Algebraic & Geometric Topology

Volume 25 (2025)

On the involutive Heegaard Floer homology of negative semidefinite plumbed 3-manifolds with $b_1 = 1$

Peter K Johnson

msp

Algebraic & Geometric Topology **25**:2 (2025) 827–885
DOI: 10.2140/agt.2025.25.827
Published: 16 May 2025

On the involutive Heegaard Floer homology of negative semidefinite plumbed 3-manifolds with $b_1 = 1$

PETER K JOHNSON

Ozsváth and Szabó (2003) used Heegaard Floer homology to define numerical invariants $d_{1/2}$ and $d_{-1/2}$ for 3-manifolds Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. We define involutive Heegaard Floer theoretic versions of these invariants analogous to the involutive d invariants $\bar{d}$ and $\underline{d}$ defined for rational homology spheres by Hendricks and Manolescu (2017). We prove their invariance under spin integer homology cobordism and use them to establish spin filling constraints and 0-surgery obstructions analogous to results by Ozsváth and Szabó for their Heegaard Floer counterparts $d_{1/2}$ and $d_{-1/2}$. We then apply calculation techniques of Dai and Manolescu (2019) and Rustamov (2004) to compute the involutive Heegaard Floer homology of some negative semidefinite plumbed 3-manifolds with $b_1 = 1$. By combining these calculations with the 0-surgery obstructions, we are able to produce an infinite family of small Seifert fibered spaces with weight 1 fundamental group and first homology $\mathbb{Z}$ which cannot be obtained by 0-surgery on a knot in S^3, extending a result of Hedden, Kim, Mark and Park (2019).

57K18, 57K31, 57K41

1 Introduction

Involutive Heegaard Floer homology is an extension of Heegaard Floer homology due to Hendricks and Manolescu [6]. It is constructed by considering the mapping cone of a naturally arising involution on the Heegaard Floer chain complex associated to a given Heegaard diagram. For certain 3-manifolds, involutive Heegaard Floer homology contains more information than Heegaard Floer homology. In particular, it has had success illuminating the structure of the integer homology cobordism group.

Over the past several years there has been significant progress in understanding how to calculate involutive Heegaard Floer homology. Some of the methods developed include the large surgery formula of Hendricks and Manolescu [6], the results on almost rational negative definite plumbings by Dai and Manolescu [2], the connected sum formula of Hendricks, Manolescu and Zemke [7], and most recently the involutive surgery exact triangle established by Hendricks, Hom, Stoffregen and Zemke [5].

To date, much of the focus of these calculation techniques and applications has been on rational homology 3-spheres. The goals of this paper are to

(1) establish topological applications of involutive Heegaard Floer homology for 3-manifolds with $b_1 = 1$, and

(2) find an efficient way to compute the involutive Heegaard Floer homology of a certain class of such manifolds.

For rational homology spheres, important topological information is encoded by the involutive d invariants $\bar{d}$ and $\underline{d}$ defined by Hendricks and Manolescu [6]. These are numerical invariants extracted from the plus (or equivalently minus) version of involutive Heegaard Floer homology with respect to a self-conjugate spin^c structure.

In this paper, we define analogous involutive d invariants $\bar{d}_{-1/2}$, $\bar{d}_{1/2}$, $\underline{d}_{-1/2}$, and $\underline{d}_{1/2}$ for 3-manifolds Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. These invariants are generalizations of the invariants $d_{-1/2}$ and $d_{1/2}$ defined by Ozsváth and Szabó [16] and also encode important topological information. In particular, they are spin integer homology cobordism invariants. Moreover, in Section 2, we prove the following theorems, which generalize [16, Theorem 9.11 and Proposition 4.11].

Theorem A *Suppose X is a smooth oriented negative semidefinite spin 4-manifold with boundary a 3-manifold Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$.*

(1) *If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial, then*
$$b_2(X) - 3 \leq 4\underline{d}_{-1/2}(Y).$$

(2) *If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is nontrivial, then*
$$b_2(X) + 2 \leq 4\underline{d}_{1/2}(Y).$$

Remark 1.1 Hypothesis (1) implies $b_2(X) \geq 1$.

Theorem B *Let M be an oriented integer homology 3-sphere and let Y and M' be the 3-manifolds obtained via 0 and $+1$ surgery, respectively, on a knot K in M. Then*

(1) $\underline{d}(M) - \frac{1}{2} \leq \underline{d}_{-1/2}(Y)$ *and* $\bar{d}(M) - \frac{1}{2} \leq \bar{d}_{-1/2}(Y)$;

(2) $\underline{d}_{1/2}(Y) - \frac{1}{2} \leq \underline{d}(M')$ *and* $\bar{d}_{1/2}(Y) - \frac{1}{2} \leq \bar{d}(M')$.

As a consequence of these theorems, we obtain the following two corollaries:

Corollary C *Suppose K is a knot in S^3 and Y is the result of 0-surgery on K. Then*

(1) $-\frac{1}{2} \leq \underline{d}_{-1/2}(Y)$;

(2) $\bar{d}_{1/2}(Y) \leq \frac{1}{2}$.

Corollary D *Suppose Y is a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. If*
$$\underline{d}_{-1/2}(Y) < -\tfrac{1}{2} \quad \text{and} \quad \underline{d}_{1/2}(Y) < \tfrac{1}{2}$$
then Y is not the boundary of any negative semidefinite spin manifold.

To put the above results to use, we need a practical way to calculate $\bar{d}_{\pm 1/2}$ and $\underline{d}_{\pm 1/2}$. The approach we take to achieve this is to adapt existing methods for computing $\bar{d}$ and $\underline{d}$ for rational homology spheres

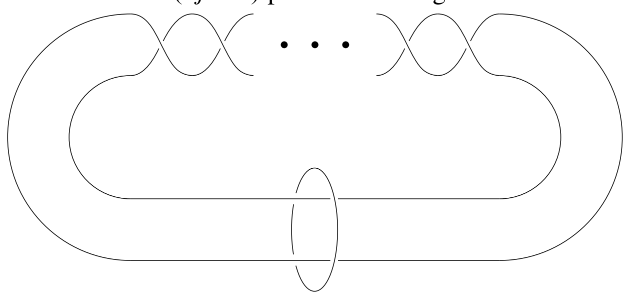


Figure 1

to the setting of 3-manifolds with $b_1 = 1$. Dai and Manolescu [2] provided a combinatorial method to compute the involutive Heegaard Floer homology of a certain class of negative definite plumbed 3-manifolds called almost rational (or AR) plumbed manifolds. In particular, their methods provide a way to compute the invariants $\bar{d}$ and $\underline{d}$ for rational homology spheres which admit such a plumbing.

Their approach utilizes the framework of lattice cohomology and graded roots introduced by Némethi [12; 13]. Lattice cohomology itself builds upon earlier work by Ozsváth and Szabó [17] in which they show how to combinatorially compute the Heegaard Floer homology of a subclass of almost rational plumbings, namely negative definite plumbings with at most one *bad* vertex. Rustamov [23] later generalized this work of Ozsváth and Szabó to the case of negative semidefinite plumbings with $b_1 = 1$ and at most one bad vertex. Subsequent works have established isomorphisms between Heegaard Floer homology and lattice (co)homology (using completed coefficients) for more general classes of plumbings, for example Ozsváth, Stipsicz and Szabó [15] and Zemke [25], the latter of which shows that the completed versions of Heegaard Floer homology and lattice homology are isomorphic for all plumbing trees.

The setting we will work in for computations of $\bar{d}_{\pm 1/2}$ and $\underline{d}_{\pm 1/2}$ is that of negative semidefinite plumbed 3-manifolds with $b_1 = 1$ and at most one bad vertex. To cohesively adapt the work of Dai and Manolescu to this setting, we first recast Rustamov's results into the language of lattice cohomology and graded roots. This requires us to slightly modify Némethi's original definition of lattice cohomology.

After establishing the above computational approach, we carry out a specific calculation of the plus version of the involutive Heegaard Floer homology of an infinite family $\{N_j\}_{j \in \mathbb{N}}$ of small Seifert fiber spaces. For $j \in \mathbb{N}$, we let

$$N_j = S^2\left(-\frac{2}{1}, \frac{-8j+1}{1}, \frac{16j-2}{8j+1}\right).$$

N_j can also be realized as surgery on a 2-component link as in Figure 1.

The family $\{N_j\}_{j \in \mathbb{N}}$ was previously studied by Hedden, Kim, Mark and Park [4]. The manifolds in this family all have first homology equal to $\mathbb{Z}$ and weight 1 fundamental groups, which are necessary conditions if said manifolds could be obtained by 0-surgery on a knot in S^3. However, by using an obstruction in terms the Rokhlin invariant, Hedden, Kim, Mark and Park [4, Theorem 7.3] proved that for all odd positive integers j, N_j cannot be obtained by 0-surgery on a knot in S^3. In the same paper, they also show that if Y is a 3-manifold that is homology cobordant to a Seifert fibered homology $S^1 \times S^2$, then Y automatically satisfies the same $d_{1/2}$ and $d_{-1/2}$ bounds as a manifold obtained by 0-surgery on a knot in S^3 (see [4, Theorem 5.2]). In other words, the noninvolutive version of Corollary C (see [16, Proposition 4.11]) cannot obstruct a Seifert fibered homology $S^1 \times S^2$ from being 0-surgery on a knot in S^3. However, it turns out that the extra information contained in involutive Heegaard Floer homology can detect Seifert fibered 0-surgery. In particular, as an application of Corollaries C and D, we are able to prove the following extension of [4, Theorem 7.3]:

Theorem E *For all positive integers j, N_j cannot be obtained by 0-surgery on a knot in S^3. In fact, N_j is not the oriented boundary of any smooth negative semidefinite spin 4-manifold.*

To provide further context for the above theorem, it is worth noting that there do exist small Seifert fiber spaces which are obtained by 0-surgery on a knot in S^3. For example, by work of Moser [10], 0-surgery on torus knots are small Seifert fibered spaces. More recently, Ichihara, Motegi and Song [8] discovered an infinite family of hyperbolic knots $\{K_n\}_{n \in \mathbb{Z} - \{0,-1,-2\}}$ with small Seifert fibered 0-surgery. These small Seifert manifolds are different from those obtained by 0-surgery on torus knots.

Interestingly, as we describe in Section 5.4,

$$HF^+(-N_1, \mathfrak{s}_0) \cong HF^+(-S^3_0(K_1), \mathfrak{s}_0)$$

where $S^3_0(K_1)$ denotes 0-surgery on K_1 and, on each side of the equation, $\mathfrak{s}_0$ is the unique self-conjugate spin^c structure. However,

$$HFI^+(-N_1, \mathfrak{s}_0) \ncong HFI^+(-S^3_0(K_1), \mathfrak{s}_0).$$

This gives a very concrete example of how involutive Heegaard Floer homology detects Seifert fibered 0-surgery whereas regular Heegaard Floer homology does not.

Organization of the paper

In Section 2, we review involutive Heegaard Floer homology and prove Theorems A and B. In Section 3, we review some basic facts about plumbed manifolds. In Section 4, we define a slightly modified version of lattice cohomology for negative semidefinite plumbings and describe how it fits with prior work of Ozsváth and Szabó, Némethi, and Rustamov. At the end of that section, we adapt [2, Theorem 3.1] to the setting of negative semidefinite plumbings with $b_1 = 1$ and at most one bad vertex. In Section 5, we compute the involutive Heegaard Floer homology of the manifolds $\{N_j\}_{j \in \mathbb{N}}$ as well as $S^3_0(K_1)$. In particular, these calculations together with the results of Section 2, enable us to prove Theorem E.

Acknowledgements

I thank my advisor Tom Mark for suggesting this project. I am deeply grateful for his guidance and countless helpful conversations. I also thank the referee for their helpful comments. I was supported by NSF RTG grant DMS-1839968.

2 Involutive Heegaard Floer homology

In this section, we briefly review the construction of involutive Heegaard Floer homology. We then recall the involutive d invariants, $\underline{d}$ and $\bar{d}$, defined by Manolescu and Hendricks for rational homology spheres and define analogous invariants, $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$, for closed oriented 3-manifolds with first homology $\mathbb{Z}$. We show that $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$ are spin integer homology cobordism invariants and use them to establish constraints on the intersection forms of negative semidefinite spin 4-manifolds whose boundary is a 3-manifold with first homology $\mathbb{Z}$. Furthermore, we establish new obstructions to a 3-manifold being realized as 0-surgery on a knot in an integer homology sphere.

We assume the reader is familiar with Heegaard Floer homology (see for example [19; 18; 21; 22]).

2.1 Notation and conventions

- We use $\mathbb{F} = \mathbb{Z}_2$ coefficients for all Heegaard Floer and involutive Heegaard Floer homology groups.
- Given a graded $\mathbb{F}[U]$-module $\mathcal{A}$, we let $\mathcal{A}[r]$ be the graded $\mathbb{F}[U]$-module defined by $\mathcal{A}[r]_k = \mathcal{A}_{k+r}$. The subscripts denote the homogeneous elements of the corresponding grading.
- We let $\mathcal{T}^+ = \mathbb{F}[U, U^{-1}]/(U \cdot \mathbb{F}[U])$ be the graded $\mathbb{F}[U]$-module where $\mathrm{gr}(U^n) = -2n$.
- We let $\mathcal{T}^+_d := \mathcal{T}^+[-d]$. In other words, $\mathcal{T}^+_d$ is the $\mathbb{F}[U]$-module $\mathcal{T}^+$ with grading shifted so that the minimal nonzero grading level is d.

2.2 Review of involutive Heegaard Floer homology

For complete details of the construction of involutive Heegaard Floer homology see [6].

Let Y be any closed, connected, oriented 3-manifold. Fix a spin^c structure $\mathfrak{s}$ on Y and let $\bar{\omega} = \{\mathfrak{s}, \bar{\mathfrak{s}}\}$ be the orbit of $\mathfrak{s}$ under the conjugation action. Let $\mathcal{H} = (H, J)$ be a Heegaard pair, ie $H = (\Sigma, \alpha, \beta, z)$ is a pointed Heegaard diagram for Y admissible with respect to $\mathfrak{s}$ and J is a generic family of almost complex structures on $\mathrm{Sym}^g(\Sigma)$. Given this setup, define

$$CF^\circ(\mathcal{H}, \bar{\omega}) = \bigoplus_{\mathfrak{t} \in \bar{\omega}} CF^\circ(\mathcal{H}, \mathfrak{t})$$

where $CF^\circ(\mathcal{H}, \mathfrak{t})$ is the usual Heegaard Floer chain complex associated to $(\mathcal{H}, \mathfrak{t})$.

We call $\overline{\mathcal{H}} = (\overline{H}, \overline{J})$ the conjugate Heegaard pair where $\overline{H} = (-\Sigma, \beta, \alpha, z)$ and where $\overline{J}$ is the corresponding conjugate family of almost complex structures. As shown by Ozsváth and Szabó [18, Theorem 2.4], there is a canonical isomorphism of chain complexes

$$\eta\colon CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\overline{\mathcal{H}}, \bar{\mathfrak{s}}).$$

Moreover, H and $\overline{H}$ both represent the same 3-manifold Y; swapping the order of the α and β curves and reversing the orientation of Σ both have the effect of reversing the orientation on Y and thus cancel each other out. One may think of $\overline{H}$ as being obtained from H by flipping the handle decomposition corresponding to H upside down.

Using naturality results of Juhász, Thurston and Zemke [9], it was observed by Hendricks and Manolescu [6, Proposition 2.3] that given two Heegaard pairs representing the same 3-manifold there is a chain homotopy equivalence between their respective Heegaard Floer chain complexes. Furthermore, these chain homotopy equivalences form a transitive system. In particular, since $\mathcal{H}$ and $\overline{\mathcal{H}}$ both represent Y, we get a chain homotopy equivalence

$$\Phi(\overline{\mathcal{H}}, \mathcal{H})\colon CF^\circ(\overline{\mathcal{H}}, \bar{\mathfrak{s}}) \to CF^\circ(\mathcal{H}, \bar{\mathfrak{s}}).$$

Taking the composition of η and Φ, we obtain a map

$$\iota = \Phi(\overline{\mathcal{H}}, \mathcal{H}) \circ \eta\colon CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\mathcal{H}, \bar{\mathfrak{s}})$$

which is uniquely determined up to chain homotopy. By swapping the roles of $\mathfrak{s}$ and $\bar{\mathfrak{s}}$ in the above discussion, we get a second map going in the opposite direction which, by an abuse of notation, we again call ι,

$$\iota\colon CF^\circ(\mathcal{H}, \bar{\mathfrak{s}}) \to CF^\circ(\mathcal{H}, \mathfrak{s}).$$

It is shown in [6] that $\iota^2\colon CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\mathcal{H}, \mathfrak{s})$ is chain homotopic to the identity.

By a further abuse of notation, we let ι also denote the direct sum of the two ι maps above, ie

$$\iota\colon CF^\circ(\mathcal{H}, \varpi) \to CF^\circ(\mathcal{H}, \varpi).$$

We then define the involutive Heegaard Floer complex, $CFI^\circ(\mathcal{H}, \varpi)$, to be the mapping cone complex

$$CF^\circ(\mathcal{H}, \varpi) \xrightarrow{Q(1+\iota)} Q \cdot CF^\circ(\mathcal{H}, \varpi)[-1].$$

Here, Q is a formal variable that shifts the grading down by 1. Therefore, as graded $\mathbb{F}[U]$-modules, $Q \cdot CF^\circ(\mathcal{H}, \varpi)[-1] \cong CF^\circ(\mathcal{H}, \varpi)$ (strictly, these are $\mathbb{Z}_2$-graded modules; there is only an absolute $\mathbb{Q}$-grading lifting the $\mathbb{Z}_2$-grading when $\mathfrak{s}$ is torsion, for example when $\mathfrak{s}$ is self-conjugate). Introducing the formal variable Q gives $CFI^\circ(\mathcal{H}, \varpi)$ the extra structure of a $\mathbb{F}[Q, U]/(Q^2)$-module rather than just an $\mathbb{F}[U]$-module. The involutive Heegaard Floer homology, $HFI^\circ(\mathcal{H}, \varpi)$, is then defined to be the homology of $CFI^\circ(\mathcal{H}, \varpi)$. It turns out that the isomorphism class of $HFI^\circ(\mathcal{H}, \varpi)$ as a graded $\mathbb{F}[Q, U]/(Q^2)$-module is independent of the choice of auxiliary data $\mathcal{H}$. Therefore, we will write $HFI^\circ(Y, \varpi)$ rather than $HFI^\circ(\mathcal{H}, \varpi)$. If $\mathfrak{s}$ is self-conjugate ($\mathfrak{s} = \bar{\mathfrak{s}}$), we write $HFI^\circ(Y, \mathfrak{s})$.

Remark 2.1 Since $HFI^\circ(Y, \overline{\omega})$ is currently only defined up to isomorphism, it is important to highlight that when one considers elements of (or maps on) HFI°, one needs to make a choice of auxiliary data. It is not known whether canonical $\mathbb{F}[Q, U]/(Q^2)$-modules can be associated to each pair $(Y, \overline{\omega})$. For that, one would need higher order naturality results. See [6, Section 2.4] for more details about this issue.

2.3 Involutive *d* invariants

Hendricks and Manolescu [6, Section 5] defined involutive d invariants, denoted by $\bar{d}$ and $\underline{d}$, for self-conjugate spin$^{\mathrm{c}}$ structures of rational homology spheres. Before recalling their definitions and generalizing them to 3-manifolds with $H_1 = \mathbb{Z}$, we need to review a few basic properties.

Proposition 2.2 [6, Proposition 4.6] *Suppose Y is a closed, connected, oriented 3-manifold and $\mathfrak{s} \in \mathrm{Spin}^{\mathrm{c}}(Y)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$. Then there exists an exact triangle of $\mathbb{F}[U]$-modules*

$$\begin{array}{ccc} HF^\circ(Y, \mathfrak{s}) & \xrightarrow{\;Q(1+\iota_*)\;} & Q \cdot HF^\circ(Y, \mathfrak{s})[-1] \\ {\scriptstyle h}\nwarrow & & \swarrow{\scriptstyle g} \\ & HFI^\circ(Y, \mathfrak{s}) & \end{array}$$

where h decreases grading by 1 and the maps $Q(1+\iota_*)$ and g preserve grading.

Corollary 2.3 *With $(Y, \mathfrak{s})$ as in the previous proposition, if $HF_r^\circ(Y, \mathfrak{s}) \cong 0$ or $\mathbb{F}$, then the map*

$$Q(1+\iota_*)\colon HF_r^\circ(Y, \mathfrak{s}) \to Q \cdot HF_r^\circ(Y, \mathfrak{s})[-1]$$

is trivial.

Proof Since ι^2 is chain homotopic to the identity, the induced map $\iota_*^2 = 1$. In particular, ι_* is an automorphism. Since the only automorphisms of $\mathbb{F}$ or 0 are the identity, if r is a grading for which $HF_r^\circ(Y, \mathfrak{s}) \cong 0$ or $\mathbb{F}$, then ι_* is the identity. Thus, $Q(1+\iota_*) = Q(1+1) = 0$. □

Next we recall a structure result for the ∞-flavor of Heegaard Floer homology. To be consistent with [18], we phrase the next theorem in terms of $\mathbb{Z}$-coefficients. However, we will only be concerned with the mod 2 reduction of this result.

Theorem 2.4 [18, Section 10] *Let Y be a closed, connected, oriented 3-manifold. If $b_1(Y) \le 2$, then there exists an equivalence class of orientation system over Y such that for any torsion spin^c structure $\mathfrak{s}$, we have*

$$HF^\infty(Y, \mathfrak{s}) \cong \mathbb{Z}[U, U^{-1}] \otimes_{\mathbb{Z}} \Lambda^* H^1(Y; \mathbb{Z})$$

as $\mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^*(H_1(Y; \mathbb{Z})/\mathrm{Tors})$-modules.

In Heegaard Floer terminology, HF^∞ is said to be *standard* if it satisfies the conclusion of the above theorem. In other words, Theorem 2.4 says that if $b_1(Y) \in \{0, 1, 2\}$, then $HF^\infty(Y, \mathfrak{s})$ is automatically standard. In particular, if $b_1(Y)=0$, ie if Y is a rational homology sphere, then $HF^\infty(Y,\mathfrak{s})\cong\mathbb{F}[U,U^{-1}]$. In this case, as graded $\mathbb{F}[U]$-modules, we have the (noncanonical) splitting

$$HF^+(Y,\mathfrak{s})\cong\mathcal{T}_d^+\oplus HF_{\mathrm{red}}^+(Y,\mathfrak{s})$$

where $d=d(Y,\mathfrak{s})$ is the usual d invariant of $(Y,\mathfrak{s})$ and $\mathcal{T}_d^+$ corresponds to the image of

$$\pi_*\colon HF^\infty(Y,\mathfrak{s})\to HF^+(Y,\mathfrak{s}).$$

Similarly, if $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$ and $\mathfrak{s}_0$ is the unique torsion spin^c structure on Y, then we have that $HF^\infty(Y,\mathfrak{s}_0)\cong\mathbb{F}[U,U^{-1}]\oplus\mathbb{F}[U,U^{-1}]$ and we get the (noncanonical) splitting

$$HF^+(Y,\mathfrak{s}_0)\cong\mathcal{T}_{d_{-1/2}}^+\oplus\mathcal{T}_{d_{1/2}}^+\oplus HF_{\mathrm{red}}^+(Y,\mathfrak{s}_0)$$

where $d_{-1/2}=d_{-1/2}(Y,\mathfrak{s}_0)$ and $d_{1/2}=d_{1/2}(Y,\mathfrak{s}_0)$ are the two d invariants for $(Y,\mathfrak{s}_0)$ and $\mathcal{T}_{d_{-1/2}}^+\oplus\mathcal{T}_{d_{1/2}}^+$ corresponds to the $\mathrm{Im}(\pi_*)$. Recall, $d_{\pm1/2}\equiv\pm1/2 \bmod 2$.

Remark 2.5 The previous paragraph applies more generally to Y with $b_1(Y)=1$, not just $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$, but to simplify the exposition we will restrict to the case $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$. Ultimately, we are concerned with 0-surgery applications, so this restriction suffices for our purposes.

Proposition 2.6 *Let Y be a closed, connected oriented 3-manifold with $b_1(Y)=0$ or $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$. If $\mathfrak{s}\in\mathrm{Spin}^c(Y)$ with $\mathfrak{s}=\bar{\mathfrak{s}}$, then we get an exact triangle of $\mathbb{F}[U]$-modules:*

$$\begin{array}{ccc} HF^\infty(Y,\mathfrak{s}) & \xrightarrow{\quad 0 \quad} & Q\cdot HF^\infty(Y,\mathfrak{s})[-1] \\ & {}_{h^\infty}\nwarrow \quad \swarrow_{g^\infty} & \\ & HFI^\infty(Y,\mathfrak{s}) & \end{array}$$

Proof By the above discussion, if r is a grading for which $HF_r^\infty(Y,\mathfrak{s})\neq0$, then $HF_r^\infty(Y,\mathfrak{s})\cong\mathbb{F}$. The proposition then follows immediately from Corollary 2.3 and Proposition 2.2. □

We now analyze the conclusion of Proposition 2.6 in the case $b_1=0$ and recall the definition of the involutive d invariants $\bar{d}$ and $\underline{d}$. After this, we consider the case $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$. To minimize confusion, for the rest of this section we use the letter M to denote rational homology spheres and the letter Y to denote 3-manifolds with $b_1=1$.

Consider a rational homology sphere M equipped with a self-conjugate spin^c structure $\mathfrak{s}$. Then the exact triangle of Proposition 2.6 decomposes into exact sequences

$$\begin{aligned} &0\to Q\cdot HF_r^\infty(M,\mathfrak{s})[-1]\xrightarrow{\sim} HFI_r^\infty(M,\mathfrak{s})\to0 && (\text{if } r\equiv d(M,\mathfrak{s}) \bmod 2),\\ &0\to HFI_r^\infty(M,\mathfrak{s})\xrightarrow{\sim} HF_{r-1}^\infty(M,\mathfrak{s})\to0 && (\text{if } r\equiv d(M,\mathfrak{s})+1 \bmod 2).\end{aligned}$$

Since the maps in the exact triangle are U-equivariant, we further get that HFI^∞ splits as a graded $\mathbb{F}[U]$-module,

$$HFI^\infty(M,\mathfrak{s})\cong Q\cdot HF^\infty(M,\mathfrak{s})[-1]\oplus HF^\infty(M,\mathfrak{s})[-1].$$

This splitting is canonical since HF_r^∞ is supported in alternating degrees. Moreover, as graded $\mathbb{F}[Q,U]/(Q^2)$-modules (up to possibly an overall grading shift) one can check that

$$HFI^\infty(M,\mathfrak{s}) \cong \mathbb{F}[Q,U,U^{-1}]/(Q^2).$$

Therefore, we may think of $HFI^\infty(M,\mathfrak{s})$ as the direct sum of two doubly infinite towers: one which is not in the image of Q, and the other which is the image of the first under multiplication by Q. Both towers have involutive grading congruent to $d(M,\mathfrak{s}) \bmod 2\mathbb{Z}$.

We now recall the definition of the involutive d invariants introduced by Hendricks and Manolescu. To make sense of the definition, it is useful to recall that

$$\operatorname{Im}(\pi_*\colon HFI^\infty(M,\mathfrak{s}) \to HFI^+(M,\mathfrak{s})) = \operatorname{Im}(U^n)$$

for $n \gg 0$ (see [19, Lemma 4.6]).

Definition 2.7 [6, Definition 5.1] Let M be an oriented rational homology 3-sphere and $\mathfrak{s} \in \mathrm{Spin}^c(M)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$. Define the lower and upper involutive correction terms of $(M,\mathfrak{s})$ to be $\underline{d}(M,\mathfrak{s})$ and $\bar{d}(M,\mathfrak{s})$, respectively, where

$$\underline{d}(M,\mathfrak{s}) = \min\{r \mid \exists\, x \in HFI_r^+(M,\mathfrak{s}), x \in \operatorname{Im}(U^n), x \notin \operatorname{Im}(U^n Q) \text{ for } n \gg 0\} - 1,$$
$$\bar{d}(M,\mathfrak{s}) = \min\{r \mid \exists\, x \in HFI_r^+(M,\mathfrak{s}), x \neq 0, x \in \operatorname{Im}(U^n Q) \text{ for } n \gg 0\}.$$

It is conceptually useful to think of $\bar{d}$ and $\underline{d}$ in terms of a splitting of HFI^+ into towers and reducible elements as follows:

Corollary 2.8 *Suppose M is an oriented rational homology 3-sphere and $\mathfrak{s} \in \mathrm{Spin}^c(M)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$. Then we get a (noncanonical) splitting as graded $\mathbb{F}[U]$-modules,*

$$HFI^+(M,\mathfrak{s}) \cong \mathcal{T}^+_{\bar{d}} \oplus \mathcal{T}^+_{\underline{d}+1} \oplus HFI^+_{\mathrm{red}}(M,\mathfrak{s}).$$

Here, $\mathcal{T}^+_{\bar{d}} \oplus \mathcal{T}^+_{\underline{d}+1}$ corresponds to $\operatorname{Im}(\pi_*)$, with $\mathcal{T}^+_{\bar{d}}$ in the image of Q.

The invariants $\underline{d}$ and $\bar{d}$ satisfy the following basic properties:

Proposition 2.9 [6, Propositions 5.1 and 5.2] *With M and $\mathfrak{s}$ as in Definition 2.7,*

(1) $\underline{d}(M,\mathfrak{s}) \le d(M,\mathfrak{s}) \le \bar{d}(M,\mathfrak{s})$;
(2) $\underline{d}(M,\mathfrak{s}) = -\bar{d}(-M,\mathfrak{s})$.

Additionally, Hendricks and Manolescu generalize [16, Theorem 9.6] to the involutive setting to obtain:

Theorem 2.10 [6, Theorem 1.2] *With M and $\mathfrak{s}$ as in Definition 2.7, if X is a smooth negative definite 4-manifold with boundary M and $\mathfrak{t}$ is a spin structure on X such that $\mathfrak{t}|_M = \mathfrak{s}$, then*

$$\operatorname{rank}(H^2(X;\mathbb{Z})) \le 4\underline{d}(M,\mathfrak{s}).$$

The method of proof of Theorem 2.10 is used to further show that $\underline{d}$ and $\bar{d}$ are spin rational homology cobordism invariants.

Now suppose Y is a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and let $\mathfrak{s}_0$ be the unique torsion spin^c-structure on Y. Then the exact triangle of Proposition 2.6 decomposes into short exact sequences

$$0 \to Q \cdot HF_r^\infty(Y,\mathfrak{s}_0)[-1] \xrightarrow{g^\infty} HFI_r^\infty(Y,\mathfrak{s}_0) \xrightarrow{h^\infty} HF_{r-1}^\infty(Y,\mathfrak{s}_0) \to 0.$$

These short exact sequences are of the form

$$0 \to \mathbb{F} \to \mathbb{F} \oplus \mathbb{F} \to \mathbb{F} \to 0.$$

Therefore, as vector spaces, we get a splitting

$$HFI_r^\infty(Y,\mathfrak{s}_0) \cong Q \cdot HF_r^\infty(Y,\mathfrak{s}_0)[-1] \oplus HF_{r-1}^\infty(Y,\mathfrak{s}_0)$$

where each summand is one-dimensional. Unlike in the $b_1 = 0$ case, this splitting is not canonical. However, we are still able to get the following structure result:

Proposition 2.11 *Suppose Y is a closed connected oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and $\mathfrak{s}_0 \in \mathrm{Spin}^c(Y)$ is the unique spin^c structure with $\mathfrak{s}_0 = \bar{\mathfrak{s}}_0$. Then, as graded $\mathbb{F}[Q,U]/(Q^2)$-modules,*

$$HFI^\infty(Y,\mathfrak{s}_0) \cong \mathbb{F}[Q,U,U^{-1}]/(Q^2) \oplus \mathbb{F}[Q,U,U^{-1}]/(Q^2)$$

where, on the right side of the equation, the first factor has gradings congruent to $1/2 \bmod 2$ and the second factor has gradings congruent to $-1/2 \bmod 2$.

Proof Fix a Heegaard pair $\mathcal{H} = (H,J)$ representing Y and admissible with respect to $\mathfrak{s}_0$. Let ∂^I be the boundary map on the involutive chain complex. We can compactly write ∂^I as $\partial^I = \partial + Q(1+\iota)$ where ∂ is the usual boundary map on the Heegaard Floer chain complex extended by Q-linearity.

By Theorem 2.4, $HF_{1/2}^\infty(\mathcal{H},\mathfrak{s}_0) \cong HF_{-1/2}^\infty(\mathcal{H},\mathfrak{s}_0) \cong \mathbb{F}$. Let $\alpha \in HF_{1/2}^\infty(\mathcal{H},\mathfrak{s}_0)$ and $\beta \in HF_{-1/2}^\infty(\mathcal{H},\mathfrak{s}_0)$ be the unique nonzero generators. Let $a, b \in CF^\infty(\mathcal{H},\mathfrak{s}_0)$ be representatives of α and β respectively. Then, the unique nonzero element in the image of

$$g^\infty\colon Q \cdot HF_{1/2}^\infty(\mathcal{H},\mathfrak{s}_0)[-1] \to HFI_{1/2}^\infty(\mathcal{H},\mathfrak{s}_0)$$

is $[Qa]$. Similarly, $[Qb]$ is the unique nonzero element in the image of

$$g^\infty\colon Q \cdot HF_{-1/2}^\infty(\mathcal{H},\mathfrak{s}_0)[-1] \to HFI_{-1/2}^\infty(\mathcal{H},\mathfrak{s}_0).$$

As we have observed above, $1 + \iota_*$ is the zero map on homology. Therefore, there exists some $x, y \in CF^\infty(\mathcal{H},\mathfrak{s}_0)$ such that $(1+\iota)a = \partial x$ and $(1+\iota)b = \partial y$. Thus, $\partial^I(a+Qx) = 0$ and $\partial^I(b+Qy) = 0$. Furthermore, we have that $Q[a+Qx] = [Qa]$ and $Q[b+Qy] = [Qb]$. Therefore, the first summand in the decomposition can be taken to be $\big(\mathbb{F}[Q,U,U^{-1}]/(Q^2)\big)[b+Qy]$ and the second to be $\big(\mathbb{F}[Q,U,U^{-1}]/(Q^2)\big)[a+Qx]$. □

The isomorphism in Proposition 2.11 is not canonical with respect to a given Heegaard pair $\mathcal{H} = (H,J)$ because the elements $[a+Qx]$ and $[b+Qy]$ depend on our choice of representatives a, b, x, y. Despite this, we can still define involutive d invariants in this situation. We only need to know the $\mathbb{F}[Q,U]/(Q^2)$-module structure of HFI^∞, regardless of a canonical isomorphism.

Definition 2.12 Let Y be a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. Let $\mathfrak{s}_0$ be the unique spin^c structure on Y with $\mathfrak{s}_0 = \bar{\mathfrak{s}}_0$. Define

$$\underline{d}_{1/2}(Y,\mathfrak{s}_0) = \min\{r \mid r \equiv -\tfrac{1}{2} \bmod 2, \exists x \in HFI_r^+(Y,\mathfrak{s}_0), x \in \mathrm{Im}(U^n), x \notin \mathrm{Im}(U^n Q) \text{ for } n \gg 0\} - 1,$$
$$\underline{d}_{-1/2}(Y,\mathfrak{s}_0) = \min\{r \mid r \equiv \tfrac{1}{2} \bmod 2, \exists x \in HFI_r^+(Y,\mathfrak{s}_0), x \in \mathrm{Im}(U^n), x \notin \mathrm{Im}(U^n Q) \text{ for } n \gg 0\} - 1,$$
$$\bar{d}_{1/2}(Y,\mathfrak{s}_0) = \min\{r \mid r \equiv \tfrac{1}{2} \bmod 2, \exists x \in HFI_r^+(Y,\mathfrak{s}_0), x \neq 0, x \in \mathrm{Im}(U^n Q) \text{ for } n \gg 0\},$$
$$\bar{d}_{-1/2}(Y,\mathfrak{s}_0) = \min\{r \mid r \equiv -\tfrac{1}{2} \bmod 2, \exists x \in HFI_r^+(Y,\mathfrak{s}_0), x \neq 0, x \in \mathrm{Im}(U^n Q) \text{ for } n \gg 0\}.$$

Remark 2.13 Since $\mathfrak{s}_0$ is unique, we will often just write $\underline{d}_{\pm 1/2}(Y)$ and $\bar{d}_{\pm 1/2}(Y)$, or $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$ if Y is clear from context.

As in the $b_1 = 0$ case, it is again useful to think of these invariants in terms of a splitting of HFI^+.

Corollary 2.14 *Suppose Y is a closed, connected, oriented 3-manifold with $H_1(Y,\mathbb{Z}) \cong \mathbb{Z}$ and $\mathfrak{s}_0 \in \mathrm{Spin}^c(Y)$ is the unique Spin^c structure with $\mathfrak{s}_0 = \bar{\mathfrak{s}}_0$. Then there exists a (noncanonical) splitting*

$$HFI^+(Y,\mathfrak{s}_0) \cong \mathcal{T}^+_{\bar{d}_{1/2}} \oplus \mathcal{T}^+_{\bar{d}_{-1/2}} \oplus \mathcal{T}^+_{\underline{d}_{1/2}+1} \oplus \mathcal{T}^+_{\underline{d}_{-1/2}+1} \oplus HFI^+_{\mathrm{red}}(Y,\mathfrak{s}_0)$$

where

$$\mathcal{T}^+_{\bar{d}_{1/2}} \oplus \mathcal{T}^+_{\bar{d}_{-1/2}} \oplus \mathcal{T}^+_{\underline{d}_{1/2}+1} \oplus \mathcal{T}^+_{\underline{d}_{-1/2}+1}$$

corresponds to $\mathrm{Im}(\pi_*)$ and $\mathcal{T}^+_{\bar{d}_{1/2}} \oplus \mathcal{T}^+_{\bar{d}_{-1/2}}$ is contained in the image of multiplication by Q.

Proposition 2.15 *The involutive correction terms $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$ satisfy the basic properties*

(1) $\underline{d}_{\pm 1/2}(Y) \leq d_{\pm 1/2}(Y) \leq \bar{d}_{\pm 1/2}(Y)$;

(2) $\underline{d}_{\pm 1/2}(Y) = -\bar{d}_{\mp 1/2}(-Y)$.

Proof The proof of (1) follows from the same arguments as the proof of [6, Proposition 5.1]. The proof of (2) follows from [6, Proposition 4.4] and the same arguments as in the proof of [6, Proposition 5.2]. □

2.4 Spin filling constraints, homology cobordism invariance, and 0-surgery obstruction

Ozsváth and Szabó [16, Theorem 9.11] established constraints in terms of $d_{\pm 1/2}$ on the intersection form of a negative semidefinite 4-manifold with boundary a given 3-manifold Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. Furthermore, Ozsváth and Szabó [16, Corollary 9.14, Proposition 4.11] established 0-surgery obstructions in terms of $d_{\pm 1/2}$. In this section, we establish the analogous results in the involutive setting.

Theorem 2.16 *Suppose X is a smooth oriented negative semidefinite spin 4-manifold with boundary a 3-manifold Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$.*

(1) *If the restriction* $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ *is trivial, then*

$$b_2(X) - 3 \le 4\underline{d}_{-1/2}(Y).$$

(2) *If the restriction* $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ *is nontrivial, then*

$$b_2(X) + 2 \le 4\underline{d}_{1/2}(Y).$$

Proof Let $\mathfrak{s}$ be a spin structure on X. In particular, $c_1^2(\mathfrak{s}) = 0$. We follow the proof strategy of [16, Theorem 9.11].

(1) Suppose the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial. First, surger out all of $b_1(X)$ without changing the nondegenerate part of the intersection form of X. Then, remove a ball from X to obtain W which we regard as a cobordism $W\colon S^3 \to Y$. As observed in the proof of [16, Theorem 9.11], the map induced from the cobordism W,

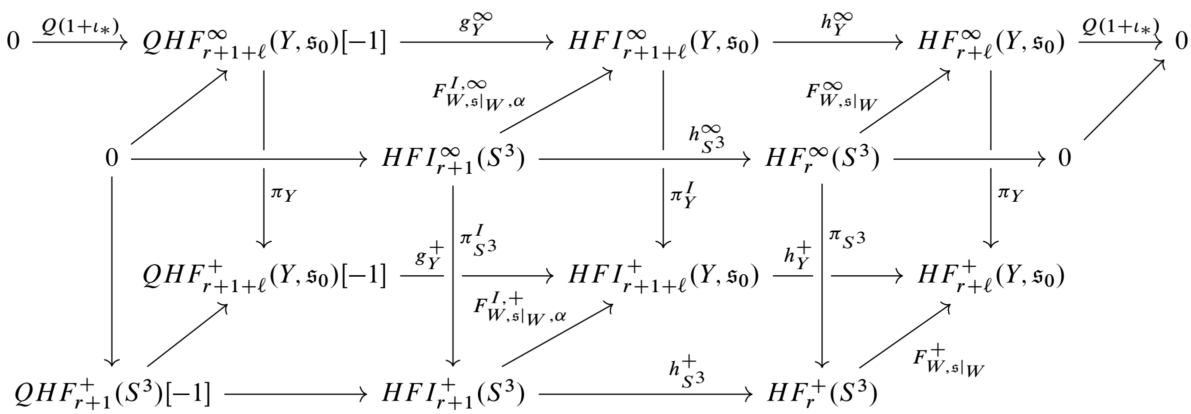
$$F^{\infty}_{W,\mathfrak{s}|_W}\colon HF^{\infty}(S^3) \to HF^{\infty}(Y,\mathfrak{s}|_Y),$$

is injective with image equal to the doubly infinite tower with degrees congruent to $-1/2 \bmod 2$ and shifts degree by $\ell = \frac{1}{4}(b_2(X) - 3)$. Also, by [6, Section 4.5] there exists an induced map

$$F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha}\colon HFI^{\infty}(S^3) \to HFI^{\infty}(Y,\mathfrak{s}_0)$$

which also shifts degree by $\ell = \frac{1}{4}(b_2(X) - 3)$. Note that the involutive cobordism map $F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha}$ depends on an additional choice of auxiliary data α.

Combining the results in [6, Section 4.5] with Proposition 2.6, we see that for every even integer r, we have the following commutative diagram with exact horizontal rows:



By definition of $\underline{d}_{-1/2}(Y)$, there exists some y^+ in $HFI^+_{\underline{d}_{-1/2}+1}(Y,\mathfrak{s}_0)$ such that $y^+ \in \mathrm{Im}(U^n)$ for $n \gg 0$ and $y^+ \notin \mathrm{Im}(U^nQ)$ for $n \gg 0$. The condition $[y^+ \in \mathrm{Im}(U^n)$ for $n \gg 0]$ is equivalent to the condition $[y^+ \in \mathrm{Im}(\pi^I_Y)]$. Therefore, there exists some $y^{\infty} \in HFI^{\infty}_{\underline{d}_{-1/2}+1}(Y,\mathfrak{s}_0)$ such that $\pi^I_Y(y^{\infty}) = y^+$. The condition $[y^+ \notin \mathrm{Im}(U^nQ)$ for $n \gg 0]$ implies that $y^{\infty} \notin \mathrm{Im}(g^{\infty}_Y)$. Therefore, by exactness, $h^{\infty}_Y(y^{\infty}) \neq 0 \in HF^{\infty}_{\underline{d}_{-1/2}}(Y,\mathfrak{s}_0)$. By assumption, the map $F^{\infty}_{W,\mathfrak{s}|_W}\colon HF^{\infty}_{\underline{d}_{-1/2}-\ell}(S^3) \to HF^{\infty}_{\underline{d}_{-1/2}}(Y,\mathfrak{s}|_Y)$

is an isomorphism. Moreover, by exactness, the map $h^{\infty}_{S^3}\colon HFI^{\infty}_{\underline{d}_{-1/2}+1-\ell}(S^3) \to HF^{\infty}_{\underline{d}_{-1/2}-\ell}(S^3)$ is also an isomorphism. Therefore, there exists some $x^{\infty} \in HFI^{\infty}_{\underline{d}_{-1/2}+1-\ell}(S^3)$ such that

$$(F^{\infty}_{W,\mathfrak{s}|_W} \circ h^{\infty}_{S^3})(x^{\infty}) = h^{\infty}_Y(y^{\infty}).$$

Let $z^{\infty} = F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha}(x^{\infty}) \in HFI^{\infty}_{\underline{d}_{-1/2}+1}(Y,\mathfrak{s}_0)$. By commutativity, $h^{\infty}_Y(z^{\infty}) = h^{\infty}_Y(y^{\infty})$. Therefore, $z^{\infty} + y^{\infty} \in \ker(h^{\infty}_Y)$. So, by exactness, there exists some $w^{\infty} \in Q \cdot HF^{\infty}_{\underline{d}_{-1/2}+1}(Y,\mathfrak{s}_0)[-1]$ such that $g^{\infty}_Y(w^{\infty}) = z^{\infty} + y^{\infty}$. If $\pi^I_Y(z^{\infty}) = 0$, then that would imply $\pi^I_Y(g^{\infty}(w^{\infty})) = y^+$. But this would be a contradiction because that would imply $y^+ \in \operatorname{Im}(U^n Q)$ for $n \gg 0$. Therefore, $\pi^I_Y(z^{\infty}) \neq 0$. Thus, $(\pi^I_Y \circ F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha})(x^{\infty}) \neq 0$. So, by commutativity, $(F^{I,+}_{W,\mathfrak{s}|_W,\alpha} \circ \pi^I_{S^3})(x^{\infty}) \neq 0$. In particular, $\pi^I_{S^3}(x^{\infty}) \neq 0$. Therefore, the element $x^+ = \pi^I_{S^3}(x^{\infty}) \in HFI^+_{\underline{d}_{-1/2}+1-\ell}(S^3)$ has the property that $x^+ \in \operatorname{Im}(U^n)$ for $n \gg 0$ and $x^+ \notin \operatorname{Im}(U^n Q)$ for $n \gg 0$. It follows that

$$\underline{d}(S^3) + 1 \le \underline{d}_{-1/2}(Y) + 1 - \ell. \tag{2.17}$$

Observing that $\underline{d}(S^3) = 0$ and rearranging/canceling the terms, we get

$$b_2(X) - 3 \le 4\underline{d}_{-1/2}(Y).$$

(2) Now suppose the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is nontrivial. Surger out the 1-dimensional homology of X until $b_1(X) = 1$ and so that the map $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is still nontrivial. Again, remove a ball from X to obtain a cobordism $W\colon S^3 \to Y$. In this case, the induced map

$$F^{\infty}_{W,\mathfrak{s}|_W}\colon HF^{\infty}(S^3) \to HF^{\infty}(Y,\mathfrak{s}|_Y)$$

is injective with image equal to the doubly infinite tower with degrees congruent to $+1/2 \bmod 2$. The degree shift of this map is now $\frac{1}{4}(b_2(X)+2)$. We then repeat the analogous diagram chase to establish the inequality. We leave the details to the reader. □

Corollary 2.18 *Suppose Y is a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. If*

$$\underline{d}_{-1/2}(Y) < -\tfrac{1}{2} \quad \text{and} \quad \underline{d}_{1/2}(Y) < \tfrac{1}{2}$$

then Y is not the boundary of any negative semidefinite spin manifold.

Proof Suppose X is a smooth negative semidefinite spin 4-manifold with boundary Y. If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial, then the map $H^1(Y;\mathbb{Z}) \to H^2(X,Y;\mathbb{Z})$ is injective. Since $H^1(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and $H^2(X,Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z})$, it follows that $b_2(X) \ge 1$. Hence, by Theorem 2.16, $-1/2 \le \underline{d}_{-1/2}(Y)$. If instead $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is nontrivial, then all we can say about $b_2(X)$ is that $b_2(X) \ge 0$. Theorem 2.16 therefore implies $1/2 \le \underline{d}_{1/2}(Y)$. The conclusion now follows. □

Proposition 2.19 *Suppose Y_1 and Y_2 are closed oriented 3-manifolds with $H_1(Y_i;\mathbb{Z}) \cong \mathbb{Z}$ for $i \in \{1,2\}$. If there exists a spin integer homology cobordism $(W,\mathfrak{s})\colon Y_1 \to Y_2$, then $\underline{d}_{\pm 1/2}(Y_1) = \underline{d}_{\pm 1/2}(Y_2)$ and $\bar{d}_{\pm 1/2}(Y_1) = \bar{d}_{\pm 1/2}(Y_2)$.*

Proof The argument is the same as in the proof of [6, Proposition 5.4], using the fact that W induces an isomorphism

$$F^{\infty}_{W,\mathfrak{s},\alpha}\colon HFI^{\infty}(Y_1,\mathfrak{s}|_{Y_1}) \to HFI^{\infty}(Y_2,\mathfrak{s}|_{Y_2}). \qquad \square$$

Theorem 2.20 *Let M be an oriented integer homology 3-sphere and let Y and M' be the 3-manifolds obtained via 0 and $+1$ surgery, respectively, on a knot K in M. Then*

(1) $\underline{d}(M) - \frac{1}{2} \le \underline{d}_{-1/2}(Y)$ *and* $\bar{d}(M) - \frac{1}{2} \le \bar{d}_{-1/2}(Y)$;

(2) $\underline{d}_{1/2}(Y) - \frac{1}{2} \le \underline{d}(M')$ *and* $\bar{d}_{1/2}(Y) - \frac{1}{2} \le \bar{d}(M')$.

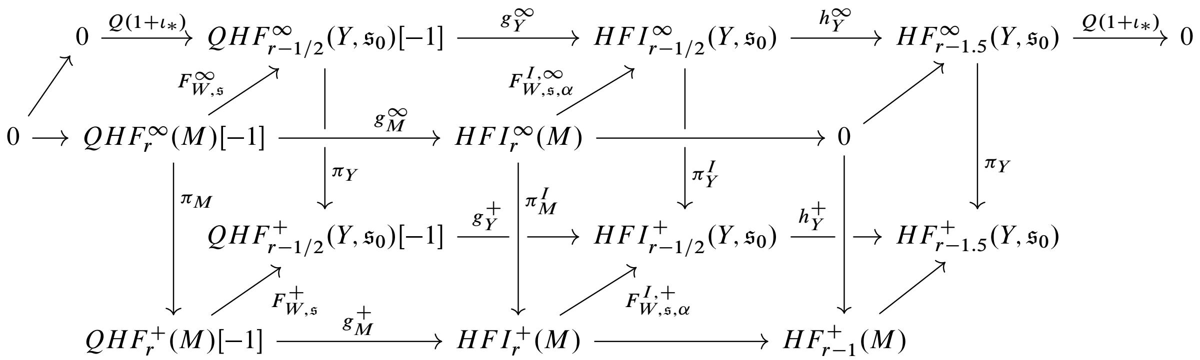
Proof First, we prove the inequalities in (1).

Let $(W,\mathfrak{s})$ be the spin cobordism from M to Y obtained by attaching a 0-framed 2-handle along K and let $\mathfrak{s}_0$ be the trivial spin^c structure on Y. Then, by [16, Proposition 9.3], the induced map

$$F^{\infty}_{W,\mathfrak{s}}\colon HF^{\infty}(M) \to HF^{\infty}(Y,\mathfrak{s}_0)$$

shifts grading by $-1/2$ and is injective with image equal to the doubly infinite tower with gradings congruent to $-1/2$ mod 2. The first inequality of (1) now follows by repeating exactly the same argument as in the proof of Theorem 2.16 where now M assumes the role of S^3 and $\ell = -1/2$ (see inequality (2.17)).

To establish the second inequality in (1), we consider the rightward continuation of the commutative diagram used in the proof of Theorem 2.16 again replacing S^3 with M. Specifically, for r even, we have the following commutative diagram with exact horizontal rows:



Now we get that g^{∞}_M is an isomorphism, and we again know that $F^{\infty}_{W,\mathfrak{s}}$ is an isomorphism. Furthermore, g^{∞}_Y is injective with $\mathrm{Im}(g^{\infty}_Y) = \ker(h^{\infty}_Y)$. Thus, $F^{I,\infty}_{W,\mathfrak{s},\alpha}$ maps $HFI^{\infty}_r(M)$ isomorphically onto $\mathrm{Im}(g^{\infty}_Y)$.

By definition of $\bar{d}_{-1/2}$, there exists some nonzero $y^+ \in HFI^+(Y,\mathfrak{s}_0)$ such that $\mathrm{gr}(y^+) = \bar{d}_{-1/2}$ and $y^+ \in \mathrm{Im}(U^n Q)$ for $n \gg 0$. This implies that there exists some element

$$y^{\infty} \in \mathrm{Im}(g^{\infty}_Y) \subset HFI^{\infty}_{\bar{d}_{-1/2}}(Y,\mathfrak{s}_0)$$

such that $\pi_Y^I(y^\infty) = y^+$. Therefore, the unique nonzero element of $HFI^\infty_{\bar{d}_{-1/2}+1/2}(M)$, which we will call x^∞, maps to y^∞ under $F^{I,\infty}_{W,\mathfrak{s},\alpha}$. Since $(\pi_Y^I \circ F^{I,\infty}_{W,\mathfrak{s},\alpha})(x^\infty) = y^+ \neq 0$, the commutativity of the diagram implies $\pi_M^I(x^\infty) \neq 0$. Additionally, $\pi_M^I(x^\infty) \in \mathrm{Im}(U^n Q)$ for $n \gg 0$. Therefore,

$$\bar{d}(M) \leq \bar{d}_{-1/2}(Y) + \tfrac{1}{2}.$$

The proofs of the inequalities in (2) follow the same arguments as the proofs of (1), except that now we consider the maps

$$F^\circ_{W',\mathfrak{s}'}\colon HF^\circ(Y,\mathfrak{s}_0) \to HF^\circ(M')$$

and

$$F^{I,\circ}_{W',\mathfrak{s}',\alpha'}\colon HFI^\circ(Y,\mathfrak{s}_0) \to HFI^\circ(M')$$

induced by the spin cobordism $(W',\mathfrak{s}')\colon Y \to M'$ obtained by attaching a 2-handle to the dual of K in Y with framing so that the resulting space is M'. Analyzing the corresponding commutative diagrams and using the fact that for all r even,

$$F^\infty_{W',\mathfrak{s}'}\colon HF^\infty_{r+1/2}(Y,\mathfrak{s}_0) \to HF^\infty_r(M')$$

is an isomorphism, we get statement (2). We leave the details to the reader. □

Corollary 2.21 *Suppose K is a knot in S^3 and Y is the result of 0-surgery on K. Then*

(1) $-\frac{1}{2} \leq \underline{d}_{-1/2}(Y)$;

(2) $\bar{d}_{1/2}(Y) \leq \frac{1}{2}$.

Proof Note that $0 = d(S^3) = \underline{d}(S^3) = \bar{d}(S^3)$. Therefore, (1) follows immediately from Theorem 2.20. For (2), let $\overline{K}$ be the mirror of K. Then 0-surgery on $\overline{K}$ is $-Y$. Thus, we have $-\frac{1}{2} \leq \underline{d}_{-1/2}(-Y,\mathfrak{s}_0)$. Now by Proposition 2.15, $\underline{d}_{-1/2}(-Y,\mathfrak{s}_0) = -\bar{d}_{1/2}(Y,\mathfrak{s}_0)$. Therefore, $\bar{d}_{1/2}(Y,\mathfrak{s}_0) \leq \frac{1}{2}$. □

3 Plumbings

We now make a digression from our discussion of involutive Heegaard Floer homology to review basic properties of plumbed 3- and 4-manifolds.

Notation 3.1 Given a graph Γ, we denote the set of vertices of Γ by $\mathcal{V}(\Gamma)$ and the set of edges by $\mathcal{E}(\Gamma)$.

Definition 3.2 A *weighted graph* is a graph Γ together with a function $m\colon \mathcal{V}(\Gamma) \to \mathbb{Z}$, called a *weight function*. Given a vertex $v \in \mathcal{V}(\Gamma)$, we call $m(v)$ the weight of v. Usually we will refer to a weighted graph as Γ and not explicitly write the weight function associated to it.

For the purposes of this paper, we will use the term *plumbing graph* to mean a weighted graph Γ such that $|\mathcal{V}(\Gamma)| < \infty$ and Γ is a forest (ie a disjoint union of trees). Plumbing graphs in general can be more complicated, however for simplicity we only consider plumbing graphs of the type just described.

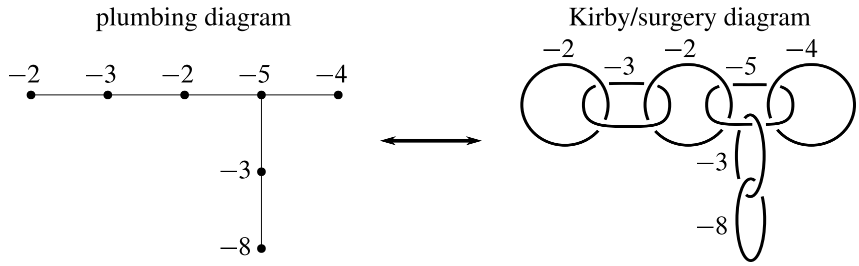


Figure 2

Given a connected plumbing graph Γ, we let $X(\Gamma)$ denote the 4-manifold obtained by plumbing disk bundles over 2-spheres according to Γ (see [3, Example 4.6.2] for details of this construction). If Γ is not connected, then we let $X(\Gamma)$ be the boundary connected sum of the plumbed 4-manifolds corresponding to the connected components of Γ. Regardless of whether Γ is connected or not, we let $Y(\Gamma)$ be the boundary of $X(\Gamma)$ and call it the *plumbed* 3-*manifold* associated to Γ.

Remark 3.3 In general, a given plumbed 3-manifold Y may bound many different plumbed 4-manifolds. Neumann [14] described a calculus for passing between different plumbing graphs that describe the same 3-manifold.

Given a plumbing graph Γ, a Kirby diagram for $X(\Gamma)$ (which is also a surgery diagram for $Y(\Gamma)$) is given by an $m(v)$-framed unknot for each $v \in \mathcal{V}(\Gamma)$ such that any pair of these unknots is either Hopf linked or unlinked depending on whether or not there is an edge between the vertices with which the unknots correspond. See Figure 2.

3.1 Algebraic topological properties of plumbings

Fix a plumbing graph Γ and let $X = X(\Gamma)$ and $Y = Y(\Gamma)$ be the associated plumbed 4- and 3-manifolds. Label the vertices of Γ by $\mathcal{V}(\Gamma) = \{v_1, \dots, v_s\}$ where $s = |\mathcal{V}(\Gamma)|$. For each $v_j \in \mathcal{V}(\Gamma)$, let $[v_j] \in H_2(X;\mathbb{Z})$ be the homology class of the 2-sphere corresponding to the 0-section of the D^2-bundle associated to v_j. Equivalently, $[v_j]$ is represented by the capped-off core of the corresponding 2-handle. In particular, it is easy to see that $H_2(X;\mathbb{Z}) \cong \bigoplus_{j=1}^{s} \mathbb{Z}[v_j]$. Given $x = \sum a_j [v_j] \in H_2(X;\mathbb{Z})$, we write $x \geq 0$ if $a_j \geq 0$ for all j. If in addition, $x \neq 0$, we write $x > 0$. Given two elements $x, y \in H_2(X;\mathbb{Z})$, we write $x \geq y$ (resp. $x > y$) if $x - y \geq 0$ (resp. $x - y > 0$).

Denote the intersection form of X by

$$(\cdot\,,\cdot)\colon H_2(X,\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}.$$

By construction,

$$([v_i],[v_j]) = \begin{cases} m(v_i) & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and there is an edge } [v_i, v_j] \text{ connecting } v_i \text{ and } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let B be the matrix of the intersection form with respect to the ordered basis $([v_1], \dots, [v_s])$. Notice, B is the incidence matrix of the graph Γ with the i^{th} diagonal entry equal to $m(v_i)$.

Definition 3.4 We define the *definiteness type* of a plumbing graph Γ to be the definiteness type of its associated intersection form $(\cdot,\cdot)$, or equivalently the definiteness type of B. For example, we say Γ is negative semidefinite if $(\cdot,\cdot)$ is negative semidefinite.

By an abuse of notation, we will also refer to the corresponding intersection pairing on cohomology as $(\cdot,\cdot)\colon H^2(X,Y;\mathbb{Z}) \times H^2(X,Y;\mathbb{Z}) \to \mathbb{Z}$. It will be useful to, in addition, consider the slightly modified intersection pairing $(\cdot,\cdot)'\colon H^2(X;\mathbb{Z}) \times H^2(X,Y;\mathbb{Z}) \to \mathbb{Z}$ with a different domain, but still defined by the usual formula: $(\alpha,\beta)' = (\alpha \cup \beta)[X]$.

Recall the set of characteristic vectors of X, denoted by $\mathrm{Char}(X)$, is defined by

$$\begin{aligned}\mathrm{Char}(X) &= \{\alpha \in H^2(X;\mathbb{Z}) \mid (\alpha,\beta)' \equiv (\beta,\beta) \bmod 2 \text{ for all } \beta \in H^2(X,Y;\mathbb{Z})\}\\ &= \{\alpha \in H^2(X;\mathbb{Z}) \mid \alpha(x) \equiv (x,x) \bmod 2 \text{ for all } x \in H_2(X;\mathbb{Z})\}.\end{aligned}$$

We now recall the relationship between the spin^c structures on X and Y and the characteristic vectors of X. The first observation is that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}^c(X) & \xrightarrow{|_Y} & \mathrm{Spin}^c(Y) \\ \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} \\ \mathrm{Char}(X) & \xrightarrow{\partial^*} & H^2(Y;\mathbb{Z}) \end{array}$$

where c_1 denotes the first Chern class of the determinant line bundle of the spin^c structure, the top horizontal map is restriction to Y and the bottom horizontal map is the restriction of the map

$$\partial^*\colon H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$$

in the long exact sequence in cohomology of the pair (X,Y). The left vertical map is a bijection since $H_1(X,\mathbb{Z})$ has no 2-torsion (see [3, page 56] for details). Therefore, c_1 provides a canonical identification of $\mathrm{Spin}^c(X)$ with $\mathrm{Char}(X)$. Furthermore, since X is simply connected, we have the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^1(Y;\mathbb{Z}) & \xrightarrow{i^*} & H^2(X,Y;\mathbb{Z}) & \xrightarrow{j^*} & H^2(X;\mathbb{Z}) & \xrightarrow{\partial^*} & H^2(Y;\mathbb{Z}) & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \\ 0 & \longrightarrow & H_2(Y;\mathbb{Z}) & \xrightarrow{i_*} & H_2(X;\mathbb{Z}) & \xrightarrow{j_*} & H_2(X,Y;\mathbb{Z}) & \xrightarrow{\partial_*} & H_1(Y;\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

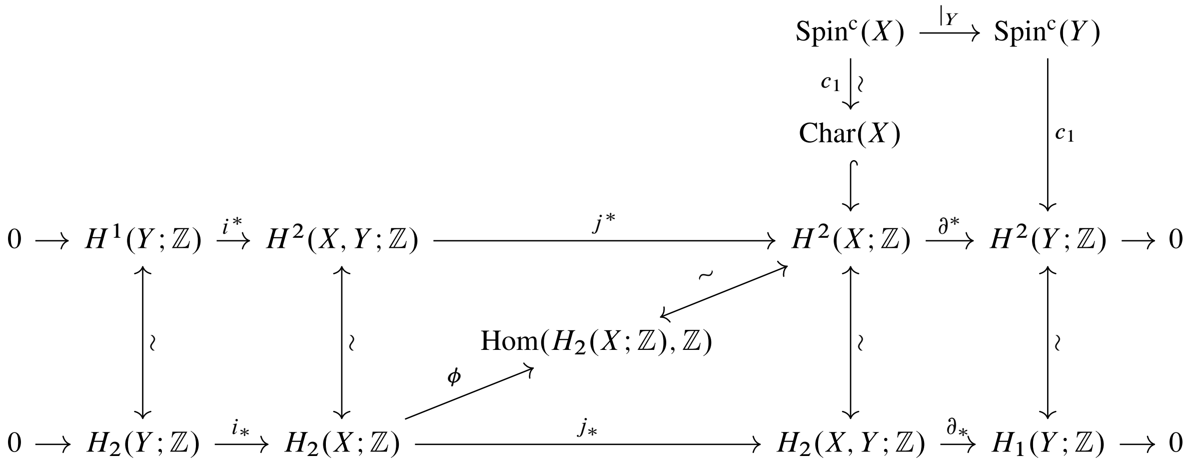
with exact rows coming from the long exact sequences in homology and cohomology of the pair (X,Y) and with vertical isomorphisms given by Poincaré/Lefschetz duality.

We have yet another commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) & \cong & H^2(X;\mathbb{Z}) \\ {\scriptstyle \phi}\uparrow & & \|\wr \\ H_2(X;\mathbb{Z}) & \xrightarrow{j_*} & H_2(X,Y;\mathbb{Z}) \end{array}$$

where the top row is the isomorphism coming from the universal coefficient theorem, the right vertical map is the Lefschetz duality isomorphism, and the map ϕ is defined by $\phi(x) = (x, \cdot)$.

Combining the three previous diagrams we get the following commutative diagram:



In addition, there is a free and transitive action of $H^2(X;\mathbb{Z})$ on $\mathrm{Char}(X)$, defined by $(\alpha, k) \mapsto k + 2\alpha$ for all $\alpha \in H^2(X;\mathbb{Z})$ and $k \in \mathrm{Char}(X)$. Restricting this action to $j^*(H^2(X, Y;\mathbb{Z}))$, we get an action of $j^*(H^2(X, Y;\mathbb{Z}))$ on $\mathrm{Char}(X)$. Let $\mathrm{Char}(X)/2j^*(H^2(X, Y; Z))$ denote the set of orbits of this action and denote the orbit $k + 2j^*(H^2(X, Y;\mathbb{Z}))$ of an element k by $[k]$.

Proposition 3.5 *The map* $\Psi\colon \mathrm{Char}(X)/2j^*(H^2(X, Y;\mathbb{Z})) \to \mathrm{Spin}^c(Y)$ *given by*

$$\Psi([k]) = c_1^{-1}(k)|_Y$$

is well defined and is a bijection.

Notation 3.6 Justified by the above proposition, we will use $[k]$ to denote both the orbit

$$k + 2j^*(H^2(X, Y;\mathbb{Z}))$$

as well as the corresponding spin^c structure $\Psi([k])$.

Remark 3.7 From the above diagram, one can see that if k is a characteristic vector, then $[k]$ is a torsion spin^c structure on Y if and only if some integer multiple of k is in the image of j^*. Equivalently, $[k]$ is torsion if and only if there exists some $z_k \in H_2(X;\mathbb{Z}) \otimes \mathbb{Q}$ such that $k(x) = (z_k, x)$ for all $x \in H_2(X;\mathbb{Z})$.

3.2 Rationality and weight conditions

We now recall some terminology that will be useful later when we discuss lattice cohomology and Heegaard Floer homology of plumbings.

If Γ is a negative definite plumbing tree, then there is a special characteristic vector K_{can} which is called the *canonical characteristic vector*. It is defined by the equation $K_{\mathrm{can}}(v) = -m(v) - 2$ for all $v \in \mathcal{V}(\Gamma)$.

Definition 3.8 A plumbing graph Γ is called *rational* if it is a negative definite tree which satisfies the following condition: if $x \in H_2(X(\Gamma); \mathbb{Z})$ and $x > 0$, then

$$-\frac{K_{\mathrm{can}}(x) - (x, x)}{2} \geq 1.$$

Némethi [12] introduced the following generalization of rational plumbings:

Definition 3.9 [12, Definition 8.1] A negative definite plumbing tree Γ is *almost rational* if there exists a vertex $v \in \mathcal{V}(\Gamma)$ and some integer $r \leq m(v)$ such that if you replace the weight of v with r, Γ becomes rational.

A further generalization of this notion is the following:

Definition 3.10 [15, Definition 2.1] A plumbing tree Γ is *type n* if there exist n vertices of Γ such that if we reduce their weights sufficiently, the plumbing becomes rational.

Remark 3.11 A type n plumbing is not required to be negative definite.

Recall the *degree*, denoted by $\delta(v)$, of a vertex $v \in \mathcal{V}(\Gamma)$ is the number of edges adjacent to v. Following the terminology introduced in [17], we say a vertex is *bad* if $m(v) > -\delta(v)$. In particular, it can be shown that a negative definite plumbing with at most one bad vertex is almost rational.

4 Heegaard Floer homology and lattice cohomology of plumbings

In this section, we review some of the key developments in the Heegaard Floer homology and lattice cohomology of plumbed 3-manifolds. We then present a modified version of lattice cohomology that involves passing to a quotient lattice. This presentation enables us to readily adapt and combine the work of Rustamov [23] and the work of Dai and Manolescu [2] to compute HFI^+ of certain negative semidefinite plumbed 3-manifolds with $b_1 = 1$ and at most one bad vertex.

4.1 Ozsváth–Szabó description of HF^+ of negative definite plumbed 3-manifolds with at most one bad vertex

In an early paper on Heegaard Floer homology, Ozsváth and Szabó [17] provided a combinatorial description of the Heegaard Floer homology of 3-manifolds plumbed along negative definite forests with at most one bad vertex. We briefly review their description.

Given a plumbing presentation Γ of a 3-manifold Y, there is a naturally associated cobordism from S^3 to Y via attaching two handles to $S^3 \times [0, 1]$ according to the plumbing graph Γ. One can turn this cobordism around and use the fact that there is an orientation-preserving diffeomorphism from $-S^3$ to S^3 to yield a cobordism $W_\Gamma \colon -Y \to S^3$. For each spin^c structure $\mathfrak{s}$ on W_Γ, we get a U-equivariant map

$$F^+_{W_\Gamma,\mathfrak{s}} \colon HF^+(-Y, \mathfrak{s}|_Y) \to HF^+(S^3).$$

It is easy to see that the spin^c structures on W_Γ correspond in a direct way to spin^c structures on the plumbed 4-manifold $X(\Gamma)$ since W_Γ is diffeomorphic to $X(\Gamma) - D^4$. Because of this we will work with spin^c structures on $X(\Gamma)$ rather than on W_Γ.

Now by the basic facts about spin^c structures and characteristic vectors described in the previous section and the fact that $HF^+(S^3) \cong \mathcal{T}^+$ as a graded $\mathbb{F}[U]$-module, we can define a map

$$T^+ \colon HF^+(-Y) \to \mathrm{Map}(\mathrm{Char}(X(\Gamma), \mathcal{T}^+)$$

via the formula

$$T^+(\xi)(c_1(\mathfrak{s})) = F^+_{W_\Gamma,\mathfrak{s}}(\xi).$$

Here $\mathrm{Map}(\mathrm{Char}(X(\Gamma), \mathcal{T}^+)$ simply denotes the set of functions from $\mathrm{Char}(X(\Gamma))$ to $\mathcal{T}^+$.

Let $\mathrm{H}^+(\Gamma) \subset \mathrm{Map}(\mathrm{Char}(X(\Gamma), \mathcal{T}^+)$ be the functions ϕ of finite support which satisfy the following adjunction relations: For each $k \in \mathrm{Char}(X(\Gamma))$ and $v_i \in \mathcal{V}(\Gamma)$, let $2n_i = k([v_i]) + ([v_i],[v_i])$. Then,

(1) if $n_i \geq 0$, we require $U^{n_i}\phi(k + 2PDj_*[v_i]) = \phi(k)$;

(2) if $n_i < 0$, we require $U^{-n_i}\phi(k) = \phi(k + 2PDj_*[v_i])$.

The set $\mathrm{H}^+(\Gamma)$ naturally inherits an $\mathbb{F}[U]$-module structure from $\mathcal{T}^+$. One can also introduce a grading on $\mathrm{H}^+(\Gamma)$ by defining $\phi \in \mathrm{H}^+(\Gamma)$ to be a homogeneous element of degree d if $\phi(k) \in \mathcal{T}^+$ is a homogeneous element of degree $d + \frac{1}{4}(k^2 + |\mathcal{V}(\Gamma)|)$ for all $k \in \mathrm{Char}(X(\Gamma))$. Furthermore, we can decompose $\mathrm{H}^+(\Gamma)$ into a direct sum over spin^c structures of Y by defining $\mathrm{H}^+(\Gamma,[k])$ to be the elements of $\mathrm{H}^+(\Gamma)$ which are supported on the *set* $[k]$. Recall $[k]$ denotes both a spin^c structure on Y as well as a subset of $\mathrm{Char}(X(\Gamma))$ (see Notation 3.6).

Remark 4.1 In [17], $\mathrm{H}^+(\Gamma)$ is instead denoted by $\mathbb{H}^+(\Gamma)$. We have changed the notation in this paper to $\mathrm{H}^+(\Gamma)$ to avoid confusion with lattice cohomology which is denoted by $\mathbb{H}^*(\Gamma)$.

The main result (Theorem 1.2) in [17] states that if Γ is a negative definite plumbing with at most one bad vertex, then $T^+ \colon HF^+(-Y(\Gamma),[k]) \to \mathrm{H}^+(\Gamma,[k])$ is an isomorphism of graded $\mathbb{F}[U]$-modules for all spin^c structures $[k]$ on $Y(\Gamma)$. Moreover, $\mathrm{H}^+(\Gamma,[k])$ can be computed combinatorially from the data encoded by the plumbing graph. Therefore, this result enables one to compute $HF^+(-Y(\Gamma),[k])$ without having to count holomorphic disks. In particular, Ozsváth and Szabó provide a relatively simple algorithm to compute $\ker(U) \subset \mathrm{H}^+(\Gamma,[k])$.

4.2 Némethi's graded roots and lattice cohomology

Building upon the work of Ozsváth and Szabó, Némethi [12] provides an algorithm to compute the entire $\mathbb{F}[U]$-module H^+ for almost rational plumbings by adapting methods of computation sequences used in the study of normal surface singularities. On the way to computing H^+, Némethi's algorithm first computes an intermediate object called a graded root whose definition we review below (see Definition 4.19). For now, we will just mention that a graded root is weighted graph associated to $Y(\Gamma)$ from which one can easily calculate H^+ and therefore HF^+. Furthermore, by using the language of graded roots, Némethi shows that [17, Theorem 1.2] holds for almost rational plumbed manifolds, a strictly larger class of plumbed 3-manifolds than the class of negative definite trees with at most one bad vertex.

Remark 4.2 We say trees in the previous sentence because strictly speaking almost rational plumbings are typically assumed to be connected. This assumption, however, is not important. The same methods apply to yield the isomorphism if you drop the connectedness assumption in the definition of almost rational.

Motivated by questions involving complex analytic normal surface singularities and the Seiberg–Witten invariant, Némethi [13] further generalized his work on negative definite plumbed 3-manifolds by introducing the broader framework of lattice cohomology. Lattice cohomology assigns to any negative definite plumbed 3-manifold and $\mathrm{spin^c}$ structure a graded $\mathbb{F}[U]$-module, which we denote by $\mathbb{H}^*$.

Némethi's original definition provides two different, but equivalent, realizations of lattice cohomology. One realization is constructed by first decomposing Euclidean space $\mathbb{R}^s = \mathbb{R} \otimes H_2(X(\Gamma); \mathbb{Z})$ into cubes using the $\mathbb{Z}$-lattice $H_2(X(\Gamma); \mathbb{Z})$ with basis $[v_1], \dots, [v_s]$. Then, one considers the usual cellular cohomology of $\mathbb{R}^s$, except with the differential modified by a set of weight functions which encode information about the intersection form of $X(\Gamma)$. The other realization is built by taking the cellular cohomology of certain sublevel sets of these weight functions on cubes.

Lattice cohomology also comes equipped with an extra $\mathbb{Z}$-grading. Namely $\mathbb{H}^*$ decomposes as

$$\mathbb{H}^* = \bigoplus_{q=0}^{\infty} \mathbb{H}^q$$

such that each $\mathbb{H}^q$ is itself a $\mathbb{Z}$-graded $\mathbb{F}[U]$-module. In particular, together with his work in [12], Némethi showed that for a negative definite almost rational plumbed 3-manifold, $Y(\Gamma)$, and $\mathfrak{s} \in \mathrm{spin^c}(Y(\Gamma))$, $\mathbb{H}^0(Y(\Gamma), \mathfrak{s})$ is isomorphic to $HF^+(-Y(\Gamma), \mathfrak{s})$ as graded $\mathbb{F}[U]$-modules (up to an overall grading shift), and, moreover, $\mathbb{H}^q(Y, \mathfrak{s}) \cong 0$ for $q \geq 1$. In general, however, it is not the case that for arbitrary negative definite plumbed 3-manifolds $\mathbb{H}^q \cong 0$ for all $q \geq 1$. For example, Némethi [13, Example 4.4.1] showed the existence of a negative definite plumbed rational homology sphere with nontrivial $\mathbb{H}^1$. Of course though, this plumbing is not almost rational.

4.3 Modified formulation of lattice cohomology

In this section, we construct a modified version of lattice cohomology in order to deal with negative semidefinite plumbings. Before defining this modified version, it is important to point out that subsequent to Némethi's original definition of lattice cohomology, other variants have been defined which apply to broader classes of plumbings than those which are negative definite. In particular, Ozsváth, Stipsicz and Szabó [15] consider lattice (co)homology with completed coefficients which apply to arbitrary plumbing trees/forests including those with negative semidefinite intersection forms. The modified construction we provide is very similar to the formulation in [15]; the main difference is that we handle degenerate plumbings by passing to a certain quotient lattice rather than using completed coefficients. As in [13], we begin by giving the constructions in general terms, without reference to plumbings.

4.3.1 Construction 1 Let A be a free finitely generated $\mathbb{Z}$-module with a specified ordered basis $(e_1, \dots, e_n)$. Let $\bar{A}$ be a quotient of A with the property that $\bar{A}$ is itself a free finitely generated $\mathbb{Z}$-module. Given $a \in A$, we write $\bar{a}$ for the corresponding element of $\bar{A}$.

We define a chain complex as follows. For each $0 \le q \le n$, let C_q be the free $\mathbb{F}$-module generated by the set $\mathcal{Q}_q = \bar{A} \times \{I \subseteq \{1, \dots, n\} \mid |I| = q\}$. Because later we will want to think of these generators as cubes in a cube complex (see Construction 2), we denote the generator of C_q and the element of $\mathcal{Q}_q$ corresponding to $(\bar{a}, I)$ by $\square(\bar{a}, I)$. We define a differential $\partial\colon C_q \to C_{q-1}$ by the formula

$$\partial\square(\bar{a}, I) = \sum_{i \in I} \bigl[\square(\bar{a}, I - \{i\}) + \square(\bar{a} + \bar{e}_i, I - \{i\})\bigr].$$

Remark 4.3 Intuitively, it may be helpful to think of this differential as a cellular boundary map on cubes. We make this point of view precise in Construction 2.

Proposition 4.4 $\partial^2 = 0.$

Proof We have

$$\partial^2\square(\bar{a}, I) = \sum_{i \in I} \sum_{j \in I - \{i\}} \bigl[\square(\bar{a}, I - \{i, j\}) + \square(\bar{a} + \bar{e}_j, I - \{i, j\})\bigr] + \sum_{i \in I} \sum_{j \in I - \{i\}} \bigl[\square(\bar{a} + \bar{e}_i, I - \{i, j\}) + \square(\bar{a} + \bar{e}_i + \bar{e}_j, I - \{i, j\})\bigr].$$

Now observe that the terms of the form $\square(\bar{a}, I - \{i, j\})$ cancel in pairs as i and j vary, as do the terms of the form $\square(\bar{a} + \bar{e}_i + \bar{e}_j, I - \{i, j\})$. Finally, the cross terms also cancel. Therefore, $\partial^2 = 0$. □

Remark 4.5 If one wanted to work over the coefficient ring $\mathbb{Z}$ instead of $\mathbb{F}$, then signs could be introduced as follows: Given a nonempty subset I of $\{1, \dots, n\}$ with $|I| = q$, let $g_I : I \to \{1, \dots, q\}$ be the unique order-preserving bijection. Define the differential via the formula

$$\partial\square(\bar{a}, I) = \sum_{i \in I} (-1)^{g_I(i)} \bigl[\square(\bar{a}, I - \{i\}) - \square(\bar{a} + \bar{e}_i, I - \{i\})\bigr].$$

One can check that we still have $\partial^2 = 0$. For the purposes of this paper, we will stick with the coefficient ring $\mathbb{F}$.

For each $0 \le q \le s$, define $\mathcal{F}^q = \mathrm{Hom}_{\mathbb{F}}(C_q, \mathcal{T}^+)$. We endow $\mathcal{F}^q$ with an $\mathbb{F}[U]$-module structure by the formula $(U^n \cdot \phi)(\square_q) = U^n \phi(\square_q)$ for all $\square_q \in \mathcal{Q}_q$. Our goal now is to define a differential, δ_w, on our cochain modules $\mathcal{F}^q$ by modifying the usual coboundary map by a set of weight functions w.

Definition 4.6 [13, Definition 3.1.4] A set of functions $w_q : \mathcal{Q}_q \to \mathbb{Z}$, for $0 \le q \le n$, is called a set of compatible weight functions if the following hold:

(1) For any integer $k \in \mathbb{Z}$, the set $w_0^{-1}((-\infty, k])$ is finite.

(2) For any $\square(\bar{a}, I) \in \mathcal{Q}_q$ and any $i \in I$,

$$w_q(\square(\bar{a}, I)) \ge w_{q-1}(\square(\bar{a}, I - \{i\})) \quad \text{and} \quad w_q(\square(\bar{a}, I)) \ge w_{q-1}(\square(\bar{a} + \bar{e}_i, I - \{i\})).$$

Fix a set of compatible weight functions w (we drop the subscript for simplicity). By using w, we are able to define a $\mathbb{Z}$-grading on our cochain modules $\mathcal{F}^q$. Specifically, we say that $\phi \in \mathcal{F}^q$ is homogeneous of degree $d \in \mathbb{Z}$ if $\phi(\square_q)$ is a homogeneous element of $\mathcal{T}^+$ of degree $d - 2w(\square_q)$ whenever $\phi(\square_q) \ne 0$.

4.3.2 The differential Mimicking the formula for the differential given in [13, Definition 3.1.4], we define $\delta_w : \mathcal{F}^q \to \mathcal{F}^{q+1}$ as follows:

- Let $\square_{q+1} \in \mathcal{Q}_{q+1}$ and write $\partial \square_{q+1} = \sum_k \square_q^k$.
- Given $\phi \in \mathcal{F}^q$, let

$$(\delta_w \phi)(\square_{q+1}) = \sum_k U^{w(\square_{q+1}) - w(\square_q^k)} \phi(\square_q^k).$$

Proposition 4.7

$$\delta_w^2 = 0.$$

Proof This follows directly from the definition and the fact that $\partial^2 = 0$. □

Definition 4.8 The homology of the cochain complex $(\mathcal{F}^*, \delta_w)$ is called the *lattice cohomology* of the triple $(\bar{A}, (e_1, \dots, e_n), w)$ and is denoted by $\mathbb{H}^*(\bar{A}, (e_1, \dots, e_n), w)$.

Remark 4.9 (1) For each q, the $\mathbb{Z}$-grading on $\mathcal{F}^q$ induces a $\mathbb{Z}$-grading on $\mathbb{H}^q$. Therefore, $\mathbb{H}^q$ is a $\mathbb{Z}$-graded $\mathbb{F}[U]$-module.

(2) If $\bar{A} = A$, then we recover the usual lattice cohomology defined by Némethi [13].

4.3.3 Construction 2 We now give a more geometric, but equivalent formulation of the lattice cohomology theory we defined in Construction 1. This is analogous to [13, Definition 3.1.11].

First, we give a geometric realization of the chain complex C_q. For each $1 \le q \le s$, let $\boldsymbol{c}_q$ be denote the q-dimensional cube $[0, 1]^q$ oriented in the standard way. Additionally, let $\boldsymbol{c}_0$ be a fixed 0-dimensional cube (ie point) oriented positively. To each $\square(\bar{a}, I) \in \mathcal{Q}_q$ we associate a distinct copy of $\boldsymbol{c}_q$. By an abuse of notation, from now on we will regard each $\square(\bar{a}, I) \in \mathcal{Q}_q$ as both a distinct copy of $\boldsymbol{c}_q$ and a generator of C_q depending on which point of view is more convenient in a given context.

We now construct a cube complex $\mathcal{C}$ whose q-dimensional cubes are precisely the elements of $\mathcal{Q}_q$ with attaching maps defined as follows:

- First, we prescribe a method for identifying each $(q-1)$-dimensional face of $\boldsymbol{c}_q$ with $\boldsymbol{c}_{q-1}$. Let $\{x_j\}_{j=1}^q$ be the standard coordinate functions on $\boldsymbol{c}_q = [0,1]^q$. Each $(q-1)$-dimensional face of $\boldsymbol{c}_q$ is defined by an equation $x_i = \epsilon$ for some $\epsilon \in \{0,1\}$. Denote this face by $f_{i,\epsilon}$. For $q \geq 2$, we identify $f_{i,\epsilon}$ with $\boldsymbol{c}_{q-1}$ via the map $(x_1,\dots,x_q) \mapsto (x_1,\dots,\hat{x}_i,\dots,x_q)$. For $q=1$, we send the point $f_{i,\epsilon}$ to the point $\boldsymbol{c}_0$.
- Given $\square(\bar{a}, I) \in \mathfrak{Q}_q$, the face $f_{i,\epsilon}$ of $\square(\bar{a}, I)$ gets glued to the cube $\square(\bar{a} + \epsilon\bar{e}_i, I - \{i\})$ via the map defined in the first bullet point.

By construction the q-dimensional cellular chain group of the cube complex $\mathscr{C}$ is equal to C_q and the cellular boundary map is equal to the differential $\partial\colon C_q \to C_{q-1}$ defined in Construction 1.

Again, fix a set of compatible weight functions w. For every integer $n \geq 1$, let S_n be the subcomplex of $\mathscr{C}$ consisting of all cubes $\square$ such that $w(\square_q) \leq n$ where q ranges over all dimensions. Let

$$m_w = \min\{w(\square_q) \mid \square_q \in \mathfrak{Q}_q,\ 0 \leq q \leq n\}.$$

Define

$$\mathbb{S}^q(\bar{A}, (e_1,\dots,e_n), w) = \bigoplus_{n \geq m_w} H^q(S_n; \mathbb{F})$$

where H^q denotes the qth-cellular cohomology. For each fixed q, we give $\mathbb{S}^q(\bar{A}, (e_1,\dots,e_n), w)$ the structure of an $\mathbb{F}[U]$-module by defining the U action to be the restriction map

$$U\colon H^q(S_{n+1}; \mathbb{Z}) \to H^q(S_n; \mathbb{Z}).$$

We additionally put a $\mathbb{Z}$-grading on $\mathbb{S}^q(\bar{A}, (e_1,\dots,e_n), w)$ by declaring the elements of $H^q(S_n, \mathbb{Z})$ to be homogeneous of degree $2n$.

Proposition 4.10 *As graded* $\mathbb{F}[U]$*-modules*, $\mathbb{H}^*(\bar{A}, (e_1,\dots,e_n), w) \cong \mathbb{S}^*(\bar{A}, (e_1,\dots,e_n), w)$.

Proof This is proved in exactly the same way as [13, Theorem 3.1.12(a)]. □

Notation 4.11 From now on we will denote lattice cohomology by $\mathbb{H}^*$ regardless of which construction we are using.

4.3.4 Lattice cohomology associated to negative semidefinite plumbings Fix a negative semidefinite plumbing graph Γ and let k be a characteristic vector of $X(\Gamma)$ such that $[k]$ is a torsion spin^c structure on $Y(\Gamma)$.

We now show how to associate a lattice cohomology module to the pair (Γ, k). Let $L = H_2(X(\Gamma); \mathbb{Z})$ and $\bar{L} = H_2(X(\Gamma); \mathbb{Z})/\ker(j_*)$. By the long exact sequence in homology, $\bar{L}$ is isomorphic to a submodule of the free finitely generated $\mathbb{Z}$-module $H_2(X, Y; \mathbb{Z})$ and therefore is itself free and finitely generated. As in Section 3, let $s = \operatorname{rank}(H_2(X; \mathbb{Z}))$. Also, let $\sigma = s - b_1(Y)$. With this notation, we have that $\bar{L} \cong \mathbb{Z}^\sigma$. Furthermore, after choosing an ordering on the vertices, the plumbing gives us an ordered basis $([v_1],\dots,[v_s])$ of L.

We now have almost all the data we need in order to get lattice cohomology. It remains to define a set of weight functions. To do this, we rely on our choice of characteristic vector k.

4.3.5 Weight functions Let $\chi_k\colon L \to \mathbb{Z}$ be the function defined by $\chi_k(x) = -\frac{1}{2}(k(x) + (x,x))$.

Proposition 4.12 *$\chi_k\colon L \to \mathbb{Z}$ descends to a well-defined function $\bar{\chi}_k\colon \bar{L} \to \mathbb{Z}$.*

Proof Since $[k]$ is assumed to be a torsion spin^c structure on Y there exists, by Remark 3.7, some $z_k \in L \otimes \mathbb{Q}$ such that $k(x) = (z_k, x)$ for all $x \in L$. Now suppose $x \in L$ and $x' \in \ker(j_*)$. Then

$$\begin{aligned}\chi_k(x+x') &= -\frac{k(x+x') + (x+x', x+x')}{2} \\ &= \chi_k(x) - \frac{k(x') + 2(x,x') + (x',x')}{2} \\ &= \chi_k(x) - \tfrac{1}{2}(z_k + 2x + x', x') \\ &= \chi_k(x) - \tfrac{1}{2} PD[j_*(x')](z_k + 2x + x') \\ &= \chi_k(x).\end{aligned}$$

□

To make it easier to state some qualitative properties of $\bar{\chi}_k$, we now consider the extension of $\bar{\chi}_k$ by scalars to the function $\bar{\chi}_k^{\mathbb{R}}\colon \bar{L} \otimes \mathbb{R} \to \mathbb{R}$. Notice that the negative semidefinite intersection form $(\cdot,\cdot)\colon L \times L \to \mathbb{Z}$ descends to a negative definite symmetric bilinear pairing on $\bar{L}$ which we denote by $(\cdot,\cdot)_{\bar{L}}$. Extending by scalars, we get a negative definite intersection form $(\cdot,\cdot)_{\bar{L}\otimes\mathbb{R}}\colon (\bar{L}\otimes\mathbb{R}) \times (\bar{L}\otimes\mathbb{R}) \to \mathbb{R}$. Therefore,

$$\bar{\chi}_k^{\mathbb{R}}(\bar{x}) = -\tfrac{1}{2}\big(k(x) + (\bar{x},\bar{x})_{\bar{L}\otimes\mathbb{R}}\big) = -\tfrac{1}{2}(\bar{z_k} + \bar{x}, \bar{x})_{\bar{L}\otimes\mathbb{R}}.$$

In particular, we see that $\bar{\chi}_k^{\mathbb{R}}$ is a positive definite quadratic form plus a linear shift. Putting these observations together yields the following proposition.

Proposition 4.13 (1) *$\bar{\chi}_k^{\mathbb{R}}$ is bounded below.*

(2) *Let $\{\bar{x}_1, \dots, \bar{x}_\sigma\}$ be any $\mathbb{R}$-basis of $\bar{L} \otimes \mathbb{R}$. Identify $\bar{L} \otimes \mathbb{R}$ with $\mathbb{R}^\sigma$ via $\bar{L} \otimes \mathbb{R} = \bigoplus_{j=1}^{\sigma} \mathbb{R}\bar{x}_j$. Then the level sets of $\bar{\chi}_k^{\mathbb{R}}\colon \mathbb{R}^\sigma \to \mathbb{R}$ are $(\sigma-1)$-dimensional ellipsoids and the sublevel sets are σ-dimensional balls bounded by these ellipsoids.*

Corollary 4.14 *$\bar{\chi}_k\colon \bar{L} \to \mathbb{Z}$ is bounded below and its sublevel sets are finite.*

Definition 4.15 Define $w_q\colon \mathcal{Q}_q \to \mathbb{Z}$ by

$$w(\square(\bar{l}, I)) = \max\left\{ \bar{\chi}_k(\bar{x}) \;\middle|\; \bar{x} = \bar{l} + \sum_{j\in J} \overline{[v_j]},\, J \subseteq I \right\}.$$

Note, $w\colon \mathcal{Q}_0 \to \mathbb{Z}$ is simply $\bar{\chi}_k$.

By Corollary 4.14, w is a valid set of weight functions.

Definition 4.16 Define $\mathbb{H}^*(\Gamma, k) = \mathbb{H}(\bar{L}, ([v_1], \dots, [v_s]), w)$.

As in the case with negative definite plumbings, different choices of representatives for $[k]$ yield isomorphic lattice cohomology up to an overall grading shift. More specifically,

Lemma 4.17 [12, Lemma 3.3.2] *If $k' = k + 2PD[j_*(l)]$ for some $l \in L$, then*

$$\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, k')[2\bar{\chi}_k(\bar{l})].$$

Remark 4.18 Némethi uses the opposite convention for grading shifts. Hence, [12, Lemma 3.3.2] is stated as $\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, k')[-2\bar{\chi}_k(\bar{l})]$.

4.4 Graded roots associated to negative semidefinite plumbings

Definition 4.19 [12, Definition 3.2] (1) Let R be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We denote by $[u, v]$ the edge with endpoints u and v. We say that R is a graded root with grading $\chi\colon \mathcal{V} \to \mathbb{Z}$ if

(a) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in \mathcal{E}$,

(b) $\chi(u) > \min\{\chi(u), \chi(w)\}$ for any $[u, v], [u, w] \in \mathcal{E}$ with $v \neq w$,

(c) χ is bounded below, $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$, and $\#\chi^{-1}(k) = 1$ if k is sufficiently large.

(2) We say that $v \in \mathcal{V}$ is a local minimum point of the graded root (R, χ) if $\chi(v) < \chi(w)$ for any edge $[v, w]$.

(3) If (R, χ) is a graded root, and $r \in \mathbb{Z}$, then we denote by $(R, \chi)[r]$ the same R with the new grading $\chi[r](v) := \chi(v) + r$. (This can be generalized for any $r \in \mathbb{Q}$ as well.)

Example 4.20 Figure 3 shows an example of a graded root.

We now show how to associate a graded root to a pair (Γ, k) where γ is a negative semidefinite plumbing and k is a characteristic vector of $X(\Gamma)$ such that $[k]$ is a torsion spin^c structure on $Y(\Gamma)$. For each $n \in \mathbb{Z}$, let $\bar{L}_{k,\leq n}$ be the graph whose vertex set is $\mathcal{V}(\bar{L}_{k,\leq n}) = \{\bar{x} \in \bar{L} \mid \bar{\chi}_k(\bar{x}) \leq n\}$ and such that there is an edge between two vertices $\bar{x}_1, \bar{x}_2$ if and only if $\bar{x}_1 - \bar{x}_2 = \pm\overline{[v_j]}$ where the v_j are as in Section 3.1. Now let $\pi_0(\bar{L}_{k,\leq n})$ denote the set of connected components of the graph $\bar{L}_{k,\leq n}$.

The graded root $(\bar{R}_k, \bar{\chi}_k)$ associated to Γ and k is constructed as follows:

- The vertex set is $\mathcal{V}(\bar{R}_k) = \bigsqcup_{n\in\mathbb{Z}} \pi_0(\bar{L}_{k,\leq n})$. By an abuse of notation, we denote the grading $\mathcal{V}(\bar{R}_k) \to \mathbb{Z}$ by $\bar{\chi}_k$ where now $\bar{\chi}_k|_{\pi_0(\bar{L}_{k,\leq n})} = n$.
- There is an edge between two vertices $v, v' \in \mathcal{V}(\bar{R}_k)$, which correspond to connected components C_v and $C_{v'}$, if and only if after possibly reordering v and v', we have $\bar{\chi}_k(v') = \bar{\chi}_k(v) + 1$ and $C_v \subset C_{v'}$.

Remark 4.21 When Γ is negative definite, $(\bar{R}_k, \bar{\chi}_k)$ is precisely the graded root, (R_k, χ_k), defined by Némethi [12, Section 4].

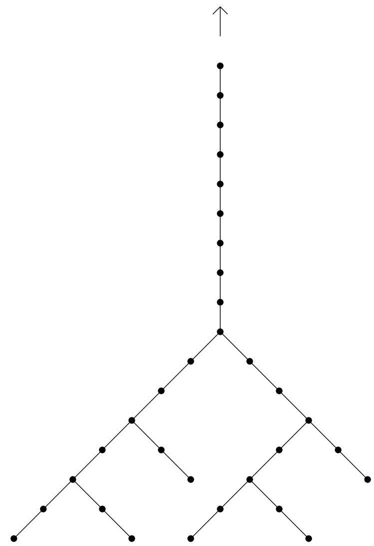


Figure 3

Remark 4.22 The graph $\bar{L}_{k,\leq n}$ is the 1-skeleton of the space S_n considered above in Construction 2 of lattice cohomology. In particular, we can think of $\pi_0(\bar{L}_{k,\leq n})$ equivalently as $\pi_0(S_n)$.

Proposition 4.23 [12, Proposition 4.3] *$(\bar{R}_k, \bar{\chi}_k)$ is a graded root.*

Proof This proof is essentially identical to the proof of [12, Proposition 4.3]. Condition (a) of Definition 4.19(1) follows immediately from the construction of $(R_k, \bar{\chi}_k)$. The proof of condition (b) is the same as in [12, Proposition 4.3]. The first two conditions of (c) follow from Corollary 4.14. The last condition of (c) follows the same argument as Némethi's proof, with mild modification. Essentially just replace the function χ_k in Némethi's proof with $\bar{\chi}_k$ and use that $\bar{\chi}_k$ has a (not necessarily unique) global minimum and that $(\cdot,\cdot)_{\bar{L}}$ is negative definite. □

Again, as in the case with negative definite plumbings, the graded roots, $(\bar{R}_k, \bar{\chi}_k)$ and $(\bar{R}_{k'}, \bar{\chi}_{k'})$ corresponding to two characteristic vectors k and k', which restrict to the same torsion spin^c structure on Y, are equal up to an overall grading shift. More specifically:

Proposition 4.24 [12, Proposition 4.4] *If $k' = k + 2PD[j_*(l)]$ for some $l \in L$ and $k \in \mathrm{Char}(X(\Gamma))$ with $[k]$ torsion, then*

$$(\bar{R}_{k'}, \bar{\chi}_{k'}) = (\bar{R}_k, \bar{\chi}_k)[\bar{\chi}_k(\bar{l})].$$

4.5 The relationship between lattice cohomology, H^+, and graded roots

In Section 4.1, we recalled the definition of the $\mathbb{F}[U]$-module $\mathrm{H}^+(\Gamma, [k])$ introduced by Ozsváth and Szabó where Γ is a negative definite plumbing and $[k]$ is a spin^c structure on $Y(\Gamma)$. The same definition makes sense for negative semidefinite plumbings and $[k]$ torsion except that we adjust the grading as follows: we say $\phi \in \mathrm{H}^+(\Gamma, [k])$ is a homogeneous element of degree d if for each $k' \in [k]$ with $\phi(k') \neq 0$, we have that $\phi(k') \in \mathcal{T}^+$ is a homogeneous element of degree

$$d + \frac{(k')^2 + |\mathcal{V}(\Gamma)| - 3b_1(Y(\Gamma))}{4}.$$

Proposition 4.25 *As graded* $\mathbb{F}[U]$*-modules,*

$$\mathrm{H}^+(\Gamma, [k]) \cong \mathbb{H}^0(\Gamma, k)\left[\frac{k^2 + |V(\Gamma)| - 3b_1(Y)}{4}\right].$$

Proof The isomorphism is induced by the map $Z\colon \mathrm{H}^+(\Gamma, [k]) \to \mathscr{F}^0$ defined by

$$Z(\phi)(\square(\bar{l}, \varnothing)) = \phi(k + 2PDj_*(l)).$$

We leave the details to the reader. □

As described in [17; 23], for calculation purposes it is convenient to consider the "dual space" of $\mathrm{H}^+(\Gamma, [k])$, which we denote by $\mathrm{K}^+(\Gamma, [k])$. To recall their definition of $\mathrm{K}^+(\Gamma, [k])$, first consider the set $\mathbb{Z}_{\geq 0} \times [k]$. Write elements $(m, k') \in \mathbb{Z}_{\geq 0} \times [k]$ as $U^m \otimes k'$. Define an equivalence relation $\sim$ on $\mathbb{Z}_{\geq 0} \times [k]$ in the following way: for each $k' \in [k]$ and $v_i \in \mathcal{V}(\Gamma)$, let $2n_i = k'([v_i]) + ([v_i], [v_i])$. Then

(1) if $n_i \geq 0$, we require $U^{n_i+m} \otimes (k' + 2PDj_*[v_i]) \sim U^m \otimes k'$;

(2) if $n_i < 0$, we require $U^m \otimes (k' + 2PDj_*[v_i]) \sim U^{m-n_i} \otimes k'$.

In other words, two elements $U^m \otimes k'$ and $U^n \otimes k''$ are equivalent if and only if there exists a finite sequence of elements $U^{m_0} \otimes k_1, \dots, U^{m_\ell} \otimes k_\ell$ such that $U^{m_0} \otimes k_1 = U^m \otimes k'$, $U^{m_\ell} \otimes k_\ell = U^n \otimes k''$ and each adjacent pair in the sequence is related by a relation of type (1) or (2) as given above. We call such a sequence a *path* connecting $U^m \otimes k'$ and $U^n \otimes k''$.

Remark 4.26 In general, there are many different paths connecting a given pair of elements $U^m \otimes k'$ and $U^n \otimes k''$.

Write the equivalence class containing $U^m \otimes k'$ as $\underline{U^m \otimes k'}$ and define $\mathrm{K}^+(\Gamma, [k])$ to be the set of these equivalence classes. $\mathrm{K}^+(\Gamma, [k])$ is the dual of $\mathrm{H}^+(\Gamma, [k])$ (or maybe more naturally $\mathrm{H}^+(\Gamma, [k])$ is the dual of $\mathrm{K}^+(\Gamma, [k])$) in the following sense:

- Define $\mathrm{K}^+(\Gamma, [k])^*$ to be the set of finitely supported functions $\phi\colon \mathrm{K}^+(\Gamma, [k]) \to \mathcal{T}^+$ such that $\phi(\underline{U^{n+m} \otimes k'}) = U^n \phi(\underline{U^m \otimes k'})$ for all $n, m \geq 0$ and $k' \in [k]$. Endow $(\mathrm{K}^+)^*$ with an $\mathbb{F}[U]$-module structure by inheriting that of $\mathcal{T}^+$.
- Define a map $F\colon \mathrm{H}^+(\Gamma, [k]) \to \mathrm{K}^+(\Gamma, [k])^*$ by

$$F(\phi)(\underline{U^m \otimes k'}) = U^m \phi(k').$$

It is straightforward to check that F is a well-defined $\mathbb{F}[U]$-module isomorphism.

We can put more structure on $\mathrm{K}^+(\Gamma,[k])$ by thinking of it as a graph. Specifically, define $g\mathrm{K}^+(\Gamma,[k])$ to be the graph whose vertices are the elements of $\mathrm{K}^+(\Gamma,[k])$ and such that there is an edge between to vertices $\underline{U^m \otimes k'}$ and $\underline{U^n \otimes k''}$ if and only if either $\underline{U^{m+1} \otimes k'} = \underline{U^n \otimes k''}$ or $\underline{U^m \otimes k'} = \underline{U^{n+1} \otimes k''}$.

Proposition 4.27 *As graphs, $g\mathrm{K}^+(\Gamma,[k])$ is isomorphic to the graded root $(\bar{R}_k, \bar{\chi}_k)$.*

Proof This proof is essentially the same as Némethi [12, Proof of Proposition 4.7]. For completeness, we provide the details here.

By definition each element $k' \in [k]$ can be written as $k' = k + 2PD[j_*(l)]$ for some $l \in L$. Let $\bar{l}_{k'} := \bar{l} \in \bar{L}$. Define a map $p\colon \mathrm{K}^+(\Gamma,[k]) \to \mathcal{V}(\bar{R}_k)$ as follows:

$$p(\underline{U^m \otimes k'}) = \text{the connected component of } \bar{L}_{k,\le \bar{\chi}_k(\bar{l}_{k'})+m} \text{ containing } \bar{l}_{k'}.$$

To show that p is well defined, let $2n_i = k'([v_i]) + ([v_i],[v_i])$. Suppose first that $n_i \ge 0$ so that we have $U^{n_i+m} \otimes (k' + 2PDj_*[v_i]) \sim U^m \otimes k'$. Let $k'' = k' + 2PDj_*[v_i]$. Then, $\bar{l}_{k''} = \bar{l}_{k'} + \overline{[v_i]}$. Thus,

$$\begin{aligned}\bar{\chi}_k(\bar{l}_{k''}) + n_i + m &= \bar{\chi}_k(\bar{l}_{k'}) + \bar{\chi}_{k'}(\overline{[v_i]}) + n_i + m \\ &= \bar{\chi}_k(\bar{l}_{k'}) - n_i + n_i + m \\ &= \bar{\chi}_k(\bar{l}_{k'}) + m.\end{aligned}$$

Therefore, $\bar{L}_{k,\le \bar{\chi}_k(\bar{l}_{k'})+m} = \bar{L}_{k,\le \bar{\chi}_k(\bar{l}_{k''})+n_i+m}$ and $\bar{l}_{k'}$ and $\bar{l}_{k''}$ are in the same connected component since they differ by $\overline{[v_i]}$. The case when $n_i < 0$ is similar. This establishes that p is well defined.

Next we define a map $q\colon \mathcal{V}(\bar{R}_k) \to \mathrm{K}^+(\Gamma,[k])$ which we will show is the inverse of p. Suppose $v \in \mathcal{V}(\bar{R}_k)$. Let C_v be the corresponding connected component in $\bar{L}_{k,\le \bar{\chi}_k(v)}$ and let $\bar{l}_v$ be some element in $\bar{L} \cap C_v$. Define

$$q(v) = \underline{U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}_v)} \otimes (k + 2PDj_*(l_v))}.$$

To show q is well defined, suppose $\bar{l}'$ is some other element in $\bar{L} \cap C_v$. It suffices to consider the case that $\bar{l}' = \bar{l}_v + \overline{[v_i]}$ for some i. First note,

$$\bar{\chi}_k(\bar{l}') = \bar{\chi}_k(\bar{l}_v + \overline{[v_i]}) = \bar{\chi}_k(\bar{l}_v) + \bar{\chi}_k(\overline{[v_i]}) - ([v_i], l_v).$$

Also,

$$(k + 2PDj_*(l_v))(v_i) + (v_i, v_i) = k(v_i) + (v_i, v_i) + 2(v_i, l_v) = -2[\bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)].$$

Hence, if $-[\bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)] \ge 0$, then

$$\begin{aligned}U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}_v)} \otimes (k + 2PDj_*(l_v)) &\sim U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}_v) - \bar{\chi}_k(\overline{[v_i]}) + (v_i, l_v)} \otimes (k + 2PDj_*(l_v + [v_i])) \\ &= U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}')} \otimes (k + 2PDj_*(l')).\end{aligned}$$

Similarly, if $-[\bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)] < 0$, then

$$\begin{aligned}U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}')} \otimes (k + 2PDj_*(l')) &\sim U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}') + \bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)} \otimes (k + 2PDj_*(l_v)) \\ &= U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}_v)} \otimes (k + 2PDj_*(l_v)).\end{aligned}$$

Therefore, q is well defined.

Now consider $qp(\underline{U^m \otimes k'})$ where $k' = k + 2PDj_*(\bar{l}_k)$. Let $v = p(\underline{U^m \otimes k'})$ and C_v be the connected component of $\bar{L}_{k,\le \bar{\chi}_k(\bar{l}_{k'})+m}$ containing $\bar{l}_{k'}$. Then, by definition,

$$q(v) = \underline{U^{\bar{\chi}_k(\bar{l}_{k'})+m-\bar{\chi}_k(\bar{l}_{k'})} \otimes k + 2PDj_*(\bar{l}_{k'})} = \underline{U^m \otimes k'}.$$

Hence, $qp = \mathrm{Id}$. The other direction, ie that $pq = \mathrm{Id}$, is tautological. Therefore, p is a bijection. To see that p takes edges to edges bijectively, let $v_1 = p(\underline{U^m \otimes k'})$ and $v_2 = p(\underline{U^{m+1} \otimes k'})$. It follows directly from the definition that $C_{v_1} \subset C_{v_2}$ and $\bar{\chi}_k(v_2) - \bar{\chi}_k(v_1) = 1$. □

Remark 4.28 It is useful to point out that under the isomorphism p constructed in the above proof, we have that

$$\mathrm{gr}(p(\underline{U^m \otimes k'})) = m - \tfrac{1}{8}((k')^2 - k^2).$$

4.6 A quick review of Rustamov's results on negative semidefinite plumbings with $b_1 = 1$

Rustamov [23] generalizes the setting in which the isomorphism T^+, described in Section 4.1, holds. In particular, Rustamov proves the following theorem:

Theorem 4.29 [23, Theorem 1.2] *Let Γ be a negative semidefinite plumbing with at most one bad vertex and with $b_1(Y(\Gamma)) = 1$. Further, let $[k]$ be a torsion* spin^c *structure. Then*

(1) $T^+ : HF^+_{\mathrm{odd}}(-Y(\Gamma), [k]) \to \mathrm{H}^+(\Gamma, [k])$ *is an isomorphism of graded $\mathbb{F}[U]$-modules;*

(2) $HF^+_{\mathrm{even}}(-Y(\Gamma), [k]) \cong \mathcal{T}^+_d$ *where $d = d_{-1/2}(-Y(\Gamma), [k])$.*

Here $HF^+_{\mathrm{odd}}(-Y(\Gamma), [k])$ and $HF^+_{\mathrm{even}}(-Y(\Gamma), [k])$ refer to the submodules generated by elements of $HF^+(-Y(\Gamma), [k])$ of degrees congruent to $1/2 \bmod 2$ and $-1/2 \bmod 2$ respectively.

Combining Rustamov's result with the observations of the previous section, we get:

Corollary 4.30 *With Γ as above,*

$$HF^+_{\mathrm{odd}}(-Y(\Gamma), [k]) \cong \mathbb{H}^0(\Gamma, k)\left[\frac{k^2 + |V(\Gamma)| - 3}{4}\right]$$

as graded $\mathbb{F}[U]$-modules. In particular, up to an overall grading shift, $\mathbb{H}^0(\Gamma, k)$ is a topological invariant of $Y(\Gamma)$.

Remark 4.31 It is likely possible that one can prove

$$\mathbb{H}^0(\Gamma, k)\left[\frac{k^2 + |V(\Gamma)| - 3}{4}\right]$$

is a topological invariant without appealing to Heegaard Floer homology, by showing invariance under Neumann moves as in the proof of [12, Proposition 4.6].

4.7 Involutions on lattice cohomology and Heegaard Floer homology

Let Γ be a negative semidefinite plumbing with at most one bad vertex and $b_1(Y(\Gamma)) = 1$. Let $[k]$ be a self-conjugate spin^c structure on $Y(\Gamma)$. In other words, $[k] = [-k]$ or, equivalently, $k = PD[j_*(l)]$ for some $l \in L$. Note that by identifying $\bar{l}$ with k, we can think of k as an element of $\bar{L}$.

As in [2, Section 2], define $J_0\colon \bar{L} \to \bar{L}$ by $J_0(\bar{x}) = -\bar{x} - \bar{l}$. Clearly, $J_0^2 = \mathrm{Id}$. We can extend J_0 to a cubical involution on the cube complex $\mathscr{C}$ considered in Construction 2 of lattice cohomology via the formula

$$J_0\square(\bar{a}, I) = \square\biggl(J_0\Bigl(\bar{a} + \sum_{i \in I} \overline{[v_i]}\Bigr), I\biggr).$$

It is straightforward to check that J_0 is compatible with the gluing of the cells. Moreover, since $\bar{\chi}_k(J_0(\bar{x})) = \bar{\chi}_k(\bar{x})$ for all $\bar{x} \in \bar{L}$, J_0 maps the subcomplex S_n of $\mathscr{C}$ to itself. Therefore, J_0 induces an involution on $H^q(S_n; \mathbb{Z})$ for each n and q, and hence on lattice cohomology. By an abuse of notation, we denote the involution on lattice cohomology again by J_0. In a similar manner, one could alternatively define J_0 by using Construction 1, but we leave the details to the reader.

Focusing our attention on the 0^{th} level of lattice cohomology, we can think of the action of J_0 on $\mathbb{H}^0$ from the dual perspective by realizing an involution on the associated graded root. More specifically, since J_0 acts continuously on S_n, J_0 also induces an involution on the connected components of S_n. Hence, J_0 induces an involution on the graded root $(\bar{R}_k, \bar{\chi}_k)$. From another perspective, under the identification of $(\bar{R}_k, \bar{\chi}_k)$ with $g\mathrm{K}^+(\Gamma, [k])$ given in Proposition 4.27, the involution J_0 sends $\underline{U^m \otimes k'}$ to $\underline{U^m \otimes -k'}$.

Dai and Manolescu [2, Theorem 3.1] showed that for negative definite almost rational plumbings, the involution J_0 on lattice cohomology is identified with the involution ι_* on Heegaard Floer homology under the isomorphism T^+ described in Section 4.1. We now show that their theorem also holds in the setting of negative semidefinite plumbed manifolds with at most one bad vertex and $b_1 = 1$.

Theorem 4.32 [2, Theorem 3.1] *Let Γ be a negative semidefinite plumbing with at most one bad vertex and $b_1(Y(\Gamma)) = 1$. If $[k]$ is a self-conjugate* spin^c *structure, then under the isomorphism T^+ given in Theorem 4.29*(1), *the maps J_0 and the restriction of ι_* to $HF^+_{\mathrm{odd}}(-Y(\Gamma), [k])$ are identified.*

Proof First note that the isomorphism $T^+\colon HF^+_{\mathrm{odd}}(-Y(\Gamma), [k]) \to \mathrm{H}^+(\Gamma, [k])$ for negative semidefinite plumbed manifolds with at most one bad vertex and $b_1 = 1$ is defined in precisely the same way as the isomorphism $T^+\colon HF^+(-Y(\Gamma), [k]) \to \mathrm{H}^+(\Gamma, [k])$ for negative definite almost rational plumbed manifolds. Therefore, as in the proof of [2, Theorem 3.1], to show that J_0 and ι_* are identified under T^+, one must show that $F^+_{W,k} = F^+_{W,-k} \circ \iota_*$. As noted in the proof of [2, Theorem 3.1], this equation follows from [20, Theorem 3.6]. □

For negative semidefinite plumbed manifolds with at most one bad vertex and $b_1 = 1$, the action of ι_* on the even part of HF^+ is less interesting. Since ι_* is U-equivariant and $HF^+_{\mathrm{even}}(-Y(\Gamma), [k]) \cong \mathscr{T}_d$ for $[k]$

self-conjugate, the restriction of ι_* to the even part must be the identity. Moreover, if one knows HF^+ and ι_*, then by using the mapping cone exact triangle in Proposition 2.2, one can completely determine HFI^+ as a graded $\mathbb{F}$-vector space.

In the context of negative definite almost rational plumbings, Dai and Manolescu [2, Sections 4 and 5]. showed that one can actually determine the entire $\mathbb{F}[U,Q]/(Q^2)$-module structure of HFI^+ just from knowing J_0. However, one encounters issues when trying to extrapolate their methods to the case of negative semidefinite plumbings with at most one bad vertex. The main difficulty is that in the negative definite almost rational case, HF^+ is supported in even gradings, whereas in the negative semidefinite case, HF^+ has both even and odd gradings which allows for the possibility of a more complicated action of ι at the chain level. Despite this issue, for negative semidefinite plumbings with at most one bad vertex whose HF^+ and ι_* are sufficiently simple, it is still possible to compute much, if not all, of the $\mathbb{F}[U,Q]/(Q^2)$-module structure of HFI^+ as well as the involutive d invariants just from the mapping cone exact triangle. We illustrate this via the examples in Section 5.

5 Small Seifert fibered space examples

In this section, we compute $HFI^+(-N_j,\mathfrak{s}_0)$ for the infinite family of small Seifert fiber spaces $\{N_j\}_{j\in\mathbb{N}}$ described in the introduction. As an application, we prove Theorem E. We also compute $HFI^+(-S^3_0(K_1),\mathfrak{s}_0)$ where $S^3_0(K_1)$ is the manifold obtained by 0-surgery on the Ichihara–Motegi–Song knot K_1 from [8]. We then compare $HFI^+(-S^3_0(K_1),\mathfrak{s}_0)$ and $HFI^+(-N_1,\mathfrak{s}_0)$.

Before computing HFI^+ of these specific manifolds, we give a brief outline in Section 5.1 of a general strategy for computing the graded root $(\bar{R}_k,\bar{\chi}_k)$, which forms a key part of our computation of HFI^+. We also, in Section 5.2, describe some combinatorial moves that will aid in the computations.

5.1 Computing the graded root

Let Γ be a negative semidefinite plumbing with at most one bad vertex and $b_1(Y(\Gamma))=1$. Let $[k]$ be a self-conjugate spin^c structure on $Y(\Gamma)$. To compute $(\bar{R}_k,\bar{\chi}_k)$, the first and main step is to determine the set

$$\mathscr{L}(\Gamma,[k]) := \{x\in \mathrm{K}^+(\Gamma,[k]) \mid x \text{ has no representative of the form } U^n\otimes k' \text{ for } n>0\}.$$

It is easy to see that the elements of $\mathscr{L}(\Gamma,[k])$ correspond to the leaves of the graded root $(\bar{R}_k,\bar{\chi}_k)$ under the isomorphism in Proposition 4.27. Moreover, from the results in Section 4.6, it follows that the leaves of $(\bar{R}_k,\bar{\chi}_k)$ correspond to a basis of the $\mathbb{F}$-vector space

$$\ker(U)\cap HF^+_{\mathrm{odd}}(-Y(\Gamma),[k]).$$

Rustamov [23, Section 3] provides an algorithm to compute $\mathscr{L}(\Gamma,[k])$ which builds on the Ozsváth and Szabó [17, Section 3] algorithm for negative definite plumbings. For our computations, rather than use

Rustamov's algorithm directly, we instead will use a simple criterion (see Proposition 5.1 below) which characterizes the elements of $\mathscr{L}(\Gamma, [k])$.

To explain this criterion, first recall from Section 4.5 that two elements $U^m \otimes k'$ and $U^n \otimes k''$ are equivalent (ie represent the same element of $\mathrm{K}^+(\Gamma, [k])$) if and only if there is a path between them. In particular, every element of $\mathscr{L}(\Gamma, [k])$ is represented by an element of the form $U^0 \otimes k'$ and every element of a path connecting $U^0 \otimes k'$ to another representative must also have 0 as the exponent on the U term. Therefore, when discussing representatives or paths for elements in $\mathscr{L}(\Gamma, [k])$, we can drop the U^0 term and instead think of a representative as an element $k' \in [k]$ and a path as a sequence of vectors $k_1, \dots, k_j \in [k]$. Furthermore, the relations defining such a path imply that for adjacent elements k_i and k_{i+1} we have that $k_{i+1} = k_i \pm 2PD[v]$ for some $v \in \mathscr{V}(\Gamma)$ with $k_i(v) = \mp m(v)$. Additionally, it follows from the definition that a representative k' of an element in $\mathscr{L}(\Gamma, [k])$ must satisfy

$$m(v) \le k'(v) \le -m(v)$$

for all $v \in \mathscr{V}(\Gamma)$. We refer to this property as $\star$ and we let $\star[k] = \{k' \in [k] \mid k' \text{ satisfies } \star\}$.

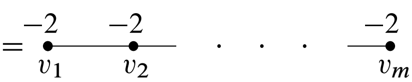
Combining these observations, we get the following proposition:

Proposition 5.1 *An element $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma, [k])$ if and only if k' satisfies $\star$ and every element on every path containing k' also satisfies $\star$.*

After using Proposition 5.1 to find elements $k_1, \dots, k_n \in [k]$ which represent the distinct elements of $\mathscr{L}(\Gamma, [k])$, it then follows that every other vertex of $(\bar{R}_k, \bar{\chi}_k)$ corresponds to an element of the form $\underline{U^m \otimes k_i}$ for some m and i. Of course, there could be relations of the form $\underline{U^{m_1} \otimes k_i} = \underline{U^{m_2} \otimes k_j}$. To determine these relations, in principle, one can write down the elements of the equivalence classes $\underline{U^{m_1} \otimes k_i}$ and $\underline{U^{m_2} \otimes k_j}$ and see whether they are equal. However, this can be quite tedious to do by hand and, in simple enough situations, there are shortcuts one can take by leveraging properties of HF^+. For example, we will use the relationship between Turaev torsion and HF^+ established in [18, Theorem 10.17] to complete the computation of $(\bar{R}_k, \bar{\chi}_k)$ for the manifolds N_j.

5.2 Moves between equivalent vectors

Let Γ be a negative semidefinite plumbing with at most one bad vertex and with $b_1 = 1$. Suppose Γ contains a linear subgraph Λ with framing -2 at each vertex, as shown below:

$\Lambda =$ 

Let $[k]$ be a self-conjugate spin^c structure on $Y(\Gamma)$. Given a characteristic vector $k' \in [k]$, let

$$k'_\Lambda = (a_1, \dots, a_m)$$

be the subvector corresponding to the vertices $v_1, \dots, v_m$. We call k'_Λ the Λ-subvector of k'.

Note, if $k' \in [k]$ and satisfies $\star$, then we must have $a_i \in \{-2, 0, 2\}$ for each $1 \le i \le m$. If there exists some i such that $a_i = \pm 2$, then $k'' = k' \pm 2PD[v_i]$ is an equivalent vector. In particular,

$$k''_\Lambda = (a_1, \dots, a_{i-1} \pm 2, \mp 2, a_{i+1} \pm 2, \dots, a_m).$$

Of course, other entries of k'' not contained in k''_Λ may also differ from those of k'. Specifically, any entry a of k' corresponding to a vertex adjacent to v_i will change from a to $a \pm 2$. We call the replacement of k' with $k'' = k' \pm 2PD[v_i]$ where $k'(v_i) = \pm 2$ a move of *type* ± 2.

Next suppose $k'_\Lambda = (a_1, \dots, a_i, 0, \dots, 0, 2, -2, a_j, \dots, a_m)$. Then, by iteratively applying type $+2$ moves to the $+2$-entry, we can convert k' into an equivalent vector k'' with

$$k''_\Lambda = (a_1, \dots, a_i, 2, -2, 0, \dots, 0, a_j, \dots, a_m).$$

We call the replacement of k' with k'' or k'' with k' a $(2, -2)$-*slide*. We define a $(-2, 2)$-*slide* analogously.

Lemma 5.2 *Let $k' \in [k]$ be a vector with $k'_\Lambda = (a_1, \dots, a_i, 0, \pm 2, 0, \dots, 0, \mp 2, a_j, \dots, a_m)$. Then k' is equivalent to a vector k'' with $k''_\Lambda = (a_1, \dots, a_i, \pm 2, 0, \dots, 0, \mp 2, 0, a_j, \dots, a_m)$.*

Proof Apply a type ± 2 move to the ± 2-entry to get an equivalent vector h' with

$$h'_\Lambda = (a_1, \dots, a_i, \pm 2, \mp 2, \pm 2, 0, \dots, 0, \mp 2, a_j, \dots, a_m).$$

Now do a rightward $(\mp 2, \pm 2)$-slide to h' to convert h' into an equivalent vector h'' with

$$h''_\Lambda = (a_1, \dots, a_i, \pm 2, 0, \dots, 0, \mp 2, \pm 2, \mp 2, a_j, \dots, a_m).$$

Finally apply a type ± 2 move to the rightmost ± 2-entry to get an equivalent vector k'' with

$$k''_\Lambda = (a_1, \dots, a_i, \pm 2, 0, \dots, 0, \mp 2, 0, a_j, \dots, a_m). \qquad \square$$

By iterating the sequence of moves described in the above proof, we can now convert any vector $k' \in [k]$ with

$$k'_\Lambda = (a_1, \dots, a_i, 0, \dots, 0, \pm 2, 0, \dots, 0, \mp 2, a_j, \dots, a_m)$$

into an equivalent vector k'' with

$$k''_\Lambda = (a_1, \dots, a_i, \pm 2, 0, \dots, 0, \mp 2, 0, \dots, 0, a_j, \dots, a_m).$$

By an abuse of notation, we also call the replacement of k' with k'' or k'' with k' via the above sequence of moves a $(\pm 2, \mp 2)$-*slide*.

Lemma 5.3 *Suppose $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma, [k])$. Then either k'_Λ is the zero vector or it has entries which alternate between 2 and -2 with possibly 0's in between.*

Proof Suppose k' represents an element of $\mathscr{L}(\Gamma, [k])$ and k'_Λ contains a subvector of the form

$$(2, \underbrace{0, \dots, 0}_{j}, 2)$$

where $j \geq 0$. Then, by doing a type $+2$ move on the leftmost $+2$-entry, k' is equivalent to a vector whose corresponding subvector is

$$(-2, 2, \underbrace{0, \dots, 0}_{j-1}, 2)$$

if $j \geq 1$ or $(-2, 4)$ if $j = 0$. In the latter case, the vector fails to satisfy $\star$ and thus we get a contradiction by Proposition 5.1. So we can assume the subvector is

$$(-2, 2, \underbrace{0, \dots, 0}_{j-1}, 2)$$

with $j \geq 1$. Now do a rightward $(-2, 2)$-slide to produce an equivalent vector whose corresponding subvector is

$$(\underbrace{0, \dots, 0}_{j-1}, -2, 2, 2).$$

Next apply a type $+2$ move to get an equivalent vector whose corresponding subvector is

$$(\underbrace{0, \dots, 0}_{j}, -2, 4).$$

We again get a contradiction for the same reason as before. Therefore, k'_Λ cannot contain a subvector of the form

$$(2, \underbrace{0, \dots, 0}_{j}, 2), \quad j \geq 0.$$

By an analogous argument, k'_Λ also cannot contain a subvector of the form

$$(-2, \underbrace{0, \dots, 0}_{j}, -2), \quad j \geq 0. \qquad \square$$

Lemma 5.4 *Suppose $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma, [k])$. Then k' is equivalent to a vector k'' such that k''_Λ is the zero vector except for possibly one nonzero entry equal to ± 2.*

Proof We induct on the number of nonzero entries of k'_Λ. Obviously the statement is true if k'_Λ is the zero vector or has only one nonzero entry. So suppose k'_Λ has $n \geq 2$ nonzero entries. Let a_i and a_{i+j} be the leftmost nonzero entries. Then by the Lemma 5.3, $a_i = \pm 2$ and $a_{i+j} = \mp 2$. For simplicity, assume $a_i = 2$. (The argument when $a_i = -2$ is identical up to sign changes.) We can write k'_Λ as

$$k'_\Lambda = (0, \dots, 0, 2, 0, \dots, 0, -2, a_{i+j+1}, \dots, a_m)$$

where there are possibly no initial 0 entries and no 0 entries between a_i and a_{i+j}. If there are initial 0 entries, then by doing a leftward $(2, -2)$-slide, k' is equivalent to a vector whose Λ-subvector is

$$(2, 0, \dots, 0, -2, 0, \dots, 0, a_{i+j+1}, \dots, a_m).$$

Now apply a type $+2$ move to the leftmost $+2$-entry to get an equivalent vector whose Λ-subvector is

$$(-2, 2, 0, \dots, 0, -2, 0, \dots, 0, a_{i+j+1}, \dots, a_m)$$

if $j > 1$, or

$$(-2, 0, \dots, 0, a_{i+2}, \dots, a_m)$$

if $j = 1$. In the latter case, we have reduced the number of nonzero entries in the Λ-subvector by 1. Hence, we can assume $j > 1$. In this case, if we do a rightward $(-2, 2)$-slide on leftmost $(-2, 2)$-pair, we get an equivalent vector whose Λ-subvector is

$$(0, \dots, 0, -2, 2, -2, 0, \dots, 0, a_{i+j+1}, \dots, a_m).$$

Finally apply a type $+2$ move to produce an equivalent vector whose Λ-subvector is

$$(0, \dots, 0, 0, -2, 0, 0, \dots, 0, a_{i+j+1}, \dots, a_m).$$

We have reduced the number of nonzero entries by 1. Therefore, by induction the result follows. □

Lemma 5.5 *Suppose $k' \in [k]$ with*

$$k'_\Lambda = (\underbrace{0, \dots, 0}_{j}, 2, \underbrace{0, \dots, 0}_{m-j-1}).$$

Then k' is equivalent to a vector k'' with

$$k''_\Lambda = (\underbrace{0, \dots, 0}_{m-j-1}, -2, \underbrace{0, \dots, 0}_{j}).$$

Proof We list the sequence of moves needed to obtain the relevant vector. In each move, we only write the resulting Λ-subvector.

(1) Type $+2$ move:
$$(\underbrace{0, \dots, 0}_{j-1}, 2, -2, 2\, \underbrace{0, \dots, 0}_{m-j-2}).$$

(2) Leftward $(2, -2)$-slide:
$$(2, -2, \underbrace{0, \dots, 0}_{j-1}, 2, \underbrace{0, \dots, 0}_{m-j-2}).$$

(3) Type $+2$ move:
$$(-2, \underbrace{0, \dots, 0}_{j}, 2, \underbrace{0, \dots, 0}_{m-j-2}).$$

(4) Rightward $(-2, 2)$-slide:
$$(\underbrace{0, \dots, 0}_{m-j-2}, -2, \underbrace{0, \dots, 0}_{j}, 2).$$

(5) Type $+2$ move:
$$(\underbrace{0, \dots, 0}_{m-j-2}, -2, \underbrace{0, \dots, 0}_{j-1}, 2, -2).$$

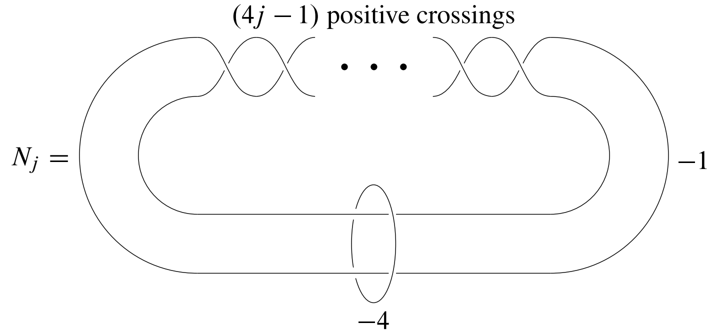


Figure 4

(6) Leftward $(2,-2)$-slide:

$$(\underbrace{0,\dots,0}_{m-j-2},-2,2,-2,\underbrace{0,\dots,0}_{j-1}).$$

(7) Type $+2$ move:

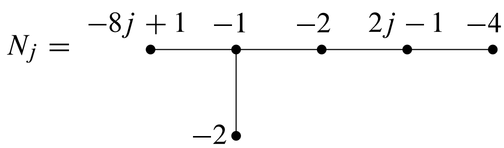
$$(\underbrace{0,\dots,0}_{m-j-1},-2,\underbrace{0,\dots,0}_{j}).$$ □

Remark 5.6 If one traces through the above sequence of moves, it is easy to see that if v is a vertex not in Λ, but is adjacent to the initial vertex v_1 or terminal vertex v_m of Λ, then $k''(v) = k'(v) + 2$.

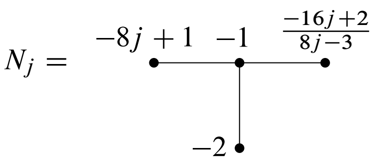
5.3 Computation of $HFI^+(-N_j, \mathfrak{s}_0)$

Recall, the 3-manifold N_j for $j \geq 1$ is given by the surgery diagram in Figure 4.

In [4, Section 7], it is shown via Kirby calculus that N_j can be represented as a plumbing as follows:



By performing two slam dunks on the rightward stem, we get:



One can further check that

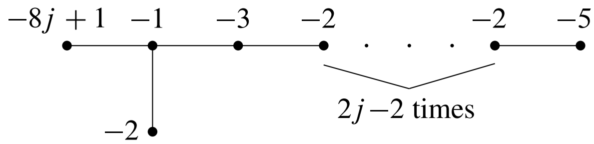
$$\frac{-16j+2}{8j-3} = -3 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{\frac{-8j+3+4r}{8j-7-4r}}}}$$

where there are r copies of -2 along the diagonal. In particular, setting $r = 2j-2$, the last term becomes

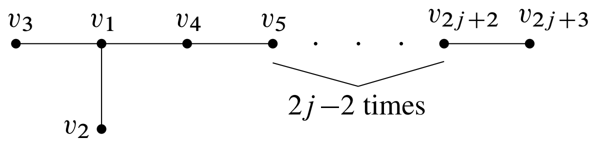
$$\frac{-8j+3+4(2j-2)}{8j-7-4(2j-2)} = -5.$$

Hence, by performing the corresponding slam dunks, we get:

$N_j =$



Let Γ_j be the above plumbing graph with vertices labeled as follows:



With respect to the ordered basis $([v_1], \dots, [v_{2j+3}])$, the matrix for the intersection form of $X(\Gamma_j)$ is

$$B_j = \begin{pmatrix} -1 & 1 & 1 & 1 & & & & & \\ 1 & -2 & & & & & & & \\ 1 & & -8j+1 & & & & & & \\ 1 & & & -3 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -5 \end{pmatrix}.$$

It is straightforward to check that B_j is negative semidefinite and $H_1(N_j; \mathbb{Z}) \cong \mathbb{Z}$, we leave this to the reader.

Note, the $\mathbb{Z}$-kernel of B_j is generated by the vector

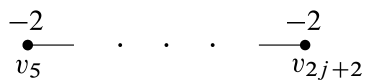
$$x = (16j-2, 8j-1, 2, 8j-3, 8j-7, 8j-11, \dots, 1).$$

Therefore, the unique self-conjugate spin^c-structure $\mathfrak{s}_0$ on N_j can be thought of as

$$[k] = \{k' \in \mathrm{Char}(X(\Gamma_j)) \mid k' \cdot x = 0\}.$$

Let Λ_j be the linear subgraph of Γ_j given by:

$\Lambda_j =$



We write vectors $k' \in [k]$ as

$$k' = (a_1, a_2, a_3, a_4, b_5, \dots, b_{2j+2}, c_{2j+3})$$

where $k'_{\Lambda_j} = (b_5, \dots, b_{2j+2})$.

Lemma 5.7 *If $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma_j, [k])$, then k' is equivalent to a vector whose Λ_j-subvector is not equal to the zero vector.*

Proof Suppose $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma_j, [k])$. For the purpose of contradiction, suppose the Λ_j-subvector of every representative of every element of $\mathscr{L}(\Gamma_j, [k])$ is zero. Then, in particular, $k'_{\Lambda_j} = 0$. Also, since k' represents an element of $\mathscr{L}(\Gamma_j, [k])$, it must satisfy $\star$. So we must have $a_4 \in \{-3, -1, 1, 3\}$. If $a_4 = \pm 3$, then by adding $\pm 2PD[v_4]$ to k' we would obtain an equivalent vector with a nonzero Λ_j-subvector. Thus, $a_4 \in \{-1, 1\}$.

Since k' must satisfy $\star$, we also have $a_1 = \pm 1$. If $a_1 = 1$ and $a_4 = 1$, then by adding $2PD[v_1]$ to k', a_4 becomes 3. But we just showed that a_4 cannot be equal to 3. Similarly, if $a_1 = -1$ and $a_4 = -1$, then by adding $-2PD[v_1]$ to k', a_4 becomes -3, which is again a contradiction. Hence, $a_1 = \pm 1$ and $a_4 = \mp 1$. By adding $-2PD[v_1]$ if necessary, we may assume $a_1 = 1$ and $a_4 = -1$. Again, by $\star$, we must have $a_2 \in \{-2, 0, 2\}$. If $a_2 = 2$, then by adding $2PD[v_1]$ to k', we get an equivalent vector with $a_2 = 4$, which contradicts Proposition 5.1. Therefore, $a_2 \in \{0, -2\}$. If $a_2 = -2$, then by adding $-2PD[v_2]$ to k' we obtain an equivalent vector with $a_1 = -1$ and $a_4 = -1$, which we already determined cannot happen. Therefore, $a_2 = 0$. Now add $2PD[v_1]$ to k'. The result is an equivalent vector with $a_1 = -1$, $a_2 = 2$, and $a_4 = 1$. Since $a_2 = 2$, we can add $2PD[v_2]$ to get an equivalent vector with $a_1 = 1, a_2 = -2$, and $a_4 = 1$, but we have already shown that we cannot have both $a_1 = 1$ and $a_4 = 1$. Therefore, we get a contradiction and hence k' must be equivalent to some vector whose Λ_j-subvector is not equal to the zero vector. □

Somewhat counterintuitively, we are now going to use the previous lemma to find a small finite set of possible representatives of $\mathscr{L}(\Gamma_j, [k])$, all of whose Λ_j-subvectors are all equal to the zero vector.

Lemma 5.8 *If $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma_j, [k])$, then k' is equivalent to a vector of the form*

$$k'' = (-1, 0, a_3, 3, 0, \dots, 0, c_{2k+3})$$

where $a_3 \in \{-8k + 1, -8k + 3, \dots, 8k + 1\}$ and $c_{2j+3} \in \{-5, -3, -1, 1, 3\}$.

Proof Suppose k' represents an element of $\mathscr{L}(\Gamma_j, [k])$. Then, by combining Lemmas 5.4, 5.5, and 5.7, we may assume

$$k'_{\Lambda_j} = (\underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell})$$

for some $0 \le \ell \le 2j - 3$. By $\star$, $a_1 = \pm 1$. If $a_1 = -1$, then we can add $-2PD[v_1]$ from k' to get an equivalent vector with $a_1 = 1$. This addition does not effect any of the entries in k'_{Λ_j}. Thus, we may assume $a_1 = 1$.

Next, by $\star$, $a_2 \in \{-2, 0, 2\}$. If $a_2 = 2$, then adding $2PD[v_1]$ to k' yields an equivalent vector with $a_2 = 4$, which violates $\star$. Therefore, $a_2 \in \{-2, 0\}$. Suppose $a_2 = -2$. Then, by adding $-2PD[v_2]$, we get an equivalent vector with $a_2 = 2$ and $a_1 = -1$. k'_{Λ_j} is unaffected by this move. If we then add $-2PD[v_1]$, we get an equivalent vector with $a_1 = 1$ and $a_2 = 0$. Again k'_{Λ_j} is unaffected. Therefore, we may assume $a_2 = 0$.

Next, with

$$k' = (1, 0, a_3, a_4, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3})$$

add $2PD[v_1]$ to k' to get the equivalent vector

$$(-1, 2, a_3 + 2, a_4 + 2, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Next, add $2PD[v_2]$ to get

$$(1, -2, a_3 + 2, a_4 + 2, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Then add another $2PD[v_1]$, to get

$$(-1, 0, a_3 + 4, a_4 + 4, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Now, if we apply the move in Lemma 5.5 and take into account Remark 5.6, one can check that we get an equivalent vector whose 4^{th} entry is $a_4 + 6$. Since we assumed k' represents an element of $\mathscr{L}(\Gamma_j, [k])$, we must therefore have that $a_4 \in \{-3, -1, 1, 3\}$ and $a_4 + 6 \in \{-3, -1, 1, 3\}$. Hence, we must have had $a_4 = -3$. To summarize, we have now shown that we can assume

$$k' = (1, 0, a_3, -3, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Next, by $\star$, $c_{2j+3} \in \{-5, -3, -1, 1, 3, 5\}$. If $c_{2j+3} = 5$, then again by applying the move from Lemma 5.5, one can check that we transform c_{2j+3} into 7, which violates $\star$. Therefore, we must have had $c_{2j+3} \in \{-5, -3, -1, 1, 3\}$.

Now add $-2PD[v_4]$ to get an equivalent vector (which we again call k') with $a_1 = -1$, $a_2 = 0$, $a_4 = 3$ and k'_{Λ_j} unchanged except for the first entry which decreases by 2. Also, c_{2k+3} remains unchanged. If $\ell = 0$, then k'_{Λ_j} is now the zero vector, so we are done. Thus, suppose $\ell > 0$. Then

$$k' = (-1, 0, a_3, 3, -2, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Now consider the following sequence of moves:

(1) Rightward $(-2, 2)$-slide:

$$(-1, 0, a_3, 3, \underbrace{0, \dots, 0}_{2j-3-\ell}, -2, \underbrace{0, \dots, 0}_{\ell-1}, 2, c_{2j+3}).$$

(2) Type $+2$ move:

$$(-1, 0, a_3, 3, \underbrace{0, \dots, 0}_{2j-3-\ell}, -2, \underbrace{0, \dots, 0}_{\ell-2}, 2, -2, c_{2j+3} + 2).$$

(3) Leftward $(2,-2)$-slide:

$$(-1, 0, a_3, 3, \underbrace{0, \dots, 0}_{2j-3-\ell}, -2, 2, -2\,\underbrace{0, \dots, 0}_{\ell-2}, c_{2j+3} + 2).$$

(4) Type $+2$ move:

$$(-1, 0, a_3, 3, \underbrace{0, \dots, 0}_{2j-2-\ell}, -2, \underbrace{0, \dots, 0}_{\ell-1}, c_{2j+3} + 2).$$

(5) Apply Lemma 5.5 and Remark 5.6:

$$(-1, 0, a_3, 1, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-2-\ell}, c_{2j+3}).$$

(6) Add $-2PD[v_1]$:

$$(1, -2, a_3 - 2, -1, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-2-\ell}, c_{2j+3}).$$

(7) Add $-2PD[v_2]$:

$$(-1, 2, a_3 - 2, -1, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-2-\ell}, c_{2j+3}).$$

(8) Add $-2PD[v_1]$:

$$(1, 0, a_3 - 4, -3, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-2-\ell}, c_{2j+3}).$$

The net effect of this sequence of moves is that the $+2$-entry in k'_{Λ_j} shifts one space to the left while every other entry, excluding a_3, remains the same. So now we can repeat the above process until $+2$-entry is in the first position of k'_{Λ_j}. Then add $-2PD[v_4]$ to get

$$(-1, 0, a_3', 3, 0, \dots, 0, c_{2j+3})$$

with $c_{jk+3} \in \{-5, -3, -1, 1, 3\}$ and, by $\star$, $a_3' \in \{-8j+1, -8j+3, \dots, 8j+1\}$. □

Proposition 5.9 *If k' represents an element of $\mathscr{L}(\Gamma_j, [k])$, then k' is equivalent to*

$$k_1 = (-1, 0, 5-4j, 3, 0, \dots, 0, -3) \quad \textit{or} \quad k_2 = (-1, 0, 3-4j, 3, 0, \dots, 0, 1).$$

In particular, $|\mathscr{L}(\Gamma_j, [k])| \le 2$.

Proof Up to this point, we have not used the fact that $k' \cdot x = 0$ where x is a generator of $\ker_{\mathbb{Z}}(B_j)$ as above. So assume k' is of the form in the previous lemma. Then

$$0 = k' \cdot x = 8j - 7 + 2a_3 + c_{2j+3}$$

where $a_3 \in \{-8j+1, -8k+3, \dots, 8j+1\}$ and $c_{2j+3} \in \{-5, -3, -1, 1, 3\}$. The only solutions to this equation with the given constraints are $(a_3, c_{2j+3}) = (5-4j, -3)$ and $(3-4j, 1)$, corresponding to k_1 and k_2, respectively. □

We have not yet proved that k_1 and k_2 represent different elements of $\mathcal{L}(\Gamma_j, [k])$. To do this we will do a similar analysis for $-N_j$ and then use Turaev torsion. However, before we undertake this task, we first compute the HF^+ grading associated to the vectors k_1 and k_2.

Corollary 5.10 $d_{1/2}(-N_j; \mathfrak{s}_0) = \frac{1}{2}$.

Proof Let

$$\alpha_1 = (-12j+6, -6j+3, -1, -6j+3, -6j, -6j-3, \dots, 2, 1, 0),$$
$$\alpha_2 = (4j+2, 2j+1, 1, 2j-1, 2j-2, 2j-3, \dots, 2, 1, 0).$$

Then $\alpha_1 B_j = k_1$ and $\alpha_2 B_j = k_2$. Thus,

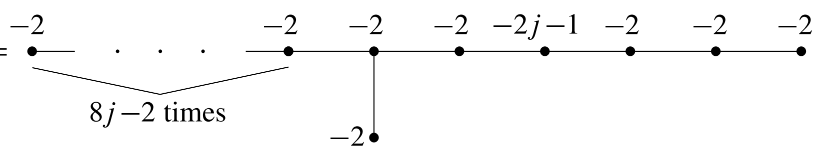
$$k_1^2 = k_1 \cdot \alpha_1 = -2j-2,$$
$$k_2^2 = k_2 \cdot \alpha_2 = -2j-2.$$

Hence, under the isomorphism from Corollary 4.30, the elements of $HF^+(-N_j, \mathfrak{s}_0)$ corresponding to k_1 and k_2 have gradings

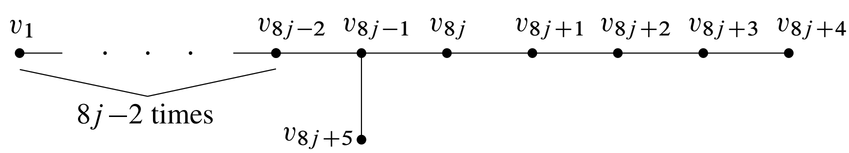
$$\mathrm{gr}(k_1) = \mathrm{gr}(k_2) = -\frac{k_2^2 + |\mathcal{V}(\Gamma_j)| - 3}{4} = -\frac{-2j-2+2j+3-3}{4} = \frac{1}{2}.$$ □

We now find a plumbing representation of $-N_j$ and then do Kirby calculus to make it negative semidefinite, as shown in Figure 5.

Now do slam dunks on the left and right vertices to get:

$-N_j =$


Let Γ_j' be the above plumbing graph with vertices labeled as follows:



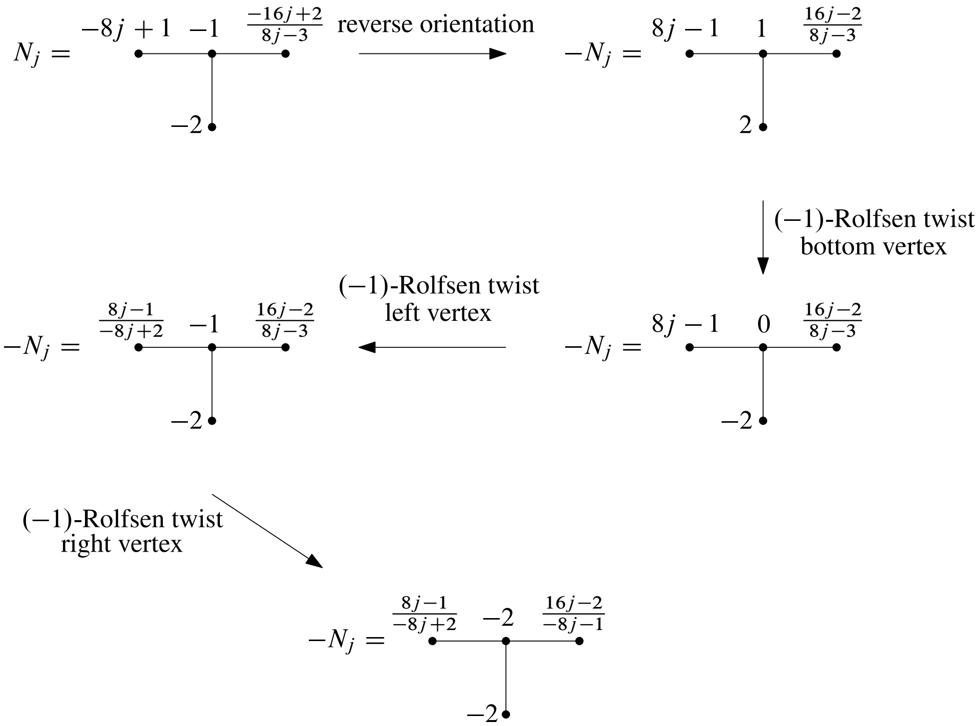


Figure 5

With respect to the ordered basis $([v_1], \ldots, [v_{8j+5}])$, the matrix for the intersection form of $X(\Gamma_j')$ is

$$B_j' = \begin{pmatrix} -2 & 1 & & & & & & & & & & \\ 1 & -2 & 1 & & & & & & & & & \\ & & & \ddots & & & & & & & & \\ & & & 1 & -2 & 1 & & & & & & \\ & & & & 1 & -2 & 1 & & & & & 1 \\ & & & & & 1 & -2 & 1 & & & & \\ & & & & & & 1 & -2j-1 & 1 & & & \\ & & & & & & & 1 & -2 & 1 & & \\ & & & & & & & & 1 & -2 & 1 & \\ & & & & & & & & & 1 & -2 & \\ & & & & & 1 & & & & & & -2 \end{pmatrix}.$$

Again, it is straightforward to check that B_j' is negative semidefinite. Also, the $\mathbb{Z}$-kernel of B_j' is generated by the vector

$$x' = (2, 4, 6, \ldots, 16j-2, 8j+1, 4, 3, 2, 1, 8j-1).$$

Let t denote a characteristic vector representing the trivial spin^c structure $\mathfrak{s}_0$. Then again, we can think of $\mathfrak{s}_0$ as

$$[t] = \{t' \in \operatorname{Char}(X(\Gamma_j')) \mid t' \cdot x' = 0\}.$$

Let Λ'_j be the linear subgraph of Γ'_j given by:

$$\Lambda'_j = \overset{-2}{\underset{v_1}{\bullet}}\!\!-\!\!-\quad \cdot \quad \cdot \quad \cdot \quad -\!\!-\!\!\overset{-2}{\underset{v_{8j}}{\bullet}}$$

We write vectors $t' \in [t]$ as

$$t' = (a_1, a_2, \dots, a_{8j}, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, d_{8j+5})$$

where $t'_{\Lambda'_j} = (a_1, a_2, \dots, a_{8j})$.

Lemma 5.11 *If $t' \in [t]$ represents an element of $\mathscr{L}(\Gamma'_j, [t])$, then t' is equivalent to a vector whose Λ'_j-subvector is of the form*

$$(0, \dots, 0, a_{8j})$$

where $a_{8j} \in \{0, 2\}$.

Proof Suppose t' represents and element of $\mathscr{L}(\Gamma'_j, [t])$. By Lemmas 5.4 and 5.5, it suffices to consider the case when

$$t'_{\Lambda'_j} = (\underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{8j-1-\ell})$$

for some $0 \le \ell \le 8j-2$. Furthermore, by considering the linear subgraph of Γ'_j whose endpoints are $v_{\ell+1}$ and v_{8j+5}, it follows from Lemma 5.3 that $d_{8j+5} \in \{0, -2\}$.

Case 1 Suppose $d_{8j+5} = -2$ and $\ell = 8j-2$. If we add $-2PD[v_{8j+5}]$, then the Λ'_j-subvector of the resulting vector is zero, so we are done.

Case 2 Suppose $d_{8j+5} = -2$ and $\ell \le 8j-3$. Consider the following sequence of moves:

(1) Add $-2PD[v_{8j+5}]$:

$$(\underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{8j-3-\ell}, -2, 0, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 2).$$

(2) Rightward $(2,-2)$-slide:

$$(\underbrace{0, \dots, 0}_{\ell+1}, 2, \underbrace{0, \dots, 0}_{8j-3-\ell}, -2, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0).$$

Note, the rightmost entry of the vector changes from 2 to 0.

(3) Type -2 move on the leftmost -2:

$$\begin{cases} (\underbrace{0, \dots, 0}_{8j-1}, 2, b_{8j+1}-2, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0) & \text{if } \ell = 8j-3, \\ (\underbrace{0, \dots, 0}_{\ell+1}, 2, \underbrace{0, \dots, 0}_{8j-4-\ell}, -2, 2, b_{8j+1}-2, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0) & \text{if } \ell \le 8j-4. \end{cases}$$

If $\ell = 8j - 3$ we are done. If $\ell = 8j - 4$, then by applying a type -2 move on the leftmost -2, we get

$$(\underbrace{0, \dots, 0}_{8j-2}, 2, 0, b_{8j+1} - 2, c_{8j+2}, c_{8j+3}, c_{8j+4}, -2).$$

Hence, we are back to case 1. Therefore, we may assume $\ell \le 8j - 5$. We now continue as follows:

(4) Leftward $(-2, 2)$-slide:

$$(\underbrace{0, \dots, 0}_{\ell+1}, 2, -2, 2\, \underbrace{0, \dots, 0}_{8j-4-\ell}, b_{8j+1} - 2, c_{8j+2}, c_{8j+3}, c_{8j+4}, -2).$$

Note, the rightmost entry of the vector now changes back to -2.

(5) Type -2 move on the leftmost -2:

$$(\underbrace{0, \dots, 0}_{\ell+2}, 2, \underbrace{0, \dots, 0}_{8j-3-\ell}, b_{8j+1} - 2, c_{8j+2}, c_{8j+3}, c_{8j+4}, -2).$$

We are now back to the vector we started with at the beginning of case 2, except that the $+2$ entry of the Λ'_j-subvector has shifted two positions to the right. Therefore, we can iterate this process until $\ell = 8j - 3$ or $8j - 4$, and we have already dealt with both of those cases.

Case 3 Suppose $d_{8j+5} = 0$ and $\ell = 8j - 2$. Add $2PD[v_{8j-1}]$ to get the equivalent vector

$$(\underbrace{0, \dots, 0}_{8j-3}, 2, -2, 2, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 2).$$

Now add $2PD[v_{8j+5}]$ to get

$$(\underbrace{0, \dots, 0}_{8j-3}, 2, 0, 2, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, -2).$$

This vector violates Lemma 5.3 and hence cannot be a representative of $\mathscr{L}(\Gamma'_j, [t])$.

Case 4 Suppose $d_{8j+5} = 0$ and $\ell \le 8j - 3$, so that we start with a vector of the form

$$(\underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{8j-1-\ell}, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0).$$

Now consider the following sequence of moves:

(1) Type $+2$ move:

$$(\underbrace{0, \dots, 0}_{\ell-1}, 2, -2, 2, \underbrace{0, \dots, 0}_{8j-2-\ell}, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0).$$

(2) Rightward $(-2, 2)$-slide:

$$(\underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{8j-3-\ell}, -2, 2, 0, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0).$$

(3) Add $2PD[v_{8j-1}]$:

$$(\underbrace{0,\dots,0}_{\ell-1},2,\underbrace{0,\dots,0}_{8j-2-\ell},-2,2,c_{8j+2},c_{8j+3},c_{8j+4},2).$$

(4) Add $2PD[v_{8j+5}]$:

$$(\underbrace{0,\dots,0}_{\ell-1},2,\underbrace{0,\dots,0}_{8j-1-\ell},2,c_{8j+2},c_{8j+3},c_{8j+4},-2).$$

Again, this vector violates Lemma 5.3 and hence cannot be a representative of $\mathscr{L}(\Gamma'_j,[t])$. □

Proposition 5.12 *If $t'\in[t]$ represents an element of $\mathscr{L}(\Gamma'_j,[t])$, then t' is equivalent to*

$$t_1=(0,\dots,0,-1,0,2,0,0) \quad \text{or} \quad t_2=(0,\dots,0,2,-1,0,0,0,-2).$$

Proof Suppose $t'\in[k]$ represents an element of $\mathscr{L}(\Gamma'_j,[t])$. By the previous lemma, we can assume

$$t'=(0,\dots,0,a_{8j},b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},d_{8j+5})$$

where $a_{8j}\in\{0,2\}$, $b_{8j+1}\in\{-2j-1,-2j+1,\dots,2j-1,2j+1\}$, $c_{8j+2},c_{8j+3},c_{8j+4}\in\{-2,0,2\}$, and $d_{8j+5}\in\{-2,0,2\}$.

Since we are assuming t' represents an element of $\mathscr{L}(\Gamma'_j,[t])$, we must have

(5.13) $$0=t'\cdot x'=(8j+1)a_{8j}+(8j-1)d_{8j+5}+4b_{8j+1}+3c_{8j+2}+2c_{8j+3}+c_{8j+4}.$$

By Lemmas 5.4 and 5.5, we can assume $(c_{8j+2},c_{8j+3},c_{8j+4})$ is the zero vector or has exactly one nonzero entry equal to $+2$. In particular, we can assume

$$3c_{8j+2}+2c_{8j+3}+c_{8j+4}\in\{0,2,4,6\}.$$

Note the moves required to put the subvector $(c_{8j+2},c_{8j+3},c_{8j+4})$ into this form only effect the entry b_{8j+1} and leave all of the others unchanged.

Now suppose $a_{8j}=2$ and $b_{8j+1}=2j+1$. Then by adding $2PD[v_{8j+1}]$ we would obtain an equivalent vector with $a_{8j}=4$. But this violates $\star$. Hence, if $a_{8j}=2$, we can assume $b_{8j+1}\le 2j-1$.

Now suppose $a_{8j}=0$ and $b_{8j+1}=2j+1$. If $(c_{8j+2},c_{8j+3},c_{8j+4})$ is not the zero vector, but rather a vector with precisely one nonzero entry equal to $+2$, then by applying the move in Lemma 5.5 and taking into account Remark 5.6, we would obtain an equivalent vector with $b_{8j+1}=2j+3$, which violates $\star$. Therefore, if $a_{8j}=0$ and $b_{8j+1}=2j+1$, we must have that $(c_{8j+2},c_{8j+3},c_{8j+4})$ is the zero vector. Plugging this into (5.13) yields

$$(8j-1)d_{8j+5}=-8j-4.$$

This clearly has no solutions with the given constraints. Therefore, we can assume $b_{8j+1}\le 2j-1$, regardless of whether $a_{8j}=0$ or 2. In particular,

$$-8j-4\le 4b_{8j+1}+3c_{8j+2}+2c_{8j+3}+c_{8j+4}\le 8j+2.$$

Case 1 Suppose $a_{8j} = 0$ and $d_{8j+5} = -2$. Then

$$0 = t' \cdot x' = -16j + 2 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \leq -8j + 4 < 0$$

which is a contradiction.

Case 2 Suppose $a_{8j} = 0$ and $d_{8j+5} = 0$. Then

$$0 = t' \cdot x' = 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4}.$$

The only solution to this equation given the constraints we have established is

$$(b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}) = (-1, 0, 2, 0)$$

which corresponds to t_1.

Case 3 Suppose $a_{8j} = 0$ and $d_{8j+5} = 2$. Then

$$0 = t' \cdot x' = 16j - 2 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \geq 8j - 6 > 0$$

which again is a contradiction.

Case 4 Suppose $a_{8j} = 2$ and $d_{8j+5} = -2$. Then

$$0 = t' \cdot x' = 4 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4}.$$

The only solution to this equation given the constraints we have established is

$$(b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}) = (-1, 0, 0, 0)$$

which corresponds to t_2.

Case 5 Suppose $a_{8j} = 2$ and $d_{8j+5} = 0$. Then

$$0 = t' \cdot x' = 16j + 2 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \geq 8j - 2 > 0$$

which again is a contradiction. Finally:

Case 6 Suppose $a_{8j} = 2$ and $d_{8j+5} = 2$. This case is ruled out by Lemma 5.3. □

Again, we have not yet proved that t_1 and t_2 represent different elements of $\mathscr{L}(\Gamma'_j, [t])$; however, we do have:

Corollary 5.14 $$d_{1/2}(N_j, \mathfrak{s}_0) = -2j + \tfrac{1}{2}.$$

Proof Let

$$\beta_1 = (2, 4, 6, \dots, 16j - 2, 8j + 1, 4, 2, 0, 0, 8j - 1) \quad \text{and} \quad \beta_2 = (0, \dots, 0, -1, 0, 0, 0, 0, 1).$$

Then $\beta_1 B_j' = t_1$ and $\beta_2 B_j' = t_2$. Thus,

$$t_1^2 = t_1 \cdot \beta_1 = -4 \quad \text{and} \quad t_2^2 = t_2 \cdot \beta_2 = -4.$$

Hence, under the isomorphism in Corollary 4.30, the elements of $HF^+(N_j, \mathfrak{s}_0)$ corresponding to t_1 and t_2 have gradings

$$\operatorname{gr}(t_1) = \operatorname{gr}(t_2) = -\frac{t_2^2 + |\mathcal{V}(\Gamma_j')| - 3}{4} = -\frac{-4 + 8j + 5 - 3}{4} = -2j + \frac{1}{2}. \qquad \square$$

Now combining Corollaries 5.10 and 5.14, and the basic fact that $d_{\pm 1/2}(-Y) = -d_{\mp 1/2}(Y)$ (see [16, Proposition 4.10]), we have

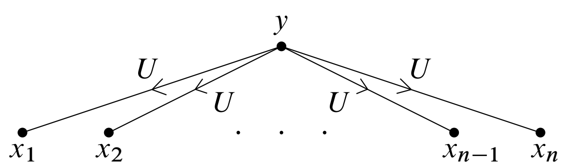
$$d_{1/2}(-N_j) = \tfrac{1}{2} \quad \text{and} \quad d_{-1/2}(-N_j) = 2j - \tfrac{1}{2}.$$

In particular, by Theorem 4.29, $HF^+_{\text{even}}(-N_j, \mathfrak{s}_0) = \mathcal{T}^+_{2j-1/2}$.

We have yet to completely determine $HF^+_{\text{odd}}(-N_j, \mathfrak{s}_0)$. So far, from Proposition 5.9, we know that $\dim_{\mathbb{F}}[\ker(U) \cap HF^+_{\text{odd}}(-N_j, \mathfrak{s}_0)] = 1$ or 2 depending on whether k_1 and k_2 represent the same element or not in $\mathcal{L}(\Gamma_j, [k])$. Therefore, as graded $\mathbb{F}[U]$-modules, we have one of the equivalences in Figure 6. Here, h_j is some positive integer depending on j which we have not yet determined.

A word of explanation is in order since on the left side of the above isomorphism we have an $\mathbb{F}[U]$-module and on the right we have one of two possible graphs. The right side is to be interpreted as follows:

- Each vertex at grading r corresponds to a basis element of the $\mathbb{F}$-vector space $HF^+_r(-N_j, \mathfrak{s}_0)$.
- If the edges emanating from a vertex y are of the form



  then $Uy = x_1 + x_2 + \cdots + x_{n-1} + x_n$. In particular, if there are no edges emanating from y, then $Uy = 0$.

We now utilize Turaev torsion. Combining our computations thus far with [18, Theorem 10.17], we see that

$$T_{N_j}(\mathfrak{s}_0) = \begin{cases} h_j + j & \text{if } k_1 \neq k_2 \in \mathcal{L}(\Gamma_j, [k]), \\ j & \text{otherwise,} \end{cases}$$

where T_{N_j} is the Turaev torsion function associated to N_j (see [24, page 119]). Therefore, to precisely determine $HF^+_{\text{odd}}(-N_j, \mathfrak{s}_0)$, it suffices to compute $T_{N_j}(\mathfrak{s}_0)$.

There are many standard ways to compute $T_{N_j}(\mathfrak{s}_0)$. For example, Turaev [24] provided a formula in terms of a surgery description. We will now give a brief outline of how to carry out the calculation using this method, but we leave the details to the reader.

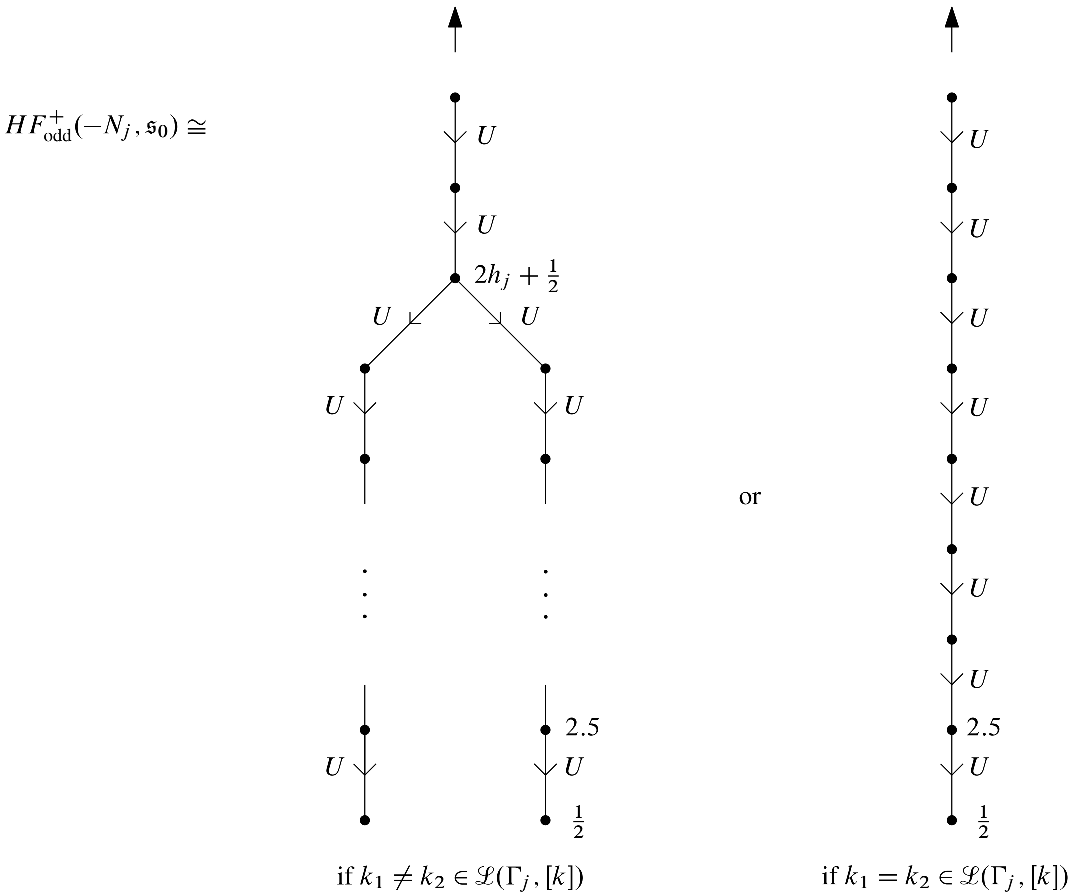


Figure 6

(1) Let $H = H_1(N_j; \mathbb{Z})$. Consider the group ring $\mathbb{Z}[H]$. Since $H \cong \mathbb{Z}$, we can think of $\mathbb{Z}[H]$ as $\mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials in the indeterminate t. Let $Q(H)$ denote the field of fractions of $\mathbb{Z}[H]$. The first step is to compute the Turaev torsion $\tau(N_j, \mathfrak{s}_0) \in Q(H)$. For this, we use the formula given in [24, VII.2, Theorem 2.2]. To apply this formula, we need to choose a surgery diagram for N_j and orient the underlying link. We use the surgery diagram in Figure 7 with underlying link L_j oriented as indicated by the arrows.

The bulk of the work in computing $\tau(N_j, \mathfrak{s}_0)$ using [24, VII.2, Theorem 2.2] is calculating the multivariable Alexander–Conway function $\nabla(L_j)$. Again, there are various approaches to computing $\nabla(L_j)$. For example, Murakami [11] provided a skein formula for ∇. Using this formula, we find that $\nabla(L_j) = yx^{4j-1} + y^{-1}x^{-4j+1}$ where the variable x corresponds to the torus knot component and the variable y corresponds to the unknot component. Plugging this into the formula for $\tau(N_j, \mathfrak{s}_0)$, we get

$$\tau(N_j, \mathfrak{s}_0) = \frac{t^{8j-1} + 1}{t^{4j-2}(t-1)^2(t+1)}.$$

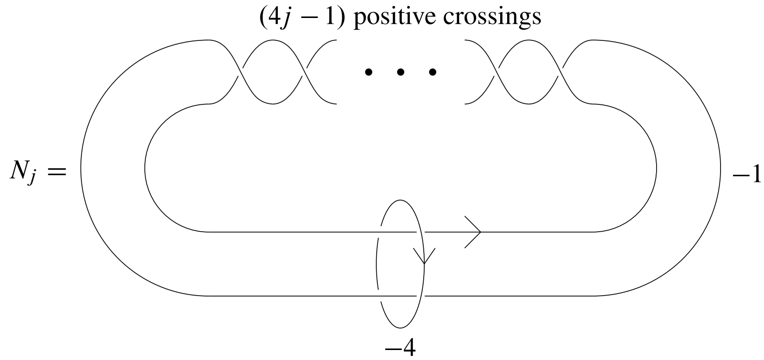


Figure 7

(2) Next, we compute $[\tau(N_j,\mathfrak{s}_0)]$ which is a Laurent polynomial obtained by truncating $\tau(N_j,\mathfrak{s}_0)$ in a certain way (see [24, page 22]). We find that

$$\begin{aligned}[\tau(N_j,\mathfrak{s}_0)] &= \frac{t^{8j-1}+1}{t^{4j-2}(t-1)^2(t+1)} - \frac{t}{(t-1)^2}\\ &= \Bigl(\sum_{i=0}^{4j-4} t^{i-4j+2}\Bigr)\Bigl(\sum_{i=1}^{2j-1} t^{2i}\Bigr) + \sum_{i=0}^{8j-4} t^{i-4j+2}\\ &= 2j + \text{nonconstant terms.}\end{aligned}$$

(3) By definition, $T_{N_j}(\mathfrak{s}_0)$ is the constant term of $[\tau(N_j,\mathfrak{s}_0)]$. Hence, $T_{N_j}(\mathfrak{s}_0) = 2j$.

Thus, we have the isomorphism of graded $\mathbb{F}[U]$-modules in Figure 8.

We now compute the involution ι_* on homology. This amounts to determining whether $-k_1$ is equivalent to k_1 or k_2. If $-k_1$ is equivalent to k_2, then the involution swaps the two legs of the left-hand graph of the above figure and leaves the right-hand graph fixed. If $-k_1$ is equivalent to k_1, then ι_* is the identity. We know show that, in fact, $-k_1$ is equivalent to k_2.

Recall, $-k_1 = (1,0,-5+4j,-3,0,\dots,0,3)$ and $k_2 = (-1,0,3-4j,3,0,\dots,0,1)$. Consider the following sequence of moves from $-k_1$ to k_2:

(1) Add $2PD[v_4]$:

$$\begin{aligned}&(-1,0,-1,3,1) = k_2 && \text{if } j=1,\\ &(-1,0,-5+4j,3,-2,\underbrace{0,\dots,0}_{2j-3},3) && \text{if } j\ge 2.\end{aligned}$$

So we can assume for the subsequent moves that $j \ge 2$.

(2) Apply Lemma 5.5 and Remark 5.6:

$$(-1,0,-5+4j,1,\underbrace{0,\dots,0}_{2j-3},2,1).$$

$HF^+(-N_j, \mathfrak{s}_0) \cong$

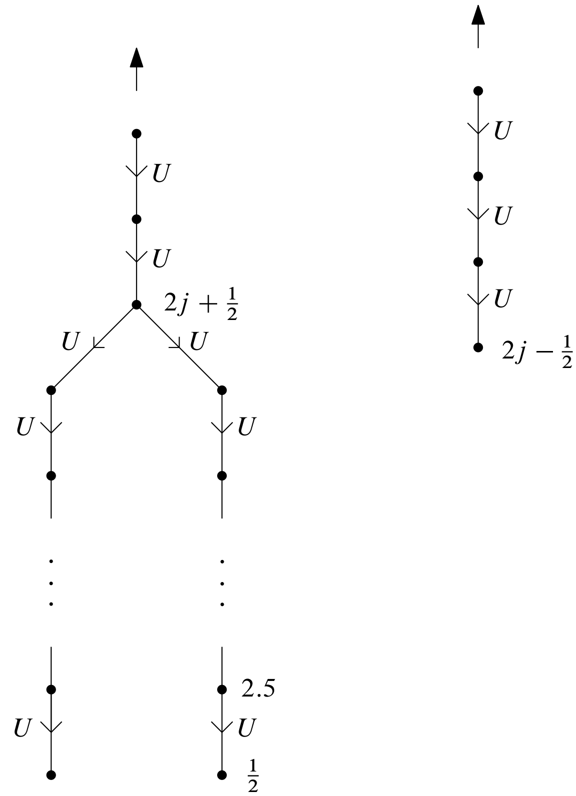


Figure 8

(3) Add $-2PD[v_1]$:

$$(1, -2, -5 + 4j - 2, -1, \underbrace{0, \dots, 0}_{2j-3}, 2, 1).$$

(4) Add $-2PD[v_2]$:

$$(-1, 2, -5 + 4j - 2, -1, \underbrace{0, \dots, 0}_{2j-3}, 2, 1).$$

(5) Add $2PD[v_1]$:

$$(1, 0, -5 + 4j - 4, -3, \underbrace{0, \dots, 0}_{2j-3}, 2, 1).$$

(6) Add $-2PD[v_4]$:

$$(-1, 0, -5 + 4j - 4, 3, -2, \underbrace{0, \dots, 0}_{2j-4}, 2, 1).$$

(7) Add $-2PD[v_5]$:

$$(-1, 0, -5 + 4j - 4, 1, 2, -2 \underbrace{0, \dots, 0}_{2j-5}, 2, 1).$$

(8) Rightward $(2,-2)$-slide:

$$(-1,0,-5+4j-4,1,\underbrace{0,\dots,0}_{2j-5},2,-22,1).$$

(9) Type -2 move:

$$(-1,0,-5+4j-4,1,\underbrace{0,\dots,0}_{2j-4},2,0,1).$$

Now notice that we are back to the same vector as in (2), except we have decreased the 3rd entry by 4 and shifted the $+2$ entry one slot to the left. Therefore, if we iterate this sequence of moves $(2j-4)$ more times, we get the vector

$$(-1,0,7-4j,1,2,\underbrace{0,\dots,0}_{2j-3},1).$$

Now consider the sequence of moves:

(1) Add $-2PD[v_1]$:

$$(1,-2,5-4j,-1,2,\underbrace{0,\dots,0}_{2j-3},1).$$

(2) Add $-2PD[v_2]$:

$$(-1,2,5-4j,-1,2,\underbrace{0,\dots,0}_{2j-3},1).$$

(3) Add $-2PD[v_1]$:

$$(1,0,3-4j,-3,2,\underbrace{0,\dots,0}_{2j-3},1).$$

(4) Add $-2PD[v_4]$:

$$(-1,0,3-4j,3,0,\dots,0,1)=k_2.$$

Theorem 5.15 *We have the isomorphism of graded $\mathbb{F}[U,Q]/(Q^2)$-modules given in Figure 9.*

Remark 5.16 The graph on the right-hand side of the isomorphism in Figure 9 should be interpreted as a graded $\mathbb{F}[U,Q]/(Q^2)$-module in a manner similar to what was described earlier in the context of $\mathbb{F}[U]$-modules, except now there are additional arrows labeled with Q to indicate the action of Q.

Proof For simplicity of exposition, we prove the statement for $j=1$. The proof for $j\geq 2$ is completely analogous and is left to the reader.

Fix an admissible Heegaard pair $\mathscr{H}=(H,J)$ for $(-N_1,\mathfrak{s}_0)$. We can choose representative cycles $a,b,c\in CF^+(\mathscr{H},\mathfrak{s}_0)$ such that

$$[a+b],[c]\in \operatorname{Im}[\pi_*\colon HF^\infty(\mathscr{H},\mathfrak{s}_0)\to HF^+(\mathscr{H},\mathfrak{s}_0)]$$

and the corresponding HF^+ homology generators are given in Figure 10.

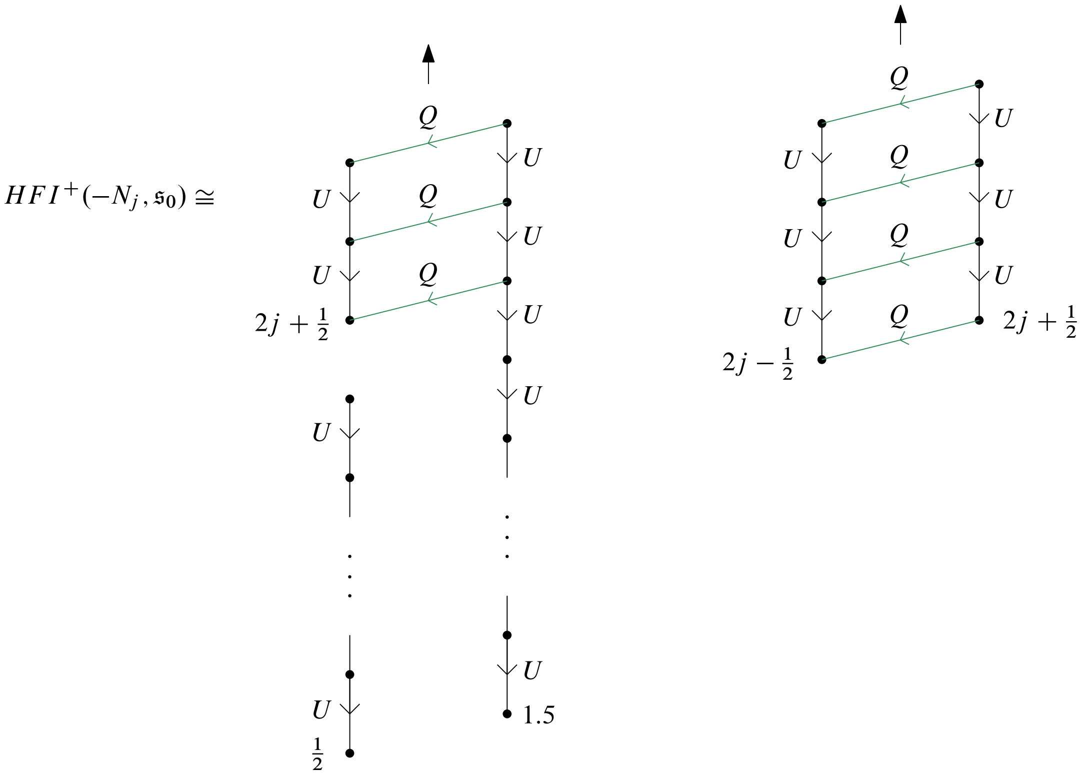


Figure 9

Since $\iota_*([a]) = \iota_*([b])$, we have that $(1+\iota_*)([a+b]) = 0$. Therefore, there exists some $d \in CF^+(\mathcal{H}, \mathfrak{s}_0)$ such that $\partial d = a+b+\iota(a+b)$. Similarly, since $(1+\iota_*)([c]) = 0$, there exists some $e \in CF^+(\mathcal{H}, \mathfrak{s}_0)$ such that $\partial e = c+\iota(c)$. It then follows from Proposition 2.2, that as graded $\mathbb{F}$-vector spaces, $HFI^+(-N_1; \mathfrak{s}_0)$ is isomorphic to that given in Figure 11.

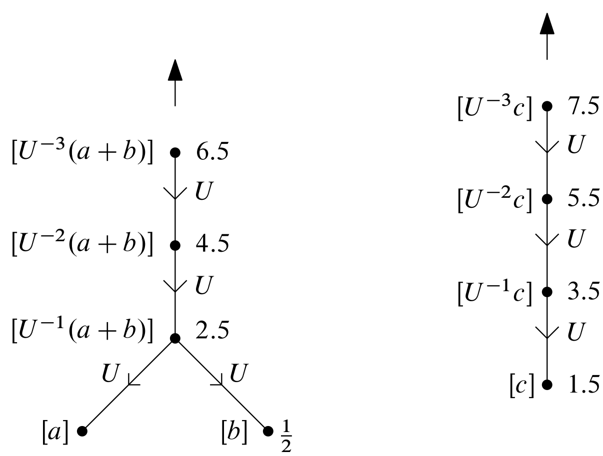


Figure 10

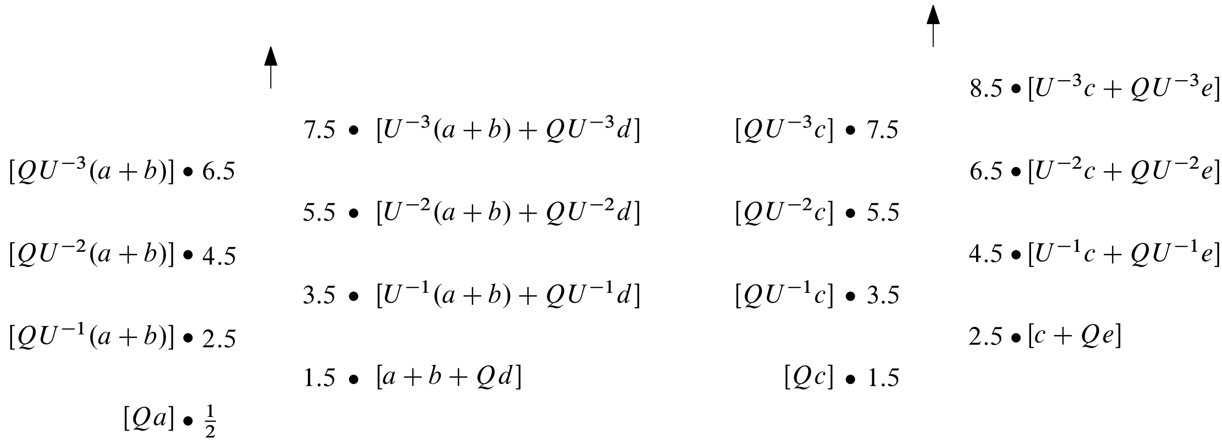


Figure 11

From this explicit description of generators, we see that for $n \geq 2$,

$$U \cdot [QU^{-n}(a+b)] = [QU^{-n+1}(a+b)],$$

and for $n \geq 1$,

$$U \cdot [U^{-n}(a+b) + QU^{-n}d] = [U^{-n+1}(a+b) + QU^{-n+1}d],$$
$$U \cdot [QU^{-n}c] = [QU^{-n+1}c],$$
$$U \cdot [U^{-n}c + QU^{-n}e] = [U^{-n+1}c + QU^{-n+1}e].$$

Next, we have

$$U \cdot [QU^{-1}(a+b)] = [Q(a+b)] = [\partial^I a] = 0.$$

Moreover, by grading considerations, we must have

$$U \cdot [Qa] = 0,$$
$$U \cdot [a+b+Qd] = 0,$$
$$U \cdot [Qc] = 0.$$

Also, either $U \cdot [c+Qe] = 0$ or $U \cdot [c+Qe] = [Qa]$. In the former case, we would have

$$\dim_{\mathbb{F}}[\ker(U \colon HFI^+(-N_j,\mathfrak{s}_0) \to HFI^+(-N_j,\mathfrak{s}_0))] = 5,$$
$$\dim_{\mathbb{F}}[\operatorname{coker}(U \colon HFI^+(-N_j,\mathfrak{s}_0) \to HFI^+(-N_j,\mathfrak{s}_0))] = 1,$$

whereas in latter we would have

$$\dim_{\mathbb{F}}[\ker(U \colon HFI^+(-N_j,\mathfrak{s}_0) \to HFI^+(-N_j,\mathfrak{s}_0)] = 4,$$
$$\dim_{\mathbb{F}}[\operatorname{coker}(U \colon HFI^+(-N_j,\mathfrak{s}_0) \to HFI^+(-N_j,\mathfrak{s}_0))] = 0.$$

Thus, by [6, Proposition 4.1], we would have either

$$\dim_{\mathbb{F}}(\widehat{HFI}(-N_j,\mathfrak{s}_0)) = 6 \text{ or } 4.$$

$HFI^+(-N_1, \mathfrak{s}_0) \cong$

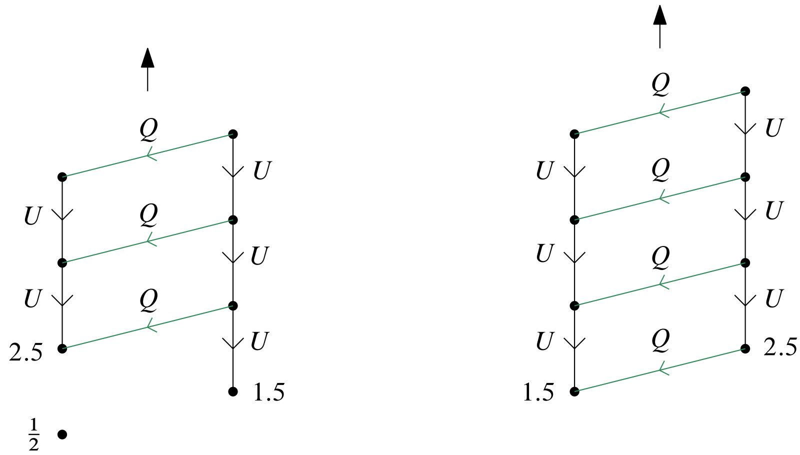


Figure 12

But by [6, Corollary 4.7] we see that

$$\begin{aligned}\dim_{\mathbb{F}}(\widehat{HFI}(-N_j, \mathfrak{s}_0)) &= \dim_{\mathbb{F}}[\ker(Q(1+\iota_*)\colon \widehat{HF}(-N_j, \mathfrak{s}_0) \to Q \cdot \widehat{HF}(-N_j, \mathfrak{s}_0))] \\ &\quad + \dim_{\mathbb{F}}[\operatorname{coker}(Q(1+\iota_*)\colon \widehat{HF}(-N_j, \mathfrak{s}_0) \to Q \cdot \widehat{HF}(-N_j, \mathfrak{s}_0))] \\ &= 3+3 \\ &= 6.\end{aligned}$$

Hence, we must have had $U \cdot [c + Qe] = 0$. We have now completely determined the U-action on $HFI^+(-N_j, \mathfrak{s}_0)$.

Next, for the Q-action, it follows from the explicit description of the generators that for $n \geq 1$,

$$Q \cdot [U^{-n}(a+b) + QU^{-1}d] = [QU^{-n}(a+b)],$$

and for $n \geq 0$,

$$Q \cdot [U^{-n}c + QU^{-n}e] = [QU^{-n}c].$$

Also,

$$Q \cdot [a+b+Qd] = [Q(a+b)] = [\partial^I a] = 0.$$

It is clear that the action of Q on all of the other generators is zero. Thus, we have the isomorphism given in Figure 12. □

Theorem 5.17 *For all positive integers j, N_j cannot be obtained by 0-surgery on a knot in S^3. In fact, N_j is not the oriented boundary of any smooth negative semidefinite spin 4-manifold.*

Proof From previous theorem, we have

$$\bar{d}_{1/2}(-N_j) = 2j + \tfrac{1}{2}, \quad \bar{d}_{-1/2}(-N_j) = 2j - \tfrac{1}{2},$$
$$\underline{d}_{1/2}(-N_j) = \tfrac{1}{2}, \qquad \underline{d}_{-1/2}(-N_j) = 2j - \tfrac{1}{2}.$$

Equivalently,

$$\underline{d}_{-1/2}(N_j) = -2j - \tfrac{1}{2}, \quad \underline{d}_{1/2}(N_j) = -2j + \tfrac{1}{2},$$
$$\bar{d}_{-1/2}(N_j) = -\tfrac{1}{2}, \qquad \bar{d}_{1/2}(N_j) = -2j + \tfrac{1}{2}.$$

The conclusion now follows immediately from Corollaries 2.18 and 2.21. □

5.4 $HFI^+(-S_0^3(K_1), \mathfrak{s}_0)$

As mentioned in the introduction, Ichihara, Motegi and Song [8] discovered an infinite family of hyperbolic knots which admit small Seifert fibered 0-surgery. In particular, for the knot K_1 in their family, they show that

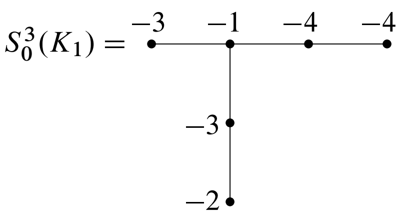
$$S_0^3(K_1) = S^2\left(\tfrac{3}{2}, -\tfrac{5}{2}, -\tfrac{15}{4}\right).$$

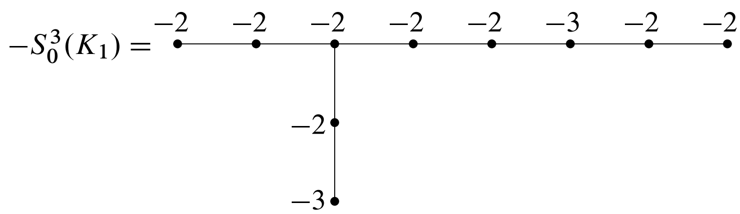
Since, by definition, $S_0^3(K_1)$ is 0-surgery on a knot in S^3, we know from Corollary 2.21 that

$$-\tfrac{1}{2} \le \underline{d}_{-1/2}(S_0^3(K_1)) \quad \text{and} \quad \bar{d}_{1/2}(S_0^3(K_1)) \le \tfrac{1}{2}.$$

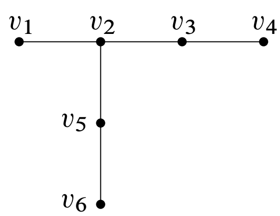
We now verify these bounds directly by computing $HFI^+(-S_0^3(K_1), \mathfrak{s}_0)$ and then we compare this to $HFI^+(-N_1, \mathfrak{s}_0)$. In the interest of brevity, we are only going to give an outline of the calculation and leave the details to the reader.

5.4.1 Step 1 We use Kirby calculus and the fact that $S_0^3(K_1)$ is a small Seifert fibered space to find negative semidefinite plumbing representations of $S_0^3(K_1)$ and $-S_0^3(K_1)$:





Label the vertices of the above left plumbing graph as:



Using the methods of the previous section, one can show that as graded $\mathbb{F}[U]$-modules, we have the isomorphism in Figure 13, where the two leaves on the left graph correspond to the representative vectors

$$z_1 = (-1, -1, 4, -2, 1, 0) \quad \text{and} \quad z_2 = (1, -1, 0, 4, 1, 0).$$

Note that $HF^+(-S_0(K_1), \mathfrak{s}_0) \cong HF^+(-N_1, \mathfrak{s}_0)$.

$HF^+(-S^3_0(K_1), \mathfrak{s}_0) \cong$

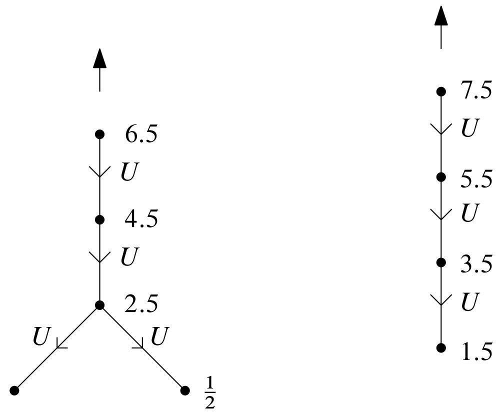


Figure 13

5.4.2 Step 2 To determine ι_*, consider the following sequence of moves starting with the vector $-z_1 = (1, 1, -4, 2, -1, 0)$:

(1) Add $-2PD[v_3]$: $(1, -1, 4, 0, -1, 0)$.

(2) Add $-2PD[v_2]$: $(-1, 1, 2, 0, -3, 0)$.

(3) Add $-2PD[v_5]$: $(-1, -1, 2, 0, 3, -2)$.

(4) Add $-2PD[v_6]$: $(-1, -1, 2, 0, 1, 2)$.

(5) Add $-2PD[v_2]$: $(-3, 1, 0, 0, -1, 2)$.

(6) Add $-2PD[v_1]$: $(3, -1, 0, 0, -1, 2)$.

(7) Add $-2PD[v_2]$: $(1, 1, -2, 0, -3, 2)$.

(8) Add $-2PD[v_5]$: $(1, -1, -2, 0, 3, 0)$.

(9) Add $-2PD[v_2]$: $(-1, 1, -4, 0, 1, 0)$.

(10) Add $-2PD[v_3]$: $(-1, -1, 4, -2, 1, 0) = z_1$.

Therefore, ι_* is the identity. In particular, unlike for $HF^+(-N_1, \mathfrak{s}_0)$, J_0 does not swap the two legs of the graded root corresponding to the odd part of HF^+. It is worth noting that this behavior of J_0 is not seen for negative definite almost rational plumbings. Specifically, in [1, Lemma 2.1] (see also [2, Section 2]) it is shown that for negative definite almost rational plumbings, the involution J_0 on the graded root of a self-conjugate spin^c structure cannot fix more than one vertex of the graded root at a given grading level. The proof of this fact relies on the result that the lattice cohomology of almost rational plumbings is concentrated in homological degree 0. However, for negative semidefinite plumbings the lattice cohomology in general will not be concentrated in homological degree 0 and hence the action of J_0 need not behave as in the almost rational case, as illustrated by this example.

5.4.3 Step 3 Applying the same methods as in the proof of Theorem 5.15, we get:

$$HFI^+(-S^3_0(K_1),\mathfrak{s}_0) \cong$$

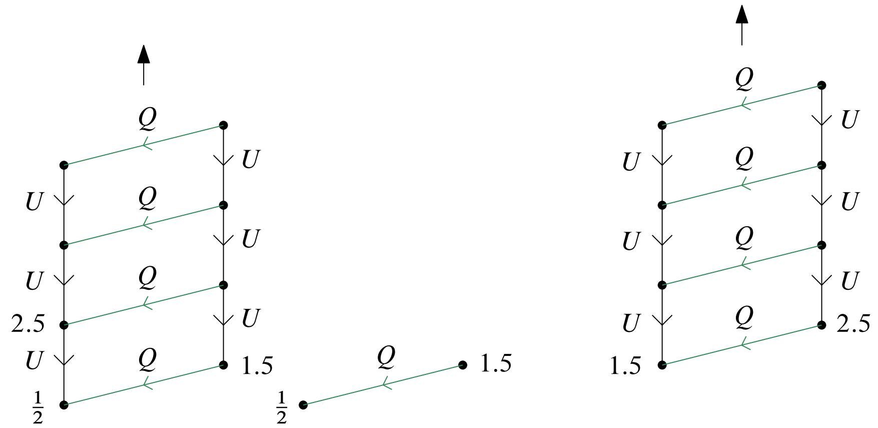


Figure 14

Theorem 5.18 *As graded* $\mathbb{F}[U,Q]/(Q^2)$*-modules, we have the isomorphism in Figure 14. In particular,*

$$\bar{d}_{1/2}(-S^3_0(K_1)) = \tfrac{1}{2}, \quad \bar{d}_{-1/2}(-S^3_0(K_1)) = 1.5,$$
$$\underline{d}_{1/2}(-S^3_0(K_1)) = \tfrac{1}{2}, \quad \underline{d}_{-1/2}(-S^3_0(K_1)) = 1.5.$$

In summary, even though

$$HF^+(-N_1,\mathfrak{s}_0) \cong HF^+(-S^3_0(K_1),\mathfrak{s}_0),$$

we see that

$$HFI^+(-N_1,\mathfrak{s}_0) \ncong HFI^+(-S^3_0(K_1),\mathfrak{s}_0).$$

References

[1] **I Dai**, *On the* Pin(2)*-equivariant monopole Floer homology of plumbed* 3*-manifolds*, Michigan Math. J. 67 (2018) 423–447 MR Zbl

[2] **I Dai**, **C Manolescu**, *Involutive Heegaard Floer homology and plumbed three-manifolds*, J. Inst. Math. Jussieu 18 (2019) 1115–1155 MR Zbl

[3] **R E Gompf**, **A I Stipsicz**, 4*-manifolds and Kirby calculus*, Graduate Studies in Math. 20, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[4] **M Hedden**, **M H Kim**, **T E Mark**, **K Park**, *Irreducible* 3*-manifolds that cannot be obtained by* 0*-surgery on a knot*, Trans. Amer. Math. Soc. 372 (2019) 7619–7638 MR Zbl

[5] **K Hendricks**, **J Hom**, **M Stoffregen**, **I Zemke**, *Surgery exact triangles in involutive Heegaard Floer homology*, preprint (2020) arXiv 2011.00113

[6] **K Hendricks**, **C Manolescu**, *Involutive Heegaard Floer homology*, Duke Math. J. 166 (2017) 1211–1299 MR Zbl

[7] **K Hendricks**, **C Manolescu**, **I Zemke**, *A connected sum formula for involutive Heegaard Floer homology*, Selecta Math. 24 (2018) 1183–1245 MR Zbl

[8] **K Ichihara**, **K Motegi**, **H-J Song**, *Seifert fibered slopes and boundary slopes on small hyperbolic knots*, Bull. Nara Univ. Ed. Natur. Sci. 57 (2008) 21–25 MR Zbl

[9] **A Juhász**, **D Thurston**, **I Zemke**, *Naturality and mapping class groups in Heegard Floer homology*, Mem. Amer. Math. Soc. 1338, Amer. Math. Soc., Providence, RI (2021) MR Zbl

[10] **L Moser**, *Elementary surgery along a torus knot*, Pacific J. Math. 38 (1971) 737–745 MR Zbl

[11] **J Murakami**, *A state model for the multivariable Alexander polynomial*, Pacific J. Math. 157 (1993) 109–135 MR Zbl

[12] **A Némethi**, *On the Ozsváth–Szabó invariant of negative definite plumbed* 3*-manifolds*, Geom. Topol. 9 (2005) 991–1042 MR Zbl

[13] **A Némethi**, *Lattice cohomology of normal surface singularities*, Publ. Res. Inst. Math. Sci. 44 (2008) 507–543 MR Zbl

[14] **W D Neumann**, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. 268 (1981) 299–344 MR Zbl

[15] **P Ozsváth**, **A I Stipsicz**, **Z Szabó**, *A spectral sequence on lattice homology*, Quantum Topol. 5 (2014) 487–521 MR Zbl

[16] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003) 179–261 MR Zbl

[17] **P Ozsváth**, **Z Szabó**, *On the Floer homology of plumbed three-manifolds*, Geom. Topol. 7 (2003) 185–224 MR Zbl

[18] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. 159 (2004) 1159–1245 MR Zbl

[19] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. 159 (2004) 1027–1158 MR Zbl

[20] **P Ozsváth**, **Z Szabó**, *Holomorphic triangles and invariants for smooth four-manifolds*, Adv. Math. 202 (2006) 326–400 MR Zbl

[21] **P Ozsváth**, **Z Szabó**, *An introduction to Heegaard Floer homology*, from "Floer homology, gauge theory, and low-dimensional topology", Clay Math. Proc. 5, Amer. Math. Soc., Providence, RI (2006) 3–27 MR Zbl

[22] **P Ozsváth**, **Z Szabó**, *Lectures on Heegaard Floer homology*, from "Floer homology, gauge theory, and low-dimensional topology", Clay Math. Proc. 5, Amer. Math. Soc., Providence, RI (2006) 29–70 MR Zbl

[23] **R Rustamov**, *On Heegaard Floer homology of plumbed three-manifolds with* $b_1 = 1$, preprint (2004) arXiv math/0405118

[24] **V Turaev**, *Torsions of* 3*-dimensional manifolds*, Progr. Math. 208, Birkhäuser, Basel (2002) MR Zbl

[25] **I Zemke**, *The equivalence of lattice and Heegaard Floer homology*, preprint (2021) arXiv 2111.14962

Department of Mathematics, University of Virginia
Charlottesville, VA, United States

pkj4vj@virginia.edu

Received: 16 July 2022 Revised: 18 September 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2025 is US $760/year for the electronic version, and $1110/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 2 (pages 645–1264) 2025