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Hyperbolic groups with logarithmic separation profile

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We prove that hyperbolic groups with logarithmic separation profiles split over cyclic groups. This shows that such groups can be inductively built from Fuchsian groups and free groups by amalgamations and HNN extensions over finite or virtually cyclic groups. However, we show that not all groups admitting such a hierarchy have logarithmic separation profile by providing an example of a surface amalgam over a cyclic group with superlogarithmic separation profile.

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1 Introduction

The separation profile was first introduced by Benjamini, Schramm and Timár [1] in 2012. It measures large scale connectivity of infinite graphs, in the spirit of the celebrated theorem of Lipton and Trajan for planar graphs [12].

Definition 1.1 (Benjamini, Schramm and Timár [1]) Given a finite graph $\Gamma = (V\Gamma, E\Gamma)$, we shall say that a set of vertices $C \subset V\Gamma$ *cuts* (or *separates*) the graph Γ if the connected components of the subgraph induced by $V\Gamma - C$ contain at most $\frac{1}{2}|V\Gamma|$ vertices.

We define the *cut* of the graph Γ, denoted $\operatorname{cut}\Gamma$, as the minimal size of a separating set.

We define the *separation profile* of a bounded degree infinite graph G as the following nondecreasing function from $\mathbb{N}^*$ to $\mathbb{N}^*$:

$$\operatorname{sep}_G(n) = \sup_{\substack{\Gamma \subset G \\ |V\Gamma| \le n}} \operatorname{cut}\Gamma.$$

We shall consider such function endowed with the partial order defined by $g \preceq h$ if and only if there exists $D > 0$ such that $g(n) \le Dh(Dn) + D$ for any $n \ge D$. We denote by $\asymp$ and $\prec$ the associated equivalence relation and strict partial order, respectively.

As noticed in [1], the factor $\frac{1}{2}$ does not play an important role in the previous definition. Replacing it by any $\beta \in (0, 1)$ would give an equivalent profile.

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The separation profile is a coarse-geometric monotone invariant (see Proposition 3.1). To our knowledge, the only such invariants that were previously defined are volume growth and asymptotic dimension; see Gromov [8]. The separation profile is a much finer invariant and has been generalized by Hume, Mackay and Tessera [10] into a spectrum of profiles called Poincaré profiles. For a survey on this topic, we refer to the first part of the thesis of the second author [11].

It is proved in [1] (see also Hume and Mackay [9]) that if a hyperbolic group has $\mathrm{sep}_G(n) \prec \log(n)$ then $\mathrm{sep}_G(n)$ is bounded and G is virtually free.

In this paper we investigate the smallest possible nonvirtually free case, namely $\mathrm{sep}_G(n) \preceq \log(n)$.

Theorem A *Let G be a hyperbolic group with $\mathrm{sep}_G(n) \preceq \log(n)$. Then G is Fuchsian or splits over finite or virtually cyclic subgroups.*

This theorem is proved in Section 2, but let us give here a sketch of proof. The first step consists in showing that the spheres of G have bounded separating sets. This is done by projecting the separating set of some suitable annulus. Then, we make these sphere separating sets converge in ∂G. This implies the existence of local cut points in ∂G, and the conclusion follows from Bowditch [3].

Corollary 1.2 *Let G be a hyperbolic group without 2-torsion. If $\mathrm{sep}_G(n) \preceq \log(n)$ then G can be inductively built from Fuchsian groups and free groups by amalgamations and HNN extensions over finite or virtually cyclic groups.*

Proof We can apply Theorem A to G. Either G is Fuchsian and we are done, or G splits over virtually cyclic groups. The edge groups are virtually cyclic, hence quasiconvex in G. This implies that the vertex groups of this splitting are quasiconvex and hence hyperbolic. By the monotonicity of the separation profile, the separation profile of the vertex groups H is $\mathrm{sep}_H(n) \preceq \mathrm{sep}_G(n) \asymp \log(n)$. Therefore, we can successively apply Theorem A to split G over virtually cyclic subgroups. Using the strong accessibility by Louder and Touikan [13] this process terminates. □

A group with conformal dimension at least one always has a separation profile bounded below by log, from [9]. Using a recent result of Carrasco and Mackay [5] giving a characterization of hyperbolic groups with conformal dimension one, we get the following corollary.

Corollary 1.3 *Let G be a one-ended hyperbolic group with no 2-torsion. If the (Ahlfors regular) conformal dimension of G is strictly greater than 1, then its separation profile is strictly greater than* log.

In this generality, to our knowledge this improves the previously known lower bounds. We do not know if this is sharp.

Lower bounds on separation profiles can be obtained from Poincaré inequalities in the boundary at infinity of hyperbolic groups, see Hume, Mackay and Tessera [10, Theorem 13]. Finding general Poincaré inequalities is an important challenge and this corollary can be seen as a step in this direction.

The following theorem shows that the converse of Theorem A (and the subsequent corollaries) is false.

Theorem B *Let S be the surface amalgam obtained by gluing two closed orientable hyperbolic surfaces along a closed filling curve in each. Then,* $\mathrm{sep}_{\pi_1 S}(n) \succ \log(n)$.

From Carrasco and Mackay [5], such a group has conformal dimension equal to 1.

From [10], a hyperbolic group with conformal dimension one always have a separation profile bounded above by any n^ϵ, with $\epsilon > 0$. To our knowledge, this is this is the first example of such a group whose separation profile is not logarithmic. This implies in particular that the conformal dimension is not attained [10, Theorem 11].

We believe that when the curves are not filling, the separation profile is actually log.

Question 1.4 Let S be a simple surface amalgam obtained by gluing two closed hyperbolic orientable surfaces along simple curves. Do we have $\mathrm{sep}_{\pi_1 S} \asymp \log$?

From Hume, Mackay and Tessera [10] study of relations between conformal dimension and separation profiles, we as well can formulate the following question:

Question 1.5 If a hyperbolic group has a separation profile bounded above by n^ϵ for every positive ϵ, does it imply that it has conformal dimension one?

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2 Proof of Theorem A

Let G be a one-ended hyperbolic group. By abuse of notation let us denote by G also the Cayley graph of G with respect to some fixed finite generating set, and assume it is δ-hyperbolic. We denote by o the identity element of G. For every $R > 0$, B_R denotes the ball, and S_R the sphere, of radius R centred at o. We denote by $A_{[R_1,R_2]}$ the annulus $B_{R_2} - B_{R_1 - 1}$.

Definition 2.1 For each $R > 0$, let $\pi_R : G - B_R \to S_R$ be a projection defined by $\pi_R(y) = [o, y] \cap S_R$, where $[o, y]$ is some choice of geodesic joining o and y.

For any $\alpha > 0$, we call an *α-step path* any family of vertices $v_1, \dots, v_k$ such that $d(v_i, v_{i+1}) \le \alpha$ for any $i = 1, \dots, k-1$.

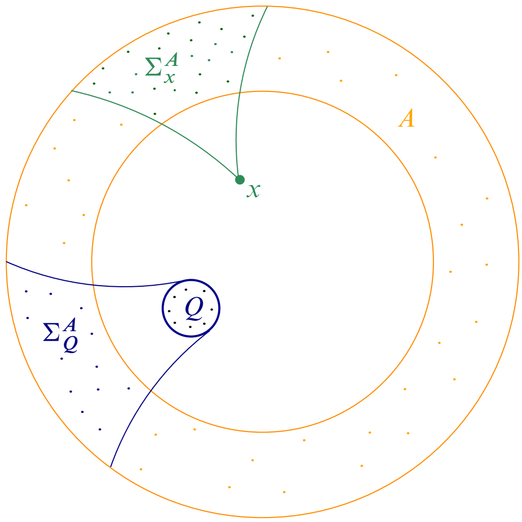

Figure 1: Shadows and sectors.

Fact 2.2 • For all $R > 0$, if γ, γ' are two geodesics from o to points $x, y \in G - B_R$ then $d(\gamma(R), \gamma'(R)) \leq d(x, y) + 2\delta$. In particular, $d(\pi_R(x), \pi_R(y)) \leq d(x, y) + 2\delta$ for all $x, y \in G - B_R$.

- If γ, γ' are two geodesic rays from o that represent the same point at infinity then $d(\gamma(R), \gamma'(R)) \leq 2\delta$.
- Since G is one-ended, there is a constant δ_1 such that the δ_1-neighbourhood of any sphere in G is connected.

Proof The first two assertions are a straightforward consequence of the δ-slimness of geodesic triangles in G. Let us prove the third assertion: Let $z_1, z_2 \in S_R$. From [2, Lemma 3.1], there is some c such that there exist infinite geodesic rays ζ_1, ζ_2 from o in G such that $d(z_i, \zeta_i(R)) \leq c$ for $i = 1, 2$. Since G is one-ended, ∂G is path connected [4; 14], and so there is a continuous path from ζ_1 to ζ_2. By extending π_R to ∂G, we can "project" any continuous path in ∂G to a $(2\delta+1)$-step path in S_R. In particular the vertices $\zeta_1(R)$ and $\zeta_2(R)$ in S_R can be joined by a $(2\delta+1)$-step path in S_R, and hence the vertices z_1, z_2 in S_R can be joined by a $(c+2\delta+1)$-step path. If we set $\delta_1 = c + 2\delta + 1$, then it follows that the δ_1-neighbourhood of any sphere of G is connected. □

From now on, we will assume that δ_1 stands for the constant of Fact 2.2.

Definition 2.3 The *shadow* Σ_x of a point $x \in G$ is that set of all points $y \in G$ such that a geodesic from o to y passes through x. For every subset $Q \subseteq G$ denote its *shadow* by $\Sigma_Q = \bigcup_{x \in Q} \Sigma_x$. For a point x and $\tau \geq 0$ denote by $\Sigma_{x,\tau} = \Sigma_{B(x,\tau)}$ its τ-*shadow* where $B(x, \tau)$ is the ball of radius τ around x in G. Similarly, for $Q \subseteq G$ and $\tau \geq 0$, denote by $\Sigma_{Q,\tau} = \Sigma_{(Q)_\tau}$ its τ-*shadow*. Finally, for a subset $A \subset X$ we will denote by Σ_x^A the intersection $\Sigma_x \cap A$, and similarly $\Sigma_Q^A, \Sigma_{x,\tau}^A, \Sigma_{Q,\tau}^A$ (see Figure 1).

We will need the following strengthening of Fact 2.2.

Fact 2.4 For every τ there exists τ' such that for every $x \in G$ and $r \geq d(o,x)$, the set $\Sigma_{x,\tau} \cap S_r$ is contained in a single connected component of $\Sigma_{x,\tau'} \cap (S_r)_{\delta_1}$.

Proof For any fixed $o' \in G$, geodesic ray $\xi : [0,\infty) \to G$ starting from o', $R \gg \delta$ and $k > 2\delta$, define $V(o',\xi,R,k)$ to be the set in ∂G of all (equivalence classes of) geodesic rays ζ from o' such that $d(\zeta(R),\xi(R)) \leq k$. The sets $\{V(o',\xi,R,k)\}_{R\gg 0}$ form a neighbourhood basis for the ideal point corresponding to ξ in ∂G. By [4; 14], the Gromov boundary ∂G is locally path connected. Therefore, for every geodesic ray ξ from o' and R there exists $L = L(o',\xi,R,k)$ such that $V(o',\xi,R+L,k+4\delta)$ is in a path component of $V(o',\xi,R,k)$. Since ∂G is compact, there exists $L' = L'(o',R,k)$ such that $V(o',\xi,R+L',k+4\delta)$ is in a component of $V(o',\xi,R,k)$ for every geodesic ray ξ starting from o'. Note that by G-equivariance, L' does not depend on o'. Our next goal is to show that L' does not depend on R as well:

For a fixed $k_0 \geq 4\delta+1$ and $R_0 \gg \tau,\delta$, let $\lambda = L'(R_0,k_0-2\delta)$, then by δ-slimness,

$$V(o,\xi,R+\lambda,k_0) \subseteq V\big(\xi(R-R_0),\xi|_{[R-R_0,\infty)},R_0+\lambda,k_0+2\delta\big) \quad \text{for all } R \gg R_0.$$

By the above,

$$V\big(\xi(R-R_0),\xi|_{[R-R_0,\infty)},R_0+\lambda,k_0+2\delta\big)$$

is in a path component of $V(\xi(R-R_0),\xi,R_0,k_0-2\delta)$ which by δ-slimness is contained in $V(o,\xi,R,k_0)$. Therefore, we have shown that there exists λ, such that $V(o,\xi,R+\lambda,k_0)$ is in a path component of $V(o,\xi,R,k_0)$ for all geodesic rays $\xi \in \partial G$ and $R \gg \delta$.

Observing the trivial inclusion $V(o,\xi,R,k_0) \subseteq V(o,\xi,R+\lambda,k_0+2\lambda)$, we get that $V(o,\xi,R+\lambda,k_0)$ is in a path component of $V(o,\xi,R+\lambda,k_0+2\lambda)$ for all sufficiently large R and geodesic ray ξ. Or equivalently, $V(o,\xi,R,k_0)$ is in a path component of $V(o,\xi,R,k_0+2\lambda)$ for all sufficiently large R and geodesic ray ξ.

We proceed as in the proof of the previous fact. Set $R = d(o,x)$, and let $r \geq R$. By [2, Lemma 3.1], there exists c such that for every two points $z_1,z_2 \in \Sigma_{x,\tau} \cap S_r$ there exist two geodesics rays ζ_1,ζ_2 from o such that $d(z_i,\zeta_i(r)) \leq c$ for $i=1,2$. By the inequality in Fact 2.2, $d(\pi_R(z_i),\zeta_i(R)) \leq c+2\delta$. Since $z_i \in \Sigma_{x,\tau}$ we get that $d(\zeta_i(R),x) \leq c+2\delta+\tau$ and hence $d(\zeta_1(R),\zeta_2(R)) \leq 2(c+2\delta+\tau)$. If we set $k_0 = 2(c+2\delta+\tau)$ we see that ζ_1,ζ_2 are in $V(o,\zeta_1,R,k_0)$. Thus, by the above, they can be connected by a continuous path in $V(o,\zeta_1,R,k_0+2\lambda)$. Projecting this path using π_r gives rise to a $(2\delta+1)$-step path in S_r between $\zeta_1(r)$ and $\zeta_2(r)$. and hence a δ_1-step path between z_1,z_2, where $\delta_1 = c+2\delta+1$ (as in Fact 2.2). Replacing each δ_1 step by a path of length δ_1 in $(S_r)_{\delta_1}$ we get a path p between z_1,z_2. Using the inequality in Fact 2.2, we see that if we further project the path p to S_R, we see that $\pi_R \circ p$ stays in the ball of radius $\tau' = k_0+2\lambda+\delta_1+2\delta$ around x. Thus, the path p is contained in $\Sigma_{x,\tau'} \cap (S_r)_{\delta_1}$. □

Fact 2.5 There exist constants $\alpha > 1$ and $\tau_0 \geq 0$ such that for all $\tau \geq \tau_0$ there exists K such that:

- For all $0 \leq R$, we have

$$K^{-1}\alpha^R \leq |S_R| \leq K\alpha^R.$$

- For all $0 \le R_1 \le R_2$, we have
$$K^{-1}(\alpha^{R_2} - \alpha^{R_1}) \le |A_{[R_1,R_2]}| \le K(\alpha^{R_2} - \alpha^{R_1}).$$
- For all $R, R' \ge 0$ and $D \subseteq S_R$, we have
$$K^{-1}\alpha^{R'}|D| \le |\Sigma_{D,\tau} \cap S_{R+R'}| \le K\alpha^{R'}|D|.$$
- For all $R \ge 0$, $R_2 \ge R_1 \ge 0$ and $D \subseteq S_R$, we have
$$K^{-1}(\alpha^{R_2} - \alpha^{R_1})|D| \le |\Sigma_{D,\tau} \cap A_{[R+R_1,R+R_2]}| \le K(\alpha^{R_2} - \alpha^{R_1})|D|.$$

Proof The first two items are immediate consequences of estimates of cardinality of balls given by Coornaert [6, théorème 7.2].

The third item is an immediate consequence of properties of the Patterson–Sullivan measure on ∂G; see Coornaert [6, Proposition 6.1]. The properties that are needed are detailed in Gouëzel, Mathéus and Maucourant [7, inequality 2.9 from the proof of Lemma 2.13, and the fact that the covering number of shadows is finite (item (1) on page 1216)].

The fourth item is obtained by summing up the inequalities given by the previous item for $R' \in [R_1, R_2]$. □

Definition 2.6 We say that G has *bounded sphere separation* if for every $\epsilon > 0$ there exists a number M such that for all R there exists a set $P_R \subseteq S_R$ such that $|P_R| \le M$, and each component of $(S_R)_{\delta_1} - (P_R)_{\delta_1}$ has size at most $\epsilon|(S_R)_{\delta_1}|$, where δ_1 is the smallest integer satisfying that the δ_1-neighbourhood of any sphere in G is connected.

Remark 2.7 According to Fact 2.2, the constant δ_1 in the definition above exists.

We are now able to state our key lemma.

Lemma 2.8 *If G is hyperbolic and $\mathrm{sep}_G(n) \preceq \log(n)$, then G has bounded sphere separation.*

Proof Let $\epsilon > 0$. Let τ_0, α be as in Fact 2.5, set $\tau = \tau_0 + 2\delta_1$ and let K be the constant of Fact 2.5. Let τ_0' and τ' be the constants of Fact 2.4 corresponding to τ_0 and τ, respectively. Without loss of generality by enlarging either τ_0' or τ', we may assume that $\tau' = \tau_0' + 2\delta$.

Let $A = A(R)$ be the annulus $A_{[2R,3R]}$. For $\beta \in (0,1)$, to be determined later, let $C \subset A(R)$ be such that each connected component of $A(R) - C$ contains at most $\beta|A(R)|$ vertices. By Fact 2.5, $|A(R)| \asymp \alpha^{3R}$, and from the assumption that $\mathrm{sep}_G \preceq \log$, we can suppose $|C| \preceq_\beta \log(|A(R)|) \preceq_\beta R$. Concretely, let

$$|C| \le cR \tag{2-1}$$

for some c (which depends on the choice of β).

Let $P_R \subset S_R$ be the set of all x such that $\Sigma_{x,\tau'} \cap (S_r)_{\delta_1} \cap C \neq \varnothing$ for all $r \in [2R+\delta_1, 3R-\delta_1]$. If $x \in P_R$, there are at least $(R-2\delta_1)/2\delta_1$ values of $r \in [2R+\delta_1, 3R-\delta_1]$ for which $(S_r)_{\delta_1}$ are disjoint, and $\Sigma^A_{x,\tau'} \cap C$ is assumed to meet all of them. Therefore we have $|\Sigma^A_{x,\tau'} \cap C| \geq (R-2\delta_1)/2\delta_1$. As long as $R \gg \delta_1$ this implies

$$|\Sigma^A_{x,\tau'} \cap C| \geq R/3\delta_1. \tag{2-2}$$

Since the τ'-shadows corresponding to vertices in S_R that are $2\tau' + 4\delta$ apart are disjoint, it follows from (2-1) and (2-2) that

$$|P_R| \leq 3c\delta_1 |B_{2\tau'+4\delta}| =: M. \tag{2-3}$$

So, there is a uniform bound M (that depends on β) on the size of P_R. It remains to show that upon choosing β small enough we can ensure that $(P_R)_{\delta_1}$ separates $(S_R)_{\delta_1}$ into components of size at most $\epsilon|(S_R)_{\delta_1}|$.

Claim 2.9 *There exists $K' > 0$ such that for every $R \gg_\beta 0$, if $x \in S_R - P_R$ then $\Sigma^A_{x,\tau} - C$ has a subset T_x of size $|T_x| \geq \frac{1}{2K'}|\Sigma^A_{x,\tau}|$ which is contained in a connected component of $A(R) - C$.*

Proof Since $x \notin P_R$, there exists $r \in [2R+\delta_1, 3R-\delta_1]$ such that $\Sigma_{x,\tau'} \cap (S_r)_{\delta_1} \cap C = \varnothing$. By Fact 2.4, $\Sigma_{x,\tau} \cap S_r$ is contained in a path component of $\Sigma_{x,\tau'} \cap (S_r)_{\delta_1}$, hence also in a path component E of $A(R) - C$. Let T_x be the intersection $E \cap \Sigma^A_{x,\tau}$.

It remains to show the lower bound on $|T_x|$. To do so, we will use τ_0-shadows of points $y \in \Sigma_{x,\tau_0} \cap S_{2R}$. Note that Σ_{y,τ_0} is contained in $\Sigma_{x,\tau}$ (since $\tau = \tau_0 + 2\delta$).

Let Q_x be the collection of all points $y \in \Sigma_{x,\tau_0} \cap S_{2R}$ such that $\Sigma^A_{y,\tau_0'} \cap C \neq \varnothing$. For each point $z \in C$ there are at most $|B_{\tau_0'+2\delta}|$ many $y \in \Sigma_{x,\tau_0} \cap S_{2R}$ such that $z \in \Sigma^A_{y,\tau_0'}$. Together with the assumption that $|C| \leq cR$ it follows that $|Q_x| \leq cR|B_{\tau_0'+2\delta}| \asymp_\beta R$. Since $|\Sigma_{x,\tau_0} \cap S_{2R}| \asymp \alpha^R$, the complementary set $Q'_x = (\Sigma_{x,\tau_0} \cap S_{2R}) - Q_x$ satisfies

$$|Q'_x| \geq \tfrac{1}{2}|\Sigma_{x,\tau} \cap S_{2R}|, \tag{2-4}$$

for any large enough $R \gg_\beta 0$.

For points $y \in Q'_x$, by Fact 2.4, Σ^A_{y,τ_0} is contained in a connected component of $\Sigma^A_{y,\tau_0'}$, and so in a component of $\Sigma^A_{x,\tau'} - C$ (since $\tau' = \tau_0' + 2\delta$) and intersects $\Sigma_{x,\tau} \cap S_r$. Thus, by the definition of T_x we have $\Sigma^A_{y,\tau_0} \subseteq T_x$ for all $y \in Q'_x$. Or equivalently, $\Sigma^A_{Q'_x,\tau_0} = \bigcup_{y \in Q'_x} \Sigma^A_{y,\tau_0} \subset T_x$. Thus, $|T_x| \geq |\Sigma^A_{Q'_x,\tau_0}|$.

Similar to Fact 2.5, we have a constant K' such that any subset $Q \subseteq \Sigma_{x,\tau_0} \cap S_{2R}$ satisfies

$$\frac{|Q|}{|\Sigma_{x,\tau} \cap S_{2R}|} \leq K' \frac{|\Sigma^A_{Q,\tau_0}|}{|\Sigma^A_{x,\tau}|}. \tag{2-5}$$

By (2-5) and (2-4) we get $|T_x| \geq \frac{1}{2K'}|\Sigma^A_{x,\tau}|$. □

Let $D' \subset (S_R)_{\delta_1} - (P_R)_{\delta_1}$ be a connected subset. We need to show $|D'| \leq \epsilon |(S_R)_{\delta_1}|$. Let D be the set of elements of S_R that are at distance at most δ_1 from D'. Define $T_D = \bigcup_{x \in D} T_x$, with T_x given by Claim 2.9.

Claim 2.10 *T_D is in a connected component of $A(R) - C$.*

Proof Since D_{δ_1} is connected, it suffices to show that for any $x, x' \in D$ at distance at most $2\delta_1$ from each other then T_x and $T_{x'}$ intersect.

Let then $x, x' \in D$ be such that $d(x, x') \leq 2\delta_1$. Then, $\Sigma_{x,\tau_0} \cap S_{2R} \subseteq \Sigma_{x,\tau} \cap \Sigma_{x',\tau} \cap S_{2R}$ and by Fact 2.5 contains $\succeq \alpha^R$ points. As in the proof of Claim 2.9 we see that if R is large enough, there exists a point $y \in \Sigma_{x,\tau} \cap \Sigma_{x',\tau} \cap S_{2R}$ which is in the complement of both Q_x and $Q_{x'}$. As before, this implies that Σ^A_{y,τ_0} is in both T_x and $T_{x'}$. □

By assumption on C, this implies that we have

(2-6) $$|T_D| \leq \beta |A(R)|.$$

Claim 2.11 *We have*

(2-7) $$\sum_{x \in D} |T_x| \leq |B_{2\tau+4\delta}| |T_D|.$$

Proof Let us show that there exists a map $\phi \colon D \to D$ such that

- $d(x, \phi(x)) \leq 2\tau$ and in particular $|\phi^{-1}(\phi(x))| \leq |B_{2\tau+4\delta}|$,
- $|T_x| \leq |T_{\phi(x)}|$, and
- if $y \neq y' \in \operatorname{Im} \phi$ then $T_y \cap T_{y'} = \varnothing$.

Assuming we have constructed such a map, the claim follows by the following inequality:

(2-8) $$\sum_{x \in D} |T_x| \leq |B_{2\tau+4\delta}| \sum_{y \in \operatorname{Im} \phi} |T_y| \leq |B_{2\tau+4\delta}| |T_D|.$$

To construct the map ϕ, let $x \in D$ be a point maximizing $|T_x|$. Let $Z \subseteq D$ be the collection of all points $x' \in D$ satisfying $T_{x'} \cap T_x \neq \varnothing$. Define ϕ on Z by $\phi(x') = x$. Note that if $d(x, x') > 2\tau + 4\delta$ then $\Sigma_{x,\tau} \cap \Sigma_{x',\tau} = \varnothing$ and hence $T_x \cap T'_x = \varnothing$. It follows that if $x' \in Z$ then $d(x, x') \leq 2\tau + 4\delta$, and by the choice of x, $|T_{x'}| \leq |T_x|$.

Remove all the points in Z from D, and iterate the construction above until ϕ is defined on all D. □

We deduce that for large enough $R \gg_\beta 0$ we have

$$\begin{aligned}
|D'| &\le |B_{\delta_1}||D| \\
&\le |B_{\delta_1}|K\alpha^{-2R}|\Sigma^A_{D,\tau}| && \text{(from Fact 2.5)} \\
&\le |B_{\delta_1}|K\alpha^{-2R}\sum_{x\in D}|\Sigma^A_{x,\tau}| \\
&\le 2K'|B_{\delta_1}|K\alpha^{-2R}\sum_{x\in D}|T_x| && \text{(from Claim 2.9)} \\
&\le 2|B_{\delta_1}||B_{2\tau+4\delta}|KK'\alpha^{-2R}|T_D| && \text{(from (2-7))} \\
&\le 2|B_{\delta_1}||B_{2\tau+4\delta}|KK'\alpha^{-2R}\beta|A(R)| && \text{(from (2-6))} \\
&\le 2|B_{\delta_1}||B_{2\tau+4\delta}|K^2K'\beta|S_R| && \text{(from Fact 2.5)} \\
&\le 2|B_{\delta_1}||B_{2\tau+4\delta}|K^2K'\beta|(S_R)_{\delta_1}|.
\end{aligned}$$

Let $w = 2|B_{\delta_1}||B_{\delta_1}||B_{2\tau+4\delta}|K^2K'$, this is a constant that depends only of G. Thus, we get

$$|D'| \le w\beta|(S_R)_{\delta_1}|. \tag{2-9}$$

For every $\epsilon > 0$, set $\beta = \frac{\epsilon}{w}$. By (2-3) there exists M such that for every R, the set $P_R \subseteq S_R$ that we constructed satisfies $|P_R| \le M$, and by (2-9) each component D' of $(S_R)_{\delta_1} - (P_R)_{\delta_1}$ has size $|D'| \le \epsilon|(S_R)_{\delta_1}|$ for large enough $R \gg_\beta 0$. We have proved the bounded sphere separation property for large enough R. This completes the proof of Lemma 2.8. □

Definition 2.12 Let X be a connected topological space. We say that a subset F *topologically separates* X if $X - F$ is not connected.

Lemma 2.13 *If G has bounded sphere separation, then ∂G has a finite topologically separating set.*

Proof Following Definition 2.6, let δ_1 is the smallest integer satisfying that the δ_1-neighbourhood of any sphere in G is connected. We start with the following claim.

Claim 2.14 *There exist $K > 0$ and $\tau_1 \ge \delta_1$ such that for any $R < R'$, if $(P_{R'})_{\delta_1}$ separates $(S_{R'})_{\delta_1}$ into connected components of size at most $\frac{1}{K}|(S_{R'})_{\delta_1}|$, then $(\pi_R(P_{R'}))_{\tau_1}$ separates $(S_R)_{\delta_1}$ into connected components of size at most $\frac{1}{2}|(S_R)_{\delta_1}|$.*

Proof For the constant τ_0 of Fact 2.5 let $\tau = \tau_0 + 2\delta_1$, and let τ' be the corresponding constant from Fact 2.4. Let $\tau_1 = \tau' + 2\delta_1$. If $x \in S_R - (\pi_R(P_{R'}))_{\tau_1}$ then $P_{R'} \cap \Sigma_{x,\tau'} = \varnothing$. It follows from Fact 2.4 that $\Sigma_{x,\tau} \cap S_{R'}$ is in a component of $\Sigma_{x,\tau'} \cap (S_{R'})_{\delta_1}$ and so in a component of $(S_{R'})_{\delta_1} - (P_{R'})_{\delta_1}$. If we have two points $d(x,y) \le 2\delta_1$ then the sets $\Sigma_{x,\tau}$ and $\Sigma_{y,\tau}$ intersect. This implies that for every connected subset D' of $(S_R)_{\delta_1} - (\pi_R(P_{R'}))_{\tau_1}$, the set $\Sigma_{D,\tau_0} \cap S_{R'}$ is connected in $(S_{R'})_{\delta_1} - (P_{R'})_{\delta_1}$, where D is the set of points in S_R at distance at most δ_1 from D'. The conclusion of claim follows from the fact that sizes of D and D' differ by some constant factor, as in Fact 2.5. □

Claim 2.15 *For each large enough R, we can choose $P_R \subset S_R$ such that*

(1) $(P_R)_{\tau_1}$ *separates* $(S_R)_{\delta_1}$ *into connected components of size at most* $\frac{1}{2}|(S_R)_{\delta_1}|$,

(2) $P_{R_1} = \pi_{R_1}(P_{R_2})$, *for every* $R_1 \leq R_2$.

Proof We can assume without any loss of generality that the projection maps are chosen so that we have $\pi_{R_1}(x) = \pi_{R_1} \circ \pi_{R_2}(x)$ for every $R_1 < R_2$ and $x \in G - B_{R_2}$.

From the assumption of bounded sphere separation, for every large enough $R' > 0$, $P_{R'} \subset S_{R'}$ of bounded size M satisfying that $(P_{R'})_{\delta_1}$ separates $(S_{R'})_{\delta_1}$ into connected components of size at most $\frac{1}{K}|(S_{R'})_{\delta_1}|$, where K is given by Claim 2.14.

From Claim 2.14, for every $R < R'$, the set $P_R := \pi_R(P_{R'})$ satisfies property (1). Now, for every $R_1 < R_2 < R'$, we have $P_{R_1} = \pi_{R_1}(P_{R_2})$ since $\pi_{R_1} = \pi_{R_1} \circ \pi_{R_2}$.

Since the spheres of G are finite, we can proceed to an extraction to obtain a sequence R'_n such that for every $R > 0$ the sequence $(\pi_R(P_{R'_n}))_{n \geq 0}$ is constant (it is only defined when $R'_n \geq R$). Without any loss of generality we can assume that we have $P_R = \pi_R(P_{R'_n})$. We finally get the desired property that $P_{R_1} = \pi_{R_1}(P_{R_2})$ for every $R_1 < R_2$. □

Now the sequence P_R has a limit $P \subseteq \partial G$ as $R \to \infty$. To complete the proof of Lemma 2.13 it remains to show that P topologically separates ∂G.

From above, there exist $\xi, \eta \in \partial G$ such that $\pi_R(\xi)$ and $\pi_R(\eta)$ are in different components of $(S_R)_{\delta_1} - (P_R)_{\delta_1}$ for all large enough R. Assume for contradiction that ξ and η are in the same component of $\partial G - P$. The boundary ∂G is path connected, let γ be a path in $\partial G - P$ connecting ξ and η. There exists $\epsilon > 0$ such that the path γ avoids the ϵ-neighbourhood of P in ∂G. Let R be big enough so that $\pi_R^{-1}(x_{2\delta_1}) \cap \partial G$ is of diameter $\leq \epsilon/2$ for each $x \in S_R$.

Thus, $\pi_R \circ \gamma$ is a $2\delta_1$-step path in S_R which avoids $(P_R)_{2\delta_1}$. Completing it with a collections of geodesic arcs of length at most δ_1, we get a path in $(S_R)_{\delta_1}$ which avoids $(P_R)_{\delta_1}$, and connects $\pi_R(\xi)$ and $\pi_R(\eta)$. A contradiction. This ends the proof of Lemma 2.13. □

Proof of Theorem A By Lemmas 2.8 and 2.13 we see that ∂G is topologically separated by a finite set of points. It therefore has a local cut point. It follows from Bowditch [3] that G splits over a virtually cyclic group or G is a Fuchsian group. □

3 Proof of Theorem B

In this section, we construct a hyperbolic group with superlogarithmic separation profile whose boundary has conformal dimension one. Let us start by giving the following proposition.

Proposition 3.1 [1, Lemma 1.3] *Let G and H be bounded degree infinite graphs such that there exists a coarse embedding $G \to H$. Then,* $\mathrm{sep}_G \preceq \mathrm{sep}_H$.

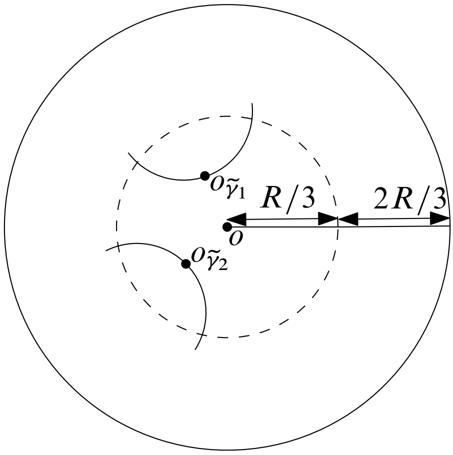

Figure 2: The set X of Proposition 3.2.

Recall that quasi-isometric embeddings are examples of coarse embeddings. This proposition implies that the separation profile is invariant under coarse equivalences and quasi-isometries. In particular we can consider separation profiles more generally for metric spaces that are coarsely equivalent to graphs of bounded degree. This is what we will do in this section for the hyperbolic plane.

Let Σ and Σ' be two closed hyperbolic orientable surfaces, and $\gamma \subset \Sigma$, $\gamma' \subset \Sigma'$ be two closed filling geodesic curves. Recall that a curve on a surface is said to be *filling* when its complementary is homeomorphic to a union of disks. Let $S = (\Sigma \sqcup \Sigma')/\gamma \simeq \gamma'$ be the space obtained by gluing Σ and Σ' along γ and γ'.

The universal cover $\widetilde{S}$ of S consists of copies of hyperbolic planes, that we will call sheets, glued together along the geodesic lines which correspond to the lifts of γ and γ'.

Let F be one of the sheets covering Σ. For a lift $\widetilde{\gamma} \subset F$ of γ let $F_{\widetilde{\gamma}}$ be the adjacent sheet covering Σ' which is glued to F along $\widetilde{\gamma}$.

Let $R > 0$. Let B_R (resp. $B_{R/3}$) be the balls of radius R (resp. $R/3$) in F centred around o. Let us consider

$$X = B_R \cup \bigcup_{\widetilde{\gamma} \cap B_{R/3} \neq \varnothing} B_{F_{\widetilde{\gamma}}}(o_{\widetilde{\gamma}}, R/3) \subset \widetilde{S}$$

where the union ranges over all lifts $\widetilde{\gamma}$ of γ that intersect $B_{R/3}$ and $B_{F_{\widetilde{\gamma}}}(o_{\widetilde{\gamma}}, R/3)$ is the ball of radius $R/3$ in the sheet $F_{\widetilde{\gamma}}$ centred at the point $o_{\widetilde{\gamma}}$ on $\widetilde{\gamma}$ which is closest to o. See Figure 2.

Proposition 3.2 *The set X satisfies* $\operatorname{cut} X \succ R$.

Let us prove how Theorem B can be deduced from this proposition.

Proof of Theorem B The fundamental group $\pi_1 S$ is quasi-isometric to the universal cover $\widetilde{S}$. Thus, we can compute the separation profile of $\widetilde{S}$ instead of that of $\pi_1 S$, and the theorem follows from Proposition 3.2. □

Proof of Proposition 3.2

Claim 3.3 *Most of the volume of X lies in the ball $B_R \subset F$:*

$$\mathrm{vol}(X) \asymp \mathrm{vol}(B_R).$$

Proof The volume of a ball $B(o,r)$ of radius r in the hyperbolic plane is

$$\mathrm{vol}(B(o,r)) = 2\pi(\cosh(r) - 1) \asymp e^r.$$

The number of lifts of a geodesic that intersect a ball $B(o,r)$ is $\preceq e^r$. Thus,

$$\mathrm{vol}(B(o,R)) \preceq \mathrm{vol}(X) \preceq \mathrm{vol}(B(o,R)) + e^{R/3}\,\mathrm{vol}(B(o,R/3)) \preceq \mathrm{vol}(B(o,R)).$$ □

Let C be a (1-thick) cutset of X, that is every connected component of $X - C$ has volume at most $\alpha \operatorname{vol} X$ for some $\alpha < 1$. Up to taking a small enough α, C has to separate the ball B_R. We want to show that C must have volume strictly bigger than $\log \mathrm{vol}(X) \asymp \log \mathrm{vol}(B(o,R)) \asymp R$. Let us assume for a contradiction that we have (up to constants), $\operatorname{vol} C \preceq R$.

Let $\Lambda = \partial C$. The total length of Λ is $O(R)$: indeed, C has volume R and can be chosen to be a union of balls of radius 1 in a given net and so the length of their boundary component has to be $O(R)$.

The components of Λ are either proper arcs or simple closed curves in B_R. Let $\widehat{\Lambda}$ be the collection of geodesics with the same endpoints as the arcs of Λ. Let C' be the union of the components of $X - \widehat{\Lambda}$ that include $C \cap \partial B_R$. See Figure 3.

Lemma 3.4 (i) $\mathrm{vol}(C \triangle C') \preceq R$.

(ii) *Every component E of $B_R - C$ of size $\succ R$ corresponds to a unique component E' of $B_R - C'$ such that* $\mathrm{vol}(E \triangle E') \preceq R$*, and vice versa.*

Proof (i) The difference between the sets C and C' has boundary in $\Lambda \cup \widehat{\Lambda}$. Then, since the total length of Λ and $\widehat{\Lambda}$ is $O(R)$, it follows from the isoperimetric inequality on the hyperbolic plane, that $\mathrm{vol}(C \triangle C') \preceq \mathrm{length}(\Lambda \cup \widehat{\Lambda}) \preceq R$.

(ii) By the isoperimetric inequality, a component E of $B_R - C$ of size $\succ R$ must intersect ∂B_R. There is a component E' of $B_R - C'$ with $E \cap \partial B_R = E' \cap \partial B_R$. The difference $E \triangle E'$ comprises of sets which are bounded by Λ and $\widehat{\Lambda}$. The volume of this difference can be bounded by $\preceq R$ again by the isoperimetric inequality. □

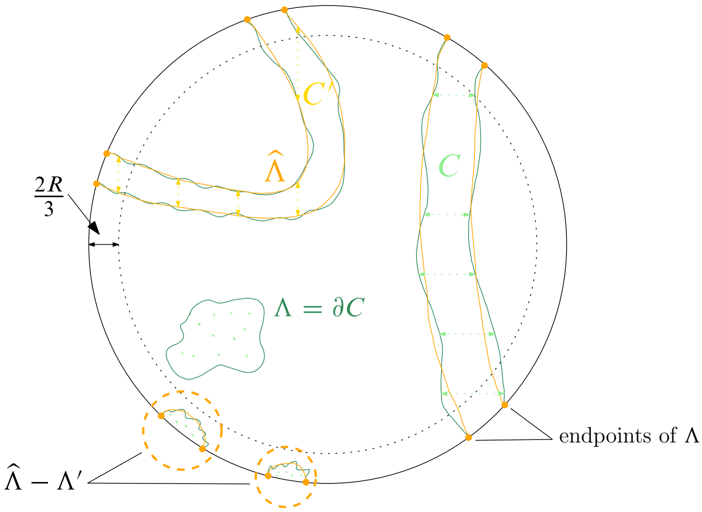

Figure 3: The separating set C of the hyperbolic ball.

Let Λ' be a set of geodesics in $\widehat{\Lambda}$ that meet $B_{R/3} = B_F(o, R/3)$. Any geodesic in Λ' must have a segment joining $\partial B_{R/3}$ and ∂B_R, and thus must have length at least $2R/3$. Since length$(\widehat{\Lambda}) \preceq R$ there are $O(1)$ many geodesics in Λ'. Let k be the number of geodesics in Λ'.

Let $m(x, y, z)$ denote the centre of the geodesic triangle spanned by a triple points $x, y, z \in \mathbb{H}^2$. Let

$$M = \{m(o, x, y) \mid x, y \in \Lambda' \cap \partial B_R\}.$$

There are at most $(2k)^2$ points in M.

Divide $B_{R/3}$ into $3((2k)^2 + 1)$ radial annuli of same width, called *layers*. By the pigeonhole principle, there exist three consecutive layers A_-, A, A_+ that do not contain a point of M.

Claim 3.5 *For R large enough, we have:*

(i) *The intersection $\Lambda' \cap A$ consists of geodesics joining the inner and outer boundaries of A.*

(ii) *If α is a component of $\Lambda' \cap A$, and α is a subarc of $\lambda \in \Lambda'$, then α is at Hausdorff distance at most δ from the arc of intersection of one of the two geodesics connecting o and $\partial\lambda$ with A.*

(iii) *If α_1, α_2 are components of $\Lambda' \cap A$, then either the Hausdorff distance $d_H(\alpha_1, \alpha_2) \le 3\delta$ or they are at distance $\sqrt{R}$ apart.*[1]

Proof (i) Otherwise, a component α of $\Lambda' \cap A$ is a geodesic arc connecting the outer boundary of A to itself. Let p be the point on α closest to o, let λ be the geodesic of Λ' to which α belongs, and let x, y be the endpoints of λ in $S_F(o, R)$. Then, $m(o, x, y) \in M$ is at distance δ from p, contradicting the assumption that $A_- \cup A \cup A_+$ does not include points of M.

[1]The function $\sqrt{R}$ can be replaced by any function $o(R)$.

(ii) Consider the geodesic triangle consisting of the geodesic λ and the two geodesics connecting its endpoints to o. By assumption, the centre of this geodesic is not in $A_- \cup A \cup A_+$, hence by slimness of triangles in $\mathbb{H}^2$ the segment α is δ-close to one of the sides.

(iii) Let α_1, α_2 be components of $\Lambda' \cap A$. Let λ_1 (resp. λ_2) be the geodesic in Λ containing α_1 (resp. α_2). By (ii), α_1 (resp. α_2) is δ-close to a radial geodesic λ'_1 (resp. λ'_2) connecting o and one of the endpoints of λ_1 (resp. λ_2). Let $\alpha'_1 = \lambda'_1 \cap A$ (resp. $\alpha'_2 = \lambda'_2 \cap A$). It suffices to prove that $d_H(\alpha'_1, \alpha'_2) \leq \delta$, or they are at distance $\sqrt{R} + 2\delta$ apart.

Let $p_1, q_1 \in \alpha'_1$ (resp. $p_2, q_2 \in \alpha'_2$) be the intersection of α'_1 (resp. α'_2) with the inner and outer boundaries of A, respectively. If $d(q_1, q_2) \leq \delta$ then $d_H(\alpha'_1, \alpha'_2) \leq \delta$ by the convexity of the metric. Similarly, if $d(p_1, p_2) \geq \sqrt{R} + 2\delta$ then α'_1, α'_2 are at least $\sqrt{R} + 2\delta$ apart. Otherwise, $d(p_1, p_2) \leq \sqrt{R} + 2\delta$ and $d(q_1, q_2) > \delta$, then the centre of the triangle with sides λ'_1, λ'_2 is at distance at most $\sqrt{R} + 3\delta$ from α'_1. For R large enough, such a point must be in $A_- \cup A \cup A_+$ in contradiction to the assumption. □

From the claim above it follows that $\Lambda' \cap A$ consists of at most $2k$ geodesic segments connecting the inner and outer boundaries of A and the relation defined by $\alpha_1 \sim \alpha_2$ if $d_H(\alpha_1, \alpha_2) \leq 3\delta$ is an equivalence relation. Let $\mathcal{W}$ be a set of representatives of the classes of this relation. We call the elements in $\mathcal{W}$ *walls*. We call the connected components of $A - \mathcal{W}$ *regions*. By the claim, the walls bounding each region are at distance $\sqrt{R}$ apart.

Claim 3.6 *Let D be a region in A, then there exists a (unique) component E of $X - C$ such that* $\mathrm{vol}(D - E) \preceq R$.

Proof Let α_1, α_2 be the walls bounding D. Let α'_1, α'_2 be the inner most arcs in D which belong to the equivalence classes of α_1, α_2, respectively. Let E' be the connected component of $X - \widehat{\Lambda}$ which includes the section of A between α'_1, α'_2. This section is contained in D, and $D - E'$ consists of two regions which are contained in the 3δ-neighbourhood of $\alpha_1 \cup \alpha_2$. Therefore, $\mathrm{vol}(D - E') \preceq R$. The set C' is bounded by geodesics in $\widehat{\Lambda}$. It cannot contain the component E', as otherwise $\mathrm{vol}(C') \succeq \mathrm{vol}(E') \succ R$. Thus E' is a component of $X - C'$. By Lemma 3.4, E' corresponds to a unique component E of $X - C$, and $\mathrm{vol}(D - E) \leq \mathrm{vol}(D - E') + \mathrm{vol}(E' - E) \preceq R$. □

Claim 3.7 *No component E of $X - C$ contains more than $\frac{2}{3}$ of the layer A.*

Proof Let E be a component of $X - C$. Assume for contradiction that $\mathrm{vol}(E \cap A) > \frac{2}{3}\,\mathrm{vol}(A)$. For every $x \in E \cap A$ consider the ray $x^* = \mathbb{R}_{\geq 1} x \cap B_R = \{tx \in B_R \mid t \geq 1\}$. Let $E_1 = \{x \in E \mid x^* \cap C = \varnothing\}$. Since $\mathrm{vol}(C)$ consists of $O(R)$ balls of radius 1, the set $E - E_1$ consists of at most $O(R)$ 1-neighbourhoods of arcs of length $O(R)$. Whence, $\mathrm{vol}(E - E_1) \preceq R^2$. Consider the set $E_1^* = \bigcup_{x \in E_1} x^*$. Thus $\mathrm{vol}(E_1) > \frac{1}{2}\,\mathrm{vol}(A)$, and therefore also $\mathrm{vol}(E_1^*) > \frac{1}{2}\,\mathrm{vol}(B_R)$. The set $E_1^* \subseteq E$, thus $\mathrm{vol}(E) > \frac{1}{2}\,\mathrm{vol}(B_R)$. We get a contradiction to the assumption that the volume of components of $X - C$ are at most $\frac{1}{2}\,\mathrm{vol}(B_R)$. □

By the previous two claims there are two regions D_+, D_- of $A - \mathcal{W}$ which correspond to two different components E_+, E_- of $X - C$. We may assume that D_1 and D_2 are adjacent, and are separated by a wall α. Let α_m be the middle third subarc of α.

Claim 3.8 *There is $k \asymp R$, and disjoint lifts $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ of γ such that $\tilde{\gamma}_i \cap \alpha_m \neq \varnothing$.*

Proof Consider the union $\Gamma = \bigcup \tilde{\gamma}$ of all the lifts $\tilde{\gamma}$ of γ to the universal cover F of Σ. Since γ is filling in Σ, the connected components of $F - \Gamma$ are one of finitely many types of convex hyperbolic nonideal polygons. Let d be the maximal diameter of these polygons. There exists an angle θ such that every geodesic line intersecting one of the polygons, forms an angle θ with at least one of its sides. Let $\mu > 0$ be such that if two geodesic lines l_1, l_2 in the hyperbolic plane intersect a third geodesic line l at points of distance $\geq \mu$ and at angles $\geq \theta$, then l_1, l_2 do not meet.

Let $\ell = \mathrm{length}(\alpha_m) \asymp R$. Every segment of length $2d$ on α_m intersects a lift $\tilde{\gamma}$ of γ in an angle $\geq \theta$. Thus, the geodesic segment α_m intersects at least $k = \ell/(\mu + 2d)$ lifts $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ of γ in an angle $\geq \theta$. Note that $k \asymp \ell \asymp R$. By the choice of μ, $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ are disjoint. □

Let $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ be the disjoint lifts as in Claim 3.8. The geodesic segments $\tilde{\gamma}_i \cap D_\pm$ have length at least $\succeq \sqrt{R}$ by Claim 3.5 and by the choice of α_m. The circles around the point $\tilde{\gamma}_i \cap \alpha_m$ in $F_{\tilde{\gamma}_i}$ form $\Theta(\sqrt{R})$ disjoint paths connecting points in D_+ to points in D_-. Considering these paths for all $\tilde{\gamma}_i$, we get $\Theta(R^{3/2})$ disjoint paths connecting D_+ to D_-. By Claim 3.6, $\mathrm{vol}(D_\pm - E_\pm) \preceq R$, and so we have at least $\succeq R^{3/2} - O(R)$ disjoint paths connecting E_+ to E_-. Since E_+ and E_- are different components of $X - C$, each of these paths meets C. We get $\mathrm{vol}(C) \succeq R^{3/2}$ which contradicts $\mathrm{vol}(C) \preceq R$. This ends the proof of Proposition 3.2. □

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