \mathfrak{C}^*_\pi(Y,K,\alpha)$,

(II) $\mathfrak{a} \in \bigcup_{z'\circ z''=z} \breve{M}_{z'}(\beta_1,\beta_3)_1 \times \breve{M}_{z''}(\beta_3,\beta_2)_0$ where $\beta_3 \in \mathfrak{C}^*_\pi(Y,K,\alpha)$,

(III) $[A] \in \breve{M}_z(\beta_1,\beta_2)$ factors through the reducible critical point θ_α.

For (I), the corresponding oriented boundary components of $(H_{\beta_1\beta_2})^{-1}(s) \cap \breve{M}^+_z(\beta_1,\beta_2)_2$ are

$$\begin{aligned}(H_{\beta_1\beta_2})^{-1}(s) \cap -\Bigl(\bigcup_{\beta_3\in\mathfrak{C}^*_\pi(Y,K,\alpha)} \bigcup_{z'\circ z''=z} \breve{M}_{z'}(\beta_1,\beta_3)_0 \times \breve{M}_{z''}(\beta_3,\beta_2)_1\Bigr) \\ = -\bigcup_{\beta_3\in\mathfrak{C}^*_\pi(Y,K,\alpha)} \bigcup_{z'\circ z''=z} \breve{M}_{z'}(\beta_1,\beta_3)_0 \times (H_{\beta_3\beta_2})^{-1}(s) \cap \breve{M}_{z''}(\beta_3,\beta_2)_1,\end{aligned}$$

since $H_{\beta_1\beta_3} = 1$. This contributes the term $-\langle vd(\beta_1),\beta_2\rangle$. For (II), the similar argument shows that this contributes to the term $\langle dv(\beta_1),\beta_2\rangle$. Case (III) requires gluing theory at the reducible. Let U be an open subset of $\breve{M}(\beta_1,\beta_2)$ which is given by

$$U = \{[A] \in \breve{M}(\beta_1,\beta_2) \mid \|A - \pi^*\theta_\alpha\|_{L^2_1((-1,1)\times(Y\setminus K))} < \epsilon\}.$$

U_z denotes the restriction of U to $M_z(\beta_1,\beta_2)$. There is the "ungluing" map

$$\breve{M}_z(\beta_1,\beta_2) \supset U_z \xrightarrow{\psi} (0,\infty) \times \bigcup_{z'\circ z''=z} \breve{M}_{z'}(\beta_1,\theta_\alpha)_0 \times S^1 \times \breve{M}_{z''}(\theta_\alpha,\beta_3)_0.$$

For $T > 0$ large enough, consider a subset $U_{z,T} = \psi^{-1}(\{(t,[A_1],s,[A_2]) \in U_z \mid t > T\})$ of U_z. Then

$$\psi(M_z(\beta_1,\beta_2) \cap U_{z,T}) = (T,\infty) \times \bigcup_{z'\circ z''=z} \breve{M}_{z'}(\beta_1,\theta_\alpha) \times \{s\} \times \breve{M}_{z''}(\theta_\alpha,\beta_2).$$

Thus corresponding boundaries of 1–manifold $H^{-1}_{\beta_1\beta_2}(s) \cap \breve{M}^+(\beta_1,\beta_2)_2$ with induced orientations are given by

$$-\bigcup_{z'\circ z''=z} \breve{M}_{z'}(\beta_1,\theta_\alpha)_0 \times \breve{M}_{z''}(\theta_\alpha,\beta_2)_0.$$

The sign counting of this contributes to the term $-\langle \delta_2\delta_1(\beta_1),\beta_2\rangle$. Finally, we obtain the relation $\langle (dv - vd - \delta_2\delta_1)(\beta_1),\beta_2\rangle = 0$. □

Next, we consider a negative definite pair $(W,S)\colon (Y,K)\to(Y',K')$ with an embedded curve $\gamma\colon[0,1]\to S$ such that $\gamma(0)=p\in K$ and $\gamma(1)=p'\in K'$. We identify γ with its image. We define

$$\gamma^+ = (-\infty,0]\times\{p\}\cup\gamma\cup[0,\infty)\times\{p'\}\subset S^+.$$

Assume that $\beta\in\mathfrak{C}^*_\pi(Y,K,\alpha)$ and $\beta'\in\mathfrak{C}^*_\pi(Y',K',\alpha)$. For each $A\in\mathcal{A}(W,S;\beta,\beta')$, taking the holonomy of A^{ad} along the path γ^+, we obtain a map

$$h^\gamma_{\beta\beta'}\colon \mathcal{B}(W,S;\beta,\beta')\to S^1,$$

and its modification

$$H^\gamma_{\beta\beta'}\colon M^+(W,S;\beta,\beta')_d\to S^1$$

so that $H^\gamma_{\beta\beta'}=1$ on 0–dimensional unparametrized broken trajectories.

Definition 3.20 We define a map $\mu=\mu_{(W,S,\gamma)}\colon C^\alpha_*(Y,K;\Delta_{\mathscr{S}})\to C^\alpha_*(Y',K';\Delta_{\mathscr{S}})$ by

$$\mu(\beta)=\sum_{\beta'\in\mathfrak{C}^*_\pi(Y',K',\alpha)}\ \sum_{z\colon\beta\to\beta'} \deg(H^\gamma_{\beta\beta'}|_{M_z(W,S;\beta,\beta')_1})\lambda^{\kappa_0-\kappa(z)}T^{\nu(z)-\nu_0}\beta'.$$

Proposition 3.21 *Let $(W,S)\colon(Y,K)\to(Y',K')$ be a negative definite pair, and let m and μ denote its corresponding maps as above. Then*

$$d'\mu+\mu d+\Delta_2\delta_1-\delta_2'\Delta_1-v'm+mv=0,$$

where the prime denotes corresponding maps for the pair (Y',K').

Proof Consider a 2–dimensional moduli space $M^+_z(W,S;\beta,\beta')_2$ and its codimension 1 faces. Firstly, there are two types of ends of $M_z(W,S;\beta,\beta')_2$ in which $[A]\in M(W,S;\beta,\beta')_2$ is broken at irreducible critical points,

(I) $$\breve{M}^+_{z'}(\beta,\beta_1)_{i-1}\times M^+_{z''}(W,S;\beta_1,\beta')_{2-i},$$

(II) $$M^+_{z'}(W,S;\beta,\beta')_{2-i}\times \breve{M}^+_{z''}(\beta_1,\beta')_{1-i},$$

where $i=1,2$. Since

$$(H^\gamma_{\beta\beta'})^{-1}(s)\cap\bigcup_{\beta_1}\bigcup_{z'\circ z''=z}\breve{M}_z(\beta,\beta_1)_0\times M^+_{z''}(W,S;\beta_1,\beta')_1$$
$$=\bigcup_{\beta_1}\bigcup_{z'\circ z''}\breve{M}_{z'}(\beta,\beta_1)_0\times(H^\gamma_{\beta_1\beta'})^{-1}(s)\cap M_{z''}(W,S;\beta_1,\beta')_1,$$

the signed counting of points in $\partial((H^{\gamma}_{\beta\beta'})^{-1}(s)\cap M^+(W,S;\beta,\beta')_2)$ which are contained in codimension 1 faces of type (I) with $i=1$ contributes to the term $-\langle \mu d(\beta),\beta'\rangle$. Next, we consider the case of type (I) with $i=2$. Since

$$(H^{\gamma}_{\beta\beta'})^{-1}(s)\cap\bigcup_{\beta_1}\bigcup_{z'\circ z''=z}\breve{M}_{z'}(\beta,\beta_1)_1\times M_z(W,S;\beta_1,\beta')_0$$
$$=\bigcup_{\beta_1}\bigcup_{z'\circ z''=z}(H_{\beta\beta_1})^{-1}(s)\cap\breve{M}_{z'}(\beta,\beta_1)_1\times M(W,S;\beta_1,\beta')_0,$$

the signed counting of points in $\partial((H^{\gamma}_{\beta\beta'})^{-1}(s)\cap M^+(W,S;\beta,\beta')_2)$ which are contained in codimension 1 faces of type (I) with $i=2$ contributes to the term $-\langle mv(\beta),\beta'\rangle$. Similarly, a collection of codimension 1 faces of type (II) contributes to the term $-\langle d'\mu(\beta),\beta'\rangle$ if $i=1$ and $\langle v'm(\beta),\beta'\rangle$ if $i=2$. Finally, we consider the ends of $M_z(W,S;\beta,\beta')_2$ which break at reducibles. Such ends are described as in the poof of Proposition 3.19 and contribute to the term $-\langle(\Delta_2\delta_1-\delta_2\Delta_1(\beta),\beta'\rangle$. □

Corollary 3.22 *We have $(\widetilde{C}^{\alpha}_*(Y,K;\Delta_{\mathscr{S}}),\tilde{d},\chi)$, where*

$$\widetilde{C}^{\alpha}_*(Y,K;\Delta_{\mathscr{S}})=C^{\alpha}_*(Y,K;\Delta_{\mathscr{S}})\oplus C^{\alpha}_{*-1}(Y,K;\Delta_{\mathscr{S}})\oplus\mathscr{S},\quad \tilde{d}=\begin{bmatrix} d & 0 & 0\\ v & -d & \delta_1\\ \delta_2 & 0 & 0\end{bmatrix}\quad\text{and}\quad \chi=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}$$

form an $\mathcal{S}$–complex. Moreover, if $(W,S)\colon(Y,K)\to(Y',K')$ is a given negative definite cobordism and α satisfies $\Delta_{(Y,K)}(e^{4\pi i\alpha})\Delta_{(Y',K')}(e^{4\pi i\alpha})\neq 0$, then

$$\widetilde{m}_{(W,S)}=\begin{bmatrix} m & 0 & 0\\ \mu & m & \Delta_2\\ \Delta_1 & 0 & \eta\end{bmatrix}$$

defines an $\mathcal{S}$–morphism $\widetilde{m}_{(W,S)}\colon\widetilde{C}^{\alpha}_*(Y,K;\Delta_{\mathscr{S}})\to\widetilde{C}^{\alpha}_*(Y',K';\Delta_{\mathscr{S}})$.

Proof The arguments in Section 3.3 and Proposition 3.19 show that $(\widetilde{C}^{\alpha}_*(Y,K;\Delta_{\mathscr{S}}),\tilde{d},\chi)$ is an $\mathcal{S}$–complex. For a generic perturbation, moduli spaces over the negative definite pair (W,S) are regular at reducible points by Proposition 2.24, and hence the counting of reducibles $\eta=\eta^{\alpha}(W,S)$ is well defined. The arguments in Section 3.2 and Propositions 3.15, and 3.21 show that $\widetilde{m}_{(W,S)}$ is an $\mathcal{S}$–morphism. □

The $\mathcal{S}$–complex $\widetilde{C}^{\alpha}_*(Y,K;\Delta_{\mathscr{S}})$ itself depends on the choices of metric and perturbation. However, the standard argument (see [9, Theorem 3.33]) shows that its $\mathcal{S}$–chain homotopy class is a topological invariant of pairs (Y,K,p) with $\Delta_{(Y,K)}(e^{4\pi i\alpha})\neq 0$, where $K\subset Y$ is an oriented knot in an integer homology 3–sphere and $p\in K$ is a basepoint. The $\mathcal{S}$–chain homotopy type of an $\mathcal{S}$–complex itself depends on the choice of basepoint, however, there is a canonical isomorphism between two homology groups of $\mathcal{S}$–complexes which are defined by different choices of basepoints.

Definition 3.23 We call

$$h^{\alpha}_{\mathscr{S}}(Y,K):=h(\widetilde{C}^{\alpha}_*(Y,K;\Delta_{\mathscr{S}}))$$

the Frøyshov invariant for (Y,K) over $\mathscr{S}$ with a holonomy parameter α.

The $\mathcal{S}$–complex $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{S}})$ admits the following connected sum theorem:

Theorem 3.24 *Let (Y, K) and (Y', K') be two oriented knots in integral homology spheres and α be a holonomy parameter such that $\Delta_{(Y,K)}(e^{4\pi i\alpha})\Delta_{(Y',K')}(e^{4\pi i\alpha}) \neq 0$. Then*

$$\widetilde{C}^{\alpha}_{*}(Y \# Y', K \# K'; \Delta_{\mathscr{S}}) \simeq \widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{S}}) \otimes_{\mathscr{S}} \widetilde{C}^{\alpha}_{*}(Y', K'; \Delta_{\mathscr{S}}),$$

where $\simeq$ denotes an $\mathcal{S}$–chain homotopy equivalence.

The strategy of proof (found in the appendix) is essentially the same as [9, Section 6].

The following corollary gives the proof of Theorem1.6:

Corollary 3.25 *Let (Y, K) and (Y', K') be knots in integral homology 3–spheres and α be a holonomy parameter such that $\Delta_{(Y,K)}(e^{4\pi i\alpha})\Delta_{(Y',K')}(e^{4\pi i\alpha}) \neq 0$. Then*

$$h^{\alpha}_{\mathscr{S}}(Y \# Y', K \# K') = h^{\alpha}_{\mathscr{S}}(Y, K) + h^{\alpha}_{\mathscr{S}}(Y', K').$$

Moreover, if there are two negative definite cobordisms

$$(W, S)\colon (Y, K) \to (Y', K') \quad \textit{and} \quad (W', S')\colon (Y', K') \to (Y, K),$$

then

$$h^{\alpha}_{\mathscr{S}}(Y, K) = h^{\alpha}_{\mathscr{S}}(Y', K').$$

Proof The first statement follows from Theorem 3.24 and Proposition 3.7. The second follows from Corollary 3.22 and Proposition 3.6. □

The filtered construction can be applied to an $\mathcal{S}$–complex for the coefficient $\mathscr{R}_\alpha$. A fixed lift $\tilde{\theta}_\alpha$ of a reducible flat connection can be identified with $1 \in \mathscr{R}_\alpha$, and $\mathscr{R}_\alpha$ itself can be identified with the set of all lifts of θ_α. We extend the $\mathbb{R}$–grading to $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})$. First, we define

$$\deg_{\mathbb{R}}(\delta) = \begin{cases} \max\{r \mid a_r \neq 0\} & \text{if } \delta \neq 0, \\ -\infty & \text{if } \delta = 0, \end{cases}$$

for $\delta = \sum_r a_r \lambda^r \in \mathscr{R}_\alpha$, $a_r \in \mathbb{Z}[T^{-1}, T]]$. Then for $(\beta, \gamma, \delta) \in C^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha}) \oplus C^{\alpha}_{*-1}(Y, K; \Delta_{\mathscr{R}_\alpha}) \oplus \mathscr{R}_\alpha$, we define

$$\widetilde{\deg}_{\mathbb{R}}(\beta, \gamma, \delta) := \max\{\deg_{\mathbb{R}}(\beta), \deg_{\mathbb{R}}(\gamma), \deg_{\mathbb{R}}(\delta)\}.$$

Obviously, we have the following proposition:

Proposition 3.26 *If we fix a lift of each critical point of the Chern–Simons functional, then the $\mathcal{S}$–complex $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})$ admits the $(\mathbb{Z}\times\mathbb{R})$–grading.*

Note that the $\mathbb{R}$–grading of $\mathcal{S}$–complexes extends to tensor products of $\mathcal{S}$–complexes in a natural way.

The filtered $\mathcal{S}$–complex $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]}$ for $R_0, R_1 \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \mathscr{C}^*$ with $R_0 < R_1$ can be defined as follows. Put $\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R]} := \{(\beta, \gamma, \delta) \in \widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha}) \mid \widetilde{\deg}_{\mathbb{R}}(\beta, \gamma, \delta) < R\}$ and

$$\widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[R_0,R_1]} := \widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R_1]} / \widetilde{C}^{\alpha}_{*}(Y, K; \Delta_{\mathscr{R}_\alpha})^{[-\infty,R_0]}$$

for $R_0 < R_1$.

3.5 Cobordism maps for immersed surfaces

Let $(W,S)\colon (Y,K)\to (Y',S')$ be a cobordism of pairs where S is possibly immersed. Blowing up all double points of S, we obtain a cobordism of pairs $(\overline{W},\overline{S})$ where $\overline{S}$ is an embedded surface.

Definition 3.27 We say (W,S) is negative definite if its blowup $(\overline{W},\overline{S})$ is negative definite. We define a cobordism map for a negative definite cobordism (W,S) where S is possibly immersed surface as

$$\widetilde{m}_{(W,S)} := \widetilde{m}_{(\overline{W},\overline{S})}.$$

We describe the relation between operations on immersed surface cobordisms and induced $\mathcal{S}$–morphisms:

Proposition 3.28 *Let $\mathscr{S}$ be an integral domain over the ring $\mathscr{R}_\alpha$. Assume that (W,S) is a negative definite pair over $\mathscr{S}$ where S is a possibly immersed surface. Let S^* be a surface obtained from S by a positive or negative twist move, or a finger move. Then $\widetilde{m}_{(W,S^*)}$ is $\mathcal{S}$–chain homotopic to $\widetilde{m}_{(W,S)}$ up to the multiplication of a unit element in $\mathscr{S}$.*

The definition of positive twist, negative twist and finger moves can be found in [17].

Proof Since the monotonicity condition cannot be assumed in our setting, we have to modify the argument in [25].

(i) **(positive twist move)** Consider the blowup at the positive self-intersection point $(\overline{W},\overline{S}^*) = (W,S)\,\#\,(\overline{\mathbb{CP}}^2,S_2)$, where S_2 is an embedded sphere whose homology class is $-2e\in H_2(\overline{\mathbb{CP}}^2;\mathbb{Z})$. Note that $\mathcal{R}_\alpha(S^3\setminus S^1,\mathrm{SU}(2))=\{\theta_\alpha\}$ for $\alpha\in\left(0,\frac{1}{2}\right)$. Assume that (W,S^*) has a metric g_T such that (S^3,S^1) has a neighborhood which is isometric to $[-T,T]\times(S^3,S^1)$, where $T>0$ is large enough. Let A_T be an instanton on $\{(W,S^*),g_T\}$ which is contained in the 0–dimensional moduli space. A_∞ denotes the limiting instanton of A_T with respect to $T\to\infty$, and A_1 and A_2 denote its restriction to components obtained by attaching cylindrical ends on (W,S) and $(\overline{\mathbb{CP}}^2,S_2)$, respectively. Then we have

$$\operatorname{ind} D_{A_1}+1+\operatorname{ind} D_{A_2}=\operatorname{ind} D_{A_\infty}\le 0.$$

The last inequality essentially follows from [27, Corollary 8.4] and our assumption. The index formula for the closed pair $(\overline{\mathbb{CP}}^2,S_2)$ shows that $\operatorname{ind} D_{A_2}\equiv -1 \bmod 4$, and we have $\operatorname{ind} D_{A_2}=-1$. By the perturbation, the instanton A_2 on $\overline{\mathbb{CP}}^2$ satisfies $H^1_{A_2}=H^2_{A_2}=0$, and the gluing along $\mathcal{R}_\alpha(S^3\setminus S^1)=\{\theta_\alpha\}$ is unobstructed. The moduli space $M(W,S^*;\beta,\beta')_0$ is diffeomorphic to

$$M(W,S;\beta,\beta')_0\times M^\alpha(\overline{\mathbb{CP}}^2,S_2)_0.$$

Note that there is a diffeomorphism $M^\alpha(\overline{\mathbb{CP}}^2\setminus D^4,S_2\setminus D^2;\theta_\alpha)_0\cong M^\alpha(\overline{\mathbb{CP}}^2,S_2)_0$ by the removable singularity theorem. Since $\operatorname{ind} D_{A_2}=-1$, A_2 is a minimal reducible. Moreover, minimal reducibles on $(\overline{\mathbb{CP}}^2,S_2)$ define elements in $M^\alpha(\overline{\mathbb{CP}}^2,S_2)_0$. Counting elements in the moduli space $M(W,S;\beta,\beta')_0$ defined by the limiting metric $\lim_{T\to\infty} g_T$ contributes the relation

$$\langle\widetilde{m}^\infty_{(\overline{W},\overline{S}^*)}\beta,\beta'\rangle=\eta^\alpha(\overline{\mathbb{CP}}^2,S_2)\langle\widetilde{m}_{(W,S)}\beta,\beta'\rangle.$$

Since

$$\eta^{\alpha}(\overline{\mathbb{CP}}^2, S_2) = \begin{cases} 1-\lambda^{4\alpha-1}T^4 & \text{if } \alpha \le \frac{1}{4}, \\ \lambda^{1-4\alpha}T^{-4}-1 & \text{if } \alpha > \frac{1}{4}, \end{cases}$$

$\eta^{\alpha}(\overline{\mathbb{CP}}^2, S_2)$ is a unit in $\mathscr{S}$. Considering the 1–parameter family of moduli spaces gives an $\mathcal{S}$–chain homotopy between $\widetilde{m}^{\infty}_{(\overline{W},\overline{S}^*)}$ and $\widetilde{m}_{(\overline{W},\overline{S}^*)}$. In particular, $\eta^{\alpha}(\overline{\mathbb{CP}}^2, S_2)$ has the top term 1, and hence the statement follows.

(ii) **(negative twist move)** In this case, we change S_2 in the above argument to an embedded sphere S_0 whose homology class is trivial. Thus we obtain $\eta^{\alpha}(\overline{\mathbb{CP}}^2, S_0) = 1$.

(iii) **(finger move)** Consider the decomposition $(W, S) = (W_1, S_1) \cup (W_2, S_2)$ where $W_2 = D^4$ and $S_2 = D^2 \sqcup D^2$. Let $(\overline{W}, \overline{S}^*) = (W_1, S_1) \cup (W_2', S_2')$ be the double blowup of (W, S^*). In this case, W_2 is a 4–manifold obtained by removing a disk from $\overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2$ and S_2 is two disjoint disks. Note that $\mathcal{R}_{\alpha} := \mathcal{R}_{\alpha}(S^3 \backslash (S^1 \sqcup S^1), \mathrm{SU}(2)) \cong [0, \pi]$ for fixed $\alpha \in \left(0, \frac{1}{2}\right)$, the interior of $\mathcal{R}_{\alpha}$ consists of irreducible flat connections and two endpoints are reducible. Moreover, the endpoint map $r_1 \colon M(W_1, S_1; \beta, \beta')_0 \to \mathcal{R}_{\alpha}$ has its image in the irreducible part of $\mathcal{R}_{\alpha}$. See [25, Lemma 3.2] for details.

We claim that the counting of the two moduli spaces $M(W_1, S_1; \beta, \beta')_0$ and $M(W, S; \beta, \beta')_0$ can be identified up to multiplication by a unit element in $\mathscr{S}$. Firstly, we define an $\mathcal{S}$–morphism $\widetilde{m}_{(W_1,S_1)}$ as

$$\langle m_{(W_1,S_1)}\beta, \beta' \rangle = \sum_{z} \# M_z(W_1, S_1; \beta, \beta')\lambda^{\kappa_0-\kappa(z)}T^{\nu(z)-\nu_0}\beta',$$

and similarly for other components in $\widetilde{m}_{(W_1,S_1)}$. Here $\kappa(z)$, κ_0, $\nu(z)$ and ν_0 are similarly defined as in Section 2.7. We have to modify the argument in [25] which is related to the unobstructed gluing along the pair $(S^3, S^1 \sqcup S^1)$. For $\rho \in \mathcal{R}_{\alpha}$ which is in the image of r_1, we take its extension A_{ρ} to $(D^4, D^2 \sqcup D^2)$. Consider the double $(S^4, S^2 \sqcup S^2) = (D^4, D^2 \sqcup D^2) \cup_{(S^3,S^1)} (D^4, D^2 \sqcup D^2)$. Then $\operatorname{ind} D_{A_{\rho}\# A_{\rho}} = 2 \operatorname{ind} D_{A_{\rho}} + 1$ by the gluing formula. Consider the pair of connected sum $(S^4, S^2 \sqcup S^2) = (S^4, S^2) \#_{(S^3,\varnothing)} (S^4, S^2)$. Then $\operatorname{ind} D_{A_{\rho}\# A_{\rho}} = 2 \operatorname{ind} D_{A_{\rho}\# A_{\rho}}|_{(S^4,S^2)} + 3$ and the left-hand side is equal to 1. Hence we have $\operatorname{ind} D_{A_{\rho}} = 0$. Thus the relation

$$\operatorname{ind} D_{A_{\rho}} + \dim H^1_{\rho} = -\dim H^0_{A_{\rho}} + \dim H^1_{A_{\rho}} - \dim H^2_{A_{\rho}}$$

tells us that $H^2_{A_{\rho}} = 0$ since $\dim H^0_{A_{\rho}} = 0$ and $\dim H^1_{A_{\rho}} = 1$. Here H^1_{ρ} is the cohomology with the local coefficient system associated with the flat connection ρ. Thus the Morse–Bott gluing of instantons over (W_1, S_1) and (W_2, S_2) is unobstructed. For a metric on (W, S) with a long neck along the cylinder $[0, 1] \times (S^3, S^1 \sqcup S^1)$, we have the diffeomorphism,

$$M(W, S; \beta, \beta')_0 \cong M(W_1, S_1; \beta, \beta')_0 \, {}_{r}\!\times_{r'} M^{\alpha}(D^4, D^2 \sqcup D^2)_1$$

where

$$r \colon M(W_1, S_1; \beta, \beta')_0 \to \mathcal{R}_{\alpha}$$

and

$$r' \colon M^{\alpha}(D^4, D^2 \sqcup D^2)_1 \to \mathcal{R}_{\alpha}$$

are restriction maps. For simplicity, we consider the case $\alpha \leq \frac{1}{4}$. Since flat connections on $(S^3, S^1 \sqcup S^1)$ uniquely extend to $(D^4, D^2 \sqcup D^2)$, the induced cobordism map has the form

$$\widetilde{m}_{(W,S)} = \Bigl(1 + \sum_{k>0} c_k Z^{-k}\Bigr)\widetilde{m}_{(W_1,S_1)},$$

where $c_k \in \mathbb{Z}$ and $Z = \lambda^{1-4\alpha} T^{-4}$. Thus $\widetilde{m}_{(W,S)}$ and $\widetilde{m}_{(W_1,S_1)}$ differ by the multiplication of a unit element in $\mathscr{S}$.

Assume that the cobordism of pairs $(\overline{W}, \overline{S}^*)$ is equipped with a metric such that $(\overline{W}, \overline{S}^*)$ has a long neck along $(S^3, S^1 \sqcup S^1)$. Then the moduli space $M(\overline{W}, \overline{S}^*; \beta, \beta')_0$ decomposes into a union of fiber products

$$M(W_1, S_1, \beta, \beta')_d \, {}_{r_1}\!\times_{r_2} M^\alpha(W_2'; S_2')_{d'}$$

with $d + d' = 1$, where

$$r_1\colon M(W_1, S_1, \beta, \beta')_d \to \mathcal{R}_\alpha \quad \text{and} \quad r_2\colon M^\alpha(W_2', S_2')_{d'} \to \mathcal{R}_\alpha$$

are restriction maps. Since $d' \equiv 1 \mod 4$ by the index formula, we have $d = 0$ and $d' = 1$. Thus there is the coefficient $c \in \mathscr{S}$ such that $\widetilde{m}_{(\overline{W}, \overline{S}^*)} = c\widetilde{m}_{(W,S)}$. Consider the special case of a finger move which is the composition of one positive twist move and one negative twist move. In this case, the coefficient c turns out to be $1 - \lambda^{4\alpha-1} T^4$ for $\alpha \leq \frac{1}{4}$ and $\lambda^{1-4\alpha} T^{-4} - 1$ for $\alpha > \frac{1}{4}$ by the argument above. Finally, we conclude that there is a unit element $c \in \mathscr{S}$ such that $\widetilde{m}_{(W,S^*)}$ and $c\widetilde{m}_{(W,S)}$ are $\mathcal{S}$–chain homotopic. □

4 Nondegeneracy of the representation variety

In this section we will discuss conjugacy classes of representations

$$\rho\colon \pi_1(Y \setminus K) \to \mathrm{SU}(2),$$

with the condition

$$\rho(\mu_K) \sim \begin{bmatrix} e^{2\pi i \alpha} & 0 \\ 0 & e^{-2\pi i \alpha} \end{bmatrix}.$$

We write $[\rho]$ for its conjugacy class to distinct elements in $\mathrm{Hom}(\pi_1(Y \setminus K), \mathrm{SU}(2))$ and $\mathcal{R}(Y \setminus K, \mathrm{SU}(2))$. Firstly, we introduce the method of taking cyclic branched coverings. Considering a knot in an integral homology 3–sphere (Y, K), we can take a cyclic branched covering $\widetilde{Y}_r(K)$ over Y branched along K. Let $N(K)$ be a tubular neighborhood of $K \subset Y$, and $V = Y \setminus \mathrm{int}(N(K))$ be its exterior. $\widetilde{V}$ denotes the r–fold unbranched covering over V with $\pi_1(\widetilde{V})$ being a kernel of $\pi_1(Y \setminus K) \to H_1(Y \setminus K, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$. $N(K)$ and $\widetilde{V}$ have a torus boundary, and let $h\colon \partial N(K) \to \partial\widetilde{V}$ be a gluing map which sends μ_K a meridian of K to its lift $\tilde{\mu}_K$. Then the r–fold cyclic branched covering over Y is defined by

$$\widetilde{Y}_r(K) = N(K) \cup_h \widetilde{V}.$$

Let $\tau\colon \widetilde{Y}_r \to \widetilde{Y}_r$ be a covering transformation. We define an induced action of $\pi_1(\widetilde{Y}_r, p)$ with a basepoint p missing the fixed point set of the τ–action on $\widetilde{Y}_r$. For this, we fix another basepoint q inside the fixed point set and a path connecting p and q. Such a choice of path defines a (noncanonical) isomorphism between $\pi_1(\widetilde{Y}_r, p)$ and $\pi_1(\widetilde{Y}_r, q)$. Since τ induces a natural action on $\pi_1(\widetilde{Y}_r, q)$, we define an induced action τ_* on $\pi_1(\widetilde{Y}_r, p)$ via the above isomorphism. The τ–action induces the action τ^* on $\mathrm{Hom}(\pi_1(\widetilde{Y}_r, p), \mathrm{SO}(3))$ by $\tau^*(\rho) = \rho \circ \tau_*$. This also defines an action on $\mathcal{R}^*(\widetilde{Y}_r, \mathrm{SO}(3))$ by $\tau^*[\rho] = [\tau^*(\rho)]$. We define the following subsets:

$$\mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)) = \{[\rho] \in \mathcal{R}^*(\widetilde{Y}_r, \mathrm{SO}(3)) \mid \tau^*[\rho] = [\rho]\},$$
$$\mathcal{R}^{*,\tau}(\widetilde{Y}, \mathrm{SU}(2)) = \{[\rho] \in \mathcal{R}^*(\widetilde{Y}, \mathrm{SU}(2)) \mid \mathrm{Ad}[\rho] \in \mathcal{R}^{*,\tau}(\widetilde{Y}, \mathrm{SO}(3))\}.$$

Since a different choice of basepoints of fundamental groups induces a canonical isomorphism on $\mathcal{R}^*(\widetilde{Y}_r, \mathrm{SO}(3))$, we may omit the choice of basepoints and a path between them from the notation here.

The aim of Section 4.1 is giving the construction of the lifting map

$$\Pi\colon \bigsqcup_{1 \le l \le r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \to \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2)),$$

which sends singular flat connections to nonsingular flat connections on a cyclic branched covering of the knot $K \subset Y$. We will see that the lifting map Π satisfies the following proposition.

Proposition 4.1 *Assume that the r–fold cyclic branched covering $\widetilde{Y}_r$ of a knot K in an integral homology 3–sphere Y is an integral homology 3–sphere. Then the lifting map Π gives a two-to-one correspondence*

$$\Pi\colon \bigsqcup_{1 \le l \le r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \to \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2)).$$

This is a generalization of the argument in [2].

Let $X(K)$ be the complement of a tubular neighborhood of the knot $K \subset S^3$. Its boundary $\partial X(K)$ is a torus. In Section 4.2, we will show that the restriction map $r\colon \mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \to \mathcal{R}(\partial X(T_{p,q}), \mathrm{SU}(2))$ is a smooth immersion of a 1–manifold without any perturbation of flat connections, using the setting of gauge theory by Herald [22] and computations of the group cohomology of π_1. In Section 4.3, we will give a proof of Theorem 1.9 using the results in Section 4.1.

4.1 The construction of the lifting map

We assign the second Stiefel–Whitney class $w \in H^2(Y, \mathbb{Z}_2)$ to $[\rho] \in \mathcal{R}(Y, \mathrm{SO}(3))$. We can construct a flat bundle $E = \widetilde{Y} \times_\rho \mathbb{R}^3$ from an SO(3)–representation ρ and define $w([\rho]) := w_2(E) \in H^2(Y, \mathbb{Z}_2)$, where $w_2(E)$ is the second Stiefel–Whitney class of E. If $w(\rho) = 0$ then the SO(3)–bundle E lifts to an SU(2)–bundle F. Let P and Q be the corresponding principal bundles of E and F, respectively. The natural map $p\colon Q \to P$ is a fiberwise double covering map. Let θ_ρ be a connection form on P which corresponds to the flat connection ρ. Then $p^*\theta_\rho$ defines a flat connection on Q. Thus each element of $\mathcal{R}(Y, \mathrm{SO}(3))$ lifts to $\mathcal{R}(Y, \mathrm{SU}(2))$ if its second Stiefel–Whitney class vanishes.

Proposition 4.2 *Let X be Y or $Y \setminus K$. Then there is an action of $H^1(X, \mathbb{Z}_2)$ on $\mathcal{R}(X, \mathrm{SU}(2))$ and the map $\mathrm{Ad}\colon \mathcal{R}(X, \mathrm{SU}(2)) \to \mathcal{R}^0(X, \mathrm{SO}(3))$ induces a bijection*

$$\mathcal{R}(X, \mathrm{SU}(2))/H^1(X, \mathbb{Z}_2) \cong \mathcal{R}^0(X, \mathrm{SO}(3)).$$

Here $\mathcal{R}^0(X, \mathrm{SO}(3))$ denotes the set of conjugacy classes of $\mathrm{SO}(3)$–representations whose second Stiefel–Whitney class vanishes.

Proof Let $\rho\colon \pi_1(X) \to \mathrm{SO}(3)$ be a representation whose second Stiefel–Whitney class vanishes and $\tilde{\rho}\colon \pi_1(X) \to \mathrm{SU}(2)$ be its $\mathrm{SU}(2)$–lift. Consider another lift $\tilde{\rho}'\colon \pi_1(X) \to \mathrm{SU}(2)$. Then there is a map $\chi\colon \pi_1(X) \to \{\pm 1\}$ such that $\tilde{\rho}'(g) = \chi(g)\tilde{\rho}(g)$ for any $g \in \pi_1(X)$. We can directly check that χ is a homomorphism and determine an element $\chi \in \mathrm{Hom}(\pi_1(X), \mathbb{Z}_2) = H^1(X, \mathbb{Z}_2)$. Conversely, two $\mathrm{SU}(2)$–representation $\sigma_1, \sigma_2\colon \pi(X) \to \mathrm{SU}(2)$ such that there exists $\chi \in \mathrm{Hom}(\pi_1(X), \mathbb{Z}_2)$ and satisfying $\sigma_1(g) = \chi(g)\sigma_2(g)$ for any $g \in \pi_1(X)$ induces the same $\mathrm{SO}(3)$–representation. We define an action of $H^1(X, \mathbb{Z}_2)$ on $\mathrm{Hom}(\pi_1(X), \mathrm{SU}(2))$ by $\sigma \mapsto \chi \cdot \sigma$, where $(\chi \cdot \sigma)(g) = \chi(g)\sigma(g)$ for $g \in \pi_1(X)$. The action of χ commutes with the conjugacy action and descends to $\mathcal{R}(X, \mathrm{SU}(2))$. □

Note that the action of $H^1(Y \setminus K, \mathbb{Z}_2)$ coincides with the flip symmetry. From Proposition 4.2, we get the following corollary:

Corollary 4.3 *For an integral homology 3–sphere Y, all elements in $\mathcal{R}(Y, \mathrm{SO}(3))$ have a unique lift in $\mathcal{R}(Y, \mathrm{SU}(2))$.*

Proof Since $H^2(Y, \mathbb{Z}_2) = 0$, the second Stiefel–Whitney class of $[\rho] \in \mathcal{R}(Y, \mathrm{SO}(3))$ vanishes, and ρ lifts to an $\mathrm{SU}(2)$–representation. By Proposition 4.2, this lift is unique since $H^1(Y, \mathbb{Z}_2) = 0$. □

If $[\rho] \in \mathcal{R}(Y \setminus K, \mathrm{SU}(2))$ satisfies

$$\rho(\mu_K) \sim \begin{bmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha}, \end{bmatrix}$$

then the induced $\mathrm{SO}(3)$–representation satisfies

$$\mathrm{Ad}\,\rho(\mu_K) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(4\pi\alpha) & -\sin(4\pi\alpha) \\ 0 & \sin(4\pi\alpha) & \cos(4\pi\alpha) \end{bmatrix}. \tag{4-1}$$

Let $\mathcal{R}_\alpha(Y \setminus K, \mathrm{SO}(3))$ be a subset of $\mathcal{R}(Y \setminus K, \mathrm{SO}(3))$ whose elements are represented by $\mathrm{SO}(3)$–representations of $\pi_1(Y \setminus K)$ such that their images of μ_K are conjugate to the right-hand side of (4-1).

Before proceeding with the argument, we introduce the orbifold fundamental group of $Y \setminus K$. (It appears in [2; 3], for example.)

Definition 4.4 The orbifold fundamental group of $Y \setminus K$ is $\pi_1^V(Y, K; r) := \pi_1(Y \setminus K)/\langle \mu_K^r \rangle$.

Proposition 4.5 *The orbifold fundamental group $\pi_1^V(Y, K; r)$ admits the split short exact sequence*

$$1 \to \pi_1(\widetilde{Y}_r) \to \pi_1^V(Y, K; r) \to \mathbb{Z}/r \to 1.$$

Proof Let $\widetilde{K} \subset \widetilde{Y}_r$ be the branched locus. Then there is the exact sequence

$$1 \to \pi_1(\widetilde{Y}_r \setminus \widetilde{K}) \to \pi_1(Y \setminus K) \to \mathbb{Z}/r \to 1.$$

since $\widetilde{Y}_r \setminus \widetilde{K} \to Y \setminus K$ is a regular covering. Applying the van Kampen theorem to $\widetilde{Y}_r \setminus \operatorname{int} N(\widetilde{K}) \cup N(\widetilde{K})$, we have $\pi_1(\widetilde{Y}_r) = \pi_1(\widetilde{Y}_r \setminus \widetilde{K})/\langle \mu_{\widetilde{K}} \rangle$. Since $\pi_1(\widetilde{Y}_r \setminus \widetilde{K}) \to \pi_1(Y \setminus K)$ maps $\mu_{\widetilde{K}}$ to μ_K^r, this induces $1 \to \pi_1(\widetilde{Y}_r) \to \pi_1^V(Y, K; r)$. Since $\pi_1(Y \setminus K) \to \mathbb{Z}/r$ maps μ_K^r to 1, this induces $\pi_1^V(Y, K; r) \to \mathbb{Z}/r$ which sends μ_K to a generator of $\mathbb{Z}/r$. The spitting $\mathbb{Z}/r \to \pi_1^V(Y, K; r)$ sends a generator of $\mathbb{Z}/r$ to μ_K. □

Lemma 4.6 *There is a natural one-to-one correspondence*

$$\mathcal{R}^*(\pi_1^V(Y, K; r), \mathrm{SO}(3)) \cong \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)).$$

Proof Let $\rho\colon \pi_1(Y \setminus K) \to \mathrm{SO}(3)$ be a representation with $[\rho] \in \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3))$. Then it factors through $\pi_1^V(Y, K; r)$. Conversely, any representation $\sigma\colon \pi_1^V(Y, K; r) \to \mathrm{SO}(3)$ satisfies

$$\sigma(\mu_K) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(4\pi\alpha) & -\sin(4\pi\alpha) \\ 0 & \sin(4\pi\alpha) & \cos(4\pi\alpha) \end{bmatrix},$$

where $\alpha = l/(2r)$ for some $0 < l < r$. Thus σ defines the desired representation of $\pi_1(Y \setminus K)$. □

Proposition 4.7 *There is a bijection*

$$\bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)) \cong \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)).$$

Proof Since there is the natural one-to-one correspondence in Lemma 4.6, we only have to construct

$$\mathcal{R}^*(\pi_1^V(Y, K; r), \mathrm{SO}(3)) \xrightarrow{\cong} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)).$$

This is induced from $\pi_1(\widetilde{Y}_r) \xrightarrow{i} \pi_1^V(Y, K; r)$ in the short exact sequence in Proposition 4.5. We claim that if $\rho\colon \pi_1^V(Y, K; r) \to \mathrm{SO}(3)$ is irreducible then $\rho \circ i$ is also irreducible. Since $\widetilde{Y}_r$ is an integral homology sphere, any reducible $\mathrm{SO}(3)$–representation of $\pi_1(\widetilde{Y}_r)$ is the trivial representation. If $\rho \circ i$ is trivial, then ρ factors through $\pi_1^V(Y, K; r)/i(\pi_1(\widetilde{Y}_r)) \cong \mathbb{Z}/r$, and hence is reducible. This is a contradiction.

We will construct the inverse correspondence of the above. Let σ be an $\mathrm{SO}(3)$–representation of $\pi_1(\widetilde{Y}_r)$ which represents an element in $\mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3))$. Since the conjugacy class of σ is fixed by the induced action of τ, there is a matrix $A \in \mathrm{SO}(3)$ such that

$$\tau^*\sigma(u) = A\sigma(u)A^{-1}$$

for any $u \in \pi_1(\widetilde{Y}_r)$. A is uniquely determined since σ is irreducible and has the trivial stabilizer $\{1\}$ in $\mathrm{SO}(3)$, and A is conjugate to the matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi l/r) & -\sin(2\pi l/r) \\ 0 & \sin(2\pi l/r) & \cos(2\pi l/r) \end{bmatrix}. \tag{4-2}$$

Since τ has order r, we get the relation

$$\sigma(u) = A^r \sigma(u) A^{-r}$$

for any $u \in \pi_1(\widetilde{Y}_r)$, and we get $A^r = 1$ using the irreducibility of σ. Thus we can assign a unique order-r element $A_\sigma \in \mathrm{SO}(3)$ for each σ. Finally, we assign a representation

$$\bar{\sigma}\colon \pi_1^V(Y, K; r) \cong \pi_1(\widetilde{Y}_r) \rtimes \mathbb{Z}/r \to \mathrm{SO}(3), \quad (u, t^k) \mapsto \sigma(u) \cdot A_\sigma^k, \tag{4-3}$$

to a given representation σ, where $t \in \mathbb{Z}/r$ is a generator. This satisfies $\bar{\sigma}(\mu_K) = A_\sigma$. The above construction gives the inverse of $\mathcal{R}^*(\pi_1^V(Y, K; r), \mathrm{SO}(3)) \ni [\rho] \mapsto [\rho \circ i] \in \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3))$. □

We write

$$\Pi'\colon \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)) \xrightarrow{\cong} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3))$$

for the bijection constructed above.

Definition 4.8 Let K be a knot in an integral homology 3–sphere Y, and assume that $\widetilde{Y}_r$ is also an integral homology 3–sphere. Then $\Pi\colon \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \to \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2))$ is given by the following composition:

$$\bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \xrightarrow{\mathrm{Ad}} \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)) \xrightarrow{\Pi'} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)) \xrightarrow{\mathrm{Ad}^{-1}} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2)).$$

We call $\Pi([\rho])$ a lift of $[\rho]$.

An $\mathrm{SU}(2)$–representation that factors through $\mathrm{Pin}(2)$ subgroups is called a binary dihedral representation. An $\mathrm{SO}(3)$–representation that factors through $O(2)$ subgroups is called a dihedral representation. Note that $O(2)$ is embedded in $\mathrm{SO}(3)$ as

$$\begin{bmatrix} A & 0 \\ 0 & \det A \end{bmatrix} \in \mathrm{SO}(3),$$

where $A \in O(2)$. The adjoint representation of a binary dihedral representation is a dihedral representation. In the proof of Proposition 4.1, which is an important property of the lift Π, we use the following lemma:

Lemma 4.9 [39] *The fixed point set of the $H^1(Y \setminus K, \mathbb{Z}_2)$–action on $\mathcal{R}(Y \setminus K, \mathrm{SU}(2))$ consists of conjugacy classes of binary dihedral representations.*

Proof Let $[\rho] \in \mathcal{R}(Y \setminus K, \mathrm{SU}(2))$ be a fixed point of the action of $H^1(Y \setminus K, \mathbb{Z}_2)$. We regard this as a representation $\rho\colon \pi_1(Y \setminus K) \to \mathrm{SU}(2)$ such that there exists $A \in \mathrm{SU}(2)$ and $(\chi \cdot \rho)(u) = A\rho(u)A^{-1}$ for any $u \in \pi_1(Y \setminus K)$. Here $\chi \in H^1(Y \setminus K)$ is a generator. Since χ has order 2, $\rho(u) = A^2\rho(u)A^{-2}$. If ρ is reducible then its image is contained in a circle in $\mathrm{SU}(2)$ and is a binary dihedral representation.

Assume that ρ is irreducible and consider two cases, $A^2 = 1$ and $A^2 = -1$. We regard $\mathrm{SU}(2)$ as the unit sphere in the quaternions. Then $\mathrm{Pin}(2) = S^1 \cup jS^1$. If $A^2 = 1$ then $A = \pm 1$ and $-\rho(u) = \rho(u)$ for some $u \in \pi_1(Y \setminus K)$. This cannot happen in $\mathrm{SU}(2)$. If $A^2 = -1$ then we can assume that $A = i$ after a conjugation, and then $\rho(u) = \pm i\rho(u)i^{-1}$ for any $u \in \pi_1(Y \setminus K)$. If $\rho(u) = i\rho(u)i^{-1}$ then $\rho(u) \in S^1 = \{a + bi\}$. If $\rho(u) = -i\rho(u)i^{-1}$ then $\rho(u) \in jS^1 = \{cj + dk\}$. Thus the image of ρ is contained in $S^1 \cup jS^1$. □

Lemma 4.10 *Let $r \in 2\mathbb{Z}$. If $\rho\colon \pi_1^V(Y, K; r) \to \mathrm{SO}(3)$ is a dihedral representation, then its pullback $\pi_1(\widetilde{Y}_r) \to \mathrm{SO}(3)$ by the orbifold exact sequence in Proposition 4.5 is a reducible representation.*

Proof Since ρ factors through $O(2)$, we have a representation $\rho'\colon \pi_1^V(Y, K; r) \to O(2)$. Composing with $\det\colon O(2) \to \mathbb{Z}/2$, we have a representation $\det \circ \rho'\colon \pi_1^V(Y, K; r) \to \mathbb{Z}/2$. Since $\det \circ \rho'$ factors through the abelianization $\pi_1^V(Y, K; r) = \pi_1(Y \setminus K)/\langle {\mu_K}^r \rangle \xrightarrow{\text{Ab}} \mathbb{Z}[\mu_K]/\langle {\mu_K}^r \rangle$, we have the diagram

$$\begin{array}{ccccccc} \pi_1(\widetilde{Y}_r) & \xrightarrow{i} & \pi_1^V(Y, K; r) & \xrightarrow{\rho'} & O(2) & \xrightarrow{\det} & \mathbb{Z}/2 \\ & & & \searrow^{\text{Ab}} & & \nearrow & \\ & & & & \mathbb{Z}/r & & \end{array}$$

where $\pi_1(\widetilde{Y}_r) \xrightarrow{i} \pi_1^V(Y, K; r)$ is the inclusion map in the orbifold exact sequence. By construction, Ab coincides with the map $\pi_1^V(Y, K; r) \to \mathbb{Z}/r$ in the orbifold exact sequence. Thus $\text{Ab} \circ i$ is the trivial representation, and hence $\det \circ \rho' \circ i$ is also the trivial representation. This implies that the image of $\rho' \circ i$ is contained in $\mathrm{SO}(2)$. Thus $\rho \circ i : \pi_1(\widetilde{Y}_r) \to \mathrm{SO}(3)$ factors through $\mathrm{SO}(2)$, and this means that $\rho \circ i$ is reducible. □

The following proposition gives the proof of Proposition 4.1:

Proposition 4.11 *Let $K \subset Y$ be a knot in an integral homology 3–sphere whose r–fold cyclic branched covering $\widetilde{Y}_r$ is also an integral homology 3–sphere. For each $[\rho] \in \mathcal{R}^{*,\tau}(\widetilde{Y}, \mathrm{SU}(2))$, $\Pi^{-1}([\rho])$ consists of two elements which correspond to each other by the flip symmetry.*

Proof Applying Proposition 4.2 to the 3–manifold $Y \setminus K$, we have a bijection

$$\mathcal{R}(Y \setminus K, \mathrm{SU}(2))/H^1(Y \setminus K, \mathbb{Z}_2) \cong \mathcal{R}(Y \setminus K, \mathrm{SO}(3)). \tag{4-4}$$

Note that $\mathcal{R}(Y \setminus K, \mathrm{SO}(3)) = \mathcal{R}^0(Y \setminus K, \mathrm{SO}(3))$ since $H^2(Y \setminus K, \mathbb{Z}_2) = 0$. We restrict this correspondence to elements with holonomy parameter $\alpha = l/(2r)$ for $l = 1, \dots, r-1$. Note that $H^1(Y \setminus K, \mathbb{Z}_2)$ acts on $\mathcal{R}^*_\alpha(Y \setminus K, \mathrm{SU}(2)) \cup \mathcal{R}^*_{1/2-\alpha}(Y \setminus K, \mathrm{SU}(2))$ since the flip symmetry changes the holonomy parameter as $\alpha \mapsto \frac{1}{2} - \alpha$, and the bijection (4-4) is restricted to

$$\Bigl[\bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2))\Bigr] \Big/ H^1(Y \setminus K, \mathbb{Z}_2) \cong \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)). \tag{4-5}$$

Fix $[\rho] \in \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2))$. The composition

$$\bigsqcup_{0<l<r} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)) \xrightarrow{\Pi'} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SO}(3)) \xrightarrow{\mathrm{Ad}^{-1}} \mathcal{R}^{*,\tau}(\widetilde{Y}_r, \mathrm{SU}(2))$$

is bijective. Thus we have a unique element $[\rho'] \in \bigsqcup_{0<l<r} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3))$ which corresponds to $[\rho]$. $\Pi^{-1}([\rho])$ is the inverse image of $[\rho']$ by the map

$$\bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2)) \xrightarrow{\mathrm{Ad}} \bigsqcup_{l=1}^{r-1} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SO}(3)).$$

Finally, we prove that $\mathrm{Ad}^{-1}([\rho'])$ consists of two elements. Let $[\sigma] \in \mathrm{Ad}^{-1}([\rho'])$ be an element contained in $\mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2))$. If r is odd then $l/(2r) \neq \frac{1}{2} - l/(2r)$, and thus $[\rho] \neq \chi[\rho]$ in $\bigsqcup_{0<l<r} \mathcal{R}^*_{l/(2r)}(Y \setminus K, \mathrm{SU}(2))$ since ρ and $\chi\rho$ have different holonomy parameters, where χ is a generator of $H^1(Y \setminus K, \mathbb{Z}_2) \cong \mathbb{Z}_2$. This means that the $H^1(Y \setminus K, \mathbb{Z}_2)$–action on $\bigcup_{0<l<r} \mathcal{R}_{l/(2r)}(Y \setminus K, \mathrm{SU}(2))$ is free and $\mathrm{Ad}^{-1}([\rho])$ consists of two elements. If r is even and $2l \neq r$, then $\mathrm{Ad}^{-1}([\rho])$ consists of two elements by the same reason. If $2l = r$ then $H^1(Y \setminus K, \mathbb{Z}_2)$ acts on $\mathcal{R}^*_{1/4}(Y \setminus K, \mathrm{SU}(2))$. The fixed points of the $H^1(Y \setminus K, \mathbb{Z}_2)$–action on $\mathcal{R}^*_{1/4}(Y \setminus K, \mathrm{SU}(2))$ are binary dihedral representations by Lemma 4.9. We show that $\Pi^{-1}([\rho])$ does not contain a binary dihedral representation. Let $\sigma' \colon \pi_1(Y \setminus K) \to \mathrm{SU}(2)$ be a binary dihedral representation with holonomy parameter $\alpha = \frac{1}{4}$. Then $\mathrm{Ad}\,\sigma'$ defines a dihedral representation $\pi_1^V(Y, K; r) \to \mathrm{SO}(3)$. Then the induced representation $\mathrm{Ad}\,\sigma' \circ i \colon \pi_1(\widetilde{Y}_r) \to \mathrm{SO}(3)$ is reducible by Lemma 4.10 and its SU(2)–lift is also reducible. This means that $\Pi^{-1}([\rho])$ does not contain any binary dihedral representation. Thus $H^1(Y \setminus K, \mathbb{Z}_2)$ acts freely on $\Pi^{-1}([\rho])$, and hence $\Pi^{-1}([\rho])$ consists of two elements which are related by the flip symmetry. □

4.2 Nondegeneracy results

The purpose of this subsection is to associate the nondegeneracy property of the critical point set $\mathfrak{C}$ of the singular Chern–Simons functional and the transversality of the moduli space of irreducible flat connections $\mathcal{R}^*(Y \setminus K, \mathrm{SU}(2))$. Let us recall the setting of the gauge theory used in [22; 23] to deal with the "pillowcase picture" of perturbed flat connections. In this subsection, Y denotes a (general) oriented closed 3–manifold and K is a knot in Y. Let E be an SU(2)–bundle over $X = Y \setminus N(K)$. We fix a Riemannian metric on X. We introduce the space of SU(2)–connections over X and $\partial X = T^2$ as follows:

$$\mathcal{A}_X = L^2_2(X, \mathfrak{su}(2) \otimes \Lambda^1), \quad \mathcal{A}_{T^2} = L^2_{3/2}(T^2, \mathfrak{su}(2) \otimes \Lambda^1).$$

Here we fix a trivialization of the SU(2)–bundle over X and ∂X, and identify the trivial connection to zero elements in each functional space. We also introduce spaces of gauge transformations:

$$\mathcal{G}_X = \{g \in \mathrm{Aut}(E) \mid g \in L^2_3\}, \quad \mathcal{G}_{T^2} = \{g \in \mathrm{Aut}(E|_{T^2}) \mid g \in L^2_{5/2}\}.$$

The action of gauge transformations on connections and $\mathfrak{su}(2)$–valued p–forms are given in obvious ways. $\mathcal{G}_X$ and $\mathcal{G}_{T^2}$ have Banach Lie group structures and act smoothly on $\mathcal{A}_X$ and $\mathcal{A}_{T^2}$, respectively. A connection whose stabilizer of gauge transformations is $\{\pm 1\}$ is called irreducible. $\mathcal{A}^*_X$ denotes the subset of irreducible connections.

We introduce the following spaces of p–forms with boundary conditions:

$$\Omega^p_\nu(X,\mathfrak{su}(2)) = \{\omega\in\Omega^p(X,\mathfrak{su}(2)) \mid *\omega|_{\partial X}=0\},\quad \Omega^p_\tau(X,\mathfrak{su}(2)) = \{\omega\in\Omega^p(X,\mathfrak{su}(2)) \mid \omega|_{\partial X}=0\}.$$

We define the L^2–inner product on $\Omega^p(X,\mathfrak{su}(2))$ by the formula

$$\langle a,b\rangle = -\int_X \operatorname{tr}(a\wedge *b).$$

For each $A\in\mathcal{A}_X$, the slice of the action of $\mathcal{G}_X$ on $\mathcal{A}_X$ is given by

$$X_A = A + \operatorname{Ker} d_A^* \cap L^2_2\Omega^1_\nu(X,\mathfrak{su}(2)).$$

For each flat connection $A\in\mathcal{A}$, the space of harmonic p–forms is given by

$$\mathcal{H}^p(X;\operatorname{ad}A) = \{\omega\in\Omega^p_\nu(X,\mathfrak{su}(2)) \mid d_A\omega=0,\ d_A^*\omega=0\},$$
$$\mathcal{H}^p(X,\partial X;\operatorname{ad}A) = \{\omega\in\Omega^p_\tau(X,\mathfrak{su}(2)) \mid d_A\omega=0,\ d_A^*\omega=0\}.$$

The holonomy perturbation h defines a compact perturbation term $V_h\colon\mathcal{A}_X\to\Omega^1(X,\mathfrak{su}(2))$ and a perturbed flat connection can be defined as a solution of the equation

$$*F_A+V_h=0. \tag{4-6}$$

$\mathcal{R}^{*,h}(X,\mathrm{SU}(2))$ denotes gauge equivalence classes of irreducible solutions for (4-6). Consider the restriction map $r\colon\mathcal{R}^{*,h}(X,\mathrm{SU}(2))\to\mathcal{R}(T^2,\mathrm{SU}(2))$. For a generic perturbation h, $\mathcal{R}^{*,h}(X,\mathrm{SU}(2))$ is a smooth 1–manifold. Moreover, the restriction map r is a smooth immersion of $\mathcal{R}^{*,h}(X,\mathrm{SU}(2))$ to the smooth part of the pillowcase. The detailed argument is contained in [22]. Put

$$S_\alpha := \{\rho\in\mathcal{R}(T^2,\mathrm{SU}(2)) \mid \operatorname{tr}\rho(\mu_K) = 2\cos(2\pi i\alpha)\}.$$

This is a vertical slice in the pillowcase. Note that $\mathcal{R}_\alpha(X,\mathrm{SU}(2)) = r^{-1}(S_\alpha)\cap\mathcal{R}(X,\mathrm{SU}(2))$ and we define $\mathcal{R}^{*,h}_\alpha(X,\mathrm{SU}(2)) := r^{-1}(S_\alpha)\cap\mathcal{R}^{*,h}(X,\mathrm{SU}(2))$.

Proposition 4.12 *Let $K\subset Y$ be a knot in a closed 3–manifold, and α be an arbitrary holonomy parameter in $\left(0,\frac{1}{2}\right)$. Assume that $[\rho]\in\mathcal{R}^*_\alpha(Y\setminus K,\mathrm{SU}(2))$ is a nondegenerate critical point. We also assume that the image of $\mathcal{R}^*(Y\setminus K,\mathrm{SU}(2))$ by the restriction map r is contained in the smooth part of the pillowcase. Then $\mathcal{R}^*(Y\setminus K,\mathrm{SU}(2))$ is smooth near $[\rho]$. Moreover, the restriction map $r\colon\mathcal{R}^*(Y\setminus K,\mathrm{SU}(2))\to\mathcal{R}(T^2,\mathrm{SU}(2))$ is an immersion to the smooth part of the pillowcase at $[\rho]$.*

For the proof of Proposition 4.12, we need gauge theory on 3–manifolds with the boundary described above.

Lemma 4.13 *Let B_0 be an abelian $\mathrm{SU}(2)$–flat connection on a torus T^2. Then the $\mathfrak{su}(2)$–valued harmonic form $h\in\mathcal{H}^1(T^2;\operatorname{ad}B_0)$ has the diagonal form*

$$h=\begin{bmatrix} ai & 0\\ 0 & -ai\end{bmatrix},$$

where $a\in\Omega^1(T^2)$.

Proof Since B_0 is an abelian flat connection, it defines a splitting of the SU(2)–bundle E over T^2 into $E = L \oplus L^*$, where L is the trivial line bundle. Then the adjoint bundle of E has a splitting $\mathfrak{g}_E = \underline{\mathbb{R}} \oplus L^{\otimes 2}$. Let $d_{B_0} = d \oplus d_C$ be the covariant derivative induced on $\Omega^p(T^2, \mathfrak{g}_E) = \Omega^p(T^2, \underline{\mathbb{R}}) \oplus \Omega^p(T^2, L^{\otimes 2})$. Any section $\omega \in \Omega^p(T^2, \mathfrak{g}_E)$ has the form

$$\omega = \begin{bmatrix} ai & b \\ -\bar{b} & -ai \end{bmatrix},$$

where $a \in \Omega^p(T^2)$ and $b \in \Omega^p(T^2, L^{\otimes 2})$. The space of harmonic forms $\mathcal{H}^p(T^2; \operatorname{ad} B_0)$ splits into $\mathcal{H}^p(T^2; \underline{\mathbb{R}}) \oplus \mathcal{H}^p(T^2; L^{\otimes 2})$ with respect to the decomposition of $(\Omega^p(T^2, \mathfrak{g}_E), d_{B_0})$. Let us compute $\mathcal{H}^1(T^2; \operatorname{ad} B_0)$ using $H^1(\pi_1(T^2); \operatorname{ad}\rho)$, where ρ is an abelian SU(2)–representation corresponding to B_0. Let μ and λ be canonical generators of $\pi_1(T^2)$. Then the space of 1–cocycles consists of the element $\gamma\colon \pi_1(T^2) \to \mathfrak{su}(2) \cong \mathbb{R}^3$ such that

$$(1 - \mathrm{Ad}_{\rho(\mu)})\gamma(\lambda) = (1 - \mathrm{Ad}_{\rho(\lambda)})\gamma(\mu),$$

since μ and λ commute. Let $F\colon \mathbb{R}^3 \oplus \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map given by

$$F(x_1, x_2) = (1 - A_\mu)x_1 - (1 - A_\lambda)x_2,$$

where $A_\mu := \mathrm{Ad}_{\rho(\mu)}$ and $A_\lambda := \mathrm{Ad}_{\rho(\lambda)}$. Since A_μ and A_λ are SO(3)–linear transformations acting on $\mathbb{R}^3$, they have 1–dimensional axes of rotation $\mathbb{R}_\mu$ and $\mathbb{R}_\lambda$, respectively. Let $\mathbb{C}_\mu$ and $\mathbb{C}_\lambda$ be their orthogonal complement spaces. Then $\mathrm{Im}(1 - A_\mu) = \mathbb{C}_\mu$ and $\mathrm{Im}(1 - A_\lambda) = \mathbb{C}_\lambda$, and hence F is surjective. Thus the space of 1–cocycles is isomorphic to $\mathbb{R}^4$. On the other hand, the space of 1–coboundaries is spanned by $\mathrm{Im}(1 - \mathrm{Ad}_{\rho(g)})$ for all $g \in \pi_1(T^2)$, and this is 2–dimensional since ρ is reducible. Thus $\mathcal{H}^1(T^2; \operatorname{ad} B_0) \cong H^1(\pi_1(T^2); \operatorname{ad}\rho) \cong \mathbb{R}^2$. Therefore $H^1(T^2; L^{\otimes 2})$ vanishes since $\mathcal{H}^1(T^2; \underline{\mathbb{R}}) \cong H^1(T^2; \mathbb{R}) \cong \mathbb{R}^2$. This means that if $\omega \in \Omega^1(T^2, \mathfrak{g}_E)$ is a harmonic form then $b = 0$. Thus $h \in \mathcal{H}^1(T^2; \operatorname{ad} B_0)$ has only diagonal components. □

Since B_0 is a reducible connection with $U(1)$–stabilizer, $\mathcal{H}^0(T^2; \operatorname{ad} B_0) = \operatorname{Ker} d_{B_0} \cong \mathbb{R}$. We fix a generator $\gamma_0 \in \mathcal{H}^0(T^2; \operatorname{ad} B_0)$.

Lemma 4.14 *There is a $\mathcal{G}_{T^2}$–invariant neighborhood N_{B_0} of $B_0 \in \mathcal{A}_{T^2}$ and $\mathcal{G}_{T^2}$–invariant map $\eta\colon N_{B_0} \to \Omega^0(T^2, \mathfrak{su}(2))$ such that*

(1) $\eta(B_0) = \gamma_0$,

(2) $\int_{T^2} \operatorname{tr}(F_B \wedge \eta(B)) = 0$ *for all* $B \in N_{B_0}$.

Proof Take a small neighborhood of B_0 in the slice of the action of $\mathcal{G}_{T^2}$ on $\mathcal{A}_{T^2}$ as

$$X_{B_0,\epsilon} = \{B_0 + b \mid b \in L^2_{3/2}\Omega^1(T^2, \mathfrak{su}(2)), d^*_{B_0} b = 0, \|b\|_{L^2_{3/2}} < \epsilon\},$$

where $\epsilon > 0$ is small enough. Firstly, we define an $\Omega^0(T^2, \mathfrak{su}(2))$–valued map η on the slice $X_{B_0,\epsilon}$ and then extend it to a gauge-invariant neighborhood. For $B = B_0 + b \in X_{B_0,\epsilon}$, define

$$\eta(B) := \gamma_0.$$

Then

$$\int_{T^2} \mathrm{tr}(F_B \wedge \gamma_0) = \int_{T^2} \mathrm{tr}((d_{B_0} b + b \wedge b) \wedge \gamma_0) = \int_{T^2} \mathrm{tr}(d_{B_0} b \wedge \gamma_0) + \int_{T^2} \mathrm{tr}(b \wedge b \wedge \gamma_0).$$

Using Stokes' theorem and the condition $d_{B_0}\gamma_0 = 0$,

$$\int_{T^2} \mathrm{tr}(d_{B_0} b \wedge \gamma_0) = \int_{T^2} d\, \mathrm{tr}(b \wedge \gamma_0) = 0.$$

Thus

$$\int_{T^2} \mathrm{tr}(F_B \wedge \eta(B)) = \int_{T^2} \mathrm{tr}(b \wedge b \wedge \gamma_0). \tag{4-7}$$

Since $d^*_{B_0} b = 0$, we have $h \in \mathcal{H}^1(T^2; \mathrm{ad}\, B_0)$ and $\omega \in \Omega^2(T^2, \mathfrak{su}(2))$ such that

$$b = h + d^*_{B_0}\omega. \tag{4-8}$$

Using (4-7) and (4-8),

$$\begin{aligned}
&\int_{T^2} \mathrm{tr}(F_B \wedge \eta(B)) \\
&\quad = \int_{T^2} \mathrm{tr}[(h + d^*_{B_0}\omega) \wedge (h + d^*_{B_0}\omega) \wedge \gamma_0] \\
&\quad = \int_{T^2} \mathrm{tr}(h \wedge h \wedge \gamma_0) + \int_{T^2} \mathrm{tr}(d^*_{B_0}\omega \wedge h \wedge \gamma_0) + \int_{T^2} \mathrm{tr}(h \wedge d^*_{B_0}\omega \wedge \gamma_0) + \int_{T^2} \mathrm{tr}(d^*_{B_0}\omega \wedge d^*_{B_0}\omega \wedge \gamma_0).
\end{aligned}$$

Note that,

$$\begin{aligned}
\int_{T^2} \mathrm{tr}(*d_{B_0} * \omega \wedge h \wedge \gamma_0) &= \int_{T^2} \mathrm{tr}(d_{B_0} * \omega \wedge *h \wedge \gamma_0) \\
&= -\int_{T^2} \mathrm{tr}(*\omega \wedge d_{B_0} * h \wedge \gamma_0) + \int_{T^2} \mathrm{tr}(*\omega \wedge *h \wedge d_{B_0}\gamma_0) = 0.
\end{aligned}$$

Here we use Stokes' theorem at the second equality. Similarly,

$$\int_{T^2} \mathrm{tr}(h \wedge d^*_{B_0}\omega \wedge \gamma_0) = -\int_{T^2} \mathrm{tr}(d_{B_0} * h \wedge *\omega \wedge \gamma_0) - \int_{T^2} \mathrm{tr}(*h \wedge *\omega \wedge d_{B_0}\gamma_0) = 0,$$

$$\int_{T^2} \mathrm{tr}(d^*_{B_0}\omega \wedge d^*_{B_0}\omega \wedge \gamma_0) = -\int_{T^2} \mathrm{tr}(d^2_{B_0} * \omega \wedge *\omega \wedge \gamma_0) - \int_{T^2} \mathrm{tr}(d_{B_0} * \omega \wedge *\omega \wedge d_{B_0}\gamma_0) = 0.$$

Hence

$$\int_{T^2} \mathrm{tr}(F_B \wedge \eta(B)) = \int_{T^2} \mathrm{tr}(h \wedge h \wedge \gamma_0).$$

Since $\gamma_0 \in \mathcal{H}^0(T^2; \mathrm{ad}\, B_0)$ is an element of the Lie algebra of the stabilizer of B_0, $\mathrm{Stab}(B_0) = U(1)$ and it has the pointwise form

$$\gamma_0(x) = \begin{bmatrix} ri & 0 \\ 0 & -ri \end{bmatrix} \in \mathfrak{su}(2),$$

where $r \in \mathbb{R}$. Similarly, $h \in \mathcal{H}^1(T^2; \mathrm{ad}\, B_0)$ has the form

$$h(x) = \begin{bmatrix} ai & 0 \\ 0 & -ai \end{bmatrix}$$

by Lemma 4.13. By the pointwise computation of $\mathrm{tr}(h \wedge h \wedge \gamma_0)$, we obtain

$$\mathrm{tr}(h \wedge h \wedge \gamma_0)(x) = \mathrm{tr}\left(\begin{bmatrix} ai 0 \\ 0 - ai \end{bmatrix} \wedge \begin{bmatrix} ai 0 \\ 0 - ai \end{bmatrix} \wedge \begin{bmatrix} ri 0 \\ 0 - ri \end{bmatrix}\right) = \mathrm{tr}\begin{bmatrix} -ra \wedge ai 0 \\ 0 ra \wedge ai \end{bmatrix} = 0.$$

Thus $\eta\colon X_{B_0,\epsilon} \to \Omega^0(T^2, \mathfrak{su}(2))$ satisfies $\int_{T^2} \operatorname{tr}(F_B \wedge \eta(B)) = 0$ for all $B \in X_{B_0,\epsilon}$. Define $N_{B_0} := \mathcal{G}_{T^2} \cdot X_{B_0,\epsilon}$ and extend η to N_{B_0} in a gauge-equivariant way (ie $\eta(g^*(B)) = g^{-1}\eta(B)g$). □

Let A_0 be a flat irreducible SU(2)–connection on X. We can assume that $A_0|_{T^2}$ is a noncentral flat connection on T^2 by the assumption of Proposition 4.12, and we write $A_0|_{T^2} = B_0$. Let U_{A_0} be a gauge invariant neighborhood of A_0 in $\mathcal{A}^*_X$. We define $\tilde{\eta}\colon U_{A_0} \to \Omega^0(X, \mathfrak{su}(2))$ as a smooth extension of η which satisfies

$$\tilde{\eta}(A)|_{\partial X} = \eta(A|_{\partial X}).$$

Here we assume that the extension $\tilde{\eta}$ satisfies $d_{A_0}\tilde{\eta}(A_0) \in \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0)$; this is possible by the following lemma:

Lemma 4.15 *For $\eta \in \mathcal{H}^0(T^2; \operatorname{ad} B_0)$ there is an extension $\tilde{\eta}$ on X such that $d_{A_0}\tilde{\eta} \in \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0)$.*

Proof For $\eta \in \mathcal{H}^0(T^2, \operatorname{ad} B_0)$, we take an arbitrary smooth extension $\tilde{\eta}$ to X. Then

$$d_{A_0}\tilde{\eta} \in \operatorname{Ker} d_{A_0}|_{\Omega^1_\tau(X,\mathfrak{su}(2))} = d_{A_0}\Omega^0_\tau(X, \mathfrak{su}(2)) \oplus \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0).$$

Let $d_{A_0}\tilde{\xi}$ be the $d_{A_0}\Omega^0_\tau(X, \mathfrak{su}(2))$–component of $d_{A_0}\tilde{\eta}$. Then $d_{A_0}(\tilde{\eta} - \tilde{\xi}) \in \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0)$ with $(\tilde{\eta} - \tilde{\xi})|_{\partial X} = \eta$. Hence we can choose an extension $\tilde{\eta}$ of η as $d_{A_0}\tilde{\eta} \in \mathcal{H}^1(X, \partial X; \operatorname{ad} A_0)$. □

We define a map

$$\Phi\colon U_{A_0} \times L^2_2\Omega^0_\tau(X, \mathfrak{su}(2)) \times \mathbb{R} \to L^2_1(X, \mathfrak{su}(2) \otimes \Lambda^1)$$

by $\Phi(A, \zeta, t) = *F_A + d_A\zeta + t d_A\tilde{\eta}(A)$. The linearized operator of Φ at $(A_0, 0, 0)$ has the form

$$D\Phi_{(A_0,0,0)}(a, \zeta, t) = *d_{A_0}a + d_{A_0}\zeta + t d_{A_0}\tilde{\eta}(A_0).$$

Coker $D\Phi_{(A_0,0,0)}$ is $\mathcal{H}^1(X, \partial X; \operatorname{ad}_{A_0}) \cap (d_{A_0}\tilde{\eta}(A_0))^\perp$ by the Hodge decomposition.

Lemma 4.16 *$\Phi(A, \zeta, t) = 0$ if only if $F_A = 0$, $\zeta = 0$ and $t = 0$.*

Proof Assume that $\Phi(A, \zeta, t) = 0$. Then

$$\|F_A\|^2_{L^2} = -\int_X \operatorname{tr}(F_A \wedge *F_A) = \int_X \operatorname{tr}(F_A \wedge d_A\zeta) + t\int_X \operatorname{tr}(F_A \wedge d_A\tilde{\eta}(A)).$$

Using Stokes' theorem and the Bianchi identity,

$$\int_X \operatorname{tr}(F_A \wedge d_A\zeta) = \int_X d\operatorname{tr}(F_A \wedge \zeta) - \int_X \operatorname{tr}(d_AF_A \wedge \zeta) = \int_{T^2} \operatorname{tr}(F_{A|_{T^2}} \wedge \zeta|_{T^2}).$$

The last term vanishes by the boundary condition on ζ. Consider the remaining term

$$\int_X \operatorname{tr}(F_A \wedge d_A\tilde{\eta}(A)). \tag{4-9}$$

Using Stokes' theorem and the Bianchi identity, this is equal to

$$\int_{T^2} \operatorname{tr}(F_B \wedge \eta(B)),$$

where $B = A|_{T^2}$. By Lemma 4.14, this is equal to zero and we have $F_A = 0$. Thus $0 = *F_A = -d_A(\zeta + t\tilde{\eta}(A))$ by our assumption. Since A is an irreducible connection, d_A has trivial kernel and $\zeta = -t\tilde{\eta}(A)$. Restricting this to the boundary T^2, we have the relation $t\eta(B) = 0$. Since $\eta(B) = \gamma_0$ is a generator of $\mathcal{H}^0(T^2; \operatorname{ad} B_0)$, we have $\eta(B) \neq 0$. Hence $t = 0$ and $\zeta = 0$ follow.

Conversely, if we assume that $F_A = 0$, $\zeta = 0$ and $t = 0$, then clearly $\Phi(A, \zeta, t) = 0$. □

Lemma 4.16 means that the two equations $F_A = 0$ and $\Phi(A, \zeta, t) = 0$ have the same zero set near an irreducible flat connection A_0. Hence $\Phi = 0$ defines the space of flat connections near A_0.

Proof of Proposition 4.12 Natural embeddings $\mu_K \hookrightarrow \partial X \hookrightarrow X$ induce maps on cohomology groups with a local coefficient system,

(4-10) $$H^1(X; \operatorname{ad}\rho) \xrightarrow{j} H^1(\partial X; \operatorname{ad}\rho) \to H^1(\mu_K; \operatorname{ad}\rho).$$

The nondegeneracy condition on $[\rho]$ is equivalent to the condition that the composition (4-10) is injective by Proposition 2.3. This implies that j is also injective. Thus the restriction map $\mathcal{R}^*(X, \mathrm{SU}(2)) \to \mathcal{R}(T^2, \mathrm{SU}(2))$ to the pillowcase is an immersion at $[\rho]$ if we show that $\mathcal{R}^*(X, \mathrm{SU}(2))$ has a smooth manifold structure near $[\rho]$. Next, we show that $\mathcal{R}^*(X, \mathrm{SU}(2))$ is a smooth manifold near $[\rho]$. Consider the long exact sequence of cohomology with local coefficient associated to the pair $(X, \partial X)$,

$$\cdots \to H^0(\partial X; \operatorname{ad}\rho) \xrightarrow{\partial} H^1(X, \partial X; \operatorname{ad}\rho) \to H^1(X; \operatorname{ad}\rho) \xrightarrow{j} H^1(\partial X; \operatorname{ad}\rho) \to \cdots.$$

The cokernel of the connecting homomorphism ∂ is zero since j is injective. Using the harmonic representative of the cohomology with local coefficient, the connecting homomorphism ∂ is given by $\gamma_0 \mapsto d_{A_0}\tilde{\eta}(A_0)$, where A_0 is an SU(2)–flat connection corresponding to ρ. Thus $\operatorname{Coker}\partial = \mathcal{H}^1(X, \partial X; \operatorname{ad}_{A_0}) \cap (d_{A_0}\tilde{\eta}(A_0))^\perp = 0$. This means that the equation $\Phi(A, \zeta, t) = 0$ has a surjective linearization map at $(A_0, 0, 0)$. Thus there is a neighborhood V_{A_0} of $A_0 \in \Phi^{-1}(0)$ which has a smooth structure by the implicit function theorem. Since A_0 is irreducible, the quotient singularity by gauge transformations $\mathcal{G}_X$ does not occur. Thus $\mathcal{R}^*(X, \mathrm{SU}(2))$ has a smooth manifold structure near $[\rho]$. □

By Proposition 2.3, the following shows that the singular Chern–Simons functional for a (p, q)–torus knot has nondegenerate irreducible critical points without perturbations:

Proposition 4.17 *For any* $[\rho] \in \mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$, *the natural map*

(4-11) $$H^1(S^3 \setminus T_{p,q}; \operatorname{ad}\rho) \to H^1(\mu_{T_{p,q}}; \operatorname{ad}\rho)$$

is injective.

Proof Firstly, we compute $H^1(S^3 \setminus T_{p,q}; \operatorname{ad}\rho)$ using the group cohomology of $\pi_1(Y \setminus K)$. Since the fundamental group $\pi_1(S^3 \setminus T_{p,q})$ has a presentation $\langle x, y \mid x^p = y^q \rangle$, 1–cocycles $\gamma : \pi_1(S^3 \setminus T_{p,q}) \to \mathfrak{su}(2)$ satisfy the relation

$$(I + A_x + \cdots + A_x^{p-1})\gamma(x) = (I + A_y + \cdots + A_y^{q-1})\gamma(y)$$

where $A_x := \mathrm{Ad}_{\rho(x)}$ and $A_y := \mathrm{Ad}_{\rho(y)}$. Since A_x and A_y are SO(3)–linear transformation acting on $\mathbb{R}^3$, they have 1–dimensional axis of rotation $\mathbb{R}_x$ and $\mathbb{R}_y$, respectively. Let $\mathbb{C}_x$ and $\mathbb{C}_y$ denote their orthogonal complement spaces. Then $\mathrm{Im}(I - A_x) = \mathbb{C}_x$ and $\mathrm{Im}(I - A_y) = \mathbb{C}_y$. Note that $\rho(x)$ and $\rho(y)$ are contained in different great circles in $\mathrm{SU}(2) \cong S^3$ since ρ is an irreducible SU(2)–representation. Thus ρ satisfies $\rho(x)^p = \rho(y)^q = \pm 1$ and hence $A_x^p = A_y^q = I$. Thus $\mathrm{Ker}(I + A_x + \cdots + A_x^{p-1}) = \mathbb{C}_x$ and $\mathrm{Ker}(I + A_y + \cdots + A_y^{q-1}) = \mathbb{C}_y$. Since ρ is irreducible, $\mathbb{R}_x$ and $\mathbb{R}_y$ are independent in $\mathbb{R}^3$. Consider a linear map $L\colon \mathbb{R}^3 \oplus \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L(x_1, x_2) = (I + A_x + \cdots + A_x^{p-1})x_1 - (I + A_y + \cdots + A_y^{q-1})x_2.$$

This has rank 2, and the space of 1–cocycles has dimension 4. On the other hand, the space of 1–coboundaries is a subspace of $\mathbb{R}^3$ spanned by $\mathrm{Im}(I - \mathrm{Ad}_{\rho(g)})$ for all $g \in \pi_1(S^3 \setminus T_{p,q})$, and this coincides with $\mathbb{R}^3$ itself. Therefore $H^1(S^3 \setminus T_{p,q}; \mathrm{ad}\,\rho) \cong \mathbb{R}^4/\mathbb{R}^3 \cong \mathbb{R}$.

Next we compute $H^1(\mu_{T_{p,q}}; \mathrm{ad}\,\rho)$. Here the space of 1–cocycles is isomorphic to $\mathbb{R}^3$ since its elements are determined by choosing $\gamma(\mu) \in \mathfrak{su}(2) \cong \mathbb{R}^3$. The space of 1–coboundaries is $\mathrm{Im}(I - \mathrm{Ad}_{\rho(\mu)}) \cong \mathbb{C}$. Thus $H^1(\mu; \mathrm{ad}\,\rho) \cong \mathbb{R}^3/\mathbb{C} \cong \mathbb{R}$.

Finally, we prove that the map (4-11) is surjective. If $\gamma\colon \pi_1(S^3 \setminus T_{p,q}) \to \mathfrak{su}(2)$ represents a nonzero element in $H^1(S^3 \setminus T_{p,q}; \mathrm{ad}\,\rho)$ then $\gamma(g) \notin \mathrm{Im}(I - \mathrm{Ad}_{\rho(g)})$ for any $g \in \pi_1(S^3 \setminus T_{p,q})$. Thus $\gamma(\mu) \notin \mathrm{Im}(I - \mathrm{Ad}_{\rho(\mu)})$ for the meridian $\mu \in \pi_1(S^3 \setminus T_{p,q})$, and this means that the image of $[\gamma] \in H^1(S^3 \setminus T_{p,q}; \mathrm{ad}\,\rho)$ in $H^1(\mu_{T_{p,q}}; \mathrm{ad}\,\rho)$ is a nonzero element. □

Consider a knot K in S^3. Note that the image of the restriction map $r\colon \mathcal{R}^*(S^3 \setminus K, \mathrm{SU}(2)) \to \mathcal{R}(T^2, \mathrm{SU}(2))$ is contained in the smooth part of the pillowcase. By Propositions 4.17 and 4.12 we get the following statement:

Corollary 4.18 *The natural restriction map $\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \to \mathcal{R}(T^2, \mathrm{SU}(2))$ to the smooth part of the pillowcase is an immersion of a smooth 1–manifold.*

In fact, it is known that the irreducible representation variety $\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ is a disjoint union of $\frac{1}{2}(p-1)(q-1)$ segments; see [24].

4.3 Levine–Tristram signature and representation variety

The following statement relates the size of the set of singular flat connections over $S^3 \setminus T_{p,q}$ and the set of flat connections over the cyclic branched covering.

Lemma 4.19 *Let p, q and r be relatively coprime positive integers and $\Sigma(p,q,r)$ be a Brieskorn homology sphere. Then*

$$2|\mathcal{R}^*(\Sigma(p,q,r), \mathrm{SU}(2))| = \sum_{l=1}^{r-1} |\mathcal{R}^*_{l/(2r)}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|.$$

Proof We apply Proposition 4.1 to the r–fold cyclic branched covering $\Sigma(p,q,r)$ of $T_{p,q} \subset S^3$. Note that the covering transformation τ induces the trivial action on $\mathcal{R}^*(\Sigma(p,q,r),\mathrm{SU}(2))$ by [2], and hence there is a two-to-one correspondence

$$\bigsqcup_l \mathcal{R}^*_{l/(2r)}(S^3 \setminus T_{p,q},\mathrm{SU}(2)) \overset{\cong}{\Rightarrow} \mathcal{R}^*(\Sigma(p,q,r),\mathrm{SU}(2)). \qquad \square$$

The nondegeneracy condition at irreducible critical points can be interpreted in the pillowcase as follows:

Lemma 4.20 *Let $\alpha \in \left(0,\frac{1}{2}\right)$ be a fixed holonomy parameter. Assume that $[\rho] \in \mathcal{R}^*_\alpha(S^3 \setminus K,\mathrm{SU}(2))$ is nondegenerate. Then S_α and the image of $\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2))$ by the restriction map*

$$r\colon \mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2)) \to \mathcal{R}(T^2,\mathrm{SU}(2))$$

intersect transversely at $r([\rho])$.

Proof Consider the natural map $p\colon \mathcal{R}(T^2,\mathrm{SU}(2)) \to \mathcal{R}(\mu_K,\mathrm{SU}(2))$ induced from the embedding $\mu_K \hookrightarrow T^2$. Let $[\sigma] \in \mathcal{R}(\mu_K,\mathrm{SU}(2))$ be an element such that $\mathrm{tr}(\sigma(\mu_K)) = 2\cos(2\alpha\pi)$. Then $p^{-1}([\sigma]) = S_\alpha \subset \mathcal{R}(T^2,\mathrm{SU}(2))$ by definition. Note that S_α is contained in the smooth part of the pillowcase, and $[\sigma]$ is also contained in the smooth part of $\mathcal{R}(\mu_K,\mathrm{SU}(2))$ since the quotient singularity by the conjugacy action of $\mathrm{SU}(2)$ does not happen when $\alpha \neq 0, \frac{1}{2}$. Thus the kernel of the map

$$dp_{[\sigma']}\colon T_{[\sigma']}\mathcal{R}(T^2,\mathrm{SU}(2)) = H^1(T^2;\mathrm{ad}\,\rho) \to T_{[\sigma]}\mathcal{R}(\mu_K,\mathrm{SU}(2)) = H^1(\mu_K;\mathrm{ad}\,\rho)$$

induced on their tangent spaces is $T_{[\sigma]}S_\alpha$, where $[\sigma'] = r([\rho])$. Note that $\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2))$ is smooth near $[\rho]$ by Proposition 4.12. The composition of the natural maps

$$T_{[\rho]}\mathcal{R}(S^3 \setminus K,\mathrm{SU}(2)) = H^1(S^3 \setminus K;\mathrm{ad}\,\rho) \xrightarrow{j} H^1(T^2;\mathrm{ad}\,\rho) \to H^1(\mu_K;\mathrm{ad}\,\rho) \tag{4-12}$$

is injective by our nondegeneracy assumption. Thus the image of $H^1(S^3 \setminus K;\mathrm{ad}\,\rho)$ in $H^1(T^2;\mathrm{ad}\,\rho)$ is independent of $\mathrm{Ker}(H^1(T^2;\mathrm{ad}\,\rho) \to H^1(\mu_K;\mathrm{ad}\,\rho))$. This means that $r(\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2)))$ and S_α intersect transversely at $r([\rho])$. $\square$

There is a relation between $\sigma_\alpha(K)$ and $\mathcal{R}^*_\alpha(S^3 \setminus K,\mathrm{SU}(2))$. We use the following inequality in the proof of Proposition 4.22:

Lemma 4.21 *Let K be a knot in S^3. Assume that $\mathcal{R}^*_\alpha(S^3 \setminus K,\mathrm{SU}(2))$ is nondegenerate. Then*

$$|\sigma_\alpha(K)| \le 2|\mathcal{R}^*_\alpha(S^3 \setminus K,\mathrm{SU}(2))|$$

for $\alpha \in \left[0,\frac{1}{2}\right]$ with $\Delta_K(e^{4\pi i\alpha}) \neq 0$.

Proof By Proposition 4.12, $\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2)) \to \mathcal{R}(T^2,\mathrm{SU}(2))$ is an immersion to the smooth part of the pillowcase. By Proposition 4.17 and Lemma 4.20, the immersed image of $\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2))$ intersects transversely to S_α. After taking a small perturbation, the image of $\mathcal{R}^{*,h}(S^3 \setminus K,\mathrm{SU}(2))$ intersects to S_α transversely and the number of intersection points do not change,

$$|\mathcal{R}^*(S^3 \setminus K,\mathrm{SU}(2)) \cap r^{-1}(S_\alpha)| = |\mathcal{R}^{*,h}(S^3 \setminus K,\mathrm{SU}(2)) \cap r^{-1}(S_\alpha)|. \tag{4-13}$$

If $\Delta_K(e^{4\pi i\alpha}) \neq 0$ and the perturbation h is chosen so that it satisfies the conditions in [23, Lemma 5.1] then the signed counting $\#\mathcal{R}^{*,h}_\alpha(S^3 \setminus K, \mathrm{SU}(2)))$ can be defined, and

$$\#\mathcal{R}^{*,h}_\alpha(S^3 \setminus K, \mathrm{SU}(2)) = -\tfrac{1}{2}\sigma_\alpha(K)$$

holds by [23, Corollary 0.2]. On the other hand, the left side of (4-13) is just the size of the set $\mathcal{R}^*_\alpha(S^3 \setminus K, \mathrm{SU}(2))$ by definition. □

Since $K = T_{p,q}$ satisfies the assumption of Lemma 4.21, $|\sigma_\alpha(T_{p,q})| \leq 2|\mathcal{R}^*_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|$ holds for $\alpha \in \left[0, \frac{1}{2}\right]$ with $\Delta_{T_{p,q}}(e^{4\pi i\alpha}) \neq 0$.

Proposition 4.22 *Let p, q and r be positive and relatively coprime integers. The formula*

$$-\tfrac{1}{2}\sigma_{l/(2r)}(T_{p,q}) = |\mathcal{R}^*_{l/(2r)}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|$$

holds for $1 \leq l \leq r-1$ with $\Delta_{T_{p,q}}(e^{2\pi i l/r}) \neq 0$.

For the proof we use the similar argument as in the proof of [2, Theorem 3.4].

Proof Consider a 4–ball B^4 and a torus knot in its boundary $T_{p,q} \subset S^3 = \partial B^4$, and take a Seifert surface S for $T_{p,q}$ as $S \subset B^4$ and $S \cap \partial B^4 = \partial S$. The r–fold cyclic branched covering of B^4 branched along S is the Milnor fiber

$$M(p,q,r) = \{(z_1,z_2,z_3) \mid z_1^p + z_2^q + z_3^r = \epsilon\} \cap B^6 \subset \mathbb{C}^3,$$

where $\epsilon > 0$ is small enough. Furthermore, $\partial M(p,q,r) = \Sigma(p,q,r)$ is an r–fold cyclic branched covering of $\partial B^4 = S^3$, branched along $T_{p,q}$. There is the following formula (see [13, Corollary 2.9]):

$$-\tfrac{1}{4}\sigma(M(p,q,r)) = |\mathcal{R}^*(\Sigma(p,q,r), \mathrm{SU}(2))|.$$

Using the signature formula in [41], Lemma 4.19 and decomposition of $\sigma(M(p,q,r))$ into the equivariant signature $\sigma(M(p,q,r); \frac{i}{r})$, we have

$$-\frac{1}{2}\sum_{l=1}^{r-1}\sigma_{l/(2r)}(T_{p,q}) = \sum_{l=1}^{r-1}|\mathcal{R}^*_{l/(2r)}(S^3 \setminus T_{p,q}, \mathrm{SU}(2))|.$$

Note that $\sigma_{l/(2r)}(T_{p,q}) \leq 0$ since $T_{p,q}$ is a positive knot. If we assume that the inequality in Lemma 4.21 is strict for some l, then $-\frac{1}{4}\sigma(M(p,q,r)) < |\mathcal{R}^*(\Sigma(p,q,r))|$, and this is a contradiction. □

Proof of Theorem 1.9 When $\alpha = 0$ or $\frac{1}{2}$, $\sigma_{T_{p,q}} = 0$ and $\mathcal{R}^*_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ is empty. So we consider the case $\alpha \in \left(0, \frac{1}{2}\right)$ with $\Delta_{T_{p,q}}(e^{4\pi i\alpha}) \neq 0$. Since the image of $\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ in the pillowcase intersects S_α transversely, there is a small $\epsilon > 0$ such that for any $\alpha' \in (\alpha - \epsilon, \alpha + \epsilon)$ we have $\Delta_{T_{p,q}}(e^{4\pi i\alpha}) \neq 0$ and

$$|\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \cap r^{-1}(S_\alpha)| = |\mathcal{R}^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \cap r^{-1}(S_{\alpha'})|.$$

Thus $|\mathcal{R}^*_\alpha(S^3\setminus T_{p,q},\mathrm{SU}(2))| = |\mathcal{R}^*_{\alpha'}(S^3\setminus T_{p,q},\mathrm{SU}(2))|$. The Levine–Tristram signature is piecewise constant and jumps at the roots of the Alexander polynomial. Hence $\sigma_{T_{p,q}}(e^{4\pi i\alpha}) = \sigma_{T_{p,q}}(e^{4\pi i\alpha'})$ if $\epsilon > 0$ is small enough. We can find a positive integer r which is coprime to p and q, and a positive integer l such that $l/(2r) \in (\alpha-\epsilon,\alpha+\epsilon)$. Then

$$-\tfrac{1}{2}\sigma_{l/(2r)}(T_{p,q}) = |\mathcal{R}^*_{l/(2r)}(S^3\setminus T_{p,q},\mathrm{SU}(2))|$$

by Proposition 4.22. Thus we have

$$-\tfrac{1}{2}\sigma_\alpha(T_{p,q}) = |\mathcal{R}^*_\alpha(S^3\setminus T_{p,q},\mathrm{SU}(2))|. \qquad \square$$

5 Properties of instanton knot invariants and their applications

In this section, we give the proof of Theorem 1.1, our main theorem. The important consequence of Section 5.1 is that the Floer chain $C^\alpha_*(T_{p,q};\Delta_{\mathscr{S}})$ is supported only on the odd graded part. The key argument is that the Frøyshov invariant of a knot $K\subset S^3$ for an appropriate choice of coefficient $\mathscr{S}$ reduces to the Levine–Tristram signature. This is a generalization of the corresponding result in [8] and the argument is parallel. Section 5.2 gives the proof of Theorem 1.1 using this specific property of the Floer chain complex $\widetilde{C}^\alpha_*(T_{p,q};\Delta_{\mathscr{S}})$ and the Frøyshov knot invariant.

5.1 The Frøyshov knot invariant and the structure theorem

Let W be a compact oriented smooth 4–manifold with $b^1(W)=b^+(W)=0$, whose boundary $\partial W = Y$ is an integral homology 3–sphere. Let $K\subset Y$ be an oriented knot and $S\subset W$ be an embedded oriented surface with $\partial S = K$. Throughout this subsection, we assume that $\mathscr{S}$ is an integral domain over $\mathscr{R}_\alpha$. We define

$$K(A) := \kappa(A) + \bigl(\alpha - \tfrac{1}{4}\bigr)\nu(A) + \alpha^2 S\cdot S \quad \text{and} \quad d^\alpha(W,S) := 4K(A_{\min}) - g(S) - \tfrac{1}{2}\sigma_\alpha(Y,K) - 1$$

for each holonomy parameter $\alpha\in\bigl(0,\tfrac{1}{2}\bigr)\cap\mathbb{Q}$. Here $A_{\min}$ is a minimal reducible, and note that $K(A_{\min})$ is independent of the choice of minimal reducibles. Moreover $d^\alpha(W,S)$ is an integer by the index theorem. The value of the Frøyshov knot invariant is evaluated by the following proposition:

Proposition 5.1 *Let (W,S) and (Y,K) be as above and $\alpha\in\bigl(0,\tfrac{1}{2}\bigr)\cap\mathbb{Q}$ satisfy $\Delta_{(Y,K)}(e^{4\pi i\alpha})\neq 0$. If $d := d^\alpha(W,S)\geq 0$ then there is a cycle $c^\alpha(W,S)\in C^\alpha_{2d+1}(Y,K;\Delta_{\mathscr{S}})$ satisfying*

$$\delta_1 v^j(c^\alpha(W,S)) = \begin{cases} 0 & \text{if } 0\leq j<d,\\ \eta^\alpha(W,S) & \text{if } j=d.\end{cases}$$

Proof We define $c^\alpha(W,S)\in C^\alpha_{2d+1}(Y,K;\Delta_{\mathscr{S}})$ by

$$\langle c^\alpha(W,S),\beta\rangle = \sum_{[A]\in M(W,S;\beta)_0} \epsilon(A)\lambda^{\kappa_0-\kappa(A)}T^{\nu(A)-\nu_0},$$

where $M(W,S,\beta)_0$ is a zero-dimensional moduli space. Since $dc^\alpha(W,S)$ corresponds to the counting of the boundary of the 1–dimensional moduli space $M^\alpha(W,S;\beta')^+_1$, we have $dc^\alpha(W,S)=0$. Let $M(W,S,\theta_\alpha)_{2d+1}$ be the moduli space of instantons A over (W,S) which are asymptotic to θ_α and satisfy $\kappa(A)=\kappa(A_{\min})$. If $d\geq 0$ then we can perturb the ASD equation so that each reducible connection in $M_z(W,S;\theta_\alpha)_{2d+1}$ has a neighborhood which is homeomorphic to the cone of $\mathbb{CP}^d$. Removing small $(2d+1)$–balls of each reducible point from $M_z(W,S;\theta_\alpha)_{2d+1}$, we get a $(2d+1)$–manifold M_z' whose boundary is $\bigsqcup(\pm\mathbb{CP}^d)$, where the sign $\pm$ is determined by the orientation of each reducible point. Cutting down M_z' by codimension 2 divisors $\{V_i\}_{1\leq i\leq d}$ associated to d points in S, $M_z'\cap V_1\cap\cdots\cap V_d$ is a 1–manifold with boundary. Then $\sum_z \#\partial(M_z'\cap V_1\cap\cdots\cap V_d)\lambda^{\kappa_0-\kappa(z)}T^{\nu(z)-\nu_0}=\eta^\alpha(W,S)$. On the other hand, $M_z'\cap V_1\cap\cdots\cap V_d$ has ends arising from the sliding end of instantons. Define $\psi\in C_1(Y,K;\Delta_{\mathscr{S}})$ by

$$\langle\beta,\psi\rangle=\sum_z\sum_{[A]}\epsilon([A])\lambda^{\kappa_0-\kappa(A)}T^{\nu(A)-\nu_0},$$

where $[A]$ runs through all elements in $\#(M_z(W,S;\beta)_{2d}\cap V_1\cap\cdots\cap V_d)$ for each z. Then ψ and $v^dc^\alpha(W,S)$ are homologous. Since $\delta_1\psi=\eta^\alpha(W,S)$, we have $\delta_1 v^d c^\alpha(W,S)=\eta^\alpha(W,S)$. If $j<d$, $M_z(W,S;\theta_\alpha)_{2j+1}$ does not contain reducible points and we have $\delta_1 v^j c^\alpha(W,S)=0$. □

Before the proof of Theorem 1.8, we state Lemma 5.2 and Proposition 5.3 related to two-bridge torus knots.

Lemma 5.2 *For any $\alpha\in\left(0,\frac{1}{2}\right)$ there exists an integer $k>0$ such that $\sigma_\alpha(T_{2,2k+1})=-2$ and $\Delta_{T_{2,2k+1}}(e^{4\pi i\alpha})\neq 0$.*

Proof Consider the case $\alpha\leq\frac{1}{4}$. By [34, Proposition 1], $\sigma_\alpha(T_{2,2k+1})$ is given by

$$\sigma_\alpha(T_{2,2k+1})=n_1-n_2,$$

where n_1 is the number of lattice points $\left\{(1,m)\mid\left(k+\frac{1}{2}\right)(1+4\alpha)<m<2k+1\right\}$ and n_2 is the number of lattice points $\left\{(1,m)\mid 0<m<\left(k+\frac{1}{2}\right)(1+4\alpha)\right\}$. Thus $\sigma_\alpha(T_{2,2k+1})=-2$ if only if $1/(8k+4)<\alpha<3/(8k+4)$. Moreover, note that the interval $(1/(8k+4),3/(8k+4))$ does not contain any root of $\Delta_{T_{2,2k+1}}(t)$. Thus, for any $\alpha\leq\frac{1}{4}$, we can find $k>0$ such that $\sigma_\alpha(T_{2,2k+1})=-2$ and $\Delta_{T_{2,2k+1}}(e^{4\pi i\alpha})\neq 0$. For the case $\alpha>\frac{1}{4}$, it follows that $\sigma_\alpha(T_{2,2k+1})=-2$ if only if $\frac{1}{2}-3/(8k+4)<\alpha<\frac{1}{2}-1/(8k+4)$ by the flip symmetry. □

Proposition 5.3 *For any $\alpha\in\mathbb{Q}\cap\left(0,\frac{1}{2}\right)$, there is an integer $k>0$ such that $\Delta_{T_{2,2k+1}}(e^{4\pi i\alpha})\neq 0$ and $h^\alpha_{\mathscr{S}}(T_{2,2k+1})=1$.*

Proof By Lemma 5.2, we can find an integer $k>0$ such that $\sigma_\alpha(T_{2,2k+1})=-2$ and $\Delta_{T_{2,2k+1}}(e^{4\pi i\alpha})\neq 0$. Consider a cobordism of pairs (W_k,S_k) obtained by the composition

$$(W_k,S_k)\colon(S^3,U)\to(S^3,T_{2,3})\to\cdots\to(S^3,T_{2,2k-1})\to(S^3,T_{2,2k+1}),$$

where $(S^3,T_{2,2i-1})\to(S^3,T_{2,2i+1})$ is obtained by the crossing change of the knot. Put $(\overline{W}_k,\overline{S}_k):=(D^4,D^2)\cup_{(S^3,U)}(W_k,S_k)$. Then it is easy to see that $b^1(\overline{W}_k)=b^+(\overline{W}_k)=0$ and $d^\alpha(\overline{W}_k,\overline{S}_k)=0$ by the similar argument as in Proposition 2.23. Applying Proposition 5.1 to the pair $(\overline{W}_k,\overline{S}_k)$, we obtain a cycle $c^\alpha(\overline{W}_k,\overline{S}_k)\in C_1^\alpha(T_{2,2k+1})$ such that $\delta_1 c^\alpha(\overline{W}_k,\overline{S}_k)\neq 0$. This implies that $h^\alpha_{\mathscr{S}}(T_{2,2k+1})\neq 0$. Since $\operatorname{rank} C_*^\alpha(T_{2,2n+1})=1$, we have $h^\alpha_{\mathscr{S}}(T_{2,2k+1})=1$. □

Proof of Theorem 1.8 Consider a knot $K \subset S^3$ and a holonomy parameter $\alpha \in \mathbb{Q} \cap \left(0, \frac{1}{2}\right)$ with $\Delta_K(e^{4\pi i\alpha}) \neq 0$. Since $K \subset S^3$ is homotopic to any knot, it can be deformed into $lT_{2,2n+1}$ by positive and negative crossing changes, where $l = -\frac{1}{2}\sigma_\alpha(K)$. This operation defines a cobordism of pairs $([0,1] \times S^3, S) \colon (S^3, K) \to (S^3, lT_{2,2n+1})$ where S is an immersed surface with normal self-intersection points. Let $S' \colon lT_{2,2n+1} \to K$ be the inverse cobordism of S. Since $\sigma_\alpha(K) = \sigma_\alpha(lT_{2,2n+1})$, two cobordisms S and S' induce negative definite cobordisms. Let $\widetilde{m}_S \colon \widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}}) \to \widetilde{C}^\alpha_*(lT_{2,2n+1}; \Delta_{\mathscr{S}})$ and $\widetilde{m}_{S'} \colon \widetilde{C}^\alpha_*(lT_{2,2n+1}; \Delta_{\mathscr{S}}) \to \widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}})$ be induced cobordism maps on $\mathcal{S}$–complexes. Since two immersed cobordisms $S' \circ S$ and $S \circ S'$ can be deformed into product cobordisms by finitely many finger moves, their induced maps $\widetilde{m}_{S' \circ S}$ and $\widetilde{m}_{S \circ S'}$ are S–chain homotopic to the identity up to the multiplication of unit elements by Proposition 3.28. By the functoriality of $\mathcal{S}$–morphisms, $\widetilde{m}_{S'} \circ \widetilde{m}_S$ and $\widetilde{m}_S \circ \widetilde{m}_S$ are $\mathcal{S}$–chain homotopic to the identity up to the multiplication of unit elements. The proof is completed by Remark 3.4. □

The proof of Theorem 1.7 immediately follows from Theorem 1.8:

Proof of Theorem 1.7 Comparing Frøyshov invariants for both sides of

$$\widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}}) \simeq C^\alpha_*(lT_{2,2k+1}; \Delta_{\mathscr{S}}),$$

we obtain $h^\alpha_{\mathscr{S}}(K) = l h^\alpha_{\mathscr{S}}(T_{2,2k+1})$ where $l = -\frac{1}{2}\sigma(K)$. Since $h^\alpha_{\mathscr{S}}(T_{2,2k+1}) = 1$ by Proposition 5.3, we obtain the desired formula. □

Remark 5.4 Since $\mathcal{S}$–chain homotopy equivalence of two $\mathcal{S}$–complexes $\widetilde{C}_* \simeq \widetilde{C}'_*$ implies chain homotopy equivalence between C_* and C'_*, assume that $\sigma_\alpha(K) \leq 0$. Then Theorems 1.5 and 1.8 imply the $\mathcal{S}$–chain homotopy equivalence $\widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}}) \simeq \widetilde{C}^\alpha_*(T_{2,2n+1}; \Delta_{\mathscr{S}})^{\otimes l}$, and hence we have the Euler characteristic formula

$$\chi(C^\alpha_*(K; \Delta_{\mathscr{S}})) = l\chi(C^\alpha_*(T_{2,2n+1}; \Delta_{\mathscr{S}})).$$

If $\sigma_\alpha(K) > 0$ then there is an $\mathcal{S}$–chain homotopy equivalence $\widetilde{C}^\alpha_*(K; \Delta_{\mathscr{S}}) \simeq \widetilde{C}^\alpha_*(-T_{2,2n+1}; \Delta_{\mathscr{S}})^{\otimes -l}$ and we have

$$\chi(C^\alpha_*(K; \Delta_{\mathscr{S}})) = -l\chi(C^\alpha_*(-T_{2,2n+1}; \Delta_{\mathscr{S}})).$$

By Proposition 5.3, $\chi(C^\alpha_*(T_{2,2n+1}; \Delta_{\mathscr{S}})) = -1$. On the other hand, $\chi(C^\alpha_*(-T_{2,2n+1}; \Delta_{\mathscr{S}})) = 1$ since if we reverse the orientation of the 3–manifold, the $\mathbb{Z}/4$–grading of the chain complex changes so that $\mathrm{gr}_{-Y}(\beta) \equiv 3 - \mathrm{gr}_Y(\beta)$, which follows from (2-4). In any case,

$$\chi(C^\alpha_*(K; \Delta_{\mathscr{S}})) = \tfrac{1}{2}\sigma_\alpha(K).$$

Note that this formula for the Euler characteristic is independent of the choice of the coefficient $\mathscr{S}$.

Proof of Theorem 1.10 Consider an arbitrary knot $K \subset S^3$. For any holonomy parameter $\alpha \in \left(0, \frac{1}{2}\right) \cap \mathbb{Q}$ with $\Delta_K(e^{4\pi i\alpha}) \neq 0$, the Floer chain complex $C^\alpha_*(K; \Delta_{\mathscr{S}})$ is defined and the relation $h^\alpha_{\mathscr{S}}(K) = -\frac{1}{2}\sigma_\alpha(K)$ holds. By the definition of the Frøyshov knot invariant, we have lower bounds of Floer homology groups

$$\operatorname{rank} I^\alpha_1(K; \Delta_{\mathscr{S}}) \geq \left\lceil -\tfrac{1}{4}\sigma_\alpha(K) \right\rceil \quad \text{and} \quad \operatorname{rank} I^\alpha_3(K; \Delta_{\mathscr{S}}) \geq \left\lfloor -\tfrac{1}{4}\sigma_\alpha(K) \right\rfloor$$

for any knot $K \subset S^3$ with $\sigma_\alpha(K) \leq 0$. In particular, $K = T_{p,q}$ satisfies this condition. Using the equality $\operatorname{rank} I_*(T_{p,q}) = -\frac{1}{2}\sigma_\alpha(T_{p,q})$, we obtain

$$\operatorname{rank} I_1^\alpha(T_{p,q};\Delta_{\mathscr{S}}) = \left\lceil -\tfrac{1}{4}\sigma_\alpha(T_{p,q})\right\rceil, \operatorname{rank} I_3^\alpha(T_{p,q};\Delta_{\mathscr{S}}) = \left\lfloor -\tfrac{1}{4}\sigma_\alpha(T_{p,q})\right\rfloor.$$

Since $I_*^\alpha(T_{p,q})$ is supported only on the odd graded part, we obtain the statement. □

5.2 An application to knot concordance

In this subsection, we complete the proof of our main theorem (Theorem 1.1).

The operators $Z^{\pm 1}$ and $U^{\pm 1}$ extend to the $\mathcal{S}$–complex $\widetilde{C}_*$ in the obvious way. We also introduce the operator

$$\mathcal{W}_{i,j,k} := \delta_1 v^i U^k Z^j : \widetilde{C}_* \to \widetilde{C}_*.$$

If $\deg_{\mathbb{R}}(Z\gamma) > \deg_{\mathbb{R}}(\gamma)$, the operator Z does not act on the filtered chain complex $C_*^{[-\infty,R]}$, and $\mathcal{W}_{i,j}$ does not directly induce a map on $\widetilde{C}_*^{[-\infty,R]}$. For this reason, we introduce the map $\mathcal{V}_{i,j,k}^{[-\infty,R]}$ on the filtered chain complex by the composition

$$\widetilde{C}_*^{[-\infty,R]} \hookrightarrow \widetilde{C}_*^{[-\infty,\infty]} \xrightarrow{\mathcal{W}_{i,j,k}} \widetilde{C}_*^{[-\infty,\infty]}.$$

We also introduce the operator $\mathcal{W}_{i,j,k}^{[R',R]}$ on the quotient filtered $\mathcal{S}$–complex $\widetilde{C}_*^{[R',R]}$ by the composition

$$\widetilde{C}_*^{[R',R]} \hookrightarrow \widetilde{C}_*^{[-\infty,\infty]} \xrightarrow{\mathcal{W}_{i,j,k}} \widetilde{C}_*^{[-\infty,\infty]} \twoheadrightarrow \widetilde{C}_*^{[R',\infty]}.$$

Here, the last map is a natural quotient map.

Proposition 5.5 *Let $S : T_{p,q} \to T_{p,q}$ be a given self-concordance. Then there is a dense subset $\mathcal{I} \subset \left(0, \frac{1}{2}\right)$ such that all elements in $\mathcal{R}_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$ extend to elements in $\mathcal{R}_\alpha((S^3 \times [0,1]) \setminus S, \mathrm{SU}(2))$ for any $\alpha \in \mathcal{I}$.*

Proof We choose a dense subset $\mathcal{I} \subset \left(0, \frac{1}{2}\right)$ such that Theorem 1.10 holds for $T_{p,q}$. Since all irreducible critical points of the Chern–Simons functional of $T_{p,q}$ are nondegenerate by Proposition 4.17, we can choose a perturbation π so that it is supported away from flat connections. In particular, we can assume that the chain complex $C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}})$ is generated by $\mathcal{R}_\alpha^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$. Since the assertion for $\alpha = \frac{1}{4}$ is proved in [8], we assume that $\alpha \neq \frac{1}{4}$. In particular, we consider the case $\alpha < \frac{1}{4}$ for a while. Since the unique flat reducible θ_α with the holonomy parameter α on $S^3 \setminus T_{p,q}$ always extends to the concordance complement, it is enough to consider the extension problem for irreducibles. We choose a field $\mathscr{S} := \mathscr{R}_\alpha \otimes \mathbb{Q}$. By Theorems 1.7 and 1.10 we have

$$h_{\mathscr{S}}^\alpha(T_{p,q}) = -\tfrac{1}{2}\sigma_\alpha(T_{p,q}) = d,$$

where $d := \operatorname{rank} C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}})$. This implies that there is a cycle $\beta_0 \in C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}})$ such that $\delta_1 v^k(\beta_0) = 0$ if $k < d-1$ and $\delta_1 v^{d-1}(\beta_0) \neq 0$. Put $\beta_i := v^i(\beta_0)$ for $0 \leq i \leq d-1$. The chain complex $C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}_\alpha})$ admits a $(\mathbb{Z}\times\mathbb{R})$–bigrading by fixing lifts $\tilde\rho_1, \dots, \tilde\rho_d$ of singular flat connections $\rho_1, \dots, \rho_d \in \mathcal{R}_\alpha^*(S^3 \setminus T_{p,q}, \mathrm{SU}(2))$. In particular, we may assume that $\deg_{\mathbb{Z}}(\tilde\rho_i) = 1$ or 3 by Theorem 1.10.

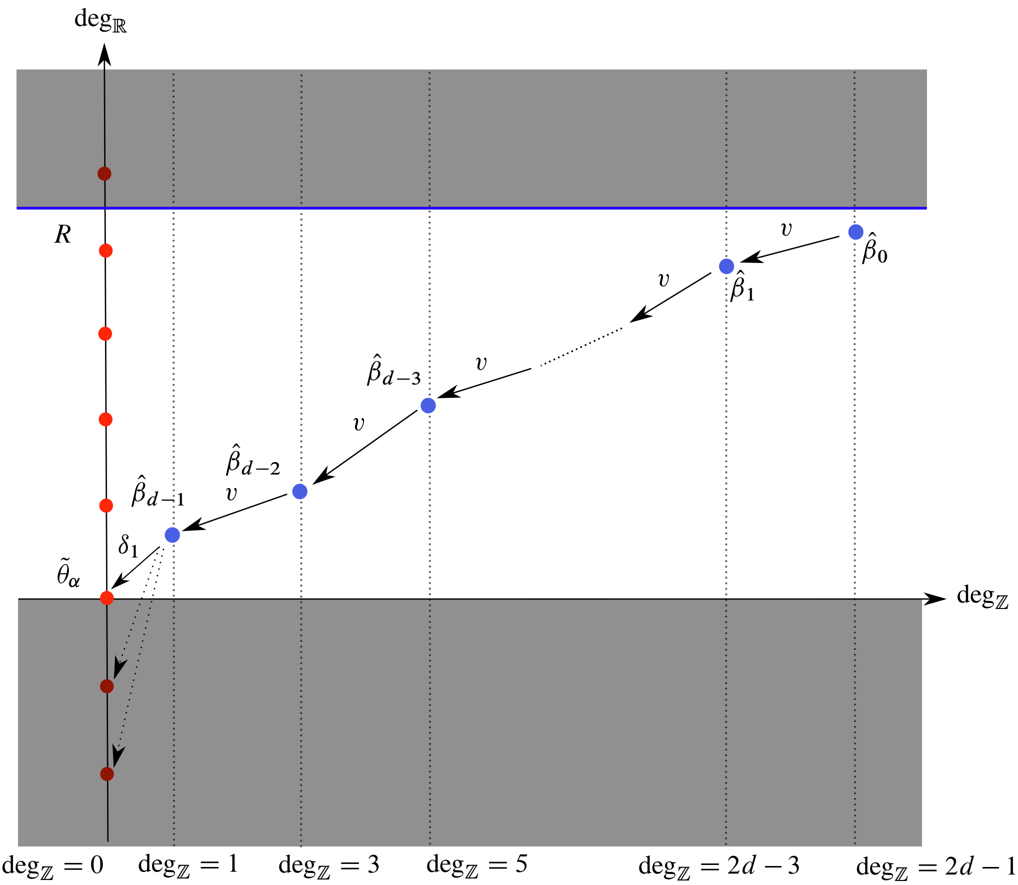

Figure 3: Elements $\hat{\beta}_0, \dots, \hat{\beta}_{d-1}$ and their $(\mathbb{Z}\times\mathbb{R})$–gradings.

Using properties of elements $\beta_0, \dots, \beta_{d-1}$, we fix elements $\hat{\beta}_0, \dots, \hat{\beta}_{d-1} \in C_*^\alpha(T_{p,q}; \Delta_{\mathscr{S}_\alpha})^{[-\infty,\infty]}$ in the following way. Firstly, there exists a cycle $\hat{\beta}_{d-1}$ such that $\deg_{\mathbb{Z}}(\hat{\beta}_{d-1}) = 1$ and satisfying

$$\delta_1(\hat{\beta}_{d-1}) = \sum_{k \le 0} c_k Z^k \tilde{\theta}_\alpha$$

with $c_0 \neq 0$ since $\delta_1(\beta_{d-1}) \neq 0$. Next, choose an element $\hat{\beta}_i$. Then $\hat{\beta}_{i-1}$ is defined as a cycle satisfying

$$v(\hat{\beta}_{i-1}) = \hat{\beta}_i.$$

Finally, we obtain cycles $\hat{\beta}_0, \dots, \hat{\beta}_{d-1}$ by induction.

Note that $\deg_{\mathbb{Z}}(\hat{\beta}_i) = 2(d-1) - 2i$ and $0 \le \deg_{\mathbb{R}}(\hat{\beta}_{i-1}) \le \dots \le \deg_{\mathbb{R}}(\hat{\beta}_0)$. We fix $R > 0$ so that it satisfies $\deg_{\mathbb{R}}(\hat{\beta}_0) < R$ and $R \notin \mathscr{C}^*$; see Figure 3.

Let $-\epsilon < 0$ be a small negative number such that an interval $[-\epsilon, 0)$ does not contain any critical value of the Chern–Simons functional CS. Our aim is to show that the cobordism map m_S on the quotient filtered chain $\widetilde{C}_*^\alpha(T_{p,q}; \Delta_{\mathscr{S}})^{[-\epsilon,R]}$ is an isomorphism.

Since $\widetilde{m}_S$ preserves the $\mathbb{Z}$–grading, it is enough to show that $\widetilde{m}_S$ is an isomorphism on $C_1^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$ and $C_3^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$. We claim that $C_1^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$ and $C_3^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$ are generated by elements of the form

$$\{Z^j U^{2k+1-(d-1)}\hat{\beta}_{2k+1} \mid k, m_{2k+1} \le j \le n_{2k+1}\} \quad \text{or} \quad \{Z^j U^{2k-(d-1)}\hat{\beta}_{2k} \mid k, m_{2k} \le j \le n_{2k}\}$$

over $\mathbb{Q}$. To see this, consider the linear combination

$$\sum_{0\le i\le d-1}\sum_{m_j\le j\le n_i} c_{i,j} Z^j U^{i-(d-1)}\hat{\beta}_i = 0, \tag{5-1}$$

where $c_{i,j}$ are rational coefficients. Then we consider applying operators $\mathcal{W}_{i,j,k}^{[-\epsilon,R]}$ to (5-1). Firstly, we apply the operator for $(i,j,k)=(0,-n_{d-1},0)$. Then we get $c_{0,n_{d-1}}\delta_1(\hat{\beta}_{d-1})=0$ with $\delta_1(\hat{\beta}_{d-1})$ nonzero. Since $\mathscr{S}$ is an integral domain, $c_{d-1,n_{d-1}}=0$. Next, we apply the operator $\mathcal{W}_{(i,j,k)}^{[-\epsilon,R]}$ for $(i,j,k)=(0,-n_{d-1}+1,0)$ to (5-1). Then we obtain $c_{d-1,n_{d-1}-1}$ using $c_{d-1,n_{d-1}}=0$. Inductively, we obtain

$$c_{d-1,n_{d-1}}=\cdots=c_{d-1,m_{d-1}}=0$$

by applying operators $\mathcal{W}_{0,-n_{d-1},0}^{[-\epsilon,R]},\dots,\mathcal{W}_{0,-m_{d-1},0}^{[-\epsilon,R]}$. We repeat a similar arguments using the operators $\{\mathcal{W}_{1,j,1}^{[-\epsilon,R]}\}_{m_{d-2}\le j\le n_{d-2}}$, and obtain

$$c_{d-2,n_{d-2}}=\cdots=c_{d-2,m_{d-2}}=0.$$

Inductively, we conclude that

$$c_{i,n_i}=\cdots=c_{i,m_i}=0$$

for all $0\le i\le d-1$. So $\{Z^jU^{i-(d-1)}\hat{\beta}_i\}$ for $0\le i\le d-1$ and $m_i\le j\le n_i$ are linearly independent. Put $\hat{\beta}_i' := m_S(\hat{\beta}_i)$. Since the induced cobordism map $\widetilde{m}_S$ on an $\mathcal{S}$–complex satisfies the relations in Proposition 3.15, the elements $\hat{\beta}_0,\dots,\hat{\beta}_{d-1}$ have the same properties, and the same technique shows that $\{Z^jU^{i-(d-1)}\hat{\beta}_i'\}$ for $0\le i\le d-1$ and $m_i\le j\le n_i$ are linearly independent. Moreover, $\deg_{\mathbb{R}}(\hat{\beta}_i)=\deg_{\mathbb{R}}(\hat{\beta}_i')$ by the construction of elements $\{\hat{\beta}_i\}$. We conclude that the map m_S is an isomorphism on $C_1^\alpha(T_{p,q};\Delta_{\mathscr{S}_\alpha})^{[-\epsilon,R]}$ and $C_3^\alpha(T_{p,q};\Delta_{\mathscr{S}_\alpha})^{[-\epsilon,R]}$.

Note that the chain complex $C_*^\alpha(T_{p,q};\Delta_{\mathscr{S}})$ is generated by those irreducible singular flat connections. Then the degree 1 part $C_1^\alpha(T_{p,q};\Delta_{\mathscr{S}})^{[-\epsilon,R]}$ of the quotient filtered chain complex is generated by elements of the forms $\{Z^j\tilde{\rho}_1\}_{m_1\le j\le n_1},\dots,\{Z^j\tilde{\rho}_l\}_{m_l\le j\le n_l}$ over $\mathbb{Q}$. We order these generators by values of the Chern–Simons functional. Then the cobordism map m_S can be represented by the form

$$\begin{bmatrix} L_1 & O & \cdots & & \\ & L_2 & O & \cdots & \\ & & \ddots & O & \cdots \\ & & & \ddots & O \\ & & & & L_k \end{bmatrix}, \tag{5-2}$$

where diagonal blocks L_i are components that correspond to the basis with the same value of the Chern–Simons functional. Note that components in L_i are defined by counting (perturbed) flat connections over the concordance complement. Since m_S is an isomorphism on the degree 1 part, the matrix (5-2) is invertible over $\mathbb{Q}$. Hence each diagonal block L_i is also invertible. In particular, they do not contain any zero-column. Since the regular condition on moduli space is an open condition with respect to choices of perturbation, all flat connections $\rho_1, \dots, \rho_l$ extend to flat connections over the concordance complement. The similar argument works for $\rho_{l+1}, \dots, \rho_d$, and thus all elements in $\mathcal{R}^*_\alpha(S^3 \setminus T_{p,q}; \Delta_{\mathscr{S}})$ extend to the concordance complement.

Finally, we consider the case $\alpha > \frac{1}{4}$. Here we only change the above argument at the following point: We apply the operator $\mathcal{W}^{[-\epsilon,R]}_{0,-m_{d-1}}$ on (5-1) the first time. Then we obtain $c_{d-1,m_{d-1}} = 0$. Next we apply the operator $\mathcal{W}^{[-\epsilon,R]}_{0,-m_{d-1}+1,0}$ and obtain $c_{d-1,m_{d-1}-1} = 0$. We inductively obtain $c_{d-1,n_{d-1}} = \cdots = c_{d-1,m_{d-1}} = 0$. The rest of the argument proceeds similarly, and finally all coefficients in (5-1) vanish. □

Proof of Theorem 1.1 Let $S\colon T_{p,q} \to K$ be a given concordance. Then we can construct a concordance $\bar{S} \circ S\colon T_{p,q} \to K \to T_{p,q}$ by the composition, where $\bar{S}$ is the opposite concordance of S. By Proposition 5.5 there exists a dense subset $\mathcal{I} \subset \left[0, \frac{1}{2}\right]$ such that there is a extension $\mathcal{R}_\alpha(S^3 \setminus T_{p,q}, \mathrm{SU}(2)) \to \mathcal{R}_\alpha((S^3 \times [0,1]) \setminus \bar{S} \circ S, \mathrm{SU}(2))$ for any $\alpha \in \mathcal{I}$. Let $\alpha \in \left[0, \frac{1}{2}\right]$ be any holonomy parameter and consider the representation $\rho\colon \pi_1(S^3 \setminus T_{p,q}) \to \mathrm{SU}(2)$ with

$$\rho(\mu_{T_{p,q}}) \sim \begin{bmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{bmatrix}.$$

Then we can choose a sequence $\{\alpha_i\} \subset \mathcal{I}$ such that $\lim_{i\to\infty} \alpha_i = \alpha$ and SU(2) representations ρ_i of $\pi_1(S^3 \setminus T_{p,q})$ with

$$\rho_i(\mu_{T_{p,q}}) \sim \begin{bmatrix} e^{2\pi i\alpha_i} & 0 \\ 0 & e^{-2\pi i\alpha_i} \end{bmatrix},$$

since ρ_i extends to an SU(2) representation $\Phi_i : \pi_1((S^3 \times [0,1]) \setminus \bar{S} \circ S) \to \mathrm{SU}(2)$ and we can choose a convergent subsequence of $\{\Phi_i\}$ with the limiting representation $\Phi_\infty\colon \pi_1((S^3 \times [0,1]) \setminus \bar{S} \circ S) \to \mathrm{SU}(2)$. (Since SU(2) is compact, we can choose a convergent subsequence $\{\Phi_i(x_j)\}_i$ for each generator x_j of $\pi_1((S^3 \times [0,1]) \setminus \bar{S} \circ S)$, and $\lim_{i\to\infty} \Phi_i(x_j)$ defines a limiting representation Φ_∞.) By restriction, we get a representation $\pi_1((S^3 \times [0,1]) \setminus S) \to \mathrm{SU}(2)$ which is the extension of ρ. □

Appendix The connected sum theorem

In this section, we give the proof of the connected sum theorem. The connected sum theorem for nonsingular settings was proved in [20], and the singular setting with $\alpha = \frac{1}{4}$ was proved in [9]. We use an argument similar to [9] to prove our connected sum theorem (Theorem 3.24). Let us recall the settings which are introduced in [9, Section 6]. Let (Y, K) and (Y', K') be two given knots in integral homology 3–spheres. Fixing basepoints $p \in K$ and $p' \in K'$, we take a pair of the connected sum $(Y \# Y', K \# K')$ at these basepoints. We also fix a basepoint $p^\# \in K \# K'$. Construct a cobordism

$$(W, S)\colon (Y \sqcup Y', K \sqcup K') \to (Y \# Y', K \# K')$$

by attaching a pair of 1–handles $(D^1 \times D^3, D^1 \times D^1)$ to the product cobordism $[0,1] \times (Y \sqcup Y', K \sqcup K')$. Let

$$(W', S') \colon (Y \# Y', K \# K') \to (Y \sqcup Y', K \sqcup K')$$

be a cobordism of the opposite direction. We define three oriented piecewise smooth paths γ, γ' and $\gamma^\#$on $S \subset W$. Assume that these paths intersect the boundaries of the cobordism only at their edge points. The path γ starts from $p \in Y$ and ends at the basepoint $p^\# \in Y \# Y'$. Similarly, γ' starts from $p' \in Y'$ and ends at $p^\#$, and $\gamma^\#$ starts from $p \in Y$ and ends at $p' \in Y'$. Let us define the paths σ, σ' and $\sigma^\#$ in S' as mirrors of γ, γ' and $\gamma^\#$, respectively. We use the notation β, β' and $\beta^\#$ (and their indexed versions) for critical points of the perturbed Chern–Simons functional on (Y, K), (Y', K') and $(Y \# Y', K \# K')$, respectively. Let θ_α, θ'_α and $\theta^\#_\alpha$ denote unique flat reducibles on (Y, K), (Y', K') and $(Y \# Y', K \# K')$, respectively. We use the reduced notation for d–dimensional moduli spaces as follows:

$$M_z(\beta, \beta'; \beta^\#)_d := M_z(W, S; \beta, \beta', \beta^\#)_d \quad \text{and} \quad M_z(\beta^\#; \beta, \beta')_d := M_z(W', S'; \beta^\#, \beta, \beta')_d.$$

We drop z from the notation above if we consider all unions of z. We define maps

$$H^\gamma \colon \mathcal{B}(W, S; \beta, \beta', \beta^\#) \to S^1 \quad \text{and} \quad H^{\gamma'} \colon \mathcal{B}(W, S; \beta, \beta', \beta^\#) \to S^1$$

as in Section 3.4. The moduli spaces cut down by these maps are defined by

$$\begin{aligned} M_{\gamma,z}(\beta, \beta'; \beta^\#)_d &:= \{[A] \in M_z(\beta, \beta'; \beta^\#)_{d+1} \mid H^\gamma([A]) = s\}, \\ M_{\gamma',z}(\beta, \beta'; \beta^\#)_d &:= \{[A] \in M_z(\beta, \beta'; \beta^\#)_{d+1} \mid H^{\gamma'}([A]) = s'\}, \\ M_{\gamma\gamma',z}(\beta, \beta'; \beta^\#)_d &:= \{[A] \in M_z(\beta, \beta'; \beta^\#)_{d+1} \mid H^\gamma([A]) = s,\ H^{\gamma'}([A]) = s'\}, \end{aligned}$$

where $s \in S^1$ is a generic point. The orientation of moduli spaces over (W, S) is defined in the following way. Let $o_W \in \mathcal{O}[W, S; \theta_{\alpha+}, \theta'_{\alpha+}, \theta^\#_{\alpha-}]$ be the canonical homology orientation of (W, S), and $o_\beta \in \mathcal{O}[\beta]$, $o_{\beta'} \in \mathcal{O}[\beta']$ and $o_{\beta^\#} \in \mathcal{O}[\beta^\#]$ be given orientations for generators. Then $o_{\beta,\beta';\beta^\#} \in \mathcal{O}[W, S; \beta, \beta'; \beta^\#]$ is fixed so that the relation

$$\Phi(o_\beta \otimes o_{\beta'} \otimes o_W) = \Phi(o_{\beta,\beta';\beta^\#} \otimes o_{\beta^\#})$$

holds.

The argument of the proof consists of the following steps:

(I) A cobordism of pairs $(W, S) \colon (Y \sqcup Y', K \sqcup K') \to (Y \# Y', K \# K')$ induces an $\mathcal{S}$–morphism

$$\widetilde{m}_{(W,S)} \colon \widetilde{C}^\alpha_*(Y, K) \otimes \widetilde{C}^\alpha_*(Y', K') \to \widetilde{C}^\alpha_*(Y \# Y', K \# K').$$

(II) A cobordism of pairs $(W', S') \colon (Y \# Y', K \# K') \to (Y \sqcup Y', K \sqcup K')$ induces an $\mathcal{S}$–morphism

$$\widetilde{m}_{(W',S')} \colon \widetilde{C}^\alpha_*(Y \# Y', K \# K') \to \widetilde{C}^\alpha_*(Y, K) \otimes \widetilde{C}^\alpha_*(Y', K').$$

(III) Put $\widetilde{C}^\# := \widetilde{C}^\alpha_*(Y \# Y', K \# K')$. The composition $\widetilde{m}_{(W,S)} \circ \widetilde{m}_{(W',S')}$ is $\mathcal{S}$–chain homotopic to $\mathrm{id}_{\widetilde{C}^\#}$ up to the multiplication of a unit element in $\mathscr{S}$.

(IV) The composition $\widetilde{m}_{(W',S')} \circ \widetilde{m}_{(W,S)}$ is $\mathcal{S}$–chain homotopic to $\mathrm{id}_{\widetilde{C}^\otimes}$.

A.1 Step I

We define a map $\widetilde{m}_{(W,S)}$ as follows. Using the decomposition of the Floer chain group

$$C^{\otimes} = (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'_*,$$

we define four maps:

$$m = [m_1, m_2, m_3, m_4]\colon (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'_* \to C^{\#}_*,$$
$$\mu = [\mu_1, \mu_2, \mu_3, \mu_4]\colon (C \otimes C')_* \oplus (C \otimes C')_{*-1} \oplus C_* \oplus C'_* \to C^{\#}_*,$$
$$\Delta_1 = [\Delta_{1,1}, \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}]\colon (C \otimes C')_0 \oplus (C \otimes C')_{-1} \oplus C_0 \oplus C'_0 \to \mathscr{S},$$
$$\Delta_2\colon \mathscr{S} \to C^{\#}_{-1}.$$

Each component of the above maps is defined as follows:

$$\langle m_1(\beta \otimes \beta'), \beta^{\#} \rangle = \sum_z \# M_{\gamma^{\#},z}(\beta, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle m_2(\beta \otimes \beta'), \beta^{\#} \rangle = \sum_z \# M_z(\beta, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle m_3(\beta), \beta^{\#} \rangle = \sum_z \# M_z(\beta, \theta'_\alpha; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \quad \langle m_4(\beta'), \beta^{\#} \rangle = \sum_z \# M_z(\theta_\alpha, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu_1(\beta \otimes \beta'), \beta^{\#} \rangle = \sum_z \# M_{\gamma\gamma',z}(\beta, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu_2(\beta \otimes \beta'), \beta^{\#} \rangle = \sum_z \# M_{\gamma,z}(\beta, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu_3(\beta), \beta^{\#} \rangle = \sum_z \# M_{\gamma,z}(\beta, \theta'_\alpha; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu_4(\beta'), \beta^{\#} \rangle = \sum_z \# M_{\gamma',z}(\theta_\alpha, \beta'; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\Delta_{1,1}(\beta \otimes \beta') = \sum_z \# M_{\gamma^{\#},z}(\beta, \beta'; \theta^{\#}_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \quad \Delta_{1,2}(\beta \otimes \beta') = \sum_z \# M_z(\beta, \beta'; \theta^{\#}_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\Delta_{1,3}(\beta) = \sum_z \# M_z(\beta, \theta'_\alpha; \theta^{\#}_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \qquad \Delta_{1,4}(\beta') = \sum_z \# M_z(\theta'_\alpha, \beta'; \theta^{\#}_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \Delta_2(1), \beta^{\#} \rangle = \sum_z \# M_z(\theta_\alpha, \theta'_\alpha; \beta^{\#})_0 \lambda^{-\kappa(z)} T^{\nu(z)}.$$

As described in [9, Remarks 6.10 and 6.11], notice that

- $H^{-1}_{\beta\beta_1}(s) \cap H^{-1}_{\beta\beta'}(s') \cap M(\beta, \beta_1)_2 = \varnothing$ for distinct regular values $s, s' \in S^1$,
- $\# M_\gamma(\beta, \beta'; \beta^{\#}) - \# M_{\gamma'}(\beta, \beta'; \beta^{\#}) = \# M_{\gamma^{\#}}(\beta, \beta'; \beta^{\#})$.

Proposition A.1 *There are the following relations*:

(A-1) $$d^{\#} \circ m = m \circ d^{\otimes},$$

(A-2) $$\delta^{\#}_1 \circ m = \Delta_1 \circ d^{\otimes} + \delta^{\otimes}_1,$$

(A-3) $$m \circ \delta^{\otimes}_2 = \delta^{\#}_2 - d^{\#} \circ \Delta_2,$$

(A-4) $$d^{\#} \circ \mu + \mu \circ d^{\otimes} = v^{\#} \circ m - m \circ v^{\otimes} + \delta^{\#}_2 \circ \Delta_1 - \Delta_2 \circ \delta^{\otimes}_1.$$

Proof The identity (A-1) decomposes into the following four relations:

(A-5) $d^\# m_1 = m_1(d\otimes 1) + m_1(\epsilon\otimes d') - m_2(\epsilon v\otimes 1) + m_3(\epsilon\otimes v') + m_3(\epsilon\otimes\delta_1') + m_4(\delta_1\otimes 1),$

(A-6) $d^\# m_2 = m_2(d\otimes 1) - m_2(\epsilon\otimes d'),$

(A-7) $d^\# m_3 = m_3(\epsilon\otimes\delta_2') + m_3 d,$

(A-8) $d^\# m_4 = -m_2(\delta_2\otimes 1) + m_4 d'.$

The identity (A-5) is obtained by counting the boundary of the compactified moduli space $M^+_{\gamma^\#,z}(\beta,\beta';\beta^\#)_1$ for each path z. In fact, the oriented boundary of $M^+_{\gamma^\#,z}(\beta,\beta';\beta^\#)_1$ consists of the following types of codimension 1 faces:

$$M_{\gamma^\#,z'}(\beta,\beta';\beta_1^\#)_0\times\breve{M}_{z''}(\beta_1^\#,\beta^\#)_0,\quad \breve{M}_{z'}(\beta,\beta_1)_0\times M_{\gamma^\#,z''}(\beta_1,\beta';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)}\breve{M}_{z'}(\beta',\beta_1')_0\times M_{\gamma,z''}(\beta,\beta_1';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)+1}(H^{-1}_{\beta\beta_1}(s)\cap M_{z'}(\beta,\beta_1)_1)\times M_{z''}(\beta_1,\beta';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)}(H^{-1}_{\beta'\beta_1'}(s)\cap M_{z'}(\beta',\beta_1')_1)\times M_{z''}(\beta,\beta_1';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)}\breve{M}_{z'}(\beta',\theta_\alpha')_0\times M_{z''}(\beta,\theta_\alpha';\beta^\#)_0,\quad \breve{M}_{z'}(\beta,\theta_\alpha)_0\times M_{z''}(\theta_\alpha,\beta';\beta^\#)_0.$$

The identities (A-6)–(A-8) are obtained by counting the compactified moduli spaces $M_z^+(\beta,\beta';\beta^\#)_1$, $M_z^+(\beta,\theta_\alpha';\beta^\#)_1$ and $M_z^+(\theta_\alpha,\beta';\beta^\#)_1$, respectively. We list up codimension 1 faces of each moduli space:

- codimension 1 faces of $\partial M_z^+(\beta,\beta';\beta^\#)_1$:
$$M_{z'}(\beta,\beta';\beta_1^\#)_0\times\breve{M}_{z''}(\beta_1^\#,\beta^\#)_0,\quad \breve{M}_{z'}(\beta,\beta_1)\times M_{z''}(\beta_1,\beta';\beta^\#)_0,$$
$$(-1)^{\mathrm{gr}(\beta)}\breve{M}_{z'}(\beta',\beta_1')_0\times M_{z''}(\beta,\beta_1';\beta^\#)_0,$$
- codimension 1 faces of $\partial M_z^+(\beta,\theta_\alpha';\beta^\#)_1$:
$$M_{z'}(\beta,\theta_\alpha';\beta_1^\#)_0\times\breve{M}_{z''}(\beta_1^\#,\beta^\#)_0,\quad (-1)^{\mathrm{gr}(\beta)}\breve{M}_{z'}(\theta_\alpha',\beta')_0\times M_{z''}(\beta,\beta';\beta^\#)_0,$$
$$\breve{M}_{z'}(\beta,\beta_1)\times M_{z''}(\beta_1,\theta_\alpha';\beta^\#)_0,$$
- codimension 1 faces of $\partial M_z^+(\theta_\alpha,\beta';\beta^\#)_1$:
$$M_{z'}(\theta_\alpha,\beta';\beta_1^\#)_0\times\breve{M}_{z''}(\beta_1^\#,\beta^\#)_0,\quad \breve{M}_{z'}(\theta_\alpha,\beta)_0\times M_{z''}(\beta,\beta';\beta^\#)_0,$$
$$\breve{M}_{z'}(\beta',\beta_1')_0\times M_{z''}(\theta_\alpha,\beta_1';\beta^\#)_0.$$

The relation (A-2) decomposes into the following four identities:

$$\delta_1^\# m_1 = \Delta_{1,1}(d\otimes 1) + \Delta_{1,1}(\epsilon\otimes d') - \Delta_{1,2}(\epsilon v\otimes 1) + \Delta_{1,2}(\epsilon\otimes v') + \Delta_{1,3}(\epsilon\otimes\delta_1') + \Delta_{1,4}(\delta_1\otimes 1),$$
$$\delta_1^\# m_2 = \Delta_{1,2}(d\otimes 1) - \Delta_{1,2}(\epsilon\otimes d'),$$
$$\delta_1^\# m_3 = \Delta_{1,2}(\epsilon\otimes\delta_2') + \Delta_{1,3} d + \delta_1,$$
$$\delta_1^\# m_4 = -\Delta_{1,2}(\delta_2\otimes 1) + \Delta_{1,4} d' + \delta_1'.$$

Each relation is obtained by counting the boundaries of the compactified 1–dimensional moduli spaces $M^+_{\gamma^\#,z}(\beta,\beta';\theta_\alpha^\#)_1$, $M_z^+(\beta,\beta';\theta_\alpha^\#)_1$, $M_z^+(\theta_\alpha,\beta';\theta_\alpha^\#)_1$ and $M_z^+(\beta,\theta_\alpha';\theta_\alpha^\#)_1$ for each path z, and the argument is similar to the previous case. Note that $M(\theta_\alpha, \theta'_\alpha; \theta^\#_\alpha)_0$ consists of the unique reducible connection. The relation (A-3) reduces to

$$m_3\delta_2 + m_4\delta'_2 = \delta^\#_2 - d^\#\Delta_2,$$

and this reduces to the counting of the boundary of $M^+_z(\theta_\alpha, \theta'_\alpha; \beta^\#)_1$ whose codimension 1 faces are

$$\breve{M}_{z'}(\theta_\alpha, \beta)_0 \times M_{z''}(\beta, \theta'_\alpha; \beta^\#)_0, \quad \breve{M}_z(\theta'_\alpha, \beta')_0 \times M_{z''}(\beta, \beta'; \theta^\#_\alpha)_0,$$
$$\breve{M}_{z'}(\theta_\alpha, \theta'_\alpha; \theta^\#_\alpha)_0 \times \breve{M}_{z''}(\theta^\#_\alpha, \beta^\#)_0, \quad M_{z'}(\theta_\alpha, \theta'_\alpha; \beta^\#_1) \times \breve{M}_{z''}(\beta^\#_1, \beta^\#).$$

The relation (A-4) reduces to four identities:

$$\begin{aligned} d^\#\mu_1 + \mu_1(d\otimes 1) + \mu_1(\epsilon\otimes d') - \mu_2(\epsilon v\otimes 1) + \mu_2(\epsilon\otimes v') + \mu_3(\epsilon\otimes\delta'_1) + \mu_4(\delta_1\otimes 1) \\ = v^\# m_1 - m_1(v\otimes 1) + \delta^\#_2\Delta_{1,1}, \end{aligned}$$
$$d^\#\mu_2 + (d\otimes 1) - \mu_2(\epsilon\otimes d') = v^\# m_2 - m_2(v\otimes 1) + m_4(\delta_1\otimes 1),$$
$$d^\#\mu_3 + \mu_2(\epsilon\otimes\delta'_2) + \mu_3 d = v^\# m_3 - m_3 v + \delta^\#_2\Delta_{1,3} - \Delta_2\delta_1,$$
$$d^\#\mu_4 - \mu_2(\delta_2\otimes 1) + \mu_4 d' = v^\# m_4 - m_1(\delta_1\otimes 1) - m_4 v' + \delta^\#_2\Delta_{1,3} - \Delta_2\delta'_1.$$

These are obtained by counting the boundaries of $M^+_{\gamma\gamma',z}(\beta, \beta'; \beta^\#)_1$, $M^+_{\gamma,z}(\beta, \beta'; \beta^\#)_1$, $M^+_{\gamma,z}(\beta, \theta'_\alpha; \beta^\#)_1$ and $M^+_{\gamma'}(\theta_\alpha, \beta'; \beta^\#)_1$; see also [9, Remark 6.11]. □

A.2 Step II

We have

$$m' = [m'_1, m'_2, m'_3, m'_4]^{\mathrm{T}} \colon C^\#_* \to (C\otimes C')_* \oplus (C\otimes C')_{*-1} \oplus C_* \oplus C'_*,$$
$$\mu' = [\mu'_1, \mu'_2, \mu'_3, \mu'_4]^{\mathrm{T}} \colon C^\#_* \to (C\otimes C')_* \oplus (C\otimes C')_{*-1} \oplus C_* \oplus C'_*,$$
$$\Delta'_1 \colon C^\#_1 \to \mathscr{S}.$$
$$\Delta'_2 = [\Delta'_{2,1}, \Delta'_{2,2}, \Delta'_{2,3}, \Delta'_{2,4}]^{\mathrm{T}} \colon \mathscr{S} \to (C\otimes C')_{-1} \oplus (C\otimes C')_{-2} \oplus C_{-1} \oplus C'_{-1},$$

Each component of the above maps is defined as follows:

$$\langle m'_1(\beta^\#), \beta\otimes\beta'\rangle = \sum_z \#M_z(\beta^\#; \beta, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle m'_2(\beta^\#), \beta\otimes\beta'\rangle = \sum_z \#M_{\sigma,z}(\beta^\#; \beta, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle m'_3(\beta^\#), \beta\rangle = \sum_z \#M_z(\beta^\#; \beta, \theta'_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \quad \langle m'_4(\beta^\#), \beta'\rangle = \sum_z \#M_z(\beta^\#; \theta_\alpha, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu'_1(\beta\otimes\beta'), \beta^\#\rangle = \sum_z \#M_{\sigma,z}(\beta^\#; \beta, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu'_2(\beta\otimes\beta'), \beta^\#\rangle = \sum_z \#M_{\sigma\sigma',z}(\beta^\#; \beta, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu'_3(\beta), \beta^\#\rangle = \sum_z \#M_{\sigma,z}(\beta^\#; \beta, \theta'_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \mu'_4(\beta'), \beta^\#\rangle = \sum_z \#M_{\sigma',z}(\beta^\#; \theta_\alpha, \beta')_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$
$$\langle \Delta'_1(1), \beta^\#\rangle = \sum_z \#M_z(\beta^\#; \theta_\alpha, \theta'_\alpha)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\Delta'_{2,1}(\beta\otimes\beta')=\sum_z \#M_z(\beta^\#;\beta,\beta')_0\lambda^{-\kappa(z)}T^{\nu(z)},\quad \Delta'_{2,2}(\beta\otimes\beta')=\sum_z \#M_{\sigma,z}(\theta^\#_\alpha;\beta,\beta')_0\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\Delta'_{2,3}(\beta)=\sum_z \#M_z(\theta^\#_\alpha;\beta,\theta'_\alpha)_0\lambda^{-\kappa(z)}T^{\nu(z)},\qquad \Delta'_{2,4}(\beta')=\sum_z \#M_z(\theta^\#_\alpha;\theta'_\alpha,\beta')_0\lambda^{-\kappa(z)}T^{\nu(z)}.$$

Proposition A.2 *There are the following relations*:

(A-9) $$d^\otimes\circ m'=m'\circ d^\#,$$

(A-10) $$\delta_1^\otimes\circ m'=\Delta'_1\circ d^\#+\delta_1^\#,$$

(A-11) $$m'\circ\delta_2^\#=\delta_2^\otimes-d^\otimes\circ\Delta'_2,$$

(A-12) $$d^\otimes\circ\mu'+\mu'\circ d^\#=v^\otimes\circ m'-m'\circ v^\#+\delta_2^\otimes\circ\Delta'_1-\Delta'_2\circ\delta_1^\#.$$

Proof The proof is similar to that of Proposition A.1. In this case, we consider the opposite cobordism (W',S'). □

A.3 Step III

Put $(W^o,S^o):=(W\circ W',S\circ S')$. We define compositions of paths $\rho^\#:=\gamma^\#\circ\sigma^\#$, $\rho:=\gamma\circ\sigma$ and $\rho':=\gamma'\circ\sigma'$; see Figure 5. We regard the configuration space of connections over (W^o,S^o) as the quotient of the space of SO(3)–adjoint connections by the determinant 1 gauge group $\mathcal{G}$. Then there is an exact sequence

$$\mathcal{G}\hookrightarrow\mathcal{G}^e\twoheadrightarrow H^1(W^o;\mathbb{Z}_2),$$

where $\mathcal{G}^e$ is an SO(3)–gauge transformation and the second map gives the obstruction to lifting an SO(3)–automorphism to an SU(2)–automorphism over the 1–skeleton. There is an action of $\mathcal{G}^e/\mathcal{G}\cong H^1(W^o,\mathbb{Z}_2)\cong\mathbb{Z}_2$ on the configuration space. In particular, there is an involution on the moduli space $M(W^o,S^o;\beta^\#,\beta_1^\#)_d$ and we denote its quotient by $M(W^o,S^o;\beta^\#,\beta_1^\#)_d^e$. We define

$$M_{\rho^\#;z}(W^o,S^o;\beta^\#,\beta_1^\#)_0^e:=\{[A]\in M_z(W^o,S^o;\beta^\#,\beta_1^\#)_1^e\mid H^{\rho^\#}([A])=s\},$$
$$M_{\rho^\#\rho;z}(W^o,S^o;\beta^\#,\beta_1^\#)_0^e:=\{[A]\in M_z(W^o,S^o;\beta^\#,\beta_1^\#)_2^e\mid H^{\rho^\#}([A])=s,\,H^{\rho}([A])=s'\}.$$

The cardinality of these moduli spaces is half of that of the usual ones. Assume that (W^o,S^o) is equipped with a Riemannian metric with a long neck along the cylinder $[0,1]\times(Y\sqcup Y',K\sqcup K')$. Then we have a good gluing relation

$$\begin{aligned}M_{\rho^\#}(W^o,S^o;\beta^\#,\beta_1^\#)_0^e=&\bigsqcup_{\beta,\beta'}M_{\sigma^\#}(W',S';\beta^\#,\beta,\beta')_0\times M(W,S;\beta,\beta';\beta_1^\#)_0\\&\sqcup\bigsqcup_{\beta,\beta'}M(W',S';\beta^\#,\beta,\beta')_0\times M_{\gamma^\#}(W,S;\beta,\beta';\beta_1^\#)_0\\&\sqcup\bigsqcup_{\beta'\in\mathfrak{C}'^*}M(W',S';\beta^\#,\theta_\alpha,\beta')_0\times M(W,S;\theta_\alpha,\beta';\beta_1^\#)_0\\&\sqcup\bigsqcup_{\beta\in\mathfrak{C}^*}M(W',S';\beta^\#,\beta,\theta_\alpha)_0\times M(W,S;\beta,\theta'_\alpha;\beta_1^\#)_0.\end{aligned}$$

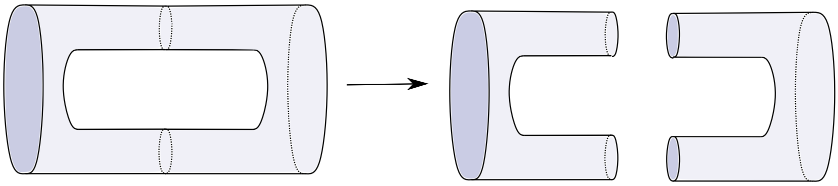

Figure 4: The family of metrics G^o.

Let $\widetilde{m}_{(W^o,S^o,\rho^\#)}\colon \widetilde{C}_*^\# \to \widetilde{C}_*^\#$ be an $\mathcal{S}$–morphism whose components m^o, μ^o, Δ_1^o and Δ_2^o are defined by

$$\langle m^o(\beta^\#), \beta_1^\#\rangle = \sum_z \#M_{\rho^\#,z}(W^o, S^o; \beta^\#, \beta_1^\#)_0^e \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle \mu^o(\beta^\#), \beta_1^\#\rangle = \sum_z \#M_{\rho^\# \rho,z}(W^o, S^o; \beta^\#, \beta_1^\#)_0^e \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\Delta_1^o(\beta^\#) = \sum_z \#M_{\rho^\#,z}(W^o, S^o; \beta^\#, \theta_\alpha^\#)_0^e \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\Delta_2^o(1)\beta^\# = \sum_z \#M_{\rho^\#,z}(W^o, S^o; \beta^\#, \theta_\alpha^\#)_0^e \lambda^{-\kappa(z)} T^{\nu(z)}.$$

Proposition A.3 *We have that $\widetilde{m}_{(W,S)} \circ \widetilde{m}_{(W',S')}$ is $\mathcal{S}$–chain homotopic to $\widetilde{m}_{(W\circ W',S\circ S';\rho^\#)}$.*

Proof Let G^o be the 1–parameter family of metrics which stretch the cobordism (W^o, S^o) as in Figure 4. We modify the definition of the $\mathcal{S}$–chain homotopy in [9, Proposition 6.16] in the following way:

$$\langle K^o(\beta^\#), \beta_1^\#\rangle = \sum_z \#\Big\{[A] \in \bigcup_{g\in G^o} M_z^g(W^o, S^o; \beta^\#, \beta_1^\#)_0^e \mid H^{\rho^\#}([A]) = s\Big\}\lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle L^o(\beta^\#), \beta_1^\#\rangle = \sum_z \#\Big\{[A] \in \bigcup_{g\in G^o} M_z^g(W^o, S^o; \beta^\#, \beta_1^\#)_0^e \mid H^{\rho^\#}([A]) = s,\, H^{\rho}([A]) = t\Big\}\lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle M_1^o(\beta^\#), 1\rangle = \sum_z \#\Big\{[A] \in \bigcup_{g\in G^o} M_z^g(W^o, S^o; \beta^\#, \theta^\#)_0^e \mid H^{\rho^\#}([A]) = s\Big\}\lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle M_2^o(1), \beta^\#\rangle = \sum_z \#\Big\{[A] \in \bigcup_{g\in G^o} M_z^g(W^o, S^o; \theta_\alpha^\#, \beta^\#)_0^e \mid H^{\rho^\#}([A]) = s\Big\}\lambda^{-\kappa(z)} T^{\nu(z)}.$$

The rest of the argument is similar to [9, Proposition 6.16], and we can check that

$$H^o = \begin{bmatrix} K^o & 0 & 0 \\ L^o & -K^o & M_2^o \\ M_1^o & 0 & 0 \end{bmatrix}$$

gives an $\mathcal{S}$–chain homotopy from $\widetilde{m}_{(W^o,S^o)}$ to $\widetilde{m}_{(W\sqcup W',S\sqcup S')}$. □

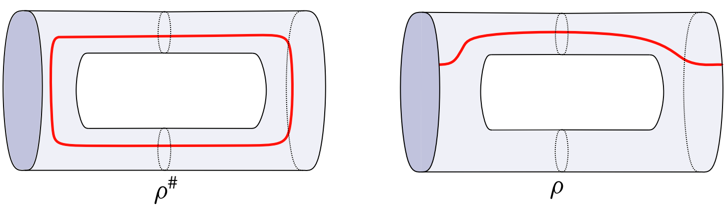

Figure 5: Paths on (W^o, S^o).

Proposition A.4 *We have that* $\widetilde{m}_{(W\circ W', S\circ S';\rho^\#)}$ *is* $\mathcal{S}$*–chain homotopic to* $\mathrm{id}_{\widetilde{C}^\#}$*, up to the multiplication of a unit element in* $\mathscr{S}$*.*

Proof As in the proof of [9, Proposition 6.17], we consider the decomposition,

$$(W^o, S^o) = (W^c, S^c) \cup (S^1 \times D^3, S^1 \times D^1)$$

along $(S^1 \times S^2, S^1 \times 2\mathrm{pt})$. We arrange the perturbation data on $(S^1 \times D^3, S^1 \times D^1)$ and the gluing region so that it is supported away from the moduli space of flat connections. Define the map $\widetilde{m}^+$ as $\widetilde{m}_{(W^o, S^o, \rho^\#)}$, but using the metric on (W^o, S^o) which is stretched along the gluing region $(S^1 \times S^2, S^1 \times 2\mathrm{pt})$. We write m^+, μ^+, Δ_1^+ and Δ_2^+ for corresponding components of $\widetilde{m}^+$.

Let μ_1 be a generator of the $S^2 \setminus 2\mathrm{pt}$ factor and μ_2 be a generator of the S^1 factor in $\pi_1(S^1 \times S^2 \setminus S^1 \times 2\mathrm{pt})$. The equivalence classes of the critical point set $\mathfrak{C}$ of the Chern–Simons functional on $(S^1 \times S^2, S^1 \times 2\mathrm{pt})$ can be identified with

$$\mathcal{R}_\alpha(S^1 \times S^2 \setminus S^1 \times 2\mathrm{pt}) = \{\beta \in \mathrm{Hom}(\pi_1, \mathrm{SU}(2)) \mid \mathrm{tr}\,\beta(\mu_1) = 2\cos(2\pi\alpha)\}/\,\mathrm{SU}(2)$$

by the holonomy correspondence. The character variety $\mathcal{R}_\alpha(S^1 \times S^2 \setminus S^1 \times 2\mathrm{pt})$ is identified with S^1 as follows. Let μ_1 be a generator of $\pi_1(S^1 \times (S^2 \setminus 2\mathrm{pt}))$ arising from the $S^2 \setminus 2\mathrm{pt}$ factor, and μ_2 be another generator arising from the S^1 factor. Since $\mathrm{tr}\,\beta(\mu_1) = 2\cos(2\pi i\alpha)$, there is an element $g_\beta \in \mathrm{SU}(2)$ with $g_\beta \beta(\mu_1) g_\beta^{-1} = e^{2\pi i\alpha} \in S^1$. Since μ_1 and μ_2 commute and $\alpha \neq 0, \frac{1}{2}$, there is $\theta(\beta) \in [0, 2\pi)$ and we have $g_\beta \beta(\mu_2) g_\beta^{-1} = e^{i\theta(\beta)} \in S^1$. The correspondence $\beta \mapsto e^{i\theta(\beta)}$ gives a bijection $\mathcal{R}_\alpha(S^1 \times S^2 \setminus S^1 \times 2\mathrm{pt}) \cong S^1$.

Let A be a singular flat connection which is the extension of $\rho \in \mathfrak{C}$ over $(S^1 \times D^3, S^1 \times D^1)$. Since all elements in $\mathfrak{C}$ have $U(1)$–stabilizer, $\dim H^0(S^1 \times D^3 \setminus S^1 \times D^1; \mathrm{ad}\,A) = 1$. Also,

$$\dim H^1(S^1 \times D^3 \setminus S^1 \times D^1; \mathrm{ad}\,A) = 1$$

by the computation of group cohomology of $\pi_1(S^1 \times (D^3 \setminus D^1))$. Thus the critical point set $\mathfrak{C} = \mathcal{R}_\alpha(S^1 \times (S^2 \setminus 2\mathrm{pt}))$ is Morse–Bott nondegenerate. Consider the closed pair $(S^1 \times S^3, S^1 \times S^1) = (S^1 \times D^3, S^1 \times D^1) \cup_{(S^1 \times S^2, S^1 \times 2\mathrm{pt})} (S^1 \times D^3, S^1 \times D^1)$. Then the gluing of the index formula is

$$2\,\mathrm{ind}\,D_A + \dim \mathfrak{C} + \dim \mathrm{Stab}(\rho) = \mathrm{ind}\,D_{A\#_\rho A}.$$

Since $\dim \mathfrak{C} = \dim \mathrm{Stab}(\rho) = 1$ and $\mathrm{ind}\,D_{A\#_\rho A} = 0$ by the index formula for a closed pair, $\mathrm{ind}\,D_A = -1$. This implies that

$$\dim H^2(S^1 \times D^3 \setminus S^1 \times D^1; \mathrm{ad}\,A) = 0,$$

and hence the gluing theory is unobstructed at the flat connection. Morse–Bott gluing theory tells us that the moduli space $M_{\rho^\#}(W^o, S^o; \beta^\#, \beta_1^\#)_0$ has the structure of the union of fiber products as follows:

$$M(W^c, S^c; \beta^\#, \beta_1^\#)_d \times_{\mathfrak{C}} M_{\rho^\#}(S^1 \times D^3, S^1 \times D^1)^{\mathrm{irred}}_{d'} \quad \text{for } d + d' = 1,$$
$$M(W^c, S^c; \beta^\#, \beta_1^\#)_1 \times_{\mathfrak{C}} M_{\rho^\#}(S^1 \times D^3, S^1 \times D^1)^{\mathrm{red}}.$$

The first case is excluded for index reasons.

Consider the restriction map

$$r' \colon M_{\rho^\#}(S^1 \times D^3, S^1 \times D^1)^{\mathrm{red}} \to \mathfrak{C}.$$

By the holonomy condition $H^{\rho^\#}([A]) = 1$ on the moduli space $M(S^1 \times D^3, S^1 \times D^1)$, the image of r' consists of two points $\theta, \theta' \in \mathfrak{C}$. Hence, if the metric on (W^o, S^o) has a long neck along $(S^1 \times S^2, S^2 \times 2\mathrm{pt})$, the moduli space $M_{\rho^\#}(W^o, S^o; \beta^\#, \beta_1^\#)$ is two copies of

$$M(W^c, S^c; \beta^\#, \theta, \beta_1^\#)_0 \times M(S^1 \times D^3, S^1 \times D^1; \theta)^{\mathrm{red}}.$$

In particular,

$$\begin{aligned}
&\sum_z \#M_{\rho^\#,z}(W^o, S^o; \beta^\#, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)} \\
&\quad = 2\sum_z \sum_{z' \circ z'' = z} \#M_{z'}(S^1 \times D^3, S^1 \times D^1; \theta)^{\mathrm{red}} \lambda^{-\kappa(z')} T^{\nu(z')} \#M_{z''}(W^c, S^c; \beta^\#, \theta, \beta_1^\#)_0 \lambda^{-\kappa(z'')} T^{\nu(z'')} \\
&\quad = 2\Big(\sum_{k \le 0} c_k Z^k\Big) \sum_{z''} \#M_{z''}(W^c, S^c; \beta^\#, \theta, \beta_1^\#)_0 \lambda^{-\kappa(z'')} T^{\nu(z'')}.
\end{aligned}$$

Since flat connections on $(S^1 \times S^2, S^1 \times 2\mathrm{pt})$ uniquely extend to $(S^1 \times D^3, S^1 \times D^1)$, we have $c_0 = 1$. Since $2\#M_{\rho^\#}(W^o, S^o; \beta^\#, \beta_1^\#)_0^e = \#M_{\rho^\#}(W^o, S^o; \beta^\#, \beta_1^\#)_0$, there is a unit element $C_1 \in \mathscr{S}$ and we have

$$\langle m^+(\beta^\#), \beta_1^\# \rangle = C_1 \sum_z \#M_z(W^c, S^c; \beta^\#, \theta, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)}.$$

The same argument with $M_{\rho^\#\rho}(W^o, S^o; \beta^\#, \theta, \beta_1^\#)_0$, $M_{\rho^\#}(W^o, S^o; \beta^\#, \theta, \theta_\alpha^\#)_0$ and $M_{\rho^\#}(W^o, S^o; \theta_\alpha^\#, \theta, \beta_1^\#)_0$ instead of $M_{\rho^\#}(W^o, S^o; \beta^\#, \beta_1^\#)_0$ gives

$$\begin{aligned}
\langle \mu^+(\beta^\#), \beta_1^\# \rangle &= C_1 \sum_z \#M_{\rho,z}(W^c, S^c; \beta^\#, \theta, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \\
\langle \Delta_1^+(\beta^\#), 1 \rangle &= C_1 \sum_z \#M_z(W^o, S^o; \beta^\#, \theta, \theta_\alpha^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)}, \\
\langle \Delta_2^+(1), \beta_1^\# \rangle &= C_1 \sum_z \#M_z(W^o, S^o; \theta_\alpha^\#, \theta, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)}.
\end{aligned}$$

Replacing the pair $(S^1 \times D^3, S^1 \times D^1)$ with $(D^2 \times S^2, D^2 \times 2\mathrm{pt})$, we obtain the product cobordism $[0,1] \times (Y \# Y', K \# K')$. By stretching the metric on $[0,1] \times (Y \# Y', K \# K')$ along the attaching domain, the moduli space $M([0,1] \times (Y \# Y', K \# K')\beta^\#, \beta_1^\#)_0$ has the structure of the union of fiber products

(A-13) $$M(W^c, S^c; \beta^\#, \beta_1^\#) \times_{\mathfrak{C}} M(D^2 \times S^2, D^2 \times 2\mathrm{pt})^{\mathrm{red}}.$$

Let A' be an extended flat connection on $(D^2 \times S^2, D^2 \times 2\mathrm{pt})$ of the flat connection θ. Such A' uniquely exists. Moreover, it can be checked that the point Θ is unobstructed as follows. Consider the closed pair

$$(S^4, S^2) := (D^2 \times S^2, D^2 \times 2\mathrm{pt}) \cup_{(S^1 \times S^2, S^1 \times 2\mathrm{pt})} (S^1 \times D^3, S^1 \times D^1)$$

and the glued reducible flat connection $A' \#_\theta A$ on (S^4, S^2). Then we have

$$\operatorname{ind} D_{A' \#_\theta A} = \operatorname{ind} D_{A'} + \dim \operatorname{Stab}(\theta) + \dim \mathfrak{C} + \operatorname{ind} D_A.$$

Since $b^1(X) = b^+(X) = 0$ and $S \cong S^2$, the index formula for a closed pair shows that $\operatorname{ind} D_{A'\#A} = -1$. Moreover, $\operatorname{ind} D_A = -1$ by the previous argument. Thus $\operatorname{ind} D_{A'} = -2$.

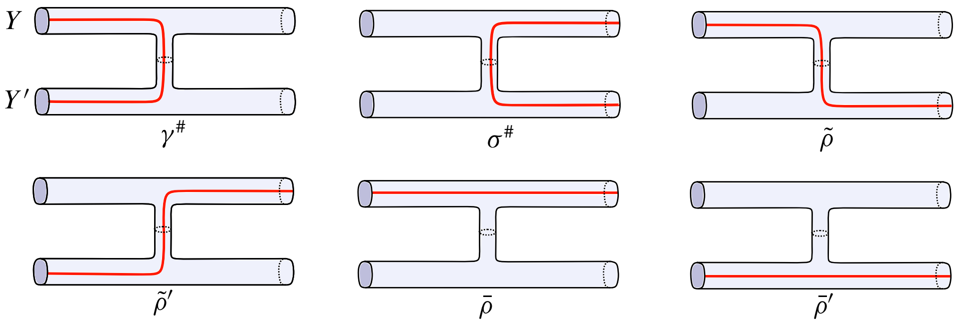

Figure 6: Paths on (W^I, S^I).

Since $\dim H^0(D^2 \times (S^2 \setminus 2\mathrm{pt}); \mathrm{ad}\, A') = 1$ and $\dim H^1(D^2 \times (S^2 \setminus 2\mathrm{pt}); \mathrm{ad}\, A') = 0$ by the computation of group cohomology, $\dim H^2(D^2 \times (S^2 \setminus 2\mathrm{pt}); \mathrm{ad}\, A')$ vanishes.

Now, the fiber product structure (A-13) implies that there is a unit element $C_2 \in \mathscr{S}$ and

$$\langle m^+(\beta^\#), \beta_1^\# \rangle = C_2 \sum_z \# M_z([0,1] \times (Y \# Y', K \# K'); \beta^\#, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)}.$$

Similarly

$$\langle \mu^+(\beta^\#), \beta_1^\# \rangle = C_2 \sum_z \# M_{\rho,z}([0,1] \times (Y \# Y', K \# K'); \beta^\#, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\Delta_1^+(\beta^\#) = C_2 \sum_z \# M_z([0,1] \times (Y \# Y', K \# K'); \beta^\#, \theta_\alpha^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)},$$

$$\langle \Delta_2^+(1), \beta^\# \rangle = C_2 \sum_z \# M_z([0,1] \times (Y \# Y', K \# K'); \theta_\alpha^\#, \beta_1^\#)_0 \lambda^{-\kappa(z)} T^{\nu(z)}.$$

Finally, there is a unit element $c \in \mathscr{S}$ and we have

$$\widetilde{m}^+ = c\widetilde{m}_{[0,1]\times(Y\#Y',K\#K')}.$$

The right-hand side is $\mathcal{S}$–chain homotopic to the identity since it is induced from the product cobordism. By construction, the unit element c has the top term 1, and hence $\widetilde{m}_{(W\circ W', S\circ S';\rho^\#)}$ is $\mathcal{S}$–chain homotopic to the identity up to the multiplication of a unit element in $\mathscr{S}$. □

A.4 Step IV

Set $\bar\rho := \sigma \circ \gamma$, $\bar\rho' := \sigma' \circ \gamma'$, $\tilde\rho := \sigma' \circ \gamma$ and $\tilde\rho' := \sigma \circ \gamma'$; see Figure 6.

Proposition A.5 *We have that $\widetilde{m}_{(W',S')} \circ \widetilde{m}_{(W,S)}$ is $\mathcal{S}$–chain homotopic to $\widetilde{m}_{(W'\circ W, S'\circ S)}$.*

Proof Let G^I be a 1–parameter family of metrics stretching $(W^I, S^I) := (W' \circ W, S' \circ S)$ along $(Y \# Y', K \# K')$. Let $\widetilde{m}_{(W^I,S^I)}$ be the cobordism map for (W^I, S^I). We claim that there is an $\mathcal{S}$–chain homotopy H^I such that

$$\tilde d^\otimes H^I + H^I \tilde d^\otimes = \widetilde{m}_{(W',S')} \circ \widetilde{m}_{(W,S)} - \widetilde{m}_{(W^I,S^I)}.$$

Let us write each components of H^I as

$$H^I = \begin{bmatrix} K^I & 0 & 0 \\ L^I & -K^I & M_2^I \\ M_1^I & 0 & 0 \end{bmatrix},$$

where K^I and L^I are 4×4 matrices. Before defining each component of these matrices, we introduce the notation

$$\mathfrak{m}^I_z(\beta,\beta',\beta_1,\beta'_1) := \#\bigcup_{g\in G^I} M^g_z(W^I,S^I,\beta,\beta';\beta'_1,\beta'_1)_{-1},$$

$$\mathfrak{m}^I_{\circ_1,\dots,\circ_d;z}(\beta,\beta',\beta_1,\beta'_1) := \#\Big\{[A]\in \bigcup_{g\in G^I} M^g_z(W^I,S^I,\beta,\beta';\beta'_1,\beta'_1)_{d-1} \mid H^{\circ_1}([A])=s_1,\dots,H^{\circ_d}([A])=s_d\Big\},$$

where $\circ_1,\dots,\circ_d$ are elements in the set of paths $\{\gamma^\#,\sigma^\#,\bar\rho,\bar\rho',\tilde\rho,\tilde\rho'\}$. Then each component of K^I, L^I, M^I_1 and M^I_2 is given as follows:

- components of K^I

$$\langle K^I_{11}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle = \sum_z \mathfrak{m}^I_{\gamma^\#;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{12}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle = \sum_z \mathfrak{m}^I_{z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{13}(\beta),\beta_1\otimes\beta'_1\rangle = \sum_z \mathfrak{m}^I_{z}(\beta,\theta'_\alpha;\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{14}(\beta'),\beta_1\otimes\beta'_1\rangle = \sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{21}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle = \sum_z \mathfrak{m}^I_{\gamma^\#,\sigma^\#;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{22}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle = \sum_z \mathfrak{m}^I_{\sigma^\#;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{23}(\beta),\beta_1\otimes\beta'_1\rangle = \sum_z \mathfrak{m}^I_{\sigma^\#;z}(\beta,\theta'_\alpha;\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{24}(\beta'),\beta_1\otimes\beta'_1\rangle = \sum_z \mathfrak{m}^I_{\sigma^\#;z}(\theta_\alpha,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{31}(\beta\otimes\beta'),\beta_1\rangle = \sum_z \mathfrak{m}^I_{\gamma^\#;z}(\beta,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{32}(\beta\otimes\beta'),\beta_1\rangle = \sum_z \mathfrak{m}^I_{z}(\beta,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{33}(\beta),\beta_1\rangle = \sum_z \mathfrak{m}^I_{z}(\beta,\theta'_\alpha;\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{34}(\beta'),\beta_1\rangle = \sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{41}(\beta\otimes\beta'),\beta'_1\rangle = \sum_z \mathfrak{m}^I_{\gamma^\#;z}(\beta,\beta';\theta_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{42}(\beta\otimes\beta'),\beta'_1\rangle = \sum_z \mathfrak{m}^I_{z}(\beta,\beta';\theta'_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{43}(\beta),\beta'_1\rangle = \sum_z \mathfrak{m}^I_{z}(\beta,\theta'_\alpha;\theta'_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

$$\langle K^I_{44}(\beta'),\beta'_1\rangle = \sum_z \mathfrak{m}^I_{z}(\theta_\alpha,\beta';\theta'_\alpha,\beta_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

- components of L^I

$$\langle L^I_{11}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\gamma^\#,\bar\rho;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{12}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{13}(\beta),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho;z}(\beta,\theta'_\alpha;\beta_1,\beta'_1)^{-\kappa(z)}_\lambda T^{\nu(z)},$$
$$\langle L^I_{14}(\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho';z}(\theta_\alpha,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{21}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#,\bar\rho;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{22}(\beta\otimes\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#,\bar\rho;z}(\beta,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{23}(\beta),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#,\bar\rho;z}(\beta,\theta'_\alpha;\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{24}(\beta'),\beta_1\otimes\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#,\bar\rho';z}(\theta_\alpha,\beta';\beta_1,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{31}(\beta\otimes\beta'),\beta_1\rangle=\sum_z \mathfrak{m}^I_{\gamma^\#,\bar\rho;z}(\beta,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{32}(\beta\otimes\beta'),\beta_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho;z}(\beta,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{33}(\beta),\beta_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho';z}(\beta,\theta'_\alpha;\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{34}(\beta'),\beta_1\rangle=\sum_z \mathfrak{m}^I_{\tilde\rho';z}(\theta_\alpha,\beta';\beta_1,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{41}(\beta\otimes\beta'),\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\gamma^\#,\bar\rho';z}(\beta,\beta';\theta_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{42}(\beta\otimes\beta'),\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\tilde\rho;z}(\beta,\beta';\theta'_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{43}(\beta),\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\tilde\rho;z}(\beta,\theta'_\alpha;\theta'_\alpha,\beta'_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle L^I_{44}(\beta'),\beta'_1\rangle=\sum_z \mathfrak{m}^I_{\bar\rho';z}(\theta_\alpha,\beta';\theta'_\alpha,\beta_1)\lambda^{-\kappa(z)}T^{\nu(z)},$$

- components of M^I_1

$$M^I_{1,1}(\beta\otimes\beta')=\sum_z \mathfrak{m}^I_{\gamma^\#;z}(\beta,\beta';\theta_\alpha,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$M^I_{1,2}(\beta\otimes\beta')=\sum_z \mathfrak{m}^I_z(\beta,\beta';\theta_\alpha,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$M^I_{1,3}(\beta)=\sum_z \mathfrak{m}^I_z(\beta,\theta'_\alpha;\theta_\alpha,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},\quad M^I_{1,4}(\beta')=\sum_z \mathfrak{m}^I_z(\theta_\alpha,\beta';\theta_\alpha,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},$$

- components of M^I_2

$$\langle M^I_{2,1}(1),\beta\otimes\beta'\rangle=\sum_z \mathfrak{m}^I_z(\theta_\alpha,\theta'_\alpha;\beta,\beta')\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle M^I_{2,2}(1),\beta\otimes\beta'\rangle=\sum_z \mathfrak{m}^I_{\sigma^\#;z}(\theta_\alpha,\theta'_\alpha;\beta,\beta')\lambda^{-\kappa(z)}T^{\nu(z)},$$
$$\langle M^I_{2,3}(1),\beta\rangle=\sum_z \mathfrak{m}^I_z(\theta_\alpha,\theta'_\alpha;\beta,\theta'_\alpha)\lambda^{-\kappa(z)}T^{\nu(z)},\quad \langle M^I_{2,4}(1),\beta'\rangle=\sum_z \mathfrak{m}^I_z(\theta_\alpha,\theta'_\alpha;\theta_\alpha,\beta')\lambda^{-\kappa(z)}T^{\nu(z)}.$$

component of (A-14)	corresponding family of moduli spaces
(1, 1)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \beta_1')_0 \mid H^{\gamma^\#}([A]) = s\}$
(1, 2)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \beta_1')_0$
(1, 3)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \theta_\alpha'; \beta_1, \beta_1')_0$
(1, 4)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \theta_\alpha, \beta'; \beta_1, \beta_1')_0$
(2, 1)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \beta_1')_2 \mid H^{\gamma^\#}([A]) = s, H^{\sigma^\#}([A]) = t\}$
(2, 2)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \beta_1')_1 \mid H^{\sigma^\#}([A]) = s\}$
(2, 3)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \theta_\alpha'; \beta_1, \beta_1')_1 \mid H^{\sigma^\#}([A]) = s\}$
(2, 4)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \theta_\alpha, \beta'; \beta_1, \beta_1')_1 \mid H^{\sigma^\#}([A]) = s\}$
(3, 1)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \theta_\alpha')_1 \mid H^{\gamma^\#}([A]) = s\}$
(3, 2)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \beta_1, \theta_\alpha')_0$
(3, 3)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \theta_\alpha'; \beta_1, \theta_\alpha')_1 \mid H^{\bar{\rho}'}([A]) = t\}$
(3, 4)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \theta_\alpha, \beta'; \beta_1, \theta_\alpha')_0$
(4, 1)	$\{[A] \in \bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \theta_\alpha, \beta_1')_1 \mid H^{\gamma^\#}([A]) = s\}$
(4, 2)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \beta'; \theta_\alpha', \beta_1')_0$
(4, 3)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \beta, \theta_\alpha'; \theta_\alpha', \beta_1')_0$
(4, 4)	$\bigcup_{g \in G^I} M_z^g(W^I, S^I; \theta_\alpha, \beta'; \theta_\alpha', \beta_1)_0$

Table 1

Then we can check that there are the following identities:

$$d^\otimes K^I + K^I d^\otimes = m'm - m^I, \tag{A-14}$$

$$v^\otimes K^I - d^\otimes L^I + \delta_2^\otimes M_1^I + L^I d^\otimes - K^I v^\otimes + M_2^I \delta_1^\otimes = \mu' m + m'\mu + \Delta_2'\Delta_1 - \mu^I, \tag{A-15}$$

$$\delta_1^\otimes K^I + M_1^I d^\otimes = \Delta_1' m + \Delta_1 - \Delta_1^I, \tag{A-16}$$

$$-d^\otimes M_2^I - K^I \delta_2^\otimes = m'\Delta_2 + \Delta_2' - \Delta_2^I. \tag{A-17}$$

The identities above are proved by counting oriented boundaries of corresponding moduli spaces. For example, such moduli spaces for identity (A-14) are given in Table 1. Other identities can be proved in similar ways. □

Proposition A.6 *We have that $\tilde{m}_{(W' \circ W, S' \circ S)}$ is $\mathcal{S}$–chain homotopic to $\mathrm{id}_{\tilde{C}^\otimes}$.*

Proof Consider a family of metrics G'^I on (W^I, S^I) which stretch the cobordism along (S^3, S^1) as in Figure 7. Let $\tilde{m}^I$ be the map defined by a long stretched metric on (W^I, S^I). The family of metrics G'^I gives an $\mathcal{S}$–chain homotopy between $\tilde{m}_{(W^I, S^I)}$ and $\tilde{m}^I$. Let $(W^{c'}, S^{c'})$ be a disjoint union

$$(Y \times [0,1] \setminus D^4, K \times [0,1] \setminus D^2) \sqcup (Y' \times [0,1] \setminus D^4, K' \times [0,1] \setminus D^2).$$

We can also define $\tilde{m}^I$ by counting instantons on $(W^{c'}, S^{c'})$. We will show that $\tilde{m}^I$ is an isomorphism of $\mathcal{S}$–complexes. We obtain a pair of cylinders $[0,1] \times (Y \sqcup Y', K \sqcup K')$ by gluing back two pairs of disks

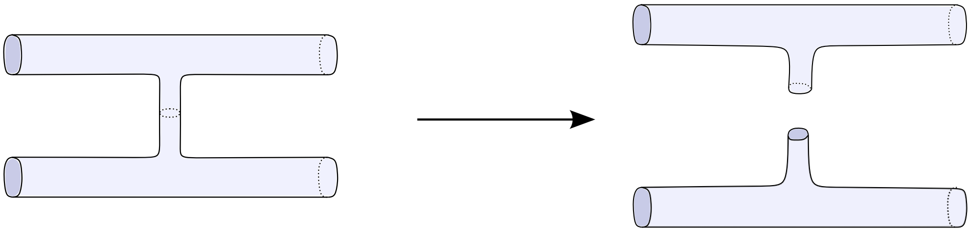

Figure 7: Family of metrics G'^I.

(D^4, D^2) to $(W^{c'}, S^{c'})$. Consider the character variety $\mathfrak{C}'$ with the holonomy parameter α on (S^3, S^1). For $0 < \alpha < \frac{1}{2}$, $\mathfrak{C}'$ is a one-point set which consists of the unique flat reducible θ_α, and the moduli space $M(D^4, D^2; \theta_\alpha)_0$ also consists of one element Θ_α which is the unique extension of θ_α to $D^4 \setminus D^2$. The computation of group cohomology of $\pi_1(S^3 \setminus S^1)$ shows that $\dim H^1(S^3 \setminus S^1; \mathrm{ad}\,\theta_\alpha) = 0$. Taking the double of (D^4, D^2), we have the relation of indices

$$2\,\mathrm{ind}\, D_{\Theta_\alpha} + 1 = \mathrm{ind}\, D_{\Theta_\alpha \# \Theta_\alpha}.$$

Moreover, $\mathrm{ind}\, D_{\Theta_\alpha \# \Theta_\alpha} = -1$ by the index formula for the closed pair (S^4, S^2). Thus $\mathrm{ind}\, D_{\Theta_\alpha} = -1$ and $H^2(D^4 \setminus D^2; \mathrm{ad}\,\Theta_\alpha) = 0$. In particular, the gluing along $\mathfrak{C}'$ is unobstructed. The Morse–Bott gluing argument shows that

$$M([0,1] \times Y, [0,1] \times K; \beta, \beta_1)_d = M([0,1] \times Y \setminus D^4, [0,1] \times K \setminus D^2; \beta, \theta_\alpha, \beta')_d,$$

and similarly for the pair (Y', K'). Thus

$$\begin{aligned}\#M_z^{g^\infty}(W^I, S^I, \beta, \beta'; \beta_1, \beta_1') &= \#M_{z'}(W^I, S^I; \beta, \theta_\alpha, \beta_1)\#M_{z''}(W^I, S^I; \beta', \theta_\alpha, \beta_1') \\ &= \#M_{z'}(Y \times [0,1], K \times [0,1]; \beta, \beta_1)\#M_{z''}(Y' \times [0,1], K' \times [0,1]; \beta', \beta_1').\end{aligned}$$

Therefore $\widetilde{m}_{(W^I, S^I)}$ is $\mathcal{S}$–chain homotopic to the morphism $\widetilde{m}_{\mathrm{prod}}$ which is induced from the product cobordism $(Y \sqcup Y', K \sqcup K') \times [0,1]$. The $\mathcal{S}$–morphism $\widetilde{m}_{\mathrm{prod}}$ is an isomorphism of $\mathcal{S}$–complexes (see [9, Lemma 6.29]), and in fact $\mathcal{S}$–chain homotopic to the identity by the formal argument. □

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On the invariance of the Dowlin spectral sequence

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Given a link L, Dowlin constructed a filtered complex inducing a spectral sequence with E_2–page isomorphic to the Khovanov homology $\overline{\mathrm{Kh}}(L)$ and E_∞–page isomorphic to the knot Floer homology $\widehat{HFK}(m(L))$ of the mirror of the link. We prove that the E_k–page of this spectral sequence is also a link invariant, for $k \geq 3$.

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1 Introduction

Dowlin [2024] associated a filtered chain complex to a link L. The spectral sequence this filtered complex gives rise to has E_2–page isomorphic to the (reduced) Khovanov homology $\overline{\mathrm{Kh}}(L)$ and converges to the knot Floer homology $\widehat{HFK}(m(L))$ of the mirror of the link. The fact that the E_2– and E_∞–pages of the spectral sequence are link invariants, independent of the diagram used to construct the filtered complex, suggests that the same may be true of all the higher pages of the spectral sequence. This is the main result of this paper.

Theorem 1.1 *For $k \geq 2$, the E_k–page of Dowlin's spectral sequence does not depend on the diagram used to construct the filtered complex, and is thus a link invariant.*

This theorem provides a whole family of link invariants $\{E_k(L)\}_{k=2}^{\infty}$. The invariance of these higher pages of the Dowlin spectral sequence helps us further decipher the connection between Khovanov homology and knot Floer homology.

This result opens several research directions. The first is to find knots (or families of knots) which have the same Khovanov homology and knot Floer homology, but are distinguished by these higher page invariants. The ranks of Khovanov homology and knot Floer homology tend to coincide for knots with few crossings [Rasmussen 2005], so finding such examples may be computationally difficult.

A second direction is to consider implications in the study of transverse links. Plamenevskaya [2006] identified an invariant of transverse links $\psi(L) \in \mathrm{Kh}(L)$, which we can think of as residing in the E_2 page of the Dowlin spectral sequence. One could hope to define a countable family of transverse link invariants $\{\psi_k(L)\}_{k=2}^{\infty}$ by taking the image of ψ on each higher page E_k for $k \geq 2$, in the style of Baldwin [2011]. It might prove interesting to compare these invariants, especially the image of ψ on the E_∞ page $\widehat{HFK}(m(L))$ with known transverse link invariants [Baldwin et al. 2013].

A third direction for future work would be the investigate potential relationships between the s invariant in Khovanov homology [Rasmussen 2010] and the τ invariant in knot Floer homology [Ozsváth and Szabó 2003]. For many knots, these invariants are related by the equation $s = 2\tau$; however, we also know of knots that break this rule [Hedden and Ording 2008]. Perhaps the spectral sequence could be used to explain this (lack of a) pattern.

Organization

We begin by reviewing the construction of $\mathrm{C}_2^-(L)$ in Section 2; as originally defined by Dowlin [2024], this filtered complex induces the spectral sequence from $\overline{\mathrm{Kh}}(L)$ to $\widehat{HFK}(m(L))$ for a given link L. In Section 3, we prove that the homotopy type of this complex is invariant under a diagrammatic change we call "relabeling vertices". We then discuss MOY moves, another set of operations on diagrams, and define maps associated to these moves in Section 4. With these in hand, we prove invariance of the higher pages of the spectral sequence in Section 5.

Conventions

There are a few homological algebra conventions that we need to establish.

- We call our complexes chain complexes, despite the fact that our differentials usually have degree 1 with respect to the homological grading.
- Our filtrations are *descending*, which is to say that $\mathscr{F}_i M \supseteq \mathscr{F}_j M$ when $i \leq j$.
- A *filtered quasi-isomorphism* $f\colon A \to B$ is a filtered chain map which induces a quasi-isomorphism between the associated graded complexes $\mathrm{gr}(f)\colon \mathrm{gr}(A) \to \mathrm{gr}(B)$. In other words, a filtered quasi-isomorphism induces a quasi-isomorphism between E_0–pages of spectral sequences, and equivalently induces isomorphisms between E_1–pages. If A and B are connected by a zigzag of filtered quasi-isomorphisms, then they have the same weak filtered homotopy type, a relationship which we denote by $A \simeq B$.
- Because the E_1–page of the filtered complex C_2^- is isomorphic to the Khovanov *complex*, and not the Khovanov *homology*, we need to work with invariance maps which are not filtered quasi-isomorphisms. Instead, they only induce quasi-isomorphisms on the E_1–pages, or equivalently induce isomorphisms on the E_2–pages. We call these maps E_1*–quasi-isomorphisms* (terminology from [Cirici et al. 2020]). As above, we write $A \simeq_1 B$ to denote that A and B are connected by a zigzag of E_1–quasi-isomorphisms.
- Since we work with two different notions of weak equivalence, we also need two different mapping cones for a filtered map $f\colon A \to B$, denoted by $\mathrm{cone}(f)$ and $\mathrm{cone}_1(f)$. Both of them have the same underlying unfiltered complex, but differ in the definition of the filtration. The former filtration is defined to be $\mathscr{F}_i(\mathrm{cone}(f)) = \mathscr{F}_i A \oplus \mathscr{F}_i B$, whereas the latter filtration is given by $\mathscr{F}_i(\mathrm{cone}_1(f)) = \mathscr{F}_i A \oplus \mathscr{F}_{i-1} B$.

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2 The spectral sequence

In this section, we review the construction of the spectral sequence from $\overline{\mathrm{Kh}}(L)$ to $\widehat{HFK}(L)$ for a link L, as originally defined by Dowlin [2024]. The spectral sequence arises from a filtered chain complex $\mathrm{C}_2^-(D)$ constructed from a *partially singular braid diagram* D associated to an unoriented link L. In Section 2.A, we define these diagrams, and in Section 2.B we associate a filtered chain complex to each such diagram. Finally, in Section 2.C, we discuss how to associate a partially singular braid diagram to an unoriented link L, and we characterize the set of moves connecting any two such partially singular braid diagrams.

2.A Partially singular braid diagrams

In this section, we define the types of diagrams we need to construct the spectral sequence.

We start by establishing some conventions regarding braid diagrams. If D is a closed braid diagram, we can consider it as a 4–valent graph embedded in $\mathbb{R}^2$ with vertices $V(D)$ the set of crossings, and edges $E(D)$ the set of arcs connecting these crossings. This agrees with the usual way of representing link diagrams as graphs. Given a graph G, recall that a *subdivision* H of G is a graph obtained by adding 2–valent vertices along edges of G.

Definition 2.1 A (*closed*) *partially singular braid diagram* is an oriented graph embedded in $\mathbb{R}^2$ which can be obtained as a subdivision of a closed braid diagram, equipped with the following extra information:

- a labeling of every 4–valent vertex as "positive", "negative", or "singular",
- a further labeling of every singular vertex as either "fixed" or "free", and
- exactly one distinguished edge, which is called the "decorated" edge.

An *open* partially singular braid diagram is defined identically to a closed one, except that it also has $2n$ 1–valent vertices (assuming n strands) corresponding to the endpoints of the strands. When drawing partially singular braid diagrams, we indicate fixed singular vertices by drawing a circle around them, as

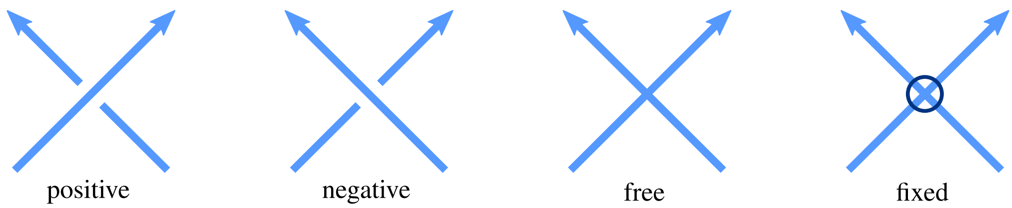


Figure 1: The different types of vertices in a partially singular braid diagram.

Figure 2: Other features that can occur in a braid diagram.

in Figure 1; 2–valent vertices are drawn simply as dots on the strands, and the decorated edge is denoted by two small lines, as in Figure 2.

Throughout, we assume the decorated edge is leftmost in the diagram. We also assume a fixed ordering of the vertices whenever we consider a partially singular braid diagram D. We let $\mathrm{Fixed}(D)$ denote the set of fixed singular vertices of D and $\mathrm{Free}(D)$ denote the set of free singular vertices of D.

A (*fully*) *singular braid diagram* is a partially singular braid diagram with no crossings. This type of diagram may arise from resolving a partially singular braid diagram D in the following sense. Let D be a partially singular braid diagram, with $c(D)$ the set of crossings of D; then a *resolution*, a function $I\colon c(D) \to \{0,1\}$, gives a fully singular braid diagram D_I by resolving each crossing according to Figure 3. In words, the 0–resolution of a positive crossing is a singular vertex, and the 1–resolution is the oriented smoothing with two subdivided edges. The 0– and 1–resolutions of a negative crossing are the 1– and 0–resolutions of a positive crossing, respectively. If a fully singular braid diagram S arises as a complete resolution of a partially singular braid diagram D, then $\mathrm{Fixed}(S) = \mathrm{Fixed}(D)$, and $\mathrm{Free}(S)$ contains all crossings in $\mathrm{Free}(D)$ as well as those which were singularized in the resolution.

2.B The filtered complex $\mathrm{C}_2^-(D)$

In this section, we recall Dowlin's construction of the filtered chain complex $\mathrm{C}_2^-(D)$ which gives rise to the spectral sequence connecting Khovanov homology to knot Floer homology. Throughout, let D be a

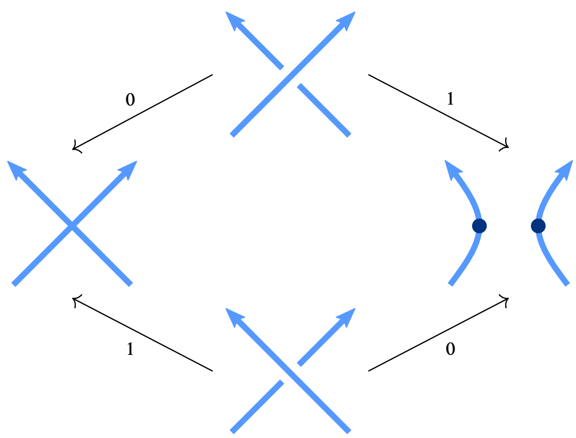


Figure 3: The 0– and 1–resolutions of positive and negative crossings.

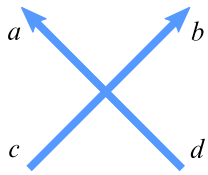


Figure 4: The local edge labels around a vertex.

partially singular braid diagram and I a resolution of D giving rise to the fully singular braid diagram D_I. We first construct $\mathrm{C}_2^-(D_I)$ for each resolution I, then combine these into a cube complex $\mathrm{C}_2^-(D)$ by adding "edge maps".

To begin, label each edge of D by a unique integer from 1 to $k = |E(D)|$, and let $R(D) = \mathbb{Q}[U_1, \dots, U_k]$ be the polynomial ring over $\mathbb{Q}$ generated by one variable for each edge. Note that, whenever crossings in D are resolved to get a diagram D', there is a natural bijection between edges in D and edges in D', so we can extend our edge labels to any resolution of D. To each vertex $v \in V(D)$ we associate two polynomials, $L(v)$ and $L^+(v)$. Label the adjacent edges to each vertex $v \in V(D)$ as in Figure 4; if we draw the vertex such that all edges are oriented upwards, then we label the edge in the top left by a, the remaining edges by b, c, and d as we traverse clockwise from the edge labeled a. Define $L(v) = U_a + U_b - U_c - U_d$ and $L^+(v) = U_a + U_b + U_c + U_d$.

One factor of $\mathrm{C}_2^-(D_I)$ does not depend on the specific resolution but only on D; we denote this factor by $\mathscr{L}_D^+$. Let

$$\mathscr{L}_D^+ := \bigotimes_{v \in \mathrm{Fixed}(D)} R(D) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(D).$$

It should be noted that $\mathscr{L}_D^+$ is not a chain complex, but rather a *matrix factorization* (or *curved complex*). A *matrix factorization* is a module M equipped with an endomorphism $\partial \colon M \to M$ such that $\partial^2 = \omega \,\mathrm{id}_M$ for some potentially nonzero scalar ω, which is called the *potential* of the matrix factorization. Despite the fact that ∂ does not square to zero, we may still refer to it as a *differential* on M; this is hopefully clear from context. In the case of $\mathscr{L}_D^+$, $\omega = \sum_{v \in \mathrm{Fixed}(D)} L(v)L^+(v)$, which is often nonzero in $R(D)$. Matrix factorizations are well-studied algebraic objects, but for our purposes we only need a few facts about them; these can be found in Section 3.

The other factor of $\mathrm{C}_2^-(D_I)$, which is not the same for every I and depends on the specific resolution, is the $R(D)$–module $Q(D_I) = R(D)/(L(D_I) + N(D_I))$. Here, $L(D_I)$ and $N(D_I)$ are two ideals of $R(D)$. The first of these is the *linear ideal* $L(D_I)$, defined as

$$L(D_I) := \sum_{v \in \mathrm{Free}(D_I)} (L(v)).$$

The second is the *nonlocal ideal* $N(D_I)$. Let Ω be a smoothly embedded disk in $\mathbb{R}^2$ that does not contain the decorated edge, and such that the boundary only intersects D transversely at edges. Let $\mathrm{In}(\Omega)$ (resp.

Out(Ω)) denote the set of edges that intersect the boundary of Ω and are oriented inward (resp. outward). We define $N(\Omega)$ to be the polynomial

$$N(\Omega) := \prod_{i \in \mathrm{Out}(\Omega)} U_i - \prod_{j \in \mathrm{In}(\Omega)} U_j.$$

The nonlocal ideal $N(D_I)$ is then generated by $N(\Omega)$ for all such embedded disks Ω:

$$N(D_I) := \sum_{\Omega} (N(\Omega)).$$

With the above definitions in hand, the complex $\mathrm{C}_2^-(D_I)$ is then defined as

$$\begin{aligned} \mathrm{C}_2^-(D_I) &:= Q(D_I) \otimes \mathscr{L}_D^+ \\ &:= R(D)/(L(D_I) + N(D_I)) \otimes \Bigg(\bigotimes_{v \in \mathrm{Fixed}(D)} R(D) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(D) \Bigg). \end{aligned}$$

It is shown in [Dowlin 2024, Lemma 2.4] that the potential ω of $\mathscr{L}_D^+$ is contained in $L(D_I) + N(D_I)$, and thus is zero in $Q(D_I)$. Thus, the endomorphism of $\mathrm{C}_2^-(D_I)$ induced by $\mathscr{L}_D^+$ squares to 0, so it is truly a differential; we denote it by d_0.

As a module, define

$$\mathrm{C}_2^-(D) := \bigoplus_{I \in \{0,1\}^{c(D)}} \mathrm{C}_2^-(D_I).$$

The differential on $\mathrm{C}_2^-(D)$ is defined as a sum $d_0 + d_1$, where d_0 is induced by the differential d_0 on the summands $\mathrm{C}_2^-(D_I)$, and d_1 is induced by edge maps that we have yet to define. In order to do so, we must first restrict the set of partially singular braid diagrams we are working with.

Definition 2.2 [Dowlin 2024, Definition 2.2] The set $\mathscr{D}^{\mathscr{R}}$ contains all partially singular braid diagrams D satisfying the following conditions for all $I \in \{0,1\}^{c(D)}$:

- D_I is connected, and
- the linear terms $L(v)$ for $v \in \mathrm{Free}(D_I)$ form a regular sequence[1] over $R(D)/N(D_I)$.

The latter condition is an algebraic restriction which is used in the proof of Theorem 3.1. It is equivalent to the existence of an ordering $v_1, \dots, v_k$ of the vertices in $\mathrm{Free}(D_I)$ such that $L(v_j)$ is not a zero divisor in $R(D)/(N(D_I) + (L(v_1), \dots, L(v_{j-1})))$ for each $1 \le j \le k$. Since $R(D)$ is a graded ring and the linear terms $L(v)$ are homogeneous of positive degree, if this condition is true for one ordering of $\mathrm{Free}(D_I)$, it is true for *every* ordering.

For the rest of the definition of $\mathrm{C}_2^-(D)$, we assume $D \in \mathscr{D}^{\mathscr{R}}$. Let I and J be two resolutions with $I \prec J$, ie I and J agree on all crossings except a single $c \in c(D)$, where $I(c) = 0$ and $J(c) = 1$. Let v be the vertex corresponding to c, and label the edges adjacent to v according to Figure 4.

[1]The $\mathscr{R}$ in $\mathscr{D}^{\mathscr{R}}$ likely stands for "regular".

The edge map $d_{I,J}: \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D_J)$ depends on whether c is a positive or negative crossing. If I and J differ at a positive crossing, let $\phi_+: Q(D_I) \to Q(D_J)$ be the unique $R(D)$–module map such that $\phi_+(1) = 1$, and define the edge map $d_{I,J}: \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D_J)$ to be $d_{I,J} = \phi_+ \otimes \mathrm{id}_{\mathscr{L}_D^+}$. Else, I and J differ at a negative crossing v. In this case, let $\phi_-: Q(D_I) \to Q(D_J)$ be the unique $R(D)$–module map such that $\phi_-(1) = U_b - U_c$, and define the edge map $d_{I,J}: \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D_J)$ to be $d_{I,J} = \phi_- \otimes \mathrm{id}_{\mathscr{L}_D^+}$. We may occasionally overload notation by referring to the edge map $d_{I,J}$ as $\phi_\pm$ when there is no risk of confusion.

Combining all of these maps together into a single map induces $d_1: \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D)$, given by

$$d_1 := \sum_{I \prec J} \epsilon(I, J) d_{I,J}.$$

Here, $\epsilon(I, J)$ is a *sign assignment*, which is a labeling of the edges of the cube of resolutions by $\{\pm 1\}$ satisfying the property that every square face has an odd number of -1–labeled edges. Such a sign assignment ensures that $(d_1)^2 = 0$, and any two choices of ϵ result in isomorphic complexes. As one example, we may let $\epsilon(I, J) = (-1)^k$, where k is the number of 1's that come before the place at which I and J differ, as in [Bar-Natan 2002]. We further abuse notation by referring to d_1, the signed sum of the edge maps $d_{I,J}$ for all $I \prec J$, itself as an edge map.

Consider $\mathrm{C}_2^-(D)$ as a chain complex with total differential $d_0 + d_1$. We filter $\mathrm{C}_2^-(D)$ by weight in the cube of resolutions, ie the filtration on $\mathrm{C}_2^-(D)$ is given by

$$\mathscr{F}_p \mathrm{C}_2^-(D) := \bigoplus_{w(I) \geq p} \mathrm{C}_2^-(D_I),$$

where $w(I) = \sum_{c \in c(D)} I(c)$ is the *weight* of I, ie the number of 1–resolved crossings of D_I. Note that d_0 preserves the weight, and d_1 increases it by 1, so the differential on $\mathrm{C}_2^-(D)$ is indeed filtered with respect to this decomposition.

Remark 2.3 We could have alternately defined $\mathrm{C}_2^-(D)$ by first defining $\mathrm{C}_2^-(S)$ for fully singular braid diagrams S, then defining $\mathrm{C}_2^-(D)$ to be the mapping cone

$$\mathrm{C}_2^-(D) := \mathrm{cone}_1(\phi \otimes \mathscr{L}_D^+) = (\mathrm{C}_2^-(D_0) \to \mathrm{C}_2^-(D_1)),$$

where D_0 and D_1 above are the 0– and 1–resolutions of a particular crossing, and $\phi: Q(D_0) \to Q(D_1)$ is the associated map of quotient modules. Iterating this construction produces a filtered complex that is isomorphic to the one that we defined previously.

2.C Diagrams associated to a link

Each partially singular braid diagram gives rise to an unoriented link by taking the *unoriented smoothing*.

Definition 2.4 Let D be a partially singular braid diagram. The *unoriented smoothing* $\mathrm{sm}(D)$ is the unoriented link obtained from D by smoothing each singular vertex in the way that does not respect the orientation.

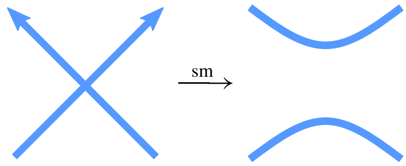


Figure 5: Unoriented smoothing of a crossing.

Figure 5 shows a local picture of smoothing a singular vertex, and Figure 6 gives an example of a partially singular braid diagram and the link obtained by taking the unoriented smoothing.

When $\mathrm{sm}(D)$ is an ℓ–component link, we can construct a "reduced" version of $\mathrm{C}_2^-(D)$. First, choose a set of edges $e_1, \ldots, e_\ell \in E(D)$ such that each e_i is on a distinct component of $\mathrm{sm}(D)$. Then, let

$$\widehat{C}_2(D) := \mathrm{C}_2^-(D) \otimes \bigotimes_{e_i} (R(D) \xrightarrow{U_{e_i}} R(D)).$$

We define the differentials given by multiplication by U_{e_i} to have weight filtration degree 1. Therefore, we get a weight filtration on $\widehat{C}_2(D)$ induced by the above definition as a tensor product of filtered complexes. This is the filtered complex that is used to define the spectral sequence relating Khovanov homology and knot Floer homology.

Theorem 2.5 [Dowlin 2024, Theorem 1.6] *Let $D \in \mathcal{D}^{\mathcal{R}}$ be a partially singular braid diagram with $\mathrm{sm}(D) = L$. The spectral sequence induced by the weight filtration on $\widehat{C}_2(D)$ has E_2–page isomorphic to $\overline{\mathrm{Kh}}(L)$ and converges to $\widehat{HFK}(L)$.*

Dowlin [2024] proves that every link can be realized as the unoriented smoothing of a diagram in $\mathcal{D}^{\mathcal{R}}$ by first considering a braid whose plat closure is the desired link, then turning that braid into a partially singular braid diagram. We go about things similarly, but instead choose a different way of embedding braid closures into $\mathcal{D}^{\mathcal{R}}$ that better fits our particular invariance proofs.

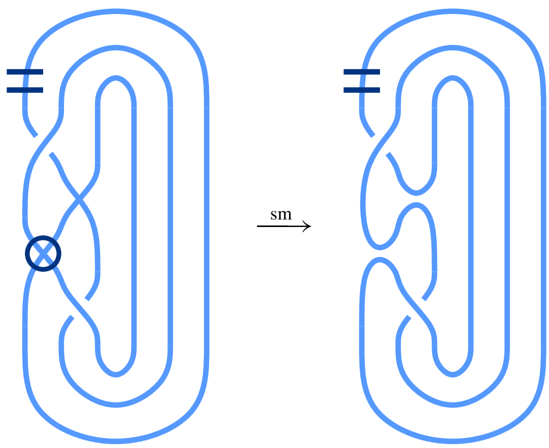


Figure 6: A diagram D and its unoriented smoothing $\mathrm{sm}(D)$.

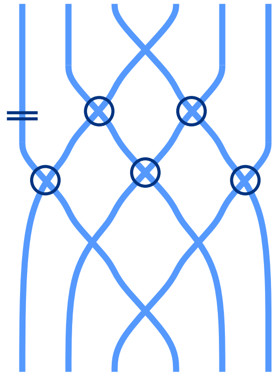


Figure 7: The partially singular braid diagram I_n in the case $n = 3$.

Proposition 2.6 *Let L be an unoriented link. There is a partially singular braid diagram $D \in \mathfrak{D}^{\mathcal{R}}$ such that* $\mathrm{sm}(D) = L$.

To prove Proposition 2.6, we make use of a special partially singular open braid diagram which we denote by I_n. This open diagram I_n consists of $2n$ upward oriented strands with $2n-1$ layers of singular vertices. The layers are symmetric, meaning layer i has singular vertices between the same strands as layer $2n-i$ for $1 \le i < n$. The first layer has a singular vertex between the strands n and $n+1$. The second layer has two singular vertices; one between strands $n-1$ and n and one between strands $n+1$ and $n+2$. In general, the i^{th} layer has i consecutive singular vertices, beginning with one between strands $n+1-i$ and $n+2-i$ and ending with one between strands $n-1+i$ and $n+i$. We let $\mathrm{Fixed}(I_n)$ be the singular vertices in layers n and $n+1$, and let $\mathrm{Free}(I_n)$ be the rest of the singular vertices. See Figure 7 for I_n in the case $n = 3$.

Definition 2.7 Given a braid $\beta \in B_n$, let $I_n(\beta)$ denote the partially singular braid diagram D built by putting n downward-oriented strands to the right of β, and putting I_n above and taking the braid closure.

Proof of Proposition 2.6 Given an unoriented link L, let β be a braid with braid closure $\mathrm{cl}(\beta)$ isotopic to L, the existence of which is guaranteed by Alexander's theorem [1923]. The unoriented smoothing $\mathrm{sm}(I_n(\beta))$ is isotopic to the braid closure $\mathrm{cl}(\beta)$ of β itself, so $D = I_n(\beta)$ is a partially singular braid diagram with $\mathrm{sm}(D)$ isotopic to L. That $D \in \mathfrak{D}^{\mathcal{R}}$ is an application of [Dowlin 2024, Lemma 7.1]. More specifically, D contains a vertically mirrored copy of the open braid diagram S_{2n} defined in [Dowlin 2024], where it is proven that any such diagram is in $\mathfrak{D}^{\mathcal{R}}$. □

See Figure 8 for an example of the process of constructing a partially singular braid diagram with specified unoriented smoothing.

Let $\mathfrak{D}^{\mathcal{B}} = \{I_n(\beta) \mid \beta \in B_n, n \in \mathbb{Z}\}$ be the set of partially singular braid diagrams constructed as above.[2] Then we have the following classification theorem.

[2]Here, the $\mathcal{B}$ in $\mathfrak{D}^{\mathcal{B}}$ stands for "braid".

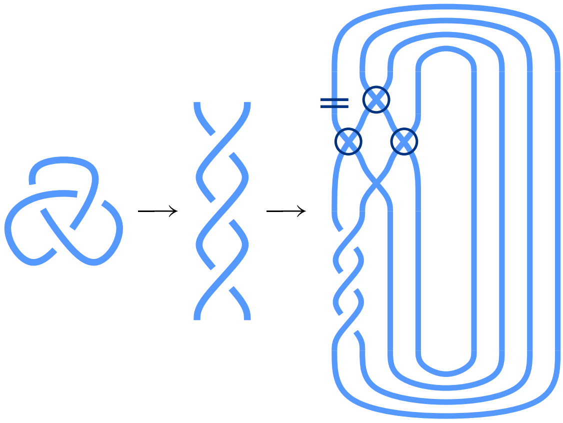


Figure 8: The process of constructing a partially singular braid diagram with smoothing isotopic to a given knot.

Theorem 2.8 *Two diagrams in $\mathscr{D}^{\mathscr{B}}$ have the same unoriented smoothing if and only if the underlying braids are connected by a finite sequence of Reidemeister II moves, Reidemeister III moves, (de)stabilizations, and conjugations.*

Proof This is just Markov's theorem [1936], repackaged. □

Since $\mathscr{D}^{\mathscr{B}} \subset \mathscr{D}^{\mathscr{R}}$, we can construct the complex $\mathrm{C}_2^-(D)$ for any $D \in \mathscr{D}^{\mathscr{B}}$. We overload notation by writing $\mathrm{C}_2^-(\beta)$ instead of $\mathrm{C}_2^-(I_n(\beta))$ for $\beta \in B_n$. We prove invariance of $\mathrm{C}_2^-(\beta)$ under the moves in Theorem 2.8 in Section 5 using maps defined in Section 4.

3 Vertex relabeling

Before we continue towards a proof of invariance, we detour to comment on a quirk of the construction of $\mathrm{C}_2^-(D)$. One natural question to ask is why $\mathrm{C}_2^-(D)$ treats fixed and free singular vertices differently. It turns out that, in order for $H_*(\mathrm{C}_2^-(D))$ to be isomorphic to $\widehat{HFK}(\mathrm{sm}(D))$, our diagram D needs to be in $\mathscr{D}^{\mathscr{R}}$, which means satisfying the regular sequence condition. This condition cannot be satisfied unless D contains sufficiently many fixed vertices in a sufficiently nice arrangement. On the other hand, we only know how to define the edge maps $d_{I,J}$ on free vertices, so we cannot make all of our vertices fixed either.

As a sort of compromise, we choose some of the vertices to be fixed and some to be free. We do not need to worry about which choice we have made when proving invariance under Reidemeister moves II and III in Section 5, since they only involve local pictures of diagrams which contain some crossings but no singular vertices. While not a local move, we define stabilization to be compatible with our vertex labeling as well. Conjugation, however, requires us to change which vertices are fixed and which are free; this is what motivates the following theorem.

While it is not immediately obvious, it turns out that the homotopy type of $\mathrm{C}_2^-(D)$ does not depend on the particular labeling of vertices as fixed or free in the following sense:

Theorem 3.1 *If $D, D' \in \mathfrak{D}^{\mathcal{R}}$ are identical partially singular braid diagrams up to relabeling of fixed and free vertices, then $\mathrm{C}_2^-(D) \simeq \mathrm{C}_2^-(D')$.*

To prove this, we need to introduce a slight variation on the technique of "excluding a variable" from [Rasmussen 2015, Lemma 3.8] or [Khovanov and Rozansky 2008a, Proposition 9]. Both sources are also good references for the relevant details on matrix factorizations, including the statement below on the effect of change of basis on matrix factorizations.

We include the necessary details on matrix factorizations below. Let R be a ring. For $a, b \in R$, let $\{a, b\}$ denote the matrix factorization

$$\{a,b\} := R \underset{a}{\overset{b}{\rightleftarrows}} R.$$

For $\vec{a}, \vec{b} \in R^n$, let

$$\{\vec{a},\vec{b}\} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$$

denote the matrix factorization

$$\{\vec{a},\vec{b}\} := \bigotimes_{i=1}^{n} \{a_i, b_i\} = \bigotimes_{i=1}^{n} R \underset{a_i}{\overset{b_i}{\rightleftarrows}} R.$$

We have already seen a matrix factorization of this form; if we let $\vec{a} = (L^+(v_1), \dots, L^+(v_n))$ and $\vec{b} = (L(v_1), \dots, L(v_n))$ for a partially singular braid diagram D with $\mathrm{Fixed}(D) = \{v_1, \dots, v_n\}$, then $\mathcal{L}_D^+ = \{\vec{a}, \vec{b}\}$. By definition, the potential ω associated to the matrix factorization $\{\vec{a}, \vec{b}\}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + \dots + a_n b_n.$$

Starting with a matrix factorization $\{\vec{a}, \vec{b}\}$, we can perform a change of basis operation to get an isomorphic one. Specifically, sending $\vec{e_i}$ to $\vec{e_i} + c\vec{e_j}$ for standard basis vectors $\vec{e_i}$ and $\vec{e_j}$ of R^n has the effect of replacing the matrix factorization by $\{\vec{a}', \vec{b}'\}$, where

$$\vec{a}'_k = \begin{cases} \vec{a}_k + c\vec{a}_j & \text{if } k = i, \\ a_k & \text{otherwise,} \end{cases}$$

and

$$\vec{b}'_k = \begin{cases} \vec{b}_k - c\vec{b}_i & \text{if } k = j, \\ b_k & \text{otherwise.} \end{cases}$$

For more details, see [Khovanov and Rozansky 2008a; Rasmussen 2015].

Let $C = \{\vec{a}, \vec{b}\}$ be any matrix factorization over R. We can decompose

$$C = C' \underset{a_1}{\overset{b_1}{\rightleftarrows}} C',$$

where $C' = \{\vec{a}', \vec{b}'\}$ is the factorization obtained by omitting the first components of $\vec{a}$ and $\vec{b}$. Define $\pi : C \to C' \otimes R/(b_1)$ by $\pi((c_1, c_2)) = c_2 \otimes 1$. Before proving Theorem 3.1, we first prove that if the potential of C is 0 and b_1 is a nonzero divisor in R, then π is a quasi-isomorphism.

Lemma 3.2 *If the potential of C is 0 and b_1 is a nonzero divisor in R, then π is a quasi-isomorphism of chain complexes.*

Proof It is clear that π is surjective; since b_1 is a nonzero divisor, multiplication by b_1 is injective, so we have the following short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C' & \xrightarrow{1} & C' & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow\downarrow & & {\scriptstyle a_1b_1}\uparrow\downarrow{\scriptstyle 1} & & {\scriptstyle a_1}\uparrow\downarrow{\scriptstyle b_1} & & \uparrow\downarrow & & \uparrow\downarrow \\ 0 & \longrightarrow & C' & \xrightarrow{b_1} & C' & \longrightarrow & C' \otimes R/(b_1) & \longrightarrow & 0 \end{array}$$

Let C'' denote the first nonzero column in this sequence, the matrix factorization

$$C' \underset{a_1b_1}{\overset{1}{\rightleftarrows}} C'.$$

By the corresponding long exact sequence in homology, it suffices to show that C'' is acyclic in order to prove that π is a quasi-isomorphism. We write C'' in matrix form, then apply our above remarks about change of basis:

$$\begin{pmatrix} a_1b_1 & 1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \sim \begin{pmatrix} a_1b_1 + a_2b_2 & 1 \\ a_2 & 0 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \sim \begin{pmatrix} \omega & 1 \\ a_2 & 0 \\ \vdots & \vdots \\ a_n & 0 \end{pmatrix}.$$

Since we know that the potential $\omega = 0$, we see that

$$C'' = \{\vec{a}', \vec{0}\} \underset{0}{\overset{1}{\rightleftarrows}} \{\vec{a}', \vec{0}\},$$

and therefore is acyclic. □

With this lemma, we can now prove that $\mathrm{C}_2^-(D)$ is independent of vertex labeling for $D \in \mathfrak{D}^{\mathcal{R}}$:

Proof of Theorem 3.1 Let $S \in \mathfrak{D}^{\mathcal{R}}$ be a fully singular braid diagram, and let $w \in \mathrm{Fixed}(S)$ be some fixed vertex such that if w were instead free, the new diagram S' would still be in $\mathfrak{D}^{\mathcal{R}}$. Note that $R(S') = R(S)$, and $Q(S') = Q(S)/(L(w))$. Since $\mathrm{C}_2^-(S) = Q(S) \otimes \mathcal{L}_S^+$, we may consider $\mathrm{C}_2^-(S)$ as the matrix factorization $\{\vec{a}, \vec{b}\}$ over $R = Q(S)$ with $\vec{a} = (L^+(v))_{v \in \mathrm{Fixed}(S)}$ and $\vec{b} = (L(v))_{v \in \mathrm{Fixed}(S)}$.

Assume without loss of generality that $b_1 = L(w)$. Since $S' \in \mathscr{D}^{\mathscr{R}}$, we know that the linear terms $L(v)$ for $v \in \mathrm{Free}(S')$ form a regular sequence over $R(S')/N(S') = R(S)/N(S)$, and in particular, $L(w)$ is a nonzero divisor in $Q(S)$, since $w \in \mathrm{Free}(S')$. We then get that

$$\begin{aligned}
\mathrm{C}_2^-(S) &= Q(S) \otimes_{R(S)} \mathscr{L}_S^+ \\
&\cong \{\vec{a}, \vec{b}\} \\
&\simeq Q(S)/(L(w)) \otimes_{Q(S)} (Q(S) \otimes_{R(S)} \{\vec{a}', \vec{b}'\}) && \text{(by Lemma 3.2)} \\
&\simeq \big(Q(S)/(L(w)) \otimes_{Q(S)} Q(S)\big) \otimes_{R(S)} \{\vec{a}', \vec{b}'\} && \text{(by associativity of } \otimes) \\
&\simeq Q(S)/(L(w)) \otimes_{R(S)} \{\vec{a}', \vec{b}'\} \\
&\cong Q(S') \otimes_{R(S)} \mathscr{L}_{S'}^+ \\
&\cong Q(S') \otimes_{R(S')} \mathscr{L}_{S'}^+ && \text{(since } R(S) = R(S')) \\
&= \mathrm{C}_2^-(S').
\end{aligned}$$

Therefore, we see that changing a fixed vertex to a free one in a fully singular diagram does not change the homotopy type of $\mathrm{C}_2^-(-)$ as long as both diagrams are in $\mathscr{D}^{\mathscr{R}}$.

Now, we need to extend this result. Let $D, D' \in \mathscr{D}^{\mathscr{R}}$ be partially singular braid diagrams that differ only on the labeling of a single vertex $w \in \mathrm{Fixed}(D) \cap \mathrm{Free}(D')$. We know that $\mathrm{C}_2^-(D_I) \simeq \mathrm{C}_2^-(D'_I)$ for all $I \in \{0,1\}^{c(D)}$. In particular, we have a map in one direction: $\pi \colon \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ is a filtered quasi-isomorphism inducing the above equivalence. Therefore, it suffices to show that π commutes with the edge map d_1, which is the sum of $d_{I,J}$. Since π is linear over $Q(S)$, we get that it is additionally $R(S)$–linear via the natural quotient map, and therefore commutes with scalar multiplication by elements of $R(S)$. Since the edge maps $d_{I,J}$ are defined via scalar multiplication by 1 or $U_b - U_c$, we see that π does in fact commute with the edge maps, and therefore extends to a filtered quasi-isomorphism $\pi \colon \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D')$ by Lemma A.4.

Given any two diagrams $D', D'' \in \mathscr{D}^{\mathscr{R}}$ that differ only by some number of vertex labels, we can construct a diagram $D \in \mathscr{D}^{\mathscr{R}}$ with $\mathrm{Fixed}(D) = \mathrm{Fixed}(D') \cup \mathrm{Fixed}(D'')$, and therefore get that

$$\mathrm{C}_2^-(D') \simeq \mathrm{C}_2^-(D) \simeq \mathrm{C}_2^-(D''),$$

thus proving the general case. □

4 MOY moves

Murakami, Ohtsuki, and Yamada [Murakami et al. 1998] studied local operations on singular diagrams ("MOY moves"). While originally formulated for oriented planar trivalent graphs, they are relevant to us because one can think of singular vertices in our braids and braid resolutions as pairs of trivalent vertices instead. Two of these moves, MOY I and MOY III, represent planar isotopy when applied to the unoriented smoothing of a diagram, and thus are useful to make up for the fact that we cannot isotope

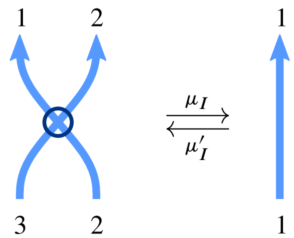


Figure 9: An MOY I move.

singularized crossings in the same ways that we can smoothed ones. The MOY II move corresponds to a cup/cap cobordism, but is rather more limited in its application. Nevertheless, these three moves suffice to define Reidemeister moves (and others) in Section 5. The maps that we choose to realize these moves are inspired by those used in [Khovanov and Rozansky 2008a; 2008b].

In this section, we construct filtered chain maps relating $\mathrm{C}_2^-(D)$ and $\mathrm{C}_2^-(D')$, where D and D' are partially singular braid diagrams connected by an MOY I, II, or III move.

4.A MOY I

Suppose D and D' are partially singular braid diagrams that differ by an MOY I move, as illustrated in Figure 9. In words, there is a fixed vertex v in D that meets the same edge e twice; without loss of generality, e is to the right of v. The diagram D' is then obtained from D by removing the edge e and relabeling v as a bivalent vertex.

Theorem 4.1 *There exist $R(D')$–linear filtered quasi-isomorphisms*

$$\mu_{\mathrm{I}}\colon \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D'), \quad \mu'_{\mathrm{I}}\colon \mathrm{C}_2^-(D') \to \mathrm{C}_2^-(D).$$

Under the identification $E_1(\mathrm{C}_2^-(-)) \cong \mathrm{CKh}^-(\mathrm{sm}(-))$, these maps induce the expected isomorphisms corresponding to planar isotopy.

First, suppose S and S' are fully singular braid diagrams that differ by an MOY I move, as illustrated in Figure 9. Specifically, S contains a fixed singular vertex v that meets the same edge twice. We would like to construct filtered chain maps $\mu_{\mathrm{I}}\colon \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ and $\mu'_{\mathrm{I}}\colon \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$. To start, let us characterize $\mathrm{C}_2^-(S)$ and $\mathrm{C}_2^-(S')$.

Without loss of generality, assume that the edge which is deleted by the MOY I move is to the right of the vertex. Label this edge with the variable U_2, label the top left edge U_1, and label the bottom left edge with U_3, again as in Figure 9.

Let R be the polynomial ring over all edges not shown in the local diagram; thus, $R(S') = R[U_1]$ and $R(S) = R(S')[U_2, U_3]$. We relate the associated quotient rings by the following proposition:

Proposition 4.2 *As $R(S')$ modules, $Q(S') \cong Q(S)/(U_1 + U_2)$.*

Proof We expand the right-hand side as a quotient of a free $R(S')$–module:

$$\begin{aligned}Q(S)/(U_1+U_2) &\cong Q(S)\otimes_{R(S)} R(S)/(U_1+U_2)\\ &\cong R(S)/(L(S)+N(S))\otimes R(S)/(U_1+U_2)\\ &\cong R(S)/(L(S)+N(S)+(U_1+U_2))\\ &\cong R(S')[U_2,U_3]/(L(S)+N(S)+(U_1+U_2))\\ &\cong R(S')[U_2,U_3]/(L(S)+\widetilde{N}(S)+(U_2-U_3)+(U_1+U_2))\\ &\cong R(S')/(L(S)+\widetilde{N}(S)).\end{aligned}$$

In the above, $\widetilde{N}(S)$ is the sum of the nonlocal relations other than U_1-U_3; this is exactly equal to $N(S')$, as any region intersecting these local diagrams can be made to avoid U_2 and any intersections with U_3 can be isotoped to intersect U_1 instead. Further, $L(S)=L(S')$. Thus,

$$Q(S)/(U_1+U_2)\cong R(S')/(L(S)+\widetilde{N}(S))=R(S')/(L(S')+N(S'))=Q(S'),$$

as desired. □

Proposition 4.3 *The chain complexes* $\mathrm{C}_2^-(S)$ *and* $\mathrm{C}_2^-(S')$ *are quasi-isomorphic as complexes over* $R(S')$.

Proof We can use Proposition 4.2 to expand $\mathrm{C}_2^-(S)$:

$$\begin{aligned}\mathrm{C}_2^-(S) &= Q(S)\otimes_{R(S)}\mathscr{L}_S^+\\ &= Q(S)\otimes\left(R(S)\underset{U_1+2U_2+U_3}{\overset{U_1-U_3}{\rightleftarrows}} R(S)\otimes\widetilde{\mathscr{L}}_S^+\right)\\ &\cong Q(S)\otimes R(S)\underset{2U_1+2U_2}{\overset{0}{\rightleftarrows}} R(S)\otimes\widetilde{\mathscr{L}}_S^+ &&\text{(using relation } U_1-U_3 \text{ in } N(S))\\ &\simeq Q(S)\otimes R(S)/(U_1+U_2)\otimes\widetilde{\mathscr{L}}_S^+ &&\text{(replacing } 2U_1+2U_2 \text{ by the cokernel)}\\ &\cong \big(Q(S)\otimes R(S)/(U_1+U_2)\big)\otimes\widetilde{\mathscr{L}}_S^+\\ &\cong Q(S')\otimes_{R(S')}\mathscr{L}_{S'}^+ &&\text{(by Proposition 4.2)}\\ &= \mathrm{C}_2^-(S').\end{aligned}$$

In the above, let

$$\widetilde{\mathscr{L}}_S^+ = \bigotimes_{w\in\mathrm{Fixed}(D)\setminus\{v\}} R(D)\underset{L^+(w)}{\overset{L(w)}{\rightleftarrows}} R(D),$$

and note $\widetilde{\mathscr{L}}_S^+=\mathscr{L}_{S'}^+$. Note that we may replace the mapping cone of $2U_1+2U_2$ by its cokernel in the fourth line only after checking that $2U_1+2U_2$ is not a zero divisor in $Q(S)$; by the logic in the proof of Proposition 4.2, we may choose a generating set of relations for $N(S)+L(S)$, none of which contain a term with a nonzero power of U_2. Therefore, $Q(S)$ is isomorphic to a free polynomial ring over U_2; since $2U_1+2U_2$ is a unit multiple (over $\mathbb{Q}$) of a monic polynomial in U_2, we therefore get that it is not a zero divisor in $Q(S)$. □

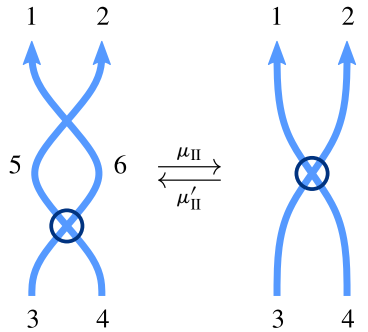


Figure 10: An MOY II move.

Let $\mu_{\mathrm{I}}\colon \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ be the quotient map implied by the above calculations. Explicitly on a simple tensor, $\mu_{\mathrm{I}}([r] \otimes (a,b) \otimes \tilde{s}) = [rb] \otimes \tilde{s}$. Let $\mu'_{\mathrm{I}}\colon \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$ be the splitting of μ_{I} given by inclusion into the first $R(S)$ summand in the equivalence of $R(S)/(U_1+U_2)$ and

$$R(S) \underset{2U_1+2U_2}{\overset{0}{\rightleftarrows}} R(S)$$

in the above proof. Explicitly on a simple tensor, $\mu'_{\mathrm{I}}([r] \otimes \tilde{s}) = [r] \otimes (0,1) \otimes \tilde{s}$.

For partially singular braid diagrams D and D' related by an MOY I move, we extend both maps to the cube of resolutions by defining $\mu_{\mathrm{I}}\colon \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ and $\mu'_{\mathrm{I}}\colon \mathrm{C}_2^-(D'_I) \to \mathrm{C}_2^-(D_I)$ as above for each $I \in \{0,1\}^{c(D)}$.

Proof of Theorem 4.1 It is clear that μ_{I} and μ'_{I} are filtered maps, since they are defined componentwise on the cube of resolutions.

We need to check that μ_{I} and μ'_{I} are chain maps, ie that they commute with the edge map d_1. Let $I, J \in \{0,1\}^{c(D)}$ with $I \prec J$. If I and J differ at a positive crossing, then $d_{I,J}$ is given by

$$\phi_+ \otimes \mathcal{L}_D^+ = 1 \otimes \mathcal{L}_D^+.$$

Otherwise, $d_{I,J}$ is given by $\phi_- \otimes \mathcal{L}_D^+ = (U_b - U_c) \otimes \mathcal{L}_D^+$. Either way, the edge maps are given by multiplication by an element of $R(D')$. Since μ_{II} and μ'_{II} were defined to be $R(D')$–linear, we get that they commute with d_1. □

4.B MOY II

Suppose D and D' are partially singular braid diagrams with D' the result of applying an MOY II move to D and reducing the number of crossings, as shown in Figure 10. In words, D contains a free vertex v_1, a fixed vertex v_2, and two edges e_5 and e_6 from v_2 to v_1. The diagram D' is obtained from D by removing e_5 and e_6 and merging v_1 and v_2 into a single fixed vertex.

Theorem 4.4 *There exists a direct sum decomposition $\mathrm{C}_2^-(D) \cong \mathrm{C}_2^-(D') \oplus \mathrm{C}_2^-(D')$ as filtered chain complexes over $R(D')$. Define $\mu_{\mathrm{II}}\colon \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D')$ to be projection onto the second summand,*

and define $\mu'_{\mathrm{II}}\colon \mathrm{C}_2^-(D') \to \mathrm{C}_2^-(D)$ *to be inclusion into the first summand. Under the identification* $E_1(\mathrm{C}_2^-(-)) \cong \mathrm{CKh}^-(\mathrm{sm}(-))$, *the maps* μ_{II} *and* μ'_{II} *induce the maps on Khovanov homology corresponding to the cobordisms which delete and introduce a circle*, *respectively.*

To start, let S and S' be fully singular braid diagrams again with S' the result of applying an MOY II move to S reducing the number of crossings. Let R be the polynomial ring over all edges not shown in the local diagrams, so that $R(S') = R[U_1, U_2, U_3, U_4]$, and $R(S) = R(S')[U_5, U_6]$.

Proposition 4.5 *As an* $R(S')$*–module*, $Q(S) \cong Q(S')\langle 1\rangle \oplus Q(S')\langle U_6\rangle$.

Proof First, note that $Q(S) = Q(S')[U_5, U_6]/(U_5 + U_6 - U_1 - U_2, U_5U_6 - U_1U_2)$. We do not need to consider any other nonlocal relations, as any region Ω intersecting these diagrams can be isotoped away from U_5 and U_6 to give an equivalent or stronger relation. We want to prove that $\{1, U_6\}$ is a basis for $Q(S)$ over $Q(S')$. To see that $\{1, U_6\}$ is a generating set, it is enough to note that in $Q(S)$, $U_5 = U_1 + U_2 - U_6$, and that $(U_1 + U_2 - U_6)U_6 - U_1U_2 = 0$, so $U_6^2 = (U_1 + U_2)U_6 - U_3U_4$. Linear independence follows from the fact that $U_6^2 - (U_1 + U_2)U_6 + U_1U_2$ is a monic polynomial of degree 2 in U_6. □

Using this proposition, we can decompose

$$\begin{aligned}
\mathrm{C}_2^-(S) &\cong Q(S) \otimes_{R(S)} \mathscr{L}_S^+ \\
&\cong Q(S) \otimes_{R(S)} \Bigl(\bigotimes_{v\in \mathrm{Fixed}(S)} R(S) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(S) \Bigr) \\
&\cong Q(S) \otimes_{R(S)} \Bigl(\bigotimes_{v\in \mathrm{Fixed}(S')} R(S) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} R(S) \Bigr) \\
&\cong \bigotimes_{v\in \mathrm{Fixed}(S')} Q(S) \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S) \\
&\cong \bigotimes_{v\in \mathrm{Fixed}(S')} Q(S')\langle 1\rangle \oplus Q(S')\langle U_6\rangle \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S')\langle 1\rangle \oplus Q(S')\langle U_6\rangle \\
&\cong \Bigl(\bigotimes_{v\in \mathrm{Fixed}(S')} Q(S')\langle 1\rangle \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S')\langle 1\rangle \Bigr) \oplus \Bigl(\bigotimes_{v\in \mathrm{Fixed}(S')} Q(S')\langle U_6\rangle \underset{L^+(v)}{\overset{L(v)}{\rightleftarrows}} Q(S')\langle U_6\rangle \Bigr) \\
&\cong (Q(S')\langle 1\rangle \otimes_{R(S')} \mathscr{L}_{S'}^+) \oplus (Q(S')\langle U_6\rangle \otimes_{R(S')} \mathscr{L}_{S'}^+) \\
&\cong \mathrm{C}_2^-(S')\langle 1\rangle \oplus \mathrm{C}_2^-(S')\langle U_6\rangle.
\end{aligned}$$

Define $\mu_{\mathrm{II}}\colon \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ to be projection onto the second summand in the above decomposition, and define $\mu'_{\mathrm{II}}\colon \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$ to be inclusion into the first summand. For partially singular braid

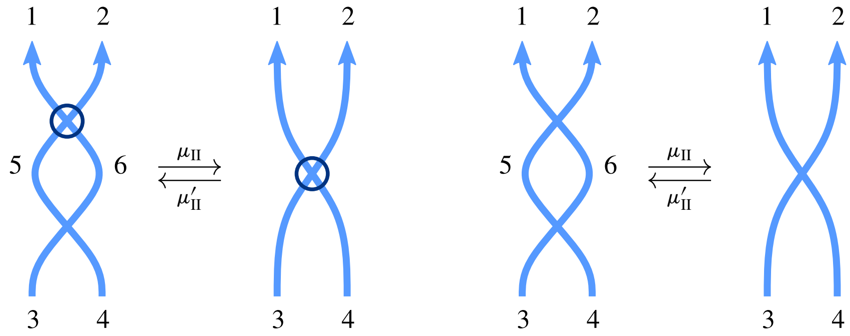


Figure 11: Variations of the MOY II move.

diagrams D and D' related by an MOY II move, we extend both maps to the cube of resolutions by defining $\mu_{\mathrm{II}}\colon \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ and $\mu'_{\mathrm{II}}\colon \mathrm{C}_2^-(D'_I) \to \mathrm{C}_2^-(D_I)$ as above for each $I \in \{0,1\}^{c(D)}$.

Proof of Theorem 4.4 It is clear that μ_{II} and μ'_{II} are filtered maps, since they are defined componentwise on the cube of resolutions. Next, we need to check that μ_{II} and μ'_{II} are chain maps, ie that they commute with the edge map d_1. Let $I, J \in \{0,1\}^{c(D)}$ with $I \prec J$. If I and J differ at a positive crossing, then $d_{I,J}$ is given by $\phi_+ \otimes \mathscr{L}_D^+ = 1 \otimes \mathscr{L}_D^+$. Otherwise, $d_{I,J}$ is given by $\phi_- \otimes \mathscr{L}_D^+ = (U_b - U_c) \otimes \mathscr{L}_D^+$. Either way, the edge maps are given by multiplication by an element of $R(D')$. Since μ_{II} and μ'_{II} were defined to be $R(D')$–linear, we get that they commute with d_1.

We used a direct sum decomposition of $\mathrm{C}_2^-(S)$ to define these maps on complete resolutions. We can see this direct sum decomposition on the cube of resolutions as well. Specifically, we have a split exact sequence:

$$0 \longrightarrow \mathrm{C}_2^-(D') \underset{1\otimes\mathscr{L}_D^+}{\overset{\mu'_{\mathrm{II}}}{\rightleftarrows}} \mathrm{C}_2^-(D) \underset{(U_6-U_1)\otimes\mathscr{L}_D^+}{\overset{\mu_{\mathrm{II}}}{\rightleftarrows}} \mathrm{C}_2^-(D') \longrightarrow 0$$

Finally, we want to show that these maps induce the correct morphisms on the Khovanov complex. The cobordism corresponding to the introduction of a circle is induced by multiplication by 1 [Bar-Natan 2005]. This should correspond to μ'_{II}, which we can see also induces multiplication by 1 on homology. The cobordism corresponding to the deletion of a circle should send $1 \mapsto 0$ and $X \mapsto 1$, where X is a variable associated to the shrinking circle. In our case, μ_{II} maps $1 \mapsto 0$ and $U_6 \mapsto 1$, inducing this same map on homology. □

One can repeat the same argument to show that we also have similar MOY II decompositions for the cases in Figure 11.

4.C MOY III

Suppose D and D' are fully singular braid diagrams with D' the result of applying an MOY III move to D and reducing the number of crossings, as shown in Figure 12. In words, D contains a fixed vertex v_1, free vertices v_2 and v_3, and edges $e_7\colon v_2 \to v_1$, $e_8\colon v_3 \to v_1$, and $e_9\colon v_3 \to v_2$. The diagram D' is obtained

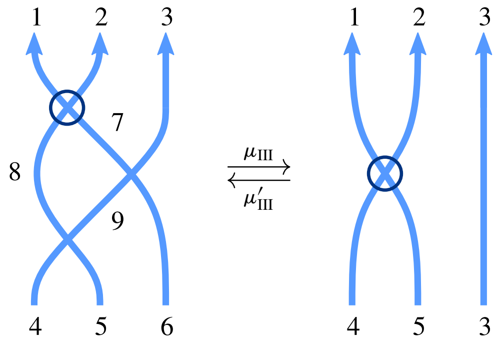


Figure 12: An MOY III move.

from D by removing the edges e_7, e_8, and e_9, merging v_1 and v_3 into a single fixed vertex, removing v_2, and merging e_6 into e_3.

Theorem 4.6 *There exist $R(D')$–linear filtered quasi-isomorphisms $\mu_{\mathrm{III}}\colon \mathrm{C}_2^-(D) \to \mathrm{C}_2^-(D')$ and $\mu'_{\mathrm{III}}\colon \mathrm{C}_2^-(D') \to \mathrm{C}_2^-(D)$. Furthermore, $\mathrm{C}_2^-(D')$ is isomorphic to a direct summand of $\mathrm{C}_2^-(D)$. Under the identification $E_1(\mathrm{C}_2^-(-)) \cong \mathrm{CKh}^-(\mathrm{sm}(-))$, these maps induce the expected isomorphisms corresponding to planar isotopy.*

Analogously to the MOY I and II cases, we again start by defining these maps on fully singular braid diagrams, then extending them to the cube of resolutions. Let S and S' be fully singular braid diagrams with S' the result of applying an MOY III move to S reducing the number of crossings as in Figure 12. Below, our goal is to prove that $\mathrm{C}_2^-(S) \cong \mathrm{C}_2^-(S') \oplus \Upsilon_L$, where Υ_L is some acyclic complex. Furthermore, the MOY III move has a nontrivial horizontal mirroring. We also prove that in the case where S and S' are connected by an MOY III move which is the mirror of Figure 12, we have $\mathrm{C}_2^-(S) \cong \mathrm{C}_2^-(S') \oplus \Upsilon_R$. While it is true that $\Upsilon_R = \Upsilon_L$, we neither need this fact nor prove it in this paper. Nevertheless, we may refer to the complex as Υ all the same.

We construct a map $\mu_{\mathrm{III}}\colon \mathrm{C}_2^-(S') \to \mathrm{C}_2^-(S)$ and another map $\mu'_{\mathrm{III}}\colon \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S')$ which splits μ_{III}, thus proving that $\mathrm{C}_2^-(S')$ is a direct summand of $\mathrm{C}_2^-(S)$. Let S'' be the fully singular braid diagram

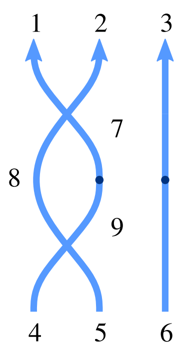


Figure 13: The fully singular diagram S'' used in the definitions of μ_{III} and μ'_{III}.

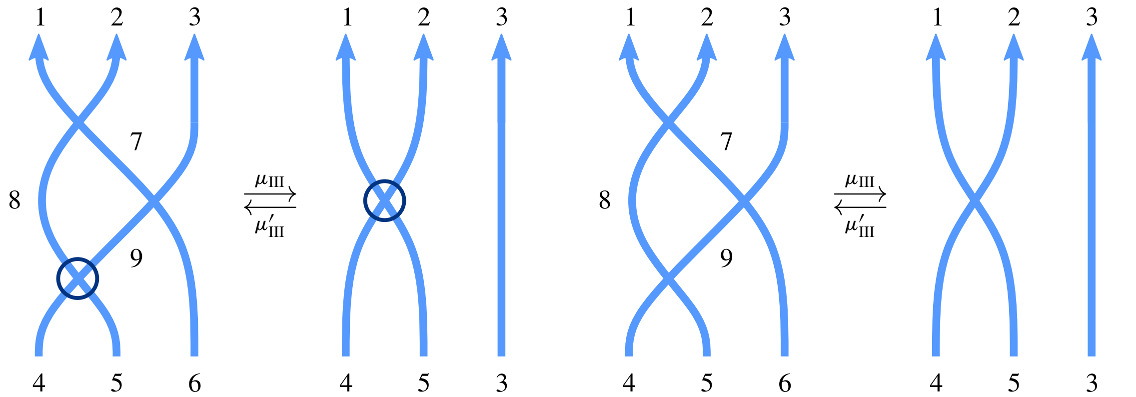


Figure 14: Variations of the MOY III move.

where the middle singular vertex v_2 is replaced by the oriented smoothing as in Figure 13, so that we may define the map $1 \otimes \mathscr{L}_S^+ \colon \mathrm{C}_2^-(S) \to \mathrm{C}_2^-(S'')$. We may then apply an MOY II move μ_{II} to the left two strands in S'' to get a map $\mu_{\mathrm{II}} \colon \mathrm{C}_2^-(S'') \to \mathrm{C}_2^-(S')$. Therefore, we define $\mu_{\mathrm{III}} = \mu_{\mathrm{II}} \circ (1 \otimes \mathscr{L}_S^+)$. We can also reverse the order of these operations to define $\mu'_{\mathrm{III}} = ((U_9 - U_3) \otimes \mathscr{L}_D^+) \circ \mu'_{\mathrm{II}}$. Note that the maps $1 \otimes \mathscr{L}_S^+$ and $(U_9 - U_3) \otimes \mathscr{L}_S^+$ are well defined since if v_2 were replaced by a positive or negative crossing, these would simply be multiples of the edge maps corresponding to resolutions of that crossing.

Proposition 4.7 *μ_{III} splits μ'_{III}, ie $\mu_{\mathrm{III}} \circ \mu'_{\mathrm{III}} = \mathrm{id}_{\mathrm{C}_2^-(S')}$.*

Proof We expand out the definitions of μ_{III} and μ'_{III} to get

$$\begin{aligned}
\mu_{\mathrm{III}} \circ \mu'_{\mathrm{III}} &= \mu_{\mathrm{II}} \circ (1 \otimes \mathscr{L}_S^+) \circ ((U_9 - U_3) \otimes \mathscr{L}_S^+) \circ \mu'_{\mathrm{II}} \\
&= \mu_{\mathrm{II}} \circ ((U_9 - U_3) \otimes \mathscr{L}_S^+) \circ \mu'_{\mathrm{II}} \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -U_3 & -U_4U_5 \\ 1 & U_4 + U_5 - U_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathscr{L}_S^+ \\
&= (1) \otimes \mathscr{L}_S^+ \\
&= \mathrm{id}_{\mathrm{C}_2^-(S')} .
\end{aligned}$$

□

Therefore, we get a direct sum decomposition $\mathrm{C}_2^-(S) \cong \Upsilon\langle 1\rangle \oplus \mathrm{C}_2^-(S')\langle U_9 - U_3\rangle$. For partially singular braid diagrams D and D' related by an MOY III move, we extend both maps to the cube of resolutions by defining $\mu_{\mathrm{III}} \colon \mathrm{C}_2^-(D_I) \to \mathrm{C}_2^-(D'_I)$ and $\mu'_{\mathrm{III}} \colon \mathrm{C}_2^-(D'_I) \to \mathrm{C}_2^-(D_I)$ as above for each $I \in \{0,1\}^{c(D)}$.

Proof of Theorem 4.6 It is clear that μ_{III} and μ'_{III} are filtered maps, since they are defined componentwise on the cube of resolutions. Furthermore, we can extend our proof of Proposition 4.7 to partially singular braid diagrams since both the edge maps and MOY II maps are defined on such diagrams, so $\mathrm{C}_2^-(D')$ really is a summand of $\mathrm{C}_2^-(D)$.

We also need to check that μ_{III} and μ'_{III} are chain maps, ie that they commute with the edge maps d_1. Let $I, J \in \{0,1\}^{c(D)}$ with $I \prec J$. If I and J differ at a positive crossing, then $d_{I,J}$ is given by $\phi_+ \otimes \mathscr{L}_D^+ = 1 \otimes \mathscr{L}_D^+$. Otherwise, $d_{I,J}$ is given by $\phi_- \otimes \mathscr{L}_D^+ = (U_b - U_c) \otimes \mathscr{L}_D^+$. Either way, the edge maps are given by multiplication by an element of $R(D')$. Since μ_{III} and μ'_{III} were defined to be $R(D')$–linear, we get that they commute with d_1. □

As before, a similar argument shows that we also have MOY III decompositions for the cases in Figure 14.

5 Invariance

In this section, we prove that $\mathrm{C}_2^-(\beta)$ is an invariant of the braid closure $\mathrm{cl}(\beta)$ by showing that it is invariant under each of the four moves of Theorem 2.8. The first two moves, Reidmeister II and III, apply to any partially singular braid diagram D, whereas the second two moves, stabilization and conjugation, are specific to diagrams of the form $D = I_n(\beta)$.

5.A Reidemeister II

We begin by proving invariance under Reidemeister II moves. There are two distinct such moves, but they are mirror images of each other, and their proofs are almost identical. We prove one of the cases in detail below.

Theorem 5.1 *If D and D' are two partially singular braid diagrams that differ by a Reidemeister II move, then $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$ over $R(D')$.*

Proof Let D and D' be the diagrams in Figure 15, with D' the result of eliminating two crossings from D' by means of a Reidemeister II move. We use Lemma A.1 to simplify $\mathrm{C}_2^-(D)$ and $\mathrm{C}_2^-(D')$ to see they have the same homotopy type. Label the edges of D with variables $U_1, \dots, U_6$ as in Figure 15, and order the crossings from top to bottom. Let $\phi_1 = \phi_+ = 1$ be the edge map corresponding to the top (positive)

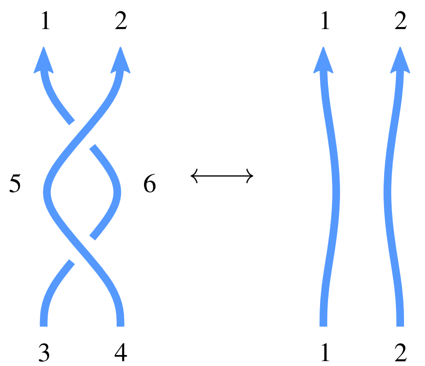


Figure 15: A Reidemeister II move.

vertex, and let $\phi_2 = \phi_- = U_6 - U_3$ be the edge map corresponding to the bottom (negative) vertex. Then, fixing a sign assignment without loss of generality, we expand the cube of resolutions for $\mathrm{C}_2^-(D)$ as:

$$\begin{array}{ccc} \mathrm{C}_2^-(D_{00}) & \xrightarrow{\phi_1} & \mathrm{C}_2^-(D_{10}) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle -\phi_2} \\ \mathrm{C}_2^-(D_{01}) & \xrightarrow{\phi_1} & \mathrm{C}_2^-(D_{11}) \end{array}$$

Note that $\mathrm{C}_2^-(D_{10})$ is isomorphic to $\mathrm{C}_2^-(D')$ via the removal of bivalent vertices, so our goal is to show that $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D_{10})$ as a filtered chain complex over $R(D')$. We work over the larger ring $R(D')[U_3, U_4]$, but do not enforce linearity with respect to U_5 or U_6. First, note that $\mathrm{C}_2^-(D_{00}) \cong \mathrm{C}_2^-(D_{11})$. We see that we can apply the MOY II decomposition from Section 4.B to write $\mathrm{C}_2^-(D_{01}) = \mathrm{C}_2^-(D_{11})\langle 1\rangle \oplus \mathrm{C}_2^-(D_{00})\langle U_6\rangle$. We compute the maps induced by ϕ_1 and ϕ_2 on these decompositions to get an isomorphic cube of resolutions:

$$\begin{array}{ccc} \mathrm{C}_2^-(D_{00}) & \xrightarrow{\quad 1 \quad} & \mathrm{C}_2^-(D_{10}) \\ \downarrow{\scriptstyle \binom{-U_3}{1}} & & \downarrow{\scriptstyle U_3 - U_2} \\ \mathrm{C}_2^-(D_{11})\langle 1\rangle \oplus \mathrm{C}_2^-(D_{00})\langle U_6\rangle & \xrightarrow{(1\ U_2)} & \mathrm{C}_2^-(D_{11}) \end{array}$$

This is the first of several times we use Lemma A.1 to simplify a cube of resolutions in this paper. This key lemma allows us to effectively cancel out isomorphisms of direct summands in cubes. In this case, it yields the E_1–quasi-isomorphic complex:

$$\begin{array}{ccc} 0 & \longrightarrow & \mathrm{C}_2^-(D_{10}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

We conclude by noting that $\mathrm{C}_2^-(D_{10}) \cong \mathrm{C}_2^-(D')$ as chain complexes over $R(D')[U_3, U_4]$. This proves invariance under one type of Reidemeister II move; the proof of the mirror-image move is analogous. □

5.B Reidemeister III

We aim to prove invariance under the Reidemeister III move shown in Figure 16, which corresponds to sliding a strand over a positive crossing. In terms of the braid group, it represents the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. All other variations of the Reidemeister III move follow from this one plus the Reidemeister II invariance result from Theorem 5.1.

Theorem 5.2 *If D and D' are two partially singular braid diagrams that differ by a Reidemeister III move, then $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$.*

Proof Let D be the diagram on the left and D' the diagram on the right in Figure 16. We aim to use Lemma A.1 to simplify $\mathrm{C}_2^-(D)$ and $\mathrm{C}_2^-(D')$ to see they have the same homotopy type.

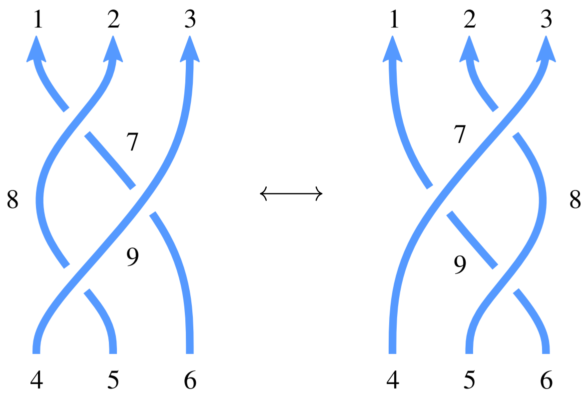
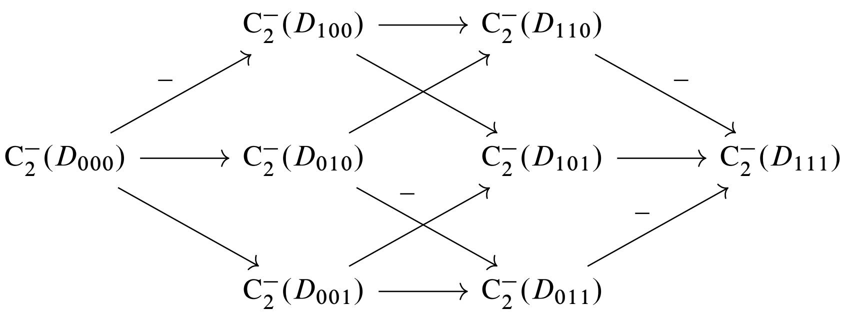


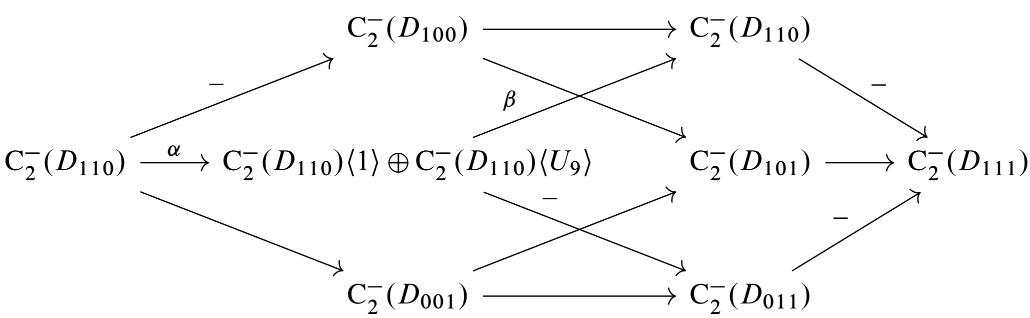
Figure 16: A Reidemeister III move.

To start, label the edges of D with variables $U_1, \dots, U_9$ as in Figure 16, and order the crossings from top to bottom. We expand the cube of resolutions for $\mathrm{C}_2^-(D)$ as:



Since our local picture of D consists of only positive crossings, all edge maps in this cube are given by $\phi_+ = 1$ up to a sign assignment, which we take to be the one in the above cube of resolutions without loss of generality.

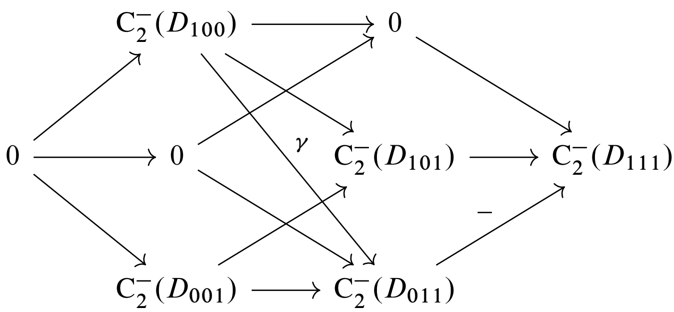
By Theorem 4.6, we note $\mathrm{C}_2^-(D_{000}) \cong \mathrm{C}_2^-(D_{110}) \oplus \Upsilon$, where Υ is acyclic. By a slight generalization of Lemma A.3, we get an E_1–quasi-isomorphic cube after replacing $\mathrm{C}_2^-(D_{000})$ by $\mathrm{C}_2^-(D_{110})$. Furthermore, Theorem 4.4 gives us that $\mathrm{C}_2^-(D_{010}) \cong \mathrm{C}_2^-(D_{110})\langle 1\rangle \oplus \mathrm{C}_2^-(D_{110})\langle U_9\rangle$. Therefore, the above cube is E_1–quasi-isomorphic to:



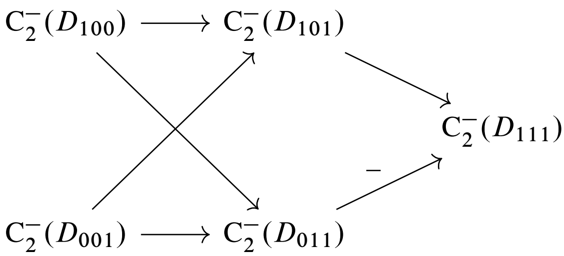
We compute the induced maps in the above cube to be

$$\alpha = \begin{pmatrix} -U_3 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & U_2 \end{pmatrix}.$$

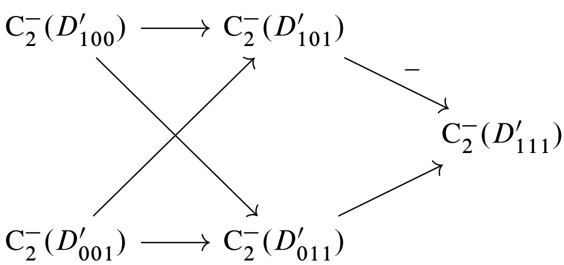
By Lemma A.1, we can cancel the isomorphisms of direct summands in the above cube to obtain the E_1–quasi-isomorphic complex:



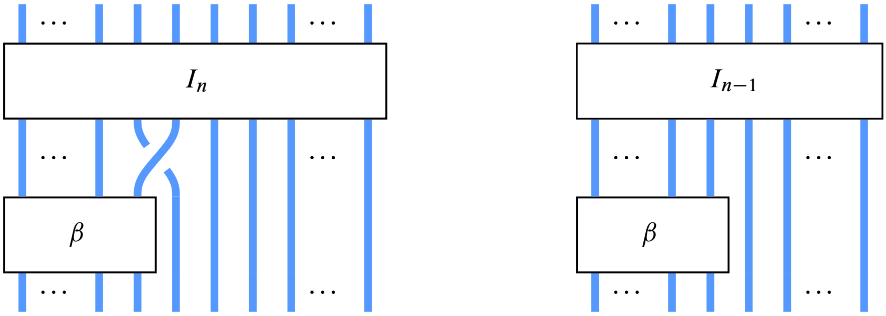
Removing the trivial complexes in the above cube, and noting that the map γ induced by cancellation is given by multiplication by 1, we get the complex:



Now, recalling that we have a second diagram D' to work with, we may go through the same steps to simplify $C_2^-(D')$ to get the complex:



We conclude by noting that the reduced complexes for $C_2^-(D)$ and $C_2^-(D')$ are isomorphic via the map that reflects the complexes about a horizontal axis, ie swaps the 100– and 001–resolutions, swaps the 101– and 011–resolutions, and fixes the 111–resolution. This map is a chain map since all the edge maps are ± 1, and therefore $C_2^-(D) \simeq_1 C_2^-(D')$. □



D: stabilized/original D': original/destabilized

Figure 17: Diagrams related by a positive stabilization. Depending on context, we either consider the diagram D and its *destabilization* D', or we consider the diagram D' and its *stabilization* D.

5.C Stabilization

Let $\beta \in B_{n-1}$ be an element of the braid group for $n \geq 2$, and consider β as an element of B_n via the natural inclusion $B_{n-1} \hookrightarrow B_n$ adjoining a straight strand to the right of β. Let $\sigma_{n-1} \in B_n$ be the generator which introduces a positive crossing between strands $n-1$ and n. The *positive stabilization* of β is the braid $\sigma_{n-1}\beta \in B_n$. Analogously, the *negative stabilization* of β is the braid $\sigma_{n-1}^{-1}\beta \in B_n$. For a braid $\beta = \sigma_{n-1}^{\pm 1}\beta' \in B_n$ in the image of one of these operations, we say that $\beta' \in B_{n-1}$ is the *destabilization* of β.

Theorem 5.3 *The E_1–homotopy type of the filtered complex $\mathrm{C}_2^-(\beta)$ is invariant under positive and negative* (de)*stabilization*, *ie* $\mathrm{C}_2^-(\sigma_{n-1}\beta) \simeq_1 \mathrm{C}_2^-(\sigma_{n-1}^{-1}\beta) \simeq_1 \mathrm{C}_2^-(\beta)$.

We note that β, $\sigma_{n-1}\beta$, and $\sigma_{n-1}^{-1}\beta$ all have isotopic braid closures. Before we prove stabilization invariance, we need to relate diagrams containing the open braid diagrams I_n and I_{n-1}, as the (de)stabilization operations alter the number of strands of our partially singular braid diagrams. Therefore, we first note that we can see I_{n-1} as a subdiagram of I_n by ignoring the rightmost vertices in every row. Equivalently, we can build I_n inductively from I_{n-1} as in Figure 18.

Consider $I_n(\sigma_{n-1}\beta)$, as shown in Figure 19. Let R be the polynomial ring over all edges not labeled in Figure 19, and label the rest of the edges accordingly, so that $R(I_n(\sigma_{n-1}\beta)) = R[U_1, U_2, U_3, U_4, U_5]$.

Let $\phi = \phi_+ = 1$ be the edge map corresponding to the positive crossing of σ_{n-1}. We write the one-dimensional cube of resolutions corresponding to resolving the crossing

$$\mathrm{C}_2^-(I_n(\sigma_{n-1}\beta)_0) \xrightarrow{\phi} \mathrm{C}_2^-(I_n(\sigma_{n-1}\beta)_1).$$

The diagrams corresponding to these resolutions are illustrated in Figure 20.

Our goal now is to use MOY moves to modify both resolutions so that they can be represented using a common diagram, tracking the effect on the complexes. By an abuse of notation, we denote this common

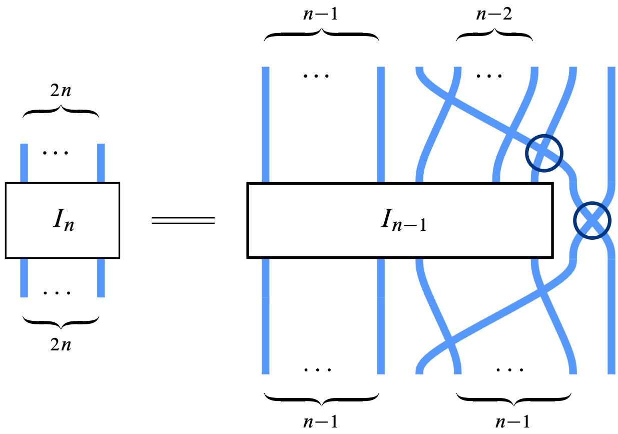


Figure 18: Constructing I_n from I_{n-1} by adding two more strands and marking two new vertices as fixed.

diagram $I'_n(\beta)$, which is gotten analogously to $I_n(\beta)$: we place a straight strand to the right of β, place n straight strands to the right of that, top this diagram with I'_n, and take the braid closure. It remains to define I'_n. We let I'_n be I_n, without the singular vertex between strands n and $n+1$ in the first layer. As for I_n, we can also define I'_n by building on I_{n-1}, as in Figure 21.

With this definition in mind, we now see that $I_n(\sigma_{n-1}\beta)_0$ is one MOY III move away from $I'_n(\beta)$, and $I_n(\sigma_{n-1}\beta)_1$ is one MOY II move away from $I'_n(\beta)$. On the one-dimensional cube of resolutions, then, we get

$$\mathrm{C}_2^-(I'_n(\beta))\langle U_3-U_5\rangle \oplus \Upsilon \xrightarrow{\phi} \mathrm{C}_2^-(I'_n(\beta))\langle 1\rangle \oplus \mathrm{C}_2^-(I'_n(\beta))\langle U_4\rangle.$$

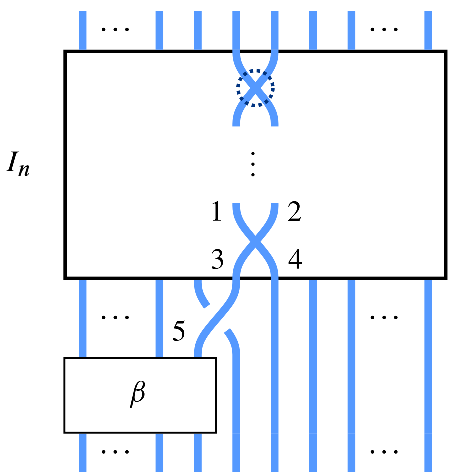


Figure 19: Relevant edge labels near σ_{n-1}. Note that the top vertex is fixed only when $n=2$, and is otherwise free for $n\geq 3$. For this reason, we use a dashed circle to indicate that the top vertex may be fixed or free.

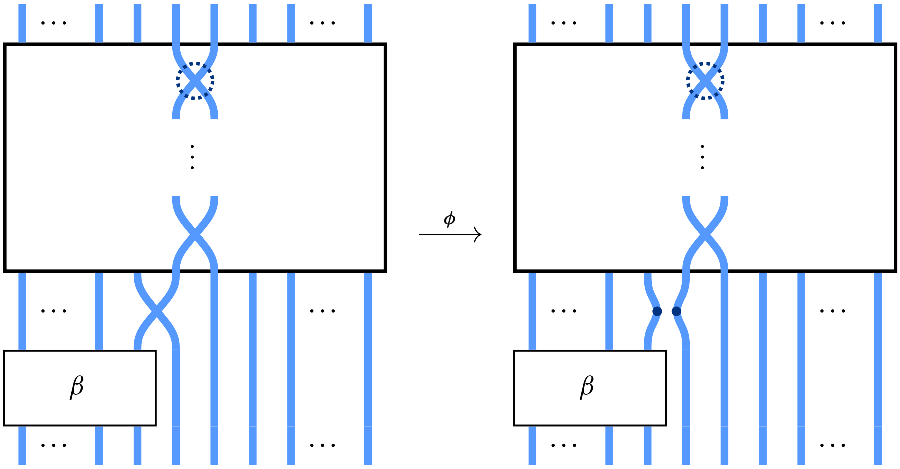


Figure 20: The mapping cone decomposition induced by σ_{n-1}.

Since Υ is acyclic, we can ignore it by Lemma A.2. We compute the map induced by ϕ on the summands as

$$\mathrm{C}_2^-(I_n'(\beta))\langle U_3 - U_5\rangle \xrightarrow{\begin{pmatrix} U_1+U_2-U_5 \\ -1 \end{pmatrix}} \mathrm{C}_2^-(I_n'(\beta))\langle 1\rangle \oplus \mathrm{C}_2^-(I_n'(\beta))\langle U_4\rangle.$$

Since the -1 entry represents an isomorphism of $\mathrm{C}_2^-(I_n'(\beta))$ summands, we can cancel it by Lemma A.1. This proves the following lemma.

Lemma 5.4 *The complexes $\mathrm{C}_2^-(\sigma_{n-1}\beta)$ and $\mathrm{C}_2^-(I_n'(\beta))$ have the same E_1–homotopy type.*

Therefore, to prove conjugation invariance, it remains to prove the following proposition.

Proposition 5.5 *The complexes $\mathrm{C}_2^-(I_n'(\beta))$ and $\mathrm{C}_2^-(\beta)$ have the same E_1–homotopy type.*

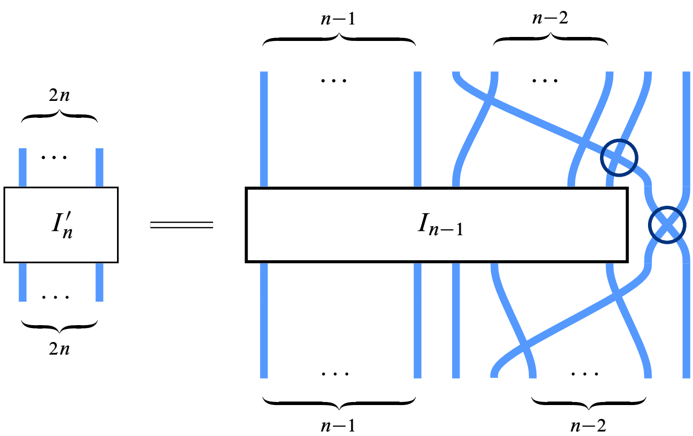


Figure 21: Building I_n' from I_{n-1}.

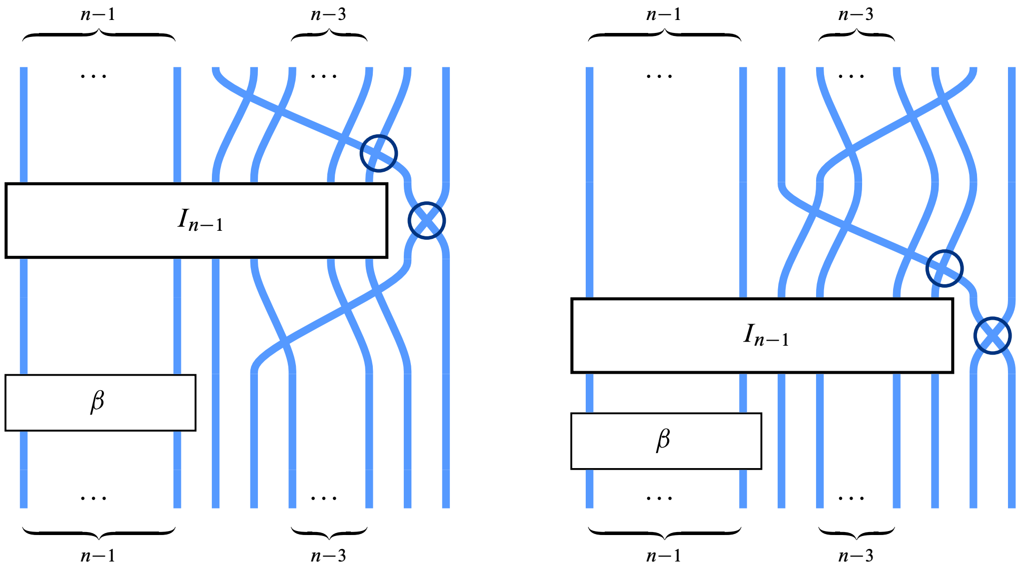


Figure 22: Shifting vertices in $I'_n(\beta)$.

Proof To begin, we note that $I'_n(\beta)$ and $I_{n-1}(\beta)$ are really braid closures, so for ease of understanding the upcoming MOY moves, we replace our usual depiction of $I'_n(\beta)$ with a shifted version, as in Figure 22.

In this shifted version, we identify the local picture on the left in Figure 23, consisting of a pair of intersecting strands and $n-2$ other strands which intersect both. We can apply $n-2$ MOY III moves to simplify this part of the diagram to the local picture on the right in Figure 23, consisting of $n-2$ straight strands and one pair of intersecting strands. By Theorem 4.6, each of these preserves the E_1–homotopy type of the complex. The global picture at this stage can be seen on the left in Figure 24. To arrive at the diagram for $I_{n-1}(\beta)$, we apply two MOY I moves to the two remaining fixed vertices outside of I_{n-1}. By Theorem 4.1, each of these preserves the E_1–homotopy type of the complex. This leads to the diagram on the right in Figure 24, which is exactly the diagram for $I_{n-1}(\beta)$. □

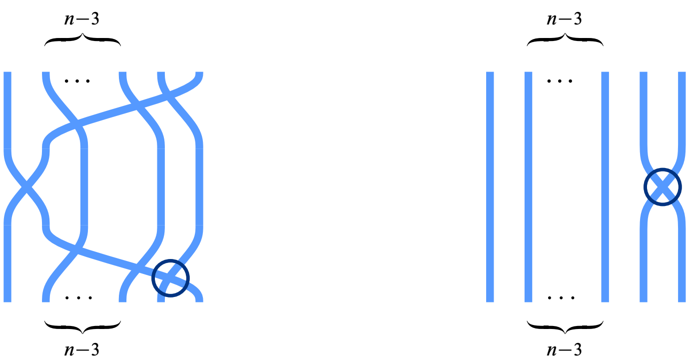


Figure 23: Local pictures of diagrams related by a sequence of $n-2$ MOY III moves.

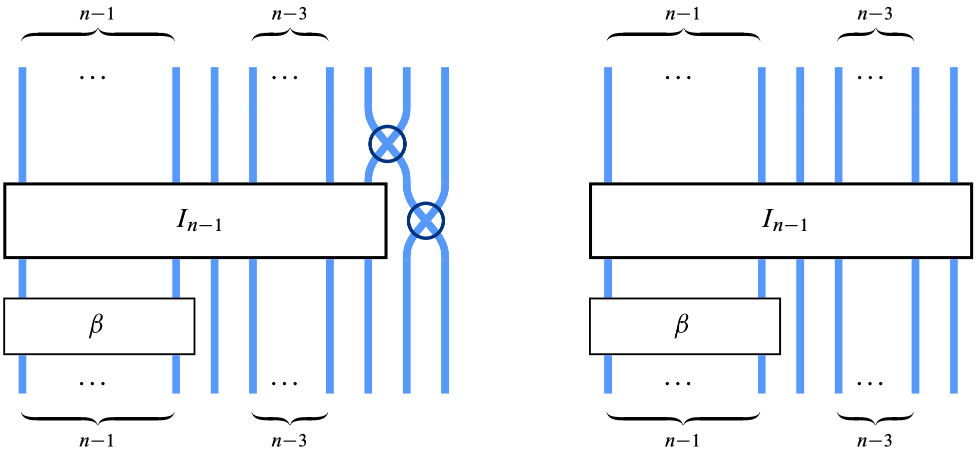


Figure 24: The last step in simplifying $I'_n(\beta)$ in the proof of Proposition 5.5.

Proof of Theorem 5.3 We have shown that $C_2^-(\sigma_{n-1}\beta) \simeq C_2^-(I'_n(\beta)) \simeq C_2^-(\beta)$. We can simplify $C_2^-(\sigma_{n-1}^{-1}\beta)$ to $C_2^-(I'_n(\beta))$ as well. Using the same edge labels and notation as before, we write the cube of resolutions for $C_2^-(\beta)$ as

$$C_2^-(I_n(\beta)_0) \xrightarrow{\phi} C_2^-(I_n(\beta)_1)$$

where this time $\phi = \phi_- = U_3 - U_5$. Applying an MOY II move to $I_n(\beta)_0$ and an MOY III move to $I_n(\beta)_1$ to write our complexes in terms of $C_2^-(I'_n(\beta))$ gives us the complex

$$C_2^-(I'_n(\beta))\langle 1\rangle \oplus C_2^-(I'_n(\beta))\langle U_4\rangle \xrightarrow{\phi} C_2^-(I'_n(\beta))\langle U_3 - U_5\rangle \oplus \Upsilon.$$

Again, excluding Υ and computing the map induced by ϕ, we get

$$C_2^-(I'_n(\beta))\langle 1\rangle \oplus C_2^-(I'_n(\beta))\langle U_4\rangle \xrightarrow{(1\ \ U_4)} C_2^-(I'_n(\beta))\langle U_3 - U_5\rangle.$$

As before, we may cancel the 1 in the above matrix to see that $C_2^-(\sigma_{n-1}^{-1}\beta) \simeq_1 C_2^-(I'_n(\beta))$ as well. The rest of the proof follows from Proposition 5.5. Since we have covered both the positive and negative cases, this suffices to show invariance under stabilization. □

5.D Conjugation

Conjugation invariance is the following statement:

Theorem 5.6 *For any $\alpha, \beta \in B_n$, we have that $C_2^-(\alpha^{-1}\beta\alpha) \simeq_1 C_2^-(\beta)$.*

To begin, we prove a lemma relating complexes associated to diagrams that locally look like the pictures in Figure 25.

Lemma 5.7 *Let A and A' be partially singular braid diagrams that are identical outside of a specific region, where they look like the diagrams in Figure 25, ie A has two opposite crossings whereas A' has oriented smoothings. Then $C_2^-(A) \simeq_1 C_2^-(A')$.*

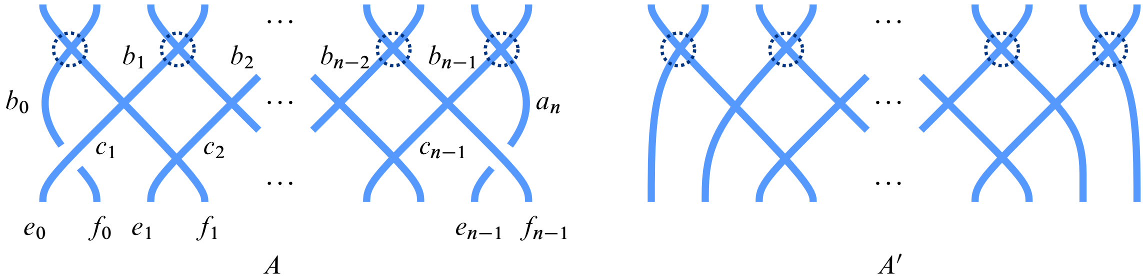


Figure 25: Local pictures of diagrams with equivalent $C_2^-(-)$. As before, the dashed circles indicate that the vertices may be fixed or free.

Proof Let $\phi_1 = \phi_+ = 1$ be the edge map corresponding to the left (positive) crossing, and let $\phi_2 = \phi_- = a_n - e_{n-1}$ be the edge map corresponding to the right (negative) crossing. We expand the cube of resolutions for $C_2^-(A)$ as follows:

$$\begin{array}{ccc} C_2^-(A_{00}) & \xrightarrow{\phi_1} & C_2^-(A_{10}) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_2} \\ C_2^-(A_{01}) & \xrightarrow{\phi_1} & C_2^-(A_{11}) \end{array}$$

The diagrams for these four partial resolutions look like Figure 26.

We can use MOY III moves to simplify three of the four corners of this cube. For A_{00}, we can start with a MOY III move on the left, simplifying the diagram. Each MOY III move we apply allows us to perform another, until we have done $n-1$ such moves moving left-to-right. We denote the resulting diagram A''_{00}; it is shown in Figure 27. By Theorem 4.6, A_{00} and A'_{00} are E_1–quasi-isomorphic.

Similarly, we can simplify A_{11} to A'_{11} by performing $n-1$ MOY III moves right-to-left, and we can simplify A_{01} to A'_{01} by performing $n-1$ MOY III moves left-to-right. In each case, Theorem 4.6 ensures

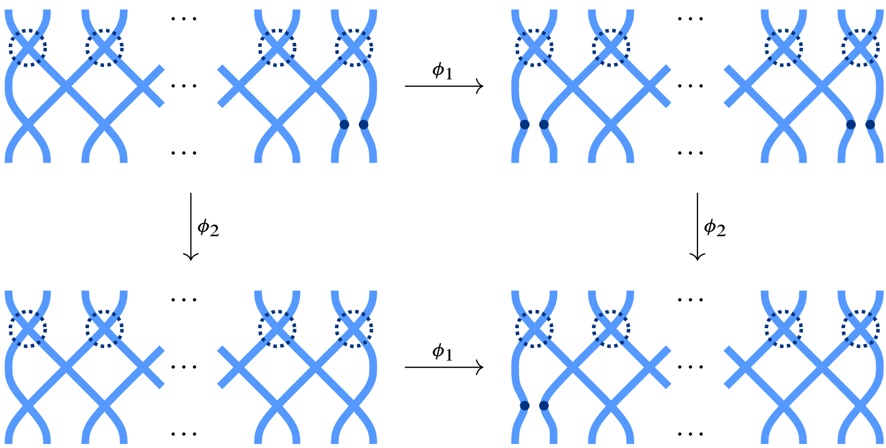


Figure 26: The cube of resolutions for diagram A in Figure 25.

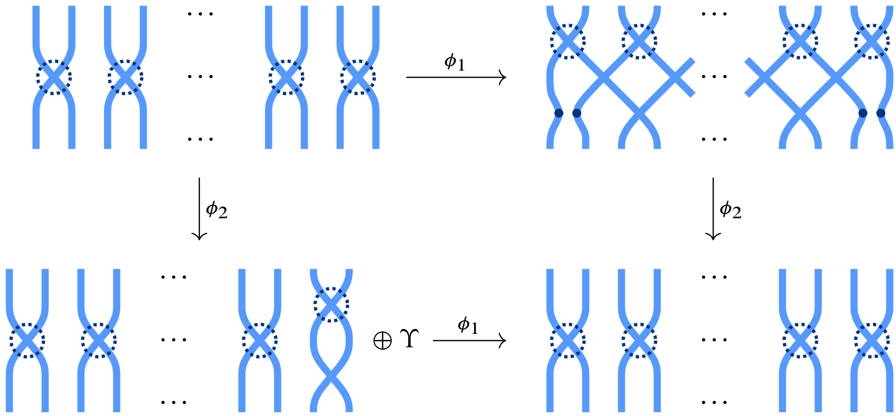


Figure 27: The reduced cube of resolutions.

we are preserving the E_1–quasi-isomorphism type. The resulting diagrammatic cube of resolutions is shown in Figure 27. Thus we obtain the complex

$$\begin{array}{ccc} C_2^-(A''_{00})\langle x\rangle & \xrightarrow{\phi_1} & C_2^-(A_{10}) \\ \downarrow{\scriptstyle\phi_2} & & \downarrow{\scriptstyle\phi_2} \\ C_2^-(A''_{01})\langle x\rangle \oplus \Upsilon & \xrightarrow{\phi_1} & C_2^-(A''_{11})\langle x\rangle \end{array}$$

This cube ignores the Υ summands in A_{00} and A_{11} by Lemma A.3, but retains the $\Upsilon := \Upsilon_1 \oplus \cdots \oplus \Upsilon_{n-1}$ summand in A_{01}. Further, as every vertex in the middle row is free, we may choose

$$x = (b_1 - c_1) \cdots (b_{n-1} - c_{n-1})$$

to be the generator for all three complexes modulo the linear ideal L.

We further decompose A''_{01} via an MOY II move on the right into two copies of A''_{00}, generated by x and $a_n x$. Additionally, we see that A''_{00} and A''_{11} are isomorphic. We can compute the maps induced by ϕ_1 and ϕ_2 and write our complex as

$$\begin{array}{ccc} C_2^-(A''_{00})\langle x\rangle & \xrightarrow{\phi_1} & C_2^-(A_{10}) \\ \downarrow{\scriptstyle \begin{pmatrix} -e_{n-1} \\ 1 \\ * \end{pmatrix}} & & \downarrow{\scriptstyle\phi_2} \\ C_2^-(A''_{00})\langle x\rangle \oplus C_2^-(A''_{00})\langle a_n x\rangle \oplus \Upsilon & \xrightarrow{\begin{pmatrix} 1 & f_{n-1} & * \end{pmatrix}} & C_2^-(A''_{00})\langle x\rangle \end{array}$$

We may cancel the 1's in the above matrices to reduce the complex by Lemma A.1 to obtain

$$\begin{array}{ccc} 0 & \longrightarrow & C_2^-(A_{10}) \\ \downarrow & & \downarrow \\ \Upsilon & \longrightarrow & 0 \end{array}$$

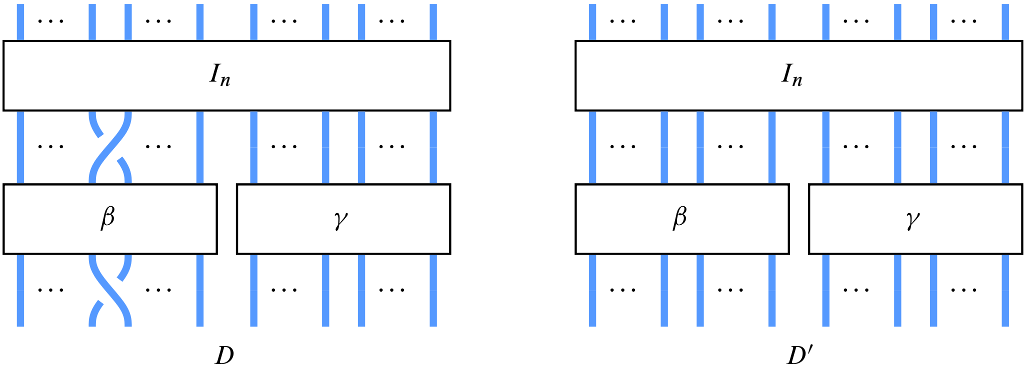


Figure 28: When $\gamma = 1$, the diagram D is the result of conjugating the braid β in D' by a generator of the braid group.

Since Υ is a direct sum of E_1–acyclic complexes, we see that the E_2–page of the above complex is isomorphic to that of $\mathrm{C}_2^-(A_{10})$, which is isomorphic to $\mathrm{C}_2^-(A')$, thereby proving Lemma 5.7 in the case of a positive crossing on the left and a negative one on the right.

The opposite case is analogous; applying the same moves (mirrored horizontally) results in the complex

$$\begin{array}{ccc} \mathrm{C}_2^-(A''_{00})\langle x\rangle & \xrightarrow{\begin{pmatrix} f_0 \\ -1 \\ * \end{pmatrix}} & \mathrm{C}_2^-(A''_{00})\langle x\rangle \oplus \mathrm{C}_2^-(A''_{00})\langle b_0 x\rangle \oplus \Upsilon \\ \Big\downarrow{\scriptstyle \phi_2} & & \Big\downarrow{\scriptstyle \begin{pmatrix} 1 & e_0 & * \end{pmatrix}} \\ \mathrm{C}_2^-(A_{01}) & \xrightarrow{\phi_1} & \mathrm{C}_2^-(A''_{00})\langle x\rangle \end{array}$$

which we can simplify to get that the E_2–page is the same as that of $\mathrm{C}_2^-(A_{01})$ and therefore $\mathrm{C}_2^-(A')$. □

With this lemma in hand, we are now prepared to prove conjugation invariance.

Proof of Theorem 5.6 It suffices to prove this in the case that $\alpha = \sigma_i^{\pm 1}$ is any generator of the braid group (or its inverse). Therefore, let $\sigma_i \in B_n$ be the generator which introduces a positive crossing between strands i and $i+1$.

Graphically, we would like to show that $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$, where D and D' are the partially singular braids depicted in Figure 28, when $\gamma = 1$. In order to prove this, we instead show that

$$\mathrm{C}_2^-(D'') \simeq_1 \mathrm{C}_2^-(D') \simeq_1 \mathrm{C}_2^-(D''')$$

for a generic $\gamma \in B_n$, where D'' and D''' are the diagrams in Figure 29.

Since we are considering the case $\alpha = \sigma_i$, note that in D'', the positive crossing occurs between strands i and $i+1$. If we decompose I_n, we see that for any i, we locally get a picture like Figure 25, where the top row of vertices is fixed if $i = 1$, and free if $i > 1$. Therefore, we may apply Lemma 5.7 directly to see

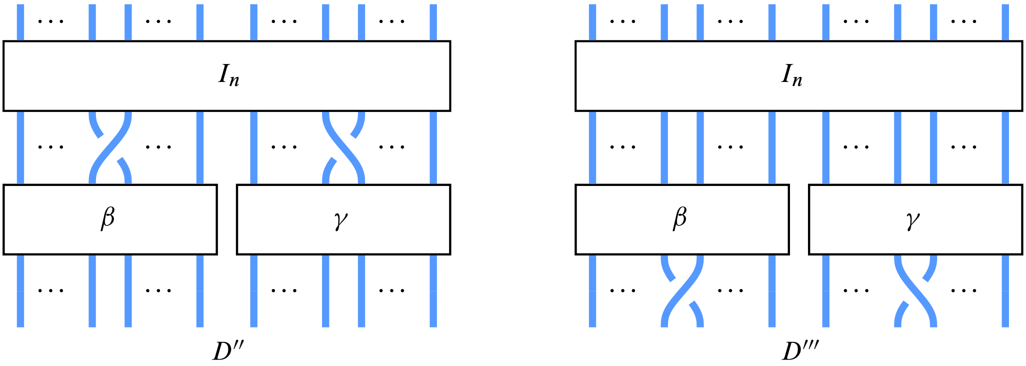


Figure 29: Alternate diagrams for proving conjugation invariance.

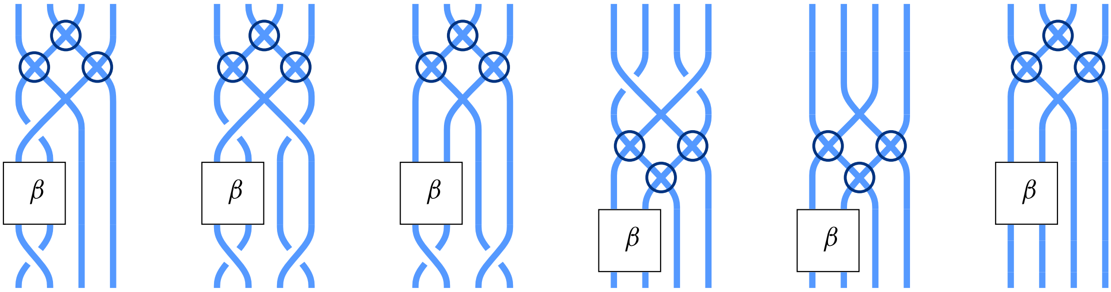


Figure 30: The steps to prove conjugation invariance for $n = 2$ and $\alpha = \sigma_1$.

that $\mathrm{C}_2^-(D'') \simeq_1 \mathrm{C}_2^-(D')$. Additionally, note that in D''', the negative crossing occurs between strands i and $i+1$. If we decompose I_n, we see that for any i, we locally get a picture like Figure 25, except that the bottom row of vertices is fixed if $i = 1$, and free if $i > 1$. In the latter case, this is not an issue and we may proceed as before to use Lemma 5.7 to prove that $\mathrm{C}_2^-(D''') \simeq_1 \mathrm{C}_2^-(D')$. If $i = 1$, then we first use Theorem 3.1 to relabel the top row of vertices as free and the bottom row as fixed; this diagram is still in $\mathscr{D}^{\mathfrak{R}}$ as it contains the open braid S_{2n} from [Dowlin 2024] as a subdiagram, so we may proceed with the rest of the proof as usual.

Therefore, we can prove the desired equivalence $\mathrm{C}_2^-(D) \simeq_1 \mathrm{C}_2^-(D')$ when $\gamma = 1$ by first performing a Reidemeister II move to add two crossings to the right side of D, then using the equivalences $\mathrm{C}_2^-(D'') \simeq_1 \mathrm{C}_2^-(D')$ and $\mathrm{C}_2^-(D''') \simeq_1 \mathrm{C}_2^-(D')$ to simplify the diagram to D'. This proves Theorem 5.6 in the case of $\alpha = \sigma_i$. We illustrate these steps for the case $n = 2$ and $\alpha = \sigma_1$ in Figure 30. The proof for $\alpha = \sigma_i^{-1}$ is analogous, from which the proof for general $\alpha \in B_n$ follows. □

Appendix Homological algebra

In this section we review a few lemmas in homological algebra to aid in our calculations. We note that all four lemmas are true for filtered complexes, replacing maps with filtered maps and quasi-isomorphisms with filtered quasi-isomorphisms. Additionally, if we instead assume that filtered maps have filtration degree 1, then these lemmas still hold, replacing $\mathrm{cone}(f)$ with $\mathrm{cone}_1(f)$ and filtered quasi-isomorphism with E_1–quasi-isomorphism.

Lemma A.1 [Bar-Natan 2007, Lemma 4.2] *If $\varphi\colon A \to B$ is an isomorphism of complexes, then the double complexes*

$$C \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} A \oplus D \xrightarrow{\begin{pmatrix}\varphi & \delta\\ \gamma & \epsilon\end{pmatrix}} B \oplus E \xrightarrow{(\mu\ \nu)} F$$

and

$$C \xrightarrow{\beta} D \xrightarrow{\epsilon - \gamma\varphi^{-1}\delta} E \xrightarrow{\nu} F$$

are quasi-isomorphic.

This lemma is proved for the very general case of additive categories in [Bar-Natan 2007], and is well known in the specific case of free modules over a ring as the "cancellation lemma" or "reduction algorithm". We require it to prove invariance in Section 5, as it greatly simplifies calculations involving cubes of resolutions.

Lemma A.2 *Let $f : A \to B$ be a map of complexes, and suppose that $A \cong A' \oplus A''$ and $B \cong B' \oplus B''$, where A'' and B'' are acyclic. Let $\iota\colon A' \hookrightarrow A$ and $\pi : B \twoheadrightarrow B'$ be the associated inclusion and projection maps, respectively. Then $\mathrm{cone}(f) \simeq \mathrm{cone}(\pi \circ f \circ \iota)$.*

Proof First, we note that $\mathrm{cone}(\iota)$ is acyclic. One way to see this is via a cancellation argument: we have that $\mathrm{cone}(\iota) \cong (A' \to A' \oplus A'')$, which is quasi-isomorphic to A'' by Lemma A.1. Similarly, we get that $\mathrm{cone}(\pi)$, being quasi-isomorphic to B'', is acyclic as well.

For any two maps $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ of complexes, we have a long exact sequence relating the homology groups of $\mathrm{cone}(\alpha)$, $\mathrm{cone}(\beta)$, and $\mathrm{cone}(\beta \circ \alpha)$ (for example, via the octahedral axiom for triangulated categories applied to the derived category of R–modules). Therefore, we get that $\mathrm{cone}(f) \simeq \mathrm{cone}(f \circ \iota) \simeq \mathrm{cone}(\pi \circ f \circ \iota)$. □

Lemma A.3 *Let A, B, C, and D be complexes, and suppose that $A \cong A' \oplus A''$ and $D \cong D' \oplus D''$, where A'' and D'' are acyclic. Let $\iota\colon A' \hookrightarrow A$ and $\pi : D \twoheadrightarrow D'$ be the associated inclusion and projection maps, respectively. Then the following two cube of resolutions complexes have the same homotopy type:*

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ \Big\downarrow{\scriptstyle g_1} & & \Big\downarrow{\scriptstyle g_2} \\ C & \xrightarrow{f_2} & D \end{array} \qquad \begin{array}{ccc} A' & \xrightarrow{f_1 \circ \iota} & B \\ \Big\downarrow{\scriptstyle g_1 \circ \iota} & & \Big\downarrow{\scriptstyle \pi \circ g_2} \\ C & \xrightarrow{\pi \circ f_2} & D' \end{array}$$

Proof We know that the inclusion $\mathrm{cone}(g_1 \circ \iota) \hookrightarrow \mathrm{cone}(g_1)$ and the projection $\mathrm{cone}(g_2) \twoheadrightarrow \mathrm{cone}(\pi \circ g_2)$ are quasi-isomorphisms by the proof of Lemma A.2:

$$\begin{array}{ccc} A' & \xrightarrow{\iota} & A \\ \Big\downarrow{\scriptstyle g_1 \circ \iota} & & \Big\downarrow{\scriptstyle g_1} \\ C & \xrightarrow{\mathrm{id}_C} & C \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\mathrm{id}_B} & B \\ \Big\downarrow{\scriptstyle g_2} & & \Big\downarrow{\scriptstyle \pi \circ g_2} \\ D & \xrightarrow{\pi} & D' \end{array}$$

We can also view the maps $f_1\colon A \to B$ and $f_2\colon C \to D$ as components of a map $f\colon \mathrm{cone}(g_1) \to \mathrm{cone}(g_2)$. Therefore, we can compose f with the inclusion and projection to get a single map

$$f'\colon \mathrm{cone}(g_1 \circ \iota) \to \mathrm{cone}(\pi \circ g_2).$$

By the same long exact sequence logic as before, the cone of this map has the same homotopy type as f, ie

$$\mathrm{cone}(f') = \mathrm{cone}(\mathrm{cone}(g_1 \circ \iota) \to \mathrm{cone}(\pi \circ g_2)) \simeq \mathrm{cone}(\mathrm{cone}(g_1) \to \mathrm{cone}(g_2)) = \mathrm{cone}(f).$$

We conclude by noting that the complex on the left in Lemma A.3 is $\mathrm{cone}(f)$, and the complex on the right is $\mathrm{cone}(f')$. □

While the above lemma is phrased only for squares, it can be iterated to reduce summands of higher-dimensional cubes as well.

Since our complexes in this paper are often constructed as mapping cones, it helps to know when a quasi-isomorphism is induced by maps on the components of the cone.

Lemma A.4 *Suppose that we have the following commutative diagram of chain maps*:

$$\begin{array}{ccc} A_0 & \xrightarrow{g} & A_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ B_0 & \xrightarrow{g'} & B_1 \end{array}$$

Let $A = \mathrm{cone}(g)$ and $B = \mathrm{cone}(g')$, so that we get a map $f\colon A \to B$ with components f_0 and f_1. If f_0 and f_1 are quasi-isomorphisms, then so is f.

Proof By properties of the mapping cone, f induces a map of short exact sequences (with grading shifts suppressed):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A & \longrightarrow & A_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B & \longrightarrow & B_0 & \longrightarrow & 0 \end{array}$$

We can look at the induced map of long exact sequences in homology

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_*(A_1) & \longrightarrow & H_*(A) & \longrightarrow & H_*(A_0) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle H_*(f_1)} & & \downarrow{\scriptstyle H_*(f)} & & \downarrow{\scriptstyle H_*(f_0)} & & \\ \cdots & \longrightarrow & H_*(B_1) & \longrightarrow & H_*(B) & \longrightarrow & H_*(B_0) & \longrightarrow & \cdots \end{array}$$

to conclude that $H_*(f)$ must be an isomorphism as well, so f is a quasi-isomorphism. □

To prove the filtered generalizations of Lemma A.4, one replaces $H_*(-)$ with $E_1(-)$ or $E_2(-)$ (see [Weibel 1994, Exercise 5.4.4]).

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Monoidal properties of Franke's exotic equivalence

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Franke's reconstruction functor $\mathcal{R}$ is known to provide examples of triangulated equivalences between homotopy categories of stable model categories, which are exotic in the sense that the underlying model categories are not Quillen equivalent. We show that, while not being a tensor-triangulated functor in general, $\mathcal{R}$ is compatible with monoidal products.

55P42; 18N55

1 Introduction

For several decades, Franke's exotic equivalence has been fascinating to homotopy theorists, as it is a rare example of a machinery that provides an equivalence up to homotopy between two model categories which are not Quillen equivalent. In practice, the known situations where Franke's construction can be applied to obtain the equivalence

$$\mathcal{R}\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{M})$$

link an algebraic model category ($\mathrm{D}^{([1],1)}(\mathcal{A})$ is the derived category of a flavour of chain complexes in a suitable abelian category $\mathcal{A}$) with a stable model category $\mathcal{M}$ which is not necessarily algebraic. Key examples include

- $\mathcal{A}$ the category of $\pi_*(R)$–modules for a ring spectrum R and $\mathcal{M}$ the category of modules over R, together with some extra assumption on the projective dimension of $\pi_*(R)$ as well as $\pi_*(R)$ being concentrated in degrees that are multiples of some $N > 1$,
- $\mathcal{A}$ the category of $E(1)_*E(1)$–comodules and $\mathcal{M}$ the category of K–local spectra at an odd prime.

In this paper, we will always assume that $\mathcal{R}$ exists and is an equivalence.

Both the algebraic side $\mathrm{D}^{([1],1)}(\mathcal{A})$ and the topological side $\mathrm{Ho}(\mathcal{M})$ are equipped with monoidal structures derived from the monoidal model category structures on $\mathrm{C}^{([1],1)}(\mathcal{A})$ and $\mathcal{M}$, so it is only natural to consider whether $\mathcal{R}$ is compatible with these. But as $\mathcal{R}$ is not derived from a Quillen functor $\mathrm{C}^{([1],1)}(\mathcal{A}) \to \mathcal{M}$, this problem requires a different approach working closely with the construction of $\mathcal{R}$ itself.

The example of K–local spectra at $p = 3$ tells us that we cannot expect $\mathcal{R}$ to be a monoidal functor in general: the preimage of the mod-3 Moore spectrum is a chain complex that is a monoid, whereas the mod-3 Moore spectrum has no associative multiplication [Ganter 2007, Remark 1.4.2]. However, we obtain the following, which is the main result of this article.

Theorem 1.0.1 *Let $(\mathcal{M},\wedge)$ be a simplicial stable monoidal model category and let $(\mathcal{A},\otimes)$ be a hereditary abelian monoidal category with enough projectives such that Franke's reconstruction functor $\mathcal{R}$ exists and is an equivalence. Then*

$$\mathcal{R}\colon (\mathrm{D}^{([1],1)}(\mathcal{A}),\otimes^{\mathbb{L}}) \to (\mathrm{Ho}(\mathcal{M}),\wedge^{\mathbb{L}})$$

commutes with the respective monoidal products up to a natural isomorphism

$$\mathcal{R}(M_*\otimes^{\mathbb{L}} N_*)\cong \mathcal{R}(M_*)\wedge^{\mathbb{L}}\mathcal{R}(N_*).$$

The reconstruction functor $\mathcal{R}$ rebuilds $\mathcal{M}$ from algebraic data in the following way. Firstly, part of the assumptions on $\mathcal{A}$ is that it splits into shifted copies of a smaller abelian category $\mathcal{B}$. This is then used to split an object of $\mathrm{C}^{([1],1)}(\mathcal{A})$ into pieces, which are placed in certain crown-shaped diagram C_N. Using this piecewise data, one then constructs a C_N–shaped diagram in $\mathcal{M}$. Finally, the homotopy colimit over C_N is applied to get to $\mathrm{Ho}(\mathcal{M})$. Specifically, $\mathcal{R}$ is the composite

$$\mathcal{R}\colon \mathrm{D}^{([1],1)}(\mathcal{A})\xrightarrow{\mathcal{Q}^{-1}}\mathcal{L}\subseteq \mathrm{Ho}(\mathcal{M}^{C_N})\xrightarrow{\mathrm{hocolim}_{C_N}}\mathrm{Ho}(\mathcal{M}).$$

We therefore take the diagram

$$\begin{array}{ccc}
\mathrm{D}^{([1],1)}(\mathcal{A})\times \mathrm{D}^{([1],1)}(\mathcal{A}) & \xrightarrow{\mathcal{R}\wedge^{\mathbb{L}}\mathcal{R}} & \mathrm{Ho}(\mathcal{M})\times\mathrm{Ho}(\mathcal{M})\\
{\scriptstyle\otimes^{\mathbb{L}}}\downarrow & & \downarrow{\scriptstyle\wedge^{\mathbb{L}}}\\
\mathrm{D}^{([1],1)}(\mathcal{A}) & \xrightarrow{\mathcal{R}} & \mathrm{Ho}(\mathcal{M})
\end{array}$$

which we would like to show to be commutative and refine it in the way below in order to deal with the different components of $\mathcal{R}$ separately:

$$(1.0.2)\qquad \begin{array}{ccccc}
\mathrm{D}^{([1],1)}(\mathcal{A})\times \mathrm{D}^{([1],1)}(\mathcal{A}) & \rightleftarrows & \mathrm{Ho}(\mathcal{M}^{C_N})\times\mathrm{Ho}(\mathcal{M}^{C_N}) & \longrightarrow & \mathrm{Ho}(\mathcal{M})\\
 & & {\scriptstyle\wedge^{\mathbb{L}}}\downarrow{\scriptstyle\cong} & \nearrow{\scriptstyle\mathrm{hocolim}} & \\
 & & \mathrm{Ho}(\mathcal{M}^{C_N\times C_N}) & & \\
{\scriptstyle -\otimes^{\mathbb{L}}-}\downarrow & & {\scriptstyle\mathbb{L}\,\mathrm{pr}_!=\mathrm{Ho}\,\mathrm{Lan}_{\mathrm{pr}}}\downarrow & \nearrow{\scriptstyle\mathrm{hocolim}_{D_N}} & \Vert\\
 & & \mathrm{Ho}(\mathcal{M}^{D_N}) & & \\
 & & {\scriptstyle i^*}\downarrow & & \\
\mathrm{D}^{([1],1)}(\mathcal{A}) & \underset{\mathcal{Q}}{\overset{\mathcal{Q}^{-1}}{\rightleftarrows}} & \mathrm{Ho}(\mathcal{M}^{C_N}) & \xrightarrow[\mathrm{hocolim}_{C_N}]{} & \mathrm{Ho}(\mathcal{M})
\end{array}$$

Here, D_N denotes a suitable modification of the crown-shaped diagram C_N together with an inclusion $i\colon C_N\to D_N$. (All the ingredients will of course be defined in detail where appropriate.) The outline of our proof roughly follows the key points of [Ganter 2007]; however, we choose to work in a setting of model categories, which makes our exposition more explicit and straightforward. Overall, we have relied on contemporary methods and we refer to modern literature. Our techniques put Ganter's theorem in firm rigorous footing and in better context with other existing literature, as well as hopefully making it more adaptable to future generalisations.

This paper is organised as follows.

In Section 2 we recall the background and tools that we need for our main result and proof, namely simplicial replacements, homotopy Kan extensions, monoidal structures on diagram model categories, a specific mapping cone construction, calculating homotopy colimits using the homology of a category with coefficients in a functor, as well as a recap of the construction of Franke's functor.

In Section 3 we will begin by setting up one of our main results, which involves working out the middle vertical part of the diagram (1.0.2). The key ingredient is given by a spectral sequence argument calculating the vertices of the functor $\mathbb{L}\,\mathrm{pr}_!(X \wedge Y)$. We will then feed this into the definition of Franke's functor $\mathcal{Q}$ in order to obtain the necessarily formulas for monoidality on the left hand side of (1.0.2), dealing with underlying graded modules of the twisted chain complexes and the differentials separately.

Section 4 now wraps up the right hand side of the diagram (1.0.2) which mostly involves standard properties of homotopy colimits. We can finally assemble these results into the proof of the main theorem and finish with some examples.

Acknowledgements

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2 Preliminaries

In this section we will introduce some of the terminology that we need for our result. We assume that the reader is familiar with the basic background regarding simplicial sets, homological algebra and model categories.

The category of simplicial sets is denoted by $s\mathrm{Set}$. For $n \geq 0$, Δ^n denotes the standard n–simplex. For an arbitrary category $\mathcal{C}$, the notation $s\mathcal{C}$ stands for the simplicial objects in $\mathcal{C}$, ie $s\mathcal{C} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$. We have $I = \Delta^1$ and $I_+ = \Delta^1 \cup *$ and $S^0 = \Delta^0 \cup *$. Similarly, S^1 stands for the simplicial circle $I/(0 \sim 1)$, that is, $\Delta^1/\partial\Delta^1$.

We will let $\mathcal{A}$ be a graded ($\mathbb{Z}$–graded) abelian category, which means that $\mathcal{A}$ possesses a shift functor [1] which is an equivalence of categories, and $[n]$ denotes the n–fold iteration of [1]. The *graded global homological dimension* of $\mathcal{A}$, $\mathrm{gl.dim}\,\mathcal{A}$, is the supremum of the projective dimensions of objects in $\mathcal{A}$. An abelian category $\mathcal{A}$ is called *hereditary* if $\mathrm{gl.dim}\,\mathcal{A} = 1$. There are other, equivalent descriptions of hereditary abelian categories but this one suits our purposes best.

2.1 Model categories

We will now set up our background on model categories. We write any cofibrant replacement functor $Q\colon \mathcal{M} \to \mathcal{M}$ that comes with a natural weak equivalence $q\colon Q \to 1_{\mathcal{M}}$.

Convention 2.1.1 We let $\mathrm{Ho}(\mathcal{M})$ denote the category $\mathcal{M}_{\mathrm{cof}}[\mathcal{W}^{-1}]$, where $\mathcal{M}_{\mathrm{cof}}$ denotes the full subcategory of cofibrant objects of $\mathcal{M}$, and we denote the set of morphisms in $\mathrm{Ho}(\mathcal{M})$ by $[X, Y]$.

Convention 2.1.1 allows us to provide a very simple description of the *left derived functor* $\mathbb{L}F$ of a left Quillen functor $F\colon \mathcal{M} \to \mathcal{N}$. Indeed, the functor

$$F|_{\mathcal{M}_{\mathrm{cof}}}\colon \mathcal{M}_{\mathrm{cof}} \to \mathcal{N}_{\mathrm{cof}}$$

preserves weak equivalences and, therefore, it induces a functor between the localization. This functor is precisely $\mathbb{L}F$ with our convention.

Finally, an important class of model categories is the class of *simplicial model categories*. These are model categories which are enriched, tensored and cotensored over $s\mathrm{Set}$ and which satisfy the pushout-product axiom (SM7). If a simplicial model category is pointed, ie the terminal object is isomorphic to the initial one, then $\mathcal{M}$ is enriched over the category $s\mathrm{Set}_*$ of pointed simplicial sets. In particular, we have functors

$$-\otimes -\colon s\mathrm{Set}_* \times \mathcal{M} \to \mathcal{M}, \quad \mathrm{Map}_{\mathcal{M}}(-,-)\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to s\mathrm{Set}_*,$$

and the adjunction

$$\mathrm{Hom}_{\mathcal{M}}(K \wedge X, Y) \cong \mathrm{Hom}_{s\mathrm{Set}}(K, \mathrm{Map}_{\mathcal{M}}(X, Y)),$$

see [Barnes and Roitzheim 2020, Definition 6.1.28; Riehl 2014, Section 11.4].

2.1.1 Diagram categories We will use model structures on diagram categories throughout the paper. Below we introduce the definition of a direct category which is a generalization of the concept of a poset; see [Hovey 1999, Definition 5.1.1] for further details.

Definition 2.1.2 Let ω denote the poset category of the ordered set $\{0, 1, 2, \ldots\}$. A small category J is called *direct* if there is a functor $f\colon J \to \omega$ that sends nonidentity morphisms to nonidentity morphisms. We refer to $f(j)$ as the *degree* of the object j. Dually, J is an *inverse* category if there is a functor $J^{\mathrm{op}} \to \omega$ that sends nonidentity morphisms to nonidentity morphisms

Any finite poset J is a direct category, and dually J^{op} is an inverse category. We provide some examples that will be useful later on.

Definition 2.1.3 Suppose $\mathcal{M}$ is a small category with small colimits, J a small category, z an object in J and J_z the category of all nonidentity morphisms with codomain z. The *latching space functor* $L_z\colon \mathcal{M}^J \to \mathcal{M}$ is the composition

$$\mathcal{M}^J \to \mathcal{M}^{J_z} \xrightarrow{\mathrm{colim}} \mathcal{M},$$

where the first arrow is the restriction functor. Equivalently the latching space of a diagram X is given by

$$L_z X = \operatorname{colim}(J_z \hookrightarrow J \xrightarrow{X} \mathcal{M}),$$

where $J_z \hookrightarrow J$ is the inclusion.

Note that we have a natural transformation $L_z X \to X_z$ for any fixed object $z \in J$.

We can now describe the *projective model structure* on $\mathcal{M}^J$; see [Hovey 1999, Theorem 5.1.3].

Proposition 2.1.4 *Given a model category $\mathcal{M}$ and a direct category J, there is a model structure on $\mathcal{M}^J$ in which a morphism $f: X \to Y$ is a weak equivalence (resp. fibration) if and only if the map $f_z: X_z \to Y_z$ is a weak equivalence (resp. fibration) for all $z \in J$. Furthermore, $f: X \to Y$ is an (acyclic) cofibration if and only if the induced map*

$$X_z \coprod_{L_z X} L_z Y \to Y_z$$

is an (acyclic) cofibration for all $z \in J$.

We will now give the finite posets J that are going to play a central role throughout this paper.

Example 2.1.5 By [1] we denote the poset $0 \leq 1$. We are aware that early in this section we also denoted the shift functor on graded objects. Both are standard notation, and from our use of the poset $0 \leq 1$ there is vanishingly little danger of confusing those two.

Example 2.1.6 Consider the poset

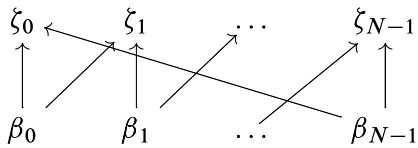
$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \\ (0,1) & & \end{array}$$

denoted by $\ulcorner$. Let $\iota: [1] \to \ulcorner$ be the map of posets which sends 0 to $(0,0)$ and 1 to $(1,0)$. In other words, ι includes the interval [1] to the top horizontal line. Furthermore, consider the product of the interval posets $[1] \times [1]$. It is the poset

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \downarrow \\ (0,1) & \longrightarrow & (1,1) \end{array}$$

and we let $i_\ulcorner: \ulcorner \to [1] \times [1]$ be the inclusion.

Example 2.1.7 Let $N \geq 2$ be a natural number. The poset C_N consists of elements $\{\beta_i, \zeta_i \mid i \in \mathbb{Z}/N\mathbb{Z}\}$ such that $\beta_i < \zeta_i$ and $\beta_i < \zeta_{i+1}$ for $i \in \mathbb{Z}/N\mathbb{Z}$, ie



Then $X \in \mathcal{M}^{C_N}$ is cofibrant if and only if the canonical map $L_z X \to X_z$ is a cofibration in $\mathcal{M}$, ie if and only if the X_{β_i}, X_{ζ_i} are cofibrant and the induced morphism

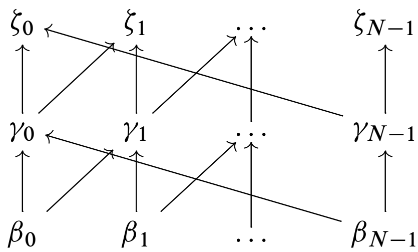
$$X_{\beta_{i-1}} \vee X_{\beta_i} \to X_{\zeta_i}$$

is a cofibration, where $\vee$ is the coproduct in $\mathcal{M}$. We will refer to an object $X \in \mathcal{M}^{C_N}$ as a *crowned diagram* due to the crown shape of the diagram C_N.

Example 2.1.8 Let D_N be the poset consisting of elements $\{\beta_n, \gamma_n, \zeta_n \mid n \in \mathbb{Z}/N\mathbb{Z}\}$ such that

$$\beta_n \leq \gamma_n \leq \zeta_n \quad \text{and} \quad \beta_n \leq \gamma_{n+1} \quad \text{and} \quad \gamma_n \leq \zeta_{n+1},$$

ie



Remark 2.1.9 In what follows, when we have a direct category I and a model category $\mathcal{M}$, the category of diagrams $\mathcal{M}^I$ will always have the model structure defined in Proposition 2.1.4 without further mention. If not, we will explicitly say so.

It follows that for any model category $\mathcal{M}$ and direct category J, there is a Quillen adjunction

$$\operatorname{colim}\colon \mathcal{M}^J \rightleftarrows \mathcal{M} : \operatorname{const}.$$

(Note that when we write an adjunction, the top arrow will always denote the left adjoint.)

Definition 2.1.10 The left derived functor of $\operatorname{colim}\colon \mathcal{M}^J \to \mathcal{M}$ is called the *homotopy colimit* and is denoted by

$$\operatorname{hocolim}\colon \operatorname{Ho}(\mathcal{M}^J) \to \operatorname{Ho}(\mathcal{M}).$$

If $J = \ulcorner$, then the homotopy colimit is called *homotopy pushout.* A particular example of homotopy pushout is the *homotopy cofiber* which is the homotopy pushout of a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow & & \\ * & & \end{array}$$

and we write

$$\mathrm{hocofib}(f) := \mathrm{hocolim}(* \leftarrow X \xrightarrow{f} Y). \tag{2.1.11}$$

In general, for notational convenience sometimes a homotopy pushout is denoted by

$$\mathrm{hocolim}(Z \leftarrow X \to Y) := Z \coprod_X^h Y.$$

2.1.2 Homotopy colimits in simplicial model categories In Definition 2.1.10 we recalled the definition of the homotopy colimit as a derived functor. Here, we will present an alternative construction via simplicial techniques. After introducing some definitions we briefly explain how this method provides a good theory of homotopy colimits; see also [Riehl 2014, Chapters 4, 5; Shulman 2006, Section 7].

Let $\mathcal{M}$ be a model category and consider the category of simplicial objects $s\mathcal{M} = \mathcal{M}^{\Delta^{\mathrm{op}}}$. We consider $s\mathcal{M}$ as a simplicial category with tensors defined objectwise, ie for $K \in s\mathrm{Set}$ and $X \in s\mathcal{M}$ we have

$$(K \otimes X)_n = K \otimes X_n.$$

Now, let $\mathcal{M}$ be a simplicial model category. Given a simplicial object $X \in s\mathcal{M}$ we can construct an object in $\mathcal{M}$ via *geometric realization*, see [Hirschhorn 2003, Definition 18.6.2].

Definition 2.1.12 (geometric realization) Let $X \in \mathcal{M}^{\Delta^{\mathrm{op}}}$. *The geometric realization* of X, denoted as $|X|$, is defined as the coequalizer

$$\mathrm{coeq}\Big(\coprod_{\sigma : [n] \to [k] \in \Delta} \Delta^k \otimes X_n \rightrightarrows \coprod_{[n] \in \Delta} \Delta^n \otimes X_n \Big).$$

This is an example of a functor tensor product (coend). In this case, the geometric realization is the functor tensor product of $X : \Delta^{\mathrm{op}} \to \mathcal{M}$ and the functor $\Delta^\bullet : \Delta \to s\mathrm{Set}$, $[n] \mapsto \Delta^n$. In other words, the realization $|X|$ is the object

$$\Delta^\bullet \otimes_{\Delta^{\mathrm{op}}} X = \int^n \Delta^n \otimes X_n.$$

The following theorem is the cornerstone of our exposition of homotopy colimits using geometric realizations; see [Goerss and Jardine 1999, VII 3.6; Hirschhorn 2003, 18.4.11; Riehl 2014, Corollary 14.3.10]. For details for the Reedy model structure on $s\mathcal{M}$, see [Goerss and Jardine 1999, Definition 2.1].

Theorem 2.1.13 *If $\mathcal{M}$ is a simplicial model category, then*

$$|-|\colon \mathcal{M}^{\Delta^{\mathrm{op}}} \to \mathcal{M}$$

is a left Quillen functor with respect to the Reedy model structure. In particular, $|-|$ sends Reedy cofibrant simplicial objects to cofibrant objects and preserves objectwise weak equivalences between them.

At this level of generality, this is the strongest result possible. It is not true that geometric realization preserves all objectwise weak equivalences. However, the above will suffice for our purposes. We can now start to work our way to the homotopy colimit of a diagram $X \in \mathcal{M}^J$ in a simplicial model category $\mathcal{M}$.

Our first definition towards this goal is the *simplicial replacement functor*. That is to say, given any diagram $F\colon I \to \mathcal{M}$ we can replace it with simplicial object in $\mathcal{M}$ with good properties.

Definition 2.1.14 (simplicial replacement) Let I be a small category and consider a diagram $X \in \mathcal{M}^I$. The *simplicial replacement* of X is the simplicial object in $\mathcal{M}$, denoted $\operatorname{srep} X$ given in simplicial degree $[n]$ by

$$(\operatorname{srep} X)_n = \coprod_{(i_0 \to i_1 \to \cdots \to i_n) \in N(I)_n} X_{i_0}.$$

The coproduct is indexed over the set of n–chains

$$\sigma = [i_0 \to i_1 \to \cdots \to i_n]$$

over the nerve of I. If $0 \le k < n$, then

$$d_k\colon (\operatorname{srep} X)_n \to (\operatorname{srep} X)_{n-1}$$

maps the term X_{i_n} indexed on σ to the term X_{i_n} indexed on

$$\sigma(k) = [i_0 \to i_1 \to i_{k-1} \to i_{k+1} \to \cdots \to i_n]$$

via the identity, while for $k = n$, the map d_n sends the term X_{i_n} to $X_{i_{n-1}}$ indexed on

$$\sigma(n) = [i_0 \to i_1 \to \cdots \to i_{n-1}]$$

via the induced map $X(i_n \to i_{n-1})$. The degeneracy maps

$$s_j\colon (\operatorname{srep} X)_n \to (\operatorname{srep} X)_{n+1}, \quad 0 \le j \le n$$

are easier to define. Each s_j sends the summand X_{i_n} corresponding to the summand

$$[i_0 \to i_1 \to \cdots \to i_n]$$

to the identical summand X_{i_n} corresponding to the chain in which one has inserted the identity map $i_j \to i_j$.

In other words, the simplicial replacement is the following simplicial object,

$$\coprod_{i_0} X_{i_0} \Lleftarrow \coprod_{i_0 \to i_1} X_{i_0} \Lleftarrow \coprod_{i_0 \to i_1 \to i_2} X_{i_0} \cdots ,$$

where degeneracy maps are omitted. Note that this is can also be found in literature as the *simplicial bar construction* or *Bousfield–Kan construction* denoted by $B(*, I, X)$.

Remark 2.1.15 The colimit of a diagram $X \in \mathcal{M}^I$, if it exists, agrees with the colimit of $\operatorname{srep}(X) \in s\mathcal{M}$. Indeed, consider the colimit of the diagram $\operatorname{srep}(X)$ as the coequilizer

$$\coprod_i X_i \leftleftarrows \coprod_{j \leftarrow i} X_i ,$$

but this is precisely the colimit of X. Therefore in this case, $\operatorname{srep}(X)$ has the augmentation

$$\operatorname{srep}(F) \to \operatorname*{colim}_I F,$$

where we regard the object $\operatorname{colim}_I F$ as a constant simplicial object.

We therefore reach the following result.

Lemma 2.1.16 *Given a diagram $X \in \mathcal{M}^I$ and its simplicial replacement* $\operatorname{srep}(X) \in \mathcal{M}^{\Delta^{\mathrm{op}}}$, *there is a canonical isomorphism*

$$\operatorname*{colim}_I X \cong \operatorname*{colim}_{\Delta^{\mathrm{op}}}(\operatorname{srep}(X)).$$

The proof can be found in [Riehl 2014, Lemma 4.4.2]. The following lemma will also be of importance, see [Riehl 2014, Lemma 5.1.2; Shulman 2006, Lemma 8.7].

Lemma 2.1.17 *Let I be a small category and let $\mathcal{M}$ be a simplicial model category. If $F \in \mathcal{M}^I$ is objectwise cofibrant*, *then* $\operatorname{srep}(F) \in s\mathcal{M}$ *is Reedy cofibrant.*

The above Lemma 2.1.16 and Theorem 2.1.13 essentially mean that geometric realization of objectwise cofibrant diagrams is a good model for calculating homotopy colimits. For details see [Riehl 2014, Theorem 6.6.1].

2.2 Homotopy Kan extensions

In this subsection we will introduce *homotopy Kan extensions*, the homotopy invariant version of ordinary Kan extensions, see eg [Hirschhorn 2003, Section 11.9].

Now, let $\mathcal{M}$ be a model category. Furthermore, let I, J be direct categories and $f: I \to J$ a functor. The pullback functor

$$f^*: \mathcal{M}^J \to \mathcal{M}^I$$

preserves weak equivalences, so it defines a functor between homotopy categories, which we denote by the same letter. Recall the functor $\operatorname{Lan}_f = f_!$, left adjoint to f^*. We have the following proposition.

Proposition 2.2.1 *Let $\mathcal{M}$ be a model category and let $f\colon I \to J$ be a map of direct categories. Then the adjunction*

$$f_!\colon \mathcal{M}^I \rightleftarrows \mathcal{M}^J : f^*$$

is a Quillen adjunction.

Proof This follows from the definition of the projective model structure; see Proposition 2.1.4. The functor f^* is a right adjoint by construction. It preserves weak equivalences and projective fibrations, which means that f^* is also a right Quillen functor. □

Thus, the derived functors of the adjoint pair $(f_!, f^*)$ define an adjoint pair on the level of homotopy categories

$$\mathbb{L}\mathrm{Lan}_f := \mathbb{L} f_!\colon \mathrm{Ho}(\mathcal{M}^I) \rightleftarrows \mathrm{Ho}(\mathcal{M}^J)\colon \mathbb{R} f^*.$$

A useful fact about homotopy Kan extensions is that they does not change the homotopy colimit of a diagram, which is similar to the properties of ordinary Kan extensions.

Corollary 2.2.2 *Let $\mathcal{M}$ be a model category, $f\colon I \to J$ a map of direct categories and let $X \in \mathcal{M}^I$. Then there is a canonical isomorphism in* $\mathrm{Ho}(\mathcal{M})$

$$\operatorname*{hocolim}_J \mathbb{L} f_! X \cong \operatorname*{hocolim}_I X.$$

Proof This follows from the fact that for every pair of left Quillen functors F and G there is a natural isomorphism

$$\mathbb{L}F \circ \mathbb{L}G \to \mathbb{L}(F \circ G),$$

see [Hovey 1999, Theorem 1.37], together with the natural isomorphism

$$\operatorname*{colim}_J \mathrm{Lan}_f X \cong \operatorname*{colim}_I X. \qquad \square$$

To conclude this section, we will shortly discuss how one calculates the values and edges of a homotopy Kan extension. Recall the notion of a *slice category* for given posets C and D and a functor $f\colon C \to D$, namely

$$f/d = \{c \in C \mid f(c) \le d\} \tag{2.2.3}$$

for $d \in D$. The following is [Cisinski 2009, Proposition 1.14], which tells us that homotopy Kan extensions can be computed pointwise.

Proposition 2.2.4 *Let $f\colon I \to J$ be a map of posets and let X be any functor $I \to \mathcal{M}$. For any object $j \in J$ there is a canonical isomorphism in* $\mathrm{Ho}(\mathcal{M})$

$$(\mathbb{L} f_! F)_j \cong \mathrm{hocolim}(f/j \xrightarrow{\pi} I \xrightarrow{X} \mathcal{M}).$$

2.3 Monoidal model categories

Let us now turn to some results concerning monoidal model categories, see eg [Hovey 1999, Definition 4.2.6], [Barnes and Roitzheim 2011, Definition 6.1.9] or [Riehl 2014, Definition 11.4.6] for definitions.

Remark 2.3.1 Let $(\mathcal{C}, \wedge)$ be a closed symmetric monoidal category and let $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ be maps in $\mathcal{C}$. The pushout-product map is the universal arrow

$$f \,\square\, g\colon X_0 \wedge Y_1 \coprod_{X_0 \wedge Y_0} X_1 \otimes Y_0 \to X_1 \wedge Y_1.$$

Another way to see the pushout-product map is as a left Kan extension. Again, consider a cocomplete, (closed) monoidal category $(\mathcal{C}, \wedge)$. Let $[1] = \{0 \leq 1\}$. Furthermore, consider the following map of posets.

$$\mathrm{pr}\colon [1] \times [1] \to [1], \qquad \begin{aligned} (0,0), (1,0), (0,1) &\mapsto 0, \\ (1,1) &\mapsto 1. \end{aligned}$$

Now let f and g be morphisms in $\mathcal{C}$. We can consider them as objects in the arrow category $f, g \in \mathcal{C}^{[1]}$. The functors $f\colon [1] \to \mathcal{C}$ and $g\colon [1] \to \mathcal{C}$ give rise to their objectwise tensor product $f \wedge g$, see Definition 2.3.2. That is, the functor

$$f \wedge g\colon [1] \times [1] \to \mathcal{C}$$

is the following commutative diagram:

$$\begin{array}{ccc} X_0 \wedge Y_0 & \longrightarrow & X_1 \wedge Y_0 \\ \downarrow & & \downarrow \\ X_0 \wedge Y_1 & \longrightarrow & X_1 \wedge Y_0 \end{array}$$

Note that the slice category $\mathrm{pr}/0$ is the poset $\ulcorner$ and the slice $\mathrm{pr}/1$ is the whole square. It follows that the map

$$\operatorname*{colim}_{\ulcorner}(f \wedge g) \to \operatorname*{colim}_{[1]\times[1]}(f \wedge g)$$

induced by the inclusion $\ulcorner \hookrightarrow [1] \times [1]$ is exactly the map

$$f \,\square\, g\colon X_0 \wedge Y_1 \coprod_{X_0 \wedge Y_0} X_1 \wedge Y_1 \to X_1 \wedge Y_1.$$

So indeed, $(\mathrm{Lan}_{\mathrm{pr}}(f \wedge g)) = \mathrm{pr}_!(f \wedge g) = f \,\square\, g$.

2.3.1 Smash products for diagram categories A monoidal category $(\mathcal{M}, \wedge)$ gives rise to more monoidal categories by considering diagrams from small categories into $\mathcal{M}$. In our next example we discuss how this is related to model category theory.

Definition 2.3.2 Let $(\mathcal{M}, \wedge)$ be a monoidal category and let I and J be direct categories. We define the external product, which is the bifunctor

$$- \wedge - : \mathcal{M}^I \times \mathcal{M}^J \to \mathcal{M}^{I \times J}$$

sending (X, Y) to the diagram

$$X \wedge Y : I \times J \to \mathcal{M}, \quad (i, j) \mapsto X_i \wedge Y_j.$$

The external product is part of a two-variable adjunction. Since we do not use the extra structure will not define the other two functors in the two-variable adjunction. We have the following proposition.

Proposition 2.3.3 *Let $(\mathcal{M}, \wedge)$ be a monoidal model category. Then, the bifunctor*

$$- \wedge - : \mathcal{M}^I \times \mathcal{M}^J \to \mathcal{M}^{I \times J}$$

is a Quillen bifunctor, that is to say, it has a total left derived functor

$$- \wedge^{\mathbb{L}} - : \mathrm{Ho}(\mathcal{M}^I) \times \mathrm{Ho}(\mathcal{M}^J) \to \mathrm{Ho}(\mathcal{M}^{I \times J}).$$

Proof Suppose that the *injective* model structures $\mathcal{M}^I_{\mathrm{inj}}$, $\mathcal{M}^J_{\mathrm{inj}}$ and $\mathcal{M}^{I \times J}_{\mathrm{inj}}$ exist, eg if $\mathcal{M}$ is a combinatorial model category. Since in the injective model structures the cofibrations are the objectwise cofibrations, the above proposition follows directly. The universal property of $- \wedge^{\mathbb{L}} -$ implies that up to canonical isomorphism both constructions give the same result. □

We have the following corollary.

Corollary 2.3.4 *In the context of Proposition 2.3.3, there is a functor isomorphism*

$$\operatorname*{hocolim}_{I \times J}(X \wedge^{\mathbb{L}} Y) \cong (\operatorname*{hocolim}_{I} X) \wedge^{\mathbb{L}} (\operatorname*{hocolim}_{J} Y).$$

Proof From Proposition 2.3.3, it follows that the external product preserves diagram cofibrant objects and preserves trivial diagram cofibrations between diagram cofibrant objects. The result now follows from the strict formula

$$\operatorname*{colim}_{I \times J}(X \wedge Y) \cong (\operatorname*{colim}_{I} X) \wedge (\operatorname*{colim}_{J} Y)$$

as all the objects involved are cofibrant. □

As a consequence of Proposition 2.3.3, we also obtain the following.

Example 2.3.5 Let $(\mathcal{M}, \wedge)$ be a monoidal model category and let J be a direct category. Consider the diagram category $\mathcal{M}^J$ with the model structure of Proposition 2.1.4. The category $\mathcal{M}^J$ inherits a monoidal structure

$$\mathcal{M}^J \times \mathcal{M}^J \to \mathcal{M}^J, \quad (X, Y) \mapsto X \wedge Y,$$

where $X \wedge Y$ is the diagram $j \mapsto X_j \wedge Y_j$. By a proof analogous to that of Proposition 2.3.3, $(\mathcal{M}^J, \wedge)$ is a monoidal model category.

Corollary 2.3.6 *Let $(\mathcal{M}, \wedge)$ be a pointed symmetric monoidal model category, and let $f\colon X \to Y$ and $g\colon U \to V$ be morphisms in $\mathcal{M}$. There is a canonical isomorphism*

$$\mathrm{hocofib}(f) \wedge^{\mathbb{L}} \mathrm{hocofib}(g) \cong \mathrm{hocofib}(f \,\square^{\mathbb{L}}\, g).$$

We will provide a proof since it is important to our exposition. A different proof can be found in [Hovey 2014, Proposition 4.1].

Proof We may assume that X, Y, U, V are cofibrant in $\mathcal{M}$. By definition,

$$\mathrm{hocofib}(f) \wedge^{\mathbb{L}} \mathrm{hocofib}(g) = \mathrm{hocolim}(* \leftarrow X \xrightarrow{f} Y) \wedge^{\mathbb{L}} \mathrm{hocolim}(* \leftarrow U \xrightarrow{g} V).$$

By Corollary 2.3.4, this is isomorphic to

$$\mathrm{hocolim}\begin{pmatrix} * & \longleftarrow & X \wedge V & \longrightarrow & Y \wedge V \\ \uparrow & & \uparrow & & \uparrow \\ * & \longleftarrow & X \wedge U & \longrightarrow & Y \wedge U \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & * & \longrightarrow & * \end{pmatrix}. \tag{2.3.7}$$

We denote the above underlying $\ulcorner\times\ulcorner$–diagram by $\mathcal{Z}$. We define the following map of posets

$$\begin{aligned} \mathrm{pr}\colon \ulcorner\times\ulcorner \to \ulcorner, \qquad & ((1,0),(1,0)) \mapsto (1,0), \\ ((0,0),(0,0)), ((0,0),(1,0)), & ((1,0),(0,0)) \mapsto (0,0), \\ & \mathrm{else} \mapsto (0,1), \end{aligned}$$

and consider the homotopy left Kan extension

$$\mathbb{L}\mathrm{pr}_!\colon \mathrm{Ho}(\mathcal{M}^{\ulcorner\times\ulcorner}) \to \mathrm{Ho}(\mathcal{M}^{\ulcorner}). \tag{2.3.8}$$

Applying the formula Proposition 2.2.4 to the diagram $\mathcal{Z}$ we obtain $(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(1,0)} = Y \wedge V$. Next, for the object $(0,0)$ the slice category $\mathrm{pr}/(0,0)$ is just the poset $\ulcorner$ and we have

$$(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(0,0)} = \mathrm{hocolim}\begin{pmatrix} X \wedge U & \xrightarrow{f\wedge 1} & Y \wedge U \\ \big\downarrow {\scriptstyle 1\wedge g} & & \\ X \wedge V & & \end{pmatrix}$$

and finally, $(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(0,1)} \cong *$. Note that

$$(\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(0,0)} \to (\mathbb{L}\mathrm{pr}_!\mathcal{Z})_{(1,0)} = f \,\square^{\mathbb{L}}\, g.$$

Hence, the homotopy left Kan extension (2.3.8) of the underlying diagram (2.3.7) is the following $\ulcorner$–diagram:

$$\begin{array}{ccc} X \wedge V \coprod_{X \wedge U} Y \wedge U & \longrightarrow & Y \wedge V \\ \downarrow & & \\ * & & \end{array}$$

It follows directly that the homotopy colimit of this diagram is

$$\mathrm{hocofib}(f \,\square^{\mathbb{L}}\, g). \qquad \square$$

2.3.2 Stable model categories and triangulated categories Recall that the homotopy category $\mathrm{Ho}(\mathcal{M})$ of a pointed model category $\mathcal{M}$ supports a *suspension* functor

$$\Sigma\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$$

given by

$$\Sigma X := \mathrm{hocolim}(* \leftarrow X \to *),$$

with a right adjoint functor

$$\Omega\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$$

given by

$$\Omega X = \mathrm{holim}(* \to X \leftarrow *).$$

Definition 2.3.9 A *stable model category* is a pointed model category for which the functors Σ and Ω are inverse equivalences.

Example 2.3.10 The prototypical example of a stable model category is the category of spectra, Sp. There are of course many variants of spectra, but as our result does not depend on a choice of suitable, monoidal model category, we will not need to specify this further.

Example 2.3.11 Let $\mathcal{A}$ be a graded abelian category with enough projectives, and let $\mathrm{C}^{([1],1)}(\mathcal{A})$ denote the category of *twisted* $([1],1)$–chain complexes or *differential* objects. An object of $\mathrm{C}^{([1],1)}(\mathcal{A})$ is a pair (M_*, d) with $M_* \in \mathcal{A}$ together with a morphism (the differential)

$$d\colon M_* \to M_*[1],$$

such that $d[1] \circ d = 0$. The category $\mathrm{C}^{([1],1)}(\mathcal{A})$ admits a stable model structure, the *projective model structure*, where the weak equivalences are the homology isomorphisms and the fibrations are the surjections. In particular, the cofibrant objects are the projective objects of $\mathcal{A}$. We let $\mathrm{D}^{([1],1)}(\mathcal{A})$ denote the homotopy category of $\mathrm{C}^{([1],1)}(\mathcal{A})$. For an object $(M_*, d) \in \mathrm{C}^{([1],1)}(\mathcal{A})$ we define the homology $H(M) = \ker d / \operatorname{im} d$, and so we have the homology functor

$$H_*\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathcal{A}.$$

In the following we will let $(\mathcal{A}, \otimes, \mathbf{1})$ be an abelian symmetric monoidal category with enough projectives. In this case $(\mathrm{C}^{([1],1)}(\mathcal{A}), \otimes)$ is a monoidal stable model category. Finally, we mention the homology functor $H_*\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathcal{A}$ is a lax symmetric monoidal functor via the Künneth morphism.

We note that our methods throughout this paper also work in a setting where $\mathcal{A}$ does not have enough projectives. In the case of $\mathcal{A} = E(1)_*E(1)$-comod, $\mathrm{C}^{([1],1)}(\mathcal{A})$ can be equipped with a model structure where the cofibrant twisted chain complexes are degreewise projective as $E(1)_*$–modules. This *relative projective model structure* is also monoidal; see [Barnes and Roitzheim 2011, Section 5].

If $\mathcal{M}$ is a pointed simplicial model category, then the suspension functor

$$\Sigma\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$$

admits a simple description. Indeed, by the simplicial model category axioms, the functor

$$S^1 \wedge -\colon \mathcal{M} \to \mathcal{M}$$

defined using the tensor with simplicial sets is a left Quillen functor. Then, Σ can be defined as the left derived functor of $S^1 \wedge -$, ie

$$\Sigma X := S^1 \wedge^{\mathbb{L}} X = S^1 \wedge QX;$$

see [Hovey 1999, 6.1.1]. Note that if $\mathcal{M}$ is stable, then the homotopy category $\mathrm{Ho}(\mathcal{M})$ is a triangulated category with Σ a shift functor; see [Barnes and Roitzheim 2011, Theorem 4.2.1; Hovey 1999, 7.1.6].

In a simplicial model category $\mathcal{M}$ we can choose a particular *model* for the homotopy cofiber (2.1.11) of a morphism, which will help with computations. It is called the *mapping cone* construction.

Definition 2.3.12 Suppose $\mathcal{M}$ is a simplicial stable model category and $f\colon X \to Y$ a morphism in $\mathcal{M}_{\mathrm{cof}}$. Let $\mathrm{cone}(f)$ be the pushout of f along the canonical morphism

$$\mathrm{incl} \otimes 1\colon S^0 \otimes X \to (I,0) \otimes X = CX,$$

that is, $\mathrm{cone}(f)$ comes with the pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{incl}\otimes 1}\downarrow & & \downarrow \\ CX & \longrightarrow & \mathrm{cone}\, f \end{array}$$

Here $CX = (I,0) \otimes X$ denotes the *cone* of X. The natural map

$$\pi\colon (I,0) \otimes X \to S^1 \otimes X$$

and the trivial map

$$*\colon Y \to S^1 \otimes X$$

induce, using the universal property of pushout, a map $\partial\colon \mathrm{cone}(f) \to S^1 \otimes X$.

The fact that the mapping cone construction represents the homotopy cofiber and further details can be found in [Barnes and Roitzheim 2020, Section 4.3].

Definition 2.3.13 Let $\mathcal{M}$ be a simplicial stable model category and $f\colon X \to Y$ a morphism in $\mathcal{M}_{\mathrm{cof}}$. The *elementary triangle* associated to f is the triangle

$$X \xrightarrow{f} Y \xrightarrow{\iota} \mathrm{cone}(f) \xrightarrow{\partial} S^1 \otimes X.$$

A triangle (f, g, h)

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

in $\mathrm{Ho}(\mathcal{M})$ is called *distinguished* if it is isomorphic to an elementary one.

2.4 Homology of a category with coefficients in a functor

In this subsection we will introduce one our main tools, namely homology of a category with coefficients in a functor. It is a particular case of functor homology that assigns the groups $\mathrm{Tor}_*^I(F, G)$ to functors $F\colon I \to \mathcal{A}$ and $G\colon I^{\mathrm{op}} \to \mathcal{A}$ with $\mathcal{A}$ an abelian category. Since we do not need such generality, we will introduce it in a more down-to-earth way using simplicial techniques that dates back to Quillen. Traditional references include [Oberst 1967; 1968], more contemporary references include [Gálvez-Carrillo et al. 2013; Richter 2020, Chapters 15, 16].

Before we define the homology of a category with coefficients in a functor we will define the associated complex of a simplicial object in an abelian category.

Definition 2.4.1 Let $D \in s\mathcal{A}$ be a simplicial object in $\mathcal{A}$. We define the *associated complex* $(C_\bullet(U), \partial) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ by

$$C_n(D) = D_n, \quad \partial_n = \sum_{i=0}^{n} (-1)^n d_i \colon C_n(D) \to C_{n-1}(D).$$

Note that the simplicial identities imply $\partial^2 = 0$, so $C_\bullet(D)$ is indeed a chain complex. Moreover, this evidently defines a functor $C\colon s\mathcal{A} \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$. In other words, the associated complex to a simplicial object $D \in s\mathcal{A}$ is the following chain complex:

$$D_0 \xleftarrow{d_0 - d_1} D_1 \xleftarrow{d_0 - d_1 + d_2} D_2 \leftarrow \cdots. \tag{2.4.2}$$

Definition 2.4.3 Let I be a small category and consider a diagram $D\colon I \to \mathcal{A}$. The *homology of the category I with coefficients in the functor D* is defined as the homology of the complex $C_\bullet(D)$, ie the homology of the associated complex of the simplicial replacement $\mathrm{srep}(D) \in s\mathcal{A}$.

So, unwinding the definition, we start by first taking the simplicial replacement $\mathrm{srep}(D)\colon \Delta^{\mathrm{op}} \to \mathcal{A}$ of D, see Definition 2.1.14, that is, the diagram

$$\bigoplus_{i_0} D_{i_0} \Longleftarrow \bigoplus_{i_0 \to i_1} D_{i_0} \Lleftarrow \bigoplus_{i_0 \to i_1 \to i_2} D_{i_0} \cdots.$$

Then, we consider the associated chain complex (2.4.2) of $C_\bullet(D)$. Then we defined $H_p(I; D)$ to be the p^{th} homology group of the chain complex $C_\bullet(D)$.

Now we will investigate how these constructions help us calculate homotopy colimits. First, recall the following.

Definition 2.4.4 We call a functor $F_*\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$ *homological* if it satisfies the following conditions:

(i) F_* is a graded functor, that is to say, it commutes with suspensions, so there are natural equivalences

$$F_*(\Sigma X) \cong F_*(X)[1] := F_{*-1}(X)$$

which are part of the structure.

(ii) F_* is additive, ie it commutes with arbitrary coproducts.

(iii) F_* converts distinguished triangles into long exact sequences.

(iv) Furthermore, if $(\mathcal{M}, \wedge)$ is a monoidal model category and $(\mathcal{A}, \otimes)$ is a monoidal abelian category, we require that F_* is lax symmetric monoidal, that is, there is a natural Künneth morphism

$$\kappa_{X,Y}\colon F_*X \otimes F_*Y \to F_*(X \wedge^{\mathbb{L}} Y).$$

Now let $\mathcal{M}$ be a simplicial stable model category, let I be a direct category and let $X \in \mathrm{Ho}(\mathcal{M}^I)$. Further, let

$$F_*\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$$

be a homological functor into an (graded) abelian category. Then there is a spectral sequence

$$E^2_{pq} = H_p(I; F_q X) \Rightarrow F_{p+q}(\operatorname*{hocolim}_J X); \tag{2.4.5}$$

see [Richter 2020, 16.3.1]. The construction of the spectral sequence (2.4.5) arises from the skeletal filtration of a simplicial object. This spectral sequence will play a central role in our calculations for the monoidal properties of $\mathcal{Q}$ in Section 3.

2.5 Franke's realization functor

In this subsection we will recall the construction of Franke's equivalence

$$\mathcal{R}\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{M}).$$

For a detailed exposition we refer to [Patchkoria 2012, Section 3.3; Roitzheim 2008]. Recall that C_N is the crown-shaped poset from Example 2.1.7, and that the category $\mathrm{D}^{([1],1)}(\mathcal{A})$ above is the derived category of twisted chain complexes from Example 2.3.11, where $\mathcal{A}$ is a graded symmetric monoidal hereditary abelian category with enough projectives, $\mathcal{M}$ is a simplicial stable model category, and $F\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$ is a homological functor. Also, we assume $\mathcal{A}$ splits into shifted copies of another abelian category $\mathcal{B}$,

$$\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{B}[i]$$

for $N > 1$. Under these assumptions, $\mathcal{R}$ exists and is an equivalence.

For an object $X \in \mathcal{M}^{C_N}$ we have the structure morphisms of X,

$$l_i : X_{\beta_i} \to X_{\zeta_i}, \quad k_i : X_{\beta_{i-1}} \to X_{\zeta_i}, \qquad i \in \mathbb{Z}/N\mathbb{Z}.$$

Furthermore, let

$$Z^{(i)}(X) = F_*(X_{\zeta_i}), \quad B^{(i)}(X) = F_*(X_{\beta_i}), \quad C^{(i)}(X) = F_*(\mathrm{cone}(k_i)),$$
$$\lambda^{(i)} := F_* l_i : B^{(i)}(X) \to Z^{(i)}(X), \quad i \in \mathbb{Z}/N\mathbb{Z},$$

where $\mathrm{cone}(k_i)$ denotes the cone construction from Definition 2.3.12. We will now list some additional assumptions that we need in order to assemble the $C^{(i)}$ into a chain complex C_*.

Definition 2.5.1 Consider the full subcategory $\mathcal{L}$ of $\mathrm{Ho}(\mathcal{M}^{C_N})$ consisting of those diagrams $X \in \mathrm{Ho}(\mathcal{M}^{C_N})$ which satisfy the following conditions:

(i) The objects X_{β_i} and X_{ζ_i} are cofibrant in $\mathcal{M}$ for any $i \in \mathbb{Z}/N\mathbb{Z}$.

(ii) The objects $F_*(X_{\beta_i})$ and $F_*(X_{\zeta_i})$ are contained in $\mathcal{B}[i]$ for any $i \in \mathbb{Z}/N\mathbb{Z}$.

(iii) The map $\lambda^{(i)} : F_*(X_{\beta_i}) \to F_*(X_{\zeta_i})$ is a monomorphism for any $i \in \mathbb{Z}/N\mathbb{Z}$.

Next we construct a functor

$$\mathcal{Q} : \mathcal{L} \to \mathrm{C}^{([1],1)}(\mathcal{A}).$$

Let X be an object of $\mathcal{L}$. As the functor

$$F_* : \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$$

is homological, the distinguished triangles

$$X_{\beta_{i-1}} \xrightarrow{k_i} X_{\zeta_i} \to \mathrm{cone}(k_i) \to \Sigma X_{\beta_{i-1}}$$

induce long exact sequences

$$\cdots \to B^{(i-1)}(X) \to Z^{(i)}(X) \xrightarrow{\iota^{(i)}} C^{(i)}(X) \xrightarrow{\rho^{(i)}} B^{(i-1)}(X)[1] \to Z^{(i)}(X)[1] \to \cdots.$$

Note that $B^{(i-1)}(X) \in \mathcal{B}[i-1]$ and $Z^{(i)}(X) \in \mathcal{B}[i]$ for all $i \in \mathbb{Z}/N\mathbb{Z}$, since $X \in \mathcal{L}$. Therefore, the morphisms $B^{(i-1)}(X) \to Z^{(i)}(X)$ and $B^{(i-1)}(X)[1] \to Z^{(i)}(X)[1]$ are zero. As a consequence, for any $i \in \mathbb{Z}/N\mathbb{Z}$ we actually obtain short exact sequence in $\mathcal{A}$,

$$0 \to Z^{(i)}(X) \xrightarrow{\iota^{(i)}} C^{(i)}(X) \xrightarrow{\rho^{(i)}} B^{(i-1)}(X)[1] \to 0. \tag{2.5.2}$$

Now consider the following objects in $\mathcal{A}$.

$$C_*(X) = C^{(0)}(X) \oplus C^{(1)}(X) \oplus \cdots \oplus C^{(N-1)}(X),$$
$$Z_*(X) = Z^{(0)}(X) \oplus Z^{(1)}(X) \oplus \cdots \oplus Z^{(N-1)}(X),$$
$$B_*(X) = B^{(0)}(X) \oplus B^{(1)}(X) \oplus \cdots \oplus B^{(N-1)}(X).$$

The morphisms $\lambda^{(i)}, \iota^{(i)}, \rho^{(i)}, i \in \mathbb{Z}/N\mathbb{Z}$, induce morphisms between the direct sums

$$\begin{aligned} \lambda\colon B_*(X) &\to Z_*(X), & \lambda &= \lambda^{(0)} \oplus \lambda^{(1)} \oplus \cdots \oplus \lambda^{(N-1)},\\ \iota\colon Z_*(X) &\to C_*(X), & \iota &= \iota^{(0)} \oplus \iota^{(1)} \oplus \cdots \oplus \iota^{(N-1)},\\ \rho\colon C_*(X) &\to B_*(X)[1], & \rho &= \rho^{(0)} \oplus \rho^{(1)} \oplus \cdots \oplus \rho^{(N-1)}. \end{aligned}$$

After summing up, we get a short exact sequence of objects in $\mathcal{A}$

$$0 \to Z_*(X) \xrightarrow{\iota} C_*(X) \xrightarrow{\rho} B_*(X)[1] \to 0. \tag{2.5.3}$$

Splicing this short exact sequence with its shifted copy gives an object in $\mathrm{C}^{([1],1)}(\mathcal{A})$. More precisely, define

$$d = \iota[1]\lambda[1]\rho\colon C_*(X) \to C_*(X)[1].$$

We have $d^2 = 0$ by construction and therefore we get a $([1],1)$–twisted complex. We have now arrived at the definition

$$\mathcal{Q}\colon \mathcal{L} \to \mathrm{C}^{([1],1)}(\mathcal{A}), \quad \mathcal{Q}(X) = \left(\bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} F_*(\mathrm{cone}(k_i)), d \right) = (C_*(X), d).$$

It can be shown that $\mathcal{Q}$ is in fact an equivalence of categories. The composite

$$\mathrm{C}^{([1],1)}(\mathcal{A}) \xrightarrow{\mathcal{Q}^{-1}} \mathcal{L} \xrightarrow{\mathrm{hocolim}} \mathrm{Ho}(\mathcal{M}). \tag{2.5.4}$$

factors over $\mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{M})$, which is Franke's realization functor $\mathcal{R}$. It follows from the construction of $\mathcal{R}$ that it commutes with suspensions and that $F_* \circ \mathcal{R} \cong H_*$.

3 Monoidal properties of $\mathcal{Q}$

In this section, we will examine properties of the bifunctor

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N})$$

via Theorem 3.1.5, which is one of the main ingredients of the diagram (1.0.2).

3.1 Preliminaries on crowned diagrams

Recall the poset C_N from Example 2.1.7 (the crown shape with two rows) and the poset D_N from Example 2.1.8 (the crown shape with three rows). We will be interested in two functors between these two categories. The first functor is the *projection functor*

$$\begin{aligned} \mathrm{pr}\colon C_N \times C_N \to D_N, \quad &(\beta_i, \beta_j) \mapsto \beta_{i+j}, \quad (\zeta_i, \zeta_j) \mapsto \zeta_{i+j},\\ &(\zeta_i, \beta_j) \mapsto \gamma_{i+j}, \quad (\beta_i, \zeta_j) \mapsto \gamma_{i+j}. \end{aligned} \tag{3.1.1}$$

Note, that we really should be writing $\beta_{i\,(\mathrm{mod}\,N)}$ and $\gamma_{i+j\,(\mathrm{mod}\,N)}$ etc, but we commit a small abuse of notation and avoid this. The other functor that we will be interested in is the functor

$$i\colon C_N \to D_N, \quad \zeta_n \mapsto \zeta_n, \quad \beta_n \mapsto \gamma_n, \tag{3.1.2}$$

which is the inclusion of the crown shape C_N into the bottom two rows of D_N. Since weak equivalences in the diagram categories are given objectwise, the functor $i^*\colon \mathcal{M}^{D_N} \to \mathcal{M}^{C_N}$ preserves weak equivalences. Thus, it defines a functor on the homotopy categories, which we denote by the same letter, that is,

$$i^*\colon \mathrm{Ho}(\mathcal{M}^{D_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N}).$$

Next, recall the external smash product for diagrams $X \in \mathcal{M}^I$ and $Y \in \mathcal{M}^J$ for I and J direct categories from Definition 2.3.2. By choosing $I = J = C_N$, it follows formally that we have the bifunctor

$$- \wedge -\colon \mathcal{M}^{C_N} \times \mathcal{M}^{C_N} \to \mathcal{M}^{C_N \times C_N}. \tag{3.1.3}$$

By Proposition 2.3.3, the external product has a total left derived functor

$$- \wedge^{\mathbb{L}} -\colon \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N \times C_N}). \tag{3.1.4}$$

Given diagrams $X, Y \in \mathrm{Ho}(\mathcal{M}^{C_N})$, we can define the homotopy left Kan extension of the external smash product $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{C_N \times C_N})$ along the projection functor $\mathrm{pr}\colon C_N \times C_N \to D_N$, that is,

$$E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N}).$$

Now that we have all the necessary ingredients we can finally state the following theorem.

Theorem 3.1.5 *The bifunctor*

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N})$$

satisfies the following. Let $X, Y \in \mathcal{L}$ such that $F_*(X_{\alpha_n}), F_*(Y_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$. Then, $i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathcal{L}$, that is to say, we have a bifunctor

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}.$$

Furthermore, there is a natural isomorphism

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y).$$

The theorem has two parts. First, we show that $i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)$ is in fact a bifunctor

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}.$$

The second part is that for any two crowned diagrams $X, Y \in \mathcal{L}$ satisfying the stated hypotheses, there is a natural isomorphism

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y).$$

The two parts combined yield that the following diagram commutes (up to natural isomorphism):

$$\begin{array}{ccc}
\mathrm{C}^{([1],1)}(\mathcal{A}) \times \mathrm{C}^{([1],1)}(\mathcal{A}) & \xleftarrow{\mathcal{Q} \times \mathcal{Q}} & \mathcal{L} \times \mathcal{L} \\
\otimes \downarrow & & \downarrow i^*\mathbb{L}\mathrm{pr}_! \\
\mathrm{C}^{([1],1)}(\mathcal{A}) & \xleftarrow[\mathcal{Q}]{} & \mathcal{L}
\end{array}$$

The first part of Theorem 3.1.5 is the content of Section 3.3 and Proposition 3.3.1. The natural isomorphism

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y)$$

is the content of Sections 3.4 and 3.5 and Proposition 3.6.6.

3.2 Slice categories of the projection functor

Again, the values of $\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)$ are given by the formula in Proposition 2.2.4. That is, the values of E at the objects of D_N are given by

$$E_{\gamma_n} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_n}(X \wedge^{\mathbb{L}} Y), \tag{3.2.1}$$

$$E_{\zeta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y), \tag{3.2.2}$$

$$E_{\beta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\beta_n}(X \wedge^{\mathbb{L}} Y). \tag{3.2.3}$$

The structure morphisms of the diagram E, $\hat{l}_n\colon E_{\gamma_n} \to E_{\zeta_n}$ and $\widehat{k}_n\colon E_{\gamma_{n+1}} \to E_{\zeta_n}$, are the edges of the homotopy Kan extension and are given by the natural maps

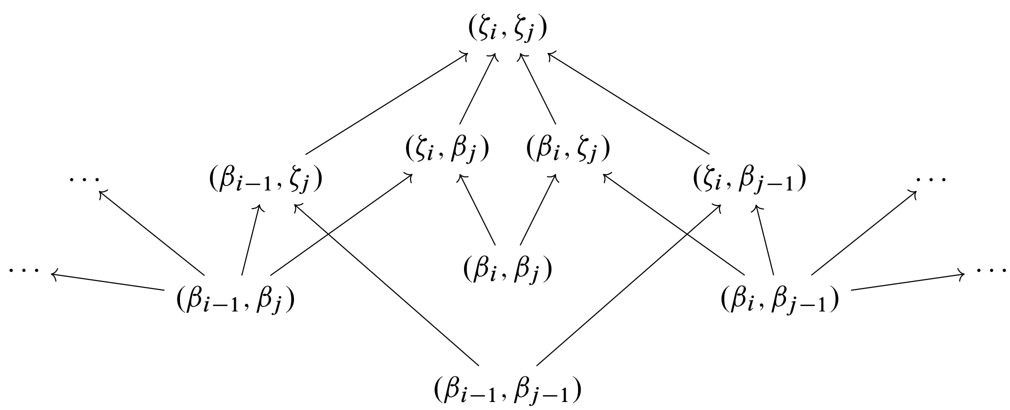
$$E_{\gamma_n} \cong \operatorname*{hocolim}_{\mathrm{pr}/\gamma_n}(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) \cong E_{\zeta_n}, \tag{3.2.4}$$

$$E_{\gamma_{n+1}} \cong \operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n+1}}(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) \cong E_{\zeta_n}, \tag{3.2.5}$$

induced by the maps of posets ϕ and ψ, respectively, see (3.2.8).

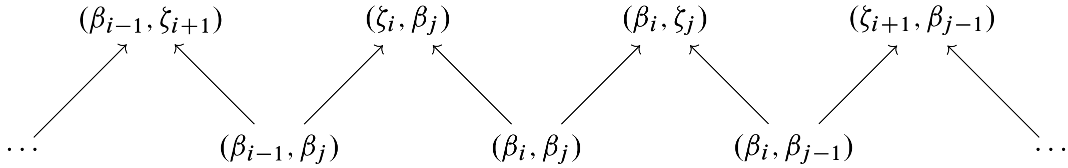
Since we are interested in the homotopy Kan extension of the functor $\mathrm{pr}\colon C_N \times C_N \to D_N$, we need to identify all the slice categories involved, ie pr/ζ_n, pr/γ_n and pr/β_n. We have the following three cases.

(i) The first case is pr/ζ_n. For $n \in \mathbb{Z}/N\mathbb{Z}$ and the object ζ_n we have the slice category pr/ζ_n



where $i + j \equiv n \pmod{N}$. Note that all the nonidentity morphisms are of the form $(1, l_i)$ or $(l_i, 1)$ and similarly $(1, k_i)$ or $(k_i, 1)$ for any $i \in \mathbb{Z}/N\mathbb{Z}$. The poset pr/ζ_n follows the same pattern to the left and to the right.

(ii) Next we have the case pr/γ_n. Let $n \in \mathbb{Z}/N\mathbb{Z}$ and consider now the slice category pr/γ_n which looks as follows,



where $i + j \equiv n \pmod{N}$. Similarly to the above all the nonidentity morphisms are of the form $(1, l_i)$ or $(l_i, 1)$ and $(1, k_i)$ or $(k_i, 1)$ for any $i \in \mathbb{Z}/N\mathbb{Z}$.

(iii) Next is the case pr/β_n. Let again $n \in \mathbb{Z}/N\mathbb{Z}$ but now we consider the slice category pr/β_n. Notice that it is

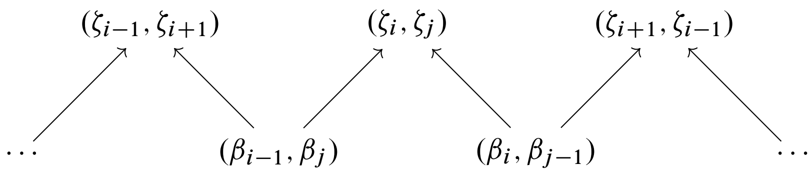
$$\cdots \qquad (\beta_{i-1}, \beta_{j+1}) \qquad (\beta_i, \beta_j) \qquad (\beta_{i+1}, \beta_{j-1}) \qquad \cdots$$

in which $i + j \equiv n \pmod{N}$. In other words, it is a discrete category. This means that

$$E_{\beta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\beta_n}(X \wedge^{\mathbb{L}} Y) \cong \bigoplus_{i+j=n} X_{\beta_i} \wedge^{\mathbb{L}} Y_{\beta_j}.$$

This is the only case that we can be explicit about the values of the homotopy left Kan extension $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)$.

(iv) Our last example is a particular subposet of pr/ζ_n and it is not strictly speaking a slice of any value. However it will be very useful for us is the following. Consider the following subposet $J_n \subseteq \mathrm{pr}/\zeta_n$ defined as follows



where $i + j \equiv n \pmod{N}$. In this poset, the nonidentity morphisms are of the form (k_i, l_i) or (l_i, k_i), unlike the examples above where one arrow was always the identity arrow.

Remark 3.2.6 Now let $\theta\colon J_n \to \mathrm{pr}/\zeta_n$ denote the inclusion of the subposet defined in (iv) into the poset in (i). We will define a map of posets

$$L\colon \mathrm{pr}/\zeta_n \to J_n,$$

where it suffices to define it for the part of the poset visible in (i) as the rest can be defined analogously. The map L is given by

$$L\colon \mathrm{pr}/\zeta_n \to J_n, \qquad \begin{aligned} (\beta_{i+1}, \beta_j) &\mapsto (\beta_{i+1}, \beta_j), \\ (\beta_i, \beta_{j+1}) &\mapsto (\beta_i, \beta_{j+1}), \\ \text{else} &\mapsto (\zeta_i, \zeta_j). \end{aligned}$$

We note that L left adjoint to θ- this can quickly be verified straight from the definition as the morphism sets in either poset are either empty or consist of exactly one element. As a consequence, since the inclusion map $\theta\colon J_n \to \mathrm{pr}/\zeta_n$ is a right adjoint, it is homotopy final, ie for any $F \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$ we have

$$\operatorname*{hocolim}_{J_n} \theta^*(F) \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} F.$$

In other words, the value E_{ζ_n} in (3.2.1) can be calculated as

$$E_{\zeta_n} \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y). \tag{3.2.7}$$

We discuss homotopy finality in more detail in Section 4.1; see Definition 4.1.3.

Given any of subposet of $C_N \times C_N$, eg pr/γ_n from example (ii), we can define the restriction of the external smash product $X \wedge Y \in \mathcal{M}^{C_N \times C_N}$ to pr/γ_n by taking the pullback along the inclusion $\nu\colon \mathrm{pr}/\gamma_n \to C_N \times C_N$, that is,

$$\nu^*\colon \mathcal{M}^{C_N \times C_N} \to \mathcal{M}^{\mathrm{pr}/\gamma_n}.$$

Notice that ν^* preserves weak equivalences so it induces a functor on homotopy categories

$$\nu^*\colon \mathrm{Ho}(\mathcal{M}^{C_N \times C_N}) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n}).$$

Moreover, we have maps between the subposets of $C_N \times C_N$. The morphisms $\gamma_n \to \zeta_n$ and $\gamma_{n+1} \to \zeta_n$ induce maps of posets

$$\psi\colon \mathrm{pr}/\gamma_n \to \mathrm{pr}/\zeta_n, \quad \phi\colon \mathrm{pr}/\gamma_{n-1} \to \mathrm{pr}/\zeta_n, \tag{3.2.8}$$

which in turn also induce pullback functors on the homotopy categories, that is,

$$\phi^*\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n}) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_{n-1}}), \quad \text{and} \quad \psi^*\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n}) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n}).$$

We conclude this section with a convention.

Convention 3.2.9 Because of the above, we will commit an abuse of notation and instead of writing, for example,

$$\phi^*(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n})$$

we will simply write

$$X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n}),$$

with the understanding that this diagram was given by a composition of restriction functors

$$\mathrm{Ho}(\mathcal{M}^{C_N \times C_N}) \xrightarrow{\pi^*} \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n}) \xrightarrow{\phi^*} \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n})$$

unless we need the extra notation for clarification.

Remark 3.2.10 Consider a diagram $F \in \mathrm{Ho}(\mathcal{M}^{C_N \times C_N})$. By Convention 2.1.1, we can assume that F is a projective cofibrant object, so in particular, it is objectwise cofibrant. The external smash product

$$- \wedge - : \mathcal{M}^{C_N} \times \mathcal{M}^{C_N} \to \mathcal{M}^{C_N \times C_N}$$

as defined in (3.1.3) is a Quillen bifunctor, so in particular it preserves cofibrant objects. This implies that $X \wedge^{\mathbb{L}} Y$ is cofibrant in $\mathcal{M}^{C_N \times C_N}$, so in particular it is objectwise cofibrant. Now, for any subposet

$$\iota : J \hookrightarrow C_N \times C_N,$$

eg any of the slice categories of the projection functor pr (3.1.1), we have the pullback functor

$$\iota^* : \mathcal{M}^{C_N \times C_N} \to \mathcal{M}^J.$$

This functor is not necessarily a left Quillen functor with respect the projective model structures; see Proposition 2.1.4. However, the diagram $\iota^*(X \wedge^{\mathbb{L}} Y)$, is objectwise cofibrant, which means that the geometric realization of the simplicial replacement still models the homotopy colimit of the diagram $\iota^*(X \wedge^{\mathbb{L}} Y)$. In particular, the skeletal filtration of all the restrictions is always Reedy cofibrant; see Lemma 2.1.17.

3.3 Spectral sequence calculations

The main result of this subsection is that given crowned diagrams $X, Y \in \mathcal{L}$ that for satisfying a simple condition, the diagram $i^*E = i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)$ is also in the subcategory $\mathcal{L}$, ie the objects X_{β_i} and X_{ζ_i} are cofibrant in $\mathcal{M}$, the objects $F_*(X_{\beta_i})$ and $F_*(X_{\zeta_i})$ are in $\mathcal{B}[i]$, and the map

$$\lambda^{(i)} : F_*(X_{\beta_i}) \to F_*(X_{\zeta_i})$$

is a monomorphism for any $i \in \mathbb{Z}/N\mathbb{Z}$; see Definition 2.5.1. Essentially, this condition is that for the given homological functor $F_* : \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$, either the crowned diagram X or Y is objectwise projective.

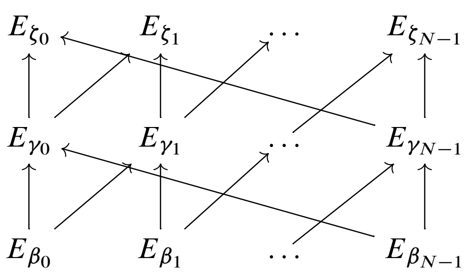
Proposition 3.3.1 *Let $X, Y \in \mathcal{L}$ such that $F_*(X_{\alpha_n}), F_*(Y_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$. Consider the homotopy left Kan extension $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N})$ of*

$$X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{C_N \times C_N})$$

along

$$\mathrm{pr} : C_N \times C_N \to D_N$$

with the values and morphisms given in (3.2.1)–(3.2.3) and (3.2.4), (3.2.5), respectively.



Then, for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$ we have $F_*(E_{\alpha_n}) \in \mathcal{B}[n]$ and the morphisms

$$F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$$

induced by $E_{\gamma_n} \to E_{\zeta_n}$ are monomorphisms.

Corollary 3.3.2 *Let X, Y be crowned diagrams satisfying the hypothesis of Proposition 3.3.1. The top two rows of the diagram $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)$ form an object in $\mathcal{L}$, that is, the diagram $i^* E \in \mathcal{L}$.*

By our assumption, for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$ the objects $F_*(X_{\alpha_n})$ and $F_*(Y_{\alpha_n})$ are projective in $\mathcal{A}$. This ensures that there are natural Künneth isomorphisms

$$F_*(X_{\alpha_n} \wedge^{\mathbb{L}} Y_{\alpha_n}) \cong F_*(X_{\alpha_n}) \otimes F_*(Y_{\alpha_n}). \tag{3.3.3}$$

Since the values E_{ζ_n}, E_{γ_n} and E_{β_n} are computed via homotopy colimits, we will use (2.4.5), the spectral sequences converging to the homology of the homotopy colimit.

Lemma 3.3.4 *There are spectral sequences*

$$E^2_{pq} = H_p(\mathrm{pr}/\gamma_n; F_q(X \wedge^{\mathbb{L}} Y)) \Rightarrow F_{p+q}\Big(\operatorname*{hocolim}_{\mathrm{pr}/\gamma_n}(X \wedge^{\mathbb{L}} Y)\Big) \cong F_{p+q}(E_{\gamma_n}) \tag{3.3.5}$$

and

$$E'^2_{pq} = H_p(\mathrm{pr}/\zeta_n; F_q(X \wedge^{\mathbb{L}} Y)) \Rightarrow F_{p+q}\Big(\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y)\Big) \cong F_{p+q}(E_{\zeta_n}) \tag{3.3.6}$$

and natural morphisms of spectral sequences $f : \{E^2_{pq}\} \to \{E'^2_{pq}\}$ induced by the map in (3.2.4).

We will now begin the proof of Proposition 3.3.1.

Proof Our claim is that $F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$ is a monomorphism, where

$$E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N})$$

as before and $F_* : \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$ is our homological functor. To obtain information on $F_*(E_{\gamma_n})$ and $F_*(E_{\zeta_n})$, we will start by working out the spectral sequence (3.3.5), which we explained is a special case of the spectral sequence (2.4.5). The proof of the proposition is divided into three parts:

- calculating the E_2–term $H_p(\mathrm{pr}/\gamma_n; F_q(X \wedge^{\mathbb{L}} Y))$,
- calculating the E_2–term $H_p(\mathrm{pr}/\zeta_n; F_q(X \wedge^{\mathbb{L}} Y))$,
- showing that the induced map of spectral sequences gives the desired isomorphism.

Step 1 $H_p(\mathrm{pr}/\gamma_n; F_q(X \wedge^{\mathbb{L}} Y))$

We will use the simplicial replacement techniques explained in Section 2.4. Recall the poset pr/γ_n which is

$$\begin{array}{ccccccccccccccccc}
 & & (\beta_{i-1},\zeta_{i+1}) & & & & (\zeta_i,\beta_j) & & & & (\beta_i,\zeta_j) & & & & (\zeta_{i+1},\beta_{j-1}) & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & (\beta_{i-1},\beta_j) & & & & (\beta_i,\beta_j) & & & & (\beta_i,\beta_{j-1}) & & & & \cdots
\end{array}$$

Here, $i+j=n \pmod N$, and so the functor $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n})$ looks as follows:

$$\begin{array}{ccccccccccccccccc}
 & & X_{\beta_{i-1}} \wedge Y_{\zeta_{i+1}} & & & & X_{\zeta_i} \wedge Y_{\beta_j} & & & & X_{\beta_i} \wedge Y_{\zeta_j} & & & & X_{\zeta_{i+1}} \wedge Y_{\beta_{j-1}} & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & X_{\beta_{i-1}} \wedge Y_{\beta_j} & & & & X_{\beta_i} \wedge Y_{\beta_j} & & & & X_{\beta_i} \wedge Y_{\beta_{j-1}} & & & & \cdots
\end{array}$$

Our goal is to compute

$$H_p(\mathrm{pr}/\gamma_n; F_q(X \wedge^{\mathbb{L}} Y)) \quad \text{for all } p \geq 0 \text{ and all } q \in \mathbb{Z},$$

which are the E^2–terms of the spectral sequence (3.3.5). In order to do so, we apply the homological functor $F_n(-)$ to the previous diagram to get the diagram $F_n(X \wedge^{\mathbb{L}} Y) \in \mathcal{A}^{\mathrm{pr}/\gamma_n}$ which, by (3.3.3), is

$$\begin{array}{ccccccccccccccccc}
 & & B^{(i-1)} \otimes \widetilde{Z}^{(j+1)} & & & & Z^{(i)} \otimes \widetilde{B}^{(j)} & & & & B^{(i)} \otimes \widetilde{Z}^{(j)} & & & & B^{(i+1)} \otimes \widetilde{Z}^{(j-1)} & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & 0 & & & & B^{(i)} \otimes \widetilde{B}^{(j)} & & & & 0 & & & & \cdots
\end{array} \tag{3.3.7}$$

We will write

$$f_{ij} = \lambda_i \otimes 1 \colon B^{(i)} \otimes \widetilde{B}^{(j)} \to Z^{(i)} \otimes \widetilde{B}^{(j)}, \tag{3.3.8}$$

$$g_{ij} = 1 \otimes \tilde{\lambda}_j \colon B^{(i)} \otimes \widetilde{B}^{(j)} \to B^{(i)} \otimes \widetilde{Z}^{(j)}, \tag{3.3.9}$$

to distinguish, for labelling purposes, the two different morphisms in the simplicial replacement below. Note that since B^i and $\widetilde{B}^j$ and projective in $\mathcal{A}$, by our convention they are automatically flat, hence the morphisms (3.3.8) and (3.3.9) are monomorphisms.

Next, we consider the simplicial replacement of the diagram $F_n(X \wedge^{\mathbb{L}} Y) \in \mathcal{A}^{\mathrm{pr}/\gamma_n}$, that is

$$\mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y)) \in \mathcal{A}^{\Delta^{\mathrm{op}}}.$$

Following Definition 2.1.14 we have that

$$\mathrm{srep}(F(X \wedge^{L} Y))_0 = \bigoplus_{i+j=n} \big((B^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (Z^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (B^{(i)} \otimes \widetilde{Z}^{(j)})\big),$$

$$\mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y))_1 = \bigoplus_{i+j=n} \big((B^{(i)} \otimes \widetilde{B}^{(j)})_{f_{ij}} \oplus (B^{(i)} \otimes \widetilde{B}^{(j)})_{g_{ij}}\big),$$

with face maps given by "source" and "target" respectively. Because of the shape of the poset pr/γ_n, for all $m \geq 2$ the simplices $\mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y))_m$ consist solely of degenerate simplices.

Now we consider the associated complex $C_*(F_n(X \wedge^{\mathbb{L}} Y))$ of this simplicial complex, see Definition 2.4.1. We briefly explain the differential of the complex $C_*(E(1)_{-n}(X \wedge^{\mathbb{L}} Y))$, namely the map

$$d = d_0 - d_1 \colon C_1(F_n(X \wedge^{\mathbb{L}} Y)) \to C_0(F_n(X \wedge^{\mathbb{L}} Y)).$$

Notice from (3.3.7), we can consider the simpler case where the diagram is

$$Z^{(i)} \otimes \widetilde{B}^{(j)} \xleftarrow{\;f_{ij}\;} B^{(i)} \otimes \widetilde{B}^{(j)} \xrightarrow{\;g_{ij}\;} B^{(i)} \otimes \widetilde{Z}^{(j)}$$

Then, the differential of the associated complex of the simplicial replacement of this diagram is

$$\begin{aligned} d_{ij} = d_0 - d_1 \colon (B^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (B^{(i)} \otimes \widetilde{B}^{(j)}) &\to (B^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (Z^{(i)} \otimes \widetilde{B}^{(j)}) \oplus (B^{(i)} \otimes \widetilde{Z}^{(j)}), \\ (x, y) &\mapsto (x + y, -f_{ij}(x), -g_{ij}(y)). \end{aligned}$$

The 0^{th} homology of the complex is just the pushout

$$B^{(i)} \otimes \widetilde{Z}^{(j)} \coprod_{B^{(i)} \otimes \widetilde{B}^{(j)}} Z^{(i)} \otimes \widetilde{B}^{(i)}.$$

The first homology is the kernel of the differential d_{ij}. Since the maps f_{ij} and g_{ij} are injective, this forces $d_{ij}(x, y) = 0$ if and only if $x = y = 0$, which implies that the first homology is trivial. It follows from the diagram (3.3.7) that the differential d on the complex $C_*(F_n(X \wedge^{\mathbb{L}} Y))$ is the direct sum of the differentials d_{ij} for $i + j = n$. Now that we know the differential of the complex $C_*(F_n(X \wedge^{\mathbb{L}} Y))$ we will compute its homology. It follows that $H_0(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y))$ is the colimit of the diagram $F_n(X \wedge^{\mathbb{L}} Y)$. By inspecting the diagram $F_n(X \wedge^{\mathbb{L}} Y)$ above we can see the colimit of the diagram is a direct sum (coproduct) of pushouts, that is,

$$H_0(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y)) = \operatorname*{colim}_{\mathrm{pr}/\gamma_n} F_n(X \wedge^{\mathbb{L}} Y) = \bigoplus_{i+j=n} \Bigl(Z^{(i)} \otimes \widetilde{B}^{(j)} \coprod_{B^{(i)} \otimes \widetilde{B}^{(j)}} B^{(i)} \otimes \widetilde{Z}^{(j)} \Bigr).$$

Similar to the simpler case, the first homology

$$H_1(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y))$$

is the kernel of the differential

$$d_0 - d_1 \colon \mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y))_1 \to \mathrm{srep}(F_n(X \wedge^{\mathbb{L}} Y))_0.$$

Since it is a direct sum of the simpler differentials d_{ij} as above, it follows that

$$H_1(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y)) = 0.$$

Of course, all the higher homologies

$$H_q(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y))$$

vanish for all $q \geq 2$.

Next, we apply the homological functor $F_{n-1}(-)$ to the diagram $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_n})$ and we have the diagram $F_{n-1}(X \wedge^{\mathbb{L}} Y) \in \mathcal{A}^{\mathrm{pr}/\gamma_n}$ which is

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & \nearrow & & \nwarrow \; \nearrow & & \nwarrow \; \nearrow & & \nwarrow \; \nearrow & & \nwarrow & \\ \cdots & & & B^{(i-1)} \otimes \widetilde{B}^{(j)} & & 0 & & B^{i} \otimes \widetilde{B}^{(j-1)} & & & \cdots \end{array}$$

Clearly,

$$H_0(\mathrm{pr}/\gamma_n; F_{n-1}(X \wedge^{\mathbb{L}} Y)) = 0,$$

and

$$H_1(\mathrm{pr}/\gamma_n; F_{n-1}(X \wedge^{\mathbb{L}} Y)) = \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)}.$$

It follows that for all $p \geq 0$ and all $m \neq -n, -n-1 \mod N$, the terms $H_p(\mathrm{pr}/\gamma_n; F_m(X \wedge^{\mathbb{L}} Y))$ all vanish. This completes the computation of the E^2 terms of the spectral sequence. It is concentrated in degrees $(0, m)$ and $(1, m-1)$ with $m \equiv n \mod N$. Therefore the spectral sequence collapses and we have a short exact sequence

$$0 \to \bigoplus_{i+j=n} \Big(Z^{(i)} \otimes \widetilde{B}^{(j)} \bigoplus_{B^{(i)} \otimes \widetilde{B}^{(j)}} B^{(i)} \otimes \widetilde{Z}^{(j)} \Big) \to F_n(E_{\gamma_n}) \to \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)} \to 0.$$

This concludes the calculation of the spectral sequence (3.3.5).

Step 2 $\boldsymbol{H_p(\mathrm{pr}/\zeta_n; F_q(X \wedge^{\mathbb{L}} Y))}$

We will now repeat the previous strategy and apply it to the spectral sequence (3.3.6). Recall the poset J_n from (iv), which is the following subposet of pr/ζ_n:

$$\begin{array}{ccccccccc} & & (\zeta_{i-1}, \zeta_{i+1}) & & (\zeta_i, \zeta_j) & & (\zeta_{i+1}, \zeta_{i-1}) & & \\ & \nearrow & & \nwarrow \; \nearrow & & \nwarrow \; \nearrow & & \nwarrow & \\ \cdots & & & (\beta_{i-1}, \beta_j) & & (b_i, b_{j-1}) & & & \cdots \end{array}$$

By Remark 3.2.6, the inclusion functor $\theta \colon J_n \to \mathrm{pr}/\zeta_n$ has a left adjoint L, and we have

$$E_{\zeta_n} \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} (X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y);$$

see (3.2.7). So, instead of the spectral sequence (3.3.6) we can compute the following spectral sequence

$$H_p\big(J_n; F_q(\theta^*(X \wedge^{\mathbb{L}} Y))\big) \Rightarrow F_{p+q}\Big(\operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y)\Big)$$

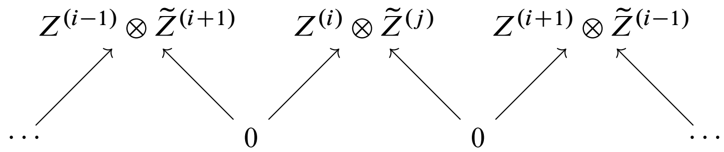
since both converge to the same target, ie the F_*–homology of E_{ζ_n},

$$F_*\Big(\operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y)\Big) \cong F_*\Big(\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} (X \wedge^{\mathbb{L}} Y)\Big) \cong F_*(E_{\zeta_n}).$$

In fact this, can be made stronger. The adjoint pair $L\colon \mathrm{pr}/\zeta_n \rightleftarrows J_n\colon \theta$ induces a natural isomorphism

$$H_*(\mathrm{pr}/\zeta_n; F_q(X \wedge^{\mathbb{L}} Y)) \cong H_*(J_n, \theta^* F_q(X \wedge^{\mathbb{L}} Y)).$$

From the diagram J_n we again only need to consider $F_n(-)$ and $F_{-n-1}(-)$. Firstly, we apply $F_n(-)$ to the diagram $\theta^*(X \wedge^{\mathbb{L}} Y)$ and we get $F_n(\theta^*(X \wedge^{\mathbb{L}} Y)) \in \mathcal{A}^{J_n}$ as



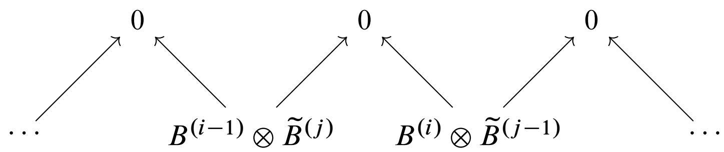
From this we get that

$$H_0\big(J_n; F_n(\theta^*(X \wedge^{\mathbb{L}} Y))\big) = \bigoplus_{i+j=n} Z^{(i)} \otimes \widetilde{Z}^{(j)}$$

and

$$H_p\big(J_n; F_n(\theta^*(X \wedge^{\mathbb{L}} Y))\big) = 0, \quad p \geq 1.$$

Next, we will apply the functor $F_{n-1}(-)$ to obtain the diagram $F_{n-1}(\theta^*(X \wedge^{\mathbb{L}} Y)) \in \mathcal{A}^{J_n}$ depicted by



From the above we get that

$$H_1\big(J_n; F_{n-1}(\theta^*(X \wedge^{\mathbb{L}} Y))\big) = \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)}$$

and

$$H_p\big(J_n; F_{n-1}(\theta^*(X \wedge^{\mathbb{L}} Y))\big) = 0 \quad \text{for } p = 0 \text{ and } p \geq 2.$$

This completes the computation of the E^2–term of the final spectral sequence. It is concentrated in degrees $(0, m)$ and $(1, m-1)$ with $m \equiv n \mod N$. Therefore, the spectral sequence collapses and we have a short exact sequence

$$0 \to \bigoplus_{i+j=n} Z^{(i)} \otimes \widetilde{Z}^{(j)} \to F_n(E_{\zeta_n}) \to \bigoplus_{i+j=n-1} B^{(i)} \otimes \widetilde{B}^{(j)} \to 0.$$

Step 3 the monomorphism $F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$

Now that we have calculated both spectral sequences we can continue with the proof that $F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$ is a monomorphism. The map of posets $\psi\colon \mathrm{pr}/\gamma_n \to \mathrm{pr}/\zeta_n$ induces morphisms on homologies of categories with coefficients $F_n(-)$ and $F_{n-1}(-)$ respectively, ie

$$H_*(\mathrm{pr}/\gamma_n; F_n(X \wedge^{\mathbb{L}} Y)) \to H_*(\mathrm{pr}/\zeta_n; F_n(X \wedge^{\mathbb{L}} Y)) \cong H_*\big(J_n; F_n(\theta^*(X \wedge^{\mathbb{L}} Y))\big),$$
$$H_*(\mathrm{pr}/\gamma_n; F_{n-1}(X \wedge^{\mathbb{L}} Y)) \to H_*(\mathrm{pr}/\zeta_n; F_{n-1}(X \wedge^{\mathbb{L}} Y)) \cong H_*\big(J_n; F_{n-1}(\theta^*(X \wedge^{\mathbb{L}} Y))\big).$$

Hence we have a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{i+j=n} \big(Z^{(i)} \otimes \widetilde{B}^{(j)} \coprod_{B^{(i)}\otimes \widetilde{B}^{(j)}} B^{(i)} \otimes \widetilde{Z}^{(j)}\big) & \longrightarrow & F_n(E_{\gamma_n}) & \longrightarrow & \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow \cong & \\
0 \longrightarrow & \bigoplus_{i+j=n} Z^{(i)} \otimes \widetilde{Z}^{(j)} & \longrightarrow & F_n(E_{\zeta_n}) & \longrightarrow & \bigoplus_{i+j=n+1} B^{(i)} \otimes \widetilde{B}^{(j)} & \longrightarrow 0
\end{array}$$

By naturality, the left vertical map is the direct sum of the pushout-product maps

$$\lambda_i \,\square\, \widetilde{\lambda}_j \colon \Big(Z^{(i)} \otimes \widetilde{B}^{(j)} \coprod_{B^{(i)}\otimes \widetilde{B}^{(j)}} B^{(i)} \otimes \widetilde{Z}^{(j)}\Big) \to Z^{(i)} \otimes \widetilde{Z}^{(j)}.$$

By Lemma 3.7.2, the map $\lambda_i \,\square\, \widetilde{\lambda}_j$ is injective which means that so is the left vertical map. The five lemma now implies that the morphism

$$F_n(E_{\gamma_n}) \to F_n(E_{\zeta_n})$$

is an injection. In particular, $F_*(E_{\gamma_n})$ and $F_*(E_{\zeta_n})$ are concentrated in the correct degrees and the induced morphisms $F_*(E_{\gamma_n}) \to F_*(E_{\zeta_n})$ are injections. This concludes the proof of the proposition. □

Corollary 3.3.2 now follows: the diagram E is indeed in the subcategory $\mathcal{L} \subseteq \mathrm{Ho}(\mathcal{M}^{C_N})$ as the vertices are in the correct degree shifts of $\mathcal{B}$, and F applied to the edges $E_{\gamma_n} \to E_{\zeta_n}$ is a monomorphism, which is precisely how $\mathcal{L}$ was defined.

3.4 Cones

In the previous section we proved that for any two crowned diagrams $X, Y \in \mathcal{L}$ which are objectwise projective, $i^*E = i^*\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathcal{L}$. In this subsection we will prove that applying the functor $\mathcal{Q}$ to the object i^*E is a good model for the tensor product $\mathcal{Q}(X) \otimes \mathcal{Q}(Y)$. This will follow as a corollary from the following proposition.

Proposition 3.4.1 *Consider $E \in \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N})$ and let i^*E be the pullback of E along $i\colon C_N \to D_N$. For every $n \in \mathbb{Z}/N\mathbb{Z}$ we have a canonical isomorphism*

$$\mathrm{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}) \cong \bigvee_{i+j=n} \mathrm{cone}(k_i) \wedge^{\mathbb{L}} \mathrm{cone}(\tilde{k}_j),$$

where $k_i\colon X_{\beta_{i-1}} \to X_{\zeta_i}$ is a structure morphism of $X \in \mathcal{L}$.

Proof This proof has three main parts. Firstly, we will work out the morphism $i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}$ by calculating the relevant values of $E_{\beta_{n-1}}$ and E_{ζ_n} using their description as homotopy colimits over slice categories; see Section 3.2. We will arrive at the conclusion that the left-hand side is actually $\mathrm{hocolim}_{\mathrm{pr}/\zeta_n}(\mathrm{cone}(\varepsilon_{X\wedge^{\mathbb{L}}Y}))$, where $\varepsilon_{X\wedge^{\mathbb{L}}Y}$ is the counit of a certain adjunction. We will then explicitly determine the map of diagrams $\varepsilon_{X\wedge^{\mathbb{L}}Y}$ in Step 2 and calculate its cone in Step 3.

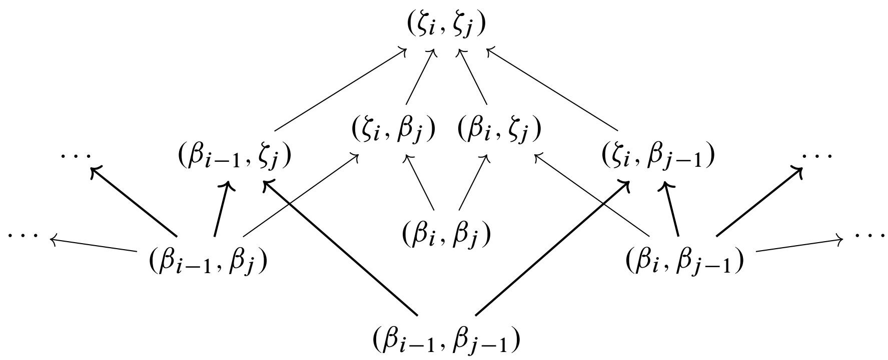
Step 1 unravelling $i^* E_{\beta_{n-1}} \to i^* E_{\zeta_n}$

Recall the slice categories of the map $\mathrm{pr}\colon C_N \times C_N \to D_N$ over ζ_n, pr/ζ_n from example (i), and pr/γ_n from example (ii). By definition of i, we have that

$$(i^* E_{\beta_{n-1}} \to i^* E_{\zeta_n}) = E_{\gamma_{n-1}} \to E_{\zeta_n}.$$

Let us begin by recalling the diagram $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$. The thick arrows show the image of the map of posets $\phi\colon \mathrm{pr}/\gamma_{n-1} \to \mathrm{pr}/\zeta_n$:

(3.4.2)



Recall from (3.2.2) that

$$E_{\zeta_n} = \mathrm{hocolim}(\mathrm{pr}/\zeta_n \xrightarrow{\pi} C_N \times C_N \xrightarrow{X \wedge^{\mathbb{L}} Y} \mathcal{M}),$$

and we committed an abuse of notation by writing

$$E_{\zeta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} \pi^*(X \wedge^{\mathbb{L}} Y).$$

Also, recall from (3.2.5) that the morphism $E_{\gamma_{n-1}} \to E_{\zeta_n}$ is the canonical morphism

$$E_{\gamma_{n-1}} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n-1}} \phi^*(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y) = E_{\zeta_n}$$

induced by the map of posets $\phi\colon \mathrm{pr}/\gamma_{n-1} \to \mathrm{pr}/\zeta_n$. The pullback functor

$$\phi^*\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n}) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_{n-1}})$$

has a left adjoint defined by the homotopy left Kan extension $\mathbb{L}\phi_!$, that is,

$$\mathbb{L}\phi_!\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_{n-1}}) \rightleftarrows \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})\colon \phi^*.$$

The counit of the derived adjunction $\varepsilon\colon \mathbb{L}\phi_!\phi^* \to \mathrm{Id}$ provides the canonical natural transformation

(3.4.3)
$$\varepsilon_{X \wedge^{\mathbb{L}} Y}\colon \mathbb{L}\phi_!\phi^*(X \wedge^{\mathbb{L}} Y) \to X \wedge^{\mathbb{L}} Y.$$

Lastly, since $\mathbb{L}\phi_!$ is a homotopy left Kan extension, there is a canonical isomorphism

$$\operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n-1}} \phi^*(X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} \mathbb{L}\phi_!\phi^*(X \wedge^{\mathbb{L}} Y).$$

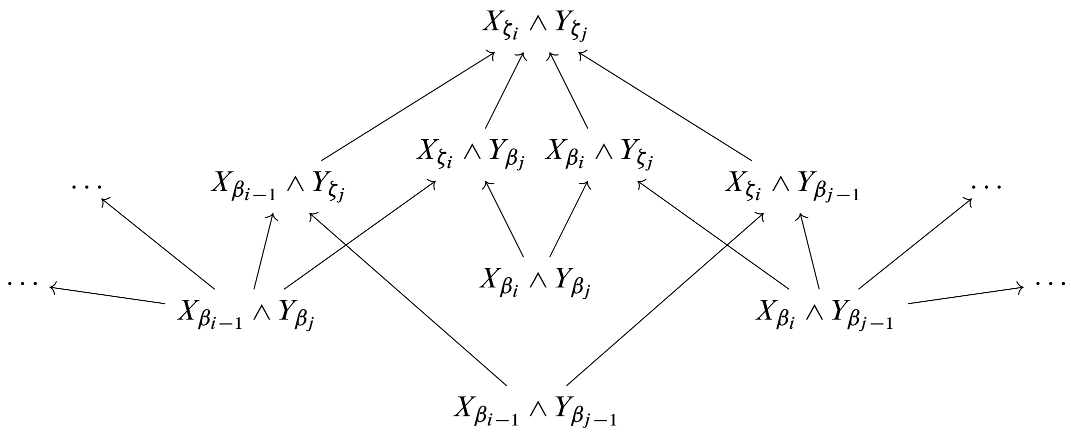
Putting all this together means that the left-hand side of Proposition 3.4.1 is

$$\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(\mathrm{cone}(\varepsilon_{X\wedge^{\mathbb{L}}Y})).$$

Step 2 working out $\boldsymbol{\varepsilon_{X\wedge^{\mathbb{L}}Y}}$

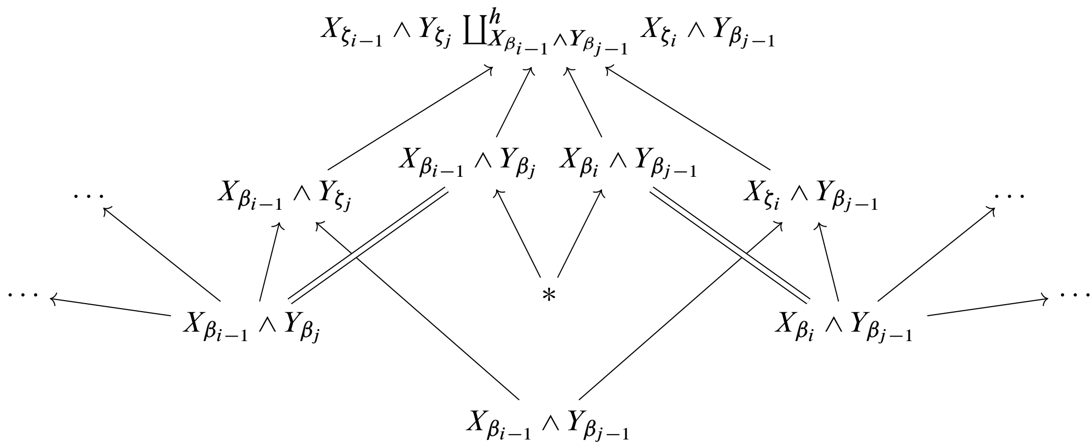
The underlying diagram $X\wedge^{\mathbb{L}} Y\in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$ is

(3.4.4)



Furthermore, the homotopy left Kan extension $\mathbb{L}\phi_!\phi^*(X\wedge^{\mathbb{L}} Y)\in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$ is

(3.4.5)



We briefly explain how we calculated the left homotopy Kan extension $\mathbb{L}\phi_!(X\wedge^{\mathbb{L}} Y)$. From the formula of Proposition 2.2.4 for calculating homotopy Kan extensions, we can calculate the homotopy left Kan extension $\mathbb{L}\phi_!\phi^*$ at an object $(\alpha_s,\alpha_t)\in \mathrm{pr}/\zeta_n$ as

$$\mathbb{L}\phi_!(X\wedge^{\mathbb{L}} Y)_{(\alpha_s,\alpha_t)}\cong \mathrm{hocolim}\big(\phi/(\alpha_s,\alpha_t)\xrightarrow{\pi}\mathrm{pr}/\gamma_{n-1}\xrightarrow{\phi^*(X\wedge^{\mathbb{L}}Y)}\mathcal{M}\big).$$

For the object (ζ_i,β_j), the slice $\phi/(\zeta_i,\beta_j)$ consists only of the object the object (β_{j-1},β_j), which implies that

$$(\mathbb{L}\phi_!)_{(\zeta_i,\beta_j)}=X_{\beta_{i-1}}\wedge Y_{\beta_j}.$$

For the object (β_i, ζ_j), the argument is the same as above. For (β_i, β_j), the slice category $\phi/(\beta_i, \beta_j)$ is empty, which means that

$$(\mathbb{L}\phi_!)_{(\beta_i,\beta_j)} \cong *.$$

For the object (ζ_i, ζ_j), the slice category $\phi/(\zeta_i, \zeta_j)$ is the poset

$$\begin{array}{ccccccc} & (\beta_{i-1}, \zeta_j) & & & & (\zeta_i, \beta_{j-1}) & \\ \nearrow & & \nwarrow & & \nearrow & & \nwarrow \\ (\beta_{i-1}, \beta_j) & & & (\beta_{i-1}, \beta_{j-1}) & & & (\beta_i, \beta_{j-1}) \end{array}$$

But the subposet

$$(\beta_{i-1}, \zeta_j) \longleftarrow (\beta_{i-1}, \beta_{j-1}) \longrightarrow (\zeta_i, \beta_{j-1})$$

is homotopy final, which means that

$$(\mathbb{L}\phi_!)_{(\zeta_i,\zeta_j)} \cong (\mathbb{L}\phi_!)_{(\zeta_i,\zeta_j)} \cong \operatorname{hocolim}\left(\begin{array}{ccc} X_{\beta_{i-1}} \wedge Y_{\beta_{j-1}} & \xrightarrow{k_i \wedge 1} & X_{\zeta_i} \wedge Y_{\beta_{j-1}} \\ \downarrow{\scriptstyle 1 \wedge \tilde{k}_j} & & \\ X_{\beta_{i-1}} \wedge Y_{\zeta_j} & & \end{array}\right).$$

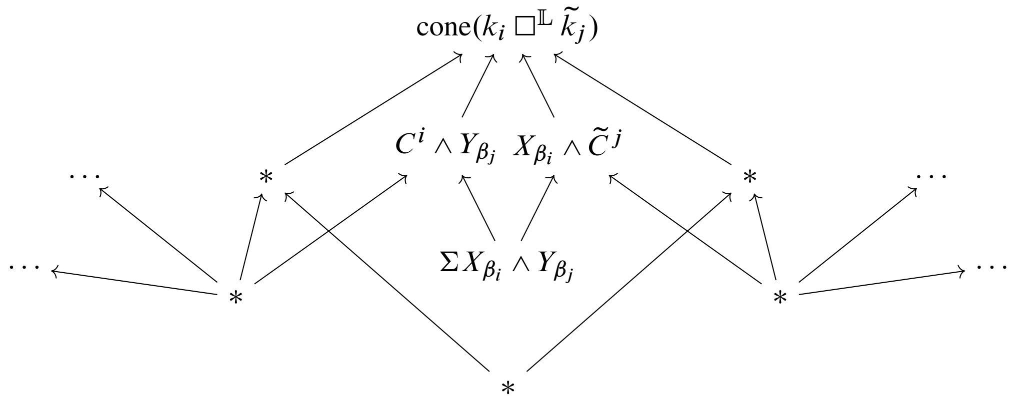
Step 3 calculating the cone in the left-hand side

Next, we calculate the cone of the natural transformation $\varepsilon_{X \wedge^{\mathbb{L}} Y}$ (3.4.3) of diagrams in $\mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$. We have the diagram $\mathrm{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y}) \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})$, which is

$$\mathrm{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y})\colon \mathrm{pr}/\zeta_n \to \mathcal{M}, \quad (\alpha_s, \alpha_t) \mapsto \mathrm{cone}\big(\phi_!(X \wedge^{\mathbb{L}} Y)_{(\alpha_s,\alpha_t)} \to (X \wedge^{\mathbb{L}} Y)_{(\alpha_s,\alpha_t)}\big).$$

In other words, we are taking objectwise cones of the canonical map from the diagram (3.4.5) to the diagram (3.4.4). This means that $\mathrm{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y})$ is

(3.4.6)



Here, we have denoted

$$C^i := \mathrm{cone}(k_i) = \mathrm{cone}(X_{i-1} \to X_{\zeta_i}), \quad \tilde{C}^j := \mathrm{cone}(\tilde{k}_j) = \mathrm{cone}(Y_{j-1} \to Y_{\zeta_i}).$$

Next, we determine the homotopy colimit of the diagram $\operatorname{cone}(\varepsilon_{X\wedge^{\mathbb{L}}Y})$. One way is to observe that the homotopy colimit of the above diagram is isomorphic in $\operatorname{Ho}(\mathcal{M})$ to the homotopy colimit of (finite) coproduct of squares

$$\begin{array}{ccc} \Sigma X_{\beta_i}\wedge Y_{\beta_j} & \longrightarrow & \operatorname{cone}(k_i)\wedge Y_{\beta_j} \\ \downarrow & & \downarrow \\ X_{\beta_i}\wedge \operatorname{cone}(\tilde{k}_j) & \longrightarrow & \operatorname{cone}(k_i \,\square^{\mathbb{L}}\, \tilde{k}_j) \end{array} \tag{3.4.7}$$

where we can consider the above as an object in $\operatorname{Ho}(\mathcal{M}^{[1]\times[1]})$. Formally this is obtained by taking the visually obvious map of posets $f\colon [1]\times[1]\to \mathrm{pr}/\zeta_n$, ie

$$(0,0)\mapsto(\beta_i,\beta_j),\quad (0,1)\mapsto(\zeta_i,\beta_j),\quad (1,0)\mapsto(\beta_i,\zeta_j),\quad (1,1)\mapsto(\zeta_i,\zeta_j)$$

and considering the pullback

$$f^*\colon \operatorname{Ho}(\mathcal{M}^{\mathrm{pr}/\zeta_n})\to \operatorname{Ho}(\mathcal{M}^{[1]\times[1]}).$$

The bottom right corner of the poset $[1]\times[1]$ is its final object, which implies that the homotopy colimit of the diagram (3.4.7) is naturally isomorphic to $\operatorname{cone}(k_i\,\square^{\mathbb{L}}\,\tilde{k}_j)$. Hence the homotopy colimit over pr/ζ_n is, up to natural isomorphism, the coproduct $\bigvee_{i+j=n}\operatorname{cone}(k_i\,\square^{\mathbb{L}}\operatorname{cone}(\tilde{k}_j))$. Another way of seeing this is by pulling back the above diagram to $\theta_n\colon J_n\to \mathrm{pr}/\zeta_n$, and we get the diagram

$$\begin{array}{ccccccccccc} & & \operatorname{cone}(k_{i-1}\,\square^{\mathbb{L}}\,k_{j+1}) & & & & \operatorname{cone}(k_i\,\square^{\mathbb{L}}\,k_j) & & & & \operatorname{cone}(k_{i+1}\,\square^{\mathbb{L}}\,k_{j-1}) & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \cdots & & & & * & & & & * & & & & \cdots \end{array}$$

All in all, we have that the homotopy colimit of the diagram (3.4.6) is

$$\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(\operatorname{cone}(\varepsilon_{X\wedge^{\mathbb{L}}Y}))\cong \bigvee_{i+j=n}\operatorname{cone}(k_i\,\square^{\mathbb{L}}\,\tilde{k}_j). \tag{3.4.8}$$

Finally, by Corollary 2.3.6, we have the canonical isomorphism

$$\operatorname{cone}(k_i\,\square^{\mathbb{L}}\,\tilde{k}_j)\cong \operatorname{cone}(k_i)\wedge^{\mathbb{L}}\operatorname{cone}(\tilde{k}_j)$$

for each pair $i,j\in\mathbb{Z}/N\mathbb{Z}$. The coproduct of these isomorphisms together with (3.4.8) gives us that

$$\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(\operatorname{cone}(\varepsilon_{X\wedge^{\mathbb{L}}Y}))\cong \bigvee_{i+j=n}\operatorname{cone}(k_i)\wedge^{\mathbb{L}}\operatorname{cone}(\tilde{k}_j).$$

Let us now gather all this information to prove Proposition 3.4.1. Calculating the homotopy cofiber (cone) of the morphisms $i^*E_{\beta_{n-1}}\to i^*E_{\zeta_n}$ is the same thing as calculating the homotopy cofiber $E_{\gamma_{n-1}}\to E_{\zeta_n}$.

We have the following natural isomorphisms:

$$\begin{aligned}
\operatorname{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}) &= \operatorname{cone}(E_{\gamma_{n-1}} \to E_{\zeta_n}) \\
&= \operatorname{cone}\Big(\operatorname*{hocolim}_{\mathrm{pr}/\gamma_{n-1}} \phi^*(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y)\Big) \\
&\cong \operatorname{cone}\Big(\operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} \phi_!\phi^*(X \wedge^{\mathbb{L}} Y) \to \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}(X \wedge^{\mathbb{L}} Y)\Big) \\
&\cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n}\Big(\operatorname{cone}\big(\phi_!\phi^*(X \wedge^{\mathbb{L}} Y) \to (X \wedge^{\mathbb{L}} Y)\big)\Big) \\
&\cong \bigvee_{i+j=n} \operatorname{cone}(k_i \,\square^{\mathbb{L}}\, \tilde{k}_j) \\
&\cong \bigvee_{i+j=n} \operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone}(\tilde{k}_j). \qquad \square
\end{aligned}$$

Corollary 3.4.9 *Let X, Y, E as before and assume furthermore that $F_*(X_{\alpha_n}), F_*(Y_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\zeta, \beta\}$. Then there is a canonical isomorphism*

$$C^{(n)}(i^*E) = F_*(\operatorname{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n})) \cong \bigoplus_{i+j=n} C^{(i)}(X) \otimes C^{(j)}(Y).$$

Proof By our assumption, for any $\alpha \in \{\zeta, \beta\}$ and any $n \in \mathbb{Z}/N\mathbb{Z}$, the object $F_*X_{\alpha_n}$ is projective. Therefore, by definition, $Z^{(n)}(X)$ and $B^{(n-1)}(X)$ are projective. The short exact sequence (2.5.2) now implies that for any $i \in \mathbb{Z}/N\mathbb{Z}$ the graded object $C^{(i)}(X)$ is projective. It follows by our assumptions that

$$F_*(\operatorname{cone} k_s \wedge^{\mathbb{L}} \operatorname{cone} \tilde{k}_t) \cong F_*(\operatorname{cone} k_s) \otimes F_*(\operatorname{cone} k_t).$$

By Proposition 3.4.1 we have

$$\operatorname{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}) \cong \bigvee_{i+j=n} \operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone}(\tilde{k}_j),$$

and applying the functor $F_*(-)$ we have

$$\begin{aligned}
F_*(\operatorname{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n})) &\cong F_*\Big(\bigvee_{i+j=n} \operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone}(\tilde{k}_j)\Big) \\
&\cong \bigoplus_{i+j=n} F_*(\operatorname{cone}(k_i) \wedge^{\mathbb{L}} \operatorname{cone} \tilde{k}_j) \\
&\cong \bigoplus_{i+j=n} F_*(\operatorname{cone} k_i) \otimes F_*(\operatorname{cone} \tilde{k}_j).
\end{aligned}$$

Shifting the above by $[n] = [i+j]$ we have

$$C^{(n)}(i^*E) \cong \bigoplus_{i+j=n} C^{(i)}(X) \otimes C^{(j)}(Y). \qquad \square$$

3.5 Differentials

In the previous subsection we proved that $C_*(i^*E) \cong C_*(X) \otimes C_*(Y)$ as objects in $\mathcal{A}$, so the diagram i^*E is a good candidate for the tensor product

$$C_*(X) \otimes C_*(Y).$$

The final step in order to show that indeed

$$\mathcal{Q}(i^*E) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y)$$

as objects in $\mathrm{C}^{([1],1)}(\mathcal{A})$ is to prove that the differential $d\colon C_*(i^*E) \to C_*(i^*E)[1]$ coincides with the differential of the tensor product $C_*(X) \otimes C_*(Y)$. That is, we have to show that

$$(C_*(i^*E), d) \cong (C_*(X) \otimes C_*(Y), d_\otimes),$$

where $d_\otimes$ is the differential of the tensor product of the dg-objects $(C_*(X), d)$ and $(C_*(Y), d)$.

3.5.1 Reduction to the case of disks We will reduce the proof to a much simpler case. Let $L_* \in \mathrm{C}^{([1],1)}(\mathcal{A})$ and choose $s \in \mathbb{Z}/N\mathbb{Z}$. Without loss of generality we will assume that L_* is degreewise projective. Consider the map of differential graded objects,

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & L_s & = & L_s & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \| & & \downarrow{\scriptstyle d^s} & & \downarrow & & \\ \cdots & \xrightarrow[d^{s+2}]{} & L_{s+1} & \xrightarrow[d^{s+1}]{} & L_s & \xrightarrow[d^s]{} & L_{s-1} & \xrightarrow[d^{s-1}]{} & L_{s-2} & \xrightarrow[d^{s-2}]{} & \cdots \end{array} \tag{3.5.1}$$

where we view the top differential graded object as an object in $\mathcal{B}[s-1] \oplus \mathcal{B}[s]$, meaning that it is concentrated in degrees $s-1$ and s modulo N. We denote this by $D^s(L_s)$, and we denote the above map of differential graded objects by $f_{L,s}\colon D^s(L_s) \to L_*$. Under the equivalence of categories $\mathcal{Q}\colon \mathcal{L} \to \mathrm{C}^{([1],1)}(\mathcal{A})$ there are crowned diagrams X and X' and a morphism $F\colon X \to X'$ such that the morphism $f_{L,s}$ is realized as $\mathcal{Q}(F)$. This means that there are isomorphisms

$$\mathcal{Q}(X) \cong D^s(L_s), \quad \mathcal{Q}(X') \cong L_*$$

and the following diagram commutes:

$$\begin{CD} \mathcal{Q}(X) @>{\mathcal{Q}(F)}>> \mathcal{Q}(X') \\ @V{\cong}VV @VV{\cong}V \\ D^s(L_s) @>>{f_{L,s}}> L_* \end{CD}$$

Now let M_* be another differential graded object, which we also assume to be degreewise projective, and let $t \in \mathbb{Z}/N\mathbb{Z}$. Similarly to (3.5.1) we have the morphism

$$g_{M,t}\colon D^t(M_t) \to M_*.$$

Again, under the equivalence $\mathcal{Q}$ there are crowned diagrams Y and Y' and a morphism $G\colon Y \to Y'$ such that

$$\mathcal{Q}(Y) \cong D^t(M_t), \quad \mathcal{Q}(Y') \cong M_*$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Q}(Y) & \xrightarrow{\mathcal{Q}(G)} & \mathcal{Q}(Y') \\ \downarrow{\cong} & & \downarrow{\cong} \\ D^t(M_t) & \xrightarrow[g_{M,t}]{} & M_* \end{array}$$

We have the morphism of dg-objects $f_{L,s} \otimes g_{M,t}\colon D^s(L_s) \otimes D^t(M_t) \to L_* \otimes M_*$, which is

(3.5.2)
$$\begin{array}{ccccccc} \cdots \longrightarrow & L_s \otimes M_t & \longrightarrow & (L_s \otimes M_t) \oplus (L_s \otimes M_t) & \longrightarrow \cdots \\ & \downarrow & & \downarrow{\scriptstyle (d^s \otimes \mathrm{id}, \mathrm{id} \otimes \tilde{d}^t)} & \\ \cdots \longrightarrow & \bigoplus_{i+j=n} L_i \otimes M_j & \longrightarrow & \bigoplus_{i+j=n+1} L_i \otimes M_j & \longrightarrow \cdots \end{array}$$

where the left vertical morphism is the inclusion of the $(s,t)^{\text{th}}$ summand and the right vertical map is the universal map out of the coproduct.

Now assume that

$$\mathcal{Q}(i^*\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y),$$

that is, we prove our claim for the case of $X \cong \mathcal{Q}^{-1}(D^s(L_s))$ and $Y \cong \mathcal{Q}^{-1}(D^t(M_t))$. The commutativity of the square (3.5.2) implies that the bottom vertical maps must also coincide with the tensor product $L_* \otimes M_*$, ie

$$Q(i^*\mathrm{pr}_!(X' \wedge^{\mathbb{L}} Y')) \cong L_* \otimes M_*$$

and the following diagram commutes degreewise:

$$\begin{array}{ccc} Q(i^*\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) & \longrightarrow & \mathcal{Q}(i^*\mathrm{pr}_!(X' \wedge^{\mathbb{L}} Y')) \\ \downarrow & & \downarrow \\ D^s(L_s) \otimes D^t(M_t) & \longrightarrow & L_* \otimes M_* \end{array}$$

The horizontal maps are indeed maps of dg-objects, so if we can show that the left hand vertical map is too, then the claim follows for the general L_* and M_*. The proof of the former will occupy the next subsection.

3.6 Differentials for disks

To prove the claim for disks, we discuss a crowned diagram that corresponds to the disks. By [Patchkoria 2012, Proposition 3.2.1], there is an object $A \in \mathrm{Ho}(\mathcal{M})$, such that $F_* A \in \mathcal{B}[s-1] = L_s$, which is due to the fact that our assumptions force the corresponding Adams spectral sequence to collapse. Consider the crowned diagram

$$X = \begin{array}{ccccccccc} & & * & & A & & * & & \cdots \\ & \nearrow & \uparrow & \nearrow & \| & \nearrow & \uparrow & \nearrow & \\ \cdots & & * & & A & & * & & \end{array}$$

where the nontrivial entries are at the $(s-1)$–spot, ie

$$X_{\beta_{s-1}} = X_{\zeta_{s-1}} = A.$$

The diagram X is in $\mathcal{L}$ since

$$B_*(X) = B^{(s-1)}(X) = F_* X_{\beta_{s-1}} = F_* A \quad \text{and} \quad Z_*(X) = Z^{(s-1)}(X) = F_* X_{\zeta_{s-1}} = F_* A.$$

Next, we calculate $(C_*(X), d) \in \mathrm{C}^{([1],1)}(\mathcal{A})$. The only nontrivial cones are $\mathrm{cone}(k_{s-1})$ and $\mathrm{cone}(k_s)$. This means that

$$\begin{aligned} C^{(s)}(X) &= F_* \,\mathrm{cone}(k_s) = F_* \,\mathrm{cone}(A \to *) = F_*(\Sigma A) \cong (F_* A)[1], \\ C^{(s-1)}(X) &= F_* \,\mathrm{cone}(k_{s-1}) = F_* \,\mathrm{cone}(* \to A) = F_* A, \\ C_*(X) &= C^{(s-1)}(X) \oplus C^{(s)}(X). \end{aligned}$$

We obtain that $\lambda\colon B_*(X) \to Z_*(X)$ is the identity map, $\iota\colon Z_*(X) \to C_*(X)$ is inclusion to the first factor and $\rho\colon C_*(X) \to B_*(X)$ is the projection to the second factor. It follows that $d\colon C_*(X) \to C_*(X)[1]$ is the identity. Similarly, $D^t(L_t)$ is mapped to a crowned diagram Y in which

$$Y_{\beta_{t-1}} = Y_{\zeta_{t-1}} = \widetilde{A},$$

where the only nontrivial morphism is the identity.

We now have the ingredients to deal with the following proposition.

Proposition 3.6.1 *Let X and Y be the objects of $\mathcal{L}$ of the form $\mathcal{Q}^{-1}(D^s(L_s))$ and $\mathcal{Q}^{-1}(D^t(M_t))$. Then*

$$\mathcal{Q}(i^* \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong (C_*(X) \otimes C_*(Y), d_\otimes),$$

where $(C_*(X) \otimes C_*(Y), d_\otimes)$ is the tensor product of $C_*(X)$ and $C_*(Y)$ in $\mathrm{C}^{([1],1)}(\mathcal{A})$.

Proof We note that the tensor product $D^s(L_s) \otimes D^t(M_t)$ is concentrated in degrees $s+t$, $s+t-1$, and $s+t-2$ modulo N. As we already know that our chain complexes agree degreewise, these are the only degrees where we have to calculate our differential. As usual, we write $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)$.

We will work out the differential in the chain complex $\mathcal{Q}(i^* \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y))$, beginning with

$$\mathcal{Q}(i^* E)^{s+t} = C^{(s+t)}(i^* E) \to C^{(s+t-1)}(i^* E) = \mathcal{Q}(i^* E)^{s+t-1},$$

and we will discuss the other degree

$$C^{(s+t-1)}(i^* E) \to C^{(s+t-2)}(i^* E)$$

afterwards. Our proof is divided into the following steps.

We start by going through the definition $\mathcal{Q}$ applied to i^*E using the descriptions given in Section 2.5, where we will arrive at the exact triangle

$$\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}} \to \Sigma E_{\zeta_{s+t-1}} \to \Sigma\mathrm{cone}(\widehat{k}_{s+t-1}).$$

The next steps separately determine $E_{\zeta_{s+t-1}}$ followed by the maps

$$\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}}, \quad E_{\gamma_{s+t-1}} \to E_{\zeta_{s+t-1}} \quad \text{and} \quad E_{\zeta_{s+t-1}} \to \mathrm{cone}(\widehat{k}_{s+t-1}).$$

Putting those together, we obtain the desired differential.

Step 1 recalling the construction of $\mathcal{Q}(i^*E)^{s+t} \to \mathcal{Q}(i^*E)^{s+t-1}$

By Proposition 3.3.1 and Proposition 3.4.1 we can construct a diagram $E = \mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{D_N})$ such that

$$i^*E \in \mathcal{L} \quad \text{and} \quad \mathrm{cone}(i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n}) \cong \bigvee_{i+j=n} \mathrm{cone}(k_i) \wedge \mathrm{cone}(\widetilde{k}_j).$$

For notational convenience we will write

$$\widehat{k}_n \colon i^*E_{\beta_{n-1}} \to i^*E_{\zeta_n} \quad \text{and} \quad \widehat{l}_n \colon i^*E_{\beta_n} \to E_{\zeta_n}$$

for the structure maps of the crowned diagram i^*E. We briefly recall the construction of the differential

$$d \colon C_*(i^*E) \to C_*(i^*E)[1], \quad d = \iota[1]\lambda[1]\rho C_*(i^*E) \to C_*(i^*E)[1].$$

Degreewise, the differential on $C^{(n)}(i^*E) \to C^{(n-1)}(i^*E)$ is given by applying $F_*(-)$ to the sequence of maps

$$\mathrm{cone}(\widehat{k}_n) \to \Sigma E_{\gamma_{n-1}} \to \Sigma E_{\zeta_{n-1}} \to \Sigma\mathrm{cone}(\widehat{k}_{n-1}). \tag{3.6.2}$$

Therefore, we have to show that for $n = s+t$ the sequence of maps (3.6.2) after applying $F_*(-)$ gives us the differential of the tensor product of disks. By Proposition 3.4.1, we have

$$\mathrm{cone}(\widehat{k}_{s+t}) \cong \mathrm{cone}(k_s) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t),$$
$$\mathrm{cone}(\widehat{k}_{s+t-1}) \cong (\mathrm{cone}(k_{s-1}) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t)) \vee (\mathrm{cone}(k_s) \wedge^{L} \mathrm{cone}(\widetilde{k}_{t-1})).$$

Recall that $A = X_{\beta_{s-1}} = X_{\zeta_{s-1}}$ and $\widetilde{A} = Y_{\beta_{s-1}} = Y_{\zeta_{s-1}}$ as before. Directly from the structure morphisms of the crowned diagrams X and Y we have

$$\mathrm{cone}(\widehat{k}_{s+t}) \cong (\Sigma A) \wedge (\Sigma \widetilde{A}), \quad \mathrm{cone}(\widehat{k}_{s+t-1}) \cong (A \wedge \Sigma \widetilde{A}) \vee (\Sigma A \wedge \widetilde{A}).$$

To analyse the sequence of maps (3.6.2) it remains to calculate $E_{\gamma_{s+t-1}}, E_{\zeta_{s+t-1}}, E_{\zeta_{s+t}}$ and the maps $E_{\gamma_{s+t-1}} \to E_{\zeta_{s+t-1}}$. The maps

$$\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}}, \tag{3.6.3}$$

$$\Sigma E_{\zeta_{s+t-1}} \to \Sigma\mathrm{cone}(\widehat{k}_{s+t-1}) \tag{3.6.4}$$

are the canonical maps that are given by construction of distinguished triangles in a simplicial stable model category. The map (3.6.3) is the canonical map

$$\mathrm{cone}(\widehat{k}_{s+t}) \to S^1 \wedge E_{\gamma_{s+t-1}};$$

see Definition 2.3.13. Similarly, the map (3.6.4) is the suspension of the canonical map

$$E_{\zeta_{s+t-1}} \to \mathrm{cone}(\widehat{k}_{s+t-1});$$

see Definition 2.3.13.

Step 2 calculating E_{ζ_n}

To compute the above, let us recall from (iv) the poset J_n with inclusion $\theta\colon J_n \hookrightarrow \mathrm{pr}/\zeta_n$ and left adjoint $L\colon \mathrm{pr}/\zeta_n \to J_n$. For $i + j \equiv n$ modulo N the poset J_n looks as follows:

$$\begin{array}{ccccccccccc}
 & & (\zeta_{i-1},\zeta_{i+1}) & & & & (\zeta_i,\zeta_j) & & & & (\zeta_{i+1},\zeta_{i-1}) & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & (\beta_{i-1},\beta_j) & & & & (\beta_i,\beta_{j-1}) & & & & \cdots
\end{array}$$

Also, recall from (ii) the slice category pr/γ_n, which for $i + j \equiv n$ is

$$\begin{array}{ccccccccccccccc}
 & & (\beta_{i-1},\zeta_{i+1}) & & & & (\zeta_i,\beta_j) & & & & (\beta_i,\zeta_j) & & & & (\zeta_{i+1},\beta_{j-1}) & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & (\beta_{i-1},\beta_j) & & & & (\beta_i,\beta_j) & & & & (\beta_i,\beta_{j-1}) & & & & \cdots
\end{array}$$

By definition of homotopy left Kan extensions, we have

$$E_{\zeta_n} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_n} X \wedge^{\mathbb{L}} Y \cong \operatorname*{hocolim}_{J_n} \theta^*(X \wedge^{\mathbb{L}} Y), \qquad E_{\gamma_n} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_n}(X \wedge^{\mathbb{L}} Y).$$

The maps

$$E_{\gamma_{n-1}} \to E_{\zeta_n} \quad \text{and} \quad E_{\gamma_{n-1}} \to E_{\zeta_{n-1}}$$

are the maps of homotopy colimits induced by the respective map of posets

$$\psi\colon \mathrm{pr}\,/\gamma_{n-1} \to \mathrm{pr}\,/\zeta_n \quad \text{and} \quad \phi\colon \mathrm{pr}\,/\gamma_{n-1} \to \mathrm{pr}\,/\zeta_{n-1}.$$

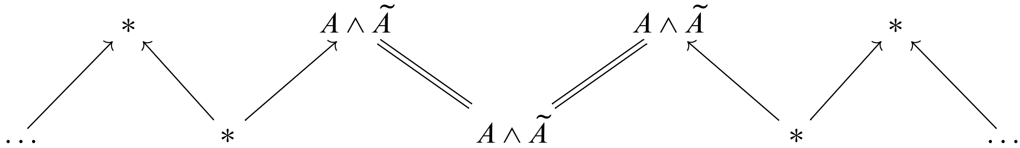
We start with calculating $E_{\zeta_{s+t-1}}$. The underlying diagram $\theta^*(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{J_{s+t-1}})$ is

(3.6.5)
$$\begin{array}{ccccccccccc}
 & & * & & & & * & & & & * & & \\
 & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
\cdots & & & & A \wedge \widetilde{A} & & & & * & & & & \cdots
\end{array}$$

where the only nontrivial entry is at $(\beta_{s-1},\beta_{t-1})$. From the diagram above we get

$$E_{\zeta_{s+t-1}} = \operatorname*{hocolim}_{\mathrm{pr}/\zeta_{s+t-1}}(X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{J_{s+t-1}} \theta^*(X \wedge^{\mathbb{L}} Y) \cong \Sigma A \wedge \widetilde{A}.$$

We do the same for $E_{\gamma_{s+t}}, E_{\gamma_{s+t-1}}$ and $E_{\gamma_{s+t-2}}$. The value $E_{\gamma_{s+t-2}}$ is the homotopy colimit of the diagram $X \wedge^{\mathbb{L}} Y \in \mathrm{Ho}(\mathcal{M}^{\mathrm{pr}/\gamma_{s+t-2}})$, which is

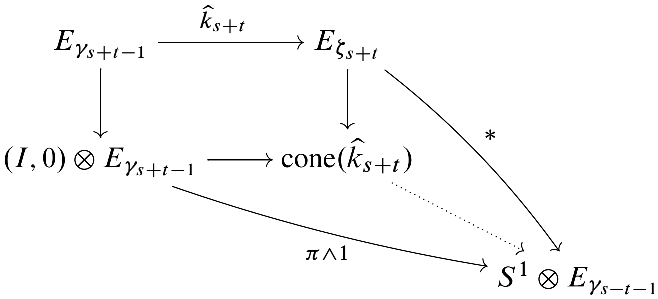


with nontrivial entries at the places $(\beta_{s-1}, \beta_{t-1})$ on the bottom, $(\zeta_{s-1}, \beta_{t-1})$ on the left and $(\beta_{s-1}, \zeta_{t-1})$ on the right. Thus, $E_{\gamma_{s+t-2}} \cong A \wedge \widetilde{A}$. Similarly, we have

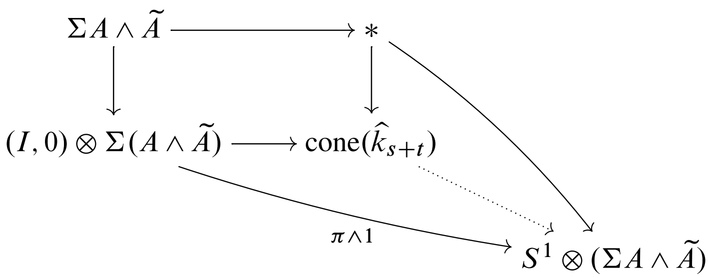
$$E_{\gamma_{s+t}} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_{s+t}} (X \wedge^{\mathbb{L}} Y) \cong *, \quad E_{\gamma_{s+t-1}} = \operatorname*{hocolim}_{\mathrm{pr}/\gamma_{s+t-1}} (X \wedge^{\mathbb{L}} Y) \cong \Sigma A \wedge \widetilde{A}.$$

Step 3 calculating $\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}}$

We move on to calculate the map $\mathrm{cone}(\widehat{k}_{s+t}) \to S^1 \otimes E_{\gamma_{s+t-1}}$. From Definition 2.3.13 we have the pushout square



which, based on our computations, is



Recall from Proposition 3.4.1, (3.4.8), and Corollary 2.3.6 that there is a series of canonical isomorphisms

$$\mathrm{cone}(\widehat{k}_{s+t}) \cong \mathrm{cone}(k_s \,\square^{\mathbb{L}}\, \widetilde{k}_t) \cong \mathrm{cone}(k_s) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t).$$

In our particular case, in which $k_s \colon A \to A$ and $\widetilde{k} \colon \widetilde{A} \to *$, this is

$$\mathrm{cone}(\widehat{k}_{s+t}) \cong \mathrm{cone}(k_s \,\square^{\mathbb{L}}\, k_t) \cong \Sigma^2 A \wedge \widetilde{A} \cong \Sigma A \wedge \Sigma \widetilde{A} \cong \mathrm{cone}(k_s) \wedge^{\mathbb{L}} \mathrm{cone}(\widetilde{k}_t).$$

This implies that the universal map out of the pushout is the identity map. Thus, the map

$$\mathrm{cone}(\widehat{k}_{s+t}) \to \Sigma E_{\gamma_{s+t-1}}$$

is the map $\Sigma A \wedge \Sigma \widetilde{A} \to \Sigma^2(A \wedge \widetilde{A})$, which is the composition of the canonical map (commutation of colimits) and the identity map.

Step 4 calculating $\hat{l}_{s+t-1} \colon E_{\gamma_{s+t-1}} \to E_{\zeta_{s+t-1}}$

From the posets above we can see directly that the map

$$\hat{l}_{s+t-1} \colon E_{\gamma_{s+t-1}} \to E_{\zeta_{s+t-1}}$$

is the identity map induced by

$$\psi \colon \mathrm{pr}/\gamma_{s+t-1} \to \mathrm{pr}/\zeta_{s+t-1}.$$

Therefore the map

$$\Sigma\hat{l}_{s+t-1} \colon E_{\gamma_{s+t-1}} \to \Sigma E_{\zeta_{s+t-1}}$$

is the identity map

$$1 \colon \Sigma^2(A \wedge \widetilde{A}) \to \Sigma^2(A \wedge \widetilde{A}).$$

Step 5 calculating $E_{\zeta_{s+t-1}} \to \mathrm{cone}(\widehat{k}_{s+t-1})$

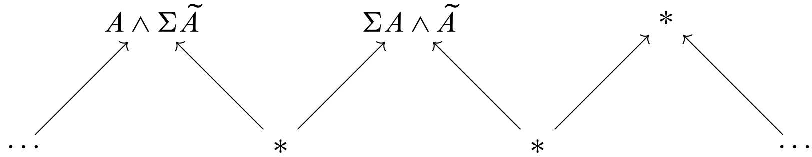
Lastly it remains to figure out the map

$$E_{\zeta_{s+t-1}} \to \mathrm{cone}(\widehat{k}_{s+t-1}).$$

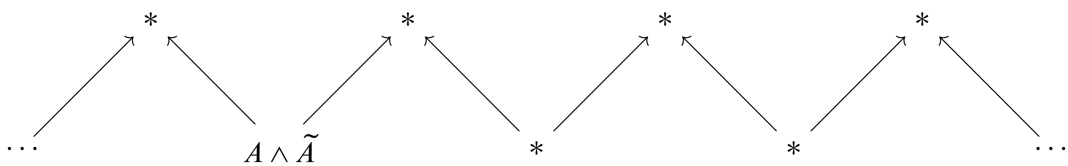
Recall from the proof of Proposition 3.4.1 that $\mathrm{cone}(\widehat{k}_{s+t-1})$ can be written as a homotopy colimit,

$$\mathrm{cone}(\widehat{k}_{s+t-1}) \cong \operatorname*{hocolim}_{\mathrm{pr}/\zeta_{s+t-1}}(\mathrm{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y})),$$

where $\phi \colon \mathrm{pr}/\gamma_{s+t-2} \to \mathrm{pr}/\zeta_{s+t+1}$, and ε is the counit of the derived adjunction $(\mathbb{L}\phi_!, \phi^*)$. Pulling back the diagram $\mathrm{cone}(\varepsilon_{X \wedge^{\mathbb{L}} Y})$ to J_{s+t-1} along the inclusion $\theta \colon J_{s+t-1} \to \mathrm{pr}/\zeta_{s+t-1}$, we obtain the diagram



with nontrivial entries at (ζ_{s-1}, ζ_t) and (ζ_s, ζ_{t-1}) respectively. Recall the following diagram from (3.6.5) $\theta^*(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{J_{s+t-1}})$,



with the only nontrivial entry at $(\beta_{s-1}, \beta_{t-1})$, left top being (ζ_{s-1}, ζ_t) and right top being (ζ_s, ζ_{t-1}). Because of the shape of the underlying posets and the map, we can safely ignore the trivial entries, so the map $E_{\gamma_{s+t-1}} \to \mathrm{cone}(\widehat{k}_{s+t-1})$ can be taken as the map of homotopy pushouts

$$\mathrm{hocolim}(* \leftarrow A \wedge \widetilde{A} \to *) \to \mathrm{hocolim}(A \wedge \Sigma\widetilde{A} \leftarrow * \to \Sigma A \wedge \widetilde{A}),$$

induced by the following map of posets:

$$\begin{array}{ccccc} * & \longleftarrow & A \wedge \widetilde{A} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ A \wedge \Sigma\widetilde{A} & \longleftarrow & * & \longrightarrow & \Sigma A \wedge \widetilde{A} \end{array}$$

Now consider the above map of diagrams and the following map at the bottom:

$$\begin{array}{ccccc} * & \longleftarrow & A \wedge \widetilde{A} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ A \wedge \Sigma\widetilde{A} & \longleftarrow & * & \longrightarrow & \Sigma A \wedge \widetilde{A} \\ \tau\downarrow & & \downarrow & & \parallel \\ \Sigma A \wedge \widetilde{A} & \longleftarrow & * & \longrightarrow & \Sigma A \wedge \widetilde{A} \end{array}$$

Here, τ is the map

$$A \wedge \Sigma\widetilde{A} = A \wedge (S^1 \wedge \widetilde{A}) \cong (A \wedge S^1) \wedge \widetilde{A} \xrightarrow{\tau} (S^1 \wedge A) \wedge \widetilde{A} \cong \Sigma A \wedge \widetilde{A}$$

and the first map is the associativity isomorphism. By Lemma 3.7.1 the induced map of homotopy colimits is, up to weak equivalence, the diagonal map

$$\mathrm{diag}\colon \Sigma A \wedge \widetilde{A} \to (\Sigma A \wedge \widetilde{A}) \vee (\Sigma A \wedge \widetilde{A}).$$

Hence, the map (3.6.2) is up to weak equivalence the diagonal map but with a sign introduced by the twist map as above. This and Corollary 3.4.9 imply that indeed the differential

$$d\colon C^{(s+t)}(i^*E) \to C^{(s+t-1)}(i^*E)$$

coincides with the differential of the tensor product of

$$((D^s L^s) \otimes (D^t \widetilde{L}^t))^{s+t} \to ((D^s L^s) \otimes (D^t \widetilde{L}^t))^{s+t-1}.$$

Step 6 $\mathcal{Q}(i^*E)^{s+t-1} \to \mathcal{Q}(i^*E)^{s+t-2}$

We do not need to do any extra work to determine the other differential, namely to check that the differential

$$C^{(s+t-1)}(i^*E) \to C^{(s+t-2)}(i^*E)$$

coincides with the differential

$$((D^s L^s) \otimes (D^t \widetilde{L}^t))^{s+t-1} \to ((D^s L^s) \otimes (D^t \widetilde{L}^t))^{s+t-2},$$

since by construction $(C_*(i^*E), d)$ is a differential graded object and that means that by necessity $d[1] \circ d = 0$ on $C_*(i^*E)$. This concludes the proof. □

To conclude this section, by combining Corollary 3.4.9 and Proposition 3.6.1 we have proved the following proposition.

Proposition 3.6.6 *Let $X, Y \in \mathcal{L}$ such that $F_*(X_{\alpha_n}), F_*(Y_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for any $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\beta, \zeta\}$. There is a natural isomorphism*

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y). \qquad \square$$

3.7 Technical lemmas

In this subsection we prove two technical lemmas that are used in the previous proofs. The first lemma is about the canonical map from the suspension of an object to the wedge product of suspensions in a stable, simplicial model category $\mathcal{M}$. The second lemma is about pushout-products of injective morphisms in a hereditary abelian category $\mathcal{A}$.

Lemma 3.7.1 *Let $\mathcal{M}$ be a stable simplicial model category and let $X \in \mathcal{M}$. Consider the following map of homotopy pushouts*

$$\mathrm{hocolim}(* \leftarrow X \rightarrow *) \rightarrow \mathrm{hocolim}(\Sigma X \leftarrow * \rightarrow \Sigma X).$$

Then the above map is, up to isomorphism in $\mathrm{Ho}(\mathcal{M})$, *the diagonal map*

$$\mathrm{diag}\colon \Sigma X \rightarrow \Sigma X \vee \Sigma X.$$

Proof Let $CX = (I, 0) \otimes X$ be the cone of X and let $i\colon X \rightarrow CX$ be the canonical inclusion, which is a cofibration. We choose a model for ΣX as the homotopy pushout

$$\Sigma X \cong \mathrm{hocolim}(CX \leftarrow X \rightarrow CX).$$

In fact, we can take this to be the ordinary pushout $\mathrm{colim}(CX \longleftarrow X \rightarrow CX)$ since $i\colon X \rightarrow CX$ is a cofibration. From this model we get directly that the induced map on pushouts

$$\begin{array}{ccccc} CX & \xleftarrow{\;i\;} & X & \xrightarrow{\;i\;} & CX \\ \downarrow{\scriptstyle \pi\otimes 1} & & \downarrow & & \downarrow{\scriptstyle \pi\otimes 1} \\ \Sigma X & \longleftarrow & * & \longrightarrow & \Sigma X \end{array}$$

where $\pi\colon I \rightarrow S^1$ is the projection is indeed the diagonal map $\mathrm{diag}\colon \Sigma X \rightarrow \Sigma X \vee \Sigma X$. Hence, the induced map of homotopy pushouts is the diagonal map up to natural isomorphism. $\square$

Lemma 3.7.2 *Let $\mathcal{A}$ be a hereditary abelian category. Let $X, Y, U, V \in \mathcal{A}_{\mathrm{proj}}$ and let $f\colon X \rightarrow Y$ and $g\colon U \rightarrow V$ be injective maps. Then the pushout-product map $f \,\square\, g$ is injective.*

Proof Since $g\colon U \rightarrow V$ is monomorphism we have the short exact sequence

$$0 \rightarrow U \xrightarrow{g} V \xrightarrow{j} \mathrm{coker}\, g \rightarrow 0.$$

Notice that the dimension of the abelian category $E(1)_*$–modules is 1, which implies that $\operatorname{coker} g$ is a projective module since it is a submodule of V. Since X is flat, $X \otimes -$ is an exact functor, which means that the sequence

$$0 \to X \otimes U \xrightarrow{1\otimes g} X \otimes V \xrightarrow{1\otimes j} X \otimes \operatorname{coker} g \to 0$$

is short exact. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X\otimes U & \xrightarrow{1\otimes g} & X\otimes V & \xrightarrow{1\otimes j} & X\otimes \operatorname{coker} g & \longrightarrow & 0\\
& & {\scriptstyle f\otimes 1}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y\otimes U & \longrightarrow & P & \longrightarrow & X\otimes\operatorname{coker} g & \longrightarrow & 0\\
& & \| & & {\scriptstyle f\square g}\downarrow & & \downarrow{\scriptstyle f\otimes 1} & & \\
0 & \longrightarrow & Y\otimes U & \xrightarrow[1\otimes g]{} & Y\otimes V & \xrightarrow[1\otimes j]{} & Y\otimes\operatorname{coker} g & \longrightarrow & 0
\end{array}$$

where P is the pushout of $1 \otimes g$ and $g \otimes 1$. Since the top left square is cocartesian, the canonical map $\operatorname{coker}(1 \otimes g) \xrightarrow{\cong} \operatorname{coker}(Y \otimes U \to P)$ is an isomorphism, so the middle row is also exact. Now note that the morphism $f \otimes 1\colon X \otimes \operatorname{coker} g \to Y \otimes \operatorname{coker} g$ is injective since $\operatorname{coker} g$ is projective. Applying the snake lemma gives us that $f \square g$ is a monomorphism. □

4 Main result

4.1 Homotopy colimit calculations

In this section we discuss how the functor $i^*\mathbb{L}\mathrm{pr}_!$ interacts with the homotopy colimits of the various diagram categories, giving us the right hand side of the main diagram (1.0.2). The main result of the section is the following.

Theorem 4.1.1 *For any pair of diagrams* $(X, Y) \in \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N})$, *the homotopy colimit of the diagram* $i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y) \in \mathrm{Ho}(\mathcal{M}^{C_N})$ *is naturally isomorphic to the smash product of the homotopy colimits of* X *and* Y, *that is,*

$$\operatorname*{hocolim}_{C_N}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \operatorname*{hocolim}_{C_N} X \wedge^{\mathbb{L}} \operatorname*{hocolim}_{C_N} Y.$$

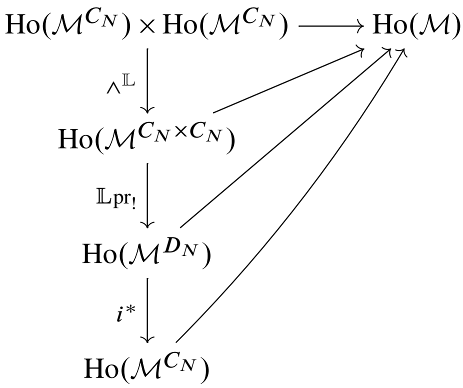
Recall that the functor

$$i^*\mathbb{L}\mathrm{pr}_!(- \wedge^{\mathbb{L}} -)\colon \mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}^{C_N})$$

is the composition

$$\mathrm{Ho}(\mathcal{M}^{C_N}) \times \mathrm{Ho}(\mathcal{M}^{C_N}) \xrightarrow{\wedge^{\mathbb{L}}} \mathrm{Ho}(\mathcal{M}^{C_N \times C_N}) \xrightarrow{\mathbb{L}\mathrm{pr}_!} \mathrm{Ho}(\mathcal{M}^{D_N}) \xrightarrow{i^*} \mathrm{Ho}(\mathcal{M}^{C_N}).$$

In order to prove Theorem 4.1.1 we will break it apart into smaller pieces. Consider the following diagram:



The top horizontal functor is the smash product of homotopy colimits of crowned diagrams, that is, $\mathrm{hocolim}_{C_N} X \wedge^{\mathbb{L}} \mathrm{hocolim}_{C_N} Y$. The three other functors are the homotopy colimit functors,

$$\operatorname*{hocolim}_{C_N \times C_N} \colon \mathrm{Ho}(\mathcal{M}^{C_N \times C_N}) \to \mathrm{Ho}(\mathcal{M}),$$
$$\operatorname*{hocolim}_{D_N} \colon \mathrm{Ho}(\mathcal{M}^{D_N}) \to \mathrm{Ho}(\mathcal{M}),$$
$$\operatorname*{hocolim}_{C_N} \colon \mathrm{Ho}(\mathcal{M}^{C_N}) \to \mathrm{Ho}(\mathcal{M}).$$

Theorem 4.1.1 asserts that the outer triangle above commutes up to isomorphism. This will follow once we show that all the small triangles commute up to isomorphism.

Lemma 4.1.2 *The top triangle and the middle triangle commute. That is,*

$$\operatorname*{hocolim}_{C_N} X \wedge^{\mathbb{L}} \operatorname*{hocolim}_{C_N} Y \cong \operatorname*{hocolim}_{C_N \times C_N}(X \wedge^{\mathbb{L}} Y)$$

and

$$\operatorname*{hocolim}_{C_N \times C_N}(X \wedge^{\mathbb{L}} Y) \cong \operatorname*{hocolim}_{D_N} \mathrm{pr}_!(X \wedge^{\mathbb{L}} Y).$$

Proof The first assertion follows from Corollary 2.3.4 as a direct application for $C = D = C_N$. The second assertion follows from the fact that the homotopy colimit of a homotopy left Kan extension of a diagram is isomorphic to the homotopy colimit of the diagram itself [Richter 2020, Proposition 4.3.2]. □

We will prove Theorem 4.1.1 by proving that the functor $i : C_N \to D_N$ satisfies the following definition; see [Riehl 2014, Definition 8.5.1].

Definition 4.1.3 A functor between small categories $K : C \to D$ is *homotopy final* (or *homotopy terminal*) if for every object $d \in D$, the simplicial set $N(d/K)$ is contractible.

A convenient way to check whether a poset is contractible is given by Quillen [1978, Section 1.5]: a poset C is *conically contractible* if there is an object $c_0 \in C$ and a map of posets $f: C \to C$ such that $c \leq f(c) \geq c_0$ for every $c \in C$. In this case one can show that the identity 1_C, the map f, and the constant map with value c_0 from C to itself are homotopic (that is to say, their realizations are homotopic), and hence C is contractible. So, given a diagram $E \in \mathrm{Ho}(\mathcal{M}^{D_N})$, to check that the canonical morphism

$$\phi_i : \operatorname*{hocolim}_{C_N} i^* E \to \operatorname*{hocolim}_{D_N} E$$

is an isomorphism it suffices to check that the slice categories α_n/i of the functor $i: C_N \to D_N$ are contractible for any $\alpha \in \{\zeta, \gamma, \beta\}$ and any $n \in \mathbb{Z}/N\mathbb{Z}$.

We will now apply this to our functor $i: C_N \to D_N$, which is the inclusion of the two-row crowned diagram into the three-row crowned diagram (3.1.2).

Lemma 4.1.4 *The functor $i: C_N \to D_N$ is homotopy final.*

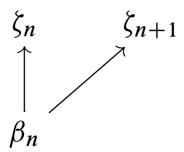
Proof We will prove the above proposition by applying Quillen's criterion of conical contractible posets. First, we identity the slice categories ζ_n/i, γ_n/i and β_n/i and then we will check that they are indeed conically contractible. We start with ζ_n/i. By definition,

$$\zeta_n/i = \{\alpha_n \in C_N \mid i(\alpha_n) \geq \zeta_n\} = \{\zeta_n\}.$$

Since this poset contains only one element it is obviously contractible. The next slice categories are of the form γ_n/i. By definition,

$$\gamma_n/i = \{\alpha_n \in C_N \mid i(\alpha_n) \geq \gamma_n\},$$

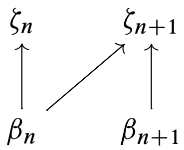
that is, γ_n/i is the poset



We choose β_n and $1: \gamma_n/i \to \gamma_n/i$. Directly from above we can see that γ_n/i is conically contractible. The last case is the slices β_n/i. By definition,

$$\beta_n/i = \{\alpha_n \in C_N \mid i(\alpha_n) \geq \beta_n\},$$

which is the poset



We choose β_n and the map of $\beta_n/i \to \beta_n/i$ as

$$\zeta_n \mapsto \zeta_n, \quad \zeta_{n+1} \mapsto \zeta_{n+1}, \quad \beta_n \mapsto \beta_n, \quad \beta_{n+1} \mapsto \zeta_{n+1}.$$

With these choices, we can see that the poset β_n/i is conically contractible. □

Finally, we obtain the commutativity of the bottom triangle of our big diagram, which also concludes the proof of Theorem 4.1.1.

Corollary 4.1.5 *The bottom triangle of (1.0.2) commutes, that is,*

$$\operatorname*{hocolim}_{C_N} i^* E \cong \operatorname*{hocolim}_{D_N} E. \qquad \square$$

4.2 Proof of main theorem

Finally, we are in a position to assemble all our work into our main theorem.

Theorem 4.2.1 *Let $\mathcal{A}$ be a hereditary abelian category, and $\mathcal{M}$ be a monoidal stable model category such that Franke's functor*

$$\mathcal{R}\colon (\mathrm{D}^{([1],1)}(\mathcal{A}), \otimes^{\mathbb{L}}) \to (\mathrm{Ho}(\mathcal{M}), \wedge^{\mathbb{L}})$$

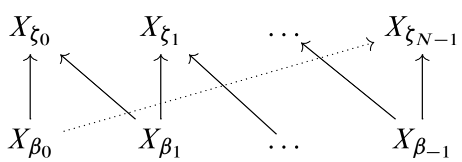
exists and is an equivalence. Then $\mathcal{R}$ preserves the monoidal products up to a natural isomorphism, that is,

$$\mathcal{R}(M_* \otimes^{\mathbb{L}} M_*) \cong \mathcal{R}(M_*) \wedge^{\mathbb{L}} \mathcal{R}(M_*).$$

Proof We assemble our proof along the lines of the diagram (1.0.2). Let M_* and N_* be objects in $\mathrm{D}^{([1],1)}(\mathcal{A})$. By Convention 2.1.1, both objects are cofibrant. Since M_* is cofibrant, the functor

$$M_* \otimes - : \mathrm{C}^{([1],1)}(\mathcal{A}) \to \mathrm{C}^{([1],1)}(\mathcal{A})$$

is left Quillen, see [Hovey 1999, Remark 4.2.3], which means it preserves cofibrant objects. Since both objects are cofibrant, the tensor product $M_* \otimes N_*$ represents the derived tensor product in $(\mathrm{D}^{([1],1)}(\mathcal{A}), \otimes^{\mathbb{L}})$ and in particular it also cofibrant. Recall from Example 2.3.11 that the cofibrant objects in $\mathrm{C}^{([1],1)}(\mathcal{A})$ are the projective objects in $\mathcal{A}$. This means, in particular, that M_*, N_* and $M_* \otimes N_*$ all belong to $\mathcal{A}_{\mathrm{proj}}$. We recall some notation from Section 3. Given a crowned diagram $X \in \mathcal{M}^{C_N}$ as



we set

$$Z^{(n)}(X) = F_*(X_{\zeta_n}), \quad B^{(n)}(X) = F_*(X_{\beta_n}), \quad C^{(n)}(X) = F_*(\mathrm{cone}(X_{\beta_{n-1}} \to X_{\zeta_n})).$$

Given $(M_*, d) \in \mathrm{C}^{([1],1)}(\mathcal{A})$, one can construct a crowned diagram X in $\mathcal{L}$ such that

$$(C_*(X), d) \cong (M_*, d), \quad Z_*(X) \cong \ker d, \quad B_*(X) \cong \operatorname{im} d.$$

By the discussion above, $M_* \in \mathcal{A}_{\mathrm{proj}}$. By assumption, $\mathcal{A}$ is a hereditary abelian category, in other words, $\mathrm{gl.dim}\, \mathcal{A} = 1$. This implies that $\ker d, \operatorname{im} d \in \mathcal{A}_{\mathrm{proj}}$ since they are submodules of M_*.

Hence, for the crowned diagram $X \cong \mathcal{Q}^{-1}(M_*)$ we have $F_*(X_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for every $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\zeta, \beta\}$. Similarly, for the dg-object (N_*, d) we get a crowned diagram $Y \cong \mathcal{Q}^{-1}(N_*)$ such that $F_*(Y_{\alpha_n}) \in \mathcal{A}_{\mathrm{proj}}$ for every $n \in \mathbb{Z}/N\mathbb{Z}$ and any $\alpha \in \{\zeta, \beta\}$.

Now, by Theorem 3.1.5,

$$\mathcal{Q}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \mathcal{Q}(X) \otimes \mathcal{Q}(Y) = M_* \otimes N_*$$

and by Theorem 4.1.1,

$$\operatorname*{hocolim}_{C_N}(i^*\mathbb{L}\mathrm{pr}_!(X \wedge^{\mathbb{L}} Y)) \cong \operatorname*{hocolim}_{C_N} X \wedge^{\mathbb{L}} \operatorname*{hocolim}_{C_N} Y.$$

Finally, we recall that Franke's realization functor (2.5.4) is defined by

$$\mathcal{R} = \operatorname*{hocolim}_{C_N} \circ\, \mathcal{Q}^{-1},$$

which concludes the proof. □

The assumptions of Theorem 4.2.1 are satisfied in the following instances.

Example 4.2.2 From [Patchkoria 2012, Corollary 5.2.1] we know that

$$\mathcal{R}\colon D(\pi_* R) \to D(R) = \mathrm{Ho}(R\text{-mod})$$

is an equivalence for a ring spectrum R with $\pi_*(R)$ concentrated in degrees that are multiples of some $N > 1$ and global dimension of $\pi_*(R)$ equal to 1. This satisfies the assumption of our Theorem 4.2.1 and applies to $R = KU$, $R = KU_{(p)}$, $R = E(1)$ (complex K–theory), and $R = k(n)$ (connective Morava K–theory).

Example 4.2.3 By [Franke 1996; Roitzheim 2008] we know that

$$\mathcal{R}\colon \mathrm{D}^{([1],1)}(\mathcal{A}) \to \mathrm{Ho}(L_1\mathcal{S})$$

is an equivalence. Here, $\mathcal{A}$ is the category of $E(1)_*E(1)$–comodules, and $L_1\mathcal{S}$ is a suitable category of spectra equipped with the K–local model structure at an odd prime. Note that as mentioned in Example 2.3.11, that while $\mathcal{A}$ does not have enough projectives, all our proofs also work when working with comodules whose underlying $E(1)_*$–module is projective; see also the first author's thesis [Nikandros 2022].

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Characterising quasi-isometries of the free group

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We introduce the notion of mixed subtree quasi-isometries, which are self-quasi-isometries of regular trees built in a specific inductive way. We then show that any self-quasi-isometry of a regular tree is at bounded distance from a mixed-subtree quasi-isometry. Since the free group is quasi-isometric to a regular tree, this provides a way to describe all self-quasi-isometries of the free group. In doing this, we also give a way of constructing quasi-isometries of the free group.

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1 Introduction

Quasi-isometries are the most fundamental maps in geometric group theory. However, for most metric spaces, very little is known about their quasi-isometry groups and there are no known tangible ways to describe all quasi-isometries, except in some cases where quasi-isometric rigidity is known. Notable exceptions to this are Baumslag–Solitar groups, which are described in [Whyte 2001] and 3–dimensional solvable Lie groups, which have been studied by Eskin, Fisher and Whyte [Eskin et al. 2007; 2012; 2013].

With this paper, we add the free group $\mathbb{F}_2$, or more generally regular trees, to the list of spaces where all quasi-isometries up to bounded distance can be described. In particular, we introduce the notion of a D–mixed subtree quasi-isometry which is a type of quasi-isometry from regular trees to themselves. While a precise definition can be found in Section 3, the main idea behind them is the following; having defined the quasi-isometry for vertices v at distance nD from the root, one next defines what the quasi-isometry does on the next level, that is, vertices at distance $(n+1)D$ from the root. Moreover, the valid choices of extending the map to the vertices at distance $(n+1)D$ only depend on which of the vertices of distance nD are mapped to the same vertex, but is otherwise independent of the choices made previously.

Our main theorem below states that a map from a regular tree to itself is a quasi-isometry if and only if it is at bounded distance from a mixed-subtree quasi-isometry.

Theorem 1.1 *Let T be a regular tree of degree at least 3, rooted at v_0. Let $f : T \to T$ be a C–quasi-isometry such that $f(v_0) = v_0$. Then there is a constant D only depending on C and a D–deep mixed subtree quasi-isometry $g : T \to T$ such that f and g are at bounded distance from each other.*

Since regular trees of degree at least 3 and nonelementary free groups are quasi-isometric, the theorem above describes quasi-isometries of the free group $\mathbb{F}_2$.

Thanks to this independence mentioned above, mixed-subtree quasi-isometries are a useful tool to construct quasi-isometries with certain desired properties. For example, this technique was used in [Goldsborough and Zbinden 2024], where the authors built a self-quasi-isometry of $\mathbb{F}_2$ with the property that the push-forward of a simple random walk by this quasi-isometry does not have a well-defined drift.

We suspect that there might be other applications of this construction. For instance, one might want to consider "random quasi-isometries" of $\mathbb{F}_2$ and properties of a "generic" quasi-isometry. Further, this characterisation might allow one to better understand the quasi-isometry group $\mathrm{QI}(\mathbb{F}_2)$.

Outline

In Section 2 we introduce the relevant notation and prove some of the technical results about quasi-isometries of trees. In particular, we extend a result of [Nairne 2023] and show that any quasi-isometry is at bounded distance from an order-preserving quasi-isometry. In Section 3 we describe mixed-subtree quasi-isometries and prove Theorem 1.1, which states that a map from a rooted tree of degree at least 3 to itself is a quasi-isometry if and only if it is at bounded distance from a mixed-subtree quasi-isometry.

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2 Preliminaries

In this section, we introduce the relevant notation and some preliminary lemmas. Throughout this paper, we will view $\mathbb{F}_2$ as a rooted tree. Therefore, our results will cover self-quasi-isometries of rooted trees.

Definition 2.1 Let (X, d) be a metric space, we say that a map $f : X \to X$ is a C*–quasi-isometric embedding* for a constant $C \geq 1$ if

$$\frac{d(x, y)}{C} - C \leq d(f(x), f(y)) \leq Cd(x, y) + C$$

for all $x, y \in X$.

Further, we say that a C–quasi-isometric embedding $f : X \to X$ is a C*–quasi-isometry* if there exists a constant D such that for all $y \in X$ there exists $x \in X$ such that $d(y, f(x)) \leq D$.

Definition 2.2 Let (X, d) be a metric space. Two maps $f, g : X \to X$ are C*–bounded* if $d(f(x), g(x)) \leq C$ for all $x \in X$. They are *bounded* if they are C–bounded for some constant C.

2.1 Notation on trees

Let T be a rooted tree and $w \in T$ a vertex. We assume throughout that trees have edge length exactly 1. We denote the subtree rooted at w by T_w. Further the subtree $T_w^k \subset T_w$ is the induced subtree of all vertices $v \in T_w$ with $d(w,v) \leq k$. Vertices $v \in T_w$ are called *descendants* of w and w is called an *ancestor* of v. Further, a vertex $v \in T_w$ is a *D–child* of w if $d(v,w) = D$ and we say that w is the *D–parent* of v. We denote the (1–)parent of a vertex $v \in T$ by $p(v)$ and say that the parent of the root is itself.

We will view a path between vertices u and v as a sequence of neighbouring vertices $u = u_0, u_1, \dots, u_n = v$, denoted by $(u_0, \dots, u_n)$. If a path $(u_0, \dots, u_n)$ is geodesic (or equivalently nonbacktracking) we also denote it by $[u_0, u_n]$.

Definition 2.3 For a subset $U \subseteq T$ of a rooted tree T based at v_0, we define the *lowest common ancestor* of U as the (unique) vertex $v \in T$ furthest away from v_0 such that every vertex $u \in U$ is a descendant of v. We will denote this vertex v as $\mathrm{LCA}(U)$.

Observe that if $v = \mathrm{LCA}(U)$, then there exists a pair of vertices $x, y \in U$ such that v lies on $[x, y]$.

Definition 2.4 Let S be a finite subtree of a rooted tree T. We say that the *boundary* of S, denoted by ∂S, is the set of vertices $v \in T \setminus S$ whose parent $p(v)$ is in S.

Remark 2.5 If T is a d–regular tree rooted at v_0, then one can easily show by induction that $|\partial S| = |S|(d-2) + 1$ if $v_0 \notin S$ and $|\partial S| = |S|(d-2) + 2$ if $v_0 \in S$.

Definition 2.6 Let T be a tree rooted at v_0. A map $f : T \to T$ is *order-preserving* if for every pair of vertices $u, v \in T$ with $v \in T_u$ we have that $f(v) \in T_{f(u)}$.

Nairne [2023] showed that every $(1, C)$–quasi-isometry between spherically homogeneous trees is at bounded distance from an order-preserving quasi-isometry. In Lemma 2.8 we extend this result and show that any C–quasi-isometry of a rooted tree to itself is at bounded distance from an order-preserving quasi-isometry.

2.2 Properties of quasi-isometries of trees

We state and prove three key technical lemmas about quasi-isometries of trees.

The following lemma states that the image of the geodesic $[u, v]$ under a quasi-isometry f coarsely surjects onto the geodesic $[f(u), f(v)]$.

Lemma 2.7 *Let T be a tree and let $f : T \to T$ be a C–quasi-isometry. For every pair of vertices $u, v \in T$ and vertex $a \in [f(u), f(v)]$ there exists a vertex $b \in [u, v]$ such that $d(f(b), a) \leq C$.*

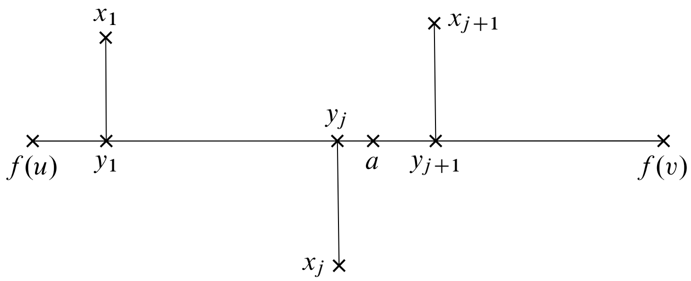

Figure 1: Images of geodesics coarsely surject onto the geodesic.

Proof Let $[u,v]=(u_0,\dots,u_n)$. For $0\le i\le n$, define $x_i=f(u_i)$ and let y_i be the closest point projection of x_i onto $[f(u),f(v)]$. This is depicted in Figure 1. Let j be the largest index such that $y_j\in[f(u),a]$. Then the path $[x_j,y_j][y_j,y_{j+1}][y_{j+1},x_{j+1}]$ is nonbacktracking and hence a geodesic from x_j to x_{j+1} going through a. Since f is a C–quasi-isometry, $d(x_j,a)+d(a,x_{j+1})=d(x_j,x_{j+1})\le 2C$. So $\min\{d(x_j,a)+d(a,x_{j+1})\}\le C$. □

The following lemma states that every quasi-isometry between a rooted tree and itself is at bounded distance from an order-preserving quasi-isometry. This extends the result of [Nairne 2023] where this is shown for $(1,C)$–quasi-isometries between spherically homogeneous trees.

Lemma 2.8 *Let T be a tree rooted at v_0 and let $f:T\to T$ be a C–quasi-isometry. The map f is at bounded distance from an order-preserving quasi-isometry. Moreover, if $f(v_0)=v_0$, then f is at K–bounded distance from an order-preserving $(2K+C)$–quasi-isometry for some K depending only on C.*

Proof It suffices to show the moreover part with $K=3C^3+2C$. Define $g\colon T\to T$ via $g(v):=\mathrm{LCA}(f(T_v))$. Clearly, g is order-preserving. It remains to show that g is at K–bounded distance from f since it then follows that g is a $(2K+C)$–quasi-isometry.

Let $u\in T$ be a vertex. We will show that $d(f(u),g(u))\le K$. We have $f(u)\in T_{g(u)}$. Thus by Lemma 2.7, there exists $w\in[v_0,u]$ such that $d(f(w),g(u))\le C$. This is depicted in Figure 2. Since $g(u)=\mathrm{LCA}(f(T_u))$, there exist vertices $x,y\in T_u$ such that $g(u)\in[f(x),f(y)]$. Again by Lemma 2.7, there exists a vertex $z\in[x,y]\subset T_u$ with $d(g(u),f(z))\le C$. In particular, $d(f(w),f(z))\le 2C$.

Observe that $u\in[w,z]$. Hence, $d(u,z)\le d(w,z)\le 3C^2$. Therefore,

$$d(g(u),f(u))\le d(g(u),f(z))+d(f(z),f(u))\le 3C^3+2C=K. \qquad \square$$

The following lemma states that if f is an order-preserving quasi-isometry and two vertices u,v have the same distance from the root, then $f(u)$ cannot be a descendant of $f(v)$, unless they are close. This lemma is a key ingredient in the proof of Lemma 3.2.

Lemma 2.9 *Let T be a tree rooted at v_0 and let $f:T\to T$ be an order-preserving C–quasi-isometry. Let $u,v\in T$ be vertices such that $d(v_0,u)=d(v_0,v)$ and $f(u)\in T_{f(v)}$. Then $d(f(u),f(v))\le K$ and $d(u,v)\le K$ for some constant K depending only on C.*

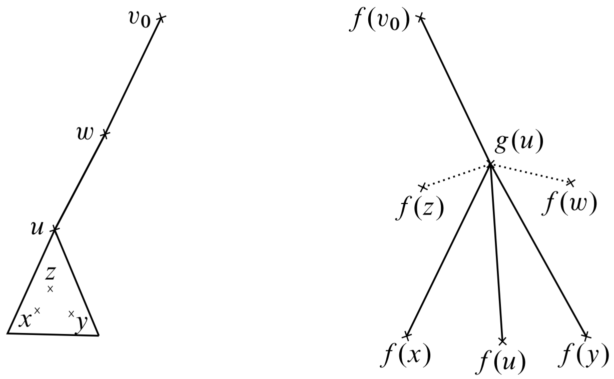

Figure 2: Quasi-isometries are at bounded distance from order-preserving quasi-isometries.

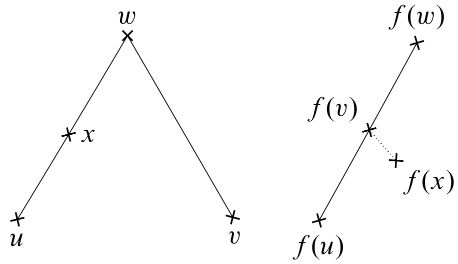

Figure 3: Illustration of the proof of Lemma 2.9.

Proof Let $w = \mathrm{LCA}(\{u, v\})$. Since f is order-preserving, $f(v)$ lies on $[f(u), f(w)]$. This is depicted in Figure 3. By Lemma 2.7, there exists a vertex $x \in [u, w]$ such that $d(f(x), f(v)) \le C$. Thus $d(w, v) \le d(x, v) \le 2C^2$. Since $d(v_0, u) = d(v_0, v)$, we have $d(u, v) = 2d(w, v)$ and hence $d(f(u), f(v)) \le 4C^3 + C$. So choosing $K = 4C^3 + C$ works. □

3 Quasi-isometries of regular trees

Notation For the rest of this section, T denotes a regular tree of degree $d \ge 3$ rooted at a vertex v_0.

In this section, we describe a way of building quasi-isometries, which we call *mixed-subtree quasi-isometries*, of regular trees to themselves. We further show that any quasi-isometry is at bounded distance from a mixed-subtree quasi-isometry. The key idea behind mixed-subtree quasi-isometries is that they are quasi-isometries which are defined iteratively for vertices further and further away from the root. Moreover, at each step, the allowed choices are in some sense independent from the choices for earlier vertices.

Construction Let $D \ge 1$ be a natural number. For all natural numbers $i \ge 0$ we inductively construct functions $f_i : T_{v_0}^{iD} \to T$. Define $f_0(v_0) = v_0$. Assuming we have defined f_i, we define f_{i+1} as follows:

- For all vertices $x \in T_{v_0}^{iD}$ define $f_{i+1}(x) = f_i(x)$.

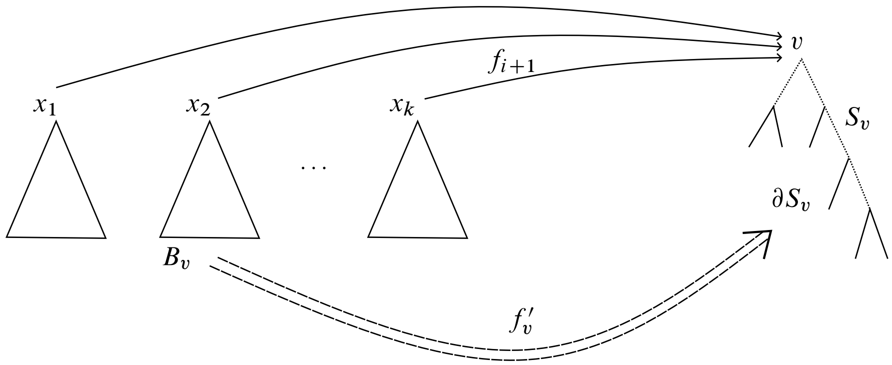

Figure 4: The definition of f'.

- Iterate through all vertices $x \in T$ with $d(v_0, x) = iD$. If we have not yet defined f_{i+1} for any descendants of x, do the following:
  - Denote $f_i(x)$ by v and let $X = \{x_1, \dots, x_k\}$ be the set of vertices that satisfy $f_i(x_j) = v$ and $d(v_0, x_j) = iD$. Define B_v as the set of all D–children of vertices $x_j \in X$. We now define $f_{i+1}(h)$ for all vertices $h \in B_v$.
  - Choose any function $f'_v\colon B_v \to T_v$ satisfying the following properties (see Figure 4):
    (1) $\mathrm{Im}(f'_v) = \partial S_v$ for some finite subtree S_v of T_v containing v.
    (2) If $f'_v(w) = f'_v(w')$, then w and w' are D–children of the same vertex $x_j \in X$.
  - Define $f_{i+1}|_{B_v} = f'_v$.
  - For all $x_j \in X$, define $f_{i+1}(w) = v$ for all vertices $w \in T^{D-1}_{x_j}$.

We first argue that there always exists at least one function f'_v satisfying (1) and (2). In other words, we have to show that there exists a subtree S_v rooted at v such that $|X| \le |\partial S_v| \le |B_v|$. If $D = 1$ and $|X| = 1$, then one can choose $S_v = \{v\}$ to get $|B_v| = |\partial S_v|$. Otherwise $|B_v| - |X| \ge d - 1$; hence by Remark 2.5 we can find a subtree S_v rooted at v with $|X| \le |\partial S_v| \le |B_v|$.

Further note that with this definition, for every $i, j \in \mathbb{N}$, f_i and f_j agree if they are both defined. Hence we can define $f\colon T \to T$ via $f(v) = f_i(v)$ for some i where v is in the domain of f_i. We call any map f constructed this way a *D–deep mixed-subtree* quasi-isometry.

The following lemma shows that mixed-subtree quasi-isometries are indeed quasi-isometries.

Lemma 3.1 *For any choice of functions f'_v, the map f constructed is an order-preserving C–quasi-isometry, where C only depends on D and T.*

Proof It follows directly from the definition that f is order-preserving. Let $K = d^D$, where d is the degree of T and let $C = 2K^2$. We will show that f is a C–quasi-isometry.

Claim 1 *If vertices b, b' are $D(i+1)$–children of v_0, then $f(b) \neq f(b')$ unless the D–parents of b and b' are the same. Furthermore, if $f(b) \neq f(b')$, then $T_{f(b)}$ and $T_{f(b')}$ are disjoint.*

Proof of Claim 1 We prove this by induction on i. For $i = 0$, b and b' have the same D–parent, namely v_0. The furthermore part follows from (1). Assume the statement is true for i; we want to show that it holds for $i+1$. Let x and x' be the D–parents of b and b' respectively. If $f(x) \neq f(x')$, then $T_{f(x)}$ is disjoint from $T_{f(x')}$ by the induction hypothesis. Hence $f(b) \neq f(b')$ and $T_{f(b)}$ is disjoint from $T_{f(b')}$. If $f(x) = f(x')$ and $x \neq x'$, then (2) implies that $f(b) \neq f(b')$. Moreover, (1) implies that $T_{f(b)}$ and $T_{f(b')}$ are disjoint. Lastly, if $x = x'$, we only have to show the furthermore part, which follows from (1). □

Claim 2 *For any i, the number of Di–children of v_0 whose images under f coincide is at most K.*

Proof of Claim 2 This follows from Claim 1 together with the fact that every vertex has at most K D–children. □

Claim 3 *If b is the D–child of a vertex x which in turn is a Di–child of v_0, then $1 \leq d(f(b), f(x)) \leq K^2$.*

Proof of Claim 3 Let $v = f(x)$. We use the notation from the construction of f_{i+1}. By Claim 2, the set B_v contains at most K^2 vertices, so $|\mathrm{Im}(f'_v)| \leq K^2$. In other words the subtree S_v from (1) has at most K^2 leaves, implying that $d(v, v') \leq K^2$ for any vertex $v' \in \partial S_v$ (see Remark 2.5). Consequently $d(f'_v(b), f(x)) \leq K^2$, which concludes the proof. □

Claim 4 *The map f is K^2–coarsely surjective.*

Proof of Claim 4 First observe that whenever a vertex v is in the image of f, there exists a Di–child x of v_0 with $f(x) = v$.

Let $v' \in T$ be a vertex. We show that $d(v', \mathrm{Im}(f)) \leq K^2$. Let v be the lowest ancestor of v' which is in the image of f. We have that $v = f(x)$ for some vertex x which is a Di child of v_0. If $v' = v$, we are done. If $v' \in S_v$, then $d(v, v') \leq K^2$ as in the proof of Claim 3. If $v' \notin S_v$, there exists $w \in \partial S_v$ which is a descendant of v and an ancestor of v'. Since $w \in \partial S_v$, it is in the image of f, a contradiction with the definition of v. □

It remains to show that
$$\frac{d(u,v)}{C} - C \leq d(f(u), f(v)) \leq C d(u,v) + C$$
for all vertices $u, v \in T$. To show the right half of the inequality, it is enough to show that for all neighbours $u, v \in T$, we have $d(f(u), f(v)) \leq C$. This follows directly from the definition of f and Claim 3. Next we show the left half of the inequality. Let $u, v \in T$ be vertices and let $n = \lfloor d(v_0, u)/D \rfloor$, $m = \lfloor d(v_0, v)/D \rfloor$. Define $u_0 = v_0$ and for $i \leq n$ define u_i as the Di–child of v_0 which is an ancestor of u. Define v_i analogously. Let k be the maximal index such that $u_k = v_k$. Claim 1, together with f being order-preserving, yields that $f(u_i)$ and $f(v_j)$ lie on the geodesic from $f(u)$ to $f(v)$ for all $k+2 \leq i \leq n$ and $k+2 \leq j \leq m$. Hence, $d(f(u), f(v)) \geq (n-k-2) + (m-k-2)$. On the other hand $d(u,v) \leq D(n-k+1) + D(m-k+1)$. The statement follows. □

We are now ready to prove the following lemma which together with Lemma 3.1 states that a map $g: T \to T$ is a quasi-isometry if and only if it is at bounded distance from a mixed-subtree quasi-isometry. The lemma is a slightly more detailed version of Theorem 1.1.

Lemma 3.2 *Let $g: T \to T$ be a C–quasi-isometry. There exist a constant $D > 0$ and a D–deep mixed subtree quasi-isometry f such that g and f are at bounded distance. Moreover, if $g(v_0) = v_0$, then D only depends on T and C.*

Proof By Lemma 2.8, which states that all quasi-isometries are at bounded distance from order-preserving quasi-isometries, it suffices to show the moreover part for an order-preserving quasi-isometry. So we assume in the following that g is order-preserving.

Let K be the constant of Lemma 2.9 and let $D = \lceil C(C+K)+1 \rceil$. We will show that there is a D–deep mixed subtree quasi-isometry f at distance $K + CD + C$ from g.

Assume that we have defined $f_i: T_{v_0}^{iD} \to T$, as in the construction, such that

(i) $d(f_i(u), g(u)) \le K$ for all u with $d(v_0, u) = Di$,

(ii) $g(u) \in T_{f_i(u)}$ for all u with $d(v_0, u) = Di$,

(iii) $d(f_i(w), g(w)) \le K + CD + C$ for all $w \in T_{v_0}^{iD}$.

We show that we can define a function f_{i+1} such that

(a) $d(f_{i+1}(u), g(u)) \le K$ for all u with $d(v_0, u) = D(i+1)$,

(b) $g(u) \in T_{f_{i+1}(u)}$ for all u with $d(v_0, u) = D(i+1)$,

(c) $d(f_{i+1}(w), g(w)) \le K + CD + C$ for all $w \in T_{v_0}^{(i+1)D}$.

Let x be a Di child of v_0, let $v = f_i(x)$ and let $X = f_i^{-1}(v)$. Observe that, for all $x' \in X$, $d(v_0, x') = Di$. Let B_v be the set of all D–children of elements of X and let $A_v = g(B_v)$. By (ii), $A_v \subset T_v$. For $b \in B_v$, define $f_v'(b)$ as the vertex $a \in A_v$ closest to v_0 which satisfies $g(b) \in T_a$. Observe that $g(b) \in T_{f_v'(b)}$; in other words, (b) is satisfied.

Note that $f_v'(b) = g(b')$ for some $b' \in B_v$. It follows from Lemma 2.9 that $d(f_v'(b), g(b)) \le K$ for all $b \in B_v$, which proves (a). Therefore, $g|_{B_v}$ and f_v' are at K–bounded distance. By (i), $d(f_i(x'), g(x')) \le K$ for all $x' \in X$. Hence for a k–child w of some $x' \in X$ for $k < D$ we have $f_{i+1}(w) = v$, and hence

$$d(f_{i+1}(w), g(w)) \le d(v, g(x')) + d(g(x'), g(w)) \le K + CD + C,$$

which, together with (iii), proves (c).

It only remains to show that f_v' as defined above is a valid choice; that is, f_v' satisfies (1) and (2). For (1), define $S_v = \{y \in T_v \mid y \notin T_a \text{ for all } a \in A_v\}$. If $w \in \partial S_v$, then $w \in T_{a_w}$ for some $a_w \in A_v$ while its parent is not in T_{a_w}. It follows that $w = a_w$. Further, for any $b \in g^{-1}(a_w)$ we have that $f'_v(b) \in [a_w, v_0]$ but $f'_v(b) \notin S_v$. Hence $f'_v(b) = a_w$ implying that $a_w \in \mathrm{Im}(f'_v)$.

Thus $\partial S_v = \mathrm{Im}(f'_v)$ is finite. If S_v is infinite, there exists a vertex $u \in S_v$ which is further away from v_0 than all points in the finite set ∂S_v. Consequently, $T_u \subseteq S_v$. Since g is a quasi-isometry (and hence coarsely surjective), there exists a vertex $u' \in T$ with $d(v_0, u') \geq (i+1)D$ and $g(u') \in T_u$. We have that u' is the descendant of some Di–child x' of v_0. By Claim 1 from the proof of Lemma 3.1 either $x' \in X$ or $T_{f_i(x')}$ is disjoint from T_v. By (ii) the latter cannot be the case. Consequently, $u' \in T_b$ for some $b \in B_v$ and since g is order-preserving, $g(u') \in T_{g(b)}$. This is a contradiction to $g(u') \in S_v$. Thus S_v is indeed finite.

In order to prove (1), it remains to show that $v \in S_v$, or in other words, that $v \notin A_v$. Let $b \in B_v$ be a D–child of some vertex $x' \in X$. By (i) and the fact that g is a C–quasi-isometry,

$$d(g(b), v) \geq d(g(b), g(x')) - d(g(x'), f_i(x')) \geq d(g(b), g(x')) - K > 0,$$

so indeed $g(b) \neq v$. Since this is true for all $b \in B_v$, it follows that $v \notin A_v$.

Next we prove (2). Let b be a D–child of x and b' be a D–child of x' with $x \neq x' \in X$. We have $d(b, b') \geq 2D$. Thus $d(g(b), g(b')) \geq 2D/C - C > 2K + C$, which implies that $d(f'_v(b), f'_v(b')) > C$. In particular, $f'_v(b) \neq f'_v(b')$. □

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