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An account is given of the compilation of the 1 847 319 428 prime knots with 20 crossings.

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1 Introduction

In the summer of 2018 the author tabulated the knots of 20 crossings. An independent tabulation was made simultaneously by B Burton (personal communication, 2018) using the software *Regina* developed by him and others [3]; as the results of the two tabulations agree, there is some confidence that the results are correct, despite the quantity and complexity of the data.

The knots are listed up to unoriented equivalence, that is to say we regard knot pairs (S^3, K) and (S^3, L) as *equivalent* if there is a homeomorphism of pairs sending (S^3, K) to (S^3, L) , and we list one representative of each equivalence class. The issue of determining which knots are amphicheiral or reversible will be addressed as a separate project.

A short historical note: knot tabulations began in earnest in the late nineteenth century with the work of P G Tait [28; 29; 30], T P Kirkman [15] and C N Little [17], Tait having being motivated by the (then current) Kelvin theory of vortex atoms. Initially, as Tait was aware, techniques were not available for distinguishing knot types rigorously; these techniques arrived shortly afterwards with the advent of the fundamental group due to Poincaré [21], whereupon M Dehn [7] and O Schreier [25] initiated the rigorous classification of knots, beginning with torus knots. A fuller account of the history, up to the classification of 16-crossing knots, is given by Hoste, Thistlewaite and Weeks [14], and, to complete the picture, Burton [2] pioneered the classification of knots of 17, 18 and 19 crossings.

A table listing the numbers of prime knots from 3 up to 20 crossings is given in the appendix.

Theorem 1.1 *The number of equivalence classes of prime knots that can be projected with 20 crossings, but not with fewer crossings, is 1 847 319 428. Of these, all but 921 are hyperbolic, the remainder comprising 915 satellites of the trefoil knot, 5 satellites of the figure-eight knot and the $(3, 10)$ -torus knot.*

The issue of primality is one that is easy to overlook, but it is important, as one has to guard against “imposter” knots that might be composite in some hidden way and are thus masquerading as prime knots. For this reason a section of this article is devoted to justifying the claim that all listed knots are prime.

Acknowledgements The author would like to thank the referee and the editors for useful suggestions that resulted in a significantly improved exposition.

It is a pleasure to acknowledge the support of Ben Burton in this endeavour; a certain error in a line of the author's earlier code would probably have persisted, had he not been able to compare his results for 17 crossings with Ben's. Historically tables of knots have been error-prone, but the probability of errors is greatly reduced by having independent tabulations to hand.

2 The tabulation

2.1 Obtaining the raw list of knots

For the most part the method is the same as that employed in [14], albeit with some minor differences. Traditionally all tabulations of knots with given crossing-number begin with a listing of the prime alternating knots with that number of crossings. This is one of the few steps in the process that is truly algorithmic: from the solutions of the various Tait conjectures, it is known that a knot with a reduced alternating n -crossing diagram cannot be projected with fewer than n crossings. Also, an alternating knot is guaranteed to be prime if its reduced alternating diagrams have the property that they do not admit a simple closed curve in the projection plane meeting the knot projection transversely in two points on distinct edges of the projection; see Figure 1 and [18]. Furthermore, any two reduced alternating diagrams represent the same link type if and only if one can transform one to the other by means of a finite sequence of *flypes*; see Figure 2.

It is precisely the failure of nonalternating knots to adhere to such desirable properties that renders their classification a challenge.

It is relatively straightforward to write a program that generates all possible reduced alternating diagrams of a given crossing-number n , choosing a representative from each flype equivalence class, although skill is required in devising a program that will run in a reasonable time. This has indeed been accomplished quite dramatically for $n \leq 23$ [22]. The present author has written a program that generates all prime

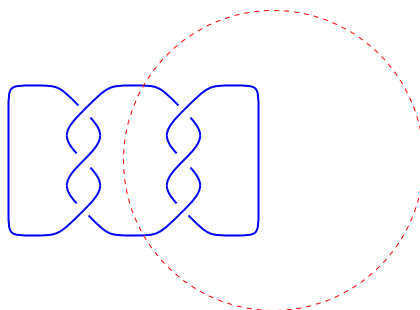


Figure 1: A composite alternating knot.

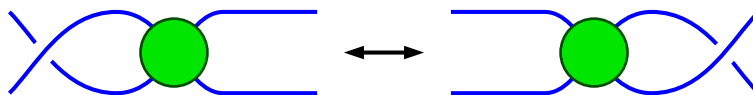


Figure 2: The “flype” transformation.

alternating links with a given number of crossings, and for the case $n = 20$ the number of prime alternating knot types turns out to be 199 631 989.

Once one has the list of alternating diagrams to hand, nonalternating diagrams can be obtained from them by means of crossing switches. It is only necessary to take one alternating diagram from each flype equivalence class, as if alternating diagrams D_1 and D_2 are flype-equivalent, then the diagrams obtained from crossing switches of D_1 are flype-equivalent to those obtained from D_2 . Switching all crossings of a knot diagram produces a knot equivalent to the original on account of being its reflected image in the projection plane, so a crude estimate of the number of diagrams to be generated in this way from a single alternating diagram is $2^{19} = 524\,288$. However, it is only on rare occasions that this number is needed, as can be seen from the following observation: if a rational tangle diagram [4] is not alternating, then there exists an isotopy of the tangle that reduces the number of crossings while keeping the four ends of the tangle fixed; see Figure 3. Therefore built into the program is a procedure that detects all nontrivial rational tangle substituents, and then we only allow crossing switches of the “base” alternating knot diagram that keep each of these tangles alternating.

The resulting nonalternating diagrams are subjected to a number of rapid viability tests to check whether the number of crossings can be reduced, and are immediately discarded upon failing any such test. A surviving diagram is then subjected to a different kind of test, specifically to see whether it can be transformed by flypes and passes (Figures 2 and 4) to a diagram whose DT code [8] is lexicographically less. As the size of an equivalence class generated by these moves can be very large, even in the tens of thousands, we declare that the diagram passes the test if it is still lexicographically minimal once some fixed number k of diagrams has been generated by the moves. Smaller values of k will entail larger redundancy, but it makes sense to keep k quite small on account of the time that would be spent on processing a large set of diagrams.

In [14] some diagram moves more “exotic” than flypes and passes were used, but this approach was avoided here as it was deemed unnecessary, quite apart from the increased danger of introducing bugs

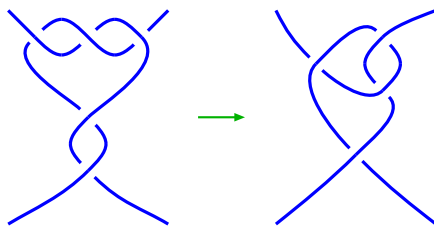


Figure 3: Reducing a nonalternating diagram of a rational tangle.

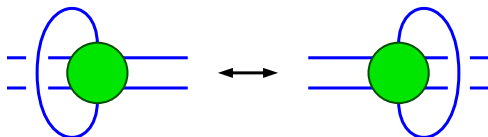


Figure 4: The “pass” transformation on diagrams.

into the program. In practice the value chosen for k was 200, and from experimentation with known tabulations with fewer crossings it was estimated that this resulted in roughly 25% redundancy overall.

2.2 Removing duplicate knots

The next task is to augment the current list of over 2 billion nonalternating diagrams by all tabulated knots with fewer crossings, and then to remove as many duplicates as possible. The extreme difficulty of achieving this “house cleaning” simply by inspecting or manipulating knot diagrams is wonderfully exhibited by the celebrated *Perko pair* [19], a pair of knots with only 10 crossings declared to be inequivalent in Little’s 1900 table, this status persisting until 1974 when K Perko finally spotted the equivalence, thereby obtaining the first correct table of 10-crossing knots.

The 1970s also saw R Riley’s [23] discovery of a hyperbolic structure on the complement of the figure-eight knot, this being one of the inspirations for W Thurston’s breakthrough work on geometric structures on 3-manifolds. This in turn led to J Weeks’ extensive program SnapPea and its more recent Python implementation SnapPy [6], one of whose many features is the ability to compute the *canonical cell decomposition* [9; 24] of a hyperbolic 3-manifold with genus-1 cusps.

The preimage of this cell structure in the universal cover can either be seen in the upper half-space model as dual to the Ford domain, or it can be seen by means of a convex hull construction in the Minkowski model. SnapPea performs a very rapid computation of a purported canonical cell decomposition by starting with a known ideal triangulation of the manifold and then implementing a heuristic optimization process that applies combinatorial moves on the triangulation without affecting the underlying topology. Because of inevitable accumulation of roundoff error, the resulting cell decomposition might on occasion not be the canonical one, but nonetheless if two hyperbolic knots produce isomorphic cell decompositions, their respective complements are proved to be homeomorphic, and from the fact that knots are determined by their complements [12], the knots are equivalent. In practice, even at the level of 20 crossings this is an effective way of removing duplicates, which otherwise could be very hard to spot.

Indeed, the current list of over 2 billion 20-crossing nonalternating diagrams was fed through SnapPea’s canonical cell decomposition procedure, and the few hundred million diagrams producing duplicate cell decompositions were discarded. During this process approximately a mere 549 491 were declared (with due caution) by SnapPea to be “apparently not hyperbolic”, and these were copied to a separate list. The

next stage is to try to distinguish by means of invariants the knots in the filtrate. There is no algorithm at work here, as there does not seem to be any way of predicting which invariants will distinguish which knots: we just throw invariants at the knots and hope for the best. However, the method is rigorous, as all computations of invariants are integer based.

2.3 Application of invariants to the list of knots

2.3.1 Description of the invariants The first invariant applied to the remaining diagrams was the Jones polynomial, for which we are fortunate in having the very fast program of [10], which on a single processor of my workstation will process a million 20-crossing knots in $2\frac{1}{2}$ minutes. This partitions the set of diagrams into relatively small equivalence classes, such that within each equivalence class all knots have the same Jones polynomial. The knots in equivalence classes of size 1 are extracted and placed in the “resolved” folder, and the remaining knots are subjected to a sequence of further invariants, the aim being to subject the partition to successive refinements so that eventually all equivalence classes have size 1.

The remaining invariants are classical, and occur in papers of R H Fox [11] and Perko [20]. They rely on the fact that knot groups seem always to have an abundance of subgroups of small index. It follows from the work of Thurston that knot groups are residually finite, but this alone does not explain why the knots in our 20-crossing list are so rich in subgroups of index less than 10. Given a subgroup H of index n of a knot group G , the group G acts transitively by left multiplication on the set of n left cosets of H , giving rise to a transitive permutation representation of degree n of G . Conversely, given a transitive permutation representation of degree n of G , the stabilizers of the n symbols are conjugate subgroups of G of index n . Using the Reidemeister–Schreier rewriting process we can obtain a presentation of such a subgroup H , and by abelianization obtain a finitely generated abelian group, which is essentially the first homology group of the covering space of the knot complement corresponding to H . We can also glue in solid tori to this covering space so that the components of the preimage of a meridian curve are spanned by cross-sectional disks, thus obtaining the first homology group of the so-called *branched covering space*.

The technique is to choose a transitive permutation representation of some group, for example the natural representation of degree 5 of the alternating group A_5 , and then find all homomorphisms of the knot group onto that group of permutations, up to composition with inner automorphisms of the image group. The multiset of abelian groups thus obtained is then an invariant of that knot type, and amazingly, together with the Jones polynomial, it was possible in this way to distinguish almost all listed 20-crossing knots from one another and from knots with fewer crossings, using only subgroups of the symmetric group S_7 . Figure 5 contains two examples of this type of invariant applied to a fairly resistant mutant pair of 20-crossing knots.

Each diagram in Figure 5 consists of an upper tangle glued to a lower tangle along four strands; the second diagram can be obtained from the first by excising the upper tangle, rotating it through a half turn in the projection plane and then gluing it back to the lower tangle.

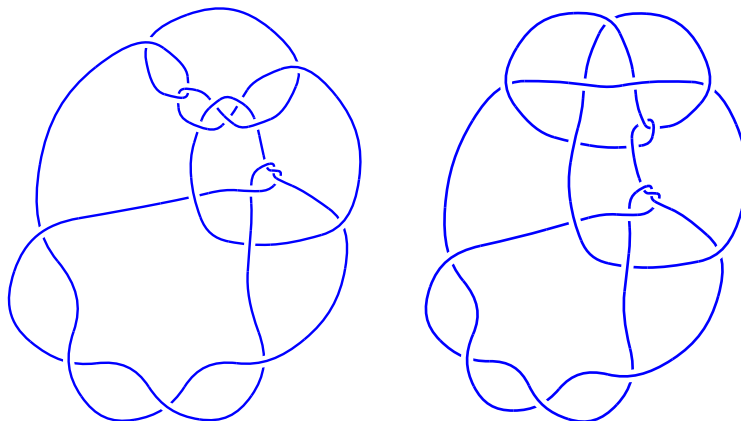


Figure 5: A pair K_1 and K_2 of mutant 20-crossing knots.

Being related by mutancy, these knots cannot be distinguished by the Jones polynomial nor indeed by the HOMFLYPT 2-variable polynomial; also they resisted homology groups associated with permutation representations of degrees 5 and 6. However they did succumb to permutation representations mapping meridians to one of the two conjugacy classes of 7-cycles in the alternating group A_7 . For K_1 there were 14 representations, producing homology groups with torsion numbers as follows:

[1083964, 14], [10873394], [117987912], [1308356, 2, 2], [13423592, 8], [155682849, 3], [2496669],
[30245222, 2], [353577, 7], [477902327], [4832310], [58290239, 7], [8694588], [909657, 7].

And for K_2 just 13 representations:

[10007522, 2], [1339604, 14], [20281751], [21298634], [24072097], [2742502, 2], [304197488, 2],
[40220460], [46137, 21], [4719806], [53620280], [56118930, 3], [6282066, 2].

Since the group A_7 admits an automorphism sending each 7-cycle to its inverse, this conjugacy class cannot be used to detect nonreversibility of knots. It was observed by H Trotter [32] that a more careful choice of target group can be effective for this purpose; indeed this was the first occasion that the existence of nonreversible knots was proved. Later R Hartley [13] used (solvable) groups of functions $x \mapsto ax + b$ (for $a \neq 0$) over finite fields to establish nonreversibility of many knots of up to 10 crossings. For the knots K_1 and K_2 of Figure 5, the sporadic Mathieu simple group M_{11} is effective in showing that they are not reversible. Specifically, we can use the irreducible permutation representation of M_{11} of degree 11, and map meridians to one of the two conjugacy classes of size 990 containing elements of cycle type $(ab)(cdefghij)$. Here are the results, with torsion numbers for each representation enclosed in square brackets as above:

K_1	[1394030, 2]
Reverse of K_1	[287520], [65322]
K_2	[14118592], [5682, 2]
Reverse of K_2	[1598572], [4161904]

It is expected that one can determine reversibility in this way for the list of 20-crossing knots, although it could be very time consuming. Determining amphicheirality is in practice easier, as almost all instances of nonamphicheirality are detected by the Jones polynomial.

2.3.2 Details of the invariants' performance There now follows details of the efficacy of the invariants used for distinguishing the 20-crossing knots from one another and from knots with fewer crossings. The enumeration stage of the classification process generated 2 229 828 372 20-crossing nonalternating diagrams; barring programming error there were no omissions in this list, and the procedure was in three main stages as described below.

Stage 1 The raw list of 20-crossing nonalternating diagrams was augmented by the list of all 352 151 858 hyperbolic knots with fewer than 20 crossings, resulting in an enlarged list containing 2 581 980 230 knots. SnapPea's canonical cell decomposition procedure was applied to each knot in the enlarged list, and the data was sorted so that duplicate cell decompositions became evident. Each cell decomposition was encoded by a string of approximately 300 bytes on average, so the amount of data involved in this step was around 775 GB. As explained in Section 2.2, it is not guaranteed that the cell decompositions output by this procedure are canonical in the sense of [9; 24], but knots with duplicate cell decompositions have homeomorphic complements, so are equivalent owing to the fact that knots are determined by their complements [12].

The canonical cell decomposition procedure declared that 549 491 knots from the list were “apparently not hyperbolic” and these were put in a separate list for further treatment. Aggressive diagram moves revealed that out of these knots 200 were the unknot, 547 611 were composite knots, and a further 482 could be drawn with fewer than 20 crossings. This left a residue of 1198 knots, which on being treated to still more stringent diagram moves were shown to belong to 921 knot types, distinct from one another and distinct from all nonhyperbolic knot types with fewer than 20 crossings. The proof that this list consisted of a single 20-crossing torus knot and 920 20-crossing satellite knots is given in Section 3.

After removing the 549 491 knots declared to be “apparently not hyperbolic” and the knots whose complements had duplicate cell decompositions, the number of knots in the refined list was 1 999 847 149. These were input into the next stage, it being expected that the only duplications were those arising from roundoff error in application of the canonical cell decomposition procedure.

Stage 2 From an accounting point of view this was the easiest stage. The Jones polynomials of the 1 999 847 149 knots output by the previous stage were computed, and the 336 548 774 knots with unique polynomials were extracted and placed in the store of “resolved” knots. The remaining 1 663 298 375 knots were input into Stage 3, which subjected them to the invariants described above, namely first homology groups of branched covering spaces corresponding to permutation representations of the knot groups.

Stage 3 Tables 1 and 2 summarize the results of this stage. The first table uses representations into alternating or symmetric groups of degrees 5 and 6, and the number of unresolved knots was reduced from

degree	cycle type	input	unique	nonunique	runtime
5	$(abcde)$	1 663 298 375	906 980 266	756 318 109	20 days
5	$(abc)(de)$	756 318 109	317 431 388	438 886 721	4 days
5	$(abcd)$	438 886 721	309 112 549	129 774 172	5 days
6	$(abc)(def)$	129 774 172	66 784 736	62 989 436	6 days
6	$(abcde)$	62 989 436	57 189 475	5 799 961	6 days
6	$(abcdef)$	5 799 961	5 071 212	728 749	1 day

Table 1

1 663 298 375 to 728 749. The column labelled “unique” gives the number of knots distinguished from all others and placed into the “resolved” store, and the column labelled “nonunique” gives the number of unresolved knots requiring further treatment. The column labelled “cycle type” gives the cycle type of the conjugacy class to which meridians of the knot group were mapped. The machine used for these computations had 160 GB of memory and 20 processing cores.

The remaining knots were then subjected to permutation representations in various specific groups, as set out in Table 2. Each of these substages took less than a day of runtime.

At this point the list of 17 528 unresolved knots were partitioned into 8755 equivalence classes, where knots within each equivalence class had resisted all invariants applied to date. It was suspected that each of these in fact represented a single knot type, and this was confirmed by a more persistent application of the canonical cell decomposition procedure: SnapPea has a convenient “random retriangulation” feature, and from this a small number of different contenders for canonicity were obtained, amongst which matching cell decompositions were found in each of the outstanding cases.

Surprisingly the last two knots to be distinguished, in the last row of Table 2, were a pair of 14-crossing two-bridged knots, with associated fractions $\frac{505}{192}$ and $\frac{505}{212}$ and respective Conway codes 2111221112 and 2211111122. These are easily distinguished by the fact that they are alternating, and also by their lens space two-fold branched covers, but for some reason they resisted polynomial invariants and the homology invariants of Tables 1 and 2 until the very last step.

group	degree	cycle type	input	unique	nonunique
$\text{PSL}(2, 7)$	7	7-cycles	728 749	572 093	156 656
$\text{PSL}(2, 11)$	11	11-cycles	156 656	117 446	39 210
$\text{PSL}(2, 13)$	14	13-cycles	39 210	15 364	23 846
$\text{PSL}(2, 17)$	18	17-cycles	23 846	5 245	18 601
A_7	7	7-cycles	18 601	1 071	17 530
$\text{PSL}(2, 19)$	20	19-cycles	17 530	2	17 528

Table 2

This concluded the task of obtaining a list of 20-crossing knots with no omissions or duplications, but it was still necessary to check that there were no “poseur” composite knots in the list. One almost hoped that some would materialize, as such examples would be noteworthy.

3 Establishing primality

A fundamental property of a hyperbolic 3-manifold is that it cannot contain an essential torus. The software *Regina* [3] confirmed that all presumed 1 847 318 507 hyperbolic 20-crossing knots in our list are indeed hyperbolic (ie there are no false positives with respect to hyperbolicity) so all are immediately known to be prime.

A different approach is needed for showing that the 921 (apparently) nonhyperbolic knots are prime. The single torus knot was easily identified, and since torus knots are prime we may restrict our attention to the remaining 920 knots in this list.

Regina [3] was also put to work by Burton to implement normal surface calculations for dealing with these 920 knots, and in this way all but eight were confirmed to be prime. *Regina* was not immediately able to certify primality of the remaining eight knots (personal communication, 2018) and this led the author to an alternative approach based on tangle decompositions, anticipating that both methods could be useful in future tabulations. Indeed the methods could be complementary, for the following reason. All prime satellite knots of up to 20 crossings have minimal diagrams that are tangle sums satisfying the hypothesis of Theorem 3.2, and although this situation is expected to continue for a while, it will not continue indefinitely. At some point a more generic technique for establishing primality will be essential, and this could be provided by programs such as *Regina*.

Figure 6 illustrates one of the eight abovementioned satellite knots. It is visibly a tangle sum of a *companion tangle* on the right (Figure 7, right) with an alternating tangle on the left, and in fact each of these eight satellites admits such a decomposition. From the main result of [18] the alternating tangle summands of these knots have no local knotting (Figure 7, left), so from Theorem 3.2 these eight knots are all prime.

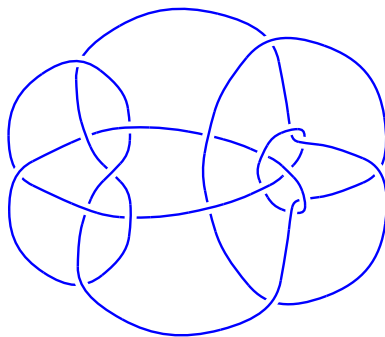


Figure 6: A 20-crossing satellite of the trefoil.

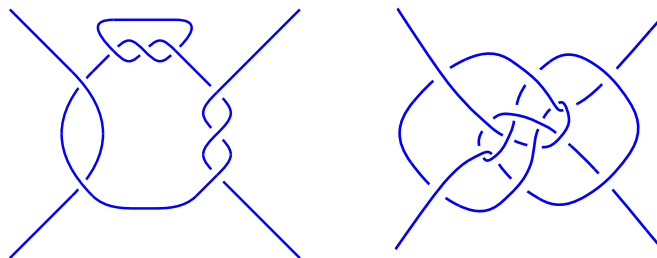


Figure 7: A locally knotted tangle (left) and a companion tangle (right).

We recall the terminology of [16]. In that paper a *tangle* is defined to be a pair (B, T) where B is a 3-ball, ie a manifold with boundary homeomorphic to the standard 3-ball B^3 , and T is a proper 1-submanifold of B consisting of two disjoint arcs (naturally we assume that we are in the piecewise linear or smooth category). Thus the boundary of T consists of four points on ∂B . This definition has some obvious generalizations; for example we might allow the number of arcs in T to be greater than 2, but the definition as given is sufficient for our purposes. Tangles (B_1, T_1) and (B_2, T_2) are *equivalent* if there is a homeomorphism of pairs from (B_1, T_1) to (B_2, T_2) , and (B, T) is *untangled* or *trivial* if it is equivalent to a product $(D, \{x, y\}) \times I$, where D is a disk and x and y are points in its interior.

In the definition of tangle equivalence given above, for a homeomorphism $h: (B, T_1) \rightarrow (B, T_2)$ there is no restriction on the effect of h on the boundary 2-sphere of B , other than the requirement that it map ∂T_1 to ∂T_2 . For example, any tangle represented as a diagram of a *rational tangle* [4] is equivalent to a tangle where T consists of two parallel line segments, ie it is trivial.

A tangle (B, T) is *locally unknotted* if each 2-sphere in B meeting T transversely in two points bounds in B a ball meeting T in an unknotted spanning arc. Otherwise we say that (B, T) is *locally knotted*; an example is illustrated in Figure 7, left. Observe that if (B, T) is locally unknotted, and there exists a properly embedded disk in B separating the arcs of T , then (B, T) is trivial.

Given tangles (B_1, T_1) and (B_2, T_2) , we may glue them together by means of some homeomorphism of (2-sphere, four points) pairs to obtain a link L of one or two components in the 3-sphere. Such a pair (S^3, L) is called a *sum* of the tangles (B_1, T_1) and (B_2, T_2) . Given two tangles drawn in the usual way as diagrams, one way of summing them is to join the diagrams by arcs in the projection plane.

If (B, T) contains a 2-sphere S exhibiting local knottedness, with knotted arc α in the ball bounded by S , then there is a well-defined nontrivial knot K obtained by joining the ends of α with an arc in S , and this knot K will persist as a connected summand of any knot formed by summing (B, T) with an arbitrary tangle. Therefore if we can sum (B, T) with a tangle so as to obtain the unknot, (B, T) is locally unknotted. In more complicated situations we have the following effective test for local unknottedness:

Proposition 3.1 *Let (B, T) be a tangle for which there exist tangles (B_1, T_1) and (B_2, T_2) , such that summing (B, T) with the (B_i, T_i) in turn produces distinct prime knots K_1 and K_2 . Then (B, T) is locally unknotted.*

Proof If (B, T) is locally knotted, there exists a nontrivial knot that is a connected summand of each of the distinct prime knots K_1 and K_2 , and this contradicts uniqueness of factorization of knots. $\square$

There is a special type of tangle that pertains to satellite knots. We define a *companion tangle* to be a tangle (B, T) where T consists of two parallel, knotted arcs in B ; an example is illustrated in Figure 7, right. Each companion tangle (B, T) contains a properly embedded annulus A in $B - T$ that “follows” the two strands of T in tube-like fashion, ie there is a homeomorphism $h: S^1 \times I \rightarrow A$ such that each section $h(S^1 \times \{t\})$ of A bounds a disk in B meeting T transversely in two points. A companion tangle cannot be locally knotted, as it is always possible to sum it with a trivial tangle so as to obtain the unknot; in Figure 7, right, this can be seen by taking two arcs in the projection plane, one joining the two left-hand ends and the other the two right-hand ends.

Let us now consider a knot (S^3, K) constructed as a sum of a companion tangle (B_1, T_1) with a locally unknotted tangle (B_2, T_2) . We may form a torus F in the complement of K as the union of the “following” annulus A_1 of (B_1, T_1) described above, with a boundary-parallel annulus A_2 in (B_2, T_2) that “swallows” T_2 . Let V be the solid torus containing K that is bounded by F ; V is the union of two “halves” $V \cap B_1$ and $V \cap B_2$ glued together along cross-sectional disks D_1 and D_2 , both in $\partial B_1 = \partial B_2$, and each meeting K in two points. The core Γ of V is a nontrivial knot in S^3 , as it is the union of a knotted arc in B_1 that is the core of A_1 with an unknotted arc in B_2 that is the core of A_2 .

Any cross-sectional disk of V not meeting K would have to separate the strands of the second tangle (B_2, T_2) , but the local unknottedness of (B_2, T_2) would force that tangle to be trivial, and we would be in the situation described above where K is the unknot. On the other hand, if there is no cross-sectional disk of V separating the strands of (B_2, T_2) , K is a satellite of Γ and the torus F is incompressible in $S^3 - K$.

The next theorem provides the method for showing that the 920 outstanding knots are prime. It is closely related to results in [26; 16; 5], as explained below; however the full proof is given here as the hypotheses are an exact fit to our situation, and moreover should be applicable to future tabulations with more than 20 crossings.

Theorem 3.2 *Let K be a knot that is a sum of a companion tangle with a locally unknotted tangle. If K is nontrivial, then K is prime.*

Proof We adopt the notation of the preceding discussion: (S^3, K) is a nontrivial knot that is the sum of a companion tangle (B_1, T_1) with a locally unknotted tangle (B_2, T_2) , $F = A_1 \cup A_2$ is the incompressible torus in $S^3 - K$ that “follows” T_1 and “swallows” T_2 , and V is the solid torus with boundary F .

Let S be a 2-sphere in S^3 meeting K transversely in two points. Before proceeding further it is useful to observe that each simple closed curve C in $S - K$ is either nullhomotopic in $S - K$ (and hence also nullhomotopic in $S^3 - K$), or else it separates the punctures. In particular, a circle on F that bounds a cross-sectional disk of V cannot lie on S .

We first consider the special case where S is contained in V . Since by hypothesis each of the constituent tangles (B_1, T_1) and (B_2, T_2) is locally unknotted, the conclusion of the theorem holds for S contained in either “half” $V \cap B_i$, and we are motivated to consider the transverse intersection of S with the two disks D_1 and D_2 along which the halves of V are glued together. We may assume that the two points of $S \cap K$ are away from the D_i . The set $S \cap (D_1 \cup D_2)$ is the union of a disjoint collection of circles on S ; let C be a circle from this collection that is innermost on S , say without loss of generality $C \subset S \cap D_1$. Then C bounds a disk $\Delta_1 \subset S$ and a disk $\Delta_2 \subset D_1$. The union of the disks Δ_i is an embedded 2-sphere Σ , bounding a ball B' contained in one of the B_i . The number n of points of $\Delta_2 \cap K$ is 0, 1 or 2, and we consider each of these cases. We can exclude the possibility $n = 2$ summarily, as in this case C would be a simple closed curve on S homotopic in $S^3 - K$ to a meridional curve of $F = \partial V$, a situation ruled out in the previous paragraph.

Suppose that $n = 0$; then Δ_1 meets K in 0 or 2 points. In the latter case, from the hypothesis of local unknottedness applied to Σ , the ball B' would meet K in an unknotted arc; we deduce from this that one of the components of $S^3 - S$ would meet K in this arc, and the conclusion of the theorem would follow. Otherwise the ball B' does not meet K . The circle C might not be innermost on D_1 , but nonetheless Δ_1 can be pushed by an isotopy through B' , taking with it all components of $S \cap B'$, reducing the number of components of $S \cap (D_1 \cup D_2)$. If $n = 1$, then from the hypothesis of local unknottedness B' meets K in an unknotted arc, so again there is an isotopy that pushes Δ_1 across B' , including if necessary another component of $S \cap B'$ meeting K in one point. Here the isotopy will move points of K along the unknotted arc, but can be assumed to fix K setwise. We conclude that there is an isotopy of S into one of the B_i without affecting transversality of $S \cap K$, whence S bounds a ball on one side meeting K in an unknotted arc, and the conclusion of the theorem follows for this special case.

For the remainder of the proof we assume that S has nonempty transverse intersection with $F = \partial V$; the proof will be completed by showing that there an isotopy of S in S^3 , maintaining transversality of S with K , that moves S to a 2-sphere contained in V .

Recall that a simple closed curve in the twice-punctured sphere $S - K$ is either nullhomotopic in $S - K$ or is homotopic to a meridian curve of K . The torus F does not contain any simple closed curve of the second type, so each component C of $S \cap F$ is a simple closed curve bounding a disk in $S - K$, and also bounding a disk in F owing to the incompressibility of F . Let us take a component C of $S \cap F$ bounding a disk $\Delta_1 \subset S - K$ whose interior does not meet F ; also let Δ_2 be the disk on F bounded by C . Then $\Delta_1 \cup \Delta_2$ is an embedded 2-sphere in $S^3 - K$, and in a manner similar to that of the special case we can perform an isotopy of S that reduces the number of components of $S \cap F$. Repeating the process will eventually move S into V , and the proof of the theorem is complete. $\square$

There is overlap between Theorem 3.2 and results in the literature, most notably H Schubert’s paper [26], where the notion of the companionship tree of a knot is introduced, and where it is shown that doubled knots and cabled knots are prime.

In [16] a tangle is called *prime* if it is nontrivial and locally unknotted, and it is proved in that paper that a sum of two prime tangles is a prime link. A companion tangle is prime according to this definition, as it is locally unknotted, and cannot be trivial as its individual strands are knotted arcs. Theorem 3.2 shows that, apart from the obvious single exception, a knot formed as a sum of a companion tangle with a trivial tangle is also prime, thus confirming a special case of the conjecture stated in [16, Section 4].

Theorem 4.4.1 of [5] deals more generally with primality of satellite knots; it includes the hypothesis that the pair (V, K) is locally unknotted, this being the conclusion of the special case dealt with in the proof of Theorem 3.2.

Recall that there are 920 knots in our tabulation that are under examination for primeness. It was suspected that five of these are satellites of the figure-eight knot, and these were easily found in the list; they are all obtained by summing the companion tangle of Figure 7, right, with a 4-crossing rational tangle. Since they are already known to be nontrivial, and rational tangles are certainly locally unknotted, application of Theorem 3.2 shows that they are prime.

Naturally one suspects that the remaining 915 knots are satellites of the trefoil knot. In order to apply Theorem 3.2 we need diagrams of these knots that show each as a sum of a companion tangle with a locally unknotted tangle. Undoubtedly it would be possible to find such diagrams directly; however, a different approach was used here. The list of 199 631 989 prime *alternating* 20-crossing knots provides, up to flype equivalence, all projections of prime nonalternating knots, and an easy search through this list found 434 projections of tangle sums of the required kind. Suitable over- and under-passes were applied to these, resulting in a refined list of 915 knot diagrams that visibly were sums of tangles (B_1, T_1) and (B_2, T_2) with (B_1, T_1) a companion tangle, and matched the tabulated 915 knots.

As the knots were already known to be nontrivial, it remained to check that in each case the tangle (B_2, T_2) was locally unknotted. In all but ten cases verification was immediate, as the diagrams of (B_2, T_2) are either alternating, in which case they are subject to [18, Theorem 1], or they are standard diagrams of arborescent tangles [1].

The ten exceptional cases come in five pairs, each pair consisting of a diagram and its reflection in the projection plane. It was only necessary to check one tangle from each pair, and they are illustrated in Figure 8.

Under mild scrutiny the individual strands of the third, fourth and fifth tangles of Figure 8 are all revealed to be unknotted, so local knots for these tangles are ruled out. One can also notice that the first two tangles are equivalent: there is a homeomorphism that interchanges the lower two tangle ends. Therefore, in order to complete the proof that all 920 satellite knots are prime, we just need to check that the first tangle is locally unknotted. This follows quickly from Proposition 3.1: summing (in the “diagrammatic” sense) with a tangle with no crossings produces a prime 8-crossing knot, and one can obtain a prime 9-crossing knot, also a prime 10-crossing knot by summing with a 2-crossing tangle.

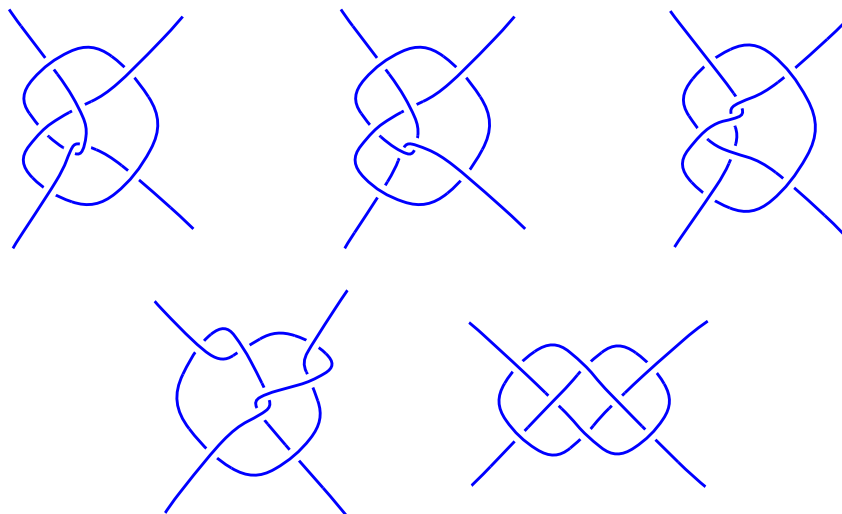


Figure 8

Appendix

A.1 The rate of growth of the number of knots

It is natural to ask whether one can estimate the number of prime knots with a given number of crossings without an actual tabulation. There are very few results in existence on this topic, but the following is known. It is stated for links rather than knots, but it suggests that the number of nonalternating knots grows exponentially faster than that of alternating knots.

Theorem A.3 (i) [27] *Let A_n denote the number of prime alternating link types with n crossings. Then*

$$\lim_{n \rightarrow \infty} A_n^{1/n} = \frac{1}{40}(101 + \sqrt{21001}) \approx 6.1479.$$

(ii) [31] *Let λ be the limit stated in (i). There exists a set $\mathcal{B}$ of prime link types, strictly containing the set of prime alternating link types, such that if B_n is the number of link types in $\mathcal{B}$ with n crossings, then $\lim_{n \rightarrow \infty} B_n^{1/n}$ exists and is strictly greater than λ .*

A.2 The number of prime knot types with n crossings, $3 \leq n \leq 20$

The first correct tabulations of knots of 17, 18 and 19 crossings were produced by Burton [2].

Table 3 gives summary data up to 20 crossings. When reading Table 3, it is worth noting that the only prime alternating nonhyperbolic knots are the $(2, n)$ -torus knots (with n necessarily odd) [18]. Thus for even n all n -crossing prime alternating knots are hyperbolic, and for odd n there is a single nonhyperbolic prime alternating knot, namely the $(2, n)$ -torus knot. Also it follows that all prime satellite knots are nonalternating.

crossings	alternating knots	nonalternating knots	hyperbolic	torus	satellites
3	1	0	0	1	0
4	1	0	1	0	0
5	2	0	1	1	0
6	3	0	3	0	0
7	7	0	6	1	0
8	18	3	20	1	0
9	41	8	48	1	0
10	123	42	164	1	0
11	367	185	551	1	0
12	1 288	888	2 176	0	0
13	4 878	5 110	9 985	1	2
14	19 536	27 436	46 969	1	2
15	85 263	168 030	253 285	2	6
16	379 799	1 008 906	1 388 694	1	10
17	1 769 979	6 283 414	8 053 363	1	29
18	8 400 285	39 866 181	48 266 380	0	86
19	40 619 385	253 511 073	294 130 212	1	245
20	199 631 989	1 647 687 439	1 847 318 507	1	920

Table 3

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