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Localization of a $\mathrm{KO}^*(\mathrm{pt})$-valued index and the orientability of the $\mathrm{Pin}^-(2)$ monopole moduli space

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It is known that the Dirac index of a Spin^c structure is localized to the characteristic submanifold. We introduce the notion of $G^{\pm}(n, s^+, s^-)$ structure on a manifold as a common generalization of the Spin^c structure and the $H_n(s)$ structure defined by D Freed and M Hopkins, and formulate a version of characteristic submanifold for the $G^{\pm}(n, s^+, s^-)$ structure. We show that the $\mathrm{KO}^*(\mathrm{pt})$-valued index associated with the $G^{\pm}(n, s^+, s^-)$ structure is localized to the characteristic submanifold. As an application, we give a topological sufficient condition for the moduli space of $\mathrm{Pin}^-(2)$ monopoles to be orientable.

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1 Introduction

In this paper, we introduce the notion of $G^{\pm}(n, s^+, s^-)$ structure, which is a generalization of the Spin structure, the Spin^c structure, the $\mathrm{Pin}^{\pm}$ structure, and the $H_s(n)$ structure due to Freed and Hopkins [5]. We construct an elliptic differential operator associated with the $G^{\pm}(n, s^+, s^-)$ structure $\mathfrak{s}$, and we have its index $\mathrm{ind}(\mathfrak{s})$ with values in $\mathrm{KO}^{s^- - s^+ \pm n}(\mathrm{pt})$. The index $\mathrm{ind}(\mathfrak{s})$ is a generalization of the Atiyah–Milnor–Singer invariant which is defined by the Dirac type operator with the Clifford action for a Spin manifold. The index $\mathrm{ind}(\mathfrak{s})$ is also a generalization of the index of the $H_s(n)$ structure with values in $\mathrm{KO}^{-n-s}(\mathrm{pt})$ defined by Freed and Hopkins.

Our main theorem is that the index above is localized to a certain submanifold which is a generalization of a characteristic submanifold of the Spin^c structure.

Main Theorem *There exists a homomorphism f such that the diagram*

$$\begin{array}{ccc} \Omega_n^{G^+(n,s^+,s^-)}(\mathrm{pt}) & \xrightarrow{f} & \Omega_{n-s^-}^{G^+(n-s^-,s^+,0)}(\mathrm{pt}) \\ & \searrow & \downarrow \\ & & \mathrm{KO}^{-n-s^++s^-}(\mathrm{pt}) \end{array}$$

commutes, where the morphisms to the KO *group are defined by the indices.*

The map f in the statement of Main Theorem is given by the localization.

Many people give generalizations of the localization of the index of the Spin^c structure to the characteristic submanifolds, for example W Zhang [18], J L Fast and S Ochanine [4], M Furuta and Y Kametani [8], and S Hayashi [9]. The methods of the localization of the index in the works of [4; 9] are to localize the topological index by using the excision theorem of K or KO theory, respectively. Our method of the localization is based on a version of the Witten deformation, which is introduced by E Witten [17] in 1982. The Witten deformation is an analytical counterpart of the excision.

The main application of the main theorem is to give a sufficient condition for the $\mathrm{Pin}^-(2)$ monopole moduli space to be orientable, which enables us to refine the $\mathrm{Pin}^-(2)$ monopole invariant. The $\mathrm{Pin}^-(2)$ monopole invariant is a variant of the Seiberg–Witten invariant introduced by N Nakamura [14; 15]. The orientability of the moduli spaces in gauge theory was originally studied for instanton by Donaldson [2; 3]. Donaldson's argument can be applied in the case of the singular instantons introduced by P B Kronheimer and T S Mrowka [12] and in the case of the $U(n)$ instantons introduced by Kronheimer [11]. In these cases, the moduli spaces are orientable. On the other hand, we show that the $\mathrm{Pin}^-(2)$ monopole moduli space may be nonorientable. Strictly speaking, we have an explicit example of a 4-manifold for which the determinant bundle on the ambient space of the moduli space is nontrivial (Corollary 5.20). We expect that our new method using the Witten deformation could be applied to other moduli spaces in gauge theory. Recently, Joyce, Tanaka and Upmeier [10] gave a new framework to deal with the orientation of moduli spaces. It is an interesting problem to understand the relation between their argument and ours.

This paper is organized as follows. In Section 2, we establish our conventions and define the index of the $G^{\pm}(n,s^+,s^-)$ structure. In Section 3, we first formulate the main theorem and give a proof in the rest of the section. The proof of the analytical details of the key localization is postponed to the appendix. In Section 4, we describe two examples in detail: the Freed–Hopkins $H_s(n)$ structure, and the $G^+(5,0,4)$ structure which we use in the next section. In Section 5, as an application, we give a topological sufficient condition for the $\mathrm{Pin}^-(2)$ monopole moduli space to be orientable and we give an example of a 4-manifold for which the determinant bundle on the ambient space which contains the $\mathrm{Pin}^-(2)$ monopole moduli space is nontrivial.

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2 Definition of the index

2.1 The $G^\pm(n, s^+, s^-)$ structures

Our purpose here is to establish our notation and conventions. We follow the notation of [13] for Clifford algebras and follow the definition of [1] for KO groups.

- Let Q_r denote the standard metric on $\mathbb{R}^r$. Let Q denote the quadratic form on $\mathbb{R}^l \oplus \mathbb{R}^m$ defined by

$$Q = Q_l - Q_m.$$

We will denote by $\mathrm{Cl}_{(l,m)}$ the Clifford algebra generated by $\{v \in \mathbb{R}^l \oplus \mathbb{R}^m\}$ subject to $v^2 = Q(v)$. We abbreviate $\mathrm{Cl}_{(n,0)}$ and $\mathrm{Cl}_{(0,n)}$ to Cl_n and Cl_{-n}, respectively. We write $\epsilon_1, \dots, \epsilon_l, e_1, \dots, e_m$ for the standard generators of $\mathrm{Cl}_{(l,m)}$.

- Note that $\mathrm{Cl}_{(l,m)}$ is naturally a $\mathbb{Z}/2$-graded algebra. Let

$$(\mathrm{Cl}_{(l,m)})^0 = \{a_1 a_2 \cdots a_{2k} \mid \forall i = 1, \dots, 2k,\ 0 \neq a_i \in \mathbb{R}^l \oplus \mathbb{R}^m\}$$

be the even part of $\mathrm{Cl}_{(l,m)}$ and

$$(\mathrm{Cl}_{(l,m)})^1 = \{a_1 a_2 \cdots a_{2k-1} \mid \forall i = 1, \dots, 2k-1, , 0 \neq a_i \in \mathbb{R}^l \oplus \mathbb{R}^m\}$$

be the odd part of $\mathrm{Cl}_{(l,m)}$. Here $\hat{\otimes}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product. Then there is an isomorphism $\mathrm{Cl}_{(l_1,m_1)} \hat{\otimes} \mathrm{Cl}_{(l_2,m_2)} \cong \mathrm{Cl}_{(l_1+l_2,m_1+m_2)}$.

- Let X be a compact Hausdorff space. A representative element of $\mathrm{KO}^{(b,a)}(X)$ is given by a four-tuple:

(1) The Clifford algebra $\mathrm{Cl}_{(b,a)}$.

(2) A separable $\mathbb{Z}/2$-graded Hilbert space H with a left $\mathrm{Cl}_{(b,a)}$ $\mathbb{Z}/2$-graded action. We denote by $\rho\colon \mathrm{Cl}_{(b,a)} \to \hom(H, H)$ the representation of the $\mathbb{Z}/2$-graded algebra. We assume this is continuous with respect to the norm topology of $\hom(H, H)$ which is a space of bounded operators on the Hilbert space H.

(3) Let us denote grading map by $\epsilon\colon H \to H$. Note that $\epsilon^2 = 1$.

(4) A continuous map $s\colon X \to \hom(H, H)$ (the topology of $\hom(H, H)$ is given by the operator norm) such that $s(x)$ is a bounded skew adjoint, odd and Fredholm operator which graded commutes with the $\mathrm{Cl}_{(b,a)}$ action for all $x \in X$.

We will write $(s, \epsilon, \mathrm{Cl}_{(b,a)}, (\rho, H))$ for this four-tuple. In cases where clarity permits, we use H as an abbreviation for (ρ, H).

- We define $(s, \epsilon, \mathrm{Cl}_{(b,a)}, H)$ and $(s', \epsilon', \mathrm{Cl}_{(b,a)}, H')$ as *equivalent* if they satisfy the following properties:
  - There exist four-tuples $(s_0, \epsilon_0, \mathrm{Cl}_{(b,a)}, H_0)$ and $(s_1, \epsilon_1, \mathrm{Cl}_{(b,a)}, H_1)$ such that both $\ker(s_0(x))$ and $\ker(s_1(x))$ are trivial for all $x \in X$. Note that this implies $s_0(x)$ and $s_1(x)$ are isomorphisms for all $x \in X$.
  - There exists an isometric linear map $f\colon H \oplus H_0 \to H' \oplus H_1$ such that $f \circ (\epsilon \oplus \epsilon_0) f^{-1} = (\epsilon' \oplus \epsilon_1)$ and $f \circ (s \oplus s_0) \circ f^{-1}$ and $s' \oplus s_1$ are homotopic through continuous maps from X to $\hom(H' \oplus H_1, H' \oplus H_1)$ which anticommute with $\epsilon' \oplus \epsilon_1$ and the $\mathrm{Cl}_{(b,a)}$ action.
  - The homotopy above anticommutes with $\epsilon' \oplus \epsilon_1$ and Clifford action.

- We define $\mathrm{KO}^{(b,a)}(X)$ by the set of the equivalence classes of four tuples $(s, \epsilon, \mathrm{Cl}_{(b,a)}, (\rho, H))$. We define

$$[(s, \epsilon, \mathrm{Cl}_{(b,a)}, (\rho, H))] + [(s', \epsilon', \mathrm{Cl}_{(b,a)}, (\rho, H'))] := \left[(s \oplus s', \epsilon \oplus \epsilon', \mathrm{Cl}_{(b,a)}, (\rho \oplus \rho', H \oplus H'))\right]$$

and it is easy to check that this operation is well defined. We can check that $\mathrm{KO}^{(b,a)}(X)$ is an abelian group under this operation $+$. Note that $[(s, \epsilon, \mathrm{Cl}_{(b,a)}, (\rho, H))] + [(-s, -\epsilon, \mathrm{Cl}_{(b,a)}, (-\rho, H))] = 0$, since we can deform $s \oplus -s$ to an odd, skew-adjoint, and isomorphic operator on $H \oplus H$ that graded commutes with $\mathrm{Cl}_{(b,a)}$.

- If s is an unbounded skew-adjoint Fredholm operator and

$$\tilde{s} = \frac{s}{\sqrt{1 + ss^*}}$$

satisfies the above properties, we write $(s, \epsilon, \mathrm{Cl}_{(b,a)}, H)$ instead of $(\tilde{s}, \epsilon, \mathrm{Cl}_{(b,a)}, H)$.

- We will denote by (ρ_1, V_1) the $\mathbb{Z}/2\mathbb{Z}$ graded representation of $\mathrm{Cl}_{(1,1)}$ which is given as follows. Let $V_1^0 = V_1^1 = \mathbb{R}$, $V_1 = V_1^0 \oplus V_1^1$ and

$$\rho(\epsilon) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \rho(e) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$

where ϵ, e ($\epsilon^2 = 1$, $e^2 = -1$) are the generators of $\mathrm{Cl}_{(1,1)}$.

- We will let $V_n := V_1^{n\hat{\otimes}}$ and define (ρ_n, V_n) to be the representation introduced by the natural isomorphism $\mathrm{Cl}_{(n,n)} \cong \mathrm{Cl}_{(1,1)}^{n\hat{\otimes}}$. Let V_n^* denote the dual space of V_n which has a natural right $\mathrm{Cl}_{(n,n)}$ module. We will denote by ϵ^n the grading operator.

- Let $v_1^0 := 1 \in V_1^0$ and we set $v_n^0 := v_1^{0^{n\otimes}} \in V_n$. Then we see that

$$\{\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}v_n^0 \in V_n \mid i_1 < i_2 < \cdots < i_k,\ 0 \le k \le n\}$$

is basis of V_n. Also $\{v_n^{0^*} \circ \epsilon_{i_k} \circ \epsilon_{i_{k-1}} \circ \cdots \circ \epsilon_{i_1} \in V_n^* \mid i_1 < i_2 < \cdots < i_k,\ 0 \le k \le n\}$ is a basis of V_n^*.

Definition 2.1 If H is a left $\mathrm{Cl}_{(b,a)}$ module, we define the right $\mathrm{Cl}_{(a,b)}$ module structure as follows: Let ϵ be the grading operator on H and $\rho\colon \mathrm{Cl}_{(b,a)} \to \hom(H,H)$ be the representation that is given by the left $\mathrm{Cl}_{(b,a)}$ module structure. Let $\epsilon_1,\dots,\epsilon_b$ and $e_1,\dots,e_a$ be orthonormal bases of $\mathbb{R}^b$ and $\mathbb{R}^a$, respectively. Then we define the right action $\rho'\colon \mathrm{Cl}^{op}_{(a,b)} \to \hom(H,H)$ by

$$\phi\cdot\rho'(\epsilon_{i_1}\cdots\epsilon_{i_k}) := \rho(e_{i_k}\cdots e_{i_1})(\epsilon^k\phi), \quad \phi\cdot\rho'(e_{j_1}\cdots e_{j_l}) := \rho(\epsilon_{j_l}\cdots\epsilon_{j_1})(\epsilon^l\phi).$$

This is well-defined and independent of the choice of the orthonormal bases of $\mathbb{R}^b$ and $\mathbb{R}^a$. If H^* is a right $\mathrm{Cl}_{(a,b)}$ module, we can define a right $\mathrm{Cl}_{(b,a)}$ structure on H^* in the same way.

Lemma 2.2 *Assume $a \ge b$. Let $(s,\epsilon,\mathrm{Cl}_{(b,a)},H)$ be a representative element of $\mathrm{KO}^{(b,a)}(X)$. Then there exists an element $(s',\epsilon',\mathrm{Cl}_{(0,a-b)},H')$ of $\mathrm{KO}^{(0,a-b)}(X)$ which satisfies the following properties: There is an isomorphism between the Hilbert spaces*

$$f\colon H \to H' \mathbin{\hat{\otimes}} V_b^*$$

such that

$$\begin{aligned} s &= f^{-1}\circ s'\otimes\epsilon^b\circ f, \\ \epsilon &= f^{-1}\circ \epsilon'\otimes\epsilon^b\circ f, \\ \epsilon_i &= f^{-1}\circ 1\otimes\epsilon_i\circ f \qquad (\text{for } i=1,\dots,b), \\ e_i &= f^{-1}\circ 1\otimes e_i\circ f \qquad (\text{for } i=1,\dots,b), \\ e_{b+i} &= f^{-1}\circ e'_i\otimes\epsilon^b\circ f \quad (\text{for } i=1,\dots,a-b), \end{aligned}$$

where $e'_1,\dots,e'_{a-b}$ are the generators of $\mathrm{Cl}_{(0,a-b)}$ and $\epsilon_1,\dots,\epsilon_b,e_1,\dots,e_b,e_{b+1},\dots,e_a$ are the generators of $\mathrm{Cl}_{(b,a)}$. Note that $\mathrm{Cl}_{(b,b)}$ action on V_n^* is the left action that is defined in Definition 2.1 by using the right $\mathrm{Cl}_{(b,b)}$ module.

Proof The subspace $H' \subset H$ is given by the intersection of $+1$-eigenspaces of $\epsilon_1e_1,\dots,\epsilon_be_b$. The actions of s,ϵ and $\mathrm{Cl}_{(0,a-b)}$ preserve H'. We write s',ϵ',e'_i $(i=1,\dots,a-b)$ for restriction of the actions of them to H'. Let us define a map g by

$$g\colon H'\mathbin{\hat{\otimes}} V_b^* \to H, \quad \phi'\otimes\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}v_n^{0^*} \mapsto \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}\phi'.$$

We can check that this is a map of the $\mathrm{Cl}_{(b,a)} \cong \mathrm{Cl}_{(0,a-b)} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(b,b)}$ module. For example,

$$s\otimes\epsilon^b\cdot\phi'\otimes\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}v_n^{0^*} = (-1)^k s\phi'\otimes\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}v_n^{0^*} \mapsto (-1)^k\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}s\phi' = s\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}\phi',$$

$$1\otimes\epsilon_j\cdot\phi'\otimes\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}v_n^{0^*} = \phi'\otimes\epsilon_j\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}v_n^{0^*} \mapsto \epsilon_j\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}s\phi'.$$

We set $f = g^{-1}$ and this proves the lemma. □

Remark 2.3 We have the option to substitute V_n for V_n^* in the assertion of Lemma 2.2. Nevertheless, in Proposition 3.11, the natural appearance is that of V_n^*. Therefore, we formulate Lemma 2.2 in the context of V_n^*.

Definition 2.4 We define a group $G^{\pm}(n, s^+, s^-)$ as

$$G^{\pm}(n,s^+,s^-) = \{g \in \mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s^+,s^-)} \mid g = v_{i_1} \cdots v_{i_{2k}},\ v_{i_j} \in \mathbb{R}^n \text{ or } \mathbb{R}^{s^+} \text{ or } \mathbb{R}^{s^-},\ |v_{i_j}| = 1\}.$$

Definition 2.5 Let

$$S(O(n) \times O(s^+) \times O(s^-)) = \{(A, B^+, B^-) \in O(n) \times O(s^+) \times O(s^-) \mid \det A \det B^+ \det B^- = 1\}.$$

The two-to-one homomorphism

$$p \colon G^{\pm}(n,s^+,s^-) \to S(O(n) \times O(s^+) \times O(s^-))$$

is defined by

$$p(g)v := gvg^{-1}$$

for $g \in G^{\pm}(n,s^+,s^-),\ v \in \mathbb{R}^n \oplus \mathbb{R}^{s^+} \oplus \mathbb{R}^{s^-}$.

We denote by p_n, p_{s^+} and p_{s^-} the compositions of p with the projections from $S(O(n)\times O(s^+)\times O(s^-))$ to each component $O(n)$, $O(s^+)$ and $O(s^-)$, respectively.

Definition 2.6 Let Y be an n dimensional Riemannian manifold. Let us denote by P_Y the orthogonal frame bundle of TY. A $G^{\pm}(n,s^+,s^-)$ structure is a tuple $(\widetilde{P}, P, \pi, o, E_+, E_-)$ such that:

- $E_{\pm}$ is an $s^{\pm}$ dimensional real vector bundle such that their structure group is $O(s^{\pm})$, respectively. We will denote by $P_{E_{\pm}}$ its frame bundle.
- o is an orientation of $TY \oplus E_+ \oplus E_-$.
- P is a principal $S(O(n)\times O(s^+)\times O(s^-))$ bundle defined as the subbundle of $P_Y \times_Y P_{E_+} \times_Y P_{E_-}$:

  $$P = \{(f_n, f_+, f_-) \in P_Y \times_Y P_{E_+} \times_Y P_{E_-} \mid f_n, f_+, f_- \text{ are compatible with the orientation } o \text{ in this order}\}.$$

- We denote by $\widetilde{P}$ a principal $G^{\pm}(n,s^+,s^-)$ bundle and $\pi \colon \widetilde{P} \to P$ is a smooth map which satisfies the following commutative diagram:

  $$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\cdot g} & \widetilde{P} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow{\cdot p(g)} & P \end{array}$$

  for all $g \in G^{\pm}(n,s^+,s^-)$.

Definition 2.7 Let $(\widetilde{P}, P, \pi, o, E_+, E_-)$ and $(\widetilde{P}', P', \pi', o', E'_+, E'_-)$ be $G^{\pm}(n, s^+, s^-)$ structures on Y. We say that they are isomorphic if there exist isomorphisms of principal bundles $f : P \to P'$ and $\tilde{f} : \widetilde{P} \to \widetilde{P}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\tilde{f}} & \widetilde{P}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ P & \xrightarrow{f} & P' \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_n} \\ P_Y & \xrightarrow{\mathrm{id}} & P_Y \end{array}$$

2.2 Definition of spinor bundles

Definition 2.8 A generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation of $G^+(n, s^+, s^-)$ is a pair (α, S) such that:

- S is a real vector space with a metric and a $\mathbb{Z}/2\mathbb{Z}$ grading $S = S_0 \oplus S_1$.
- α is a representation $\alpha : G^{\pm}(n, s^+, s^-) \to O(S)$ such that for all $g \in G^{\pm}(n, s^+, s^-)$, $\alpha(g)$ preserves the $\mathbb{Z}/2\mathbb{Z}$ grading of S.
- The representation space S has a $G^{\pm}(n, s^+, s^-)$ equivariant products
$$c'_n : \mathbb{R}^n \times S \to S, \quad c'_{s+} : \mathbb{R}^{s^+} \times S \to S, \quad c'_{s-} : \mathbb{R}^{s^-} \times S \to S,$$
such that they are anticommutative each other and odd. Here, the action of $G^{\pm}(n, s^+, s^-)$ on $\mathbb{R}^n, \mathbb{R}^{s^+}, \mathbb{R}^{s^-}$ is the left adjoint representations p_n, p_+, p_- as defined in Definition 2.4. Note that $p_n(g)v := gvg^{-1}$ for $g \in G^+(n, s^+, s^-)$ and $v \in \mathbb{R}^n$. We have similar formula for p_+ and p_-.
- The multiplications above give a $\mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)}$ module structure on S.

Moreover, if S has an additional left Clifford action of $\mathrm{Cl}_{(b,a)}$ and its generators anticommutes with c'_n, c'_{s+}, c'_{s-}, we call (α, S) a generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation with $\mathrm{Cl}_{(b,a)}$ action.

Definition 2.9 We define $(\rho, \mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)})$ to be the generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation with left $\mathrm{Cl}_{\mp n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s-,s+)}$ action as follows: Take $g \in G^{\pm}(n, s^+, s^-)$ and $\phi \in \mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)}$. We define $\rho(g)\phi := g\phi$, where the right hand side is a multiplication of $\mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)}$. The representation ρ preserves the $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)}$. We define
$$c'_n : \mathbb{R}^n \times \mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)} \to \mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)}$$
by $c'_n(v)\phi = v\phi$. Replacing $\mathbb{R}^n$ with $\mathbb{R}^{s\pm}$, we define c'_{s_+}, c'_{s_-} similarly. We define the additional left $\mathrm{Cl}_{\mp n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s-,s+)}$ action in the following way. Let ϵ be the $\mathbb{Z}/2\mathbb{Z}$ grading operator of $\mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)}$. We define $v \cdot \phi := (\epsilon\phi)v$ for $v \in \mathbb{R}^n \oplus \mathbb{R}^{s^+} \oplus \mathbb{R}^{s^-}$, where the right hand side is a multiplication of $\mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)}$. This defines a left $\mathrm{Cl}_{\mp n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s-,s+)}$ action because the right $\mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s+,s-)}$ action is odd and v anticommutes with ϵ.

Definition 2.10 Let $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^{\pm}(n, s^+, s^-)$ structure on Y and let (α, S) be a generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation of $G^+(n, s^+, s^-)$. Then they define the vector bundle

$$\not{S} = \widetilde{P} \times_{\alpha} S.$$

Let $\not{S}_0$ and $\not{S}_1$ be subbundles of $\not{S}$ defined by S_0 and S_1, respectively. We define the Clifford multiplication on $\not{S}$ using c'_n, c'_{s^+}, c'_{s^-}:

$$c_{TY} \colon TY \times \not{S} \to \not{S}, \quad c_{E_\pm} \colon E_\pm \times \not{S} \to \not{S}.$$

We call $\not{S}$ a generalized $\mathbb{Z}/2$ graded spinor bundle. If (α, S) is a generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation with $\mathrm{Cl}_{(b,a)}$ action, we call $\not{S}$ a generalized $\mathbb{Z}/2$ graded spinor bundle with $\mathrm{Cl}_{(b,a)}$ action.

Remark 2.11 From Definition 2.1, if $\not{S}$ has a left $\mathrm{Cl}_{(b,a)}$ action, we have a right $\mathrm{Cl}_{(a,b)}$ action on $\not{S}$ as we defined in Definition 2.1. Note that this right action commutes with $c_{TY}, c_{E_\pm}$ even if odd elements in $\mathrm{Cl}_{(a,b)}$.

Definition 2.12 Let $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^{\pm}(n, s^+, s^-)$ structure on Y. We will denote by $\not{\mathfrak{S}} = \not{\mathfrak{S}}_0 \oplus \not{\mathfrak{S}}_1$ the generalized $\mathbb{Z}/2$ graded spinor bundle with $\mathrm{Cl}_{\mp n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s^-, s^+)}$ action defined by the generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation with $\mathrm{Cl}_{\mp n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s^-, s^+)}$ action $(\rho, \mathrm{Cl}_{\pm n} \mathbin{\widehat{\otimes}} \mathrm{Cl}_{(s^+, s^-)})$. We call $\not{\mathfrak{S}}$ the standard spinor bundle of $\mathfrak{s}$.

Definition 2.13 Let $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^{\pm}(n, s^+, s^-)$ structure on Y and $\not{\mathfrak{S}}$ the standard spinor bundle of $\mathfrak{s}$. We define a Dirac type operator $\not{D}$ on $\Gamma(\not{\mathfrak{S}})$ by using the Clifford multiplication c_{TY}. We will denote by ϵ the $\mathbb{Z}/2\mathbb{Z}$ grading operator on $\not{\mathfrak{S}}$.

Let us denote by $\mathrm{ind}(\mathfrak{s})$ the element in $\mathrm{KO}^{-n \pm (s^- - s^+)}(\mathrm{pt})$ defined by the representative element

$$(\not{D}, \epsilon, \mathrm{Cl}_{\mp n} \otimes \mathrm{Cl}_{(s^-, s^+)}, L^2(Y, \not{\mathfrak{S}})).$$

We call $\mathrm{ind}(\mathfrak{s})$ the index of the $G^{\pm}(n, s^+, s^-)$ structure $\mathfrak{s}$.

Remark 2.14 The index defined above coincides with the so called Atiyah–Milnor–Singer invariant index of Spin structures when $s^+ = s^- = 0$. This index is a generalization of the mod 2 index of Spin structures on a riemannian surface and the $\hat{A}$ genus. This is defined explicitly by Lawson and Michelsohn [13, Chapter II, Section 7]. In some cases, the index coincides with the index of $H_s(n)$ structure introduced in [5].

3 Proof of Main Theorem

3.1 Statement of Main Theorem

Main Theorem *There exists a homomorphism f such that the diagram*

$$\begin{array}{ccc} \Omega_n^{G^+(n,s^+,s^-)}(\mathrm{pt}) & \xrightarrow{\ f\ } & \Omega_{n-s^-}^{G^+(n-s^-,s^+,0)}(\mathrm{pt}) \\ & \searrow & \downarrow \\ & & \mathrm{KO}^{-n-s^++s^-}(\mathrm{pt}) \end{array}$$

commutes, where the morphisms to the KO *group are defined by the indices.*

Remark 3.1 We can find a similar commutative diagram of G^- bordism groups. In this case, the codomain target of f is $\Omega_{n-s^+}^{G^-(n-s^+,0,s^-)}(\mathrm{pt})$ instead of $\Omega_{n-s^-}^{G^+(n-s^-,s^+,0)}(\mathrm{pt})$. The proof is parallel to that of the G^+ cases. Moreover, we can prove $G^+(n,s^+,s^-) \cong G^-(n,s^-,s^+)$ and hence we will study only the G^+ cases.

First, we will construct the morphism f in the main theorem in this subsection. Second, we will prove a localization theorem of the indices in Section 3.2. Some analytic lemmas are proved in the appendix. Finally, we prove the commutativity of the diagram using the localization theorem in the final part of this section and we will complete the proof of the main theorem.

Lemma 3.2 *Let* $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ *be a* $G^{\pm}(n,s^+,s^-)$ *structure on* Y*. Then there exists a canonical* Spin *structure of the vector bundle*

$$(\det TY \otimes \det E_+) \oplus TY \oplus E_+ \oplus \det E_- \oplus E_-$$

induced by $\mathfrak{s}$*. Conversely, if the vector bundle above has an* Spin *structure, it gives a canonical* $G^+(n,s^+,s^-)$ *structure on* Y*.*

Proof Let $\epsilon_1,\dots,\epsilon_{n+s^+}$ is a standard basis of $\mathbb{R}^n \oplus \mathbb{R}^{s^+}$ and $e_0', e_0, e_1, \dots, e_{n+s^++s^-}$ are the generators of $\mathrm{Cl}(0,(n+1)+(s^++1)+s^-)$. There is an embedding $\mathrm{Cl}(n+s^+,s^-) \to \mathrm{Cl}(0,(n+1)+(s^++1)+s^-)$ given by mapping $\epsilon_1,\dots,\epsilon_{n+s^+}$ to $e_0'e_0e_1,\dots,e_0'e_0e_{n+s^+}$. Restricting this map, we have an embedding $G^+(n,s^+,s^-) \to \mathrm{Spin}(1+n+s^++1+s^-)$. Let π be the projection $\mathrm{Spin}(1+n+s^++1+s^-) \to \mathrm{SO}(1+n+s^++1+s^-)$. The image $\pi(G^+(n,s^+,s^-))$ is isomorphic to $S(O(n)\times O(s^+)\times O(s^-))$. This image gives the representation of $S(O(n)\times O(s^+)\times O(s^-))$ whose associated vector bundle is $\det TY \otimes \det E_+ \oplus TY \oplus E_+ \oplus \det E_- \oplus E_-$. Here we define that

$$\langle e_0\rangle, \quad \langle e_1,\dots,e_n\rangle, \quad \langle e_{n+1},\dots,e_{n+s^+}\rangle, \quad \langle e_0'\rangle \quad \text{and} \quad \langle e_{n+s^++1},\dots,e_{n+s^++s^-}\rangle$$

are the representations such that the associated vector bundle of the representations are $\det TY \otimes \det E_+$, TY, E_+, $\det E_-$ and E_- respectively. Then we construct a Spin structure as desired. We have the second half of the statement by reversing the proof above. □

Remark 3.3 The isomorphism class of the $G^+(n,s^+,s^-)$ structure given by the second half of the Lemma 3.2 uniquely determined by the isomorphism class of the Spin structure of the vector bundle $(\det TY \otimes \det E_+) \oplus TY \oplus E_+ \oplus \det E_- \oplus E_-$. On the other hand, if we change the $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$ by an isomorphism of the $G^+(n,s^+,s^-)$ structure, the isomorphism class of the Spin structure given by the first half of Lemma 3.2 may change because there may exist an automorphism of the $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$ which covers a nontrivial isomorphism of $E_\pm$. We use this lemma to define the cobordism class of the $G^+(n-s^-,s^+,0)$ structure $\mathfrak{s}_C$. If two $G^+(n,s^+,s^-)$ structures $\mathfrak{s}$ and $\mathfrak{s}'$ are isomorphic they are cobordant. From the Lemma 3.9, $\mathfrak{s}_C$ and $\mathfrak{s}_C'$ are cobordant therefore this ambiguity does not matter to define the map f in the statement of the main theorem.

We will prove some lemmas in a general setting. In the following three lemmas, let M be a manifold and F, E_0, E_1 be oriented real vector bundles on M with fiber metrics.

Lemma 3.4 *An orientation preserving isometry $\alpha\colon E_0 \to E_1$ determines a canonical* Spin *structure on $E_0 \oplus E_1$.*

Proof Let n be the rank of E_0. The structure group of the subbundle

$$\{(f, \alpha(f)) \mid f \text{ is an oriented orthonormal frame of } E_0\}$$

of the frame bundle of $E_0 \oplus E_1$ is a subgroup of $\mathrm{SO}(2n)$ which is the image of the diagonal embedding $j\colon \mathrm{SO}(n) \to \mathrm{SO}(n)\times\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(2n)$. The map $j_*\colon \pi_1(\mathrm{SO}(n)) \to \pi_1(\mathrm{SO}(2n))$ is trivial, and therefore there exists a homomorphism $\tilde{j}\colon \mathrm{SO}(n) \to \mathrm{Spin}(2n)$ which covers j. Let$\{g_{\alpha\beta}\}$ be the transition functions of $E_0 \oplus E_1$. We define the Spin structure on $E_0 \oplus E_1$ by using $\{\tilde{j}(g_{\alpha\beta})\}$ as transition functions. It is easy to check that this Spin structure does not depend on how to take transition functions of $E_0 \oplus E_1$. □

Lemma 3.5 *Let F, F' denote oriented real vector bundles on M with metrics. If there are* Spin *structures on $F \oplus F'$ and F', these* Spin *structures determine a canonical* Spin *structure on F.*

Proof We will denote by $\{g_{\alpha\beta}\}, \{g'_{\alpha\beta}\}$ transition functions of F and F', respectively. Let $\{\tilde{g}'_{\alpha\beta}\}$ be the lifts of $\{g'_{\alpha\beta}\}$ to the Spin group defined by the Spin structure on F'. The lift of $(g_{\alpha\beta}, g'_{\alpha\beta})$ defined by the Spin structure of $F \oplus F'$ is expressed by $[(\tilde{g}_{\alpha\beta}, \tilde{g}'_{\alpha\beta})] \in \mathrm{Spin}(n)\times\mathrm{Spin}(m)/(\mathbb{Z}/2\mathbb{Z})$, where $\tilde{g}_{\alpha\beta}$ is a lift of $g_{\alpha\beta}$. The lift $\tilde{g}_{\alpha\beta}$ is unique because the second component is fixed to be $\tilde{g}'_{\alpha\beta}$.

The lift of the transition functions $\{\tilde{g}_{\alpha\beta}\}$ satisfy the cocycle condition because $\{[(\tilde{g}_{\alpha\beta}, \tilde{g}'_{\alpha\beta})]\}$ and $\{\tilde{g}'_{\alpha\beta}\}$ do. We define the Spin structure on F by using $\{\tilde{j}(g_{\alpha\beta})\}$ as transition functions. Again it is easy to check that this Spin structure does not depend on how to take transition functions of F. □

The lemma below is clear from Lemmas 3.4 and 3.5.

Lemma 3.6 *Let $F \oplus E_0 \oplus E_1$ be a vector bundle with orientation o and* Spin *structure $\widetilde{P}$. We assume that E_1 is oriented and there is an isometry $\phi\colon E_0 \to E_1$. We will denote by o_E the orientation of E_0. Then F has a natural orientation and a* Spin *structure defined by $o, o_E, \widetilde{P}$ and ϕ.*

We use the lemmas above in our setting.

Definition 3.7 Let Y be an n dimensional Riemannian manifold and $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^+(n, s^+, s^-)$ structure on Y. We will denote by $h \in \Gamma(E_-)$ an transverse section. Let $C = h^{-1}(0)$. We call C a characteristic submanifold of $\mathfrak{s}$.

The theorem below is a generalization of the theorem that is an existence of a canonical Spin structure on a characteristic submanifold of Spin^c structure.

Theorem 3.8 *Let Y be an n dimensional Riemannian manifold and $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^+(n, s^+, s^-)$ structure on Y and C be a characteristic submanifold of $\mathfrak{s}$. Then we have a natural $G^+(n - s^-, s^+, 0)$ structure on C.*

Proof We restrict the vector bundle

$$\det TY \otimes \det E_+ \oplus TY \oplus E_+ \oplus \det E_- \oplus E_-$$

to C. This is naturally isomorphic to the vector bundle

$$\det TY|_C \otimes \det E_+|_C \oplus TC \oplus N \oplus E_+|_C \oplus \det E_-|_C \oplus E_-|_C,$$

where N is the normal bundle of C. We define the isomorphism $\psi\colon N \to E_-$ by using dh. By the definition of $G^+(n, s^+, s^-)$ structure, there exists an isomorphism $i\colon \det TY|_C \otimes \det E_+|_C \cong \det E_-|_C$. Let $F = TC \oplus E_+|_C$, $E_0 = \det TY|_C \otimes \det E_+|_C \oplus N$, $E_1 = \det E_-|_C \oplus E_-|_C$, and $\phi = i \oplus \psi$. They satisfy the assumption of Lemma 3.6, and hence we have the Spin structure of $TC \oplus E_+|_C$ as desired. From the latter statement of Lemma 3.2, this Spin structure defines the $G^+(n - s^-, s^+, 0)$ structure on C as desired. □

Lemma 3.9 *As an element of bordism group $\Omega_{n-s^-}^{G^+(n-s^-, s^+, 0)}(\mathrm{pt})$, $[(C, \mathfrak{s}_C(h))]$ is independent of h. Moreover, if the $G^+(n, s^+, s^-)$ structure $\mathfrak{s}_0$ on a closed manifold Y_0 and $\mathfrak{s}_1$ on a closed manifold Y_1 are $G^+(n, s^+, s^-)$ cobordant, $G^+(n - s^-, s^+, 0)$ bordism class of $[(C, \mathfrak{s}_C(h))]$ given by $\mathfrak{s}_0$ and $\mathfrak{s}_1$ are same.*

Proof We first prove the first half of the statement. Let h, h' be transverse sections of E_- and let $C = h^{-1}(0)$, $C' = h^{-1}(0)$. We define a natural $G^+(n + 1, s^+, s^-)$ structure on $Y \times [0, 1]$ by using $G^+(n, s^+, s^-)$ structure on Y. Let $\tilde{h}$ be a transverse section on $Y \times [0, 1]$ such that $\tilde{h}|_{Y \times \{0\}} = h$, $\tilde{h}|_{Y \times \{1\}} = h'$. Then $\tilde{h}^{-1}(0)$ has a $G^+(n + 1 - s^-, s^+, 0)$ structure by Theorem 3.8. This manifold with $G^+(n + 1 - s^-, s^+, 0)$ structure gives the cobordism from $[(C, \mathfrak{s}_C(h))]$ to $[(C', \mathfrak{s}'_C(h'))]$. A slight change to the above proof actually shows the second half of the statement. □

We now define the map f of Main Theorem.

Definition 3.10 Let Y be an n dimensional closed manifold with $G^+(n, s^+, s^-)$ structure given by $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$. Set

$$f([Y, \mathfrak{s}]) = [C, \mathfrak{s}_C].$$

3.2 Localization by Witten deformation

In this subsection, we will prove the commutativity of the diagram in Main Theorem. We prove it by using Witten deformation. We prove some lemmas in a more general setting than the setting in Main Theorem. The proofs in this section and in the appendix are based on the localization argument of Furuta [7] and Fukaya, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita [6]. We simplify and modify these arguments to suit the situation in this paper.

In the previous subsection, we define the $G^+(n-s^-,s^+,0)$ structure $\mathfrak{s}_C$ on a characteristic submanifold $C=h^{-1}(0)$ from a $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$ on Y in Theorem 3.8. Now we consider the relation between the standard spinor bundle of $\mathfrak{s}_C$ and the restriction of the standard spinor bundle $\mathfrak{S}$ of $\mathfrak{s}$ to C. First, we consider $\mathfrak{S}|_C$, the restriction of the standard $\mathbb{Z}/2\mathbb{Z}$ graded spinor bundle of $\mathfrak{s}$ to C.

Proposition 3.11 *Let Y be an n dimensional manifold with $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$. We will denote by $\mathfrak{S}$ the standard $\mathbb{Z}/2\mathbb{Z}$ graded spinor bundle of $\mathfrak{s}$. Let $C=h^{-1}(0)$ and let $\mathfrak{s}_C$ be the $G^+(n-s^-,s^+,0)$ structure on C defined in Theorem 3.8. We define a Clifford action on $\mathfrak{S}|_C$ by restricting the Clifford action $(c_{TY},c_{E_+})\colon (TY\oplus E_+)\otimes\mathfrak{S}\to\mathfrak{S}$ to the vector bundle $(TC\oplus E_+|_C)\otimes\mathfrak{S}|_C$ on C. We will denote by $(\tilde{c}_{TC},\tilde{c}_{E_+})$ this restricted Clifford action. Note that the image of $(\tilde{c}_{TC},\tilde{c}_{E_+})$ is $\mathfrak{S}|_C$. Then there exists an isomorphism Φ which has following properties: Let $\mathfrak{S}_C$ be the standard $\mathbb{Z}/2\mathbb{Z}$ graded spinor bundle of $\mathfrak{s}_C$. The isomorphism*

$$\Phi\colon \mathfrak{S}|_C\to \mathrm{Cl}_+(N(C))\mathbin{\hat{\otimes}}\mathfrak{S}_C\mathbin{\hat{\otimes}} V^*_{s^-}$$

preserves the Clifford action of $TC\oplus E_+|_C$ when we define a Clifford action to right hand side by

$$c_{TC}(x)\cdot\omega\otimes\phi\otimes v:=(-1)^{|\omega|}\omega\otimes\tilde{c}_{TC}(x)\phi\otimes v,$$
$$c_{E_+}(w)\cdot\omega\otimes\phi\otimes v:=(-1)^{|\omega|}\omega\otimes\tilde{c}_{E_+}(w)\phi\otimes v$$

for $\omega\otimes\phi\otimes v\in\mathrm{Cl}_+(N(C))\mathbin{\hat{\otimes}}\mathfrak{S}_C\mathbin{\hat{\otimes}}V^*_{s^-}$, $x\in TC$ and $w\in E_+|_C$. Moreover, Φ preserves the right $\mathrm{Cl}_{(n+s^+,s^-)}$ action if we define the right $\mathrm{Cl}_{(n+s^+,s^-)}$ action to $\mathrm{Cl}_+(N(C))\mathbin{\hat{\otimes}}\mathfrak{S}_C\mathbin{\hat{\otimes}}V^*_{s^-}$ by

$$\omega\otimes\phi\otimes v\cdot a\otimes b:=(-1)^{|a||v|}\omega\otimes\phi\cdot a\otimes vb$$

for $\omega\otimes\phi\otimes v\in\mathrm{Cl}_+(N(C))\mathbin{\hat{\otimes}}\mathfrak{S}_C\mathbin{\hat{\otimes}}V^*_{s^-}$ and $a\otimes b\in\mathrm{Cl}_{(n+s^+-s^-,0)}\mathbin{\hat{\otimes}}\mathrm{Cl}_{(s^-,s^-)}$.

Proof From the proof of Theorem 3.8, we see the structure group of the Spin structure of

$$\det N(C)\oplus\det E_-\oplus N(C)\oplus E_-$$

reduces the subgroup of $\mathrm{Spin}(2s^-+2)$ which is isomorphic to $O(s^-)$. If we write $O(s^-)$ for the subgroup, then the structure group of $\mathfrak{S}|_C$ is $G^+(n-s^-,s^+,0)\times O(s^-)$. Write $G=G^+(n-s^-,s^+,0)\times O(s^-)$.

Let us denote by $S=\mathrm{Cl}_{(n+s^+,s^-)}$ a representation space whose associated vector bundle is $\mathfrak{S}$. We describe the action of G on S. We define $\tilde{g}_v\in O(s^-)$ to be the natural lift of g_v to $\mathrm{Spin}(2s^-+2)$ given in Lemma 3.4 where g_v is the composition of the reflections of vectors$(e,0,e,0),(0,v,0,v)\in\det\mathbb{R}^{s^-}\oplus\mathbb{R}^{s^-}\oplus\det\mathbb{R}^{s^-}\oplus\mathbb{R}^{s^-}$ ($|e|=|v|=1$). The element g_v is independent of the choice of e. Elements in the group $O(s^-)\subset\mathrm{Spin}(2s^-+2)$ are the products of finitely many elements of the form $\tilde{g}_v$ for some v and ±1. The Clifford multiplications c'_n, c'_- are G equivariant and hence the action of $\tilde{g}_v$ is given by $\pm c'_n(v)c'_-(v)$.

The sign of $\tilde{g}_v=\pm c'_n(v)c'_-(v)$ is determined by how to embed the group $G^+(n,s^+,s^-)$ to a Spin group. In our convention of Lemma 3.2, the image of $c'_n(e_i)c'_-(e_i)$ of the embedding is $e_0e_ie'_0e'_i$ and this coincides with $\tilde{g}_{e_i}$ which is a lift of g_v given in Lemma 3.4. Thus we have $\tilde{g}_v=c'_n(v)c'_-(v)$.

We will denote by S_C a subspace of S which is the intersection of $+1$-eigenspaces of $\tilde{g}_v = c'_n(v)c'_-(v)$ for all $v \in S(\mathbb{R}^{s^-})$. The subspace S_C coincides with the intersections of $+1$-eigenspaces of $c'_n(e_i)c'_-(e_i)$ $(i = 1, \dots, s^-)$, where $\{e_1, \dots, e_{s^-}\}$ is an orthonormal basis of $\mathbb{R}^{s^-}$. The action of $G^+(n-s^-, s^+, 0)$ preserves S_C because this action commutes with $c'_n(v)c'_-(v)$ for all v. The definition of S_C immediately implies that the action of $O(s^-)$ on S_C is trivial.

Let us show that S_C coincides with $\mathrm{Cl}_{(n+s^+-s^-,0)} \subset S$. Recall that the standard spinor bundle $\mathfrak{S}$ is the associated bundle with the representation $\mathrm{Cl}_{(n+s^+,s^-)} = \mathrm{Cl}_{(n-s^-+s^+,0)} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^-,s^-)}$. An element $\tilde{g}_v = c'_n(v)c'_-(v)$ acts on $\mathrm{Cl}_{(n-s^-+s^+,0)}$ trivially and on $\mathrm{Cl}_{(s^-,s^-)}$ by left multiplication of $\mathrm{Cl}_{(s^-,s^-)}$. We will denote by $\widetilde{V}$ the intersection of the $+1$-eigenspaces of $c'_n(v)c'_-(v)$ in $\mathrm{Cl}_{(s^-,s^-)}$ for all $v \in S(\mathbb{R}^{s^-})$. The dimension of $\widetilde{V}$ is 2^{s^-}. The action of $c'_n(v)c'_-(v)$ commutes with the right $\mathrm{Cl}_{(s^-,s^-)}$ action. The only irreducible representation of $\mathrm{Cl}_{(s^-,s^-)}$ is V_{s^-} and its dimension is 2^{s^-}. Hence there is an isomorphism $\psi\colon V^*_{s^-} \to \widetilde{V}$. Using ψ, we have the isomorphism $\mathrm{Cl}_{(n+s^+-s^-,0)} \mathbin{\hat{\otimes}} V^*_{s^-} \to S_C$ by $\xi \otimes v \mapsto \xi\psi(v)$. If we give the G action on $\mathrm{Cl}_{(s^-,0)}$ by $g \cdot x \mapsto gxg^{-1}$, we have the G equivariant isomorphism

$$\Psi_0\colon \mathrm{Cl}_{(s^-,0)} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(n+s^+-s^-,0)} \mathbin{\hat{\otimes}} V^*_{s^-} \to \mathrm{Cl}_{(n+s^+,s^-)}, \quad x \otimes \xi \otimes v \mapsto x \cdot \xi \cdot \psi(v).$$

The associated vector bundles of $\mathrm{Cl}_{(s^-,0)}, \mathrm{Cl}_{(n+s^+-s^-,0)}$ and $V^*_{s^-}$ is $\mathrm{Cl}_+(N(C))$, $\mathfrak{S}_C$ and the trivial bundle $V^*_{s^-}$, respectively. Thus we have the map

$$\Psi\colon \mathrm{Cl}_+(N(C)) \mathbin{\hat{\otimes}} \mathfrak{S}_C \mathbin{\hat{\otimes}} V^*_{s^-} \to \mathfrak{S}|_C.$$

We define the map Φ to be the inverse of the map Ψ. It is easy to check to see that Φ preserves the Clifford action of $TC \otimes E_+|_C$ and right $\mathrm{Cl}_{(n+s^+,s^-)}$ action. □

In this section we identify $\mathfrak{S}|_C$ with $\mathrm{Cl}_+(N(C)) \mathbin{\hat{\otimes}} \mathfrak{S}_C \mathbin{\hat{\otimes}} V^*_{s^-}$ by the map Φ.

Next we prove the localization of analytic index in Proposition 3.22. We identify a tubular neighborhood of C, denoted by $U(C)$, with the open disk bundle of the normal bundle of C,

$$B(N(C)) = \{v \in N(C) \mid |v| < 1\}.$$

We will denote by $\pi\colon N(C) \to C$ the projection of the normal bundle. Let us outline the proof of the localization.

Step 1 We first formulate the index of a Dirac type operator acting on the sections of the vector bundle $\pi^*(\mathfrak{S}|_C)$ over $N(C)$. Since $N(C)$ is not closed, we need behavior of its end. We will see the index of $N(C)$ coincides with the index of the $G^+(n, s^-, s^+)$ structure $\mathfrak{s}_C$ on C.

Step 2 The analytic index of the $G^+(n, s^+, s^-)$ structure $\mathfrak{s}$ on Y coincides with the index of **Step 1**. In this argument we deform the Dirac type operator in Definition 2.13. We prove some analytic lemmas and propositions of this step in the appendix.

3.2.1 We begin with Step 1 First, we consider the trivial $G^+(n,0,n)$ structure on $\mathbb{R}^n$. This structure appears in a fiberwise way when we consider the operator on $N(C)$. Let $\mathfrak{s}_0$ be a trivial $G^+(n,0,n)$ structure on $\mathbb{R}^n$. We will denote by $h(x)=x$ the section of $E_-=\mathbb{R}^n\times\mathbb{R}^n$ and we have the isomorphism from $T\mathbb{R}^n$ to E_- by using dh. We have the reduction of the structure group of $\mathfrak{s}_0$ to the subgroup $G\subset G^-(n,0,n)$ which is naturally isomorphic to $O(n)$ by using Lemma 3.4. Let $\mathfrak{S}_f$ denote the standard $\mathbb{Z}/2$ graded spinor bundle of $\mathfrak{s}_0$. Let $C=h^{-1}(0)=\{0\}$, be a characteristic submanifold. From Proposition 3.11 we identify $\mathfrak{S}_f|_C$ with $\mathrm{Cl}_+(\mathbb{R}^n)\mathbin{\hat\otimes}\underline{\mathbb{R}}\mathbin{\hat\otimes}V_n^*$. Note that $\mathfrak{S}_C\cong C\times\mathbb{R}=\underline{\mathbb{R}}$ is a vector bundle on a point. Let L be a trivial vector bundle on $\mathbb{R}^n$.

Lemma 3.12 *We will denote by $\mathcal{S}$ a set of rapidly decreasing sections of $\mathfrak{S}_f\otimes L$. We will denote by D'_m a differential operator acting on sections of $\Gamma(\mathfrak{S}_f\otimes L)$ given by*

$$D'_m=\sum_{i=1}^{n}\Bigl(c_{T\mathbb{R}^n}(dx^i)\frac{\partial}{\partial x^i}+mc_{E_-}(e_i)x^i\Bigr).$$

Note that D'_m preserves the subspace $\mathcal{S}\subset\Gamma(\mathfrak{S}_f\otimes L)$. The operator D'_m is independent of the choice of the orthonormal basis of $\mathbb{R}^n$. This operator commutes with the right $\mathrm{Cl}_{(n,n)}$ action, and there is an isomorphism between $\ker D'_m\cap\mathcal{S}$ and $\underline{\mathbb{R}}\mathbin{\hat\otimes}V_n^*\otimes L$ as right $\mathrm{Cl}_{(n,n)}$ modules.

Proof We see at once that D'_m is independent of choice of the orthonormal basis of $\mathbb{R}^n$ by direct calculation.

Let ϕ be a rapidly decreasing section. We have $D'_m\phi=0$ if and only if $(D'_m)^2\phi=0$ because D'_m is a skew-symmetric operator. We have

$$(D'_m)^2=\sum_{i=1}^{n}\Bigl(\Bigl(\frac{\partial}{\partial x^i}\Bigr)^2-m^2(x^i)^2+mc_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)\Bigr)=-H+\sum_{i=1}^{n}(mc_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)),$$

where H is a harmonic oscillator acting on smooth functions $\mathbb{R}^n\to\mathbb{R}$. It is well known that H has only discrete spectrum and each eigenspace is 1-dimensional. The eigenvalues are $nm,2nm,3nm,\dots$ and each eigenfunction is rapidly decreasing. In particular, the eigenfunction of nm is $e^{-m|x|^2/2}$ and this does not depend on the choice of orthonormal basis of $\mathbb{R}^n$. The eigenvalues of $mc_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)$ are $\pm m$ for all i. Hence the kernel of $(D'_m)^2$ is in the intersection of the $+1$ eigenspace of $c_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)$ and nm eigenspace of H. The intersection of the $+1$ eigenspace of $c_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)$ for all i coincides with $\underline{\mathbb{R}}\mathbin{\hat\otimes}V_n^*\otimes L$ from the construction of the map Φ on Proposition 3.11. Thus we have

$$\ker D'_m\cap\mathcal{S}=\underline{\mathbb{R}}\mathbin{\hat\otimes}V_n^*\otimes L\cdot e^{-m|x|^2/2}.\qquad\square$$

Definition 3.13 We will denote by $\tilde u_0$ a function on $\mathbb{R}^n$ given by

$$\frac{e^{-m|x|^2/2}}{\|e^{-m|x|^2/2}\|_{L^2(\mathbb{R}^n)}}.$$

Definition 3.14 We will follow the notation of the statement and the proof of Proposition 3.11. We deform a metric of Y if necessary and we identify a tubular neighborhood $U(C)$ of C with $B(N(C)) = \{v \in N(C) \mid |v| < 1\}$ as a Riemannian manifold. Let us denote by $\pi\colon N(C) \to C$ the projection. We define a $G^+(n,s^+,s^-)$ structure on $B(N(C))$ by restricting the $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$ on Y. By abuse of notation, we use the same letter $\mathfrak{s}$ for this $G^+(n,s^+,s^-)$ structure on $B(N(C))$ and we write $\mathfrak{S}$ instead of $\mathfrak{S}|_{U(C)}$. The structure group of $\mathfrak{s}$ reduces to $G = G^+(n-s^-,s^+,0) \times O(s^-)$ by using the isomorphism $dh\colon N(C) \cong E_-$. Let $\not{D}_m = \not{D} + mc_-(h)$ where $\not{D}$ is a Dirac operator of $\mathfrak{S}$. We abbreviate $c_-(h)$ to h. Note that $\not{D}_m$ is an antisymmetric operator. Let A be a connection such that $c_{TY} \circ A = \not{D}$. Perturbing A, we may assume that $\pi^*(A|_C) = A$ on $B(N(C))$.

Lemma 3.15 *Let $\not{D}'_C$ be the Dirac operator defined by the Clifford action $c_{TC}\colon TC \otimes \mathfrak{S}|_C \to \mathfrak{S}|_C$. Then we have a decomposition*

$$\not{D} = D_C + D_f$$

with the following property:

- *On $U \times \mathbb{R}^{s^-}$, which is a trivialization of $N(C)$,*
$$D_C = \not{D}'_C|_U, \quad D_f = \sum_{k=1}^{s^-} c_{TY}(d\xi_i)\frac{\partial}{\partial \xi_i}.$$
Here c_{TY} is the restriction of the Clifford multiplication of $\mathfrak{s}$ to $\pi^* N(C) \subset TN(C)$ and $(\xi_1,\dots,\xi_{s^-})$ are coordinates of $\mathbb{R}^{s^-}$, which is a fiber of $N(C)$.
- *D_C and D_f anticommute.*
- *Both D_C and D_f are antisymmetric with respect to the L^2 inner product on $N(C)$.*

Proof It is easy to see that we have the decomposition of $\not{D}$ on each trivialization of $N(C)$. The operator

$$\sum_{k=1}^{s^-} c_{TY}(d\xi_i)\frac{\partial}{\partial \xi_i}$$

is $O(n)$ invariant. Hence the operator above on each trivialization coincides with the corresponding one on another trivialization. Thus we have the operator D_f on the whole of $N(C)$. Let $D_C := \not{D} - D_f$. On each trivialization, the operator D_f coincides with $\sum_{k=1}^{s^-} c_{TY}(d\xi_i)\frac{\partial}{\partial \xi_i}$ and D_C coincides with $\not{D}'_C|_U$. We will see at once anticommutativity of these operators on each trivialization on $N(C)$. We see these operators are antisymmetric with respect to inner product L^2. It is sufficient to prove that D_f is antisymmetric because the operator $\not{D}$ is antisymmetric. To see this, we decompose an integral on $N(C)$ by $\int_{N(C)} = \int_C \int_{\text{fiber}}$, where $\int_{\text{fiber}}$ is the integration on each fiber. The operator D_f is antisymmetric with respect to the L^2 inner product on each fiber and we have D_f is antisymmetric with respect to the L^2 norm on $N(C)$. □

Definition 3.16 We will denote by u_0 the function on $N(C)$ whose restriction to each fiber of $N(C)$ coincides with the function $\tilde{u}_0$.

Lemma 3.17 *We will denote by $h \in \Gamma(\pi^* E_-)$ the tautological section of $\pi^* E_- \cong \pi^* N(C)$. Let $\not{D}_C$ be a Dirac operator on $\mathfrak{S}_C \mathbin{\hat{\otimes}} V^*_{s^-}$. Let m be a positive real number and let $a \in \Gamma(\mathfrak{S}_C \mathbin{\hat{\otimes}} V^*_{s^-})$. We have*

$$(\not{D} + mh)\pi^* a \cdot u_0 = \pi^*(\not{D}_C a) \cdot u_0.$$

Proof Recall the decomposition $\not{D} = D_C + D_f$. The restriction of $D_f + mh$ to each trivialization of $N(C)$ coincides with the operator D'_m in Lemma 3.12. Thus we have $(D_f + mh)(\pi^* a \cdot u_0) = 0$. It is sufficient to consider the operator D_C. The operator D_C coincides with the operator $\not{D}'_C$ on each trivialization and this operator is a pullback of an operator on C. Hence we have $D_C(\pi^* a \cdot u_0) = \pi^*(\not{D}_C a) \cdot u_0$. □

Lemma 3.18 *Let $H = \{\pi^* a \cdot u_0 \in \Gamma(\mathfrak{S}) \mid a \in \Gamma(\mathfrak{S}_C \mathbin{\hat{\otimes}} V^*_{s^-}),\ \not{D}_C a = 0\}$. We will denote by λ_C the minimum of the absolute value of nonzero eigenvalues of $\not{D}_C$. Let m be a positive number such that $m \geq |\lambda_C|$. Let $\phi \in \Gamma(N(C), \mathfrak{S})$ be a section whose restriction to each fiber of $N(C)$ is rapidly decreasing function. If ϕ is orthogonal to H in the L^2 inner product, We have $\|\not{D}_m \phi\| \geq \lambda_C \|\phi\|$, where $\|\cdot\|$ is the L^2 norm on $N(C)$.*

Proof We decompose ϕ into $\phi = \phi_0 + \pi^* b \cdot u_0$, where $b \in \Gamma(\mathfrak{S}_C) \mathbin{\hat{\otimes}} V^*_{s^-}$ and ϕ_0 is a fiberwise rapidly decreasing section such that $\int_{N(C)_x} \langle \phi_0, \pi^* a \cdot u_0 \rangle = 0$ for all $x \in C$, $a \in (\mathfrak{S}_C \mathbin{\hat{\otimes}} V^*_{s^-})_x$. From Lemma 3.12 and for L to be $(\mathfrak{S}_C)_x$, ϕ_0 is orthogonal to the kernel of $D_f + mh$ on each fiber. The section b is orthogonal to $\ker \not{D}_C$ in the L^2 inner product on C because ϕ is orthogonal to H. From the above decomposition, we have

$$(\not{D} + mh)\phi = (\not{D} + mh)\phi_0 + \pi^*(\not{D}_C b) \cdot u_0.$$

It is easy to see that

$$\int_{N(C)} \big\langle (\not{D} + mh)\phi_0, \pi^*(\not{D}_C b) \cdot u_0 \big\rangle = 0 \quad \text{and} \quad \|(\not{D} + mh)\phi\|^2 = \|(\not{D} + mh)\phi_0\|^2 + \|\pi^*(\not{D}_C b) \cdot u_0\|^2.$$

It is straightforward to see D_C and $D_f + mh$ are anticommutative.

Thus we have

$$\begin{aligned}\int_{N(C)} |(\not{D} + mh)\phi_0|^2 &= \int_{N(C)} |D_C \phi_0|^2 + \int_{N(C)} |(D_f + mh)\phi_0|^2 \\ &\geq \int_C \int_{\text{fiber}} |(D_f + mh)\phi_0|^2 \geq \int_C m^2 \int_{\text{fiber}} |\phi_0|^2 \geq \int_C \lambda_C^2 \int_{\text{fiber}} |\phi_0|^2\end{aligned}$$

and

$$\int_{N(C)} |\pi^*(\not{D}_C b) \cdot u_0|^2 = \int_{N(C)} |\pi^*(\not{D}_C b) \cdot u_0|^2 \geq \lambda_C^2 \int_{N(C)} |\pi^* b \cdot u_0|^2.$$

Hence we have the inequality $\|\not{D}_m \phi\| \geq \lambda_C \|\phi\|$ as desired. □

Definition 3.19 The finite-dimensional vector space H has a natural $\mathbb{Z}/2\mathbb{Z}$ grading and a left $\mathrm{Cl}_{(n+s^+, s^-)}$ action induced by $\mathfrak{S}$. Let ϵ be the $\mathbb{Z}/2\mathbb{Z}$ grading operator. The four-tuple $(0, \epsilon, \mathrm{Cl}_{(n+s^+, s^-)}, H)$ defines an element of $\mathrm{KO}^{s^- - n - s^+}(\mathrm{pt})$. We write $\mathrm{ind}(N(C), \mathfrak{S})$ for this element in $\mathrm{KO}^{s^- - n - s^+}(\mathrm{pt})$.

Remark 3.20 By the definition of H and from Lemma 2.2, we have

$$\mathrm{ind}(\mathfrak{s}_C) = \big[(\not{D}_C, \epsilon, \mathrm{Cl}_{(n+s^+,s^-)}, L^2(C, \not{S}_C \mathbin{\hat{\otimes}} V^*_{s^-}))\big] = \mathrm{ind}(N(C), \not{S}) \in \mathrm{KO}^{s^- - n - s^+}(\mathrm{pt}).$$

Remark 3.21 The notion of the index of open manifold is introduced by Mikio Furuta in [7]. When the index $\mathrm{ind}(N(C), \not{S})$ values in $\mathrm{KO}^0(\mathrm{pt})$, it coincides with the index of the pair $(N(C), [h])$.

3.2.2 Now Step 2

We prove $\mathrm{ind}(\mathfrak{s})$, an analytic index of $G^+(n, s^+, s^-)$ structure on Y, is equal to $\mathrm{ind}(N(C), \not{S})$.

We introduce some notation.

- If necessary we perturb h so that h satisfies $|h| < 1$ on $U(C)$ and $|h| = 1$ on the complement set of $U(C)$.
- We will denote by $\mathcal{H}^\lambda_m$ the direct sum of eigenspaces of $\not{D}_m \colon \Gamma(Y, \not{S}) \to \Gamma(Y, \not{S})$ such that the absolute value of each eigenvalue is less than λ^2. This is a finite-dimensional subspace of $L^2(Y, \not{S})$. $\mathcal{H}^\lambda_m$ has a natural $\mathbb{Z}/2\mathbb{Z}$ grading and a left $\mathrm{Cl}_{(n+s^+,s^-)}$ action.
- Let ρ be a smooth function on Y supported in $U(C)$ such that
$$\rho(z) = \begin{cases} 1 & \text{if } |z| \le \frac{1}{2}, \\ 0 & \text{if } |z| \ge \frac{2}{3} \end{cases}$$
for $z \in U(C) \cong B(N(C)) \subset N(C)$ and monotone decreasing on $\frac{1}{2} < |z| < \frac{2}{3}$ in $|z|$, where $|\cdot|$ is a norm of $N(C)$. (We identify $U(C)$ and $B(N(C))$.)

The following proposition is proved by the general theory of Witten deformation. We prove this proposition in the appendix.

Proposition 3.22 *Assume that a positive constant λ is smaller than a constant determined by the principal symbol of D and the differential of ρ and suppose that $m > \lambda$. Let Π' be the orthogonal projection from $L^2(N(C), \not{S})$ to H. Then the map*

$$\mathcal{H}^\lambda_m \to H, \quad \phi \mapsto \Pi'(\rho\phi)$$

is an isomorphism and preserves the $\mathbb{Z}/2$ grading and the $\mathrm{Cl}_{(n+s^+,s^-)}$ action.

Proof of Main Theorem From the definition of the index of $\mathfrak{s}$,

$$\mathrm{ind}(\mathfrak{s}) = [(\not{D}_m, \epsilon, \mathrm{Cl}_{(s^-,s^++n)}, \mathcal{H}^\lambda_m)]$$

and we see

$$[(0, \epsilon, \mathrm{Cl}_{(s^-,s^++n)}, H)] = [(\not{D}_m, \epsilon, \mathrm{Cl}_{(s^-,s^++n)}, \mathcal{H}^\lambda_m)] \in \mathrm{KO}^{s^- - n - s^+}(\mathrm{pt})$$

from Proposition 3.22. We have $[(0, \epsilon, \mathrm{Cl}_{(s^-,s^++n)}, H)] = \mathrm{ind}(\mathfrak{s}_C)$ from Remark 3.20. Thus we have proved Main Theorem. ∎

4 Examples

4.1 Freed–Hopkins $H_n(s)$

In this subsection, we consider a family of groups $H_n(s)$ defined by Freed and Hopkins [5]. The group $H_n(s)$ is given in Table 1.

Lemma 4.1 *In the case $s = 0, 1, 2, 3$, we have $H_n(s) \cong G^+(n, s, 0)$ and in the case $s = -1, -2, -3$, we have $H_n(s) \cong G^+(n, 0, -s)$.*

Proof It is obvious for the case $s = 0$ therefore it is sufficient to prove the case of $|s| = 1, 2, 3$. First we consider the case $|s| = 3$. We use the identification $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ and $(\mathrm{Cl}_{+n} \mathbin{\hat{\otimes}} (\mathbb{R} \oplus \mathbb{R}\Gamma))_0 \cong \mathrm{Cl}_{\pm n}$, where $\Gamma = e_1 e_2 e_3 \in \mathrm{Cl}_s$ and $\pm$ is the sign of s. We have an isomorphism $G^+(n, s, 0) \to H_n(s)$ (or $G^+(n, 0, -s) \to H_n(s)$) given by

$$gu \mapsto \begin{cases} [g, u] & \text{if } g \in \mathrm{Spin}(n),\ u \in \mathrm{Spin}(3), \\ [g \otimes \Gamma, \Gamma^{-1} u] & \text{otherwise} \end{cases}$$

for $g \in \mathrm{Pin}^+(n),\ u \in \mathrm{Pin}^{\pm}(3)$.

We consider the case $|s| = 1, 2$. We will denote by e a generator of Cl_s with $|e| = 1$. We use the identification $\mathrm{Cl}_{+n} \mathbin{\hat{\otimes}} (\mathbb{R} \oplus \mathbb{R}e) \cong \mathrm{Cl}_{\pm n}$ where $\pm$ is the sign of s. We have an isomorphism $G^+(n, s, 0) \to H_n(s)$ (or $G^+(n, 0, -s) \to H_n(s)$) given by

$$gu \mapsto \begin{cases} [g, u] & \text{if } g \in \mathrm{Spin}(n),\ u \in \mathrm{Spin}(|s|), \\ [g \otimes e, e^{-1} u] & \text{otherwise} \end{cases}$$

for $g \in \mathrm{Pin}^+(n),\ u \in \mathrm{Pin}^{\pm}(|s|)$. □

Remark 4.2 An index of $H_n(s)$ structure is defined in [5]. From the lemma above, we see the index of $H_n(s)$ coincides with our index of $G^+(n, s, 0)$ (or $G^+(n, 0, -s)$) structure.

s	$H_n(s)$
0	$\mathrm{Spin}(n)$
-1	$\mathrm{Pin}^+(n)$
-2	$\mathrm{Pin}^+(n) \ltimes_{\mathbb{Z}/2\mathbb{Z}} U(1)$
-3	$\mathrm{Pin}^-(n) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{SU}(2)$
4	$\mathrm{Spin}(n) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{SU}(2)$
3	$\mathrm{Pin}^+(n) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{SU}(2)$
2	$\mathrm{Pin}^-(n) \ltimes_{\mathbb{Z}/2\mathbb{Z}} U(1)$
1	$\mathrm{Pin}^-(n)$

Table 1: Groups $H_n(s)$ defined by Freed and Hopkins [5].

Remark 4.3 In the case $s=4$, $H_n(4)$ is isomorphic to a subgroup of $G^+(n,0,4)$ and we can see the index of $H_n(4)$ structure defined in [5] coincides with our index if the structure group of a $G^+(n,0,4)$ structure reduces to $H_n(4)$.

We identify $\mathrm{Spin}(4)$ with $\mathrm{SU}(2)\times\mathrm{SU}(2)$. We give an embedding of $H_n(4)$ to $G^+(n,0,4)$ by

$$\mathrm{Spin}(n)\times_{\mathbb{Z}/2\mathbb{Z}}\mathrm{SU}(2)\ni[g,u]\mapsto[g,\mathrm{diag}(u,u)]\in\mathrm{Spin}(n)\times_{\mathbb{Z}/2\mathbb{Z}}\mathrm{Spin}(4)\subset G^+(n,0,4).$$

The following proposition is a consequence of Main Theorem, Lemma 4.1 and the above remarks.

Proposition 4.4 *We assume* $s=-1,-2,-3,$ *or* 4. *Then there exists an isomorphism* f *such that the following diagram commutes*:

$$\begin{array}{ccc}\Omega_n^{H_n(s)}(\mathrm{pt}) & \xrightarrow{\ f\ } & \Omega_{n-|s|}^{\mathrm{Spin}}(\mathrm{pt})\\ & \searrow & \downarrow\\ & & \mathrm{KO}^{-n-s}(\mathrm{pt})\end{array}$$

4.2 The case of $G^+(5,0,4)$ structure

We consider the index of the $G^+(5,0,4)$ structure especially because it is important to see the orientability of $\mathrm{Pin}^-(2)$ monopole moduli space in the next section. The following arguments can be easily generalized to the case of the index of the $G^+(8k+5,0,4)$ structure. We use only the $k=0$ case in the next section; therefore we only consider this case to avoid complications.

Definition 4.5 Let Z be a 5 dimensional closed manifold and $\mathfrak{s}$ be a $G^+(5,0,4)$ structure on Z. Let $\mathfrak{S}$ be the standard spinor bundle of $\mathfrak{s}$. Note that $\mathfrak{S}$ has a natural right $\mathrm{Cl}_{(5,4)}$ action. Let $\epsilon_0,\epsilon_1,\dots,\epsilon_4$ be generators of Cl_{+5}. and let $e_1,\dots,e_4$ be generators of Cl_{-4}. We will denote by $\widetilde{S}$ a subbundle of $\mathfrak{S}$ such that the intersection of $+1$-eigenspaces of $\epsilon_1e_1,\dots,\epsilon_4e_4$. We call $\widetilde{S}$ the spinor bundle of $\mathfrak{s}$. The Clifford multiplication of TZ and E_- preserves $\widetilde{S}$ because they commute with the right $\mathrm{Cl}_{(5,4)}$ action. We define a skew-adjoint Dirac operator $\widetilde{D}$ on $\widetilde{S}$ by using Clifford action of TZ.

We have the following lemma from Lemma 2.2:

Lemma 4.6 *The spinor bundle* $\widetilde{S}$ *is a generalized* $\mathbb{Z}/2$ *graded spinor bundle with left* Cl_{-1} *action. The index* $\mathrm{ind}(\mathfrak{s})$ *of* $\mathfrak{s}$ *coincides with* $(0,\epsilon,\mathrm{Cl}_{-1},\ker\widetilde{D})\in\mathrm{KO}^{-1}(\mathrm{pt})$. *Moreover, under the isomorphism* $\mathrm{KO}^{-1}(\mathrm{pt})\cong\mathbb{Z}/2\mathbb{Z}$, *we have* $\mathrm{ind}(\mathfrak{s})=\dim\ker(\widetilde{D}|_{\widetilde{S}^+})\bmod 2$ *where* $\widetilde{S}^+$ *is the even part of* $\widetilde{S}$.

We provide a specific construction of $\widetilde{S}$ when the structure group of $\mathfrak{s}$ is reduced from $G^+(5,0,4)$ to $\mathrm{Spin}(5)\times\mathrm{Spin}(4)/\mathbb{Z}/2\mathbb{Z}$. This construction is useful to consider the orientability of $\mathrm{Pin}^-(2)$ monopole moduli space.

Definition 4.7 We will denote by $\epsilon_0, \dots, \epsilon_4$ the generators of Cl_{+5} and by $e_1, \dots, e_4$ the generators of Cl_{-4}.

- Let $\mathbb{H}$ be the quaternion ring. We use the convention that $ijk = 1$.
- Let $\mathbb{H}(2)$ be the matrix ring of 2×2 matrices with entries in the ring $\mathbb{H}$.
- We define an isomorphism $\alpha\colon \mathrm{Cl}_{-4} \to \mathbb{H}(2)$ by
$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 \mapsto \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad e_4 \mapsto \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$
- Let $\Gamma = e_1 e_2 e_3 e_4 \in \mathrm{Cl}_{-4}$. Note that $\Gamma^2 = 1$ and
$$\alpha(\Gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
- We define an isomorphism $\beta\colon \mathrm{Cl}_{+5} \to \mathbb{H}(2) \oplus \mathbb{H}(2)$ by
$$\epsilon_0 \mapsto (\alpha(\Gamma), -\alpha(\Gamma)), \qquad \epsilon_i \mapsto (\alpha(\Gamma)\alpha(e_i), -\alpha(\Gamma)\alpha(e_i)), \quad i = 1, 2, 3, 4.$$
- We define an isomorphism $f\colon \mathrm{Cl}_{(5,4)} \to (\mathbb{H}(2) \oplus \mathbb{H}(2)) \otimes \mathbb{H}(2)$ by
$$\epsilon_p \mapsto \beta(\epsilon_p) \otimes \alpha(\Gamma), \quad e_q \mapsto 1 \otimes \alpha(e_q),$$
where $p = 0, \dots, 4,\ q = 1, \dots, 4$.
- Using isomorphisms α, β, we define the spinor representations of $\mathrm{Cl}_{+5}, \mathrm{Cl}_{-4}$ whose representation spaces are $\mathbb{H}^2 \oplus \mathbb{H}^2$, $\mathbb{H}^2$, respectively. By abuse of notation, we let α, β stand for these spinor representations, respectively. Note that
$$\beta(\epsilon_i) \cdot (\phi_0, \phi_1) = (\alpha(\Gamma)\alpha(e_i)\phi_0, -\alpha(\Gamma)\alpha(e_i)\phi_1) \quad \text{for } i = 0, \dots, 4.$$

Remark 4.8 In our notations, the $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathbb{H}(2) \oplus \mathbb{H}(2)$ induced by the isomorphism β is given as follows: The subspace $\{(A, A) \mid A \in \mathbb{H}(2)\}$ is the even part and $\{(A, -A) \mid A \in \mathbb{H}(2)\}$ is the odd part. The $\mathbb{Z}/2\mathbb{Z}$ grading operator is a map $(A, B) \mapsto (B, A)$.

The $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathbb{H}(2)$ induced by the $\mathbb{Z}/2\mathbb{Z}$ grading of Cl_{-4} using the map α is given by declaring the even part consists of diagonal matrices and the odd part of off-diagonal one. The $\mathbb{Z}/2\mathbb{Z}$ grading operator is given by $A \mapsto \alpha(\Gamma)A\alpha(\Gamma)^{-1} = \mathrm{Ad}(\alpha(\Gamma))(A)$.

Definition 4.9 We assume that the structure group of $\mathfrak{s}$ reduces to the group $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2)$.

Let $S = S_0 \oplus S_1,\ S_0 = S_1 = \mathbb{H}^2$ be a representation space of the spinor representation β of Cl_{+5}. We define the representation ρ by
$$\rho([q, (u_0, u_1)])\phi = \beta(q) \cdot (\phi_0 u_0^{-1}, \phi_1 u_1^{-1})$$

for

$$\phi = (\phi_0, \phi_1) \in S \cong \mathbb{H}^2 \oplus \mathbb{H}^2,$$
$$[q, (u_0, u_1)] \in \mathrm{Spin}(5) \times (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/(\mathbb{Z}/2) \cong \mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2).$$

We denote by S' the associated vector bundle of the representation ρ. We give a $\mathbb{Z}/2\mathbb{Z}$ grading on $S' \oplus S'$ by declaring the subspace $\{(\varphi, \varphi) \in S' \oplus S'\}$ is the even part and the subspace $\{(\varphi, -\varphi) \in S' \oplus S'\}$ is the odd part. We give the following right Cl_{+1} action: let ϵ_0 be the generator of Cl_{+1} and we define the right action of ϵ_0 by $(\varphi_0, \varphi_1) \cdot \epsilon_0 = (\varphi_0, -\varphi_1)$ for $(\varphi_0, \varphi_1) \in S' \oplus S'$. The left Cl_{-1} action is given by $e_0 \cdot \phi := (\epsilon \phi) \cdot \epsilon_0$, where ϵ is the $\mathbb{Z}/2\mathbb{Z}$ grading operator. We define the Clifford multiplication c'_n by the left Clifford multiplication of $\mathrm{Cl}_{(5,4)}$ given by $\epsilon_j \cdot (\varphi_0, \varphi_1) = (\beta(\epsilon_j)\varphi_0, -\beta(\epsilon_j)\varphi_1)$ for $j = 0, \dots, 4$. Then $S' \oplus S'$ is $\mathbb{Z}/2\mathbb{Z}$ graded generalized spinor bundle with left Cl_{-1} action of $\mathfrak{s}$.

Remark 4.10 We can define the c_{E_-} Clifford action on $S' \oplus S'$ such that that action is preserved by the isomorphism in the Lemma below. But we will not use c_- in Section 5.

Lemma 4.11 *As a $\mathbb{Z}/2\mathbb{Z}$ graded generalized spinor bundle with left Cl_{-1} action of $\mathfrak{s}$, $S' \oplus S' \cong \widetilde{S}$. In particular, $\widetilde{S}^+ \cong S'$.*

Proof We identify $\mathrm{Cl}_{(5,4)}$ with $(\mathbb{H}(2) \oplus \mathbb{H}(2)) \otimes_{\mathbb{R}} \mathbb{H}(2)$ by using the isomorphism f in Definition 4.7. Let Φ be a map

$$(\mathbb{H}(2) \oplus \mathbb{H}(2)) \otimes_{\mathbb{R}} \mathbb{H}(2) \ni (A_0, A_1) \otimes B \mapsto (A_0 B^*, A_1\, \mathrm{Ad}(\alpha(\Gamma))(B^*)) \in \mathbb{H}(2) \oplus \mathbb{H}(2),$$

where B^* is transpose matrix of $\bar{B}$. (We will denote by $\bar{\cdot}$ the quaternion conjugate.) We define the $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z})$ action on the space left hand side by

$$[q, (u_0, u_1)](A_0, A_1) = \left(q A_0 \begin{pmatrix} u_0^{-1} & 0 \\ 0 & u_1^{-1} \end{pmatrix}, q A_1 \begin{pmatrix} u_0^{-1} & 0 \\ 0 & u_1^{-1} \end{pmatrix} \right)$$

for $[q, (u_0, u_1)] \in \mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Spin}(5) \times (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/(\mathbb{Z}/2\mathbb{Z})$. For this action, we see Φ is a $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z})$ equivariant map because $\mathrm{diag}(u_0^{-1}, u_1^{-1})$ commutes with $\alpha(\Gamma)$.

The map Φ is invariant under the right multiplication of $f(\epsilon_i e_i)$ for $i = 1, 2, 3, 4$. Therefore, the -1-eigenspaces of $f(\epsilon_i e_i)$ for some $i = 1, 2, 3, 4$ are the subspace of $\ker \Phi$. We see at once Φ is surjective and the dimension of $\mathbb{H}(2) \oplus \mathbb{H}(2)$ is equal to the dimension of $\widetilde{S}$. Thus we have that the restriction of Φ to the representation space $\widetilde{S}$ is a $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z})$ equivariant isomorphism.

We see the $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z})$ representation $\mathbb{H}(2) \oplus \mathbb{H}(2)$ is equivalent to the representation $S' \oplus S'$ through an isomorphism given by identifying each column of the matrix with an element of S'_0, S'_1. The $\mathbb{Z}/2\mathbb{Z}$ grading of $\widetilde{S}$ is defined by the $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathrm{Cl}_{(5,4)}$. It is easy to see that this $\mathbb{Z}/2\mathbb{Z}$ grading of $\widetilde{S}$ coincides with the $\mathbb{Z}/2\mathbb{Z}$ grading of $S' \oplus S'$. Moreover, we see at once Φ preserves the Clifford multiplication c_{TZ} and $f(\epsilon_0)$. This completes the proof. □

Remark 4.12 If we change the basis of $S' \oplus S'$ by the isomorphism $(\phi_0, \phi_1) \mapsto \big(\frac{1}{2}(\phi_0+\phi_1), \frac{1}{2}(\phi_0-\phi_1)\big)$, we change the $\mathbb{Z}/2\mathbb{Z}$ grading of $S' \oplus S'$ and the Clifford multiplication c_{TZ}. The $\mathbb{Z}/2\mathbb{Z}$ grading is given by $S' \oplus 0$ is even and $0 \oplus S'$ is odd. The Clifford multiplication c_{TZ} and the right ϵ_0 action are given by the matrix

$$c_{TZ}(\epsilon_j) = \begin{pmatrix} 0 & \beta(\epsilon_j) \\ \beta(\epsilon_j) & 0 \end{pmatrix}, \quad \epsilon_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that c_{TZ} and right ϵ_0 action commute.

5 The orientability of $\mathrm{Pin}^-(2)$ monopole moduli space

5.1 The orientability and mod 2 indices

In Section 4.1, we consider the spinor bundle and index of the $G^+(5,0,4)$ structure. In this section, we translate the orientability of the $\mathrm{Pin}^-(2)$ monopole moduli space into a mod 2 index on a five dimensional manifold.[1] From Main Theorem, we give a topological criterion for the determinant line bundle on the ambient space of the $\mathrm{Pin}^-(2)$ monopole moduli space to be trivial.

Definition 5.1 We define a group $\mathrm{Spin}^{c-}(4)$ by

$$\mathrm{Spin}^{c-}(4) := \mathrm{Spin}(4) \times \mathrm{Pin}^-(2)/(\mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{Z}/2\mathbb{Z} \cong \{(1,1),(-1,-1)\} \in \mathrm{Spin}(4) \times \mathrm{Pin}^-(2)$.

Remark 5.2 By the definition above, we see that $\mathrm{Spin}^{c-}(4)$ is a subgroup of $G^+(4,0,3)$ by using the natural embedding $\mathrm{Spin}(4) \subset \mathrm{Cl}_{(+4)}$.

Definition 5.3 We define a Spin^{c-} structure on Riemannian 4-manifold X as a $G^+(4,0,3)$ structure whose structure group reduces to Spin^{c-}.

Definition 5.4 Let X be a 4-manifold and $\mathfrak{s}_X$ be a Spin^{c-} structure on X. We will denote by $\widetilde{P}$ a principal $\mathrm{Spin}^{c-}(4)$ bundle given by the Spin^{c-} structure $\mathfrak{s}_X$.

- We identify $\mathbb{R}^4$ with $\mathbb{H}$. We define the representation of $\mathrm{Spin}^{c-}(4)$

$$\Delta^{\pm}\colon \mathrm{Spin}^{c-}(4) \to \mathrm{GL}(4,\mathbb{R}) \quad \text{and} \quad \rho\colon \mathrm{Spin}^{c-}(4) \to \mathrm{GL}(4,\mathbb{R})$$

by

$$\Delta^{\pm}([q^+,q^-,u])\phi = q^{\pm}\phi u^{-1}, \quad \rho([q^+,q^-,u])v = q^+ v (q^-)^{-1}.$$

We will denote by $S^{\pm}$ the associated vector bundle given by the representation $\Delta^{\pm}$ and the principal Spin^{c-} bundle $\widetilde{P}$. The associated vector bundle given by the representation ρ is TX.

[1] Originally, Mikio Furuta translated the orientability of $\mathrm{Pin}^-(2)$ monopole moduli space into a mod 2 index associated with a 3 dimensional submanifold with $G^+(3,0,2)$ structure. Nobuhiro Nakamura wrote Furuta's idea in the note [16]. The argument in that note is the neck stretching argument. In this paper, we use Witten deformation to determine the orientability of $\mathrm{Pin}^-(2)$ monopole moduli space.

- Let $\rho^*([q^+, q^-, u])v' := q^- v (q^+)^{-1}$. We observe that there is a natural isomorphism $\widetilde{P} \times_{\rho^*} \mathbb{H} \cong T^*X$ and we have an isomorphism $TX \to T^*X$ that is given by

$$TX \cong \widetilde{P} \times_\rho \mathbb{H} \ni [\tilde{p}, v] \mapsto [\tilde{p}, \bar{v}] \in \widetilde{P} \times_{\rho^*} \mathbb{H} \cong T^*X.$$

We denote the image of $v \in T_x X$ under this mapping as $\bar{v}$.

- We define a Clifford multiplication

$$c\colon TX \otimes S^+ \to S^-$$

by $c(v \otimes \phi) = \bar{v}\phi$. We will denote by $c(v)\phi$ this Clifford action. We define

$$c^*\colon TX \otimes S^- \to S^+$$

by $c^*(v \otimes \phi) = v\phi$. By abuse of notation, we use the same letter $c(v)\phi$. We have $c(v)^2 = |v|^2$ for $v \in T_x X$.

- A connection A on $\widetilde{P}$ is a Spin^{c-} connection if it satisfies

$$\nabla^A(c(X)\phi) = c(\nabla^{LC} X)\phi + c(X)\nabla^A \phi$$

for $X \in \Gamma(TX)$, $\phi \in \Gamma(S^+)$, where ∇^{LC} is the Levi-Civita connection of X.

- Let A be a Spin^{c-} connection. We define a Dirac operator $D_A^\pm$ associated to A by the composition of following maps:

$$\Gamma(S^\pm) \xrightarrow{\nabla^A} \Gamma(TX^* \otimes S^\pm) \cong \Gamma(TX \otimes S^\pm) \xrightarrow{c} \Gamma(S^\mp).$$

The notion of the Spin^{c-} structure is introduced by Nobuhiro Nakamura [14; 15] to define the Pin$^-$(2) monopole equations and the Pin$^-$(2) monopole invariant. The Pin$^-$(2) monopole invariant is a $\mathbb{Z}/2\mathbb{Z}$-valued invariant. The matter of when we determine the orientability of the Pin$^-$(2) monopole moduli space is whether a gauge transformation preserves an orientation of $\operatorname{ind} D_A$.

To define a gauge transformation, we introduce some associated vector bundles and their Clifford actions.

Definition 5.5 Let X be a 4-manifold and $\mathfrak{s}_X$ be a Spin^{c-} structure on X. Let $\widetilde{P}$ be the principal Spin^{c-}(4) bundle given by the Spin^{c-} structure $\mathfrak{s}_X$.

- We identify $\mathbb{R}^2$ with the subspace $\{z \in \mathbb{H} \mid z = x + iy,\ x, y \in \mathbb{R}\} \cong \mathbb{C}$. We define the real Spin^{c-}(4) representation

  $$\rho'_0\colon \mathrm{Spin}^{c-}(4) \to \mathrm{GL}(2, \mathbb{R})$$

  by $\rho'_0([q^+, q^-, u])z = uzu^{-1}$, where the multiplication is that of $\mathbb{H}$.

- We identify $\mathbb{R}^2$ with the subspace $\{jw \in \mathbb{H} \mid w \in \mathbb{C}\}$. We define the real Spin^{c-}(4) representation

  $$\rho'_1\colon \mathrm{Spin}^{c-}(4) \to \mathrm{GL}(2, \mathbb{R})$$

  by $\rho'_1([q^+, q^-, u])jw = ujwu^{-1}$, where the multiplication is that of $\mathbb{H}$.

It is easy to check the well-definedness of the above definition. We will denote by $\widetilde{\mathbb{C}}$, E the associated vector bundles given by $\widetilde{P}$ and the representations ρ_0', ρ_1', respectively.

Definition 5.6 We define the multiplication $S \otimes (\widetilde{\mathbb{C}} \oplus E) \to S$ as follows. Note that the representation space of $\rho_0 \oplus \rho_1$ and $\Delta^+ \oplus \Delta^-$ is $\mathbb{H}$ and $\mathbb{H}^2 = \mathbb{H}_+ \oplus \mathbb{H}_-$, respectively. We define a multiplication of elements in these representation space by

$$\phi \cdot v := \phi v$$

for $v \in \mathbb{H}$ and $\phi \in \mathbb{H}^2$. This is a $\mathrm{Spin}^{c-}(4)$ equivariant multiplication. This defines a multiplication $S = S^+ \oplus S^-$ and $\widetilde{\mathbb{C}} \oplus E$. We define the multiplication $(\widetilde{\mathbb{C}} \oplus E) \otimes (\widetilde{\mathbb{C}} \oplus E) \to \widetilde{\mathbb{C}} \oplus E$ in the same way.

Remark 5.7 Regarding the Spin^{c-} structure $\mathfrak{s}_X$ as a $G^+(4,0,3)$ structure, the vector bundle $\det E \oplus E$ is the vector bundle E_- associated with $\mathfrak{s}_X$ (see Definition 2.6). From the definition of $\widetilde{\mathbb{C}}$ and E, we have $\mathrm{Im}(\widetilde{\mathbb{C}}) \cong \det E$. From now on, we fix an isomorphism $\mathrm{Im}(\widetilde{\mathbb{C}}) \cong \det E$ and identify $\mathrm{Im}(\widetilde{\mathbb{C}})$ with $\det E$. If we take another isomorphism, arguments do not change.

Definition 5.8 We call an element of $\{u \in \Gamma(\widetilde{\mathbb{C}}) \mid |u| = 1\}$ a gauge transformation.

Now we state the topological method of evaluating the orientability of $\mathrm{Pin}^-(2)$ monopole space.

Theorem 5.9 *Let u be a gauge transformation. We perturb u if necessary and we may assume that -1 is a regular value of u. Let h be a section of E such that h is transverse to the zero section and its submanifold $u^{-1}(-1) \subset X$. Then there is a natural* Spin *structure on $h^{-1}(0) \cap u^{-1}(-1)$. This defines an element of* 1 *dimensional* Spin *bordism group $\Omega_1^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}$ and we denote by* t-ind$(\mathfrak{s}_X, u)$ *this element. Then the following statements are equivalent*:

1. *The gauge transformation u preserves the orientation of the index bundle $\{\mathrm{ind}(D_A)\}_A$ on the configuration space.*
2. *The element* t-ind$(\mathfrak{s}_X, u)$ *of $\Omega_1^{\mathrm{Spin}}(\mathrm{pt})$ is trivial.*

Corollary 5.10 If we have t-ind$(\mathfrak{s}_X, u) = 0$ for all gauge transformation u, the $\mathrm{Pin}^-(2)$ monopole moduli space is orientable.

We begin the preparation of the proof of Theorem 5.9.

Definition 5.11 Let X be a 4-manifold and $\mathfrak{s}_X$ be a Spin^{c-} structure on X. Let $S = S^+ \oplus S^-$ be the spinor bundle of $\mathfrak{s}_X$ and u be a gauge transformation.

- We define the vector bundle L on $X \times S^1$ as follows. Let $\pi : [0,1] \times X \to X$ be the projection. We introduce an equivalence relation on $\pi^*\widetilde{\mathbb{C}}$ by $(0, z) \sim (1, zu)$ for $z \in \widetilde{\mathbb{C}}$ and L is the quotient of this equivalence relation $L = \pi^*\widetilde{\mathbb{C}}/\sim$. Note that L has the canonical left $\pi^*\widetilde{\mathbb{C}}$ action.

- Let $\widetilde{\pi}\colon S^1 \times X \to X$ be the projection. By abuse of notation, we use the same letter $\widetilde{\mathbb{C}}$ for $\widetilde{\pi}^*\widetilde{\mathbb{C}}$.
- $V = V^+ \oplus V^-$ is the $\mathbb{Z}/2\mathbb{Z}$ graded vector bundle given by $V = \widetilde{\pi}^* S \otimes_{\widetilde{\mathbb{C}}} (L \oplus \widetilde{\mathbb{C}})$, $V^+ = \widetilde{\pi}^* S$ and $V^- = \widetilde{\pi}^* S \otimes_{\widetilde{\mathbb{C}}} L$, where $\otimes_{\widetilde{\mathbb{C}}}$ is a tensor product of $\widetilde{\mathbb{C}}$ modules.

Lemma 5.12 *We define a skew-adjoint Dirac type operator D on V as follows*:

- *Let A be a* Spin^{c-} *connection on X and D_A is the Dirac operator on X given by A.*
- *Let A_t be a one-parameter family of* Spin^{c-} *connection on X such that $A_t = A$ for $t < \frac{1}{3}$ and $A_t = u^* A$ for $t > \frac{2}{3}$.*
- *Let ϵ be the $\mathbb{Z}/2\mathbb{Z}$ grading operator of S. Let ϵ' be the operator on $L \oplus \widetilde{\mathbb{C}}$ given by $1 \oplus (-1)$.*
- *We define the Dirac type operator on $X \times S^1$ by*

$$D = D_t \otimes \mathrm{pr}_L + D_A \otimes \mathrm{pr}_{\widetilde{\mathbb{C}}} + \epsilon \partial_t \otimes \epsilon',$$

  where t is the coordinate of S^1 and pr_L, $\mathrm{pr}_{\widetilde{\mathbb{C}}}$ are the projections to $L, \widetilde{\mathbb{C}}$, respectively.

Then the operator D is well defined on $S^1 \times X$. It is equivalent that u preserves an orientation of the index bundle $\{\mathrm{ind}(D_A)\}$ on the configuration space and $\dim\ker D \bmod 2 = 0$.

Proof The four-tuple $(D_t \otimes \mathrm{pr}_L + D_A \otimes \mathrm{pr}_{\widetilde{\mathbb{C}}}, \epsilon \otimes \epsilon', 0, L^2(X \times S^1, V))$ defines a family index in $\mathrm{KO}^0(S^1, \mathrm{pt})$. This family index coincides with $\mathrm{ind}(D_t) - \mathrm{ind}(D_A)$ when we use the definition of $\mathrm{KO}^0(S^1, \mathrm{pt})$ as the subgroup of the Grothendieck group of real vector bundles on S^1. We see at once the family index $\mathrm{ind}(D_t) - \mathrm{ind}(D_A)$ is trivial if and only if u preserves an orientation of $\mathrm{ind}(D_A)$ by the definition of V and D_t.

We see that this family index coincides with $\dim\ker D \bmod 2 \in \mathbb{Z}/2\mathbb{Z} \cong \mathrm{KO}^0(S^1, \mathrm{pt})$ from index theory. □

Definition 5.13 We define the $G^+(5,0,4)$ structure $\mathfrak{s}_Z$ on $Z = X \times S^1$ as follows: Let $\widetilde{P}$ be the principal $\mathrm{Spin}^{c-}(4)$ bundle on X given by Spin^{c-} structure $\mathfrak{s}_X$ and $\pi\colon [0,1] \times X \to X$ be the projection. We will denote by ι an embedding

$$\mathrm{Spin}^{c-}(4) \cong \mathrm{Spin}(4) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Pin}^-(2) \to \mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(4) \cong \mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} (\mathrm{Sp}(1) \times \mathrm{Sp}(1))$$

given by $[q, u'] \mapsto [i(q), (u', u')]$, where i is a embedding $\mathrm{Spin}(4) \to \mathrm{Spin}(5)$ which is a lift of a map $A \mapsto \mathrm{diag}(1, A)$. Let u be a gauge transformation of $\mathfrak{s}_X$. We define a principal $\mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(4)$ bundle $\widetilde{P}_Z$ on Z by $\pi^* \widetilde{P} \times_\iota (\mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(4))/\sim$, where $\sim$ is an equivalent relation given by

$$(0, p) \sim (1, pu), \ \ p \in \widetilde{P} \times_\iota (\mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(4)).$$

It is easy to see that $\widetilde{P}_Z$ defines a Spin structure of $TZ \oplus L \oplus \pi^* E$. We will denote by $\mathfrak{s}_Z$ a $G^+(5,0,4)$ structure on Z given by $\widetilde{P}_Z$.

Lemma 5.14 *The* mod 2 *index* $\dim\ker D \bmod 2$ *of the skew-adjoint operator D coincides with* $\mathrm{ind}(\mathfrak{s}_Z) \in \mathrm{KO}^{-1}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}$ *where* $\mathrm{ind}(\mathfrak{s}_Z)$ *is the index of* $\mathfrak{s}_Z$.

Proof By the definition of V, we see that V is a spinor bundle S' of $G^+(5,0,4)$ structure $\mathfrak{s}_Z$ given in Section 4.2. This lemma follows from Lemma 4.11 and Remark 4.12. □

We have the following proposition from Main Theorem.

Proposition 5.15 *Let X be a closed Riemannian 4-manifold and $\mathfrak{s}$ be a* Spin^{c-} *structure on X. Take a gauge transformation $u \in \Gamma(\widetilde{\mathbb{C}})$. Then the following statements are equivalent*:

1. *The gauge transformation u preserves the orientation of the index bundle $\{\mathrm{ind}(D_A)\}$.*
2. *The* Spin *structure induced on the zero locus of a transverse section of the vector bundle $L \oplus \pi^* E$ on $X \times S^1$ from Theorem 3.8 is trivial.*

We consider the Spin structure on the zero locus of the transverse section of $L \oplus \pi^* E$.

Lemma 5.16 *If it is necessary, we perturb u by homotopy and we may assume that -1 is a regular value of u. Let h be a section of E such that h is transverse to the zero section and its submanifold $u^{-1}(-1) \subset X$. Let $C = h^{-1}(0) \cap u^{-1}(-1)$ and $U(C)$ is its tubular neighborhood. From Theorem 3.8, we have a spin structure $\mathfrak{s}_C$ on $C \subset U(C)$ introduced by the section $\mathrm{Im}(u) \oplus h \in \Gamma((\det E \oplus E)|_{U(C)})$. Then there exists a transverse section of the vector bundle $L \oplus \pi^* E$ whose zero locus is $(h^{-1}(0) \cap u^{-1}(-1)) \times \{\frac{1}{2}\} \subset X \times \{\frac{1}{2}\}$ and the* Spin *structure on $C \times \{\frac{1}{2}\}$ given in Theorem 3.8 coincides with $\mathfrak{s}_C$.*

Proof The transversality of $\mathrm{Im}(u) \oplus h \in \Gamma((\det E \oplus E)|_{U(C)})$ follows from the assumptions on u and h. Then $\mathfrak{s}_C$ is well defined.

To define a section of $L \times \pi^* E$, it is sufficient to take a section $s = (s_0, s_1) \in \Gamma((\widetilde{\mathbb{C}} \oplus E) \times [0,1])$ such that $s_0(0) \cdot u = s_0(1)$. We define a section $s = (s_0, s_1)$ as

$$s_0(t) = (1-t) + tu, \quad s_1(t) = h.$$

From the definition of s, we see that $s^{-1}(0) = (h^{-1}(0) \cap u^{-1}(-1)) \times \{\frac{1}{2}\} \subset X \times \{\frac{1}{2}\}$ and s is transverse to the zero section. The normal bundle of $s^{-1}(0)$ splits into the $[0,1]$ direction and the X direction. The $[0,1]$ direction is trivial rank one vector bundle, and the X direction is isomorphic to the vector bundle $\det E \oplus E$. On $s^{-1}(0)$, the real part of s_0 is a section of a summand of the normal bundle which is the $[0,1]$ direction, and $\mathrm{Im}(s_0) \oplus s_1 = \mathrm{Im}(u)/2 \oplus h$ is a section of $\det E \oplus E$. Then we have a Spin structure on $s^{-1}(0)$ given by s and $\mathfrak{s}_Z$ coincides with $\mathfrak{s}_C$. □

Proof of Theorem 5.9 From Lemma 5.16, a gauge transformation u preserves the orientation of the index bundle $\{\mathrm{ind}(D_A)\}_A$ on the configuration space if and only if the index of the Spin structure $\mathfrak{s}_C$ on $h^{-1}(0) \cap u^{-1}(-1)$ is trivial. The index gives the isomorphism $\Omega_1^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}$. □

Remark 5.17 A slight change in the proof of Lemma 5.16 shows that there is a $G^+(3,0,2)$ structure ($\mathrm{Pin}^{\tilde{c}}_+$ structure) on $Y := u^{-1}(-1) \subset X$ induced by $\mathfrak{s}_X$. The vector bundle $E|_Y$ coincides with E_- given by this $G^+(3,0,2)$ structure. Thus we have that the index of the Spin structure on $h^{-1}(0) \cap Y$ coincides with the index of this $G^+(3,0,2)$ structure on Y from Main Theorem. From Theorem 5.9, u preserves an orientation of moduli space if and only if the index of this $G^+(3,0,2)$ structure on $Y = u^{-1}(-1)$ is trivial. This is the statement proved by Mikio Furuta.

5.2 Examples

In this section, we give an example of four manifold with Spin^{c-} structure such that there exists a gauge transformation which reverses an orientation of $\{\mathrm{ind} D_A\}$.

Proposition 5.18 *Let Y be a 3 dimensional closed Riemannian manifold and $\mathfrak{s}$ be a $G^+(3,0,2)$ structure ($\mathrm{Pin}^{\tilde{c}}_+$ structure) on Y such that the index of $\mathfrak{s}$ is nontrivial. We denote by X a four manifold given by gluing two copies of the disk bundle of $\det TY$ along boundaries. There is a Spin^{c-} structure on X and a gauge transformation which reverse the orientation of $\{\mathrm{ind} D_A\}$.*

Proof First, we construct a Spin^{c-} structure on X.

- Let $l = \det TY$ and $\pi : l \to Y$ be the projection. We will denote by E a vector bundle E_- associated with the $G^+(3,0,2)$ structure $\mathfrak{s}$. Note that $\det E \cong l$ and $\mathfrak{s}$ is given by a Spin structure of

$$\det TY \oplus TY \oplus \det E \oplus E.$$

- We will denote by $\mathfrak{s}'$ a Spin^{c-} structure on the total space l given by the Spin structure of

$$\pi^* l \oplus TY \oplus \pi^* l \oplus \pi^* E$$

induced by $\mathfrak{s}$.

- Let $D(l)$ be the disk bundle of l and $S(l)$ be the sphere bundle of l. We choose a canonical trivialization of $\pi^* l$ on $S(l)$. Hence we have that E and $TS(l)$ are orientable on $S(l)$. Thus the restriction of $\mathfrak{s}'$ to $S(l)$ induces a Spin^c structure.
- $S(l)$ is a double cover of Y and the covering transformation τ reverses the orientation of $S(l)$. We glue two copies of $D(l)$ along $S(l)$ by the map τ. Let $X = D(l) \cup_\tau D(l)$. X is an oriented closed manifold.
- We glue Spin^c structure on each $S(l)$ to give a Spin^{c-} structure $\mathfrak{s}_X$ on X such that the restriction of $\mathfrak{s}_X$ to each $D(l)$ coincides with $\mathfrak{s}'$.

Second, we give a gauge transformation u which reverses the orientation of $\mathrm{ind} D_A$.

- We will denote by f the tautological section of $\pi^* l$ on l.
- On the open subset $l \setminus Y$, $\pi^* l$ has the canonical trivialization. In this trivialization, we have $f(v) = |v|$ for $v \in l \setminus Y$.

- Deform f in the area $|v| \geq \frac{1}{2}$ and we assume that $f(v) = 1$ for $|v| \geq \frac{2}{3}$.
- We define $s(v) = -\exp(i\pi f(v))$ as follows: There is the natural isomorphism $l \otimes l \cong \mathbb{R}$ and using this isomorphism we have $f^{2n} \in \mathbb{R}$, $f^{2n+1} \in l$. We define $\exp(i\pi f(v)) \in S(\mathbb{R} \otimes \sqrt{-1}l)$ using Taylor expansion of exponential function.
- Note that $s = 1$ around $S(l)$ and we extend on X by 1 on another $D(l)$. We have $s^{-1}(-1) = Y$.
- The index of $\mathfrak{s}$ on Y is nontrivial and from Theorem 5.9 and Remark 5.17, u reverses the orientation of $\operatorname{ind} D_A$. □

We give an explicit example of Y in Proposition 5.18.

Lemma 5.19 *There is a $G^+(3,0,2)$ structure $\mathfrak{s}_0$ on $\mathbb{R}P^2 \times S^1$ whose index is nontrivial.*

Proof For simplicity of notation, we omit the notation of the pull-back of projections. From Lemma 3.4, we give a Spin structure on the vector bundle

$$\det T\mathbb{R}P^2 \oplus TS^1 \oplus T\mathbb{R}P^2 \oplus \det T\mathbb{R}P^2 \oplus T\mathbb{R}P^2$$

and we give the $G^+(3,0,2)$ structure $\mathfrak{s}_0$ by Lemma 3.2. Note that $E_- = T\mathbb{R}P^2$. We take a transverse section of $T\mathbb{R}P^2$ on $\mathbb{R}P^2$ whose zero locus is a single point on $\mathbb{R}P^2$. By pulling back this section, we have a transverse section h of E_- whose zero locus is $S^1 \times \{\mathrm{pt}\}$. We immediately see that the Spin structure on $h^{-1}(0) \cong S^1$ induced by h from Theorem 3.8 is given by the product $S^1 \times \mathrm{Spin}(1)$. This is the nontrivial element in the 1-dimensional Spin bordism group. From Main Theorem, we have that the index of $\mathfrak{s}_0$ is nontrivial. □

From Proposition 5.18 and Lemma 5.19, we deduce following corollary.

Corollary 5.20 We set $X = (\mathbb{R}P^3 \sharp \mathbb{R}P^3) \times S^1$ and $\mathfrak{s} = \mathfrak{s}_0$. The determinant bundle on the ambient space of the moduli space is not orientable.

This manifold is diffeomorphic to $P\gamma \times S^1$, where γ is the tautological bundle of $\mathbb{R}P^2$ and $P\gamma$ is its projectivization.

Remark 5.21 The gluing construction in the proof of Proposition 5.18 can be generalized in the case of gluing two 4 dimensional Spin^{c-} manifolds with boundary. If the restrictions of the Spin^{c-} structures to their boundaries induce Spin^c structures and their boundaries are diffeomorphic by a map which preserves the orientation and the Spin^c structure, our construction works.

Remark 5.22 In the case $Y = \mathbb{R}P^2 \times S^1$ and $\mathfrak{s} = \mathfrak{s}_0$, we have $S(l) = S^2 \times S^1$. We cannot glue $D(l)$ by $D^3 \times S^1$ along $S(l)$ because the first Chern class of Spin^c structure on $S(l)$ is the Euler class of TS^2 and this cannot be extended to $D^3 \times S^1$.

Appendix

Here we prove Proposition 3.22. We follow the notation of Section 3.2.

Lemma A.1 *We assume that $\phi \in \Gamma(Y, \mathfrak{S})$ satisfies $\|\not{D}_m\phi\| \le \lambda\|\phi\|$. There are functions $A_h(m,\lambda)$, $B_h(m,\lambda)$ of positive real numbers m, λ depending on h such that if we fix a value λ, they satisfy*

$$A_h(m,\lambda) \to 0, \quad B_h(m,\lambda) \to 1$$

when $m \to \infty$. The functions $A_h(m,\lambda)$, $B_h(m,\lambda)$ satisfy the inequalities

$$\int_V |\phi|^2 \le A_h(m,\lambda) \int_Y |\phi|^2, \quad B_h(m,\lambda) \int_Y |\phi|^2 \le \int_Y |\rho\phi|^2,$$

where $V = \{z \in U(C) \cong B(N(C)) \mid |z| > \frac{1}{2}\} \cup U(C)^c$.

Proof We have the following estimate:

$$\begin{aligned}
\lambda^2 \int_Y |\phi|^2 &\ge \int_Y |\not{D}_m\phi|^2 \\
&= \int_Y \langle -\not{D}_m^2\phi, \phi\rangle \\
&= \int_Y \langle (-(\not{D})^2 - m\{\not{D}, h\} - m^2h^2)\phi, \phi\rangle \\
&= \int_Y |\not{D}\phi|^2 + \int_Y m^2|h\phi|^2 - \int_Y m\langle\{\not{D}, h\}, \phi\rangle \\
&\ge \int_V m^2|h\phi|^2 - \int_Y mC_0\|dh\|_\infty|\phi|^2 \\
&\ge \frac{m^2}{4}\int_V |\phi|^2 - mC_0\|dh\|_\infty \int_Y |\phi|^2.
\end{aligned}$$

We define $A_h(m,\lambda) = 2(mC_0\|dh\|_\infty + \lambda^2)/m^2$ and we have the first inequality, where C_0 is a constant only depending on the principal symbol of D (Clifford action). We have the second inequality by the following estimate:

$$\begin{aligned}
\left(\int_Y |\rho\phi|^2\right)^{\frac{1}{2}} &\ge \left(\int_Y |\phi|^2\right)^{\frac{1}{2}} - \left(\int_Y |(1-\rho)\phi|^2\right)^{\frac{1}{2}} \\
&\ge \left(\int_Y |\phi|^2\right)^{\frac{1}{2}} - \left(\int_V |\phi|^2\right)^{\frac{1}{2}} \\
&\ge \left(\int_Y |\phi|^2\right)^{\frac{1}{2}} - \left(A_h(m,\lambda)\int_Y |\phi|^2\right)^{\frac{1}{2}} \\
&= \left(1 - \sqrt{A_h(m,\lambda)}\right)\left(\int_Y |\phi|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Setting $B_h(m,\lambda) = \left(1 - \sqrt{A_h(m,\lambda)}\right)^2$, the proof is completed. □

A slight change of the proof of the above lemma actually shows the following lemma.

Lemma A.2 *We will denote by $\mathcal{S}(N(C), \mathfrak{S})$ a set of rapidly decreasing sections of $\mathfrak{S}$ on the total space of the vector bundle $N(C)$. We assume a section $\psi \in \mathcal{S}(N(C), \mathfrak{S})$ satisfies $\|\not{D}_m\psi\| \le \lambda\|\psi\|$. There are functions $A'(m,\lambda)$, $B'(m,\lambda)$ of positive real numbers m, λ such that if we fix a value λ, they satisfy*

$$A'(m,\lambda) \to 0, \quad B'(m,\lambda) \to 1$$

when $m \to \infty$. The functions $A'(m,\lambda)$, $B'(m,\lambda)$ satisfy the inequalities

$$\int_{V'} |\psi|^2 \le A'(m,\lambda) \int_{N(C)} |\psi|^2, \quad B'(m,\lambda) \int_{N(C)} |\psi|^2 \le \int_{N(C)} |\rho\psi|^2,$$

where $V' = \{z \in N(C) \mid |z| > \frac{1}{2}\}$.

Note that the perturbation term h of $\not{D}_m$ on $N(C)$ satisfies $|\{\not{D}, h\}| = 2$.

Lemma A.3 *We assume that λ is smaller than a constant given by the principal symbol of D and the differentiation of ρ. We assume that m is large enough. Let Π' be the orthogonal projection from $L^2(N(C), \mathfrak{S})$ to H. Then the map*

$$\mathcal{H}_m^\lambda \to H, \quad \phi \mapsto \Pi'(\rho\phi)$$

is injective.

Proof If a section $\phi \in \mathcal{H}_m^\lambda$ satisfies $\|\phi\|_{L^2(Y,\mathfrak{S})} = 1$ and $\Pi'(\rho\phi) = 0$, we have

$$\|\not{D}_m\rho\phi\|_{L^2(N(C),\mathfrak{S})} \ge \lambda_C \|\rho\phi\|_{L^2(N(C),\mathfrak{S})}$$

from Lemma 3.18 when m is large enough. The support of the function ρ is contained in

$$U = \{z \in N(C) \mid |z| < 1\}$$

and we identify U with $U(C) \subset Y$. We regard $\rho\phi$ as a section on Y and we have

$$\|\not{D}_m\rho\phi\|_{L^2(Y,\mathfrak{S})} \ge \lambda_C \|\rho\phi\|_{L^2(Y,\mathfrak{S})}.$$

But we have the following estimate from Lemma A.1:

$$\begin{aligned}
\int_Y |D_m\rho\phi|^2 &\le \int_Y |[\not{D}, \rho]\phi|^2 + \int_Y |\rho D_m\phi|^2 \\
&\le C_0\|d\rho\|_\infty^2 \int_{U\cap V} |\phi|^2 + \int_Y |D_m\phi|^2 \\
&\le C_0\|d\rho\|_\infty^2 A_h(m,\lambda) \int_Y |\phi|^2 + \lambda^2 \int_Y |\phi|^2 \\
&\le C_0\|d\rho\|_\infty^2 (A_h(m,\lambda) + \lambda^2) B_h(m,\lambda)^{-1} \int_Y |\rho\phi|^2.
\end{aligned}$$

Provided m is large enough, the coefficient of $\int_Y |\rho\phi|^2$ tends to $C_0\|d\rho\|_\infty^2 \lambda^2$. If this constant is smaller than λ_C, the above estimate contradicts the inequality $\|D_m\rho\phi\| \ge \lambda_C\|\rho\phi\|$. □

We fix the value λ so that $2C\|d\rho\|_\infty^2\lambda^2 < \lambda_C$.

Lemma A.4 *We assume that m is large enough. Let Π_m^λ be the orthogonal projection from $L^2(Y, \not{S})$ to $\mathcal{H}_m^\lambda$. The map*

$$H \to \mathcal{H}_m^\lambda, \quad \psi \mapsto \Pi_m^\lambda(\rho\psi)$$

is injective.

Proof If the map above is not injective, we have

$$\|D_m\rho\psi\|_{L^2(Y,\not{S})} \geq \lambda\|\rho\psi\|_{L^2(Y,\not{S})}$$

for some $\psi \in H$. On the other hand, elements in H are rapidly decreasing sections; hence we use Lemma A.2. Thus we have the following estimate by a similar argument to the proof of the Lemma A.3:

$$\int_{N(C)} |D_m\rho\psi|^2 \leq C\|d\rho\|_\infty^2 A'(m,0)B'(m,0)^{-1}\int_{N(C)} |\rho\psi|^2.$$

If m is large enough, the above estimate contradicts the inequality $\|D_m\rho\psi\| \geq \lambda\|\rho\psi\|$. □

Proof of Proposition 3.22 We have $\dim H \geq \dim \mathcal{H}_m^\lambda$ from Lemma A.3 and we have $\dim H \leq \dim \mathcal{H}_m^\lambda$ from Lemma A.4. Thus the maps of Lemmas A.3 and A.4 are isomorphisms. In particular, the map in Proposition 3.22 is the same as the map in Lemma A.4 and it is an isomorphism. Moreover, it is easy to see that this map preserves the $\mathbb{Z}/2$ gradings and the left $\mathrm{Cl}_{(s^-,n+s^+)}$ actions. □

References

[1] **M F Atiyah**, **I M Singer**, *Index theory for skew-adjoint Fredholm operators*, Inst. Hautes Études Sci. Publ. Math. 37 (1969) 5–26 MR Zbl

[2] **S K Donaldson**, *The orientation of Yang–Mills moduli spaces and 4-manifold topology*, J. Differential Geom. 26 (1987) 397–428 MR Zbl

[3] **S K Donaldson**, **P B Kronheimer**, *The geometry of four-manifolds*, Oxford Univ. Press (1990) MR Zbl

[4] **J L Fast**, **S Ochanine**, *On the KO characteristic cycle of a* Spin^c *manifold*, Manuscripta Math. 115 (2004) 73–83 MR Zbl

[5] **D S Freed**, **M J Hopkins**, *Reflection positivity and invertible topological phases*, Geom. Topol. 25 (2021) 1165–1330 MR Zbl

[6] **H Fukaya**, **M Furuta**, **S Matsuo**, **T Onogi**, **S Yamaguchi**, **M Yamashita**, *The Atiyah–Patodi–Singer index and domain-wall fermion Dirac operators*, Comm. Math. Phys. 380 (2020) 1295–1311 MR Zbl

[7] **M Furuta**, *Index theorem, I*, Transl. Math. Monogr. 235, Amer. Math. Soc., Providence, RI (2007) MR Zbl

[8] **M Furuta**, **Y Kametani**, *Equivariant version of Rochlin-type congruences*, J. Math. Soc. Japan 66 (2014) 205–221 MR Zbl

[9] **S Hayashi**, *Localization of Dirac operators on* $4n+2$ *dimensional open* Spin^c *manifolds*, preprint (2013) arXiv 1306.0389

[10] **D Joyce**, **Y Tanaka**, **M Upmeier**, *On orientations for gauge-theoretic moduli spaces*, Adv. Math. 362 (2020) art. id. 106957 MR Zbl

[11] **P B Kronheimer**, *Four-manifold invariants from higher-rank bundles*, J. Differential Geom. 70 (2005) 59–112 MR Zbl

[12] **P B Kronheimer**, **T S Mrowka**, *Knot homology groups from instantons*, J. Topol. 4 (2011) 835–918 MR Zbl

[13] **H B Lawson, Jr**, **M-L Michelsohn**, *Spin geometry*, Princeton Math. Ser. 38, Princeton Univ. Press (1989) MR Zbl

[14] **N Nakamura**, $\mathrm{Pin}^-(2)$*-monopole equations and intersection forms with local coefficients of four-manifolds*, Math. Ann. 357 (2013) 915–939 MR Zbl

[15] **N Nakamura**, $\mathrm{Pin}^-(2)$*-monopole invariants*, J. Differential Geom. 101 (2015) 507–549 MR Zbl

[16] **N Nakamura**, *A note on the orientability of the moduli space of the* $\mathrm{Pin}^-(2)$ *monopole*, preprint (2018) In Japanese. Available at http://kansai-gauge.squares.net/misc/pin2-ori.pdf

[17] **E Witten**, *Supersymmetry and Morse theory*, J. Differential Geom. 17 (1982) 661–692 MR Zbl

[18] **W Zhang**, Spin^c*-manifolds and Rokhlin congruences*, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993) 689–692 MR Zbl

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