Circular-orderability of 3-manifold groups

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We initiate the study of circular-orderability of 3-manifold groups, motivated by the L-space conjecture. We show that a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold has a circularly orderable fundamental group if and only if there exists a finite cyclic cover with left-orderable fundamental group, which naturally leads to a "circular-orderability version" of the L-space conjecture. We also show that the fundamental groups of almost all graph manifolds are circularly orderable, and contrast the behaviour of circular-orderability and left-orderability with respect to the operations of Dehn surgery and taking cyclic branched covers.

1 Introduction

For an irreducible, rational homology 3-sphere M, the L-space conjecture posits a relationship between the properties of M admitting a coorientable taut foliation, M being *not* an L-space, and M having a left-orderable fundamental group (see Conjecture 3.3). While this conjecture is known to hold for some classes of manifolds, for example graph manifolds, new techniques are needed to tackle more general classes of manifolds, or, indeed, to tackle the conjecture in full generality.

With this conjecture in mind, several of the most successful techniques developed in recent years to tackle left-orderability of $\pi_1(M)$ have shared a common theme: they all begin with an action on the circle, and use cohomological techniques to pass to an action on the real line. For instance, in studying manifolds arising from Dehn surgery on a knot K in $\mathbb{S}^3$, a common technique is to study one-parameter families of representations $\rho_t : \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{R})$ that are built so as to provide representations that factor through the quotient groups $\pi_1(\mathbb{S}^3_{p/q}(K))$ for certain values of $p/q \in \mathbb{Q} \cup \{\infty\}$. Controlling the Euler classes of these representations allows one to construct lifts $\tilde{\rho}_t : \pi_1(\mathbb{S}^3_{p/q}(K)) \to \widetilde{\mathrm{PSL}}(2, \mathbb{R})$, and these lifts show that $\pi_1(\mathbb{S}^3_{p/q}(K))$ is left-orderable since they have left-orderable image; see Boyer, Gordon and Watson [12], Hakamata and Teragaito [29; 30], Motegi and Teragaito [42], Culler and Dunfield [23] and Gao [26]. This technique has also been used to study left-orderability of cyclic branched covers of knots by Hu [34], Tran [51], Turner [52] and Gordon and Lidman [28].

In a similar vein, if one starts with an irreducible rational homology 3-sphere M admitting a coorientable taut foliation $\mathcal{F}$, Thurston's universal circle construction yields a representation $\rho_{\mathrm{univ}} : \pi_1(M) \to \mathrm{Homeo}_+(\mathbb{S}^1)$. With appropriate restrictions on $\mathcal{F}$, one can control the Euler class of ρ_{univ} and guarantee

the existence of a lift $\widetilde{\rho}_{\mathrm{univ}} \colon \pi_1(M) \to \widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$, again yielding left-orderability of $\pi_1(M)$ for similar reasons; see Calegari and Dunfield [16] and Boyer and Hu [13].

Motivated by the utility of actions on the circle by homeomorphisms in addressing the L-space conjecture, this work is a first step toward directly addressing the question of when the fundamental group of a 3-manifold acts on the circle by homeomorphisms — though we take an algebraic approach to the problem. Just as the existence of left-ordering of $\pi_1(M)$ captures whether or not there is an embedding $\rho \colon \pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R})$, we approach the problem by studying the existence of an orientation cocycle $c \colon \pi_1(M)^3 \to \{\pm 1, 0\}$ that is compatible with the group operation, called a circular ordering of $\pi_1(M)$. The existence of such a map determines whether or not there exists an embedding $\rho \colon \pi_1(M) \to \mathrm{Homeo}_+(\mathbb{S}^1)$, analogous to the case of left-orderings. It should be noted that, throughout, we adopt the convention that the trivial group is left-orderable. We show:

Theorem 1.1 *Suppose that M is a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold. Then $\pi_1(M)$ is circularly orderable if and only if M admits a finite cyclic cover with left-orderable fundamental group.*

Our contribution here is not the existence of a finite-index left-orderable subgroup, as this fact already appears implicitly in [16], but that there is a normal, left-orderable subgroup that yields a finite cyclic group upon passing to the quotient. This motivates an obvious "circular-orderability" version of the L-space conjecture (see Conjecture 3.4), which mirrors the usual L-space conjecture up to finite cyclic covers.

This theorem is in fact a special case of a new algebraic result. Associated to every circular ordering c of G is a cohomology class $[f_c] \in H^2(G; \mathbb{Z})$, called the Euler class of the circular ordering. We show that, when a group G admits a circular ordering whose Euler class has order k in $H^2(G; \mathbb{Z})$, it also admits a left-orderable normal subgroup N such that $G/N \cong \mathbb{Z}/k\mathbb{Z}$; see Theorem 2.6.

From here we begin an exploration of exactly which fundamental groups admit circular orderings. We first tackle the case of Seifert fibred manifolds, providing the details of a claim of Calegari [15]. Note that circular-orderability of finite groups is well understood (a finite group is circularly orderable if and only if it is cyclic; see Proposition 2.5), and so we focus on infinite fundamental groups. If G is a group with circular ordering c, we use $\mathrm{rot}_c(g)$ to denote the rotation number of $g \in G$; see Section 2.

Theorem 1.2 *Let M be a compact, connected Seifert fibred space and let h denote the class of a regular fibre in $\pi_1(M)$.*

(1) *If $\pi_1(M)$ is infinite, then there exists a circular ordering c of $\pi_1(M)$ such that $\mathrm{rot}_c(h) = 0$; in particular, $\pi_1(M)$ is circularly orderable whenever it is infinite.*

(2) *If $\pi_1(M)$ is infinite and M is orientable and has nonorientable base orbifold, then every circular ordering c of $\pi_1(M)$ satisfies $\mathrm{rot}_c(h) \in \{0, \frac{1}{2}\}$.*

(3) *If $\pi_1(M)$ is left-orderable and M is orientable and has base orbifold $\mathbb{S}^2(\alpha_1, \ldots, \alpha_n)$ with $n \geq 3$, then, for every $p \in \mathbb{N}_{>0}$, there exists a circular ordering c of $\pi_1(M)$ such that $\mathrm{rot}_c(h) = 1/p$.*

(4) *If M is orientable and has no exceptional fibres, then, for every $r \in \mathbb{R}/\mathbb{Z}$, there exists a circular ordering c of $\pi_1(M)$ such that $\operatorname{rot}_c(h) = r$.*

This leads naturally to the study of graph manifolds, where we show that an analogous fact holds.

Theorem 1.3 *Suppose that W is a graph manifold whose JSJ decomposition has Seifert fibred pieces $M_1, \ldots, M_n$. Further suppose that, for each $1 \leq i \leq n$, if ∂M_i is a single torus boundary component, then there is no slope $\alpha \in H_1(\partial M_i; \mathbb{Z})/\{\pm 1\}$ such that $\pi_1(M_i(\alpha))$ is finite. Then, if $\pi_1(W)$ is infinite, it is circularly orderable.*

If W is not a rational homology sphere, then the first Betti number $b_1(W)$ is positive, so $\pi_1(W)$ is left-orderable by Boyer, Rolfsen and Wiest [14, Theorem 3.2]. On the other hand, if W is a rational homology sphere, then Theorem 1.3 is in fact a special case of a stronger, more technical result; see Theorem 6.6 and Corollary 6.7. We conjecture that, with appropriate generalizations of the techniques developed here, one can prove that the fundamental group of a graph manifold is circularly orderable whenever it is infinite. See Conjecture 6.11 and the preceding discussion for details.

Our approach to this proof is to mirror the technique of "slope detection" developed by Boyer and Clay [9] for the case of left-orderings of fundamental groups of graph manifolds; in Theorem 4.3 we develop a result in the case of circular orderings that is analogous to the main tool of Clay, Lidman and Watson [20]. This tool provides sufficient conditions that a manifold $W = M_1 \cup_\phi M_2$ have circularly orderable fundamental group, by requiring that the gluing map ϕ identify slopes on M_1 and M_2 whose fillings yield fundamental groups admitting compatible circular orderings. Using this technique, it turns out that, in many cases, it is sufficient to study fillings along rational longitudes to conclude that $W = M_1 \cup_\phi M_2$ has circularly orderable fundamental group; see Proposition 5.6.

We also deal with several notable cases not covered by Theorem 6.6 or Theorem 1.3; for instance, we also show:

Theorem 1.4 *The fundamental group of a compact, connected Sol manifold is circularly orderable.*

We close with a discussion of circular-orderability of fundamental groups of hyperbolic 3-manifolds. There is a well-known example of a hyperbolic 3-manifold whose fundamental group is not circularly orderable, which is the Weeks manifold; see Calegari and Dunfield [16, Theorem 9]. Therefore, we cannot expect the fundamental groups of hyperbolic 3-manifolds to be circularly orderable whenever they are infinite, as in the case of Seifert fibred manifolds.

Two approaches to the question of left-orderability of fundamental groups of hyperbolic 3-manifolds that have enjoyed success are via cyclic branched covers and via Dehn surgery. In both of these cases, advancements in Heegaard Floer techniques have provided guidance as to the expected behaviour of left-orderability with respect to these constructions. Over the course of several examples, including several infinite families of hyperbolic 3-manifolds having circularly orderable but non-left-orderable fundamental

groups, we find that none of the behaviour exhibited by left-orderability with respect to these familiar topological constructions is shared with circular-orderability.

For example, it is suspected that, if the n-fold cyclic branched cover of a knot in $\mathbb{S}^3$ has left-orderable fundamental group, then so does the m-fold cyclic branched cover for all $m \geq n$. This does not hold for circular-orderability; see Propositions 7.1 and 7.2. Similarly, it is conjectured that the double branched cover of a quasialternating knot always has non-left-orderable fundamental group, but our examples show that there is no apparent relationship when left-orderability is replaced with circular-orderability: there exist alternating links (or, more generally, quasialternating links) whose double branched covers have non-circularly orderable fundamental groups (see Section 7.2), while other alternating (or quasialternating) links yield double branched covers with circularly orderable fundamental groups. Similar observations hold for the behaviour of circular-orderability with respect to Dehn surgery on a knot in $\mathbb{S}^3$.

We organize this paper as follows: Section 2 contains background and results relating to circular-orderability and left-orderability of groups in general. In Section 3 we relate these facts to 3-manifold fundamental groups, discuss the L-space conjecture and prove Theorem 1.1. In Section 4 we introduce our tools that are analogous to slope detection by left-orderings, and in Section 5 we show how these results can be applied to fillings along rational longitudes. In Section 6 we study circular-orderability of the fundamental groups of Seifert fibred manifolds and graph manifolds. Finally, in Section 7 we discuss circular-orderability of the fundamental groups of manifolds arising as the cyclic branched covers of links, and manifolds arising from Dehn surgery.

2   Left- and circular-orderability

A strict total order $<$ on a group G is said to be a *left-ordering* if, for every $f, g, h \in G$, if $g < h$ then $fg < fh$. A group G is called *left-orderable* if it admits a left-ordering. Every left-ordering of G determines a subset $P = \{g \in G \mid g > \mathrm{id}\}$ called the *positive cone* of the ordering, it satisfies

  (i) $P \cdot P \subset P$, and

  (ii) $P \sqcup P^{-1} = G \setminus \{\mathrm{id}\}$.

Conversely any subset $P \subset G$ satisfying (i) and (ii) determines a left-ordering of G via the prescription

$$g < h \iff g^{-1}h \in P.$$

A *left-circular ordering* of a group G is a map $c \colon G^3 \to \{\pm 1, 0\}$ satisfying:

  (1)   If $(g_1, g_2, g_3) \in G^3$, then $c(g_1, g_2, g_3) = 0$ if and only if $\{g_1, g_2, g_3\}$ are not all distinct.

  (2)   For all $g_1, g_2, g_3, g_4 \in G$, we have

$$c(g_1, g_2, g_3) - c(g_1, g_2, g_4) + c(g_1, g_3, g_4) - c(g_2, g_3, g_4) = 0.$$

(3)   For all $g, g_1, g_2, g_3 \in G$, we have

$$c(g_1, g_2, g_3) = c(gg_1, gg_2, gg_3).$$

If G admits such a map, then G is called *left-circularly orderable*. When no confusion will arise from doing so, we will write *circular ordering* for short and *circularly orderable*.

Every left-orderable group is circularly orderable, for if $<$ is a left-ordering of G then we may define $c \colon G^3 \to \{\pm 1, 0\}$ by $c(g_1, g_2, g_3) = \mathrm{sign}(\sigma)$ when $\{g_1, g_2, g_3\}$ are distinct and $c(g_1, g_2, g_3) = 0$ otherwise; here σ is the unique permutation such that $g_{\sigma(1)} < g_{\sigma(2)} < g_{\sigma(3)}$. When a circular ordering c of a left-orderable group G arises in this way, we will say that c is a *secret left-ordering*.

Every circular ordering c of G is evidently a homogeneous cocycle. However, from each circular ordering c, we can define an associated inhomogeneous cocycle $f_c \colon G^2 \to \{0, 1\}$ by

$$f_c(g, h) = \begin{cases} 0 & \text{if } g = \mathrm{id} \text{ or } h = \mathrm{id}, \\ 1 & \text{if } gh = \mathrm{id} \text{ and } g \neq \mathrm{id}, \\ \frac{1}{2}(1 - c(\mathrm{id}, g, gh)) & \text{otherwise}; \end{cases}$$

we call $[f_c] \in H^2(G; \mathbb{Z})$ the *Euler class* of the circular ordering c.

Construction 2.1   [54]   Associated to $[f_c]$ is a central extension $\widetilde{G}_c$ of G, which is constructed by equipping the set $\mathbb{Z} \times G$ with the operation $(a, g)(b, h) = (a + b + f_c(g, h), gh)$.[1] The central extension $\widetilde{G}_c$ is easily seen to be left-orderable, as one can check that the set $P = \{(a, g) \mid a \geq 0\} \setminus \{(0, \mathrm{id})\}$ defines the positive cone of a left-ordering, which we denote by $<_c$. We call $\widetilde{G}_c$ the *left-ordered central extension associated to the circularly ordered group G with ordering c*.

Recall that a subset S of a left-ordered group $(G, <)$ is $<$-*cofinal* if, for every $g \in G$, there exist elements $s, t \in S$ such that $s < g < t$. An element $g \in G$ is called $<$-cofinal (or simply cofinal if the ordering is understood) whenever the cyclic subgroup $\langle g \rangle$ is $<$-cofinal as a set. The central element $(1, \mathrm{id}) \in \widetilde{G}_c$ is positive and cofinal in the left-ordering $<_c$ of $\widetilde{G}_c$ and is called the *canonical positive, cofinal, central element of $\widetilde{G}_c$*.

Construction 2.2   [54]   The above construction is reversible, in a categorical sense made precise in [17]; the basic construction is as follows. Suppose that G is a left-ordered group with ordering $<$, and there is a central element $z \in G$ which is positive and $<$-cofinal. Then the quotient $G/\langle z \rangle$ inherits a circular ordering defined as follows. For each $g\langle z \rangle \in G/\langle z \rangle$, define the *minimal representative* $\bar{g}$ to be the unique coset representative of $g\langle z \rangle$ satisfying $\mathrm{id} \leq \bar{g} < z$. Then define a circular ordering $c_<$ on $G/\langle z \rangle$ by

$$c_<(g_1\langle z \rangle, g_2\langle z \rangle, g_3\langle z \rangle) = \mathrm{sign}(\sigma),$$

where σ is the unique permutation satisfying $\bar{g}_{\sigma(1)} < \bar{g}_{\sigma(2)} < \bar{g}_{\sigma(3)}$.

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[1]This is just an application of the standard construction associating elements of $H^2(G; \mathbb{Z})$, represented by inhomogeneous 2-cocycles, to equivalence classes of central extensions $1 \to \mathbb{Z} \to \widetilde{G} \to G \to 1$.

When G admits a circular ordering c with $[f_c] = \mathrm{id} \in H^2(G; \mathbb{Z})$, then G is left-orderable, because the left-orderable central extension $\widetilde{G}_c$ is isomorphic to $G \times \mathbb{Z}$ (though the ordering c need not be a secret left-ordering for this to happen). It happens that c is a secret left-ordering if and only if $[f_c] = \mathrm{id} \in H_b^2(G; \mathbb{Z})$, where $H_b^2(G; \mathbb{Z})$ is the second bounded cohomology group (see [5] for details).

We also recall the notion of *rotation number* of an orientation-preserving homeomorphism $f: \mathbb{S}^1 \to \mathbb{S}^1$, which is connected to a circular ordering and the lift $\widetilde{G}_c$ as follows. For an orientation-preserving homeomorphism $f: \mathbb{S}^1 \to \mathbb{S}^1$, one may choose a preimage $\widetilde{f} \in \widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$ of $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$ and define the rotation number of f to be

$$\lim_{n \to \infty} \frac{\widetilde{f}^n(0)}{n} \quad \mathrm{mod}\ \mathbb{Z}.$$

We can define the rotation number for an element g of a circularly ordered group (G, c), following [2], in a similar way. Let $z = (1, \mathrm{id}) \in \widetilde{G}_c$ denote the cofinal, central element of $\widetilde{G}_c$ relative to the ordering $<_c$. Choose an element $\widetilde{g} \in \widetilde{G}_c$ such that $q(\widetilde{g}) = g$, where $q: \widetilde{G}_c \to G$ is the quotient map. For each $n \in \mathbb{Z}$, let a_n denote the unique integer such that

$$z^{a_n} \leq \widetilde{g}^n < z^{a_n + 1},$$

and define

$$\mathrm{rot}_c(g) = \lim_{n \to \infty} \frac{a_n}{n} \quad \mathrm{mod}\ \mathbb{Z}.$$

Note that this limit always exists by Fekete's lemma, as one can check that the sequence $\{a_n\}_{n=1}^\infty$ is superadditive. Is it not difficult, though rather tedious, to show that this notion of rotation number agrees with the "traditional definition" if one uses the circular ordering c of G to create a dynamical realization $\rho_c: G \to \mathrm{Homeo}_+(\mathbb{S}^1)$ such that the circular ordering c of G agrees with the circular ordering of the orbit of 0 inherited from the natural circular ordering of $\mathbb{S}^1$ (see [19, Sections 2.2 and 2.4]). In particular, this implies that rotation number is invariant under conjugation, and is a homomorphism from $A \to S^1$ when restricted to any abelian subgroup $A \subset G$. Moreover, the induced homomorphism $A/\ker(\mathrm{rot}_c) \to \mathbb{S}^1$ is order-preserving with left-ordered kernel; see [27, Propositions 5.3 and 6.17; 21, Section 2].

A fundamental tool in constructing circular orderings on a given group G is the lexicographic construction, which we use often throughout this work.

Proposition 2.3 *Let*

$$1 \to K \to G \xrightarrow{q} H \to 1$$

be a short exact sequence of groups.

(1) *If K and H are left-orderable, then G is left-orderable.*

(2) *If K is left-orderable and H admits a circular ordering d, then G admits a circular ordering c satisfying $\mathrm{rot}_d(q(g)) = \mathrm{rot}_c(g)$ for all $g \in G$, and whose restriction to K is secretly a left-ordering.*

Proof   Claim (1) is a straightforward exercise and is common in the literature. Claim (2) is less common, so we outline a lexicographic construction following [15, Lemma 2.2.12] and verify the claimed properties

of the resulting circular ordering. Suppose K is equipped with the left-ordering $<$. Define a circular ordering $c\colon G \to \{0, \pm 1\}$ as follows. Given three distinct elements $g_1, g_2, g_3 \in G$:

Case 1 If $q(g_i)$ are all distinct, set $c(g_1, g_2, g_3) = d(q(g_1), q(g_2), q(g_3))$.

Case 2 If exactly two of $\{q(g_1), q(g_2), q(g_3)\}$ are equal, we may (by cyclically permuting the arguments and relabelling if necessary) assume that $q(g_1) = q(g_2)$, in which case we declare $c(g_1, g_2, g_3) = 1$ if $g_1^{-1} g_2 > \mathrm{id}$ and $c(g_1, g_2, g_3) = -1$ otherwise.

Case 3 If all of $\{q(g_1), q(g_2), q(g_3)\}$ are equal, then declare $c(g_1, g_2, g_3) = 1$ if and only if $\mathrm{id} < g_1^{-1} g_2 < g_1^{-1} g_3$, up to cyclic permutation.

Note that, if $g_1, g_2, g_3 \in K$, then we define $c(g_1, g_2, g_3)$ by appealing to Case 3, so $c(g_1, g_2, g_3) = 1$ if and only if $g_1 < g_2 < g_3$ up to cyclic permutation. Thus the circular ordering c is a secret left-ordering upon restriction to K.

That $\mathrm{rot}_d(q(g)) = \mathrm{rot}_c(g)$ for all $g \in G$ is proved in [2, Proof of Proposition 4.10]. □

Left-orderability and circular-orderability are also well behaved with respect to free products:

Proposition 2.4 *Let $\{G_i\}_{i \in I}$ be a family of groups. Then*:

(1) [53] *The free product $\bigast_{i \in I} G_i$ is left-orderable if and only if each group G_i is left-orderable. Moreover, if $<_i$ is a left-ordering of G_i for each $i \in I$, then there exists a left-ordering of $\bigast_{i \in I} G_i$ extending the orderings $<_i$.*

(2) [3, Theorem 4.2] *The free product $\bigast_{i \in I} G_i$ is circularly orderable if and only if each group G_i is circularly orderable. Moreover, if c_i is a circular ordering of G_i for each $i \in I$, then there exists a circular ordering of $\bigast_{i \in I} G_i$ extending the orderings c_i.*

Free products with amalgamation are much more finicky, with necessary and sufficient conditions that a free product with amalgamation be left-orderable (resp. circularly orderable) appearing in [6] (resp. [17]).

Tools to obstruct circular-orderability are somewhat rarer than the tools commonly used to obstruct left-orderability. One of the basic tools in this regard is the following fact:

Proposition 2.5 *A finite group is circularly orderable if and only if it is cyclic.*

For a proof from an algebraic point of view, see [18, Proposition 2.8]. A circular ordering c of G may also give rise to a left-orderable subgroup of G, depending on whether or not the corresponding Euler class $[f_c] \in H^2(G; \mathbb{Z})$ has finite order. This allows one to obstruct circular-orderability of a group G by reducing the problem to obstructing left-orderability of certain finite-index subgroups. Calegari and Dunfield [16] use a variant of this theorem, for instance, to show that the fundamental group of the Weeks manifold is not circularly orderable.

Theorem 2.6 *Suppose that c is a circular ordering of G whose Euler class $[f_c] \in H^2(G; \mathbb{Z})$ has order k. Then G contains a left-orderable normal subgroup H such that $G/H \cong \mathbb{Z}/k\mathbb{Z}$.*

Proof Consider the cocycle $kf_c \colon G^2 \to \{0, k\}$ defined by taking k times the inhomogeneous cocycle f_c, and the corresponding central extension $\widetilde{G}_{kf_c}$ constructed as $\mathbb{Z} \times G$ with multiplication $(a, g)(b, h) = (a + b + kf_c(g, h), gh)$. Define a map $\phi \colon \widetilde{G}_c \to \widetilde{G}_{kf_c}$ by $\phi(a, g) = (ka, g)$; one can check that this is an injective homomorphism. Since we assume $[f_c]$ has order k, we know that $[kf_c] = \mathrm{id} \in H^2(G; \mathbb{Z})$, and thus there exists a map $\eta \colon G \to \mathbb{Z}$ satisfying $\eta(\mathrm{id}) = 0$ and $kf_c(g, h) = \eta(g) - \eta(gh) + \eta(h)$ for all $g, h \in G$. This map η allows us to define an isomorphism $\psi \colon \widetilde{G}_{kf_c} \to \mathbb{Z} \times G$ by $\psi(a, g) = (a + \eta(g), g)$.

Let H denote the subgroup $(\psi \circ \phi)(\widetilde{G}_c) \cap (\{0\} \times G)$ of $\{0\} \times G$, meaning that

$$H = \{(0, g) \mid \exists a \in \mathbb{Z} \text{ such that } ka + \eta(g) = 0\}.$$

Then H is clearly left-orderable since it is the image under an injective map of a left-orderable group.

Let $q_k \colon \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ denote the quotient map. To the equation $\eta(gh) = \eta(g) + \eta(h) - kf_c(g, h)$ we apply the homomorphism q_k to arrive at $(q_k \circ \eta)(gh) = (q_k \circ \eta)(g) + (q_k \circ \eta)(h)$ and conclude that $q_k \circ \eta \colon G \to \mathbb{Z}/k\mathbb{Z}$ is a homomorphism. Moreover, $g \in \ker(q_k \circ \eta)$ if and only if $\eta(g) \equiv 0 \bmod k$, meaning that $(0, g) \in H$. Thus the obvious isomorphism $\{0\} \times G \cong G$ carries H to $\ker(q_k \circ \eta)$.

As $q_k \circ \eta$ has image in $\mathbb{Z}/k\mathbb{Z}$, it remains to argue that $q_k \circ \eta$ is surjective. Suppose the order of $\mathrm{im}(q_k \circ \eta)$ is m; then m divides k and $\frac{k}{m}$ is the largest divisor of k such that $\eta(g) \in \frac{k}{m}\mathbb{Z}$ for all $g \in G$. This means that the function $\zeta(g) = \frac{m}{k}\eta(g)$ satisfies $\zeta(g) - \zeta(gh) + \zeta(h) = mf_c(g, h)$, meaning that $[f_c] \in H^2(G; \mathbb{Z})$ has order dividing m. But $m \leq k$ and the order of $[f_c]$ is k, so this forces $m = k$, implying that $q_k \circ \eta$ is surjective. $\square$

3 Fundamental groups of 3-manifolds and orderability

First, we note that every compact 3-manifold other than $\mathbb{S}^3$ admits a decomposition

$$M \cong M_1 \# M_2 \# \cdots \# M_n$$

into prime 3-manifolds, and, as such, the fundamental group of a nonprime 3-manifold M can be expressed as a free product

$$\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_n).$$

In light of Proposition 2.4(2), the question of circular-orderability of fundamental groups of 3-manifolds reduces to considering the fundamental groups of prime 3-manifolds. In fact, since the only reducible orientable prime 3-manifold is $\mathbb{S}^1 \times \mathbb{S}^2$, whose fundamental group is clearly circularly orderable, in the case of orientable 3-manifolds it suffices to consider only the fundamental groups of irreducible orientable 3-manifolds. In the case of a nonorientable 3-manifold M, the first Betti number is positive whenever M is $\mathbb{P}^2$-irreducible, so M has circularly orderable fundamental group [14, Theorem 1.1].

Remark 3.1 On the other hand, if M is $\mathbb{P}^2$-reducible, then we do not know whether or not $\pi_1(M)$ is circularly orderable in general. Our techniques for producing circular orderings of 3-manifold groups make frequent use of algebraic properties that depend heavily on the situation at hand. For example, our main tools use left-orderability of infinite-index subgroups of $\pi_1(M)$ as in Proposition 3.2, or a decomposition of $\pi_1(M)$ as a free product with amalgamation of groups whose circular orderings are well understood as in Theorem 4.3 (whose proof also uses Proposition 3.2). Neither of these properties hold for the fundamental groups of $\mathbb{P}^2$-reducible manifolds; in particular, their fundamental groups always contain a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [25, Theorem 8.2], and so any argument that relies on left-orderability of infinite-index subgroups will fail.

It is well known that left-orderability behaves in a very special way with respect to fundamental groups of irreducible 3-manifolds. One of the key results in the area is the following, which we present alongside a generalization to the case of circular-orderability:

Proposition 3.2 *Let M be a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold, let G be a nontrivial group and suppose there exists an epimorphism $\rho\colon \pi_1(M) \to G$.*

(1) *If G is left-orderable, then $\pi_1(M)$ is left-orderable.*

(2) *If G is infinite and circularly orderable, then $\pi_1(M)$ is circularly orderable.*

Proof The first claim is [14, Theorem 3.2]. For the second, since $G = \rho(\pi_1(M))$ is infinite, $K = \ker(\rho)$ is an infinite-index normal subgroup of $\pi_1(M)$. From [14, Proof of Theorem 3.2], the subgroup K is therefore locally indicable and thus left-orderable, so we can use the short exact sequence $1 \to K \to \pi_1(M) \to G \to 1$ to construct a lexicographic circular ordering of $\pi_1(M)$ as in Proposition 2.3(2). $\square$

This implies, for instance, that $\pi_1(M)$ is left-orderable whenever M satisfies the hypotheses of Proposition 3.2 and $H_1(M; \mathbb{Z})$ is infinite. The case of interest is therefore when $H_1(M; \mathbb{Z})$ is finite, where the L-space conjecture posits a connection between left-orderability of $\pi_1(M)$, the existence of coorientable taut foliations in M, and whether or not M is a Heegaard Floer homology L-space (that is, a manifold whose Heegaard Floer homology is of minimal rank).

Conjecture 3.3 (the L-space conjecture [12; 37]) If M is an irreducible, rational homology 3-sphere other than $\mathbb{S}^3$, then the following are equivalent:[2]

(1) The fundamental group of M is left-orderable.

(2) The manifold M supports a coorientable taut foliation.

(3) The manifold M is not an L-space.

Surrounding this conjecture, there are many tools and techniques to obstruct left-orderability of fundamental groups — see [16; 24; 22; 1], to name a few — and many more to prove that fundamental groups are left-orderable. Our main contribution in this section is to connect circular-orderability to this conjecture

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[2]We require the caveat that $M \neq \mathbb{S}^3$ because, by our definition, the trivial group is left-orderable.

as in Theorem 1.1 of the introduction, by applying Theorem 2.6 and Proposition 3.2. Recall that a *finite cyclic cover* is, by definition, a regular covering space for which the group of deck transformations is a finite cyclic group.

Proof of Theorem 1.1   If $H_1(M; \mathbb{Z})$ is infinite, then there exists a surjection $\pi_1(M) \to \mathbb{Z}$. Therefore $\pi_1(M)$ is left-orderable by [14, Theorem 1.1]. It follows that $\pi_1(M)$ is circularly orderable, and all finite cyclic covers (including the trivial cover) have left-orderable fundamental group, so there is nothing to prove in this case.

On the other hand, suppose $H_1(M; \mathbb{Z})$ is finite and $\pi_1(M)$ is circularly orderable with circular ordering c. Note that, by a standard Euler characteristic argument (see for example [14, Lemma 3.3]), either M is closed and orientable, or M has nonempty boundary containing only $\mathbb{S}^2$ and $\mathbb{P}^2$ components. Since M is $\mathbb{P}^2$-irreducible, the latter case does not occur.

We conclude $H_1(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z})$ by Poincaré duality, and then $H^2(M; \mathbb{Z}) \cong H^2(\pi_1(M); \mathbb{Z})$ since M is irreducible. Thus $[f_c]$ has finite order; say it has order k. Then, by Theorem 2.6, $\pi_1(M)$ admits a normal, left-orderable subgroup H such that $\pi_1(M)/H$ is cyclic. In this case, the cover $\widetilde{M}$ of M with $\pi_1(\widetilde{M}) = H$ has the desired properties.

To show the converse, suppose that $p \colon \widetilde{M} \to M$ is a finite cyclic cover with left-orderable fundamental group. Then there is a short exact sequence

$$1 \to p_*(\pi_1(\widetilde{M})) \to \pi_1(M) \to \mathbb{Z}/k\mathbb{Z} \to 1$$

for some $k \geq 1$, where the kernel is left-orderable and the quotient (if nontrivial) is circularly orderable. If the quotient is trivial, this means that $\pi_1(M)$ itself is left-orderable, and the conclusion follows. If the quotient is nontrivial, then Proposition 2.3(2) finishes the proof.   □

Circular-orderability is therefore one possible approach to the L-space conjecture, by first tackling the conjecture up to finite cyclic covers. That is, we have the following "circular-orderability" version of Conjecture 3.3:

Conjecture 3.4   (the L-space conjecture, circular-orderability version)   If M is an irreducible, rational homology 3-sphere that is not a lens space, then the following are equivalent:[3]

  (1)   The fundamental group of M is circularly orderable.

  (2)   There exists a finite cyclic cover $\widetilde{M}$ of M that supports a coorientable taut foliation.

  (3)   There exists a finite cyclic cover $\widetilde{M}$ of M that is not an L-space.

Note that this has conjectural implications beyond what would follow from the "left-orderability" version of the conjecture. For instance, the following theorem connects certain topological properties directly to circular-orderability (without passing via left-orderability):

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[3]If we allow M to be a lens space, then the conjecture as stated would not be true, again because the trivial group is left-orderable by our conventions.

Theorem 3.5 [16, Theorem 3.1, Corollary 3.9] *If M is an orientable, atoroidal 3-manifold containing a very full tight essential lamination or supporting a pseudo-Anosov flow, then $\pi_1(M)$ is circularly orderable.*

Conjecture 3.4 therefore predicts that one can obstruct the existence of a pseudo-Anosov flow on a rational homology 3-sphere M, or the existence of a very full tight essential lamination in M, by showing, for example, that all finite cyclic covers are L-spaces or that no finite cyclic cover supports a coorientable taut foliation.

4 Slope detection and 3-manifolds having nontrivial JSJ decomposition

One of the key techniques in the analysis of manifolds admitting a nontrivial JSJ decomposition is that of *slope detection* by left-orderings. This idea first appeared in a basic form in [20, Definition 2.6], was further developed in [9] as a key technique in proving Conjecture 3.3 for graph manifolds [9; 31], and appears also in [10; 11]. We recall the central technique from [20] in the following theorem:

Theorem 4.1 [20, Theorem 2.7] *Suppose that M_1 and M_2 are 3-manifolds with incompressible torus boundaries ∂M_i, and that $\phi \colon \partial M_1 \to \partial M_2$ is a homeomorphism such that $W = M_1 \cup_\phi M_2$ is irreducible. If there exists a slope α such that both $\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$ are left-orderable, then $\pi_1(W)$ is left-orderable.*

In what follows, we develop a generalization of this technique that applies to circular-orderability and present applications to various classes of manifolds. To prepare, we recall the following construction.

If (G, c) and (H, d) are circularly ordered groups and $\phi \colon G \to H$ is a homomorphism satisfying

$$c(g_1, g_2, g_3) = d(\phi(g_1), \phi(g_2), \phi(g_3)) \quad \text{for all } g_1, g_2, g_3 \in G,$$

then we say that ϕ is *order-preserving* or *compatible with the pair* (c, d). Note that in this case ϕ is necessarily injective. Then we can define $\widetilde{\phi} \colon \widetilde{G}_c \to \widetilde{H}_d$ by $\widetilde{\phi}(n, g) = (n, \phi(g))$, so that $\widetilde{\phi}$ is order-preserving with respect to the left-orderings $<_c$ and $<_d$ of $\widetilde{G}_c$ and $\widetilde{H}_d$, respectively (or *compatible with the pair* $(<_c, <_d)$).

Proposition 4.2 *Suppose that $\{(G_i, c_i)\}_{i \in I}$ are circularly ordered groups each containing a subgroup H_i. If (D, d) is a cyclic circularly ordered group and $\phi_i \colon D \to H_i$ is an isomorphism compatible with the pair (d, c_i) for every i, then the free product with amalgamation $\bigstar_{i \in I} G_i (D \cong_{\phi_i} H_i)$ is circularly orderable.*

Proof The homomorphism $\phi_i \phi_j^{-1}$ is compatible with the pair (c_j, c_i) for all $i, j \in I$, and therefore the map $\widetilde{\phi_i \phi_j^{-1}}$ is also compatible with the pair $(<_{c_j}, <_{c_i})$ for all $i, j \in I$.

Now suppose that $D \cong \mathbb{Z}$, in which case $\widetilde{D}_d \cong \mathbb{Z} \times \mathbb{Z}$ and the images $\widetilde{\phi}_i(\widetilde{D}_d)$ and $\widetilde{\phi}_j(\widetilde{D}_d)$ are bounded neither from above nor below in $\widetilde{(G_i)}_{c_i}$ and $\widetilde{(G_j)}_{c_j}$, respectively, since each image contains the cofinal central element of the respective extension. We may therefore apply [17, Proposition 5.6] to conclude that

the group $\divideontimes_{i \in I} (\widetilde{G_i})_{c_i} (\widetilde{D}_d \cong_{\widetilde{\phi}_i} (\widetilde{H_i})_{c_i})$ is left-orderable, and then apply [17, Theorem 1] to conclude that $\divideontimes_{i \in I} G_i (D \cong_{\phi_i} H_i)$ is circularly orderable in this case.

Next suppose that D is finite, in which case $\widetilde{D}_d$ is infinite cyclic. Then $\divideontimes_{i \in I} (\widetilde{G_i})_{c_i} (\widetilde{D}_d \cong_{\widetilde{\phi}_i} (\widetilde{H_i})_{c_i})$ is an amalgamation of left-orderable groups along a cyclic subgroup, which is always left-orderable [6]. As above, that $\divideontimes_{i \in I} G_i (D \cong_{\phi_i} H_i)$ is circularly orderable then follows from [17, Theorem 1]. $\square$

In preparation for the next theorem, suppose that G is a group and let $g \in G$. We denote by $\langle\!\langle g \rangle\!\rangle$ the normal closure of $g \in G$, that is, the smallest (with respect to inclusion) normal subgroup of G containing g.

Theorem 4.3 *Let M_1 and M_2 be two 3-manifolds with incompressible torus boundaries, and let $\phi \colon \partial M_1 \to \partial M_2$ be a homeomorphism such that $M = M_1 \cup_\phi M_2$ is $\mathbb{P}^2$-irreducible. If there exists a slope $\alpha \in H_1(\partial M_1; \mathbb{Z})/\{\pm 1\}$ such that $\pi_1(M_1(\alpha))$ and $\pi_1(M_2(\phi_*(\alpha)))$ are infinite circularly orderable groups, and either*

(1) *at least one of $\pi_1(\partial M_1) \subset \langle\!\langle \alpha \rangle\!\rangle$ or $\pi_1(\partial M_2) \subset \langle\!\langle \phi_*(\alpha) \rangle\!\rangle$ holds, or*

(2) *$\pi_1(M_1(\alpha))$ and $\pi_1(M_2(\phi_*(\alpha)))$ admit circular orderings c_1 and c_2, respectively, such that the induced map*

$$\overline{\phi}_* \colon \pi_1(\partial M_1)/(\langle\!\langle \alpha \rangle\!\rangle \cap \pi_1(\partial M_1)) \to \pi_1(\partial M_2)/(\langle\!\langle \phi_*(\alpha) \rangle\!\rangle \cap \pi_1(\partial M_2))$$

is an isomorphism between nontrivial groups which is compatible with the pair (c_1, c_2),

then $\pi_1(M)$ is circularly orderable.

Proof Suppose first that there exists a slope α satisfying condition (1) of the theorem; without loss of generality we assume that $\pi_1(\partial M_1) \subset \langle\!\langle \alpha \rangle\!\rangle$. To simplify notation, let $G_i = \pi_1(M_i)$ for $i = 1, 2$, each equipped with an inclusion homomorphism $f_i \colon \mathbb{Z} \oplus \mathbb{Z} \to G_i$ that identifies the peripheral subgroup $\pi_1(\partial M_i)$ with $\mathbb{Z} \oplus \mathbb{Z}$, satisfying $\phi_* \circ f_1 = f_2$, and let $q_1 \colon G_1 \to G_1/\langle\!\langle \alpha \rangle\!\rangle$ and $q_2 \colon G_2 \to G_2/\langle\!\langle \phi_*(\alpha) \rangle\!\rangle$ be the quotient maps.

In this case, there exists a unique map f such that the following diagram commutes:

As the image of the map f is an infinite circularly orderable group, that $\pi_1(M) \cong G_1 *_{\phi_*} G_2$ is circularly orderable follows from Proposition 3.2(2).

Suppose there exists a slope α satisfying condition (2) of the theorem. Let $\overline{\phi}_*$ denote the map induced by ϕ, as in the statement of the theorem. There are two possibilities for the subgroup $\langle\!\langle \alpha \rangle\!\rangle \cap \pi_1(\partial M_1)$ of

$\pi_1(\partial M_1) \cong \mathbb{Z} \oplus \mathbb{Z}$: it is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \oplus n\mathbb{Z}$ for some $n > 1$. In both cases, $\bar{\phi}_*$ is an isomorphism between cyclic subgroups of G_1 and G_2 that is compatible with (c_1, c_2), and so $G_1/\langle\!\langle \alpha \rangle\!\rangle *_{\bar{\phi}_*} G_2/\langle\!\langle \phi_*(\alpha) \rangle\!\rangle$ is circularly orderable by Proposition 4.2.

Then, as before, there is a unique map f such that the following diagram commutes:

Now, since the groups $\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$ are infinite, so is $G_1/\langle\!\langle \alpha \rangle\!\rangle *_{\bar{\phi}_*} G_2/\langle\!\langle \phi_*(\alpha) \rangle\!\rangle$. That $\pi_1(M) \cong G_1 *_{\phi_*} G_2$ is circularly orderable follows from Proposition 3.2. □

Note that this recovers Theorem 4.1 in the case that the quotients are both left-orderable.

Recall that $\mathrm{rot}_c \colon G \to \mathbb{S}^1$ is an order-preserving homomorphism upon restriction to any abelian subgroup, so long as it is injective. From this observation and Theorem 4.3, we arrive at the following:

Corollary 4.4 *Let M_1, M_2, $\phi \colon \partial M_1 \to \partial M_2$ and $M = M_1 \cup_\phi M_2$ be as in Theorem 4.3, and $\alpha \in H_1(\partial M_1; \mathbb{Z})/\{\pm 1\}$ be such that $\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$ are infinite circularly orderable groups. Let $\beta \in \pi_1(\partial M_1)$ denote a dual class to α, and let $q_1 \colon \pi_1(M_1) \to \pi_1(M_1(\alpha))$ and $q_2 \colon \pi_1(M_2) \to \pi_1\big(M_2(\phi_*(\alpha))\big)$ denote the quotient maps. If there exist circular orderings c_1 and c_2 of $\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$, respectively, such that*

$$c_1(q_1(\beta^j), q_1(\beta^k), q_1(\beta^l)) = c_2\big(q_2(\phi_*(\beta^j)), q_2(\phi_*(\beta^k)), q_2(\phi_*(\beta^l))\big)$$

for all $j, k, l \in \mathbb{Z}$, then $\pi_1(M)$ is circularly orderable. In particular, if

$$\mathrm{rot}_{c_1}(q_1(\beta)) = \mathrm{rot}_{c_2}\big(q_2(\phi_*(\beta))\big)$$

and rot_{c_i} are injective, then $\pi_1(M)$ is circularly orderable.

Applications of Theorem 4.3 or Corollary 4.4 therefore hinge upon being able to construct circular orderings of fundamental groups of 3-manifolds where certain elements have prescribed rotation number.

5 Rational longitudes and knot manifolds

Recall that a knot manifold is a compact, connected, irreducible and orientable 3-manifold with boundary an incompressible torus. In this section we demonstrate a technique for creating circular orderings of fundamental groups of knot manifolds, where the cyclic subgroup generated by a class dual to the rational longitude has a prescribed circular ordering.

Lemma 5.1 *Suppose that C is a cyclic group, D is a nontrivial subgroup of C, and c is a circular ordering of D. Then there exists a circular ordering c' of C such that*

$$c'(r, s, t) = c(r, s, t) \quad \text{for all } r, s, t \in D.$$

Proof We first consider the case where $C = \mathbb{Z}$ is infinite cyclic and $D = k\mathbb{Z}$. Let c be an arbitrary circular ordering of $k\mathbb{Z}$ and let d denote the standard circular ordering of $\mathbb{S}^1$. Consider the rotation number homomorphism $\mathrm{rot}_c : k\mathbb{Z} \to \mathbb{S}^1$ corresponding to the circular ordering c. Suppose that $\mathrm{rot}_c(k) = \exp(2\pi i \alpha)$ with $\alpha \in [0, 1]$, and define $\phi : \mathbb{Z} \to \mathbb{S}^1$ by $\phi(1) = \exp(2\pi i \alpha / k)$. There are two cases to consider.

First, if rot_c is injective, then $c(r, s, t) = d(\mathrm{rot}_c(r), \mathrm{rot}_c(s), \mathrm{rot}_c(t))$ for all $r, s, t \in k\mathbb{Z}$. Note that ϕ is injective because rot_c is injective, so we can define a circular ordering c' on $\mathbb{Z}$ by $c'(r, s, t) = d(\phi(r), \phi(s), \phi(t))$, which clearly extends c as required.

On the other hand, suppose that rot_c is not injective; say $\alpha = p/q$ with $\gcd(p, q) = 1$. Then c arises lexicographically from a short exact sequence

$$1 \to H \to k\mathbb{Z} \xrightarrow{\mathrm{rot_c}} \mathbb{Z}/q\mathbb{Z} \to 1,$$

where $\mathbb{Z}/q\mathbb{Z}$ is identified naturally with the qth roots of unity and equipped with the restriction of the natural circular ordering d of $\mathbb{S}^1$. In this case ϕ yields a short exact sequence

$$1 \to H \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/qk\mathbb{Z} \to 1,$$

where $\mathbb{Z}/qk\mathbb{Z}$ is again equipped with the natural circular ordering arising from the natural embedding into $\mathbb{S}^1$. Thus, if we use the latter short exact sequence to lexicographically define a circular ordering c' of $\mathbb{Z}$, using the same left-ordering of H as in the former short exact sequence, then c' will be an extension of the given circular ordering c of $k\mathbb{Z}$.

When C is finite, rot_c is injective, so we can use the same construction as in the first case above. $\square$

The next lemma is a standard result, but it is essential to our arguments and so we include a proof.

Lemma 5.2 *Let M be a compact, connected, orientable 3-manifold with a torus boundary, and $\mathbb{F}$ be a field. If $i : \partial M \to M$ denotes the inclusion map, then the image of the map*

$$i_*^1 : H_1(\partial M; \mathbb{F}) \to H_1(M; \mathbb{F})$$

is of rank one.

Proof Considering the pair $(M, \partial M)$, we have the long exact sequence

$$\cdots \to H_1(\partial M; \mathbb{F}) \xrightarrow{i_*^1} H_1(M; \mathbb{F}) \xrightarrow{p_*^1} H_1(M, \partial M; \mathbb{F}) \xrightarrow{\partial_1} H_0(\partial M; \mathbb{F}) \xrightarrow{i_*^0} H_0(M; \mathbb{F}) \xrightarrow{\partial_0} 0.$$

Since M is connected, it is also path-connected, and hence $H_0(M, \partial M; \mathbb{F}) = 0$. By exactness, we have $\mathrm{im}(i_*^1) = \ker(p_*^1)$, $\mathrm{im}(p_*^1) = \ker(\partial_1)$, $\mathrm{im}(\partial_1) = \ker(i_*^0)$ and $\mathrm{im}(i_*^0) = H_0(M; \mathbb{F})$. By the rank–nullity theorem,

$$\dim_{\mathbb{F}}(\mathrm{im}(i_*^1)) = \dim_{\mathbb{F}}(H_1(M; \mathbb{F})) - \dim_{\mathbb{F}}(H_1(M, \partial M; \mathbb{F})) + \dim_{\mathbb{F}}(H_0(\partial M; \mathbb{F})) - \dim_{\mathbb{F}}(H_0(M; \mathbb{F})).$$

Since M is compact, by universal coefficient theorems and duality, we have

$$\dim_{\mathbb{F}}(H_1(M, \partial M; \mathbb{F})) = \dim_{\mathbb{F}}(H_2(M; \mathbb{F})).$$

Hence,

$$\dim_{\mathbb{F}}(\mathrm{im}(i_*^1)) = \dim_{\mathbb{F}}(H_1(M; \mathbb{F})) - \dim_{\mathbb{F}}(H_1(M, \partial M; \mathbb{F})) + \dim_{\mathbb{F}}(H_0(\partial M; \mathbb{F})) - \dim_{\mathbb{F}}(H_0(M; \mathbb{F}))$$

$$= -\chi(M) + \dim_{\mathbb{F}}(H_3(M; \mathbb{F})) + \dim_{\mathbb{F}}(H_0(\partial M; \mathbb{F})).$$

Since the boundary of M is not empty, $\dim_{\mathbb{F}}(H_3(M; \mathbb{F})) = 0$. Thus,

$$\dim_{\mathbb{F}}(\mathrm{im}(i_*^1)) = -\chi(M) + \dim_{\mathbb{F}}(H_0(\partial M; \mathbb{F})).$$

Since $\chi(M) = \frac{1}{2}\chi(\partial M) = 0$, we conclude that $\dim_{\mathbb{F}}(\mathrm{im}(i_*^1)) = \dim_{\mathbb{F}}(H_0(\partial M; \mathbb{F})) = 1$. $\qquad\square$

It follows that the image of $i_*^1 \colon H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ is also of rank one when M is orientable. The unique primitive element in $H_1(\partial M; \mathbb{Z})$ whose image is of finite order in $H_1(M; \mathbb{Z})$ is referred to as the *rational longitude* of M, and is denoted λ_M.

Remark 5.3 Consider the long exact sequence, with M as in the previous lemma,

$$\cdots \to H_1(\partial M; \mathbb{Z}) \xrightarrow{i_*^1} H_1(M; \mathbb{Z}) \xrightarrow{p_*^1} H_1(M, \partial M; \mathbb{Z}) \xrightarrow{\partial_1} H_0(\partial M; \mathbb{Z}) \xrightarrow{i_*^0} H_0(M; \mathbb{Z})$$
$$\xrightarrow{\partial_0} H_0(M, \partial M; \mathbb{Z}) \to 0.$$

Since M and ∂M are both connected, they are also path-connected, and hence $H_0(M, \partial M; \mathbb{Z}) = 0$, $H_0(\partial M; \mathbb{Z}) \cong \mathbb{Z}$ and $H_0(M; \mathbb{Z}) = \mathbb{Z}$. Hence i_*^0 is an isomorphism, and $\mathrm{im}(\partial_1) = \ker(i_*^0) = 0$. This implies that $\mathrm{im}(p_*^1) = \ker(\partial_1) = H_1(M, \partial M; \mathbb{Z})$, and that i_*^1 is surjective if and only if $H_1(M, \partial M; \mathbb{Z}) = 0$. Thus we do not assume surjectivity of i_*^1, which necessitates the use of Lemma 5.1 in the proofs below.

Remark 5.4 If M is a compact, nonorientable, $\mathbb{P}^2$-irreducible 3-manifold, then $\pi_1(M)$ is left-orderable by [14, Lemma 3.3 and Theorem 3.1]; on the other hand, if M is $\mathbb{P}^2$-reducible, then our techniques do not apply in general; see Remark 3.1. Hence, from now on, we assume that all 3-manifolds are orientable.

Proposition 5.5 *Suppose that M is a knot manifold, let μ be the class of any closed curve in ∂M that is dual to λ_M and let q denote the quotient map $q \colon \pi_1(M) \to \pi_1(M(\lambda_M))$. If $M(\lambda_M)$ is irreducible, then, for every circular ordering c' of the cyclic subgroup $\langle q(\mu) \rangle$, there exists a circular ordering c of $\pi_1(M(\lambda_M))$ such that*

$$c'(q(\mu^j), q(\mu^k), q(\mu^l)) = c(q(\mu^j), q(\mu^k), q(\mu^l))$$

for all $j, k, l \in \mathbb{Z}$.

Proof For such a manifold M, it follows that $|H_1(M(\lambda_M); \mathbb{Z})|$ is infinite, with the class of μ being of infinite order by Lemma 5.2. Therefore there exists a map $\psi \colon \pi_1(M(\lambda_M)) \to \mathbb{Z}$ with the image of μ being nontrivial; say $\psi(\mu) = k$. Let c' be a given circular ordering of $\langle q(\mu) \rangle$.

Denote the standard circular ordering of $\mathbb{S}^1$ by d and suppose that $\mathrm{rot}_{c'}(q(\mu)) = r$. Define a map $\phi \colon k\mathbb{Z} \to \mathbb{S}^1$ by $\phi(k) = \exp(2\pi i r)$, and first suppose that $\mathrm{rot}_{c'}$ is injective. Then ϕ is injective, so we

may define a circular ordering c'' of $k\mathbb{Z}$ by $c''(r, s, t) = d(\phi(r), \phi(s), \phi(t))$. Next suppose that $\mathrm{rot}_{c'}$ is not injective, so that $r = s/t \in \mathbb{Q}$ (with s/t in lowest terms) and the circular ordering c' of $\langle q(\mu) \rangle$ is lexicographic with respect to the short exact sequence

$$0 \to \ker(\mathrm{rot}_{c'}) \to \langle q(\mu) \rangle \to \mathbb{Z}/t\mathbb{Z} \to 0$$

for some choice of left-ordering of $\ker(\mathrm{rot}_{c'})$. Then ϕ is not injective, and we may use Proposition 2.3(2) to create a circular ordering c'' of $k\mathbb{Z}$ from the sequence $1 \to K \to k\mathbb{Z} \xrightarrow{\phi} \mathbb{S}^1$, where K is the kernel of ϕ, such that our choice of left-ordering of K agrees with the left-ordering of $\ker(\mathrm{rot}_{c'})$ under the isomorphism $k\mathbb{Z} \cong \langle q(\mu) \rangle$ given by $k \mapsto q(\mu)$.

In either case, by Lemma 5.1, there is a circular ordering $\hat{c}$ of $\psi\big(\pi_1(M(\lambda_M))\big) \cong \mathbb{Z}$ that extends c''. That $\pi_1(M(\lambda_M))$ admits a circular ordering c with the required property now follows from Proposition 2.3(2) and the proof of Proposition 3.2(2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Following [50, pages 428–431], recall that a Seifert fibred space is a 3-manifold which is foliated by circles, called fibres, such that each circle S has a closed tubular neighbourhood which is a union of fibres and is isomorphic to a fibred solid torus if S preserves the orientation, or a fibred solid Klein bottle if S reverses the orientation. We use h throughout to denote the class of the regular fibre.

Proposition 5.6 *Suppose that M_1 and M_2 are compact, connected and orientable 3-manifolds with torus boundary, $\phi: \partial M_1 \to \partial M_2$ is a homeomorphism, and $M = M_1 \cup_\phi M_2$ is irreducible.*

(1) *If $\mathrm{rk}(H_1(M_1; \mathbb{Q})) \geq 2$, then $\pi_1(M)$ is left-orderable.*

(2) *Suppose $\mathrm{rk}(H_1(M_1; \mathbb{Q})) = 1$ and let λ_1 denote the rational longitude of M_1. If $\pi_1\big(M_2(\phi_*(\lambda_1))\big)$ is infinite and circularly orderable and either $M_1(\lambda_1)$ is irreducible or M_1 is Seifert fibred with incompressible boundary, then $\pi_1(M)$ is circularly orderable.*

Proof In what follows, let $i: \partial M_1 \to M_1$ denote the inclusion map.

To prove (1), assume $\mathrm{rk}(H_1(M_1; \mathbb{Q})) \geq 2$. In this case, the image of $H_1(\partial M_1; \mathbb{Z}) \xrightarrow{i_*} H_1(M_1; \mathbb{Z})$ has rank one by Lemma 5.2. Therefore we may compose the Hurewicz map $\pi_1(M_1) \to H_1(M; \mathbb{Z})$ with the quotient $H_1(M; \mathbb{Z}) \to H_1(M; \mathbb{Z})/\mathrm{im}(i_*)$ and obtain a map $\psi: \pi_1(M_1) \to A$, where A is an abelian group of positive rank, satisfying $\psi\big(i_*(\pi_1(\partial M_1))\big) = 0$. In this case, since M is a free product with amalgamation of $\pi_1(M_1)$ and $\pi_1(M_2)$, there exists a surjective homomorphism $\pi_1(M) \to A$ induced by $\psi: \pi_1(M_1) \to A$ and the zero homomorphism $\pi_1(M_2) \to A$. Thus $\pi_1(M)$ admits a torsion-free abelian quotient, so $\pi_1(M)$ is left-orderable by [14, Theorem 3.2].

For (2), suppose that $\mathrm{rk}(H_1(M_1; \mathbb{Q})) = 1$ and let $q_1: \pi_1(M_1) \to \pi_1(M_1(\lambda_1))$ and $q_2: \pi_1(M_2) \to \pi_1\big(M_2(\phi_*(\lambda_1))\big)$ denote the quotient maps. Equip $\pi_1\big(M_2(\phi_*(\lambda_1))\big)$ with a circular ordering c_2, and let μ_2 denote the class of a curve dual to $\phi(\lambda_1)$. Let μ_1 denote the class of a curve dual to λ_1. In this case,

if $M_1(\lambda_1)$ is irreducible, then, by Proposition 5.5, we can equip $\pi_1(M_1(\lambda_1))$ with a circular ordering c_1 that agrees with c_2 under the identification of $\langle q_1(\mu_1)\rangle$ and $\langle q_2(\mu_2)\rangle$ induced by ϕ_*. The result then follows from Corollary 4.4.

On the other hand, if M_1 is Seifert fibred, then $M_1(\lambda_1)$ is irreducible by [33] unless λ_1 is the class of a regular fibre, so we need only consider the case that λ_1 is the class of a regular fibre. For λ_1 to be the class of a regular fibre, M_1 must have nonorientable base orbifold. In this case, since $\mathrm{rk}(H_1(M_1;\mathbb{Q}))=1$ we know the base orbifold must be a once-punctured projective plane and we compute

$$\pi_1(M_1) = \langle a, \gamma_1, \ldots, \gamma_n, \mu, h \mid aha^{-1}=h^{-1}, [\gamma_i, h]=1, [\mu, h]=1, \gamma_i^{\alpha_i}=h^{\beta_i}, a^2\mu\gamma_1\ldots\gamma_n=1\rangle,$$

where μ is a class dual to $\lambda_1 = h$ on ∂M_1. Therefore

$$\pi_1(M_1(\lambda_1)) = \langle a, \gamma_1, \ldots, \gamma_n, \mu \mid \gamma_i^{\alpha_i}=1, a^2\mu\gamma_1\ldots\gamma_n=1\rangle,$$

which is isomorphic to the free product with amalgamation

$$\langle \gamma_1 \mid \gamma_1^{\alpha_1}=1\rangle * \cdots * \langle \gamma_n \mid \gamma_n^{\alpha_n}=1\rangle * \langle \mu \rangle *_{\mu\gamma_1\ldots\gamma_n=a^{-2}} \langle a \rangle.$$

In this case, we can first equip the infinite cyclic group $\langle \mu \rangle$ with a circular ordering c that agrees with c_2 under the identification of $\langle \mu \rangle$ and $\langle q_2(\mu_2)\rangle$ induced by ϕ_*. By Proposition 2.4, this circular ordering extends to the free product $\langle \gamma_1 \mid \gamma_1^{\alpha_1}=1\rangle * \cdots * \langle \gamma_n \mid \gamma_n^{\alpha_n}=1\rangle * \langle \mu \rangle$, which in turn extends to a circular ordering of $\pi_1(M_1(\lambda_1))$ by combining Lemma 5.1 and Proposition 4.2. Thus the claim follows from Corollary 4.4. $\qquad\square$

6 Seifert fibred manifolds and graph manifolds

Aside from this last tool, there are other situations where we can control the rotation numbers of certain elements in circular orderings of fundamental groups. We next investigate Seifert fibred manifolds, where our goal is to show that the fundamental group of a Seifert fibred space is always circularly orderable whenever it is infinite, and to describe the possible circular orderings in terms of the rotation numbers of the class of a regular fibre.

6.1 Seifert fibred manifolds

It was first claimed by Calegari [15, Remark 4.3.2] that, if M is Seifert fibred and $\pi_1(M)$ is infinite, then it is circularly orderable. We provide the details of this claim below.

Recall that, when M is an orientable Seifert fibred manifold over an orientable closed surface of genus $g \geq 0$, the fundamental group has presentation

$$\pi_1(M) = \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n, h \mid h \text{ central}, \gamma_i^{\alpha_i}=h^{\beta_i}, [a_1, b_1]\ldots[a_g, b_g]\gamma_1\ldots\gamma_n=h^b\rangle$$

and, if the surface is nonorientable, then

$$\pi_1(M) = \langle a_1, \ldots, a_g, \gamma_1, \ldots, \gamma_n, h \mid a_j h a_j^{-1}=h^{-1}, [\gamma_i, h]=1, \gamma_i^{\alpha_i}=h^{\beta_i}, a_1^2\ldots a_g^2\gamma_1\ldots\gamma_n=h^b\rangle.$$

In either case, quotienting by the normal subgroup $\langle h \rangle$ (the cyclic subgroup generated by the class of the regular fibre) yields the orbifold fundamental group of the underlying orbifold. Consequently we observe the following lemma:

Lemma 6.1 *Suppose that $\alpha_1, \ldots, \alpha_n \geq 2$ are integers and that $\mathcal{B}$ is an orbifold of type $\mathbb{S}^2(\alpha_1, \ldots, \alpha_n)$ or $\mathbb{P}^2(\alpha_1, \ldots, \alpha_n)$. Assume that*

(1) $n \geq 3$ *and* $\sum_{i=1}^{n} 1/\alpha_i < n - 2$ *if* $\mathcal{B} = \mathbb{S}^2(\alpha_1, \ldots, \alpha_n)$, *and*

(2) $n \geq 2$ *if* $\mathcal{B} = \mathbb{P}^2(\alpha_1, \ldots, \alpha_n)$.

Then $\pi_1^{\mathrm{orb}}(\mathcal{B})$ is circularly orderable.

Proof (1) With the assumptions in (1), as $\mathcal{B}$ is hyperbolic, the group $\pi_1^{\mathrm{orb}}(\mathcal{B})$ embeds in $\mathrm{PSL}_2(\mathbb{R})$. As $\mathrm{PSL}_2(\mathbb{R})$ embeds in $\mathrm{Homeo}_+(\mathbb{S}^1)$, it is circularly orderable, so the result follows.

(2) In this case, for the group $\mathbb{Z}/\alpha_1\mathbb{Z} \ast \cdots \ast \mathbb{Z}/\alpha_n\mathbb{Z}$, suppose that the generator of $\mathbb{Z}/\alpha_i\mathbb{Z}$ is x_i, and use x to denote the product $x_1 \ldots x_n$. Then

$$\pi_1^{\mathrm{orb}}(\mathcal{B}) = (\mathbb{Z}/\alpha_1\mathbb{Z} \ast \cdots \ast \mathbb{Z}/\alpha_n\mathbb{Z}) \ast_{\langle x \rangle = 2\mathbb{Z}} \mathbb{Z},$$

and the group $\mathbb{Z}/\alpha_1\mathbb{Z} \ast \cdots \ast \mathbb{Z}/\alpha_n\mathbb{Z}$ is circularly orderable by Proposition 2.4. That $\pi_1^{\mathrm{orb}}(\mathcal{B})$ is circularly orderable then follows from Proposition 4.2. $\qquad\square$

Last, we note the following holds for all left-ordered groups admitting a cofinal, central element:

Proposition 6.2 *Suppose that $(G, <)$ is a left-ordered group, and that $z \in G$ is a cofinal, central element. Then, for every $p \in \mathbb{N}_{>0}$, the group G admits a circular ordering c such that $\mathrm{rot}_c(z) = 1/p$.*

Proof Let $p \in \mathbb{N}_{>0}$ be given, and first construct a circular ordering d of $G/\langle z^p \rangle$ by mimicking Construction 2.2 as follows. Define the minimal representative $\bar{g}$ of each $g\langle z^p \rangle \in G/\langle z^p \rangle$ to be the unique coset representative satisfying $\mathrm{id} \leq \bar{g} < z^p$, and set

$$d(g_1\langle z^p \rangle, g_2\langle z^p \rangle, g_3\langle z^p \rangle) = \mathrm{sign}(\sigma),$$

where σ is the unique permutation such that the minimal representatives satisfy $\bar{g}_{\sigma(1)} < \bar{g}_{\sigma(2)} < \bar{g}_{\sigma(3)}$. Then construct a circular ordering c of G by using the short exact sequence

$$1 \to \langle z^p \rangle \to G \to G/\langle z^p \rangle \to 1,$$

the orderings $<$ and d of the kernel and quotient, respectively, and Proposition 2.3. The circular ordering c satisfies $\mathrm{rot}_c(z) = 1/p$ (see [2, Proof of Theorem 6.2]). $\qquad\square$

Proposition 6.3 *Suppose that M is a compact, connected, orientable Seifert fibred space with base orbifold $\mathbb{S}^2(\alpha_1, \ldots, \alpha_n)$ with $n \geq 3$, and let h denote the class of a regular fibre in $\pi_1(M)$. If $\pi_1(M)$ is left-orderable, then $\pi_1(M)$ admits a left-ordering relative to which h is cofinal.*

Proof This result is essentially a restatement of [9, Proposition 4.7]. $\qquad\square$

Proof of Theorem 1.2 We first prove (1). By [14, Theorem 1.3], $\pi_1(M)$ is left-orderable whenever the first Betti number $b_1(M)$ is positive and $M \not\cong \mathbb{P}^2 \times \mathbb{S}^1$. When $M = \mathbb{P}^2 \times \mathbb{S}^1$, the fundamental group $\pi_1(M)$ clearly admits a circular ordering of the required kind, via a lexicographic construction. This proves the claim for Seifert fibred manifolds M with $b_1(M) > 0$.

Thus we can assume that $b_1(M) = 0$, which implies that M is closed and orientable (see eg [14, Lemma 3.3]), and the base orbifold of M is either $\mathbb{S}^2(\alpha_1, \alpha_2, \ldots, \alpha_n)$ or $\mathbb{P}^2(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Assume that either condition (1) or (2) of Lemma 6.1 holds. Since we also assume that $\pi_1(M)$ is infinite, the class of the fibre h is of infinite order (see eg [14, Proposition 4.1(1)]). Therefore, from the short exact sequence

$$1 \to \langle h \rangle \to \pi_1(M) \to \pi_1^{\mathrm{orb}}(\mathcal{B}) \to 1,$$

we can lexicographically construct the required circular ordering of $\pi_1(M)$ using Lemma 6.1 and Proposition 2.3, completing the proof in these cases.

For the remaining cases, first suppose that $\mathcal{B} = \mathbb{S}^2(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\sum_{i=1}^{n} 1/\alpha_i \geq n - 2$. Note that necessarily $n \leq 4$, and our assumption that $\pi_1(M)$ is infinite and $b_1(M) = 0$ rules out $n = 0$, $n = 1$ and $n = 2$. When $n = 3$ and $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 > 1$, the group $\pi_1(M)$ is finite, so we need not consider this case. On the other hand, if $\sum_{i=1}^{n} 1/\alpha_i = n - 2$, then $\pi_1(M)$ has infinite abelianization [32, Proposition 2; 36, VI.13 Example], and so is left-orderable. Last, if M has base orbifold $\mathbb{P}^2$ or $\mathbb{P}^2(\alpha_1)$, then $\pi_1(M)$ is finite, $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$ [36, VI.11(c)]. Since we have assumed that $\pi_1(M)$ is infinite, $\pi_1(M)$ is either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$, which is circularly orderable in both cases. We construct a circular ordering of $\pi_1(M)$ for which $\mathrm{rot}(h)$ is zero in either case as follows: when $\pi_1(M)$ is $\mathbb{Z}$, we equip $\mathbb{Z}$ with a secret left-ordering, and, when $\pi_1(M) \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle a, h \mid a^2 = \mathrm{id}, aha^{-1} = h^{-1} \rangle$, we circularly order $\pi_1(M)$ lexicographically using

$$1 \to \langle h \rangle \to \pi_1(M) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

We observe that (2) follows from the defining relations of the fundamental group $\pi_1(M)$. When M has nonorientable base orbifold and admits at least one exceptional fibre, the relation $\gamma h \gamma^{-1} = h^{-1}$ holds in $\pi_1(M)$. As rotation number is invariant under conjugation, we get that $\mathrm{rot}_c(h) = \mathrm{rot}_c(h^{-1})$ for every circular ordering c of $\pi_1(M)$, from which it follows that $\mathrm{rot}_c(h) \in \left\{0, \frac{1}{2}\right\}$.

To prove (3), suppose that $\pi_1(M)$ is left-orderable. Using Proposition 6.3, choose a left-ordering of $\pi_1(M)$ relative to which the class of the fibre is cofinal. The result now follows from Proposition 6.2.

Finally, (4) follows from the short exact sequence of the fibration

$$1 \to \pi_1(F) \to \pi_1(M) \to \pi_1(\mathbb{S}^1) \to 1$$

and the observation that we may use any circular ordering of $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ we please in applying Proposition 2.3. $\qquad\square$

6.2 Graph manifolds

We begin with a few preliminaries to establish notation and some well-known facts. Recall that every compact, orientable, irreducible 3-manifold M admits a unique minimal family of disjoint incompressible tori $\mathcal{T}$ such that $M \setminus \mathcal{T}$ consists of Seifert fibred 3-manifolds and atoroidal 3-manifolds, called the JSJ decomposition. By a graph manifold, we mean a compact, connected, orientable, irreducible 3-manifold admitting a JSJ decomposition into Seifert fibred pieces. By using the tori of the JSJ decomposition to cut our graph manifolds into Seifert fibred pieces, we know that the collection of such tori is minimal for each graph manifold, and each torus is incompressible and thus π_1-injective.

Note that it follows from Proposition 3.2 that graph manifolds with infinite first homology always have left-orderable fundamental group; in particular, their fundamental groups are always circularly orderable. Thus, when it comes to circular-orderability, we will only consider the case of rational homology sphere graph manifolds.

Lemma 6.4 *Let W be a graph manifold with a torus boundary that is not homeomorphic to a Seifert fibred manifold. If $\alpha \in H_1(\partial W; \mathbb{Z})/\{\pm 1\}$ is not the slope of a regular fibre, then $W(\alpha)$ is a graph manifold.*

Proof Let $M_1, M_2, \ldots, M_p$ be the Seifert pieces of the JSJ decomposition of W. Assume that the JSJ decomposition of W has only two pieces, M_1 and M_2. Without loss of generality, assume that the torus boundary of W is contained in M_2. Since α is not the slope of a regular fibre, $M_2(\alpha)$ is a Seifert fibred space with boundary $\partial M_2 \setminus \partial W$, by [33; 8, Theorem 5.1]. Therefore, $M_2(\alpha)$ is irreducible (because it is not $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$ [14, Proposition 4.1(3)]). Hence, $W(\alpha)$ is a graph manifold with Seifert pieces M_1 and $M_2(\alpha)$. Now assume that $p > 2$. Let M_k be the Seifert piece of W containing the torus boundary of W. Since α is not the slope of a regular fibre, $M_k(\alpha)$ is a Seifert fibred space with boundary $\partial M_k \setminus \partial W$, by [33; 8, Theorem 5.1]. Therefore, $M_k(\alpha)$ is irreducible (because it is not $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$ [14, Proposition 4.1(3)]). Hence $W(\alpha)$ is a graph manifold with Seifert pieces $M_1, M_2, \ldots, M_k(\alpha), \ldots, M_p$. $\square$

Lemma 6.5 *Suppose that W is a graph manifold and that ∂W is a torus. Suppose W has Seifert fibred pieces $M_1, \ldots, M_l$ and that $\partial W \subset M_1$. Then, if $W(\alpha)$ is reducible, α must be the slope of the regular fibre in a Seifert fibration of M_1 unless W is a Seifert fibred space with base orbifold $\mathcal{B}_1(\alpha_1, \ldots, \alpha_{s_1})$ such that $\alpha_1 = \cdots = \alpha_{s_1} = 1$ or $s_1 = 0$, and the geometric intersection number satisfies $\Delta(\alpha, h) = 1$, where h is the slope of a regular fibre.*

Proof Let $\alpha \in H_1(\partial W; \mathbb{Z})/\{\pm 1\}$ be a slope. We have two cases:

Case 1 Assume that α is not the slope of a regular fibre. Then the geometric intersection number satisfies $\Delta(\alpha, h) = d > 0$, where h is the slope of a regular fibre in M_1. Hence, if the base orbifold of M_1 is $\mathcal{B}_1(\alpha_1, \ldots, \alpha_{s_1})$, then the base orbifold of $M_1(\alpha)$ is $\mathcal{B}'_1(\alpha_1, \ldots, \alpha_{s_1}, d)$, where $\mathcal{B}'$ is obtained from $\mathcal{B}$ by filling a disk. We have two subcases:

Subcase 1 Assume that $M_1 = W$; that is, W is a Seifert fibred space. Then $W(\alpha)$ is a Seifert fibred space by [33; 8, Theorem 5.1]. Hence $W(\alpha)$ is irreducible unless it is $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$ [14, Proposition 4.1(3)]. (This is possible only in the case where $\alpha_1 = \cdots = \alpha_{s_1} = 1$ or $s_1 = 0$, and $d = 1$.)

Subcase 2 If W is not a Seifert fibred space, then $W(\alpha)$ is a graph manifold by Lemma 6.4. Hence, $W(\alpha)$ is irreducible by definition of a graph manifold.

Case 2 If α is the slope of a regular fibre, then $W(\alpha)$ may be reducible [33; 8, Theorem 5.1]. □

Given a rational homology sphere graph manifold W, note that the underlying graph must be a tree. In this situation, for a fixed JSJ torus T, note that $W \setminus T$ has two components. We denote the closure of these components by W_1 and W_2, and by $\phi \colon \partial W_1 \to \partial W_2$ the homeomorphism such that $W = W_1 \cup_\phi W_2$. We use this fixed notation for the discussion and proof below.

Define a class $\mathscr{C}$ of rational homology sphere graph manifolds to be the smallest collection of 3-manifolds satisfying:

(1)   All connected, irreducible rational homology sphere Seifert fibred manifolds belong to $\mathscr{C}$.

(2)   A rational homology sphere graph manifold W is in $\mathscr{C}$ if and only if, for every JSJ torus $T \subset W$,

    (a)   at least one of $W_1(\phi_*^{-1}(\lambda_{W_2}))$ and $W_2(\phi_*(\lambda_{W_1}))$ has infinite fundamental group, and

    (b)   every irreducible manifold of the form $W_i(\alpha)$ satisfying $|H_1(W_i(\alpha); \mathbb{Z})| < \infty$ lies in $\mathscr{C}$.

Theorem 6.6 *For $W \in \mathscr{C}$, if $\pi_1(W)$ is infinite, then it is circularly orderable.*

Proof For every graph manifold W, let n_W denote the minimal number of tori required to cut W into Seifert fibred pieces (ie the number of tori in its JSJ decomposition). When $n_W = 0$, if $\pi_1(W)$ is infinite, then it is circularly orderable by Theorem 1.2. For induction, assume that $k \geq 0$ and that $\pi_1(W)$ is circularly orderable for all $W \in \mathscr{C}$ with $n_W \leq k$, and consider the case of $n_W = k + 1$.

For a manifold $W \in \mathscr{C}$ with $n_W = k + 1$ and infinite fundamental group, choose a JSJ torus T such that $W \setminus T$ results in two pieces W_1 and W_2 such that W_1 is Seifert fibred.

First suppose that $W_2(\phi_*(\lambda_{W_1}))$ has infinite fundamental group and let $M_1, \ldots, M_l$ denote the Seifert fibred pieces of the JSJ decomposition of W_2; assume that $\partial W_2 \subset M_1$.

Next suppose that $W_2(\phi_*(\lambda_{W_1}))$ is irreducible. If $H_1\big(W_2(\phi_*(\lambda_{W_1})); \mathbb{Z}\big)$ is finite, then we conclude $W_2(\phi_*(\lambda_{W_1})) \in \mathscr{C}$ by property (2)(b). Then $\phi_*(\lambda_{W_1})$ is not the slope of a regular fibre in the outermost piece of M_1 and the JSJ components of $W_2(\phi_*(\lambda_{W_1}))$ are precisely the Seifert fibred manifolds $M_1(\phi_*(\lambda_{W_1})), M_2, \ldots, M_l$ by Lemma 6.4. Thus $W_2(\phi_*(\lambda_{W_1}))$ is a graph manifold in $\mathscr{C}$ with $n_{W_2(\phi_*(\lambda_{W_1}))} < n_W$, so $\pi_1\big(W_2(\phi_*(\lambda_{W_1}))\big)$ is circularly orderable by induction. On the other hand, if $H_1\big(W_2(\phi_*(\lambda_{W_1})); \mathbb{Z}\big)$ is infinite, then $\pi_1\big(W_2(\phi_*(\lambda_{W_1}))\big)$ is left-orderable, and hence circularly orderable, by [14, Theorem 3.2]. In either case it follows that $\pi_1(W)$ is circularly orderable by Proposition 5.6(2).

Now suppose that $W_2(\phi_*(\lambda_{W_1}))$ is reducible, in which case $\phi_*(\lambda_{W_1})$ must be the slope of a regular fibre in ∂M_1, or $W_2(\phi_*(\lambda_{W_1}))$ is $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$, by Lemma 6.5. If $W_2(\phi_*(\lambda_{W_1}))$ is $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$, then, as each has circularly orderable fundamental group, $\pi_1(W)$ is circularly orderable by Proposition 5.6(2).

Next, in the case that $\phi_*(\lambda_{W_1})$ is the slope of a regular fibre in ∂M_1, recall that $H_1(W; \mathbb{Z})$ is finite and so the surface underlying the base orbifold of M_1 has genus zero. Further suppose that M_1 has r exceptional fibres and boundary tori $T, T_1, \ldots, T_m$, and that $W_2 \setminus M_1$ has components $Y_1, \ldots, Y_m$, where each Y_j is a graph manifold with torus boundary. Let $\phi_j : \partial Y_j \to T_j \subset M_1$ for $j = 1, \ldots, m$ denote the gluing maps that recover W_2 from the pieces $M_1, Y_1, \ldots, Y_m$. In this case, by [8, Theorem 5.1; 33], filling M_1 along $\phi_*(\lambda_{W_1})$ yields

$$M_1(\phi_*(\lambda_{W_1})) \cong L_1 \# \cdots \# L_r \# (\mathbb{S}^1 \times \mathbb{D}^2) \# \cdots \# (\mathbb{S}^1 \times \mathbb{D}^2)$$

or

$$M_1(\phi_*(\lambda_{W_1})) \cong L_1 \# \cdots \# L_r \# (\mathbb{S}^1 \times \mathbb{D}^2) \# \cdots \# (\mathbb{S}^1 \times \mathbb{D}^2) \# (\mathbb{S}^1 \times \mathbb{S}^2),$$

depending on whether or not the underlying manifold of the base orbifold is a punctured $\mathbb{S}^2$ or $\mathbb{P}^2$. Here, $L_1, \ldots, L_r$ are lens spaces, there are m copies of $\mathbb{S}^1 \times \mathbb{D}^2$, each arising from a torus component of $\partial M_1 \setminus T$, and each $\partial \mathbb{D}^2$ is path-homotopic to (the image of) a regular fibre in the Dehn filled manifold $M_1(\phi_*(\lambda_{W_1}))$.

Therefore, if we denote the slope of a regular fibre on ∂M_1 by h, it follows that

$$W_2(\phi_*(\lambda_{W_1})) \cong L_1 \# \cdots \# L_r \# Y_1((\phi_1^{-1})_*(h)) \# \cdots \# Y_m((\phi_m^{-1})_*(h))$$

or

$$W_2(\phi_*(\lambda_{W_1})) \cong L_1 \# \cdots \# L_r \# Y_1((\phi_1^{-1})_*(h)) \# \cdots \# Y_m((\phi_m^{-1})_*(h)) \# (\mathbb{S}^1 \times \mathbb{S}^2),$$

again depending on whether or not the base orbifold is orientable.

In either case, we may proceed as follows. Suppose that every manifold $Y_j((\phi_j^{-1})_*(h))$ for $j = 1, \ldots, m$ has infinite fundamental group. If Y_j is homeomorphic to a Seifert fibred manifold, it follows that its fundamental group of $Y_j((\phi_j^{-1})_*(h))$ is circularly orderable by Theorem 1.2. On the other hand, if Y_j is not homeomorphic to a Seifert fibred manifold, then, as $(\phi_j^{-1})_*(h)$ is not the slope of a regular fibre in the outermost Seifert fibred piece of Y_j, we know that $Y_j((\phi_j^{-1})_*(h))$ is irreducible by Lemma 6.4. Thus either $\left| H_1\big(Y_j((\phi_j^{-1})_*(h)); \mathbb{Z}\big) \right| = \infty$ or $\left| H_1\big(Y_j((\phi_j^{-1})_*(h)); \mathbb{Z}\big) \right| < \infty$ and $Y_j((\phi_j^{-1})_*(h)) \in \mathcal{C}$ by property (2)(b). In the former case, the fundamental group of $Y_j((\phi_j^{-1})_*(h))$ is circularly orderable since it is in fact left-orderable by [14, Theorem 3.2]. In the latter case, we may argue that $n_{Y_j((\phi_j^{-1})_*(h))} < n_W$, so the fundamental group of $Y_j((\phi_j^{-1})_*(h))$ is circularly orderable by induction.

It follows that $\pi_1\big(W_2(\phi_*(\lambda_{W_1}))\big)$ is a free product of circularly orderable groups, and thus is circularly orderable. Note it is also infinite by assumption. Last, note that λ_{W_1} is not the slope of a regular fibre in ∂W_1, since it is glued via ϕ_* to the slope of a regular fibre in ∂M_1, and thus $W_1(\lambda_{W_1})$ is irreducible by [33]. Therefore $\pi_1(W)$ is circularly orderable by Proposition 5.6(2).

Now suppose that there exists j_0 such that the fundamental group of $Y_{j_0}((\phi_{j_0}^{-1})_*(h))$ is finite, in which case Y_{j_0} is Seifert fibred and $(\phi_{j_0}^{-1})_*(h)$ is not equal to the rational longitude $\lambda_{Y_{j_0}}$ of Y_{j_0}. One of the components of $W \setminus T_{j_0}$ is Y_{j_0}; we will call the other component W'', so that $W = Y_{j_0} \cup_{\phi_{j_0}} W''$. Note that $W''((\phi_{j_0})_*(\lambda_{Y_{j_0}}))$ is irreducible by Lemma 6.4, since $(\phi_{j_0})_*(\lambda_{Y_{j_0}})$ is not the slope a regular fibre in the outermost piece of W''. Thus, either the homology of $W''((\phi_{j_0})_*(\lambda_{Y_{j_0}}))$ is infinite and so it has left-orderable fundamental group, or the homology is finite and so $W''((\phi_{j_0})_*(\lambda_{Y_{j_0}})) \in \mathscr{C}$. In the latter case, $W''((\phi_{j_0})_*(\lambda_{Y_{j_0}}))$ has infinite fundamental group since it is an irreducible graph manifold whose JSJ decomposition consists of two or more Seifert fibred pieces, and its fundamental group is circularly orderable by induction. Thus we may apply Proposition 5.6(2) with Y_{j_0} in place of M_1 and W'' in place of M_2 in order to conclude that $\pi_1(W)$ is circularly orderable.

Last, suppose $W_2(\phi_*(\lambda_{W_1}))$ has finite fundamental group (and thus W_2 is Seifert fibred), so that $W_1(\phi_*^{-1}(\lambda_{W_2}))$ is infinite by property (2)(a). Then $\pi_1\big(W_1(\phi_*^{-1}(\lambda_{W_2}))\big)$ is circularly orderable by Theorem 1.2. Thus the result follows from Proposition 5.6(2). □

We can more precisely codify the manifolds covered by Theorem 6.6 as follows. The possible base orbifolds of a Seifert fibred manifold admitting a single incompressible torus boundary component and a finite filling are

$$\mathscr{A} := \{\mathbb{D}^2(p,q), \mathbb{D}^2(2,2,r), \mathbb{D}^2(2,3,3), \mathbb{D}^2(2,3,4), \mathbb{D}^2(2,3,5) \mid r \geq 1, p \geq 2, q \geq 2\}.$$

Proof of Theorem 1.3 First, if W is not a rational homology sphere, then the first Betti number $b_1(W)$ is positive, so $\pi_1(W)$ is left-orderable by [14, Theorem 3.2]. Second, if W is a graph manifold satisfying the assumptions of Theorem 1.3 and W is Seifert fibred, then Theorem 1.2 finishes the proof.

From here, we complete the proof by showing that, if W is a rational homology sphere graph manifold satisfying the hypotheses of Theorem 1.3, then $W \in \mathscr{C}$, so Theorem 6.6 applies.

To do this, we induct on the number n_W of tori in the JSJ decomposition of W, noting that, if W satisfies the hypotheses of Theorem 1.3 and $n_W = 0$, then $W \in \mathscr{C}$ by definition. Next suppose that every rational homology sphere graph manifold W satisfying the hypotheses of Theorem 1.3 with $n_W < k$ lies in $\mathscr{C}$, and consider W with $n_W = k$.

Suppose W has Seifert fibred pieces $M_1, \ldots, M_l$. Choose an arbitrary JSJ torus $T \subset W$ and cut along T to arrive at $W = W_1 \cup_\phi W_2$.

Considering $W_1(\phi_*^{-1}(\lambda_{W_2}))$ and $W_2(\phi_*(\lambda_{W_1}))$, if either admits a JSJ decomposition with one or more JSJ tori, then its fundamental group is infinite. On the other hand, if either is Seifert fibred, then again the fundamental group is infinite since the hypotheses of Theorem 1.3 imply the base orbifold of M_i is not in $\mathscr{A}$ for $i = 1, \ldots, l$. In any event, W satisfies property (2)(a) in the definition of $\mathscr{C}$.

Next suppose that $W_1(\alpha)$ is irreducible for some $\alpha \in H_1(W_1; \mathbb{Z})/\{\pm 1\}$ and that $H_1(W(\alpha); \mathbb{Z})$ is finite. If W_1 is Seifert fibred, then $W_1(\alpha)$ is Seifert fibred and so lies in $\mathscr{C}$ by definition. On the other hand, if W_1 is not Seifert fibred, then, by Lemma 6.5, α must not be the slope of the regular fibre in the outermost piece

of W_1 and so $W_1(\alpha)$ is a graph manifold by Lemma 6.4. Moreover, $n_{W_1(\alpha)} < k$ and so $W_1(\alpha) \in \mathscr{C}$ by our induction assumption. As the same arguments hold for W_2, we conclude that W satisfies property (2)(b) in the definition of $\mathscr{C}$, and thus $W \in \mathscr{C}$. $\square$

In fact, suppose W is an arbitrary rational homology sphere graph manifold admitting a JSJ decomposition $W = M_1 \cup_\phi M_2$ such that the base orbifold of M_i lies in $\mathscr{A}$. If $\pi_1(W)$ is circularly orderable for every such W, then Theorem 6.6 and its proof can be used, mutatis mutandis, to show that the fundamental group of *every* rational homology sphere graph manifold is circularly orderable whenever it is infinite. We thus make explicit exactly which graph manifolds having two pieces in their JSJ decomposition are covered by Theorem 6.6.

Let $\mathscr{E}$ be the set of graph manifolds whose JSJ decomposition has at least one Seifert piece with base orbifold in $\mathscr{A}$. Further, set

$$\mathscr{F} = \{\mathbb{D}^2(2,2), \mathbb{D}^2(2,3), \mathbb{D}^2(3,3), \mathbb{D}^2(3,4), \mathbb{D}^2(3,5)\} \subset \mathscr{A}.$$

Corollary 6.7 *Let W be a rational homology sphere graph manifold whose JSJ decomposition has only two Seifert pieces M_1 and M_2 with base orbifolds $\mathscr{B}_1(p_1^1, \ldots, p_{s_1}^1)$ and $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2)$, respectively. Let $\phi\colon \partial M_1 \to \partial M_2$ denote the gluing map that recovers W from the pieces M_1 and M_2, and let λ_i and h_i denote the rational longitude and the slope of a regular fibre on ∂M_i for each of $i = 1, 2$. Suppose that W satisfies*

(1) $W \notin \mathscr{E}$; *or*

(2)   *the base orbifold of M_1 lies in $\mathscr{A}$ and the base orbifold of M_2 is not in $\mathscr{A}$; or*

(3)   *the base orbifolds of both M_1 and M_2 lie in $\mathscr{A}$ with $\mathscr{B}_1(p_1^1, \ldots, p_{s_1}^1) \notin \mathscr{F}$, and*

    (a)   *if $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2) \notin \mathscr{F}$, then either $\Delta(\phi_*(\lambda_1), h_2) > 1$ or $\Delta(\phi_*^{-1}(\lambda_2), h_1) > 1$;*

    (b)   *if $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2) \in \mathscr{F}$, then either*

       • $\Delta(\phi_*^{-1}(\lambda_2), h_1) > 1$, *or*

       • $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2) = \mathbb{D}^2(2,3)$ *and* $\Delta(\phi_*(\lambda_1), h_2) > 5$, *or*

       • $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2) = \mathbb{D}^2(3,3)$ *and* $\Delta(\phi_*(\lambda_1), h_2) > 2$, *or*

       • $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2) = \mathbb{D}^2(3,4)$ *and* $\Delta(\phi_*(\lambda_1), h_2) > 2$, *or*

       • $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2) = \mathbb{D}^2(3,5)$ *and* $\Delta(\phi_*(\lambda_1), h_2) > 2$;

then $\pi_1(W)$ is circularly orderable.

Proof   We prove only case (3)(b), as the other statements claimed are all consequences of Theorem 6.6, since any such manifold lies in $\mathscr{C}$.

Assume that $\mathscr{B}_1(p_1^1, \ldots, p_{s_1}^1) \notin \mathscr{F}$, $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2) \in \mathscr{F}$ and $\Delta(\phi_*^{-1}(\lambda_2), h_1) > 1$. Let $\varphi = \phi^{-1}$. Since $\mathscr{B}_1(p_1^1, \ldots, p_{s_1}^1) \notin \mathscr{F}$ and $\Delta(\phi_*^{-1}(\lambda_2), h_1) > 1$, $\pi_1(M_1(\varphi(\lambda_2)))$ is infinite and circularly orderable. Since $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2) \in \mathscr{F}$, M_2 is Seifert fibred with incompressible boundary. Therefore, $\pi_1(M)$ is circularly orderable by Proposition 5.6.

Next assume that $\mathscr{B}_2(p_1^2, \ldots, p_{s_2}^2) = \mathbb{D}^2(2,3)$ (resp. $\mathbb{D}^2(3,3)$, $\mathbb{D}^2(3,4)$, $\mathbb{D}^2(3,5)$) and $\Delta(\phi_*(\lambda_1), h_2) > 5$ (resp. $\Delta(\phi_*(\lambda_1), h_2) > 2$). Then the base orbifold of $M_2(\phi(\lambda_1))$ is $\mathbb{S}^2(2,3,a)$ (resp. $\mathbb{S}^2(3,3,a)$, $\mathbb{S}^2(3,4,a)$, $\mathbb{S}^2(3,5,a)$) with $a \geq 6$ (resp. $a \geq 3$). Hence, $\pi_1(M_2(\phi(\lambda_1)))$ is infinite and circularly orderable. Since $\mathscr{B}_1(p_1^1, \ldots, p_{s_1}^1) \in \mathcal{A}$, M_1 is Seifert fibred with incompressible boundary. Therefore, $\pi_1(M)$ is circularly orderable by Proposition 5.6. $\qquad\square$

6.3 Graph manifolds with two pieces

Our goal in this section is to show that we can circularly order many of the fundamental groups of graph manifolds having two pieces in their JSJ decomposition that are not covered by Corollary 6.7.

Lemma 6.8 *Let W be a 3-manifold obtained by gluing two knot exteriors in some integer homology 3-spheres on their torus boundary by some orientation-reversing homeomorphism. Then $H_1(W, \mathbb{Z})$ is cyclic.*

Proof Let K_1 and K_2 be two knots in some integer homology 3-spheres W_1 and W_2, respectively. Let M_1 and M_2 be the knot exteriors of K_1 and K_2, respectively. Let W be the 3-manifold obtained by identifying ∂M_1 and ∂M_2 by some orientation-reversing homeomorphism φ. Let (μ_1, λ_1) and (μ_2, λ_2) be the meridian–longitude slope pairs of ∂M_1 and ∂M_2, respectively. Let $\varphi_*(\mu_1) = a\mu_2 + b\lambda_2$ and $\varphi_*(\lambda_1) = c\mu_2 + d\lambda_2$.

Let $T = \partial M_1 \cong \partial M_2$. We have the Mayer–Vietoris sequence

$$\cdots \to H_1(T; \mathbb{Z}) \xrightarrow{\phi_1} H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z}) \xrightarrow{\psi_1} H_1(W; \mathbb{Z}) \xrightarrow{\partial_1} H_0(T; \mathbb{Z})$$
$$\xrightarrow{\phi_0} H_0(M_1; \mathbb{Z}) \oplus H_0(M_2; \mathbb{Z}) \xrightarrow{\psi_0} H_0(W; \mathbb{Z}) \to 0.$$

We have that $\lambda_1 = 0$ in $H_1(M_1; \mathbb{Z})$, and $H_1(M_1; \mathbb{Z})$ is generated by μ_1. We have also that $\lambda_2 = 0$ in $H_1(M_2; \mathbb{Z})$, and $H_1(M_2; \mathbb{Z})$ is generated by μ_2. We consider $\{\mu_1, \lambda_1\}$ to be the generators of $H_1(T; \mathbb{Z})$. Hence, $\phi_1(\mu_1) = i_T(\mu_1) \oplus -i_T(\mu_1) = (\mu_1, -a\mu_2)$ and $\phi_1(\lambda_1) = i_T(\lambda_1) \oplus -i_T(\lambda_1) = (0, -c\mu_2)$. By exactness, $\operatorname{im}(\phi_1) = \ker(\psi_1)$, $\operatorname{im}(\psi_1) = \ker(\partial_1)$ and $\operatorname{im}(\partial_1) = \ker(\phi_0)$. Since T, M_1 and M_2 are connected and hence path-connected, ϕ_0 is injective. Therefore, $\operatorname{im}(\partial_1) = 0$ and $\operatorname{im}(\psi_1) = \ker(\partial_1) = H_1(W; \mathbb{Z})$. So $H_1(W; \mathbb{Z}) = \operatorname{im}(\psi_1) \cong H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z})/\ker(\psi_1) = H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z})/\operatorname{im}(\phi_1) = \mathbb{Z} \oplus \mathbb{Z}/\operatorname{im}(\phi_1)$. We have that $\operatorname{im}(\phi_1) = \langle(\mu_1, -a\mu_2), (0, -c\mu_2)\rangle \cong \langle(1, -a), (0, -c)\rangle$. Consider the matrix $\begin{pmatrix} 1 & 0 \\ -a & -c \end{pmatrix}$. By adding a times the first row to the second row, we obtain $\begin{pmatrix} 1 & 0 \\ 0 & -c \end{pmatrix}$. Hence $\operatorname{im}(\phi_1) = \langle(\mu_1, -a\mu_2), (0, -c\mu_2)\rangle \cong \langle(1, -a), (0, -c)\rangle \cong \langle(1, 0), (0, -c)\rangle \cong \mathbb{Z} \oplus |c|\mathbb{Z}$, and so

$$H_1(W; \mathbb{Z}) = \operatorname{im}(\psi_1) \cong H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z})/\ker(\psi_1) = H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z})/\operatorname{im}(\phi_1),$$

with this final group being isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/(\mathbb{Z} \oplus |c|\mathbb{Z}) \cong \mathbb{Z}_{|c|}$. $\qquad\square$

If W is a rational homology sphere graph manifold, we can construct a graph called the *splice diagram* $\Gamma(W)$ as follows: nodes are in one-to-one correspondence with the Seifert pieces of W. Two nodes are
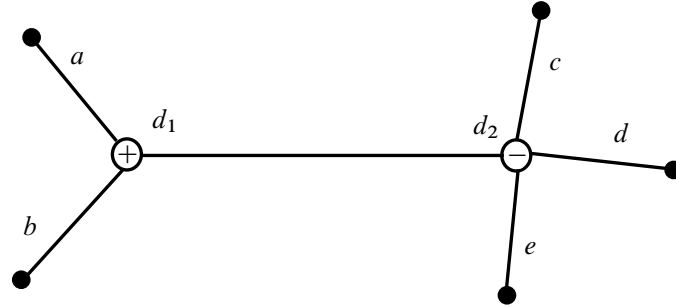
Figure 1: Example of a splice diagram.

connected by an edge if the corresponding Seifert fibred pieces are glued together along a common torus boundary component. To each node, one attaches a leaf for each singular fibre of the corresponding Seifert fibred piece.

We then decorate the graph so constructed as follows: Each edge corresponding to a leaf is labelled with the multiplicity of the corresponding singular fibre. Let v be a node of $\Gamma(W)$ and e be an edge of $\Gamma(W)$ connecting v. Let N and K be the two pieces of W obtained by cutting W along the torus corresponding to e such that K does not contain M_v, where M_v is the Seifert piece of W corresponding to v. Let D be the manifold obtained by Dehn filling K with a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ by identifying a regular fibre of ∂K with a meridian of $\mathbb{D}^2 \times \mathbb{S}^1$. Let $d_v = |H_1(D)|$, and take d_v to be the label of the edge e at the node v. We also decorate the nodes of $\Gamma(W)$ with signs $+$ or $-$ corresponding to the sign of the linking number of two nonsingular fibres in the Seifert fibration (see [46, Section 2] for more details).

Proposition 6.9 *Suppose that M_1 and M_2 are Seifert fibred spaces with incompressible boundaries and base orbifolds $\mathbb{D}^2(a_1, \ldots, a_s)$ and $\mathbb{D}^2(b_1, \ldots, b_t)$, respectively, and that each is the exterior of a knot in an integer homology sphere. Let W be a graph manifold obtained by gluing M_1 and M_2 along their torus boundaries by some orientation-reversing homeomorphism. If $\gcd(a_i, a_l) = 1$ and $\gcd(b_j, b_k) = 1$ for $1 \leq i \neq l \leq s$ and $1 \leq j \neq k \leq t$, and $\gcd(a_i, b_k) = 1$ for all $i = 1, \ldots, s$ and $k = 1, \ldots t$, then $\pi_1(W)$ is circularly orderable.*

Proof   With the restrictions on a_i and b_k as in the statement of the theorem, the manifold W has a corresponding splice diagram $\Gamma(W)$ with two nodes. The edge labels around any node in the diagram are all pairwise coprime, so, by [46, Corollary 6.3], the universal abelian cover of W is an integer homology sphere graph manifold. Hence, the commutator group $[\pi_1(W), \pi_1(W)]$ is left-orderable by [20]. Since $H_1(W)$ is cyclic by Lemma 6.8, $\pi_1(W)$ is circularly orderable by Proposition 2.3. $\square$

In particular, Proposition 6.9 applies to manifolds $W = M_1 \cup_\phi M_2$ where M_i are torus knot exteriors whose cone points have relatively coprime orders. Manifolds of this form are not completely covered by Corollary 6.7 or Theorem 6.6. We can further deal with other special cases of interest not covered by these theorems.

Proposition 6.10 *Suppose that $W = M_1 \cup_\phi M_2$ where each M_i is a twisted I-bundle over the Klein bottle. Then $\pi_1(W)$ is circularly orderable.*

Proof The fundamental group of W is an amalgamated free product of two Klein bottle groups $K_1 = \langle a, b \mid a^2 = b^2 \rangle$ and $K_2 = \langle c, d \mid c^2 = d^2 \rangle$, whose peripheral subgroups are $\langle a^2, ab \rangle$ and $\langle c^2, cd \rangle$, respectively. The amalgamation is with respect to an isomorphism $\phi \colon \langle a^2, ab \rangle \to \langle c^2, cd \rangle$. Observe that K_1 admits a lexicographic circular ordering arising from the short exact sequence

$$1 \to \langle a^2, ab \rangle \to K_1 \to \mathbb{Z}/2\mathbb{Z} \to 1,$$

and that K_2 admits a similar lexicographic circular ordering, with the choice of left-ordering on the subgroup $\langle a^2, ab \rangle$ (resp. $\langle c^2, cd \rangle$) being arbitrary. As such, we can construct circular orderings of K_1 and K_2 so that the homomorphism ϕ is order-preserving, and the subgroups $\langle a^2, ab \rangle$ and $\langle c^2, cd \rangle$ are convex.[4] Then, by [17, Proposition 1.1], $\pi_1(W)$ is circularly orderable. $\qquad\square$

Proof of Theorem 1.4 Suppose M is such a manifold. By [14, Theorem 1.7(1)], if the boundary of M is not empty, or M is nonorientable, or M is a torus bundle over the circle, then $\pi_1(M)$ is left-orderable. Thus $\pi_1(M)$ is circularly orderable. By [14, Section 9], the only case which is left to check is when M is orientable and the union of two twisted I-bundles over the Klein bottle K, which are glued together along their torus boundaries. Therefore Proposition 6.10 finishes the proof. $\qquad\square$

It seems, however, that the special case of a graph manifold consisting of two Seifert fibred pieces, each admitting finite fillings, is out of reach of our current technology. A general notion of "slope detection by a circular ordering" is likely needed to deal with these last few cases, though our results thusfar contribute ample evidence for the truth of the following conjecture:

Conjecture 6.11 *Suppose W is a rational homology sphere graph manifold. If $\pi_1(W)$ is infinite, then it is circularly orderable.*

7 Cyclic branched covers and Dehn surgery

With respect to certain well-known geometric constructions, left-orderability is conjectured to exhibit certain predictable behaviours. In this section we contrast the conjectured "predictable behaviours" of left-orderability with the behaviour of circular-orderability, which is strikingly different and whose expected behaviour at this time is completely unknown.

7.1 Cyclic branched covers

We recall the standard construction of the cyclic covers and cyclic branched covers of a knot in $\mathbb{S}^3$ in order to establish notation.

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[4]Convexity here means that the quotient group inherits a circular ordering. For a more general definition and discussion of convexity in the context of circular orderings, see [17].

Let K be an oriented knot in $\mathbb{S}^3$. Let M_K be the exterior of K and S be a Seifert surface for K. Isotope S so that $S \cap \partial M_K$ is a longitude of K and let $F = S \cap M_K$. Let C be a tubular neighbourhood of F in M_K, so that C is homeomorphic to $F \times [-1, 1]$.

Set $Y = M_K \setminus F \times (-1, 1)$. The boundary of Y contains two copies of F, which we denote by $F^- \cong F \times \{-1\}$ and $F^+ \cong F \times \{1\}$; use these to create a triple (Y, F^+, F^-). Consider n copies of this triple, denoted by (Y_i, F_i^+, F_i^-) for $i = 0, \ldots, n-1$, and glue them together by identifying $F_0^+ \subset Y_0$ with $F_1^- \subset Y_1$, $F_1^+ \subset Y_1$ with $F_2^- \subset Y_2, \ldots, F_{n-2}^+ \subset Y_{n-2}$ with $F_{n-1}^- \subset Y_{n-1}$ and $F_{n-1}^+ \subset Y_{n-1}$ with $F_0^- \subset Y_0$. Call the resulting space X_n.

There is a regular covering map $g: X_n \to M_K$ and its group of deck transformations is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. The manifold X_n is called the n-fold cyclic cover of M_K and its fundamental group is isomorphic to $\ker(\pi_1(M_K) \to \mathbb{Z}/n\mathbb{Z})$. To construct the n-fold cyclic branched cover $\Sigma_n(K)$, we glue a solid torus $V \cong D^2 \times \mathbb{S}^1$ to Y_n by identifying the meridian $\partial D^2 \times \{1\}$ of V with the preimage of the meridian μ of ∂M_K under the map $g: X_n \to M_K$. The manifold $\Sigma_n(K)$ that results is a closed, oriented 3-manifold. For any $n \in \mathbb{N}$, let $q_n: X_n \to \Sigma_n(K)$ be the inclusion map. The map q_n induces a homomorphism $(q_n)_*: \pi_1(X_n) \to \pi_1(\Sigma_n(K))$ and $\ker(q_n)_* = \langle\!\langle \mu^n \rangle\!\rangle$. Therefore we have a short exact sequence $1 \to \langle\!\langle \mu^n \rangle\!\rangle \to \pi_1(X_n) \to \pi_1(\Sigma_n(K)) \to 1$, which identifies the fundamental group of $\Sigma_n(K)$ as the quotient of a certain subgroup of the knot group $\pi_1(M_K)$.

If $L \subset \mathbb{S}^3$ is an oriented link, then the n-fold cyclic branched cover of L, $\Sigma_n(L)$, can be also constructed [7].

Set

$$\mathcal{LO}_{\mathrm{br}}(K) = \{n \geq 2 \mid \pi_1(\Sigma_n(K)) \text{ is left-orderable}\}$$

and

$$\mathcal{CO}_{\mathrm{br}}(K) = \{n \geq 2 \mid \pi_1(\Sigma_n(K)) \text{ is circularly orderable}\}.$$

Note that $\mathcal{LO}_{\mathrm{br}}(K) \subset \mathcal{CO}_{\mathrm{br}}(K)$.

Motivated by the L-space conjecture, in [52, Question 1.8; 7], the authors ask whether or not the set $\mathcal{LO}_{\mathrm{br}}(K)$ is always of the form $\{n \mid n \geq N\}$ for some $N \geq 2$, or empty. In contrast, circular-orderability does not behave this way, with this behaviour being evident upon examining the torus knots. For example, considering the trefoil, we have the following:

Proposition 7.1 *With notation as above, $\mathcal{CO}_{\mathrm{br}}(3_1) = \{2\} \cup \{n \mid n \geq 6\}$.*

Proof The double branched cover of 3_1 is the lens space $L(3, 1)$, so its fundamental group is circularly orderable. On the other hand, the 3-, 4- and 5-fold branched cyclic covers of 3_1 have fundamental group the quaternion group, the binary tetrahedral group, and the binary icosahedral group, respectively, all of which are finite and noncyclic. It follows from Proposition 2.5 that none of these groups are circularly orderable. For $n \geq 6$ we know that $n \in \mathcal{LO}_{\mathrm{br}}(3_1) \subset \mathcal{CO}_{\mathrm{br}}(3_1)$ by [28, Theorem 1.2(i)]. $\square$

This behaviour is not confined to torus knots.

Proposition 7.2 *With notation as above,*

$$2 \in \mathcal{CO}_{\mathrm{br}}(5_2) \quad \text{and} \quad 3 \notin \mathcal{CO}_{\mathrm{br}}(5_2), \quad \text{and} \quad \{n \mid n \geq 9\} \subset \mathcal{CO}_{\mathrm{br}}(5_2).$$

Proof The knot 5_2 is a two-bridge knot, corresponding to the fraction $\frac{7}{4}$. As such, its double branched cover is the lens space $L(7, 4)$, which has fundamental group $\mathbb{Z}/7\mathbb{Z}$, and is therefore circularly orderable. On the other hand, the fundamental group of the Weeks manifold W is not circularly orderable by [16, Theorem 9.2], yet W is homeomorphic to $\Sigma_3(5_2)$ by [39, Main result].

Last, $n \in \mathcal{LO}_{\mathrm{br}}(5_2) \subset \mathcal{CO}_{\mathrm{br}}(5_2)$ for all $n \geq 9$ by [34]. □

There are also examples of knots for which $\mathcal{LO}_{\mathrm{br}}(K)$ is empty. Notable examples are the two-bridge knots $p/q = 2m + 1/(2k)$, or $L_{[2k,2m]}$ with $k, m > 0$ in Conway's notation [24]. However, for these knots, $\mathcal{CO}_{\mathrm{br}}(K)$ is never empty, because the double branched cover is the lens space $L(p, q)$, for which $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$, and is thus circularly orderable. Indeed, for the figure eight knot 4_1, which is $L_{[2,2]}$, we can show that infinitely many of the cyclic branched covers have circularly orderable fundamental group. We first require a proposition.

Proposition 7.3 *Let K be a prime knot in $\mathbb{S}^3$. If $n \geq 2$ and $\pi_1(\Sigma_n(K))$ is circularly orderable and infinite, then $\pi_1(\Sigma_m(K))$ is circularly orderable for all m divisible by n.*

Proof By [28, Lemma 2.11], there exists a surjective group homomorphism

$$q_{m,n} : \pi_1(\Sigma_m(K)) \to \pi_1(\Sigma_n(K))$$

for any positive integer m divisible by n. By [47], the manifolds $\Sigma_m(K)$ are irreducible and so Proposition 3.2(2) applies; we conclude that $\pi_1(\Sigma_m(K))$ is circularly orderable. □

Proposition 7.4 *If n is divisible by 3, then $\pi_1(\Sigma_n(4_1))$ is circularly orderable. In particular, $3\mathbb{N} \subset \mathcal{CO}_{\mathrm{br}}(4_1)$.*

Proof The manifold $\Sigma_3(4_1)$ is homeomorphic to the Hantzsche–Wendt manifold, which is a Seifert fibred manifold with infinite fundamental group. Therefore $\pi_1(\Sigma_3(4_1))$ is circularly orderable by Theorem 1.2. The result now follows from Proposition 7.3. □

These observations naturally lead to the following questions:

Question 7.5 What subsets of $\mathbb{N}$ can occur as $\mathcal{CO}_{\mathrm{br}}(K)$ for a knot K in $\mathbb{S}^3$?

Question 7.6 Is there a knot K in $\mathbb{S}^3$ for which $\mathcal{CO}_{\mathrm{br}}(K) = \varnothing$?

7.2 Double branched covers

In this section we study the circular-orderability of the double branched cover of links, in particular the case of double branched covers of alternating links. We start with the observation that, if the L-space conjecture is true, then the fundamental group of the double branched cover of a quasialternating knot is never left-orderable (the double branched cover of a quasialternating link is an L-space [45]).

In contrast to this, there are alternating Montesinos links whose double branched cover are prism manifolds by [49; 4], so their fundamental groups are not circularly orderable by Proposition 2.5. Similarly, it turns out that the Weeks manifold is homeomorphic to $\Sigma_2(9_{49})$ [39, Main result], and so the double branched cover of 9_{49} has non-circularly orderable fundamental group. Yet 9_{49} is a quasialternating knot [35].

On the other hand, there are many examples of alternating and quasialternating knots whose double branched covers do have circularly orderable fundamental groups, as the next two results show.

7.2.1 Generalized Fibonacci groups
Let $M(k, m)$ denote the double branched cover of the alternating link which is the closure of the 3-strand braid $(\sigma_1^k \sigma_2^{-k})^m$, where m and k are positive integers. By [38, page 169], the fundamental group of $M(k, m)$ is isomorphic to the generalized Fibonacci group

$$F_{2m}^k = \langle x_1, \dots, x_{2m} \mid x_i x_{i+1}^k = x_{i+2} \text{ for any } i = 1, \dots, 2m \rangle,$$

where the indices are taken mod $2m$.

Proposition 7.7 *For any $k \geq 2$, the fundamental group of $M(k, m)$ is circularly orderable.*

Proof For any $k \geq 2$, $M(k, m)$ is irreducible by [38, Lemma 6 and page 171]. Moreover we can define an epimorphism $\rho : F_{2m}^k \to \mathbb{Z}_k * \mathbb{Z}_k$ by $\rho(x_{2i}) \mapsto x$ and $\rho(x_{2i+1}) \mapsto y$, where x and y are the generators of $\mathbb{Z}_k * \mathbb{Z}_k$. Since $\mathbb{Z}_k * \mathbb{Z}_k$ is circularly orderable by Proposition 2.4(2), $\pi_1(M(k, m))$ is circularly orderable by Proposition 3.2(2). □

Remark 7.8 For $k \geq 2$, the manifolds $M(k, 2)$ are obtained by identifying two Seifert fibred spaces along a common torus boundary [38, page 171]. Thus $M(k, 2)$ is a graph manifold. For $k \geq 2$ and $m \geq 3$, the manifolds $M(k, m)$ are irreducible, Haken and atoroidal hyperbolic 3-manifolds by [38, Lemma 6].

7.2.2 Generalized Takahashi manifolds
Fix two positive integers n and m and a collection of integers $\{p_{k,j}, q_{k,j}, r_{k,j}, s_{k,j}\}$ satisfying $\gcd(p_{k,j}, q_{k,j}) = 1$, $\gcd(r_{k,j}, s_{k,j}) = 1$ and $p_{k,j}, r_{k,j} \geq 0$ for all $1 \leq k \leq n$ and $1 \leq j \leq m$. The generalized Takahashi manifold $T_{n,m}(p_{k,j}/q_{k,j}; r_{k,j}/s_{k,j})$ is the double branched cover of $\mathbb{S}^3$, branched over the closure of the braid appearing in Figure 2 [43, Theorem 3], first defined in [43, Section 2]. We will denote the closure of this braid by $L_{n,m}(p_{k,j}/q_{k,j}; r_{k,j}/s_{k,j})$.

This family of manifolds contains many well-known 3-manifolds, such as all n-fold cyclic branched covers of 2-bridge knots, the Weeks manifold, and some graph manifolds. When $p_{k,j} = p_j$, $q_{k,j} = q_j$, $r_{k,j} = r_j$ and $s_{k,j} = s_j$ for all $1 \leq k \leq n$ and $1 \leq j \leq m$, the manifold $T_{n,m}(p_j/q_j; r_j/s_j)$ is called a generalized periodic Takahashi manifold; correspondingly, it is the double branched cover of the link $L_{n,m}(p_j/q_j; r_j/s_j)$.

Each generalized periodic Takahashi manifold can also be viewed as a cyclic branched cover over a knot in a connected sum of lens spaces.
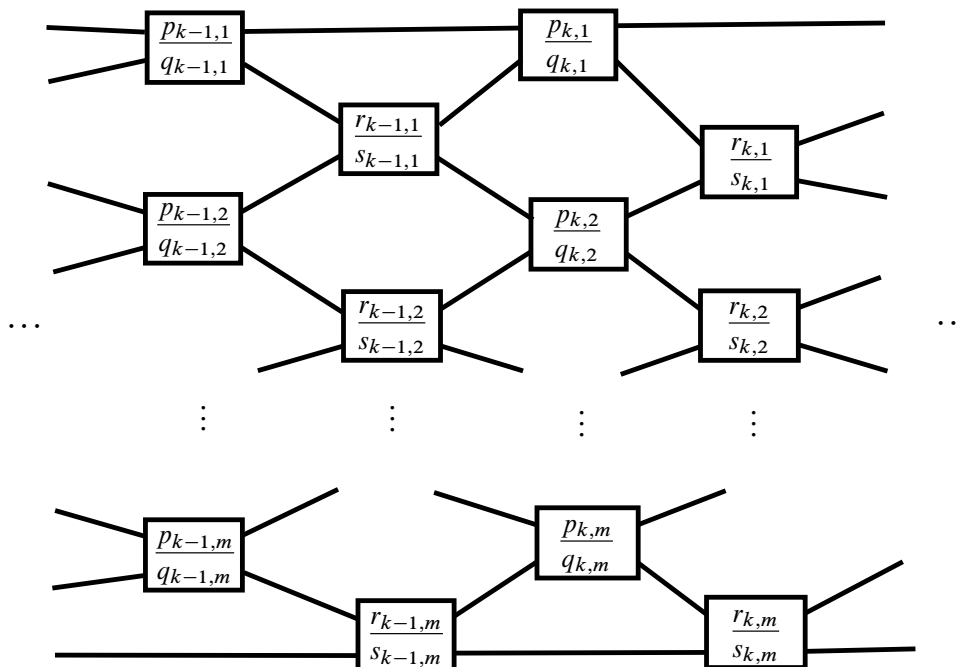
Figure 2: The braid that defines the generalized Takahashi manifold $T_{n,m}(p_{k,j}/q_{k,j}; r_{k,j}/s_{k,j})$. The fraction used to label each box determines the rational tangle used in that box to create $L_{n,m}(p_{k,j}/q_{k,j}; r_{k,j}/s_{k,j})$.

Theorem 7.9 [43, Theorem 6] *The generalized periodic Takahashi manifold $T_{n,m}(p_j/q_j; r_j/s_j)$ is the n-fold cyclic branched cover of the connected sum of $2m$ lens spaces*

$$L(p_1, q_1) \# L(r_1, s_1) \# \cdots \# L(p_m, q_m) \# L(r_m, s_m)$$

branched over a knot which does not depend on n.

We use this as follows:

Theorem 7.10 *The fundamental group of a generalized periodic Takahashi manifold $T_{n,m}(p_j/q_j; r_j/s_j)$ is circularly orderable if the set $\{p_j, r_j \mid 1 \leq j \leq m\}$ contains at least two elements different from 1 and the link $L_{n,m}(p_j/q_j; r_j/s_j)$ is prime.*

Proof Since the link $L_{n,m}(p_j/q_j; r_j/s_j)$ is prime, $T_{n,m}(p_j/q_j; r_j/s_j)$ is irreducible by the equivariant sphere theorem [40] and the positive answer of the Smith conjecture [41]. By Theorem 7.9 and [28, Lemma 2.11], there exists a surjective homomorphism from the fundamental group of the generalized periodic Takahashi manifold $\pi_1(T_{n,m}(p_j/q_j; r_j/s_j))$ to the free product $\mathbb{Z}_{p_1} * \mathbb{Z}_{r_1} * \cdots * \mathbb{Z}_{p_m} * \mathbb{Z}_{r_m}$. Therefore, if the set $\{p_j, r_j \mid 1 \leq j \leq m\}$ contains at least two elements different from 1, this free product is infinite, and hence, by Proposition 3.2, $\pi_1(T_{n,m}(p_j/q_j; r_j/s_j))$ is circularly orderable. $\square$

Remark 7.11 The family of generalized Fibonacci manifolds is a subfamily of the family of generalized periodic Takahashi manifolds.

Question 7.12 Is it possible to characterize the knots $K \subset \mathbb{S}^3$ for which $\pi_1(\Sigma_2(K))$ is circularly orderable?

7.3 Dehn surgery

In this brief section, we point out that circular-orderability of manifolds arising from Dehn surgery on a knot in an integer homology 3-sphere has already appeared in the literature under a different guise, from which we already observe different behaviour than left-orderability with respect to Dehn surgery.

Recall that, for a knot K in an irreducible integer homology 3-sphere M, the result of p/q Dehn surgery on M is denoted by $M_{p/q}(K)$. The L-space conjecture predicts that, for a knot K in $\mathbb{S}^3$, if $\pi_1(\mathbb{S}^3_{p/q}(K))$ is non-left-orderable for some $p/q > 0$, then in fact $\pi_1(\mathbb{S}^3_{p/q}(K))$ is non-left-orderable precisely when $p/q \geq 2g(K) - 1$, where $g(K)$ is the genus of K [44, Proposition 2.1; 48, Theorem 1].

To contrast this with circular-orderability, we first observe that, in light of Theorem 1.1:

Proposition 7.13 *Suppose that M is a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold and $H_1(M; \mathbb{Z})$ is cyclic. Then $\pi_1(M)$ is circularly orderable if and only if the universal abelian cover of M has left-orderable fundamental group.*

Consequently, for an irreducible integer homology 3-sphere M, since we have $H_1(M_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, we precisely need to investigate the universal abelian covers of these manifolds in order to know whether or not their fundamental groups are circularly orderable. This is precisely what is done in [13].

When K is fibred, we use h to denote its monodromy and $c(h)$ the fractional Dehn twist coefficient of h; see [13, Section 4] for details.

Theorem 7.14 [13] *Suppose that K is a fibred hyperbolic knot in an irreducible integer homology 3-sphere M. Given coprime p and q with $p \geq 1$, the universal abelian cover of $M_{p/q}(K)$ has left-orderable fundamental group whenever*

(1) $pc(h) \in \mathbb{Z}$ *and* $q \neq pc(h)$, *or*

(2) $pc(h) \notin \mathbb{Z}$ *and* $q \notin \{\lfloor pc(h) \rfloor, \lfloor pc(h) \rfloor + 1\}$.

Consequently, for any fibred knot K in an irreducible integer homology 3-sphere M, the result of p/q surgery is a manifold with circularly orderable fundamental group whenever the surgery coefficient p/q satisfies either condition (1) or (2) of Theorem 7.14.

Question 7.15 Fix a knot K in an irreducible integer homology 3-sphere M. Is it true that the set $\{p/q \mid M_{p/q}(K)$ is not circularly orderable$\}$ is always finite?

References

[1] **I Ba**, *L-spaces, left-orderability and two-bridge knots*, J. Knot Theory Ramifications 28 (2019) art. id. 1950019 MR Zbl

[2] **I Ba**, **A Clay**, *The space of circular orderings and semiconjugacy*, J. Algebra 586 (2021) 582–606 MR Zbl

[3] **H Baik**, **E Samperton**, *Spaces of invariant circular orders of groups*, Groups Geom. Dyn. 12 (2018) 721–763 MR Zbl

[4] **W Ballinger**, **C C-Y Hsu**, **W Mackey**, **Y Ni**, **T Ochse**, **F Vafaee**, *The prism manifold realization problem*, Algebr. Geom. Topol. 20 (2020) 757–815 MR Zbl

[5] **J Bell**, **A Clay**, **T Ghaswala**, *Promoting circular-orderability to left-orderability*, Ann. Inst. Fourier (Grenoble) 71 (2021) 175–201 MR Zbl

[6] **V V Bludov**, **A M W Glass**, *Word problems, embeddings, and free products of right-ordered groups with amalgamated subgroup*, Proc. Lond. Math. Soc. 99 (2009) 585–608 MR Zbl

[7] **M Boileau**, **S Boyer**, **C M Gordon**, *Branched covers of quasi-positive links and L-spaces*, J. Topol. 12 (2019) 536–576 MR Zbl

[8] **S Boyer**, *Dehn surgery on knots*, from "Handbook of geometric topology" (R J Daverman, R B Sher, editors), North-Holland, Amsterdam (2002) 165–218 MR Zbl

[9] **S Boyer**, **A Clay**, *Foliations, orders, representations, L-spaces and graph manifolds*, Adv. Math. 310 (2017) 159–234 MR Zbl

[10] **S Boyer**, **A Clay**, *Order-detection of slopes on the boundaries of knot manifolds*, Groups Geom. Dyn. 18 (2024) 1317–1348 MR Zbl

[11] **S Boyer**, **C M Gordon**, **Y Hu**, *Slope detection and toroidal 3-manifolds*, preprint (2021) arXiv 2106.14378

[12] **S Boyer**, **C M Gordon**, **L Watson**, *On L-spaces and left-orderable fundamental groups*, Math. Ann. 356 (2013) 1213–1245 MR Zbl

[13] **S Boyer**, **Y Hu**, *Taut foliations in branched cyclic covers and left-orderable groups*, Trans. Amer. Math. Soc. 372 (2019) 7921–7957 MR Zbl

[14] **S Boyer**, **D Rolfsen**, **B Wiest**, *Orderable 3-manifold groups*, Ann. Inst. Fourier (Grenoble) 55 (2005) 243–288 MR Zbl

[15] **D Calegari**, *Circular groups, planar groups, and the Euler class*, from "Proceedings of the Casson Fest", Geom. Topol. Monogr. 7, Geom. Topol. Publ., Coventry (2004) 431–491 MR Zbl

[16] **D Calegari**, **N M Dunfield**, *Laminations and groups of homeomorphisms of the circle*, Invent. Math. 152 (2003) 149–204 MR Zbl

[17] **A Clay**, **T Ghaswala**, *Free products of circularly ordered groups with amalgamated subgroup*, J. Lond. Math. Soc. 100 (2019) 775–803 MR Zbl

[18] **A Clay**, **T Ghaswala**, *Circularly ordering direct products and the obstruction to left-orderability*, Pacific J. Math. 312 (2021) 401–419 MR Zbl

[19] **A Clay**, **T Ghaswala**, *Cofinal elements and fractional Dehn twist coefficients*, Int. Math. Res. Not. 2024 (2024) 7401–7420 MR Zbl

[20] **A Clay**, **T Lidman**, **L Watson**, *Graph manifolds, left-orderability and amalgamation*, Algebr. Geom. Topol. 13 (2013) 2347–2368 MR Zbl

[21]   **A Clay**, **K Mann**, **C Rivas**, *On the number of circular orders on a group*, J. Algebra 504 (2018) 336–363 MR Zbl

[22]   **A Clay**, **L Watson**, *Left-orderable fundamental groups and Dehn surgery*, Int. Math. Res. Not. 2013 (2013) 2862–2890 MR Zbl

[23]   **M Culler**, **N M Dunfield**, *Orderability and Dehn filling*, Geom. Topol. 22 (2018) 1405–1457 MR Zbl

[24]   **M K Dąbkowski**, **J H Przytycki**, **A A Togha**, *Non-left-orderable 3-manifold groups*, Canad. Math. Bull. 48 (2005) 32–40 MR Zbl

[25]   **D B A Epstein**, *Projective planes in 3-manifolds*, Proc. Lond. Math. Soc. 11 (1961) 469–484 MR Zbl

[26]   **X Gao**, *Orderability of homology spheres obtained by Dehn filling*, Math. Res. Lett. 29 (2022) 1387–1427 MR Zbl

[27]   **É Ghys**, *Groups acting on the circle*, Enseign. Math. 47 (2001) 329–407 MR Zbl

[28]   **C Gordon**, **T Lidman**, *Taut foliations, left-orderability, and cyclic branched covers*, Acta Math. Vietnam. 39 (2014) 599–635 MR Zbl

[29]   **R Hakamata**, **M Teragaito**, *Left-orderable fundamental group and Dehn surgery on the knot* 5_2, Canad. Math. Bull. 57 (2014) 310–317 MR Zbl

[30]   **R Hakamata**, **M Teragaito**, *Left-orderable fundamental groups and Dehn surgery on genus one 2-bridge knots*, Algebr. Geom. Topol. 14 (2014) 2125–2148 MR Zbl

[31]   **J Hanselman**, **J Rasmussen**, **S D Rasmussen**, **L Watson**, *L-spaces, taut foliations, and graph manifolds*, Compos. Math. 156 (2020) 604–612 MR Zbl

[32]   **W Heil**, *Almost sufficiently large Seifert fiber spaces*, Michigan Math. J. 20 (1973) 217–223 MR Zbl

[33]   **W Heil**, *Elementary surgery on Seifert fiber spaces*, Yokohama Math. J. 22 (1974) 135–139 MR Zbl

[34]   **Y Hu**, *Left-orderability and cyclic branched coverings*, Algebr. Geom. Topol. 15 (2015) 399–413 MR Zbl

[35]   **S Jablan**, *Tables of quasi-alternating knots with at most 12 crossings*, preprint (2014) arXiv 1404.4965

[36]   **W Jaco**, *Lectures on three-manifold topology*, CBMS Region. Conf. Ser. Math. 43, Amer. Math. Soc., Providence, RI (1980) MR Zbl

[37]   **A Juhász**, *A survey of Heegaard Floer homology*, from "New ideas in low dimensional topology", Ser. Knots Everything 56, World Sci., Hackensack, NJ (2015) 237–296 MR Zbl

[38]   **C Maclachlan**, **A W Reid**, *Generalised Fibonacci manifolds*, Transform. Groups 2 (1997) 165–182 MR Zbl

[39]   **A Mednykh**, **A Vesnin**, *Visualization of the isometry group action on the Fomenko–Matveev–Weeks manifold*, J. Lie Theory 8 (1998) 51–66 MR Zbl

[40]   **W Meeks, III**, **L Simon**, **S T Yau**, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. 116 (1982) 621–659 MR Zbl

[41]   **J W Morgan**, **H Bass** (editors), *The Smith conjecture*, Pure Appl. Math. 112, Academic, Orlando, FL (1984) MR Zbl

[42]   **K Motegi**, **M Teragaito**, *Generalized torsion elements and bi-orderability of 3-manifold groups*, Canad. Math. Bull. 60 (2017) 830–844 MR Zbl

[43]   **M Mulazzani**, **A Vesnin**, *Generalized Takahashi manifolds*, Osaka J. Math. 39 (2002) 705–721 MR Zbl

[44] **P Ozsváth**, **Z Szabó**, *On knot Floer homology and lens space surgeries*, Topology 44 (2005) 1281–1300 MR Zbl

[45] **P Ozsváth**, **Z Szabó**, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. 194 (2005) 1–33 MR Zbl

[46] **H M Pedersen**, *Splice diagram determining singularity links and universal abelian covers*, Geom. Dedicata 150 (2011) 75–104 MR Zbl

[47] **S P Plotnick**, *Finite group actions and nonseparating 2-spheres*, Proc. Amer. Math. Soc. 90 (1984) 430–432 MR Zbl

[48] **J Rasmussen**, *Lens space surgeries and L-space homology spheres*, preprint (2007) arXiv 0710.2531

[49] **J Remigio-Juárez**, **Y Rieck**, *The link volumes of some prism manifolds*, Algebr. Geom. Topol. 12 (2012) 1649–1665 MR Zbl

[50] **P Scott**, *The geometries of 3-manifolds*, Bull. Lond. Math. Soc. 15 (1983) 401–487 MR Zbl

[51] **A T Tran**, *On left-orderability and cyclic branched coverings*, J. Math. Soc. Japan 67 (2015) 1169–1178 MR Zbl

[52] **H Turner**, *Left-orderability, branched covers and double twist knots*, Proc. Amer. Math. Soc. 149 (2021) 1343–1358 MR Zbl

[53] **A A Vinogradov**, *On the free product of ordered groups*, Mat. Sb. 25/67 (1949) 163–168 MR Zbl In Russian

[54] **S D Zheleva**, *Cyclically ordered groups*, Sibirsk. Mat. Zh. 17 (1976) 1046–1051 MR Zbl In Russian; translated in Siberian Math. J. 17 (1976) 773–777

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