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Property (QT) for 3-manifold groups

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According to Bestvina, Bromberg and Fujiwara, a finitely generated group is said to have property (QT) if it acts isometrically on a finite product of quasitrees so that orbital maps are quasi-isometric embeddings. We prove that the fundamental group $\pi_1(M)$ of a compact, connected, orientable 3-manifold M has property (QT) if and only if no summand in the sphere-disc decomposition of M supports either Sol or Nil geometry. In particular, all compact, orientable, irreducible 3-manifold groups with nontrivial torus decomposition and not supporting Sol geometry have property (QT). In the course of our study, we establish property (QT) for the class of Croke–Kleiner admissible groups and for relatively hyperbolic groups under natural assumptions on the peripheral subgroups.

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1 Introduction

1.1 Background and motivation

The study of group actions on quasitrees has recently received a great deal of interest. A *quasitree* means here a possibly locally infinite connected graph that is quasi-isometric to a simplicial tree. Groups acting on (simplicial) trees have been well understood thanks to Bass–Serre theory. On the one hand, quasitrees have the obvious advantage of being more flexible; hence, many groups can act unboundedly on quasitrees but act on any trees with global fixed points. Many hyperbolic groups with Kazhdan's property (T) and mapping class of groups are among many examples that belong to this category (see Manning [38; 39] for other examples). In effect, these are sample applications of a powerful axiomatic construction of quasitrees proposed in the work of Bestvina, Bromberg and Fujiwara [5]. This construction will be fundamental in this paper.

We say that a finitely generated group G has *property* (*QT*) if it acts isometrically on a finite product $X = T_1 \times T_2 \times \cdots \times T_n$ of quasitrees with the L^2-metric such that for any basepoint $o \in X$, the induced orbit map

$$g \in G \mapsto go \in X$$

is a quasi-isometric embedding of G equipped with some (or any) word metric d_G to X. Informally speaking, property (QT) gives an undistorted picture of the ambient group in a reasonably good space.

Here, the direct product structure usually comes from the independence of several negatively curved layers endowed on the group. Such a hierarchy structure has emerged from the study of mapping class groups since Masur and Minsky [40]. In addition, property (QT) is a commensurability invariant as observed by Bestvina, Bromberg and Fujiwara [6] and Button [14], and could be thought of as a stronger property than the finiteness of asymptotic dimension.

Extending their earlier results of [5], Bestvina, Bromberg and Fujiwara [6] recently showed that residually finite hyperbolic groups and mapping class groups have property (QT). It is known that Coxeter groups have property (QT) (see Dranishnikov and Januszkiewicz [23]), and thus every right-angled Artin group has property (QT) (see [6, Induction 2.2]).

In 3-manifold theory, the study of the fundamental groups of 3-manifolds is one of the most central topics. Determining property (QT) of finitely generated 3-manifold groups is the main task of the present paper.

1.2 Property (QT) of 3-manifold groups

Let M be a 3-manifold with finitely generated fundamental group. Since property (QT) is a commensurability invariant, we can assume that M is compact and orientable by considering the Scott core of M and a double cover of M (if M is nonorientable).

In recent years, the theory of special cube complexes — see Haglund and Wise [30] — has led to a deep understanding of 3-manifold groups; see Agol [3] and Wise [53]. By definition, the fundamental group of a compact special cube complex is undistorted in a right-angled Artin group, and then has property (QT) by [23]. However, 3-manifolds without nonpositively curved Riemannian metrics cannot be cubulated by Przytycki and Wise [46] and certain cubulated 3-manifold groups are not virtually compact special (see Hagen and Przytycki [28] and Tidmore [51]). Thus it was left still open to determine the property (QT) for all 3-manifold groups.

By the sphere-disc decomposition, a compact oriented 3-manifold M is a connected sum of prime summands M_i ($1 \le i \le n$) with incompressible boundary. It is an easy observation that if a group has property (QT) then every nontrivial element is undistorted (see Lemma 2.5), and hence if M_i supports Sol or Nil geometry from the eight Thurston geometries, then $\pi_1(M_i)$ fails to have property (QT). Our first main result is the following characterization of property (QT) for all 3-manifold groups.

Theorem 1.1 *Let M be a connected, compact, orientable 3-manifold. Then $\pi_1(M)$ has property (QT) if and only if no summand in its sphere-disk decomposition supports either Sol or Nil geometry.*

By standard arguments, we are reduced to the case where M is a compact, connected, orientable, irreducible 3-manifold with empty or tori boundary. Theorem 1.1 actually follows from the following theorem.

Theorem 1.2 *Let M be a compact orientable irreducible 3-manifold with empty or tori boundary, and with nontrivial torus decomposition which does not support the Sol geometry. Then $\pi_1(M)$ has property (QT).*

A 3-manifold M as in Theorem 1.2 is called a *graph manifold* if all the pieces in its torus decomposition are Seifert fibered spaces; otherwise M is called a *mixed manifold*. It is well known that the fundamental group of a mixed 3-manifold is hyperbolic relative to a collection of abelian groups and graph manifold groups. As a result, to prove Theorem 1.2, we actually only need to determine the property (QT) of *Croke–Kleiner admissible groups*, and of relatively hyperbolic groups, which will be discussed in detail in the following subsections. These results include but are much more general than the fundamental groups of graph manifolds and mixed manifolds.

1.3 Property (QT) of Croke–Kleiner admissible groups

We first address property (QT) of graph manifolds. Our approach is based on a study of a particular class of graph of groups introduced by Croke and Kleiner [21] which they called *admissible groups* and generalized the fundamental groups of graph manifolds. We say that an admissible group G is a *Croke–Kleiner admissible group* or a *CKA group* if it acts properly discontinuously, cocompactly and by isometries on a complete proper CAT(0) space X. Such an action $G \curvearrowright X$ is called a *CKA action* and the space X is called a *CKA space*. The CKA action is modeled on the JSJ structure of graph manifolds where the Seifert fibration is replaced by the following central extension of a general hyperbolic group H_v:

$$1 \to Z(G_v) \to G_v \to H_v \to 1 \tag{1}$$

where $Z(G_v) = \mathbb{Z}$. It is worth pointing out that CKA groups encompass a much more general class of groups and can be used to produce interesting groups by a "flip" trick from any finite number of hyperbolic groups (see Example 2.14).

The notion of an *omnipotent* group was introduced by Wise [52] and has found many applications in subgroup separability. We refer the reader to Definition 4.6 for its definition and note here that free groups [52], surface groups (see Bajpai [4]), and the more general class of virtually special hyperbolic groups [53] are omnipotent. Nguyen and Yang [43] proved property (QT) for a special class of CKA actions under flip conditions (see Definition 2.18). One of the main contributions of this paper is to remove this assumption and prove the following result in full generality.

Theorem 1.3 *Let $G \curvearrowright X$ be a CKA action where for every vertex group the central extension (1) has omnipotent hyperbolic quotient group. Then G has property (QT).*

Remark 1.4 It is a long-standing problem whether every hyperbolic group is residually finite. Wise [52, Remark 3.4] noted that if every hyperbolic group is residually finite, then any hyperbolic group is omnipotent.

Let us comment on the relation between this work and the previous one [43]. As in [43], the ultimate goal is to utilize Bestvina, Bromberg and Fujiwara's projection complex machinery to obtain actions on quasitrees. The common starting point is the class of special paths developed in [43] that record the distances of X. However, the flip assumption (see Definition 2.18) on CKA actions was crucially used there: the fiber lines coincide with boundary lines in adjacent vertex pieces when crossing the boundary plane, roughly speaking. Hence, a straightforward gluing construction works in that case but fails in our general setting. In this paper, we use a completely different projection system to achieve the same purpose with a more delicate analysis.

It is worth mentioning the following fact frequently invoked by many authors: the fundamental group of any graph manifold is quasi-isometric to the fundamental group of some flip manifold as defined by Kapovich and Leeb [34]. This simplification, however, is useless to address property (QT), as such a quasi-isometry does not respect the group actions. Conversely, our direct treatment of any graph manifolds (closed or with nonempty boundary) is new, and we believe it will potentially allow for further applications.

We now explain how we apply Theorem 1.3 to graph manifolds. If M is a graph manifold with nonempty boundary then it always admits a Riemannian metric of nonpositive curvature (see Leeb [35]). In particular, $\pi(M) \curvearrowright \widetilde{M}$ is a CKA action, and thus property (QT) of $\pi_1(M)$ follows immediately from Theorem 1.3. However, closed graph manifolds may not support any Riemannian metric of nonpositive curvature [35], so property (QT) in this case does not follow immediately from Theorem 1.3. We have to make certain modifications on some steps to run the proof of Theorem 1.3 for the fundamental groups of closed graph manifolds (see Section 8.2.1 for details).

1.4 Property (QT) of relatively hyperbolic groups

When M is a mixed 3-manifold, $\pi_1(M)$ is hyperbolic relative to the finite collection $\mathcal{P}$ of fundamental groups of maximal graph manifold components, isolated Seifert components, and isolated JSJ tori (see Bigdely and Wise [8] and Dahmani [22]). Therefore, we need to study property (QT) for relatively hyperbolic groups.

Our main result in this direction is a characterization of property (QT) for residually finite relatively hyperbolic groups, which generalizes the corresponding results of [6] on Gromov-hyperbolic groups.

Theorem 1.5 *Suppose that a finitely generated group H is hyperbolic relative to a finite set of subgroups $\mathbb{P}$. Assume that each $P \in \mathbb{P}$ acts by isometry on finitely many quasitrees T_i $(1 \le i \le n_P)$ such that the induced diagonal action on $\prod_{i=1}^{n_P} T_i$ has property (QT). If H is residually finite, then H has property (QT).*

Remark 1.6 Since maximal parabolic subgroups are undistorted, each $P \in \mathcal{P}$ obviously has property (QT) if G has property (QT). A nonequivariant version of this result was proven by Mackay and Sisto [37].

Remark 1.7 It is well known that mixed 3-manifold groups $G = \pi_1(M)$ are hyperbolic relative to a collection $\mathbb{P}$ of abelian groups and graph manifold groups $P = \pi_1(M_i)$. However, it is still insufficient to derive directly via Theorem 1.5 the property (QT) of G from that of graph manifold groups P asserted in Theorem 1.3, since P may not preserve factors in the finite product of quasitrees. Of course, passing to an appropriate finite-index subgroup $P' < P$ preserves the factors, but it is not clear at all whether P' are peripheral subgroups of a finite-index subgroup G' of G. In order to find such a G', a stronger assumption must be satisfied so that every finite-index subgroup of each P is separable in G. This requires the notion of a *full profinite topology* induced on subgroups (see the precise definition before Theorem 3.5 and a relevant discussion of Reid [47]). See Theorem 3.5 for the precise statement. In the setting of a mixed 3-manifold, Lemma 8.5 verifies that each peripheral subgroup $P \in \mathbb{P}$ of $\pi_1(M)$ satisfies this assumption. Therefore, all mixed 3-manifolds are proven to have property (QT).

We now explain a few algebraic and geometric consequences for groups with property (QT).

Similar to trees, any isometry on quasitrees must be either elliptic or loxodromic [38]. Hence, if a finitely generated group acts properly (in a metric sense) on a finite product of quasitrees, then every nontrivial element is undistorted (Lemma 2.5). Moreover, property (QT) allows one to characterize virtually abelian groups among subexponential growth groups and solvable groups.

Theorem 1.8 *Let G be a finitely generated group.*

1. *Assume that G has subexponential growth. Then G has property* (*QT*) *if and only if G is virtually abelian.*
2. *Suppose that G is solvable with finite virtual cohomological dimension. Then G has property* (*QT*) *if and only if it is virtually abelian.*

By Theorem 1.5, this yields as a consequence that nonuniform lattices in $\mathrm{SU}(n,1)$ and $\mathrm{Sp}(n,1)$ for $n \geq 2$ fail to act properly on finite products of quasitrees.

Corollary 1.9 *A nonuniform lattice in $\mathrm{SU}(n,1)$ for $n \geq 2$ or $\mathrm{Sp}(n,1)$ for $n \geq 1$ does not have property* (*QT*), *while any lattice of $\mathrm{SO}(n,1)$ has property* (*QT*) *for $n \geq 2$.*

Overview

The paper is structured as follows. In Section 2, we recall the preliminary materials about Croke–Kleiner admissible groups, axiomatic constructions of quasitrees, and we collect a few preliminary observations to prove Theorem 1.8 and to disprove property (QT) for the fundamental groups of 3-manifolds with Sol or Nil geometry. Section 3 contains a proof of Theorem 1.5 and its variant Theorem 3.5. The next four sections aim to prove Theorem 1.3: Section 4 first recalls a cone-off construction of CKA actions from [43] and then outlines the steps executed in Sections 5, 6, and 7 to prove property (QT) for CKA actions. Sections 5 and 6 explain in detail the construction of projection systems of fiber lines and then prove the corresponding distance formula. We finish the proof of Theorem 1.3 in Section 7. In Section 8, we present the applications of the previous results for 3-manifold groups and prove Theorem 1.2 as well as Theorem 1.1.

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2 Preliminaries

This section reviews concepts property (QT), Croke–Kleiner admissible actions, and the construction of quasitrees. Several observations are made to determine property (QT) of the fundamental groups of 3-manifolds with Sol or Nil geometry. This includes the fact that every element is undistorted in groups with property (QT) and some attempts to characterize by property (QT) the class of virtually abelian groups in solvable/subexponential growth groups.

In the sequel, we use the notion $a \preceq_K b$ if the exists $C = C(K) > 0$ such that $a \le Cb + C$, and $a \sim_K b$ if $a \preceq_K b$ and $b \preceq_K a$. Also, when we write $a \asymp_K b$ we mean that $a/C \le b \le Ca$. If the constant C is universal from context, the subindex $\preceq_K$ shall be omitted.

2.1 Property (QT)

Definition 2.1 We say that a finitely generated group G has *property* (*QT*) if it acts isometrically on a finite product $X = T_1 \times T_2 \times \cdots \times T_n$ of quasitrees with L^2-metric such that for any basepoint $o \in X$, the induced orbit map

$$g \in G \mapsto go \in X$$

is a quasi-isometric embedding of G equipped with some (or any) word metric d_G to X with the product metric d.

Remark 2.2 A group with property (QT) acts properly on a finite product of quasitrees in a metric sense: $d(o, go) \to \infty$ as $d_G(1, g) \to \infty$. We would emphasize that all consequences of the property (QT) in the present paper use merely the existence of a metric proper action.

By definition, a quasitree is assumed to be a graph quasi-isometric to a simplicial tree. This does not lose generality as any geodesic metric space (with an isometric action) is quasi-isometric to a graph (with an equivariant isometric action) by taking the 1-*skeleton of its Rips complex*: the vertex set consists of all points and two points with distance less than 1 are connected by an edge.

The first part of the following lemma allows one to pass to finite-index subgroups in the study of property (QT) of groups, as explained in [6, Section 2.2]. The second part of Lemma 2.3 is an immediate consequence of the definition of property (QT).

Lemma 2.3 (1) *Let $H \le G$ be a finite-index subgroup of G. Then G has property (QT) if and only if H has property (QT).*

(2) *Let $H \le G$ be an undistorted subgroup of G. Suppose that G has property (QT), then H also has property (QT).*

Below is a corollary of the de Rham decomposition theorem [26, Theorem 1.1] which will be utilized in the subsequent discussions.

Corollary 2.4 *A finite product $X = T_1 \times T_2 \times \cdots \times T_n$ of quasitrees must have de Rham decomposition*

$$X = \mathbb{R}^k \times T_{k+1} \times \cdots \times T_n$$

if the first k quasitrees ($k \ge 0$) are all real lines among $\{T_i \mid 1 \le i \le n\}$.

A finite product $\prod_{i=1}^n T_i$ of quasitrees has no $\mathbb{R}$*-factor* if no T_i is isometric to $\mathbb{R}$ or a point. In this case, the Euclidean factor $\mathbb{R}^k$ will disappear. In what follows, we present some general results about groups with property (QT).

Lemma 2.5 *Assume that G has property (QT). Then the subgroup generated by an element $g \in G$, is undistorted in G.*

Proof Let $X = \mathbb{R}^k \times T_{k+1} \times \cdots \times T_n$ be the de Rham decomposition of a finite product of quasitrees. By [26, Corollary 1.3], up to passage to finite-index subgroups, G acts by isometries on each factor $\mathbb{R}^k$, and T_i for $k+1 \le i \le n$. Let $g \in G$ be an infinite order element. If the image of g is an isometry on the Euclidean space $\mathbb{R}^k$, then it either fixes a point or preserves an axis. If the image of g is an isometry on a quasitree T_i then by [39, Corollary 3.2], it has either a bounded orbit or a quasi-isometrically embedded orbit.

Fix a basepoint $o = (o_k, o_{k+1}, \dots, o_n) \in X$. If the action of G on X is proper, then by the first paragraph, there must exist an unbounded action of $\langle g \rangle$ on some factor $Y = \mathbb{R}^k$ or $Y = T_i$, so we have $m \le \lambda |o_k - g^m o_k|_Y + c$ for some $\lambda, c > 0$. Since every isometric orbital map is Lipschitz, we have $|o - g^m o|_X \le C|1 - g^m|_G$ for some $C > 0$. Noting that $|o - g^m o|_Y \le |o - g^m o|_X$, we have that the map $m \mapsto g^m$ is a quasi-isometric embedding of $\langle g \rangle \cong \mathbb{Z}$ into G. □

Note that the Sol group embeds quasi-isometrically into a product of two hyperbolic planes (for example, see [19, Section 9]). However, the Sol lattice contains exponentially distorted elements by [41, Lemma 5.2]; as a consequence, we have the following:

Corollary 2.6 *The fundamental group of a 3-manifold with Sol geometry does not have property (QT).*

Corollary 2.7 *The Baumslag–Solitar group* $\mathrm{BS}(1,n)$ *for* $n>1$ *does not have property (QT).*

2.2 Subexponential growth and solvable groups with property (QT)

The fundamental group of a 3-manifold M with Nil geometry also fails to have property (QT) since it contains quadratically distorted elements (for example, see [41, Proposition 1.2]). Generalizing results about property (QT) of 3-manifolds with Sol or Nil geometry, in the rest of this subsection, we provide a characterization of subexponential growth groups and solvable groups with property (QT) and give the proof of Theorem 1.8.

In the next results, we apply the general conclusions in [17] about the isometric actions on hyperbolic spaces to the actions on quasitrees. By Gromov, unbounded isometric group actions can be classified into the following four types:

(1) *horocyclic* if it has no loxodromic element;

(2) *lineal* if it has a loxodromic element and any two loxodromic elements have the same fixed points in the Gromov boundary;

(3) *focal* if it has a loxodromic element which is not lineal, and any two loxodromic elements have one common fixed point;

(4) *general type* if it has two loxodromic elements without common fixed point.

Proposition 2.8 *Assume that* G *has property (QT). Then there exists a finite-index subgroup* $\dot G$ *of* G *which acts on a Euclidean space* $\mathbb{R}^k$ *with* $k\ge 0$ *and finitely many quasitrees* T_i *for* $1\le i\le n$ *with lineal or focal or general type action such that the orbital map of* $\dot G$ *into* $\mathbb{R}^k\times\prod_{i=1}^n T_i$ *is a quasi-isometric embedding.*

Moreover, the action on each T_i *can be chosen to be cobounded.*

Proof By Corollary 2.4, the finite product of quasitrees given by property (QT) has the above form of de Rham decomposition. By [26, Corollary 1.3],

$$1\to \mathrm{Isom}(\mathbb{R}^k)\times\prod_{i=k+1}^{n}\mathrm{Isom}(Y_i)\to\mathrm{Isom}(X)\to F\to 1$$

where F is a subgroup of the permutation group on the indices $\{k+1,\dots,n\}$. Thus, there exists a finite-index subgroup $\dot G$ of G acting on each de Rham factor such that $\dot G\subset\mathrm{Isom}(\mathbb{R}^k)\times\prod_{i=1}^n\mathrm{Isom}(Y_i)$ for $k\ge 0$ and $i\ge k+1$.

First of all, we can assume that the actions of $\dot G$ on $\mathbb{R}^k$ and each T_i is unbounded. Otherwise, we can remove $\mathbb{R}^k$ or T_i with bounded actions from the product without affecting the property (QT).

We now consider the action on T_i for $k+1 \le i \le n$. We then need to verify that the action of $\dot{G}$ on T_i cannot be horocyclic. By way of contradiction, assume that the action of $\dot{G}$ on given T_i is horocyclic.

Note that the proof of [17, Proposition 3.1] shows that the intersection of any orbit of $\dot{G}$ on T_i with any quasigeodesic is bounded. By [39, Corollary 3.2], any isometry on a quasitree T_i has either bounded orbits or a quasigeodesic orbit. Thus, we conclude that any orbit of $\langle h \rangle$ for every $h \in \dot{G}$ on T_i is bounded. We are then going to prove that the action of $\dot{G}$ on T_i has bounded orbits. This is a well-known fact and we present the proof for completeness.

By δ-hyperbolicity of T_i, each $h \in \dot{G}$ (with bounded orbits) has a *quasicenter* $c_h \in T_i$: there exists a constant $D > 0$ depending only on δ such that $|c_h - h^i c_h|_{T_i} \le D$ for $i \in \mathbb{Z}$. Moreover, for any $x \in c_h$ and any $y \in T_i$, the Gromov product $\langle y, hy \rangle_x$ is bounded by a constant C depending only on D. As a consequence, the union Z of quasicenters $\{c_h \mid h \in \dot{G}\}$ has finite diameter. Indeed, note that $\langle y, h_1 y \rangle_x$ and $\langle x, h_2^{-1} x \rangle_y$ are bounded by C for any $x \in c_{h_1}$ and $y \in c_{h_2}$. If there exist two elements, h_1 and h_2, such that the distance $|c_{h_1} - c_{h_2}|_{T_i}$ is sufficiently large relative to C, then the piecewise geodesic path connecting points $(h_1 h_2)^n x$ for $n \in \mathbb{Z}$ would be a sufficiently long local quasigeodesic, so it is a global quasigeodesic. By the previous paragraph, we obtain a contradiction, so the $\dot{G}$-invariant set Z is bounded. Since the action on T_i is assumed to be unbounded, we thus proved that the action on T_i cannot be horocyclic.

At last, it remains to prove the "moreover" statement. By Manning's bottleneck criterion [39], any geodesic is contained in a uniform neighborhood of every path with the same endpoints. Thus, any connected subgraph of a quasitree is uniform quasiconvex and thus is a uniform quasitree. Since G is a finitely generated group, by taking the image of the Cayley graph, we can thus construct a connected subgraph on each quasitree T_i such that the action on the subgraph (quasitree) is cobounded. □

We are able to characterize subexponential groups with property (QT) as follows.

Proposition 2.9 *Let G be a finitely generated group with subexponential growth. Then G has property (QT) if and only if G is virtually abelian.*

Proof We first observe that $\mathbb{R}^k$ in Proposition 2.8 can be replaced by a finite product of real lines. Indeed, consider the action of $\dot{G}$ on Euclidean space $\mathbb{R}^k$. By assumption, $\dot{G}$ is of subexponential growth. It is well known that the growth of any finitely generated group dominates that of quotients, so the image $\Gamma \subset \mathrm{Isom}(\mathbb{R}^k)$ of $\dot{G}$ acting on $\mathbb{R}^k$ has subexponential growth. Since finitely generated linear groups do not have intermediate growth, Γ must be virtually nilpotent. It is well known that virtually nilpotent subgroups in $\mathrm{Isom}(\mathbb{R}^k)$ must be virtually abelian. Thus, Γ contains a finite-index subgroup $\mathbb{Z}^l$ for $1 \le l \le k$. By taking the preimage of $\mathbb{Z}^l$ in $\dot{G}$, we can assume further that $\dot{G}$ acts on $\mathbb{R}^k$ through $\mathbb{Z}^l$. It is clear that $\mathbb{Z}^l$ acts on l real lines $\mathbb{R}_1, \mathbb{R}_2, \dots, \mathbb{R}_l$ such that the product action is geometric. We thus replace $\mathbb{R}^k$ by the product $\prod_{1 \le i \le l} \mathbb{R}_i$ where $\dot{G}$ admits a lineal action on each $\mathbb{R}_i$ by translation.

By Proposition 2.8 the action of $\dot{G}$ on T_i is either lineal or focal or general type. In the latter two cases, $\dot{G}$ contains a free (semi)group by [17, Lemma 3.3], contradicting the subexponential growth of $\dot{G}$. Thus, the action of $\dot{G}$ on each T_i is lineal. By Proposition 2.8, we can assume that T_i is a quasiline.

By [39, Lemma 3.7], a quasiline T admits a $(1, C)$-quasi-isometry ϕ (with a quasi-inverse ψ) to $\mathbb{R}$ for some $C > 0$. A lineal action of G on T is then conjugated to a quasiaction of G on $\mathbb{R}$ sending $g \in G$ to a $(1, C')$-quasi-isometry $\phi g \psi$ on $\mathbb{R}$ for some $C' = C'(C) > 0$. By taking an index at most 2 subgroup, we can assume that every element in G fixes pointwise the two ends of T. Note that a $(1, C')$-quasi-isometry $\phi g \psi$ on $\mathbb{R}$ fixing the two ends of $\mathbb{R}$ is uniformly bounded away from a translation on $\mathbb{R}$. So, for any $x \in \mathbb{R}$, the orbital map $g \mapsto \phi g \psi(x)$ is a quasimorphism $G \to \mathbb{R}$. It is well known that for any amenable group, any quasimorphism must be a homomorphism up to bounded error. We conclude that any $[G, G]$-orbit on T stays in a bounded set.

Therefore, any $[G, G]$-orbit on $\left(\prod_{1 \le i \le l} \mathbb{R}_i\right) \times \left(\prod_{1 \le i \le n} T_i\right)$ is bounded, so the proper action on X implies that $[\dot{G}, \dot{G}]$ is a finite group. It is well known that if a group has a finite commutator subgroup, then it is virtually abelian [11, Lemma II.7.9]. □

It would be interesting to ask whether Proposition 2.9 holds within the class of solvable groups. In Proposition 2.11 below, we are able to give a positive answer to the previous question when the solvable group has finite virtual cohomological dimension. To this end, we need the following fact.

Lemma 2.10 *Any unbounded isometric action of a meta-abelian group on a quasitree must be lineal.*

Recall that a meta-abelian group is a group whose commutator subgroup is abelian.

Proof Indeed, the abelian group $\Gamma = [G, G]$ (of possibly infinite rank) cannot contain free semigroups, so by [17, Lemma 3.3], the action of Γ on a quasitree T must be bounded or lineal.

Assume first that Γ has a bounded orbit K in T. Since G/Γ is abelian, we have that $g^m h^n K = h^n g^m K$ for any $n, m \in \mathbb{Z}$ and $g, h \in G$, and thus $gh^n K = h^n gK$ has finite Hausdorff distance to $h^n K$ for any $n \in \mathbb{Z}$. Assume that g and h are loxodromic. Then $\{h^n K, n \in \mathbb{Z}\}$ is quasi-isometric to a line. Hence, we obtain that the fixed points of g and h at the Gromov boundary must coincide. This means the action of G on T is lineal.

In the lineal case, Γ preserves some bi-infinite quasigeodesic γ up to finite Hausdorff distance. Since Γ is a normal subgroup in G, we see that every loxodromic element in G also preserves γ up to a finite Hausdorff distance. Thus, the action of G on T is also lineal. □

By Lemma 2.5, a group with property (QT) is *translation proper* in the sense of Conner [18]: the translation length of any nontorsion element is positive. If G is solvable and has finite virtual cohomological dimension, then Conner shows that G is virtually meta-abelian.

Proposition 2.11 *Suppose that a solvable group G has finite virtual cohomological dimension. If G has property (QT) then it is virtually abelian.*

Proof Passing to finite-index subgroups, assume that G is meta-abelian so any quotient of G is meta-abelian. By Lemma 2.10, the action of G on each T_i is lineal.

After possibly passing to an index 2 subgroup, a lineal action of any amenable group G on a quasiline T can be quasiconjugated to an isometric action on $\mathbb{R}$. Indeed, by the proof of Proposition 2.9, conjugating the original action by almost isometries gives a quasiaction of G on $\mathbb{R}$ such that any orbital map induces a quasimorphism of G to $\mathbb{R}$. For amenable groups, any quasimorphism differs from a homomorphism by a uniformly bounded constant. Thus, up to quasiconjugacy, the lineal action of G on T can be promoted to become an isometric action on $\mathbb{R}$.

Consequently, we can quasiconjugate the action of a solvable group G on a finite product of quasitrees to a proper action on a Euclidean space. Thus, G must be virtually abelian. □

Proof of Theorem 1.8 The proof is a combination of Propositions 2.9 and 2.11. □

2.3 CKA groups

Admissible groups, first introduced in [21], are a particular class of graph of groups that includes fundamental groups of 3-dimensional graph manifolds. In this section, we review admissible groups and their properties that will used throughout the paper.

Let $\mathcal{G}$ be a connected graph. We often consider oriented edges from e_- to e_+ and write $e = [e_-, e_+]$. Then $\bar{e} = [e_+, e_-]$ denotes the oriented edge with reversed orientation. Denote by $\mathcal{G}^0$ the set of vertices and by $\mathcal{G}^1$ the set of all oriented edges.

Definition 2.12 A graph of groups $\mathcal{G}$ is *admissible* if the following hold:

(1) $\mathcal{G}$ is a finite graph with at least one edge.

(2) Each vertex group G_v has center $Z(G_v) \cong \mathbb{Z}$, $H_v := G_v/Z(G_v)$ is a nonelementary hyperbolic group, and every edge subgroup G_e is isomorphic to $\mathbb{Z}^2$.

(3) Let e_1 and e_2 be distinct directed edges entering a vertex v, and for $i = 1, 2$, let $K_i \subset G_v$ be the image of the edge homomorphism $G_{e_i} \to G_v$. Then for every $g \in G_v$, gK_1g^{-1} is not commensurable with K_2, and for every $g \in G_v - K_i$, gK_ig^{-1} is not commensurable with K_i.

(4) For every edge group G_e, if $\alpha_i : G_e \to G_{v_i}$ is the edge monomorphism, then the subgroup generated by $\alpha_1^{-1}(Z(G_{v_1}))$ and $\alpha_2^{-1}(Z(G_{v_1}))$ has finite index in G_e.

A group G is *admissible* if it is the fundamental group of an admissible graph of groups.

Definition 2.13 We say that an admissible group G is a *Croke–Kleiner admissible group* or *CKA group* if it acts properly discontinuously, cocompactly and by isometries on a complete proper CAT(0) space X. Such an action $G \curvearrowright X$ is called a *CKA action* and the space X is called a *CKA space*.

Example 2.14 (1) Let M be a nongeometric graph manifold that admits a nonpositively curved metric. Lift this metric to the universal cover $\widetilde{M}$ of M, and denote it by d. Then the action $\pi_1(M) \curvearrowright (\widetilde{M}, d)$ is a CKA action.

(2) Let T be the torus complexes constructed in [20]. Then $\pi_1(T) \curvearrowright \widetilde{T}$ is a CKA action.

(3) One may build Croke–Kleiner admissible groups algebraically from any finite number of hyperbolic CAT(0) groups. The following example is for $n = 2$ but the same principle works for any $n \geq 2$. Let H_1 and H_2 be two torsion-free hyperbolic groups that act geometrically on CAT(0) spaces X_1 and X_2 respectively. Then $G_i = H_i \times \langle t_i \rangle$ (with $i = 1, 2$) acts geometrically on the CAT(0) space $Y_i = X_i \times \mathbb{R}$. Any primitive hyperbolic element h_i in H_i gives rise to a totally geodesic torus T_i in the quotient space Y_i/G_i with basis $([h_i], [t_i])$. We rescale Y_i so that the translation length of h_i is equal to that of t_i for each i. Let $f : T_1 \to T_2$ be a *flip* isometry respecting these lengths, that is, an orientation-reversing isometry mapping $[h_1]$ to $[t_2]$ and $[t_1]$ to $[h_2]$. Let M be the space obtained by gluing Y_1 to Y_2 by the isometry f. There is a metric on the space M that gives rise to a locally CAT(0) space (see eg [11, Proposition II.11.6]). By the Cartan–Hadamard theorem, the universal cover $\widetilde{M}$ with the induced length metric from M is a CAT(0) space. Let G be the fundamental group of M. Then the action $G \curvearrowright \widetilde{M}$ is geometric, and G is an example of a Croke–Kleiner admissible group.

Remark 2.15 All graph 3-manifold groups are admissible, but there are closed graph 3-manifold groups that are not CAT(0) groups (see [33]), and thus are not CKA groups. The following is another example. Take two nonvirtually split central extensions of hyperbolic groups by $\mathbb{Z}$ (eg $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ lattices) and amalgamate them over $\mathbb{Z}^2$ to obtain an admissible group. This group cannot act properly on CAT(0) spaces, since central extensions acting on CAT(0) spaces must virtually split as direct products [11, Theorem II.7.1].

A collection of subgroups $\{K_1, \ldots, K_n\}$ in a group H is called *almost malnormal* if $\#(gK_i g^{-1} \cap K_j) = \infty$ implies $i = j$ and $g \in K_i$. It is well known that a hyperbolic group is hyperbolic relative to any almost malnormal collection of quasiconvex subgroups [10].

Lemma 2.16 *Let K_e be the image of an edge group G_e into G_v and $\overline{K}_e$ be its projection in H_v under $G_v \to H_v = G_v/Z(G_v)$. Then $\mathbb{P} := \{\overline{K}_e \mid e_- = v, e \in \mathcal{G}^1\}$ is an almost malnormal collection of virtually cyclic subgroups in H_v.*

In particular, H_v is hyperbolic relative to $\mathbb{P}$.

Proof Since $Z(G_v) \subset K_e \cong \mathbb{Z}^2$, we have $\overline{K}_e = K_e/Z(G_v)$ is virtually cyclic. The almost malnormality follows from noncommensurability of K_e in G_v. Indeed, assume that $\overline{K}_e \cap h\overline{K}_{e'}h^{-1}$ contains an infinite order element by the hyperbolicity of H_v. If $g \in G_v$ is sent to h, then $K_e \cap gK_{e'}g^{-1}$ is sent to $\overline{K}_e \cap h\overline{K}_{e'}h^{-1}$. Thus, $K_e \cap gK_{e'}g^{-1}$ contains an abelian group of rank 2. The noncommensurability of K_e in G_v implies that $e = e'$ and $g \in K_e$. This shows that $\mathbb{P}$ is almost malnormal. □

Let $G \curvearrowright X$ be a CKA action where G is the fundamental group of an admissible graph of groups $\mathcal{G}$, and let $G \curvearrowright T$ be the action of G on the associated Bass–Serre tree T of $\mathcal{G}$ (we refer the reader to [21, Section 2.5] for a brief discussion). Let T^0 and T^1 be the vertex and edge sets of T. By CAT(0) geometry:

(1) For every vertex $v \in T^0$, the minimal set $Y_v := \bigcap_{g \in Z(G_v)} \mathrm{Minset}(g)$ of X splits as metric product $\overline{Y}_v \times \mathbb{R}$ where $Z(G_v)$ acts by translation on the $\mathbb{R}$-factor and $H_v = G_v/Z(G_v)$ acts geometrically on the Hadamard space $\overline{Y}_v$. Since H_v is a hyperbolic group, it follows that $\overline{Y}_v$ is a hyperbolic space.

(2) For every edge $e \in T^1$, the minimal set $Y_e := \bigcap_{g \in G_e} \mathrm{Minset}(g)$ of X splits as $\overline{Y}_e \times \mathbb{R}^2 \subset Y_v$ where $\overline{Y}_e$ is a compact Hadamard space and $G_e = \mathbb{Z}^2$ acts cocompactly on the Euclidean plane $\mathbb{R}^2$.

We note that the assignments $v \to Y_v$ and $e \to Y_e$ are G-equivariant with respect to the natural G actions.

We summarize results in [21, Section 3.2] that will be used in this paper.

Lemma 2.17 *Let $G \curvearrowright X$ be a CKA action. Then there exists a constant $D > 0$ such that*

(1) $X = \bigcup_{v \in T^0} N_D(Y_v) = \bigcup_{e \in T^1} N_D(Y_e)$;

(2) *if $\sigma, \sigma' \in T^0 \cup T^1$ and $N_D(Y_\sigma) \cap N_D(Y_{\sigma'}) \neq \varnothing$ then $|\sigma - \sigma'|_T < D$.*

We shall refer to $\widetilde{Y}_v = N_D(Y_v)$ and $\widetilde{Y}_e = N_D(Y_e)$ as vertex and edge spaces for X.

2.3.1 Strips in CKA spaces [21, Section 4.2] We first choose, in a G-equivariant way, a plane $F_e \subset Y_e$ (which we will call *boundary plane*) for each edge $e \in T^1$. For every pair of adjacent edges e_1 and e_2, we choose, again equivariantly, a minimal geodesic from F_{e_1} to F_{e_2}; by the convexity of $Y_v = \overline{Y}_v \times \mathbb{R}$ where $v := e_1 \cap e_2$, this geodesic determines a Euclidean strip $\mathcal{S}_{e_1 e_2} := \gamma_{e_1 e_2} \times \mathbb{R}$ (possibly of width zero) for some geodesic segment $\gamma_{e_1 e_2} \subset \overline{Y}_v$.

Note that $\mathcal{S}_{e_1 e_2} \cap F_{e_i}$ is an axis of $Z(G_v)$. Hence if $e_1, e_2, e \in E$ and $e_i \cap e = v_i \in V$ are distinct vertices, then the angle between the geodesics $\mathcal{S}_{e_1 e} \cap F_e$ and $\mathcal{S}_{e_2 e} \cap F_e$ is bounded away from zero. If $\langle f_1 \rangle = Z(G_{v_1})$ and $\langle f_2 \rangle = Z(G_{v_2})$ then $\langle f_1, f_2 \rangle$ generates a finite-index subgroup of G_e. We remark that the intersection of two strips $\mathcal{S}_{e_1 e}$ and $\mathcal{S}_{e_2 e}$ is a point. Indeed, we have $\mathcal{S}_{e_1 e} \cap \mathcal{S}_{e_2 e} = (\mathcal{S}_{e_1 e} \cap F_e) \cap (\mathcal{S}_{e_2 e} \cap F_e)$. As two lines $\mathcal{S}_{e_1 e} \cap F_e$ and $\mathcal{S}_{e_2 e} \cap F_e$ in the plane F_e are axes of $\langle f_{v_1} \rangle = Z(G_{v_1})$ and $\langle f_{v_1} \rangle = Z(G_{v_2})$, respectively, and $\langle f_1, f_2 \rangle$ generates a finite-index subgroup of G_e, it follows that these two lines are not parallel, and hence their intersection must be a single point.

We note that the intersection of a boundary plane F_e of Y_v with the hyperbolic space $\overline{Y}_v$ is a line. The *boundary lines* $\mathbb{L}_v$ of the hyperbolic space $\overline{Y}_v$ is the collection of lines $\mathbb{L}_v = \{\ell_e := F_e \cap \overline{Y}_v \mid e_- = v\}$.

Definition 2.18 If for each edge $e := [v, w] \in T$, the boundary line $\ell = \overline{Y}_v \cap F_e$ is parallel to the $\mathbb{R}$-line in $Y_w = \overline{Y}_w \times \mathbb{R}$, then the CKA action is called *flip*.

In the sequel, it will be useful to make the following specific choices.

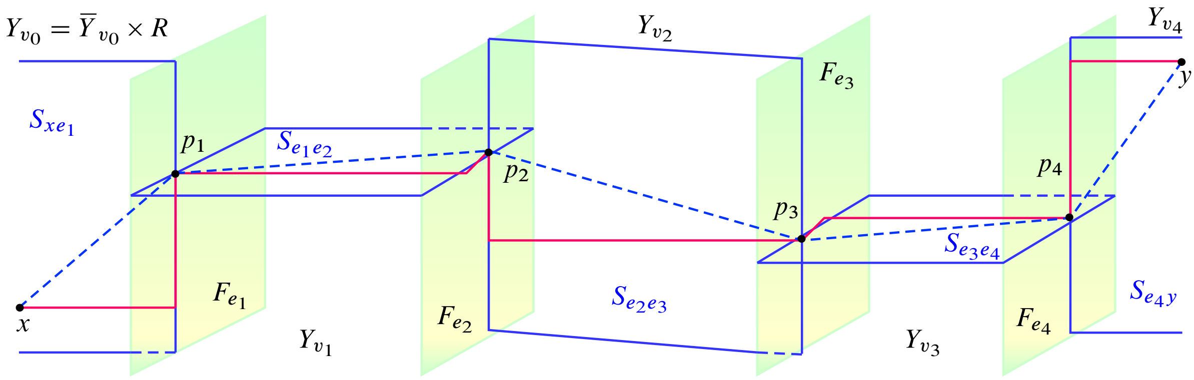

Figure 1: The dotted and blue path from x to y is a special path, and the red path is one L^1-version of it.

Definition 2.19 An *indexed map* $\rho\colon X \to T^0$ is a G-equivariant coarsely Lipschitz map such that $x \in \widetilde{Y}_{\rho(x)}$ for all $x \in X$.

If G acts freely on X, such a map ρ can be constructed as follows. Choose a fundamental set Σ such that Σ contains exactly one point from each orbit. Define $\rho\colon \Sigma \to T^0$ so that $x \in \widetilde{Y}_{\rho(x)}$, and extend ρ equivariantly to the whole space X. By Lemma 2.17.(2), one can show that ρ is a coarsely Lipschitz map: $|\rho(x)-\rho(y)|_T \le L|x-y|_X + L$ for some $L > 0$. See [21, Section 3.3] for more details.

If G acts only geometrically on X, we could replace X with a G-orbit Go for a basepoint o with trivial stabilizer. This does not matter much as we are only interested in the coarse geometry hereafter. By modifying X, we can always assume such a basepoint o exists. Indeed, this can be achieved by attaching a Euclidean cone to a point o such that its nontrivial but finite stabilizer acts freely on its boundary circle. Then we do the modification equivariantly for all translates in Go.

2.3.2 Special paths in CKA spaces

Let $G \curvearrowright X$ be a CKA action. We now introduce the class of *special paths* in X.

Definition 2.20 (special paths in X) Let $\rho\colon X \to T^0$ be the indexed map given by Definition 2.19. Let x and y be two points in X. If $\rho(x) = \rho(y)$, a *special path* in X connecting x to y is the geodesic $[x, y]$. Otherwise, let $e_1 \cdots e_n$ be the geodesic edge path connecting $\rho(x)$ to $\rho(y)$ and let $p_i = \mathcal{S}_{e_{i-1}e_i} \cap \mathcal{S}_{e_ie_{i+1}}$ be the intersection point of adjacent strips, where $e_0 := x$ and $e_{n+1} := y$. A *special path* connecting x to y is the concatenation of the geodesics

$$[x, p_1][p_1, p_2] \cdots [p_{n-1}, p_n][p_n, y].$$

Remark 2.21 By definition, except for $[x, p_1]$ and $[p_n, y]$, the special path depends only on the geodesic $e_1 \cdots e_n$ in T, the choice of planes F_e and the indexed map ρ.

Proposition 2.22 [43, Proposition 3.8] *There exists a constant $\mu > 0$ such that every special path γ in X is a (μ,μ)-quasigeodesic.*

Assume that $v_0 = \rho(x), v_{2n} = \rho(y) \in \mathcal{V}$ are such that $d(v_0, v_{2n}) = 2n$ for $n \geq 0$. If γ is a special path between x and y, then we define

$$|x-y|_X^{\mathrm{hor}} := \sum_{i=0}^{2n} |p_i - p_{i+1}|_{Y_{v_i}}^{\mathrm{hor}}, \quad |x-y|_X^{\mathrm{ver}} := \sum_{i=0}^{2n} |p_i - p_{i+1}|_{Y_v}^{\mathrm{ver}} \tag{2}$$

where $p_0 := x$ and $p_{n+1} := y$. By Proposition 2.22, we have

$$|x-y|_X \sim |x-y|_X^{\mathrm{hor}} + |x-y|_X^{\mathrm{ver}}.$$

By definition, the system of special paths is G-invariant, so the symmetric functions $d^h(x,y)$ and $d^v(x,y)$ are G-invariant for any $x, y \in X$.

We partition the vertex set T^0 of the Bass–Serre tree into two disjoint classes of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ such that if v and v' are in $\mathcal{V}_i$ then $d_T(v,v')$ is even.

Lemma 2.23 [43, Lemma 4.6] *There exists a subgroup $\dot{G}$ of index at most 2 in G preserving $\mathcal{V}_i$ for $i = 1, 2$ such that $G_v \subset \dot{G}$ for any $v \in T^0$.*

2.4 Projection axioms

In this subsection, we briefly recall the work of Bestvina, Bromberg and Fujiwara [5] on constructing a quasitree of spaces.

Definition 2.24 (projection axioms) Let $\mathbb{Y}$ be a collection of geodesic spaces equipped with projection maps

$$\{\pi_Y : \mathbb{Y} - \{Y\} \to \mathcal{P}(Y)\}_{Y \in \mathbb{Y}}$$

where $\mathcal{P}(Y)$ is the power set of Y. Write $d_Y(X,Z) = \mathrm{diam}(\pi_Y(X) \cup \pi_Y(Z))$ for $X \neq Y \neq Z \in \mathbb{Y}$. The pair $(\mathbb{Y}, \{\pi_Y\}_{Y \in \mathbb{Y}})$ satisfies *projection axioms* for a *projection constant* $\xi \geq 0$ if the following hold:

(1) $\mathrm{diam}(\pi_Y(X)) \leq \xi$ when $X \neq Y$.

(2) If X, Y and Z are distinct and $d_Y(X,Z) > \xi$ then $d_X(Y,Z) \leq \xi$.

(3) For $X \neq Z$, the set $\{Y \in \mathbb{Y} \mid d_Y(X,Z) > \xi\}$ is finite.

The following is a useful example to keep in mind throughout the paper. For further details, we refer the reader to the introduction of [5]. In this example, the collection of metric spaces $\mathbb{Y}$ consists of subspaces of a singe metric space; however, we emphasize that this need not be the case in general.

Example 2.25 Let G be a discrete group of isometries of $\mathbb{H}^2$, and $\gamma \in G$ be a loxodromic element with axis γ. Let $\mathbb{Y}$ be the set of all G-translates of γ. Given $Y \in \mathbb{Y}$, let π_Y denote the shortest projection map in $\mathbb{H}^2$. Since all translates of γ are convex, this is a well-defined 1-Lipschitz map. One may check that $(\mathbb{Y}, \{\pi_Y\}_{Y \in \mathbb{Y}})$ satisfies the projection axioms for some constant ξ.

Remark 2.26 Let $(\mathbb{Y}, \{\pi_Y\}_{Y \in \mathbb{Y}})$ satisfy the projection axioms. By [7, Theorem 4.1 and Lemma 4.13], there exists a variant π'_Y of π_Y such that π_Y and π'_Y are uniformly close in Hausdorff distance, and $(\mathbb{Y}, \{\pi'_Y\}_{Y \in \mathbb{Y}})$ satisfies strong projection axioms, ie the axioms are the same as projection axioms except for replacing (2) in Definition 2.24 with the following stronger statement: if X, Y, Z are distinct and $d_Y(X, Z) > \xi$ then $\pi_X(Y) = \pi_X(Z)$ for a projection constant ξ' depending only on ξ.

The following results from [5] will be used in this paper.

- Fix $K > 0$. In [5], a quasitree of spaces $\mathcal{C}_K(\mathbb{Y})$ is constructed for given $(\mathbb{Y}, \{\pi_Y\}_{Y \in \mathbb{Y}})$ which satisfies the projection axioms with constant ξ.
- If $K > 4\xi$ and $\mathbb{Y}$ is a collection of uniform quasilines, then $\mathcal{C}_K(\mathbb{Y})$ is a unbounded quasitree. If $\mathbb{Y}$ admits a group action of G such that $\pi_{gY} = g\pi_Y$ for any $g \in G$ and $Y \in \mathbb{Y}$, then G acts by isometry on $\mathcal{C}_K(\mathbb{Y})$.

Set $[t]_K = t$ if $t \geq K$, otherwise $[t]_K = 0$. Let $x \in X$ and $z \in Z$ for $X, Z \in \mathbb{Y}$. If $X \neq Y \neq Z$, we define $d_Y(x, z) = d_Y(X, Z)$. If $Y = X$ and $Y \neq Z$, then define $d_Y(x, z) = \mathrm{diam}(\pi_Y(x, Z))$. If $X = Y = Z$, let $d_Y(x, z)$ be the distance in Y. The following distance formula from [7] is crucial in what follows.

Proposition 2.27 [7, Theorem 6.3] *Let $(\mathbb{Y}, \{\pi_Y\}_{Y \in \mathbb{Y}})$ satisfy the strong projection axioms with constant ξ. Then for any $x, y \in \mathcal{C}_K(\mathbb{Y})$,*

$$\frac{1}{4} \sum_{Y \in \mathbb{Y}} [d_Y(x, y)]_K \leq |x - y|_{\mathcal{C}_K(\mathbb{Y})} \leq 2 \sum_{Y \in \mathbb{Y}} [d_Y(x, y)]_K + 3K$$

for all $K \geq 4\xi$.

Definition 2.28 (acylindrical action [9; 44]) Let G be a group acting by isometries on a metric space (X, d). The action of G on X is called *acylindrical* if for any $r \geq 0$, there exist constants $R, N \geq 0$ such that for any pair $a, b \in X$ with $|a - b|_X \geq R$ we have

$$\#\{g \in G \mid |ga - a|_X \leq r \text{ and } |gb - b|_X \leq r\} \leq N.$$

By [9], any nontrivial isometry of acylindrical group action on a hyperbolic space is either elliptic or loxodromic. A (λ, c)-quasigeodesic γ for some $\lambda, c > 0$ is referred to as a *quasiaxis* for a loxodromic element g if γ and $g\gamma$ have finite Hausdorff distance depending only on λ and c.

A group is called *nonelementary* if it is neither finite nor virtually cyclic.

Proposition 2.29 [6] *Assume that a nonelementary hyperbolic group H acts acylindrically on a hyperbolic space $\overline{Y}$. For a loxodromic element $g \in H$, consider the set $\mathbb{A}$ of all H-translates of a given (λ, c)-quasiaxis of g for given $\lambda, c > 0$. Then there exists a constant $\theta = \theta(\lambda, c) > 0$ such that for any $\gamma \in \mathbb{A}$, the set*

$$\{h \in G \mid \mathrm{diam}(\pi_\gamma(h\gamma)) \geq \theta\}$$

is a finite union of double $E(g)$-cosets.

In particular, there are only finitely many distinct pairs $(\gamma, \gamma') \in \mathbb{A} \times \mathbb{A}$ satisfying $\mathrm{diam}(\pi_\gamma(\gamma')) > \theta$ up to the action of H.

Lemma 2.30 [54, Lemma 2.14] *Let H be a nonelementary group admitting a cobounded and acylindrical action on a δ-hyperbolic space $(\overline{Y}, d)$. Fix a basepoint o. Then there exists a set $F \subset H$ of three loxodromic elements and $\lambda, c > 0$ with the following property.*

For any $h \in H$ there exists $f \in F$ such that hf is a loxodromic element and the bi-infinite path

$$\gamma = \bigcup_{i \in \mathbb{Z}} (hf)^i([o, ho][ho, hfo])$$

is a (λ, c)-quasigeodesic.

Convention 2.31 When speaking of quasilines in hyperbolic spaces with actions satisfying Lemma 2.30, we always mean (λ, c)-quasigeodesics where $\lambda, c > 0$ depend on F and δ.

3 Property (QT) of relatively hyperbolic groups

In this section, we are going to prove Theorem 1.5. The notion of relatively hyperbolic groups can be formulated from a number of equivalent ways. Here we shall present a quick definition due to Bowditch [10] and recall the relevant facts we shall need without proofs.

Let H be a finitely generated group with a finite collection of subgroups $\mathcal{P}$. Fixing a finite generating set S, we consider the corresponding Cayley graph $\mathrm{Cay}(H, S)$ equipped with path metric d and we denote by $|h|_H = d(1, h)$ the word length.

Denote by $\mathbb{P} = \{hP \mid h \in H, P \in \mathcal{P}\}$ the collection of peripheral cosets. Let $\widehat{H}(\mathbb{P})$ be the *coned-off Cayley graph* obtained from $\mathrm{Cay}(H, S)$ as follows. A *cone point* denoted by $c(P)$ is added for each peripheral coset $P \in \mathbb{P}$ and is joined by *half edges* to each element in P. The union of two half edges at a cone point is called a *peripheral edge*. Denote by $\hat{d}$ the induced path metric after coning-off and $|h|_{\widehat{H}} = \hat{d}(1, h)$.

The pair $(G, \mathcal{P})$ is said to be *relatively hyperbolic* if the coned-off Cayley graph $\widehat{H}(\mathbb{P})$ is hyperbolic and *fine*: any edge is contained in finitely many simple circles with uniformly bounded length.

By [9, Lemma 3.3; 44, Proposition 5.2], the action of H on $\widehat{H}(\mathbb{P})$ is acylindrical.

Let π_P denote the shortest projection in word metric to $P \in \mathbb{P}$ in H and $d_P(x,y)$ the $|\cdot|_H$-diameter of the projections of the points x, y to P. Since $\mathbb{P}$ has the strongly contracting property with bounded intersection property, the projection axioms with a constant $\xi > 0$ hold for $\mathbb{P}$ (see [48]).

3.1 Thick distance formula

A geodesic edge path β in the coned-off Cayley graph $\hat{H}(\mathbb{P})$ is K*-bounded* for $K > 0$ if the end points of every peripheral edge have d-distance at most K.

By definition, a geodesic $\beta = [x, y]$ can be subdivided into maximal K-bounded nontrivial segments α_i $(0 \le i \le n)$ separated by peripheral edges e_j $(0 \le j \le m)$ where $|(e_j)_- - (e_j)_+|_H > K$. It is possible that $n = 0$: β consists of only peripheral edges.

Define

$$|\beta|_K := \sum_{0 \le i \le n} [\mathrm{Len}(\alpha_i)]_K,$$

which sums up the lengths of K-bounded subpaths of length at least K. It is possible that $n = 0$, so $|\beta|_K = 0$. Define the K*-thick distance*

$$|x - y|^K_{\hat{H}} = \max\{|\beta|_K\} \tag{3}$$

over all relative geodesics β between x and y. Thus, $|x-y|^K_{\hat{H}}$ is H_v-invariant.

A relative path without backtracking in $\hat{H}(\mathbb{P})$ admits nonunique *lifts* in $\mathrm{Cay}(H, S)$ which are obtained by replacing the peripheral edge by a geodesic in $\mathrm{Cay}(H, S)$ with the same endpoints. The distance formula follows from the fact that the lift of a relative quasigeodesic is a quasigeodesic (see [25; 27, Proposition 6.1]). The following formula is made explicit in [48, Theorem 0.1].

Lemma 3.1 *For any sufficiently large $K > 0$ and for any $x, y \in H$,*

$$|x - y|_H \sim_K |x - y|^K_{\hat{H}} + \sum_{P \in \mathbb{P}} [d_P(x, y)]_K. \tag{4}$$

The following result is proved in [43, Lemma 5.5] under the assumption that H is hyperbolic relative to a set of virtually cyclic subgroups. However, the same proof works for any relatively hyperbolic group.

Lemma 3.2 *For any sufficiently large $K > 0$, there exists an H-finite collection $\mathbb{A}$ of quasilines in $\hat{H}$ and a constant $N = N(K, \hat{H}, \mathbb{A}) > 0$ such that for any two vertices $x, y \in \hat{H}$,*

$$|x - y|^K_{\hat{H}} \sim_N \sum_{\ell \in \mathbb{A}} [\hat{d}_\ell(x, y)]_K. \tag{5}$$

A group H endowed with the *profinite topology* is a topological group such that the set of all finite-index subgroups is a (closed/open) neighborhood base of the identity. A subgroup P is called *separable* if it is closed in the profinite topology. Equivalently, it is the intersection of all finite-index subgroups containing P. A group is called *residually finite* if the trivial subgroup is closed.

A maximal abelian subgroup of a residually finite group is separable (see [31, Proposition 1]). Note that a maximal elementary (ie virtually cyclic) group E in a relatively hyperbolic group H contains a maximal abelian group (of rank 1) as a finite-index subgroup. If H is residually finite, then E is a finite union of closed subsets; hence it is closed and thus separable.

We will use the following corollary in the proof of Theorem 1.5.

Corollary 3.3 *Assume that H is a residually finite relatively hyperbolic group. Then for any $K \gg 0$, there exists a finite-index subgroup $\dot{H} \leq H$ acting on finitely many quasitrees T_i $(1 \leq i \leq n)$ such that every orbital map of the $\dot{H}$-action on $\prod_{i=1}^{n} T_i$ is a quasi-isometric embedding from $(\dot{H}, |\cdot|_{\hat{H}}^{K})$ to $\prod_{i=1}^{n} T_i$.*

This corollary is essentially proved in [43], inspired by the arguments in the setting of mapping class groups [6]. We sketch the proof for the convenience of the reader.

Sketch of proof Recall that for any $\theta > 0$, a set $\mathbb{T}$ of (uniform) quasilines in a hyperbolic space with θ-bounded projection satisfies the projection axioms for a projection constant $\xi = \xi(\theta) > 0$. Let λ and c be the constants given by Lemma 2.30 with respect to the acylindrical action $H \curvearrowright \hat{H}$. For our purpose, we will choose θ to be the constant given by Proposition 2.29. Then the distance formula for the quasitree $\mathcal{C}_K(\mathbb{T})$ constructed from $\mathbb{T}$ holds for any $K \geq 4\xi$.

For a fixed large constant K, Lemma 3.2 provides an H-finite set of quasilines $\mathbb{A}$ such that (5) holds. We then use the separability to find a finite-index subgroup $\dot{H}$ of H such that $\mathbb{A}$ decomposes as a finite union of $\dot{H}$-invariant $\mathbb{T}_i$ each of which satisfies the projection axioms with projection constant ξ. To be precise, the stabilizer E of a quasiline ℓ in $\mathbb{A}$ is a maximal elementary subgroup of H and thus is separable in H if H is residually finite (since a maximal abelian group in a residually finite group is separable). By Proposition 2.29 and the paragraph after Lemma 2.1 in [6], the separability of E allows one to choose a finite-index subgroup $\dot{H}$ containing E such that any $\dot{H}$-orbit $\mathbb{T}_i$ in the collection of quasilines $H\ell$ satisfies the projection axioms with projection constant ξ. We take a common finite-index subgroup $\dot{H}$ for finitely many quasilines ℓ in $\mathbb{A}$ up to H-orbits and therefore have found all $\dot{H}$-orbit $\mathbb{T}_i$ such that their union covers $\mathbb{A}$.

Finally, it is straightforward to verify that the right-hand term of (5) coincides with the sum of distances over the finitely many quasitrees $T_i := \mathcal{C}_K(\mathbb{T}_i)$. Thus, the thick distance $d_{\hat{H}}^{K}(x, y)$ is quasi-isometric to the distance on a finite product of quasitrees. □

All our discussion generalizes to the geometric action of H on a geodesic metric space Y, since there exists an H-equivariant quasi-isometry between $\mathrm{Cay}(H, S)$ and Y. Therefore, replacing $\mathrm{Cay}(H, S)$ with Y, we have the same thick distance formula. This is the setup for CKA actions in next sections.

In next subsection, we obtain property (QT) for relatively hyperbolic groups provided peripheral subgroups do so.

3.2 Proof of property (QT) of relatively hyperbolic groups

Proof of Theorem 1.5 Recall that $\mathcal{P}$ is a finite set of subgroups. For each $P \in \mathcal{P}$, choose a full set E_P of left P-coset representatives in H such that $1 \in E_P$. For given P and $1 \le i \le n_P$, we define the collection of quasitrees

$$\mathbb{T}_P^i := \{fT_i \mid f \in E_P\}$$

where T_i are quasitrees associated to P given by assumption. Then H preserves $\mathbb{T}_P^i$ by the following action: for any point $f(x) \in fT_i$ and $h \in H$,

$$h \cdot f(x) := f'p(x) \in f'T_i$$

where $p \in P$ is given by $hf = f'p$ for $f' \in E_P$.

We are now going to define projection maps $\{\pi_{fT_i}\}$ as follows.

By assumption, we fix an orbital embedding ι_P^i of P into T_i such that the induced map

$$\prod_{i=1}^{n_P} \iota_P^i : P \to \prod_{i=1}^{n_P} T_i$$

is a quasi-isometric embedding. We then define an equivariant family of orbital maps $\iota_{fP}^i : fP \to fT_i$ such that

$$\iota_{fP}^i(x) := f\iota_P^i(f^{-1}x) \quad \text{for all } x \in fP.$$

Then for any $h \in H$ and $x \in fP$, $h \cdot \iota_{fP}^i(x) = \iota_{f'P}^i(hx)$ where $f' \in E_P$ with $hf = f'p$ and $p \in P$.

Let π_{fP} be the shortest projection to the coset fP in H with respect to the word metric. For any two distinct $fT_i, f'T_i \in \mathbb{T}_P^i$, we set

$$\pi_{fT_i}(f'T_i) := \iota_{fP}^i(\pi_{fP}(f'P)).$$

Recall that $\mathbb{P} = \{fP \mid f \in H, P \in \mathcal{P}\}$ satisfies the projection axioms with shortest projection maps $\{\pi_{fP}\}$. It is readily checked that the projection axioms pass to the collection $\mathbb{T}_P^i$ under equivariant Lipschitz maps $\{\iota_{fP}^i\}_{fP \in \mathbb{P}}$.

We can therefore build the projection complex for $\mathbb{T}_P^i$ for a fixed $K \gg 0$. By Proposition 2.27, the following distance holds for any $x', y' \in \mathcal{C}_K(\mathbb{T}_P^i)$:

$$|x' - y'|_{\mathcal{C}_K(\mathbb{T}_P^i)} \sim_K \sum_{T \in \mathbb{T}_P^i} [d_T(x', y')]_K. \tag{6}$$

Note that $\prod_{i=1}^{n_P} \iota_P^i : P \to \prod_{i=1}^{n_P} T_i$ is a quasi-isometric embedding for each $P \in \mathcal{P}$. Thus, for any $x, y \in G$ and $P \in \mathbb{P}$,

$$d_P(x, y) = |\pi_P(x) - \pi_P(y)|_P \sim \sum_{i=1}^{n_P} |\iota_P^i(\pi_P(x)) - \iota_P^i(\pi_P(y))|_{T_i}. \tag{7}$$

Setting $x' = \iota_P^i(\pi_P(x))$ and $y' = \iota_P^i(\pi_P(y))$ in (7), we deduce from (6) that

$$d_P(x,y) \preceq_K \sum_{i=1}^{n_P} |\iota_P^i(\pi_P(x)) - \iota_P^i(\pi_P(y))|_{\mathscr{C}_K(\mathbb{T}_P^i)}. \tag{8}$$

Recall from Lemma 3.1 that for any $x, y \in H$, we have

$$|x-y|_H \sim_K |x-y|_{\hat{H}}^K + \sum_{P \in \mathbb{P}} [d_P(x,y)]_K.$$

Note that the orbital map of any isometric action is Lipschitz. To prove property (QT) of H, it suffices to give an upper bound of $|x-y|_H$. Taking account of (8), it remains to construct a finite product of quasitrees to bound $|x-y|_{\hat{H}}^K$.

Since H is residually finite, by Corollary 3.3, there exists a finite-index subgroup, still denoted by H, and a finite product Y of quasitrees such that the orbital map Π_0 from H to Y gives a quasi-isometric embedding of H equipped with $|\cdot|_{\hat{H}}^K$-function into Y.

Recall that π_P is the shortest projection to $P \in \mathbb{P}$. For $1 \le i \le n_P$, define

$$\Pi_i : H \to \mathscr{C}_K(\mathbb{T}_P^i)$$

by sending an element $h \in H$ to $\iota_P^i(\pi_P(h))$. We then have n equivariant maps Π_i from H to quasitrees after reindexing, where $n := \sum_{P \in \mathscr{P}} n_P$.

Let $\Pi := \Pi_0 \times \prod_{i=1}^n \Pi_i$ be the map from H to $Y \times \prod_{i=1}^n \mathscr{C}_K(\mathbb{T}_P^i)$, where Y is the finite product of quasitrees as in the previous paragraphs. As previously mentioned, the product map Π gives an upper bound on $d_H(x,y)$, so is a quasi-isometric embedding of H. Therefore, H has property (QT). □

Remark 3.4 An immediate corollary of Theorem 1.5 is that the fundamental group of a finite volume hyperbolic 3-manifold has property (QT). An alternative proof is that $\pi_1(M)$ is virtually compact special by deep theorems of Agol [3] and Wise [53], and thus $\pi_1(M)$ has property (QT).

We say that the profinite topology on H induces a *full profinite topology* on a subgroup P if every finite-index subgroup of P contains the intersection of P with a finite-index subgroup of H.

Theorem 3.5 *Suppose that H is residually finite and each $P \in \mathscr{P}$ is separable. Assume furthermore that H induces the full profinite topology on each $P \in \mathscr{P}$. If each $P \in \mathscr{P}$ acts by isometry on a finite product of quasitrees without $\mathbb{R}$-factor such that orbital maps are quasi-isometric embeddings, then H has property* (*QT*).

Proof By [26, Corollary 1.3], there is a finite-index subgroup $\dot{P}$ of P acting on each quasitree T_i such that the diagonal action of $\dot{P}$ on $\prod_{i=1}^n T_i$ induces a quasi-isometric embedding orbital map $\prod_{i=1}^n \iota_{\dot{P}}^i$.

By the assumption, H induces the full profinite topology on $P \in \mathscr{P}$, so every finite-index subgroup of a separable subgroup P is also separable. Thus, there are finite-index subgroups $\dot{H}_P$ of H for $P \in \mathscr{P}$ such that $\dot{P} = \dot{H}_P \cap P$.

Consider the finite-index normal subgroup $\dot{H} := \bigcap\{h\dot{H}_P h^{-1} \mid P \in \mathcal{P}\}$ in H. Since $\dot{H}$ is normal in H, we see that $\dot{H} \cap hPh^{-1} \subset h\dot{P}h^{-1}$ is equivalent to $\dot{H} \cap P \subset \dot{P}$. The later holds by the choice of $\dot{H}_P$. Hence, for every $h \in H$, $\dot{H} \cap hPh^{-1}$ preserves the factors of the product decomposition. Note that $\dot{H}$ is hyperbolic relative to $\{\dot{H} \cap hPh^{-1} \mid h \in H\}$. The conclusion follows from Theorem 1.5. □

In subsequent sections (Sections 4, 5, 6 and 7), the proof of property (QT) of CKA groups will be discussed, which may be considered as the technical heart of this paper.

4 Coning-off CKA spaces

In this section, we recapitulate the content of [43, Section 5] and give an outline of the proof of Theorem 1.3.

Let $G \curvearrowright X$ be a CKA action where G is the fundamental group of an admissible graph of groups $\mathcal{G}$ (see Section 2.3), and let $G \curvearrowright T$ be the action of G on the associated Bass–Serre tree T of $\mathcal{G}$. Let T^0 and T^1 be the vertex and edge sets of T.

Let $\{F_e\}$ be the collection of boundary planes of the space Y_v (see Section 2.3). We note that the intersection of a boundary plane F_e of Y_v with the hyperbolic space $\overline{Y}_v$ is a line. We define the collection of lines $\mathbb{L}_v$ of the hyperbolic space $\overline{Y}_v$ as

$$\mathbb{L}_v = \{\ell_e := F_e \cap \overline{Y}_v \mid e_- = v\},$$

which shall be referred as *boundary lines*.

4.1 Construction of coned-off spaces

Recall that $T^0 = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_i$ consists of vertices in T with pairwise even distances. Let $\dot{G} < G$ be the subgroup of index at most 2 preserving $\mathcal{V}_1$ and $\mathcal{V}_2$ given by Lemma 2.23.

Fix a large $r > 0$. A *hyperbolic r-cone* by definition is the metric completion of the (incomplete) universal cover of a punctured hyperbolic disk of radius r. Let $\mathbb{Y}_i = \{\overline{Y}_v \mid v \in \mathcal{V}_i\}$ be the collection of hyperbolic spaces and $\dot{\mathbb{Y}}_i = \{\dot{Y}_v \mid v \in \mathcal{V}_i\}$ be their coned-off spaces (which are uniformly hyperbolic for $r \gg 0$) by attaching hyperbolic r-cones along the boundary lines of $\overline{Y}_v$.

Note that $\dot{G}$ preserves $\mathbb{Y}_i$ and $\dot{\mathbb{Y}}_i$ by the action on the index $gY_v = Y_{gv}$ for any $g \in \dot{G}$. For each $w \in T^0$, let $\mathrm{St}(w)$ be the star of w in T with adjacent vertices as *extremities*. Then $\mathrm{St}(w)$ admits the action of G_w so that the stabilizers of the extremities are the corresponding edge groups.

Define $\dot{\mathcal{X}}_i$ to be the space obtained from the disjoint union of coned-off spaces $\dot{Y}_v$ ($v \in \mathcal{V}_i$) with cone points identified with the extremities of the stars $\mathrm{St}(w)$ with $v \in \mathrm{Lk}(w)$. Endowed with induced length metric, the space $\dot{\mathcal{X}}_i$ is a Gromov-hyperbolic space.

Lemma 4.1 *Fix a sufficiently large $r > 0$ and $i \in \{1, 2\}$. The space $\dot{\mathcal{X}}_i$ is a δ-hyperbolic space where $\delta > 0$ only depends on the hyperbolicity constants of $\dot{Y}_v$ ($v \in \mathcal{V}_i$).*

The subgroup $\dot{G}$ acts on $\dot{\mathcal{X}}_i$ with the following properties:

(1) *for each $v \in \mathcal{V}_i$, the stabilizer of $\dot{Y}_v$ is isomorphic to G_v and H_v acts coboundedly on $\dot{Y}_v$, and*

(2) *for each $w \in T^0 - \mathcal{V}_i$, G_w acts on* $\mathrm{St}(w)$ *in the same manner as the action on the Bass–Serre tree T.*

Proof Note that the stabilizers of the cone points of $\dot{Y}_v$ under the action of G_v on $\dot{Y}_v$ are the same as that of the extremities of stars $\mathrm{St}(w)$, which are both the edge groups G_e for $e = [v, w]$. By construction, the cone points of $\dot{Y}_v$ are identified with the extremities of stars $\mathrm{St}(w)$, so the actions of G_v on $\dot{Y}_v$ ($v \in \mathcal{V}_i$) and of G_w on $\mathrm{St}(w)$ ($w \in T^0 - \mathcal{V}_i$) extend over $\dot{\mathcal{X}}_i$, and hence $\dot{G}$ acts by isometries on $\dot{\mathcal{X}}_i$. □

Remark 4.2 Our construction of coned-off spaces is slightly different from the one in [43, Section 5.1], where the cone points are identified directly between different spaces $\dot{Y}_v$ and $\dot{Y}_{v'}$. Thus certain assumption on vertex groups is necessary in [43] to ensure an action on the coned-off space.

We now define the thick distance on $\dot{\mathcal{X}}_i$ ($i = 1, 2$) by taking the sum of thick distances through $\dot{Y}_v$ as follows.

If x is a point in a coned-off space $\dot{Y}_v \subset \dot{\mathcal{X}}_i$, we denote $\rho(x)$ by v (by abuse of notation). By the above tree-like construction, any path between $x, y \in \dot{\mathcal{X}}_i$ has to pass through in order a pair of boundary lines ℓ_v^- and ℓ_v^+ of $\overline{Y}_v$ for each $v \in [\rho(x), \rho(y)]$. By abuse of language, if x is not contained in a hyperbolic cone, set $\ell_v^- = x$ for $v = \rho(x)$. Similarly, if y is not contained in a hyperbolic cone, set $\ell_v^+ = y$ for $v = \rho(y)$. Let (x_v, y_v) be a pair of points in the boundary lines (ℓ_v^-, ℓ_v^+) such that $[x_v, y_v]$ is orthogonal to ℓ_v^- and ℓ_v^+. Recall that $|x_v - y_v|_{\dot{Y}_v}^K$ is the K-cut-off thick distance defined in (3).

Definition 4.3 For any $K \geq 0$, the *K-thick distance* between x and y is defined by

$$|x - y|_{\dot{\mathcal{X}}_i}^K := \sum_{v \in [\rho(x), \rho(y)] \cap \mathcal{V}_i} |x_v - y_v|_{\dot{Y}_v}^K. \tag{9}$$

Since $|\cdot|_{\dot{Y}_v}^K$ is H_v-invariant, we see that $|x - y|_{\dot{\mathcal{X}}_i}^K$ is $\dot{G}$-invariant.

Remark 4.4 The definition of $|\cdot|_{\dot{\mathcal{X}}_i}^K$ is designed to ignore the parts in hyperbolic cones between different pieces. One consequence is that perturbing x and y in hyperbolic cones does not change their K-thick distance.

4.2 Construct the collection of quasilines in $\dot{\mathcal{X}}_i$

If $E(\ell)$ denotes the stabilizer in H_v of a boundary line ℓ of $\overline{Y}_v$, then $E(\ell)$ is virtually cyclic and almost malnormal. Since $\{E(\ell)\}$ is H_v-finite by conjugacy, let $\mathbb{E}_v$ be a complete finite set of conjugacy representatives. By Lemma 2.16, H_v is hyperbolic relative to peripheral subgroups $\mathbb{E}_v$. Hence, the results in Section 3 apply here.

Let $\lambda, c > 0$ be the universal constants given by Lemma 2.30 applied to the actions of H_v on $\dot{Y}_v$ for all $v \in T^0$ (since there are only finitely many actions up to conjugacy). By convention, the quasilines in coned-off spaces are understood as (λ, c)-quasigeodesics in $\dot{\mathcal{X}}_i$ and $\dot{Y}_v$.

The coning-off construction has the following consequence [43, Lemma 5.14]: the shortest projection of any quasiline α in $\dot{Y}_v$ to a quasiline β in $\dot{Y}_{v'}$ has to pass through the cone point attached to $\dot{Y}_{v'}$, and thus has uniformly bounded diameter by $\theta = \theta(\lambda, c) > 0$.

For simplicity, we also assume that $\theta = \theta(\lambda, c) > 0$ satisfies the conclusion of Proposition 2.29. Consequently, this determines a constant $\xi = \xi(\theta) > 0$ such that any set of quasilines with θ-bounded projection satisfies the projection axioms with projection constant ξ.

Fix $K > \max\{4\xi, \theta\}$. For each $v \in \mathcal{V}$, there exists an H_v-finite collection of quasilines $\mathbb{A}_v$ in $\dot{Y}_v$ and a constant $N = N(\mathbb{A}_v, K)$ such that the $d^K_{\hat{H}_v}$-distance formula holds by Lemma 3.2.

Since $\dot{G}$ acts cofinitely on $\mathcal{V}_1$ and $\mathcal{V}_2$, we can assume $\mathbb{A}_w = g\mathbb{A}_v$ if $w = gv$ for $g \in \dot{G}$. Let

$$\mathbb{A}_i := \bigcup_{v \in \mathcal{V}_i} \mathbb{A}_v$$

for $i = 1, 2$, which are both $\dot{G}$-invariant. We now equip $\mathbb{A}_i$ with projection maps as the shortest projection maps between two quasilines in $\dot{\mathcal{X}}_i$ for $i = 1, 2$.

If γ is a quasiline in $\dot{\mathcal{X}}_i$ for $i = 1, 2$, denote by $\dot{d}_\gamma(x, y)$ the $|\cdot|_{\dot{\mathcal{X}}_i}$-diameter of the shortest projection of $x, y \in \dot{\mathcal{X}}_i$ to γ.

The following result shows that the thick distance is captured by the projections of $\mathbb{A}_i$. Recall that r is the radius of the hyperbolic cones in constructing $\dot{\mathcal{X}}_i$.

Proposition 4.5 [43, Proposition 5.9] *For any $x, y \in \dot{\mathcal{X}}_i$,*

$$|x - y|^K_{\dot{\mathcal{X}}_i} \sim_{r,K} \sum_{\gamma \in \mathbb{A}_i} [\dot{d}_\gamma(x, y)]_K + |\rho(x) - \rho(y)|_T. \tag{10}$$

In the next subsection, we construct a suitable finite subgroup of G such that it acts isometrically on a finite product of quasitrees $T_1, \dots, T_n$ under some assumptions on vertex groups. This allows rewriting the right-hand side of the distance formula (10) as the product distance of the T_i.

4.3 Isometric action of a suitable finite-index subgroup of G

In a group, two elements are *independent* if they do not have conjugate powers (see [52, Definition 3.2]).

Definition 4.6 A group H is *omnipotent* if for any nonempty set of pairwise independent elements $\{h_1, \dots, h_r\}$ $(r \geq 1)$ there is a integer $p \geq 1$ such that for every choice of positive natural numbers $\{n_1, \dots, n_r\}$, there is a finite quotient $H \to \hat{H}$ such that $\hat{h}_i$ has order $n_i p$ for each i.

Let $G \curvearrowright X$ be a CKA action, where G is the fundamental group of the admissible graph of groups $\mathcal{G}$ such that every vertex group G_v is a central extension of an omnipotent hyperbolic group. By Lemma 4.1, the finite-index subgroup $\dot{G}$ acts on $\dot{\mathcal{X}}_1 \times \dot{\mathcal{X}}_2 \times T$ which is equipped with the $\dot{G}$-invariant function $|\cdot|^K_{\dot{\mathcal{X}}_1} \times |\cdot|^K_{\dot{\mathcal{X}}_2} \times |\cdot|_T$. The main result of this subsection is the following.

Proposition 4.7 *The group $\dot{G}$ admits finitely many isometric actions on quasitrees T_i for $1 \le i \le n$ such that there exists a $\dot{G}$-equivariant quasi-isometric embedding from $\dot{\mathscr{X}}_1 \times \dot{\mathscr{X}}_2 \times T$ to $T_1 \times T_2 \times \cdots \times T_n \times T$.*

We emphasize here that $|\cdot|^K_{\dot{\mathscr{X}}_1} \times |\cdot|^K_{\dot{\mathscr{X}}_2} \times |\cdot|_T$ on the domain for the quasi-isometric embedding is not a distance function, but the target is equipped with product distance.

By [11, Theorem II.6.12], G_v contains a subgroup K_v intersecting trivially with $Z(G_v)$ such that the direct product $K_v \times Z(G_v)$ is a finite-index subgroup. Thus, the image of K_v in $G_v/Z(G_v)$ is of finite index in H_v and K_v acts geometrically on hyperbolic spaces $\overline{Y}_v$. Since H_v is omnipotent and then is residually finite, we can assume that K_v is torsion-free.

Recall the $\dot{G}$-invariant collection of quasilines in Section 4.2,

$$\mathbb{A}_i = \bigcup_{v \in \mathscr{V}_i} \mathbb{A}_v,$$

where $\mathbb{A}_v$ is the collection of quasilines such that d^K-distance formula holds by Lemma 3.2. By the residual finiteness of K_v, there exists a finite-index subgroup $\dot{K}_v$ such that $\mathbb{A}_v$ is partitioned into $\dot{K}_v$-invariant subcollections with projection constants ξ.

To prepare the proof, we need to introduce a compatible condition of gluing finite-index subgroups. A collection of finite-index subgroups $\{G'_e, G'_v \mid v \in \mathscr{G}^0, e \in \mathscr{G}^1\}$ is called *compatible* if whenever $v = e_-$, we have

$$G_v \cap G'_e = G'_v \cap G_e.$$

By [24, Theorem 7.51], a compatible collection of finite-index subgroups gives a finite-index subgroup of G. The following result says that upon taking finite-index subgroups, we can assume that each vertex group is a direct product in a CKA group.

Lemma 4.8 *Let $\{\dot{K}_v < K_v \mid v \in \mathscr{G}^0\}$ be a collection of finite-index subgroups. Then there exist finite-index subgroups $\ddot{K}_v$ of $\dot{K}_v$, G'_e of G_e and Z_v of $Z(G_v)$ such that the collection of finite-index subgroups $\{G'_e, G'_v = \ddot{K}_v \times Z_v \mid v \in \mathscr{G}^0, e \in \mathscr{G}^1\}$ is compatible.*

Assuming Lemma 4.8, we now complete the proof of Proposition 4.7.

Proof of Proposition 4.7 We pass to further finite-index subgroups $\ddot{K}_v < \dot{K}_v$ satisfying compatible conditions, which then gives a further indexed subgroup $\ddot{G} \subset \dot{G}$. For $i = 1, 2$, let us partition $\mathbb{A}_i = \bigcup_{k=1}^{n_i} \mathbb{A}^i_k$ into $\ddot{G}$-obits $\mathbb{A}^i_k$. By the construction of $\ddot{G}$, we know that $\ddot{G}$ intersects each vertex group G_v of the Bass–Serre tree in a (conjugate) subgroup $\ddot{K}_v$. Thus, for each k, $\mathbb{A}^i_k$ are the union of certain $\ddot{K}_v$-invariant subcollections where v are varied in $\mathscr{V}_i$.

Recall that $\mathbb{A}^i$ for $i = 1, 2$ satisfies the projection axioms with a uniform projection constant ξ in Section 4.2. We can then build the quasitrees $T^i_k := \mathscr{C}_K(\mathbb{A}^i_k)$ where $1 \le k \le n_i$. Setting $n = n_1 + n_2$, this thus yields isometric group actions of $\ddot{G}$ on quasitrees T_i $(1 \le i \le n)$.

We first construct a $\ddot{G}$-equivariant map Φ from $\dot{\mathscr{X}}_1 \times \dot{\mathscr{X}}_2 \times T$ to $T_1 \times T_2 \times \cdots \times T_n \times T$. By equivariance, it suffices to fix a basepoint in each $\mathscr{X}_1$, $\mathscr{X}_2$, T and T_i so that Φ sends basepoints to basepoints. The quasi-isometric embedding property follows from the distance formula (10), where the right-hand side is now replaced by the distance in the corresponding quasitrees.

Note that $\ddot{G}$ is of finite index in $\dot{G}$. By taking more copies of quasitrees T_i in the target, the map Φ can be made $\dot{G}$-equivariant. Indeed, if a finite-index subgroup $H \subset G$ acts on some space X then G acts on a finite product of $[G : H]$ copies of X without preserving the factors. The map Φ can be extended to these copies as well. □

Proof of Lemma 4.8 Assume that $\langle f_v \rangle = Z(G_v)$ for any $v \in \mathscr{G}^0$. Then for an oriented edge $e = [v, w]$ from v to w, the subgroup $\langle f_v, f_w \rangle$ is of finite index in G_e.

Note that $G_e \cong \mathbb{Z}^2$ admits a base $\{\hat{f}_v, \hat{b}_e\}$ where $\hat{f}_v$ is primitive so that f_v is some power of $\hat{f}_v$. Let $\pi_v \colon G_v \to H_v = G_v/Z(G_v)$. Thus, $\pi_v(G_e)$ is a direct product of a torsion group with $\langle b_e \rangle$ in H_v, where $b_e = \pi_v(\hat{b}_e)$ is a loxodromic element.

Similarly, let $\hat{f}_w, \hat{b}_{\bar{e}} \in G_e$ such that $\langle \hat{f}_w, \hat{b}_{\bar{e}} \rangle = G_e$. Keep in mind that for any integer $n \neq 0$,

$$\langle \hat{f}_v^n, \hat{b}_e^n \rangle = \langle \hat{b}_{\bar{e}}^n, \hat{f}_w^n \rangle$$

is of finite index in G_e.

We choose an integer $m \neq 0$ such that $\hat{b}_e^m \in \dot{K}_v$ for every vertex $v \in \mathscr{G}^0$ and every oriented edge e from $e_- = v$. Such an integer m exists since $\dot{K}_v$ injects into H_v as a finite-index subgroup, and $\mathscr{G}$ is a finite graph of groups.

Apply the omnipotence of H_v to the independent set of elements $\{b_e \mid e_- = v\}$. Let p_v be the constant given by Definition 4.6. Set

$$s := m \prod_{v \in \mathscr{G}^0} p_v.$$

Set $l_v = s/p_v$. Thus, for the collection $\{b_e \mid e_- = v\}$, there exists a finite quotient $\xi_v \colon H_v \to \overline{H}_v$ such that $\xi_v(b_e)$ has order $s = l_v p_v$ and $b_e^s \in \ker(\xi_v)$. Then $\ddot{K}_v := \dot{K}_v \cap \pi_v^{-1} \ker(\xi_v)$ is of finite index in $\dot{K}_v$. Recall that $\pi_v|_{K_v} \colon K_v \to H_v$ is injective (see the paragraph before Lemma 4.8). Since $\pi_v(\hat{b}_e^s) = b_e^s$ is loxodromic in H_v and $\hat{b}_e^s \in \dot{K}_v$ for $m|s$, we have that $\hat{b}_e^s$ is a loxodromic element in $\ddot{K}_v$.

For each oriented edge $e = [v, w] \in \mathscr{G}^1$, define

$$G'_v := \langle \hat{f}_v^s \rangle \times \ddot{K}_v, \quad G'_w := \langle \hat{f}_w^s \rangle \times \ddot{K}_w, \quad G'_e := \langle \hat{f}_v^s, \hat{b}_e^s \rangle = \langle \hat{b}_{\bar{e}}^s, \hat{f}_w^s \rangle < G'_v.$$

Let $g \in G_e \cap G'_v$ be any element so we can write $g = \hat{f}_v^{sm} k$ for some $m \in \mathbb{Z}$ and $k \in \ddot{K}_v$. Recall that $\pi_v(G_e)$ is a direct product of $\langle b_e \rangle$ and a torsion group, and $\ddot{K}_v$ is torsion-free. So

$$\pi_v(g) = \pi_v(k) \in \pi_v(G_e) \cap \pi_v(\ddot{K}_v)$$

is some power of b_e; $\pi_v(k) = b_e^l$ for some $l \in \mathbb{Z}$. Note that $b_e^l = \pi_v(k) \in \ker(\xi_v)$ so omnipotence implies that $s|l$, ie $l = ns$ for some $n \in \mathbb{Z}$. Since $b_e = \pi_v(\hat{b}_e)$ and $\pi_v \colon \ddot{K}_v \to H_v$ is injective, we obtain that $k = \hat{b}_e^{ns}$. Therefore, $g = \hat{f}_v^{sm}\hat{b}_e^{ns} \in G'_e$ which implies

$$G_v \cap G'_e = G'_v \cap G_e.$$

Therefore, the collection $\{G'_v, G'_e \mid v \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ is verified to be compatible. □

4.4 Outline of the proof of Theorem 1.3

Let $G \curvearrowright X$ be a CKA action where G is the fundamental group of an admissible graph of groups $\mathcal{G}$ such that for every vertex group the central extension (1) has omnipotent hyperbolic quotient group. Recall that property (QT) is preserved undertaking finite-index subgroups (see Lemma 2.3). Upon passing to further indexed subgroups in Lemma 4.8, we can assume that $G_v = H_v \times \mathbb{Z}$, where H_v acts geometrically on $\overline{Y}_v$ and also we can assume $\dot{G} = G$. To show the property (QT) of G, we must find not only a suitable action on a finite product of quasitrees, but also ensure the distance of points in the image can recover word distance in the ambient group. We briefly describe here the strategy of the proof. Details are given in Sections 5 and 6.

Thanks to Proposition 4.7, we know that there exists a G-equivariant quasi-isometric embedding (note that $\dot{G} = G$)

$$\dot{\mathcal{X}}_1 \times \dot{\mathcal{X}}_2 \times T \to T_1 \times T_2 \times \cdots \times T_n \times T.$$

Here T_i (with $i \in \{1, 2, \ldots, n\}$) is a quasitree. As the geometry of space $\dot{\mathcal{X}}_1 \times \dot{\mathcal{X}}_2 \times T$ does not capture the distance from vertical parts of X, there is no way finding a quasi-isometric embedding from the orbit Go to $\dot{\mathcal{X}}_1 \times \dot{\mathcal{X}}_2 \times T$. To overcome this obstacle, in Section 5, we will construct two additional quasitrees, denoted by $\mathcal{C}_K(\mathbb{F}_1)$ and $\mathcal{C}_K(\mathbb{F}_2)$, and will show that there is indeed a G-equivariant quasi-isometric embedding

$$\Phi \colon Go \to \mathcal{C}_K(\mathbb{F}_1) \times \mathcal{C}_K(\mathbb{F}_2) \times \dot{\mathcal{X}}_1 \times \dot{\mathcal{X}}_2 \times T$$

(Section 6 is devoted to constructing Φ and verifying G-equivariant quasi-isometric embedding of Φ). As a consequence, we obtain the desirable G-equivariant quasi-isometric embedding

$$Go \to \mathcal{C}_K(\mathbb{F}_1) \times \mathcal{C}_K(\mathbb{F}_2) \times T_1 \times T_2 \times \cdots \times T_n \times T$$

which entails property (QT) of G.

5 Projection system of fiber lines

Recall we partition $T^0 = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_i$ consists of vertices in T with pairwise even distances. For convenience, we sometimes write $\mathcal{V} = \mathcal{V}_1$ and $\mathcal{W} = \mathcal{V}_2$. We note that property (QT) of a group is preserved under taking a finite-index subgroup (see Lemma 2.3). Thus passing to a finite-index subgroup (see Lemma 2.23) if necessary we could assume that G is torsion-free and preserves $\mathcal{V}_i$ with $i = 1, 2$.

Note that $e = [w, v]$ is an oriented edge from w towards v, and $\bar{e} = [v, w]$ is the oriented edge from v towards w. For each oriented edge e, let F_e be the corresponding boundary plane. It is clear that $F_e = F_{\bar{e}}$ does not depend on the orientation.

5.1 Desired quasilines

By Lemma 2.17, the CKA space X decomposes as the union of vertex spaces $\widetilde{Y}_v = N_D(Y_v)$ for $v \in T^0$, on which the vertex groups G_v act geometrically. The center $Z(G_v) \simeq \mathbb{Z}$ allows us to split Y_v as a metric product $\overline{Y}_v \times \mathbb{R}$. Upon passing to further finite-index subgroups in Lemma 4.8, we can assume that $G_v = H_v \times \mathbb{Z}$, where H_v acts geometrically on $\overline{Y}_v$. If the CKA action $G \curvearrowright X$ is not flip (as in [43]), the system of fiber lines $\mathbb{R}$ in $Y_v = \overline{Y}_v \times \mathbb{R}$ does not behave well with respect to the G-action. We introduce better geometric models for vertex subgroups in order to resolve the G-action of fiber lines. As in [29], these models are the metric product of $\overline{Y}_v$ with a quasiline.

We first explain the construction of the quasiline obtained from a quasimorphism. The following lemma is cited from Lemma 4.2 and the proof of Corollary 4.3 in [29]. We present their proof as it is short and crucial for our discussion.

Lemma 5.1 *Let H be a hyperbolic group relative to a finite collection of virtually cyclic subgroups $\{E_i \mid 1 \le i \le n\}$. Consider $G = H \times \mathbb{Z}$ and fix a set of elements $c_i \in E_i \times \mathbb{Z}$ for each $1 \le i \le n$ such that $\langle c_i \rangle$ has unbounded projection to E_i. Then there exist a generating set S of G and a (λ, λ)-quasi-isometry $\varphi\colon \mathrm{Cay}(G, S) \to \mathbb{R}$ such that the following holds.*

(1) *If $g\langle c_i \rangle$ and $g'\langle c_i \rangle$ are two $\langle c_i \rangle$-cosets for $g, g' \in E_i \times \mathbb{Z}$, then*

$$\lambda^{-1}|g\langle c_i \rangle - g'\langle c_i \rangle|_G - \lambda \le |\varphi(g\langle c_i \rangle) - \varphi(g'\langle c_i \rangle)| \le \lambda|g\langle c_i \rangle - g'\langle c_i \rangle|_G + \lambda$$

where $|g\langle c_i \rangle - g'\langle c_i \rangle|_G$ denotes the distance between two subsets in G equipped with a word metric relative to a finite generating set (so not the distance on $\mathrm{Cay}(G, S)$*).*

(2) *With the natural action of $G \to H$, the diagonal action of $G = H \times \mathbb{Z}$ on $H \times \mathrm{Cay}(G, S)$ is metrically proper and cobounded, where $\mathbb{Z} \subset G$ acts loxodromically on* $\mathrm{Cay}(G, S)$ *but $\langle c_i \rangle$ acts boundedly.*

In applications, the choice of elements c_i shall come from the fiber generator of the adjacent pieces. See Lemma 5.2 below.

Proof Let $\pi_H\colon G = H \times \mathbb{Z} \to H$ and $\pi_{\mathbb{Z}}\colon G = H \times \mathbb{Z} \to \mathbb{Z}$ be the natural projections. Let $t_i = \pi_H(c_i) \in E_i$ be the projection to H of the element c_i. We then choose a quasimorphism $\phi_i : H \to \mathbb{R}$ by [32] such that $\phi_i(t_i) = 1$ but $\phi_i(E_k) = 0$ if $E_k \ne E_i$. Define the quasimorphism of $G \to \mathbb{R}$ as follows: for any $x \in G$,

$$\varphi(x) := \pi_{\mathbb{Z}}(x) - \sum_{i=1}^{n} \pi_{\mathbb{Z}}(c_i) \cdot (\phi_i \circ \pi_H(x)).$$

By definition, φ takes the constant value on $\langle c_i \rangle$-cosets. Moreover, the distance $|g\langle c_i \rangle - g'\langle c_i \rangle|_G$ is bi-Lipschitz to $|\varphi(g\langle c_i \rangle) - \varphi(g'\langle c_i \rangle)|$ with a constant depending only on $\langle c_i \rangle$.

To find the generating set S, notice that the homogenization of φ (still denoted by φ) has a bounded distance to the original one. As φ is unbounded, there exists $h \in G$ such that $\{\varphi(h^n) = n\varphi(h) \mid n \in \mathbb{Z}\}$ is an infinite cyclic subgroup. Let $S := \varphi^{-1}([0, 2\varphi(h)])$ be a (possibly infinite) subset of G. One can prove that S generates G, and $\varphi \colon G \to \mathbb{R}$ induces a desired quasi-isometry $\varphi \colon \mathrm{Cay}(G, S) \to \mathbb{R}$. See [1, Lemma 4.15] for details. □

5.2 New geometric model for vertex spaces

Recall that G acts on the Bass–Serre tree T with finitely many vertex orbits. Let $\{v_1, v_2, \dots, v_n\} \subset T$ be the full set of vertex representatives, and let S_{v_i} be the (infinite) generating set for G_{v_i} given by Lemma 5.1. Then G_{v_i} acts on the quasiline $\mathfrak{fl}(v_i) := \mathrm{Cay}(G_{v_i}, S_{v_i})$. Let v be an arbitrary vertex in T, so that $v = g v_i$ for some $g \in G$ and $i \in \{1, 2, \dots, n\}$. By equivariance, we define the quasiline $\mathfrak{fl}(v) := g\mathfrak{fl}(v_i) = g\,\mathrm{Cay}(G_{v_i}, S_{v_i})$, and the action of $G_{g v_i} = g G_{v_i} g^{-1}$ on $g\mathfrak{fl}(v_i)$ is induced from the action of G_{v_i} on $\mathfrak{fl}(v_i)$.

Consider the word metric on G given by a finite generating set of G including a finite generating set of G_{v_i} for each representative vertex v_i. Equipping each vertex group G_v with a word metric, the inclusion of G_v into G is a quasi-isometric embedding since Y_v is quasi-isometrically embedded in the CAT(0) space X.

Write $X_v := \overline{Y}_v \times \mathfrak{fl}(v)$ for the new geometric model for G_v. By Lemma 5.1, the diagonal action $G_v \curvearrowright X_v$ is metrically proper and cobounded, and hence the induced orbital map

$$G_v \to G_v o' \subset X_v$$

is a G_v-equivariant quasi-isometry for any basepoint $o' = (o'_1, o'_2) \in X_v$.

Let us fix a basepoint $o = (o_1, o_2) \in Y_v$. As G_v acts freely and geometrically on $Y_v = \overline{Y}_v \times \mathbb{R}$, let

$$G_v o \to G_v$$

be a bijective G_v-equivariant quasi-isometry, a quasi-inverse to the orbital map of $G_v \curvearrowright Y_v$.

Choose the same first coordinate $o_1 = o'_1$ for the above basepoints o and o'. Define a G_v-equivariant map $\Lambda_v \colon Y_v \to X_v$ as the composite of the above two G-equivariant maps

$$\Lambda_v \colon Y_v = \overline{Y}_v \times \mathbb{R} \to G_v \to X_v = \overline{Y}_v \times \mathfrak{fl}(v).$$

Define the horizontal and vertical projection maps

$$\Lambda_v^{\mathrm{hor}} \colon Y_v \to \overline{Y}_v, \quad \Lambda_v^{\mathrm{ver}} \colon Y_v \to \mathfrak{fl}(v) \tag{11}$$

as the composites of the map Λ_v with the projections to the factor $\overline{Y}_v$ and $\mathfrak{fl}(v)$ respectively. For the product space $X_v = \overline{Y}_v \times \mathfrak{fl}(v)$, we define similarly the horizontal distance and vertical distances $|\cdot|^{\mathrm{hor}}_{X_v}$ and $|\cdot|^{\mathrm{ver}}_{X_v}$. In terms of these notations, we have for any $x, y \in Y_v$,

$$|\Lambda_v(x) - \Lambda_v(y)|^{\mathrm{hor}}_{X_v} = |\Lambda^{\mathrm{hor}}_v(x) - \Lambda^{\mathrm{hor}}_v(y)|_{\overline{Y}_v},$$
$$|\Lambda_v(x) - \Lambda_v(y)|^{\mathrm{ver}}_{X_v} = |\Lambda^{\mathrm{ver}}_v(x) - \Lambda^{\mathrm{ver}}_v(y)|_{\mathfrak{fl}(v)}.$$

We now derive a few important facts from Lemma 5.1 about Λ_v.

Recall that each piece Y_v of the CKA space X splits as a metric product $\overline{Y}_v \times \mathbb{R}$. In this context, a *fiber line* in Y_v refers to a subset $\{x\} \times \mathbb{R}$ of Y_v where $x \in \overline{Y}_v$.

Let $\pi_v \colon Y_v \to \overline{Y}_v$ be the natural projection map coming from the splitting $Y_v = \overline{Y}_v \times \mathbb{R}$. We remark that π_v and Λ^{hor}_v are not the same.

Lemma 5.2 *There exists a uniform constant $\lambda > 0$ such that Λ_v is a (λ, λ)-quasi-isometry: for any $x, y \in Y_v$,*

$$\frac{1}{\lambda}|\Lambda_v(x) - \Lambda_v(y)|_{X_v} - \lambda \le |x - y|_{Y_v} \le \lambda|\Lambda_v(x) - \Lambda_v(y)|_{X_v} + \lambda.$$

Moreover, let Y_w be the adjacent piece of Y_v in the CKA space X. Let ℓ and ℓ' be lines in the plane $P = Y_v \cap Y_w$ such that ℓ and ℓ' are fibers in Y_w. Then the following hold:

(1) $\mathrm{diam}(\Lambda^{\mathrm{ver}}_v(\ell)) \le \lambda$. *In other words, $\Lambda_v(\ell) \subset \overline{Y}_v \cap B(a, \lambda)$ in $\overline{Y}_v \times \mathfrak{fl}(v)$ for some $a \in \mathfrak{fl}(v)$.*

(2) *Let $p \in Y_v = \overline{Y}_v \times \mathbb{R}$ be any point and $\pi_v(p)$ be the projection of p into the factor $\overline{Y}_v$. Then $|\pi_v(p) - \Lambda^{\mathrm{hor}}_v(p)|_{\overline{Y}_v} \le \lambda$.*

(3) *Denote by $|\ell - \ell'|_{Y_v}$ the distance between ℓ and ℓ' in Y_v. Then*

$$\lambda^{-1}|\ell - \ell'|_{Y_v} - \lambda \le \mathrm{diam}_{\mathfrak{fl}(v)}(\Lambda^{\mathrm{ver}}_v(\ell) \cup \Lambda^{\mathrm{ver}}_v(\ell')) \le \lambda|\ell - \ell'|_{Y_v} + \lambda.$$

Proof We first prove (2). Choose the fixed basepoints $o = (o_1, o_2)$ in Y_v and $o' = (o'_1, o'_2)$ in $\overline{Y}_v \times \mathfrak{fl}(v)$ such that their projections into the factor $\overline{Y}_v$ are the same: $o_1 = o'_1 \in \overline{Y}_v$. Take any point $p = (a, t)$ in $Y_v = \overline{Y}_v \times \mathbb{R}$, so $\pi_v(p) = a$. By our definition of the G_v-equivariant quasi-inverse $Y_v \to G_v$, there exists a group element $g \in G_v$ such that $|go - p|_{Y_v} \le \lambda$ for some uniform constant λ. We write $g = (h, n)$ in $H_v \times \mathbb{Z}$. Note that G_v acts on $\overline{Y}_v \times \mathfrak{fl}(v)$ diagonally; thus the image of the group element $g = (h, n)$ under the composition map

$$G_v \to \overline{Y}_v \times \mathfrak{fl}(v) \to \overline{Y}_v$$

is $h \cdot o_1$, where the first one is the orbital map and the second one is the projection map. If Y_v is equipped with L^1-metric, it follows that $|ho_1 - a|_{\overline{Y}_v} \le |go - p|_{Y_v} \le \lambda$. As the map Λ_v descends to the map $\overline{Y}_v \to \overline{Y}_v$ sending a to $h(o_1)$, our claim is confirmed:

$$|\Lambda^{\mathrm{hor}}_v(x) - \Lambda^{\mathrm{hor}}_v(y)|_{\overline{Y}_v} - \lambda \le d^h(x, y) \le \lambda + |\Lambda^{\mathrm{hor}}_v(x) - \Lambda^{\mathrm{hor}}_v(y)|_{\overline{Y}_v}.$$

For part (1), as there are only finitely many isometric types of Y_v of X, we only need to prove that $\operatorname{diam}(\Lambda_v^{\mathrm{ver}}(\ell)) \le \lambda$ for one given Y_v. Indeed, recall that $\Lambda_v^{\mathrm{ver}}\colon Y_v \to \mathfrak{fl}(v)$ factors through

$$Y_v \to X_v = \overline{Y}_v \times \mathfrak{fl}(v)$$

as the natural projection $X_v \to \mathfrak{fl}(v)$. The latter agrees with the quasimorphism $\varphi\colon G_v \to \mathbb{R}$ up to a bounded error in the proof of Lemma 5.1, vanishing on the center $Z(G_w)$. If $B(a,\lambda)$ denotes the ball at some element $a \in \mathfrak{fl}(v)$ with radius λ, it follows that $Z(G_w)o \subset \overline{Y}_v \times B(a,\lambda)$. Every fiber line ℓ in Y_w lies in a uniform neighborhood of the orbit of a $Z(G_w)$-coset. Our second claim is thus verified.

Part (3) is clear from our construction. □

5.3 Projection maps

Recall $T^0 = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_i$ consists of vertices in T with pairwise even distances. Let

$$\mathbb{F}_1 = \{\mathfrak{fl}(v) \mid v \in \mathcal{V}_1\}, \quad \mathbb{F}_2 = \{\mathfrak{fl}(w) \mid w \in \mathcal{V}_2\}.$$

It remains to define a family of projection maps for them.

Definition 5.3 (projection maps in $\mathbb{F}_i$) Let $e_1 = [v,w]$, $e_2 = [w,v_2]$ denote the first two (oriented) edges in $[v,v']$. Let $F_{e_1} = Y_v \cap Y_w$ and $F_{e_2} = Y_{v_2} \cap Y_w$ be the two boundary planes of Y_w. Let $\mathcal{S}_{e_1e_2}$ be the strip in Y_w joining two boundary plane F_{e_1} and F_{e_2} of Y_w (see Section 2.3.1 for the definition of strips). We note that $\mathcal{S}_{e_1e_2} \cap F_{e_1}$ is a line in F_{e_1} that is parallel to a fiber in Y_w. We then define the *projection from* $\mathfrak{fl}(v')$ *into* $\mathfrak{fl}(v)$ to be

$$\Pi_{\mathfrak{fl}(v)}(\mathfrak{fl}(v') := \Lambda_v^{\mathrm{ver}}(\mathcal{S}_{e_1e_2} \cap F_{e_1}),$$

where Λ_v^{ver} defined in (11) is the vertical projection to the quasiline in $X_v = \overline{Y}_v \times \mathfrak{fl}(v)$.

Lemma 5.4 *Let $\lambda > 0$ be the constant given by Lemma 5.2. Let a, b and c be distinct vertices in $\mathcal{V}_i$ with $i = 1,2$. If $d_T(a,[b,c]) \ge 2$ then $\Pi_{\mathfrak{fl}(a)}(\mathfrak{fl}(c)) = \Pi_{\mathfrak{fl}(a)}(\mathfrak{fl}(b)) \le \lambda$.*

Proof Let $[b,a]$ and $[c,a]$ be the geodesics in the tree T connecting b and c to w respectively. Let $e \cdot e'$ be the last two edges in $[b,a]$ (that is also the last two edges in $[c,a]$). Let $\mathcal{S}_{ee'}$ be the strip in Y_{e_+} connecting two boundary planes F_e and $F_{e'}$ of Y_{e_+} By our definition of projection maps, we have that $\Pi_{\mathfrak{fl}(a)}(\mathfrak{fl}(c)) = \Pi_{\mathfrak{fl}(a)}(\mathfrak{fl}(b)) = \Lambda_a^{\mathrm{ver}}(\mathcal{S}_{ee'} \cap P_{e'}) \le \lambda$. □

5.4 Projection axioms

We are now going to verify that $\mathbb{F}_i$ $(i = 1,2)$ with the above-defined projection maps in Definition 5.3 satisfy the projection axioms (see Definition 2.24). For each vertex $v \in T$, let $\mathbb{L}_v$ be the collection of boundary lines in the hyperbolic space $\overline{Y}_v$ defined at the beginning of Section 4. Let ℓ_1, ℓ_2 and ℓ_3 be three distinct boundary lines in $\mathbb{L}_v$. We write

$$d_{\ell_1}(\ell_2,\ell_3) = \operatorname{diam}(\pi_{\ell_1}(\ell_2) \cup \pi_{\ell_1}(\ell_3))$$

where $\pi_{\ell_i}(\ell_j)$ is the shortest projection of ℓ_j to ℓ_i in the CAT(0) hyperbolic space $\overline{Y}_v$ (note that $\overline{Y}_v$ is a hyperbolic space since H_v acts geometrically on $\overline{Y}_v$ and H_v is a nonelementary hyperbolic group). Recall that

$$d_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_2),\mathfrak{fl}(v_3)) := \operatorname{diam}\big(\Pi_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_2)) \cup \Pi_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_3))\big).$$

Lemma 5.5 *There exists a uniform constant $\lambda > 0$ such that the following holds. Let v_1, v_2 and v_3 be distinct vertices in $\mathcal{V}_1$ such that v_1, v_2 and v_3 are in $\mathrm{Lk}(o)$ for some vertex o in $\mathcal{V}_2$. Let e_i denote the edge $[v_i, o]$ with $i = 1, 2, 3$ and let F_{e_i} be the plane in X associated to e_i. For each $i = 1, 2, 3$, let ℓ_i denote the boundary line of $\overline{Y}_o$ that is the projection of F_{e_i} into $\overline{Y}_o$. Then*

$$\frac{1}{\lambda} d_{\ell_1}(\ell_2,\ell_3) - \lambda \le d_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_2),\mathfrak{fl}(v_3)) \le \lambda d_{\ell_1}(\ell_2,\ell_3) + \lambda.$$

Proof Let $\mathcal{S}_{e_1e_2}$ and $\mathcal{S}_{e_1e_3}$ be the strips in Y_o connecting the planes F_{e_1} to F_{e_2} and F_{e_1} to F_{e_3} respectively. We denote the line $\mathcal{S}_{e_1e_2} \cap F_{e_1}$ by ℓ and denote the line $\mathcal{S}_{e_1e_3} \cap F_{e_1}$ by ℓ'. Note that both lines ℓ and ℓ' are fibers in Y_o. Recall that by our definition of projection maps, we have $\Pi_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_2)) = \Lambda^{\mathrm{ver}}_{v_1}(\ell)$ and $\Pi_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_3)) = \Lambda^{\mathrm{ver}}_{v_1}(\ell')$. By part (3) of Lemma 5.2, for some $\lambda > 0$, we have that

$$\frac{1}{\lambda}|\ell - \ell'| - \lambda \le \operatorname{diam}\big(\Pi_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_2)) \cup \Pi_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_3))\big) \le \lambda|\ell - \ell'| + \lambda.$$

Note that $|\ell - \ell'| = d_{\ell_1}(\ell_2,\ell_3)$ (indeed, let α and β be the shortest geodesics joining ℓ_2 to ℓ_1 and ℓ_3 to ℓ_1 respectively; then ℓ and ℓ' are the product $\alpha_+ \times \mathbb{R}$ and $\beta_+ \times \mathbb{R}$ of endpoints of α and β, respectively, with the $\mathbb{R}$ direction in $Y_o = \overline{Y}_o \times \mathbb{R}$). Combining the above inequalities, we obtain a constant $\lambda' = \lambda'(\lambda) > 0$ still denoted by λ such that

$$\frac{1}{\lambda} d_{\ell_1}(\ell_2,\ell_3) - \lambda \le \operatorname{diam}\big(\Pi_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_2)) \cup \Pi_{\mathfrak{fl}(v_1)}(\mathfrak{fl}(v_3))\big) \le \lambda d_{\ell_1}(\ell_2,\ell_3) + \lambda. \qquad \square$$

We are now going to prove the following.

Lemma 5.6 *There exists a constant $\xi > 0$ such that for each $i \in \{1,2\}$, the collection $\mathbb{F}_i$ with projection maps $\pi_{\mathfrak{fl}(v)}$ satisfies the projection axioms with projection constant ξ.*

Proof We verify in order the projection axioms (see Definition 2.24) for the projection maps defined on $\mathbb{F}_1$. The case for $\mathbb{F}_2$ is symmetric. The constant ξ will be defined explicitly during the proof.

Axiom 1 Let $\lambda > 0$ be the constant given by Lemma 5.2. Since $\mathcal{S}_{e_1e_2} \cap F_{e_1}$ is a fiber line in Y_w, it follows from Lemma 5.2 that $\operatorname{diam} \Lambda^{\mathrm{ver}}_v(\mathcal{S}_{e_1e_2} \cap F_{e_1}) \le \lambda$. Thus $\operatorname{diam}\big(\Pi_{\mathfrak{fl}(v)}(\mathfrak{fl}(v'))\big) \le \lambda$. Axiom 1 in Definition 2.24 is verified.

Axiom 2 Let u, v and w be distinct vertices in $\mathcal{V}_1$. We will show that there exists $\xi \ge 0$ sufficiently large that if $d_{\mathfrak{fl}(w)}(\mathfrak{fl}(u),\mathfrak{fl}(v)) > \xi$, then $d_{\mathfrak{fl}(u)}(\mathfrak{fl}(w),\mathfrak{fl}(v)) \le \xi$ or $d_{\mathfrak{fl}(v)}(\mathfrak{fl}(w),\mathfrak{fl}(u)) \le \xi$. The constant ξ will be defined explicitly during the proof. Since $d_{\mathfrak{fl}(w)}(\mathfrak{fl}(u),\mathfrak{fl}(v)) > \xi$, it follows from Lemma 5.4 that there is some restriction on w, ie w is either lies on $[u,v]$ or $d_T(w,[u,v]) = 1$.

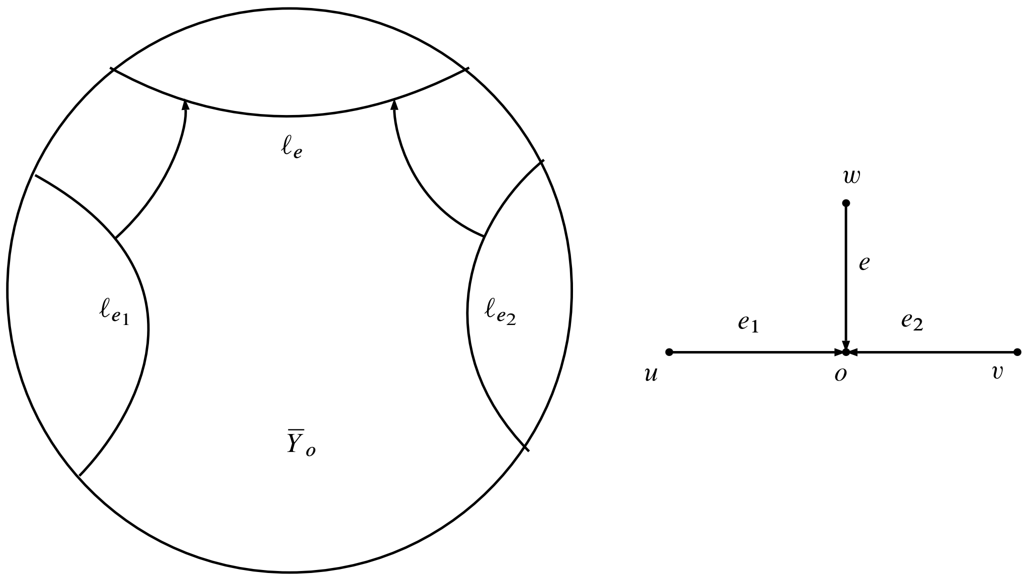

Figure 2: Verification of Axiom 2.

Case 1 Suppose w lies on $[u,v]$. Since $u,w,v \in \mathcal{V}_1$, we have $d_T(u,[w,v]) \geq 2$ and $d_T(v,[u,w]) \geq 2$. Axiom 2 thus follows from Lemma 5.4.

Case 2 Suppose $d_T(w,[u,v]) = 1$. Without loss of generality, we can assume that u, v and w lie in the same link $\mathrm{Lk}(o)$ for some vertex o in $\mathcal{V}_2$. Indeed, let $o \in [u,v]$ be adjacent to w and $u', v' \in \mathrm{Lk}(o) \cap [u,v]$. It is clear by definition that $\pi_{\mathfrak{fl}(u)}(\mathfrak{fl}(v')) = \pi_{\mathfrak{fl}(u)}(\mathfrak{fl}(v))$ and $\pi_{\mathfrak{fl}(v)}(\mathfrak{fl}(u')) = \pi_{\mathfrak{fl}(v)}(\mathfrak{fl}(u))$. As a result, we can thus assume that $u = u'$ and $v = v'$ lie in the link $\mathrm{Lk}(o)$.

Recall that $\overline{Y}_o$ is a δ-hyperbolic space whose boundary lines $\mathbb{L}_o$ satisfy the projection axioms for a constant ξ_0 [48]. We claim that $\xi = \xi_0$ is the desired constant for Axiom 2.

Write $e = [w,o]$, $e_1 = [u,o]$ and $e_2 = [v,o]$. Let ℓ_e, ℓ_{e_1} and ℓ_{e_2} be the corresponding boundary lines of $\overline{Y}_o$ to the oriented edges e, e_1 and e_2. By Lemma 5.5, we have

$$\frac{1}{\lambda} d_{\ell_e}(\ell_{e_1},\ell_{e_2}) - \lambda \leq d_{\mathfrak{fl}(w)}(\mathfrak{fl}(u),\mathfrak{fl}(v)) \leq \lambda d_{\ell_e}(\ell_{e_1},\ell_{e_2}) + \lambda.$$

As $\mathbb{L}_o$ satisfies the projection axioms, we see that if $d_{\ell_e}(\ell_{e_1},\ell_{e_2}) > \xi_0$, then $d_{\ell_{e_1}}(\ell_e,\ell_{e_2}) \leq \xi_0$. Using Lemma 5.5 again, we have that

$$\frac{1}{\lambda} d_{\ell_{e_1}}(\ell_e,\ell_{e_2}) - \lambda \leq d_{\mathfrak{fl}(u)}(\mathfrak{fl}(w),\mathfrak{fl}(v)) \leq \lambda d_{\ell_{e_1}}(\ell_e,\ell_{e_2}) + \lambda.$$

Let ξ be a constant such that $\xi > \lambda\xi_0 + \lambda$. It follows from the above inequalities that

$$d_{\mathfrak{fl}(u)}(\mathfrak{fl}(w),\mathfrak{fl}(v)) = \mathrm{diam}\big(\Pi_{\mathfrak{fl}(u)}(\mathfrak{fl}(w)) \cup \Pi_{\mathfrak{fl}(u)}(\mathfrak{fl}(v))\big) \leq \xi,$$

so Axiom 2 is verified.

Axiom 3 For $u \neq v \in \mathcal{V}_1$, the set

$$\{w \in \mathcal{V}_1 \mid d_{\mathfrak{fl}(w)}(\mathfrak{fl}(u), \mathfrak{fl}(v)) > \xi\}$$

is a finite set.

Indeed, by Lemma 5.4, such a w is either contained in the interior of $[u, v]$ or $d(w, [u, v]) = 1$. The first case yields only $(d(u, v) - 1)$ choices for w. We now consider the case $d(w, [u, v]) = 1$. Since u, v and w have pairwise even distance, there exists $o \in \mathcal{W} \cap [u, v]^0$ and two vertices u' and v' on $[u, v]$ adjacent to o such that $u', v', w \in \mathrm{Lk}(o)$. By the projection axioms of boundary lines $\mathbb{L}_o$ of $\overline{Y}_o$, the set of w satisfying $d_{\mathfrak{fl}(w)}(\mathfrak{fl}(u), \mathfrak{fl}(v)) > \xi$ is finite. Thus, in both cases, the set of such w is finite. □

Lemma 5.7 *For each $i = 1, 2$, the collection $\mathbb{F}_i = \{\mathfrak{fl}(v) \mid v \in \mathcal{V}_i\}$ admits an action of the group G such that*

$$\Pi_{g\,\mathfrak{fl}(v)}(g\mathfrak{fl}(u)) = g\Pi_{\mathfrak{fl}(v)}(\mathfrak{fl}(u))$$

for any $v, u \in \mathcal{V}_i$ and any $g \in G$.

Proof First, let us recall some discussion in the beginning of Section 5.2. Recall that $\{v_1, v_2, \dots, v_n\} \subset T$ is the full set of vertex representatives of T and for each representative vertex $v_1, v_2, \dots, v_n$ of T, the quasiline $\mathfrak{fl}(v_j)$ is the Cayley graph $\mathrm{Cay}(G_{v_j}, S_{v_j})$ for some generating set S_{v_j} of G_{v_j} (see Lemma 5.1). Let v be an arbitrary vertex in T; then $v = gv_i$ for some $g \in G$ and $i \in \{1, 2, \dots, n\}$. The quasiline $\mathfrak{fl}(v)$ is given by $g\mathfrak{fl}(v_i) = g\,\mathrm{Cay}(G_{v_i}, S_{v_i})$, and the action of $G_{gv_i} = gG_{v_i}g^{-1}$ on $g\mathfrak{fl}(v_i)$ is induced from the action of G_{v_i} on $\mathfrak{fl}(v_i)$. We are now going to show that

$$\Pi_{g\,\mathfrak{fl}(v)}(g\mathfrak{fl}(u)) = g\Pi_{\mathfrak{fl}(v)}(\mathfrak{fl}(u)).$$

Recall that the family of maps $\Lambda^{\mathrm{ver}}_{gv} \colon Y_{gv} = gY_v \to g\mathfrak{fl}(v)$ are G-equivariant: $\Lambda^{\mathrm{ver}}_{gv}(gx) = g\Lambda^{\mathrm{ver}}_v(x)$ for all $x \in Y_v$. Let e_1 and e_2 be the first two edges in the geodesic $[v, u]$ with $v = (e_1)_-$ and $(e_1)_+ = (e_2)_-$. By Definition 5.3 of projection map, we have that

$$\begin{aligned}
\Pi_{\mathfrak{fl}(gv)}(\mathfrak{fl}(gu)) &= \mathrm{diam}\big(\Lambda^{\mathrm{ver}}_{gv}(\mathcal{S}_{ge_1ge_2} \cap F_{ge_1})\big) \\
&= \mathrm{diam}\big(\Lambda^{\mathrm{ver}}_{gv}(g(\mathcal{S}_{e_1e_2} \cap F_{e_1}))\big) \\
&= \mathrm{diam}\big(g\Lambda^{\mathrm{ver}}_{v}(\mathcal{S}_{e_1e_2} \cap F_{e_1})\big) \\
&= g\,\mathrm{diam}\big(\Lambda^{\mathrm{ver}}_{v}(\mathcal{S}_{e_1e_2} \cap F_{e_1})\big) = g\Pi_{\mathfrak{fl}(v)}(\mathfrak{fl}(u))
\end{aligned}$$

for any $g \in G$. □

Definition 5.8 Let $\xi > 0$ be the projection constant given by Lemma 5.6, so the collection of quasilines $\mathbb{F}_i = \{\mathfrak{fl}(v) \mid v \in \mathcal{V}_i\}$ with $i = 1, 2$ satisfies the projection axioms. For any fixed $K > 4\xi$, we obtain the unbounded quasitrees of metric spaces $\mathcal{C}_K(\mathbb{F}_1)$ and $\mathcal{C}_K(\mathbb{F}_2)$ (see Section 2.4). Combining Lemma 5.7 with [5, Section 4.4], the spaces $\mathcal{C}_K(\mathbb{F}_1)$ and $\mathcal{C}_K(\mathbb{F}_2)$ are quasitrees and admit unbounded isometric actions $G \curvearrowright \mathcal{C}_K(\mathbb{F}_1)$ and $G \curvearrowright \mathcal{C}_K(\mathbb{F}_2)$. The quasitrees $\mathcal{C}_K(\mathbb{F}_1)$ and $\mathcal{C}_K(\mathbb{F}_2)$ are called *vertical quasitrees* hereafter.

6 Distance formulas in the CKA space X

Let $\mathscr{C}_K(\mathbb{F}_1)$ and $\mathscr{C}_K(\mathbb{F}_2)$ be the vertical quasitrees in Definition 5.8. Let $\dot{\mathscr{X}}_1$ and $\dot{\mathscr{X}}_2$ be the coned-off spaces defined in Section 4.1. According to the outline of the proof of Theorem 1.3 in Section 4.4, the last step to prove property (QT) of G is to show that there is a G-equivariant quasi-isometric embedding

$$\Phi\colon Go \to \mathscr{C}_K(\mathbb{F}_1) \times \mathscr{C}_K(\mathbb{F}_2) \times \dot{\mathscr{X}}_1 \times \dot{\mathscr{X}}_2 \times T.$$

This section is devoted to constructing such a desired map Φ and verifying it is a quasi-isometric embedding.

We list here notation that will be used in the rest of this section.

- We fix an edge $[v_0, w_0]$ in the Bass–Serre tree T such that $v_0 \in \mathscr{V}_1$. Let $o \in X$ be a basepoint in the common boundary plane $F_{[v_0,w_0]}$ between two pieces Y_{v_0} and Y_{w_0}.
- Assume that $x = o \in Y_{v_0}$ and $y = go \in Y_{v_{2n}}$ for some $g \in G$ and $v_{2n} = go$. We list the vertices on the geodesic $[v_0, v_{2n}]$ by $\{v_0, v_1, \dots, v_{2n}\}$ where $v_{2i} \in \mathscr{V}_1$ and $v_{2i+1} \in \mathscr{V}_2$. Let $e_{i+1} = [v_i, v_{i+1}]$ be the oriented edge towards v_{i+1}. By definition of special paths, let $p_i := \mathscr{S}_{e_{i-1}e_i} \cap \mathscr{S}_{e_i e_{i+1}}$ be the intersection of two strips with $p_0 : x = o$ and $p_{2n+1} = y = go$.
- Let α be the geodesic edge path in the Bass–Serre tree T connecting v_0 to v_{2n}. And let $w_1 \in \mathscr{V}_2$ be a vertex adjacent to v_{2n}. Set

$$\tilde{\alpha} := e_0 \cup \alpha \cup e_{2n+1}$$

  where $e_0 = [w_0, v_0]$ and $e_{2n+1} = [v_{2n}, w_1]$. It is possible that $e_0 = \bar{e}_1$ and $e_{2n+1} = \bar{e}_{2n}$, ie $\tilde{\alpha}$ contains backtracking at e_0 and e_{2n}.

6.1 Construction of the desired map Φ

It is a product of the following four maps with the index map ρ in Definition 2.19.

- We define $\vartheta_1\colon Go \to \mathscr{C}_K(\mathbb{F}_1)$ as follows. Recall that each quasiline $\mathfrak{fl}(v)$ for $v \in \mathscr{V}_1$ embeds as a convex subset into $\mathscr{C}_K(\mathbb{L}_1)$ and $\Lambda_v^{\mathrm{ver}}\colon G_v o \to \mathfrak{fl}(v)$ is a G_v-equivariant map. For every $g \in G$, we set $\vartheta_1(go) := \Lambda_{gv_0}^{\mathrm{ver}}(go) = g\Lambda_{v_0}^{\mathrm{ver}}(o)$. The second equality follows by G_v-equivariance.
- Similarly, define $\vartheta_2\colon Go \to \mathscr{C}_K(\mathbb{F}_2)$ by $\vartheta_2(go) := \Lambda_{gw_0}^{\mathrm{ver}}(go) = g\Lambda_{w_0}^{\mathrm{ver}}(o)$ for every $g \in G$.
- Define $\vartheta_3(o) := \pi_{\overline{Y}_{v_0}}(o)$ and extend the definition by equivariance so that $\vartheta_3(go) := g\vartheta_3(o)$ for any $g \in G$. We thus obtain a G-equivariant map $\vartheta_3\colon Go \to \dot{\mathscr{X}}_1$.
- Choose $\vartheta_4(o)$ to be the cone point of the hyperbolic cone attached to the boundary line $\ell_{[v_0,w_0]}$ of $\overline{Y}_{w_0}$. We then extend $\vartheta_4(go) = g\vartheta_4(o)$ for any $g \in G$ so that $g\vartheta_4(o)$ is the corresponding cone point to $\ell_{[gv_0,gw_0]}$ of $\overline{Y}_{gw_0}$. We thus obtain a G-equivariant map $\vartheta_4\colon Go \to \dot{\mathscr{X}}_2$.

We then define

(♣) $$\Phi\colon Go \to \mathscr{C}_K(\mathbb{F}_1)\times\mathscr{C}_K(\mathbb{F}_2)\times(\dot{\mathscr{X}}_1, d^K_{\dot{\mathscr{X}}_1})\times(\dot{\mathscr{X}}_2, d^K_{\dot{\mathscr{X}}_2})\times T$$

by

$$\Phi := \vartheta_1\times\vartheta_2\times\vartheta_3\times\vartheta_4\times\rho$$

where $\dot{\mathscr{X}}_i$ for $i=1,2$ are equipped with the K-thick distance $d^K_{\dot{\mathscr{X}}_i}$ (not genuine distance) defined in (9), and the other three spaces are equipped with length metric. By abuse of language, we call the sum of the distances over the factors the L^1-metric on the product space.

The remainder of this section is to verify the following.

Proposition 6.1 *The map Φ in (♣) is a G-equivariant quasi-isometric embedding.*

Idea of the proof of Proposition 6.1 Since the orbital map of any isometric action is Lipschitz (see eg [11, Lemma I.8.18]), we will only need to give a linear upper bound on $|x-y|_X$. Recall from (2) in Section 2.3.2, for any $x, y \in X$,

$$|x-y|_X \sim |x-y|_X^{\mathrm{hor}} + |x-y|_X^{\mathrm{ver}}$$

where $|x-y|_X^{\mathrm{hor}} = \sum_{i=0}^{2n} |p_i - p_{i+1}|^{\mathrm{hor}}_{Y_{v_i}}$ and $|x-y|_X^{\mathrm{ver}} = \sum_{i=0}^{2n} |p_i - p_{i+1}|^{\mathrm{ver}}_{Y_{v_i}}$.

Recall from Section 5.2, we build a new geometric model X_v of Y_v for each vertex v in the Bass–Serre tree T. Namely, we have a G_v-equivariant quasi-isometric map $\Lambda_v\colon Y_v = \overline{Y}_v\times\mathbb{R}\to X_v = \overline{Y}_v\times\mathfrak{fl}(v)$. For $x, y\in Go$, we shall accordingly replace $|x-y|_X^{\mathrm{ver}}$ by the quantity

(12) $$V(x,y) := \sum_{0\le i\le 2n} |\Lambda^{\mathrm{ver}}_{v_i}(p_i) - \Lambda^{\mathrm{ver}}_{v_i}(p_{i+1})|_{\mathfrak{fl}(v_i)}.$$

To be precise, we first prove in Lemma 6.2 that

$$|x-y|_X \le \epsilon\big(|\rho(x)-\rho(y)|_T + |x-y|_X^{\mathrm{hor}} + V(x,y)\big),$$

and then we find suitable upper bounds of $V(x,y)$ (see Proposition 6.4) and $|x-y|_X^{\mathrm{hor}}$ (see Lemma 6.6).

6.2 Verifying Φ is a quasi-isometric embedding

In this section, we will verify that the map Φ in (♣) is a quasi-isometric embedding.

6.2.1 Upper bound of the distance $|x-y|_X$ on X

Lemma 6.2 *Let $x, y\in Go$. The exists a constant $\epsilon > 0$ such that*

(13) $$|x-y|_X \le \epsilon\big(|\rho(x)-\rho(y)|_T + |x-y|_X^{\mathrm{hor}} + V(x,y)\big).$$

Proof Recall that $p_0 = x$ and $p_{2n+1} = y$. Using the triangle inequality we have

$$|x-y|_X \le \sum_{i=0}^{2n} |p_i - p_{i+1}|_{Y_{v_i}}.$$

Note that $2n = |\rho(x)-\rho(y)|_T$ and $|x-y|_X^{\mathrm{hor}} = \sum_{i=0}^{2n} |p_i - p_{i+1}|_{Y_{v_i}}^{\mathrm{hor}}$. The proof is then complete by summing over $0 \le i \le 2n$ the following inequality (14).

Claim *There exists a uniform constant $\epsilon' > 0$ such that for any $i \in \{0, 1, \dots, 2n\}$,*

$$|p_i - p_{i+1}|_{Y_{v_i}} \le \epsilon' + \epsilon' |p_i - p_{i+1}|_{Y_{v_i}}^{\mathrm{hor}} + \epsilon' |\Lambda_{v_i}^{\mathrm{ver}}(p_i) - \Lambda_{v_i}^{\mathrm{ver}}(p_{i+1})|_{\mathfrak{fl}(v_i)}. \tag{14}$$

Proof of the claim Indeed, since $\Lambda_{v_i} : Y_{v_i} \to X_{v_i}$ is a quasi-isometry by Lemma 5.2, we then have

$$|p_i - p_{i+1}|_{Y_{v_i}} \sim_\lambda |\Lambda_{v_i}(p_i) - \Lambda_{v_i}(p_{i+1})|_{X_{v_i}}.$$

Using part (2) of Lemma 5.2 we have that

$$|\Lambda_{v_i}^{\mathrm{hor}}(p_i) - \Lambda_{v_i}^{\mathrm{hor}}(p_{i+1})|_{\overline{Y}_{v_i}} \sim_\lambda |p_i - p_{i+1}|_{Y_{v_i}}^{\mathrm{hor}}.$$

It implies that

$$\begin{aligned} |\Lambda_{v_i}(p_i)) - \Lambda_{v_i}(p_{i+1})|_{X_{v_i}} &\sim_{\sqrt{2}} |\Lambda_{v_i}^{\mathrm{hor}}(p_i) - \Lambda_{v_i}^{\mathrm{hor}}(p_{i+1})|_{\overline{Y}_{v_i}} + |\Lambda_{v_i}^{\mathrm{ver}}(p_i) - \Lambda_{v_i}^{\mathrm{ver}}(p_{i+1})|_{\mathfrak{fl}(v_i)} \\ &\sim_\lambda |p_i - p_{i+1}|_{Y_{v_i}}^{\mathrm{hor}} + |\Lambda_{v_i}^{\mathrm{ver}}(p_i) - \Lambda_{v_i}^{\mathrm{ver}}(p_{i+1})|_{\mathfrak{fl}(v_i)} \end{aligned}$$

where the first coarse equality holds by definition of $\Lambda_{v_i}^{\mathrm{hor}}$ and $\Lambda_{v_i}^{\mathrm{ver}}$. Hence there exists a uniform constant $\epsilon' > 0$ such that the inequality (14) holds. □

The lemma is proved. □

6.2.2 Preparation for upper bounds of $V(x, y)$ and $|x-y|_X^{\mathrm{hor}}$

Fix $K \ge 4\xi$ where the constant $\xi > \lambda$ is given by Lemma 5.6. Let $\mathscr{C}_K(\mathbb{F}_1)$ and $\mathscr{C}_K(\mathbb{F}_2)$ be the vertical quasitrees given by Definition 5.8. With $i \in \{1, 2\}$, Proposition 2.27 gives the distance formula

$$|\vartheta_i(x) - \vartheta_i(y)|_{\mathscr{C}_K(\mathbb{F}_i)} \sim_K \sum_{\mathfrak{fl}(w) \in \mathbb{F}_i} [d_{\mathfrak{fl}(w)}(\vartheta_i(x), \vartheta_i(y))]_K. \tag{$\circledast$}$$

To give an appropriate upper bound of $V(x, y)$, we need the following two technical lemmas (Lemmas 6.3 and 6.5).

Lemma 6.3 *For any $v_{2i} \in [v_0, v_{2n}]$ with $0 \le i \le n$, we have*

$$d_{\mathfrak{fl}(v_{2i})}(\vartheta_1(x), \vartheta_1(y)) \sim_\lambda |\Lambda_{v_{2i}}^{\mathrm{ver}}(p_{2i}) - \Lambda_{v_{2i}}^{\mathrm{ver}}(p_{2i+1})|_{\mathfrak{fl}(v_{2i})}. \tag{15}$$

For any $0 \le i \le n-1$, we have

$$d_{\mathfrak{fl}(v_{2i+1})}(\vartheta_2(x), \vartheta_2(y)) \sim_\lambda |\Lambda_{v_{2i+1}}^{\mathrm{ver}}(p_{2i+1}) - \Lambda_{v_{2i+1}}^{\mathrm{ver}}(p_{2i+2})|_{\mathfrak{fl}(v_{2i+1})}. \tag{16}$$

Proof We first prove (15) for the case $0 < i < n$. The cases $i = 0$ or $i = n$ are similar.

Note that $\ell_1 := \mathscr{S}_{e_{2i-1}e_{2i}} \cap F_{e_{2i}}$ is a fiber line of $Y_{v_{2i-1}}$ containing p_{2i}, and similarly,

$$\ell_2 := \mathscr{S}_{e_{2i+1}e_{2i+2}} \cap F_{e_{2i+1}}$$

contains p_{2i+1}. By Definition 5.3 of projection maps, we have

$$\Pi_{\mathfrak{fl}(v_{2i})}(\mathfrak{fl}(v_0)) = \Lambda^{\mathrm{ver}}_{v_{2i}}(\ell_1) \quad \text{and} \quad \Pi_{\mathfrak{fl}(v_{2i})}(\mathfrak{fl}(v_{2n})) = \Lambda^{\mathrm{ver}}_{v_{2i}}(\ell_2).$$

Let $\lambda > 0$ be the constant given by Lemma 5.2, so the fiber lines ℓ_1 and ℓ_2 are sent by $\Lambda^{\mathrm{ver}}_{v_{2i}}$ into $\mathfrak{fl}(v_{2i})$ as subsets of diameter at most λ:

$$\operatorname{diam} \Lambda^{\mathrm{ver}}_{v_{2i}}(\ell_1), \operatorname{diam} \Lambda^{\mathrm{ver}}_{v_{2i}}(\ell_2) \le \lambda.$$

By definition of ϑ_1, we have $\vartheta_1(x) = \Lambda^{\mathrm{ver}}_{v_0}(x) \in \mathfrak{fl}(v_0)$ and $\vartheta_1(y) = \Lambda^{\mathrm{ver}}_{v_{2n}}(y) \in \mathfrak{fl}(v_{2n})$. Thus,

$$d_{\mathfrak{fl}(v_{2i})}(\vartheta_1(x), \vartheta_1(y)) \sim_\lambda d_{\mathfrak{fl}(v_{2i})}(\mathfrak{fl}(v_0), \mathfrak{fl}(v_{2n})).$$

As $p_{2i} \in \ell_1$ and $p_{2i+1} \in \ell_2$, we obtain

$$d_{\mathfrak{fl}(v_{2i})}(\mathfrak{fl}(v_0), \mathfrak{fl}(v_{2n})) \sim_\lambda |\Lambda^{\mathrm{ver}}_{v_{2i}}(p_{2i}) - \Lambda^{\mathrm{ver}}_{v_{2i}}(p_{2i+1})|_{\mathfrak{fl}(v_{2i})},$$

completing the proof of (15).

We are now going to prove (16). If $w_0 \ne v_1$ or $1 \le i \le n-1$, the same proof for (15) proves (16). We now consider $w_0 = v_1$ and $i = 0$. In this case, we note that $e_0 = \bar{e}_1$. By definition, we have that $\vartheta_2(x) = \vartheta_2(o) = \Lambda^{\mathrm{ver}}_{w_0}(o) \in \mathfrak{fl}(w_0)$, so we obtain $\Pi_{\mathfrak{fl}(v_1)}(\vartheta_2(x)) = \vartheta_2(x)$. Recall that $\mathcal{S}_{xe_1}$ is the strip in Y_{v_0} over the shortest arc from x to F_{e_1} (see construction of special path). As $x \in F_{e_0} = F_{e_1}$, we have $\ell_1 := \mathcal{S}_{xe_1}$ is a fiber line of Y_{v_0} that passes through x and also p_1. Thus, $\vartheta_2(x) \in \Pi_{\mathfrak{fl}(v_1)}(\ell_1)$.

Recall that $\mathcal{S}_{xe_1}$ is the strip in Y_{v_0} over the shortest arc from x to F_{e_1} (see construction of special path). As $x \in F_{e_0} = F_{e_1}$, we have that $\ell_1 := \mathcal{S}_{xe_1}$ is a fiber line of Y_{v_0} that passes through x and also p_1. Thus, $\vartheta_2(x) \in \Pi_{\mathfrak{fl}(v_1)}(\ell_1)$. Let $\ell_2 = \mathcal{S}_{e_2e_3} \cap F_{e_2}$ be the fiber line on Y_{v_2} that passes through p_2. If $w_1 = v_1$, then $\alpha = [v_0, v_1][v_1, v_2]$ and $y \in F_{e_2}$. By the same reason, ℓ_2 passes through y, so $\vartheta_2(y) \in \Pi_{\mathfrak{fl}(v_2)}(\ell_2)$. If $w_1 \ne v_1$, the projection $\Pi_{\mathcal{L}_{v_1}}(\vartheta_2(y))$ must be contained in $\Pi_{\mathfrak{fl}(v_1)}(\ell_2)$. In both cases, we have

$$d_{\mathfrak{fl}(v_1)}(\vartheta_2(x), \vartheta_2(y)) \sim_\lambda \operatorname{diam}(\Lambda^{\mathrm{ver}}_{v_1}(\ell_1) \cup \Lambda^{\mathrm{ver}}_{v_1}(\ell_2))$$

where we use $\operatorname{diam}(\Lambda^{\mathrm{ver}}_{v_1}(\ell_1)), \operatorname{diam}(\Lambda^{\mathrm{ver}}_{v_1}(\ell_2)) \le \lambda$ by Lemma 5.2. For $p_1 \in \ell_1$ and $p_2 \in \ell_2$, we obtain

$$\operatorname{diam}(\Lambda^{\mathrm{ver}}_{v_1}(\ell_1) \cup \Lambda^{\mathrm{ver}}_{v_1}(\ell_2)) \sim_\lambda |\Lambda^{\mathrm{ver}}_{v_1}(p_1) - \Lambda^{\mathrm{ver}}_{v_1}(p_2)|_{\mathfrak{fl}(v_1)},$$

concluding the proof of (16). □

Let us recall the notation from Section 2.4. Let $x \in \mathfrak{fl}(v), z \in \mathfrak{fl}(u) \in \mathbb{F}_i$.

If $\mathfrak{fl}(v) \ne \mathfrak{fl}(u) \ne \mathfrak{fl}(w)$ then

$$d_{\mathfrak{fl}(w)}(x, z) := d_{\mathfrak{fl}(w)}(\mathfrak{fl}(v), \mathfrak{fl}(u)).$$

If $\mathfrak{fl}(w) = \mathfrak{fl}(v)$ and $\mathfrak{fl}(w) \ne \mathfrak{fl}(u)$, define $d_{\mathfrak{fl}(w)}(x, z) := \operatorname{diam}(\pi_{\mathfrak{fl}(w)}(x, \mathfrak{fl}(u)))$.

If $\mathfrak{fl}(v) = \mathfrak{fl}(u) = \mathfrak{fl}(w)$, let $d_{\mathfrak{fl}(w)}(x, z)$ be the distance in $\mathfrak{fl}(w)$.

6.2.3 Upper bound of $V(x,y)$ Let ϑ_1 and ϑ_2 be the maps defined in Section 6.1. We now have prepared all ingredients for the proof of the following result.

Proposition 6.4 *Let $x, y \in Go$ and $\alpha := [\rho(x), \rho(y)]$ be the geodesic in T. Then*

$$V(x,y) \preceq_K \sum_{j=1,2} \Bigg(\sum_{v \in \alpha \cap \mathscr{V}_j} [d_{\mathfrak{fl}(v)}(\vartheta_j(x), \vartheta_j(y))]_K \Bigg) + d_T(\rho(x), \rho(y)). \tag{17}$$

Proof The goal is to recover the sum on the right side of (12), that is

$$V(x,y) = \sum_{0 \le i \le 2n} |\Lambda^{\mathrm{ver}}_{v_i}(p_i) - \Lambda^{\mathrm{ver}}_{v_i}(p_{i+1})|_{\mathfrak{fl}(v_i)}$$

via the maps ϑ_1 and ϑ_2. By Lemma 6.3, we have the desired inequalities (15) for even indices $v_{2i} \in [v_0, v_{2n}] \cap \mathscr{V}_1$ with $0 \le i \le n$, that is

$$d_{\mathfrak{fl}(v_{2i})}(\vartheta_1(x), \vartheta_1(y)) \sim_\lambda |\Lambda^{\mathrm{ver}}_{v_{2i}}(p_{2i}) - \Lambda^{\mathrm{ver}}_{v_{2i}}(p_{2i+1})|_{\mathfrak{fl}(v_{2i})}.$$

By Lemma 6.3, the inequalities (16) recover the odd indices $v_{2i+1} \in [v_0, v_{2n}] \cap \mathscr{V}_2$ with $0 \le i \le n-1$ in (12), that is,

$$d_{\mathfrak{fl}(v_{2i+1})}(\vartheta_2(x), \vartheta_2(y)) \sim_\lambda |\Lambda^{\mathrm{ver}}_{v_{2i+1}}(p_{2i+1}) - \Lambda^{\mathrm{ver}}_{v_{2i+1}}(p_{2i+2})|_{\mathfrak{fl}(v_{2i+1})}.$$

Plugging inequalities (15) and (16) into (12), and using the term $|\rho(x) - \rho(y)|_T$ to count the additive errors in this process completes the proof of the desired inequality (17). Applying then the K-cutoff function $[\cdot]_K$ does not affect the inequalities. □

6.2.4 Upper bound of $|x-y|^{\mathrm{hor}}_X$ The horizontal distance d^h defined in (2) of the special path γ from x to y records the totality of the projected distances to the base hyperbolic spaces $\overline{Y}_v$:

$$\begin{aligned} |x-y|^{\mathrm{hor}}_X &= |x-p_1|^{\mathrm{hor}}_{Y_{v_1}} + |p_1 - p_2|^{\mathrm{hor}}_{Y_{v_2}} + \cdots + |p_{2n} - y|^{\mathrm{hor}}_{Y_{v_{2n}}} \\ &= |\vartheta_3(x) - F_{e_1}|_{Y_{v_0}} + \sum_{i=1}^{2n-1} |F_{e_i} - F_{e_{i+1}}|_{Y_{v_i}} + |F_{e_{2n}} - \vartheta_3(y)|_{Y_{v_{2n}}} \end{aligned}$$

where the map ϑ_3 defined in Section 6.1 sends a point in $Y_v = \overline{Y}_v \times \mathbb{R}$ to the hyperbolic base $\overline{Y}_v$.

Before moving on, let us introduce more notation to represent the horizontal distance. Let $x_0 = \vartheta_3(x)$, $y_0 \in F_{e_1}$ and $x_{2n} \in F_{e_{2n}}$, $y_{2n} = \vartheta_3(y)$ be such that $[x_0, y_0]$ is orthogonal to F_{e_1}, and $[x_{2n}, y_{2n}]$ to $F_{e_{2n}}$. Choose $x_i \in F_{e_i}$ and $y_i \in F_{e_{i+1}}$ so that $[x_i, y_i]$ is a geodesic in $\overline{Y}_{v_i}$ orthogonal to F_{e_i} and $F_{e_{i+1}}$. Thus,

$$|x-y|^{\mathrm{hor}}_X = \sum_{i=0}^{2n} |x_i - y_i|_{\overline{Y}_{v_i}}. \tag{18}$$

Recall that $\dot{\mathscr{X}}_1$ and $\dot{\mathscr{X}}_2$ are the coned-off spaces defined in Section 4.1. By Definition 4.3 of the K-thick distance of $\dot{\mathscr{X}}_j$ for any $K > 0$, and the remark after it, we have

$$\sum_{i=0}^{2n} |x_i - y_i|^K_{\dot{Y}_{v_i}} = |\vartheta_3(x) - \vartheta_3(y)|^K_{\dot{\mathscr{X}}_1} + |\vartheta_4(x) - \vartheta_4(y)|^K_{\dot{\mathscr{X}}_2} \tag{19}$$

where $|\cdot|^K_{\dot{Y}_{v_i}}$ defined in (3) is the K-thick distance on the coned-off space $\dot{Y}_{v_i}$. The map ϑ_4 defined in Section 6.1 sends a point go in Go to the hyperbolic cone point to the boundary line $\ell_{g[v_0,w_0]}$ (recall that o is chosen on a common boundary plane $F_{[v_0,w_0]}$).

Hence, the K-thick distance (19) differs from the horizontal distance (18) by the amount coned-off on boundary lines. The purpose of this subsection is to recover the loss in the coned-off from the projection system of fiber lines.

To prove Lemma 6.6, we need the following lemma.

Lemma 6.5 *Let ϑ_2 be the map given by Section 6.1. Let v be a vertex in $\mathrm{Lk}(v_{2i})-\tilde{\alpha}$ and let $e=[v,v_{2i}]$. Let ℓ_e, $\ell_{e_{2i}}$, and $\ell_{\bar{e}_{2i+1}}$ be the boundary lines of $\widetilde{F}_{v_{2i}}$ associated to distinct edges e, e_{2i} and $\bar{e}_{2i+1}$ respectively. Then we have*

$$d_{\mathfrak{fl}(v)}(\vartheta_2(x),\vartheta_2(y)) \sim_\lambda d_{\ell_e}(\ell_{e_{2i}},\ell_{\bar{e}_{2i+1}}). \tag{20}$$

Proof Note that ℓ_e, $\ell_{e_{2i}}$ and $\ell_{\bar{e}_{2i+1}}$ are the projection of planes F_e, $F_{e_{2i}}$ and $F_{e_{2i+1}}$ of $Y_{v_{2i}}$ into the factor $Y_{v_{2i}}$. We prove (20) case by case, according to the configuration of e_0, e_{2n+1} with α.

Case 1 Suppose $0<i<n$. By Definition 5.3, the projection of $\vartheta_2(x)=\Lambda^{\mathrm{ver}}_{w_0}(o)\in\mathfrak{fl}(w_0)$ to $\mathfrak{fl}(v)$ is the same as that of $\mathfrak{fl}(v_1)$ to $\mathfrak{fl}(v)$, and the projection of $\vartheta_2(y)\in\mathfrak{fl}(w_1)$ to $\mathfrak{fl}(v)$ is the same as that of $\mathfrak{fl}(v_{2n-1})$ to $\mathfrak{fl}(v)$. That is to say, $d_{\mathfrak{fl}(v)}(\vartheta_2(x),\vartheta_2(y))=d_{\mathfrak{fl}(v)}(\mathfrak{fl}(v_1),\mathfrak{fl}(v_{2n-1}))$. Hence, (20) follows by Lemma 5.5: $d_{\mathfrak{fl}(v)}(\mathfrak{fl}(v_1),\mathfrak{fl}(v_{2n-1}))\sim_\lambda d_{\ell_e}(\ell_{e_{2i}},\ell_{\bar{e}_{2i+1}})$ for any $v\in\mathrm{Lk}(v_{2i})-\tilde{\alpha}$.

Case 2 Suppose $i=0$ or $i=n$. We only consider the case $i=0$ and analyze the configuration of w_0 with α. The analyze for the case for $i=n$ and w_1 is symmetric.

Case 2.1 Suppose $w_0\neq v_1$. In this case $e_0\cdot\alpha$ is a geodesic from w_0 to v_{2n}. By Definition 5.3 of projection maps, no matter whether $\bar{e}_{2n+1}=e_{2n}$ holds or not, the projection of $\vartheta_2(x)\in\mathfrak{fl}(w_0)$ to $\mathfrak{fl}(v)$ is the same as that of $\mathfrak{fl}(w_0)$ to $\mathfrak{fl}(v)$, and the projection of $\vartheta_2(y)\in\mathfrak{fl}(w_1)$ to $\mathfrak{fl}(v)$ is the same as that of $\mathfrak{fl}(v_{2n-1})$ to $\mathfrak{fl}(v)$. By Lemma 5.5, we have $d_{\mathfrak{fl}(v)}(\mathfrak{fl}(w_0),\mathfrak{fl}(v_{2n-1}))\sim_\lambda d_{\ell_e}(\ell_{e_{2i}},\ell_{\bar{e}_{2i+1}})$.

Case 2.2 Suppose $w_0=v_1$. No matter whether $w_0=w_1$ or not, we have

$$d_{\mathfrak{fl}(v)}(\vartheta_2(x),\vartheta_2(y))\le\Pi_{\mathfrak{fl}(v)}(\mathfrak{fl}(w_0))\le\xi$$

where ξ is the projection constant given by Lemma 5.6. On the right side of (20), $d_{\ell_e}(\ell_{e_{2i}},\ell_{\bar{e}_{2i+1}})$ is bounded above by ξ for $i=0$ (as $e_0=\bar{e}_1$). Thus (20) holds as well in this case. □

Lemma 6.6 *For any $x,y\in Go$, we have*

$$|x-y|^{\mathrm{hor}}_X \preceq_K \sum_{i=0}^{2n}|x_i-y_i|^K_{\dot{Y}_{v_i}}+\sum_{i=0}^{2n}\sum_{w\in\mathrm{Lk}(v_i)-\alpha}[d_{\mathfrak{fl}(w)}(\vartheta_j(x),\vartheta_j(y))]_K$$

where the index $j=1$ is chosen if i is odd, otherwise $j=2$.

Proof We consider (18) for the horizontal distance $|x-y|_X^{\mathrm{hor}}$. Let $\mathbb{L}_{v_i}$ be the set of boundary lines of $\overline{Y}_{v_i}$ corresponding to the set of oriented edges $e \in \mathrm{St}(v_i)$ (ie the collection $\{F_e \cap \overline{Y}_{v_i} \mid e \in \mathrm{St}(v_i)\}$). By Lemma 3.1, for each $0 \le i \le 2n$, we have

$$|x_i - y_i|_{\overline{Y}_{v_i}} \sim_K |x_i - y_i|^K_{\dot{Y}_{v_i}} + \sum_{\ell_e \in \mathbb{L}_{v_i}} [d_{\ell_e}(x_i, y_i)]_K \tag{21}$$

for any sufficiently large $K \gg 0$.

Let $e = [w, v_i] \in \mathrm{St}(v_i)$ and $\ell_e \in \mathbb{L}_{v_i}$ be the corresponding boundary line of $\overline{Y}_{v_i}$. Set $j = 1$ if i is odd, otherwise $j = 2$.

If $e = e_i$ or $e = \bar{e}_{i+1}$ for $1 \le i \le 2n-1$, then

$$d_{\ell_e}(x_i, y_i) \le \xi$$

since $[x_i, y_i]$ is orthogonal to ℓ_e.

We remark that when $i = 0$ (the case $i = 2n$ is similar), it is possible that $[x_0, y_0]$ may not be perpendicular to ℓ_e. However, we have

$$d_{\ell_e}(x_0, y_0) \preceq d_{\mathfrak{fl}(w_0)}(\vartheta_2(x), \vartheta_2(y)).$$

Otherwise, if $e \ne e_i$ and $e \ne \bar{e}_{i+1}$ for $1 \le i \le 2n-1$, we have $e \notin \alpha$ for which the following holds by Lemma 6.5 for $j = 2$ and by Lemma 5.5 for $j = 1$:

$$d_{\mathfrak{fl}(w)}(\vartheta_j(x), \vartheta_j(y)) \sim d_{\ell_e}(x_i, y_i).$$

Note that $A \le \lambda B + C$ with $B \ge K \ge C$ implies $[A]_K \preceq_K [B]_K$. Thus, for each $0 \le i \le 2n$, we deduce from (21) that

$$|x_i - y_i|_{\overline{Y}_{v_i}} \sim_K |x_i - y_i|^K_{\dot{Y}_{v_i}} + \sum_{w \in \mathrm{Lk}(v_i) - \alpha} [d_{\mathfrak{fl}(w)}(\vartheta_j(x), \vartheta_j(y))]_K \tag{22}$$

for any $K \gg 0$, where $j = 1$ if i is odd, and $j = 2$ otherwise. We sum up (22) over $v_i \in \alpha$ to get the horizontal distance $d^h(x, y)$ in (18):

$$|x-y|_X^{\mathrm{hor}} = \sum_{i=0}^{2n} |x_i - y_i|_{\overline{Y}_{v_i}} \preceq_K \sum_{i=0}^{2n} |x_i - y_i|^K_{\dot{Y}_{v_i}} + \sum_{i=0}^{2n} \sum_{w \in \mathrm{Lk}(v_i) - \alpha} [d_{\mathfrak{fl}(w)}(\vartheta_j(x), \vartheta_j(y))]_K. \qquad \square$$

We now have prepared all ingredients in the proof of Proposition 6.1.

Proof of Proposition 6.1 Since ρ and ϑ_i (with $i \in \{1, 2, 3, 4\}$) are G-equivariant maps, it follows that Φ is a G-equivariant map. Since the orbital map of any isometric action is Lipschitz (see eg [11, Lemma I.8.18]), it suffices to give an upper bound on $d(x, y)$.

Let $\epsilon > 0$ be the constant given by Lemma 6.2, so that

$$|x - y|_X \le \epsilon\big(|\rho(x) - \rho(y)|_T + |x - y|_X^{\mathrm{hor}} + V(x, y)\big).$$

Appropriate upper bounds of the vertical distance $V(x,y)$ and the horizontal distance $|x-y|_X^{\mathrm{hor}}$ have been already treated in Proposition 6.4 and Lemma 6.6 respectively. They are

$$V(x,y) \preceq_K \sum_{j=1,2}\Bigg(\sum_{v\in\alpha\cap\mathcal{V}_j}[d_{\mathfrak{fl}(v)}(\vartheta_j(x),\vartheta_j(y))]_K\Bigg)+|\rho(x)-\rho(y)|_T$$

and

$$|x-y|_X^{\mathrm{hor}} \preceq_K \sum_{i=0}^{2n}|x_i-y_i|^K_{\dot{Y}_{v_i}}+\sum_{i=0}^{2n}\sum_{w\in\mathrm{Lk}(v_i)-\alpha}[d_{\mathfrak{fl}(w)}(\vartheta_j(x),\vartheta_j(y))]_K$$

where the index j depends on i: $j=1$ if i is odd, otherwise $j=2$. The above two inequalities yield

$$|x-y|_X^{\mathrm{hor}}+V(x,y) \preceq_K |\rho(x)-\rho(y)|_T+\sum_{i=0}^{2n}|x_i-y_i|^K_{\dot{Y}_{v_i}}+\sum_{i=0}^{2n}\sum_{w\in\mathrm{Lk}(v_i)}[d_{\mathfrak{fl}(w)}(\vartheta_j(x),\vartheta_j(y))]_K.$$

By (⊛), we have

$$\sum_{i=0}^{2n}\sum_{w\in\mathrm{Lk}(v_i)}[d_{\mathfrak{fl}(w)}(\vartheta_j(x),\vartheta_j(y))]_K \preceq_K |\vartheta_1(x)-\vartheta_1(x)|_{\mathscr{C}_K(\mathbb{F}_1)}+|\vartheta_2(x)-\vartheta_2(x)|_{\mathscr{C}_K(\mathbb{F}_2)}.$$

It follows that

$$|x-y|_X^{\mathrm{hor}}+V(x,y) \preceq_K |\rho(x)-\rho(y)|_T+\sum_{i=0}^{2n}|x_i-y_i|^K_{\dot{Y}_{v_i}}+\sum_{i=1}^{2}|\vartheta_i(x)-\vartheta_i(x)|_{\mathscr{C}_K(\mathbb{F}_i)}.$$

Plugging the thick distance formula (19) into the above inequality, we obtain

$$\begin{aligned}|x-y|_X^{\mathrm{hor}}+V(x,y) \preceq_K |\rho(x)-\rho(y)|_T+|\vartheta_3(x)-\vartheta_3(y)|^K_{\dot{\mathscr{X}}_1}+|\vartheta_4(x)-\vartheta_4(y)|^K_{\dot{\mathscr{X}}_2}\\ +|\vartheta_1(x)-\vartheta_1(x)|_{\mathscr{C}_K(\mathbb{F}_1)}+|\vartheta_2(x)-\vartheta_2(x)|_{\mathscr{C}_K(\mathbb{F}_2)}.\end{aligned}$$

As $|x-y|_X \le \epsilon(|\rho(x)-\rho(y)|+|x-y|_X^{\mathrm{hor}}+V(x,y))$, it is a consequence from the above inequality that the map $\Phi=\vartheta_1\times\vartheta_2\times\vartheta_3\times\vartheta_4\times\rho$ in (♣) is a G-equivariant quasi-isometric embedding from X to $\mathscr{C}_K(\mathbb{F}_1)\times\mathscr{C}_K(\mathbb{F}_2)\times\dot{\mathscr{X}}_1\times\dot{\mathscr{X}}_2\times T$. □

7 Proof of Theorem 1.3

Let $G \curvearrowright X$ be a CKA action such that for every vertex group the central extension (1) has omnipotent hyperbolic quotient group. Let $\dot{G}<G$ be the subgroup of the index at most 2 preserving $\mathcal{V}_1$ and $\mathcal{V}_2$ given by Lemma 2.23. Upon passing to further finite-index subgroups in Lemma 4.8, we obtain a finite-index subgroup G' of $\dot{G}$ such that the results in Sections 5 and 6 hold for G'. We caution the reader that at the beginning of Section 5 we assume that each vertex group of G is a direct product, this assumption may not hold for the original G, but holds in the finite-index subgroup G' of G.

As G' is a subgroup of $\dot{G}$, it follows from Proposition 4.7 that there exists a G'-equivariant quasi-isometric embedding

$$\eta\colon(\dot{\mathscr{X}}_1\times\dot{\mathscr{X}}_2\times T, d^K_{\dot{\mathscr{X}}_1}\times d^K_{\dot{\mathscr{X}}_2}\times d_T)\to T_1\times T_2\times\cdots\times T_n\times T.$$

Applying Proposition 6.1 to G', we have a G'-equivariant quasi-isometric embedding

$$\Phi\colon G'o \to \mathscr{C}_K(\mathbb{F}_1) \times \mathscr{C}_K(\mathbb{F}_2) \times (\dot{\mathscr{X}}_1, d^K_{\dot{\mathscr{X}}_1}) \times (\dot{\mathscr{X}}_2, d^K_{\dot{\mathscr{X}}_2}) \times T.$$

It implies that $(\mathrm{id}_{\mathscr{C}_K(\mathbb{F}_1)} \times \mathrm{id}_{\mathscr{C}_K(\mathbb{F}_2)} \times \eta) \circ \Phi$ is a G'-equivariant quasi-isometric embedding from $G' \cdot o$ to the finite product of quasitrees $\mathscr{C}_K(\mathbb{F}_1) \times \mathscr{C}_K(\mathbb{F}_1) \times T_1 \times T_2 \times \cdots \times T_n \times T$. Thus G' has property (QT), implying G has property (QT).

8 Applications: property (QT) of 3-manifold groups

In this section, we apply results obtained in previous sections to give a complete characterization of property (QT) of all finitely generated 3-manifold groups (Theorem 1.1). Note that property (QT) is a commensurability invariant. Hence, we can always assume that all 3-manifolds are compact and orientable (by taking Scott's compact core and double cover).

Let M be a compact, connected, orientable, irreducible 3-manifold with empty or tori boundary. M is called *geometric* if its interior admits geometric structures in the sense of Thurston; those are S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, Nil and Sol. If M is not geometric, then M is called a *nongeometric 3-manifold*. By geometric decomposition of 3-manifolds, there is a nonempty minimal union $\mathscr{T} \subset M$ of disjoint essential tori and Klein bottles, unique up to isotopy, such that each component of $M \setminus \mathscr{T}$ is either a Seifert fibered piece or a hyperbolic piece. M is called *graph manifold* if all the pieces of $M \setminus \mathscr{T}$ are Seifert fibered pieces, otherwise it is a *mixed manifold*.

We remark here that the geometric decomposition is slightly different from the torus decomposition, but they are closely related (if M has no decomposing Klein bottle, then these two decompositions agree with each other). Such a difference can be got rid of by passing to some finite cover of M. Since we are only interested in virtual properties of 3-manifolds in this paper, we can always assume that these two decompositions agree with each other (on some finite cover of M). For this reason, we will only use the term torus decomposition in the remainder of this section.

8.1 Property (QT) of geometric 3-manifolds

Proposition 8.1 *The fundamental group $\pi_1(M)$ of a geometric 3-manifold M has property (QT) if and only if M does not support Sol or Nil geometry.*

Proof We are going to prove the necessity. Assume that $\pi_1(M)$ has property (QT). By Lemma 2.5, $\pi_1(M)$ does not contain any distorted element, while the fundamental group of a 3-manifold with Nil geometry or Sol geometry contains quadratically/exponentially distorted elements (for example, see [41, Proposition 1.2]). Hence, M does not support Sol or Nil geometry.

Now, we are going to prove sufficiency. If M supports geometry $\mathbb{E}^3$, S^3 or $S^2 \times \mathbb{R}$, then $\pi_1(M)$ is virtually abelian so has property (QT). If the geometry of M is $\mathbb{H}^2 \times \mathbb{R}$ then M is virtually covered by $\Sigma \times S^1$ for some hyperbolic surface Σ. Note that $\pi_1(\Sigma)$ is a residually finite hyperbolic group so it has property (QT) by [6, Theorem 1.1]. Hence, $\pi_1(\Sigma) \times \mathbb{Z}$ has property (QT). Since $\pi_1(\Sigma) \times \mathbb{Z}$ is a finite-index subgroup of $\pi_1(M)$, it follows that $\pi_1(M)$ has property (QT) by Lemma 2.3. If M supports geometry $\mathbb{H}^3$, $\pi_1(M)$ is virtually compact special by deep theorems of Agol [3] and Wise [53]; thus $\pi_1(M)$ has property (QT) since it is undistorted in a right-angled Artin group. Note that if the boundary of M is empty, then $\pi_1(M)$ is a residually finite hyperbolic group. As a result, it can be inferred that $\pi_1(M)$ possesses property (QT) as an alternative argument, according to [6, Theorem 1.1].

Finally, we need to show that if M supports $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry then $\pi_1(M)$ has property (QT). To see this, by passing to a finite cover if necessary, we could assume that M is a nontrivial circle bundle over a closed surface Σ with $\chi(\Sigma) < 0$. Let $1 \to K \to \pi_1(M) \to \pi_1(\Sigma) \to 1$ be the short exact sequence associated with the circle bundle where K is the normal cyclic subgroup of $\pi_1(M)$ generated by a fiber. Let $\pi : \pi_1(M) \to \pi_1(\Sigma)$ be the surjective homomorphism in the above short exact sequence. Note that the short exact sequence does not split since M is supporting $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry. According to the first paragraph in the proof of [29, Corollary 4.3], there exists a generating set $\mathcal{S}$ of $G = \pi_1(M)$ such that $\mathcal{L} := \mathrm{Cay}(G, \mathcal{S})$ is a quasiline. Moreover, the diagonal action of G on $\pi_1(\Sigma) \times \mathcal{L}$ is metrically proper and cobounded, and thus its orbital map is a quasi-isometry. Since $\pi_1(\Sigma)$ is a residually finite hyperbolic group, it follows from [6] that $\pi_1(\Sigma)$ has property (QT). Hence there exists a finite product of quasitrees $\prod_{i=1}^n T_i$ such that $\pi_1(\Sigma) \curvearrowright \prod_{i=1}^n T_i$ such that its orbital map is a quasi-isometric embedding. It is easy to see that the orbital map of the diagonal action $G \curvearrowright \prod_{i=1}^n T_i \times \mathcal{L}$ of G on the product $\prod_{i=1}^n T_i \times \mathcal{L}$ is a quasi-isometric embedding. Therefore $\pi_1(M)$ has property (QT). □

8.2 Property (QT) of nongeometric 3-manifolds

In this section, we are going to prove Theorem 1.2. Recall that a nongeometric 3-manifold is either a graph manifold or a mixed manifold.

8.2.1 Property (QT) of graph manifolds Let M be a graph manifold. Since property (QT) is preserved under taking finite-index subgroups (see Lemma 2.3), we only need to show that a finite cover of M has property (QT). By passing to a finite cover, we can assume that each Seifert fibered piece in the JSJ decomposition of M is a trivial circle bundle over a hyperbolic surface of genus at least 2, and the intersection numbers of fibers of adjacent Seifert pieces have absolute value 1 (see [34, Lemma 2.1]). Also we can assume that the underlying graph of the graph manifold M is bipartite since any nonbipartite graph manifold is double covered by a bipartite one.

We note that $\pi_1(M)$ is an admissible group in the sense of Definition 2.12. However, it is not always true that $\pi_1(M)$ can act geometrically on a CAT(0) space, so property (QT) in this case does not follow immediately from Theorem 1.3. Indeed, if M is a graph manifold with nonempty boundary then it always admits a Riemannian metric of nonpositive curvature (see [35]). In particular, $\pi(M) \curvearrowright \widetilde{M}$ is a CKA action, and thus property (QT) of $\pi_1(M)$ follows from Theorem 1.3. However, many closed graph manifolds are shown to not support any Riemannian metric of nonpositive curvature (see [35]).

We remark here that the CAT(0) metric on the CKA space X in Sections 5 and 6 is not really essential in the proofs. Below we will make certain modifications on some steps to run the proof of Theorem 1.3 for closed graph manifolds.

Metrics on M

We now are going to describe a convenient metric on M introduced by Kapovich and Leeb [34]. For each Seifert component $M_v = F_v \times S^1$ of M, we choose a hyperbolic metric on the base surface F_v such that all boundary components are totally geodesic of unit length, and then equip each Seifert component $M_v = F_v \times S^1$ with the product metric d_v such that the fibers have length one. Metrics d_v on M_v induce the product metrics on $\widetilde{M}_v$ which by abuse of notations is also denoted by d_v.

Let M_v and M_w be adjacent Seifert components in the closed graph manifold M, and let $T \subset M_v \cap M_w$ be a JSJ torus. Each metric space $(\widetilde{T}, d_v)$ and $(\widetilde{T}, d_w)$ is a Euclidean plane. After applying a homotopy to the gluing map, we may assume that at each JSJ torus T, the gluing map ϕ from the boundary torus $\overleftarrow{T} \subset M_v$ to the boundary torus $\overrightarrow{T} \subset M_w$ is affine in the sense that the identity map $(\widetilde{T}, d_v) \to (\widetilde{T}, d_w)$ is affine. We now have a product metric on each Seifert component $M_v = F_v \times S^1$. These metrics may not agree with each other on the JSJ tori but the gluing maps are bilipschitz (since they are affine). The product metrics on the Seifert components induce a length metric on the graph manifold M denoted by d (see [12, Section 3.1] for details). Moreover, there exists a positive constant L such that on each Seifert component $M_v = F_v \times S^1$ we have

$$\frac{1}{L} d_v(x, y) \le d(x, y) \le L d_v(x, y)$$

for all x and y in M_v. (See [45, Lemma 1.8] for a detailed proof of the last claim.) Metric d on M induces metric on $\widetilde{M}$, which is also denoted by d (by abuse of notation). Then for all x and y in $\widetilde{M}_v$ we have

$$\frac{1}{L} d_v(x, y) \le d(x, y) \le L d_v(x, y).$$

Remark 8.2 Note that the space $(\widetilde{M}, d)$ may not be a CAT(0) space but $\pi_1(M)$ acts geometrically on $(\widetilde{M}, d)$ via deck transformations.

In Section 2.3.2, we define special paths on a CAT(0) space X. In this section, although $(\widetilde{M}, d)$ is no longer a CAT(0) space, we are still able to define special paths in $(\widetilde{M}, d)$. The construction is similar to Section 2.3.2 with slight changes.

Special paths on $\widetilde{M}$

Lift the JSJ decomposition of the graph manifold M to the universal cover $\widetilde{M}$, and let T be the tree dual to this decomposition of $\widetilde{M}$ (ie the Bass–Serre tree of $\pi_1(M)$). For every pair of adjacent edges e_1 and e_2 in T, let v be the common vertex of e_1 and e_2. Let ℓ and ℓ' be two boundary lines of $\widetilde{F}_v$ corresponding to the edges e_1 and e_2 respectively. Let $\gamma_{e_1e_2}$ be the shortest geodesic joining ℓ and ℓ' in $(\widetilde{M}_v, d_v)$. This geodesic determines an Euclidean strip $\mathcal{S}_{e_1e_2} := \gamma_{e_1e_2} \times \mathbb{R}$ in $(\widetilde{M}_v, d_v)$. Let x be a point in $(\widetilde{M}_v, d_v)$ and e be an edge with an endpoint v. The minimal geodesic from x to the plane F_e also define a strip $\mathcal{S}_{xe} := \gamma_{xe} \times \mathbb{R}$ in $(\widetilde{M}_v, d_v)$ where $\gamma_{xe} \subset \widetilde{F}_v$ is the projection to $\widetilde{F}_v$ of this minimal geodesic.

Now, let x and y be any two points in the universal cover $\widetilde{M}$ of M such that x and y belong to the interiors of pieces $\widetilde{M}_v$ and $\widetilde{M}'_v$ respectively. If $v = v'$ then we define a *special path* in X connecting x and y to be the geodesic $[x, y]$ in $(\widetilde{M}, d)$. Otherwise, let $e_1 \cdots e_n$ be the geodesic edge path connecting v and v'. For notational purpose, we write $e_0 := x$ and $e_{n+1} := y$. Let $p_i \in F_{e_i}$ be the intersection point of the strips $\mathcal{S}_{e_{i-1}e_i}$ and $\mathcal{S}_{e_ie_{i+1}}$. The *special path* connecting x and y is the concatenation of the geodesics

$$[x, p_1] \cdot [p_1, p_2] \cdots [p_n, y].$$

We label $p_0 := x$ and $p_{n+1} := y$.

Proposition 8.3 *If M is a graph manifold, then $\pi_1(M)$ has property (QT).*

Proof If M is a nonpositively curved graph manifold (for example, when M has nonempty boundary) then the fact that $\pi_1(M)$ has property (QT) is followed from Theorem 1.3. The only case that does not follow directly from Theorem 1.3 is when M is a closed graph manifold (recall many closed graph manifolds are nonpositively curved but many are not). Since the metric d on $\widetilde{M}$ restricted to each piece $\widetilde{M}_v$ is L-bilipschitz equivalent to d_v, so the inequalities in Section 6 are slightly changed by a uniform multiplicative constant. For example, the statement $a \asymp_K b$ (or $a \preceq_K b$) in Section 6 will be changed to $a \asymp_{K'} b$ (or $a \preceq'_K b$) for some constant K' depending on K. Thus, the proof, in this case, is performed along the same lines as the proof of Theorem 1.3. □

8.2.2 Property (QT) of mixed 3-manifolds Recall that a nongeometric 3-manifold with empty or tori boundary is either a graph manifold or a mixed 3-manifold. The case of graph manifold has been addressed in Section 8.2.1. In this section, we address the mixed 3-manifold case.

Proposition 8.4 *The fundamental group of a mixed 3-manifold has property (QT).*

The fundamental group of a mixed 3-manifold has a natural relatively hyperbolic structure as follows: Let $M_1, \ldots, M_k$ be the maximal graph manifold pieces, isolated Seifert fibered components of the JSJ-decomposition of M, and $S_1, \ldots, S_l$ be the tori in M not contained in any M_i. The fundamental group $G = \pi_1(M)$ is hyperbolic relative to the set of parabolic subgroups

$$\mathcal{P} = \{\pi_1(M_p) \mid 1 \leq p \leq k\} \cup \{\pi_1(S_q) \mid 1 \leq q \leq l\}$$

(see [8; 22]).

The following lemma provides many separable subgroups in $\pi_1(M)$, generalizing [50, Lemma 3.3]. The proof uses a recent result of the second author and Sun in [41] where the authors show that separability and distortion of subgroups in 3-manifold groups are closely related.

Lemma 8.5 *Let M be a compact, orientable, irreducible 3-manifold with empty or tori boundary, with nontrivial torus decomposition and that does not support the Sol geometry. If H is a finitely generated, undistorted subgroup of $\pi_1(M)$, then H is separable in $\pi_1(M)$.*

Proof Let M_H be the covering space of M corresponding to $H \le \pi_1(M)$. Generalizing a notion called "almost fiber part" in [36], an embedded (possibly disconnected) subsurface $\Phi(H)$ in M_H called an "almost fiber surface" is introduced in [49]. Sun [49, Theorem 1.3] shows that all information about the separability of H can be obtained by examining the almost fibered surface.

In [41], the authors introduce a notion called "modified almost fibered surface" (denoted by $\hat{\Phi}(H)$) that slightly modifies the original definition of almost fibered surface in [49] and show that information about the distortion of H in G can be also be obtained by examining the "modified almost fibered surface". We refer the reader to [49] for the definition of the almost fiber surface and to [41] for the definition of the modified almost fiber surface. The precise definitions are not needed here, so we only state here some facts from [41] that will be used later in the proof.

The torus decomposition of M induces the torus decomposition of $\Phi(H)$. Let $\Phi(H)$ and $\hat{\Phi}(H)$ be the almost fiber surface and modified almost fiber surface of H respectively.

(1) Both the almost fiber surface $\Phi(H)$ and the modified almost fiber surface $\hat{\Phi}(H)$ are (possibly disconnected) subsurfaces of M_H.
(2) The almost fiber surface $\Phi(H)$ has some piece that is homeomorphic to the annulus and parallel to the boundary of $\Phi(H)$. We delete these annulus pieces from $\Phi(H)$ to get the modified almost fiber surface, and we denote it by $\hat{\Phi}(H)$.

The surface $\Phi(H)$ (resp. $\hat{\Phi}(H)$) has a natural graph of spaces structure with the dual graph denoted by $\Gamma_{\Phi(H)}$ (resp. $\Gamma_{\hat{\Phi}(H)}$). By [41, Theorem 1.4], every component S of the modified almost fiber surface $\hat{\Phi}(H)$ must contain only one piece (otherwise, the distortion of H in $\pi_1(M)$ is at least quadratic, this contradicts the fact that H is undistorted in $\pi_1(M)$). This fact combined with (2) implies that the graph $\Gamma_{\Phi(H)}$ is a union of trees. By [49, Theorem 1.3] (or see also [50, Theorem 3.2] for a statement) tells us that whenever $\Gamma_{\Phi(H)}$ does not contain a simple cycle then H is separable. As shown above, we are in this case; hence we conclude that the subgroup H is separable in $\pi_1(M)$. □

Proof of Proposition 8.4 Let $M_1, \dots, M_k$ be the collection of maximal graph manifold components and Seifert fibered pieces in the geometric decomposition of M. Let $S_1, \dots, S_\ell$ be the tori in the boundary of M that bound a hyperbolic piece, and let $T_1, \dots, T_m$ be the tori in the JSJ decomposition of M that separate two hyperbolic components of the JSJ decomposition. Then $\pi_1(M)$ is hyperbolic relative to

$$\mathbb{P} = \{\pi_1(M_p)\}_{p=1}^{k} \cup \{\pi_1(S_q)\}_{q=1}^{\ell} \cup \{\pi_1(T_r)\}_{r=1}^{m}$$

(see [8; 22]).

We are going to show that $G = \pi_1(M)$ satisfies all conditions in Theorem 1.5.

Claim 1 *$\pi_1(M)$ induces the full profinite topology on each $P \in \mathcal{P}$.*

Indeed, it is well known that the fundamental groups of all compact 3-manifolds are residually finite; thus $\pi_1(M)$ is residually finite. Since each peripheral subgroup P is undistorted in $\pi_1(M)$, it follows from Lemma 8.5 that P is separable in $\pi_1(M)$. Again, by Lemma 8.5, each finite-index subgroup of P is also separable in $\pi_1(M)$. By [47, Lemma 2.8], $\pi_1(M)$ induces the full profinite topology on P.

Claim 2 *For each peripheral subgroup $P \in \mathbb{P}$, there exists a finite-index subgroup P' of P acting isometrically on a finite number of quasitrees so that the diagonal action of P' on the finite product of these quasitrees induces quasi-isometric embeddings on orbital maps.*

Indeed, if $P = \pi_1(T_r)$ or $P = \pi_1(S_q)$ for some r or q then $\pi_1(P) = \mathbb{Z}^2$; we let $P' := P$. If $P = \pi_1(M_j)$ for some Seifert component $M_j = F_j \times S^1$ then $P = \pi_1(F_j) \times \mathbb{Z}$. In this case, as F_j is a hyperbolic surface with nonempty boundary, $\pi_1(F_j)$ is a free group; hence we choose $P' = P$ as $\pi_1(F_j)$ is a quasitree. The last case we must consider is that $P = \pi_1(M_j)$ where M_j is a maximal graph manifold component. Passing to an appropriate finite cover $M_j' \to M_j$ we can assume that $\pi_1(M_j')$ acts on a finite number of quasitrees (but they are not quasilines) $T_1, T_2, \ldots, T_n$ so that the orbital map induced from the diagonal action $\pi_1(M_j) \curvearrowright \prod_{1=1}^n T_i$ is a quasi-isometric embedding (see Proposition 8.3). Claim 2 is confirmed. We then repeat the proof of Theorem 3.5 (the second and third paragraph) to show that P satisfies the hypothesis of Theorem 1.5.

In summary, we have verified the hypotheses in Theorem 1.5 for $G = \pi_1(M)$, so mixed 3-manifold groups have property (QT). □

Proof of Theorem 1.2 Let M be a compact orientable irreducible 3-manifold with empty or tori boundary, with nontrivial torus decomposition, and that does not support the Sol geometry. Such a 3-manifold M is either a graph manifold or a mixed manifold. The graph manifold case and mixed manifold case have been addressed in Propositions 8.3 and 8.4, respectively, and hence the theorem is proved. □

8.3 Property (QT) of finitely generated 3-manifolds

Proposition 8.6 *Let M be a compact, orientable, irreducible, ∂-irreducible 3-manifold such that it has a boundary component of genus at least 2. Then $\pi_1(M)$ has property (QT).*

Proof We consider the following two cases:

Case 1 M has trivial torus decomposition. In this case, M supports a geometrically finite hyperbolic structure with infinite volume. We paste hyperbolic 3-manifolds with totally geodesic boundaries to M to get a finite volume hyperbolic 3-manifold N. By the covering theorem (see [16]) and the subgroup tameness theorem (see [2; 15]), a finitely generated subgroup of the finite volume hyperbolic 3-manifold N is either a virtual fiber surface subgroup or undistorted. By the construction of N, the subgroup $\pi_1(M) \leq \pi_1(N)$ could not be a virtual fiber surface subgroup, and thus $\pi_1(M)$ must be undistorted in $\pi_1(N)$. Since $\pi_1(N)$ has property (QT), it follows that $\pi_1(M)$ has property (QT) (see Lemma 2.3).

Case 2 We now assume that M has nontrivial torus decomposition. By [49, Section 6.3], we paste hyperbolic 3-manifolds with totally geodesic boundaries to M to get a 3-manifold N with empty or tori boundary. The new manifold N satisfies the following properties.

(1) M is a submanifold of N with incompressible tori boundary.

(2) The torus decomposition of M also gives the torus decomposition of N.

(3) Each piece of M with a boundary component of genus at least 2 is contained in a hyperbolic piece of N.

In particular, it follows from (2) and (3) that N is a mixed 3-manifold. The subgroup $\pi_1(M)$ sits nicely in $\pi_1(N)$. By the proof of Case 1.2 in the proof of [41, Theorem 1.3], we have that $\pi_1(M)$ is undistorted in $\pi_1(N)$ (even more than that, $\pi_1(M)$ is strongly quasiconvex in $\pi_1(N)$ (see [42]). Note that $\pi_1(N)$ has property (QT) by Proposition 8.4. Since $\pi_1(M)$ is undistorted in $\pi_1(N)$ and $\pi_1(N)$ has property (QT), it follows that $\pi_1(M)$ has property (QT). □

We now give the proof of Theorem 1.1 which gives a complete characterization of property (QT) for finitely generated 3-manifolds groups.

Proof of Theorem 1.1 Since M is a compact, orientable 3-manifold, it decomposes into irreducible, ∂-irreducible pieces $M_1, \dots, M_k$ by the sphere-disc decomposition. In particular, $\pi_1(M)$ is the free product $\pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_k) * F_r$ for some free group F_r. We remark here that $\pi_1(M)$ is hyperbolic relative to the collection $\mathbb{P} = \{P_1, \dots, P_k, F_r\}$ where $P_i := \pi_1(M_i)$.

We are going to prove the necessity. Assume that $\pi_1(M)$ has property (QT). Since $\pi_1(M_i)$ is undistorted in $\pi_1(M)$, it follows that $\pi_1(M_i)$ has property (QT) (see Lemma 2.3). By Proposition 8.1, M_i does not support Sol and Nil geometry.

Now, we are going to prove sufficiency. Assume that there is no piece M_i that supports either Sol or Nil geometry. We would like to show that $\pi_1(M)$ has property (QT). In this case, we observe that each peripheral subgroup $P \in \mathbb{P}$ has property (QT). Indeed, a free group $P = F_r$ of course has property (QT), so let us now assume that $P = \pi_1(M_i)$ for some $1 \le i \le k$. If M_i has a boundary component of genus at least 2 then property (QT) of $\pi_1(M_i)$ follows from Proposition 8.6. Otherwise, M_i has empty or tori boundary. Then the property (QT) of $\pi_1(M_i)$ follows from Proposition 8.1 for geometric manifolds, Proposition 8.3 for graph manifolds, and Proposition 8.4 for mixed graph manifolds.

We are going to show that $G = \pi_1(M)$ satisfies all conditions in Theorem 1.5. The proof is similar to the proof of Proposition 8.4 with minor changes.

Claim 1 *$\pi_1(M)$ induces the full profinite topology on each $P_i \in \mathcal{P}$.*

It is well known that the fundamental groups of all compact 3-manifolds are residually finite, thus $\pi_1(M)$ is residually finite and its finite-index subgroups are residually finite as well. Any finite-index subgroup

H of $P_i = \pi_1(M_i)$ is separable in the free product $G = P_1 * P_2 * \cdots * P_k * F_r$ by [13, Theorem 1.1]. Hence it follows from [47, Lemma 2.8] that G induces the full profinite topology on P_i.

Claim 2 *For each peripheral subgroup $P \in \mathbb{P}$, there exists a finite-index subgroup P' of P acting isometrically on a finite number of quasitrees, so that the diagonal action of P' on the finite product of these quasitrees induces quasi-isometric embeddings on orbital maps.*

Indeed, the claim obviously holds for $P = F_r$ or $P = \mathbb{Z}^2$. The claim also holds for $P = \pi_1(M_i)$ where M_i is a geometric 3-manifold. The case of graph manifolds is proved in Claim 2 of the proof of Proposition 8.4. The only case left is when M_i is a mixed 3-manifold or M_i has a boundary component with genus at least 2. It has been shown in Proposition 8.6 that if M_i has a boundary component with genus at least 2 then it is an undistorted subgroup in a mixed 3-manifold. Therefore it suffices to consider only the mixed 3-manifold case. Recall that in the proof of Proposition 8.4, we show that there exists a finite-index subgroup of $\pi_1(M_i)$ such that it is a relatively hyperbolic group, satisfying the conditions of Theorem 1.5, and thus Claim 2 is confirmed.

With Claims 1 and 2, we use the same argument as in the proof of Theorem 3.5 (see the second and third paragraph) to find a finite-index normal subgroup G' of G such that G' is hyperbolic relative to a collection of subgroups satisfying the hypotheses in Theorem 1.5, and thus G' has property (QT). Therefore, $\pi_1(M)$ has property (QT) since G' is a finite-index subgroup of $\pi_1(M)$ and G' does have property (QT). □

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