AGT

Algebraic & Geometric Topology

Volume 25 (2025)

A cubulation with no factor system

SAM SHEPHERD

msp

Algebraic & Geometric Topology **25**:1 (2025) 255–266
DOI: 10.2140/agt.2025.25.255
Published: 5 March 2025

A cubulation with no factor system

SAM SHEPHERD

The primary method for showing that a given cubulated group is hierarchically hyperbolic is by constructing a factor system on the cube complex. We show that such a construction is not always possible, namely we construct a cubulated group for which the cube complex does not have a factor system. We also construct a cubulated group for which the induced action on the contact graph is not acylindrical.

20F65; 20F67

1 Introduction

A *cubulated group* $G \curvearrowright X$ is a group G together with a proper cocompact action of G on a CAT(0) cube complex X (and if G is fixed then each such action is called a *cubulation of* G). Numerous groups can be cubulated, including small cancellation groups, finite volume hyperbolic 3-manifold groups and many Coxeter groups — see Wise [15] for further background and examples. In turn, many cubulated groups are examples of hierarchically hyperbolic groups (HHGs), a class of groups that includes hyperbolic groups, relatively hyperbolic groups and mapping class groups among others — see Behrstock, Hagen and Sisto [3; 4] for relevant definitions and background. The primary method for showing that a given cubulated group $G \curvearrowright X$ is an HHG is by constructing a certain family of subcomplexes of X, called a factor system, which we define below following [3]. Many cubulated groups are known to have factor systems, including virtually special cubulated groups [3, Proposition B] — see also Hagen and Susse [9].

Definition 1.1 Let X be a CAT(0) cube complex. Each hyperplane H in X has an associated carrier $H \times [-1,1] \subset X$, and we call the convex subcomplexes $H \times \{\pm 1\}$ *combinatorial hyperplanes*. For a convex subcomplex $K \subset X$ we let $\mathfrak{g}_K : X \to K$ denote the closest point projection to K. A collection $\mathfrak{F}$ of subcomplexes of X is called a *factor system* if it satisfies the following:

(1) $X \in \mathfrak{F}$.
(2) Each $F \in \mathfrak{F}$ is a nonempty convex subcomplex of X.
(3) There exists $\Delta \geq 0$ such that for all $x \in X^{(0)}$, at most Δ elements of $\mathfrak{F}$ contain x.
(4) Every nontrivial convex subcomplex parallel to a combinatorial hyperplane of X is in $\mathfrak{F}$.
(5) There exists $\xi \geq 0$ such that for any pair of subcomplexes $F, F' \in \mathfrak{F}$, either $\mathfrak{g}_F(F') \in \mathfrak{F}$ or $\operatorname{diam}(\mathfrak{g}_F(F')) < \xi$.

Our first theorem is as follows, which answers a question of Behrstock, Hagen and Sisto [3, Question 8.13].

Theorem 1.2 *There is a cubulated group $G \curvearrowright X$ such that X does not have a factor system.*

Hagen and Susse [9, Theorem A] provided three separate sufficient conditions for a cubulated group $G \curvearrowright X$ to admit a factor system:

(1) the action is rotational,

(2) it satisfies the weak finite height condition for hyperplanes, and

(3) it satisfies the essential index condition together with the Noetherian intersection of conjugates condition on hyperplane stabilizers.

Theorem 1.2 gives the first known example of a cubulated group that fails all of these conditions (see Remark 2.4 for more on this). The example behind Theorem 1.2 also contains pairs of hyperplanes that are L-well-separated but not $(L-1)$-well-separated for arbitrarily large L (Remark 2.3), which provides a negative answer to a question of Genevois [7, Question 6.69 (first part)].

Associated to a CAT(0) cube complex X is the *contact graph* $\mathscr{C}X$: the vertices are the hyperplanes of X, and edges correspond to pairs of hyperplanes whose carriers intersect (equivalently, pairs of hyperplanes that are not separated by a third hyperplane). The contact graph is always a quasitree — see Hagen [8] — so in particular it is hyperbolic. Moreover, the contact graph is a key ingredient of the HHG structure that one usually builds for cubulated groups. More precisely, if a cubulated group $G \curvearrowright X$ has a G-invariant factor system $\mathfrak{F}$, then one can build an HHG structure for G by taking the contact graph $\mathscr{C}F$ of each $F \in \mathfrak{F}$ and coning off certain subgraphs of $\mathscr{C}F$ that correspond to smaller elements of $\mathfrak{F}$ — see [3] for details. The existence of a factor system for $G \curvearrowright X$ also implies that the induced action of G on $\mathscr{C}X$ is acylindrical [3, Theorem D]. (Recall that the action of a group G on a metric space (M, d) is *acylindrical* if for all $\epsilon > 0$ there exist $R, N > 0$ such that $d(x, y) \geq R$ implies that there are at most N elements $g \in G$ satisfying $d(x, gx), d(y, gy) < \epsilon$.) The following theorem is therefore a strengthening of Theorem 1.2.

Theorem 1.3 *There is a cubulated group $G \curvearrowright X$ for which the induced action on the contact graph $\mathscr{C}X$ is not acylindrical.*

This theorem is even more surprising in light of Genevois [6, Theorem 1.1], which implies that every cubulated group $G \curvearrowright X$ has a *nonuniformly acylindrical* action on $\mathscr{C}X$ (*nonuniformly* meaning that "at most N elements" is replaced by "finitely many elements" in the definition of acylindrical).

Briefly, the construction for Theorem 1.2 is to take a free cocompact action of a group Γ on a product of trees $T_1 \times T_2$ that contains an antitorus, and then Γ-equivariantly attach infinite strips to $T_1 \times T_2$ along antitori axes. The details are in Section 2. The construction for Theorem 1.3 builds on this by defining a certain HNN extension $\Lambda = \Gamma *_{\mathbb{Z}}$, and an action of Λ on a cube complex that splits as a tree of spaces, with vertex spaces being copies of $T_1 \times T_2$ and edge spaces corresponding to the infinite strips described above. The arguments for this are in Section 3. Although Γ admits a cubulation without a factor system, it is still an HHG because $\Gamma \curvearrowright T_1 \times T_2$ is another cubulation that does have a factor system. On the other hand, we do not know whether our second group Λ admits a cubulation with a factor system, and we do not know whether Λ is an HHG. In particular, the question of whether all cubulated groups are HHGs is still open [4, Question A]. One possible strategy is to find a cubulated group with no largest acylindrical action (see Definition 3.4), since all HHGs have a largest acylindrical action — see Abbott, Behrstock and Durham [2]. This does not work for the group Λ however, as we prove in Proposition 3.6 that Λ does have a largest acylindrical action.

Acknowledgements I am grateful for Mark Hagen's suggestion of considering antitori, which made my construction for Theorem 1.2 more general. Thanks go to Anthony Genevois for his comments, in particular regarding Remark 3.3. And I thank the referee for their comments and corrections. I am also thankful for the support of the Institut Henri Poincaré (UAR 839 CNRS-Sorbonne Université), and LabEx CARMIN (ANR-10-LABX-59-01).

2 Example with no factor system

Let T_1 and T_2 be locally finite trees, and let Γ be a group acting freely and cocompactly on $T_1 \times T_2$. Suppose that elements $g_1, g_2 \in \Gamma$ form an *antitorus*, meaning firstly that they translate nontrivially along intersecting axes $\ell_1 \times \{p_2\}, \{p_1\} \times \ell_2 \subset T_1 \times T_2$ respectively (so $p_1 \in \ell_1$ and $p_2 \in \ell_2$), and secondly that no nonzero powers of g_1 and g_2 commute. In addition, suppose that g_1 and g_2 are not proper powers in Γ. The condition that no powers of g_1 and g_2 commute is equivalent to saying that the flat $\Pi = \ell_1 \times \ell_2$ is not periodic. We also note that the existence of an antitorus implies that Γ is *irreducible* [10, Lemma 18], meaning it does not have a finite-index subgroup that splits as a product $\Gamma_1 \times \Gamma_2$ with Γ_i acting trivially on T_{3-i}. Examples of antitori were constructed by Wise [14], Janzen and Wise [10] and Rattaggi [12]. The smallest example is in [10], where $(T_1 \times T_2)/\Gamma$ consists of one vertex, four edges and four 2-cells. See [5] for more about antitori and irreducible lattices in products of trees.

Choose orientations for the edges in the finite quotient $(T_1 \times T_2)/\Gamma$, and label them with distinct letters from an alphabet A. Lift this labeling to $T_1 \times T_2$. Each finite edge path in $T_1 \times T_2$ or its quotient is thus labeled by some word w on $A^{\pm}$, and we denote the length of w by $|w|$. The axes $\ell_1 \times \{p_2\}$ and $\{p_1\} \times \ell_2$ descend to loops in $(T_1 \times T_2)/\Gamma$ based at $\Gamma \cdot (p_1, p_2)$; say these loops are labeled by words w_1 and w_2 respectively. Lifts of the w_1-loop to $T_1 \times T_2$ will be referred to as w_1*-geodesics* (equivalently, these are Γ-translates of $\ell_1 \times \{p_2\}$).

Lemma 2.1 *For any $n \geq 1$ there exists a w_1-geodesic whose intersection with Π is a finite path of the form $\gamma \times \{y\} \subset \ell_1 \times \ell_2$, with $p_1 \in \gamma$ and γ of length at least n.*

Proof For each $i \geq 1$, consider the rectangle in Π with two sides labeled by w_1^n and w_2^i that meet at the vertex (p_1, p_2), as shown in Figure 1. Note that the bottom side is a subpath of the axis $\ell_1 \times \{p_2\}$, while

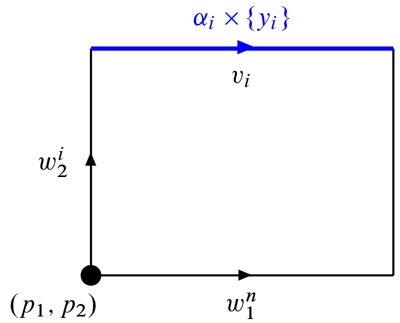

Figure 1: Rectangle in Π with two sides labeled by w_1^n and w_2^i that meet at the vertex (p_1, p_2).

the left-hand side is a subpath of the axis $\{p_1\} \times \ell_2$. Let $\alpha_i \times \{y_i\}$ denote the top side, and suppose that it is labeled by the word v_i. There are only finitely many words of length $|w_1^n|$, so $v_i = v_{i+j}$ for some $i, j \geq 1$. Since $\alpha_i \times \{y_i\}$ and $\alpha_{i+j} \times \{y_{i+j}\}$ are both labeled by v_i, the element g_2^j maps $\alpha_i \times \{y_i\}$ to $\alpha_{i+j} \times \{y_{i+j}\}$. Moreover, g_2^j preserves the axis $\{p_1\} \times \ell_2$, so it maps the rectangle shown in Figure 1 to another rectangle in Π. Restricting to the bottom sides of the rectangles, we see that g_2^j maps the subpath of $\ell_1 \times \{p_2\}$ labeled w_1^n to $\alpha_j \times \{y_j\}$, so $v_j = w_1^n$.

The path $\alpha \times \{y\} := \alpha_j \times \{y_j\}$ extends to a unique w_1-geodesic. Let $\gamma \times \{y\} \subset \ell_1 \times \ell_2$ denote the intersection of this w_1-geodesic with Π. Since $p_1 \in \alpha \subset \gamma$, and since α has length $|w_1^n| \geq n$, it remains to show that γ is finite. Say that p_1 splits γ into subpaths γ_1 and γ_2, with α being an initial segment of γ_2. We will show that γ_2 is finite — finiteness of γ_1 follows by a similar argument.

For $k \geq n$, consider the rectangle in Π with two sides labeled by w_1^k and w_2^j that meet at the vertex (p_1, p_2), as shown in Figure 2. Let $\beta_k \times \{y\}$ denote the top side. Note that $\alpha \times \{y\}$ is an initial segment of $\beta_k \times \{y\}$. Say the right-hand side is labeled by the word v'_k. The same argument we used earlier in the proof shows that $v'_k = w_2^j$ for some k. For this k, we then argue that γ_2 has length less than $|w_1^k|$. Indeed, otherwise $\gamma_2 \times \{y\}$ would contain $\beta_k \times \{y\}$ as an initial segment, so $\beta_k \times \{y\}$ would be labeled by w_1^k, but then the labels on the rectangle would imply that g_1^k and g_2^j commute, contradicting the fact that g_1 and g_2 form an antitorus. Thus γ_2 is finite, as required. □

To construct a cubulation of Γ with no factor system we first take the quotient $(T_1 \times T_2)/\Gamma$, then we attach an annulus by gluing one boundary component along the edge loop labeled by w_1, and then we let X be the universal cover. If the annulus is subdivided into $|w_1|$ squares then X is a CAT(0) cube complex. Attaching the annulus to $(T_1 \times T_2)/\Gamma$ doesn't change the fundamental group, so Γ acts on X by deck transformations. The picture upstairs is that X is obtained from $T_1 \times T_2$ by attaching an infinite strip to each w_1-geodesic (and only one strip since g_1 is not a proper power). We already remarked that $(T_1 \times T_2)/\Gamma$ has only four 2-cells for the example in [10]; moreover the word w_1 has length two in this case, so the cube complex X/Γ would be a $\mathcal{VH}$-complex consisting of just six 2-cells.

Theorem 2.2 *X has no factor system.*

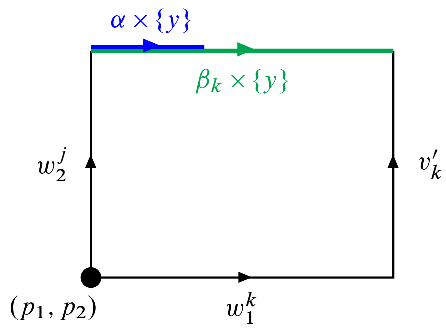

Figure 2: Rectangle in Π with two sides labeled by w_1^k and w_2^j that meet at the vertex (p_1, p_2).

Proof Suppose for contradiction that X has a factor system $\mathfrak{F}$. In X, there is an infinite strip glued to each w_1-geodesic, and there is a hyperplane that runs along the middle of each infinite strip (shown as dotted red lines in Figure 3). Hence each w_1-geodesic is a combinatorial hyperplane in X, and is an element of $\mathfrak{F}$ by Definition 1.1(4). In particular $F := \ell_1 \times \{p_2\} \in \mathfrak{F}$.

Choose an integer $n \geq 1$ and apply Lemma 2.1. This provides us with a w_1-geodesic F' whose intersection with Π is a finite path of the form $\gamma \times \{y\} \subset \ell_1 \times \ell_2 = \Pi$, with $p_1 \in \gamma$ and γ of length at least n. The projection $\mathfrak{g}_F \colon X \to F$ maps $\gamma \times \{y\}$ to $\gamma \times \{p_2\}$, and it maps the rest of F' to the endpoints of $\gamma \times \{p_2\}$. By Definition 1.1(5), the image $\mathfrak{g}_F(F') = \gamma \times \{p_2\}$ is in $\mathfrak{F}$ for sufficiently large n. But we have $(p_1, p_2) \in \mathfrak{g}_F(F')$ for all n, so we contradict Definition 1.1(3). □

Remark 2.3 If H and H' are the hyperplanes that run along the strips glued to F and F' from the above proof (see Figure 3), then the hyperplanes that are transverse to both H and H' are precisely the hyperplanes that cross the path $\gamma \times \{y\}$. Moreover, this collection of hyperplanes has no facing triples, so if γ has length L then H and H' are L-well-separated but not $(L-1)$-well-separated. We can choose H and H' so that L is arbitrarily large, so this provides a negative answer to a question of Genevois [7, Question 6.69 (first part)].

Remark 2.4 Hagen and Susse [9, Theorem A] provided three separate sufficient conditions for a cubulated group to admit a factor system. Since X has no factor system, we know that $\Gamma \curvearrowright X$ does not satisfy any of these three conditions. We describe what these conditions are below, and outline some direct arguments for why they fail for $\Gamma \curvearrowright X$:

(1) A cubulated group $G \curvearrowright Z$ is *rotational* if for each hyperplane B there is a finite-index subgroup $K_B < \mathrm{Stab}_G(B)$ such that, for any hyperplane A disjoint from B and all $k \in K_B$, the carriers of A and kA are either equal or disjoint.

For our cubulated group $\Gamma \curvearrowright X$, we can consider $B = H$ to be the hyperplane shown in Figure 3, and we can show that for any $1 \neq k \in \mathrm{Stab}_\Gamma(H)$ there is a hyperplane A disjoint from H such that the carriers of A and kA are distinct but not disjoint. Indeed, for $1 \neq k \in \mathrm{Stab}_\Gamma(H) = \langle g_1 \rangle$ we know that

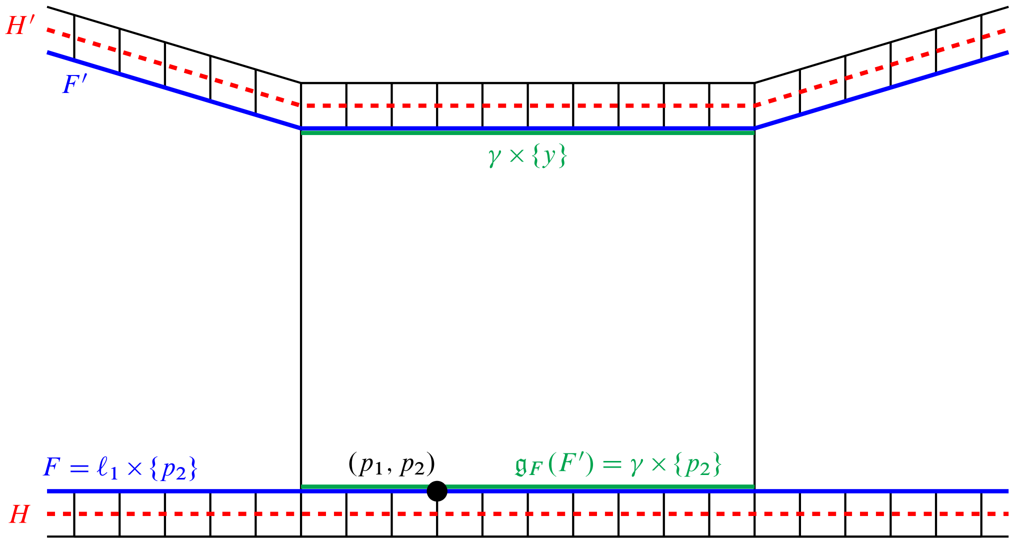

Figure 3: The w_1-geodesics F and F' with their attached strips.

$k(\{p_1\} \times \ell_2) \cap \Pi$ is a finite path (else k would commute with some power of g_2 by a similar argument to the proof of Lemma 2.1, contradicting the fact that g_1 and g_2 form an antitorus), so there must be a vertex $x \in \ell_2$ incident to an edge $e \subset \ell_2$ such that $k(p_1, x) \in \ell_1 \times \{x\}$ but $k(\{p_1\} \times e) \not\subseteq \Pi$. With A being the hyperplane dual to $\{p_1\} \times e$, the intersection of the carriers of A and kA is $T_1 \times \{x\}$, which is a proper subset of the carrier of A (it is one of the combinatorial hyperplanes of A).

(2) A cubulated group $G \curvearrowright Z$ satisfies the *weak finite height condition for hyperplanes* if the following holds for each hyperplane A and its stabilizer $K = \mathrm{Stab}_G(A)$: if $\{g_i\} \subset G$ is an infinite set such that $K \cap \bigcap_{i \in J} K^{g_i}$ is infinite for all finite $J \subset I$, then there exist distinct g_i and g_j such that $K \cap K^{g_i} = K \cap K^{g_j}$.

This condition fails for our cubulated group $\Gamma \curvearrowright X$ (in fact it also fails for $\Gamma \curvearrowright T_1 \times T_2$) by considering an edge $e \subset \ell_2$ and taking A to the hyperplane dual to $\{p_1\} \times e$. Then the stabilizer $K = \mathrm{Stab}_\Gamma(A)$ is just the stabilizer of e with respect to the action $\Gamma \curvearrowright T_2$. For any $i \geq 1$ we know that $g_1^i(\{p_1\} \times \ell_2) \cap \Pi$ is finite (as in case (1)), or equivalently $g_1^i \ell_2 \cap \ell_2 \subset T_2$ is finite. The element g_2 translates along ℓ_2, so for any power g_2^j the conjugate $K^{g_2^j}$ is the Γ-stabilizer of the edge $g_2^j e \subset \ell_2$. Thus $g_1^i \notin K^{g_2^j}$ for all sufficiently large $j \geq 1$. Since T_2 is locally finite, K and $K^{g_2^j}$ are commensurable in Γ for any j, and $\langle g_1 \rangle \cap K^{g_2^j}$ is infinite since g_1 fixes the vertex $p_2 \in T_2$. Hence, we may construct increasing sequences of positive integers (i_k) and (j_k) such that $g_1^{i_k}$ is not in $K^{g_2^{j_k}}$ but $g_1^{i_{k+1}}$ is. Therefore,

$$K \cap K^{g_2^{j_1}} \supsetneq K \cap K^{g_2^{j_2}} \supsetneq K \cap K^{g_2^{j_3}} \supsetneq \cdots$$

is a strictly descending chain of commensurable (hence infinite) subgroups of Γ, which means the weak finite height condition for hyperplanes does not hold for $\Gamma \curvearrowright X$.

(3) The third condition has two parts. A cubulated group $G \curvearrowright Z$ satisfies the *essential index condition* if there is a constant ζ such that for any $F \in \mathfrak{F}$ (where $\mathfrak{F}$ is the smallest collection of convex subcomplexes of Z that contains Z, contains all combinatorial hyperplanes, and is closed under closest point projection) the G-stabilizer of F has index at most ζ in the G-stabilizer of the essential core of F. A cubulated group $G \curvearrowright Z$ satisfies the *Noetherian intersection of conjugates* (*NIC*) *condition on hyperplane stabilizers* if the following holds for each hyperplane stabilizer K: given $\{g_i\} \subset G$ such that $K_n = K \cap \bigcap_{i=0}^{n} K^{g_i}$ is infinite for all n, there exists l such that K_n and K_l are commensurable for $n \geq l$.

Our cubulated group $\Gamma \curvearrowright X$ satisfies the NIC condition for hyperplane stabilizers (because all hyperplane stabilizers are either cyclic or commensurated) but fails the essential index condition as follows. Suppose the geodesic $\ell_1 \times \{p_2\}$ (from Figure 3) crosses the hyperplanes $H_1, H_2, H_3, \ldots$, respectively, when starting at (p_1, p_2) and moving in the direction of translation of g_1. Let F_i be the combinatorial hyperplane of H_i that is on the same side of H_i as (p_1, p_2). Let $\mathfrak{F}$ be as described above for our cubulated group $\Gamma \curvearrowright X$, and consider the subcomplexes $\mathfrak{g}_{F_1}(F_i) \in \mathfrak{F}$. Given an integer $n \geq 1$, as in the proof of Theorem 2.2 we can apply Lemma 2.1 to obtain a w_1-geodesic F' whose intersection with Π is a finite path of the form $\gamma \times \{y\} \subset \ell_1 \times \ell_2 = \Pi$, with $p_1 \in \gamma$ and γ of length at least n. Moreover, it follows from the proof of Lemma 2.1 that we can take $(p_1, y) = g_2^k(p_1, p_2)$ for some $k \neq 0$, and we may assume that the subpath of $\gamma \times \{y\}$ that starts at (p_1, y) and moves in the direction of translation of $g_2^k g_1 g_2^{-k}$ has length at least n. Let e and e' be the edges in F_1 incident at vertices (p_1, p_2) and (p_1, y), respectively, that cross the hyperplanes H and H' respectively (again from Figure 3). Note that $g_2^k e = e'$. One can show that $e \subset \mathfrak{g}_{F_1}(F_i)$ for all i. On the other hand, $e' \subset \mathfrak{g}_{F_1}(F_i)$ for $1 \leq i \leq n$ but for only finitely many i in total. One can then argue that g_2^k is in the Γ-stabilizer of $\mathfrak{g}_{F_1}(F_i)$ for $1 \leq i \leq n$ but for only finitely many i in total. This can be done for any $n \geq 1$, so the Γ-stabilizers of the $\mathfrak{g}_{F_1}(F_i)$ is a descending sequence of subgroups that never terminates. Meanwhile, all the $\mathfrak{g}_{F_1}(F_i)$ have essential core $\{p_1\} \times T_2$, so the essential index condition fails.

3 Example with nonacylindrical action on the contact graph

We now construct a free cocompact action of a group Λ on a CAT(0) cube complex Y, such that the induced action on the contact graph $\mathscr{C}Y$ is not acylindrical. We start with the action of Γ on $T_1 \times T_2$ from Section 2, with elements $g_1, g_2 \in \Gamma$ forming an antitorus. We retain all the notation from Section 2, so in particular g_1 translates along an axis $\ell_1 \times \{p_2\} \subset T_1 \times T_2$ which descends to a loop in $(T_1 \times T_2)/\Gamma$ labeled by w_1. Next we attach an annulus to $(T_1 \times T_2)/\Gamma$ by gluing both its boundary components (with matching orientations) along the loop labeled by w_1. We make this a nonpositively curved cube complex by subdividing the annulus into $|w_1|$ squares, and we let Y be the universal cover. Unlike for the construction of X in Section 2, we have glued *both* boundary components of the annulus to $(T_1 \times T_2)/\Gamma$, so the gluing changes the fundamental group from Γ to the HNN extension

$$\Lambda := \Gamma *_{\langle g_1 \rangle} = \langle \Gamma, t \mid t g_1 t^{-1} = g_1 \rangle, \tag{3-1}$$

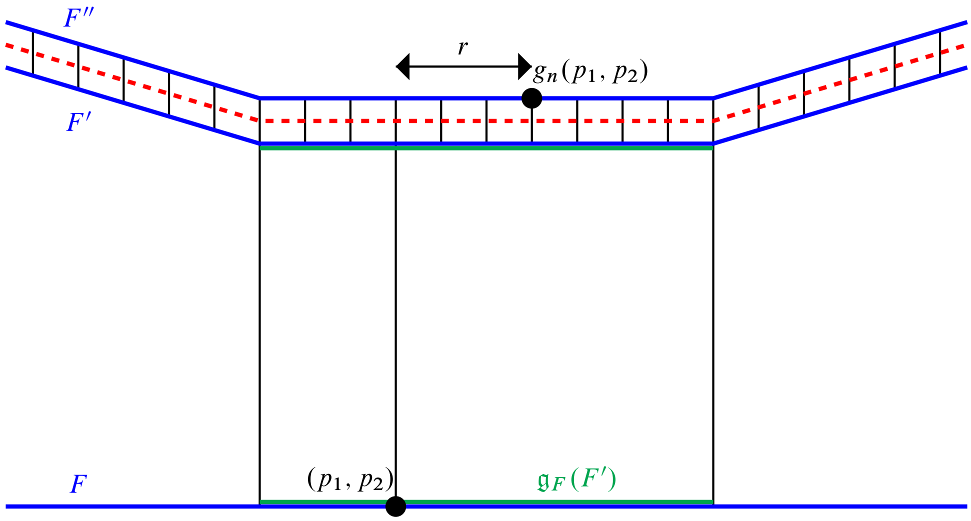

Figure 4: The positioning of $g_n(p_1,p_2)$.

and Λ acts on Y by deck transformations. Observe that Y has the structure of a tree of spaces, where the vertex spaces are copies of $T_1\times T_2$, and the edge spaces are infinite strips. The edge labeling of $(T_1\times T_2)/\Gamma$ induces an edge labeling of Y/Λ (apart from the edges that cross the annulus) and we can lift this to an edge labeling of Y. As in Section 2, lifts of the w_1-loop in Y/Λ to Y will be referred to as w_1-*geodesics*. Each w_1-geodesic in Y is attached to two edge spaces since the w_1-loop in Y/Λ is attached to both boundary components of the annulus.

Theorem 3.1 *The action of Λ on the contact graph $\mathscr{C}Y$ is not acylindrical.*

Proof We will show that for any $R,N>0$ there exist $H,H'\in\mathscr{C}Y$ such that $d_{\mathscr{C}Y}(H,H')\geq R$ and there are more than N elements $g\in\Lambda$ satisfying $d_{\mathscr{C}Y}(H,gH), d_{\mathscr{C}Y}(H',gH')\leq 2$.

Consider a vertex space in Y, and identify it with $T_1\times T_2$. Given an integer $n\geq 1$, we can choose w_1-geodesics F and F' as in the proof of Theorem 2.2 such that the projection $\mathfrak{g}_F(F')$ is a finite path of length at least n that contains the vertex (p_1,p_2). Now take one of the edge spaces in Y glued to F', and let F'' be the geodesic on the other side of the edge space. F'' is in a different vertex space, but it is again a w_1-geodesic, so it contains vertices in the orbit $\Lambda\cdot(p_1,p_2)$. Moreover, the spacing between these vertices is at most $|w_1|$, so we can choose $g_n\in\Lambda$ such that $g_n(p_1,p_2)$ lies on F'' but is shifted to the right relative to (p_1,p_2) by an integer $0<r\leq|w_1|$, as shown in Figure 4.

Note that $g_nF=F''$. Furthermore, applying powers of g_n to F and F' produces a staircase-like picture as shown in Figure 5, where each step has depth r. Each pair g_n^iF, g_n^iF' lies in a different vertex space of Y. If H is the hyperplane that runs along the edge space between $g_n^{-1}F'$ and F, then the hyperplanes g_n^iH run along a sequence of edge spaces that connect the aforementioned sequence of vertex spaces.

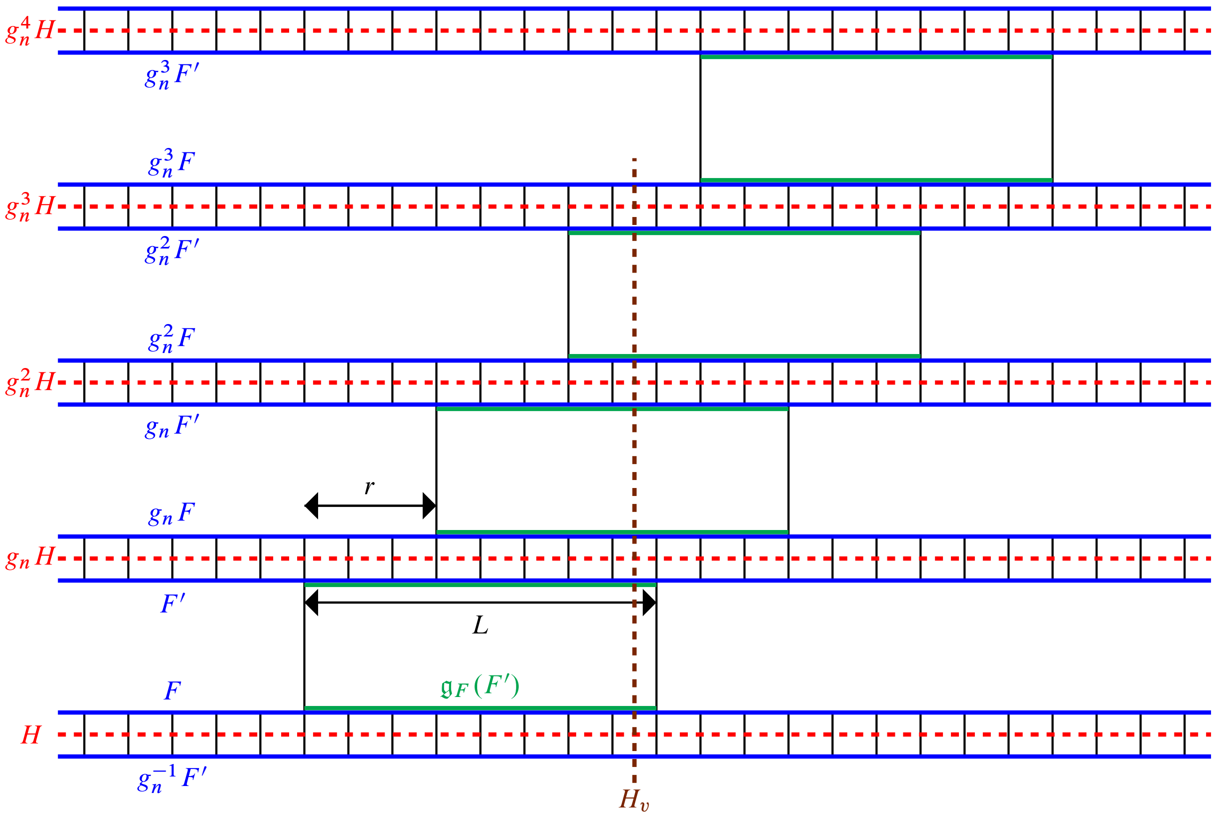

Figure 5: The arrangement of the w_1-geodesics $g_n^i F$ and $g_n^i F'$.

Suppose that the path $\mathfrak{g}_F(F')$ has length L (remembering that $L \geq n$). If a hyperplane intersects more than one of the hyperplanes $g_n^i H$ then it must cross some $\langle g_n \rangle$-translate of the projection $\mathfrak{g}_F(F')$. The hyperplane H_v that is dual to the last edge of $\mathfrak{g}_F(F')$ intersects exactly $M := \lceil L/r \rceil + 1$ of the hyperplanes $g_n^i H$, and no other hyperplane intersects more of them. In particular, for $1 \leq i < M$, H_v intersects H and $g_n^i H$, so the distance between H and $g_n^i H$ in the contact graph $\mathscr{C}Y$ is

$$d_{\mathscr{C}Y}(H, g_n^i H) = 2. \tag{3-2}$$

On the other hand, $d_{\mathscr{C}Y}(H, g_n^p H) \to \infty$ as $p \to \infty$. Indeed, suppose the geodesic in $\mathscr{C}Y$ from H to $g_n^p H$ consists of hyperplanes $H = H_0, H_1, \dots, H_d = g_n^p H$. We know that H and $g_n^p H$ are separated by the hyperplanes $g_n^i H$ for each $0 < i < p$, so each of these $g_n^i H$ either equals one of the H_j or intersects one of the H_j. But we know from earlier that each H_j intersects at most M of the $g_n^i H$; hence

$$d = d_{\mathscr{C}Y}(H, g_n^p H) \geq \frac{p}{M}.$$

Putting $H' = g_n^p H$, we have $d_{\mathscr{C}Y}(H, H') \geq R$ provided $p \geq MR$. But (3-2) implies that H and H' are both moved at most distance 2 in $\mathscr{C}Y$ by the elements $\{1, g_n, g_n^2, \dots, g_n^{M-1}\} \subset \Lambda$; and $M = \lceil L/r \rceil + 1 > n/r$, so if we choose $n \geq rN$ then we get more than N elements $g \in \Lambda$ satisfying $d_{\mathscr{C}Y}(H, gH), d_{\mathscr{C}Y}(H', gH') \leq 2$, as required. So we conclude that the action of Λ on $\mathscr{C}Y$ is not acylindrical. □

Remark 3.2 The staircase in Figure 5 is not a staircase as defined in [9] (visually speaking, the former looks like it has empty space below the staircase whereas the latter does not). However, they are both obstructions to the existence of a factor system. Indeed, the existence of staircases as in Figure 5 for arbitrarily large ratios L/r is the key to proving Theorem 3.1, which in turn implies that Y has no factor system. Meanwhile, the existence of just a single convex staircase in the sense of [9] rules out the possibility of a factor system. It remains an open question whether any cubulation of a group contains a convex staircase in the sense of [9].

Remark 3.3 Genevois [7] defined a metric δ_K for a CAT(0) cube complex X as the maximal number of pairwise K-well-separated hyperplanes separating two given vertices. The space (X, δ_K) is hyperbolic for all K, and it is quasi-isometric to the contact graph for $K = 0$. Moreover, if $G \curvearrowright X$ is a cubulated group, then the induced action on (X, δ_K) is nonuniformly acylindrical for all K. However, for our cubulated group $\Lambda \curvearrowright Y$, the action of Λ on (Y, δ_K) is not acylindrical for any K. The argument is similar to that in the proof of Theorem 3.1: you can define the element $g_n \in \Lambda$ in the same way, and show that g_n acts loxodromically on (Y, δ_K), and you can exhibit points on the axis of g_n that are arbitrarily far apart but are moved at most distance 2 by many powers of g_n — with the number of such powers tending to infinity as $n \to \infty$.

As discussed in the introduction, we do not know whether Λ is an HHG. A possible strategy to prove that Λ is not an HHG would be to show that Λ has no largest acylindrical action (definition below), because all HHGs possess a largest acylindrical action [2].

Definition 3.4 Let G be a group that acts on metric spaces R and S. We say that $G \curvearrowright R$ is *dominated* by $G \curvearrowright S$, written $G \curvearrowright R \preceq G \curvearrowright S$, if there exist $r \in R$, $s \in S$ and a constant C such that

$$d_R(r, gr) \leq C d_S(s, gs) + C$$

for all $g \in G$. The actions $G \curvearrowright R$ and $G \curvearrowright S$ are *equivalent* if $G \curvearrowright R \preceq G \curvearrowright S$ and $G \curvearrowright S \preceq G \curvearrowright R$. We denote the equivalence class by $[G \curvearrowright R]$. The relation $\preceq$ defines a partial order on the set of equivalence classes of actions of G on metric spaces. The *largest acylindrical action* of G (if it exists) is the largest element of the set of equivalence classes of cobounded acylindrical actions of G on hyperbolic metric spaces.

Alas, we show below in Proposition 3.6 that Λ does have a largest acylindrical action, which is defined as follows. Let T be the Bass–Serre tree for the splitting $\Lambda = \Gamma *_{\langle g_1 \rangle}$. We say that edges $e_1, e_2 \in ET$ are *equivalent* if the stabilizers Λ_{e_1} and Λ_{e_2} are commensurable. Each equivalence class defines a subtree of T called a *cylinder*. The *tree of cylinders* T_c is the bipartite tree with vertex set $V_0 T_c \sqcup V_1 T_c$, where

V_0T_c are the vertices of T and V_1T_c is the set of cylinders. The edges of T_c are of the form (v, C) where v is a vertex in T that lies in the cylinder $C \subset T$. The action of Λ on T induces an action on T_c.

To help us prove Proposition 3.6 we will use the following lemma, which is equivalent to [1, Corollary 4.14].

Lemma 3.5 *Let G act cocompactly on a connected graph Δ and let $G \curvearrowright R$ be another cobounded action on a metric space. If the vertex stabilizers of Δ have bounded orbits in R, then $G \curvearrowright R \preceq G \curvearrowright \Delta$.*

Proposition 3.6 *The largest acylindrical action of Λ is its action on the tree of cylinders $\Lambda \curvearrowright T_c$.*

Proof First we show that the action of Λ on T_c is acylindrical. It suffices to show that only the identity element of Λ fixes a path in T_c of the form v_1, C_1, u, C_2, v_2, where $u, v_i \in V_0T_c$ and $C_i \in V_1T_c$. Indeed if $g \in \Lambda$ fixes such a path, then it must also fix any pair of edges $e_1, e_2 \in ET$ that lie on the geodesics $[u, v_1]$ and $[u, v_2]$ respectively. It follows that e_1 and e_2 lie in the cylinders C_1 and C_2, and so the stabilizers Λ_{e_1} and Λ_{e_2} are not commensurable. But these stabilizers are infinite cyclic (they are conjugates of $\langle g_1\rangle$), hence they have trivial intersection, so $g \in \Lambda_{e_1} \cap \Lambda_{e_2}$ is trivial.

The action $\Lambda \curvearrowright T_c$ is clearly cobounded and T_c is obviously hyperbolic, so it remains to show that $\Lambda \curvearrowright T_c$ dominates all other cobounded acylindrical actions of Λ on hyperbolic spaces. Let $\Lambda \curvearrowright R$ be such an action. By Lemma 3.5, it suffices to show that the vertex stabilizers of T_c have bounded orbits in R.

First consider $v \in V_0T_c$. If the stabilizer Λ_v does not have bounded orbits in R then there must exist a loxodromic element $g \in \Lambda_v$ by [11, Theorem 1.1], and g must be contained in a virtually cyclic hyperbolically embedded subgroup of Λ_v by [11, Theorem 1.4]. It then follows from [13, Theorem 1] that g is Morse in Λ_v. But this is impossible since $\Lambda_v \cong \Gamma$ is quasi-isometric to a product of trees.

Now consider $C \in V_1T_c$. Without loss of generality we may assume that C contains the edge that corresponds to the subgroup $\langle g_1\rangle < \Gamma < \Lambda$. We know that Λ_C contains the element g_1 as well as the stable letter t from the presentation (3-1), so Λ_C is not virtually cyclic. If Λ_C does not have bounded orbits in R then there must exist a loxodromic element $g \in \Lambda_C$ by [11, Theorem 1.1]. Fix a point $r \in R$. The element g_1 lies in the V_0T_c-vertex stabilizer $\Gamma < \Lambda$, so g_1 is elliptic in R by the previous paragraph, and the orbit $\langle g_1\rangle \cdot r$ lies in some ϵ-ball about r. By definition of C, for any integer k the subgroups $\langle g_1\rangle$ and $g^k\langle g_1\rangle g^{-k}$ are commensurable, so have infinite intersection; and any element h of this intersection satisfies $d_R(r, hr), d_R(g^kr, hg^kr) < \epsilon$. But $d(r, g^kr) \to \infty$ as $k \to \infty$ since g is loxodromic, which contradicts the acylindricity of the action $\Lambda \curvearrowright R$. □

References

[1] **C Abbott**, **S H Balasubramanya**, **D Osin**, *Hyperbolic structures on groups*, Algebr. Geom. Topol. 19 (2019) 1747–1835 MR Zbl

[2] **C Abbott**, **J Behrstock**, **M G Durham**, *Largest acylindrical actions and stability in hierarchically hyperbolic groups*, Trans. Amer. Math. Soc. Ser. B 8 (2021) 66–104 MR Zbl

[3] **J Behrstock**, **M F Hagen**, **A Sisto**, *Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups*, Geom. Topol. 21 (2017) 1731–1804 MR Zbl

[4] **J Behrstock**, **M Hagen**, **A Sisto**, *Hierarchically hyperbolic spaces, II: Combination theorems and the distance formula*, Pacific J. Math. 299 (2019) 257–338 MR Zbl

[5] **P-E Caprace**, *Finite and infinite quotients of discrete and indiscrete groups*, from "Groups St Andrews 2017 in Birmingham", Lond. Math. Soc. Lect. Note Ser. 455, Cambridge Univ. Press (2019) 16–69 MR Zbl

[6] **A Genevois**, *Acylindrical hyperbolicity from actions on* CAT(0) *cube complexes: a few criteria*, New York J. Math. 25 (2019) 1214–1239 MR Zbl

[7] **A Genevois**, *Hyperbolicities in* CAT(0) *cube complexes*, Enseign. Math. 65 (2019) 33–100 MR Zbl

[8] **M F Hagen**, *Weak hyperbolicity of cube complexes and quasi-arboreal groups*, J. Topol. 7 (2014) 385–418 MR Zbl

[9] **M F Hagen**, **T Susse**, *On hierarchical hyperbolicity of cubical groups*, Israel J. Math. 236 (2020) 45–89 MR Zbl

[10] **D Janzen**, **D T Wise**, *A smallest irreducible lattice in the product of trees*, Algebr. Geom. Topol. 9 (2009) 2191–2201 MR Zbl

[11] **D Osin**, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016) 851–888 MR Zbl

[12] **D Rattaggi**, *Anti-tori in square complex groups*, Geom. Dedicata 114 (2005) 189–207 MR Zbl

[13] **A Sisto**, *Quasi-convexity of hyperbolically embedded subgroups*, Math. Z. 283 (2016) 649–658 MR Zbl

[14] **D T Wise**, *Complete square complexes*, Comment. Math. Helv. 82 (2007) 683–724 MR Zbl

[15] **D T Wise**, *From riches to raags:* 3*-manifolds, right-angled Artin groups, and cubical geometry*, CBMS Regional Conf. Ser. Math. 117, Amer. Math. Soc., Providence, RI (2012) MR Zbl

Institut für Mathematische Logik und Grundlagenforschung
Fachbereich Mathematik und Informatik, Universität Münster
Münster, Germany

sam.shepherd@uni-muenster.de

Received: 27 February 2023 Revised: 13 June 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2025 is US $760/year for the electronic version, and $1110/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 1 (pages 1–644) 2025