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Relative h-principle and contact geometry

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We show that if $F(M)$ is a space of holonomic solutions with space of formal solutions $F^f(M)$ that satisfies a certain relative h-principle, then the nonrelative map $F(M) \to F^f(M)$ admits a section up to homotopy. We apply this to the relative h-principle for overtwisted contact structures proved by Borman, Eliashberg and Murphy to find infinite cyclic subgroups in the homotopy groups of contactomorphism groups.

53D35, 57R17; 58D99

1 Introduction

Gromov [12] showed that if M is an open manifold then the inclusion $\mathrm{Cont}(M) \to \mathrm{AlmCont}(M)$ is a weak equivalence, where $\mathrm{Cont}(M)$ is the space of contact structures on M and $\mathrm{AlmCont}(M)$ is the space of almost contact structures on M. The case of compact manifolds is not so simple. For example, there exist contact structures on closed 3-manifolds that are formally homotopic but not homotopic; see Bennequin [1]. In [2], Matthew Borman, Yakov Eliashberg and Emmy Murphy advanced the field of contact geometry by first extending the definition of an overtwisted contact manifold from 3-dimensional manifolds to manifolds of dimension $2n+1 \geq 3$, and then proving an h-principle result for overtwisted contact manifolds. Essentially, an overtwisted contact manifold is a contact manifold M that contains an embedded overtwisted disk, ie an embedded $2n$-disk Δ with a certain model germ of a contact structure on a neighborhood of Δ (see [2, Definition 3.6]). If $\mathrm{Cont}^{\mathrm{OT}}(M, \Delta)$ and $\mathrm{AlmCont}(M, \Delta)$ denote, respectively, the spaces of contact and formal contact structures that are overtwisted with fixed disk Δ, then the main result of [2] is that

$$\mathrm{Cont}^{\mathrm{OT}}(M, \Delta) \to \mathrm{AlmCont}(M, \Delta)$$

is a weak equivalence. However, it is known that in general the map $\mathrm{Cont}^{\mathrm{OT}}(M) \to \mathrm{AlmCont}(M)$ from overtwisted contact structures to almost contact structures is not a weak equivalence; see for example Vogel [18]. Given this, one may wonder how much can be learned about the maps $\mathrm{Cont}^{\mathrm{OT}}(M) \to \mathrm{AlmCont}(M)$ and $\mathrm{Cont}(M) \to \mathrm{AlmCont}(M)$ using the h-principle when one fixes a disk. In fact, there is a much more general question here about relative h-principles, motivated by this example.

Question Let Δ be some subset of M and γ be the germ of some holonomic solution on Δ. Let $F(M \text{ rel}(\Delta, \gamma))$ denote the set of all holonomic solutions that have germ γ on Δ, and $F^f(M \text{ rel}(\Delta, \gamma))$

denote the set of formal solutions that have germ γ on Δ. If the map

$$F(M \operatorname{rel}(\Delta, \gamma)) \to F^f(M \operatorname{rel}(\Delta, \gamma))$$

is a weak equivalence for all pairs $(\Delta, \gamma) \in \mathcal{W}$ for some collection $\mathcal{W}$, what can be said about the map

$$F(M) \to F^f(M)?$$

One of the main results of this paper is a partial answer to this question. Suppose M is a manifold and $A \subset M$ is a (possibly empty) closed subset of M such that $M \setminus A$ is a manifold without boundary. By assumption $F^f(M)$ is the space of sections of some bundle $\pi\colon E \to M$, and furthermore we suppose the fibers of E are path connected. Let $\mathcal{W}$ be a *sufficiently separated collection* for the bundle E relative to A (see Definition 2.2), such that each germ in the collection is holonomic. Finally, let ξ_0 be a holonomic solution near A. Using this, we construct a semisimplicial space $F_\bullet(M \operatorname{rel}(A, \xi_0); \mathcal{W})$ with an augmentation to $F(M)$, and prove the following theorem:

Theorem A *If the map*

$$F\big(M \operatorname{rel}(\Delta_1, \gamma_1), \dots, (\Delta_k, \gamma_k), (A, \xi_0)\big) \to F^f(M \operatorname{rel}(\Delta_1, \gamma_1), \dots, (\Delta_k, \gamma_k), (A, \xi_0))$$

is a weak equivalence for all finite sets of disjoint elements $\{(\Delta_i, \gamma_i)\}_{i=1}^k \subset \mathcal{W}$, $k \geq 1$, *then the following diagram commutes*:

$$\begin{array}{ccc} \|F_\bullet(M \operatorname{rel}(A, \xi_0); \mathcal{W})\| & \longrightarrow & F(M \operatorname{rel}(A, \xi_0)) \\ & \searrow^{\simeq} & \downarrow \\ & & F^f(M \operatorname{rel}(A, \xi_0)) \end{array}$$

and the map $\|F_\bullet(M \operatorname{rel}(A, \xi_0); \mathcal{W})\| \to F^f(M \operatorname{rel}(A, \xi_0))$ *is a weak equivalence.*

Remark 1.1 As the above theorem is stated, the so-called "nonrelative map" is still relative to the set A. This is because there are two notions of relative h-principle here, one that gives an h-principle relative to fairly arbitrary sets and functions near those sets, and one that gives an h-principle relative to very specific sets and functions near those sets. The former usually includes the empty set and is thus strictly stronger than the nonrelative statement, while the latter should be thought of as asking for the existence of some sort of nice local model. So, the above theorem should be thought of as saying that if one has an h-principle relative to some nice local models, then even when you do not fix any models the map admits a section up to homotopy.

This has some immediate consequences in contact geometry, as the above theorem allows us to find a subgroup of $\pi_k \operatorname{Cont}^{\mathrm{OT}}(M)$ isomorphic to $\pi_k \operatorname{AlmCont}(M)$, induced by a map of spaces, for all k. This is an improvement on the current tool used to analyze the difference between the homotopy groups of $\operatorname{Cont}^{\mathrm{OT}}(M)$ and $\operatorname{AlmCont}(M)$, the *overtwisted group* (see Casals, del Pino and Presas [4, Proposition 1] or Fernández and Gironella [8, Definition 10]), which only allows one to realize $\pi_k \operatorname{AlmCont}(M)$ as a subgroup of $\pi_k \operatorname{Cont}^{\mathrm{OT}}(M)$ when $1 \leq k \leq 2n$. Furthermore, we show that these subgroups agree when

$1 \le k \le 2n$. Finally, we use these results to help study certain homotopy groups of the contactomorphism group of an overtwisted contact manifold. Let $\mathcal{C}_0(M, \xi_{\mathrm{OT}} \operatorname{rel} \partial M) \subset \mathrm{Diff}_0(M \operatorname{rel} \partial M)$ denote the space of contactomorphisms of the contact manifold (M, ξ_{OT}) which are smoothly isotopic to the identity and agree with the identity near the boundary of M.

Theorem B *If (M, ξ_{OT}) is a compact, coorientable, overtwisted contact manifold of dimension $2n+1$, then $\pi_k \mathcal{C}_0(M, \xi_{\mathrm{OT}} \operatorname{rel} \partial M)$ contains an infinite cyclic subgroup whenever*

$$\pi_k \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) \otimes \mathbb{Q} \neq 0, \quad \textit{for } k \le \phi^{\mathbb{Q}}(\mathbb{D}^{2n}) - 1,\ k \neq 0.$$

Here $\phi^{\mathbb{Q}}(\mathbb{D}^{2n})$ is the rational concordance stable range for $\mathbb{D}^{2n}$. The stability theorem of Igusa [13, page 6] implies $\phi^{\mathbb{Q}}(M^d) \ge \min\big(\frac{1}{2}(d-7), \frac{1}{3}(d-4)\big)$ for any compact d-dimensional manifold M. Better lower bounds exist in different cases, for example Corollary C of Goodwillie, Krannich and Kupers [11]. For $d \ge 10$ it is known that $\phi^{\mathbb{Q}}(\mathbb{D}^d) = d - 4$ by Corollary B of Kranniich and Randal-Williams [15]. Also, it is known that $\pi_k \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) \otimes \mathbb{Q} \neq 0$ for many k in this range, for example Krannich [14, Corollary B] computes $\pi_k \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) \otimes \mathbb{Q}$ for $k < 2n - 4$ in terms of the algebraic K-theory of the integers.

Overview of the paper

In Section 2, we show that for a fiber bundle $E \to M$ with space of continuous sections $\Gamma(E)$, if $\mathcal{W}$ is a *sufficiently separated collection* relative to A of pairs (Δ, γ) for E (see Definition 2.2) then the space $\Gamma(E \operatorname{rel}(A, \xi_0))$ is weakly equivalent to the geometric realization of a certain semisimplicial space built from the spaces of relative sections. Then in Section 3, we use this result to prove Theorem A. In Section 4, we show that the collection of all overtwisted disks on a manifold is sufficiently separated, and so the map $\mathrm{Cont}^{\mathrm{OT}}(M \operatorname{rel} A) \to \mathrm{AlmCont}(M \operatorname{rel} A)$ admits a section up to homotopy. Then we use this to show that in the range it is defined, the usual overtwisted group agrees with the image of the map induced by this section. In Section 5 we use these results to prove Theorem B, which finds infinite cyclic subgroups in the homotopy groups of the contactomorphism groups of compact overtwisted contact manifolds, in degrees different than those found in [8]. Finally, in Section 6 we note other applications of Theorem A, specifically coming from Engel geometry.

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2 Semisimplicial resolutions of section spaces

For technical reasons, suppose we are in the category of compactly generated spaces. Let M be a d-dimensional manifold, possibly with boundary. Let X be a path connected space and $\pi\colon E \to M$ be a fiber bundle with fiber X. Let $\Gamma(E)$ denote the space of sections of E.

Definition 2.1 Let $A \subset M$ be some subset of M. A *germ of a section on* A is a pair (γ, U), where U is an open neighborhood of A and γ is a section on U, with the equivalence relation that two germs are the same if they agree on some neighborhood of A.

For convenience we usually omit the neighborhood U and just let γ denote the germ, with the understanding that γ is defined on some arbitrarily small neighborhood of A. If $A_1, \dots, A_k \subset M$ and γ_i is a germ of a section on A_i for $1 \le i \le k$, we will let $\Gamma(E \operatorname{rel} (A_1, \gamma_1), \dots, (A_k, \gamma_k))$ denote the space of sections of E which agree with γ_i near A_i for all $1 \le i \le k$. By abuse of notation, we will let $\Gamma(E \operatorname{rel} A_1, \dots, A_k)$ denote the same space, and more generally we will use this convention for many function spaces appearing in this paper.

Definition 2.2 A *sufficiently separated collection relative to* A for the bundle $\pi\colon E \to M$ is a collection $\mathcal{W}$ of pairs (Δ, γ) of (a) a contractible compact subset $\Delta \subset M \setminus A$ and (b) a germ of a section near Δ. These are required to satisfy:

(1) For each $(\Delta, \gamma) \in \mathcal{W}$, there exists some neighborhood $D \cong \mathbb{D}^d$ such that Δ is contained in the interior of D, $D \subset M \setminus A$, and the inclusion map $\iota\colon \Delta \to D$ is a closed cofibration.

(2) Given any finite collection $(\Delta_i, \gamma_i)_{i=1}^k$ in $\mathcal{W}$, there exists some $(\Delta', \gamma') \in \mathcal{W}$ so that Δ' is contained in the interior of a closed ball $D \cong \mathbb{D}^d$ that is disjoint from each Δ_i, $D \subset M \setminus A$, and the inclusion $\iota\colon \Delta' \to D$ is a closed cofibration.

We say that two elements (Δ_1, γ_1), (Δ_2, γ_2) of $\mathcal{W}$ are *disjoint* if $\Delta_1 \cap \Delta_2 = \varnothing$, and a collection of elements is *disjoint* if each pair of elements in the collection is disjoint. Next, let $A \subset M$ and ξ_0 be some germ of a section on A. Suppose $\mathcal{W}$ is a sufficiently separated collection relative to A for the bundle E. Then let $\Gamma_\bullet(E \operatorname{rel} A; \mathcal{W})$ be the semisimplicial space defined by

$$\Gamma_p(E \operatorname{rel} A; \mathcal{W}) := \coprod_{\substack{\text{tuples } ((\Delta_0,\gamma_0),\dots,(\Delta_p,\gamma_p)) \\ \text{of } p+1 \text{ disjoint elements of } \mathcal{W}}} \Gamma(E \operatorname{rel} \Delta_0, \dots, \Delta_p, A).$$

The i^{th} face map is given by forgetting that the sections agree with γ_i near Δ_i. Our goal is to compare the space $\|\Gamma_\bullet(E \operatorname{rel} A; \mathcal{W})\|$ to $\Gamma(E \operatorname{rel} A)$. In order to do this, we need a few lemmas. First, let $(\Delta, \gamma) \in \mathcal{W}$ and $\Gamma(E, \Delta \operatorname{rel} A)$ denote the space of sections s of E which agree with ξ_0 near A and satisfy $s|_\Delta = \gamma|_\Delta$. Then we have the following lemma, which will allow us to forget about germs and just work with spaces of sections that have fixed values on subsets.

Lemma 2.3 *The map $\Gamma(E \operatorname{rel} \Delta, A) \to \Gamma(E, \Delta \operatorname{rel} A)$ given by inclusion is a weak equivalence.*

Proof By definition, $\Gamma(E \text{ rel } \Delta, A)$ is topologized as the colimit

$$\Gamma(E \text{ rel } \Delta, A) := \varinjlim \Gamma(E, B_i \text{ rel } A),$$

where the B_i are some neighborhood basis of Δ. Let $\alpha\colon \mathbb{D}^n \to \Gamma(E, \Delta \text{ rel } A)$ be a continuous map such that we have the following diagram:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\partial\alpha} & \Gamma(E \text{ rel } \Delta, A) \\ \downarrow & & \downarrow \\ \mathbb{D}^n & \xrightarrow{\alpha} & \Gamma(E, \Delta \text{ rel } A) \end{array}$$

We will show there exists a lift $\beta\colon \mathbb{D}^n \to \Gamma(E \text{ rel } \Delta, A)$ of α up to homotopy relative to the boundary. By Lemma 3.6 of [17], any map from a compact space to a colimit of closed inclusions factors over one of the inclusions, so $S^{n-1} \to \Gamma(E \text{ rel } \Delta, A)$ factors as

$$S^{n-1} \to \Gamma(E, B_i \text{ rel } A) \to \Gamma(E \text{ rel } \Delta, A)$$

for some neighborhood B_i. We can choose B_i to be contractible and such that the inclusion $B_i \to M$ is a closed cofibration. We can also ensure B_i is disjoint from A. Then it suffices to show that $\Gamma(E, B_i \text{ rel } A) \to \Gamma(E, \Delta \text{ rel } A)$ is a weak equivalence, which holds due to the following commutative diagram; the rows are fiber sequences and $\Gamma(E|_{B_i}) \to \Gamma(E|_\Delta)$ is a weak equivalence since both Δ and B_i are contractible:

$$\begin{array}{ccccc} \Gamma(E, B_i \text{ rel } A) & \longrightarrow & \Gamma(E \text{ rel } A) & \longrightarrow & \Gamma(E|_{B_i}) \\ {\scriptstyle\iota}\downarrow & & {\scriptstyle\text{id}}\downarrow & & {\scriptstyle\text{res}}\downarrow \\ \Gamma(E, \Delta \text{ rel } A) & \longrightarrow & \Gamma(E \text{ rel } A) & \longrightarrow & \Gamma(E|_\Delta) \end{array}$$

□

Remark 2.4 A similar argument can be used if one replaces (Δ, γ) with finitely many disjoint elements of $\mathcal{W}$.

This allows us to use the following abuse of notation. Let $\Gamma_\bullet(E \text{ rel } A; \mathcal{W})$ denote the semisimplicial space given by

$$\Gamma_p(E \text{ rel } A; \mathcal{W}) := \coprod_{\substack{\text{tuples } ((\Delta_0,\gamma_0),\dots,(\Delta_p,\gamma_p)) \\ \text{of } p+1 \text{ disjoint elements of } \mathcal{W}}} \Gamma(E, \Delta_0, \dots, \Delta_p \text{ rel } A),$$

where again the face maps just forget the fixed sets. We can do this because Lemma 2.3 implies the resulting semisimplicial space is levelwise weakly equivalent to the one constructed before, and so the geometric realizations of both are weakly equivalent. We can also consider the semisimplicial space $W_\bullet$ given by

$$W_p := \coprod_{\substack{\text{tuples } ((\Delta_0,\gamma_0),\dots,(\Delta_p,\gamma_p)) \\ \text{of } p+1 \text{ disjoint elements of } \mathcal{W}}} \{\star\},$$

where the face maps are given by forgetting elements of $\mathcal{W}$. Here $\{\star\}$ is just a singleton set, so W_p contains a point for each tuple of $p+1$ disjoint elements of $\mathcal{W}$, with the discrete topology. These semisimplicial spaces are related via the following lemma.

Lemma 2.5 *The space $\|\Gamma_\bullet(E \operatorname{rel} A;\mathcal{W})\|$ is homeomorphic to the subspace of $\Gamma(E \operatorname{rel} A)\times\|W_\bullet\|$ given by*

$$\{(f,\vec{w},\vec{t}) \mid f(m)=\gamma_i(m) \text{ whenever } m\in\Delta_i \text{ and } t_i\neq 0\}/\sim,$$

where (Δ_i,γ_i) is the i^{th} component of $\vec{w}$, and $\sim$ is just the usual geometric realization equivalence on the second factor.

Proof It is clear these are the same as sets, so we just need to show that this map is a homeomorphism onto some subspace. First, since the quotient of a subspace is naturally a subspace of the quotient in compactly generated spaces, we have

$$\|\Gamma_\bullet(E \operatorname{rel} A;\mathcal{W})\| \subset \|\Gamma(E \operatorname{rel} A)\times W_\bullet\|.$$

On the other hand, by [7, page 2106] we have that

$$\|\Gamma(E \operatorname{rel} A)\times W_\bullet\| = \|\Gamma(E \operatorname{rel} A)\otimes W_\bullet\| \cong \|\Gamma(E \operatorname{rel} A)\|\times\|W_\bullet\| \cong \Gamma(E \operatorname{rel} A)\times\|W_\bullet\|,$$

where we are treating $\Gamma(E \operatorname{rel} A)$ as a semisimplicial space with only 0-simplices in order to use the exterior product defined in [7, page 2103]. So, $\|\Gamma_\bullet(E \operatorname{rel} A;\mathcal{W})\|$ is homeomorphic to the subspace of $\Gamma(E \operatorname{rel} A)\times\|W_\bullet\|$ described above. □

We can now introduce the main result of this section:

Theorem 2.6 *Let $A\subset M$ be a closed subset of a manifold M and ξ_0 be a germ of a section on A, such that $M\setminus A$ is a manifold without boundary. Furthermore, let $\pi\colon E\to M$ be a fiber bundle with connected fiber X, and $\mathcal{W}$ be a sufficiently separated collection relative to A for the bundle E. Let $\|\Gamma_\bullet(E \operatorname{rel} A;\mathcal{W})\|$ be as before. Then the map $\|\Gamma_\bullet(E \operatorname{rel} A;\mathcal{W})\|\to\Gamma(E \operatorname{rel} A)$ given by forgetting the fixed subsets is a weak equivalence.*

Proof We will prove this by showing that the relative homotopy groups of the map are zero. So, let $\alpha\colon\mathbb{D}^n\to\Gamma(E \operatorname{rel} A)$ be a continuous map such that we have the following diagram:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\partial\alpha} & \|\Gamma_\bullet(E \operatorname{rel} A;\mathcal{W})\| \\ \downarrow & & \downarrow \\ \mathbb{D}^n & \xrightarrow{\alpha} & \Gamma(E \operatorname{rel} A) \end{array}$$

where by abuse of notation $\partial\alpha$ is the map $S^{n-1}\to\|\Gamma_\bullet(E \operatorname{rel} A;\mathcal{W})\|,\ p\mapsto(\alpha(p),\vec{w}_p,\vec{t}_p)$ for some finite ordered set of elements $\vec{w}_p$ of $\mathcal{W}$ and weights $\vec{t}_p$ (where of course we just exclude any elements when their weight goes to zero). Then we need to show that there exists a continuous map $\alpha'\simeq\alpha$ relative to the boundary, and a lift $\beta\colon\mathbb{D}^n\to\|\Gamma_\bullet(E \operatorname{rel} A;\mathcal{W})\|$ of α' such that the resulting diagram still commutes.

First, consider the section

$$\alpha_1(p)(m) := \begin{cases} \alpha(2p)(m), & 0 \le |p| \le \frac{1}{2}, \\ \alpha(p/|p|), & \frac{1}{2} \le |p| \le 1. \end{cases}$$

So α_1 is just α compressed to a smaller ball, with the boundary extended to an annulus. It is clear that $\alpha_1 \simeq \alpha$ relative to the boundary, so we can work with α_1, which gives us a buffer away from the boundary. Next, consider the set

$$\widetilde{W} := \{m \in M \mid \text{for some } p \in S^{n-1} \text{ and some positive integer } i, m \in \Delta_{p,i}, \\ \text{where } (\Delta_{p,i}, \gamma_{p,i}) \text{ is the } i^{\text{th}} \text{ component of } \vec{w}_p\}.$$

Note that α_1 lifts to $\|\Gamma_\bullet(E \text{ rel } A; \mathcal{W})\|$ on the annulus, and furthermore if we only change α_1 away from $\widetilde{W}$ then it will still lift in the annulus. Now, using the natural projection map from $\|\Gamma_\bullet(E \text{ rel } A; \mathcal{W})\| \to \|W_\bullet\|$ given in Lemma 2.5, we can consider the map

$$S^{n-1} \to \|\Gamma_\bullet(E \text{ rel } A; \mathcal{W})\| \to \|W_\bullet\|,$$

where $p \mapsto (\vec{w}_p, \vec{t}_p)$. Since S^{n-1} is compact we know that the image of this map is compact, and so it hits finitely many cells of $\|W_\bullet\|$. Also, each cell consists of finitely many elements of $\mathcal{W}$, so the set of all elements of $\mathcal{W}$ that are a component of $\vec{w}_p$ for some p is finite. But, if $\{(\Delta_1, \gamma_1), \dots, (\Delta_k, \gamma_k)\}$ is that set, then clearly $\widetilde{W} = \Delta_1 \cup \dots \cup \Delta_k$. Let $(\Delta, \gamma) \in \mathcal{W}$ be such that there is some neighborhood $D \cong \mathbb{D}^d$ of Δ contained in $M \setminus A$ that is disjoint from $\widetilde{W}$. Since a map $\mathbb{D}^n \to \Gamma(E \text{ rel } A)$ is the same data as a section of the bundle $\mathbb{D}^n \times E \to \mathbb{D}^n \times M$, $(p, e) \mapsto (p, \pi(e))$ we will from now on consider α, α_1 as maps from $\mathbb{D}^n \times M \to \mathbb{D}^n \times E$ so that composition with the projection map is the identity. Now, let $g_0 = \alpha_1|_{\mathbb{D}^n \times D}$, $\mathbb{D}^n_{3/4} := \{p \in \mathbb{D}^n \mid |p| \le \frac{3}{4}\}$. Since $\mathbb{D}^n \times D$ is a contractible submanifold of $\mathbb{D}^n \times M$, we know that the restriction of the bundle to $\mathbb{D}^n \times D$ is trivial. So, g_0 and any other sections on (a subset of) $\mathbb{D}^n \times D$ just become maps from (a subset of) $\mathbb{D}^n \times D$ to X. Consider the homotopy

$$(\mathbb{D}^n_{3/4} \times \Delta) \cup \partial(\mathbb{D}^n \times D) \times [0,1] \to X$$

given by

$$H(p,m,s) := \begin{cases} \alpha_1(p,m), & p \in \partial\mathbb{D}^n, \\ \alpha_1(p,m), & m \in \partial D, \\ h(p,m,s), & (p,m) \in \mathbb{D}^n_{3/4} \times \Delta, \end{cases}$$

where $h(p,m,s)$ is a homotopy between $\alpha_1|_{\mathbb{D}^n_{3/4} \times \Delta}$ and the map $(p,m) \mapsto \gamma(m)$. Such an h exists since $\mathbb{D}^n_{3/4} \times \Delta$ is contractible and X is path connected, which implies any two maps are homotopic. Since the inclusion of Δ into D is a cofibration, and the inclusion of $\mathbb{D}^n_{3/4}$ into $\mathbb{D}^n$ is a cofibration, the inclusion of $\mathbb{D}^n_{3/4} \times \Delta$ into $\mathbb{D}^n \times D$ also is. This implies $\big(\mathbb{D}^n \times D, (\mathbb{D}^n_{3/4} \times \Delta) \cup \partial(\mathbb{D}^n \times D)\big)$ satisfies the homotopy extension property, so there exists a homotopy

$$\mathbb{D}^n \times D \times [0,1] \to X$$

between g_0 and a function $g_1\colon \mathbb{D}^n \times D \to X$, relative to the boundary, so that $g_1 = g_0$ on $\partial\mathbb{D}^n$, ∂D, and $(p,m) \mapsto \gamma(m)$ on $\mathbb{D}^n_{3/4} \times \Delta$. Now, we can extend this to a section on all of M by letting

$$\alpha_2(p,m) := \begin{cases} \alpha_1(p,m), & m \notin D, \\ g_1(p,m), & m \in D. \end{cases}$$

Clearly $\alpha_2 \simeq \alpha_1$ relative to the boundary, $\alpha_2(p,m) = \gamma(m)$ when $|p| \le \frac{3}{4}$, $m \in \Delta$, and for $m \in \widetilde{W}$ we have $\alpha_2(p,m) = \alpha_1(p,m) = \gamma_{p/|p|,i}(m)$ when $|p| \ge \frac{1}{2}$. So, all we need to do now is show that α_2 lifts. Let us again view these as maps from $\mathbb{D}^n \to \Gamma(E \operatorname{rel} A)$, and consider the map $\beta\colon \mathbb{D}^n \to \|\Gamma_\bullet(E \operatorname{rel} A; \mathcal{W})\|$ given by

$$\beta(p) := \begin{cases} (\alpha_2(p), (\Delta,\gamma), 1), & 0 \le |p| \le \frac{1}{2}, \\ \bigl(\alpha_2(p), (\vec{w}_{p/|p|}, (\Delta,\gamma)), ((4|p|-2)\vec{t}_{p/|p|}, 3-4|p|)\bigr), & \frac{1}{2} < |p| \le \frac{3}{4}, \\ (\alpha_2(p), \vec{w}_{p/|p|}, \vec{t}_{p/|p|}), & \frac{3}{4} < |p| \le 1. \end{cases}$$

Clearly β is a lift of α_2, as required. □

Remark 2.7 In particular, the above theorem applies when M has no boundary and $A = \varnothing$, and when M has boundary and $A = \partial M$.

3 *h*-principle

Let M be a d-dimensional manifold and A be a (possibly empty) closed subset of M such that $M \setminus A$ is a manifold without boundary. Suppose $F(M)$ is a space of holonomic solutions with space of formal solutions $F^f(M)$, and ξ_0 is a germ of a holonomic solution on A. We would like to use Theorem 2.6 to show that the map $F(M \operatorname{rel} A) \to F^f(M \operatorname{rel} A)$ from the space of holonomic solutions relative to A to the space of formal solutions relative to A admits a section up to homotopy under the right conditions. We assume there exists some bundle $E \to M$ such that $F^f(M \operatorname{rel} A)$ is the space $\Gamma(E \operatorname{rel} A)$ of sections of E relative to A. Suppose furthermore that the fibers of E are path connected, and let $\mathcal{W}$ be a sufficiently separated collection for E relative to A. In addition, for each $(\Delta,\gamma) \in \mathcal{W}$, suppose γ is the germ of a holonomic solution. Then we can define semisimplicial spaces $F_\bullet(M \operatorname{rel} A; \mathcal{W})$ and $F^f_\bullet(M \operatorname{rel} A; \mathcal{W})$ by letting

$$F_p(M \operatorname{rel} A; \mathcal{W}) := \coprod_{\substack{\text{tuples } ((\Delta_0,\gamma_0),\dots,(\Delta_p,\gamma_p)) \\ \text{of } p+1 \text{ disjoint elements of } \mathcal{W}}} F(M \operatorname{rel} A, \Delta_0, \dots, \Delta_p),$$

$$F^f_p(M \operatorname{rel} A; \mathcal{W}) := \coprod_{\substack{\text{tuples } ((\Delta_0,\gamma_0),\dots,(\Delta_p,\gamma_p)) \\ \text{of } p+1 \text{ disjoint elements of } \mathcal{W}}} F^f(M \operatorname{rel} A, \Delta_0, \dots, \Delta_p),$$

where as before the face maps come from forgetting that the functions agree with the given germs near the given sets. There is a map of semisimplicial spaces

$$F_\bullet(M \operatorname{rel} A; \mathcal{W}) \to F^f_\bullet(M \operatorname{rel} A; \mathcal{W})$$

induced by the maps

$$F(M \text{ rel } A, \Delta_0, \dots, \Delta_p) \to F^f(M \text{ rel } A, \Delta_0, \dots, \Delta_p).$$

Also, there are natural maps $\|F_\bullet(M \text{ rel } A; \mathcal{W})\| \to F(M \text{ rel } A)$ and $\|F^f_\bullet(M \text{ rel } A; \mathcal{W})\| \to F^f(M \text{ rel } A)$ which forget the elements of $\mathcal{W}$.

Theorem 3.1 *If the natural inclusion map*

$$F(M \text{ rel } A, \Delta_0, \dots, \Delta_p) \to F^f(M \text{ rel } A, \Delta_0, \dots, \Delta_p)$$

is a weak equivalence for all finite sets of disjoint elements $\{(\Delta_i, \gamma_i)\}_{i=1}^k \subset \mathcal{W}$, $k \geq 1$, *then the composition map* $\|F_\bullet(M \text{ rel } A; \mathcal{W})\| \to F(M \text{ rel } A) \to F^f(M \text{ rel } A)$ *is a weak homotopy equivalence.*

Proof First, we are given that

$$F(M \text{ rel } A, \Delta_0, \dots, \Delta_p) \to F^f(M \text{ rel } A, \Delta_0, \dots, \Delta_p)$$

is a weak equivalence for all tuples of elements of $\mathcal{W}$, so the map

$$\|F_\bullet(M \text{ rel } A; \mathcal{W})\| \to \|F^f_\bullet(M \text{ rel } A; \mathcal{W})\|$$

is a levelwise weak equivalence and hence a weak equivalence. Also, by Theorem 2.6,

$$\|F^f_\bullet(M \text{ rel } A; \mathcal{W})\| \to F^f(M \text{ rel } A)$$

is a weak equivalence, so we get the following commutative diagram:

$$\begin{array}{ccc} \|F_\bullet(M \text{ rel } A; \mathcal{W})\| & \longrightarrow & F(M \text{ rel } A) \\ \downarrow \simeq & & \downarrow \\ \|F^f_\bullet(M \text{ rel } A; \mathcal{W})\| & \xrightarrow{\simeq} & F^f(M \text{ rel } A) \end{array}$$

which gives the required weak homotopy equivalence. □

Remark 3.2 A more common formulation of such a relative h-principle result is that there is an h-principle relative to any fixed closed set A and one fixed subset Δ, where Δ is from some special collection. Such a result will usually imply the result for many fixed subsets $\Delta_1, \dots, \Delta_k$ in such a collection, since we can take A to be the union of the first $k-1$ such sets, and use Δ_k as our fixed subset Δ.

4 Improved h-principle for contact geometry

Let us briefly recall some basic definitions from contact geometry. A cooriented contact structure on a connected, orientable, $(2n+1)$-dimensional manifold M is a "maximally nonintegrable" hyperplane distribution $\xi = \ker(\alpha)$ for a 1-form α. Maximally nonintegrable means $\alpha \wedge d\alpha^n \neq 0$. Unless otherwise stated we assume all contact structures are cooriented. This naturally induces a reduction of the structure group of M to $U(n) \times 1$, and so an *almost contact structure* is just a reduction of the structure group of

M to $U(n)\times 1$. Equivalently, an almost contact structure is a triple (ξ, J, R), where ξ is a hyperplane distribution, J is a complex structure on ξ, and R is a trivial sub-line-bundle of TM such that $\xi\oplus R = TM$. We let $\mathrm{Cont}(M)$ denote the space of contact structures on M and $\mathrm{AlmCont}(M)$ denote the space of almost contact structures on M.

Next, we recall the notion of an overtwisted contact structure. An *overtwisted disk* in a manifold M is a pair (Δ,γ), where $\Delta\subset M$ is an embedded $2n$-dimensional disk and γ is a certain model germ of a contact structure on Δ. Then a contact manifold (M,ξ) is said to be overtwisted if there exists an embedding of an overtwisted disk (Δ,γ) such that the contact germ γ agrees with ξ on some neighborhood of Δ (ie the embedding is a contact embedding). In this case we say that Δ is overtwisted for ξ. The details of this definition in any dimension can be found in [2, Definition 3.6]. Let $\mathrm{Cont}^{\mathrm{OT}}(M)$ denote the space of overtwisted contact structures on M. It has been shown by [2, Theorem 1.2] that if M is a closed $2n+1$-dimensional manifold, A is a closed subset of M such that $M\setminus A$ is connected, (Δ,γ) is an overtwisted disk in $M\setminus A$, and ξ_0 is an almost contact structure on M that is a genuine contact structure on a neighborhood of A, then the map

$$\mathrm{Cont}^{\mathrm{OT}}(M \text{ rel } A, \Delta)\to \mathrm{AlmCont}(M \text{ rel } A, \Delta)$$

is a weak equivalence. Here $\mathrm{Cont}^{\mathrm{OT}}(M \text{ rel } A, \Delta)$ is the space of contact structures on M that agree with ξ_0 in a neighborhood of A and are overtwisted with disk Δ, and $\mathrm{AlmCont}(M \text{ rel } A, \Delta)$ is the corresponding space of almost contact structures. Next, let $\mathcal{W}$ be the collection of all overtwisted disks in $M\setminus A$, and let $\mathrm{Cont}^{\mathrm{OT}}_\bullet(M \text{ rel } A;\mathcal{W})$ and $\mathrm{AlmCont}_\bullet(M \text{ rel } A;\mathcal{W})$ be the semisimplicial spaces defined by

$$\mathrm{Cont}^{\mathrm{OT}}_p(M \text{ rel } A;\mathcal{W}) := \coprod_{\substack{\text{tuples } ((\Delta_0,\gamma_0),\dots,(\Delta_p,\gamma_p))\\ \text{of } p+1 \text{ disjoint elements of } \mathcal{W}}} \mathrm{Cont}^{\mathrm{OT}}(M \text{ rel } A,\Delta_0,\dots,\Delta_p),$$

$$\mathrm{AlmCont}_p(M \text{ rel } A;\mathcal{W}) := \coprod_{\substack{\text{tuples } ((\Delta_0,\gamma_0),\dots,(\Delta_p,\gamma_p))\\ \text{of } p+1 \text{ disjoint elements of } \mathcal{W}}} \mathrm{AlmCont}(M \text{ rel } A,\Delta_0,\dots,\Delta_p),$$

where as usual the face maps just forget the overtwisted disks. From this, we would like to prove the following theorem:

Theorem 4.1 *Let M be a $(2n+1)$-dimensional manifold and $A\subset M$ be a closed subset of M such that $M\setminus A$ is connected and without boundary. Then the composition*

$$\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M \text{ rel } A;\mathcal{W})\|\to \mathrm{Cont}^{\mathrm{OT}}(M \text{ rel } A)\to \mathrm{AlmCont}(M \text{ rel } A)$$

is a weak homotopy equivalence.

Proof It is known (see [8, page 191]) that $\mathrm{AlmCont}(M)$ is naturally the section space of a certain fiber bundle $\pi\colon E\to M$ with connected fiber $\mathrm{SO}(2n+1)/U(n)$, which comes from viewing $\mathrm{AlmCont}(M)$ as the space of reductions of the structure group of M to $U(n)\times 1$, so $\mathrm{AlmCont}(M \text{ rel } A)$ is naturally $\Gamma(E \text{ rel } A)$. Also, if we have some collection of disjoint overtwisted disks $\Delta_1,\dots,\Delta_k$, we can let $A' = A\cup\Delta_1\cup\dots\cup\Delta_{k-1}$ and Δ_k be the overtwisted disk, so the h-principle given in [2] implies an

h-principle of the form required in Theorem 3.1. So, we just need to show that the collection of all overtwisted disks on $M \setminus A$ is sufficiently separated relative to A. Clearly such an embedded disk is closed, contractible and compact. Also, by finding a sufficiently small regular neighborhood, it is clear that any embedded $2n$-dimensional disk is a neighborhood deformation retract of a $(2n+1)$-dimensional ball in $M \setminus A$ containing it, so the inclusion is a cofibration and condition (1) of Definition 2.2 is satisfied. Finally, it is clear that a finite collection of embedded overtwisted disks in $M \setminus A$ can't cover $M \setminus A$, so given some finite set of overtwisted disks in $M \setminus A$ there will always be some point $m \in M \setminus A$ that is not in any of them, and since an overtwisted disk is just some embedded disk with a local germ, we can always introduce a new overtwisted disk at the point m that doesn't intersect the rest of the overtwisted disks. Again, we can choose a sufficiently small regular neighborhood of the disk that doesn't intersect the given overtwisted disks or the set A, so that the inclusion is a cofibration and condition (2) of Definition 2.2 is satisfied. □

Remark 4.2 This argument also shows that the map $\mathrm{Cont}(M \text{ rel } A) \to \mathrm{AlmCont}(M \text{ rel } A)$ admits a section up to homotopy, but this section factors through the overtwisted contact structures.

Remark 4.3 In particular, this holds if M is closed and connected and A is empty.

From Theorem 4.1 we get as an immediate consequence that $\pi_k \,\mathrm{AlmCont}(M \text{ rel } A)$ is isomorphic to a subgroup of $\pi_k \,\mathrm{Cont}^{\mathrm{OT}}(M \text{ rel } A)$ for all k. For the remainder of this section, suppose M is a closed manifold and $A = \varnothing$. Then our result is an improvement on the current overtwisted group $\mathrm{OT}_k(M)$ (see [4, Proposition A.2]), which gives an isomorphism between $\pi_k \,\mathrm{AlmCont}(M)$ and a subgroup of $\pi_k \,\mathrm{Cont}^{\mathrm{OT}}(M)$ when $1 \le k \le 2n$. In fact, when $1 \le k \le 2n$ the image of $\pi_k \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M; \mathcal{W})\|$ in $\pi_k \,\mathrm{Cont}^{\mathrm{OT}}(M)$ is $\mathrm{OT}_k(M)$:

Theorem 4.4 *The overtwisted group* $\mathrm{OT}_k(M)$ *is the image of* $\pi_k \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M; \mathcal{W})\|$ *induced by the natural forgetful map* $\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M; \mathcal{W})\| \to \mathrm{Cont}^{\mathrm{OT}}(M)$ *when* $1 \le k \le 2n$.

Before we can prove this, we need to define some intermediary spaces that will help us understand the relationship between $\pi_k \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M; \mathcal{W})\|$ and $\mathrm{OT}_k(M)$. Recall that an element of $\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M; \mathcal{W})\|$ is a contact structure, along with a list of disks and weights, such that each disk is overtwisted for the contact structure as long as its weight is nonzero. Since we defined this for an arbitrary section space and an arbitrary sufficiently separated collection, we did not make use of any topology on the space of disks. So, in $\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M; \mathcal{W})\|$ the fixed disks are not allowed to move through M, and the only way to change the disks is to introduce a new overtwisted disk somewhere, or delete a disk by letting its weight go to zero. However, the space of overtwisted disks does have a topology coming from the space of embeddings of $\mathbb{D}^{2n}$ into M. We can use this to define new semisimplicial spaces that are more clearly related to $\mathrm{OT}_k(M)$.

Remark 4.5 Strictly speaking, overtwisted disks are only piecewise smooth, so instead of embeddings of the standard disk into M we want to take a specific piecewise structure coming from the model overtwisted disk (see [2, Definition 3.6]) and consider the space of embeddings of this into M that preserve the piecewise smooth structure. However, none of our arguments depend on this distinction, so by abuse of notation we just denote this space as $\mathrm{Emb}(\mathbb{D}^{2n}, M)$.

Definition 4.6 Let $\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C$ be the semisimplicial space defined by

$$\mathrm{Cont}^{\mathrm{OT}}_p(M;\mathcal{W})^C \subset \mathrm{Cont}^{\mathrm{OT}}(M) \times \mathrm{Emb}(\mathbb{D}^{2n}, M)^{p+1},$$

the set of all $(\xi, \Delta_0, \dots, \Delta_p)$ such that each Δ_i is an overtwisted disk for ξ, and Δ_i, Δ_j are disjoint when $i \neq j$. The face maps d_i are given by forgetting the i^{th} disk.

Similarly, let $\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C$ be defined in the same way except with $\mathrm{AlmCont}(M)$ instead of $\mathrm{Cont}^{\mathrm{OT}}(M)$. The geometric realizations of these differ from $\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\|$ and $\|\mathrm{AlmCont}_\bullet(M;\mathcal{W})\|$ since continuous maps into $\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C\|$ and $\|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\|$ can also deform disks along families inside of M. For example, if $\alpha\colon S^k \to \mathrm{Cont}^{\mathrm{OT}}(M)$ is a family of contact structures, and $\Delta\colon S^k \to \mathrm{Emb}(\mathbb{D}^{2n}, M)$ is a certificate of overtwistedness for α, then $(\alpha, \Delta, 1)$ is naturally a continuous map $S^k \to \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C\|$. Furthermore, since constant embeddings are still continuous embeddings, there are natural maps

$$\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\| \to \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C\| \quad \text{and} \quad \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})\| \to \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\|$$

given by viewing the fixed disks as constant embeddings. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\| & \longrightarrow & \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C\| & \longrightarrow & \mathrm{Cont}^{\mathrm{OT}}(M) \\ \downarrow & & \downarrow & & \downarrow \\ \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})\| & \longrightarrow & \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\| & \longrightarrow & \mathrm{AlmCont}(M) \end{array}$$

We already know the map $\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\| \to \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})\|$ is a weak equivalence by the h-principle given in [2]. Also, we have the following lemma, which is a direct consequence of [2].

Lemma 4.7 *The map*

$$\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C\| \to \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\|$$

induced by $\mathrm{Cont}^{\mathrm{OT}}(M) \to \mathrm{AlmCont}(M)$ *is a weak equivalence.*

Proof First, $\mathrm{Cont}^{\mathrm{OT}}_p(M;\mathcal{W})^C \to \mathrm{AlmCont}_p(M;\mathcal{W})^C$ is a weak equivalence as a direct consequence of Theorem 1.6 of [2]. Indeed, suppose we have the following diagram:

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\partial\alpha} & \mathrm{Cont}^{\mathrm{OT}}_p(M;\mathcal{W})^C \\ \downarrow & & \downarrow \\ \mathbb{D}^k & \xrightarrow{\alpha} & \mathrm{AlmCont}_p(M;\mathcal{W})^C \end{array}$$

where $\alpha(t) = (\xi(t), \Delta_0(t), \dots, \Delta_p(t))$. First, we can homotope α relative to the boundary by extending the boundary to an annulus, so that α is genuine on a neighborhood of the boundary. Then we can consider $V = M \times \mathbb{D}^k$, so that ξ can be viewed as a leafwise almost contact structure on V. If we let $A = S^{k-1} \times M \subset V$, $\xi_0 = \xi|_A$, and $h_i = \Delta_i$ for $0 \le i \le p$, we have that

$$\xi \in \mathrm{AlmCont}(V; A, \xi_0, h_0, \dots, h_p),$$

where $\mathrm{AlmCont}(V; A, \xi_0, h_0, \dots, h_p)$ is the space of leafwise almost contact structures that agree with ξ_0 near A, with overtwisted basis $\{h_i\}_{i=0}^p$ (see [2, Theorem 1.6 and definitions immediately preceding]). Then ξ is a representative of an element in $\pi_0 \mathrm{AlmCont}(V; A, \xi_0, h_0, \dots, h_p)$. But, if $\mathrm{Cont}(V; A, \xi_0, h_0, \dots, h_p)$ is the space of leafwise contact structures that agree with ξ_0 near A, with overtwisted basis $\{h_i\}_{i=0}^p$, then by [2, Theorem 1.6] we have that

$$\pi_0 \mathrm{Cont}(V; A, \xi_0, h_0, \dots, h_p) \to \pi_0 \mathrm{AlmCont}(V; A, \xi_0, h_0, \dots, h_p)$$

is an isomorphism, so there is a path from ξ to $\tilde{\xi}$ in $\mathrm{AlmCont}(V; A, \xi_0, h_0, \dots, h_p)$ for some $\tilde{\xi} \in \mathrm{Cont}(V; A, \xi_0, h_0, \dots, h_p)$. But, a path in this space is a homotopy of ξ relative to the boundary and relative to the families of overtwisted disks $\Delta_0, \dots, \Delta_p$. Clearly such a homotopy is a homotopy of $\alpha\colon \mathbb{D}^k \to \mathrm{AlmCont}_p(M; \mathcal{W})^C$ relative to the boundary to a map $\beta = (\tilde{\xi}, \Delta_0, \dots, \Delta_p)$, which has image in $\mathrm{Cont}_p^{\mathrm{OT}}(M; \mathcal{W})^C$. So indeed $\mathrm{Cont}_p^{\mathrm{OT}}(M; \mathcal{W})^C \to \mathrm{AlmCont}_p(M; \mathcal{W})^C$ is a weak equivalence and so $\|\mathrm{Cont}_\bullet^{\mathrm{OT}}(M; \mathcal{W})^C\| \to \|\mathrm{AlmCont}_\bullet(M; \mathcal{W})^C\|$ is also a weak equivalence. □

Furthermore, we have the following, which relates $\|\mathrm{AlmCont}_\bullet(M; \mathcal{W})^C\|$ to $\mathrm{AlmCont}(M)$:

Lemma 4.8 *The map* $\|\mathrm{AlmCont}_\bullet(M; \mathcal{W})^C\| \to \mathrm{AlmCont}(M)$ *is a weak equivalence.*

Proof The proof is similar to the proof of Theorem 2.6, so we will omit some technical details that were included there. Let $\alpha\colon \mathbb{D}^k \to \mathrm{AlmCont}(M)$ be a continuous map such that we have a commutative diagram

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\partial\alpha} & \|\mathrm{AlmCont}_\bullet(M; \mathcal{W})^C\| \\ \downarrow & & \downarrow \\ \mathbb{D}^k & \xrightarrow{\alpha} & \mathrm{AlmCont}(M) \end{array}$$

where by abuse of notation $\partial\alpha$ is the map $S^{k-1} \to \|\mathrm{AlmCont}_\bullet(M; \mathcal{W})^C\|$, $p \mapsto (\alpha(p), \vec{w}_p, \vec{t}_p)$, where $\vec{w}_p = (w_1(p), \dots w_\ell(p))$ is a finite ordered set of overtwisted disks for $\alpha(p)$ and $\vec{t}_p = (t_1(p), \dots, t_\ell(p))$ are their corresponding weights. We can assume all of these weights appearing are nonzero.

There is only one part of the proof of Theorem 2.6 that does not go through immediately, which is finding a disk that is disjoint from all the disks $w_i(p)$ for all p, i. The problem is that since the embedded disks are no longer locally constant but rather can vary from point to point, we have S^{k-1} families of disks instead of finitely many, so it is possible that they cover all of M. However, we can get around this as follows. First, we can make a buffer away from the boundary by replacing α with a radial compression to the disk of radius $\frac{1}{2}$, which is homotopic to α relative to the boundary. Again by abuse of notation we will let α denote this new map. Let A be the annulus of radius $\frac{1}{2}$, so now the map α lifts to $\|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\|$ in A. Let $S^{k-1}_{1/2}$ be the sphere of radius $\frac{1}{2}$. By arguing as in Section 6.2 of [9], we can assume that for each $p \in S^{k-1}_{1/2}$ there is a contractible open neighborhood V_p of $p \in \mathbb{D}^k$ such that there is some $m \in M$ so that m is not contained in any $w_i(q)$ for any $q \in V_p \cap A$ and $1 \le i \le \ell(q)$. Furthermore, we can let $U_p \subset V_p$ be a slightly smaller contractible open neighorhood of p so that the closure of U_p, $\mathrm{cl}(U_p)$, is contractible and contained in V_p. These can be chosen so that the inclusion of $\mathrm{cl}(U_p)$ into V_p is a closed cofibration, for example by choosing them as small balls, so we assume that this is the case. Since spheres are compact, we can find a finite subcover by these smaller neighborhoods, $U_1, \dots, U_j$, which also gives us a cover by the larger neighborhoods $V_1, \dots, V_j$. By construction the disks $w_i(q)$ for $q \in V_a \cap A$ don't cover M for any given $1 \le a \le j$, so in particular we can find an embedded disk $\Delta_1 \subset M$ and a regular neighborhood D_1 of Δ_1 in M, such that D_1 is disjoint from all such disks $w_i(q)$, $q \in V_1 \cap A$. Finally, we can pick some $V_0 \subset \mathrm{int}(\mathbb{D}^k_{1/2})$ and some slightly smaller U_0 so that $U_1, \dots, U_j, U_0, A$ cover $\mathbb{D}^k$.

With all of this set up, we can now do the following. Since $\mathrm{cl}(U_1) \times \Delta_1$ is contractible, if we restrict α to this we can homotope it to agree with the overtwisted germ that comes with Δ_1. Also, we can use the homotopy extension property to extend this homotopy to one on $V_1 \times D_1$, such that on ∂D_1, ∂V_1 the homotopy is just α. Then we can extend the homotopy by α to all of $\mathbb{D}^k$, M so that we have a new map $\alpha_1 \colon \mathbb{D}^k \to \mathrm{AlmCont}(M)$ that is homotopic to α, agrees with α away from $V_1 \times D_1$, and satisfies that Δ_1 is overtwisted for $\alpha_1(p)$ for all $p \in U_1$. Also, $w_i(q)$ is still overtwisted for $\alpha_1(q)$ for all $q \in A$, since in V_1, Δ_1 is away from all of the $w_i(q)$, and outside of V_1, $\alpha_1 = \alpha$. We can repeat this on U_2, now being careful to choose Δ_2, D_2 so that D_2 is disjoint from Δ_1 as well as $w_i(q)$ for $q \in V_2 \cap A$. Then we can find α_2 homotopic to α_1 so that $\alpha_2 = \alpha_1$ outside of $V_2 \times D_2$ and Δ_2 is overtwisted for $\alpha_2(p)$ for all $p \in \mathrm{cl}(U_2)$. So, by the same reasoning Δ_1 is still overtwisted for $\alpha_2(p)$ for $p \in U_1$ and $w_i(q)$ is still overtwisted for $\alpha_2(q)$ for all $q \in A$. Repeat this for $U_3, \dots, U_j$, and get α_j, which still has the same overtwisted disks in A as well as Δ_a is overtwisted for α_j on U_a. Finally, since V_0 is disjoint from the annulus A by definition, on U_0 we just need to find an overtwisted disk Δ_0 with regular neighborhood D_0 disjoint from $\Delta_1, \dots, \Delta_j$, which we can always do. Then do the same homotopy trick as before to get α_0 that agrees with α_j outside of $V_0 \times D_0$ and has overtwisted disk Δ_0 on $\mathrm{cl}(U_0)$. Finally, let $s_0, \dots, s_j, s_A$ be a partition of unity subordinate to the cover $U_0, U_1, \dots, U_j, A$. Then we have a map $\beta \colon \mathbb{D}^k \to \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\|$ given by

$$p \mapsto \big(\alpha_0(p), \Delta_0, \Delta_1, \dots, \Delta_j, \vec{w}_{p/|p|}, s_0(p), s_1(p), \dots, s_j(p), s_A(p)\vec{t}_{p/|p|}\big).$$

By construction the specified disk is an overtwisted disk at p precisely when its weight is nonzero, all of the weights add up to one, and this is clearly a lift of α_0 which is homotopic to α relative to the boundary. □

Using this lemma, we can now prove Theorem 4.4; $\mathrm{OT}_k(M)$ is the image of $\pi_k\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\|$ induced by the natural forgetful map $\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\| \to \mathrm{Cont}^{\mathrm{OT}}(M)$ when $1 \le k \le 2n$.

Proof Recall the commutative diagram relating the different semisimplicial spaces:

$$\begin{array}{ccccc} \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\| & \longrightarrow & \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C\| & \longrightarrow & \mathrm{Cont}^{\mathrm{OT}}(M) \\ \downarrow & & \downarrow & & \downarrow \\ \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})\| & \longrightarrow & \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\| & \longrightarrow & \mathrm{AlmCont}(M) \end{array}$$

First, we will show that $\mathrm{OT}_k(M)$ is a subgroup of the image of $\pi_k\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\|$. By abuse of notation we will use specific representatives of elements of homotopy groups when we mean their homotopy classes, so we are really working up to homotopy relative to the basepoint. Let $\alpha\colon S^k \to \mathrm{Cont}^{\mathrm{OT}}(M)$ be an element of $\mathrm{OT}_k(M)$ and $\Delta\colon S^k \to \mathrm{Emb}(\mathbb{D}^{2n}, M)$ be a certificate of overtwistedness for α. Then $(\alpha, \Delta, 1) \in \pi_k\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C\|$ maps to α. However, we know that the maps

$$\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\| \to \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})\| \quad \text{and} \quad \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C\| \to \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\|$$

in the above diagram are weak equivalences. Furthermore, by the previous lemma

$$\|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\| \to \mathrm{AlmCont}(M)$$

is a weak equivalence, and since we know that $\|\mathrm{AlmCont}_\bullet(M;\mathcal{W})\| \to \mathrm{AlmCont}(M)$ is also a weak equivalence by Theorem 2.6 we have that $\|\mathrm{AlmCont}_\bullet(M;\mathcal{W})\| \to \|\mathrm{AlmCont}_\bullet(M;\mathcal{W})^C\|$ is a weak equivalence. Combining these equivalences with the previous commutative diagram, we have that $\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\| \to \|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})^C\|$ is a weak equivalence, and so there exists some $\beta \in \pi_k\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\|$ such that $\beta \mapsto (\alpha, \Delta, 1)$ and hence maps to α. So indeed, $\mathrm{OT}_k(M)$ is a subgroup of the image of $\pi_k\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\|$. However, we know that the isomorphisms

$$\mathrm{OT}_k(M) \to \pi_k\,\mathrm{AlmCont}(M) \quad \text{and} \quad \pi_k\|\mathrm{Cont}^{\mathrm{OT}}_\bullet(M;\mathcal{W})\| \to \pi_k\,\mathrm{AlmCont}(M)$$

are both induced by the natural inclusion $\mathrm{Cont}^{\mathrm{OT}}(M) \to \mathrm{AlmCont}(M)$, so we have a group isomorphism that remains an isomorphism when restricted to a subgroup. This is only possible if the subgroup is the whole group, so indeed $\mathrm{OT}_k(M)$ is this image. □

5 Infinite cyclic subgroups in the homotopy groups of the contactomorphism group

We can now use Theorem 4.1 to generalize the results from [8]. Let (M, ξ_{OT}) be a compact, connected, cooriented, overtwisted contact manifold of dimension $2n+1$ with (possibly empty) boundary, and let $\mathcal{C}_0(M, \xi_{\mathrm{OT}} \operatorname{rel} \partial M) \subset \mathrm{Diff}_0(M \operatorname{rel} \partial M)$ be the group of contactomorphisms of (M, ξ_{OT}) relative to the boundary, ie all diffeomorphisms of M that are isotopic to the identity, agree with the identity near the boundary, and preserve the contact structure ξ_{OT}. Then from [10, Lemma 1.1] we have a fiber sequence

$$\mathcal{C}_0(M, \xi_{\mathrm{OT}} \operatorname{rel} \partial M) \to \mathrm{Diff}_0(M \operatorname{rel} \partial M) \to \mathrm{Cont}(M \operatorname{rel} \partial M),$$

where $\mathrm{Diff}_0(M \text{ rel } \partial M) \to \mathrm{Cont}(M \text{ rel } \partial M)$ is given by $f \mapsto f^\star \xi_{\mathrm{OT}}$, which induces a long exact sequence of homotopy groups.

Remark 5.1 In the literature the fibration is given by pushforward not pullback, ie the map $f \mapsto f_\star \xi_{\mathrm{OT}}$. While this may be more natural geometrically, since we are using diffeomorphisms pullback is just pushforward by the inverse, so our map is still a fibration. Also, we will see later that it is convenient to factor this map through something more general, where pushforward is no longer well defined but pullback is.

We would like to find infinite cyclic subgroups inside $\pi_k(\mathcal{C}_0(M, \xi_{\mathrm{OT}} \text{ rel } \partial M), \mathrm{id})$, ie we want to find nonzero elements of the rational homotopy groups of $\mathcal{C}_0(M, \xi_{\mathrm{OT}} \text{ rel } \partial M)$. To do this, we will prove a few lemmas. Let $\mathrm{Bun}_\partial(TM)$ denote the space of all pairs $(f, \delta f)$, where $f : M \to M$ is a smooth map which agrees with the identity map near the boundary, and $\delta f : TM \to f^\star TM$ is some vector bundle map over M that agrees with the identity near the boundary and is a fiberwise isomorphism, ie the space of bundle isomorphisms of TM.

Lemma 5.2 *The map*

$$\mathrm{Diff}_0(M \text{ rel } \partial M) \to \mathrm{Cont}(M \text{ rel } \partial M) \to \mathrm{AlmCont}(M \text{ rel } \partial M)$$

factors through the space of bundle isomorphisms of TM as

$$\begin{array}{ccc} \mathrm{Diff}_0(M \text{ rel } \partial M) & \xrightarrow{f \mapsto f^\star \xi_{\mathrm{OT}}} & \mathrm{Cont}(M \text{ rel } \partial M) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_\partial(TM) & \xrightarrow{(f,\delta f) \mapsto (f,\delta f)^\star \xi_{\mathrm{OT}}} & \mathrm{AlmCont}(M \text{ rel } \partial M) \end{array}$$

where the left vertical map is the derivative $f \mapsto (f, df)$, and $(f, \delta f)^* \xi_{\mathrm{OT}}$ is the almost contact structure obtained by realizing $f^* \xi_{\mathrm{OT}} \subset f^* TM$ as a subbundle of TM via the isomorphism $\delta f : TM \to f^* TM$.

Proof It is clear the diagram commutes as long as the bottom map is well defined, so to verify this, we just need to ensure that if $(f, \delta f) \in \mathrm{Bun}_\partial(TM)$ then $(f, \delta f)^\star \xi_{\mathrm{OT}}$ is still an almost contact structure on M, which agrees with ξ_{OT} near the boundary. First, $f^\star \xi_{\mathrm{OT}}$ and $f^\star R$ are always a hyperplane bundle and line bundle on M respectively, for any smooth function $f : M \to M$. Also, the Whitney sum decomposition and the almost complex structure are naturally preserved by pullback. Since both f and δf agree with the identity near the boundary, the pullback agrees with ξ_{OT} near the boundary, so the only possible obstruction is that $f^\star \xi_{\mathrm{OT}}$ and $f^\star R$ are not necessarily isomorphic to subbundles of TM for an arbitrary smooth function. However, we are given a fiberwise isomorphism $\delta f : TM \to f^\star TM$ which allows us to realize these pullbacks as subbundles of the tangent bundle, as required. □

Next, we need some results about how the diffeomorphism group of a disk glued into M maps to $\mathrm{Cont}(M \text{ rel } \partial M)$. First, we can use a recent result in [6, Theorem 1.4] which says that the inclusion map

$$\mathrm{Diff}_0(\mathbb{D}^{2n+1} \text{ rel } \partial \mathbb{D}^{2n+1}) \to \mathrm{Diff}_0(M \text{ rel } \partial M)$$

is injective on rational homotopy in degrees k in the rational concordance range $k \le \phi^{\mathbb{Q}}(\mathbb{D}^{2n}) - 1$, $k \neq 0$. So, if we can find something nontrivial in $\pi_k \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) \otimes \mathbb{Q}$ that maps to zero in $\pi_k \operatorname{Cont}(M \operatorname{rel} \partial M) \otimes \mathbb{Q}$ in this range, then by the injectivity result we will have a nontrivial element of $\pi_k \mathrm{Diff}_0(M \operatorname{rel} \partial M) \otimes \mathbb{Q}$ that maps to zero in $\pi_k \operatorname{Cont}(M \operatorname{rel} \partial M) \otimes \mathbb{Q}$, which will give us a nontrivial element of $\pi_k \mathcal{C}_0(M, \xi_{\mathrm{OT}} \operatorname{rel} \partial M) \otimes \mathbb{Q}$ by exactness. Let $\mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) \to \operatorname{Cont}(M \operatorname{rel} \partial M)$ be given by composing through $\mathrm{Diff}_0(M \operatorname{rel} \partial M)$.

Lemma 5.3 *Let $\alpha \in \pi_k \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1})$ be such that $\alpha \mapsto 0$ under the map*

$$\pi_k \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) \to \pi_k \operatorname{Cont}(M \operatorname{rel} \partial M) \to \pi_k \operatorname{AlmCont}(M \operatorname{rel} \partial M).$$

Then we also have that $\alpha \mapsto 0$ under the map

$$\pi_k \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) \to \pi_k \operatorname{Cont}(M \operatorname{rel} \partial M).$$

Proof Let Δ be an overtwisted disk for ξ_{OT}, and let $D \simeq \mathbb{D}^{2n+1}$ be an embedded disk in the interior of M disjoint from a neighborhood of Δ. Then since we get a diffeomorphism of M by extending a diffeomorphism of the disk by the identity, we have that Δ is overtwisted for $f^\star \xi_{\mathrm{OT}}$ for all $f \in \mathrm{Diff}_0(D \operatorname{rel} \partial D)$. Then we have that the map $\mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) \to \operatorname{Cont}(M \operatorname{rel} \partial M)$ factors through $\|\operatorname{Cont}^{\mathrm{OT}}_\bullet(M \operatorname{rel} \partial M; \mathcal{W})\|$, where $f \mapsto (f^\star \xi_{\mathrm{OT}}, \Delta, 1)$, and of course the map

$$\|\operatorname{Cont}^{\mathrm{OT}}_\bullet(M \operatorname{rel} \partial M; \mathcal{W})\| \to \operatorname{Cont}(M \operatorname{rel} \partial M)$$

just comes from forgetting the overtwisted disks. So, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) & \longrightarrow & \operatorname{Cont}(M \operatorname{rel} \partial M) \\ \downarrow & \nearrow & \downarrow \\ \|\operatorname{Cont}^{\mathrm{OT}}_\bullet(M \operatorname{rel} \partial M; \mathcal{W})\| & \longrightarrow & \operatorname{AlmCont}(M \operatorname{rel} \partial M) \end{array}$$

But, as we saw in a previous section the map $\|\operatorname{Cont}^{\mathrm{OT}}_\bullet(M \operatorname{rel} \partial M; \mathcal{W})\| \to \operatorname{AlmCont}(M \operatorname{rel} \partial M)$ is a weak equivalence, so indeed anything mapping to zero in $\pi_k \operatorname{AlmCont}(M \operatorname{rel} \partial M)$ must map to zero in $\pi_k \|\operatorname{Cont}^{\mathrm{OT}}_\bullet(M \operatorname{rel} \partial M; \mathcal{W})\|$ and thus in $\pi_k \operatorname{Cont}(M \operatorname{rel} \partial M)$ as well. □

Now, we have enough to prove the following lemma:

Lemma 5.4 *The map $\mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) \to \operatorname{Cont}(M \operatorname{rel} \partial M)$ is trivial on rational homotopy groups.*

Proof From Lemma 5.2, we can factor the map through bundle isomorphisms, as follows:

$$\begin{array}{ccccc} \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial \mathbb{D}^{2n+1}) & \longrightarrow & \mathrm{Diff}_0(M \operatorname{rel} \partial M) & \longrightarrow & \operatorname{Cont}(M \operatorname{rel} \partial M) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Bun}_\partial(\mathbb{D}^{2n+1}) & \longrightarrow & \operatorname{Bun}_\partial(TM) & \longrightarrow & \operatorname{AlmCont}(M \operatorname{rel} \partial M) \end{array}$$

where the left vertical map is the derivative

$$d\colon \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial\mathbb{D}^{2n+1}) \to \Omega^{2n+1}\,\mathrm{SO}(2n+1).$$

But, this map is zero on rational homotopy groups, see for example [5, page 9]. So, the composition through the bottom of the diagram to $\mathrm{AlmCont}(M \operatorname{rel} \partial M)$ is zero on rational homotopy. So, since the diagram is commutative, we can go through the top, and use Lemma 5.3 to conclude $\mathrm{Diff}_\partial(\mathbb{D}^{2n+1} \operatorname{rel} \partial\mathbb{D}^{2n+1}) \to \mathrm{Diff}_0(M \operatorname{rel} \partial M) \to \mathrm{Cont}(M \operatorname{rel} \partial M)$ is trivial on rational homotopy groups, as required. □

Corollary 5.5 *If (M, ξ_{OT}) is a compact, cooriented, overtwisted contact manifold of dimension $2n+1$, then $\pi_k \mathcal{C}_0(M, \xi_{\mathrm{OT}} \operatorname{rel} \partial M)$ contains an infinite cyclic subgroup whenever*

$$\pi_k \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial\mathbb{D}^{2n+1}) \otimes \mathbb{Q} \neq 0, \quad \text{for } k \leq \phi^{\mathbb{Q}}(\mathbb{D}^{2n}) - 1,\ k \neq 0.$$

Proof By the injectivity result of [6], every nonzero element of

$$\pi_k \mathrm{Diff}_0(\mathbb{D}^{2n+1} \operatorname{rel} \partial\mathbb{D}^{2n+1}) \otimes \mathbb{Q}$$

has nonzero image in $\pi_k \mathrm{Diff}_0(M \operatorname{rel} \partial M) \otimes \mathbb{Q}$, which is then mapped to zero in $\pi_k \mathrm{Cont}^{\mathrm{OT}}(M \operatorname{rel} \partial M) \otimes \mathbb{Q}$ by Lemma 5.4. By exactness, there must be a nonzero element in $\pi_k \mathcal{C}_0(M, \xi_{\mathrm{OT}} \operatorname{rel} \partial M) \otimes \mathbb{Q}$. □

6 Further applications

Another immediate application of Theorem 3.1 appears in Engel geometry, where it was recently shown that there is a notion of overtwistedness parallel to contact overtwistedness, and one still gets an h-principle with a fixed overtwisted disk; see [16, Theorem 1.1 and Corollary 1.2]. All of the relevant properties of contact overtwistedness also apply to Engel overtwistedness; the collection of all overtwisted Engel disks is still sufficiently separated, and Theorem 1.1 of [16] gives a strong enough relative h-principle that we can get an h-principle for any number of fixed overtwisted disks. We can apply Theorem 3.1 to conclude that for a 4-manifold M, $\mathcal{E}(M) \to \mathcal{E}^f(M)$ admits a section up to homotopy, where $\mathcal{E}(M), \mathcal{E}^f(M)$ are the spaces of Engel and formal Engel structures on M, respectively. This shows that $\pi_k \mathcal{E}^f(M)$ is a subgroup of $\pi_k \mathcal{E}(M)$ for all k via the map from the semisimplicial realization. Furthermore, using the foliated version of this h-principle from [16, Theorem 6.25], this subgroup agrees with the subgroup found using a certificate of overtwistedness in degrees $k \leq 3$ in [16], using the same arguments used in Theorem 4.4. It was already known that the map $\mathcal{E}(M) \to \mathcal{E}^f(M)$ is surjective on homotopy groups by [3], and one of the main results of [4] shows $\pi_k \mathcal{E}^f(M)$ is a subgroup of $\pi_k \mathcal{E}(M)$ for all k. The natural question this raises is whether the subgroup of $\pi_k \mathcal{E}(M)$ found in [4] using loose Engel structures is the same as the subgroup one gets using semisimplicial realization via overtwisted disks. Understanding this may help to understand how loose and overtwisted Engel structures interact, which is currently poorly understood.

References

[1] **D Bennequin**, *Entrelacements et équations de Pfaff*, from "Third Schnepfenried Geometry Conference, I", Astérisque 107–108, Soc. Math. France, Paris (1983) 87–161 MR Zbl

[2] **M S Borman**, **Y Eliashberg**, **E Murphy**, *Existence and classification of overtwisted contact structures in all dimensions*, Acta Math. 215 (2015) 281–361 MR Zbl

[3] **R Casals**, **J L Pérez**, **Á del Pino**, **F Presas**, *Existence h-principle for Engel structures*, Invent. Math. 210 (2017) 417–451 MR Zbl

[4] **R Casals**, **Á del Pino**, **F Presas**, *Loose Engel structures*, Compos. Math. 156 (2020) 412–434 MR Zbl

[5] **D Crowley**, **T Schick**, **W Steimle**, *The derivative map for diffeomorphism of disks: an example*, Geom. Topol. 27 (2023) 3699–3713 MR Zbl

[6] **J Ebert**, *Diffeomorphisms of odd-dimensional discs, glued into a manifold*, Algebr. Geom. Topol. 23 (2023) 2329–2345 MR Zbl

[7] **J Ebert**, **O Randal-Williams**, *Semisimplicial spaces*, Algebr. Geom. Topol. 19 (2019) 2099–2150 MR Zbl

[8] **E Fernández**, **F Gironella**, *A remark on the contactomorphism group of overtwisted contact spheres*, C. R. Math. Acad. Sci. Paris 358 (2020) 189–196 MR Zbl

[9] **S Galatius**, **O Randal-Williams**, *Stable moduli spaces of high-dimensional manifolds*, Acta Math. 212 (2014) 257–377 MR Zbl

[10] **E Giroux**, **P Massot**, *On the contact mapping class group of Legendrian circle bundles*, Compos. Math. 153 (2017) 294–312 MR Zbl

[11] **T Goodwillie**, **M Krannich**, **A Kupers**, *Stability of concordance embeddings*, Proc. Roy. Soc. Edinburgh Sect. A (online publication March 2023)

[12] **M L Gromov**, *Stable mappings of foliations into manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. 33 (1969) 707–734 MR Zbl In Russian; translated in Math. USSR-Izv. 3 (1969) 671–694

[13] **K Igusa**, *The stability theorem for smooth pseudoisotopies*, K-Theory 2 (1988) vi+355 MR Zbl

[14] **M Krannich**, *A homological approach to pseudoisotopy theory, I*, Invent. Math. 227 (2022) 1093–1167 MR Zbl

[15] **M Krannich**, **O Randal-Williams**, *Diffeomorphisms of discs and the second Weiss derivative of $B\mathrm{Top}(-)$*, preprint (2021) arXiv 2109.03500

[16] **Á del Pino**, **T Vogel**, *The Engel–Lutz twist and overtwisted Engel structures*, Geom. Topol. 24 (2020) 2471–2546 MR Zbl

[17] **N P Strickland**, *The category of CGWH spaces*, preprint (2009) Available at `https://ncatlab.org/nlab/files/StricklandCGHWSpaces.pdf`

[18] **T Vogel**, *Non-loose unknots, overtwisted discs, and the contact mapping class group of S^3*, Geom. Funct. Anal. 28 (2018) 228–288 MR Zbl

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