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An exotic presentation of $\mathbb{Z} \times \mathbb{Z}$ and the Andrews–Curtis conjecture

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We prove that the presentations $\langle x, y \mid [x, y], 1\rangle$ and $\langle x, y \mid [x, [x, y^{-1}]]^2 y[y^{-1}, x]y^{-1}, [x, [[y^{-1}, x], x]]\rangle$ are not Q^*-equivalent even though their standard complexes have the same simple homotopy type.

20F05, 20F65, 57M05, 57Q10

1 Introduction

A finite presentation $\langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_m\rangle$ of a group G can be transformed into another presentation of G by performing one of the following:

(i) Changing a relator r_j by $r_j r_i$ for some $i \neq j$.

(ii) Changing a relator r_j by r_j^{-1}.

(iii) Changing a relator r_j by a conjugate $g r_j g^{-1}$ for some g in the free group $F(x_1, x_2, \dots, x_n)$.

(iv) Changing each relator r_j by $\phi(r_j)$, where ϕ is an automorphism of $F(x_1, x_2, \dots, x_n)$.

(v) Adding a generator x_{n+1} and a relator r_{n+1} which coincides with x_{n+1}.

(vi) The inverse of (v), when possible.

Moves (i) to (iii) are called Q-transformations, (i) to (iv) are Q^*-transformations and moves (i) to (vi) are called Q^{**}-transformations. Two finite presentations are said to be Q-equivalent (Q^* or Q^{**}) if one can be obtained from the other by performing a sequence of Q-transformations (Q^* or Q^{**}). The Andrews–Curtis conjecture [1] states that any two presentations of the trivial group with n generators and $m = n$ relators are Q-equivalent. The weak version of the Andrews–Curtis conjecture states that in the same situation the two presentations are just Q^{**}-equivalent. The latter is equivalent to the statement that any contractible finite 2-dimensional CW-complex 3-deforms to a point. The so-called generalized Andrews–Curtis conjecture [13, Section 4.1] says that any two presentations $\mathcal{P}$ and $\mathcal{Q}$ with simple homotopy equivalent standard complexes $K_{\mathcal{P}}$ and $K_{\mathcal{Q}}$ are Q^{**}-equivalent. These three conjectures are open.

Although the original conjecture says that in some cases any Q^{**}-equivalence is a Q-equivalence, there are known examples of presentations which are Q^{**}-equivalent but Q^*-inequivalent. The first were probably those given by Zieschang [24, page 36]: $\langle x, y \mid x^3 y^5\rangle$ and $\langle x, y \mid x^3 y^3 x^3 y^2\rangle$, and, with more generality, by McCool and Pietrowski [18]: $\langle x, y \mid x^k y^{pt+1}\rangle$ and $\langle x, y \mid (x^k y^t)^p y\rangle$ for $k, p, t \geq 2$.

In each case the presentations are Q^{**}-equivalent. The authors only prove that they present the same group, but it is easy to translate their proof to the language of Q^{**}-transformations. On the other hand, Whitehead's algorithm can be used to prove that the relators of all these presentations are of minimal length in $\mathbb{F}_2 = F(x, y)$, ie in their orbit under the $\mathrm{Aut}(\mathbb{F}_2)$ action. Since in each example the relators have different length, there is no automorphism of $\mathbb{F}_2$ taking one to the other or its inverse. Thus, these one-relator presentations are not Q^*-equivalent. In fact, the construction of [18] shows that for every $m > 0$ there are m presentations which are pairwise Q^{**}-equivalent and Q^*-inequivalent. By a result of Magnus, the normal closures $N(r)$ and $N(s)$ of two elements r and s in a free group coincide if and only if r is a conjugate of s or its inverse. Thus, these one-relator examples are not Q^*-equivalent for an elementary reason: there is no automorphism of $\mathbb{F}_2$ taking the normal closure of one relator to the normal closure of the other. In [19], Metzler gives an example of a different nature. He shows that the presentations $\langle x, y \mid x^5, y^5, [x, y]\rangle$ and $\langle x, y \mid x^5, y^5, [x^2, y]\rangle$ are Q^{**}-equivalent and not Q^*-equivalent. In this case the normal closures of both sets of relators coincide.

McCool–Pietrowski's family and Metzler's example occur at minimal Euler characteristic (ie the Euler characteristic of the standard complexes of those presentations are minimal among all possible presentations of the same groups), though the one-relator examples can be stabilized (by adding trivial relations) to obtain Q^{**}-equivalent and Q^*-inequivalent presentations (the same argument can be used) arbitrarily far from minimal Euler characteristic.

In this article we construct presentations $\mathcal{P}$ and $\mathcal{Q}$ of $\mathbb{Z} \times \mathbb{Z}$ each with two generators and two relators, having simple homotopy equivalent standard complexes, but such that $\mathcal{P}$ and $\mathcal{Q}$ are not Q^*-equivalent. The normal closures of both relator sets coincide and the Euler characteristic is one above the minimal level.

Non-homotopy-equivalent presentations of the same group G with equal Euler characteristic have been constructed by using stably free nonfree $\mathbb{Z}[G]$-modules. Although every finitely generated projective $\mathbb{Z}[G]$-module is free when $G = \mathbb{Z} \times \mathbb{Z}$, in our construction we need to distinguish Q-equivalence classes, and we use a more subtle idea: an exotic basis of $\mathbb{Z}[G]^2$ which is not obtained from the standard basis by elementary row operations. The basis change matrix, which is not a product of elementary and diagonal matrices, was found by Evans in [9]. The second ingredient of our proof is a new invariant called the winding invariant. The present article was meant to be a section in the paper [3], which introduces this invariant along with applications. We believe that it is better to present this example in a separate article.

There are known examples of presentations $\mathcal{P}_1$ and $\mathcal{P}_2$ which are

(a) not simple homotopy equivalent but homotopy equivalent, or

(b) not homotopy equivalent, but such that the stabilized presentations $\mathcal{P}_1^{(1)}$ and $\mathcal{P}_2^{(1)}$, obtained by adding one trivial relator, become Q-equivalent.

These examples have minimal Euler characteristic. Our presentations $\mathcal{P}$ and $\mathcal{Q}$ are above minimal Euler characteristic and become Q-equivalent after one stabilization.

After the first version of this paper was finished, X Fernández [10] proved that the presentations $\mathcal{P}$ and $\mathcal{Q}$ are in fact Q^{**}-equivalent. Also, at that time we were not aware of Zieschang, McCool–Pietrowski and Metzler's examples. Although the results of the present version and the original are essentially the same, we have added comments suggested by two anonymous referees. The results about the tree of Q-equivalence classes were motivated by one of the referees' comments. We are grateful to the referees for their suggestions.

Projecting to $\mathbb{F}_2'/\mathbb{F}_2''$ Previously known methods for proving that the presentations $\langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$ and $\langle x_1, x_2, \ldots, x_n \mid s_1, s_2, \ldots, s_m \rangle$ are not Q or Q^{**}-equivalent were developed by Browning [6], Hog-Angeloni and Metzler [12, Section 2.2] and Borovik, Lubotzky and Myasnikov [5]. As explained by Hog-Angeloni and Metzler in [14, Section 2.2], the idea is to define a homomorphism q from $F(x_1, x_2, \ldots, x_n)$ to a test group G^* and prove that the m-tuples $(q(r_1), q(r_2), \ldots, q(r_m))$ and $(q(s_1), q(s_2), \ldots, q(s_m))$ are not equivalent. The results of Hog-Angeloni and Metzler [13, Theorems 2.3 and 2.4] show that solvable groups are not useful as test groups to distinguish Q^{**}-equivalence. In Borovik, Lubotzky and Myasnikov [5], Browning [7] and Hog-Angeloni and Metzler [14], there are results concerning the use of finite groups to distinguish Q and Q^{**}-equivalences. Our methods appeared as a natural application when studying the winding invariant. However, they can be seen through this perspective as an application of solvable groups to distinguish Q-equivalence. Concretely we use $G^* = \mathbb{F}_2/\mathbb{F}_2''$, the free metabelian group of rank 2.

The presentations $\mathcal{P}$ and $\mathcal{Q}$ present $\mathbb{Z} \times \mathbb{Z}$, so their Whitehead group is trivial. The torsion $\tau(f) \in \mathrm{Wh}(K_{\mathcal{P}})$ of any homotopy equivalence $f\colon K_{\mathcal{Q}} \to K_{\mathcal{P}}$ is therefore trivial. To distinguish Q^*-equivalence classes we will work in $\mathrm{GL}_2(\mathbb{Z}[\pi_1(\mathcal{P})])/\mathrm{GE}_2(\mathbb{Z}[\pi_1(\mathcal{P})])$ instead of $\mathrm{Wh}(K_{\mathcal{P}}) = \mathrm{GL}(\mathbb{Z}[\pi_1(\mathcal{P})])/\mathrm{GE}(\mathbb{Z}[\pi_1(\mathcal{P})])$. We recall the definitions of these concepts.

Given a ring R, denote by $E_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ generated by the elementary matrices. Recall that $E \in \mathrm{GL}_n(R)$ is elementary if all the diagonal coefficients are $1 \in R$ and all the other coefficients but one are $0 \in R$. We call $D_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ of diagonal matrices and $\mathrm{GE}_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ generated by $E_n(R)$ and $D_n(R)$. Note that $D_n(R)$ normalizes $E_n(R)$, so $\mathrm{GE}_n(R) = D_n(R)E_n(R) = E_n(R)D_n(R)$. If R is a Euclidean ring, then $\mathrm{GE}_n(R) = \mathrm{GL}_n(R)$ for every $n \geq 1$. A ring R is said to be generalized Euclidean if $\mathrm{GE}_n(R) = \mathrm{GL}_n(R)$ for every n. It was proved by Bachmuth and Mochizuki [2, Theorem 1] that $R = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]$ is not generalized Euclidean. Evans [9, Theorem C] gives a concrete example of a 2 by 2 invertible matrix over that ring which is not in $E_2(R)$.

From now on R will denote the ring $\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]$.

Theorem 1 (Evans) *The matrix*

$$\begin{pmatrix} 1 - 2(X-1)Y^{-1} & 4Y^{-1} \\ -(X-1)^2 Y^{-1} & 1 + 2(X-1)Y^{-1} \end{pmatrix}$$

is in $\mathrm{GL}_2(R)$ *but not in* $\mathrm{GE}_2(R)$.

The commutator subgroup of $\mathbb{F}_2$ is denoted by $[\mathbb{F}_2, \mathbb{F}_2]$ or $\mathbb{F}_2'$. Recall that $w \in \mathbb{F}_2$ lies in $[\mathbb{F}_2, \mathbb{F}_2]$ if and only if the total exponents of x and of y in w are both 0.

Definition 2 Let $w \in \mathbb{F}_2$. Then $w = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_t^{\epsilon_t}$, where $x_i \in \{x, y\}$ and $\epsilon_i \in \{1, -1\}$ for each i. The word w determines a path γ_w in $\mathbb{R}^2$ which begins in $(0,0)$ and is a concatenation of paths $\gamma_1, \gamma_2, \ldots, \gamma_l$. The path γ_i moves one unit parallel to the axis x_i and with positive or negative direction depending on the sign ϵ_i. The image of γ_w is contained in the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$. The ending point of γ_w is (k, l), where k is the total exponent of x in w and l is the total exponent of y. Suppose $w \in [\mathbb{F}_2, \mathbb{F}_2]$, so γ_w finishes in $(0,0)$. For each $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, let $a_{i,j}$ be the winding number $\mathrm{w}\big(\gamma_w, i + \frac{1}{2}, j + \frac{1}{2}\big)$ of γ_w around the point $p = \big(i + \frac{1}{2}, j + \frac{1}{2}\big)$. Define the *winding invariant* $P_w \in R = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]$ of w to be the Laurent polynomial $P_w = \sum a_{i,j} X^i Y^j$.

Our notation for commutators is $[u, v] = uvu^{-1}v^{-1}$. So, for instance, the winding invariant of $[x, y]$ is $P_{[x,y]} = 1 \in R$ and $P_{[y^{-1},x]} = Y^{-1}$. The path γ_w is just the lift of w to the Cayley graph $\Gamma(\mathbb{Z} \times \mathbb{Z}, \{x, y\}) = \mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$, and the winding invariant $\mathbb{F}_2' \to R$ can be seen as the projection $N \to N/N'$ of the normal closure N of $[x, y]$ in $\mathbb{F}_2$ onto the relation module N/N' of the presentation $\langle x, y \mid [x, y] \rangle$; see [3, Section 8]. If $w \in \mathbb{F}_2'$, then $\psi(\partial w/\partial x) = (1 - Y)P_w$ and $\psi(\partial w/\partial y) = (X - 1)P_w$; see [3, Section 10]. Here $\partial/\partial x, \partial/\partial y : \mathbb{Z}[\mathbb{F}_2] \to \mathbb{Z}[\mathbb{F}_2]$ are the Fox derivatives, $\psi : \mathbb{Z}[\mathbb{F}_2] \to \mathbb{Z}[\mathbb{F}_2/\mathbb{F}_2']$ is the canonical projection, and $\mathbb{Z}[\mathbb{F}_2/\mathbb{F}_2']$ is identified with R via the isomorphism which maps the class of x to X and the class of y to Y. Thus, P_w is essentially the Alexander polynomial of the group $\langle x, y \mid w \rangle$ (which is only defined up to a multiplication by a unit of R and a change of basis of $\mathbb{Z}^2$). The geometric nature of our definition is useful to understand the intuition behind the algebraic arguments.

The proof of the next result is clear from the definition (see [3, Proposition 7]), the comments above about relation modules or the relation with Fox derivatives.

Lemma 3 *Let $w, w' \in [\mathbb{F}_2, \mathbb{F}_2]$, $u \in \mathbb{F}_2$. Then the following hold*:

(i) $P_{ww'} = P_w + P_{w'}$.

(ii) $P_{w^{-1}} = -P_w$.

(iii) $P_{uwu^{-1}} = X^k Y^l P_w$, *where k and l are the total exponents of x and y in u.*

(iv) $P_{[u,w]} = (X^k Y^l - 1) P_w$.

Call a presentation $\mathcal{P} = \langle x, y \mid r_1, r_2, \ldots, r_m \rangle$ *cocommutative* if each relator r_j lies in $[\mathbb{F}_2, \mathbb{F}_2]$.

Remark 4 Every presentation Q^*-equivalent to a cocommutative presentation is also cocommutative. We can associate to a cocommutative presentation $\mathcal{P}$ the vector $\Lambda(\mathcal{P}) = (P_{r_1}, P_{r_2}, \ldots, P_{r_m}) \in R^m$. This can also be seen as the first column in the Alexander matrix $A \in R^{m \times 2}$ of $\langle x, y \mid r_1, r_2, \ldots, r_m \rangle$ divided by $(1 - Y)$. The effect on $\Lambda(\mathcal{P})$ of performing a Q-transformation on $\mathcal{P}$ is to change a polynomial P_{r_j} by $P_{r_j} + P_{r_i}$ for certain $i \neq j$, or by $-P_{r_j}$, or by $X^k Y^l P_{r_j}$ for certain $k, l \in \mathbb{Z}$. Therefore, if $\mathcal{P}$ is a cocommutative presentation and $\mathfrak{Q}$ is Q-equivalent to $\mathcal{P}$, then $\Lambda(\mathfrak{Q})^t = E \Lambda(\mathcal{P})^t$ for some $E \in \mathrm{GE}_m(R)$. Here $\Lambda(\mathcal{P})^t, \Lambda(\mathfrak{Q})^t \in R^{m \times 1}$ denote the column vectors.

Figure 1: Left, the graphical representation of the curve γ_{r_1} which begins in the black dot. The represented curve and the actual curve are homotopic in the plane with the centers of the squares removed. In the interior of the squares we see the corresponding winding numbers. Right, the curve γ_{r_2}.

If $\mathcal{P} = \langle x, y \mid r_1, r_2, \ldots, r_m\rangle$ and $\mathcal{Q} = \langle x, y \mid s_1, s_2, \ldots, s_m\rangle$ are cocommutative presentations such that the normal closure of $\{r_1, r_2, \ldots, r_m\}$ coincides with the normal closure of $\{s_1, s_2, \ldots, s_m\}$, then each s_j is a product of conjugates of r_i's and inverses of r_i's. By Lemma 3, P_{s_j} is an R-linear combination of the P_{r_i}. Therefore $\Lambda(\mathcal{Q})^t = M\Lambda(\mathcal{P})^t$ for certain $M \in R^{m\times m}$. Symmetrically, $\Lambda(\mathcal{P})^t = M'\Lambda(\mathcal{Q})^t$ for some $M' \in R^{m\times m}$.

Let

$$\mathcal{P} = \langle x, y \mid [x,y], 1\rangle \quad\text{and}\quad \mathcal{Q} = \langle x, y \mid [x,[x,y^{-1}]]^2 y[y^{-1},x]y^{-1}, [x,[[y^{-1},x],x]]\rangle.$$

We state the main result of the article.

Theorem 5 *The standard complexes $K_{\mathcal{P}}$ and $K_{\mathcal{Q}}$ are simple homotopy equivalent, while $\mathcal{P}$ and $\mathcal{Q}$ are not Q^*-equivalent. Moreover, the normal closures of both relator sets coincide. Also, these examples occur at Euler characteristic one above the minimal level.*

Since the Whitehead group of $\pi_1(K_{\mathcal{P}}) = \mathbb{Z}\times\mathbb{Z}$ is trivial [4], to prove simple homotopy equivalence we only need to prove homotopy equivalence. This can be achieved by standard methods and we postpone this part to the end of the proof.

We begin by proving that $\mathcal{P}$ and $\mathcal{Q}$ are not Q-equivalent. We compute first $\Lambda(\mathcal{Q})$. Let

$$r_1 = [x,[x,y^{-1}]]^2 y[y^{-1},x]y^{-1} \quad\text{and}\quad r_2 = [x,[[y^{-1},x],x]]$$

be the relators of $\mathcal{Q}$. By Lemma 3 the winding invariant of r_1 is $2(X-1)P_{[x,y^{-1}]} + YP_{[y^{-1},x]}$. Since $P_{[y^{-1},x]} = Y^{-1}$ and $P_{[x,y^{-1}]} = -P_{[y^{-1},x]} = -Y^{-1}$, it follows that $P_{r_1} = 1-2(X-1)Y^{-1}$; see Figure 1. Similarly, $P_{r_2} = (X-1)P_{[[y^{-1},x],x]} = -(X-1)^2 P_{[y^{-1},x]} = -(X-1)^2Y^{-1}$. Thus,

$$\Lambda(\mathcal{Q}) = (1-2(X-1)Y^{-1}, -(X-1)^2Y^{-1})$$

is the first column of matrix M in Theorem 1.

On the other hand it is easy to see that $\Lambda(\mathcal{P}) = (1,0) \in R^2$. If $\mathcal{P}$ and $\mathcal{Q}$ are Q-equivalent, by Remark 4 there exists a matrix $E \in \mathrm{GE}_2(R)$ such that $M\binom{1}{0} = \Lambda(\mathcal{Q})^t = E\Lambda(\mathcal{P})^t = E\binom{1}{0}$. Then $E^{-1}M\binom{1}{0} = \binom{1}{0}$.

Thus $E^{-1}M$ is a matrix of the form

$$\begin{pmatrix} 1 & A \\ 0 & B \end{pmatrix}$$

for some $A, B \in R$. Moreover, since $E, M \in \mathrm{GL}_2(R)$, $B \in R$ is a unit, so $E_2 E_1 E^{-1} M = \mathrm{Id}$ for the diagonal and elementary matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & B^{-1} \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix}.$$

Then $M = E E_1^{-1} E_2^{-1} \in \mathrm{GE}_2(R)$, which contradicts Theorem 1. This completes the proof that $\mathcal{P}$ and $\mathcal{Q}$ are not Q-equivalent. The last lines of our proof are implicit in the comments of [8, page 115] about unimodular columns. The matrix M was used by Myasnikov, Myasnikov and Shpilrain in [20, Theorem 1.6] to study Q-transformations of m-tuples in a nonfree group G. As we mentioned above, the idea of our proof naturally appeared when studying applications of the winding invariant. We discovered later that our methods are very similar to those used in the proof of [20, Proposition 5.1].

The fact that $\mathcal{P}$ and $\mathcal{Q}$ are not Q^*-equivalent, either, will follow from the next key lemma.

Lemma 6 *The normal closure N of $\{r_1, r_2\}$ is $[\mathbb{F}_2, \mathbb{F}_2]$.*

Before we give a proof the lemma, we show how to use it to prove that $\mathcal{P}$ and $\mathcal{Q}$ are not Q^*-equivalent. Suppose they are. Then there is an automorphism ϕ of $\mathbb{F}_2$ such that $\phi\mathcal{P} = \langle x, y \mid \phi([x, y]), 1 \rangle$ and $\mathcal{Q}$ are Q-equivalent. Since Q-transformations preserve the normal closure of the relators, the normal closure of $\phi([x, y])$ is $[\mathbb{F}_2, \mathbb{F}_2] = N([x, y])$ by Lemma 6. By a well-known result of Magnus, $\phi([x, y])$ is a conjugate of $[x, y]$ or $[x, y]^{-1}$. In any case, $\phi\mathcal{P}$ is Q-equivalent to $\mathcal{P}$, so $\mathcal{P}$ and $\mathcal{Q}$ are Q-equivalent, a contradiction.

Alternatively, that $\mathcal{P}$ and $\mathcal{Q}$ are not Q^*-equivalent follows from the fact that there is no matrix $E \in \mathrm{GE}_2(R)$ such that $E\Lambda(\mathcal{P})^t = \Lambda(Q)^t$, and the following. If $\phi \in \mathrm{Aut}(\mathbb{F}_2)$, then the winding invariant $P_{\phi([x,y])}$ is a unit of R, so there exists $E' \in \mathrm{GE}_2(R)$ such that $E'\Lambda(\mathcal{P})^t = \Lambda(\phi\mathcal{P})^t$; see [3]. Then $\phi\mathcal{P}$ and $\mathcal{Q}$ cannot be Q-equivalent.

Proof of Lemma 6 It is clear that $r_1, r_2 \in [\mathbb{F}_2, \mathbb{F}_2]$, so we only need to show that $[x, y] = 1$ in $\mathbb{F}_2/N$. Let $d = [x, y^{-1}] = xy^{-1}x^{-1}y$. Since

$$1 = r_2 = [x, [d^{-1}, x]] = xd^{-1}xdx^{-1}x^{-1}xd^{-1}x^{-1}d = xd^{-1}xdx^{-1}d^{-1}x^{-1}d$$

in $\mathbb{F}_2/N$, it follows that $xdx^{-1}d^{-1} = dx^{-1}d^{-1}x = x^{-1}(xdx^{-1}d^{-1})x$. Therefore $e = [x, d]$ commutes with x in $\mathbb{F}_2/N$.

On the other hand $1 = r_1 = [x, d]^2 yd^{-1}y^{-1} = e^2 yd^{-1}y^{-1}$, so

$$d = y^{-1}e^2 y. \tag{1}$$

Finally, by definition, $e = xdx^{-1}d^{-1} = x(y^{-1}e^2y)x^{-1}d^{-1}$. But since $d = xy^{-1}x^{-1}y$, we have $dy^{-1}x = xy^{-1}$. We use this in the previous equation to obtain

$$e = dy^{-1}xe^2x^{-1}yd^{-1}d^{-1}.$$

We use now that e and x commute to deduce $e = dy^{-1}e^2yd^{-2} = ddd^{-2} = 1$ in $\mathbb{F}_2/N$. The last equality follows from (1). By (1) again we deduce $d = 1$ in $\mathbb{F}_2/N$. Thus, $d = [x, y^{-1}] \in N$, so $[x, y] \in N$. □

To finish the proof of the theorem we need to prove that $K_{\mathcal{P}}$ and $K_{\mathcal{Q}}$ are homotopy equivalent. We have done the most important part already in Lemma 6. It implies that $K_{\mathcal{P}}$ and $K_{\mathcal{Q}}$ have isomorphic fundamental groups with an isomorphism $\pi_1(K_{\mathcal{Q}}) \to \pi_1(K_{\mathcal{P}})$ induced by a map which is the identity on 1-skeletons. Then $K_{\mathcal{P}}$ and $K_{\mathcal{Q}}$ have isomorphic fundamental groups and the same Euler characteristic. In general this does not imply homotopy equivalence, but it does in our case since $\pi_1(K_{\mathcal{P}})$ is free abelian of rank 2. This is explained by Harlander in [11]: Suppose K and L are finite connected 2-dimensional complexes with $\pi_1(K) \cong \pi_1(L) \cong \mathbb{Z}\times\mathbb{Z}$ and $\chi(K) = \chi(L)$. Since $\mathbb{Z}\times\mathbb{Z}$ is aspherical, $H_2(\widetilde{K})$ and $H_2(\widetilde{L})$ are projective R-modules by the generalized Schanuel lemma. By the Quillen–Suslin theorem, these R-modules are free; see Swan's comments [23] on how to go from polynomials to Laurent polynomials. Since $\chi(K) = \chi(L)$, it follows that $H_2(\widetilde{K})$ and $H_2(\widetilde{L})$ have the same rank, so they are isomorphic. Once again, since $\mathbb{Z}\times\mathbb{Z}$ is aspherical, $H^3(\mathbb{Z}\times\mathbb{Z}, H_2(\widetilde{K})) = H^3(\mathbb{Z}\times\mathbb{Z}, H_2(\widetilde{L})) = 0$, so K and L have isomorphic algebraic 2-types. By Mac Lane–Whitehead's theorem [22, Theorem 4.9], K and L are homotopy equivalent. This finishes the proof of Theorem 5.

An alternative and simpler way to see that $K_{\mathcal{P}}$ and $K_{\mathcal{Q}}$ are homotopy equivalent is to show that there exists a homomorphism $\Phi\colon C_2(\widetilde{K}_{\mathcal{Q}}) \to C_2(\widetilde{K}_{\mathcal{P}})$ of R-modules that makes the following diagram commutative

$$\begin{array}{ccc} C_2(\widetilde{K}_{\mathcal{Q}}) & \xrightarrow{d_2} & C_1(\widetilde{K}_{\mathcal{Q}}) \\ \downarrow \Phi & & \| \\ C_2(\widetilde{K}_{\mathcal{P}}) & \xrightarrow{d_2'} & C_1(\widetilde{K}_{\mathcal{P}}) \end{array}$$

A computation of Fox derivatives, or our comments right after Definition 2, show that d_2 has matrix representation

$$\begin{pmatrix} (1-Y)(1-2(X-1)Y^{-1}) & (X-1)^2(1-Y^{-1}) \\ (X-1)(1-2(X-1)Y^{-1}) & -(X-1)^3Y^{-1} \end{pmatrix} = \begin{pmatrix} 1-Y \\ X-1 \end{pmatrix} \Lambda(\mathcal{Q}),$$

and d_2' is represented by

$$\begin{pmatrix} 1-Y & 0 \\ X-1 & 0 \end{pmatrix} = \begin{pmatrix} 1-Y \\ X-1 \end{pmatrix} \Lambda(\mathcal{P}).$$

Therefore the map $\Phi\colon C_2(\widetilde{K}_{\mathcal{Q}}) \to C_2(\widetilde{K}_{\mathcal{P}})$ represented by the transpose of the (invertible) matrix M in Theorem 1 is an isomorphism satisfying $d_2'\Phi = d_2$, and by [22, Theorem 3.9], there is a homotopy equivalence $f\colon K_{\mathcal{Q}} \to K_{\mathcal{P}}$.

The fact that the Euler characteristic of $\mathscr{P}$ and $\mathscr{Q}$ is one level above the minimal follows from the general bound $\chi(K) \geq 1 - \dim H_1(G;\mathbb{Q}) + \dim H_2(G;\mathbb{Q})$ that holds for any finite 2-dimensional complex K with $\pi_1(K) \cong G$, which in turn can be deduced from the fact that there is an epimorphism $H_2(K;\mathbb{Q}) \to H_2(G;\mathbb{Q})$ induced by the inclusion of K into a $K(G,1)$.

The tree of Q-equivalence Let G be a group. If K and L are finite 2-dimensional complexes with fundamental group G, there exist $k, l \geq 0$ such that $K \vee \bigvee_{i=1}^{k} S^2 \simeq L \vee \bigvee_{i=1}^{l} S^2$. In fact for some k, l, one of these complexes 3-deforms into the other [13, page 28]. The tree of homotopy types of finite 2-dimensional complexes with fundamental group G has a directed edge from the homotopy type of K to the homotopy type of $K \vee S^2$. The classification problem consists in understanding this tree for each group G. This has been achieved in few examples (free groups, finite abelian, and few others), and interesting features of the tree have been found in other cases. The complexes of minimal Euler characteristic (minimal among all the 2-complexes with fundamental group G) are roots of this tree, but there can be roots above minimal characteristic. Very recently Nicholson [21] proved that for every $k \geq 0$ there exists a group G and (infinitely many) distinct homotopy types of 2-complexes with fundamental group G and Euler characteristic equal to the minimal plus k. It is an open problem whether there exist 2-complexes X and Y such that $X \vee S^2 \not\simeq Y \vee S^2$ while $X \vee S^2 \vee S^2 \simeq Y \vee S^2 \vee S^2$.

The trees of simple homotopy types and 3-deformation types are defined similarly. If two finite presentations $\mathscr{P}_1 = \langle X \mid R \rangle, \mathscr{P}_2 = \langle X \mid S \rangle$ with same generator set X have their relators with equal normal closure $N(R) = N(S)$, then there exists $k, l \geq 0$ such that the presentations $\mathscr{P}_1^{(k)}$ and $\mathscr{P}_2^{(l)}$ obtained from $\mathscr{P}_1$ and $\mathscr{P}_2$ by adding k and l trivial relators respectively, are Q-equivalent. In fact, k and l can be taken as $|S \setminus R|$ and $|R \setminus S|$, respectively. Given a group G, a finite set X and a normal subgroup $N \trianglelefteq F(X)$ such that $F(X)/N \cong G$, the tree of Q-equivalence classes of finite presentations of G with respect to X and N has a directed edge from the Q-equivalence class of a presentation $\mathscr{P}_1 = \langle X \mid R \rangle$ with $N(R) = N$ to the class of $\mathscr{P}_1^{(1)}$. The example of Metzler of presentations of $\mathbb{Z}_5 \times \mathbb{Z}_5$ with minimal Euler characteristic shows that the tree of Q-equivalence classes of $\mathbb{Z}_5 \times \mathbb{Z}_5$ with respect to $\{x, y\}$, $N(x^5, y^5, [x, y])$ has two roots $\langle x, y \mid R \rangle$ and $\langle x, y \mid S \rangle$ of minimal Euler characteristic, which are adjacent to a same type, since $|R \setminus S| = |S \setminus R| = 1$; see Figure 2. Our example shows that a different situation may happen. Our presentations $\mathscr{P}$ and $\mathscr{Q}$ have Euler characteristic one above the minimal level. In fact $\mathscr{P}$ is not a root, but $\mathscr{Q}$ is. Moreover, the tree of Q-equivalence classes of presentations of $\mathbb{Z} \times \mathbb{Z}$ with respect to $\{x, y\}$ and $\mathbb{F}_2'$

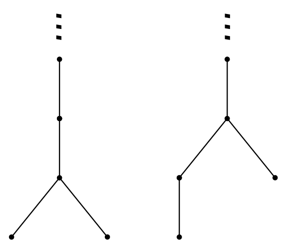

Figure 2: Part of the trees of Q-equivalence classes of Metzler's example and ours.

has a unique class with minimal Euler characteristic, which is $\langle x, y \mid [x, y]\rangle$. We have that $\mathcal{P}^{(1)}$ and $\mathcal{Q}^{(1)}$ are Q-equivalent.

Note that the Andrews–Curtis conjecture says that the tree of Q-equivalence classes of presentations of the trivial group with respect to $\{x_1, x_2, \dots, x_n\}$ and $\mathbb{F}_n$ has a unique class with minimal Euler characteristic.

We mention other known examples, from the perspective of the tree of Q-equivalence. In [16], Lustig proves that the presentations

$$\mathcal{P}_1 = \langle x, y, z \mid y^3, yx^{10}y^{-1}x^{-5}, [x^7, z]\rangle,$$
$$\mathcal{P}_2 = \langle x, y, z \mid y^3, yx^{10}y^{-1}x^{-5}, x^{14}zx^{14}z^{-1}x^{-7}zx^{-21}z^{-1}\rangle,$$

are homotopy equivalent and not simple homotopy equivalent. Moreover, he proves that the normal closures of relators coincide. Thus $\mathcal{P}_1^{(1)}$ is Q-equivalent to $\mathcal{P}_2^{(1)}$.

The twisted presentations of a finite abelian group (see Latiolais [15]) can be used to show that for any k there exists a group G and presentations $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ of G with minimal Euler characteristic whose standard complexes are pairwise not homotopy equivalent but such that $\mathcal{P}_i^{(1)}$ and $\mathcal{P}_j^{(1)}$ are Q-equivalent for every $1 \le i, j \le k$.

In [17] Mannan and Popiel give an example of two presentations $\mathcal{P}_1 = \langle x, y \mid y^2x^{-7}, xyxy^{-1}\rangle$ and $\mathcal{P}_2 = \langle x, y \mid y^2x^{-7}, y^{-1}xyx^2y^{-1}x^{-2}yx^{-3}\rangle$ of the quaternion group Q_{28} with nonisomorphic second homotopy modules for any identification of fundamental groups (and thus not homotopy equivalent). Again they have minimal Euler characteristic. It is proved that both relator sets have equal normal closures. Thus, $\mathcal{P}_1^{(1)}$ and $\mathcal{P}_2^{(1)}$ are Q-equivalent.

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