Algebraic & Geometric Topology

Volume 25 (2025)

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Algebraic & Geometric Topology **25**:1 (2025) 489–561
DOI: 10.2140/agt.2025.25.489
Published: 24 March 2025

Mapping tori of A_∞-autoequivalences and Legendrian lifts of exact Lagrangians in circular contactizations

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We study mapping tori of quasi-autoequivalences $\tau\colon \mathcal{A} \to \mathcal{A}$ which induce a free action of $\mathbb{Z}$ on objects. More precisely, we compute the mapping torus of τ when it is strict and acts bijectively on hom-sets, or when the A_∞-category $\mathcal{A}$ is directed and there is a bimodule map $\mathcal{A}(-,-) \to \mathcal{A}(-,\tau(-))$ satisfying some hypotheses. Then we apply these results in order to link together the Fukaya A_∞-category of a family of exact Lagrangians, and the Chekanov–Eliashberg DG-category of Legendrian lifts in the circular contactization.

53D42, 53D37, 18G70

Introduction

Legendrian contact homology was introduced by Chekanov [8] and Eliashberg [20], and it fits into the symplectic field theory as introduced by Eliashberg, Givental and Hofer [21]. It has been rigorously defined in the contactization of a Liouville manifold by Ekholm, Etnyre and Sullivan in [16] following [14]. The importance of Legendrian contact homology goes beyond its applications to the Legendrian isotopy problem: for example, it was used by Bourgeois, Ekholm and Eliashberg in [5] to compute symplectic invariants of Weinstein manifolds, and in a different way by Chantraine, Dimitroglou Rizell, Ghiggini and Golovko in [7] to prove a generation result for the wrapped Fukaya category of Weinstein manifolds.

The motivation for this paper is the study of Legendrian contact homology in subcritically fillable and Boothby–Wang contact manifolds, the latter being named after [4]. This has been done combinatorially in dimension three by Ekholm and Ng in [18] for the subcritically fillable case, and by Sabloff in [34] for the Boothby–Wang case. The importance of the first kind of manifolds comes from the fact that every Weinstein manifold is obtained from a subcritical Weinstein manifold (of the form $\mathbb{C} \times P$ for some Weinstein manifold P) by attaching handles along Legendrian submanifolds in its boundary at infinity. The importance of the second kind of manifolds comes from a theorem of Donaldson in [11], which states that any integral symplectic manifold (X, ω) admits a symplectic submanifold $D \subset X$ of codimension 2, such that $X \setminus D$ is a Liouville manifold whose boundary at infinity is a Boothby–Wang contact manifold. The first step before attacking both cases presented above is to study Legendrian contact homology in the circular contactization of a Liouville manifold. In fact, both subcritically fillable and Boothby–Wang contact manifolds can be seen as compactifications of such spaces. This paper links together the Fukaya A_∞-category of a family of connected compact exact Lagrangians in a Liouville manifold (P, λ), and the Chekanov–Eliashberg DG-category of Legendrian lifts in the circular contactization $(S^1 \times P, \ker(d\theta - \lambda))$.

The strategy we follow is to lift the situation to the usual contactization $\mathbb{R} \times P$ which has been much more studied. This naturally leads to consider an A_∞-category whose objects are the lifts in $\mathbb{R} \times P$ of our starting Legendrians, and morphisms spaces are generated by Reeb chords. Moreover, the deck transformations of the cover $\mathbb{R} \to S^1$ induce an A_∞-autoequivalence of this category. The rest of the proof has two main ingredients:

(1) Functorial properties of the Legendrian invariants, which are used to bring us in a situation where we can apply the correspondence result of Dimitroglou Rizell [10] between discs in the symplectization $\mathbb{R} \times \mathbb{R} \times P$ and polygons in P.

(2) Two algebraic results of independent interest about mapping tori of A_∞-autoequivalences, that allow us to bridge the gaps between the algebraic invariants we are interested in.

We now proceed to describe the organization of the paper and state our main results.

Algebra In Section 1, we briefly recall the definitions of A_∞-(co)categories and give references for standard notions that we do not recall, such as (co)bar, graded dual and Koszul dual constructions. On the other hand, we discuss in some detail the notions of modules over A_∞-categories, as well as the Grothendieck construction and homotopy pushout associated to a diagram of A_∞-categories following Ganatra, Pardon and Shende [24, Section A.4]. We use it to introduce the notion of "cylinder object for an A_∞-category", which is supposed to mimic the corresponding notion in homotopy theory.

Mapping torus of an A_∞-autoequivalence In Section 2,[1] we define the mapping torus associated to a quasi-autoequivalence τ of an A_∞-category $\mathcal{A}$ as the A_∞-category

$$\mathrm{MT}(\tau) := \operatorname{hocolim}\begin{pmatrix} \mathcal{A} \sqcup \mathcal{A} & \longrightarrow & \mathcal{A} \\ \big\downarrow & & \\ \mathcal{A} & & \end{pmatrix}.$$

[1] In Section 2, A_∞-categories are always assumed to be *strictly unital* (see Paragraph (2a) in Seidel's work[36]).

Observe that this terminology was also used by Kartal in [26], but we do not know if the two notions coincide. When considering an A_∞-autoequivalence $\tau\colon \mathcal{A} \to \mathcal{A}$, we always assume that $\mathcal{A}$ is equipped with a $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$ compatible with τ, which is a bijection

$$\mathbb{Z} \times \mathcal{E} \xrightarrow{\sim} \mathrm{ob}(\mathcal{A}), \quad (n, E) \mapsto X^n(E),$$

such that $\tau(X^n(E)) = X^{n+1}(E)$ for every $n \in \mathbb{Z}$ and $E \in \mathcal{E}$ (see Definition 2.2). This naturally turns $\mathcal{A}$ into an Adams-graded A_∞-category, where the Adams degree of a morphism in $\mathcal{A}(X^i(E), X^j(E'))$ is defined to be $j - i$. It then follows that the mapping torus of τ is also Adams-graded.

Section 2 contains two results about mapping tori of A_∞-autoequivalences: we choose to only state the most important ones in this introduction. We denote by $\mathbb{F}[t_m]$ the augmented Adams-graded associative algebra generated by a variable t_m of bidegree $(m, 1)$. Observe that if $\mathcal{C}$ is a subcategory of an A_∞-category $\mathcal{D}$ with $\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{D})$, then $\mathcal{C} \oplus (t_m \mathbb{F}[t_m] \otimes \mathcal{D})$ is naturally an Adams-graded A_∞-category, where the Adams degree of $t_m^k \otimes x$ equals k. Besides, if $\mathcal{C}$ is an A_∞-category equipped with a $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{C})$, we denote by $\mathcal{C}^0$ the full A_∞-subcategory of $\mathcal{C}$ whose set of objects corresponds to $\{0\} \times \mathcal{E}$. Finally, we use the functor $\mathcal{C} \mapsto \mathcal{C}_m$ of Definition 1.27.

Theorem A *Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$, weakly directed with respect to some compatible $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$. Assume that there exists a closed degree 0 bimodule map $f\colon \mathcal{A}_m(-,-) \to \mathcal{A}_m(-,\tau(-))$ such that $f\colon \mathcal{A}_m(X^i(E), X^j(E')) \to \mathcal{A}_m(X^i(E), X^{j+1}(E'))$ is a quasi-isomorphism for every $i < j$ and $E, E' \in \mathcal{E}$. Then there is a quasi-equivalence of Adams-graded A_∞-categories*

$$\mathrm{MT}(\tau) \simeq \mathcal{A}_m^0 \oplus (t_m \mathbb{F}[t_m] \otimes \mathcal{A}_m[f(\mathrm{units})^{-1}]^0).$$

Remark (1) In Ganatra's work [22], the chain complex of $\mathcal{A}$-bimodule maps from the diagonal bimodule $\mathcal{A}(-,-)$ to some $\mathcal{A}$-bimodule $\mathcal{B}$ is called the two-pointed complex for Hochschild cohomology of $\mathcal{A}$ with coefficients in $\mathcal{B}$. According to [22, Proposition 2.5], this complex is quasi-isomorphic to the (ordinary) Hochschild cochain complex of $\mathcal{A}$ with coefficients in $\mathcal{B}$. In particular, the bimodule map f in Theorem A defines a class in the Hochschild cohomology of $\mathcal{A}_m$ with coefficients in $\mathcal{A}_m(-,\tau(-))$.

(2) The A_∞-category which computes the mapping torus in Theorem A is very similar to the categories studied by Seidel in [35], with main difference the presence of curvature in Seidel's setting.

(3) The use of the functor $\mathcal{C} \mapsto \mathcal{C}_m$ in Theorem A is not of any deep importance. It was convenient for us to introduce it here for our application to Legendrian contact homology (see Theorem B).

Chekanov–Eliashberg DG-algebra In Section 3, we recall the definition and functorial properties of the Chekanov–Eliashberg DG-category associated to a family of Legendrians in a hypertight contact manifold.

Legendrian lifts of exact Lagrangians in the circular contactization In Section 4, we start with a family

$$\boldsymbol{L} = (L(E))_{E \in \mathcal{E}}, \quad \mathcal{E} = \{1, \dots, N\},$$

of mutually transverse compact connected exact Lagrangian submanifolds in a Liouville manifold (P,λ), and we study a Legendrian lift of $\boldsymbol{L}$ in the circular contactization $(S^1\times P, \ker(d\theta-\lambda))$. More precisely, we assume[2] that there are primitives $f_E\colon L(E)\to\mathbb{R}$ of $\lambda|_{L(E)}$ such that $0\le f_1<\cdots<f_N\le\frac{1}{2}$, and we consider the family of Legendrians

$$\boldsymbol{\Lambda}^\circ := (\Lambda^\circ(E))_{E\in\mathcal{E}}, \quad \text{where } \Lambda^\circ(E)=\{(f_E(x),x)\in(\mathbb{R}/\mathbb{Z})\times P \mid x\in L(E)\}.$$

We denote by $\mathrm{CE}(\boldsymbol{\Lambda}^\circ)$ the Chekanov–Eliashberg category of $\boldsymbol{\Lambda}^\circ$, by $\mathcal{F}uk(\boldsymbol{L})$ the full subcategory of $\mathcal{F}uk(P)$ (see for example [36, Chapter 2]) with objects the Lagrangians $L(E)$, and by $\overrightarrow{\mathcal{F}uk}(\boldsymbol{L})$ its directed subcategory (see [36, Paragraph (5n)]).

In order for the latter algebraic objects to be $\mathbb{Z}$-graded, we assume that $H_1(P)$ is free, that the first Chern class of P (equipped with any almost complex structure compatible with $(-d\lambda)$) is 2-torsion, and that the Maslov class of the Lagrangians $L(E)$ vanish. As explained in Section 3.1, the grading on $\mathrm{CE}(\boldsymbol{\Lambda}^\circ)$ depends on the choice of a symplectic trivialization of the contact structure along a fiber $h_0=S^1\times\{a_0\}$. We denote by $\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)$ the Chekanov–Eliashberg DG-category of $\boldsymbol{\Lambda}^\circ$ with grading induced by the trivialization

$$(\xi^\circ|_{h_0}, d\alpha^\circ)\xrightarrow{\sim}(h_0\times\mathbb{C}^n, dx\wedge dy), \quad ((\theta,a_0),(\lambda_{a_0}(v),v))\mapsto((\theta,a_0),e^{2i\pi r\theta}\psi(v)),$$

where $\psi\colon(T_{a_0}P,-d\lambda_{a_0})\xrightarrow{\sim}(\mathbb{C}^n,dx\wedge dy)$ is a symplectic isomorphism.

In this setting, $\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)$ is augmented (with the trivial augmentation) and Adams-graded (by the number of times a Reeb chord winds around the fiber). As above, we denote by $\mathbb{F}[t_m]$ the augmented Adams-graded associative algebra generated by a variable t_m of bidegree $(m,1)$. Moreover, we denote by $E(-)=B(-)^\#$ (graded dual of bar construction) the Koszul dual functor (see work by Lu, Palmieri, Wu and Zhang [29, Section 2] or Ekholm and Lekili [17, Section 2.3]). We say that Koszul duality holds for an augmented Adams-graded A_∞-category A if the natural map $A\to E(E(A))$ is a quasi-isomorphism (see [29, Theorem 2.4] or [17, Definition 17]).

Theorem B *Koszul duality holds for* $\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)$*, and there is a quasi-equivalence of augmented Adams-graded* A_∞*-categories*

$$E(\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ))\simeq\overrightarrow{\mathcal{F}uk}(\boldsymbol{L})\oplus(t_{2r}\mathbb{F}[t_{2r}]\otimes\mathcal{F}uk(\boldsymbol{L})).$$

Remark Koszul duality has many important consequences, see for example [29] or [17]. In particular, Theorem B implies that there is a quasi-equivalence of augmented Adams-graded DG-categories

$$\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)\simeq E\big(\overrightarrow{\mathcal{F}uk}(\boldsymbol{L})\oplus(t_{2r}\mathbb{F}[t_{2r}]\otimes\mathcal{F}uk(\boldsymbol{L}))\big).$$

Observe that in the particular case when the Lagrangians are spheres, this formula is closely related to Conjecture 6.3 in [35], which was also discussed by Ganatra and Maydanskiy in the appendix of [5].

[2] This can always be achieved by applying the Liouville flow in backwards time.

We now give a corollary of the latter result. If B is a (unpointed) space, we consider its one-point compactification B^* and view it as a pointed space (with basepoint the point at infinity). If moreover X is a pointed space, we consider the half-smash product of B and X,

$$X \rtimes B := X \wedge B^*$$

(where $\wedge$ denotes the smash product of pointed spaces). Finally, if Y is a pointed space, we denote by ΩY its based loop space.

Corollary *If L is a connected compact exact Lagrangian and Λ° is a Legendrian lift of L in the circular contactization, then there is a quasi-equivalence of augmented DG-algebras*

$$\mathrm{CE}^1_{-*}(\Lambda^\circ) \simeq C_{-*}(\Omega(\mathbb{CP}^\infty \rtimes L)).$$

Acknowledgments This work is part of my PhD thesis that I did at Nantes Université under the supervision of Paolo Ghiggini and Vincent Colin, who I thank for their guidance and support. I also thank Baptiste Chantraine, Georgios Dimitroglou Rizell and Tobias Ekholm for helpful discussions. Finally, I thank the anonymous referee for suggestions that helped improve the exposition of the paper. The work presented in this article is supported by Villum Fonden, Villum Investigator grant 37814.

1 Algebra

In the following, $\mathbb{F}$ denotes the field $\mathbb{Z}/2\mathbb{Z}$. Vector spaces are always over $\mathbb{F}$.

Definition 1.1 An A_∞-category $\mathcal{A}$ is the data of

(1) a collection of objects $\mathrm{ob}\,\mathcal{A}$,

(2) for every objects X, Y, a graded vector space of morphisms $\mathcal{A}(X, Y)$,

(3) a family of degree $2-d$ linear maps

$$\mu^d : \mathcal{A}(X_0, X_1) \otimes \cdots \otimes \mathcal{A}(X_{d-1}, X_d) \to \mathcal{A}(X_0, X_d)$$

indexed by the sequences of objects $(X_0, \dots, X_d)$, $d \geq 1$, such that

$$\sum_{0 \leq i < j \leq d} \mu^{d-(j-i)+1} \circ (\mathbf{1}^i \otimes \mu^{j-i} \otimes \mathbf{1}^{d-j}) = 0,$$

for all $d \geq 1$.

Definition 1.2 An A_∞-cocategory $\mathcal{C}$ is the data of

(1) a collection of objects $\mathrm{ob}\,\mathcal{C}$,

(2) for every objects X, Y, a graded vector space of morphisms $\mathcal{C}(X, Y)$,

(3) a family of degree $2-d$ linear maps

$$\delta^d : \mathcal{C}(X_0, X_d) \to \bigoplus_{d \geq 1} \bigoplus_{X_1, \dots, X_{d-1}} \mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{d-1}, X_d)$$

indexed by the sequences of objects $(X_0, \dots, X_d)$, $d \ge 1$, such that

- for all $d \ge 1$,
$$\sum_{0 \le i < j \le d} (\mathbf{1}^i \otimes \delta^{j-i} \otimes \mathbf{1}^{d-j}) \circ \delta^{d-(j-i)+1} = 0,$$
- the map
$$C \to \prod_{d \ge 1} C^{\otimes d}, \quad x \mapsto (\delta^d(x))_{d \ge 1},$$
factors through the inclusion $\bigoplus_{d \ge 1} C^{\otimes d} \to \prod_{d \ge 1} C^{\otimes d}$.

Remark If $\mathcal{E}$ is some set, denote by $\mathbb{F}_{\mathcal{E}}$ the semisimple algebra over $\mathbb{F}$ generated by elements e_X, $X \in \mathcal{E}$, such that
$$e_X \cdot e_Y = \begin{cases} e_X & \text{if } X = Y, \\ 0 & \text{if } X \neq Y. \end{cases}$$
To any A_∞-category $\mathcal{A}$ with $\mathrm{ob}(\mathcal{A}) = \mathcal{E}$, we can associate an A_∞-algebra over $\mathbb{F}_{\mathcal{E}}$ where

- the underlying graded vector space is $\bigoplus_{X,Y \in \mathcal{E}} \mathcal{A}(X, Y)$,
- given $x \in \mathcal{A}(X_0, Y_0)$,
$$e_X \cdot x = \begin{cases} x & \text{if } X = X_0, \\ 0 & \text{if } X \neq X_0, \end{cases} \qquad \text{and} \qquad x \cdot e_Y = \begin{cases} x & \text{if } Y = Y_0, \\ 0 & \text{if } Y \neq Y_0, \end{cases}$$
- operations are the same as on $\mathcal{A}$.

Conversely, to any A_∞-algebra over $\mathbb{F}_{\mathcal{E}}$, one can associate an A_∞-category with $\mathrm{ob}(\mathcal{A}) = \mathcal{E}$. Note that the above discussion also applies to A_∞-cocategories. As a result, the theory of A_∞-(co)categories with $\mathcal{E}$ as set of objects is equivalent to the theory of A_∞-(co)algebras over $\mathbb{F}_{\mathcal{E}}$.

In this paper, we will appeal to several standard notions in the theory of A_∞-(co)categories that we choose not to recall: instead, we list them and give corresponding references.

- For A_∞-(co)maps, (co)augmentations and (co)bar, graded dual, Koszul dual constructions, see [17, Section 2] (where everything is written in the language of A_∞-(co)algebras over $\mathbb{F}_{\mathcal{E}}$).
- For general definitions and results about A_∞-categories (in particular about homotopy between A_∞-functors, homological perturbation theory, directed (sub)categories and twisted complexes), see [36, Chapter 1].
- For quotient of A_∞-categories, see [30], and for localization of A_∞-categories, see [23, Section 3.1.3].

Finally, we will use the following notion.

Definition 1.3 An Adams-graded vector space is a $\mathbb{Z}\times\mathbb{Z}$-graded vector space: if x is an element in the (i, j) component, we say that i is the cohomological degree of x, and j is the Adams degree of x. An Adams-graded A_∞-(co)category is an A_∞-(co)category enriched over Adams-graded vector spaces, where the operations are required to be of degree 0 with respect to the Adams grading. See [29] for a treatment of Koszul duality in the context of Adams-graded A_∞-algebras.

1.1 Modules over A_∞-categories

Let $\mathcal{C},\mathcal{D}$ be two A_∞-categories, and let $\mathcal{A},\mathcal{B}$ be two full subcategories of $\mathcal{C},\mathcal{D}$, respectively.

Definition 1.4 A $(\mathcal{C},\mathcal{D})$-bimodule $\mathcal{M}$ consists of the following data:

(1) for every pair $(X,Y)\in\mathrm{ob}(\mathcal{C})\times\mathrm{ob}(\mathcal{D})$, a graded vector space $\mathcal{M}(X,Y)$,

(2) a family of degree $1-p-q$ linear maps

$$\mu_{\mathcal{M}}:\mathcal{C}(X_0,X_1)\otimes\cdots\otimes\mathcal{C}(X_{p-1},X_p)\otimes\mathcal{M}(X_p,Y_q)\otimes\mathcal{D}(Y_q,Y_{q-1})\otimes\cdots\otimes\mathcal{D}(Y_1,Y_0)\to\mathcal{M}(X_0,Y_0)$$

indexed by the sequences

$$(X_0,\dots,X_p,Y_0,\dots,Y_q)\in\mathrm{ob}(\mathcal{C})^{p+1}\times\mathrm{ob}(\mathcal{D})^{q+1},$$

which satisfy the relations

$$\sum\mu_{\mathcal{M}}(\dots,\mu_{\mathcal{C}}(\dots),\dots,u\dots)+\sum\mu_{\mathcal{M}}(\dots,\mu_{\mathcal{M}}(\dots,u,\dots),\dots)+\sum\mu_{\mathcal{M}}(\dots,u,\dots,\mu_{\mathcal{D}}(\dots),\dots)=0.$$

A degree s morphism $t:\mathcal{M}_1\to\mathcal{M}_2$ between two $(\mathcal{C},\mathcal{D})$-bimodules consists of a family of degree $s-p-q$ linear maps

$$t:\mathcal{C}(X_0,X_1)\otimes\cdots\otimes\mathcal{C}(X_{p-1},X_p)\otimes\mathcal{M}_1(X_p,Y_q)\otimes\mathcal{D}(Y_q,Y_{q-1})\otimes\cdots\otimes\mathcal{D}(Y_1,Y_0)\to\mathcal{M}_2(X_0,Y_0)$$

indexed by the sequences

$$(X_0,\dots,X_p,Y_0,\dots,Y_q)\in\mathrm{ob}(\mathcal{C})^{p+1}\times\mathrm{ob}(\mathcal{D})^{q+1}.$$

The differential of such a morphism is defined by

$$\mu^1_{\mathrm{Mod}_{\mathcal{C},\mathcal{D}}}(t)(\dots,u,\dots)=\sum t(\dots,\mu_{\mathcal{C}}(\dots),\dots,u,\dots)+\sum t(\dots,\mu_{\mathcal{M}_1}(\dots,u,\dots),\dots)+\sum t(\dots,u,\dots,\mu_{\mathcal{D}}(\dots),\dots)+\sum\mu_{\mathcal{M}_2}(\dots,t(\dots,u,\dots),\dots).$$

Finally, the composition of $t_1:\mathcal{M}_1\to\mathcal{M}_2$ and $t_2:\mathcal{M}_2\to\mathcal{M}_3$ is such that

$$\mu^2_{\mathrm{Mod}_{\mathcal{C}}}(t_1,t_2)(\dots,u,\dots)=\sum t_2(\dots,t_1(\dots,u,\dots),\dots).$$

We denote by $\mathrm{Mod}_{\mathcal{C},\mathcal{D}}$ the DG-category of $(\mathcal{C},\mathcal{D})$-bimodules.

Definition 1.5 Let $\Phi_1,\Phi_2:\mathcal{C}\to\mathcal{D}$ be two A_∞-functors. Then there is a $\mathcal{C}$-bimodule $\mathcal{D}(\Phi_1(-),\Phi_2(-))$ defined as follows:

(1) On objects, it sends (X_1,X_2) to $\mathcal{D}(\Phi_1X_1,\Phi_2X_2)$.

(2) On morphisms, it sends a sequence $(\dots,y,\dots)$ in

$$\mathcal{C}(X_0,X_1)\times\cdots\times\mathcal{C}(X_{p-1},X_p)\times\mathcal{D}(\Phi_1X_p,\Phi_2X_{p+1})\times\mathcal{C}(X_{p+1},X_{p+2})\times\cdots\times\mathcal{C}(X_{p+q},X_{p+q+1})$$

to

$$\mu_{\mathcal{D}(\Phi_1(-),\Phi_2(-))}(\dots,y,\dots)=\sum\mu_{\mathcal{D}}\big(\Phi_1(\dots),\dots,\Phi_1(\dots),y,\Phi_2(\dots),\dots,\Phi_2(\dots)\big).$$

In the following, we will focus on *left* $\mathcal{C}$-modules, which correspond to $(\mathcal{C}, \mathbb{F})$-bimodules. We denote by $\mathrm{Mod}_{\mathcal{C}}$ the DG-category of (left) $\mathcal{C}$-modules.

Definition 1.6 Let $t\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a degree 0 closed $\mathcal{C}$-module map. We say that t is a quasi-isomorphism if the induced chain map $t\colon \mathcal{M}_1(X) \to \mathcal{M}_2(X)$ is a quasi-isomorphism for every object X in $\mathcal{C}$. (See [24, Section A.2] for a discussion on quasi-isomorphisms between A_∞-modules.)

Definition 1.7 Let $t, t'\colon \mathcal{M}_1 \to \mathcal{M}_2$ be two degree 0 closed morphisms of $\mathcal{C}$-modules. A homotopy between t and t' is a $\mathcal{C}$-module map $h\colon \mathcal{M}_1 \to \mathcal{M}_2$ such that

$$t + t' = \mu^1_{\mathrm{Mod}_{\mathcal{C}}}(h).$$

Definition 1.8 (see [36, Paragraph (1l); 24, Section A.1]) There is an A_∞-functor

$$\mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}, \quad Y \mapsto \mathcal{C}(-, Y),$$

called the Yoneda A_∞-functor, defined as follows. For every object X,

$$\mathcal{C}(-, Y)(X) = \mathcal{C}(X, Y).$$

Also, a sequence

$$(x_0, \dots, x_{d-1}) \in \mathcal{C}(X_0, X_1) \times \cdots \times \mathcal{C}(X_{d-1}, X_d)$$

acts on an element u in $\mathcal{C}(X_d, Y)$ via the operations

$$\mu_{\mathcal{C}(-,Y)}(x_0, \dots, x_{d-1}, u) = \mu_{\mathcal{C}}(x_0, \dots, x_{d-1}, u).$$

Finally, let

$$\boldsymbol{y} = (y_0, \dots, y_{p-1}) \in \mathcal{C}(Y_0, Y_1) \times \cdots \times \mathcal{C}(Y_{p-1}, Y_p)$$

be a sequence of morphisms in $\mathcal{C}$. Then the Yoneda functor gives a morphism of $\mathcal{C}$-modules

$$t_{\boldsymbol{y}} : \mathcal{C}(-, Y_0) \to \mathcal{C}(-, Y_p)$$

which sends every sequence $(x_0, \dots, x_{d-1}, u)$ as above to

$$\mu_{\mathcal{C}}(x_0, \dots, x_{d-1}, u, y_0, \dots, y_{p-1}) \in \mathcal{C}(X_0, Y_p).$$

We have the following important result.

Proposition 1.9 (Yoneda lemma) *The Yoneda A_∞-functor*

$$\mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}, \quad Y \mapsto \mathcal{C}(-, Y),$$

is cohomologically full and faithful.

Proof This is Lemma 2.12 in [36], and also Lemma A.1 in [24]. □

The Yoneda lemma has the following easy consequence. We state it for future reference.

Corollary 1.10 *Every closed $\mathcal{C}$-module map $f : \mathcal{C}(-, X) \to \mathcal{C}(-, Y)$ is homotopic to the $\mathcal{C}$-module map $t_{f(e_X)}$ induced by $f(e_X) \in \mathcal{C}(X, Y)$. (see Definition 1.8).*

Proof According to the Yoneda lemma, f is homotopic to t_x for some closed x in $\mathcal{C}(X,Y)$. Thus, there exists a $\mathcal{C}$-module map $h\colon \mathcal{C}(-,X)\to\mathcal{C}(-,Y)$ such that

$$f = t_x + \mu^1_{\mathrm{Mod}_{\mathcal{C}}}(h).$$

Evaluating the latter relation at the unit $e_X\in\mathcal{C}(X,X)$ gives

$$f(e_X) = x + \mu^1_{\mathcal{C}}(he_X).$$

Therefore, x is homotopic to $f(e_X)$, and this implies that t_x is homotopic to $t_{f(e_X)}$ by the Yoneda lemma. Finally, we have that f is homotopic to $t_{f(e_X)}$. □

Pullback of A_∞-modules

Definition 1.11 (see [36, Paragraph (1k)]) Let $\Phi\colon\mathcal{C}\to\mathcal{D}$ be an A_∞-functor. Then there is a DG-functor

$$\Phi^*\colon \mathrm{Mod}_{\mathcal{D}}\to\mathrm{Mod}_{\mathcal{C}},\quad \mathcal{N}\mapsto\Phi^*\mathcal{N},$$

defined as follows. Let $\mathcal{N}$ be a $\mathcal{D}$-module. For every object X,

$$\Phi^*\mathcal{N}(X) = \mathcal{N}(\Phi X).$$

Also, a sequence

$$(x_0,\dots,x_{d-1})\in\mathcal{C}(X_0,X_1)\times\cdots\times\mathcal{C}(X_{d-1},X_d)$$

acts on an element $u\in\Phi^*\mathcal{N}(X_d)$ via the operations

$$\mu_{\Phi^*\mathcal{N}}(x_0,\dots,x_{d-1},u) = \sum\mu_{\mathcal{N}}(\Phi(x_0,\dots,x_{i_1-1}),\dots,\Phi(x_{d-i_r},\dots,x_{d-1}),u).$$

Finally, let $t\colon\mathcal{N}_1\to\mathcal{N}_2$ be a $\mathcal{D}$-module map. Then the above functor gives a $\mathcal{C}$-module map

$$\Phi^*t\colon\Phi^*\mathcal{N}_1\to\Phi^*\mathcal{N}_2$$

which sends every sequence $(x_0,\dots,x_{d-1},u)$ as above to

$$\Phi^*t(x_0,\dots,x_{d-1},u) = \sum t(\Phi(x_0,\dots,x_{i_1-1}),\dots,\Phi(x_{d-i_r},\dots,x_{d-1}),u).$$

Remark 1.12 Let $\Phi\colon\mathcal{C}\to\mathcal{D}$ be an A_∞-functor, and let $\Psi\colon\mathcal{D}\to\mathcal{E}$ be another A_∞-functor towards a third A_∞-category $\mathcal{E}$. Then $\Phi^*\circ\Psi^* = (\Psi\circ\Phi)^*$ as DG-functors.

Definition 1.13 Let Y be an object of $\mathcal{C}$, and let $\Phi\colon\mathcal{C}\to\mathcal{D}$ be an A_∞-functor. Then there is a degree 0 closed $\mathcal{C}$-module map $t_\Phi\colon\mathcal{C}(-,Y)\to\Phi^*\mathcal{D}(-,\Phi(Y))$ which sends any sequence

$$(x_0,\dots,x_{d-1},u)\in\mathcal{C}(X_0,X_1)\times\cdots\times\mathcal{C}(X_{d-1},X_d)\times\mathcal{C}(X_d,Y)$$

to

$$t_\Phi(x_0,\dots,x_{d-1},u) = \Phi(x_0,\dots,x_{d-1},u)\in\mathcal{D}(\Phi X_0,\Phi Y).$$

Quotient of A_∞-modules

Definition 1.14 (see [23, Section 3.1.3]) There is a DG-functor

$$\mathrm{Mod}_{\mathcal{C}} \to \mathrm{Mod}_{\mathcal{C}/\mathcal{A}}, \quad \mathcal{M} \mapsto {}_{\mathcal{A}\backslash}\mathcal{M},$$

defined as follows. Let $\mathcal{M}$ be a $\mathcal{C}$-module. For every object X,

$${}_{\mathcal{A}\backslash}\mathcal{M}(X) = \mathcal{M}(X) \oplus \Biggl(\bigoplus_{\substack{p \geq 1 \\ A_1,\dots,A_p \in \mathcal{A}}} \mathcal{C}(X, A_1)[1] \otimes \cdots \otimes \mathcal{C}(A_{p-1}, A_p)[1] \otimes \mathcal{M}(A_p) \Biggr).$$

Also, a sequence

$$\boldsymbol{x}_i = (x_i^0, \dots, x_i^{p_i - 1}) \in \mathcal{C}/\mathcal{A}(X_i, X_{i+1}) \quad (0 \leq i \leq d-1)$$

acts on an element

$$\boldsymbol{u} = (x_d^0, \dots, x_d^{p_d - 1}, u) \in {}_{\mathcal{A}\backslash}\mathcal{M}(X_d)$$

via the operations

$$\begin{aligned} \mu_{{}_{\mathcal{A}\backslash}\mathcal{M}}&(\boldsymbol{x}_0, \dots, \boldsymbol{x}_{d-1}, \boldsymbol{u}) \\ &= \sum_{\substack{0 \leq i \leq p_0,\, 1 \leq j \leq p_d \\ i<j \text{ if } d=0}} x_0^0 \otimes \cdots \otimes x_0^{i-1} \otimes \mu_{\mathcal{C}}(x_0^i, \dots, x_d^{j-1}) \otimes x_d^j \otimes \cdots \otimes x_d^{p_d-1} \otimes u \\ &\qquad + \sum_{0 \leq i \leq p_0} x_0^0 \otimes \cdots \otimes x_0^{i-1} \otimes \mu_{\mathcal{M}}(x_0^i, \dots, x_d^{p_d-1}, u). \end{aligned}$$

Finally, let $t\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a $\mathcal{C}$-module map. Then the above functor gives a $\mathcal{C}/\mathcal{A}$-module map ${}_{\mathcal{A}\backslash}t\colon {}_{\mathcal{A}\backslash}\mathcal{M}_1 \to {}_{\mathcal{A}\backslash}\mathcal{M}_2$ which sends every sequence $(\boldsymbol{x}_0, \dots, \boldsymbol{x}_{d-1}, \boldsymbol{u})$ as above to

$${}_{\mathcal{A}\backslash}t(\boldsymbol{x}_0, \dots, \boldsymbol{x}_{d-1}, \boldsymbol{u}) = \sum_{0 \leq i \leq p_0} x_0^0 \otimes \cdots \otimes x_0^{i-1} \otimes t(x_0^i, \dots, x_d^{p_d-1}, u).$$

Relations between pullback and quotient of A_∞-modules

Definition 1.15 Let $\Phi\colon \mathcal{C} \to \mathcal{D}$ be an A_∞-functor such that $\Phi(\mathcal{A})$ is contained in $\mathcal{B}$, and let X be a fixed object of $\mathcal{C}$. Then, for each $\mathcal{D}$-module $\mathcal{N}$, there is a chain map ${}_{\mathcal{A}\backslash}(\Phi^*\mathcal{N})(X) \to {}_{\mathcal{B}\backslash}\mathcal{N}(\Phi X)$ which sends an element

$$\boldsymbol{u} = (x^0, \dots, x^{p-1}, u) \in {}_{\mathcal{A}\backslash}(\Phi^*\mathcal{N})(X)$$

to

$$\sum \Phi(x^0, \dots, x^{i_1 - 1}) \otimes \cdots \otimes \Phi(x^{i_r}, \dots, x^{p-1}) \otimes u \in {}_{\mathcal{B}\backslash}\mathcal{N}(\Phi X).$$

This defines a natural transformation between the functors $\mathcal{N} \mapsto {}_{\mathcal{A}\backslash}(\Phi^*\mathcal{N})(X)$ and $\mathcal{N} \mapsto {}_{\mathcal{B}\backslash}\mathcal{N}(\Phi X)$ from $\mathrm{Mod}_{\mathcal{D}}$ to Ch. In other words, for every $\mathcal{D}$-module map $t\colon \mathcal{N}_1 \to \mathcal{N}_2$, the following diagram of chain complexes commutes:

$$\begin{array}{ccc} {}_{\mathcal{A}\backslash}(\Phi^*\mathcal{N}_1)(X) & \longrightarrow & {}_{\mathcal{B}\backslash}\mathcal{N}_1(\Phi X) \\ \big\downarrow {\scriptstyle {}_{\mathcal{A}\backslash}(\Phi^* t)} & & \big\downarrow {\scriptstyle {}_{\mathcal{B}\backslash}t} \\ {}_{\mathcal{A}\backslash}(\Phi^*\mathcal{N}_2)(X) & \longrightarrow & {}_{\mathcal{B}\backslash}\mathcal{N}_2(\Phi X) \end{array}$$

Remark 1.16 Let Y be an object of $\mathcal{C}$, and let $\Phi\colon \mathcal{C} \to \mathcal{D}$ be an A_∞-functor such that $\Phi(\mathcal{A})$ is contained in $\mathcal{B}$. Let $\widetilde{\Phi}\colon \mathcal{C}/\mathcal{A} \to \mathcal{D}/\mathcal{B}$ be the A_∞-functor induced by Φ (see [30, Section 3]). Localize the morphism $t_\Phi\colon \mathcal{C}(-,Y) \to \Phi^*\mathcal{D}(-,\Phi Y)$ of Definition 1.13 at $\mathcal{A}$ and evaluate at X to get a chain map

$$\mathcal{C}/\mathcal{A}(X,Y) = {}_{\mathcal{A}\backslash}\mathcal{C}(-,Y)(X) \xrightarrow{\mathcal{A}\backslash t_\Phi} {}_{\mathcal{A}\backslash}(\Phi^*\mathcal{D}(-,\Phi Y))(X).$$

Then the composition of this map with the chain map

$${}_{\mathcal{A}\backslash}(\Phi^*\mathcal{D}(-,\Phi Y))(X) \to {}_{\mathcal{B}\backslash}\mathcal{D}(-,\Phi Y)(\Phi X) = \mathcal{D}/\mathcal{B}(\Phi X, \Phi Y)$$

of Definition 1.15 is the chain map $\widetilde{\Phi}\colon \mathcal{C}/\mathcal{A}(X,Y) \to \mathcal{D}/\mathcal{B}(\Phi X, \Phi Y)$.

Proposition 1.17 *Let $\Phi\colon \mathcal{C}_1 \to \mathcal{C}_2$ be an A_∞-functor such that $\Phi(\mathcal{A}_1)$ is contained in $\mathcal{A}_2$, and let $\widetilde{\Phi}\colon \mathcal{C}_1/\mathcal{A}_1 \to \mathcal{C}_2/\mathcal{A}_2$ be the A_∞-functor induced by Φ.*

Let Y_1 be an object of $\mathcal{C}_1$ and set $Y_2 := \Phi(Y_1)$. Assume that there exists a $\mathcal{C}_i$-module $\mathcal{M}_{\mathcal{C}_i}$, a degree 0 closed $\mathcal{C}_i$-module map $t_{\mathcal{C}_i}\colon \mathcal{C}_i(-,Y_i) \to \mathcal{M}_{\mathcal{C}_i}$ and a degree 0 closed $\mathcal{C}_1$-module map $t_0\colon \mathcal{M}_{\mathcal{C}_1} \to \Phi^*\mathcal{M}_{\mathcal{C}_2}$ such that the following diagram of $\mathcal{C}_1$-modules commutes:

$$\begin{array}{ccc} \mathcal{C}_1(-,Y_1) & \xrightarrow{t_\Phi} & \Phi^*\mathcal{C}_2(-,Y_2) \\ \downarrow{\scriptstyle t_{\mathcal{C}_1}} & & \downarrow{\scriptstyle \Phi^* t_{\mathcal{C}_2}} \\ \mathcal{M}_{\mathcal{C}_1} & \xrightarrow{t_0} & \Phi^*\mathcal{M}_{\mathcal{C}_2} \end{array}$$

(see Definition 1.13 for the map t_Φ). Then for every object X in $\mathcal{C}_1$, there is a chain map

$$u\colon {}_{\mathcal{A}_1\backslash}\mathcal{M}_{\mathcal{C}_1}(X) \to {}_{\mathcal{A}_2\backslash}\mathcal{M}_{\mathcal{C}_2}(\Phi X)$$

such that the following diagram of chain complexes commutes:

$$\begin{array}{ccc} \mathcal{C}_1/\mathcal{A}_1(X,Y_1) & \xrightarrow{\widetilde{\Phi}} & \mathcal{C}_2/\mathcal{A}_2(\Phi X, Y_2) \\ \downarrow{\scriptstyle \mathcal{A}_1\backslash t_{\mathcal{C}_1}} & & \downarrow{\scriptstyle \mathcal{A}_2\backslash t_{\mathcal{C}_2}} \\ {}_{\mathcal{A}_1\backslash}\mathcal{M}_{\mathcal{C}_1}(X) & \xrightarrow{u} & {}_{\mathcal{A}_2\backslash}\mathcal{M}_{\mathcal{C}_2}(\Phi X) \\ \uparrow & & \uparrow \\ \mathcal{M}_{\mathcal{C}_1}(X) & \xrightarrow{t_0} & \mathcal{M}_{\mathcal{C}_2}(\Phi X) \end{array}$$

(the two lowest vertical maps are the inclusions). If, moreover,

1. *for every objects A in $\mathcal{A}_i$, the complexes $\mathcal{M}_{\mathcal{C}_i}(A)$ are acyclic,*
2. *the maps ${}_{\mathcal{A}_i\backslash}t_{\mathcal{C}_i}\colon {}_{\mathcal{A}_i\backslash}\mathcal{C}_i(X,Y_i) \to {}_{\mathcal{A}_i\backslash}\mathcal{M}_{\mathcal{C}_i}(X)$ are quasi-isomorphisms, and*
3. *the map $t_0\colon \mathcal{M}_{\mathcal{C}_1}(X) \to \Phi^*\mathcal{M}_{\mathcal{C}_2}(X)$ is a quasi-isomorphism,*

then the map $\widetilde{\Phi}\colon \mathcal{C}_1/\mathcal{A}_1(X,Y_1) \to \mathcal{C}_2/\mathcal{A}_2(\Phi X, Y_2)$ is a quasi-isomorphism.

Proof We apply the functor $\mathcal{P} \mapsto {}_{\mathcal{A}_1\backslash}\mathcal{P}$ to the first diagram, we evaluate at X and we use the natural map of Definition 1.15 to get the commutative diagram of chain complexes

$$\begin{array}{ccccc} {}_{\mathcal{A}_1\backslash}\mathcal{C}_1(-,Y_1)(X) & \xrightarrow{{}_{\mathcal{A}_1\backslash}t_\Phi} & {}_{\mathcal{A}_1\backslash}(\Phi^*\mathcal{C}_2(-,Y_2))(X) & \longrightarrow & {}_{\mathcal{A}_2\backslash}\mathcal{C}_2(-,Y_2)(\Phi X) \\ \big\downarrow {\scriptstyle {}_{\mathcal{A}_1\backslash}t_{\mathcal{C}_1}} & & \big\downarrow {\scriptstyle {}_{\mathcal{A}_1\backslash}(\Phi^* t_{\mathcal{C}_2})} & & \big\downarrow {\scriptstyle {}_{\mathcal{A}_2\backslash}t_{\mathcal{C}_2}} \\ {}_{\mathcal{A}_1\backslash}\mathcal{M}_{\mathcal{C}_1}(X) & \xrightarrow{{}_{\mathcal{A}_1\backslash}t_0} & {}_{\mathcal{A}_1\backslash}(\Phi^*\mathcal{M}_{\mathcal{C}_2})(X) & \longrightarrow & {}_{\mathcal{A}_2\backslash}\mathcal{M}_{\mathcal{C}_2}(\Phi X) \end{array}$$

Then we compose the horizontal maps and we use Remark 1.16 to get a commutative diagram of chain complexes

$$\begin{array}{ccc} \mathcal{C}_1/\mathcal{A}_1(X,Y_1) & \xrightarrow{\widetilde{\Phi}} & \mathcal{C}_2/\mathcal{A}_2(\Phi X,Y_2) \\ \big\downarrow {\scriptstyle {}_{\mathcal{A}_1\backslash}t_{\mathcal{C}_1}} & & \big\downarrow {\scriptstyle {}_{\mathcal{A}_2\backslash}t_{\mathcal{C}_2}} \\ {}_{\mathcal{A}_1\backslash}\mathcal{M}_{\mathcal{C}_1}(X) & \xrightarrow{u} & {}_{\mathcal{A}_2\backslash}\mathcal{M}_{\mathcal{C}_2}(\Phi X) \end{array}$$

This proves the first part of the proposition because the following diagram of chain complexes commutes:

$$\begin{array}{ccc} {}_{\mathcal{A}_1\backslash}\mathcal{M}_{\mathcal{C}_1}(X) & \xrightarrow{u} & {}_{\mathcal{A}_2\backslash}\mathcal{M}_{\mathcal{C}_2}(\Phi X) \\ \big\uparrow & & \big\uparrow \\ \mathcal{M}_{\mathcal{C}_1}(X) & \xrightarrow{t_0} & \mathcal{M}_{\mathcal{C}_2}(\Phi X) \end{array}$$

The second part of the proposition follows directly with [23, Lemma 3.13]. □

Cone of module maps

Definition 1.18 Let $t\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a degree 0 closed morphism of $\mathcal{C}$-modules. We denote by

$$\mathrm{Cone}(\mathcal{M}_1 \xrightarrow{t} \mathcal{M}_2) = \begin{bmatrix} \mathcal{M}_1 \\ \big\downarrow {\scriptstyle t} \\ \mathcal{M}_2 \end{bmatrix}$$

the $\mathcal{C}$-module $\mathcal{M}$ defined as follows. For every object X in $\mathcal{C}$,

$$\mathcal{M}(X) = \mathcal{M}_1(X)[1] \oplus \mathcal{M}_2(X)$$

as graded vector space, and any sequence

$$(x_0,\dots,x_{d-1}) \in \mathcal{C}(X_0,X_1) \times \dots \times \mathcal{C}(X_{d-1},X_d)$$

acts on an element $u_1 \oplus u_2$ in $\mathcal{M}(X_d)$ via the operations

$$\mu_{\mathcal{M}}(x_0,\dots,x_{d-1},u_1\oplus u_2) = \mu_{\mathcal{M}_1}(x_0,\dots,x_{d-1},u_1) \oplus \big(\mu_{\mathcal{M}_2}(x_0,\dots,x_{d-1},u_2) + t(x_0,\dots,x_{d-1},u_1)\big).$$

If we have two $\mathcal{C}$-module maps $t\colon \mathcal{M}_1 \to \mathcal{M}_2$ and $t'\colon \mathcal{M}_1 \to \mathcal{M}_2'$, then we set

$$\begin{bmatrix} & \mathcal{M}_1 & \\ \swarrow {\scriptstyle t} & & \searrow {\scriptstyle t'} \\ \mathcal{M}_2 & & \mathcal{M}_2' \end{bmatrix} := \begin{bmatrix} \mathcal{M}_1 \\ \big\downarrow {\scriptstyle (t,t')} \\ \mathcal{M}_2 \oplus \mathcal{M}_2' \end{bmatrix}.$$

Proposition 1.19 *Consider a diagram of $\mathcal{C}$-modules*

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{t_1} & \mathcal{M}_2 \\ \downarrow{\scriptstyle t_1'} & & \downarrow{\scriptstyle t_2} \\ \mathcal{M}_2' & \xrightarrow{t_2'} & \mathcal{M}_3 \end{array}$$

where all the morphisms are of degree 0 *and closed. Then any homotopy $h\colon \mathcal{M}_1 \to \mathcal{M}_3$ between*

$$t := \mu^2_{\mathrm{Mod}_{\mathcal{C}}}(t_1, t_2) \quad \text{and} \quad t' := \mu^2_{\mathrm{Mod}_{\mathcal{C}}}(t_1', t_2')$$

induces a degree 0 *closed $\mathcal{C}$-module map*

$$t_h \colon \left[\begin{array}{ccccc} & & \mathcal{M}_1 & & \\ & \swarrow{\scriptstyle t_1} & & \searrow{\scriptstyle t_1'} & \\ \mathcal{M}_2 & & & & \mathcal{M}_2' \end{array} \right] \to \mathcal{M}_3$$

defined by

$$t_h(x_0, \dots, x_{d-1}, u_1 \oplus u_2 \oplus u_2') = h(x_0, \dots, x_{d-1}, u_1) + t_2(x_0, \dots, x_{d-1}, u_2) + t_2'(x_0, \dots, x_{d-1}, u_2').$$

Proof The only thing to check is that $\mu^1_{\mathrm{Mod}_{\mathcal{C}}}(t_h) = 0$, which is straightforward. □

Remark If $t\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a degree 0 closed $\mathcal{C}$-module map, then

$${}_{\mathcal{A}\backslash}\mathrm{Cone}(\mathcal{M}_1 \xrightarrow{t} \mathcal{M}_2) = \mathrm{Cone}({}_{\mathcal{A}\backslash}\mathcal{M}_1 \xrightarrow{\mathcal{A}\backslash t} {}_{\mathcal{A}\backslash}\mathcal{M}_2).$$

1.2 Grothendieck construction and homotopy pushout

An exposition on Grothendieck constructions and homotopy colimits in the context of A_∞-categories can be found in [24, Appendix A]. We recall here definitions and basic facts that will serve us. In this section, A_∞-categories are always assumed to be *strictly unital* (see [36, Paragraph (2a)]).

Definition 1.20 Consider a diagram of A_∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi_1} & \mathcal{D}_1 \\ {\scriptstyle \Phi_2}\downarrow & & \\ \mathcal{D}_2 & & \end{array}$$

The Grothendieck construction of this diagram is the A_∞-category $\mathcal{G}$ such that:

(1) The set of objects is $\mathrm{ob}(\mathcal{C}) \sqcup \mathrm{ob}(\mathcal{D}_1) \sqcup \mathrm{ob}(\mathcal{D}_2)$.

(2) The space of morphisms between two objects X and Y is given by

$$\mathcal{G}(X,Y) = \begin{cases} \mathcal{C}(X,Y) & \text{if } X, Y \in \mathrm{ob}(\mathcal{C}), \\ \mathcal{D}_i(X,Y) & \text{if } X, Y \in \mathrm{ob}(\mathcal{D}_i), \\ \mathcal{D}_i(\Phi_i X, Y) & \text{if } X \in \mathrm{ob}(\mathcal{C}) \text{ and } Y \in \mathrm{ob}(\mathcal{D}_i), \\ 0 & \text{otherwise.} \end{cases}$$

(3) The operations involving only objects of $\mathcal{C}$, respectively of $\mathcal{D}_i$, are the same as in $\mathcal{C}$, respectively in $\mathcal{D}_i$, and for every sequence

$$(x_0,\dots,x_{p-1},y,z_0,\dots,z_{q-1}) \\ \in \mathcal{C}(X_0,X_1)\otimes\cdots\otimes\mathcal{C}(X_{p-1},X_p)\otimes\mathcal{G}(X_p,Y_0)\otimes\mathcal{D}_i(Y_0,Y_1)\otimes\cdots\otimes\mathcal{D}_i(Y_{q-1},Y_q),$$

we have

$$\mu_{\mathcal{G}}(x_0,\dots,x_{p-1},y,z_0,\dots,z_{q-1}) \\ = \sum \mu_{\mathcal{D}_i}(\Phi_i(x_0,\dots,x_{i_1-1}),\dots,\Phi_i(x_{p-i_r},\dots,x_{p-1}),y,z_0,\dots,z_{q-1}).$$

An *adjacent unit* of $\mathcal{G}$ is any morphism in $\mathcal{G}(X,\Phi_i(X))$ which corresponds to the unit in $\mathcal{D}_i(\Phi_i(X),\Phi_i(X))$. The *homotopy colimit* $\mathcal{H}$ of the above diagram is the localization of $\mathcal{G}$ at its adjacent units.

Proposition 1.21 *Let $\mathcal{G}$ be the Grothendieck construction of a diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi_1} & \mathcal{D}_1 \\ {\scriptstyle\Phi_2}\downarrow & & \\ \mathcal{D}_2 & & \end{array}$$

Then any strictly commutative square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi_1} & \mathcal{D}_1 \\ {\scriptstyle\Phi_2}\downarrow & & \downarrow{\scriptstyle\Psi_1} \\ \mathcal{D}_2 & \xrightarrow{\Psi_2} & \mathcal{E} \end{array}$$

induces a functor $\sigma\colon \mathcal{G}\to\mathcal{E}$ defined as follows. On the objects, σ acts on $\mathcal{D}_i$ as Ψ_i, and on $\mathcal{C}$ as $\Psi_1\circ\Phi_1=\Psi_2\circ\Phi_2$; on the morphisms, σ acts on $\mathcal{D}_i$ as Ψ_i, on $\mathcal{C}$ as $\Psi_1\circ\Phi_1=\Psi_2\circ\Phi_2$, and it sends any sequence

$$(x_0,\dots,x_{p-1},y,z_0,\dots,z_{q-1}) \\ \in \mathcal{C}(X_0,X_1)\otimes\cdots\otimes\mathcal{C}(X_{p-1},X_p)\otimes\mathcal{G}(X_p,Y_0)\otimes\mathcal{D}_i(Y_0,Y_1)\otimes\cdots\otimes\mathcal{D}_i(Y_{q-1},Y_q)$$

to

$$\sigma(x_0,\dots,x_{p-1},y,z_0,\dots,z_{q-1}) \\ = \sum \Psi_i\big(\Phi_i(x_0,\dots,x_{i_1-1}),\dots,\Phi_i(x_{p-i_r},\dots,x_{p-1}),y,z_0,\dots,z_{q-1}\big).$$

Proof This is a straightforward verification. □

Proposition 1.22 [24, Lemma A.5] *A strictly commutative diagram of A_∞-categories*

$$\begin{array}{ccccc} \mathcal{B}_1 & \longleftarrow & \mathcal{A} & \longrightarrow & \mathcal{B}_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{D}_1 & \longleftarrow & \mathcal{C} & \longrightarrow & \mathcal{D}_2 \end{array}$$

induces an A_∞-functor from the Grothendieck construction of the top line to the Grothendieck construction of the bottom line which preserves adjacent units. If moreover each vertical arrow is a quasi-equivalence, then the induced functor

$$\operatorname{hocolim}\begin{pmatrix} \mathcal{A} \longrightarrow \mathcal{B}_1 \\ \big\downarrow \\ \mathcal{B}_2 \end{pmatrix} \to \operatorname{hocolim}\begin{pmatrix} \mathcal{C} \longrightarrow \mathcal{D}_1 \\ \big\downarrow \\ \mathcal{D}_2 \end{pmatrix}$$

is a quasi-equivalence.

Proposition 1.23 *Consider two diagrams of A_∞-categories*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi_1} & \mathcal{D}_1 \\ {\scriptstyle \Phi_2}\big\downarrow & & \\ \mathcal{D}_2 & & \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{\Psi_1} & \mathcal{D}_1 \\ {\scriptstyle \Psi_2}\big\downarrow & & \\ \mathcal{D}_2 & & \end{array}$$

If Φ_i and Ψ_i (for $i \in \{1,2\}$) are homotopic (see [36, Paragraph (1h)]), *then the homotopy colimits of the diagrams above are quasi-equivalent.*

Proof Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be the Grothendieck constructions of the above diagrams.

Let T_i be a homotopy from Φ_i to Ψ_i. This means that

$$\Phi_i+\Psi_i=\sum T_i(\dots,\mu_{\mathcal{C}}(\dots),\dots)+\sum \mu_{\mathcal{D}_i}\big(\Psi_i(\dots),\dots,\Psi_i(\dots),T_i(\dots),\Phi_i(\dots),\dots,\Phi_i(\dots)\big).$$

We consider the functor $\kappa\colon \mathcal{G}_0 \to \mathcal{G}_1$ such that

$$\kappa|_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}, \quad \kappa|_{\mathcal{D}_i} = \mathrm{id}_{\mathcal{D}_i},$$

and which sends every sequence

$$(\dots,y,\dots) \in \mathcal{C}(X_0,X_1)\times\cdots\times\mathcal{C}(X_{p-1},X_p)\times\mathcal{G}_0(X_p,Y_0)\times\mathcal{D}_i(Y_0,Y_1)\times\cdots\times\mathcal{D}_i(Y_{q-1},Y_q)$$

to

$$\kappa(\dots,y,\dots)=\sum \mu_{\mathcal{D}_i}(\Psi_i(\dots),\dots,\Psi_i(\dots),T_i(\dots),\Phi_i(\dots),\dots,\Phi_i(\dots),y,\dots)$$

if p is positive, and to

$$\Phi(y,\dots)=\mathrm{id}_{\mathcal{D}_i}(y,\dots)$$

otherwise. Using the facts that Φ_i,Ψ_i are A_∞-functors, that T_i is a homotopy from Φ_i to Ψ_i, and gathering the terms depending on if they contain $T_i^k(\dots)$ or y, we conclude that κ satisfies the A_∞-relations. This proves the result because κ is a quasi-equivalence sending the adjacent units of $\mathcal{G}_0$ onto those of $\mathcal{G}_1$. □

1.3 Cylinder object and homotopy

Let $\mathcal{A}_\bot$, $\mathcal{A}_I$ and $\mathcal{A}_\top$ be three copies of an A_∞-category $\mathcal{A}$. We denote by $\mathcal{C}$ the Grothendieck construction of the diagram

$$\begin{array}{ccc} \mathcal{A}_I & \xrightarrow{\text{id}} & \mathcal{A}_\top \\ {\scriptstyle\text{id}}\downarrow & & \\ \mathcal{A}_\bot & & \end{array}$$

and we let $\iota_\bot, \iota_I, \iota_\top : \mathcal{A} \to \mathcal{C}$ be the strict inclusions with images $\mathcal{A}_\bot$, $\mathcal{A}_I$, $\mathcal{A}_\top$ respectively. Finally, we denote by $W_\mathcal{C}$ the set of adjacent units in $\mathcal{C}$, and we let $\mathcal{C}yl_\mathcal{A} = \mathcal{C}[W_\mathcal{C}^{-1}]$ be the homotopy colimit of the diagram above. We say that $\mathcal{C}yl_\mathcal{A}$ is a cylinder object for $\mathcal{A}$.

We denote by $\pi : \mathcal{C} \to \mathcal{A}$ the A_∞-functor induced by the commutative square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{id}} & \mathcal{A} \\ {\scriptstyle\text{id}}\downarrow & & \downarrow{\scriptstyle\text{id}} \\ \mathcal{A} & \xrightarrow{\text{id}} & \mathcal{A} \end{array}$$

(see Proposition 1.21).

Proposition 1.24 *The following diagram of A_∞-categories commutes*:

$$\mathcal{A} \sqcup \mathcal{A} \xrightarrow{\iota_\bot \sqcup \iota_\top} \mathcal{C} \xrightarrow{\pi} \mathcal{A}, \qquad \pi \circ (\iota_\bot \sqcup \iota_\top) = \text{id} \sqcup \text{id}$$

Moreover, π sends $W_\mathcal{C}$ to the set of units in $\mathcal{A}$, and the induced A_∞-functor $\tilde{\pi} : \mathcal{C}yl_\mathcal{A} \to \mathcal{A}[\{\text{units}\}^{-1}]$ is a quasi-equivalence.

Proof The facts that $\pi \circ (\iota_\bot \sqcup \iota_\top) = \text{id} \sqcup \text{id}$ and that π sends $W_\mathcal{C}$ to the set of units in $\mathcal{A}$ are clear. We now show that $\tilde{\pi} : \mathcal{C}yl_\mathcal{A} \to \mathcal{A}[\{\text{units}\}^{-1}]$ is a quasi-equivalence.

First observe that it is enough to show that the map

$$\tilde{\pi} : \mathcal{C}yl_\mathcal{A}(X, Y) \to \mathcal{A}[\{\text{units}\}^{-1}](\pi X, \pi Y)$$

is a quasi-isomorphism for every objects X, Y in $\mathcal{A}_\bot$ because every object of $\mathcal{C}$ can be related to one of $\mathcal{A}_\bot$ by a zigzag of morphisms in $W_\mathcal{C}$, which are quasi-isomorphisms in $\mathcal{C}yl_\mathcal{A}$ (see [23, Lemma 3.12]). Our strategy is to apply Proposition 1.17. Let $Y = \iota_\bot(Z)$ be an object in $\mathcal{A}_\bot$. For the $\mathcal{C}$-module we take

$$\mathcal{M}_\mathcal{C} = \left[\begin{array}{ccc} & \mathcal{C}(-, \iota_I(Z)) & \\ \swarrow^{t_{I\bot}} & & \searrow^{t_{I\top}} \\ \mathcal{C}(-, \iota_\bot(Z)) & & \mathcal{C}(-, \iota_\top(Z)) \end{array} \right],$$

where

$$t_{I\triangle} : \mathcal{C}(-, \iota_I(Z)) \to \mathcal{C}(-, \iota_\triangle(Z)), \quad \triangle \in \{\bot, \top\},$$

is the $\mathcal{C}$-module map induced by the adjacent unit in $\mathcal{C}(\iota_I(Z), \iota_\triangle(Z))$ (see Definition 1.8)). For the $\mathcal{A}$-module we simply take $\mathcal{A}(-, Z)$. Besides, we let $t_\mathcal{C} : \mathcal{C}(-, \iota_\bot(Z)) \to \mathcal{M}_\mathcal{C}$ be the $\mathcal{C}$-module inclusion, and we let $t_{\mathcal{A}}\colon \mathcal{A}(-,Z)\to\mathcal{A}(-,Z)$ be the identity map. We now define the morphism $t_0\colon \mathcal{M}_{\mathcal{C}}\to\pi^*\mathcal{A}(-,Z)$. Consider the diagram of $\mathcal{C}$-modules

$$\begin{array}{ccc} \mathcal{C}(-,\iota_I(Z)) & \xrightarrow{t_{I\perp}} & \mathcal{C}(-,\iota_\top(Z)) \\ \big\downarrow{\scriptstyle t_{I\top}} & & \big\downarrow{\scriptstyle t_\pi} \\ \mathcal{C}(-,\iota_\perp(Z)) & \xrightarrow{t_\pi} & \pi^*\mathcal{A}(-,Z) \end{array}$$

Observe that this diagram is commutative, and thus it induces a strict $\mathcal{C}$-module map $t_0\colon\mathcal{M}_{\mathcal{C}}\to\pi^*\mathcal{A}(-,Z)$ according to Proposition 1.19. It is then easy to see that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}(-,\iota_\triangle(Z)) & \xrightarrow{t_\pi} & \pi^*\mathcal{A}(-,Z) \\ \big\downarrow{\scriptstyle t_{\mathcal{C}}} & & \big\downarrow{\scriptstyle \pi^* t_{\mathcal{A}}} \\ \mathcal{M}_{\mathcal{C}} & \xrightarrow{t_0} & \pi^*\mathcal{A}(-,Z) \end{array}$$

To conclude the proof, it suffices to check the three items of Proposition 1.17. Observe that the pair $(\mathcal{A}(-,Z),t_{\mathcal{A}})$ trivially satisfies the two first items.

We check that $\mathcal{M}_{\mathcal{C}}$ satisfies the first item of Proposition 1.17. Let Z' be an object in $\mathcal{A}$ and let w be the adjacent unit in $\mathcal{C}(\iota_I(Z'),\iota_\perp(Z'))$ (the proof is the same for the adjacent unit in $\mathcal{C}(\iota_I(Z'),\iota_\top(Z'))\cap W_{\mathcal{C}}$). Then

$$\mathcal{M}_{\mathcal{C}}(\operatorname{Cone} w)=\operatorname{Cone}\big(\mathcal{M}_{\mathcal{C}}(\iota_\perp(Z'))\xrightarrow{\mu^2_{\mathcal{C}}(w,-)}\mathcal{M}_{\mathcal{C}}(\iota_I(Z'))\big)=\operatorname{Cone}\big(\mathcal{C}(\iota_\perp(Z'),\iota_\perp(Z))\xrightarrow{\mu^2_{\mathcal{C}}(w,-)}K\big),$$

where

$$K=\left[\begin{array}{ccc} & \mathcal{C}(\iota_I(Z'),\iota_I(Z)) & \\ \swarrow{\scriptstyle t_{I\perp}} & & \searrow{\scriptstyle t_{I\top}} \\ \mathcal{C}(\iota_I(Z'),\iota_\perp(Z)) & & \mathcal{C}(\iota_I(Z'),\iota_\top(Z)) \end{array}\right].$$

Observe that $\mu^2_{\mathcal{C}}(w,-)\colon\mathcal{C}(\iota_\perp(Z'),\iota_\perp(Z))\to K$ is injective so its cone is quasi-isomorphic to its cokernel, which is the cone of $t_{I\top}\colon\mathcal{C}(\iota_I(Z'),\iota_I(Z))\to\mathcal{C}(\iota_I(Z'),\iota_\top(Z))$. The latter map is a quasi-isomorphism, so $\mathcal{M}_{\mathcal{C}}(\operatorname{Cone} w)$ is acyclic.

We now check that $(\mathcal{M}_{\mathcal{C}},t_{\mathcal{C}})$ satisfies the second item of Proposition 1.17. Observe that

$$_{W_{\mathcal{C}}^{-1}}\mathcal{M}_{\mathcal{C}}=\left[\begin{array}{ccc} & _{W_{\mathcal{C}}^{-1}}\mathcal{C}(-,\iota_I(Z)) & \\ \swarrow{\scriptstyle _{W_{\mathcal{C}}^{-1}}t_{I\perp}} & & \searrow{\scriptstyle _{W_{\mathcal{C}}^{-1}}t_{I\top}} \\ _{W_{\mathcal{C}}^{-1}}\mathcal{C}(-,\iota_\perp(Z)) & & _{W_{\mathcal{C}}^{-1}}\mathcal{C}(-,\iota_\top(Z)) \end{array}\right]$$

and $_{W_{\mathcal{C}}^{-1}}t_{\mathcal{C}}\colon {_{W_{\mathcal{C}}^{-1}}}\mathcal{C}(-,\iota_\perp(Z))\to {_{W_{\mathcal{C}}^{-1}}}\mathcal{M}_{\mathcal{C}}$ is the inclusion. Thus if X is some object of $\mathcal{C}$, the cone of $_{W_{\mathcal{C}}^{-1}}t_{\mathcal{C}}\colon \mathcal{C}yl_{\mathcal{A}}(X,\iota_\perp(Z))\to {_{W_{\mathcal{C}}^{-1}}}\mathcal{M}_{\mathcal{C}}(X)$ is quasi-isomorphic to the cone of the multiplication in $\mathcal{C}yl_{\mathcal{A}}$ by an element of $W_{\mathcal{C}}$, which is a quasi-isomorphism. Thus the map $_{W_{\mathcal{C}}^{-1}}t_{\mathcal{C}}\colon {_{W_{\mathcal{C}}^{-1}}}\mathcal{C}(-,\iota_\perp(Z))\to {_{W_{\mathcal{C}}^{-1}}}\mathcal{M}_{\mathcal{C}}$ indeed is a quasi-isomorphism.

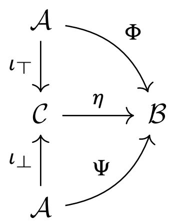
It remains to check the third item of Proposition 1.17, which is that the map $t_0\colon \mathcal{M}_{\mathcal{C}}(X) \to \pi^*\mathcal{A}(-,Z)(X)$ is a quasi-isomorphism when X is in $\mathcal{A}_\perp$. This is true because $\mathcal{M}_{\mathcal{C}}(\iota_\perp(Z')) = \mathcal{C}(\iota_\perp(Z'), \iota_\perp(Z)) = \mathcal{A}(Z', Z)$, and

$$t_0\colon \mathcal{A}(Z', Z) = \mathcal{M}_{\mathcal{C}}(\iota_\perp(Z')) \to \pi^*\mathcal{A}(-,Z)(\iota_\perp(Z')) = \mathcal{A}(Z', Z)$$

is the identity. □

Remark Proposition 1.24 can be thought as saying that $\mathcal{C}yl_{\mathcal{A}}$ is a cylinder object for $\mathcal{A}$.

Proposition 1.25 *If two A_∞-functors $\Phi, \Psi\colon \mathcal{A} \to \mathcal{B}$ are homotopic* (*see* [36, Paragraph (1h)]), *then there is an A_∞-functor $\eta\colon \mathcal{C} \to \mathcal{B}$ which sends the adjacent units of $\mathcal{C}$ to the units in $\mathcal{B}$ and such that the following diagram commutes*:



Proof On the objects, we set $\eta(X_\triangle) = \Phi(X) = \Psi(X)$ for every object X of $\mathcal{A}$ and $\triangle \in \{\perp, I, \top\}$. On the morphisms, we set

$$\eta|_{\mathcal{A}_\perp} = \eta|_{\mathcal{A}_I} = \Psi, \quad \eta|_{\mathcal{A}_\top} = \Phi$$

and ask for the restriction of η to

$$\mathcal{A}_I(X_0, X_1) \otimes \cdots \otimes \mathcal{A}_I(X_{p-1}, X_p) \otimes \mathcal{C}(X_p, X_{p+1}) \otimes \mathcal{A}_\perp(X_{p+1}, X_{p+2}) \otimes \cdots \otimes \mathcal{A}_\perp(X_{p+q}, X_{p+q+1})$$

to be Ψ. It remains to define η for

$$(\ldots, x, \ldots) \\ \in \mathcal{A}_I(X_0, X_1) \otimes \cdots \otimes \mathcal{A}_I(X_{p-1}, X_p) \otimes \mathcal{C}(X_p, X_{p+1}) \otimes \mathcal{A}_\top(X_{p+1}, X_{p+2}) \otimes \cdots \otimes \mathcal{A}_\top(X_{p+q}, X_{p+q+1}).$$

For this we take a homotopy T between Φ and Ψ, which means that

$$\Phi + \Psi = \sum T(\ldots, \mu_{\mathcal{A}}(\ldots), \ldots) + \sum \mu_{\mathcal{B}}\big(\Phi(\ldots), \ldots, \Phi(\ldots), T(\ldots), \Psi(\ldots), \ldots, \Psi(\ldots)\big).$$

Then we let

$$\eta(\ldots, x, \ldots) \\ = \sum \mu_{\mathcal{B}}\big(\Phi(\ldots), \ldots, \Phi(\ldots), T(\ldots), \Psi(\ldots), \ldots, \Psi(\ldots), \Psi(\ldots, x, \ldots)\Psi(\ldots), \ldots, \Psi(\ldots)\big)$$

if p is positive, and $\eta(x, \ldots) = \Psi(x, \ldots)$ otherwise. □

1.4 Adjunctions between Adams-graded and non-Adams-graded

We end this section by describing specific adjunctions between the category of Adams-graded A_∞-categories concentrated in nonnegative Adams degree and the category of (non-Adams-graded) A_∞-categories.

Definition 1.26 If V is an Adams-graded vector space and m is an integer, we denote by V_m the graded vector space whose degree n component is the direct sum of the bidegree (p,k) components of V, where the sum is over the set of couples $(p,k) \in \mathbb{Z} \times \mathbb{Z}$ such that $p - mk = n$.

Definition 1.27 If $\mathcal{C}$ is an Adams-graded A_∞-category, we denote by $\mathcal{C}_m$ the (non-Adams-graded) A_∞-category obtained from $\mathcal{C}$ by changing the grading so that

$$\mathcal{C}_m(X_0, X_1) = \mathcal{C}(X_0, X_1)_m$$

Observe that any A_∞-functor $\Phi\colon \mathcal{C}_1 \to \mathcal{C}_2$ between two Adams-graded A_∞-categories induces an A_∞-functor from $(\mathcal{C}_1)_m$ to $(\mathcal{C}_2)_m$ (that we still denote by Φ) which acts exactly as Φ on objects and morphisms. This defines a functor $\mathcal{C} \mapsto \mathcal{C}_m$ from the category of Adams-graded A_∞-categories to the category of (non-Adams-graded) A_∞-categories.

We denote by $\mathbb{F}[t_m]$ the augmented Adams-graded associative algebra generated by a variable t_m of bidegree $(m,1)$, and by $t_m\mathbb{F}[t_m]$ its augmentation ideal (or equivalently, the ideal generated by t_m).

Definition 1.28 If $\mathcal{D}$ is a (non-Adams-graded) A_∞-category, we denote by $\mathbb{F}[t_m] \otimes \mathcal{D}$ the Adams-graded A_∞-category such that

(1) the objects of $\mathbb{F}[t_m] \otimes \mathcal{D}$ are those of $\mathcal{D}$,

(2) the space of morphisms from Y_1 to Y_2 is $\mathbb{F}[t_m] \otimes \mathcal{D}(Y_1, Y_2)$, and if $y \in \mathcal{D}(Y_1, Y_2)$ is of degree j, $t_m^k \otimes y$ is of bidegree $(j + mk, k)$,

(3) the operations send any sequence $(t_m^{k_0} \otimes y_0, \dots, t_m^{k_{d-1}} \otimes y_{d-1})$ of morphisms to

$$\mu_{\mathbb{F}[t_m]\otimes\mathcal{D}}(t_m^{k_0} \otimes y_0, \dots, t_m^{k_{d-1}} \otimes y_{d-1}) = t_m^{k_0+\dots+k_{d-1}} \otimes \mu_{\mathcal{D}}(y_0, \dots, y_{d-1}).$$

Observe that any A_∞-functor $\Psi\colon \mathcal{D}_1 \to \mathcal{D}_2$ between (non-Adams-graded) A_∞-categories induces an A_∞-functor $\mathbb{F}[t_m] \otimes \mathcal{D}_1 \to \mathbb{F}[t_m] \otimes \mathcal{D}_2$ which acts as Ψ on objects, and which sends any sequence $(t_m^{k_0} \otimes y_0, \dots, t_m^{k_{d-1}} \otimes y_{d-1})$ of morphisms to $t_m^{k_0+\dots+k_{d-1}} \otimes \Psi(y_0, \dots, y_{d-1})$. This defines a functor $\mathcal{D} \mapsto \mathbb{F}[t_m] \otimes \mathcal{D}$ from the category of (non-Adams-graded) A_∞-categories to the category of Adams-graded A_∞-categories.

Definition 1.29 Let $\mathcal{C}$ be an Adams-graded A_∞-category concentrated in nonnegative Adams degree, and let $\mathcal{D}$ be a (non-Adams-graded) A_∞-category. To any A_∞-functor $\Psi_m\colon \mathcal{C}_m \to \mathcal{D}$, we associate an A_∞-functor $\Psi\colon \mathcal{C} \to \mathbb{F}[t_m] \otimes \mathcal{D}$ which sends a sequence $(x_0, \dots, x_{d-1})$, where x_j is of bidegree (i_j, k_j), to

$$\Psi(x_0, \dots, x_{d-1}) = t_m^{k_0+\dots+k_{d-1}} \otimes \Psi_m(x_0, \dots, x_{d-1}).$$

This defines an adjunction between the category of Adams-graded A_∞-categories concentrated in nonnegative Adams degree and the category of (non-Adams-graded) A_∞-categories.

2 Mapping torus of an A_∞-autoequivalence

In this section, we introduce the notion of mapping torus for a quasi-autoequivalence of an A_∞-category, by analogy with the mapping torus associated to an automorphism of a topological space. This terminology was also used in [26], but we do not know if the two notions coincide. The two main theorems of this section allow us to compute this mapping torus under different hypotheses.

Remark In this section, A_∞-categories are always assumed to be *strictly unital* (see [36, Paragraph (2a)]).

2.1 Definitions and main results

2.1.1 Definitions

Definition 2.1 Let τ be a quasi-autoequivalence of an Adams-graded A_∞-category $\mathcal{A}$. The mapping torus of τ is the A_∞-category

$$\mathrm{MT}(\tau) := \operatorname{hocolim}\left(\begin{array}{ccc} \mathcal{A} \sqcup \mathcal{A} & \xrightarrow{\mathrm{id}\sqcup\tau} & \mathcal{A} \\ {\scriptstyle \mathrm{id}\sqcup\mathrm{id}}\downarrow & & \\ \mathcal{A} & & \end{array}\right)$$

(see Definition 1.20).

Remark (1) We use the terminology "mapping torus" by analogy with the analogous situation in the category of topological spaces. Indeed, if f is an automorphism of some topological space X, then the mapping torus of f

$$M_f = (X \times [0,1])/((x,0) \sim (f(x),1))$$

is the homotopy colimit of the diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\mathrm{id}\sqcup f} & X \\ {\scriptstyle \mathrm{id}\sqcup\mathrm{id}}\downarrow & & \\ X & & \end{array}$$

(2) The terminology "mapping torus of an autoequivalence of A_∞-categories" also appears in [26], where the corresponding DG-category is denoted by M_τ, and it is used in [25] to distinguish open symplectic mapping tori. According to [25, Appendix A], M_τ is equivalent to the homotopy colimit of

$$\mathcal{A} \underset{i_0\otimes\mathrm{id}}{\overset{i_\infty\otimes\tau}{\rightrightarrows}} \mathcal{O}(\mathbb{P}^1)\otimes\mathcal{A}$$

whereas $\mathrm{MT}(\tau)$ should rather be equivalent to the homotopy colimit of

$$\mathcal{A} \underset{\mathrm{id}}{\overset{\tau}{\rightrightarrows}} \mathcal{A}$$

(we did not define $\mathrm{MT}(\tau)$ using the latter diagram because [24] only defines homotopy colimits of diagrams indexed by posets).

(3) The mapping torus of a quasi-autoequivalence is also Adams-graded, because it is the localization of an Adams-graded A_∞-category at morphisms of Adams degree 0.

Definition 2.2 Let $\mathcal{A}$ be an A_∞-category. A $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$ is a bijection

$$\mathbb{Z} \times \mathcal{E} \xrightarrow{\sim} \mathrm{ob}(\mathcal{A}), \quad (n, E) \mapsto X^n(E),$$

where $\mathcal{E}$ is some set. If such a splitting has been chosen, we define the Adams-grading of a homogeneous element $x \in \mathcal{A}(X^i(E), X^j(E))$ to be $j - i$. This turns $\mathcal{A}$ into an Adams-graded A_∞-category.

Let τ be a quasi-autoequivalence of $\mathcal{A}$. We say that a $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$ is compatible with τ if

$$\tau(X^n(E)) = X^{n+1}(E)$$

for every $n \in \mathbb{Z}$ and $E \in \mathcal{E}$.

We say that $\mathcal{A}$ is weakly directed with respect to a $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$ if

$$\mathcal{A}(X^i(E), X^j(E')) = 0$$

for every $i > j$ and $E, E' \in \mathcal{E}$ (we use the term "weakly directed" A_∞-category because the notion is slightly more general than that of directed A_∞-category defined by Seidel in [36, Paragraph (5m)]).

Remark Compatible $\mathbb{Z}$-splittings naturally arise in the context of $\mathbb{Z}$-actions. A strict $\mathbb{Z}$-action on an A_∞-category $\mathcal{A}$ is a family of A_∞-endofunctors $(\tau_n)_{n\in\mathbb{Z}}$ such that $\tau_0 = \mathrm{id}_{\mathcal{A}}$ and $\tau_{i+j} = \tau_i \circ \tau_j$ (see [36, Paragraph (10b)]). If the induced $\mathbb{Z}$-action on $\mathrm{ob}(\mathcal{A})$ is free, then any section σ of the projection $\mathrm{ob}(\mathcal{A}) \to \mathcal{E}$, where $\mathcal{E}$ is the set of equivalence classes of objects in $\mathcal{A}$ under the $\mathbb{Z}$-action, gives a $\mathbb{Z}$-splitting

$$\mathbb{Z} \times \mathcal{E} \xrightarrow{\sim} \mathrm{ob}(\mathcal{A}), \quad (n, E) \mapsto \tau_n(\sigma(E)),$$

which is compatible with the automorphism τ_1.

2.1.2 Main results

First result

Definition 2.3 Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$. Assume that τ is strict, ie $\tau^d = 0$ for $d \geq 2$, and acts bijectively on hom-sets. In this case, we define an Adams-graded A_∞-category $\mathcal{A}_\tau$ as:

(1) The set of objects of $\mathcal{A}_\tau$ is $\mathcal{E}$.

(2) The space of morphisms $\mathcal{A}_\tau(E, E')$ is the Adams-graded vector space given by

$$\mathcal{A}_\tau(E, E') = \Bigl(\bigoplus_{i,j\in\mathbb{Z}} \mathcal{A}(X^i(E), X^j(E')) \Bigr) \Big/ (\tau(x) \sim x).$$

(3) The operations are the unique linear maps such that for every sequence

$$(x_0, \dots, x_{d-1}) \in \mathcal{A}(X^{i_0}(E_0), X^{i_1}(E_1)) \times \cdots \times \mathcal{A}(X^{i_{d-1}}(E_{d-1}), X^{i_d}(E_d)),$$

we have

$$\mu_{\mathcal{A}_\tau}([x_0], \dots, [x_{d-1}]) = [\mu_{\mathcal{A}}(x_0, \dots, x_{d-1})],$$

where $[-]\colon \mathcal{A}(X^i(E), X^j(E')) \to \mathcal{A}_\tau(E, E')$ denotes the projection. (It is not hard to see that such operations exist and satisfy the A_∞-relations.)

Remark When $\mathcal{A}$ is a DG-category, the latter construction is known as the orbit category, see [27; 28, Section 4.9]. In [26, Section 4], it is denoted by $\mathcal{A}\#\mathbb{Z}$ (considering that τ induces a $\mathbb{Z}$-action on $\mathcal{A}$).

Theorem 2.4 *Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$. Assume that τ is strict and acts bijectively on hom-sets. Then there is a quasi-equivalence of Adams-graded A_∞-categories*

$$\mathrm{MT}(\tau) \simeq \mathcal{A}_\tau.$$

Remark (1) The A_∞-category $\mathcal{A}_\tau$ is the (ordinary) colimit of the diagram used to define $\mathrm{MT}(\tau)$. Thus, Theorem 2.4 can be thought of as a "homotopy colimit equals colimit" result.

(2) In [26], given a DG-category $\mathcal{A}$ and an autoequivalence τ acting bijectively on hom-sets, the author defines a DG-category $M_\tau := (\mathcal{O}(\widetilde{\mathcal{T}}_0)_{\mathrm{dg}} \otimes \mathcal{A})\#\mathbb{Z}$ (see [26] for the notation). In the case where τ moreover induces a free $\mathbb{Z}$-action on objects, Theorem 2.4 says that relating $\mathrm{MT}(\tau)$ and M_τ amounts to comparing $\mathcal{A}\#\mathbb{Z}$ and $(\mathcal{O}(\widetilde{\mathcal{T}}_0)_{\mathrm{dg}} \otimes \mathcal{A})\#\mathbb{Z}$.

Second result We denote by $\mathbb{F}[t_m]$ the augmented Adams-graded associative algebra generated by a variable t_m of bidegree $(m, 1)$. Observe that if $\mathcal{C}$ is a subcategory of an A_∞-category $\mathcal{D}$ with $\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{D})$, then $\mathcal{C} \oplus (t\mathbb{F}[t] \otimes \mathcal{D})$ is naturally an Adams-graded A_∞-category, where the Adams degree of $t^k \otimes x$ equals k. Besides, if $\mathcal{C}$ is an A_∞-category equipped with a $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{C})$, we denote by $\mathcal{C}^0$ the full A_∞-subcategory of $\mathcal{C}$ whose set of objects corresponds to $\{0\} \times \mathcal{E}$. Finally, we use the functor $\mathcal{C} \mapsto \mathcal{C}_m$ of Definition 1.27.

Theorem 2.5 (Theorem A in the introduction) *Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$, weakly directed with respect to some compatible $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$. Assume that there exists a closed degree 0 bimodule map $f : \mathcal{A}_m(-,-) \to \mathcal{A}_m(-,\tau(-))$ such that*

$$f : \mathcal{A}_m(X^i(E), X^j(E')) \to \mathcal{A}_m(X^i(E), X^{j+1}(E'))$$

is a quasi-isomorphism for every $i < j$ and $E, E' \in \mathcal{E}$. Then there is a quasi-equivalence of Adams-graded A_∞-categories

$$\mathrm{MT}(\tau) \simeq \mathcal{A}_m^0 \oplus (t_m\mathbb{F}[t_m] \otimes \mathcal{A}_m[f(\mathrm{units})^{-1}]^0).$$

Outline of the section In Section 2.2, we consider an A_∞-category $\mathcal{A}$ equipped with a $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$ and a choice of a closed degree 0 morphism $c_n(E) \in \mathcal{A}(X^n(E), X^{n+1}(E))$ for every $n \in \mathbb{Z}$ and every $E \in \mathcal{E}$. We give technical results about specific modules associated to this data. This will be used in the proof of Theorem 2.5 with $c_n(E) = f(e_{X^n(E)})$.

In Section 2.3, we consider the Grothendieck construction $\mathcal{G}$ of a slightly different diagram than the one in Definition 2.1, together with a set $W_\mathcal{G}$ of closed degree 0 morphisms. The idea is that the localization

$\mathcal{H} = \mathcal{G}[W_{\mathcal{G}}^{-1}]$ is the homotopy colimit of a diagram obtained from the one in Definition 2.1 by a cofibrant replacement of the diagonal functor $\mathcal{A} \sqcup \mathcal{A} \to \mathcal{A}$. Thus it is not surprising that $\mathcal{H}$ is quasi-equivalent to the mapping torus of τ. Moreover, we prove technical results about specific modules over $\mathcal{G}$ that will be used in the proofs of Theorems 2.4 and 2.5.

In Section 2.4, we prove Theorem 2.4. We first define an A_∞-functor $\Phi\colon \mathcal{G} \to \mathcal{A}_\tau$ which sends $W_{\mathcal{G}}$ to the set of units in $\mathcal{A}_\tau$. Then we prove that the induced A_∞-functor $\widetilde{\Phi}\colon \mathcal{H} \to \mathcal{A}_\tau[\{\text{units}\}^{-1}]$ is a quasi-equivalence. To do that, our strategy is to apply Proposition 1.17 using the results of Section 2.3 about the specific $\mathcal{G}$-modules.

In Section 2.5, we prove Theorem 2.5. We use the fact that $\mathcal{G}$ is "big enough" (there are more objects and morphisms than in the Grothendieck construction of the diagram in Definition 2.1) in order to define an A_∞-functor $\Psi_m\colon \mathcal{G}_m \to \mathcal{A}_m$ (see Definition 1.27). This induces an A_∞-functor

$$\widetilde{\Psi}\colon \mathcal{H} \to \mathbb{F}[t_m] \otimes \mathcal{A}_m[f(\{\text{units}\})^{-1}].$$

Then we prove that for every Adams degree $j \geq 1$, and for every objects X, Y in $\mathcal{H}$, the map

$$\widetilde{\Psi}\colon \mathcal{H}(X,Y)^{*,j} \to (\mathbb{F}[t_m] \otimes \mathcal{A}_m[f(\{\text{units}\})^{-1}])(\Psi X, \Psi Y)^{*,j}$$

is a quasi-isomorphism (if V is an Adams-graded vector space, $V^{*,j}$ denotes the subspace of Adams degree j elements). To do that, we apply once again Proposition 1.17 using the results of Sections 2.2 and 2.3 about the specific modules over $\mathcal{A}_m$ and $\mathcal{G}$ respectively. This allows us to finish the proof of Theorem 2.5.

2.2 Results about specific modules

In this section, we give technical results that will allow us to apply Proposition 1.17 in the proof of Theorem 2.5.

Let $\mathcal{A}$ be an A_∞-category equipped with a $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$. Assume that we have chosen, for every $n \in \mathbb{Z}$ and every $E \in \mathcal{E}$, a closed degree 0 morphism $c_n(E) \in \mathcal{A}(X^n(E), X^{n+1}(E))$. Moreover, assume that we have chosen a set $W_{\mathcal{A}}$ of closed degree 0 morphisms which contains the morphisms $c_n(E)$.

Remark According to Definition 2.2, the $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$ naturally induces an Adams-grading on $\mathcal{A}$. However in this section, we do not consider $\mathcal{A}$ as being Adams-graded.

In the following, we fix some element $E_\diamond \in \mathcal{E}$. When we write an object X^n or a morphism c_n without specifying the element of $\mathcal{E}$, we mean $X^n(E_\diamond)$ or $c_n(E_\diamond)$ respectively. Recall that

$$t_{c_n}\colon \mathcal{A}(-, X^n) \to \mathcal{A}(-, X^{n+1})$$

denotes the $\mathcal{A}$-module map induced by $c_n \in \mathcal{A}(X^n, X^{n+1})$ (see Definition 1.8).

Definition 2.6 We set $\mathcal{M}_{\mathcal{A}}$ to be the $\mathcal{A}$-module

$$\mathcal{M}_{\mathcal{A}} := \begin{bmatrix} \cdots & & \mathcal{A}(-,X^0) & & \mathcal{A}(-,X^1) & & \cdots \\ & \searrow^{t_{c_{-1}}} & \downarrow \mathrm{id} & \searrow^{t_{c_0}} & \downarrow \mathrm{id} & \searrow^{t_{c_1}} & \\ \cdots & & \mathcal{A}(-,X^0) & & \mathcal{A}(-,X^1) & & \cdots \end{bmatrix} = \begin{bmatrix} \bigoplus_{i\in\mathbb{Z}} \mathcal{A}(-,X^i) \\ \downarrow \bigoplus_{i\in\mathbb{Z}}(\mathrm{id},t_{c_i}) \\ \bigoplus_{i\in\mathbb{Z}} \mathcal{A}(-,X^i) \end{bmatrix}$$

(see Definition 1.18). Besides, we set $t^n_{\mathcal{A}}\colon \mathcal{A}(-,X^n) \to \mathcal{M}_{\mathcal{A}}$ to be the $\mathcal{A}$-module inclusion for every $n \in \mathbb{Z}$.

The first result highlights a key property of the module $\mathcal{M}_{\mathcal{A}}$.

Lemma 2.7 *For every $n \in \mathbb{Z}$, the closed $\mathcal{A}$-module map $t^{n+1}_{\mathcal{A}} \circ t_{c_n}\colon \mathcal{A}(-,X^n) \to \mathcal{M}_{\mathcal{A}}$ is homotopic to $t^n_{\mathcal{A}}\colon \mathcal{A}(-,X^n) \to \mathcal{M}_{\mathcal{A}}$.*

Proof Consider the degree -1 strict $\mathcal{A}$-module map $s\colon \mathcal{A}(-,X^n) \to \mathcal{M}_{\mathcal{A}}$ which sends a morphism in $\mathcal{A}(X,X^n)$ to the corresponding shifted element in $\mathcal{A}(X,X^n)[1]$. Then an easy computation gives

$$\mu^1_{\mathrm{Mod}_{\mathcal{A}}}(s) = t^{n+1}_{\mathcal{A}} \circ t_{c_n} + t^n_{\mathcal{A}}. \qquad \square$$

In the proof of the two results below, we will use specific $\mathcal{A}$-modules. If p is a fixed nonnegative integer, we set

$$K_p = \begin{bmatrix} \cdots & & \mathcal{A}(-,X^{p-1}) & & \mathcal{A}(-,X^p) \\ & \searrow^{t_{c_{p-2}}} & \downarrow \mathrm{id} & \searrow^{t_{c_{p-1}}} & \downarrow \mathrm{id} \\ \cdots & & \mathcal{A}(-,X^{p-1}) & & \mathcal{A}(-,X^p) \end{bmatrix}$$

and

$$\widetilde{K}_p = \begin{bmatrix} \mathcal{A}(-,X^p) & & \mathcal{A}(-,X^{p+1}) & & \cdots \\ & \searrow^{t_{c_p}} & \downarrow \mathrm{id} & \searrow^{t_{c_{p+1}}} & \\ & & \mathcal{A}(-,X^{p+1}) & & \cdots \end{bmatrix}.$$

Moreover, we will consider the sequences of $\mathcal{A}$-modules $(F^q_p)_{q\geq 0}$, $(\widetilde{F}^q_p)_{q\geq 0}$ starting at $F^0_p = \widetilde{F}^0_p = 0$ and with

$$F^q_p = \begin{bmatrix} \mathcal{A}(-,X^{p-q+1}) & & \cdots & & \mathcal{A}(-,X^p) \\ \downarrow \mathrm{id} & \searrow^{t_{c_{p-q+1}}} & & \searrow^{t_{c_{p-1}}} & \downarrow \mathrm{id} \\ \mathcal{A}(-,X^{p-q+1}) & & \cdots & & \mathcal{A}(-,X^p) \end{bmatrix}$$

and

$$\widetilde{F}^q_p = \begin{bmatrix} \mathcal{A}(-,X^p) & & \cdots & & \\ & \searrow^{t_{c_p}} & & \searrow^{t_{c_{p+q-1}}} & \\ & & \cdots & & \mathcal{A}(-,X^{p+q}) \end{bmatrix}$$

for $q \geq 1$.

The following lemma is mostly technical. It will be used in the proofs of Lemmas 2.9 and 2.21.

Lemma 2.8 *Assume that for every $i < j$ and that for every $E \in \mathcal{E}$ the chain map*

$$\mu^2_{\mathcal{A}}(-, c_j)\colon \mathcal{A}(X^i(E), X^j) \to \mathcal{A}(X^i(E), X^{j+1})$$

is a quasi-isomorphism. Then for every $k < n$ and for every $E \in \mathcal{E}$, the inclusion $\mathcal{A}(X^k(E), X^n) \hookrightarrow \mathcal{M}_{\mathcal{A}}(X^k(E))$ is a quasi-isomorphism.

Proof The cone of the inclusion $\mathcal{A}(X^k(E), X^n) \hookrightarrow \mathcal{M}_{\mathcal{A}}(X^k(E))$ is quasi-isomorphic to its cokernel, which is $K_{n-1}(X^k(E)) \oplus \widetilde{K}_n(X^k(E))$.

We have to show that these complexes are acyclic. Observe that

$$\big(F^q_{n-1}(X^k(E))\big)_{q\geq 0} \quad \text{and} \quad \big(\widetilde{F}^q_n(X^k(E))\big)_{q\geq 0}$$

are increasing, exhaustive, and bounded from below filtrations of $K_{n-1}(X^k(E))$ and $\widetilde{K}_n(X^k(E))$, respectively. For every $q \geq 1$, we have

$$F^q_{n-1}(X^k(E))/F^{q-1}_{n-1}(X^k(E)) = \left[\begin{array}{c} \mathcal{A}(X^k(E), X^{n-q}) \\ \big\downarrow \mathrm{id} \\ \mathcal{A}(X^k(E), X^{n-q}) \end{array}\right]$$

and

$$\widetilde{F}^q_n(X^k(E))/\widetilde{F}^{q-1}_n(X^k(E)) = \left[\begin{array}{ccc} \mathcal{A}(X^k(E), X^{n+q-1}) & & \\ & \searrow^{t_{c_{n+q-1}}} & \\ & & \mathcal{A}(X^k(E), X^{n+q}) \end{array}\right].$$

The first of the two latter complexes is clearly acyclic, and the second one is acyclic by assumption on the morphisms c_j. Thus the entire complex $K_{n-1}(X^k(E)) \oplus \widetilde{K}_n(X^k(E))$ is acyclic, which is what we needed to prove. □

The following two lemmas will be used later in order to apply Proposition 1.17.

Lemma 2.9 *Assume that for every $i < j < k$ and for every $E \in \mathcal{E}$ that the chain maps*

$$\begin{aligned} \mu^2_{\mathcal{A}}(-, c_j)\colon \quad & \mathcal{A}(X^i(E), X^j) \to \mathcal{A}(X^i(E), X^{j+1}), \\ \mu^2_{\mathcal{A}}(c_j(E), -)\colon & \mathcal{A}(X^{j+1}(E), X^{k+1}) \to \mathcal{A}(X^j(E), X^{k+1}) \end{aligned}$$

are quasi-isomorphisms. Then for every $(n, E) \in \mathbb{Z} \times \mathcal{E}$, the complex $\mathcal{M}_{\mathcal{A}}(\mathrm{Cone}\, c_n(E))$ is acyclic.

Proof We have

$$\mathcal{M}_{\mathcal{A}}(\mathrm{Cone}\, c_n(E)) = \mathrm{Cone}\big(\mathcal{M}_{\mathcal{A}}(X^{n+1}(E)) \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{A}}}(c_n(E), -)} \mathcal{M}_{\mathcal{A}}(X^n(E))\big),$$

so we have to prove that $\mu^2_{\mathcal{M}_{\mathcal{A}}}(c_n(E), -)\colon \mathcal{M}_{\mathcal{A}}(X^{n+1}(E)) \to \mathcal{M}_{\mathcal{A}}(X^n(E))$ is a quasi-isomorphism. Observe that we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}_{\mathcal{A}}(X^{n+1}(E)) & \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{A}}}(c_n(E),-)} & \mathcal{M}_{\mathcal{A}}(X^n(E)) \\
\big\uparrow & & \big\uparrow \\
\mathcal{A}(X^{n+1}(E), X^{n+2}) & \xrightarrow{\mu^2_{\mathcal{A}}(c_n(E),-)} & \mathcal{A}(X^n(E), X^{n+2})
\end{array}$$

The bottom horizontal map is a quasi-isomorphism by assumption on the morphisms $c_j(E)$. Moreover, the vertical maps are quasi-isomorphisms according to Lemma 2.8. This implies that $\mu^2_{\mathcal{M}_{\mathcal{A}}}(c_n(E), -)$ is indeed a quasi-isomorphism. □

Lemma 2.10 *The $\mathcal{A}$-module map ${}_{W_{\mathcal{A}}^{-1}}t^n_{\mathcal{A}}\colon {}_{W_{\mathcal{A}}^{-1}}\mathcal{A}(-, X^n) \to {}_{W_{\mathcal{A}}^{-1}}\mathcal{M}_{\mathcal{A}}$ is a quasi-isomorphism for every $n \in \mathbb{Z}$.*

Proof Let X be some object of $\mathcal{A}$. We want to prove that the chain map

$${}_{W_{\mathcal{A}}^{-1}}t^n_{\mathcal{A}}\colon {}_{W_{\mathcal{A}}^{-1}}\mathcal{A}(X, X^n) \to {}_{W_{\mathcal{A}}^{-1}}\mathcal{M}_{\mathcal{A}}(X)$$

is a quasi-isomorphism. Observe that

$${}_{W_{\mathcal{A}}^{-1}}\mathcal{M}_{\mathcal{A}}(X) = \left[\begin{array}{ccccc}
\cdots & \mathcal{A}[W_{\mathcal{A}}^{-1}](X, X^0) & & \mathcal{A}[W_{\mathcal{A}}^{-1}](X, X^1) & \cdots \\
\searrow & \big\downarrow \mathrm{id} & \searrow^{{}_{W_{\mathcal{A}}^{-1}}t_{c_0}} & \big\downarrow \mathrm{id} & \searrow^{{}_{W_{\mathcal{A}}^{-1}}t_{c_1}} \\
\cdots & \mathcal{A}[W_{\mathcal{A}}^{-1}](X, X^0) & & \mathcal{A}[W_{\mathcal{A}}^{-1}](X, X^1) & \cdots
\end{array}\right]$$

and the chain map ${}_{W_{\mathcal{A}}^{-1}}t^n_{\mathcal{A}}\colon {}_{W_{\mathcal{A}}^{-1}}\mathcal{A}(X, X^n) \to {}_{W_{\mathcal{A}}^{-1}}\mathcal{M}_{\mathcal{A}}(X)$ is the inclusion. The cone of the latter chain map is then quasi-isomorphic to its cokernel, which is ${}_{W_{\mathcal{A}}^{-1}}K_{n-1}(X) \oplus {}_{W_{\mathcal{A}}^{-1}}\widetilde{K}_n(X)$. Observe that $({}_{W_{\mathcal{A}}^{-1}}F^q_{n-1}(X))_{q\geq 0}$, $({}_{W_{\mathcal{A}}^{-1}}\widetilde{F}^q_n(X))_{q\geq 0}$ are increasing, exhaustive, and bounded from below filtrations of ${}_{W_{\mathcal{A}}^{-1}}K_{n-1}(X)$, ${}_{W_{\mathcal{A}}^{-1}}\widetilde{K}_n(X)$, respectively. For every $q \geq 1$, we have

$${}_{W_{\mathcal{A}}^{-1}}F^q_{n-1}(X)/{}_{W_{\mathcal{A}}^{-1}}F^{q-1}_{n-1}(X) = \left[\begin{array}{c}
\mathcal{A}[W_{\mathcal{A}}^{-1}](X, X^{n-q}) \\
\big\downarrow \mathrm{id} \\
\mathcal{A}[W_{\mathcal{A}}^{-1}](X, X^{n-q})
\end{array}\right]$$

and

$${}_{W_{\mathcal{A}}^{-1}}\widetilde{F}^q_n(X)/{}_{W_{\mathcal{A}}^{-1}}\widetilde{F}^{q-1}_n(X) = \left[\begin{array}{cc}
\mathcal{A}[W_{\mathcal{A}}^{-1}](X, X^{n-1+q}) & \\
 & \searrow^{{}_{W_{\mathcal{A}}^{-1}}t_{c_{n-1+q}}} \\
 & \mathcal{A}[W_{\mathcal{A}}^{-1}](X, X^{n+q})
\end{array}\right].$$

The first of the two latter complexes is clearly acyclic, and the second one is acyclic because c_{n-1+q} belongs to the set $W_{\mathcal{A}}$ by which we localized (see [23, Lemma 3.12]). Thus the entire complex ${}_{W_{\mathcal{A}}^{-1}}K_{n-1}(X) \oplus {}_{W_{\mathcal{A}}^{-1}}\widetilde{K}_n(X)$ is acyclic, which is what we needed to prove. □

2.3 The A_∞-category and modules for the mapping torus

In this section, we consider an A_∞-category $\mathcal{G}$, together with a set $W_\mathcal{G}$ of closed degree 0 morphisms. We prove that $\mathcal{H} = \mathcal{G}[W_\mathcal{G}^{-1}]$ is quasi-equivalent to the mapping torus of τ, and we prove technical results about specific $\mathcal{G}$-modules that will allow us to apply Proposition 1.17 in the proofs of Theorems 2.4 and 2.5.

Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$. If $\mathcal{A}_\triangle$ is a copy of $\mathcal{A}$, we denote by $X^n_\triangle(E)$ the object of $\mathcal{A}_\triangle$ corresponding to $(n, E) \in \mathbb{Z} \times \mathcal{E}$.

2.3.1 The Grothendieck construction $\mathcal{G}$ The A_∞-category $\mathcal{G}$ will be the Grothendieck construction of a slightly different diagram than the one in Definition 2.1. The idea is to introduce an A_∞-category $\mathcal{C}$ together with a set of closed degree 0 morphisms $W_\mathcal{C}$ such that the localization $\mathcal{C}[W_\mathcal{C}^{-1}]$ is a cylinder object for $\mathcal{A}$. Observe that this kind of cofibrant replacement is common in homotopy colimits computation, and indeed we need it to prove Theorem 2.5.

Definition 2.11 Let $\mathcal{A}_\perp$, $\mathcal{A}_I$ and $\mathcal{A}_\top$ be three copies of $\mathcal{A}$. We denote by $\mathcal{C}$ the Grothendieck construction (see Definition 1.20) of the diagram

$$\begin{array}{ccc} \mathcal{A}_I & \xrightarrow{\mathrm{id}} & \mathcal{A}_\top \\ {\scriptstyle \mathrm{id}}\downarrow & & \\ \mathcal{A}_\perp & & \end{array}$$

and we let $\iota_\perp, \iota_I, \iota_\top : \mathcal{A} \to \mathcal{C}$ be the strict inclusions with images $\mathcal{A}_\perp$, $\mathcal{A}_I$, $\mathcal{A}_\top$, respectively. Finally, we denote by $W_\mathcal{C}$ the set of adjacent units in $\mathcal{C}$, and we let $\mathcal{C}yl_\mathcal{A} = \mathcal{C}[W_\mathcal{C}^{-1}]$ be the homotopy colimit of the diagram above.

Definition 2.12 Let $\mathcal{A}_-$, $\mathcal{A}_+$, $\mathcal{A}_\bullet$ be three copies of $\mathcal{A}$. We denote by $\mathcal{G}$ the Grothendieck construction of the diagram

$$\begin{array}{ccc} \mathcal{A}_- \sqcup \mathcal{A}_+ & \xrightarrow{\mathrm{id} \sqcup \tau} & \mathcal{A}_\bullet \\ {\scriptstyle \iota_\perp \sqcup \iota_\top}\downarrow & & \\ \mathcal{C} & & \end{array}$$

Also, we denote by $W_\mathcal{G}$ the union of $W_\mathcal{C}$ and the set of adjacent units in $\mathcal{G}$, and we set

$$\mathcal{H} := \mathcal{G}[W_\mathcal{G}^{-1}].$$

According to Proposition 1.24, $\mathcal{C}yl_\mathcal{A}$ can be thought as a cylinder object for $\mathcal{A}$. Therefore, the following result should not be surprising.

Lemma 2.13 *The mapping torus of τ is quasi-equivalent to $\mathcal{H}$.*

Proof Let $\pi : \mathcal{C} \to \mathcal{A}$ be the A_∞-functor induced by the commutative diagram

$$\begin{array}{ccc} \mathcal{A}_I & \xrightarrow{\text{id}} & \mathcal{A}_\top \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} \\ \mathcal{A}_\perp & \xrightarrow{\text{id}} & \mathcal{A} \end{array}$$

(see Proposition 1.21). We get a commutative diagram

$$\begin{array}{ccccc} \mathcal{C} & \xleftarrow{\iota_\perp \sqcup \iota_\top} & \mathcal{A}_- \sqcup \mathcal{A}_+ & \xrightarrow{\text{id} \sqcup \tau} & \mathcal{A}_\bullet \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} \\ \mathcal{A} & \xleftarrow[\text{id} \sqcup \text{id}]{} & \mathcal{A}_- \sqcup \mathcal{A}_+ & \xrightarrow[\text{id} \sqcup \tau]{} & \mathcal{A}_\bullet \end{array}$$

which induces an A_∞-functor χ from $\mathcal{G}$ to the Grothendieck construction of the bottom line (see Proposition 1.22). Observe that χ sends $W_\mathcal{C}$ to the set U of units in $\mathcal{A}$. Now, according to Proposition 1.24, the A_∞-functor $\widetilde{\pi} : \mathcal{C}yl_\mathcal{A} = \mathcal{C}[W_\mathcal{C}^{-1}] \to \mathcal{A}[U^{-1}]$ is a quasi-equivalence. According to Lemma A.6 in [24] (called "localization and homotopy colimits commute"), this implies that the A_∞-functor induced by χ,

$$\text{hocolim} \left(\begin{array}{ccc} \mathcal{A}_- \sqcup \mathcal{A}_+ & \xrightarrow{\text{id} \sqcup \tau} & \mathcal{A}_\bullet \\ \downarrow{\scriptstyle \iota_\perp \sqcup \iota_\top} & & \\ \mathcal{C} & & \end{array} \right) [W_\mathcal{C}^{-1}] \xrightarrow{\widetilde{\chi}} \text{hocolim} \left(\begin{array}{ccc} \mathcal{A}_- \sqcup \mathcal{A}_+ & \xrightarrow{\text{id} \sqcup \tau} & \mathcal{A}_\bullet \\ \downarrow{\scriptstyle \text{id} \sqcup \text{id}} & & \\ \mathcal{A} & & \end{array} \right) [U^{-1}],$$

is a quasi-equivalence. This completes the proof because the source of $\widetilde{\chi}$ is exactly $\mathcal{H}$, and its target is quasi-equivalent to the mapping torus of τ. □

2.3.2 Modules over $\mathcal{G}$ In the following, we fix some element $E_\diamond \in \mathcal{E}$. When we write an object $X^n_\triangle$ without specifying the element of $\mathcal{E}$, we mean $X^n_\triangle(E_\diamond)$. Moreover, we denote by

$$t^n_{\triangle\square} : \mathcal{G}(-, X^n_\triangle) \to \mathcal{G}(-, X^{n+\delta_{\triangle\square}}_\square)$$

the $\mathcal{G}$-module map induced by the adjacent unit in $\mathcal{G}(X^n_\triangle, X^{n+\delta_{\triangle\square}}_\square)$ (see Definition 1.8), where

$$\delta_{\triangle\square} = \begin{cases} 1 & \text{if } (\triangle, \square) = (+, \bullet), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.14 We denote by $\mathcal{M}_\mathcal{G}$ the $\mathcal{G}$-module defined by

$$\mathcal{M}_\mathcal{G} = \left[\begin{array}{cccccccccc} \cdots & & \mathcal{G}(-, X^0_-) & & \mathcal{G}(-, X^0_I) & & \mathcal{G}(-, X^0_+) & & \mathcal{G}(-, X^1_-) & \cdots \\ & \searrow^{t^{-1}_{+\bullet}} & \downarrow{\scriptstyle t^0_{-\bullet}} & \searrow^{t^0_{-\perp}} & \downarrow{\scriptstyle t^0_{I\perp}} & \searrow^{t^0_{I\top}} & \downarrow{\scriptstyle t^0_{+\top}} & \searrow^{t^0_{+\bullet}} & \downarrow{\scriptstyle t^1_{-\bullet}} & \searrow^{t^1_{-\perp}} \\ \cdots & & \mathcal{G}(-, X^0_\bullet) & & \mathcal{G}(-, X^0_\perp) & & \mathcal{G}(-, X^0_\top) & & \mathcal{G}(-, X^1_\bullet) & \cdots \end{array} \right]$$

(see Definition 1.18). For practical reasons, we also consider the $\mathcal{G}$-modules

$$\mathcal{M}^n_\star := \left[\begin{array}{ccccc} & & \mathcal{G}(-, X^n_I) & & \\ & \swarrow^{t^n_{I\perp}} & & \searrow^{t^n_{I\top}} & \\ \mathcal{G}(-, X^n_\perp) & & & & \mathcal{G}(-, X^n_\top) \end{array} \right], \quad n \in \mathbb{Z}.$$

We denote by $t_\mathcal{G} : \mathcal{G}(-, X^0_\bullet) \to \mathcal{M}_\mathcal{G}$ the $\mathcal{G}$-module inclusion.

Remark We can write

$$\mathcal{M}_{\mathcal{G}} = \left[\begin{array}{c} \bigoplus_{n\in\mathbb{Z}} \mathcal{G}(-, X^n_-) \oplus \mathcal{G}(-, X^n_I) \oplus \mathcal{G}(-, X^n_+) \\ \big\downarrow \bigoplus_{n\in\mathbb{Z}} (t^n_{-\bullet}, t^n_{-\perp}, t^n_{I\perp}, t^n_{I\top}, t^n_{+\top}, t^n_{+\bullet}) \\ \bigoplus_{n\in\mathbb{Z}} \mathcal{G}(-, X^n_\bullet) \oplus \mathcal{G}(-, X^n_\perp) \oplus \mathcal{G}(-, X^n_\top) \end{array}\right]$$

and also

$$\mathcal{M}_{\mathcal{G}} = \left[\begin{array}{ccc} & \bigoplus_{n\in\mathbb{Z}} (\mathcal{G}(-, X^n_-) \oplus \mathcal{G}(-, X^n_+)) & \\ \swarrow \bigoplus_{n\in\mathbb{Z}} (t^n_{-\perp}, t^n_{+\top}) & & \searrow \bigoplus_{n\in\mathbb{Z}} (t^n_{-\bullet}, t^n_{+\bullet}) \\ \bigoplus_{n\in\mathbb{Z}} \mathcal{M}^n_\star & & \bigoplus_{n\in\mathbb{Z}} \mathcal{G}(-, X^n_\bullet) \end{array}\right].$$

The following two lemmas are analogs of Lemmas 2.9 and 2.10, respectively. They will be used later in order to apply Proposition 1.17.

Lemma 2.15 *For every w in $W_{\mathcal{G}}$, the complex $\mathcal{M}_{\mathcal{G}}(\mathrm{Cone}\, w)$ is acyclic.*

Proof Let w be the morphism in $W_{\mathcal{G}} \cap \mathcal{G}(X^k_I(E), X^k_\top(E))$ (the proof is analogous for the morphism in $W_{\mathcal{G}} \cap \mathcal{G}(X^k_I(E), X^k_\perp(E))$). Then

$$\begin{aligned} \mathcal{M}_{\mathcal{G}}(\mathrm{Cone}\, w) &= \mathrm{Cone}\big(\mathcal{M}_{\mathcal{G}}(X^k_\top(E)) \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)} \mathcal{M}_{\mathcal{G}}(X^k_I(E))\big) \\ &= \bigoplus_n \mathrm{Cone}\big(\mathcal{G}(X^k_\top(E), X^n_\top) \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)} \mathcal{M}^n_\star(X^k_I(E))\big). \end{aligned}$$

We want to prove that $\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)\colon \mathcal{G}(X^k_\top(E), X^n_\top) \to \mathcal{M}^n_\star(X^k_I(E))$ is a quasi-isomorphism for every n. Observe that the following diagram of chain complexes is commutative:

$$\begin{array}{ccc} \mathcal{G}(X^k_\top(E), X^n_\top) & \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)} & \mathcal{M}^n_\star(X^k_I(E)) \\ \| & & \uparrow \\ \mathcal{G}(X^k_\top(E), X^n_\top) & \xrightarrow{\mu^2_{\mathcal{G}}(w,-)} & \mathcal{G}(X^k_I(E), X^n_\top) \end{array}$$

The rightmost vertical arrow is injective, so its cone is quasi-isomorphic to its cokernel, which is the cone of $t^n_{I\perp}\colon \mathcal{G}(X^k_I(E), X^n_I) \to \mathcal{G}(X^k_I(E), X^n_\perp)$. Since the latter map is a quasi-isomorphism, the cone of $\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)\colon \mathcal{G}(X^k_\top(E), X^n_\top) \to \mathcal{M}^n_\star(X^k_I(E))$ is quasi-isomorphic to the cone of

$$\mu^2_{\mathcal{G}}(w,-)\colon \mathcal{G}(X^k_\top(E), X^n_\top) \to \mathcal{G}(X^k_I(E), X^n_\top).$$

The latter map is a quasi-isomorphism, so we conclude that $\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)\colon \mathcal{G}(X^k_\top(E), X^n_\top) \to \mathcal{M}^n_\star(X^k_I(E))$ is a quasi-isomorphism for every n, and thus $\mathcal{M}_{\mathcal{G}}(\mathrm{Cone}\, w)$ is acyclic.

Now let w be the morphism in $W_{\mathcal{G}} \cap \mathcal{G}(X^k_+(E), X^k_\top(E))$ (the proof is analogous for the morphism in $W_{\mathcal{G}} \cap \mathcal{G}(X^k_-(E), X^k_\perp(E))$). Then

$$\begin{aligned} \mathcal{M}_{\mathcal{G}}(\mathrm{Cone}\, w) &= \mathrm{Cone}\big(\mathcal{M}_{\mathcal{G}}(X^k_\top(E)) \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)} \mathcal{M}_{\mathcal{G}}(X^k_+(E))\big) \\ &= \bigoplus_n \mathrm{Cone}\big(\mathcal{G}(X^k_\top(E), X^n_\top) \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)} K^n\big), \end{aligned}$$

where

$$K^n = \left[\begin{array}{ccc} & \mathcal{G}(X^k_+(E), X^n_+) & \\ \swarrow^{t^n_{+\top}} & & \searrow^{t^n_{+\bullet}} \\ \mathcal{G}(X^k_+(E), X^n_\top) & & \mathcal{G}(X^k_+(E), X^{n+1}_\bullet) \end{array}\right].$$

Observe that $\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)\colon \mathcal{G}(X^k_\top(E), X^n_\top) \to K^n$ is injective (it is basically an inclusion once we unravel the definitions), so its cone is quasi-isomorphic to its cokernel, which is the cone of

$$t^n_{+\bullet}\colon \mathcal{G}(X^k_+(E), X^n_+) \to \mathcal{G}(X^k_+(E), X^{n+1}_\bullet).$$

The latter map is a quasi-isomorphism because τ is a quasi-equivalence. This implies that the cone of $\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)\colon \mathcal{G}(X^k_\top(E), X^n_\top) \to K^n$ is acyclic for every n, and thus $\mathcal{M}_\mathcal{G}(\operatorname{Cone} w)$ is acyclic.

It remains to consider a morphism w in $W_\mathcal{G} \cap \mathcal{G}(X^k_+(E), X^{k+1}_\bullet(E))$ (the proof is analogous for the morphism in $W_\mathcal{G} \cap \mathcal{G}(X^k_-(E), X^k_\bullet(E))$). Then

$$\begin{aligned} \mathcal{M}_\mathcal{G}(\operatorname{Cone} w) &= \operatorname{Cone}\big(\mathcal{M}_\mathcal{G}(X^{k+1}_\bullet(E)) \xrightarrow{\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)} \mathcal{M}_\mathcal{G}(X^k_+(E))\big) \\ &= \bigoplus_n \operatorname{Cone}\big(\mathcal{G}(X^{k+1}_\bullet(E), X^n_\bullet) \xrightarrow{\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)} K^n\big), \end{aligned}$$

where

$$K^n = \left[\begin{array}{ccc} & \mathcal{G}(X^k_+(E), X^{n-1}_+) & \\ \swarrow^{t^{n-1}_{+\top}} & & \searrow^{t^{n-1}_{+\bullet}} \\ \mathcal{G}(X^k_+(E), X^{n-1}_\top) & & \mathcal{G}(X^k_+(E), X^n_\bullet) \end{array}\right].$$

Observe that $\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)\colon \mathcal{G}(X^{k+1}_\bullet(E), X^n_\bullet) \to K^n$ is injective, so its cone is quasi-isomorphic to its cokernel, which is the cone of $t^{n-1}_{+\top}\colon \mathcal{G}(X^k_+(E), X^{n-1}_+) \to \mathcal{G}(X^k_+(E), X^{n-1}_\top)$. The latter map is a quasi-isomorphism, so we conclude that the cone of $\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)\colon \mathcal{G}(X^{k+1}_\bullet(E), X^n_\bullet) \to K^n$ is acyclic for every n, and thus $\mathcal{M}_\mathcal{G}(\operatorname{Cone} w)$ is acyclic. $\square$

Lemma 2.16 *The $\mathcal{H}$-module map ${}_{W_\mathcal{G}^{-1}}t_\mathcal{G}\colon {}_{W_\mathcal{G}^{-1}}\mathcal{G}(-, X^0_\bullet) \to {}_{W_\mathcal{G}^{-1}}\mathcal{M}_\mathcal{G}$ is a quasi-isomorphism.*

Proof We fix an object X in $\mathcal{G}$, and we want to prove that ${}_{W_\mathcal{G}^{-1}}t_\mathcal{G}\colon {}_{W_\mathcal{G}^{-1}}\mathcal{G}(X, X^0_\bullet) \to {}_{W_\mathcal{G}^{-1}}\mathcal{M}_\mathcal{G}(X)$ is a quasi-isomorphism. Observe that

$${}_{W_\mathcal{G}^{-1}}\mathcal{M}_\mathcal{G} := \left[\begin{array}{ccccc} \cdots & \mathcal{G}[W_\mathcal{G}^{-1}](-, X^{-1}_+) & & \mathcal{G}[W_\mathcal{G}^{-1}](-, X^0_-) & \cdots \\ \searrow & \downarrow {}_{W_\mathcal{G}^{-1}}t^{-1}_{+\top} & \searrow {}_{W_\mathcal{G}^{-1}}t^{-1}_{+\bullet} & \downarrow {}_{W_\mathcal{G}^{-1}}t^0_{-\bullet} & \searrow {}_{W_\mathcal{G}^{-1}}t^0_{-\perp} \\ \cdots & \mathcal{G}[W_\mathcal{G}^{-1}](-, X^{-1}_\top) & & \mathcal{G}[W_\mathcal{G}^{-1}](-, X^0_\bullet) & \cdots \end{array}\right]$$

and that the chain map ${}_{W_{\mathcal{G}}^{-1}}t_{\mathcal{G}}\colon {}_{W_{\mathcal{G}}^{-1}}\mathcal{G}(X, X^0_\bullet) \to {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X)$ is the inclusion. The cone of the latter chain map is then quasi-isomorphic to its cokernel, which can be written $K' \oplus K''$, where

$$K' = \begin{bmatrix} \cdots & & \mathcal{G}[W_{\mathcal{G}}^{-1}](X, X_I^{-1}) & & \mathcal{G}[W_{\mathcal{G}}^{-1}](X, X_+^{-1}) \\ & \searrow & \Big\downarrow {\scriptstyle {}_{W_{\mathcal{G}}^{-1}}t_{I\perp}^{-1}} & \searrow^{{}_{W_{\mathcal{G}}^{-1}}t_{I\top}^{-1}} & \Big\downarrow {\scriptstyle {}_{W_{\mathcal{G}}^{-1}}t_{+\top}^{-1}} \\ \cdots & & \mathcal{G}[W_{\mathcal{G}}^{-1}](X, X_\perp^{-1}) & & \mathcal{G}[W_{\mathcal{G}}^{-1}](X, X_\top^{-1}) \end{bmatrix}$$

and

$$K'' = \begin{bmatrix} \mathcal{G}[W_{\mathcal{G}}^{-1}](X, X_-^0) & & \mathcal{G}[W_{\mathcal{G}}^{-1}](X, X_I^0) & & \cdots \\ & \searrow^{{}_{W_{\mathcal{G}}^{-1}}t_{-\perp}^{0}} & \Big\downarrow {\scriptstyle {}_{W_{\mathcal{G}}^{-1}}t_{I\perp}^{0}} & \searrow^{{}_{W_{\mathcal{G}}^{-1}}t_{I\top}^{0}} & \\ & & \mathcal{G}[W_{\mathcal{G}}^{-1}](X, X_\perp^0) & & \cdots \end{bmatrix}.$$

Observe that the maps defining the chain complex structures in K' and K'' are all quasi-isomorphisms (see [23, Lemma 3.12]). Thus it is not difficult to show using an increasing exhaustive and bounded from below filtration of K' and K'' that these complexes are acyclic (compare the proof of Lemma 2.10). This implies that the map ${}_{W_{\mathcal{G}}^{-1}}t_{\mathcal{G}}\colon {}_{W_{\mathcal{G}}^{-1}}\mathcal{G}(X, X^0_\bullet) \to {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X)$ is a quasi-isomorphism. □

2.4 Proof of the first result

Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$. Assume that τ is strict and acts bijectively on hom-sets.

Observe that there is a strict A_∞-functor $\sigma\colon \mathcal{A} \to \mathcal{A}_\tau$ which sends $X^n(E)$ to E, and which sends $x \in \mathcal{A}(X^i(E_1), X^j(E_2))$ to $[x] \in \mathcal{A}_\tau(E_1, E_2)$. Besides, let $\pi : \mathcal{C} \to \mathcal{A}$ be the A_∞-functor induced by the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{id}} & \mathcal{A} \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{A} & \xrightarrow{\mathrm{id}} & \mathcal{A} \end{array}$$

(see Proposition 1.21). Then the diagram of Adams-graded A_∞-categories

$$\begin{array}{ccc} \mathcal{A}_- \sqcup \mathcal{A}_+ & \xrightarrow{\mathrm{id}\sqcup\tau} & \mathcal{A}_\bullet \\ {\scriptstyle \iota_\perp \sqcup \iota_\top}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ \mathcal{C} & \xrightarrow{\sigma\circ\pi} & \mathcal{A}_\tau \end{array}$$

is commutative because $\sigma \circ \tau = \sigma$. Moreover, the induced A_∞-functor $\Phi\colon \mathcal{G} \to \mathcal{A}_\tau$ is strict, and it sends $W_{\mathcal{G}}$ to the set of units in $\mathcal{A}_\tau$. Let

$$\widetilde{\Phi}\colon \mathcal{H} \to \mathcal{A}_\tau[\{\text{units}\}^{-1}]$$

be the A_∞-functor induced by Φ.

According to Lemma 2.13, $\mathcal{H}$ is quasi-equivalent to the mapping torus of τ. Moreover, $\mathcal{A}_\tau[\{\text{units}\}^{-1}]$ is quasi-equivalent to $\mathcal{A}_\tau$. Thus, Theorem 2.4 will follow if we prove that $\widetilde{\Phi}$ is a quasi-equivalence. Our strategy is to apply Proposition 1.17. Observe that it suffices to prove that

$$\widetilde{\Phi}\colon \mathcal{H}(X,Y) \to \mathcal{A}_\tau[\{\text{units}\}^{-1}](\Phi X, \Phi Y)$$

is a quasi-isomorphism for every object X, Y in $\mathcal{A}^0_\bullet$ (recall that $\mathcal{A}^0_\bullet$ denotes the subcategory of $\mathcal{A}_\bullet$ generated by the objects $X^0_\bullet(E)$, $E \in \mathcal{E}$) because every object of $\mathcal{G}$ can be related to one of $\mathcal{A}^0_\bullet$ by a zigzag of morphisms in $W_{\mathcal{G}}$, which are quasi-isomorphisms in $\mathcal{H}$ (see [23, Lemma 3.12]).

In the following, we fix some element $E_\diamond \in \mathcal{E}$. When we write an object $X^n_\triangle$ without specifying the element of $\mathcal{E}$, we mean $X^n_\triangle(E_\diamond)$. We consider the corresponding $\mathcal{G}$-module $\mathcal{M}_{\mathcal{G}}$ and the $\mathcal{G}$-module map $t_{\mathcal{G}}\colon \mathcal{G}(-, X^0_\bullet) \to \mathcal{G}$ of Definition 2.14. Moreover, we set

$$\mathcal{M}_{\mathcal{A}_\tau} := \mathcal{A}_\tau(-, E_\diamond) \quad \text{and} \quad t_{\mathcal{A}_\tau} := \mathrm{id}\colon \mathcal{A}_\tau(-, E_\diamond) \to \mathcal{M}_{\mathcal{A}_\tau}.$$

Lemma 2.17 *There exists a $\mathcal{G}$-module map $t_0\colon \mathcal{M}_{\mathcal{G}} \to \Phi^*\mathcal{M}_{\mathcal{A}_\tau}$ (see Definition 1.11 for the pullback functor) such that:*

(1) *The following diagram of $\mathcal{G}$-modules commutes:*

$$\begin{array}{ccc} \mathcal{G}(-, X^0_\bullet) & \xrightarrow{t_\Phi} & \Phi^*\mathcal{A}_\tau(-, E_\diamond) \\ \downarrow{\scriptstyle t_{\mathcal{G}}} & & \downarrow{\scriptstyle \Phi^* t_{\mathcal{A}_\tau} = \mathrm{id}} \\ \mathcal{M}_{\mathcal{G}} & \xrightarrow{t_0} & \Phi^*\mathcal{M}_{\mathcal{A}_\tau} \end{array}$$

(see Definition 1.13 for the map t_Φ).

(2) *For every $E \in \mathcal{E}$, the map $t_0\colon \mathcal{M}_{\mathcal{G}}(X^0_\bullet(E)) \to \Phi^*\mathcal{M}_{\mathcal{A}_\tau}(X^0_\bullet(E))$ is a quasi-isomorphism.*

Proof Observe that the diagram of $\mathcal{G}$-modules

$$\begin{array}{ccc} \mathcal{G}(-, X^n_I) & \xrightarrow{t^n_{I\top}} & \mathcal{G}(-, X^n_\top) \\ \downarrow{\scriptstyle t^n_{I\bot}} & & \downarrow{\scriptstyle t_\Phi} \\ \mathcal{G}(-, X^n_\bot) & \xrightarrow{t_\Phi} & \Phi^*\mathcal{M}_{\mathcal{A}_\tau} \end{array}$$

is commutative, so that it induces a $\mathcal{G}$-module map $\mathcal{M}^n_\star \to \Phi^*\mathcal{M}_{\mathcal{A}_\tau}$ (see Proposition 1.19). Now observe that the following diagram of $\mathcal{G}$-modules commutes:

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{Z}}(\mathcal{G}(-, X^n_-) \oplus \mathcal{G}(-, X^n_+)) & \xrightarrow{\bigoplus_{n\in\mathbb{Z}}(t^n_{-\bullet} \oplus t^n_{+\bullet})} & \bigoplus_{n\in\mathbb{Z}} \mathcal{G}(-, X^n_\bullet) \\ \downarrow{\scriptstyle \bigoplus_{n\in\mathbb{Z}}(t^n_{-\bot} \oplus t^n_{+\top})} & & \downarrow{\scriptstyle t_\Phi} \\ \bigoplus_{n\in\mathbb{Z}} \mathcal{M}^n_\star & \longrightarrow & \Phi^*\mathcal{M}_{\mathcal{A}_\tau} \end{array}$$

We let $t_0\colon \mathcal{M}_{\mathcal{G}} \to \Phi^*\mathcal{M}_{\mathcal{A}_\tau}$ be the induced $\mathcal{G}$-module map. It is then easy to verify that the following diagram of $\mathcal{G}$-modules is commutative:

$$\begin{array}{ccc} \mathcal{G}(-, X^0_\bullet) & \xrightarrow{t_\Phi} & \Phi^*\mathcal{A}_\tau(-, E_\diamond) \\ \downarrow{\scriptstyle t_{\mathcal{G}}} & & \downarrow{\scriptstyle \Phi^* t_{\mathcal{A}_\tau} = \mathrm{id}} \\ \mathcal{M}_{\mathcal{G}} & \xrightarrow{t_0} & \Phi^*\mathcal{M}_{\mathcal{A}_\tau} \end{array}$$

We now prove the second part of the lemma. We have

$$\mathcal{M}_{\mathcal{G}}(X^0_\bullet(E)) = \bigoplus_n \mathcal{G}(X^0_\bullet(E), X^n_\bullet) = \bigoplus_n \mathcal{A}(X^k(E), X^n)$$

and

$$t_0\colon \bigoplus_n \mathcal{A}(X^k(E), X^n) = \mathcal{M}_{\mathcal{G}}(X^0_\bullet(E)) \to \Phi^*\mathcal{M}_{\mathcal{A}_\tau}(X^0_\bullet(E)) = \mathcal{A}_\tau(E, E_\diamond)$$

is the sum of the projections, which is an isomorphism. □

Lemma 2.18 *For every $E \in \mathcal{E}$, the chain map*

$$\widetilde{\Phi}\colon \mathcal{H}(X^0_\bullet(E), X^0_\bullet) \to \mathcal{A}_\tau[\{\text{units}\}^{-1}](E, E_\diamond)$$

is a quasi-isomorphism.

Proof According to Lemmas 2.15, 2.16 and 2.17, the assumptions of Proposition 1.17 are satisfied. □

As explained above, Theorem 2.4 follows from Lemma 2.18 since $\mathcal{H}$ is quasi-equivalent to the mapping torus of τ (see Lemma 2.13) and $\mathcal{A}_\tau[\{\text{units}\}^{-1}]$ is quasi-equivalent to $\mathcal{A}_\tau$.

2.5 Proof of the second result

Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$. Assume that the following holds:

(1) $\mathcal{A}$ is weakly directed with respect to the $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$ (see Definition 2.2).

(2) There exists a closed degree 0 bimodule map $f\colon \mathcal{A}_m(-,-) \to \mathcal{A}_m(-, \tau(-))$ (see Definitions 1.4 and 1.5) such that $f\colon \mathcal{A}_m(X^i(E), X^j(E')) \to \mathcal{A}_m(X^i(E), X^{j+1}(E'))$ is a quasi-isomorphism for every $i < j$ and $E, E' \in \mathcal{E}$.

Remark It follows from Corollary 1.10 and τ being a quasi-equivalence that the chain maps

$$\begin{aligned} \mu^2_{\mathcal{A}_m}(-, f(e_{X^j(E)}))\colon &\quad \mathcal{A}_m(X^i(E'), X^j(E)) \to \mathcal{A}_m(X^i(E'), X^{j+1}(E)), \\ \mu^2_{\mathcal{A}_m}(f(e_{X^j(E)}), -)\colon &\quad \mathcal{A}_m(X^{j+1}(E), X^{k+1}(E')) \to \mathcal{A}_m(X^j(E), X^{k+1}(E')), \end{aligned}$$

are quasi-isomorphisms for every $i < j < k$ and $E, E' \in \mathcal{E}$.

In the following, we set

$$c_n(E) := f(e_{X^n(E)})$$

for every $n \in \mathbb{Z}$, $E \in \mathcal{E}$, and

$$W_{\mathcal{A}_m} := \{c_n(E) \mid n \in \mathbb{Z}, E \in \mathcal{E}\} \cup \{\text{units of } \mathcal{A}_m\}.$$

2.5.1 Generalized homotopy Recall that we introduced a functor $\mathcal{B} \mapsto \mathcal{B}_m$ from the category of Adams-graded A_∞-categories to the category of (non-Adams-graded) A_∞-categories. Also, recall that we introduced Adams-graded A_∞-categories $\mathcal{C}$ and $\mathcal{G}$ in Definitions 2.11 and 2.12, respectively. Observe that $\mathcal{C}_m$ and $\mathcal{G}_m$ are the Grothendieck constructions of the diagrams

$$\begin{array}{ccc} (\mathcal{A}_I)_m & \xrightarrow{\text{id}} & (\mathcal{A}_\top)_m \\ {\scriptstyle \text{id}}\downarrow & & \\ (\mathcal{A}_\bot)_m & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (\mathcal{A}_-)_m \sqcup (\mathcal{A}_+)_m & \xrightarrow{\text{id}\sqcup\tau} & (\mathcal{A}_\bullet)_m \\ {\scriptstyle \iota_\bot\sqcup\iota_\top}\downarrow & & \\ \mathcal{C}_m & & \end{array}$$

respectively. We denote by $W_{\mathcal{C}_m}$ the set of adjacent units in $\mathcal{C}_m$, and by $W_{\mathcal{G}_m}$ the union of $W_{\mathcal{C}_m}$ and the set of adjacent units in $\mathcal{G}_m$.

We would like to define an A_∞-functor $\Psi_m\colon \mathcal{G}_m \to \mathcal{A}_m$ which sends $W_{\mathcal{G}_m}$ to $W_{\mathcal{A}_m}$. According to Proposition 1.21, it is enough to prove the following result.

Lemma 2.19 *There exists an A_∞-functor $\eta\colon \mathcal{C}_m \to \mathcal{A}_m$ which sends $W_{\mathcal{C}_m}$ to $W_{\mathcal{A}_m}$, and such that*

$$\eta \circ \iota_I = \eta \circ \iota_\bot = \text{id}, \quad \eta \circ \iota_\top = \tau.$$

Proof We first define η to be id on $(\mathcal{A}_\bot)_m$, $(\mathcal{A}_I)_m$. and to be τ on $(\mathcal{A}_\top)_m$. Observe that this completely defines η on the objects. Also, we ask for η to act as the identity on the sequences involving an adjacent morphism from $(\mathcal{A}_I)_m$ to $(\mathcal{A}_\bot)_m$.

It remains to define η on the sequences involving an adjacent morphism from $(\mathcal{A}_I)_m$ to $(\mathcal{A}_\top)_m$. Consider a sequence of morphisms

$$\begin{aligned}(x_0, \dots, x_{p+q}) \\ \in \mathcal{C}_m(X_I^{i_0}(E_0), X_I^{i_1}(E_1)) \times \dots \times \mathcal{C}_m(X_I^{i_{p-1}}(E_{p-1}), X_I^{i_p}(E_p)) \times \mathcal{C}_m(X_I^{i_p}(E_p), X_\top^{i_{p+1}}(E_{p+1})) \\ \times \mathcal{C}_m(X_\top^{i_{p+1}}(E_{p+1}), X_\top^{i_{p+2}}(E_{p+2})) \times \dots \times \mathcal{C}_m(X_\top^{i_{p+q}}(E_{p+q}), X_\top^{i_{p+q+1}}(E_{p+q+1})).\end{aligned}$$

Observe that

$$\mathcal{C}_m(X_I^i(E), X_I^j(E')) = \mathcal{C}_m(X_I^i(E), X_\top^j(E')) = \mathcal{C}_m(X_\top^i(E), X_\top^j(E')) = \mathcal{A}_m(X^i(E), X^j(E)).$$

Then we set

$$\eta(x_0, \dots, x_{p+q}) := f(x_0, \dots, x_{p-1}, x_p, x_{p+1}, \dots, x_{p+q+1}) \in \mathcal{A}_m(X^{i_0}(E_0), \tau X^{i_{p+q+1}}(E_{p+q+1})).$$

The functor η we defined satisfies the A_∞-relations because $f\colon \mathcal{A}_m(-,-) \to \mathcal{A}_m(-,\tau(-))$ is a closed degree 0 bimodule map. Moreover, η sends $W_{\mathcal{C}_m}$ to $W_{\mathcal{A}_m}$ by construction. □

Remark First observe that

$$\mathcal{C}yl_{\mathcal{A}_m} = \mathcal{C}_m[W_{\mathcal{C}_m}^{-1}] = (\mathcal{C}[W_{\mathcal{C}}^{-1}])_m = (\mathcal{C}yl_{\mathcal{A}})_m.$$

According to Lemma 2.19, the functor η induces an A_∞-functor $\widetilde{\eta}\colon \mathcal{C}yl_{\mathcal{A}_m} \to \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}]$. Moreover, Lemma 2.19 implies that the following diagram commutes:

$$\begin{array}{ccc}
(\mathcal{A}_+)_m & & \\
\Big\downarrow{\scriptstyle \lambda_{\mathcal{C}_m}\circ\iota_\top} & \overset{\lambda_{\mathcal{A}_m}\circ\tau}{\searrow} & \\
\mathcal{C}yl_{\mathcal{A}_m} & \xrightarrow{\ \widetilde{\eta}\ } & \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}] \\
\Big\uparrow{\scriptstyle \lambda_{\mathcal{C}_m}\circ\iota_\perp} & \underset{\lambda_{\mathcal{A}_m}}{\nearrow} & \\
(\mathcal{A}_-)_m & &
\end{array}$$

($\lambda_{\mathcal{A}_m}\colon \mathcal{A}_m \to \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}]$ and $\lambda_{\mathcal{C}_m}\colon \mathcal{C}_m \to \mathcal{C}_m[W_{\mathcal{C}_m}^{-1}]$ denote the localization functors). Since $\mathcal{C}yl_{\mathcal{A}_m}$ should be thought of as a cylinder object for $\mathcal{A}_m$ (see Proposition 1.24), we should think that the functors $\lambda_{\mathcal{A}_m}$ and $\lambda_{\mathcal{A}_m}\circ\tau$ are homotopic (even if they do not act the same way on objects) and that $\widetilde{\eta}$ is a generalized homotopy between them (see Proposition 1.25 for a justification of this terminology).

2.5.2 Relation between $\mathcal{G}$ and $\mathcal{A}_m$ Using the A_∞-functor $\eta\colon \mathcal{C}_m \to \mathcal{A}_m$ of Lemma 2.19, we get a commutative diagram of (non-Adams-graded) A_∞-categories

$$\begin{array}{ccc}
(\mathcal{A}_-)_m \sqcup (\mathcal{A}_+)_m & \xrightarrow{\ \mathrm{id}\sqcup\tau\ } & (\mathcal{A}_\bullet)_m \\
\Big\downarrow{\scriptstyle \iota_\perp\sqcup\iota_\top} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\
\mathcal{C}_m & \xrightarrow{\quad \eta \quad} & \mathcal{A}_m
\end{array}$$

and the induced A_∞-functor $\Psi_m\colon \mathcal{G}_m \to \mathcal{A}_m$ (see Proposition 1.21) sends $W_{\mathcal{G}_m}$ to $W_{\mathcal{A}_m}$ (see Lemma 2.19). Let

$$\widetilde{\Psi}_m\colon \mathcal{H}_m = \mathcal{G}_m[W_{\mathcal{G}_m}^{-1}] \to \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}]$$

be the A_∞-functor induced by Ψ_m. Observe that, since $\mathcal{A}$ is assumed to be weakly directed and since the Adams degree of $\mathcal{A}$ comes from the $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$ (see Definition 2.2), $\mathcal{H}$ is concentrated in nonnegative Adams degree. In particular, we can apply the adjunction of Definition 1.29 to $\widetilde{\Psi}_m$, which gives an A_∞-functor

$$\widetilde{\Psi}\colon \mathcal{H} = \mathcal{G}[W_{\mathcal{G}}^{-1}] \to \mathbb{F}[t_m] \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}].$$

We would like to prove that for every Adams degree $j \geq 1$, and for every objects X, Y in $\mathcal{A}_\bullet^0$ (recall that $\mathcal{A}_\bullet^0$ denotes the subcategory of $\mathcal{A}_\bullet$ generated by the objects $X_\bullet^0(E)$, $E \in \mathcal{E}$), the map

$$\widetilde{\Psi}\colon \mathcal{H}(X,Y)^{*,j} \to (\mathbb{F}[t_m] \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}])(\Psi X, \Psi Y)^{*,j} = \mathbb{F}_m^j \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}](\Psi X, \Psi Y)$$

is a quasi-isomorphism, ($\mathbb{F}_m^j$ is the vector space generated by t_m^j; also recall that if V is an Adams-graded vector space, $V^{*,j}$ denotes the subspace of Adams degree j elements). Our strategy is once again to apply Proposition 1.17.

In the following we fix some element $E_\diamond \in \mathcal{E}$. When we write $X^n_\triangle$ or c_n without specifying the element of $\mathcal{E}$, we mean $X^n_\triangle(E_\diamond)$ or $c_n(E_\diamond)$, respectively. We consider the corresponding $\mathcal{G}$-module $\mathcal{M}_\mathcal{G}$, and the $\mathcal{G}$-module map $t_\mathcal{G}\colon \mathcal{G}(-, X^n_\triangle) \to \mathcal{G}$ of Definition 2.14. Moreover, we consider the $\mathcal{A}_m$-module $\mathcal{M}_{\mathcal{A}_m}$ and the $\mathcal{A}_m$-module maps

$$t^n_{\mathcal{A}_m}\colon \mathcal{A}_m(-, X^n) \to \mathcal{M}_{\mathcal{A}_m}, \quad n \in \mathbb{Z},$$

associated to the morphisms $(c_n)_{n\in\mathbb{Z}}$ as in Definition 2.6.

The following result is a first step in order to define a $\mathcal{G}_m$-module map $(t_0)_m\colon (\mathcal{M}_\mathcal{G})_m \to \Psi^*_m \mathcal{M}_{\mathcal{A}_m}$ as in Proposition 1.17.

Lemma 2.20 *For every $n \in \mathbb{Z}$, the diagram of $\mathcal{G}_m$-modules*

$$\begin{array}{ccc} \mathcal{G}_m(-, X^n_I) & \xrightarrow{t^n_{I\top}} & \mathcal{G}_m(-, X^n_\top) \\ \Big\downarrow{\scriptstyle t^n_{I\perp}} & & \Big\downarrow{\scriptstyle \Psi^*_m t^{n+1}_{\mathcal{A}_m} \circ t_{\Psi_m}} \\ \mathcal{G}_m(-, X^n_\perp) & \xrightarrow{\Psi^*_m t^n_{\mathcal{A}_m} \circ t_{\Psi_m}} & \Psi^*_m \mathcal{M}_{\mathcal{A}_m} \end{array}$$

commutes up to homotopy.

Proof First observe that $\iota^*_I \mathcal{G}_m(-, X^n_I) = \mathcal{A}_m(-, X^n)$, and $\iota^*_I \Psi^*_m \mathcal{M}_{\mathcal{A}_m} = \mathcal{M}_{\mathcal{A}_m}$ because $\Psi_m \circ \iota_I = \mathrm{id}$ (see Remark 1.12). Moreover, it suffices to show that the $\mathcal{A}_m$-module maps

$$\iota^*_I(\Psi^*_m t^n_{\mathcal{A}_m} \circ t_{\Psi_m} \circ t^n_{I\perp}) = t^n_{\mathcal{A}_m} \circ \iota^*_I(t_{\Psi_m} \circ t^n_{I\perp})\colon \mathcal{A}_m(-, X^n) \to \mathcal{M}_{\mathcal{A}_m}$$

and

$$\iota^*_I(\Psi^*_m t^{n+1}_{\mathcal{A}_m} \circ t_{\Psi_m} \circ t^n_{I\top}) = t^{n+1}_{\mathcal{A}_m} \circ \iota^*_I(t_{\Psi_m} \circ t^n_{I\top})\colon \mathcal{A}_m(-, X^n) \to \mathcal{M}_{\mathcal{A}_m}$$

are homotopic because

$$\mathcal{G}_m(X^k_\triangle, X^n_I) = 0 \quad \text{if } \triangle \neq I.$$

On the one hand,

$$t^n_{\mathcal{A}_m} \circ \iota^*_I(t_{\Psi_m} \circ t^n_{I\perp}) = t^n_{\mathcal{A}_m}.$$

On the other hand, $\iota^*_I(t_{\Psi_m} \circ t^n_{I\top})\colon \mathcal{A}_m(-, X^n) \to \mathcal{A}_m(-, X^{n+1})$ is closed (as composition and pullback of closed module maps), and

$$\iota^*_I(t_{\Psi_m} \circ t^n_{I\top})(e_{X^n}) = \eta(e_{X^n}) = c_n$$

according to Lemma 2.19. Therefore, $\iota^*_I(t_{\Psi_m} \circ t^n_{I\top})$ is homotopic to t_{c_n} according to Corollary 1.10, and thus $t^{n+1}_{\mathcal{A}_m} \circ \iota^*_I(t_{\Psi_m} \circ t^n_{I\top})$ is homotopic to $t^{n+1}_{\mathcal{A}_m} \circ t_{c_n}$. Now according to Lemma 2.7, $t^{n+1}_{\mathcal{A}_m} \circ t_{c_n}$ is homotopic to $t^n_{\mathcal{A}_m}$. □

We can now state the result establishing the existence of a $\mathcal{G}_m$-module map $(t_0)_m\colon (\mathcal{M}_\mathcal{G})_m \to \Psi^*_m \mathcal{M}_{\mathcal{A}_m}$ as in Proposition 1.17.

Lemma 2.21 *There exists a $\mathcal{G}_m$-module map $(t_0)_m\colon (\mathcal{M}_\mathcal{G})_m \to \Psi^*_m \mathcal{M}_{\mathcal{A}_m}$ such that the following holds:*

(1) *The following diagram of $\mathcal{G}_m$-modules commutes*:

$$\begin{array}{ccc} \mathcal{G}_m(-, X^0_\bullet) & \xrightarrow{t_{\Psi_m}} & \Psi_m^* \mathcal{A}_m(-, X^0) \\ \big\downarrow t_{\mathcal{G}_m} & & \big\downarrow \Psi_m^* t^0_{\mathcal{A}_m} \\ (\mathcal{M}_{\mathcal{G}})_m & \xrightarrow{(t_0)_m} & \Psi_m^* \mathcal{M}_{\mathcal{A}_m} \end{array}$$

(2) *For every $E \in \mathcal{E}$ and $j \geq 1$, the induced map $t_0 \colon \mathcal{M}_{\mathcal{G}}(X^0_\bullet(E)) \to \mathbb{F}[t_m] \otimes \Psi_m^* \mathcal{M}_{\mathcal{A}_m}(X^0_\bullet(E))$ (see Definition 1.29) is a quasi-isomorphism in each positive Adams degree.*

Proof Using Lemma 2.20 and Proposition 1.19, we get a $\mathcal{G}_m$-module map $t^n_\star \colon (\mathcal{M}^n_\star)_m \to \Psi_m^* \mathcal{M}_{\mathcal{A}_m}$ for every $n \in \mathbb{Z}$ (see Definition 2.14 for the $\mathcal{G}$-modules $\mathcal{M}^n_\star$). Observe that the diagram of $\mathcal{G}_m$-modules

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{Z}} (\mathcal{G}_m(-, X^n_-) \oplus \mathcal{G}_m(-, X^n_+)) & \xrightarrow{\bigoplus_{n\in\mathbb{Z}} (t^n_{-\bullet} \oplus t^n_{+\bullet})} & \bigoplus_{n\in\mathbb{Z}} \mathcal{G}_m(-, X^n_\bullet) \\ \big\downarrow \bigoplus_{n\in\mathbb{Z}} (t^n_{-\perp} \oplus t^n_{+\top}) & & \big\downarrow \bigoplus_{n\in\mathbb{Z}} \Psi_m^* t^n_{\mathcal{A}_m} \circ t_{\Psi_m} \\ \bigoplus_{n\in\mathbb{Z}} (\mathcal{M}^n_\star)_m & \xrightarrow{\bigoplus_{n\in\mathbb{Z}} t^n_\star} & \Psi_m^* \mathcal{M}_{\mathcal{A}_m} \end{array}$$

is commutative (the composition is id for $-$-terms and τ for $+$-terms), so that it induces a $\mathcal{G}_m$-module map $(t_0)_m \colon (\mathcal{M}_{\mathcal{G}})_m \to \Psi_m^* \mathcal{M}_{\mathcal{A}_m}$. It is then easy to verify that the following diagram of $\mathcal{G}_m$-modules is commutative:

$$\begin{array}{ccc} \mathcal{G}_m(-, X^0_\bullet) & \xrightarrow{t_{\Psi_m}} & \Psi_m^* \mathcal{A}_m(-, X^0) \\ \big\downarrow t_{\mathcal{G}_m} & & \big\downarrow \Psi_m^* t^0_{\mathcal{A}_m} \\ (\mathcal{M}_{\mathcal{G}})_m & \xrightarrow{(t_0)_m} & \Psi_m^* \mathcal{M}_{\mathcal{A}_m} \end{array}$$

It remains to show that the map $t_0 \colon \mathcal{M}_{\mathcal{G}}(X^0_\bullet(E))^{*,j} \to \mathbb{F}^j_m \otimes \mathcal{M}_{\mathcal{A}_m}(X^0(E))$ is a quasi-isomorphism for every $E \in \mathcal{E}$ and $j \geq 1$. Note that

$$\mathcal{M}_{\mathcal{G}}(X^0_\bullet(E))^{*,j} = \mathcal{G}(X^0_\bullet(E), X^j_\bullet) = \mathcal{A}(X^0, X^j)$$

and the map

$$\mathcal{A}(X^0(E), X^j) = \mathcal{M}_{\mathcal{G}}(X^0_\bullet(E))^{*,j} \xrightarrow{t_0} \mathbb{F}^j_m \otimes \mathcal{M}_{\mathcal{A}_m}(X^0(E))$$

is the inclusion. Now observe that the following diagram of chain complexes commutes:

$$\begin{array}{ccc} \mathcal{A}(X^0(E), X^j) & \xrightarrow{t_0} & \mathbb{F}^j_m \otimes \mathcal{M}_{\mathcal{A}_m}(X^0(E)) \\ \big\| & & \big\uparrow \\ \mathbb{F}^j_m \otimes \mathcal{A}_m(X^0(E), X^j) & = & \mathbb{F}^j_m \otimes \mathcal{A}_m(X^0(E), X^j) \end{array}$$

The inclusion $\mathcal{A}_m(X^0(E), X^j) \hookrightarrow \mathcal{M}_{\mathcal{A}_m}(X^0(E))$ is a quasi-isomorphism according to Lemma 2.8 (observe that it is important here that j is *strictly* greater than 0). Therefore the map

$$t_0 \colon \mathcal{A}(X^0(E), X^j) \to \mathbb{F}^j_m \otimes \mathcal{M}_{\mathcal{A}_m}(X^0(E))$$

is a quasi-isomorphism, which is what we needed to prove. □

Lemma 2.22 *For every $E \in \mathcal{E}$ and $j \geq 1$, the map*

$$\widetilde{\Psi}\colon \mathcal{H}(X^0_\bullet(E), X^0_\bullet)^{*,j} \to (\mathbb{F}[t_m] \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}])(X^0(E), X^0)^{*,j} = \mathbb{F}_m^j \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}](X^0(E), X^0)$$

is a quasi-isomorphism.

Proof Using the first part of Lemma 2.21 and Proposition 1.17, we know that there exists a chain map $u_m\colon {}_{W_{\mathcal{G}m}^{-1}}(\mathcal{M}_\mathcal{G})_m(X^0_\bullet(E)) \to {}_{W_{\mathcal{A}m}^{-1}}\mathcal{M}_{\mathcal{A}_m}(X^0)$ such that the following diagram of chain complexes commutes:

$$\begin{array}{ccc}
\mathcal{H}_m(X^0_\bullet(E), X^0_\bullet) & \xrightarrow{\widetilde{\Psi}_m} & \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}](X^0(E), X^0) \\
\Big\downarrow {\scriptstyle W_{\mathcal{G}m}^{-1} t_{\mathcal{G}m}} & & \Big\downarrow {\scriptstyle W_{\mathcal{A}m}^{-1} t^0_{\mathcal{A}m}} \\
{}_{W_{\mathcal{G}m}^{-1}}(\mathcal{M}_\mathcal{G})_m(X^0_\bullet(E)) & \xrightarrow{u_m} & {}_{W_{\mathcal{A}m}^{-1}}\mathcal{M}_{\mathcal{A}_m}(\Psi_m X) \\
\Big\uparrow & & \Big\uparrow \\
(\mathcal{M}_\mathcal{G})_m(X^0_\bullet(E)) & \xrightarrow{(t_0)_m} & \mathcal{M}_{\mathcal{A}_m}(\Psi_m X)
\end{array}$$

Observe that

$$\begin{aligned}
\mathcal{H}_m(X^0_\bullet(E), X^0_\bullet) &= \mathcal{H}(X^0_\bullet(E), X^0_\bullet)_m, \\
{}_{W_{\mathcal{G}m}^{-1}}(\mathcal{M}_\mathcal{G})_m(X^0_\bullet(E)) &= {}_{W_\mathcal{G}^{-1}}\mathcal{M}_\mathcal{G}(X^0_\bullet(E))_m, \\
(\mathcal{M}_\mathcal{G})_m(X^0_\bullet(E)) &= \mathcal{M}_\mathcal{G}(X^0_\bullet(E))_m.
\end{aligned}$$

Applying the adjunction of Definition 1.29 to the last diagram, we get the commutative diagram of Adams-graded chain complexes

$$\begin{array}{ccc}
\mathcal{H}(X^0_\bullet(E), X^0_\bullet) & \xrightarrow{\widetilde{\Psi}} & \mathbb{F}[t_m] \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}](X^0(E), X^0) \\
\Big\downarrow {\scriptstyle W_\mathcal{G}^{-1} t_\mathcal{G}} & & \Big\downarrow {\scriptstyle \mathrm{id} \otimes {}_{W_{\mathcal{A}m}^{-1}} t^0_{\mathcal{A}m}} \\
{}_{W_\mathcal{G}^{-1}}\mathcal{M}_\mathcal{G}(X^0_\bullet(E)) & \xrightarrow{u} & \mathbb{F}[t_m] \otimes {}_{W_{\mathcal{A}m}^{-1}}\mathcal{M}_{\mathcal{A}_m}(\Psi_m X) \\
\Big\uparrow & & \Big\uparrow \\
\mathcal{M}_\mathcal{G}(X^0_\bullet(E)) & \xrightarrow{t_0} & \mathbb{F}[t_m] \otimes \mathcal{M}_{\mathcal{A}_m}(\Psi_m X)
\end{array}$$

Specializing to the components of fixed Adams degree $j \geq 1$, we get the commutative diagram of chain complexes

$$\begin{array}{ccc}
\mathcal{H}(X^0_\bullet(E), X^0_\bullet)^{*,j} & \xrightarrow{\widetilde{\Psi}} & \mathbb{F}_m^j \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}](X^0(E), X^0) \\
\Big\downarrow {\scriptstyle W_\mathcal{G}^{-1} t_\mathcal{G}} & & \Big\downarrow {\scriptstyle \mathrm{id} \otimes {}_{W_{\mathcal{A}m}^{-1}} t^0_{\mathcal{A}m}} \\
{}_{W_\mathcal{G}^{-1}}\mathcal{M}_\mathcal{G}(X^0_\bullet(E))^{*,j} & \xrightarrow{u} & \mathbb{F}_m^j \otimes {}_{W_{\mathcal{A}m}^{-1}}\mathcal{M}_{\mathcal{A}_m}(\Psi_m X) \\
\Big\uparrow & & \Big\uparrow \\
\mathcal{M}_\mathcal{G}(X^0_\bullet(E))^{*,j} & \xrightarrow{t_0} & \mathbb{F}_m^j \otimes \mathcal{M}_{\mathcal{A}_m}(\Psi_m X)
\end{array}$$

Using Lemmas 2.15, 2.16, and [23, Lemma 3.13], we know that all the vertical maps on the left are quasi-isomorphisms. Similarly, using Lemmas 2.9, 2.10, and [23, Lemma 3.13], we know that all the vertical maps on the right are quasi-isomorphisms. Moreover, the second part of Lemma 2.21 states that the bottom horizontal map is a quasi-isomorphism. Thus, the chain map

$$\widetilde{\Psi}\colon \mathcal{H}(X^0_\bullet(E), X^0_\bullet)^{*,j} \to \mathbb{F}^j_m \otimes \mathcal{A}_m[W^{-1}_{\mathcal{A}_m}](X^0(E), X^0)$$

is a quasi-isomorphism. □

2.5.3 End of the proof We end the section with the proof of Theorem 2.5. Now that we have proved Lemma 2.22 which takes care of the *positive* Adams degrees, we have to treat the zero Adams degree part (recall that $\mathcal{H}$ is concentrated in nonnegative Adams degree because $\mathcal{A}$ is assumed to be weakly directed).

Let $\mathcal{I}$ be the (nonfull) A_∞-subcategory of $\mathcal{H}$ with

$$\mathrm{ob}(\mathcal{I}) = \{X^0_\bullet(E) \mid E \in \mathcal{E}\} \quad \text{and} \quad \mathcal{I}(X,Y) = \mathcal{G}(X,Y) \oplus \Big(\bigoplus_{j\geq 1} \mathcal{H}(X,Y)^{*,j}\Big)$$

(recall that if V is an Adams-graded vector space, we denote by $V^{*,j}$ its component of Adams degree j).

Lemma 2.23 *The inclusion $\mathcal{I} \hookrightarrow \mathcal{H}$ is a quasi-equivalence.*

Proof Observe that the inclusion $\mathcal{I} \hookrightarrow \mathcal{H}$ is cohomologically essentially surjective because every object of $\mathcal{H}$ can be related to one of $\mathcal{I}$ by a zigzag of morphisms in $W_{\mathcal{G}}$, which are quasi-isomorphisms in $\mathcal{H}$ (see [23, Lemma 3.12]). Therefore, it suffices to show that the inclusion

$$\mathcal{G}(X^0_\bullet(E), X^0_\bullet(E_\diamond)) \hookrightarrow \mathcal{H}(X^0_\bullet(E), X^0_\bullet(E_\diamond))^{*,0}$$

is a quasi-isomorphism for every $E, E_\diamond \in \mathcal{E}$.

Let $E_\diamond$ be an element of $\mathcal{E}$. When we write an object $X^n_\bullet$ without specifying the element of $\mathcal{E}$, we mean $X^n_\bullet(E_\diamond)$. Recall that we introduced a pair $(\mathcal{M}_{\mathcal{G}}, t_{\mathcal{G}})$ in Definition 2.14. According to Lemmas 2.15, 2.16 and [23, Lemma 3.13], the inclusion $\mathcal{M}_{\mathcal{G}}(X^0_\bullet(E)) \hookrightarrow {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X^0_\bullet(E))$ and the map ${}_{W_{\mathcal{G}}^{-1}}t_{\mathcal{G}}\colon \mathcal{H}(X^0_\bullet(E), X^0_\bullet) \to {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X^0_\bullet(E))$ are quasi-isomorphisms for every $E \in \mathcal{E}$. Also, observe that

$$\mathcal{M}_{\mathcal{G}}(X^0_\bullet(E))^{*,0} = \mathcal{G}(X^0_\bullet(E), X^0_\bullet).$$

The result then follows from the commutativity of the diagram

$$\begin{array}{ccccc}
\mathcal{G}(X^0_\bullet(E), X^0_\bullet) & = & \mathcal{G}(X^0_\bullet(E), X^0_\bullet) & = & \mathcal{G}(X^0_\bullet(E), X^0_\bullet) \\
\| & & \downarrow & & \downarrow \\
\mathcal{M}_{\mathcal{G}}(X^0_\bullet(E))^{*,0} & \xhookrightarrow{\sim} & {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X^0_\bullet(E))^{*,0} & \xleftarrow[{}_{W_{\mathcal{G}}^{-1}}t_{\mathcal{G}}]{\sim} & \mathcal{H}(X^0_\bullet(E), X^0_\bullet)^{*,0}
\end{array}$$

□

The following diagram of Adams-graded A_∞-categories is commutative:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\tilde{\Psi}} & \mathbb{F}[t_m] \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}] \\ \uparrow\wr & & \uparrow \\ \mathcal{I} & \xrightarrow{\tilde{\Psi}} & \mathcal{A}_m^0 \oplus (t\mathbb{F}[t_m] \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}]^0) \end{array}$$

(recall that if $\mathcal{C}$ is an A_∞-category equipped with a splitting $\mathrm{ob}(\mathcal{C}) \simeq \mathbb{Z} \times \mathcal{E}$, then we denote by $\mathcal{C}^0$ the full A_∞-subcategory of $\mathcal{C}$ whose set of objects corresponds to $\{0\} \times \mathcal{E}$). Moreover, since $\mathcal{A}$ is assumed to be weakly directed with respect to the $\mathbb{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$, Lemma 2.22 implies that the bottom horizontal A_∞-functor is a quasi-equivalence. Therefore we have

$$\mathcal{H} \simeq \mathcal{A}_m^0 \oplus (t\mathbb{F}[t_m] \otimes \mathcal{A}_m[W_{\mathcal{A}_m}^{-1}]^0).$$

Recall that $W_{\mathcal{A}_m} = f(\{\text{units}\}) \cup \{\text{units}\}$, so that

$$\mathcal{A}_m[W_{\mathcal{A}_m}^{-1}] \simeq \mathcal{A}_m[f(\{\text{units}\})^{-1}].$$

This concludes the proof of Theorem 2.5, since $\mathcal{H}$ is quasi-equivalent to the mapping torus of τ (see Lemma 2.13).

3 Chekanov–Eliashberg DG-category

Recall the following terminology.

Definition 3.1 A contact form is said to be *hypertight* if its Reeb vector field has no contractible periodic orbits.

In this section, we recall the definition of the Chekanov–Eliashberg DG-category associated to a family of Legendrians in a contact manifold equipped with a hypertight contact form α. We also describe the behavior of the Chekanov–Eliashberg DG-category under change of data.

In the following, (V, ξ) is a contact manifold of dimension $2n+1$. In order to have well defined gradings in $\mathbb{Z}$, we assume that $H_1(V)$ is free and that the first Chern class of ξ (equipped with any compatible almost complex structure) is 2-torsion. We will need the following definition.

Definition 3.2 We say that a Legendrian submanifold Λ in (V, ξ) is *chord generic* with respect to a contact form α if

(1) for every Reeb chord $c\colon [0, T] \to V$ of Λ, the space $D\varphi_{R_\alpha}^T(T_{c(0)}\Lambda)$ is transverse to $T_{c(T)}\Lambda$ in ξ,

(2) different Reeb chords belong to different Reeb trajectories.

3.1 Conley–Zehnder index

Let α be a hypertight contact form on (V,ξ) and let Λ be a chord generic Legendrian submanifold of (V,α). In the following, we define the Conley–Zehnder index of a Reeb chord of Λ starting and ending on the same connected component (such chords are called *pure*).

We briefly recall what is the Maslov index of a loop in the Grassmannian of Lagrangian subspaces in $\mathbb{C}^n$. We refer to [33] for a precise exposition. Fix a Lagrangian subspace K, and denote by $\Sigma_k(K)$ the set of Lagrangian subspaces in $\mathbb{C}^n$ whose intersection with K is k dimensional. Consider the *Maslov cycle*

$$\Sigma = \Sigma_1(K) \cup \cdots \cup \Sigma_n(K).$$

This is an algebraic variety of codimension one in the Lagrangian Grassmannian. Now if Γ is a loop in the Lagrangian Grassmannian, its Maslov index $\mu(\Gamma) \in \mathbb{Z}$ is the intersection number of Γ with Σ. The contribution of an intersection instant t_0 is computed as follows. Choose a Lagrangian complement W of K in $\mathbb{C}^n$. Then for each v in $\Gamma(t_0) \cap K$, there exists a vector $w(t)$ in W such that $v + w(t)$ is in $\Gamma(t)$ for every t near t_0. Consider the quadratic form

$$Q(v) = \frac{d}{dt}\omega(v, w(t))\Big|_{t=t_0}$$

on $\Gamma(t_0) \cap K$. Without loss of generality, Q can be assumed to be nonsingular and the contribution of t_0 to $\mu(\Gamma)$ is the signature of Q.

Recall that $H_1(V)$ is assumed to be free. We choose a family $(h_1, \dots, h_r)$ of embedded circles in V which represent a basis of $H_1(V)$, and a symplectic trivialization of ξ over each h_i. If γ is some loop in Λ, there is a unique family $(a_1, \dots, a_r)$ of integers such that $\left[\gamma_c - \sum_i a_i h_i\right]$ is zero in $H_1(V)$. Choose a surface Σ_γ in V such that

$$\partial\Sigma_\gamma = \gamma - \sum_i a_i h_i.$$

There is a unique trivialization of ξ over Σ_γ which extends the chosen trivializations over h_i. Thus we get a trivialization $\gamma^{-1}\xi \simeq S^1 \times \mathbb{C}^n$ (where n is the dimension of Λ). We denote by Γ the loop of Lagrangian planes in $\mathbb{C}^n$ corresponding, via the latter trivialization, to the loop $t \mapsto T_{\gamma(t)}\Lambda$. The Maslov index of Γ does not depend on the choice of the surface Σ_γ because we assumed $2c_1(\xi) = 0$. This construction defines a morphism $H_1(\Lambda, \mathbb{Z}) \to \mathbb{Z}$, and the *Maslov number* $m(\Lambda)$ of Λ is the generator of its image. In the following, we assume that the Maslov number of Λ is zero.

Now, let c be a pure Reeb chord of Λ (a Reeb chord is called pure if it starts and ends on the same connected component of the Legendrian). We choose a path $\gamma_c\colon [0,1] \to \Lambda$ which starts at the endpoint of c, and ends at its starting point (γ_c is called a *capping path* of c). We denote by $\bar{\gamma}_c$ the loop obtained by concatenating γ and c. Let $(a_1, \dots, a_r)$ be the unique family of integers such that $\left[\bar{\gamma}_c - \sum_i a_i h_i\right]$ is zero in $H_1(V)$, and choose a surface Σ_c in V such that

$$\partial\Sigma_c = \bar{\gamma}_c - \sum_i a_i h_i.$$

There is a unique trivialization of ξ over Σ_c which extends the chosen trivializations over h_i. Thus we get a trivialization $\overline{\gamma}_c^{-1}\xi \simeq S^1 \times \mathbb{C}^n$ (where n is the dimension of Λ). We denote by Γ_c the path of Lagrangian planes in $\mathbb{C}^n$ corresponding, via the latter trivialization, to the concatenation of $t \mapsto T_{\gamma(t)}\Lambda$ and $t \mapsto D\varphi^t_{R_\alpha}(T_{c(0)}\Lambda)$. Since Λ is chord generic, Γ_c is not a loop: we close it in the following way. Let I be a complex structure on $\mathbb{C}^n$ which is compatible with the standard symplectic form on $\mathbb{C}^n$ and such that $I(\Gamma_c(1)) = \Gamma_c(0)$. Then we let $\overline{\Gamma}_c$ be the loop of Lagrangian subspaces obtained by concatenating Γ_c and the path $t \in \left[0, \frac{\pi}{2}\right] \mapsto e^{tI}\Gamma_c(1)$. The Conley–Zehnder index of c is the Maslov index of $\overline{\Gamma}_c$:

$$\mathrm{CZ}(c) := \mu(\overline{\Gamma}_c).$$

The Conley–Zehnder index of a Reeb chord does not depend on the choice of Σ_c because the first Chern class of ξ is 2-torsion, and it does not depend on the choice of γ_c because the Maslov number of Λ vanishes.

Remark In the case where $c(V,\xi)$ (where $c(V,\xi)$ is the positive generator of $\langle 2c_1(\xi), H_1(V)\rangle$) or $m(\Lambda)$ is nonzero, the Conley–Zehnder index is well defined in $\mathbb{Z}/d\mathbb{Z}$, where

$$d = \gcd(c(V,\xi), m(\Lambda)).$$

3.2 Moduli spaces

Recall that (V,ξ) is a contact manifold such that $H_1(V)$ is free and the first Chern class of ξ (equipped with any compatible almost complex structure) is 2-torsion. Let α be a hypertight contact form on (V,ξ) and let Λ be a chord generic Legendrian submanifold of (V,α) with vanishing Maslov number. In the following, we introduce the moduli spaces needed to define the Chekanov–Eliashberg category of Λ.

Definition 3.3 A Riemann $(d+1)$-pointed disk is a triple $(D, \boldsymbol{\zeta}, j)$ such that

(1) D is a smooth oriented manifold-with-boundary diffeomorphic to the closed unit disk in $\mathbb{C}$,

(2) $\boldsymbol{\zeta} = (\zeta_d, \dots, \zeta_1, \zeta_0)$ is a cyclically ordered family of distinct points on ∂D,

(3) j is an integrable almost complex structure on D which induces the given orientation on D.

If $(D, \boldsymbol{\zeta}, j)$ is a Riemann pointed disk, we denote by $\Delta := D \setminus \{\zeta_d, \dots, \zeta_1, \zeta_0\}$ the corresponding punctured disk.

Definition 3.4 A family of Riemann $(d+1)$-pointed discs is a bundle $\mathcal{S} \to \mathcal{R}$ with

(1) a family $\boldsymbol{\zeta} = (\zeta_d, \dots, \zeta_1, \zeta_0)$ of nonintersecting sections $\zeta_k : \mathcal{R} \to \mathcal{S}$ and

(2) a section $j : \mathcal{R} \to \mathrm{End}(T\mathcal{S})$

such that $(\mathcal{S}_r, \boldsymbol{\zeta}(r), j(r))$ is a Riemann $(d+1)$-pointed disk for every $r \in \mathcal{R}$.

Definition 3.5 Let $\mathcal{S} \to \mathcal{R}$ be a family of Riemann $(d+1)$-pointed discs. A choice of strip-like ends for $\mathcal{S} \to \mathcal{R}$ is a family of sections

$$\epsilon_d, \dots, \epsilon_1 \colon \mathcal{R} \times \mathbb{R}_{\leq 0} \times [0,1] \to \Delta_r, \quad \epsilon_0 \colon \mathcal{R} \times \mathbb{R}_{\geq 0} \times [0,1] \to \Delta_r,$$

such that

(1) $\epsilon_d(r), \dots, \epsilon_1(r), \epsilon_0(r)$ are proper embeddings with

$$\epsilon_k(r)(\mathbb{R}_{\leq 0} \times \{0,1\}) \subset \partial\Delta_r \quad \text{and} \quad \epsilon_0(r)(\mathbb{R}_{\geq 0} \times \{0,1\}) \subset \partial\Delta_r,$$

(2) $\epsilon_d(r), \dots, \epsilon_1(r), \epsilon_0(r)$ satisfy the asymptotic conditions

$$\epsilon_k(r)(s,t) \xrightarrow[s \to -\infty]{} \zeta_k(r) \quad \text{and} \quad \epsilon_0(r)(s,t) \xrightarrow[s \to +\infty]{} \zeta_0(r),$$

(3) $\epsilon_d(r), \dots, \epsilon_1(r), \epsilon_0(r)$ are $(i, j(r))$-holomorphic, where i is the standard complex structure on $\mathbb{C}$.

As explained in [36, Section (9c)], there is a universal family $\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$ of Riemann $(d+1)$-pointed discs when $d \geq 2$, which means that any other family $\mathcal{S} \to \mathcal{R}$ is isomorphic to the pullback of $\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$ by a map $\mathcal{R} \to \mathcal{R}^{d+1}$. In the following, we fix a choice of strip-like ends for the universal family $\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$.

Definition 3.6 Let J be an almost complex structure on ξ compatible with $(d\alpha)|_\xi$. We denote by J^α the unique almost complex structure on $\mathbb{R}_\sigma \times V$ which sends ∂_σ to R_α and which restricts to J on ξ. Let $c_d, \dots, c_1, c_0$ be Reeb chords of Λ, where $c_k \colon [0, T_k] \to V$.

(1) If $d = 1$, we denote by $\widetilde{\mathcal{M}}_{c_1,c_0}(\mathbb{R} \times \Lambda, J, \alpha)$ the set of equivalence classes of maps $u \colon \mathbb{R} \times [0,1] \to \mathbb{R} \times V$ such that

- u maps the boundary of $\mathbb{R} \times [0,1]$ to $\mathbb{R} \times \Lambda$,
- u satisfies the asymptotic conditions

$$u(s,t) \xrightarrow[s \to -\infty]{} (-\infty, c_1(T_1 t)) \text{ and } u(s,t) \xrightarrow[s \to +\infty]{} (+\infty, c_0(T_0 t)),$$

- u is (i, J^α)-holomorphic,

where two maps u and u' are identified if there exists $s_0 \in \mathbb{R}$ such that $u'(\cdot,\cdot) = u(\cdot + s_0, \cdot)$.

(2) If $d \geq 2$, we denote by $\widetilde{\mathcal{M}}_{c_d,\dots,c_1,c_0}(\mathbb{R} \times \Lambda, J, \alpha)$ the set of pairs (r, u) such that

- $r \in \mathcal{R}^{d+1}$ and $u \colon \Delta_r \to \mathbb{R} \times V$ maps the boundary of Δ_r to $\mathbb{R} \times \Lambda$,
- u satisfies the asymptotic conditions

$$(u \circ \epsilon_k(r))(s,t) \xrightarrow[s \to -\infty]{} (-\infty, c_k(T_k t)) \quad \text{and} \quad (u \circ \epsilon_0(r))(s,t) \xrightarrow[s \to +\infty]{} (+\infty, c_0(T_0 t)),$$

- u is (i, J^α)-holomorphic.

Observe that $\mathbb{R}$ acts on $\widetilde{\mathcal{M}}_{c_d,\dots,c_1,c_0}(\mathbb{R} \times \Lambda, J, \alpha)$ by translation in the $\mathbb{R}_\sigma$-coordinate. We set

$$\mathcal{M}_{c_d,\dots,c_1,c_0}(\mathbb{R} \times \Lambda, J, \alpha) := \widetilde{\mathcal{M}}_{c_d,\dots,c_1,c_0}(\mathbb{R} \times \Lambda, J, \alpha)/\mathbb{R}.$$

The moduli space $\widetilde{\mathcal{M}}_{c_d,\dots,c_1,c_0}(\mathbb{R}\times\Lambda, J,\alpha)$ can be realized as the zero-set of a section $\bar{\partial}\colon \mathcal{B}\to\mathcal{E}$ of a Banach bundle $\mathcal{E}\to\mathcal{B}$ (see for example [16]). We say that $\widetilde{\mathcal{M}}_{c_d,\dots,c_1,c_0}(\mathbb{R}\times\Lambda, J)$ is transversely cut out if $\bar{\partial}$ is transverse to the 0-section.

Definition 3.7 We say that J is *regular* (with respect to α and Λ) if the moduli spaces

$$\widetilde{\mathcal{M}}_{c_d,\dots,c_1,c_0}(\mathbb{R}\times\Lambda, J,\alpha)$$

are all transversely cut out.

Proposition 3.8 [9, Proposition 3.13] *The set of regular almost complex structures on ξ is Baire. Moreover, the dimension of a transversely cut out moduli space is*

$$\dim\widetilde{\mathcal{M}}_{c_d,\dots,c_1,c_0}(\mathbb{R}\times\Lambda, J,\alpha)=\mathrm{CZ}(a)-\left(\sum_{k=1}^{d}\mathrm{CZ}(b_k)\right)+d-1.$$

3.3 Chekanov–Eliashberg DG-category

Recall that (V,ξ) is a contact manifold such that $H_1(V)$ is free and the first Chern class of ξ (equipped with any compatible almost complex structure) is 2-torsion. Let α be a hypertight contact form on (V,ξ) and let $\mathbf{\Lambda}=(\Lambda(E))_{E\in\mathcal{E}}$ be a family of Legendrian submanifolds of (V,ξ). We set $\Lambda:=\bigcup_{E\in\mathcal{E}}\Lambda(E)$ and we assume that Λ is chord generic with vanishing Maslov number. Moreover, we denote by $\mathcal{C}(\Lambda(E),\Lambda(E'))$ the graded vector space generated by the words of Reeb chords $c_1\cdots c_d$, $d\geq 1$, where c_1 starts on $\Lambda(E)$, c_d ends on $\Lambda(E')$, and the ending component of c_i is the starting component of c_{i+1} for every $1\leq i\leq d-1$, with grading

$$|c_1\cdots c_d|:=\sum_{i=1}^{d}(\mathrm{CZ}(c_i)-1).$$

Finally, let J be a regular almost complex structure on ξ.

Definition 3.9 We denote by $\mathrm{CE}_*(\mathbf{\Lambda})=\mathrm{CE}_*(\mathbf{\Lambda},J,\alpha)$ the graded category defined as follows:

(1) The objects are the Legendrians $\Lambda(E)$, $E\in\mathcal{E}$.

(2) The space of morphisms from $\Lambda(E)$ to $\Lambda(E')$ is

$$\mathcal{C}(\Lambda(E),\Lambda(E'))\quad\text{if } E\neq E',\qquad \mathbb{F}\oplus\mathcal{C}(\Lambda(E),\Lambda(E'))\quad\text{if } E=E'$$

(the summand $\mathbb{F}$ corresponds to the "empty word").

(3) The composition is given by concatenation of words.

If c_0 is a Reeb chord in $\mathrm{CE}_*(\mathbf{\Lambda})$, we set

$$\partial(c_0):=\sum_{c_d,\dots,c_1}\#\mathcal{M}_{c_d,\dots,c_1,c_0}(\mathbb{R}\times\Lambda,J,\alpha)c_d\cdots c_1,$$

where $\#\mathcal{M} \in \mathbb{F}$ denotes the number of elements modulo 2 in $\mathcal{M}$ if $\mathcal{M}$ is finite, and 0 otherwise. Finally, we extend ∂ to $\mathrm{CE}_*(\mathbf{\Lambda})$ so that it is linear and satisfies the Leibniz rule with respect to the concatenation product.

Theorem 3.10 $\partial\colon \mathrm{CE}_*(\mathbf{\Lambda}) \to \mathrm{CE}_*(\mathbf{\Lambda})$ *decreases the grading by* 1 *and satisfies* $\partial \circ \partial = 0$*. As a result*, $(\mathrm{CE}_{-*}(\mathbf{\Lambda}), \partial)$ *is a DG-category.*

Proof This follows from Proposition 3.8, SFT compactness (see [1; 6], in particular [1, Theorem 3.20]) and pseudoholomorphic gluing. See [12; 14; 16] for details. □

Augmentations and Legendrian A_∞-(co)category Let $\mathbb{F}_\mathcal{E}$ be the category with $\mathcal{E}$ as set of objects, and morphism space from E to E' equal to $\mathbb{F}$ if $E = E'$, or 0 if $E \neq E'$. Assume that we have an augmentation of $\mathrm{CE}_{-*}(\mathbf{\Lambda})$, ie a DG-functor $\varepsilon\colon \mathrm{CE}_{-*}(\mathbf{\Lambda}) \to \mathbb{F}_\mathcal{E}$. Denote by ϕ_ε the automorphism of $\mathrm{CE}_{-*}(\mathbf{\Lambda})$ defined by

$$\phi_\varepsilon(c) = c + \varepsilon(c)$$

for every Reeb chord c of Λ. We denote by $\mathrm{CE}^\varepsilon_{-*}(\mathbf{\Lambda})$ the DG-category whose underlying graded category is the same as for $\mathrm{CE}_{-*}(\mathbf{\Lambda})$, but the differential is $\partial_\varepsilon = \phi_\varepsilon \circ \partial \circ \phi_\varepsilon^{-1}$. Now let $\overline{\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda})}$ be the graded precategory (no composition) with

(1) objects the set of Legendrians $\{\Lambda(E) \mid E \in \mathcal{E}\}$,

(2) morphisms from $\Lambda(E)$ to $\Lambda(E')$ the vector space generated by (individual, not words of) Reeb chords c which start on $\Lambda(E)$ and end on $\Lambda(E')$, with grading

$$|c| := -\mathrm{CZ}(c).$$

Observe that, as a graded precategory, we have

$$\mathrm{CE}^\varepsilon_{-*}(\mathbf{\Lambda}) = \mathbb{F}_\mathcal{E} \oplus \Bigl(\bigoplus_{d\geq 1} \overline{\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda})}[-1]^{\otimes d}\Bigr).$$

If we write

$$(\partial_\varepsilon)|_{\overline{\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda})}} = \sum_{d\geq 0} \partial^d_\varepsilon \quad \text{with } \partial^d_\varepsilon\colon \overline{\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda})} \to \overline{\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda})}^{\otimes d},$$

then $\partial^0_\varepsilon = \varepsilon \circ \partial = 0$. Moreover, the operations $(\partial^d_\varepsilon)_{d\geq 1}$ make $\overline{\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda})}$ a (noncounital) A_∞-cocategory (see Definition 1.2). We define the coaugmented A_∞-cocategory of $(\mathbf{\Lambda}, \varepsilon)$ to be

$$\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda}) := \mathbb{F}_\mathcal{E} \oplus \overline{\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda})}$$

(the A_∞-cooperations are naturally extended so that $1 \in \mathbb{F}_\mathcal{E}(E, E)$, $E \in \mathcal{E}$ are counits). Now observe that, as a DG-category,

$$\mathrm{CE}^\varepsilon_{-*}(\mathbf{\Lambda}) = \Omega(\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda}))$$

(see [17, Section 2.2] for the cobar construction). Finally, we define the augmented A_∞-category of $(\mathbf{\Lambda}, \varepsilon)$ to be the graded dual (see [17, Section 2.1.3]) of $\mathrm{LC}^\varepsilon_*(\mathbf{\Lambda})$:

$$\mathrm{LA}^*_\varepsilon(\mathbf{\Lambda}) = \mathrm{LC}^\varepsilon_*(\mathbf{\Lambda})^\#.$$

3.4 Functoriality

Recall that (V,ξ) is a contact manifold such that $H_1(V)$ is free and the first Chern class of ξ (equipped with any compatible almost complex structure) is 2-torsion. Let $\boldsymbol{M} = (M(E))_{E\in\mathcal{E}}$ be a family of n-dimensional manifolds. When we write a map $\boldsymbol{\Lambda}\colon \boldsymbol{M} \to V$, we mean that $\boldsymbol{\Lambda}$ is a family of maps $\Lambda(E)\colon M(E) \to V$ indexed by $\mathcal{E}$, and we set

$$\Lambda = \bigsqcup_{E\in\mathcal{E}} \Lambda(E)\colon \bigsqcup_{E\in\mathcal{E}} M(E) \to V.$$

Definition 3.11 Let α be a hypertight contact form on (V,ξ). We denote by $\mathcal{L}_{\boldsymbol{M}}(\alpha)$ the bicategory where:

(1) Objects are the pairs $(\boldsymbol{\Lambda}, J)$, where $\boldsymbol{\Lambda}\colon \boldsymbol{M} \to V$ is a family of Legendrian embedding such that Λ is chord generic with vanishing Maslov number, and J is a regular almost complex structure on ξ.

(2) Morphisms from $(\boldsymbol{\Lambda}_0, J_0)$ to $(\boldsymbol{\Lambda}_1, J_1)$ are the smooth paths $\Phi = (\boldsymbol{\Lambda}_t, J_t)_{0\le t\le 1}$ going from $(\boldsymbol{\Lambda}_0, J_0)$ to $(\boldsymbol{\Lambda}_1, J_1)$, where $\boldsymbol{\Lambda}_t\colon \boldsymbol{M} \to V$ is a family of Legendrian embeddings and J_t is an almost complex structure on ξ.

(3) Homotopies from a morphism $\Phi = (\boldsymbol{\Lambda}_t, J_t)_{0\le t\le 1}\colon (\boldsymbol{\Lambda}_0, J_0) \to (\boldsymbol{\Lambda}_1, J_1)$ to another morphism $\Phi' = (\boldsymbol{\Lambda}'_t, J'_t)_{0\le t\le 1}\colon (\boldsymbol{\Lambda}_0, J_0) \to (\boldsymbol{\Lambda}_1, J_1)$ are the smooth families $(\boldsymbol{\Lambda}_{s,t}, J_{s,t})_{0\le s\le S,\, 0\le t\le 1}$, where $\boldsymbol{\Lambda}_{s,t}\colon \boldsymbol{M} \to V$ is a family of Legendrian embeddings, $J_{s,t}$ is an almost complex structure on ξ, and

$$(\boldsymbol{\Lambda}_{s,0}, J_{s,0}) = (\boldsymbol{\Lambda}_0, J_0), \quad (\boldsymbol{\Lambda}_{s,1}, J_{s,1}) = (\boldsymbol{\Lambda}_1, J_1), \quad (\boldsymbol{\Lambda}_{0,t}, J_{0,t}) = (\boldsymbol{\Lambda}_t, J_t), \quad (\boldsymbol{\Lambda}_{S,t}, J_{S,t}) = (\boldsymbol{\Lambda}'_t, J'_t).$$

Definition 3.12 Let α, α' be hypertight contact forms on (V,ξ), and let φ be a contactomorphism of (V,ξ) such that $\varphi^*\alpha = \alpha'$. If $\Phi = (\boldsymbol{\Lambda}_t, J_t)_{0\le t\le 1}$ is a morphism in $\mathcal{L}_{\boldsymbol{M}}(\alpha)$, we denote by

$$\varphi^*\Phi = (\varphi^{-1}(\boldsymbol{\Lambda}_t), \varphi^* J_t)_{0\le t\le 1}$$

the corresponding morphism in $\mathcal{L}_{\boldsymbol{M}}(\alpha')$, and by

$$f^{\varphi}_{(\boldsymbol{\Lambda}_t, J_t)}\colon \mathrm{CE}_{-*}(\boldsymbol{\Lambda}_t, J_t, \alpha) \to \mathrm{CE}_{-*}(\varphi^{-1}(\boldsymbol{\Lambda}_t), \varphi^* J_t, \alpha')$$

the DG-functor which sends a Reeb chord c to $\varphi^{-1}(c)$.

Definition 3.13 Let α be a hypertight contact form on (V,ξ), and let $\Phi = (\boldsymbol{\Lambda}_t, J_t)_{0\le t\le 1}$ be a morphism in $\mathcal{L}_{\boldsymbol{M}}(\alpha)$. A *handle slide instant* in Φ is a time t_0 where $\boldsymbol{\Lambda}_{t_0}$ is chord generic and has Reeb chords $c_d, \dots, c_1, c_0$ such that the moduli space $\widetilde{\mathcal{M}}_{c_d,\dots,c_1,c_0}(\mathbb{R}\times\Lambda_{t_0}, J_{t_0}, \alpha)$ is not transversely cut out.

Theorem 3.14 *There exist functors $\mathcal{F}_\alpha$ from $\mathcal{L}_{\boldsymbol{M}}(\alpha)$ to the bicategory*[3] *of DG-categories such that*:

(1) *$\mathcal{F}_\alpha$ sends an object $(\boldsymbol{\Lambda}, J)$ to $\mathrm{CE}_{-*}(\boldsymbol{\Lambda}, J, \alpha)$.*

[3] Homotopies between DG-maps are DG-homotopies, see for example [31, Section 2.1].

(2) *$\mathcal{F}_\alpha$ sends a morphism to a homotopy equivalence.*

(3) *If φ is a contactomorphism of (V,ξ) such that $\varphi^*\alpha = \alpha'$ and if $\Phi = (\boldsymbol{\Lambda}_t, J_t)_{0\le t\le 1}$ is a morphism in $\mathcal{L}_{\boldsymbol{M}}(\alpha)$, then*
$$\mathcal{F}_{\alpha'}(\varphi^*\Phi) = f^{\varphi}_{(\boldsymbol{\Lambda}_1,J_1)} \circ \mathcal{F}_\alpha(\Phi) \circ (f^{\varphi}_{(\boldsymbol{\Lambda}_0,J_0)})^{-1}.$$

(4) *If $(\varphi_t)_{0\le t\le 1}$ is a contact isotopy of (V,ξ) satisfying $\varphi_t^*\alpha = \alpha'$ for every t, and if $(\boldsymbol{\Lambda}, J)$ is an object of $\mathcal{L}_{\boldsymbol{M}}(\alpha)$ such that there is neither birth/death of Reeb chords nor handle slide instants in the path $\Phi' = (\varphi_t^{-1}(\boldsymbol{\Lambda}), \varphi_t^* J)_t$, then*
$$\mathcal{F}_{\alpha'}(\Phi') = f^{\varphi_1}_{(\boldsymbol{\Lambda},J)} \circ (f^{\varphi_0}_{(\boldsymbol{\Lambda},J)})^{-1}.$$

Proof The existence of such functors at the category level (without homotopies) has been established in [14; 16] for the case $(V,\alpha) = (\mathbb{R}\times P, dz-\lambda)$. Statements in the general case can be found in [12, Section 4; 19, Section 5]. □

Note that I proved a weaker version of this result in my thesis by generalizing methods of [14; 16; 31]. The following is the only particular case of Theorem 3.14 that we will use in this paper.

Theorem 3.15 [32, Theorem 3.8] *Theorem 3.14 holds if we replace the categories $\mathcal{L}_{\boldsymbol{M}}(\alpha)$ by the subcategories $\mathcal{L}^0_{\boldsymbol{M}}(\alpha)$ where*

(1) *objects are the pairs $(\boldsymbol{\Lambda}, J)$ such that Λ has finitely many Reeb chords,*

(2) *morphisms from $(\boldsymbol{\Lambda}_0, J_0)$ to $(\boldsymbol{\Lambda}_1, J_1)$ are the families $\Phi = (\boldsymbol{\Lambda}_t, J_t)_{0\le t\le 1}$ such that Λ_t is chord generic and has finitely many Reeb chords for every t,*

(3) *homotopies from a morphism $\Phi = (\boldsymbol{\Lambda}_t, J_t)_{0\le t\le 1}\colon (\boldsymbol{\Lambda}_0, J_0) \to (\boldsymbol{\Lambda}_1, J_1)$ to another morphism $\Phi' = (\boldsymbol{\Lambda}'_t, J'_t)_{0\le t\le 1}\colon (\boldsymbol{\Lambda}_0, J_0) \to (\boldsymbol{\Lambda}_1, J_1)$ are the families $(\boldsymbol{\Lambda}_{s,t}, J_{s,t})_{0\le s\le S,\, 0\le t\le 1}$ such that $\Lambda_{s,t}$ is chord generic and has finitely many Reeb chords for every s,t.*

Remark We expect that the finiteness of Reeb chords condition in Theorem 3.15 (which is very restrictive) can be easily dropped using (homotopy) colimits of DG-categories diagrams. On the other hand, studying birth/death of Reeb chords phenomena is a more serious issue that we will address in future work.

4 Legendrian lifts of exact Lagrangians in the circular contactization

In this section, we start with a family $\boldsymbol{L}$ of mutually transverse compact connected exact Lagrangian submanifolds in a Liouville manifold, and we study a Legendrian lift $\boldsymbol{\Lambda}^\circ$ of $\boldsymbol{L}$ in the circular contactization. For the standard contact form, each point on a Legendrian gives rise to a (countable) infinite set of Reeb chords, and thus $\boldsymbol{\Lambda}^\circ$ is not chord generic. In Section 4.1, we explain how we perturb the contact form and we state our main result, which relates the Chekanov–Eliashberg DG-category of $\boldsymbol{\Lambda}^\circ$ and the Fukaya A_∞-category of $\boldsymbol{L}$.

4.1 Setting

Let (P,λ) be a Liouville manifold, and let

$$\boldsymbol{L} = (L(E))_{E\in\mathcal{E}}, \quad \mathcal{E} = \{1,\dots,N\},$$

be a family of mutually transverse compact connected exact Lagrangian submanifolds in (P,λ) such that there are primitives $f_E\colon L(E)\to\mathbb{R}$ of $\lambda|_{L(E)}$ satisfying $0\le f_1<\dots<f_N\le\frac{1}{2}$. We consider the contact manifold

$$(V^\circ,\xi^\circ) = (S^1\times P, \ker\alpha^\circ), \quad \text{where } S^1=\mathbb{R}_\theta/\mathbb{Z},\ \alpha^\circ = d\theta-\lambda,$$

and the family of Legendrian submanifolds

$$\boldsymbol{\Lambda^\circ} := (\Lambda^\circ(E))_{E\in\mathcal{E}}, \quad \text{where } \Lambda^\circ(E) = \{(f_E(x),x)\in(\mathbb{R}/\mathbb{Z})\times P \mid x\in L(E)\}.$$

In order for the Chekanov–Eliashberg category of $\boldsymbol{\Lambda^\circ}$ and the Fukaya category of $\boldsymbol{L}$ to be $\mathbb{Z}$-graded, we assume that $H_1(P)$ is free, that the first Chern class of P (equipped with any almost complex structure compatible with $(-d\lambda)$) is 2-torsion, and that the Maslov classes of the Lagrangians $L(E)$ vanish.

4.1.1 Reeb chords Observe that $\Lambda^\circ=\bigcup_{E\in\mathcal{E}}\Lambda(E)$ is not chord generic for α° (see Definition 3.2). We will choose a compactly supported function $H\colon P\to\mathbb{R}$, and consider the perturbed contact form

$$\alpha_H^\circ = e^H\alpha^\circ.$$

The Reeb vector field of α_H° is then

$$R_{\alpha_H^\circ} = e^{-H}\begin{pmatrix}1+\lambda(X_H)\\ X_H\end{pmatrix},$$

where X_H is the unique vector field on P satisfying $\iota_{X_H}\,d\lambda=-dH$.

We fix a compact neighborhood K of L which is contained in a Weinstein neighborhood of L in P. It is not hard to see that for every positive integer N, the space of smooth functions H on P supported in K, such that the $R_{\alpha_H^\circ}$-chords of Λ° with action less than N are generic, is open and dense in $C_K^\infty(P)$. Therefore, the space of functions $H\in C_K^\infty(P)$ such that Λ° is chord generic with respect to α_H° is a Baire subset of $C_K^\infty(P)$. In particular, the latter is dense in $C_K^\infty(P)$. In the following, we choose $H\in C_K^\infty(P)$ such that

(1) Λ° is chord generic with respect to α_H°,

(2) H is sufficiently close to 0 so that

$$d\theta(R_{\alpha_H^\circ}) = e^{-H}(1+\lambda(X_H))\ge\tfrac{1}{2}.$$

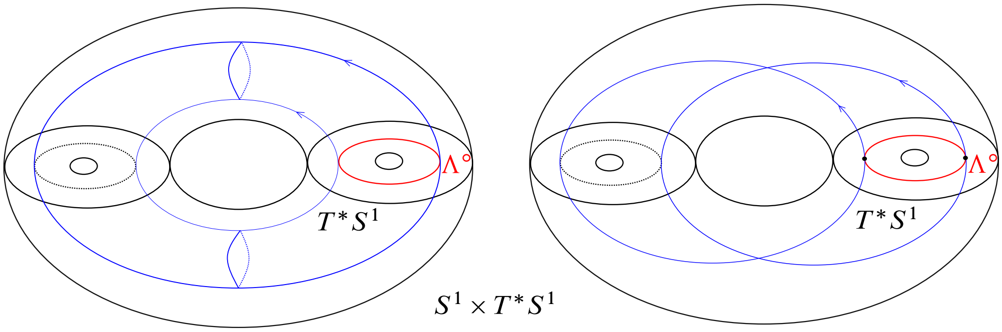


Figure 1: Reeb chords (in blue) of $\Lambda^\circ = \{0\} \times 0_{S^1}$. Left: for α°. Right: for α°_H.

Example 4.1 Assume that we are in the case

$$(P, \lambda) = (T^*M, p\,dq), \quad L = 0_M, \quad \text{and} \quad H(q,p) = h(q),$$

where $h\colon M \to \mathbb{R}$ is a Morse function (we present this example in order to see what happens, even if H is not compactly supported in T^*M). The Reeb vector field of α°_H is

$$R_{\alpha^\circ_H} = e^{-h}\begin{pmatrix} 1 \\ 0 \\ -dh \end{pmatrix},$$

and therefore the Reeb flow satisfies

$$\varphi^t_{R_{\alpha^\circ_H}}(\theta, (q,p)) = \big(\theta + te^{-h(q)}, (q, p - te^{-h(q)}\,dh(q))\big).$$

Thus, the $R_{\alpha^\circ_H}$-chords of Λ° are the paths $c\colon [0,T] \to S^1 \times T^*M$ of the form

$$c(t) = (te^{-h(q_0)}, (q_0, 0)), \quad \text{with } Te^{-h(q_0)} \in \mathbb{Z}_{\geq 1} \text{ and } q_0 \in \operatorname{Crit} h.$$

Observe that these Reeb chords are transverse but lie on top of each other. See Figure 1, where we illustrate this perturbation when $M = S^1$.

Conley–Zehnder index In order to define the Conley–Zehnder index (see Section 3.1), we need to choose a family $(h_0, h_1, \dots, h_s)$ of embedded circles in $V^\circ = S^1 \times P$ which represent a basis of $H_1(V^\circ)$, and a symplectic trivialization of ξ° over each h_i. We let $h_0 = S^1 \times \{a_0\}$ be some fiber of $S^1 \times P \to P$, and we fix $(h_1, \dots, h_s)$ to be any family of embedded circles in P which represent a basis of $H_1(P)$. We choose a symplectic isomorphism $\psi\colon (T_{a_0}P, -d\lambda_{a_0}) \xrightarrow{\sim} (\mathbb{C}^n, dx \wedge dy)$, and then we choose the symplectic trivialization

$$(\xi^\circ|_{h_0}, d\alpha^\circ) \xrightarrow{\sim} (h_0 \times \mathbb{C}^n, dx \wedge dy), \quad ((\theta, a_0), (\lambda_{a_0}(v), v)) \mapsto ((\theta, a_0), e^{2i\pi r\theta}\psi(v)),$$

where r is some integer, that we call *r-trivialization of ξ° over the fiber*. Finally, we choose some trivialization of ξ° over each h_i, $1 \leq i \leq s$.

Example 4.2 We compute the Conley–Zehnder index of a Reeb chord in the case of Example 4.1, ie when

$$(P,\lambda)=(T^*M,p\,dq),\quad L=0_M,\quad\text{and}\quad H(q,p)=h(q),$$

where $h\colon M\to\mathbb{R}$ is a Morse function. In this case, the Reeb flow is given by

$$\varphi^t_{R_{\alpha^\circ_H}}(\theta,(q,p))=\big(\theta+te^{-h(q)},(q,p-te^{-h(q)}\,dh(q))\big).$$

Let $c\colon[0,T]\to V^\circ$ be a Reeb chord of Λ°. Then there exists a positive integer k and a critical point q_0 of h such that

$$c(t)=(te^{-h(q_0)},(q_0,0))\quad\text{and}\quad Te^{-h(q_0)}=k.$$

Observe that $c(0)=c(T)$, and thus there is no need to choose a capping path for c. Besides, for every u in $T_{q_0}M$, we have

$$D\varphi^t_{R_{\alpha^\circ_H}}(c(0))(0,u,0)=(0,u,-te^{-h(q_0)}D^2h(q_0)u).$$

In order to compute the index of c, we first choose coordinates $(x_1,\dots,x_n)$ around $q_0\in M$ in which

$$h=h(q_0)+\frac{1}{2}\sum_{j=1}^{\dim(M)}\sigma_jx_j^2,\quad\text{where } \sigma_j=\pm1,$$

and we extend it to symplectic coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$ around $(q_0,0)\in T^*M$ by setting

$$y_j(q,p)=\Big\langle p,\frac{\partial}{\partial x_j}(q)\Big\rangle.$$

Our choice of trivialization for a fiber of $S^1\times P\to P$ induces the trivialization

$$e^{2i\pi rkt/T}(dx+i\,dy)\colon c^{-1}\xi^\circ\xrightarrow{\sim}(\mathbb{R}/T\mathbb{Z})\times\mathbb{C}^n$$

(observe that $\xi^\circ_{c(t)}=\{0\}\times T_{(q_0,0)}(T^*M)$). Accordingly, the path $t\mapsto D\varphi^t_{R_{\alpha^\circ_H}}(T_{c(0)}\Lambda^\circ)$ induces a path of Lagrangians

$$\Gamma_c\colon t\in[0,T]\mapsto\big\{\big(e^{2i\pi mkt/T}(u_j-ite^{-h(q_0)}\sigma_ju_j)\big)_{1\le j\le n}\mid u\in\mathbb{R}^n\big\}\subset\mathbb{C}^n.$$

We close this path using a counterclockwise rotation Γ, and call the resulting loop $\overline{\Gamma}_c$. In order to compute the Conley–Zehnder index of c, we have to look at how $\overline{\Gamma}_c$ intersects the Lagrangian $i\mathbb{R}^n$ (as explained in [14, Section 2.2]). Observe that Γ_c intersects $i\mathbb{R}^n$ positively $2rk$ times, so that Γ_c contributes $2rk$ to the Conley–Zehnder index of c. Moreover, since Γ is a counterclockwise rotation bringing

$$\big\{(u_j-iTe^{-h(q_0)}\sigma_ju_j)_{1\le j\le n}\mid u\in\mathbb{R}^n\big\}\quad\text{to } \mathbb{R}^n,$$

the contributions to the intersection between Γ and $i\mathbb{R}^n$ come from the negative eigenvalues σ_j. The computation done in [15, Lemma 3.4] implies that Γ contributes $\mathrm{ind}(q_0)$ to the Conley–Zehnder index of c. We conclude that the Conley–Zehnder index of c is

$$\mathrm{CZ}(c)=\mu(\overline{\Gamma}_c)=2rk+\mathrm{ind}(q_0).$$

4.1.2 Main result Let j be an almost complex structure on P compatible with $(-d\lambda)$, and let J° be its lift to a complex structure on ξ°. Recall from Section 3.3 the definition of the Chekanov–Eliashberg DG-category of a family of Legendrians. In our situation, $\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ) = \mathrm{CE}_{-*}(\boldsymbol{\Lambda}^\circ, J^\circ, \alpha^\circ_H)$ (with grading induced by the r-trivialization of ξ° over the fiber) is an Adams-graded DG-algebra, where the Adams degree of a Reeb chord c is the number of times c winds around the fiber. Besides, the map $\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ) \to \mathbb{F}$ which sends every Reeb chord to zero (and preserves units) defines an augmentation of $\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)$.

Remark In the case of Example 4.1, the cohomological degree of a Reeb chord c in $\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)$ corresponding to a positive integer k and a critical point q_0 is

$$1 - \mathrm{CZ}(c) = 1 - 2rk - \mathrm{ind}(q_0)$$

(see Example 4.2).

Besides, we denote by $\mathcal{F}uk(\boldsymbol{L})$ the Fukaya category with objects being the set of Lagrangians $\{L(E) \mid E \in \mathcal{E}\}$ (see for example [36, Chapter 2]), and by $\overrightarrow{\mathcal{F}uk}(\boldsymbol{L})$ its directed subcategory:

$$\hom_{\overrightarrow{\mathcal{F}uk}(\boldsymbol{L})}(L(E), L(E')) = \begin{cases} \langle L(E) \cap L(E') \rangle & \text{if } E < E', \\ \mathbb{F} & \text{if } E = E', \\ 0 & \text{if } E > E'; \end{cases}$$

see [36, Paragraph (5n)].

Let $\mathbb{F}[t_m]$ be the augmented Adams-graded associative algebra generated by a variable t_m of bidegree $(m, 1)$. Observe that if $\mathcal{C}$ is a subcategory of an A_∞-category $\mathcal{D}$ with $\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{D})$, then $\mathcal{C} \oplus (t_m \mathbb{F}[t_m] \otimes \mathcal{D})$ is naturally an Adams-graded A_∞-category, where the Adams degree of $t_m^k \otimes x$ equals k. Moreover, we denote by $E(-) = B(-)^\#$ (graded dual of bar construction) the Koszul dual functor (see [17, Section 2.3] or [29, Section 2]). We say that Koszul duality holds for an augmented Adams-graded A_∞-category A if the natural map $A \to E(E(A))$ is a quasi-isomorphism (see [29, Theorem 2.4] or [17, Definition 17]).

Theorem 4.3 (Theorem B in the introduction) *Koszul duality holds for* $\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)$, *and there is a quasi-equivalence of augmented Adams-graded A_∞-categories*

$$E(\mathrm{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)) \simeq \overrightarrow{\mathcal{F}uk}(\boldsymbol{L}) \oplus (t_{2r}\mathbb{F}[t_{2r}] \otimes \mathcal{F}uk(\boldsymbol{L})).$$

Corollary 4.4 *If L is a connected compact exact Lagrangian and Λ° is a Legendrian lift of L in the circular contactization, then there is a quasi-equivalence of augmented DG-algebras,*

$$\mathrm{CE}^1_{-*}(\Lambda^\circ) \simeq C_{-*}(\Omega(\mathbb{CP}^\infty \rtimes L)).$$

Proof Let x_0 be the basepoint of $\mathbb{CP}^\infty$, and set $P := \mathbb{CP}^\infty \setminus \{x_0\}$. Observe that

$$(P \times L)^* = P^* \wedge L^* = \mathbb{CP}^\infty \wedge L^* = \mathbb{CP}^\infty \rtimes L.$$

We have

$$\mathbb{F} \oplus (t_2\mathbb{F}[t_2] \otimes CF^*(L)) \simeq \mathbb{F} \oplus (t_2\mathbb{F}[t_2] \otimes C^*(L)) \simeq C^*((P \times L)^*) \simeq C^*(\mathbb{CP}^\infty \rtimes L).$$

Thus, it follows from Theorem 4.3 that

$$E(\mathrm{CE}^1_{-*}(\Lambda^\circ)) \simeq C^*(\mathbb{CP}^\infty \rtimes L).$$

Since Koszul duality holds for $\mathrm{CE}^1_{-*}(\Lambda^\circ)$,

$$\mathrm{CE}^1_{-*}(\Lambda^\circ) \simeq E(C^*(\mathbb{CP}^\infty \rtimes L)).$$

Observe that the graded algebra $H^*(\mathbb{CP}^\infty \rtimes L)$ is locally finite (ie each degree component is finitely generated) and simply connected (ie its augmentation ideal is concentrated in components of degree strictly greater than 1). Thus, according to the homological perturbation lemma (see [36, Proposition 1.12]), we can assume that $C^*(\mathbb{CP}^\infty \rtimes L)$ is a locally finite and simply connected A_∞ model for the DG-algebra of cochains on $\mathbb{CP}^\infty \rtimes L$. Therefore, [17, Lemma 10] implies that

$$\mathrm{CE}^1_{-*}(\Lambda^\circ) \simeq \Omega(C_{-*}(\mathbb{CP}^\infty \rtimes L)).$$

Now, since $\mathbb{CP}^\infty \rtimes L$ is simply connected, Adams' result (see [2; 3; 17]) yields

$$\Omega(C_{-*}(\mathbb{CP}^\infty \rtimes L)) \simeq C_{-*}(\Omega(\mathbb{CP}^\infty \rtimes L)). \qquad \square$$

4.1.3 Strategy of proof We explain the strategy to compute $E(\mathrm{CE}^r_{-*}(\mathbf{\Lambda}^\circ))$. Recall from the last paragraph of Section 3.3 that there is a coaugmented A_∞-cocategory $\mathrm{LC}_*(\mathbf{\Lambda}^\circ)$ such that

$$\mathrm{CE}^r_{-*}(\mathbf{\Lambda}^\circ) = \Omega(\mathrm{LC}_*(\mathbf{\Lambda}^\circ)).$$

$\mathrm{LC}_*(\mathbf{\Lambda}^\circ)$ inherits an Adams-grading from $\mathrm{CE}^r_{-*}(\mathbf{\Lambda}^\circ)$ (the same), and we denote by $\mathrm{LA}^*(\mathbf{\Lambda}^\circ)$ its graded dual (see [17, Section 2.1.3]). In our situation, $\mathrm{LA}^*(\mathbf{\Lambda}^\circ)$ is an augmented Adams-graded A_∞-category whose augmentation ideal is generated by the Reeb chords of $\mathbf{\Lambda}^\circ$ (and the Adams degree of a Reeb chord c is the number of times c winds around the fiber). Since there is a quasi-isomorphism $B(\Omega C) \simeq C$ for every A_∞-cocategory C (see [17, Section 2.2.2]), it follows that

$$E(\mathrm{CE}^r_{-*}(\mathbf{\Lambda}^\circ)) = B(\mathrm{CE}^r_{-*}(\mathbf{\Lambda}^\circ))^\# \simeq \mathrm{LC}_*(\mathbf{\Lambda}^\circ)^\# = \mathrm{LA}^*(\mathbf{\Lambda}^\circ)$$

(graded dual preserves quasi-isomorphisms).

Remark In the case of Example 4.1, the cohomological degree of a Reeb chord c in $\mathrm{LA}^*(\mathbf{\Lambda}^\circ)$ corresponding to a positive integer k and a critical point q_0 is

$$\mathrm{CZ}(c) = 2rk + \mathrm{ind}(q_0)$$

(see Example 4.2).

In order to compute $\mathrm{LA}^*(\mathbf{\Lambda}^\circ)$, we lift the problem to the contact manifold

$$(V, \xi) = (\mathbb{R}_\theta \times P, \ker(d\theta - \lambda)),$$

and introduce the following objects.

Definition 4.5 Let $\boldsymbol{M} = (M^n(E))_{(n,E)\in\mathbb{Z}\times\mathcal{E}}$ be a family of Legendrian submanifolds in (V,ξ), K an almost complex structure on ξ, and β a hypertight contact form on (V,ξ) for which $\boldsymbol{M}$ is chord-generic. We denote by $\mathcal{A}(\boldsymbol{M}, K, \beta)$ the A_∞-category defined as follows:

(1) The objects of $\mathcal{A}(\boldsymbol{M}, K, \beta)$ are the Legendrians $M^n(E)$, $(n,E)\in\mathbb{Z}\times\mathcal{E}$.

(2) The space of morphisms from $M^i(E)$ to $M^j(E')$ is either generated by the R_β-chords from $M^i(E)$ to $M^j(E')$ if $(i,E)<(j,E')$, or $\mathbb{F}$ if $(i,E)=(j,E')$, or 0 otherwise.

(3) The operations are such that $1\in\mathcal{A}(\boldsymbol{M}, K, \beta)(M^n(E), M^n(E))$ is a strict unit, and for every sequence $(i_0,E_0)<\cdots<(i_d,E_d)$, for every sequence of Reeb chords

$$(c_1,\dots,c_d)\in\mathcal{R}(M^{i_0}(E_0), M^{i_1}(E_1))\times\cdots\times\mathcal{R}(M^{i_{d-1}}(E_{d-1}), M^{i_d}(E_d)),$$

we have

$$\mu_{\mathcal{A}(\boldsymbol{M},K,\beta)}(c_1,\dots,c_d) = \sum_{c_0\in\mathcal{R}(M^{i_0}(E_0),M^{i_d}(E_d))} \#\mathcal{M}_{c_d,\dots,c_1,c_0}(\mathbb{R}\times M, K, \beta)c_0$$

(see Definition 3.6 for the moduli spaces).

Definition 4.6 Consider a path $(\boldsymbol{M}_t)_{0\le t\le 1}$, where $\boldsymbol{M}_t = (M_t^n(E))_{(n,E)\in\mathbb{Z}\times\mathcal{E}}$ is a family of Legendrian submanifolds in (V,ξ), such that $M_1^{n-1}(E) = M_0^n(E) =: M^n(E)$. Let K be an almost complex structure on ξ, and β a hypertight contact form on (V,ξ) for which $\boldsymbol{M} = (M^n(E))_{(n,E)\in\mathbb{Z}\times\mathcal{E}}$ is chord-generic. We denote by $\tau_{(\boldsymbol{M}_t)_t,K,\beta}\colon \mathcal{A}(\boldsymbol{M}, K, \beta)\to\mathcal{A}(\boldsymbol{M}, K, \beta)$ the A_∞-functor defined as follows:

(1) On objects, $\tau_{(\boldsymbol{M}_t)_t,K,\beta}$ sends $M^n(E) = M_0^n(E)$ to $M^{n+1}(E) = M_1^n(E)$.

(2) On morphisms, the map

$$\tau_{(\boldsymbol{M}_t)_t,(K_t)_t,\beta}\colon \mathcal{A}(\boldsymbol{M}, K, \beta)(M^{i_0}(E_0), M^{i_1}(E_1))\otimes\cdots\otimes\mathcal{A}(\boldsymbol{M}, K, \beta)(M^{i_{d-1}}(E_{d-1}), M^{i_d}(E_d)) \\ \to \mathcal{A}(\boldsymbol{M}, K, \beta)(M^{i_0+1}(E_0), M^{i_d+1}(E_d))$$

is obtained by dualizing the components of the DG-isomorphism

$$\mathrm{CE}_{-*}((M_1^n)_{i_0\le n\le i_d}, K_1, \beta)\to\mathrm{CE}_{-*}((M_0^n)_{i_0\le n\le i_d}, K_0, \beta)$$

induced by the path $((M_{1-t}^n)_{i_0\le n\le i_d}, K_{1-t})_{0\le t\le 1}$ (see Theorem 3.14).

Remark (1) The A_∞-functor $\tau_{(\boldsymbol{M}_t)_t,K,\beta}$ is a quasi-equivalence because it is defined by dualizing the components of a DG-isomorphism.

(2) The $\mathbb{Z}$-splitting

$$\mathbb{Z}\times\mathcal{E}\xrightarrow{\sim}\mathrm{ob}(\mathcal{A}(\boldsymbol{M}, K, \beta)),\quad (n,E)\mapsto M^n(E),$$

is compatible with the quasi-autoequivalence $\tau_{(\boldsymbol{M}_t)_t,K,\beta}$ in the sense of Definition 2.2. As explained there, this turns $\mathcal{A}(\boldsymbol{M}, K, \beta)$ into an Adams-graded A_∞-category: the Adams degree of a morphism c from $M^i(E)$ to $M^j(E')$ is $j-i$.

In Section 4.2, we lift the data used to define $\mathrm{LA}^*(\mathbf{\Lambda}^\circ)$ (Legendrian $\mathbf{\Lambda}^\circ$, almost complex structure J°, contact form α_H°) to $\mathbb{R} \times P$. This gives us a path $(\mathbf{\Lambda}_t)_t$, an almost complex structure J and a contact form α_H for which we can prove, using Theorem 2.4, that

$$\mathrm{LA}^*(\mathbf{\Lambda}^\circ) \simeq \mathrm{MT}(\tau_{(\mathbf{\Lambda}_t)_t, J, \alpha_H}).$$

In Section 4.3, we use a contactomorphism ϕ_H satisfying $\phi_H^* \alpha_H = (d\theta - \lambda) =: \alpha$ to change our data into $(\mathbf{\Lambda}_{H,t})_t$, J_H and α. We then prove that

$$\mathrm{MT}(\tau_{(\mathbf{\Lambda}_t)_t, J, \alpha_H}) \simeq \mathrm{MT}(\tau_{(\mathbf{\Lambda}_{H,t})_t, J_H, \alpha}).$$

In Section 4.4, we change the almost complex structure J_H to the original one J, and use Theorem 3.15 to prove that

$$\mathrm{MT}(\tau_{(\mathbf{\Lambda}_{H,t})_t, J_H, \alpha}) \simeq \mathrm{MT}(\tau_{(\mathbf{\Lambda}_{H,t})_t, J, \alpha}).$$

In Section 4.5 we project our data to P, so that we get a path $(\boldsymbol{L}_{H,t})_t$ of Lagrangians in P and the almost complex structure j. We use these new data to define an A_∞-category $\mathcal{O}$ and a quasi-autoequivalence $\gamma\colon \mathcal{O} \to \mathcal{O}$. Then we use [10, Theorem 2.1] to prove that

$$\mathrm{MT}(\tau_{(\mathbf{\Lambda}_{H,t})_t, J, \alpha}) \simeq \mathrm{MT}(\gamma).$$

Finally in Section 4.6, we use Theorem 2.5 (Theorem A in the introduction) to conclude.

4.2 Lift to $\mathbb{R} \times P$

In the following we consider the contact manifold

$$(V, \xi) = (\mathbb{R}_\theta \times P, \ker(\alpha)), \quad \text{where } \alpha = d\theta - \lambda,$$

and the family of Legendrian submanifolds

$$\mathbf{\Lambda} := (\Lambda^n(E))_{(n,E) \in \mathbb{Z} \times \mathcal{E}}, \quad \text{where } \Lambda^\theta(E) = \{(f_E(x) + \theta, x) \in \mathbb{R} \times P \mid x \in L(E)\}.$$

Recall from Section 4.1.1 that we chose a compactly supported function $H\colon P \to \mathbb{R}$ such that

(1) $\mathbf{\Lambda}^\circ$ is chord generic with respect to α_H°,

(2) H is sufficiently close to 0 so that

$$d\theta(R_{\alpha_H^\circ}) = e^{-H}(1 + \lambda(X_H)) \geq \tfrac{1}{2}.$$

We consider the contact form

$$\alpha_H := e^H \alpha,$$

with Reeb vector field

$$R_{\alpha_H} = e^{-H} \begin{pmatrix} 1 + \lambda(X_H) \\ X_H \end{pmatrix}.$$

Moreover, we denote by J the lift of J° to an almost complex structure on ξ.

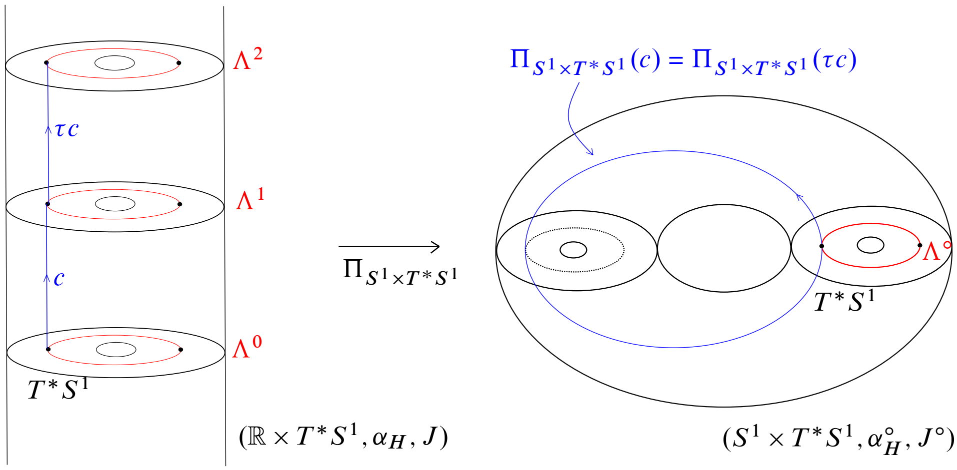


Figure 2: Action of the projection $\Pi_{S^1\times T^*S^1}$.

Definition 4.7 Consider the path of Legendrians $(\mathbf{\Lambda}_t)_{0\le t\le 1}$, where $\Lambda^n_t(E) = \Lambda^{n+t}(E)$. We set

$$\mathcal{A} := \mathcal{A}(\mathbf{\Lambda}, J, \alpha_H) \quad \text{and} \quad \tau := \tau_{(\mathbf{\Lambda}_t)_t, J, \alpha_H}$$

(see Definitions 4.5 and 4.6).

Relation between $\mathrm{LA}^*(\mathbf{\Lambda}^\circ)$ and $(\mathcal{A},\tau)$ We now explain how $\mathrm{LA}^*(\mathbf{\Lambda}^\circ)$ and $(\mathcal{A},\tau)$ are related. See Figure 2, where we illustrate the action of the projection $\Pi_{S^1\times P}$ in the case

$$(P,\lambda) = (T^*S^1, p\,dq), \quad L = 0_{S^1}, \quad \text{and} \quad H(q,p) = h(q),$$

where $h\colon S^1\to\mathbb{R}$ is a Morse function.

Lemma 4.8 *The A_∞-functor τ is strict, and it sends a Reeb chord $t\mapsto(\theta(t),x(t))$ in $\mathcal{A}(\Lambda^i(E),\Lambda^j(E'))$ to the Reeb chord $t\mapsto(\theta(t)+1,x(t))$ in $\mathcal{A}(\Lambda^{i+1}(E),\Lambda^{j+1}(E'))$. In particular, τ acts bijectively on hom-sets.*

Proof Recall that $\alpha_H = e^H\alpha$, with H a function defined on the base manifold P. In particular, the flow $\varphi^t_{\partial_\theta}$ of ∂_θ is a strict contactomorphism of (V,α_H). Moreover, since J is the lift of an almost complex structure j on P, we have

$$((\Lambda^{n+1-t})_{i_0\le n\le i_d}, J) = \big(((\varphi^t_{\partial_\theta})^{-1}\Lambda^{n+1})_{i_0\le n\le i_d}, (\varphi^t_{\partial_\theta})^*J\big).$$

The result follows from Theorem 3.15. □

We denote by $\mathcal{A}_\tau$ the Adams-graded A_∞-category associated to τ as in Definition 2.3.

Lemma 4.9 *There is a quasi-isomorphism of Adams-graded A_∞-categories*

$$\mathrm{LA}^*(\mathbf{\Lambda}^\circ) \simeq \mathcal{A}_\tau.$$

Proof Consider the map which sends a Reeb chord $c \in \mathcal{R}(\Lambda^i(E), \Lambda^j(E'))$ to the corresponding chord $\Pi_{S^1\times P}(c) \in \mathcal{R}(\Lambda^\circ(E), \Lambda^\circ(E'))$ (where $\Pi_{S^1\times P} \colon \mathbb{R} \times P \to S^1 \times P$ is the projection). According to Lemma 4.8, $\Pi_{S^1\times P}(\tau c) = \Pi_{S^1\times P}(c)$, and thus the map $c \mapsto \Pi_{S^1\times P}(c)$ induces a map $\psi \colon \mathcal{A}_\tau \to \mathrm{LA}^*(\Lambda^\circ)$. Moreover, observe that ψ is a bijection on hom-spaces. It remains to prove that ψ is an A_∞-map. This follows from the fact that the map

$$u = (\sigma, v) \mapsto (\sigma, \Pi_{S^1\times P} \circ v)$$

induces a bijection

$$\mathcal{M}_{c_d,\dots,c_1,c_0}(\mathbb{R} \times \Lambda, J, \alpha_H) \xrightarrow{\sim} \mathcal{M}_{\psi(c_d),\dots,\psi(c_1),\psi(c_0)}(\mathbb{R} \times \Lambda^\circ, J^\circ, \alpha_H^\circ). \qquad \square$$

Lemma 4.10 *The Adams-graded A_∞-category* $\mathrm{LA}^*(\boldsymbol{\Lambda}^\circ)$ *is quasi-equivalent to the mapping torus of* $\tau \colon \mathcal{A} \to \mathcal{A}$ *(see Definition 2.1).*

Proof This follows directly from Theorem 2.4 using Lemmas 4.8 and 4.9. □

4.3 Rectification of the contact form

Now that we are in the usual contactization, we have the following result.

Lemma 4.11 *There exists a contactomorphism ϕ_H of (V, ξ) such that*

$$\phi_H^* \alpha_H = \alpha.$$

Proof Recall that $\alpha_H = e^H \alpha$, with H a compactly supported function on the base manifold P such that $e^{-H}(1 + \lambda(X_H)) \geq \frac{1}{2}$.

Assume that there is a contact isotopy $(\phi_t)_{0\leq t\leq 1}$ such that $\phi_0 = \mathrm{id}$ and

$$\phi_t^* \alpha_{tH} = \alpha \tag{1}$$

for every t. Let $(F_t)_t$ be the family of functions on V such that

$$\frac{d}{dt}\phi_t = Y_{F_t} \circ \phi_t,$$

where, for each fixed t, Y_{F_t} is the vector field on V satisfying

$$\alpha(Y_{F_t}) = F_t, \quad \iota_{Y_{F_t}} d\alpha = dF_t(R_\alpha)\alpha - dF_t.$$

Let us prove that Y_{F_t} satisfies

$$\alpha_{tH}(Y_{F_t}) = e^{tH} F_t, \quad \iota_{Y_{F_t}} d\alpha_{tH} = d(e^{tH} F_t)(R_{\alpha_{tH}})\alpha_{tH} - d(e^{tH} F_t).$$

Observe that the first equality is clear, and that it is enough to show that the second equality holds on $\xi = \ker(\alpha)$. Now for every $Z \in \xi$, we have

$$\begin{aligned}
\iota_{Y_{F_t}} d\alpha_{tH}(Z) &= (d(e^{tH}) \wedge \alpha)(Y_{F_t}, Z) + e^{tH} d\alpha(Y_{F_t}, Z) \\
&= -\alpha(Y_{F_t}) d(e^{tH})(Z) - e^{tH} dF_t(Z) && \text{(because } Z \in \xi\text{)} \\
&= -F_t d(e^{tH})(Z) - e^{tH} dF_t(Z) && \text{(because } \alpha(Y_{F_t}) = F_t\text{)} \\
&= -d(e^{tH} F_t)(Z).
\end{aligned}$$

Taking the derivative of (1) with respect to t, and using what we just proved, we get

$$H + d(e^{tH} F_t)(R_{\alpha_{tH}}) = 0. \tag{2}$$

Besides, we deduce from

$$R_{\alpha_{tH}} = e^{-tH} \begin{pmatrix} 1 + t\lambda(X_H) \\ tX_H \end{pmatrix}, \quad \iota_{X_H} d\lambda = -dH,$$

that

$$dH(R_{\alpha_{tH}}) = 0.$$

Then (2) gives

$$dF_t(R_{\alpha_{tH}}) = -He^{-tH}. \tag{3}$$

Conversely, if $(F_t)_t$ is a family of functions on V satisfying (3), then the contact isotopy $(\phi_t)_t$ defined by

$$\phi_0 = \mathrm{id} \quad \text{and} \quad \frac{d}{dt}\phi_t = Y_{F_t} \circ \phi_t$$

satisfies

$$\frac{d}{dt}(\phi_t^* \alpha_{tH}) = 0,$$

and thus $\phi_H := \phi_1$ gives the desired result.

Therefore, it remains to find a family $(F_t)_t$ satisfying (3). First recall that

$$R_{\alpha_{tH}} = e^{-tH} \begin{pmatrix} 1 + t\lambda(X_H) \\ tX_H \end{pmatrix}.$$

By assumption on H, the function $d\theta(R_{\alpha_{tH}})$ is greater than $\frac{1}{2}$ for every $t \in [0,1]$. Thus, for every $t \in [0,1]$ and every (θ, x) in V, there exists a unique real number $\rho_t(\theta, x)$ such that

$$\varphi_{R_{\alpha_{tH}}}^{-\rho_t(\theta,x)}(\theta, x) \in \{0\} \times P.$$

Then we let

$$F_t := -\rho_t H e^{-tH}.$$

For every real number t, we have

$$F_t \circ \varphi_{R_{\alpha_{tH}}}^t = -(\rho_t \circ \varphi_{R_{\alpha_{tH}}}^t) He^{-tH} \quad \text{because } dH(R_{\alpha_{tH}}) = 0.$$

But the map $\varphi_{R_{\alpha_{tH}}}^{-\rho_t \circ \varphi^t_{R_{\alpha_{tH}}} + t}$ takes its values in $\{0\} \times P$ by definition of ρ_t, so by uniqueness we have

$$\rho_t \circ \varphi_{R_{\alpha_{tH}}}^t = \rho_t + t.$$

Then we have

$$F_t \circ \varphi^t_{R_{\alpha_{tH}}} = -(\rho_t + t)He^{-tH},$$

and thus

$$dF_t(R_{\alpha_{tH}}) = -He^{-tH}. \qquad \square$$

Example 4.12 Assume that we are in the case

$$(P,\lambda) = (T^*M, p\,dq), \quad L = 0_M, \quad \text{and} \quad H(q,p) = h(q),$$

where $h\colon M \to \mathbb{R}$ is a Morse function. Then the diffeomorphism ϕ_H defined by

$$\phi_H^{-1}(\theta,(q,p)) = \big(\theta e^{h(q)}, (q, e^{h(q)}p + \theta e^{h(q)}\,dh(q))\big)$$

satisfies $\phi_H^*\alpha_H = \alpha$. With this choice of ϕ_H, we have in particular

$$\phi_H^{-1}(\{\theta\} \times 0_M) = j^1(\theta e^h) \subset \mathbb{R} \times T^*M.$$

In the following, we fix a contactomorphism ϕ_H as in Lemma 4.11. We define a pair $(\mathcal{A}_1, \tau_1)$, which is roughly obtained by pulling back the data of $(\mathcal{A}, \tau)$ by ϕ_H.

Definition 4.13 Let

$$\mathbf{\Lambda}_H := (\Lambda^n_H(E))_{(n,E)\in\mathbb{Z}\times\mathcal{E}}, \quad \text{where } \Lambda^\theta_H(E) := \phi_H^{-1}(\Lambda^\theta(E)), \text{ and } J_H := \phi_H^* J.$$

Consider the path of Legendrians $(\mathbf{\Lambda}_{H,t})_{0\le t\le 1}$, where $\Lambda^n_{H,t}(E) = \Lambda^{n+t}_H(E)$. We set

$$\mathcal{A}_1 := \mathcal{A}(\mathbf{\Lambda}_H, J_H, \alpha) \quad \text{and} \quad \tau_1 := \tau_{(\mathbf{\Lambda}_{H,t})_t, J_H, \alpha}$$

(see Definitions 4.5 and 4.6).

Relation between $(\mathcal{A}, \tau)$ and $(\mathcal{A}_1, \tau_1)$ We now explain how the pairs $(\mathcal{A}, \tau)$ and $(\mathcal{A}_1, \tau_1)$ (Definitions 4.7 and 4.13) are related. See Figure 3, where we illustrate the action of the contactomorphism ϕ_H^{-1} in the case

$$(P,\lambda) = (T^*S^1, p\,dq), \quad L = 0_{S^1}, \quad \text{and} \quad H(q,p) = h(q),$$

where $h\colon S^1 \to \mathbb{R}$ is a Morse function.

Lemma 4.14 *There is a strict A_∞-isomorphism $\zeta_1\colon \mathcal{A} \to \mathcal{A}_1$ defined as follows:*

1. *On objects, $\zeta_1(\Lambda^n(E)) = \Lambda^n_H(E)$.*
2. *On morphisms, ζ_1 sends a Reeb chord c in $\mathcal{A}(\Lambda^i(E), \Lambda^j(E'))$ to the Reeb chord*

$$\zeta_1(c) = \phi_H^{-1} \circ c$$

in $\mathcal{A}_1(\Lambda^i_H(E), \Lambda^j_H(E'))$.

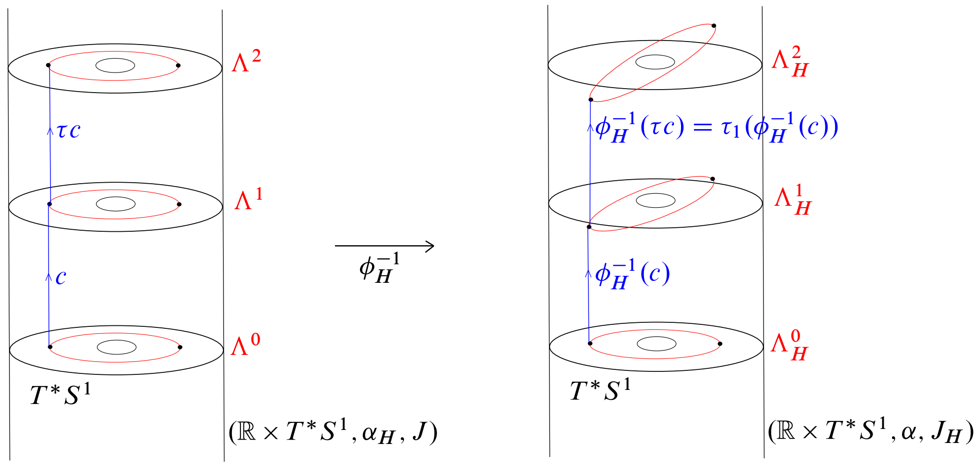


Figure 3: Action of the contactomorphism ϕ_H^{-1}.

Proof We have to show that ζ_1 is an A_∞-map. This follows from the fact that the map

$$u=(\sigma,v)\mapsto(\sigma,\phi_H^{-1}\circ v)$$

induces a bijection

$$\mathcal{M}_{c_d,\dots,c_1,c_0}(\mathbb{R}\times\Lambda,J,\alpha_H)\xrightarrow{\sim}\mathcal{M}_{\phi_H^{-1}(c_d)\dots\phi_H^{-1}(c_1),\phi_H^{-1}(c_0)}(\mathbb{R}\times\Lambda_H,J_H,\alpha).\qquad\square$$

Lemma 4.15 $$\tau_1=\zeta_1\circ\tau\circ\zeta_1^{-1}.$$

Proof This follows from Theorem 3.15 using that $\phi_H^*\alpha_H=\alpha$ and

$$((\Lambda_H^{n+1-t})_{i_0\le n\le i_d},J_H)=((\phi_H^{-1}\Lambda^{n+1-t})_{i_0\le n\le i_d},\phi_H^*J).\qquad\square$$

Lemma 4.16 *The mapping torus of $\tau:\mathcal{A}\to\mathcal{A}$ is quasi-equivalent to the mapping torus of $\tau_1:\mathcal{A}_1\to\mathcal{A}_1$ (see Definition 2.1).*

Proof According to Lemma 4.15 the following diagram of Adams-graded A_∞-categories is commutative:

$$\begin{array}{ccccc}\mathcal{A} & \xleftarrow{\mathrm{id}\sqcup\mathrm{id}} & \mathcal{A}\sqcup\mathcal{A} & \xrightarrow{\mathrm{id}\sqcup\tau} & \mathcal{A}\\ \downarrow{\zeta_1} & & \downarrow{\zeta_1\sqcup\zeta_1} & & \downarrow{\zeta_1}\\ \mathcal{A}_1 & \xleftarrow{\mathrm{id}\sqcup\mathrm{id}} & \mathcal{A}_1\sqcup\mathcal{A}_1 & \xrightarrow{\mathrm{id}\sqcup\tau_1} & \mathcal{A}_1\end{array}$$

Moreover, each vertical arrow is a quasi-equivalence according to Lemma 4.14. Thus the result follows from Proposition 1.22. □

4.4 Back to the original almost complex structure

In this section, we introduce a pair $(\mathcal{A}_2,\tau_2)$ defined using the same data as $(\mathcal{A}_1,\tau_1)$ (Definition 4.13), except we are using the almost complex structure J instead of J_H.

Definition 4.17 We set

$$\mathcal{A}_2 := \mathcal{A}(\mathbf{\Lambda}_H, J, \alpha) \quad \text{and} \quad \tau_2 := \tau_{(\mathbf{\Lambda}_{H,t})_t, J, \alpha}$$

(see Definitions 4.5 and 4.6).

Relation between $(\mathcal{A}_1, \tau_1)$ and $(\mathcal{A}_2, \tau_2)$

Lemma 4.18 *Choose a generic path $(J_t^{12})_{0\le t\le 1}$ such that $J_0^{12} = J$ and $J_1^{12} = J_H$. There is an A_∞-isomorphism $\zeta_{12}\colon \mathcal{A}_1 \to \mathcal{A}_2$ defined as follows:*

(1) *On objects, $\zeta_{12}(\Lambda_H^n(E)) = \Lambda_H^n(E)$.*

(2) *On morphisms, the map*

$$\zeta_{12}\colon \mathcal{A}_1(\Lambda_H^{i_0}(E_0), \Lambda_H^{i_1}(E_1)) \otimes \cdots \otimes \mathcal{A}_1(\Lambda_H^{i_{d-1}}(E_{d-1}), \Lambda_H^{i_d}(E_d)) \to \mathcal{A}_2(\Lambda_H^{i_0}(E_0), \Lambda_H^{i_d}(E_d))$$

is obtained by dualizing the components of the DG-isomorphism

$$\mathrm{CE}_{-*}((\Lambda_H^n)_{i_0\le n\le i_d}, J, \alpha) \to \mathrm{CE}_{-*}((\Lambda_H^n)_{i_0\le n\le i_d}, J_H, \alpha)$$

induced by the path $((\Lambda_H^n)_{i_0\le n\le i_d}, J_t^{12})_{0\le t\le 1}$ (see Theorem 3.15).

Proof We have to prove that ζ_{12} is an isomorphism. This follows from the fact that it is defined by dualizing the components of a DG-isomorphism. □

Lemma 4.19 *The A_∞-functor τ_2 is homotopic to $\zeta_{12} \circ \tau_1 \circ \zeta_{12}^{-1}$ (see* [36, Paragraph (1h)]).

Proof First recall that τ_1 is obtained by dualizing the components of the DG-map

$$\mathrm{CE}_{-*}((\Lambda_H^{n+1})_{i_0\le n\le i_d}, J_H, \alpha) \to \mathrm{CE}_{-*}((\Lambda_H^n)_{i_0\le n\le i_d}, J_H, \alpha)$$

induced by the path $((\Lambda_H^{n+1-t})_{i_0\le n\le i_d}, J_H)_{0\le t\le 1}$. Thus, $\zeta_{12}\circ\tau_1$ is obtained by dualizing the components of the composition

$$\mathrm{CE}_{-*}((\Lambda_H^{n+1})_{i_0\le n\le i_d}, J, \alpha) \to \mathrm{CE}_{-*}((\Lambda_H^{n+1})_{i_0\le n\le i_d}, J_H, \alpha) \to \mathrm{CE}_{-*}((\Lambda_H^n)_{i_0\le n\le i_d}, J_H, \alpha).$$

On the other hand, τ_2 is obtained by dualizing the components of the DG-map

$$\mathrm{CE}_{-*}((\Lambda_H^{n+1})_{i_0\le n\le i_d}, J, \alpha) \to \mathrm{CE}_{-*}((\Lambda_H^n)_{i_0\le n\le i_d}, J, \alpha)$$

induced by the path $((\Lambda_H^{n+1-t})_{i_0\le n\le i_d}, J)_{0\le t\le 1}$. Thus, $\tau_2 \circ \zeta_{12}$ is obtained by dualizing the components of the composition

$$\mathrm{CE}_{-*}((\Lambda_H^{n+1})_{i_0\le n\le i_d}, J, \alpha) \to \mathrm{CE}_{-*}((\Lambda_H^n)_{i_0\le n\le i_d}, J, \alpha) \to \mathrm{CE}_{-*}((\Lambda_H^n)_{i_0\le n\le i_d}, J_H, \alpha).$$

According to Theorem 3.15, the DG-maps used to define $\zeta_{12}\circ\tau_1$ and $\tau_2\circ\zeta_{12}$ are DG-homotopic. Therefore the A_∞-functors $\zeta_{12} \circ \tau_1$ and $\tau_2 \circ \zeta_{12}$ are homotopic. □

Lemma 4.20 *The mapping torus of $\tau_1\colon \mathcal{A}_1\to\mathcal{A}_1$ is quasi-equivalent to the mapping torus of $\tau_2\colon \mathcal{A}_2\to\mathcal{A}_2$ (see Definition 2.1).*

Proof Let $\tau_{12}:=\zeta_{12}\circ\tau_1\circ\zeta_{12}^{-1}$. Consider the commutative diagram of Adams-graded A_∞-categories

$$\begin{array}{ccccc} \mathcal{A}_1 & \xleftarrow{\mathrm{id}\sqcup\mathrm{id}} & \mathcal{A}_1\sqcup\mathcal{A}_1 & \xrightarrow{\mathrm{id}\sqcup\tau_1} & \mathcal{A}_1 \\ \big\downarrow{\scriptstyle\zeta_{12}} & & \big\downarrow{\scriptstyle\zeta_{12}\sqcup\zeta_{12}} & & \big\downarrow{\scriptstyle\zeta_{12}} \\ \mathcal{A}_2 & \xleftarrow{\mathrm{id}\sqcup\mathrm{id}} & \mathcal{A}_2\sqcup\mathcal{A}_2 & \xrightarrow{\mathrm{id}\sqcup\tau_{12}} & \mathcal{A}_2 \end{array}$$

Each vertical arrow is a quasi-equivalence according to Lemma 4.18, so it follows from Proposition 1.22 that the mapping torus of τ_1 is quasi-equivalent to the mapping torus of τ_{12}. Now according to Lemma 4.19, τ_{12} is homotopic to τ_2. Thus the result follows from Proposition 1.23. □

4.5 Projection to P

4.5.1 The A_∞-category $\mathcal{O}$ In order to define the A_∞-category $\mathcal{O}$, we need to introduce moduli spaces of pseudoholomorphic discs in P.

Definition 4.21 Let $\boldsymbol{L}=(L^n(E))_{(n,E)\in\mathbb{Z}\times\mathcal{E}}$ be a family of mutually transverse connected compact exact Lagrangians in (P,λ). Consider a sequence of integers $i_0<\cdots<i_d$, and a family of intersection points $(x_0,x_1,\dots,x_d)$, where

$$x_0\in L^{i_0}(E_0)\cap L^{i_d}(E_d)\qquad\text{and}\qquad x_k\in L^{i_{k-1}}(E_{k-1})\cap L^{i_k}(E_k),\quad 1\le k\le d.$$

(1) If $d=1$, we denote by $\mathcal{M}_{x_1,x_0}(\boldsymbol{L},j)$ the set of equivalence classes of maps $u\colon\mathbb{R}\times[0,1]\to P$ such that

- u maps $\mathbb{R}\times\{0\}$ to $L^{i_0}(E_0)$ and $\mathbb{R}\times\{1\}$ to $L^{i_1(E_1)}$,
- u satisfies the asymptotic conditions
$$u(s,t)\xrightarrow[s\to-\infty]{}x_1\quad\text{and}\quad u(s,t)\xrightarrow[s\to+\infty]{}x_0,$$
- u is (i,j)-holomorphic,

where two maps u and u' are identified if there exists $s_0\in\mathbb{R}$ such that $u'(\cdot,\cdot)=u(\cdot+s_0,\cdot)$.

(2) If $d\ge 2$, we denote by $\mathcal{M}_{x_d,\dots,x_1,x_0}(\boldsymbol{L},j)$ the set of pairs (r,u) such that

- $r\in\mathcal{R}^{d+1}$ and $u\colon\Delta_r\to P$ maps the boundary arc (ζ_{k+1},ζ_k) of Δ_r to $L^{i_k}(E_k)$,
- u satisfies the asymptotic conditions
$$(u\circ\epsilon_k(r))(s,t)\xrightarrow[s\to-\infty]{}x_k\quad\text{and}\quad(u\circ\epsilon_0(r))(s,t)\xrightarrow[s\to+\infty]{}x_0,$$
- u is (i,j)-holomorphic.

Recall that we have chosen a contactomorphism ϕ_H as in Lemma 4.11. We set

$$L_H^n:=\Pi_P(\Lambda_H^n(E))\subset P\quad\text{and}\quad\boldsymbol{L}_H:=(L_H^n(E))_{(n,E)\in\mathbb{Z}\times\mathcal{E}}.$$

Definition 4.22 We denote by $\mathcal{O}$ the A_∞-category defined as follows:

(1) The objects of $\mathcal{O}$ are the Lagrangians $L^n_H(E)$, $(n,E)\in\mathbb{Z}\times\mathcal{E}$.

(2) The space of morphisms from $L^i_H(E)$ to $L^j_H(E')$ is either generated by $L^i_H(E)\cap L^j_H(E')$ if $(i,E)<(j,E')$, or $\mathbb{F}$ if $(i,E)=(j,E')$, or 0 otherwise.

(3) The operations are such that $1\in\mathcal{O}(L^n_H(E),L^n_H(E))$ is a strict unit, and for every sequence $(i_0,E_0)<\cdots<(i_d,E_d)$, for every sequence of intersection points

$$(x_1,\dots,x_d)\in(L^{i_0}_H(E_0)\cap L^{i_1}_H(E_1))\times\cdots\times(L^{i_{d-1}}_H(E_{d-1})\cap L^{i_d}_H(E_d)),$$

we have

$$\mu_{\mathcal{O}}(x_1,\dots,x_d)=\sum_{x_0\in L^{i_0}_H(E_0)\cap L^{i_d}_H(E_d)}\#\mathcal{M}_{x_d,\dots,x_1,x_0}(\boldsymbol{L}_H,j)x_0.$$

4.5.2 The quasi-autoequivalence $\boldsymbol{\gamma}$ Before defining the A_∞-functor $\gamma\colon\mathcal{O}\to\mathcal{O}$, we recall Legendrian contact homology as defined in [16]. To each generic Legendrian Λ in $\mathbb{R}\times P$, the authors associate a semifree DG-algebra $A=A(\Lambda,j)$ generated by the self-intersection points of $\Pi_P(\Lambda)$, with a differential $\partial\colon A\to A$ defined using j-holomorphic discs in P. In our case, the differential of $A\big(\bigsqcup_k\Lambda^k_H(E),j\big)$ on a generator $x_0\in L^{i_0}_H(E_0)\cap L^{i_d}_H(E_d)$ is given by

$$\partial x_0=\sum_{(x_1,\dots,x_d)}\#\mathcal{M}_{x_d,\dots,x_1,x_0}(\boldsymbol{L}_H,j)x_d\cdots x_1,$$

where the sum is over the sequences

$$(x_1,\dots,x_d)\in(L^{i_0}_H(E_0)\cap L^{i_1}_H(E_1))\times\cdots\times(L^{i_{d-1}}_H(E_{d-1})\cap L^{i_d}_H(E_d)).$$

According to [10, Theorem 2.1], Legendrian contact homology as defined in [16] coincides with the version exposed in Section 3:

$$A(\Lambda,j)=\mathrm{CE}_*(\Lambda,(D\Pi_P)|_\xi^*j,\alpha).$$

We introduced this version only because it makes clearer the fact that some operations are defined using pseudoholomorphic polygons in the base P.

Definition 4.23 We denote by $\gamma\colon\mathcal{O}\to\mathcal{O}$ the A_∞-functor defined as follows:

(1) On objects, $\gamma(L^n_H(E))=L^{n+1}_H(E)$.

(2) On morphisms, the map

$$\gamma\colon\mathcal{O}(L^{i_0}_H(E_0),L^{i_1}_H(E_1))\otimes\cdots\otimes\mathcal{O}(L^{i_{d-1}}_H(E_{d-1}),L^{i_d}_H(E_d))\to\mathcal{O}(L^{i_0+1}_H(E_0),L^{i_d+1}_H(E_d))$$

is obtained by dualizing the components of the DG-isomorphism

$$A\Big(\bigsqcup_{k=i_0}^{i_d}\Lambda_H^{k+1}, j\Big) = \mathrm{CE}_{-*}\Big(\mathbb{R}\times\bigsqcup_{k=i_0}^{i_d}\Lambda_H^{k+1}, (D\Pi_P)|_\xi^* j, \alpha\Big) \to \mathrm{CE}_{-*}\Big(\mathbb{R}\times\bigsqcup_{k=i_0}^{i_d}\Lambda_H^{k}, (D\Pi_P)|_\xi^* j, \alpha\Big) = A\Big(\bigsqcup_{k=i_0}^{i_d}\Lambda_H^{k}, j\Big)$$

induced by the Legendrian isotopy $\big(\bigsqcup_{k=i_0}^{i_d}\Lambda_H^{k+1-t}\big)_{0\le t\le 1}$ (see Theorem 3.15).

Remark (1) The A_∞-functor $\gamma\colon\mathcal{O}\to\mathcal{O}$ is a quasi-equivalence because it is defined by dualizing the components of a DG-isomorphism.

(2) The $\mathbb{Z}$-splitting

$$\mathbb{Z}\times\mathcal{E}\xrightarrow{\sim}\mathrm{ob}(\mathcal{O}),\quad (n,E)\mapsto L_H^n(E),$$

is compatible with the quasi-autoequivalence γ in the sense of Definition 2.2. As explained there, this turns $\mathcal{O}$ into an Adams-graded A_∞-category.

4.5.3 Relation with the previous invariants We now explain how the pairs $(\mathcal{A}_2,\tau_2)$ (Definition 4.17) and $(\mathcal{O},\gamma)$ are related. See Figure 4, where we illustrate the action of the projection Π_P in the case

$$(P,\lambda) = (T^*S^1, p\,dq),\quad L = 0_{S^1},\quad \text{and}\quad H(q,p) = h(q),$$

where $h\colon S^1\to\mathbb{R}$ is a Morse function.

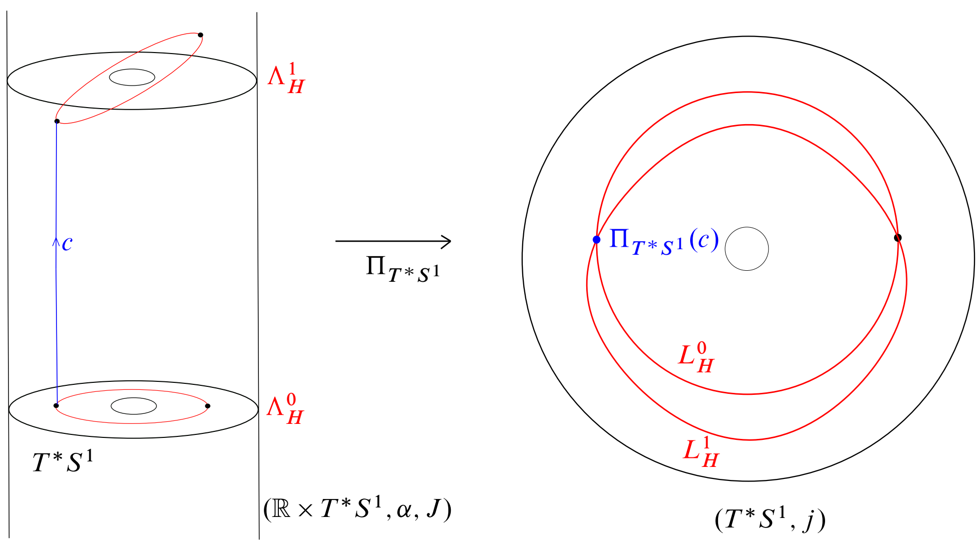


Figure 4: Action of the projection $\Pi_{T^*S^1}$.

Lemma 4.24 *There is a strict A_∞-isomorphism $\zeta_2\colon \mathcal{A}_2 \to \mathcal{O}$ defined as follows:*

(1) *On objects, $\zeta_2(\Lambda^n_H(E)) = L^n_H(E)$.*

(2) *On morphisms, ζ_2 sends a Reeb chord c in $\mathcal{A}_2(\Lambda^i_H(E), \Lambda^j_H(E'))$ to the intersection point*

$$\zeta_2(c) = \Pi_P(c)$$

in $\mathcal{O}(L^i_H(E), L^j_H(E'))$.

Proof We have to show that ζ_2 is an A_∞-map. Since $J = (D\Pi_P)|^*_\xi j$, it follows from [10, Theorem 2.1] that the map

$$u = (\sigma, v) \mapsto \Pi_P \circ v$$

induces a bijection

$$\mathcal{M}_{c_d,\dots,c_1,c_0}(\mathbb{R} \times \Lambda_H, J, \alpha) \xrightarrow{\sim} \mathcal{M}_{\Pi_P(c_d)\dots\Pi_P(c_1),\Pi_P(c_0)}(\boldsymbol{L}_H, j).$$

This implies the result. □

Lemma 4.25
$$\gamma = \zeta_2 \circ \tau_2 \circ \zeta_2^{-1}.$$

Proof This follows from the definitions of τ_2, γ, ζ_2 and the fact that $J = (D\Pi_P)|^*_\xi j$. □

Lemma 4.26 *The mapping torus of $\tau_2\colon \mathcal{A}_2 \to \mathcal{A}_2$ is quasi-equivalent to the mapping torus of $\gamma\colon \mathcal{O} \to \mathcal{O}$ (see Definition 2.1).*

Proof According to Lemma 4.25 the following diagram of Adams-graded A_∞-categories is commutative:

$$\begin{array}{ccccc}
\mathcal{A}_2 & \xleftarrow{\mathrm{id}\sqcup\mathrm{id}} & \mathcal{A}_2 \sqcup \mathcal{A}_2 & \xrightarrow{\mathrm{id}\sqcup\tau_2} & \mathcal{A} \\
\downarrow{\scriptstyle \zeta_2} & & \downarrow{\scriptstyle \zeta_2\sqcup\zeta_2} & & \downarrow{\scriptstyle \zeta_2} \\
\mathcal{O} & \xleftarrow{\mathrm{id}\sqcup\mathrm{id}} & \mathcal{O} \sqcup \mathcal{O} & \xrightarrow{\mathrm{id}\sqcup\gamma} & \mathcal{O}
\end{array}$$

Moreover, each vertical arrow is a quasi-equivalence according to Lemma 4.24. Thus, the result follows from Proposition 1.22. □

4.6 Mapping torus of γ

In this section, we show that we can apply Theorem 2.5 (Theorem A in the introduction) in order to compute the mapping torus of $\gamma\colon \mathcal{O} \to \mathcal{O}$. This allows us to finish the proof of Theorem 4.3.

Recall that we have fixed a contactomorphism ϕ_H of V such that $\phi_H^*\alpha_H = \alpha$. Also recall that if θ is some real number, then

$$\Lambda^\theta(E) = \{(f_E(x) + \theta, x) \mid x \in L\}, \quad \Lambda^\theta_H(E) = \phi_H^{-1}(\Lambda^\theta(E)), \quad \text{and} \quad L^\theta_H(E) = \Pi_P(\Lambda^\theta_H(E)).$$

4.6.1 Continuation elements We denote by $\mathcal{O}_{2r}$ the A_∞-category obtained from $\mathcal{O}$ by applying the functor of Definition 1.27. We denote by

$$\Gamma = \left\{c_n(E) \in \mathcal{O}_{2r}(L_H^n(E), L_H^{n+1}(E)) \mid (n, E) \in \mathbb{Z} \times \mathcal{E}\right\}$$

the set of continuation elements in $\mathcal{O}_{2r}$ induced by the exact Lagrangian isotopies $(L_H^{n+t})_{0 \le t \le 1}$ (see for example [23, Section 3.3]).

Recall that if $\mathcal{C}$ is an A_∞-category equipped with a $\mathbb{Z}$-splitting of ob($\mathcal{C}$), we denote by $\mathcal{C}^0$ the full A_∞-subcategory of $\mathcal{C}$ whose set of objects corresponds to $\{0\} \times \mathcal{E}$.

Lemma 4.27 *There are quasi-equivalences of A_∞-categories*

$$\mathcal{O}_{2r}^0 \simeq \overrightarrow{\mathcal{F}uk}(\boldsymbol{L}_H) \quad \text{and} \quad \mathcal{O}_{2r}[\Gamma^{-1}]^0 \simeq \mathcal{F}uk(\boldsymbol{L}_H).$$

Proof First observe that we actually have $\mathcal{O}_{2r}^0 = \overrightarrow{\mathcal{F}uk}(\boldsymbol{L}_H)$.

The second equivalence follows from the results of [37, Lecture 10; 23] about Fukaya categories and localization of A_∞-categories. More precisely, consider the subcategory $\mathcal{F}$ of $\mathcal{F}uk(P)$ with objects the Lagrangians $L^n(E)$. There is a trivial A_∞-functor $\mathcal{O}_{2r} \to \mathcal{F}$ (which is the identity on objects and on morphisms in $\mathcal{O}(L_H^i(E), L_H^j(E'))$ whenever $(i, E) < (j, E')$). Moreover, this functor sends continuation elements of $\mathcal{O}_{2r}$ to quasi-invertible morphisms in $\mathcal{F}$, and therefore induces an A_∞-functor $\mathcal{O}_{2r}[\Gamma^{-1}] \to \mathcal{F}$. Since the map

$$\mathcal{O}_{2r}(L_H^i(E), L_H^j(E')) \to \mathcal{O}_{2r}[\Gamma^{-1}](L_H^i(E), L_H^j(E'))$$

is a quasi-isomorphism whenever $(i, E) < (j, E')$, it follows that the functor $\mathcal{O}_{2r}[\Gamma^{-1}] \to \mathcal{F}$ is a quasi-equivalence. Thus we get

$$\mathcal{O}_{2r}[\Gamma^{-1}]^0 \simeq \mathcal{F}^0 = \mathcal{F}uk(\boldsymbol{L}_H). \qquad \square$$

4.6.2 The $\mathcal{O}_{2r}$-bimodule map In order to apply Theorem 2.5, we need a degree 0 closed $\mathcal{O}_{2r}$-module map $f : \mathcal{O}_{2r}(-,-) \to \mathcal{O}_{2r}(-, \gamma(-))$ such that the elements in f(units) satisfy certain hypotheses. As usual, we would like to find such an f geometrically, ie using some Lagrangian (or Legendrian) isotopy. However, here the unit $1 = e_{L_H^k(E)} \in \mathcal{O}(L_H^k(E), L_H^k(E))$, which is not an intersection point between Lagrangians, is supposed to be sent by f to something in $\mathcal{O}(L_H^k(E), L_H^{k+1}(E))$, which is generated by the intersection points between $L_H^k(E)$ and $L_H^{k+1}(E)$. Therefore, we need to somehow replace this unit by some intersection point between Lagrangians. The idea is that we will slightly perturb $L_H^k(E)$ to $L_H^{k+\delta}(E)$, and then replace $e_{L_H^k(E)}$ by the continuation element in the vector space generated by $L_H^k(E) \cap L_H^{k+\delta}(E)$.

Observe that if δ is small enough, $L_H^{k+\delta}(E)$ is a small perturbation of $L_H^k(E)$. Therefore, in a Weinstein neighborhood of $L_H^k(E)$, the Lagrangian $L_H^{k+\delta}(E)$ is the graph of $dh_{\delta,k,E}$, where $h_{\delta,k,E}$ is some Morse function on $L(E)$. In particular, the intersection points between $L_H^k(E)$ and $L_H^{k+\delta}(E)$ correspond to the critical points of $h_{\delta,k,E}$. Moreover, the continuation element in the vector space generated by $L_H^k(E) \cap L_H^{k+\delta}(E)$ corresponds to the sum of the minima of $h_{\delta,k,E}$.

Example 4.28 Assume that we are in the case

$$(P,\lambda) = (T^*M, p\,dq), \quad L = 0_M, \quad \text{and} \quad H(q,p) = h(q),$$

where $h\colon M \to \mathbb{R}$ is a Morse function. As explained in Example 4.12, in this case we have

$$L_H^\theta = \Pi_{T^*M}(j^1(\theta e^h)) = \operatorname{graph}(d(\theta e^h)).$$

Thus, $L_H^{k+\delta}$ is the graph of $d(\delta e^h)$ over L_H^k.

We will need the following result about moduli spaces of discs with boundary on small perturbations of the Lagrangians.

Lemma 4.29 *Let $g := -d\lambda(-, j-)$ be the metric on P induced by j and $(-d\lambda)$. For every positive integer n, there exists $\delta_n > 0$ such that the following holds for every $\delta \in {]0,\delta_n]}$. For every sequence*

$$(-n, E_0) \le (j_0, E_0) < \dots < (j_p, E_p) \le (\ell_0, E_0') < \dots < (\ell_q, E_q') \le (n, E_q'), \quad p, q \ge 0,$$

the rigid j-holomorphic discs in P with boundary on

$$L_H^{j_0}(E_0) \cup \dots \cup L_H^{j_p}(E_p) \cup L_H^{\ell_0+\delta}(E_0') \cup \dots \cup L_H^{\ell_q+\delta}(E_q')$$

are

(1) *in bijection with the rigid j-holomorphic discs in P with boundary on*

$$L_H^{j_0}(E_0) \cup \dots \cup L_H^{j_p}(E_p) \cup L_H^{\ell_0}(E_0') \cup \dots \cup L_H^{\ell_q}(E_q')$$

if $(j_p, E_p) < (\ell_0, E_0')$, or

(2) *in bijection with the rigid j-holomorphic discs in P with boundary on*

$$L_H^{j_0}(E_0) \cup \dots \cup L_H^{j_{p-1}}(E_{p-1}) \cup L_H^{\ell_0}(E_0') \cup L_H^{\ell_1}(E_1') \cup \dots \cup L_H^{\ell_q}(E_q')$$

with a flow line of $(-\nabla_g h_{\delta,k,E_0'})$ attached on the component in $L_H^{\ell_0}(E_0')$ if $(j_p, E_p) = (\ell_0, E_0')$.

Proof The case $j_p < \ell_0$ follows from transversality of the moduli spaces in consideration. The case $j_p = \ell_0$ follows from the main analytic theorem of [13] (Theorem 3.6). □

In order to define the $\mathcal{O}_{2r}$-bimodule map f properly, we will use Lemma 4.29 to modify the A_∞-category $\mathcal{O}_{2r}$. In the following, we fix a *decreasing* sequence of positive real numbers $(\delta_n)_{n\ge1}$ such that, for every n,

(1) Lemma 4.29 holds with δ_n, and

(2) δ_n is small enough so that there is no handle slide instant in the Legendrian isotopy

$$\bigcup_{\ell=-n}^{n} \Lambda_H^{\ell+\delta_n t} = \bigcup_{\ell=-n}^{n} \bigcup_{E\in\mathcal{E}} \Lambda_H^{\ell+\delta_n t}(E), \quad t \in [0,1].$$

We define two families of A_∞-categories $(\mathcal{O}_{n,k})_{n,k}$ and $(\widetilde{\mathcal{O}}_{n,k})_{n,k}$ indexed by the couples (n,k), where $n \ge 1$ and $-n \le k \le n$. The A_∞-category $\mathcal{O}_{n,k}$ is basically obtained from $\mathcal{O}_{2r}$ by restricting to objects $L_H^i(E)$, $-n \le i \le n$, and adding a copy of the object $L_H^k(E)$.

Definition 4.30 For every $(j,E)\in\mathbb{Z}\times\mathcal{E}$, let $\bar{L}^j_H(E)$ be a copy of $L^j_H(E)$. We denote by $\mathcal{O}_{n,k}$ the A_∞-category defined as follows:

(1) The set of objects of $\mathcal{O}_{n,k}$ is
$$\mathrm{ob}(\mathcal{O}_{n,k})=\{L^j_H(E)\mid -n\le j\le k,\ E\in\mathcal{E}\}\cup\{\bar{L}^\ell_H(E)\mid k\le \ell\le n,\ E\in\mathcal{E}\}.$$

(2) The spaces of morphisms in $\mathcal{O}_{n,k}$ are the corresponding spaces of morphisms in $\mathcal{O}_{2r}$ when we replace $\bar{L}^\ell_H(E),\ k\le\ell\le n$, by $L^\ell_H(E)$, except that
$$\mathcal{O}_{n,k}(\bar{L}^k_H(E),L^k_H(E))=\{0\}.$$

(3) The operations are the same as in $\mathcal{O}_{2r}$.

The A_∞-category $\widetilde{\mathcal{O}}_{n,k}$ is obtained from $\mathcal{O}_{n,k}$ by perturbing the objects $\bar{L}^\ell_H(E),\ k\le\ell\le n$, to $L^{\ell+\delta_n}_H(E)$.

Definition 4.31 Let
$$\Theta_{n,k}:=\{-n,\dots,k\}\cup\{\ell+\delta_n\mid k\le\ell\le n\}\subset\mathbb{R},\quad\text{and}\quad \widetilde{\boldsymbol{L}}_H:=(L^\theta_H(E))_{(\theta,E)\in\Theta_{n,k}\times\mathcal{E}}.$$

We denote by $\widetilde{\mathcal{O}}_{n,k}$ the A_∞-category defined as follows:

(1) The objects of $\widetilde{\mathcal{O}}_{n,k}$ are the Lagrangians $L^\theta_H(E)$, $(\theta,E)\in\Theta_{n,k}\times\mathcal{E}$.

(2) The space of morphisms from $L^\theta_H(E)$ to $L^{\theta'}_H(E')$ is either generated by $L^\theta_H(E)\cap L^{\theta'}_H(E')$ if $(\theta,E)<(\theta',E')$, or $\mathbb{F}$ if $(\theta,E)=(\theta',E')$, or 0 otherwise.

(3) The operations are such that $e_{L^\theta_H(E)}=1\in\widetilde{\mathcal{O}}_{n,k}(L^\theta_H(E),L^\theta_H(E))$ is a strict unit, and for every sequence $(\theta_0,E_0)<\dots<(\theta_d,E_d)$, for every sequence of intersection points
$$(x_1,\dots,x_d)\in(L^{\theta_0}_H(E_0)\cap L^{\theta_1}_H(E_1))\times\dots\times(L^{\theta_{d-1}}_H(E_{d-1})\cap L^{\theta_d}_H(E_d)),$$
we have
$$\mu_{\widetilde{\mathcal{O}}_{n,k}}(x_1,\dots,x_d)=\sum_{x_0\in L^{\theta_0}_H(E_0)\cap L^{\theta_d}_H(E_d)}\#\mathcal{M}_{x_d,\dots,x_1,x_0}(\widetilde{\boldsymbol{L}}_H,j)x_0.$$

These A_∞-categories being defined, Lemma 4.29 implies the following result.

Lemma 4.32 *There is a strict A_∞-functor $\rho_{n,k}\colon\mathcal{O}_{n,k}\to\widetilde{\mathcal{O}}_{n,k}$ defined as follows:*

(1) *On objects, $\rho_{n,k}(L^j_H(E))=L^j_H(E)$ if $-n\le j\le k$ and $\rho_{n,k}(\bar{L}^\ell_H(E))=L^{\ell+\delta_n}_H(E)$ if $k\le\ell\le n$.*

(2) *On morphisms, $\rho_{n,k}$ sends the unit of $\mathcal{O}_{n,k}(L^k_H(E),\bar{L}^k_H(E))=\mathbb{F}$ to the continuation element in $\widetilde{\mathcal{O}}_{n,k}(L^k_H(E),L^{k+\delta_n}_H(E))$, and it sends any other morphism of $\mathcal{O}_{n,k}$ to the corresponding one in $\widetilde{\mathcal{O}}_{n,k}$.*

Proof Consider a sequence $(x_0,\dots,x_{d-1})$ of morphisms in $\mathcal{O}_{n,k}$. If in this sequence there is no morphism from $L^k_H(E)$ to $\bar{L}^k_H(E)$, then the relation
$$\mu_{\widetilde{\mathcal{O}}_{n,k}}(\rho_{n,k}x_0,\dots,\rho_{n,k}x_d)=\rho_{n,k}(\mu_{\mathcal{O}_{n,k}}(x_0,\dots,x_d))$$

follows directly from the first item of Lemma 4.29. Now assume that there is $p \in \{0, \dots, d-1\}$ such that $x_p = e_{L^k_H(E)} \in \mathcal{O}_{n,k}(L^k_H(E), \bar{L}^k_H(E))$. Recall that the continuation element in $\widetilde{\mathcal{O}}_{n,k}(L^k_H(E), L^{k+\delta_n}_H(E))$ corresponds to the sum of the minima of $h_{\delta_n,k,E}$. Then the second item of Lemma 4.29 implies that

$$\mu_{\widetilde{\mathcal{O}}_{n,k}}(\rho_{n,k}x_0, \dots, \rho_{n,k}x_d) = \begin{cases} \rho_{n,k}x_1 & \text{if } d = 1 \text{ and } p = 0, \\ \rho_{n,k}x_0 & \text{if } d = 1 \text{ and } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the A_∞-relation for $\rho_{n,k}$ is still satisfied according to the behavior of the operations $\mu_{\mathcal{O}_{n,k}}$ with respect to the unit $e_{L^k_H(E)}$. □

We can now define geometrically an A_∞-functor that will finally allow us to define the $\mathcal{O}_{2r}$-bimodule map f.

Definition 4.33 We denote by $\nu_{n,k} : \widetilde{\mathcal{O}}_{n,k} \to \mathcal{O}_{2r}$ the A_∞-functor defined as follows:

(1) On objects, $\nu_{n,k}(L^j_H(E)) = L^j_H(E)$ if $-n \le j \le k$, and $\nu_{n,k}(L^{\ell+\delta_n}_H(E)) = L^{\ell+1}_H(E)$ if $k \le \ell \le n$.

(2) On morphisms, $\nu_{n,k}$ is obtained by dualizing the components of the DG-isomorphism

$$A\Big(\bigsqcup_{i=-n}^{n+1} \Lambda^i_H\Big) \xrightarrow{\sim} A\Big(\bigsqcup_{j=-n}^{k} \Lambda^j_H \sqcup \bigsqcup_{\ell=k}^{n} \Lambda^{\ell+\delta_n}_H\Big).$$

induced by the Legendrian isotopy

$$\Big(\bigsqcup_{j=-n}^{k} \Lambda^j_H\Big) \sqcup \Big(\bigsqcup_{\ell=k}^{n} \Lambda^{\ell+1-t(1-\delta_n)}_H\Big), \quad t \in [0,1]$$

(see Theorem 3.15 or [16, Proposition 2.6]).

Remark 4.34 We point out some properties of the A_∞-functor

$$\sigma_{n,k} := \nu_{n,k} \circ \rho_{n,k} : \mathcal{O}_{n,k} \to \mathcal{O}_{2r}.$$

(1) Let $n \le p$ be two positive integers, and let $k \in \{-n, \dots, n\}$. Recall that we have chosen δ_n small enough so that there is no handle slide instant in the Legendrian isotopy

$$\bigsqcup_{\ell=-n}^{n} \Lambda^{\ell+\delta_n t}_H, \quad 0 \le t \le 1.$$

Since $\delta_p \le \delta_n$, neither is there any handle slide instant in the Legendrian isotopy

$$\bigsqcup_{\ell=-n}^{n} \Lambda^{\ell+\delta_p t}_H, \quad 0 \le t \le 1.$$

Therefore, $\sigma_{p,k}$ agrees with $\sigma_{n,k}$ on $\mathcal{O}_{n,k} \subset \mathcal{O}_{p,k}$.

(2) Consider a sequence of integers

$$-n \le j_0 < \dots < j_p \le k_1 < k_2 \le \ell_0 < \dots < \ell_q \le n,$$

and a sequence of morphisms

$$\begin{aligned}(x_0, \dots, x_{p-1}, u, y_0, \dots, y_{q-1}) \\ \in \mathcal{O}_{n,k_i}(L_H^{j_0}(E_0), L_H^{j_1}(E_1)) \times \dots \times \mathcal{O}_{n,k_i}(L_H^{j_{p-1}}(E_{p-1}), L_H^{j_p}(E_p)) \times \mathcal{O}_{n,k_i}(L_H^{j_p}(E_p), \bar{L}_H^{\ell_0}(E_0')) \\ \times \mathcal{O}_{n,k_i}(\bar{L}_H^{\ell_0}(E_0'), \bar{L}_H^{\ell_1}(E_1')) \times \dots \times \mathcal{O}_{n,k_i}(\bar{L}_H^{\ell_{q-1}}(E_{q-1}'), \bar{L}_H^{\ell_q}(E_q')).\end{aligned}$$

Since the Legendrian isotopy defining ν_{n,k_i} is

$$\Big(\bigsqcup_{j=-n}^{k_i} \Lambda_H^j\Big) \sqcup \Big(\bigsqcup_{\ell=k_i}^{n} \Lambda_H^{\ell+1-t(1-\delta_n)}\Big), \quad t \in [0,1],$$

we have

$$\begin{aligned}&\sigma_{n,k_1}(x_0, \dots, x_{p-1}) = \delta_{1p} x_0, \\ &\sigma_{n,k_2}(y_0, \dots, y_{q-1}) = \gamma(y_0, \dots, y_{q-1}), \\ &\sigma_{n,k_2}(x_0, \dots, x_{p-1}, u, y_0, \dots, y_{q-1}) = \sigma_{n,k_1}(x_0, \dots, x_{p-1}, u, y_0, \dots, y_{q-1}).\end{aligned}$$

(3) By construction, the A_∞-functor $\nu_{n,k}$ sends the continuation element in $\widetilde{\mathcal{O}}_{n,k}(L_H^k(E), L_H^{k+\delta_n}(E))$ (corresponding to the sum of the minima of $h_{\delta_n,k,E}$) to the continuation element

$$c_k(E) \in \mathcal{O}_{2r}(L_H^k(E), L_H^{k+1}(E)).$$

In other words, $\sigma_{n,k}$ sends the unit $e_{L_H^k(E)} \in \mathcal{O}_{n,k}(L_H^k(E), \bar{L}_H^k(E))$ to $c_k(E)$.

(4) The map $\sigma_{n,k} \colon \mathcal{O}_{n,k}(L_H^j(E), \bar{L}_H^k(E')) \to \mathcal{O}_{2r}(L_H^j(E), L_H^{k+1}(E'))$ is a quasi-isomorphism for every $j < k$ and $E, E' \in \mathcal{E}$.

We can now state and prove the desired result.

Lemma 4.35 *There exists a degree 0 closed $\mathcal{O}_{2r}$-bimodule map $f \colon \mathcal{O}_{2r}(-,-) \to \mathcal{O}_{2r}(-,\gamma(-))$ which sends the unit $e_{L_H^k(E)} \in \mathcal{O}_{2r}(L_H^k(E), L_H^k(E))$ to the continuation element*

$$c_k(E) \in \mathcal{O}_{2r}(L_H^k(E), L_H^{k+1}(E)) \cap \Gamma,$$

and such that $f \colon \mathcal{O}_{2r}(L_H^j(E), L_H^k(E')) \to \mathcal{O}_{2r}(L_H^j(E), L_H^{k+1}(E'))$ is a quasi-isomorphism for every $j < k$ and $E, E' \in \mathcal{E}$.

Proof Consider a sequence

$$(j_0, E_0) < \dots < (j_p, E_p) \le (k, E) = (\ell_0, E_0') < \dots < (\ell_q, E_q'),$$

and a sequence of morphisms

$$\begin{aligned}(x_0,\dots,x_{p-1},u,y_0,\dots,y_{q-1})\\ \in \mathcal{O}_{2r}(L_H^{j_0}(E_0),L_H^{j_1}(E_1))\times\dots\times\mathcal{O}_{2r}(L_H^{j_{p-1}}(E_{p-1}),L_H^{j_p}(E_p))\times\mathcal{O}_{2r}(L_H^{j_p}(E_p),L_H^{k}(E_0'))\\ \times\mathcal{O}_{2r}(L_H^{k}(E_0'),L_H^{\ell_1}(E_1'))\times\dots\times\mathcal{O}_{2r}(L_H^{\ell_{q-1}}(E_{q-1}'),L_H^{\ell_q}(E_q')).\end{aligned}$$

We choose $n\ge 1$ such that $-n\le j_0\le \ell_q\le n$, and we set

$$f(x_0,\dots,x_{p-1},u,y_0,\dots,y_{q-1}):=\sigma_{n,k}(x_0,\dots,x_{p-1},u,y_0,\dots,y_{q-1})\in\mathcal{O}_{2r}(L_H^{j_0}(E_0),\gamma L_H^{\ell_q}(E_q')),$$

where on the right-hand side we consider that

$$\begin{aligned}(x_0,\dots,x_{p-1},u,y_0,\dots,y_{q-1})\\ \in \mathcal{O}_{n,k}(L_H^{j_0}(E_0),L_H^{j_1}(E_1))\times\dots\times\mathcal{O}_{n,k}(L_H^{j_{p-1}}(E_{p-1}),L_H^{j_p}(E_p))\times\mathcal{O}_{n,k}(L_H^{j_p}(E_p),\bar{L}_H^{k}(E_0'))\\ \times\mathcal{O}_{n,k}(\bar{L}_H^{k}(E_0'),\bar{L}_H^{\ell_1}(E_1'))\times\dots\times\mathcal{O}_{n,k}(\bar{L}_H^{\ell_{q-1}}(E_{q-1}'),\bar{L}_H^{\ell_q}(E_q')).\end{aligned}$$

Observe that f is well defined (it does not depend on the choice of n) according to the first item of Remark 4.34.

We now verify that f is closed. According to Definition 1.4, we have

$$\begin{aligned}&\mu^1_{\mathrm{Mod}_{\mathcal{C},\mathcal{C}}}(f)(x_0,\dots,x_{p-1},u,y_0,\dots,y_{q-1})\\ &=\sum\sigma_{n,k}(\dots,\mu_{\mathcal{O}_{2r}}(\dots),\dots,u,\dots)+\sum\sigma_{n,\ell_s}(\dots,\mu_{\mathcal{O}_{2r}}(x_r,\dots,x_{p-1},u,y_0,\dots,y_{s-1}),\dots)\\ &\quad+\sum\sigma_{n,k}(\dots,u,\dots,\mu_{\mathcal{O}_{2r}}(\dots),\dots)+\sum\mu_{\mathcal{O}_{2r}}(\dots,\sigma_{n,k}(\dots,u,\dots),\gamma(\dots),\dots,\gamma(\dots)).\end{aligned}$$

Now according to the second item of Remark 4.34, we have

$$\begin{aligned}&\sum\sigma_{n,\ell_s}(\dots,\mu_{\mathcal{O}_{2r}}(x_r,\dots,x_{p-1},u,y_0,\dots,y_{s-1}),\dots)\\ &\qquad=\sum\sigma_{n,k}(\dots,\mu_{\mathcal{O}_{2r}}(x_r,\dots,x_{p-1},u,y_0,\dots,y_{s-1}),\dots)\end{aligned}$$

and

$$\begin{aligned}&\sum\mu_{\mathcal{O}_{2r}}(\dots,\sigma_{n,k}(\dots,u,\dots),\gamma(\dots),\dots,\gamma(\dots))\\ &\qquad=\sum\mu_{\mathcal{O}_{2r}}\big(\sigma_{n,k}(\dots),\dots,\sigma_{n,k}(\dots),\sigma_{n,k}(\dots,u,\dots),\sigma_{n,k}(\dots),\dots,\sigma_{n,k}(\dots)\big).\end{aligned}$$

Therefore, we get

$$\mu^1_{\mathrm{Mod}_{\mathcal{C},\mathcal{C}}}(f)(x_0,\dots,x_{p-1},u,y_0,\dots,y_{q-1})=0$$

from the fact that $\sigma_{n,k}$ is an A_∞-functor.

Now f sends the unit $e_{L_H^k(E)}\in\mathcal{O}_{2r}(L_H^k(E),L_H^k(E))$ to the continuation element

$$c_k(E)\in\mathcal{O}_{2r}(L_H^k(E),L_H^{k+1}(E))\cap\Gamma$$

according to the third item of Remark 4.34. Finally, the map

$$f\colon \mathcal{O}_{2r}(L_H^j(E), L_H^k(E')) \to \mathcal{O}_{2r}(L_H^j(E), L_H^{k+1}(E'))$$

is a quasi-isomorphism for every $j < k$ and $E, E' \in \mathcal{E}$ according to the last item of Remark 4.34. □

4.6.3 Proof of the main result We end the section with the proof of Theorem 4.3 (Theorem B in the introduction).

Recall that we denote by $\mathbb{F}[t_m]$ the augmented Adams-graded associative algebra generated by a variable t_m of bidegree $(m, 1)$, and by $t_m\mathbb{F}[t_m]$ its augmentation ideal (or equivalently, the ideal generated by t_m). The key result is the following.

Lemma 4.36 *The mapping torus of γ is quasi-equivalent to the Adams-graded A_∞-category*

$$\overrightarrow{\mathcal{F}uk}(\boldsymbol{L}) \oplus (t_{2r}\mathbb{F}[t_{2r}] \otimes \mathcal{F}uk(\boldsymbol{L})).$$

Proof Let $f\colon \mathcal{O}_{2r}(-,-) \to \mathcal{O}_{2r}(-,\gamma(-))$ be the degree 0 closed bimodule map of Lemma 4.35. According to the latter, the hypotheses of Theorem 2.5 are satisfied, and $f(\text{units}) = \Gamma$. Thus the mapping torus of γ is quasi-equivalent to the Adams-graded A_∞-algebra $\mathcal{O}_{2r}^0 \oplus (t_{2r}\mathbb{F}[t_{2r}] \otimes \mathcal{O}_{2r}[\Gamma^{-1}]^0)$ (recall that if $\mathcal{C}$ is an A_∞-category equipped with a $\mathbb{Z}$-splitting $\mathbb{Z} \times \mathcal{E} \simeq \text{ob}(\mathcal{C})$, we denote by $\mathcal{C}^0$ the full A_∞-subcategory of $\mathcal{C}$ whose set of objects corresponds to $\{0\} \times \mathcal{E}$). According to Lemma 4.27 we have

$$\mathcal{O}_{2r}^0 \simeq \overrightarrow{\mathcal{F}uk}(\boldsymbol{L}_H) \quad \text{and} \quad \mathcal{O}_{2r}[\Gamma^{-1}]^0 \simeq \mathcal{F}uk(\boldsymbol{L}_H).$$

The result follows from invariance of the Fukaya category (see [36, Section (10a)])

$$\overrightarrow{\mathcal{F}uk}(\boldsymbol{L}_H) \simeq \overrightarrow{\mathcal{F}uk}(\boldsymbol{L}) \quad \text{and} \quad \mathcal{F}uk(\boldsymbol{L}_H) \simeq \mathcal{F}uk(\boldsymbol{L}).$$ □

We now give the proof of Theorem 4.3 (Theorem B in the introduction). According to [29, Theorem 2.4], Koszul duality holds for the augmented Adams-graded DG-algebra $\text{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)$ because it is *Adams connected* (see [29, Definition 2.1]). Indeed, recall from Section 4.1.2 that the Adams degree in $\text{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)$ of a Reeb chord c is the number of times c winds around the fiber. Besides, recall from Section 4.1.2 that there is a coaugmented Adams-graded A_∞-cocategory $\text{LC}_*(\boldsymbol{\Lambda}^\circ)$ such that

$$\text{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ) = \Omega(\text{LC}_*(\boldsymbol{\Lambda}^\circ)) \quad \text{and} \quad \text{LA}^*(\boldsymbol{\Lambda}^\circ) = \text{LC}_*(\boldsymbol{\Lambda}^\circ)^\#.$$

Since there is a quasi-isomorphism $B(\Omega C) \simeq C$ for every A_∞-cocategory C (see [17, Section 2.2.2]), it follows that

$$E(\text{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ)) = B(\text{CE}^r_{-*}(\boldsymbol{\Lambda}^\circ))^\# \simeq \text{LC}_*(\boldsymbol{\Lambda}^\circ)^\# = \text{LA}^*(\boldsymbol{\Lambda}^\circ)$$

(graded dual preserves quasi-isomorphisms). Now the quasi-equivalence

$$\text{LA}^*(\boldsymbol{\Lambda}^\circ) \simeq \overrightarrow{\mathcal{F}uk}(\boldsymbol{L}) \oplus (t_{2r}\mathbb{F}[t_{2r}] \otimes \mathcal{F}uk(\boldsymbol{L}))$$

follows from Lemmas 4.10, 4.16, 4.20, 4.26 and 4.36. This concludes the proof.

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Received: 15 December 2022 Revised: 24 August 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 1 (pages 1–644) 2025