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Topology and geometry of flagness and beltness of simple handlebodies

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We consider a class of right-angled Coxeter orbifolds, called simple handlebodies, which are a generalization of right-angled Coxeter simple polytopes. We generalize the notions of flag and belt in the setting of simple polytopes into the setting of simple handlebodies, and prove the following two topological properties characterized in terms of combinatorics: a simple handlebody is orbifold-aspherical if and only if it is flag; and the orbifold fundamental group of a simple handlebody contains a rank-two free abelian subgroup if and only if this simple handlebody contains an $\square$-belt. Furthermore, together with some results of geometry, it is shown that the existence of some curvatures on manifold double over a simple handlebody can be also characterized in terms of combinatorics.

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1 Introduction

A polytope is called *simple* if its each codimension-k face is the intersection of exact k *facets* (i.e. codimension-one faces). Simple polytopes give rise to many interesting and beautiful connections among topology, geometry, combinatorics and so on.

The story originated from Pogorelov and Andreev. Pogorelov's theorem implies that a 3-dimensional simple polytope (except for tetrahedra) can be embedded into hyperbolic space $\mathbb{H}^3$ with right dihedral angles if and only if the polytope satisfies certain combinatorial conditions (i.e. containing no prismatic 3-circuit and prismatic 4-circuit); see Pogorelov [52]. Furthermore, Andreev's theorem gives a complete characterization of 3-dimensional compact hyperbolic (simple) polytopes having nonobtuse dihedral angles in term of pure combinatorial conditions; see Andreev [2] and Roeder, Hubbard and Dunbar [53].

The theorems of Pogorelov and Andreev play an important role in Thurston's hyperbolization theorem for Haken 3-manifolds [58]. An orbifold version of the hyperbolization theorem was given by Otal [49]. The hyperbolization theorem implies that the main obstructions to hyperbolic structure on closed Haken 3-manifolds (or 3-orbifolds) are asphericity (i.e. π_k trivial for each $k \geq 2$) and atoroidalness (i.e. π_1 contains no subgroup $\mathbb{Z} \oplus \mathbb{Z}$) which is related to the combinatorics of simple polytopes.

The combinatorial characterizations of asphericity and atoroidalness for cubical complexes were given by Gromov [32] and have played fundamental roles in geometric group theory.

(a) A piecewise Euclidean cubical complex is locally CAT(0) (hence, aspherical) if and only if the link of its each vertex is flag (i.e. contains no Δ).

(b) A piecewise Euclidean cubical complex is locally CAT(-1) (hence, atoroidal) if and only if the link of its each vertex is flag and contains no $\square$.

It follows that a right-angled Coxeter group is always CAT(0), and it is atoroidal if and only if it satisfies no $\square$ condition. These results have been strengthened by Moussong to conclude that every Coxeter group is CAT(0), and it is CAT(-1) if and only if it does not contain a copy of $\mathbb{Z} \oplus \mathbb{Z}$; see Moussong [46] or Davis [18, Corollary 12.6.3].

It's worth noting that each n-dimensional simple polytope admits a natural right-angled Coxeter orbifold structure, that is locally modeled on the quotient $\mathbb{R}^n/(\mathbb{Z}_2)^k$ of the $(\mathbb{Z}_2)^k$-reflective action on $\mathbb{R}^n$. A combinatorial characterization of orbifold-asphericity for right-angled Coxeter simple polytopes is derived from a theorem of Davis, Januszkiewicz and Scott [24]. That is, a right-angled Coxeter simple polytope is orbifold-aspherical if and only if the dual of its boundary is a flag complex. On the other hand, the orbifold fundamental group of a right-angled Coxeter simple polytope is a right-angled Coxeter group. Thus, a combinatorial characterization of atoroidalness for right-angled Coxeter simple polytopes is implied in Gromov's result. Right-angled Coxeter simple polytopes (and more general Coxeter simple polytopes) have played an important role in geometric group theory and in constructing high-dimensional hyperbolic manifolds (e.g. see Davis [18], Everitt, Ratcliffe and Tschantz [26], Garrison and Scott [30], and Gromov [32]).

Beyond that, simple polytopes have played an important role in theory of toric varieties, toric geometry, toric topology, etc (e.g. see Barreto, López de Medrano and Verjovsky [5], Buchstaber, Erokhovets, Masuda, Panov and Pak [8; 9], Danilov [15; 16], Davis and Januszkiewicz [22; 23], Fulton [29], Kuroki, Masuda and Yu [38], Lü and Masuda [42], Notbohm [48], and Wu and Yu [61]). There are also various relative works with other topics and viewpoints of simple polytopes (e.g. see Bahri, Bendersky, Cohen and Gitler [3], Bosio and Meersseman [6], Cao and Lü [10], Chen, Lü and Yu [11], Choi and Park [14], Gitler and López de Medrano [31], and Lü and Tan [43]).

In this paper, we consider the combinatorial characterizations of orbifold-asphericity and atoroidalness for a *simple handlebody*, which can be obtained from a right-angled Coxeter simple n-polytope by gluing some specific disjoint facets. Specifically, each simple handlebody Q with dimension $n \geq 3$ satisfies the following conditions.

(a) As an orbifold, the underlying space $|Q|$ is an n-dimensional handlebody of genus $\mathfrak{g} \geq 0$ that is a tubular neighborhood of the wedge sum of $\mathfrak{g}$ circles in $\mathbb{R}^n$ (of course, an n-dimensional handlebody of genus 0 is exactly an n-ball).

(b) The *nerve* of Q, denoted by $\mathcal{N}(Q)$, is a triangulation of the boundary $\partial|Q|$, where $\mathcal{N}(Q)$ is the abstract simplicial complex with a vertex for each *facet* (i.e. codimension-one face) of Q and a $(k-1)$-simplex for each nonempty k-fold intersection.

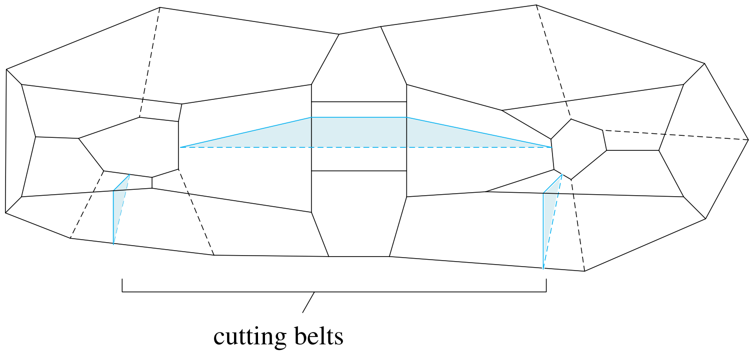

Figure 1: A simple 3-handlebody of genus 2.

(c) Each facet in Q is a simple polytope.

(d) Q can be cut into a simple polytope P_Q along some codimension-one B-belts (called *cutting belts*; for the notion of B-belts, see Definition 3.1).

Conditions (c) and (d) are restrictive conditions of the proof, and will be automatically omitted for $n = 3$; see Lemma 7.14. An example of simple 3-handlebody is shown in Figure 1.

We shall carry out our work from the following aspects:

(I) We generalize the notions of belt and flag in the setting of simple polytopes into the setting of simple handlebodies (see Definitions 3.1 and 3.3). Indeed, there is quite a difference because the underlying space of a simple handlebody is not contractible. As we shall see, the flagness of a simple handlebody Q cannot be defined by the flagness of its nerve $\mathcal{N}(Q)$ in general. Actually, its definition is given in such a different way that Q contains no Δ^k-belt for any $k \geq 2$. Here a Δ^k-belt of Q is an essential embedding suborbifold given by $(\mathbb{Z}_2)^{k+1}$-torus action on S^k.

(II) To understand the implicit structure of a simple handlebody, we introduce the notions of "right-angled Coxeter cells" and "right-angled Coxeter cellular complexes". Then we see that for a right-angled Coxeter cellular complex X, its orbifold fundamental group $\pi_1^{\mathrm{orb}}(X)$ is isomorphic to the orbifold fundamental group of its 2-skeleton (see Proposition 2.9). For a simple handlebody Q, we can give an explicit right-angled Coxeter cellular decomposition of Q, so that we can obtain an explicit presentation of $\pi_1^{\mathrm{orb}}(Q)$, which is just an iterative HNN-extension over some right-angled Coxeter group. Indeed, generally $\pi_1^{\mathrm{orb}}(Q)$ will not be the right-angled Coxeter group of Q, given by only reflections on facets of Q, and it actually contains torsion-free generators.

(III) We will use a "basic construction" of Davis [18, Chapter 5], which plays an important role on our work. This basic construction tells us that each simple n-handlebody Q can be finitely covered by a closed n-manifold M_Q with an action of some 2-torus group G, which is called a *manifold double* over Q. By the theory of orbifold covering,

$$\pi_k^{\mathrm{orb}}(Q) \cong \pi_k(M_Q), \quad k \geq 2.$$

Thus, Q is orbifold-aspherical if and only if M_Q is aspherical. It follows from [24, Theorem 2.2.5] that a simple polytope P is orbifold-aspherical if and only if P is flag; so in general, a simple n-handlebody Q may not be orbifold-aspherical although $|Q|$ is aspherical. In addition, using the basic construction, we can also use $\pi_1^{\mathrm{orb}}(Q)$ and P_Q to construct the orbifold universal cover $\widetilde{Q}$ of Q.

(IV) Based upon (I), (II) and (III), together with the Cartan–Hadamard theorem and the work of Gromov [32, Section 4.2] on nonpositive curvature, we obtain a combinatorial characterization of orbifold-asphericity of simple handlebodies (see Theorem A). Making use of Tits' theorem of Coxeter groups [18, Theorem 3.4.2] and the normal form theorem of HNN-extensions [44, Theorem 2.1, page 182], we also obtain a combinatorial characterization of atoroidalness of simple handlebodies (see Theorem B).

Now let us state our main results as follows.

Theorem A *Let Q be a simple handlebody of dimension $n \geq 3$. Then Q is orbifold-aspherical if and only if it is flag, that is, Q contains no Δ^k-belt for any $k \geq 2$.*

Remark 1.1 Theorem A is a "combinatorial sphere theorem" of simple handlebodies, which can be viewed as a generalization of Davis, Januszkiewicz and Scott [24, Theorem 2.2.5] for simple polytopes. Theorem A tells us that if Q is not flag, then there must exist a Δ^k-belt for some $k \geq 2$ in Q, so that the pullback of the embedding $\Delta^k \hookrightarrow Q$ via the projection $M_Q \to Q$ gives an (equivariant) embedding $S^k \hookrightarrow M_Q$ which represents a nontrivial element in $\pi_k(M_Q)$, as shown in the following diagram:

$$\begin{tikzcd}
S^k \arrow[r, hook] \arrow[d] & M_Q \arrow[d] \\
\Delta^k \arrow[r, hook] & Q
\end{tikzcd}$$

Our other main result characterizes the rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$ in terms of combinatorics of Q.

Theorem B *Let Q be a simple handlebody of dimension $n \geq 3$. Then there is a rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$ if and only if Q contains a $\square$-belt.*

Remark 1.2 Theorem B is a "combinatorial flat torus theorem" of simple handlebodies. Similar to the pullback way in Remark 1.1, the existence of a $\square$-belt in Q actually means that there exists an essential embedding of 2-dimensional torus T^2 in M_Q, which is an obstacle of the existence of hyperbolic structure or negative curvature on M_Q. For the flat torus theorem of nonpositively curved spaces, one can refer to Bridson and Haefliger [7, Charter 7, Part II] or Lawson and Yau [39].

Next, as further consequences of our two main results, we discuss the topology and geometry of covering spaces over a simple handlebody. Together with some important results in geometry from Davis [18, Proposition I.6.8], Kapovich [37], Kuroki, Masuda and Yu [38, Theorem 1.2], Otal [49, Chapter 7], and

Wu and Yu [61, Proposition 4.9], for a simple handlebody Q, we obtain some relations between the existence of some curvatures on M_Q and the combinatorics of Q.

Corollary 1.3 *Let Q be a simple handlebody of dimension $n \geq 2$, M_Q be the smooth manifold double over Q, and $\widetilde{Q}$ be the orbifold universal cover of Q. Then*:

- (i) *The following statements are equivalent*:
  - (1) *M_Q is nonpositively curved*;
  - (2) *$\widetilde{Q}$ is CAT*(0);
  - (3) *Q is flag* (*this is equivalent to Q being orbifold-aspherical*);
  - (4) *M_Q is aspherical.*
- (ii) *If M_Q admits a strictly negative curvature then Q is flag and contains no $\square$-belt. In particular*, *if Q is a simple polytope P, then M_P admits a strictly negative curvature if and only if P is flag and contains no $\square$-belt.*

In the 3*-dimensional case*,

- (iii) *M_Q is hyperbolic if and only if it is flag and contains no $\square$-belt. Moreover*, *Q admits a right-angled hyperbolic structure if and only if it is flag and contains no $\square$-belt. In this case*, $\widetilde{Q} \approx \mathbb{H}^3$.
- (iv) *When Q is a simple* 3*-polytope*, *M_Q admits a positive scalar curvature if and only if every* 2*-dimensional belt in Q is Δ^2*, *or Q is just a tetrahedron.*

Remark 1.4 The proof of Corollary 1.3 will mainly be finished in Section 7.

- Corollary 1.3(i)–(ii) are based on Gromov's results; see [32]. A metric space is said to be of curvature k if it is locally a CAT(k) space. A comparison theorem in [7, Theorem 1A.6, page 173] tells us that a smooth Riemannian manifold has curvature $\leq k$ if and only if it has sectional curvature $\leq k$. Hence, under the condition that M_Q admits a smooth Riemannian metric, the curvature in the statements of Corollary 1.3(i)–(ii) can be replaced by sectional curvature. See [38, Theorem 1.2] for simple polytopes.

- There are examples of closed orientable 3-manifolds that are aspherical but do not support a Riemannian metric with nonpositive sectional curvature (see Leeb [40]).

- Corollary 1.3(iii) is the hyperbolization theorem on simple 3-handlebodies. The hyperbolization theorem on general right-angled Coxeter 3-orbifolds was considered by Otal [49]. An *irreducible and atoroidal* 3-manifold Q with corners, defined by Otal [49, page 168], implies essentially that all involved Δ^2 and $\square$ suborbifolds in Q are not belts. This is actually equivalent to saying that Q is flag and contains no $\square$-belt. Here our statement is more combinatorial.

- Corollary 1.3(iv) is also a restatement of a result of [61]. A vc(k) in [61] is equivalent to the simple 3-polytope in Corollary 1.3(iv).

- All 2-dimensional right-angled Coxeter orbifolds can be classified by their orbifold Euler numbers; see Thurston [58].

- With a bit of additional argument, the "simple" condition in 3-dimensional case can be generalized to the case of a right-angled Coxeter 3-handlebody whose nerve is an ideal triangulation of its boundary, where the concept of ideal triangulation can be found in Fomin, Shapiro and Thurston [27, Section 2]. In this case, there may exist bad 3-handlebodies, that is, as right-angled Coxeter orbifolds, they cannot be covered by 3-manifolds. So these bad orbifolds cannot admit any hyperbolic metric. See Lemma 7.16. Although so, we can obtain that a right-angled Coxeter 3-handlebody with ideal nerve is hyperbolic if and only if it is very good, flag and contains no $\square$-belt; see Corollary 7.18. More generally, if the nerve of Q is not a triangulation or an ideal triangulation of $\partial|Q|$, then only the flag condition and no $\square$-belt condition can prevent the (right-angled) hyperbolicity of Q. An example is given in Example 7.20.

Structure of the paper

In Section 2, we review the notions of (right-angled Coxeter) orbifolds and manifolds with corners. We introduce the right-angled Coxeter cellular decomposition of right-angled Coxeter orbifolds, and discuss their orbifold fundamental groups. In addition, we also introduce the theory of fundamental domain. In Section 3 we generalize the notions of B-belts and flags from simple polytopes to simple handlebodies. In Section 4, we give a right-angled Coxeter cellular decomposition of a simple n-handlebody Q, so that we can explicitly give a presentation of orbifold fundamental group $\pi_1^{\mathrm{orb}}(Q)$. We further show that this presentation of $\pi_1^{\mathrm{orb}}(Q)$ is an iterative HNN-extension of some right-angled Coxeter group. Moreover, the orbifold universal cover of Q is constructed by using $\pi_1^{\mathrm{orb}}(Q)$ and the simple polytope P_Q associated to Q. In Section 5, we review the work of Gromov, and compute the homology groups of the manifold double and universal cover of a simple handlebody Q by Davis' method, which are useful in the proof of Theorem A. Then we prove Theorem A. In Section 6, we show that the existence of a rank-two free abelian subgroup in the orbifold fundamental group of a simple handlebody Q is characterized by a $\square$-belt in Q (Theorem B). In Section 7, applying Theorems A and B and some results of geometry, we discuss the existence of some curvatures on a smooth manifold double over a simple handlebody Q in terms of the combinatorics of Q.

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2 Preliminaries

2.1 Orbifold

As a generalization of manifolds, an n-dimensional *orbifold* $\mathbb{O}$ is a singular space which is locally modeled on the quotient of a finite group acting on an open subset of $\mathbb{R}^n$. For any $p \in \mathbb{O}$, there is an *orbifold chart*

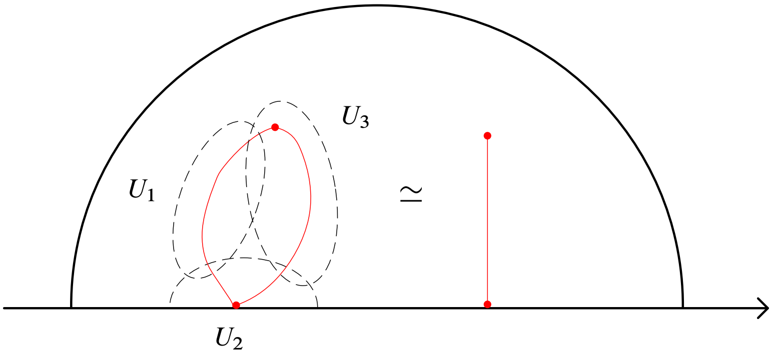


Figure 2: An orbifold loop.

(U, G, ψ) satisfying that U is an n-ball centered at origin O and $\psi^{-1}(p) = O$, where $\psi\colon U \to U/G$ is the projection map. In particular, the origin O is fixed by G. We called G the *local group* at p.

Definition 2.1 (Thurston [58, Definition 13.2.2]) A *covering orbifold* of an orbifold $\mathbb{O}$ is an orbifold $\widetilde{\mathbb{O}}$ with a projection $\pi\colon \widetilde{\mathbb{O}} \to \mathbb{O}$, satisfying that:

- Every $x \in \mathbb{O}$ has a neighborhood V which is identified with an open subset U of $\mathbb{R}^n$ modulo a finite group G_x, such that each component V_i of $\pi^{-1}(V)$ is homeomorphic to U/Γ_i, where $\Gamma_i < G_x$ is some subgroup;
- $\pi|_{V_i}\colon V_i \to V$ corresponds to the natural projection $U/\Gamma_i \to U/G_x$.

An orbifold is *good* (resp. *very good*) if it can be covered (resp. finitely covered) by a manifold. Otherwise it is *bad*. Any orbifold $\mathbb{O}$ has an universal cover $\widetilde{\mathbb{O}}$; see [58, Proposition 13.2.4].

In general, the *orbifold fundamental group* of an orbifold is defined as the deck transformation group of its universal cover; see [58, Definition 13.2.5]. Another equivalent definition uses the notion of based orbifold loops, that is, the orbifold fundamental group is defined by the homotopy classes of based orbifold loops. For more details, see [12, Section 3].

Example 2.2 Let D^2 be the unit disk in $\mathbb{R}^2$. A transformation r on D^2 via $r(x, y) = (x, -y)$ gives a reflective $\mathbb{Z}_2$-action on D^2. The orbit space $D^2/\mathbb{Z}_2$ has a natural orbifold structure. Any $(x, 0) \in D^2/\mathbb{Z}_2$ is a singular point with local group $\mathbb{Z}_2$. Since D^2 is contractible, $\pi_1^{\mathrm{orb}}(D^2/\mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by the transformation r.

In the viewpoint of orbifold loops, any path between $(x_1, 0)$ and (x_2, y_2) with $y_2 > 0$ can be viewed as a nontrivial orbifold loop. It is clear that $D^2/\mathbb{Z}_2 \cong D^1 \times D^1/\mathbb{Z}_2 \simeq D^1/\mathbb{Z}_2$. Hence, $\pi_1^{\mathrm{orb}}(D^2/\mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by a based orbifold loop $D^1/\mathbb{Z}_2$; see Figure 2.

Example 2.3 [1] If a discrete group G acts properly discontinuously on a manifold M, then the orbit space M/G canonically inherits an orbifold structure. Here M/G is called the *quotient orbifold* by G acting on M.

Let $p\colon M \to \mathbb{O}$ be a regular orbifold cover over a good orbifold $\mathbb{O}$, where M is a manifold. Then by orbifold covering theory [12], the orbifold homotopy group of $\mathbb{O}$ is isomorphic to the homotopy group of M,

$$\pi_k^{\mathrm{orb}}(\mathbb{O}) \cong \pi_k(M), \quad k \geq 2.$$

Thus a good orbifold is orbifold-aspherical if and only if its covering manifold is aspherical.

See [1; 12; 13; 19; 54] for more details of orbifold homotopy theory.

2.2 Right-angled Coxeter orbifolds, manifolds with corners and their manifold covers

Following [21; 22], a *right-angled Coxeter n-orbifold Q* is a special n-orbifold locally modeled on the quotient $\mathbb{R}^n/(\mathbb{Z}_2)^n$ of the standard $(\mathbb{Z}_2)^n$-action on $\mathbb{R}^n$ by reflections across the coordinate hyperplanes. A *stratum* of codimension k is the closure of a component of the subspace of $|Q|$ consisting of all points with local group $(\mathbb{Z}_2)^k$, where $|Q|$ denotes the underlying space of Q. It is easy to see that $\mathbb{R}^n/(\mathbb{Z}_2)^n$ possesses the following properties:

- Topologically and combinatorially, $\mathbb{R}^n/(\mathbb{Z}_2)^n$ is the standard simplicial cone
$$\mathscr{C}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ 1 \leq i \leq n\}$$
in $\mathbb{R}^n$.
- The local group at $x = (x_1, \dots, x_n) \in \mathbb{R}^n/(\mathbb{Z}_2)^n$ is the subgroup $(\mathbb{Z}_2)^{c(x)}$, where $c(x)$ is the number of those coordinates $x_i = 0$ in x, called the *codimension* of x.
- For $0 \leq k \leq n$, $(\mathbb{Z}_2)^k$ as a local group determines $\binom{n}{k}$ strata of codimension k, each of which is isomorphic to $\mathbb{R}^{n-k}/(\mathbb{Z}_2)^{n-k}$.

Davis [17, Section 6] (or see [18, Chapter 10, page 180]) defined *n-manifolds with corners*, each of which is a Hausdorff space X together with a maximal atlas of local charts onto open subsets of the standard simplicial cone $\mathscr{C}^n$ such that the overlap maps are homeomorphisms of preserving codimension, where for any chart $\varphi\colon U \to \mathscr{C}^n$, the codimension of any $x \in U$ is defined as $c(\varphi(x))$, also denoted by $c(x)$, and it is independent of the chart. An *open face* of codimension k is a component of $\{x \in X \mid c(x) = k\}$. A *face* is the closure of such a component.

A right-angled Coxeter orbifold Q naturally inherits the structure of a manifold with corners. On the other hand, since the topological and combinatorial structure of $\mathscr{C}^n$ is compatible with that of right-angled Coxeter orbifold on $\mathbb{R}^n/(\mathbb{Z}_2)^n$, an n-manifold with corners naturally admits a right-angled Coxeter orbifold structure. Furthermore, all strata in a right-angled Coxeter orbifold Q bijectively correspond to all faces in Q as a manifold with corners. A stratum or face of codimension one is called a *facet*.

In this paper we are mainly concerned with a special class of right-angled Coxeter orbifolds, i.e. simple handlebodies, as defined in the beginning of this paper. Let Q be a simple n-handlebody with facet set $\mathscr{F}(Q) = \{F_1, \dots, F_m\}$. For some $n \leq k \leq m$, the map

$$\lambda\colon \mathscr{F}(Q) \to (\mathbb{Z}_2)^k$$

is called a *characteristic map* if for each l-face f^l in Q (so there are exactly $n-l$ facets, say $F_{i_1},\dots,F_{i_{n-l}}$, whose intersection is f^l since Q is simple), $\lambda(F_{i_1}),\dots,\lambda(F_{i_{n-l}})$ are independent in $(\mathbb{Z}_2)^k$. Clearly, each l-face f^l in Q determines a subgroup G_{f^l} generated by $\lambda(F_{i_1}),\dots,\lambda(F_{i_{n-l}})$ via λ. Note that each $x\in\partial|Q|$ always lies in the relative interior of a unique face f. Then there is a *manifold cover* of Q defined as

$$\mathcal{U}(Q,(\mathbb{Z}_2)^k)=Q\times(\mathbb{Z}_2)^k/{\sim}, \tag{2-1}$$

where

$$(x,g)\sim(y,h)\quad\Longleftrightarrow\quad\begin{cases}x=y \text{ and } g=h & \text{if } x\in\mathrm{Int}(|Q|),\\ x=y \text{ and } gh^{-1}\in G_f & \text{if } x\in f\subset\partial|Q|.\end{cases}$$

Essentially this is a special case of the "basic construction" of Davis [18, Chapter 5]. It follows from [18, Proposition 10.1.10] that $\mathcal{U}(Q,(\mathbb{Z}_2)^k)$ is an n-dimensional closed manifold and naturally admits an action of $(\mathbb{Z}_2)^k$ with quotient orbifold Q. So a simple handlebody is a very good orbifold.

Lemma 2.4 *A simple handlebody Q is the quotient orbifold of $(\mathbb{Z}_2)^k$ acting on $\mathcal{U}(Q,(\mathbb{Z}_2)^k)$.*

If $k=n$, then $\mathcal{U}(Q,(\mathbb{Z}_2)^n)$ is called *small manifold cover* over Q which is a generalization of small covers over simple polytopes (see [22]), but it may not exist even if Q is a simple polytope (see also [22, Nonexamples 1.22]). However, if $k=m$, we can take $\lambda(F_i)=e_i$ for each facet F_i of Q where $\{e_1,\dots,e_m\}$ is the standard basis of $(\mathbb{Z}_2)^m$, such that there always exists such $\mathcal{U}(Q,(\mathbb{Z}_2)^m)$, called the *manifold double* (see [21, Proposition 2.4]) over Q, which is a generalization of real moment angled manifolds over simple polytopes (see [9]). In this case, for simplicity, we use M_Q to replace $\mathcal{U}(Q,(\mathbb{Z}_2)^m)$.

2.3 The right-angled Coxeter cellular decomposition

Now let us introduce the right-angled Coxeter orbifold cellular decomposition for right-angled Coxeter orbifolds, which will play an important role on the calculation of the orbifold fundamental groups of right-angled Coxeter orbifolds. The more general notion of cellular decomposition of certain orbifolds was considered as q-cellular complexes (or, q-CW complexes) in [4; 51].

Let $r_i:\mathbb{R}^n\to\mathbb{R}^n$ be the i^{th} standard reflection defined by

$$r_i(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n)=(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_n).$$

All standard reflections in $\mathbb{R}^n$ induce a standard $(\mathbb{Z}_2)^n$-action on the closed unit n-ball B^n with a right-angled corner $B^n/(\mathbb{Z}_2)^n$ as its orbit space. Of course, $\mathrm{Int}\,B^n$ is $(\mathbb{Z}_2)^n$-equivariantly homeomorphic to $\mathbb{R}^n$.

Definition 2.5 (right-angled Coxeter cells) Let Γ be a group generated by some standard reflections in $\mathbb{R}^n$. Then the quotient B^n/Γ is called a *right-angled Coxeter n-ball*, and the quotient $\mathrm{Int}\,B^n/\Gamma$ is called an *open right-angled Coxeter n-ball*. Note that if Γ is not a trivial group, then the right-angled Coxeter n-ball B^n/Γ is an n-orbifold with boundary $\partial B^n/\Gamma$.

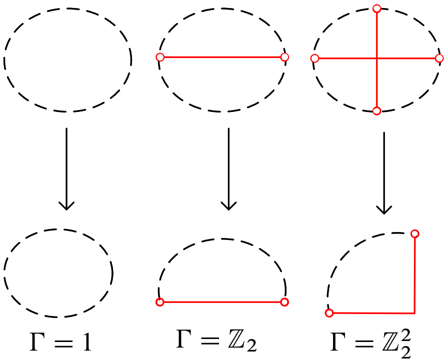


Figure 3: Right-angled Coxeter 2-cells.

If e^n is Γ-equivariantly homeomorphic to Int B^n, then the quotient e^n/Γ is called a *right-angled Coxeter n-cell*, and its closure is call a *closed right-angled Coxeter n-cell*.

For example, a right-angled Coxeter 1-cell is either a connected open interval or a semiopen and semiclosed interval whose closed endpoint gives a local group $\mathbb{Z}_2$. A right-angled Coxeter 2-cell has three possible types, with local group being the trivial group, $\mathbb{Z}_2$ or $(\mathbb{Z}_2)^2$, as shown in Figure 3.

An n-dimensional *right-angled Coxeter cellular complex* (or *Coxeter CW complex*) X can be constructed in the same way as CW complex (see [36, page 5]). A key point is that the attaching map of every n-dimensional right-angled Coxeter cell e^n_α/Γ to the $(n-1)$-skeleton X^{n-1},

$$\phi_\alpha\colon \partial\bar{e}^n_\alpha/\Gamma \to X^{n-1}, \tag{2-2}$$

is required to *preserve the local group of each point in $\partial\bar{e}^n_\alpha/\Gamma$*.

Here the attaching maps $\{\phi_\alpha\}$ of right-angled Coxeter cells with nontrivial local groups have a much stronger restriction than those in CW complexes. Actually, ϕ_α preserving local groups implies that singular points and nonsingular points of each embedding right-angled Coxeter n-cell are still singular and nonsingular, respectively, in X.

Remark 2.6 (right-angled Coxeter cubical cellular complex) Recall that a cubical cellular complex is a CW complex X whose cells are cubes, with the property that for two cubes c and c' of X, $c \cap c'$ is a common face of c and c'; in other words, cubes are glued in X via combinatorial isometries of their faces. Similarly, a *right-angled Coxeter cubical cellular complex* can be defined in the same way whose cells are all right-angled Coxeter cubical cells, that is, the orbits of standard reflections on an n-cube $[-1,1]^n$. For example, the standard cubical cellular decomposition of a simple polytope P (i.e. the cone of the barycentric subdivision of $\mathcal{N}(P)$) gives a right-angled Coxeter cubical cellular complex structure of P. Of course, right-angled Coxeter cubical cellular complexes form a special class of right-angled Coxeter cellular complexes.

Proposition 2.7 *Each simple handlebody has a finite right-angled Coxeter cellular complex structure.*

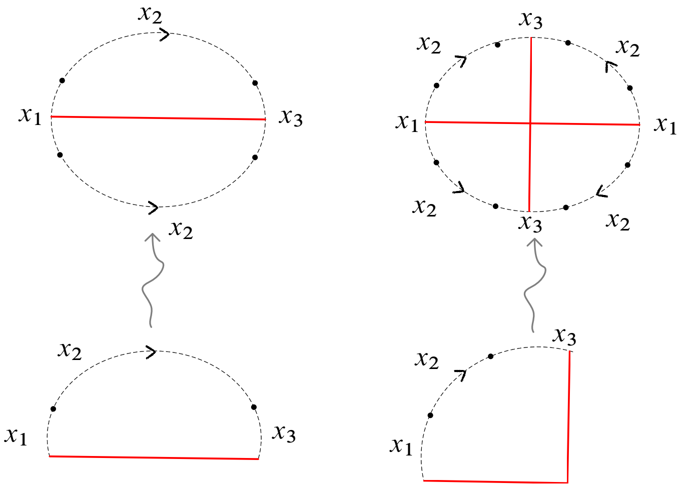


Figure 4: Relations determined by right-angled Coxeter 2-cells in the case $n = 3$.

Proof Let Q be a simple n-handlebody with the associated simple polytope P_Q. Then the standard cubical subdivision of P_Q induces a right-angled Coxeter cellular decomposition of Q. More details will be shown in Section 4. □

Remark 2.8 It should be pointed out that each simple handlebody still has a right-angled Coxeter cubical cellular complex structure. This can be seen in Section 7.1.

In general, a right-angled Coxeter cellular complex is naturally an orbispace. Here its orbifold fundamental group is defined by the homotopy classes of based orbifold loops. For more details, see [12, Section 3]. Although a right-angled Coxeter cell with nontrivial local group is not contractible in the sense of orbifolds, all attaching maps $\{\phi_\alpha\}$ preserving local groups ensure that the orbifold fundamental group of a right-angled Coxeter cellular complex is isomorphic to the orbifold fundamental group of its 2-skeleton.

Proposition 2.9 *Let X be a right-angled Coxeter cellular complex. Then*

$$\pi_1^{\mathrm{orb}}(X^2) \cong \pi_1^{\mathrm{orb}}(X),$$

where X^2 is the 2-skeleton of X.

Proof The argument can be proved in a similar way as shown by Hatcher [36, Proposition 1.26]. The only thing to note is that the local group information of each right-angled Coxeter n-cell can be inherited by the boundary orbifold of its closure in X^{n-1}. □

Remark 2.10 We can easily read out the generators and relations of $\pi_1^{\mathrm{orb}}(X) \cong \pi_1^{\mathrm{orb}}(X^2)$ from the 2-skeleton of a right-angled Coxeter cellular complex X. Let us look at a right-angled Coxeter 2-cell with nontrivial local group in X. Assume that the boundary of a right-angled Coxeter 2-cell with nontrivial local group consists of $x_1, x_2, \dots, x_n$, where each x_i is a closed oriented orbifold loop in X, and only one endpoint of x_1 and x_n has nontrivial local group. Regard these closed orbifold loops as generators. Then $x_1^2 = x_n^2 = 1$ and the right-angled Coxeter 2-cells with local group $\mathbb{Z}_2$ give a relation $x_1 x_2 \cdots x_n \cdot x_{n-1}^{-1} \cdots x_2^{-1} = 1$, while the right-angled Coxeter 2-cells with local group $\mathbb{Z}_2^2$ give a relation $(x_1 x_2 \cdots x_n \cdot x_{n-1}^{-1} \cdots x_2^{-1})^2 = 1$. This can intuitively be seen from Figure 4 when $n = 3$.

Example 2.11 Let P be a simple polytope with facet set $\mathcal{F}(P)$. Regard P as a right-angled Coxeter orbifold. The standard cubical subdivision of P is a right-angled Coxeter cellular decomposition of P. Calculating the orbifold fundamental group of P by the 2-skeleton of its right-angled Coxeter cellular decomposition, $\pi_1^{\mathrm{orb}}(P)$ can be represented by the right-angled Coxeter group W_P of P:

$$\pi_1^{\mathrm{orb}}(P) \cong W_P = \langle s_F, F \in \mathcal{F}(P) \mid s_F^2 = 1 \text{ for all } F; (s_F s_{F'})^2 = 1 \text{ for } F \cap F' \neq \varnothing \rangle.$$

2.4 Group action and fundamental domain ([18, page 64] or [59, pages 159–161])

Suppose that a discrete group G acts properly on a connected topological space X. A closed subset $D \subset X$ is a *fundamental domain* for the G-action on X if each G-orbit intersects with D and if for each point x in the interior of D, $G(x) \cap D = \{x\}$. In other words, $\{gD \mid g \in G\}$ forms a locally finite cover for X, such that no two of $\{gD \mid g \in G\}$ have common interior points. Such $\{gD \mid g \in G\}$ is called a *decomposition* for X so

$$X = \bigcup_{g \in G} gD$$

and each gD is called a *chamber* of G on X.

Throughout the following, the fundamental domain of G acting on X will be taken as a simple polytope D. Then each $g \in G$ gives a self-homeomorphism of X

$$\phi_g \colon X \to X$$

by mapping chamber hD to $g \cdot hD$ for any $h \in G$. If two chambers gD and hD have a nonempty intersection which includes some facets of gD and hD, then there is a homeomorphism $\phi_{hg^{-1}}$ that maps gD to hD. Hence, for two facets F and F' from gD and hD, respectively, that are glued together in X, naturally we can assign hg^{-1} and gh^{-1} to F and F', respectively. This means that the action of G on X gives a *characteristic map* on the facets set of D,

$$\lambda \colon \mathcal{F}(D) \to G.$$

For each facet F of D, $\lambda(F) \in G$ is called a *coloring* on F. Each $\lambda(F) \in G$ naturally determines a self-homeomorphism $\phi_{\lambda(F)} \in \mathrm{Homeo}(X)$, which is called an *adjacency transformation* on X with respect to F. Such $\phi_{\lambda(F)}$ maps each chamber into an adjacent chamber such that the facet F is contained in the intersection of those two chambers. Each adjacency transformation has an inverse adjacency transformation corresponding to a facet F' of D. Of course, $F = F'$ is allowed. In this case, we call F a *mirror* of X associated with G, and the corresponding adjacency transformation is called a *reflection* of X with respect to F.

Remark 2.12 It should be pointed out that two adjacency transformations determined by different facets of D are viewed as being different, although they may correspond to the same self-homeomorphism of X. The inverse adjacency transformation of an adjacency transformation determined by a facet F is exactly determined by another facet F' which is identified with F in X.

All inverse adjacency transformations give an equivalence relation $\sim$ on $\mathscr{F}(D)\times G$, where $(F,g)\sim(F',h)$ if and only if

$$\lambda(F)\cdot g=h,\quad \lambda(F')\cdot h=g. \tag{2-3}$$

In other words, if two chambers gD and hD are attached together by identifying a facet F of gD with a facet F' of hD in X, then $\lambda(F)\cdot\lambda(F')=1$, which gives a *pair relation* for G. When F is a mirror, the pair relation is $\lambda(F)\cdot\lambda(F)=1$.

Remark 2.13 The equivalence relation $\sim$ on $\mathscr{F}(D)\times G$ gives an equivalence relation $\sim'$ on $\mathscr{F}(D)$ via the projection $\mathscr{F}(D)\times G\to\mathscr{F}(D)$ as follows: for $F,F'\in\mathscr{F}(D)$,

$$F\sim' F' \iff (F,g)\sim(F',h) \text{ for } g,h\in G \text{ satisfying relation (2-3).}$$

Thus, we can obtain a quotient orbifold $D/{\sim'}$ by attaching some facets on the boundary of D via the equivalence relation $\sim'$ on $\mathscr{F}(D)$.

On the contrary, giving a simple polytope D and a characteristic map satisfying relation (2-3), we can construct a space X with G-action by

$$X=D\times G/{\sim}, \tag{2-4}$$

where the equivalence relation is defined in relation (2-3).

The construction of X gives a natural polyhedral cellular decomposition of X, denoted by $\mathscr{P}(X)$. The dual complex of $\mathscr{P}(X)$ is denoted by $\mathscr{C}(X)$. If each codimension-k face of D in X intersects with exactly 2^k chambers, then each cell of $\mathscr{C}(X)$ is a cube, which is exactly one induced by the standard cubical decomposition of the simple polytope D. Furthermore, if $\mathscr{C}(X)$ is a cubical complex, then the link of each vertex in $\mathscr{C}(X)$ is a simplicial complex which is exactly the boundary complex of the dual of D. The 1-skeleton of $\mathscr{C}(X)$ is exactly the Cayley graph of G with generator set consisting of adjacency transformations determined by all facets of D. Therefore, one has that:

Lemma 2.14 [59, page 160] *The group G is generated by all adjacency transformations.*

To simplify notation, let $\lambda(F_i)=s_i$ or s_{F_i} for each $F_i\in\mathscr{F}(D)$. Then for each $g\in G$, ϕ_g can be decomposed into the composition of some adjacency transformations,

$$g=s_{i_1}s_{i_2}\cdots s_{i_k}.$$

The relations with $s_{i_1}s_{i_2}\cdots s_{i_k}=1$, except pair relations, are called *Poincaré relations*.

Lemma 2.15 [59, page 161] *The Poincaré relations together with the pair relations form a set of relations of the group* G.

For each codimension-2 face of D, there is a Poincaré relation with form $s_k s_{k-1} \cdots s_1 = 1$ (alternatively, $s'_1 \cdots s'_k = 1$, where $s'_i = (s_i)^{-1}$ for each i).

Define a group G_D with generators consisting of all adjacency transformations determined by $\mathscr{F}(D)$ and relations formed by all pair relations and Poincaré relations determined by all codimension 2 faces in D:

$$G_D = \langle s_i \text{ for } F_i \in \mathscr{F}(D) \mid s_i s_j = 1 \text{ for } F_i \sim' F_j ; s_{i_1} s_{i_2} \cdots s_{i_k} = 1 \text{ for each codim-2 face in } D \rangle \tag{2-5}$$

For the sake of preciseness, suppose again that each codimension-k face of D in X intersects with exactly 2^k chambers. Then the cubical subdivision of D induces a right-angled Coxeter cellular decomposition for the quotient orbifold X/G. It is not difficult to see that $D/{\sim'}$ is isomorphic to X/G as orbifolds. According to Proposition 2.9, G_D is isomorphic to the orbifold fundamental group of the quotient space X/G. Therefore, we have the following lemma.

Lemma 2.16 *The orbifold fundamental group of* $D/{\sim'} \cong X/G$ *is isomorphic to* G_D.

There is a natural quotient map $\lambda_* \colon G_D \to G$, and the image of λ_* on each adjacency transformation s_F is the coloring on corresponding facet F. Then the fundamental group of X is isomorphic to the kernel of λ_*.

Proposition 2.17 *Let* G *be a discrete group which acts properly discontinuously on a manifold* X. *Suppose* X *is decomposed into* $X = \bigcup_{g \in G} gD = D \times G/{\sim}$, *where* D *is a simple polytope and each codimension-*k *face of* D *in* X *intersects with exactly* 2^k *chambers. Let* G_D *be the group defined as in expression* (2-5), *and* λ_* *be the quotient map from* G_D *to* G *induced by the characteristic map* $\lambda \colon \mathscr{F}(D) \to G$. *Then there is a short exact group sequence*

$$1 \to \pi_1(X) \to G_D \xrightarrow{\lambda_*} G \to 1$$

which is induced by the orbifold covering $\pi \colon X \to X/G$.

Proof We refer to Chen [12, pages 40–49]. Here it is only necessary to show that $G_D \cong \pi_1^{\mathrm{orb}}(X/G)$, which is exactly Lemma 2.16. □

Given a simple convex polytope D and a discrete group G, assume that there exists a characteristic map $\lambda \colon \mathscr{F}(D) \to G$ such that $X = D \times G/{\sim}$ is a G-manifold, where $(F, g) \sim (F', h)$ for any $F, F' \in \mathscr{F}(D)$ and $g, h \in G$ if and only if relation (2-3) holds. Then we have the following result.

Corollary 2.18 *Under the assumption of Proposition 2.17,* X *is simply connected if and only if* $G \cong G_D$.

Example 2.19 Let $\square$ be a square with faces F_1, F_2, F_3 and F_4 colored by e_1, e_2, e_1 and e_2 respectively, where $e_1 = (-1, 1)$ and $e_2 = (1, -1)$ are generators of $(\mathbb{Z}_2)^2$. Then $X = \square \times (\mathbb{Z}_2)^2/{\sim} \cong T^2$ is a small cover over $\square$ (see [22]), and

$$G_\square = \langle s_1, s_2, s_3, s_4 \mid s_i^2 = 1; (s_1 s_2)^2 = (s_2 s_3)^2 = (s_3 s_4)^2 = (s_4 s_1)^2 = 1 \rangle \cong (\mathbb{Z}_2 * \mathbb{Z}_2) \oplus (\mathbb{Z}_2 * \mathbb{Z}_2)$$

is the right-angled Coxeter group determined by $\square$. Then $\pi_1(X) \cong \ker \lambda_* = \mathbb{Z}^2$ is a normal subgroup of $G_\square$ generated by Poincaré relations $s_1 s_3$ and $s_2 s_4$.

2.5 Right-angled Coxeter group and HNN-extension

In this subsection, we refer to [18, Chapter 3] and [44, Chapter 4].

Let $w = s_1 s_2 \cdots s_m$ be a word in a right-angled Coxeter group $W = \langle S \mid R \rangle$. An *elementary operation* on w is one of the following two types of operations:

(i) **Length-reducing** Delete a subword of ss.

(ii) **Braid (commutation)** Replace a subword of the form st with ts if $(st)^2 = 1$ in the relations set R of W.

A word is *reduced* if it cannot be shorten by a sequence of elementary operations.

Theorem 2.20 (Tits [18, Theorem 3.4.2]) *Two reduced words x and y are the same in a right-angled Coxeter group if and only if one of both x and y can be transformed into the other one by a sequence of elementary operations of type (ii).*

Definition 2.21 (Higman–Neumann–Neumann extension [44, page 179]) Let G be a group with presentation $G = \langle S \mid R \rangle$, and let $\phi \colon A \to B$ be an isomorphism between two subgroups of G. Let t be a new symbol out of S. Then the HNN-extension of G relative to ϕ is defined as

$$G *_\phi = \langle S, t \mid R, t^{-1} g t = \phi(g), g \in A \rangle.$$

Let $\omega = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \cdots g_{n-1} t^{\epsilon_n} g_n$ $(n \geq 0)$ be an expression in $G *_\phi$, where each g_i is an element in G (probably g_i may be taken as the unit element 1 in G), and ϵ_i is either number 1 or -1. Then ω is said to be *t-reduced* if there is no consecutive subword $t^{-1} g_i t$ or $t g_j t^{-1}$ with $g_i \in A$ and $g_j \in B$.

A *normal form* of an element in $G *_\phi$ is a word $\omega = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \cdots g_{n-1} t^{\epsilon_n} g_n$ $(n \geq 0)$ where

(i) g_0 is an arbitrary element of G;

(ii) if $\epsilon_i = -1$, then g_i is a representative of a coset of A in G;

(iii) if $\epsilon_i = +1$, then g_i is a representative of a coset of B in G;

(iv) there is no consecutive subword $t^\epsilon 1 t^{-\epsilon}$.

Theorem 2.22 (the normal form theorem for HNN-extensions [44, Theorem 2.1, page 182]) *Let* $G*_{\phi} = \langle G, t \mid t^{-1}gt = \phi(g), g \in A\rangle$ *be an HNN-extension. Then there are two equivalent statements*:

(I) *The group* G *is embedded in* $G*_{\phi}$ *by the map* $g \mapsto g$. *If* $\omega = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n = 1$ *in* $G*_{\phi}$, *then* ω *is not reduced.*

(II) *Every element* ω *of* $G*_{\phi}$ *has a unique representation* $\omega = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$ *which is a normal form.*

A *t-reduction* of $\omega = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$ is one of the following two operations:

- replace a subword of the form $t^{-1}gt$, where $g \in A$, by $\phi(g)$;
- replace a subword of the form tgt^{-1}, where $g \in B$, by $\phi^{-1}(g)$.

A finite number of t-reductions leads from $\omega = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$ to a normal form.

3 Flagness and beltness of simple handlebodies

3.1 B-belts and flagness

Assume that Q is a simple n-handlebody with nerve $\mathcal{N}(Q)$. Denote by Q^* the *dual* of Q, whose facial structure is given by $\mathcal{N}(Q)$.

Definition 3.1 (B-belts) Let $i: B \hookrightarrow Q$ be an embedding closed simple k-suborbifold whose underlying space is a k-ball. We say that $i(B)$ is a *B-belt* of Q if

- i preserves codimensions, i.e. i maps each codimension-d face f of B to a codimension-d face F_f of Q;
- the intersection $\bigcap f_\alpha = \varnothing$ for some facets f_α in B if and only if either $\bigcap F_{f_\alpha} = \varnothing$ or $\bigcup F_{f_\alpha}$ cannot deformatively retract onto B in $|Q|$.

Remark 3.2 The orbifold embedding $i: B \hookrightarrow Q$ preserving codimension is equivalent to that i restricting on the local group of each point in B induces an identity. The statement that $\bigcup F_{f_\alpha}$ cannot deformatively retract onto B in $|Q|$ is equivalent to that there is at least a hole in the area surrounded by $\{F_{f_\alpha}\}$ and B.

A simple polytope P itself is a P-belt. For a B-belt in a simple polytope P, the intersection $\bigcap f_\alpha = \varnothing$ for some facets f_α in B if and only if $\bigcap F_{f_\alpha} = \varnothing$. And each B-belt is *π-injective* in the sense of Lemma 7.1, which is an analogue of π_1-injective surfaces in 3-dimensional manifolds.

A 2-dimensional B-belt in a simple 3-handlebody Q is a k-gon. Traditionally, such a B-belt is also called a *k-belt* of Q. In the case of dimension three, any simple 3-polytope except the tetrahedron has a 2-dimensional B-belt.

Next, we want to generalize the definition of flagness to simple handlebodies in terms of B-belts defined above. Recall that a simplicial complex K with vertex set V is a *flag* complex if every finite subset of V, which is pairwise joined by edges, spans a simplex. Let X be a cubical complex equipped with a piecewise Euclidean structure. Then Gromov's lemma (see [32]) tells us that X is nonpositively curved if and only if the link of each vertex in X is a flag simplicial complex. Furthermore, by the Cartan–Hadamard theorem, a nonpositively curved space is aspherical.

A simple polytope P is *flag* if the boundary complex of its dual is a flag simplicial complex. Let $M \to P$ be a small cover or a real moment angled manifold over P. Then we know from [24, Theorem 2.2.5] that M is aspherical if and only if P is flag. Equivalently, P, as a right-angled Coxeter orbifold, is orbifold-aspherical if and only if it is flag.

Naturally, the flagness of a simple handlebody Q should still be closely related to the orbifold-asphericity of Q. The right-angled Coxeter orbifold structure of Q induces a facial structure of Q which can be carried by the nerve $\mathcal{N}(Q)$ of Q as a manifold with corners. The combinatorial obstruction of orbifold-asphericity of Q contains some quotient orbifolds of S^k for $k \geq 2$ by reflective actions of $(\mathbb{Z}_2)^l$ for $1 \leq l \leq k+1$. Moreover, since Q is a simple handlebody, $S^k/(\mathbb{Z}_2)^{k+1} \cong \Delta^k$ is the unique possible combinatorial obstruction of orbifold-asphericity of Q. Notice that the orbifold-asphericity of Q is determined by both $\mathcal{N}(Q)$ and $|Q|$.

Definition 3.3 A simple handlebody Q is said to be *flag* if it contains no $\triangle^k$-belt for any $k \geq 2$.

Remark 3.4 The notion of B-belt and flagness can be generalized to a right-angled Coxeter orbifold whose underlying space is an arbitrary compact manifold with nonempty boundary. We call such an orbifold a *simple orbifold* if it satisfies conditions (b) and (c) in the definition of simple handlebody. A natural conjecture arises as follows:

Conjecture 3.5 *A simple orbifold Q is orbifold-aspherical if and only if its underlying space $|Q|$ is aspherical as a manifold and Q contains no Δ^k-belt for any $k \geq 2$.*

Davis' results in [18, Theorem 9.1.4] and [20, Theorem 3.5] tell us that the conjecture is true in the cases where $|Q|$ is acyclic or Q has a corner structure defined in [20, Section 3.1]. Our Theorem A also proves the case that Q is a simple handlebody. All of these support the conjecture.

A simple handlebody Q is a simple polytope or there exist finitely many disjoint B-belts of codimension one, called *cutting belts*, such that Q can be cut open into a simple polytope P_Q along those cutting belts. Here the cutting operation is similar to a hierarchy of Haken manifolds (or Haken orbifolds). Note that a simple handlebody is not a Haken orbifold except that it is flag. Refer to [28; 60] for Haken 3-manifolds and generalized Haken manifolds.

Let Q be a simple handlebody. We see that some vertices $F_1, F_2, \dots, F_k$ of $\mathcal{N}(Q)$ span a simplex $\triangle^{k-1}$ in $\mathcal{N}(Q)$ if and only if the associated vertices span a simplex in $\mathcal{N}(P_Q)$, and they span an *empty simplex* (that is, $\partial\triangle^{k-1} \subset \mathcal{N}(Q)$ but $\triangle^{k-1}$ itself is not in $\mathcal{N}(Q)$) whose interior is contained in the interior of Q^*

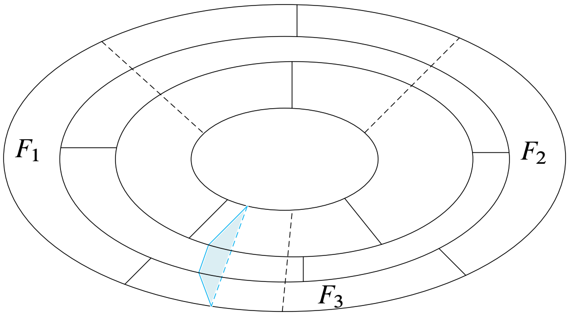


Figure 5: A flag simple 3-handlebody whose nerve is not a flag simplicial complex.

if and only if associated vertices span an empty simplex in $\mathcal{N}(P_Q)$. Specifically, those empty simplices correspond to some $\triangle^k$-belts in Q. Hence, we have the following result.

Lemma 3.6 *A simple handlebody Q is flag if and only if the associated simple polytope P_Q is flag* (*in other words*, $\mathcal{N}(P_Q)$ *is a flag simplicial complex*).

Remark 3.7 Notice that a flag simple handlebody defined above may contain an empty simplex whose interior cannot be embedded in its dual Q^*, as shown in Figure 5 for three pairwise intersected faces F_1, F_2 and F_3 in a flag simple solid torus. Therefore, the statement that $\mathcal{N}(Q)$ is a flag simplicial complex is not equivalent to that Q is a flag simple handlebody.

3.2 $\square$-belts in a simple handlebody

Definition 3.8 A $\square$-belt in a simple handlebody Q is a B-belt where B is a two-disk with a square boundary.

Remark 3.9 (1) In Gromov's paper [32, Section 4.2], *Siebenmann's no $\square$-condition* for a flag simplicial complex K means no *empty square* in K, where an empty square in K must make sure that neither pair of opposite vertices is connected by an edge, which is a special case in our definition.

(2) A prismatic 3-circuit (see [53]) in a simple 3-polytope P^3 determines a Δ^2-belt in P^3. If there is no prismatic 3-circuit in P^3, then P^3 is a flag polytope or a tetrahedron. Similarly for a prismatic 4-circuit (see [53]) in a flag simple 3-polytope, it determines a $\square$-belt in P in our definition.

Next, we give two lemmas as the preliminary of the proof of Theorem B.

Let Q be a simple handlebody, and $B_\square$ be a $\square$-belt in Q with four ordered edges f_1, f_2, f_3 and f_4, any two of which have a nonempty intersection except for pairs $\{f_1, f_3\}$ and $\{f_2, f_4\}$. Assume that each f_i is contained in a facet F_i of Q. Then we may claim that $\{F_i \mid i = 1, 2, 3, 4\}$ must be different from each other. More precisely, we have the following lemma.

Lemma 3.10 *Let Q be a simple handlebody, and $B_\square$ be a $\square$-belt in Q. Then*

- *two adjacent edges of $B_\square$ cannot be contained in the same facet of Q;*
- *two disjoint edges of $B_\square$ cannot be contained in the same facet of Q.*

Proof Assume that the edges $\{f_1, f_2, f_3, f_4\}$ of $B_\square$ are contained in four ordered facets $\{F_1, F_2, F_3, F_4\}$ of Q, respectively. If there are two adjacent edges of $B_\square$ contained in the same facet of Q — without loss of generality, suppose that $F_1 = F_2$ — then $f_1 \cap f_2 \neq \varnothing$ implies that F_1 has a self-intersection, which is equivalent to there being a 1-simplex which bounds a single vertex in $\mathcal{N}(Q)$. This contradicts that Q is simple.

Similarly, if there are two disjoint edges of $B_\square$ contained in the same facet of Q, then one can assume that $F_1 = F_3$. This happens only for the case where the genus of Q is more than zero since $B_\square$ is a $\square$-belt in Q. Thus there are some holes between F_1 and $B_\square$. However, F_2 is contractible, so this induces that $F_2 \cap F_1$ is disconnected. In other words, there are two 1-simplices which bound the same two vertices in $\mathcal{N}(Q)$. This is also impossible since Q is simple. □

Lemma 3.10 tells us that in a simple handlebody Q, a $\square$-belt can be presented as four different vertices $\{F_1, F_2, F_3, F_4\}$ in $\mathcal{N}(Q)$, which satisfies the following two conditions:

(I) $\{F_1, F_2, F_3, F_4\}$ bounds a square with its interior located in the interior of Q^* and with its edges contained in the 1-skeleton of $\mathcal{N}(Q)$.

(II) The full subcomplex spanned by $\{F_1, F_2, F_3, F_4\}$ in $\mathcal{N}(Q)$ is either a square or a nonsquare subcomplex (containing two 2-simplices gluing along an edge). Here the latter "a nonsquare subcomplex" may happen only when the genus of Q is more than zero.

Example 3.11 (squares in the dual of a simple handlebody) Let Q be a simple handlebody, and Q^* be its dual. There are some possible cases of squares and nonsquares in Q^*, listed in Figure 6, where all vertices and edges are considered in $\mathcal{N}(Q)$. Diagrams (a) and (b) are not squares in Q^*, while (c) and (d) are. Notice that (d) is not an empty square in $\mathcal{N}(Q)$, which is different from the case of Siebenmann's no $\square$-condition, as stated in Remark 3.9(1).

Lemma 3.12 *Let $B_\square$ be a $\square$-belt in a simple n-handlebody Q, and B be a cutting belt of Q. Then either $B_\square$ and B can be separated in Q, or B intersects transversely with only a pair of disjoint edges of $B_\square$.*

Proof Assume that the four ordered edges f_1, f_2, f_3 and f_4 of $B_\square$ are contained in four facets F_1, F_2, F_3 and F_4 of Q, respectively. Since $B_\square$ and B are contractible, we see that B and $B_\square$ can be separated if and only if their boundaries can be separated.

First we assume that ∂B and $\partial B_\square$ intersect transversely, meaning that $\partial B \cap \partial B_\square$ is a set of isolated points cyclically ordered on the boundary of $B_\square$, which is denoted by $\mathcal{V}$. Then $\mathcal{V}$ contains at least two points if $\mathcal{V}$ is nonempty.

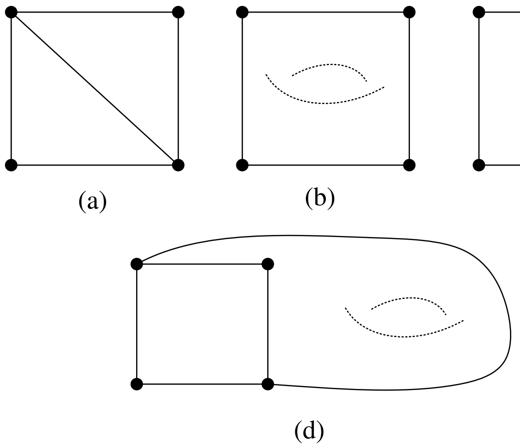


Figure 6: Squares and nonsquares.

Let v and v' be two adjacent points in $\mathcal{V}$. Then there are the following cases:

(i) v and v' are located in the same edge of $B_\square$;

(ii) v and v' are located in two adjacent edges of $B_\square$;

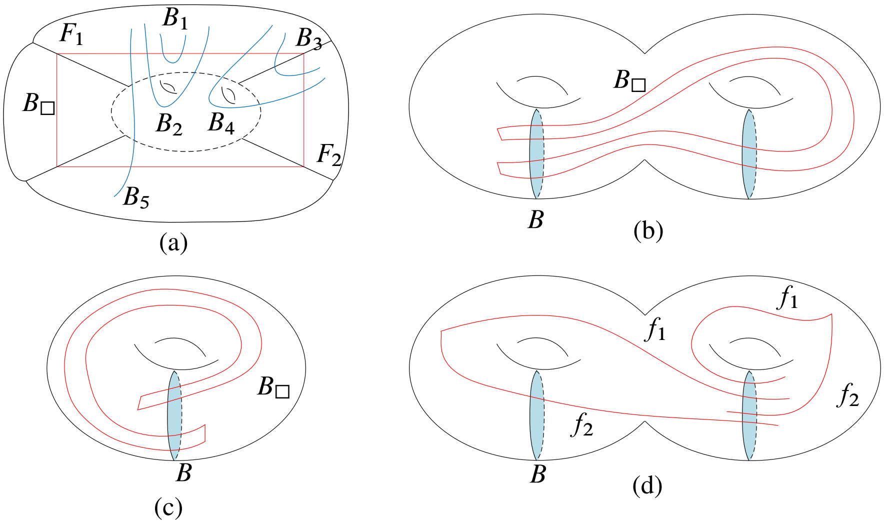
(iii) v and v' are located in two disjoint edges of $B_\square$.

In the case (i), without loss of generality, suppose that $v, v' \in \operatorname{int}(f_1)$. Now if v and v' are contained in the same connected component of $F_1 \cap B$ (without a loss of generality, assume that B is regarded as B_1 of (a) in Figure 7), then we can deform the interior of f_1 such that $f_1 \cap \partial B = \varnothing$ will not contain v and v'. If v and v' are contained in two connected components of $F_1 \cap B$, without loss of generality, assume that B is regarded as B_2 of (a) in Figure 7. Since B is a B-belt, there is a hole surrounded by f_1 and B. This case is allowed (also see (b) and (c) in Figure 7).

In the case (ii), without loss of generality, assume that B intersects with f_1 and f_2. Now if $B \cap F_1 \cap F_2 \neq \varnothing$ (regard B as B_3 of (a) in Figure 7), then we can move vertex $f_1 \cap f_2$ in $F_1 \cap F_2$ such that $\partial B_\square \cap \partial B$ does not contain v and v'.

Repeating this operation, we can assume that any two adjacent points v and v' in $\mathcal{V}$ cannot remove. This means that $B \cap F_1 \cap F_2 = \varnothing$ in the case (ii), so we may regard B as B_4 of (a) in Figure 7. Then by the definition of B-belt, there is a hole in the area surrounded by B, f_1 and f_2 (see (d) in Figure 7). If $|\mathcal{V}| = 2$, then $B_\square$ will not be contractible. This is a contradiction. If $|\mathcal{V}| > 2$, let v'' be a point after v' by the cyclic order of all isolated points in $\mathcal{V}$. If v' and v'' belong to the same edge f of $B_\square$, then there must be a hole surrounded by f and B. If v' and v'' belong to two adjacent edges f' and f'' of $B_\square$, then there is also a hole surrounded by f', f'' and B. If v' and v'' belong to two disjoint edges f' and f'' of $B_\square$, then there is still a hole surrounded by f, f'' and B, where f is the edge containing v. Whichever of all possible cases above happens implies that $\partial B_\square$ is not contractible in $|Q|$, but this is impossible.

The case (iii) is allowed; see B_5 of (a) in Figure 7. So the conclusion holds. □



(a) (b) (c) (d)

Figure 7: $\square$-belt and cutting belt.

We see that if there are some cutting belts that intersect with $B_\square$, then one can do some deformations such that those cutting belts either do not intersect with $B_\square$ or intersect transversely with only a pair of disjoint edges of $B_\square$.

4 The orbifold fundamental groups of simple handlebodies

4.1 The right-angled Coxeter cellular decomposition of simple handlebodies

Let Q be a simple n-handlebody of genus $\mathfrak{g}$ with facet set $\mathcal{F}(Q)=\{F_1,\dots,F_m\}$. Then we can cut Q into a simple polytope P_Q along $\mathfrak{g}$ cutting belts $B_1,\dots,B_{\mathfrak{g}}$, each of which intersects transversely with some facets of Q and is a simple $(n-1)$-polytope. Two copies of B_i in P_Q, denoted by B_i^+ and B_i^-, are two disjoint facets of P_Q. Since they share the common belt B_i in Q, by $B_i^+\sim B_i^-$ we denote this share between them. The number of facets of P_Q around B_i^+ is the same as the number of facets of P_Q around B_i^-. In addition, each facet F of P_Q around B_i^+ also uniquely corresponds to a facet F' of P_Q around B_i^- such that F and F' share a common facet in Q, so by $F\cap B_i^+\sim F'\cap B_i^-$ we mean this share between F and F' via the belt B_i of Q.

Let $\mathcal{F}(P_Q)$ denote the set of all facets in P_Q and $\mathcal{F}_B$ denote the set of those facets in P_Q produced by cutting belts of Q, so $\mathcal{F}_B$ contains $2\mathfrak{g}$ facets of P_Q, appearing in pairs.

P_Q is viewed as a right-angled Coxeter orbifold with boundary consisting of all facets in $\mathcal{F}_B$. By attaching all pairs $B^+\sim B^-$ in $\mathcal{F}_B$ and all corresponding pairs (F,F') with $F\cap B^+\sim F'\cap B^-$ together, we can recover Q from P_Q. Thus Q can be regarded as a quotient $P_Q/{\sim}$, and we denote the quotient map by

$$q\colon P_Q\to Q. \tag{4-1}$$

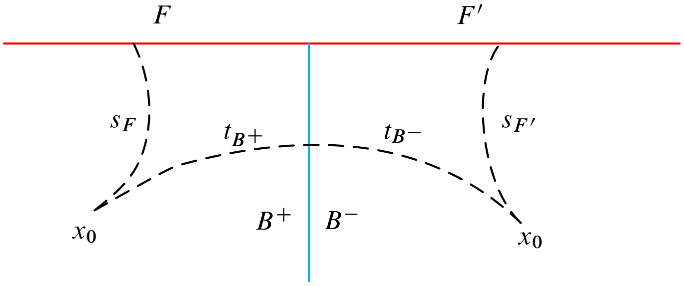


Figure 8: The right-angled Coxeter 2-cell nearby B-belt.

There is a canonical right-angled Coxeter cubical cellular decomposition $\mathscr{C}(P_Q)$ of P_Q, whose cells consist of

- all cubes in the standard cubical decomposition of P_Q;
- all cubes in the standard cubical decomposition of all boundary components of P_Q in $\mathscr{F}_B$.

Moreover, $\mathscr{C}(P_Q)$ induces a right-angled Coxeter cellular decomposition on Q by attaching some cubical cells of the copies of B-belts. Let c be a k-cube in $\mathscr{C}(P_Q)$ and $B \in \mathscr{F}_B$.

- If $c \cap B = \varnothing$, then we may take c as a right-angled Coxeter cubical cell for Q. Such a c corresponds to a codimension-k face in P_Q which is determined by k facets in $\mathscr{F}(P_Q) - \mathscr{F}_B$, so c is of the form $e^k/(\mathbb{Z}_2)^k$.
- If c is a k-cube in $\mathscr{C}(B^+) \subset \mathscr{C}(P_Q)$, then there is also another k-cube $c' \in \mathscr{C}(B^-) \subset \mathscr{C}(P_Q)$. Both c and c' are codimension-one faces of two $(k+1)$-cubes in $\mathscr{C}(P_Q)$. Gluing those two $(k+1)$-cubes by identifying c with c', we obtain a right-angled Coxeter cubical cell with form $e^{k+1}/(\mathbb{Z}_2)^k$.

Finally, we obtain a right-angled Coxeter cellular decomposition of Q, denoted by $\mathscr{C}(Q)$, whose cells are right-angled Coxeter cubes. Of particular note is that $\mathscr{C}(Q)$ is not cubical. This is because there exists the cubical cell glued by two cells c and c' in $\mathscr{C}(P_Q)$ as above, which has a self-intersection, namely the cone point x_0, as shown in Figure 8. The cone point is the only 0-cell in $\mathscr{C}(Q)$, which will be chosen as the basepoint when we calculate the orbifold fundamental group of Q.

4.2 The orbifold fundamental groups of simple handlebodies

Following the above notations, by Proposition 2.9, we can directly write out a presentation of orbifold fundamental group of Q.

Proposition 4.1 *Let Q be a simple handlebody of genus $\mathfrak{g}$, and P_Q be the associated simple polytope with copies of cutting belts $\mathscr{F}_B$. Then $\pi_1^{\mathrm{orb}}(Q)$ has a presentation with generators s_F indexed by $F \in \mathscr{F}(P_Q)$, satisfying the relations*

(1) $s_F^2 = 1$ *for* $F \in \mathscr{F}(P_Q) - \mathscr{F}_B$,

(2) $t_{B^+} t_{B^-} = 1$ *for two* B^+ *and* B^- *with* $B^+ \sim B^-$ *in* $\mathscr{F}_B$,

(3) $(s_F s_{F'})^2 = 1$ *for* $F, F' \in \mathscr{F}(P_Q) - \mathscr{F}_B$ *with* $F \cap F' \neq \varnothing$,

(4) $s_F t_{B^+} = t_{B^+} s_{F'}$ *for* $B^+ \sim B^-$ *in* $\mathscr{F}_B$ *and* $F, F' \in \mathscr{F}(P_Q) - \mathscr{F}_B$ *with* $F \cap B^+ \sim F' \cap B^-$,

where the basepoint of $\pi_1^{\text{orb}}(Q)$ *is the cone point* x_0 *in the interior of* Q.

On the other hand, we show here that $\pi_1^{\text{orb}}(Q)$ is actually an iterative HNN-extension on $W(P_Q, \mathscr{F}_B)$, where $W(P_Q, \mathscr{F}_B)$ is a right-angled Coxeter group determined by facial structure of P_Q by ignoring the facets in $\mathscr{F}_B$:

$$W(P_Q, \mathscr{F}_B) = \langle s_F,\ F \in \mathscr{F}(P_Q) - \mathscr{F}_B \mid s_F^2 = 1 \text{ for all } F;\ (s_F s_{F'})^2 = 1 \text{ for } F \cap F' \neq \varnothing \rangle,$$

which can be regarded as the orbifold fundamental group of P_Q as a right-angled Coxeter orbifold with boundary consisting of the disjoint union of all facets in $\mathscr{F}_B$.

Let B be a cutting belt in Q, and $B^+, B^- \in \mathscr{F}_B$ be two copies of B. Set

$$\mathscr{F}^{B^+} = \{F \in \mathscr{F}(P_Q) - \mathscr{F}_B \mid F \cap B^+ \neq \varnothing\},$$
$$\mathscr{F}^{B^-} = \{F \in \mathscr{F}(P_Q) - \mathscr{F}_B \mid F \cap B^- \neq \varnothing\}.$$

The associated right-angled Coxeter groups W_{B^+} and W_{B^-} are isomorphic since B^+ and B^- are combinatorially equivalent as simple polytopes.

Lemma 4.2 *The maps* $i_{B^+} \colon W_{B^+} \to W(P_Q, \mathscr{F}_B)$ *and* $i_{B^-} \colon W_{B^-} \to W(P_Q, \mathscr{F}_B)$ *induced by inclusions* $B^+ \hookrightarrow P_Q$ *and* $B^- \hookrightarrow P_Q$ *are monomorphisms.*

Proof According to the definition of B-belt, i_{B^+} and i_{B^-} are obviously well defined. There are two group homomorphisms $j_{B^+} \colon W(P_Q, \mathscr{F}_B) \to W_{B^+}$ and $j_{B^-} \colon W(P_Q, \mathscr{F}_B) \to W_{B^-}$, defined by reducing modulo the normal subgroups generated by facets being not in $\mathscr{F}^{B^+}$ and $\mathscr{F}^{B^-}$, such that $j_{B^+} \circ i_{B^+} = \mathrm{id}_{W_{B^+}}$ and $j_{B^-} \circ i_{B^-} = \mathrm{id}_{W_{B^-}}$. The result follows from this. □

Thus, W_{B^+} and W_{B^-} can also be regarded as two isomorphic subgroups of $W(P_Q, \mathscr{F}_B)$ generated by $\{s_F,\ F \in \mathscr{F}^{B^+}\}$ and $\{s_{F'},\ F' \in \mathscr{F}^{B^-}\}$, respectively. Define $\phi_B \colon W_{B^-} \to W_{B^+}$ by $\phi_B(s_{F'}) = s_F$ with $F' \cap B^- \sim F \cap B^+$. Then ϕ_B is a well-defined isomorphism. Furthermore, attaching two facets on P_Q corresponding to the cutting belt B is equivalent to doing one HNN-extension on its orbifold fundamental group $\pi_1^{\text{orb}}(P_Q) = W(P_Q, \mathscr{F}_B)$, giving new elements t_{B^+} and t_{B^-} with certain conditions in $\pi_1^{\text{orb}}(Q)$. By doing an induction on the genus of Q and repeating the use of HNN-extension, the orbifold fundamental group of Q is isomorphic to doing $\mathfrak{g}$ HNN-extensions on the right-angled Coxeter group $W(P_Q, \mathscr{F}_B)$,

$$(Q_{\mathfrak{g}}, B_{\mathfrak{g}}) \longrightarrow \cdots \longrightarrow (Q_1, B_1) \longrightarrow Q_0 = P_Q$$

$$\xrightarrow{\text{cutting}}$$
$$\xleftarrow{\text{HNN-extension}}$$

$$G_{\mathfrak{g}} = \pi_1^{\text{orb}}(Q) \longleftarrow \cdots \longleftarrow G_1 \longleftarrow G_0 = W(P_Q, \mathscr{F}_B)$$

where each Q_k is the simple handlebody of genus k obtained from Q_{k+1} by cutting open along the $(k+1)^{\text{st}}$ belt B_{k+1}, which is a right-angled Coxeter orbifold with boundary consisting of double copies of $\{B_{k+1},\dots,B_{\mathfrak{g}}\}$, and each G_k is the orbifold fundamental group of Q_k which is obtained from an HNN-extension on G_{k-1}.

Proposition 4.3 *Let Q be a simple handlebody of genus $\mathfrak{g}$ with cutting belts $B_1,\dots,B_{\mathfrak{g}}$. Then*

$$\pi_1^{\mathrm{orb}}(Q)\cong(\cdots((W(P_Q,\mathscr{F}_B)*_{\phi_{B_1}})*_{\phi_{B_2}})\cdots)*_{\phi_{B_{\mathfrak{g}}}}.$$

Notice that the expression $(\cdots((W(P_Q,\mathscr{F}_B)*_{\phi_{B_1}})*_{\phi_{B_2}})\cdots)*_{\phi_{B_{\mathfrak{g}}}}$ in Proposition 4.3 is independent of orders of ϕ_{B_i}. In addition, the presentation of $\pi_1^{\mathrm{orb}}(Q)$ in Proposition 4.1 can be simplified by deleting all generators t_{B^-} and relations $t_{B^+}t_{B^-}=1$, replaced by only all t_B. It should be pointed out that the right-angled Coxeter group W_Q determined by the facial structure of Q is not a subgroup of $\pi_1^{\mathrm{orb}}(Q)$ in general. Actually, W_Q is the quotient group of $\pi_1^{\mathrm{orb}}(Q)$ with respect to the normal group generated by all t_B.

Remark 4.4 In [25, Theorem 4.7.2], Davis, Januszkiewicz and Scott give a similar form. However, all generators in their paper lifted into the universal space as homeomorphisms onto itself are involutions, i.e. $t_B^2=1$. Here, with a little difference, we require that the lifted action of t_B is free. In particular, the last relation in Proposition 4.1 belongs to a kind of *Baumslag–Solitar relations*, which are related to the HNN-extension. In other words, pasting pairs of facets corresponding to cutting belts of the polytope P_Q can be viewed as a topological explanation for the HNN-extension of their orbifold fundamental groups. More precisely, for a cutting belt B, there are two copies B^+ and B^- in P_Q, and the composite map

$$W_B\cong W_{B^+}\xrightarrow{i_{B^+}}W(P_Q,\mathscr{F}_B)\xrightarrow{i_1}G_1\xrightarrow{i_2}\cdots\xrightarrow{i_g}G_{\mathfrak{g}}=\pi_1^{\mathrm{orb}}(Q)$$

embeds W_B into $\pi_1^{\mathrm{orb}}(Q)$, where i_k is defined by $i_k(h)=h\in G_k$ for $h\in G_{k-1}$. Both W_{B^+} and W_{B^-} are linked in $\pi_1^{\mathrm{orb}}(Q)$ by an isomorphism and the injectivity of i_k is followed by the normal form theorem of HNN-extension (see Theorem 2.22).

4.3 The orbifold universal covers of simple handlebodies

Let Q be a simple handlebody with cutting belts $\{B_1,\dots,B_{\mathfrak{g}}\}$, and P_Q be the simple polytope given by cutting Q with the quotient map $q\colon P_Q\to Q$. Let $\pi_1^{\mathrm{orb}}(Q)$ be the orbifold fundamental group with the presentation in Proposition 4.1. Define a *characteristic map* on the facet set of P_Q,

$$\lambda\colon\mathscr{F}(P_Q)\to\pi_1^{\mathrm{orb}}(Q),$$

given by $\lambda(F)=s_F$ for $F\in\mathscr{F}(P_Q)-\mathscr{F}_B$, and $\lambda(B)=t_B$ for $B\in\mathscr{F}_B$. Then we construct the space

$$\widetilde{Q}=P_Q\times\pi_1^{\mathrm{orb}}(Q)/{\sim},\tag{4-2}$$

where $(x,g)\sim(y,h)$ if and only if either

- $x=y\in F\in\mathscr{F}(P_Q)-\mathscr{F}_B$ and $gs_F=h$, or
- $(x,y)\in(B,B')$, $B,B'\in\mathscr{F}_B$, $q(x)=q(y)$ and $t_B\cdot g=h$.

The orbit space of the action of $\pi_1^{\mathrm{orb}}(Q)$ on $\widetilde{Q}$ is Q, so the polytope P_Q can be viewed as the fundamental domain of $\pi_1^{\mathrm{orb}}(Q)$ acting on $\widetilde{Q}$. According to Corollary 2.18:

Lemma 4.5 *$\widetilde{Q}$ is the universal orbifold cover of Q.*

5 Proof of Theorem A

This section is devoted to giving the proof of Theorem A.

5.1 Nonpositively curved cubical complex

A geodesic metric space X is *nonpositively curved* if it is a locally CAT(0) space. The Cartan–Hadamard theorem implies that nonpositively curved spaces are aspherical; see [7; 18; 32].

Definition 5.1 (the links in a cubical complex, see [7, Section 7.15] or [18, page 508]) Let K be a cubical complex. For each vertex $v \in K$, its (*geometric*) *link*, denoted by $\mathrm{Lk}(v)$, is a simplicial complex defined by all cubes in K that properly contain v with respect to the inclusion. A d-cube c of K that properly contains v determines a $(d-1)$-simplex $s(c)$ in $\mathrm{Lk}(v)$.

Proposition 5.2 (Gromov lemma; see [32] or [18, Corollary I.6.3]) *A piecewise Euclidean cubical complex is nonpositively curved if and only if the link of its each vertex is a flag complex.*

5.2 Homology groups of manifold covers over simple handlebodies

Let M_Q and $\widetilde{Q}$ be the manifold double and orbifold universal cover over a simple handlebody Q with m facets, respectively. In this subsection, we discuss the homology groups of M_Q and $\widetilde{Q}$.

5.2.1 Homology groups of M_Q By Davis' result (see [18, Theorem 8.12]),

$$H_*(M_Q) \cong \bigoplus_{g \in (\mathbb{Z}_2)^m} H_*(|Q|, \mathcal{F}_g) \tag{5-1}$$

where $\mathcal{F}_g = \bigcup_{s_i \in S(g)} F_i \subset \partial |Q|$, $F_i \in \mathcal{F}(Q)$ and $S(g) = \{s_i \mid l(s_i \cdot g) = l(g) - 1\}$ for a reduced word g of length $l(g)$ in $(\mathbb{Z}_2)^m$. If Q is a simple handlebody of genus $\mathfrak{g} \geq 0$, then $|Q| \simeq \bigvee_{\mathfrak{g}} S^1$. By the long exact sequence of homology groups of $(|Q|, \mathcal{F}_g)$, if $* \geq 3$, then

$$H_*(|Q|, \mathcal{F}_g) \cong H_{*-1}(\mathcal{F}_g) \cong H_{*-1}(K_g)$$

where $K_g \simeq \mathcal{F}_g$ is the dual simplicial complex of $\mathcal{F}_g$, which is a subcomplex of $\mathcal{N}(Q)$. Hence for $* \geq 3$,

$$H_*(M_Q) \cong \bigoplus_{J \subset \mathcal{F}(Q)} H_{*-1}(K_J)$$

where J is the set of those facets F_i corresponding to all $s_i \in S(g)$.

For $* = 1, 2$, we have

$$0 \to H_2(|Q|, \mathcal{F}_g) \to H_1(\mathcal{F}_g) \xrightarrow{(i_g)_*} H_1(|Q|) \cong \mathbb{Z}^{\mathfrak{g}} \to H_1(|Q|, \mathcal{F}_g) \to 0$$

where $i_g : \mathcal{F}_g \to |Q|$ is an inclusion. Then

$$H_1(M_Q) \cong \bigoplus_{g \in (\mathbb{Z}_2)^m} \operatorname{coker}(i_g)_* \quad \text{and} \quad H_2(M_Q) \cong \bigoplus_{g \in (\mathbb{Z}_2)^m} \ker(i_g)_*.$$

Remark 5.3 The formula (5-1) is actually Hochster's formula in the setting of simple handlebodies. When Q is a simple polytope P, Hochster's formula [9, Proposition 3.2.11] can also be expressed as

$$H^{l(g)-i-1}(\mathcal{F}_g) \cong \operatorname{Tor}^{-i, l(g)}_{\mathbb{Z}[v_1, \dots, v_m]}(\mathcal{K}(P), \mathbb{Z})$$

where $\mathcal{K}(P)$ is the Stanley–Reisner face ring of P.

5.2.2 Homology groups of $\widetilde{Q}$ Davis' [18, Theorem 8.12] cannot be directly applied to give the homology groups of $\widetilde{Q}$ since $\pi_1^{\mathrm{orb}}(Q)$ is not a Coxeter group when the genus of Q is more than zero. However, we can employ the method of Davis in [18, Chapter 8] to calculate of the homology groups of $\widetilde{Q}$.

Let Q be a simple handlebody with nerve $\mathcal{N}(Q)$, and P_Q be the associated simple polytope. Let $G = \pi_1^{\mathrm{orb}}(Q)$ be the orbifold fundamental group of Q. We have known that G is an iterative HNN-extension on a right-angled Coxeter group $W(P_Q, \mathcal{F}_B)$. Namely,

$$G = \pi_1^{\mathrm{orb}}(Q) \cong (\cdots((W(P_Q, \mathcal{F}_B) *_{\phi_{B_1}}) *_{\phi_{B_2}}) \cdots) *_{\phi_{B_{\mathfrak{g}}}}$$

where $\mathfrak{g}$ is the genus of Q. For any $w \in G$, consider the reduced normal form

$$w = g_0 t_1 g_1 \cdots g_{m-1} t_m g_m$$

where each g_i is reduced in $W(P_Q, \mathcal{F}_B)$, and each t_i is one of $\{t_B^{\pm 1}\}$ which determines an isomorphism of $\{\phi_B^{\pm 1}\}$ on some subgroups of $\pi_1^{\mathrm{orb}}(Q)$. Denote the generator set of G by

$$\mathcal{S} = \{s_F ; F \in \mathcal{F}(P_Q) - \mathcal{F}_B\} \cup \{t_B ; B \in \mathcal{F}_B\}.$$

For any word $w \in G$, put

$$S(w) = \{s \in \mathcal{S} \mid l(ws) < l(w)\},$$

where $l(w)$ is the word length of the reduced normal form of w in G (i.e. the shortest length between 1 and w in the Cayley graph of G associated with the generator set $\mathcal{S}$). For each subset T of $\mathcal{S}$, let P_Q^T be the subcomplex of P_Q defined by

$$P_Q^T = \bigcup_{t \in T} F_t,$$

where $F_{s_F} = F$ for $s_F \in \mathcal{F}(P_Q) - \mathcal{F}_B$ and $F_{t_B} = B'$ for $B \in \mathcal{F}_B$ with $B \sim B'$.

Let $\widetilde{Q} = P_Q \times G/\sim$ be the universal cover of Q defined as in (4-2). Then we have the following conclusion which generalizes Davis' theorem in [18, Theorem 8.12].

Proposition 5.4 *The homology of $\widetilde{Q}$ is isomorphic to the direct sum*

$$H_*(\widetilde{Q}) \cong \bigoplus_{w \in G} H_*(P_Q, P_Q^{S(w)}),$$

where $G = \pi_1^{\mathrm{orb}}(Q)$ has the presentation in Proposition 4.1.

Remark 5.5 It should be emphasized that here P_Q is not used as a mirrored space in the sense of Davis in [18] although it is a simple polytope. Actually, here we just put P_Q and $\pi_1^{\mathrm{orb}}(Q)$ together to construct the orbifold universal cover $\widetilde{Q}$, but $\pi_1^{\mathrm{orb}}(Q)$ is not a Coxeter group except that the genus of Q is zero.

Corollary 5.6 *If there is an empty k-simplex $\triangle^k$ in $\mathcal{N}(P_Q)$, then $H_k(\widetilde{Q}) \neq 0$.*

Proof Assume that the vertex set of a empty k-simplex $\triangle^k$ in $\mathcal{N}(P_Q)$ is

$$T = \{F_1, F_2, \dots, F_{k+1}\}$$

which does not contain the facet in $\mathcal{F}_B$ (in fact, any facet in $\mathcal{F}_B$ is not the vertex of any empty simplex of $\mathcal{N}(P_Q)$; this is guaranteed by the definition of B-belt). Let $w = s_1 s_2 \cdots s_{k+1}$. Regard T as $\{s_1, \dots, s_{k+1}\}$. Then $S(w) = T$. Moreover, $P_Q^{S(w)} = P_Q^T = \bigcup_{i=1}^{k+1} F_i \simeq \partial\triangle^k \simeq S^{k-1}$. Since P_Q is a contractible ball, by the long exact homology group sequence of pair (P_Q, P_Q^T), we have

$$H_k(P_Q, P_Q^T) \cong H_{k-1}(P_Q^T) \cong H_{k-1}(S^{k-1}) \neq 0.$$

Therefore, by Proposition 5.4, $H_k(\widetilde{Q}) \neq 0$. □

5.2.3 Proof of Proposition 5.4 Before we prove Proposition 5.4, we first give some notation (see [18]).

A subset T of $\mathcal{S}$ is called *spherical* if the subgroup generated by T is a finite subgroup of G. Each s_F in a spherical subset T exactly corresponds to a facet $F \in \mathcal{F}(P) - \mathcal{F}_B$, and $F \cap F' \neq \varnothing$ for any s_F and $s_{F'}$ in a spherical set T. Let W_T be the group generated by a spherical subset T. Then $W_T \cong (\mathbb{Z}_2)^{\#T}$, where $\#T$ denotes the number of all elements in T.

If the set T is the union of a spherical set T_S and a t_B for $B \in \mathcal{F}_B$, then

$$W_T = W_{T_S} \cup t_{B'} W_{T_S},$$

where B' is the facet which is identified with B in Q.

Lemma 5.7 *Let G be the orbifold fundamental group of a simple handlebody with generator set $\mathcal{S}$. Then for each $w \in G$, $S(w)$ is either a spherical subset of $\mathcal{S}$ or the union of a t_B and a spherical subset in $\mathcal{S}$.*

Proof Let $w = g_0 t_1 g_1 \cdots g_{m-1} t_m g_m$ be a reduced normal form in G. We might as well assume that this expression of w is a normal form in the opposite direction for each t_B, that is, each g_i is a representative of a coset of $W_{B_{i+1}}$ or $W_{B'_{i+1}}$ in G, for $i = 0, \dots, m-1$.

It is easy to see that for $F \in \mathscr{F}(P) - \mathscr{F}_B$, $s_F \in S(w)$ if and only if $s_F \in S(t_m g_m)$. If there is a $B \in \mathscr{F}_B$ such that $t_B \in S(w)$, then $g_m t_B = t_B g'_m$ where $g'_m = \phi_B(g_m)$, and the last t_m is t_B^{-1}. For another $t_{B'} \neq t_B$, it cannot reduce the length of w. Thus the conclusion holds. □

For a spherical set $T = S(w)$, we define an element in $\mathbb{Z}W_T \subset \mathbb{Z}W(P_Q, \mathscr{F}_B)$ by the formula

$$\beta_T = \sum_{w \in W_T} (-1)^{l(w)} w.$$

Consider a natural cellular decomposition of P_Q given by its facial structure. Let $C_*(P_Q)$ and $C_*(\widetilde{Q})$ denote the cellular chain complexes of P_Q and $\widetilde{Q}$, respectively, and let $H_*(P_Q)$ and $H_*(\widetilde{Q})$ be their respective homology groups. Since G acts cellularly on $\widetilde{Q}$, $C_*(\widetilde{Q})$ is a $\mathbb{Z}(G)$-module.

Let T be a spherical set. Multiplication by β_T defines a homomorphism $\beta_T \colon C_*(P_Q) \to C_*(W_T P_Q)$.

Lemma 5.8 *$C_*(P_Q^T)$ is contained in the kernel of $\beta_T \colon C_*(P_Q) \to C_*(W_T P_Q)$.*

Proof Suppose τ is a cell in P_Q^T. If T is a spherical set, then τ lies in some $F \in \mathscr{F}(P_Q) - \mathscr{F}_B$ such that $s_F \in T$. Let $\mathscr{B}$ be a subset of W_T such that $W_T = \mathscr{B} \cup s_F \mathscr{B}$; then we can write β_T as

$$\beta_T = \sum_{w \in W_T} (-1)^{l(w)} w = \sum_{v \in \mathscr{B}} (-1)^{l(v)} (v - v s_F).$$

Since vF is identified with $v s_F F$ in $\widetilde{Q}$, we have that

$$\beta_T \tau = \sum (-1)^{l(v)} (v - v s_F)\tau = \sum (-1)^{l(v)} (v\tau - v\tau) = 0.$$

Thus, $C_*(P_Q^T) \subset \ker \beta_T$. □

Hence, β_T induces a chain map $C_*(P_Q, P_Q^T) \to C_*(W_T P_Q)$, still denoted by β_T.

For each $w \in G$:

- If $T = S(w)$ is a spherical set, we then define a map

  $$\rho^w = w\beta_T \colon C_*(P_Q, P_Q^T) \xrightarrow{\beta_T} C_*(W_T P_Q) \xrightarrow{w} C_*(wW_T P_Q).$$

  Hence, we have a map

  $$\rho^w_* \colon H_*(P_Q, P_Q^T) \to H_*(wW_T P_Q).$$

- If $T = S(w) = \{t_B\} \cup T_S$ where T_S is a spherical set, $t_B s = s t_B$ for any $s \in T_S$ implies that $W_{T_S} < W_B$, i.e. $B \cap F_s \neq \varnothing$ for any $s \in T_S$. So B' does not intersect any F_s; hence for $k > 1$ we have

  $$H_k(P_Q, P_Q^T) \cong H_{k-1}(P_Q^T) \cong H_{k-1}\Bigl(P_Q^{T_S} \coprod B'\Bigr) \cong H_{k-1}(P_Q^{T_S}) \cong H_k(P_Q, P_Q^{T_S})$$

  where P_Q and B' are contractible simple polytopes. Now put

  $$\rho^w_* \colon H_k(P_Q, P_Q^T) \cong H_k(P_Q, P_Q^{T_S}) \xrightarrow{\beta_{T_S}} H_*(W_{T_S} P_Q) \xrightarrow{i_*} H_*(W_T P_Q) \xrightarrow{\times w} H_*(wW_T P_Q).$$

Next, order the elements of G,

$$w_1, w_2, \dots$$

so that $l(w_i) \le l(w_{i+1})$. For each $n \ge 1$, put

$$X_n = \bigcup_{i=1}^{n} w_i P_Q.$$

To simplify notation, set $w = w_n$.

Lemma 5.9 $$X_{n-1} \cap wP = wP^{S(w)}.$$

Proof Notice that X_{n-1} contains a subgraph of Cayley graph of G associated with the generator set $\mathcal{S}$, where the length between each vertex and the unit element is less than or equal to $l(w)$. Then

$$l(ws) = \begin{cases} l(w) - 1 & \text{if } s \in S(w), \\ l(w) + 1 & \text{if } s \in \mathcal{S} - S(w). \end{cases}$$

A chamber $w_i P_Q$ $(i < n)$ in X_{n-1} intersects with wP_Q in the facet wF if and only if either $w_i \cdot s_F = w$ for $F \in \mathcal{F}(P) - \mathcal{F}_B$ or $w_i \cdot t_F^{-1} = w$ for $F \in \mathcal{F}_B$ where t_F is a torsion-free generator in $\mathcal{S}$; in other words, either $l(ws_F) = l(w) - 1$ or $l(wt_F) = l(w) - 1$. Therefore, $X_{n-1} \cap wP_Q = wP_Q^{S(w)}$. □

Finally let us finish the proof of Proposition 5.4.

Proof of Proposition 5.4 We know from Lemma 5.9 that $X_{n-1} \cap wP_Q = wP_Q^{S(w)}$. Hence, the excision theorem gives an isomorphism

$$H_*(X_n, X_{n-1}) \xrightarrow{\cong} H_*(wP_Q, wP_Q^{S(w)}).$$

Consider the exact sequence of the pair (X_n, X_{n-1}),

$$\cdots \to H_*(X_{n-1}) \xrightarrow{j_*} H_*(X_n) \xrightarrow{k_*} H_*(X_n, X_{n-1}) \to \cdots.$$

We claim that the map k_* is a split epimorphism, which is equivalent to the map

$$k_*^w \colon H_*(X_n) \to H_*(P_Q, P_Q^{S(w)})$$

being a split epimorphism, where k_*^w denotes the composition of k_* with the excision isomorphism and left translation by w^{-1}. Consider the map ρ_*^w on $H_*(P_Q, P_Q^{S(w)})$ whose image is contained in $H_*(wW_{S(w)}P_Q)$. For every $v \neq 1$ in $W_{S(w)}$, we have $l(wv) < l(w)$; hence, $wW_{S(w)}P_Q \subset X_n$. Hence the image of ρ_*^w is contained in $H_*(X_n)$. All these can be seen from the commutative diagram

$$\begin{tikzcd}
H_*(X_n, X_{n-1}) \arrow[r, "\cong"] & H_*(wP_Q, wP_Q^{S(w)}) \arrow[d, "\times w^{-1}"] \\
H_*(X_n) \arrow[u, "k_*"] \arrow[r, bend left, "k_*^w"] & H_*(P_Q, P_Q^{S(w)}) \arrow[l, bend left, "\rho_*^w"] \arrow[d, "\beta_*"] \\
H_*(wW_{S(w)}P_Q) \arrow[u, "i_*"] & H_*(W_{S(w)}P_Q) \arrow[l, "\times w"']
\end{tikzcd}$$

where β_* is induced by multiplication by $\beta_{S(w)}$ when $S(w)$ is a spherical set, and is the composition

$$\beta_{T_S} \circ i_* \colon H_k(P_Q, P_Q^T) \cong H_k(P_Q, P_Q^{T_S}) \xrightarrow{\beta_{T_S}} H_*(W_{T_S} P_Q) \xrightarrow{i_*} H_*(W_T P_Q)$$

when $S(w)$ is the union of a $\{t_B\}$ and a spherical set T_S.

Since $\widetilde{Q}$ is the universal cover of Q, $H_1(\widetilde{Q}) \cong 0$. For $* > 1$, it can be see that $k_*^w \circ \rho_*^w$ is the identity on $H_*(P_Q, P_Q^{S(w)})$ by above diagram. Hence there is the splitting short exact sequence

$$0 \to H_*(X_{n-1}) \xrightarrow{j_*} H_*(X_n) \xrightarrow{k_*^w} H_*(P_Q, P_Q^{S(w)}) \to 0.$$

This implies that

$$H_*(X_n) \cong H_*(X_{n-1}) \oplus H_*(P_Q, P_Q^{S(w)})$$

where $H_*(X_1) = H_*(P_Q) = 0$. Since $\widetilde{Q}$ is the increasing union of the X_n, we have

$$H_*(\widetilde{Q}) = \lim_{n \to \infty} H_*(X_n) \cong \bigoplus_{w \in G} H_*(P_Q, P_Q^{S(w)}). \qquad \square$$

5.3 Proof of Theorem A

Let Q be a simple handlebody and $q \colon P_Q \to Q$ be the quotient map by gluing all paired facets in $\mathcal{F}_B$. Then the orbifold universal cover $\pi \colon \widetilde{Q} \to Q$ of Q can be constructed by (4-2).

Let $\mathcal{C}(P_Q)$ be the standard cubical cellular decomposition of P_Q. For each cube $c \in \mathcal{C}(P_Q)$, each component of $\pi^{-1}(c)$ is a cube in $\widetilde{Q}$. Then $\mathcal{C}(P_Q)$ determines a cubical cellular decomposition of $\widetilde{Q}$, denoted by $\mathcal{C}(\widetilde{Q})$, such that the link of each point v in $\mathcal{C}(\widetilde{Q})$ is exactly the nerve $\mathcal{N}(P_Q)$ of P_Q. Hence, if Q is flag, then P_Q is flag, and so is $\mathcal{N}(P_Q)$. By the Gromov lemma, $\widetilde{Q}$ is nonpositively curved. In fact, $\widetilde{Q}$ is a CAT(0) space. Then by the Cartan–Hadamard theorem, $\widetilde{Q}$ is aspherical. Therefore, Q is orbifold-aspherical.

On the contrary, if Q is orbifold-aspherical, then $\widetilde{Q}$ is contractible. Using an idea of Davis in [20, Section 8.2], we shall show that if P_Q is not flag then $\widetilde{Q}$ is not contractible. Indeed, if P_Q is not flag, then $\mathcal{N}(P_Q)$ contains an empty k-simplex for some $k \geq 2$. The dual of this empty k-simplex gives an essential embedding sphere in $\widetilde{Q}$. By Corollary 5.6, the fundamental class of such a sphere is nontrivial in $H_k(\widetilde{Q})$, which contradicts that $\widetilde{Q}$ is contractible. $\square$

6 Proof of Theorem B

The purpose of this section is to characterize the rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$ in terms of a $\square$-belt in Q. Here $\pi_1^{\mathrm{orb}}(Q)$ is an iterative HNN-extension over a right-angled Coxeter group.

Theorem B *Suppose that Q is a simple n-handlebody. Then there is a rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$ if and only if Q contains a $\square$-belt.*

Remark 6.1 The "simple" condition of a handlebody is necessary in above proposition. In fact, it is easy to see that the orbifold fundamental group of a two-dimensional annulus as a right-angled Coxeter orbifold is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}_2 * \mathbb{Z}_2)$, which contains a rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$. Consider a right-angled Coxeter 3-handlebody Q with a π_1-injective annulus-suborbifold B such that B is a π_1-injective suborbifold; it provides a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in its orbifold fundamental group. Of course, such a Q is not simple. All of these results are the generalization of [7, Lemma 5.22] which is related to the flat torus theorem in [7, Chapter II.7].

Example 6.2 (squares of Example 3.11) We show that each $\square$ in (c) and (d) of Example 3.11 determines a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$, whereas cases (a) and (b) do not.

In (a), the four facets F_1, F_2, F_3 and F_4 correspond to a suborbifold B which is a quadrilateral in Q^*, but it is not a $\square$-belt in Q. In fact,

$$i_*(\pi_1^{\mathrm{orb}}(B)) \cong W_\square/\langle (s_1s_3)^2\rangle \cong (\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_2 * \mathbb{Z}_2) < \pi_1^{\mathrm{orb}}(Q)$$

and s_1s_3, s_2s_4 generate a subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}$ in $i_*(\pi_1^{\mathrm{orb}}(B)) < \pi_1^{\mathrm{orb}}(Q)$, where $i_*\colon \pi_1^{\mathrm{orb}}(B) \to \pi_1^{\mathrm{orb}}(Q)$ is induced by the inclusion $i\colon B \hookrightarrow Q$. Thus, there is no subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $i_*(\pi_1^{\mathrm{orb}}(B))$.

In (b), $\{F_1, F_2, F_3, F_4\}$ does not determine a quadrilateral suborbifold. Without loss of generality, assume that $\{F_1, F_2, F_3, F_4\}$ bounds only one hole of Q^*. Then there are at least 5 generators in $\pi_1^{\mathrm{orb}}(Q)$ associated to five facets in P_Q, denoted by $\{F_1, F_2, F_3, F_4, F_1'\}$ with $F_1 \cap B^+ \sim F_1' \cap B^-$, where B is the cutting belt of Q and cut F_1 into two facets in P_Q. Thus, (b) induces a subgroup of $\pi_1^{\mathrm{orb}}(Q)$ as follows:

$$\begin{aligned} W_b &:= \langle s_1, s_2, s_3, s_4, s_1', t \mid (s_i)^2 = 1, \forall i; (s_1s_2)^2 = (s_2s_3)^2 = (s_3s_4)^2 = (s_4s_1')^2 = 1; s_1' = ts_1t\rangle \\ &= \langle s_1, s_2, s_3, s_4, t \mid (s_i)^2 = 1, \forall i; (s_1s_2)^2 = (s_2s_3)^2 = (s_3s_4)^2 = (s_4ts_1t^{-1})^2 = 1\rangle \end{aligned}$$

which contains no subgroup $\mathbb{Z} \oplus \mathbb{Z}$.

In (c) or (d), $\{F_1, F_2, F_3, F_4\}$ determines a $\square$-belt $B_\square$ of Q. If $B_\square$ does not intersect with any cutting belt, then $B_\square$ is kept in P_Q, so there is a subgroup $\mathbb{Z} \oplus \mathbb{Z} < W(P_Q, \mathcal{F}_B) < \pi_1^{\mathrm{orb}}(Q)$. If there are some cutting belts $B_1, B_2, \dots, B_k$ intersecting transversely with only a pair of disjoint edges of $B_\square$, without loss of generality, assume that $B_1, B_2, \dots, B_k$ intersect with two disjoint edges f_1 and f_3 of $B_\square$, where some cutting belts may cut f_1 and f_3 many times; see (c) in Figure 7. Then there is also a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ generated by s_1s_3 and $s_2t_1t_2\cdots t_ks_4t_k^{-1}\cdots t_2^{-1}t_1^{-1}$ where each t_i is one of $\{t_B^{\pm 1}\}$. Also see Figure 9.

6.1 The special case where Q is a simple polytope

When Q is a simple polytope, Theorem B can be followed by Moussong's theorem; see [46] or [18, Corollary 12.6.3].

Theorem 6.3 (Moussong's theorem; see [46] or [18, Corollary 12.6.3]) *If W_P is the right-angled Coxeter group of a simple polytope P, then there exists a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup in W_P if and only if there is a $\square$-belt in P.*

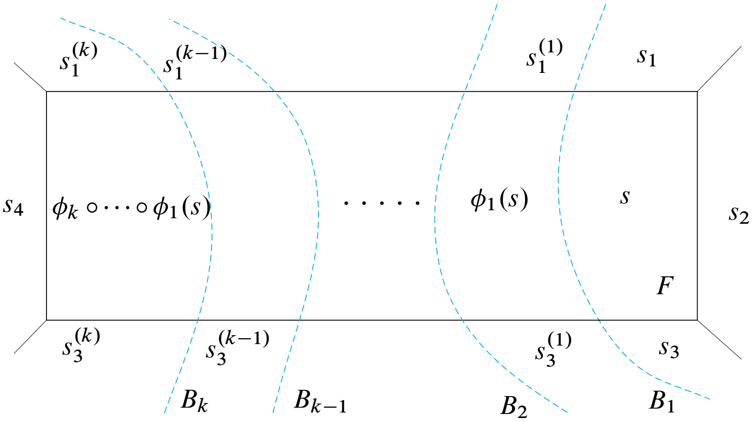


Figure 9: $\square$-belt and $\mathbb{Z} \oplus \mathbb{Z}$.

Next let us deal with the case of a simple handlebody. Let Q be a simple handlebody of genus $\mathfrak{g} > 0$, and P_Q be the associated simple polytope obtained by cutting Q open along cutting belts $\{B_i \mid i = 1, 2, \ldots, \mathfrak{g}\}$.

6.2 Proof of the sufficiency of Theorem B

Assume that there is a $\square$-belt $B_\square$ given by $\{F_1, F_2, F_3, F_4\}$ in $\mathcal{N}(Q)$. After cutting Q open along cutting belts B_i, $i = 1, 2, \ldots, \mathfrak{g}$, by Lemma 3.12, there are the following two cases.

- The $B_\square$ is still kept in P_Q. Then $B_\square$ gives a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $W(P_Q, \mathcal{F}_B) < \cdots < \pi_1^{\mathrm{orb}}(Q)$, which is generated by $s_1 s_3$ and $s_2 s_4$.
- The $B_\square$ is not kept in P_Q. Then there is only one situation in which some cutting belts B_i intersect transversely with a pair of disjoint edges of $B_\square$, say F_1 and F_3. If $B_\square$ intersects transversely with cutting belts $B_1, B_2, \ldots, B_k$ in turn, then $s_1 s_3$ and $s_2 t_1 \cdots t_k s_4 t_k^{-1} \cdots t_1^{-1}$ generate a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$, as in cases (c) or (b) on Example 3.11. See also Example 6.2. $\square$

6.3 Proof of the necessity of Theorem B

Cutting Q open along a cutting belt B, we get a right-angled Coxeter n-handlebody of genus $\mathfrak{g} - 1$, denoted by $Q_{\mathfrak{g}-1}$. Conversely, Q can be recovered from $Q_{\mathfrak{g}-1}$ by gluing its two disjoint boundary facets associated with B, which implies that the orbifold fundamental group of Q is an HNN-extension on $\pi_1^{\mathrm{orb}}(Q_{\mathfrak{g}-1})$. Write $G_{\mathfrak{g}-1} = \pi_1^{\mathrm{orb}}(Q_{\mathfrak{g}-1})$, and let W_{B^+} and W_{B^-} be two isomorphic subgroups of $G_{\mathfrak{g}-1}$ determined by two copies of B. Then we have

$$\pi_1^{\mathrm{orb}}(Q) \cong G_{\mathfrak{g}-1} *_\phi = \langle G_{\mathfrak{g}-1}, t \mid t^{-1} a t = \phi(a), a \in W_{B^-} \rangle \tag{6-1}$$

where $\phi\colon W_{B^-} \to W_{B^+}$ is an isomorphism by mapping $s' \in W_{B^-}$ into $s \in W_{B^+}$. Generally, $\pi_1^{\mathrm{orb}}(Q)$ is isomorphic to $\mathfrak{g}$ HNN-extensions on the right-angled Coxeter group $W(P_Q, \mathcal{F}_B)$ as we have seen in the proof of Proposition 4.3:

$$\pi_1^{\mathrm{orb}}(Q) \leftarrow G_{\mathfrak{g}-1} \leftarrow \cdots \leftarrow G_1 \leftarrow G_0 = W(P_Q, \mathcal{F}_B),$$

where each G_k is also an HNN-extension over G_{k-1} for $1 \le k \le \mathfrak{g}-1$, and $G_0 = W(P_Q, \mathcal{F}_B)$ is a right-angled Coxeter group.

According to the normal form theorem of HNN-extensions (see Theorem 2.22), each element x in $\pi_1^{\mathrm{orb}}(Q)$ has a unique iterative normal form. First, write

$$x = g_0 t_{\mathfrak{g}}^{\epsilon_1} g_1 t_{\mathfrak{g}}^{\epsilon_2} \cdots g_{n-1} t_{\mathfrak{g}}^{\epsilon_n} g_n$$

as a normal form for $t_{\mathfrak{g}}$ where $g_i \in G_{\mathfrak{g}-1}$. Next, inductively each g_i is also a normal form in G_k for $1 \le k \le \mathfrak{g}-1$. More generally, x has a unique form

$$x = g_0 t_1 g_1 \cdots g_{m-1} t_m g_m \tag{6-2}$$

where each g_i is reduced in $G_0 = W(P_Q, \mathcal{F}_B)$, and each t_i is one of $\{t_B^{\pm 1}\}$ which determines an isomorphism of $\{\phi_B^{\pm 1}\}$ on some subgroups of $\pi_1^{\mathrm{orb}}(Q)$. This expression of x is a normal form with respect to all possible t_B. The expression in (6-2) is called a *reduced normal form* of x in $\pi_1^{\mathrm{orb}}(Q)$. The number m is called the (*total*) *t-length of x*.

By applying the Tits theorem (see Theorem 2.20) and the normal form theorem of HNN-extensions (see Theorem 2.22), we have the following conclusion.

Lemma 6.4 *Two reduced words x and y are the same in $\pi_1^{\mathrm{orb}}(Q)$ if and only if one of x and y can be transformed into the other one by a sequence of commutations of right-angled Coxeter groups and t-reductions of HNN-extensions.*

Next, we prove two lemmas.

Lemma 6.5 *If there is a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$, then one generator of $\mathbb{Z} \oplus \mathbb{Z}$ can be presented as a cyclically reduced word in $W(P_Q, \mathcal{F}_B)$.*

Proof Assume that there is a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$, which is generated by two reduced normal forms as in (6-2),

$$x = g_0 t_1 g_1 \cdots g_{m-1} t_m g_m, \quad y = h_0 t_1' h_1 \cdots h_{n-1} t_n' h_n.$$

Then $xy = yx$ in $\pi_1^{\mathrm{orb}}(Q)$. By Lemma 6.4, xy and yx have the same reduced normal form as in (6-2). We do t-reductions on

$$xy = g_0 t_1 g_1 \cdots g_{m-1} t_m g_m \cdot h_0 t_1' h_1 \cdots h_{n-1} t_n' h_n,$$
$$yx = h_0 t_1' h_1 \cdots h_{n-1} t_n' h_n \cdot g_0 t_1 g_1 \cdots g_{m-1} t_m g_m.$$

Since x and y are reduced normal forms, xy and yx have the same tails. Without loss of generality, assume that $m \ge n$. Write $\tilde{y} = t_1' h_1 \cdots h_{n-1} t_n' h_n = h_0^{-1} y$. Then x can be written as

$$x = g_0 t_1 g_1 \cdots t_{m-n} g_{m-n} \tilde{y} = g_0 t_1 g_1 \cdots t_{m-n} g_{m-n} \cdot h_0^{-1} y.$$

Since x and y generate $\mathbb{Z} \oplus \mathbb{Z}$, both y and xy^{-1} do so. The word xy^{-1} has a shorter t-length. We further do t-reductions on xy^{-1} to get a normal form, also denoted by x.

We can always continue to do this algorithm, so that we can take either x or y from $W(P_Q,\mathscr{F}_B)$. Suppose $y=h\in W(P_Q,\mathscr{F}_B)$.

Furthermore, we can assume that h is a cyclically reduced word in $W(P_Q,\mathscr{F}_B)$. In fact, if h is not cyclically reduced, without loss of generality, assume that h is of the form $w^{-1}h'w$, where w is an arbitrary word and h' is a cyclically reduced word in $W(P_Q,\mathscr{F}_B)$. Then we replace h by h', such that h' and wxw^{-1} generate a $\mathbb{Z}\oplus\mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$. □

Lemma 6.6 *Let $x=g_0\cdot t_1\cdots t_k$ be a reduced normal form, where $g_0\in W(P_Q,\mathscr{F}_B)$ and each t_i is one of $\{t_B^{\pm 1}, B\in\mathscr{F}_B\}$, and h be a cyclically reduced word in $W(P_Q,\mathscr{F}_B)$. Then x,h cannot generate a $\mathbb{Z}\oplus\mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$.*

Proof If x,h generate a $\mathbb{Z}\oplus\mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$, then

$$x\cdot h=g_0\cdot t_1\cdots t_k\cdot h=g_0h'\cdot t_1\cdots t_k,$$

where $h'=\phi_1\circ\cdots\circ\phi_k(h)$ is the image of the composition of some ϕ_i on h.

We first claim that h' is reduced in $W(P_Q,\mathscr{F}_B)$, and the word length of h' and h are equal. In fact, for each i, ϕ_i is an isomorphism from some W_{B^-} to W_{B^+} which maps generators to generators, and all W_{B^+} and W_{B^-} are subgroups of $W(P_Q,\mathscr{F}_B)$.

Next, we claim that $h=h'$. In fact, $xh=hx$ implies that $g_0h'=hg_0$, that is $h'=g_0^{-1}hg_0$. Let $h=s_1\cdots s_n$. If there exists a letter s in g_0 such that:

- $l(hs)=l(h)+1$ and $hs\neq sh$, then $l(h')>l(h)$ which is a contradiction.
- $l(hs)=l(h)-1$, then there is a letter $s_k=s$ in h such that $s_js=ss_j\neq 1$ for any $j>k$. So h equal to a new word $s_1\cdots s_{k-1}s_{k+1}\cdots s_ns$. Moreover, $g_0h'=hg_0$ implies that there is a letter $s_i=s$ in hg_0 can be moved to the head of word hg_0.
  - If $i=k$, then s commutes with all letters except s_k in h. Since there are two s canceled in hg_0 by the relation $s^2=1$, there are two canceled s in g_0h'. Since g_0 is reduced, s commutes with all letters except s in g_0 and there is an s that can be moved to the head of word h'. Now we consider new $\bar{h}':=sh'$ and $\bar{h}:=sh$. Then there is no s in sh and s commutes with all letters in sh.
  - Otherwise, $i\neq k$. Then h equals to a word with form $s\bar{h}s$, which contradicts that h is cyclically reduced.

Hence all letters in g_0 commute with h. That is, $g_0h=hg_0=g_0h'$, Thus $h=h'=\phi_1\circ\cdots\circ\phi_k(h)$.

If $\phi_1\circ\cdots\circ\phi_k=\mathrm{id}$, then the associated sequence $t_1\cdots t_k=1$, which contradicts that x is reduced. If $\phi_1\circ\cdots\circ\phi_k\neq\mathrm{id}$ has a fixed point in all letters in h, then there is a letter s_0 in h such that $\phi_1\circ\cdots\circ\phi_k(s_0)=s_0$. Furthermore, $s_0,\phi_k(s_0),\dots,\phi_1\circ\cdots\circ\phi_{k-1}(s_0)$ determine a noncontractible facet in Q, which contradicts that Q is simple. More generally, if $\phi_1\circ\cdots\circ\phi_k\neq\mathrm{id}$ has no fixed point, then there is a generator s_1 as a letter in h, such that $s_2=\phi_1\circ\cdots\circ\phi_k(s_1)\neq s_1$. Continue this procedure; one can get a sequence

$s_1, s_2, s_3, \ldots$, such that each s_i is a generator as a letter in h and $s_i = \phi_1 \circ \cdots \circ \phi_k(s_{i-1})$. However, the word length of h is finite; thus there must be two same elements in the sequence. Geometrically, this means that there is a noncontractible facet in Q, which contradicts that Q is simple. □

Now let us give the proof of the necessity of Theorem B in the general case.

Proof of the necessity of Theorem B Suppose that there are two elements x and y in $\pi_1^{\mathrm{orb}}(Q)$ which generate a rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$. Our arguments are divided into the following steps.

Step 1 *Simplify two generators x and y of $\mathbb{Z} \oplus \mathbb{Z}$ by doing t-reductions.*

Lemma 6.5 tells us that one of x or y can be chosen as a cyclically reduced word h in $W(P_Q, \mathscr{F}_B)$, say $y = h$. Now if x is also a word in $W(P_Q, \mathscr{F}_B)$ (i.e. the t-length of x is zero), then by Moussong's theorem (see Theorem 6.3), there is a $\square$-belt in P_Q which can appear in Q, as desired.

Next let us consider the case in which the t-length of x is greater than zero. Let

$$x = g_0 t_1 g_1 \cdots g_{m-1} t_m g_m$$

be a reduced normal form in $\pi_1^{\mathrm{orb}}(Q)$. Then $xh = hx$ implies that

$$\begin{aligned} g_m h &= h g_m, & t_m \cdot h &= \phi_m(h) \cdot t_m, \\ g_{m-1} \cdot \phi_m(h) &= \phi_m(h) \cdot g_{m-1}, & t_{m-1} \cdot \phi_m(h) &= \phi_{m-1} \circ \phi_m(h) \cdot t_{m-1}, \\ &\vdots & & \\ g_0 \cdot \phi_1 \circ \cdots \circ \phi_m(h) &= \phi_1 \circ \cdots \circ \phi_m(h) \cdot g_0, & & \end{aligned}$$

where each $\phi_i : W_{B_i} \to W_{B_i'}$ is an isomorphism determined by some $B_i \in \mathscr{F}_B$, each $\phi_i \circ \cdots \circ \phi_m(h)$ is an expression in $W_{B_i'} \cap W_{B_{i-1}}$ for $i = 2, \ldots, m$ and $\phi_1 \circ \cdots \circ \phi_m(h) \in W_{B_1'}$ for $h \in W_{B_m}$. Here two B_i and B_j may correspond to the same $B \in \mathscr{F}_B$.

Step 2 *Find facets F_1 and F_3 around B or B' in P_Q.*

Without loss of generality, $h \in W(P_Q, \mathscr{F}_B)$ is a cyclically reduced word. Since h is a free element in $W_{B_m} \cap W(P_Q, \mathscr{F}_B)$, we can take two generators s_1 and s_3 in h corresponding to two disjoint facets F_1 and F_3 of P_Q such that F_1 and F_3 intersect with B_m. In particular, $s_1 s_3$ is a free element in $W_{B_m} < W(P_Q, \mathscr{F}_B) = G_0 < \cdots < G_{\mathfrak{g}-1} < G_{\mathfrak{g}} = \pi_1^{\mathrm{orb}}(Q)$.

Step 3 *Find the facet F_2 which intersects F_1 and F_3.*

If $g_m \neq 1$, since x is a normal form, then g_m is a representative of a coset of W_{B_m} in $\pi_1^{\mathrm{orb}}(Q)$. Thus there is a generator $s_2 \notin S(W_{B_m})$ in g_m such that $h s_2 = s_2 h$, where $S(W_{B_m})$ is the generator set of W_{B_m}. This generator s_2 determines a facet F_2 in P_Q, as desired.

If $g_m = 1$, then $x \cdot h = g_0 t_1 g_1 \cdots t_m \cdot h = g_0 t_1 g_1 \cdots t_{m-1} g_{m-1} \cdot \phi_m(h) \cdot t_m$. A similar argument shows that either there is a $s_2 \notin S(W_{B_{m-1}})$ as desired, or

$$x \cdot h = g_0 t_1 g_1 \cdots t_{m-1} \cdot \phi_m(h) \cdot t_m = g_0 t_1 g_1 \cdots t_{m-2} g_{m-2} \cdot \phi_{m-1} \circ \phi_m(h) \cdot t_{m-1} t_m.$$

We can continuously carry out the above procedure. Finally we can arrive at two possible cases:

- There exists some $g_i \neq 1$ for $i > 0$. Then there must be a letter s_2 in g_i which determines the required F_2.
- x is of the form $x = g_0 \cdot t_1 \cdots t_m$, where $g_0 \in W(P_Q, \mathscr{F}_B)$ and $t_1 \cdots t_m$ is a word formed by letters in $\{t_B^{\pm 1}\}$. By Lemma 6.6, x and h cannot generate a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\mathrm{orb}}(Q)$, so $x = g_0 \cdot t_1 \cdots t_m$ is impossible.

Thus, we can always find a facet F_2 from a nontrivial g_i in the reduced form (6-2) of x where $i > 0$.

Step 4 *Find a facet F_4 such that F_1, F_2, F_3 and F_4 determine a $\square$-belt in Q.*

We proceed our argument as follows.

(I) If there is only a $g_i \neq 1$ (i.e. $g_j = 1$ for any $j \neq i$) in the expression of x, then $x = t_1 \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m$, where i must be more than zero by Lemma 6.6. Now $xh = hx$ implies that $t_1 \cdots t_i \cdot t_{i+1} \cdots t_m = 1$. Actually, if $t_1 \cdots t_i \cdot t_{i+1} \cdots t_m \neq 1$, then $\phi_1 \circ \cdots \circ \phi_m(h) = h$ implies that there is a noncontractible facet in Q, which is impossible (also see the proof of Lemma 6.6). Thus, $t_1 \cdots t_i = (t_{i+1} \cdots t_m)^{-1}$, so $x = t_1 \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m = (t_{i+1} \cdots t_m)^{-1} g_i (t_{i+1} \cdots t_m)$. Since x, h generate a $\mathbb{Z} \oplus \mathbb{Z}$, we see that $g_i, \phi_{i+1} \circ \cdots \circ \phi_m(h)$ generate a $\mathbb{Z} \oplus \mathbb{Z}$ in $W(P_Q, \mathscr{F}_B)$. Then by Moussong's theorem (see Theorem 6.3), there is a $\square$-belt in Q.

(II) If there are at least two nontrivial $g_i, g_j \neq 1$ in x where $0 < j < i \leq m$ but $g_k = 1$ for all $k > j$ and $k \neq i$, then one may write $x = \cdots t_j \cdot g_j \cdot t_{j+1} \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m$. So we have

$$\begin{aligned} xh &= \cdots t_j \cdot g_j \cdot t_{j+1} \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m \cdot h \\ &= \cdots t_j \cdot g_j \cdot t_{j+1} \cdots t_i \cdot g_i \cdot h' t_{i+1} \cdots t_m \\ &= \cdots t_j \cdot g_j \cdot t_{j+1} \cdots t_i \cdot h' g_i \cdot t_{i+1} \cdots t_m \\ &= \cdots t_j \cdot g_j \cdot h'' \cdot t_{j+1} \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m \end{aligned} \tag{6-3}$$

where $h' = \phi_{i+1} \circ \cdots \circ \phi_m(h)$ and $h'' = \phi_{j+1} \circ \cdots \circ \phi_m(h)$. Since $xh = hx$, we have that $g_j h'' = h'' g_j$, so we can take a generator s_4 in g_j (not in $S(W_{B_j})$) such that $h'' s_4 = s_4 h''$. Similarly, here s_4 determines a facet F_4 of P_Q such that $F_4 \cap F_1'' \neq \varnothing$ and $F_4 \cap F_3'' \neq \varnothing$ where F_1'' and F_3'' are two facets of P_Q determined by the images of $\phi_{j+1} \circ \cdots \circ \phi_m$ on s_1 and s_3. In particular, $F_2 \neq F_4$ in P_Q. Otherwise, the intersection of $q(F_1)$ and $q(F_2)$ in Q is disconnected where $q \colon Q \to P_Q$ is the quotient map defined in (4-1), which contradicts that Q is simple. Hence, we get a $\square$-belt in Q.

(III) If there are only g_0 and g_i that are nontrivial in x where $i > 0$, then one may write

$$x = g_0 t_1 \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m.$$

Without loss of generality, assume that g_0 and g_i are two reduced words in $W(P_Q, \mathscr{F}_B)$. Now if $x = g_0 t_1 \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m = t_1 \cdots t_i \cdot g_0' g_i \cdot t_{i+1} \cdots t_m$ where $g_0' = \phi_i^{-1} \circ \cdots \circ \phi_1^{-1}(g_0)$, then by the proof of Lemma 6.6, $xh = hx$ implies that $t_1 \cdots t_i \cdot t_{i+1} \cdots t_m = 1$. As in case (I), $g_0' g_i$ and $h' = \phi_{i+1} \circ \cdots \circ \phi_m(h)$ generate a $\mathbb{Z} \oplus \mathbb{Z}$ in $W(P_Q, \mathscr{F}_B)$. Hence we can find a $\square$ in Q. If

$$x = g_0 t_1 \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m = g_0' t_1 \cdots t_j g_0'' t_{j+1} \cdots t_i \cdot g_0''' g_i \cdot t_{i+1} \cdots t_m$$

where g_0'' cannot cross t_{j+1} and $g_0 = g_0' \cdot \phi_1 \circ \cdots \circ \phi_j(g_0'') \cdot \phi_1 \circ \cdots \circ \phi_i(g_0''')$, then as in case (II), there is a generator s_4 in g_0'' which is not in $S(W_{B'_{j+1}})$. Then s_4 determines a facet F_4 of P_Q such that F_4 intersects with F_1 and F_3 in Q. So there is a $\square$-belt in Q.

Together with all arguments above, we complete the proof. $\square$

7 Applications

Throughout the following, we always assume that Q is a genus $\mathfrak{g}$ simple handlebody with m facets and M is the manifold double over Q. Moreover, for the sake of discussion, we always assume that M is a smooth manifold. In this section, we shall show that some B-belts in Q can play a role in the obstruction of the existence of some Riemannian metrics on M. First we can see that every B-belt of Q is a π*-injective* suborbifold in the sense of the following Lemma 7.1.

Lemma 7.1 *Let $i\colon B \hookrightarrow Q$ be a belt of Q. Then $i_*\colon \pi_1^{\mathrm{orb}}(B) \to \pi_1^{\mathrm{orb}}(Q)$ is an injection. Moreover, if B is not orbifold-aspherical, then Q is not orbifold-aspherical.*

Proof If B is a cutting belt of Q, then by Lemma 4.2, B is π_1-injective.

If B is a belt of Q which is disjoint with any cutting belts of Q, then B can be embedded into P_Q, so the induced map $\pi_1^{\mathrm{orb}}(B) \to \pi_1^{\mathrm{orb}}(P_Q)$ is an injection. Thus $i_*\colon \pi_1^{\mathrm{orb}}(B) \to \pi_1^{\mathrm{orb}}(P_Q) \to \pi_1^{\mathrm{orb}}(Q)$ is an injection.

If B intersects with some cutting belts, then we can always do some deformation on B such that it intersects transversely with those cutting belts. Thus, we may assume that B is split into $B_1, B_2, \ldots, B_k$ by cutting belts $E_1, E_2, \ldots, E_{k-1}$ such that for each i, B_i and B_{i+1} exactly intersect with E_i since all E_i are simple polytopes and $|B|$ is a ball. In addition, it is also easy to see that for each i, $\pi_1^{\mathrm{orb}}(B_i)$ is a right-angled Coxeter group. Now

$$\pi_1^{\mathrm{orb}}(B) = \pi_1^{\mathrm{orb}}(B_1) * \pi_1^{\mathrm{orb}}(B_2) * \cdots * \pi_1^{\mathrm{orb}}(B_k) / \langle R(E_i) \mid i = 1, \ldots, k-1 \rangle$$

where $R(E_i)$ is a relation set consisting of all equations $s = t$, each of which is associated with $F_s \sim_{E_i} F_t$ where $F_s \in \mathscr{F}(B_i)$ and $F_t \in \mathscr{F}(B_{i+1})$.

Now for $g \in \pi_1^{\mathrm{orb}}(B_i)$, $i_*(g) = t_{i-1}^{-1} \cdots t_1^{-1} g t_1 \cdots t_{i-1}$. Define $\kappa\colon \pi_1^{\mathrm{orb}}(Q) \to \pi_1^{\mathrm{orb}}(B)$,

$$\kappa(s) = \begin{cases} s & \text{if } F_s \in \bigcup \mathscr{F}(B_i), \\ 1 & \text{if } F_s \in \mathscr{F}(P_Q) - \bigcup \mathscr{F}(B_i), \end{cases}$$

where all torsion free generators are mapped to 1. Then it is clear that κ is well defined and $\kappa \circ i_* = \mathrm{id}$. Hence, i_* is an injection.

Let $\widetilde{Q}$ and $\widetilde{B}$ be the universal cover of Q and B, respectively. If B is not orbifold-aspherical, then there is an integer $k \geq 2$ such that $\pi_k(\widetilde{B}) \neq 0$ and $\pi_i(\widetilde{B}) = 0$ for any $1 \leq i < k$. By the Hurewicz theorem, $H_k(\widetilde{B}) \cong \pi_k(\widetilde{B}) \neq 0$. Hence by Proposition 5.4 there is a Δ^k-belt in B. So there is a Δ^k-belt of Q. Hence by Theorem A, Q is not orbifold-aspherical. □

Lemma 7.2 *The manifold double of a simple handlebody is orientable.*

Proof Let Q be a simple n-handlebody with m facets and cutting belts $\{B_1, \dots, B_{\mathfrak{g}}\}$, P_Q be the associated simple polytope, and $M = Q \times (\mathbb{Z}_2)^m/{\sim}$ be the manifold double over Q, as defined in (2-1). It suffices to prove that $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$. We shall use the proof method of Nakayama and Nishimura (see [47, Theorem 1.7]). The combinatorial structure of P_Q defines a natural cellular decomposition of M. We denote by $\{(C_k(M), \partial_k)\}$ the chain complex associated with this cellular decomposition. In particular, $C_n(M)$ and $C_{n-1}(M)$ are the free abelian groups generated by $\{P_Q\} \times (\mathbb{Z}_2)^m = \{(P_Q, g) \mid g \in (\mathbb{Z}_2)^m\}$ and $\mathcal{F}(P_Q) \times (\mathbb{Z}_2)^m/{\sim'} = \{[F, g] \mid F \in \mathcal{F}(P_Q), g \in (\mathbb{Z}_2)^m\}$, respectively, where the equivalence class of $\mathcal{F}(P_Q) \times (\mathbb{Z}_2)^m$ is defined by the equivalence relation

$$\begin{aligned} (F, g) &\sim' (F, g \cdot e_F) && \text{if } F \in \mathcal{F}(P_Q) - \mathcal{F}_B, \\ (B^+, g) &\sim' (B^-, g) && \text{if } B^+, B^- \in \mathcal{F}_B. \end{aligned}$$

It should be pointed out that actually there is a characteristic map $\lambda\colon \mathcal{F}(Q) \to (\mathbb{Z}_2)^m$ in the construction of $M = Q \times (\mathbb{Z}_2)^m/{\sim}$ such that $\{\lambda(F) = e_F \mid F \in \mathcal{F}(Q)\}$ is the standard basis $\{e_i \mid i = 1, \dots, m\}$ of $(\mathbb{Z}_2)^m$. For any facet F' in $\mathcal{F}(P_Q) - \mathcal{F}_B$, there must be a facet F in $\mathcal{F}(Q)$ such that $F' = F$ or $F' \subsetneqq F$, so F' and F are colored by the same element e_F of $(\mathbb{Z}_2)^m$. For any B in $\mathcal{F}_B$, since $\operatorname{Int} B \subset \operatorname{Int}(Q)$, we adopt the convention that B is colored by the unit element $\mathbf{e}_0$ of $(\mathbb{Z}_2)^m$. In other words, the characteristic map $\lambda\colon \mathcal{F}(Q) \to (\mathbb{Z}_2)^m$ induces a compatible characteristic map $\lambda'\colon \mathcal{F}(P_Q) \to (\mathbb{Z}_2)^m$ such that for any $F' \in \mathcal{F}(P_Q) - \mathcal{F}_B$, $e_{F'} = \lambda'(F') = \lambda(F) = e_F$ where $F \in \mathcal{F}(Q)$ with $F' \subset F$, and for B in $\mathcal{F}_B$, $\lambda'(B) = \mathbf{e}_0$.

We give an orientation on each facet F_i and $B_i^{\pm}$ such that the orientation of B_i^+ is exactly the inverse orientation of B_i^-, so

$$\partial P_Q = \sum_{F \in \mathcal{F}(P_Q)} F = F_1 + \dots + F_{m'} + B_1^+ + \dots + B_{\mathfrak{g}}^+ + B_1^- + \dots + B_{\mathfrak{g}}^- = \sum_{F \in \mathcal{F}(P_Q) - \mathcal{F}_B} F$$

where m' is the number of all facets in $\mathcal{F}(P_Q) - \mathcal{F}_B$.

Let $c_n = \sum_{g \in (\mathbb{Z}_2)^m} n_g (P, g)$ be an n-cycle of $C_n(M)$ where $n_g \in \mathbb{Z}$. Then

$$\partial(c_n) = \left[\sum_{g \in (\mathbb{Z}_2)^m} n_g \sum_{F \in \mathcal{F}(P_Q)} (F, g) \right] = \sum_{[F, g] \in ((\mathcal{F}(P_Q) - \mathcal{F}(B)) \times (\mathbb{Z}_2)^m)/{\sim'}} (n_g + n_{g e_F})[F, g] = 0,$$

which induces that $n_g = -n_{ge_F}$ for any facet $F \in \mathcal{F}(P_Q) - \mathcal{F}_B$ and $g \in (\mathbb{Z}_2)^m$. Let $l(g)$ denote the word length of g presented by $\{e_F\}$. For any $g \in (\mathbb{Z}_2)^m$, there exists a subset $\mathcal{I}_g = \{F_{i_1}, \dots, F_{i_k}\}$ of $\mathcal{F}(P_Q) - \mathcal{F}_B$ such that $g = \prod_{F \in \mathcal{I}_g} e_F$. Then we see easily that

$$n_g = -n_{ge_{F_{i_1}}} = n_{ge_{F_{i_1}} e_{F_{i_2}}} = \cdots = (-1)^{l(g)} n_{g \prod_{F \in \mathcal{I}_g} e_F} = (-1)^{l(g)} n_{\mathrm{e}_0}$$

so $c_n = n_{\mathrm{e}_0} \sum_{g \in (\mathbb{Z}_2)^m} (-1)^{l(g)} (P, g)$. Then we obtain that $H_n(M; \mathbb{Z}) = \ker \partial_n \cong \mathbb{Z}$ is generated by $\sum_{g \in (\mathbb{Z}_2)^m} (-1)^{l(g)} (P, g)$; hence M is orientable. □

7.1 Nonpositive curvature

Recall that a geodesic metric space X is *nonpositively curved* if it is a locally CAT(0) space. In general, X is said to be of *curvature* $\leq k$ (in the sense of Alexandrov) if it is locally a CAT(k) space. See [7, Definition 1.2, page 159].

Theorem 7.3 [7, Theorem 1A.6, page 173] *A smooth Riemannian manifold N is of curvature $\leq k$ in the sense of Alexandrov if and only if the sectional curvature of N is $\leq k$.*

Proposition 7.4 [50, Corollary 6.2.4] *If (N, g) is a complete Riemannian manifold of nonpositive curvature, then the fundamental group is torsion free.*

Let Q be a simple handlebody of dimension $n \geq 3$ and genus $\mathfrak{g}$, and $M \to Q$ be the smooth manifold double over Q, as defined in (2-1). Let P_Q be the simple polytope obtained from Q by cutting open along $\mathfrak{g}$ disjoint cutting belts $B_1, \dots, B_{\mathfrak{g}}$ in Q. More precisely, P_Q can be obtained in the following way: For each belt B_i, choose a regular neighborhood $N(B_i)$ of B_i that is homeomorphic to $B_i \times [-1, 1]$ as manifolds with corners. Clearly $N(B_i)$ is identified with a simple polytope, and it can also be understood as the disk D^1-bundle of the trivial normal bundle of B_i in Q. Then we get P_Q by removing the interiors of trivial D^1-bundles $B_i \times [-1, 1]$ of all B_i.

In order to use the Gromov lemma as above, we need a cubical cellular structure of the manifold double M over Q. For this, we perform the following procedure:

1. First we decompose Q into more pieces,
$$Q = P_Q \bigcup_{i=1}^{\mathfrak{g}} N^+(B_i) \cup N^-(B_i),$$
where $N^+(B_i) = B_i \times [0, 1]$ and $N^-(B_i) = B_i \times [-1, 0]$ satisfy $N(B_i) = N^+(B_i) \cup N^-(B_i)$.
2. Next, the standard cubical decompositions of P_Q and all $N^\pm(B_i)$ determine a right-angled Coxeter cubical cellular decomposition of Q, denoted by $\mathscr{C}(Q)$. Specifically, all cone points of P_Q and all $N^\pm(B_i)$ will be 0-cells with trivial local group in $\mathscr{C}(Q)$. There are two kinds of $k(>0)$-cubes in the cubical decompositions of P_Q and all $N^\pm(B_i)$, each of which either intersects transversely with an $(n-k)$-face $f^{n-k} = F_{i_1} \cap \cdots \cap F_{i_k}$ or intersects transversely with an $(n-k)$-face $f^{n-k} = F_{i_1} \cap \cdots \cap F_{i_{k-1}} \cap B_i$. The first type of cubes determine right-angled Coxeter cubical

cells of the form $e^k/(\mathbb{Z}_2)^k$ in $\mathscr{C}(Q)$, and the second type of cubes determine right-angled Coxeter cubical cells of the form $e^k/(\mathbb{Z}_2)^{k-1}$. Then, $\mathscr{C}(Q)$ is obtained by attaching each pair associated with B_i of the second type of cubes together. It is clear that $\mathscr{C}(Q)$ is a right-angled Coxeter cubical cellular decomposition of Q.

(3) Finally, by pulling back $\mathscr{C}(Q)$ to M via the covering map $p\colon M \to Q$, one can obtain a cubical cellular decomposition of M, denoted by $\mathscr{C}(M)$, such that each cube in $\mathscr{C}(M)$ is a connected component of $p^{-1}(c)$ for c in $\mathscr{C}(Q)$. In particular, all vertices in $\mathscr{C}(M)$ exactly consist of the liftings of all cone points in $\mathscr{C}(Q)$.

Lemma 7.5 *Let v be a vertex in $\mathscr{C}(M)$. Then $\mathrm{Lk}(v)$ in $\mathscr{C}(M)$ is combinatorially isomorphic to one of the nerves $\mathcal{N}(P_Q)$, $\mathcal{N}(N^+(B_i))$ or $\mathcal{N}(N^-(B_i))$.*

Proof In fact, if $p(v)$ is the cone point of P_Q, then each $k(>0)$-cube adjacent to v gives a $(k-1)$-simplex in $\mathcal{N}(P_Q)$, which corresponds to an $(n-k)$-face of P_Q. Therefore, $\mathrm{Lk}(v) \cong \mathcal{N}(P_Q)$. The same argument can be applied to the case where $p(v)$ is the cone point of $N^+(B_i)$ or $N^-(B_i)$. □

Proposition 7.6 *M is nonpositively curved if and only if Q is flag.*

Proof Let $\mathscr{C}(M)$ be the cubical cellular decomposition of M discussed above. The Gromov lemma (see Proposition 5.2) tells us that M is nonpositively curved if and only if the link of each vertex in the cubical cellular decomposition of M is flag. By Lemma 7.5, the latter of the above statement means that $\mathcal{N}(P_Q)$ and all $\mathcal{N}(N^{\pm}(B_i))$ are flag, so P_Q and all $N^{\pm}(B_i)$ are flag simple polytopes. This is also equivalent to saying that Q is flag. Thus, if a simple handlebody Q is flag, then its manifold double M is nonpositively curved.

Conversely, if M is nonpositively curved, then by the Cartan–Hadamard theorem, M is aspherical. Then Q is orbifold-aspherical. By Theorem A, Q is flag. □

7.2 Strictly negative curvature

Proposition 7.7 (Preissmann [50, Theorem 6.2.6]) *If (N, g) is a compact manifold of negative curvature, then any abelian subgroup of the fundamental group is cyclic.*

Proposition 7.8 (Gromov [32]) *Let $\mathscr{C}$ be a cubical complex. Suppose that the link of each vertex in $\mathscr{C}$ is flag and contains no $\square$. Let us give $\mathscr{C}$ a $(N, -\epsilon)$ geometry, where each cube in $\mathscr{C}$ is isomorphic to the unit cube in the hyperbolic space of curvature $-\epsilon$. If ϵ is sufficiently small then $K(\mathscr{C}) \le -\epsilon$.*

Lemma 7.9 *Let Q be a simple handlebody with m facets, and M be the manifold double over Q. Then $\mathbb{Z} \oplus \mathbb{Z} < \pi_1(M)$ if and only if $\mathbb{Z} \oplus \mathbb{Z} < \pi_1^{\mathrm{orb}}(Q)$.*

Proof If $\mathbb{Z} \oplus \mathbb{Z} < \pi_1(M)$, then the short exact sequence

$$1 \to \pi_1(M) \to \pi_1^{\mathrm{orb}}(Q) \xrightarrow{\lambda} (\mathbb{Z}_2)^m \to 1$$

induces that $\mathbb{Z}\oplus\mathbb{Z} < \pi_1(M) < \pi_1^{\mathrm{orb}}(Q)$. Conversely, if $\mathbb{Z}\oplus\mathbb{Z} < \pi_1^{\mathrm{orb}}(Q)$, then by Theorem B, there is a $\square$-belt in Q and $\mathbb{Z}\oplus\mathbb{Z} < \pi_1^{\mathrm{orb}}(Q)$ is generated by two pairs of disjoints facets in $\square$. Denote the generators of $\mathbb{Z}\oplus\mathbb{Z}$ by $x = s_1s_3$ and $y = s_2t_1t_2\cdots t_ks_4t_k^{-1}\cdots t_2^{-1}t_1^{-1}$ where each t_i is one of $\{t_B^{\pm1}\}$; then $\lambda(x^2) = \lambda(y^2) = 1$. Hence $x^2, y^2 \in \ker\lambda \cong \pi_1(M)$. So $\mathbb{Z}\oplus\mathbb{Z} < \pi_1(M)$. □

Proposition 7.10 *If M admits a strictly negative curvature, then Q is flag and contains no $\square$-belt. Specially, if Q is a simple polytope, then M admits a strictly negative curvature if and only if Q is flag and contains no $\square$-belt.*

Proof If M admits a strictly negative curvature, then by Proposition 7.6, Q is flag. Furthermore, if there is a $\square$-belt in Q, then by Theorem B, there is a rank-two abelian subgroup in $\pi_1^{\mathrm{orb}}(Q)$. By Lemma 7.9, there is a subgroup $\mathbb{Z}\oplus\mathbb{Z}$ in $\pi_1(M)$. By Preissmann's result (Proposition 7.7), M cannot admit a strictly negative curvature. Hence, Q contains no $\square$-belt.

When Q is a simple polytope, then the standard cubical cellular decomposition of Q induces a cubical cellular decomposition of M, denoted by $\mathscr{C}(M)$, such that each point $v \in \mathscr{C}(M)$ has link $\mathcal{N}(Q)$. By a result of Gromov (Proposition 7.8), if $\mathcal{N}(Q)$ is flag and contains no $\square$-belt, then M admits a strictly negative curvature. □

Remark 7.11 We are inclined to think that M admits a strict negative curvature if a simple handlebody Q is flag and contains no $\square$-belt. But we cannot find a suitable cubical decomposition of M so as to use Gromov's result. Of course, this is related to the weak hyperbolization conjecture: "Let N be a closed aspherical manifold. Then either $\pi_1(N)$ contains $\mathbb{Z}\oplus\mathbb{Z}$ or $\pi_1(N)$ is Gromov-hyperbolic". See [37, Conjecture 20.12].

7.3 Hyperbolic curvature

When Q is a simple 3-polytope, the Pogorelov theorem, which is right-angled case of the Andreev theorem (see [2; 53]), states that Q admits a right-angled hyperbolic structure if and only if it is flag and contains no 4-belt, where "right-angled" means that all dihedral angles are $\pi/2$. This gives a combinatorial equivalent description of the hyperbolicity of simple 3-polytopes as a right-angled Coxeter orbifold. Now Q is also called a (right-angled) hyperbolic polyhedra in $\mathbb{H}^3$.

As a generalization of 3-dimensional hyperbolic polyhedra, a 3-manifold with corners is *hyperbolic* if its interior admits a hyperbolic metric which can be extended to the boundary such that its all faces are totally geodesic (or locally convex). Moreover we say that a 3-manifold with corners is *right-angled hyperbolic* if its all dihedral angles are $\pi/2$.

Notice that a hyperbolic 3-manifold with corners is not right-angled in general. A 3-manifold with corners can be equipped with many different orbifold structures. A hyperbolic structure on a 3-manifold with corners should be compatible with an orbifold structure on it. Hence there may be different hyperbolic structures on a 3-manifold with corners. The hyperbolization of 3-manifolds with corners corresponds to the generalization of the Andreev theorem [2; 53]. This question is still open now. However, the hyperbolic structure of a hyperbolic closed 3-orbifold (or 3-manifold) is unique by the Mostow rigidity theorem (see [45]).

Here we mainly consider the right-angled hyperbolicity of simple 3-manifolds with corners, where a simple 3-manifold with corners is given by forgetting the orbifold structure on a simple 3-orbifold. Then one can obtain the same understanding for right-angled hyperbolicity from the following two geometric objects:

(1) a right-angled hyperbolic simple 3-manifold with corners;

(2) a hyperbolic simple 3-orbifold (as a right-angled Coxeter 3-orbifold).

Thus, a right-angled hyperbolic simple 3-manifold with corners is a hyperbolic simple 3-orbifold, and vice versa.

Proposition 7.12 *Let Q be a 3-dimensional simple handlebody. Then M is hyperbolic if and only if Q is flag and contains no $\square$-belt.*

Proof Together with Perelman's work, Thurston's hyperbolization theorem implies that a closed orientable 3-manifold is hyperbolic if and only if it is aspherical and atoroidal, where if there is no subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$, then M is atoroidal. By Lemma 7.2, M is always orientable. Together with Theorems A and B and Lemma 7.9, we know that M is aspherical and atoroidal if and only if Q is flag and contains no $\square$-belt. □

By 3-dimensional hyperbolic manifold theory (see [49]), M is hyperbolic if and only if Q is hyperbolic. Hence, we have:

Corollary 7.13 *Q is right-angled hyperbolic as a 3-manifold with corners if and only if Q is flag and contains no $\square$-belt.*

For a higher-dimensional simple handlebody, the fundamental group of a closed hyperbolic manifold is a discrete convex-cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, and contains no subgroup $\mathbb{Z} \oplus \mathbb{Z}$. Hence, if M is hyperbolic, then Q must be flag and contains no $\square$-belt. By a result in [18, Corollary 6.11.6] there must exist a triangle or quadrilateral face in a simple polytope of dimension greater than 4. So

- a simple handlebody with dimension greater than 4 cannot be hyperbolic.

In the 4-dimensional case, it is not clear whether only 120-cells are hyperbolic simple 4-polytopes; see [30].

7.3.1 3-handlebodies with simplicial nerve A 3-*handlebody with simplicial nerve* is just a simple 3-orbifold whose underlying space is a 3-handlebody. Let Q be a 3-handlebody with simplicial nerve and with genus $\mathfrak{g} > 0$. Next we show that Q can always be cut into a simple polytope along some codimension-one B-belts.

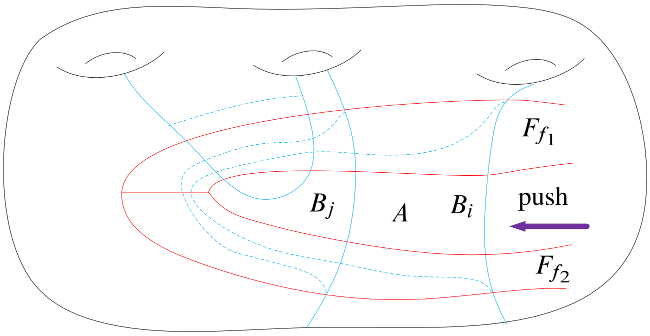


Figure 10: Modifying the boundary of B_i.

Lemma 7.14 *A* 3*-handlebody with simplicial nerve is a simple* 3*-handlebody.*

Proof Let Q be a 3-handlebody of genus $\mathfrak{g} \geq 0$ with simplicial nerve. If $\mathfrak{g} = 0$, then it is easy to check that Q is a simple 3-polytope by the Steinitz theorem and [41, Proposition 3.4]. If $\mathfrak{g} > 0$, let

$$\{(D_i^2, \partial D_i^2) \hookrightarrow (|Q|, \partial|Q|) \mid i = 1, 2, \ldots, \mathfrak{g}\}$$

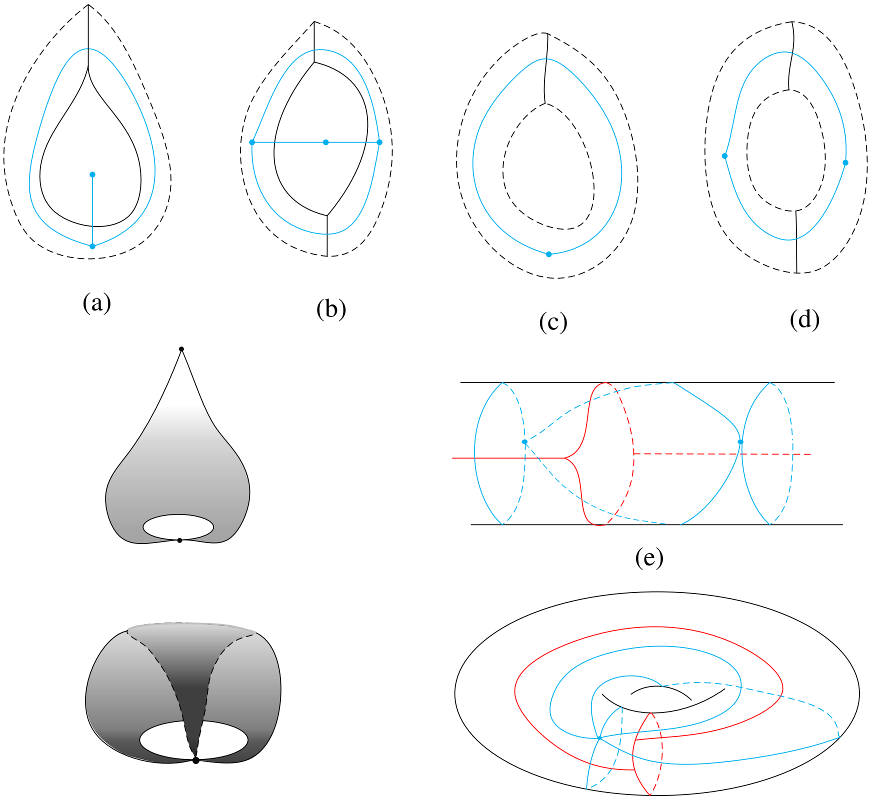
be some disjoint compressing 2-disks in $|Q|$ such that $|Q|$ is cut into a connected 3-ball along those compressing 2-disks. Considering the facial structure determined by the triangulation $\mathcal{N}(Q)$ of $\partial|Q|$, we can always do some slight deformations for the boundaries of compressing 2-disks on faces of Q, so that $\{D_i^2\}$ can be modified into some embedded suborbifolds $\{B_i\}$ of preserving codimension in Q. Each B_i is a polygon.

Given a B_i, by the definition of B-belts, we see that B_i is not a B-belt if and only if there exist two nonadjacent edges f_1 and f_2 in B_i such that

(i) $F_{f_1} \cap F_{f_2} \neq \varnothing$ (probably F_{f_1} and F_{f_2} can even be the same face of Q);

(ii) $F_{f_1} \cup F_{f_2}$ can deformatively retract onto B_i in $|Q|$ (in fact, $F_{f_1} \cup F_{f_2}$ can deformatively retract onto ∂B in $\partial|Q|$);

where F_{f_1} and F_{f_2} are two 2-faces of Q that contain f_1 and f_2 respectively. So, if B_i is not a B-belt, then there is no hole in the area A in $\partial|Q|$ surrounded by F_{f_1}, F_{f_2} and B_i. Then we can modify the boundary of B_i by pushing the retract of $F_{f_1} \cup F_{f_2}$ into F_{f_1} and F_{f_2}, and throwing some edges of B_i away, as shown in Figure 10, so that one can obtain a new B_i' with fewer edges which intersects transversely with $F_{f_1} \cap F_{f_2}$. In particular, if $F_{f_1} = F_{f_2}$, then f_1 and f_2 will become the same edge in B_i', and if $F_{f_1} \neq F_{f_2}$ then f_1 is adjacent to f_2 in B_i'. In addition, if there is also another suborbifold B_j which intersects with the area A in $\partial|Q|$, this means that B_j is not a belt, too. The above "pushing" process will move the boundary of B_j out from the area A and modify B_j into B_j' with fewer edges such that $B_i' \cap B_j' = \varnothing$. Since B_i is a polygon with finite edges, this process can end after a finite number of steps until one has modified B_i into an B-belt which does not intersect with other B_j.



(a) (b) (c) (d)

(e)

ideal triangles

(f)

Figure 11: Ideal nerves.

We can perform the same procedure to other non-B-belts in $\{B_j\}_{j\neq i}$. Finally one can obtain a set of disjoint cutting belts such that Q is cut open into a simple 3-polytope along those cutting belts, implying that Q is a simple 3-handlebody. □

Hence, Proposition 7.12 still holds for 3-handlebodies with simplicial nerve.

Proposition 7.15 *A* 3*-handlebody with simplicial nerve* (*the existence of cutting belts is intrinsic*) *is right-angled hyperbolic as a* 3*-manifold with corners if and only if* Q *is flag and contains no* $\square$*-belt.*

7.3.2 3-handlebodies with ideal nerve We say that Q is a 3-*handlebody with ideal nerve* if Q is a right-angled Coxeter 3-orbifold such that its underlying space $|Q|$ is a 3-handlebody and its nerve is an ideal triangulation of the boundary $\partial|Q|$.

Now let Q be a 3-handlebody with ideal nerve. Then, by the definition of ideal triangulation [27, Definition 2.6], the interior of each face of Q is also contractible. On the facial structure of Q, there are three possible cases:

- Some 2-faces of Q are henagons (i.e. 2-faces with only one point of codimension 3 in Q; see (a) in Figure 11) or digons (i.e. 2-faces with only two points of codimension 3 in Q; see (b) in Figure 11).
- There may be some 2-faces with self-intersection (see (c) in Figure 11).
- The intersection of two 2-faces may be not connected (see (d) in Figure 11).

If there is a henagon 2-face of Q, then it gives a self-folded ideal triangle. For example, see the blue part of (a) in Figure 11. In general, if there is a henagon 2-suborbifold in Q, then the nerve of associated faces may give some ideal triangles, such as (e) and (f) in Figure 11. In particular, the nerve of (f) contains only one vertex and two ideal triangles glued along their three edges as shown in Figure 11. All those cases agree with the definition of ideal triangulations in [27, Definition 2.6].

Lemma 7.16 *Let Q be a 3-handlebody with ideal nerve. Then Q is very good if and only if it does not contain a henagon 2-suborbifold.*

Proof By applying a theorem of Morgan or Kato (see [37, Theorem 6.14]), each compact locally reflective 3-orbifold that contains no bad 2-suborbifolds is very good. This means that if there is no henagon 2-suborbifold in Q, then Q is very good. Conversely, if there is a henagon 2-suborbifold in Q, then it is obvious that Q is bad. □

Hence, if there is a henagon 2-suborbifold of Q, then Q cannot be hyperbolic.

Suppose that Q contains no henagon 2-suborbifolds. Then, by Lemma 7.16, Q can be covered finitely by a closed 3-manifold M. In general, Q is not nice in the sense of Davis [18, page 180]; thus there is no natural manifold double defined as in (2-1) for Q.

A digon 2-suborbifold in Q is said to be *essential* if its two vertices are not contained in a unique edge of Q. If there is an essential digon 2-suborbifold in Q, then its nerve $\mathcal{N}(Q)$ will contain two simplices with common vertices. See (d) in Figure 11.

Lemma 7.17 *Let Q be a 3-handlebody with ideal nerve. Assume that there is no henagon suborbifold in Q, and M is a covering manifold over Q. If Q contains an essential digon suborbifold, then M is reducible.*

Proof Assume that two edges of a digon are contained in two 2-faces F_1 and F_2 of Q. Then we consider the double cover of Q, denoted by D_Q, which is obtained by gluing two copies of Q along F_1. At the same time, two copies of F_2 are also glued along $F_1 \cap F_2$, giving an annulus in D_Q. Let M' be a manifold double over D_Q. Then M' can be decomposed into the connected sum of some 3-manifolds, which implies that M' is reducible. Hence, D_Q and Q are reducible. So M is reducible. □

A digon 2-face of Q will give an essential digon 2-suborbifold in Q unless that Q is a trihedron ($S^3/(\mathbb{Z}_2)^3$). Thus in this case Q is reducible as well. Therefore, if there is a henagon 2-suborbifold or an essential digon 2-suborbifold in Q, then Q cannot be hyperbolic.

Next, suppose that Q is not a trihedron and contains no henagon and essential digon 2-suborbifolds. If there are some 2-faces with self-intersection or the intersection of two 2-faces is not connected, then we can always construct some orbifold covers of Q. In fact, we can use some copies of Q to construct a covering space of Q as follows. First we cut open each of copies by using a fixed 2-suborbifold B, and then form a connected handlebody $\widehat{Q}$ by attaching them together along those new facets produced by B. If necessary, we can choose enough copies of Q so as to make sure that this connected handlebody is simple, and is exactly the required covering space of Q. Applying Theorem A gives:

Corollary 7.18 *A 3-handlebody with ideal nerve is hyperbolic if and only if it is not trihedron, tetrahedron and contains no $\triangle^2$, $\square$-belts and no henagon or essential digon 2-suborbifolds.*

Remark 7.19 Let Q be a 3-handlebody with ideal nerve. We can define henagon 2-suborbifolds and essential digons 2-suborbifolds in Q as 1- and 2-belts of Q, respectively. Then by Lemma 7.16, Q is very good if and only if Q contains no 1-belts. An easy argument gives that a very good Q is flag if and only if it is not trihedron or tetrahedron (i.e. $S^3/(\mathbb{Z}_2)^3$ or $S^3/(\mathbb{Z}_2)^4$) and contains no 2- or 3-belts (i.e. π_1-injective $S^2/(\mathbb{Z}_2)^2$- or $S^2/(\mathbb{Z}_2)^3$-suborbifolds). Furthermore, a very good flag Q is hyperbolic if and only if it contains no 4-belts (i.e. π_1-injective $T^2/(\mathbb{Z}_2)^2$-suborbifolds). Thus, a right-angled Coxeter 3-handlebody with ideal nerve except trihedron and tetrahedron is hyperbolic if and only if it contains no 1-, 2-, 3- or 4-belts.

7.3.3 Example of a nonsimple 3-handlebody

Example 7.20 Let P be the product of a pentagon and $[0,1]$. Gluing two opposite pentagons of P together such that its diagonal vertices coincide with each other gives a right-angled Coxeter 3-orbifold with its underlying space as a solid torus, denoted by Q. Then Q is a Seifert 3-orbifold. Thus it cannot be hyperbolic. This is because each embedding annulus 2-facet is an obstruction.

7.4 Positive (scalar) curvature

A simple polytope P is *two-neighborly* if any two facets of P has a nonempty intersection. So it is clear that P is two-neighborly if and only if it contains no I-belt. In addition, we also know from [38, Proposition 2.1] that P is two-neighborly if and only if its manifold double M is simply connected. Thus, we have that:

Lemma 7.21 *Let P be a simple polytope. Then the following statements are equivalent.*

- *P is two-neighborly.*
- *P contains no I-belt.*
- *Its manifold double M is simply connected.*

Remark 7.22 Similarly, for a simple handlebody Q, we may define it to be *two-neighborly* if any two vertices in its nerve $\mathcal{N}(Q)$ are connected by an edge. However, if the genus of Q is greater than zero, then we can always find an I-belt between two facets with nonempty intersection such that this I-belt cannot be deformed onto the intersection of two facets in Q. Hence the existence of I-belts cannot be used to detect whether Q is two-neighborly.

Lemma 7.23 *Let N be a triangulable closed n-manifold with $n > 1$. If $\pi_1(N)$ is nontrivial, then the 1-skeleton of any triangulation of N cannot be a complete graph.*

Proof Assume that K is a triangulation of N whose 1-skeleton K^1 is a complete graph. Fixed a vertex x in K, let $N(x)$ be the union of those n-simplices in K which contain x. Then $N(x) \cong D^n$ whose boundary $\partial N(x)$ is a two-neighborly simplicial $(n-1)$-sphere.

Let Δ^n be an arbitrary simplex which does not contain x. Since K^1 is a complete graph, all vertices of Δ^n are contained in $\partial N(x)$. Hence, K^1 is a subcomplex of $N(x)$. So any closed loop in K^1 is contractible in $N(x)$. This means that N is simply connected, giving a contradiction. □

Corollary 7.24 *A (flag) simple handlebody with genus >0 cannot be two-neighborly. In other words, a two-neighborly simple handlebody must be a two-neighborly simple polytope as a manifold with corners.*

Proposition 7.25 (Hopf–Rinow, Myers, [50, Corollary 6.3.2 and Theorem 6.3.3]) *If an n-dimensional closed manifold N admits a complete Riemannian metric of positive sectional curvature, then $\pi_1(N)$ is finite. Specially, $\pi_1(N)$ is 0 or $\mathbb{Z}_2$ if n is even, and N is orientable if n is odd.*

If N admits a complete Riemannian metric of positive Ricci curvature, then $\pi_1(N)$ is finite, too.

For a simple handlebody Q, by the short exact sequence

$$1 \to \pi_1(M) \to \pi_1^{\mathrm{orb}}(Q) \xrightarrow{\lambda} (\mathbb{Z}_2)^m \to 1,$$

if the genus of Q is greater than zero, then any torsion-free generator t determined by a cutting belt is mapped to $1 \in (\mathbb{Z}_2)^m$ via λ. Hence t gives a torsion-free element in $\pi_1(M)$. So we have:

Lemma 7.26 *If the genus of Q is greater than zero, then $\pi_1(M)$ is not finite.*

As a direct consequence of Proposition 7.25 and Lemmas 7.21 and 7.26, we have:

Corollary 7.27 *If M admits a complete Riemannian metric of positive sectional or Ricci curvature, then Q must be a two-neighborly simple polytope, that is, there is no I-belt in Q.*

Conversely, the existence of positive sectional curvature and positive Ricci curvature of a closed (or compact) n-manifold is a very hard question, which is involved in many conjectures and open questions. For example, we know that the real moment angle manifold over $\Delta^2 \times \Delta^2$ is $S^2 \times S^2$. However, it is well known that the existence of positive sectional curvature on $S^2 \times S^2$ is just the Hopf conjecture.

On the other hand, there are some results about the existence of positive scalar curvature. One can refer to some works of Gromov and Lawson [34; 33; 35], Schoen and Yau [55; 56], and Stolz [57]. By Gromov and Lawson [35], a compact manifold of nonpositive sectional curvature cannot carry a metric of positive sectional curvature. So

- if M admits a positive scalar curvature, then Q is not flag.

Moreover, it is reasonable to conjecture that:

Conjecture 7.28 *If a simple polytope Q is two-neighborly, then M admits a positive scalar curvature.*

In the 3-dimensional case, Wu and Yu [61] gave a combinatorial description for the case of real moment-angled manifolds with positive scalar curvature over simple 3-polytopes.

Proposition 7.29 [61, Corollary 4.10] *A real moment-angle manifold (or small cover) over a simple 3-polytope P can admit a Riemannian metric with positive scalar curvature if and only if P is combinatorially equivalent to a polytope obtained from Δ^3 by a sequence of vertex cuts.*

Let P be a simple polytope obtained from Δ^3 by a sequence of vertex cuts. Then except for $P = \Delta^3$, any 2-dimensional belt in P is a Δ^2-belt. Conversely, assume that every 2-dimensional belt is only a Δ^2-belt in a simple polytope P, then it is easy to see that P is a tetrahedron or P can be decomposed into the connected sum of some Δ^3's. This is equivalent to saying that P can be obtained from Δ^3 by a sequence of vertex cuts. Hence, we get an equivalent description of Proposition 7.29 in terms of Δ^2-belts and Δ^3-belts.

Corollary 7.30 *Let P be a simple 3-polytope. Then its manifold double M admits a Riemannian metric with positive scalar curvature if and only if every 2-dimensional belt in P is Δ^2, or P is just a tetrahedron.*

We summarize what we have discussed in Table 1.

M Q	2-neighborly	flag	Pogorelov	description
Sec < 0	⬏ not	⬏	⬏ ↲ dim 3	Proposition 7.10
hyperbolic	⬏ not	⬏	⬏ ↲ dim 3	dim 3: Proposition 7.12; dim 4: not clear; dim ≥ 5: none
Sec ≤ 0, NPC	⬏ not	⬏ ↲	↲	Proposition 7.6
flat	⬏ not	⬏	⬏ not	[38, Theorem 1.2]
spherical	⬏	⬏ not	⬏ not	[38, Theorem 1.2]
Sec, Ric > 0	⬏	⬏ not	⬏ not	not clear (e.g. Hopf conjecture)
scalar > 0	↲ Conjecture 7.28	⬏ not	⬏ not	dim 3: [61]; dim > 3: not clear

Table 1: A simple handlebody is called *Pogorelov* if it is flag and contains no □-belt.

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