**Toward a topological description of
Legendrian contact homology
of unit conormal bundles**

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For a smooth compact submanifold K of a Riemannian manifold Q, its unit conormal bundle Λ_K is a Legendrian submanifold of the unit cotangent bundle of Q with a canonical contact structure. Using pseudoholomorphic curve techniques, the Legendrian contact homology of Λ_K is defined when, for instance, $Q = \mathbb{R}^n$. Aiming at giving another description of this homology, we define a graded $\mathbb{R}$-algebra for any pair (Q, K) with orientations from a perspective of string topology and prove its invariance under smooth isotopies of K. We conjecture that it is isomorphic to the Legendrian contact homology of Λ_K with coefficients in $\mathbb{R}$ in all degrees. This is a reformulation of a homology group, called string homology, introduced by Cieliebak, Ekholm, Latschev and Ng when the codimension of K is 2, though the coefficient is reduced from the original $\mathbb{Z}[\pi_1(\Lambda_K)]$ to $\mathbb{R}$. We compute our invariant (i) in all degrees for specific examples, and (ii) in the 0th degree when the normal bundle of K is a trivial 2-plane bundle.

1 Introduction

Convention Throughout this paper, all manifolds are of class C^∞ without boundary and second countable, and all submanifolds are of class C^∞ without boundary, unless otherwise specified.

Background Let Q be a manifold with a Riemannian metric, and K be a compact submanifold of Q. To any pair (Q, K), one can associate the unit cotangent bundle UT^*Q of Q and the unit conormal bundle Λ_K of K. It is well known that UT^*Q has a canonical contact structure and Λ_K is a Legendrian submanifold of UT^*Q.

As an invariant of Legendrian submanifolds, the *Legendrian contact homology* has been studied for pairs (M, Λ) of a contact manifold M and its compact Legendrian submanifold Λ. It is the homology of a differential graded algebra generated by Reeb chords of Λ, and was introduced by Chekanov [4] and Eliashberg [12]. The differential is defined by using pseudoholomorphic curves in the symplectization of M. A rigorous definition was given by Ekholm, Etnyre and Sullivan [9; 11] when there is a diffeomorphism from M to the contactization of a Liouville manifold which preserves contact forms. As is mentioned in [11, Section 5.1], this included the case of $M = UT^*\mathbb{R}^n$. (The definition of [11] is given by pseudoholomorphic curves in the Liouville manifold. These curves can be lifted to pseudoholomorphic curves in the symplectization of M. See Dimitroglou Rizell [7].)

Suppose conceptually that we have an algebraic invariant in symplectic or contact topology defined by using pseudoholomorphic curves, and apply it to an object related to the cotangent bundle T^*Q. For instance, we consider the symplectic homology of T^*Q, or the wrapped Floer homology of the conormal bundle L_K of K in T^*Q. In this case, it is known by the following results that these invariants have another view from the topology of the loop or path space of Q, without using pseudoholomorphic curves (here we assume that Q is a closed spin manifold and all homology groups have $\mathbb{Z}$-coefficients):

- The symplectic homology $SH_*(T^*Q)$ of T^*Q is isomorphic to the singular homology of the free loop space of Q; see Abbondandolo and Schwarz [2], Abouzaid [3] and Viterbo [20].

- The wrapped Floer homology $WF_*(L_K, L_K)$ of L_K is isomorphic to the singular homology of the space of paths in Q with endpoints in K; see Abbondandolo, Portaluri and Schwarz [1].

These results lead us to an expectation that if the Legendrian contact homology of a pair (UT^*Q, Λ_K) is defined, it has another description in terms of the topology of the path space of Q. This expectation has already been confirmed in particular cases. When the codimension of K is 2, Cieliebak, Ekholm, Latschev and Ng [6] defined a graded $\mathbb{Z}[\pi_1(\Lambda_K)]$-algebra, called *string homology*, which is inspired by string topology of the path space of Q. They showed that when Q is equal to $\mathbb{R}^3$ with the standard metric and K is a knot, the 0th degree part of this algebra is isomorphic to the 0th degree part of the *fully noncommutative* Legendrian contact homology of $(UT^*\mathbb{R}^3, \Lambda_K)$ with coefficients in $\mathbb{Z}[\pi_1(\Lambda_K)]$. However, such topological descriptions have not yet been defined in higher degrees or for K with codim $K \neq 2$.

Main results Let Q be an oriented manifold and K be its compact oriented submanifold of codimension $d \geq 1$. The main purpose of this paper is to define a graded $\mathbb{R}$-algebra $H_*^{\text{string}}(Q, K)$ and observe its basic properties. This graded $\mathbb{R}$-algebra can be regarded as a reformulation of the string homology of [6], whose coefficient is reduced from $\mathbb{Z}[\pi_1(\Lambda_K)]$ to $\mathbb{R}$. The feature of our formulation is that $H_*^{\text{string}}(Q, K)$ is defined for K of an arbitrary codimension and in all degrees, compared to the string homology defined for K of codimension 2 and generated by singular chains of degree less than or equal to 2. The two main differences from the string homology in its construction are the reduction of the coefficient and the substitution of singular chains by *de Rham chains* explained below.

The construction of $H_*^{\mathrm{string}}(Q, K)$ can be briefly summarized as follows: We first choose auxiliary data including a complete Riemannian metric on Q. As a graded $\mathbb{R}$-vector space, it is defined to be

$$(1) \qquad H_*^{\mathrm{string}}(Q, K) := \varinjlim_{a \to \infty} \varprojlim_{\substack{(\varepsilon, \delta) \in \mathcal{T}_a \\ \varepsilon \to 0}} H_*^{<a}(\varepsilon, \delta),$$

where $H_*^{<a}(\varepsilon, \delta)$ for $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ and $(\varepsilon, \delta) \in \mathcal{T}_a$ is the homology of a chain complex

$$(2) \qquad \left(C_*^{<a}(\varepsilon) := \bigoplus_{m=0}^{\infty} C_{*-m(d-2)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0), D_\delta \right).$$

An explanation for each piece of the above definition is the following:

(A) $C_*^{\mathrm{dR}}(X, A)$ is the $\mathbb{R}$-vector space of de Rham chains defined for a pair of differentiable spaces (X, A). Together with the boundary operator

$$\partial \colon C_*^{\mathrm{dR}}(X, A) \to C_{*-1}^{\mathrm{dR}}(X, A),$$

$(C_*^{\mathrm{dR}}(X, A), \partial)$ becomes a chain complex. De Rham chains can be used as substitutions of singular chains over $\mathbb{R}$. Their basic properties are summarized in Section 2. The advantage is that the fiber product of de Rham chains can be defined in a natural way. The main references are Irie [15; 16].

(B)   For $a \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 1}$, Σ_m^a is a differentiable space of sequences $(\gamma_1, \ldots, \gamma_m)$ of paths $\gamma_k \colon [0, T_k] \to Q$ for $k = 1, \ldots, m$ with endpoints in K. It includes all $(\gamma_k)_{k=1,\ldots,m}$ whose total length is less than a. For the precise definition, see Section 3.1. Exceptionally, Σ_0^a is the one-point set for $a > 0$ and Σ_0^0 is the empty set.

(C) D_δ is defined by $D_\delta(x) := \partial x + \sum_{k=1}^m (-1)^{p+kd+1} f_{k,\delta}(x)$ for $x \in C_{p-m(d-2)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$. Here

$$f_{k,\delta} \colon C_*^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) \to C_{*+1-d}^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon}, \Sigma_{m+1}^0) \quad \text{for } k = 1, \ldots, m$$

are operators which play the key role in our construction. The idea comes from an operation of string topology explained by three steps:

(i)   Fix a pair of short paths $(\sigma_i \colon [0, \varepsilon_i] \to N_\varepsilon)_{i=1,2}$ in a tubular neighborhood N_ε of K such that $\sigma_1(\varepsilon_1), \sigma_2(0) \in K$ and $\sigma_1(0) = \sigma_2(\varepsilon_2)$.

(ii)   For any sequence of m paths $(\gamma_k)_{k=1,\ldots,m}$, we split the kth path $\gamma_k \colon [0, T_k] \to Q$ at a time, say τ, if the image $\gamma_k(\tau)$ coincides with $\sigma_1(0)$. We then concatenate $\gamma_k|_{[0,\tau]}$ (resp. $\gamma_k|_{[\tau, T_k]}$) with σ_1 (resp. σ_2) to get a new sequence of $m + 1$ paths

$$(\gamma_1, \ldots, \gamma_{k-1}, (\gamma_k|_{[0,\tau]} \cdot \sigma_1), (\sigma_2 \cdot \gamma_k|_{[\tau, T_k]}), \gamma_{k+1}, \ldots, \gamma_m).$$

(iii)   We extend the procedures (i)–(ii) for families (or chains) of paths parametrized over manifolds.

For the precise definition, we need to take fiber products of chains. See Setions 3.3 and 3.4. The operator $f_{k,\delta}$ depends on a chain $\delta \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$, where S_ε for $\varepsilon > 0$ is a differentiable space of pairs

of short paths in N_ε introduced in Section 3.1. For $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$, where $\mathcal{L}(K)$ is a closed subset of Lebesgue measure 0, a set $\mathcal{T}_a$ consisting of pairs (ε, δ) is defined in Definition 4.4. It is necessary to prove $D_\delta \circ D_\delta = 0$ for any $(\varepsilon, \delta) \in \mathcal{T}_a$, which we do in Proposition 4.1, to define the chain complex (2).

(D) The inverse limit in (1) is defined from an inverse system

$$\left(\{H_*^{<a}(\varepsilon, \delta)\}_{(\varepsilon, \delta) \in \mathcal{T}_a}, \{k_{(\varepsilon', \delta'),(\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon}\right).$$

Its construction is given in Section 4.3. To define the linear map $k_{(\varepsilon', \delta'),(\varepsilon, \delta)} \colon H_*^{<a}(\varepsilon', \delta') \to H_*^{<a}(\varepsilon, \delta)$, we need to factor through another homology group constructed from "$[-1, 1]$-modeled de Rham chains". Furthermore, to check its well-definedness and a claim about composition, we need one more homology group constructed from "$[-1, 1]^2$-modeled de Rham chains". These variants of de Rham chains are introduced in Section 3.5.

(E) The inverse limit is denoted by $H_*^{<a}(Q, K) := \varprojlim_{\varepsilon \to 0} H_*^{<a}(\varepsilon, \delta)$. The direct limit in (1) is defined from $\left(\{H_*^{<a}(Q, K)\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}, \{I^{a,b}\}_{a \leq b}\right)$, where $I^{a,b} \colon H_*^{<a}(Q, K) \to H_*^{<b}(Q, K)$ is induced by the inclusion maps $\Sigma_m^{a+m\varepsilon} \to \Sigma_m^{b+m\varepsilon}$ for all $m \in \mathbb{Z}_{\geq 0}$. See Section 4.4.

(F) A graded associative product structure on $H_*^{\mathrm{string}}(Q, K)$ is induced by natural maps $\Sigma_m^a \times \Sigma_{m'}^{a'} \to \Sigma_{m+m'}^{a+a'}$ for all $m, m' \in \mathbb{Z}_{\geq 0}$. The unit comes from $1 \in \mathbb{R} = C_0^{\mathrm{dR}}(\Sigma_0^a, \Sigma_0^0)$ for $a > 0$. See Section 4.4.2.

A fundamental property of $H_*^{\mathrm{string}}(Q, K)$ is the invariance under C^∞ isotopies of K.

Theorem 1.1 *The unital graded $\mathbb{R}$-algebra $H_*^{\mathrm{string}}(Q, K)$ is independent up to isomorphism of the auxiliary data and invariant under changing the orientation of K. Moreover, it is invariant under C^∞ isotopies of K.* (See Proposition 4.20.)

We also give nontrivial computations when $Q = \mathbb{R}^{2d-1}$ for $d \geq 2$. For two specific submanifolds in $\mathbb{R}^{2d-1}$ both of which are diffeomorphic to $S^{d-1} \sqcup S^{d-1}$, we prove that our invariant is isomorphic to the homology of a finitely generated differential graded algebra. Using this computation, we obtain the next result.

Theorem 1.2 *For every $d \geq 2$, there are two nonisotopic oriented submanifolds K and K' in $\mathbb{R}^{2d-1}$ of codimension d such that Λ_K is isotopic to $\Lambda_{K'}$ as a C^∞ submanifold with a spin structure in $UT^*\mathbb{R}^{2d-1}$, while $H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K) \not\cong H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K')$.* (See Corollary 5.10.)

The spin structure on Λ_K for any submanifold K in a spin manifold Q is explained in Proposition 5.11.

Another purpose of this paper is to shed light on the relation to Legendrian contact homology. The following result is nontrivial from the construction.

Theorem 1.3 *When the codimension of K is 2 and the normal bundle of K is trivial, $H_0^{\mathrm{string}}(Q, K)$ is isomorphic to the cord algebra of (Q, K) over $\mathbb{R}$.* (See Theorem 6.11.)

If K is connected, the cord algebra over $\mathbb{R}$ we consider in this paper is a reduction of the cord algebra over $\mathbb{Z}[H_1(\Lambda_K)]$ defined by Ng [19]. Combined with a result of Ekholm, Etnyre, Ng and Sullivan [8], the cord algebra for a knot K in $\mathbb{R}^3$ was proved to be isomorphic to the 0th degree part of the Legendrian contact homology of $(UT^*\mathbb{R}^3, \Lambda_K)$. Later, another direct proof was given in [6].

The author makes the following more radical conjecture when $Q = \mathbb{R}^n$.

Conjecture 1.4 *For any compact oriented submanifold K in $\mathbb{R}^n$, $H_*^{\text{string}}(\mathbb{R}^n, K)$ is isomorphic to the Legendrian contact homology of $(UT^*\mathbb{R}^n, \Lambda_K)$ with coefficients in $\mathbb{R}$.*

The Legendrian contact homology with coefficients in $\mathbb{R}$ is an invariant of Legendrian submanifolds with a spin structure; see Ekholm, Etnyre and Sullivan [10; 11]. If Conjecture 1.4 is true, then our invariant can be applied to study the contact topology of $UT^*\mathbb{R}^n$. For instance, assuming this conjecture, Theorem 1.2 would imply that Λ_K is not isotopic to $\Lambda_{K'}$ as a Legendrian submanifold with a spin structure.

Organization of paper In Section 2, general notions of a differentiable space and its de Rham chain complex are introduced. In Section 3.1, the differentiable spaces Σ_m^a and S_ε are defined. Their de Rham chain complexes are observed in Section 3.2. Through Sections 3.3 and 3.4, the operator $f_{k,\delta}$ is defined. In Section 3.5, $[-1, 1]$-modeled and $[-1, 1]^2$-modeled de Rham chains for path spaces are introduced. In Section 4.1, we define the chain complexes (2) and give a couple of computations. In Section 4.2, we consider their variants using those chains in Section 3.5. They are necessary to define the map $k_{(\varepsilon', \delta'),(\varepsilon, \delta)}$ in Section 4.3. The definition of $H_*^{\text{string}}(Q, K)$ is given in Section 4.4. The independence on auxiliary data is checked in Section 4.5, from which the isotopy invariance follows immediately. In Section 5, we examine the algebraic structure of $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K)$ when K is a higher-dimensional generalization of the Hopf link or the unlink in $\mathbb{R}^3$. In Section 6.1, referring to [6], we define the cord algebra and give another description as the 0th degree part of the string homology. In Section 6.2, we construct a graded map from the string homology to $H_*^{\text{string}}(Q, K)$. In Section 6.3, this map is proved to be an isomorphism on the 0th degree part.

2 Differentiable space and de Rham chains

In this section, the notions of differentiable spaces and de Rham chains are introduced. We also summarize results applied in the latter sections.

Remark 2.1 The notion of differentiable space goes back to [5] by K-T Chen. The notion of de Rham chains was proposed by Fukaya [14], and later Irie gave the definition in [15; 16]. We mainly refer, especially about sign conventions, to [16]. As is mentioned in [15, Remark 4.1], the definition of plots (elements of a differentiable structure) in this paper is different from that of [5].

2.1 Notation and conventions

For $m, N \in \mathbb{Z}_{\geq 0}$, let $\mathcal{U}_{m,N}$ be the set of oriented m-dimensional submanifolds of $\mathbb{R}^N$. We then define $\mathcal{U} := \bigcup_{m,N \in \mathbb{Z}_{\geq 0}} \mathcal{U}_{m,N}$. Let us fix a few conventions about orientations. If we write $\mathbb{R}^n$ for $n \in \mathbb{Z}_{\geq 1}$, this means the manifold $\mathbb{R}^n \in \mathcal{U}_{n,n}$ whose orientation is given so that $dx_1 \wedge \cdots \wedge dx_n$ is a positive volume form when $(x_1, \ldots, x_n)$ is the standard coordinate of $\mathbb{R}^n$. If we write $\{0\}$, this means $\{0\} \in \mathcal{U}_{0,0}$ with a positive sign assigned.

Let us think about the orientation of fiber products of oriented manifolds. For $U, V, M \in \mathcal{U}$, suppose that there are two C^∞ maps $f : U \to M$ and $g : V \to M$. We also assume that g is a submersion. (Hereafter, all submersions are of class C^∞.) Then the fiber product

$$U \; {}_f\times_g V := \{(u, v) \in U \times V \mid f(u) = g(v)\}$$

is a C^∞ submanifold of $U \times V$. In order to determine the orientation at $(u, v) \in U \; {}_f\times_g V$, we take a right inverse $s : T_{g(v)}M \to T_v V$ of $(dg)_v$ (ie $(dg)_v \circ s = \mathrm{id}_{T_{g(v)}M}$). Then there are two isomorphisms

$$T_{g(v)}M \times \mathrm{Ker}(dg)_v \to T_v V, \qquad\qquad (z, y) \mapsto s(z) + y,$$
$$T_u U \times \mathrm{Ker}(dg)_v \to T_{(u,v)}(U \; {}_f\times_g V), \quad (x, y) \mapsto (x, y + s \circ (df)_u(x)).$$

The orientations of $\mathrm{Ker}(dg)_v$ and $T_{(u,v)}(U \; {}_f\times_g V)$ are determined so that the above isomorphisms preserve orientations. Of course, when X and Y are oriented $\mathbb{R}$-vector spaces, we assign the product orientation on $X \times Y$. In particular, when $M = \{0\}$, this gives the orientation of the product manifold $U \times V$.

For $U \in \mathcal{U}$, $\Omega_c^p(U)$ is the vector space of compactly supported C^∞ differential p-forms on U. When $p < 0$ or $p > \dim U$, we define $\Omega_c^p(U) := 0$. For $U, U' \in \mathcal{U}$ and a submersion $\pi : U' \to U$, we have an $\mathbb{R}$-linear map

$$\pi_! : \Omega_c^p(U') \to \Omega_c^{p - (\dim U' - \dim U)}(U),$$

called the *integration along fibers*. When $U' = \mathbb{R}^d \times \mathbb{R}^k$, $U = \mathbb{R}^k$ and $\pi(t, x) = x$ for $(t, x) \in U'$, this map is characterized by the following: for $f \in \Omega_c^0(U')$, $1 \leq i_1 < \cdots < i_a \leq d$ and $1 \leq j_1 < \cdots < j_b \leq k$, if we take $\omega := f(dt_{i_1} \wedge \cdots \wedge dt_{i_a} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_b})$, then, for every $x \in U$,

$$(\pi_!(\omega))_x = \begin{cases} 0 & \text{if } a < d, \\ \left(\int_{\mathbb{R}^d} f(\cdot, x) \, dt_1 \wedge \cdots \wedge dt_d\right)(dx_{i_1} \wedge \cdots \wedge dx_{i_a})_x & \text{if } a = d. \end{cases}$$

For an arbitrary submersion $\pi : U' \to U$, $\pi_!$ is defined by taking local charts and a partition of unity on U.

2.2 De Rham chain complex

2.2.1 Differentiable space
We proceed to the definition of differentiable spaces.

Definition 2.2 Let X be a set and P_X be a set of pairs (U, φ) of $U \in \mathcal{U}$ and a map $\varphi \colon U \to X$. We say P_X is a *differentiable structure* on X if it satisfies the following condition:

- For any $(U, \varphi) \in P_X$, $U' \in \mathcal{U}$ and a submersion $\pi \colon U' \to U$, the pair $(U', \varphi \circ \pi)$ is also an element of P_X.

We call such a pair (X, P_X) a *differentiable space*. An element of P_X is called a *plot* of (X, P_X).

Example 2.3 Let M be a manifold. There are two types of canonical differentiable structures on M:

$$P_M := \{(U, \varphi) \mid \varphi \colon U \to M \text{ is a } C^\infty \text{ map}\}, \quad P_M^{\text{reg}} := \{(U, \varphi) \mid \varphi \colon U \to M \text{ is a submersion}\}.$$

Clearly, (M, P_M) and (M, P_M^{reg}) are differentiable spaces. The latter is denoted by M^{reg}. We consider the differentiable structure P_M for any manifold M, unless we declare we use M^{reg}.

Definition 2.4 Let (X, P_X) and (Y, P_Y) be differentiable spaces and Z be a subset of X. Denote the projection map from $X \times Y$ to X (resp. Y) by pr_X (resp. pr_Y), and the inclusion map from Z to X by ι_Z.

(a) We define differentiable structures on $X \times Y$ and Z by

$$P_{X \times Y} := \{(U, \varphi) \mid (U, \text{pr}_X \circ \varphi) \in P_X \text{ and } (U, \text{pr}_Y \circ \varphi) \in P_Y\}, \quad P_Z := \{(U, \varphi) \mid (U, \iota_Z \circ \varphi) \in P_X\}.$$

(b) Let $f \colon X \to Y$ be a map. We say f is a *smooth map* if $(U, f \circ \varphi) \in P_Y$ for any $(U, \varphi) \in P_X$.

In the case of the above definition, we simply call (Z, P_Z) a *subspace* of (X, P_X). Note that, given a set W and two maps $f \colon X \to W$ and $g \colon Y \to W$, the fiber product $X \,_f \times_g Y$ becomes a differentiable space as a subspace of $(X \times Y, P_{X \times Y})$.

2.2.2 De Rham chains
Next we introduce the notion of de Rham chain complex. Hereafter, if we say that X is a differentiable space, this means that X is equipped with a differentiable structure, denoted by P_X.

Let X be a differentiable space. We consider a graded $\mathbb{R}$-vector space

$$A_*(X) := \bigoplus_{(U, \varphi) \in P_X} \Omega_c^{\dim U - *}(U).$$

For $(U, \varphi) \in P_X$ and $\omega \in \Omega_c^{\dim U - *}(U)$, let (U, φ, ω) denote the element of $A_*(X)$ whose component for $(V, \psi) \in P_X$ is

$$(U, \varphi, \omega)_{(V, \psi)} = \begin{cases} \omega & \text{if } (V, \psi) = (U, \varphi), \\ 0 & \text{if } (V, \psi) \neq (U, \varphi). \end{cases}$$

We take a linear subspace $Z_*(X)$ of $A_*(X)$ generated by

$$\{(U', \varphi \circ \pi, \omega) - (U, \varphi, \pi_! \omega) \mid (U, \varphi) \in P_X \text{ and } \pi \colon U' \to U \text{ is a submersion}\}.$$

Then we define a quotient vector space

$$C_*^{\text{dR}}(X) := A_*(X)/Z_*(X).$$

The equivalence class of $(U, \varphi, \omega) \in A_*(X)$ in $C_*^{\mathrm{dR}}(X)$ is denoted by $[U, \varphi, \omega]$. We also define an $\mathbb{R}$-linear map $\partial \colon C_*^{\mathrm{dR}}(X) \to C_{*-1}^{\mathrm{dR}}(X)$ of degree -1 by

$$\partial[U, \varphi, \omega] := (-1)^{|\omega|+1}[U, \varphi, d\omega].$$

This map is well defined and $\partial \circ \partial = 0$. $(C_*^{\mathrm{dR}}(X), \partial)$ is called the *de Rham chain complex* of a differentiable space X, and its elements are called *de Rham chains* of X. By taking its homology, we obtain

$$H_*^{\mathrm{dR}}(X) := H_*(C_*^{\mathrm{dR}}(X), \partial).$$

In addition, a functoriality holds. Namely, any smooth map $f \colon X \to Y$ induces a chain map

$$f_* \colon C_*^{\mathrm{dR}}(X) \to C_*^{\mathrm{dR}}(Y), \quad [U, \varphi, \omega] \mapsto [U, f \circ \varphi, \omega].$$

Remark 2.5 The following are fundamental techniques to compute de Rham chains:

(a) For $[U, \varphi, \omega] \in C_*^{\mathrm{dR}}(X)$, suppose that $V \subset U$ is an open subset containing $\mathrm{supp}\,\omega$. Then $[U, \varphi, \omega] = [V, \varphi|_V, \omega|_V] \in C_*^{\mathrm{dR}}(X)$.

(b) If $(\mathbb{R} \times U, \varphi) \in P_X$ satisfies $\varphi(s, \cdot) = \varphi_0$ when $s \leq 0$ and $\varphi(s, \cdot) = \varphi_1$ when $s \geq 1$ for some $(U, \varphi_0), (U, \varphi_1) \in P_X$, then

$$\partial[\mathbb{R} \times U, \varphi, (-1)^{|\omega|}\chi \times \omega] = [U, \varphi_1, \omega] - [U, \varphi_0, \omega] \in C_*^{\mathrm{dR}}(X)$$

for a closed form $\omega \in \Omega_c^{\dim U - *}(U)$ and $\chi \colon \mathbb{R} \to [0, 1]$ such that $\mathrm{supp}\,\chi$ is compact and $\chi(s) = 1$ for every $s \in [0, 1]$.

Example 2.6 Let M be an oriented smooth manifold. The de Rham chain complex of M^{reg} is naturally isomorphic to $(\Omega_c^{\dim M - *}(M), d)$ through the map

$$C_p^{\mathrm{dR}}(M^{\mathrm{reg}}) \to \Omega_c^{\dim M - p}(M), \quad [U, \varphi, \omega] \mapsto (-1)^{s(p)}\varphi_!\omega.$$

Here $s(p) := \frac{1}{2}(p - \dim M)(p - \dim M - 1)$. Hence $H_*^{\mathrm{dR}}(M^{\mathrm{reg}})$ is isomorphic to the compactly supported de Rham cohomology $H_{c,\mathrm{dR}}^{\dim M - *}(M)$.

Let us define the de Rham chain complex for a pair of differentiable spaces. A smooth map $f \colon X \to \mathbb{R}$ is said to be *approximately smooth* if there exists a decreasing sequence $(f_j)_{j \in \mathbb{Z}_{\geq 1}}$ of smooth maps from X to $\mathbb{R}$ such that $\lim_{j \to \infty} f_j(x) = f(x)$ for every $x \in X$. The following lemma is proved in [15, Lemma 4.11].

Lemma 2.7 *For an approximately smooth function $f \colon X \to \mathbb{R}$, let $X^a := f^{-1}((-\infty, a))$ for every $a \in \mathbb{R} \cup \{\infty\}$. Then, for $a, b \in \mathbb{R} \cup \{\infty\}$ with $a \leq b$, the linear map $i_* \colon C_*^{\mathrm{dR}}(X^a) \to C_*^{\mathrm{dR}}(X^b)$, which is induced by the inclusion map $i \colon X^a \to X^b$, is injective.*

In the setting of the above lemma, we define a quotient complex

$$C_*^{\mathrm{dR}}(X^b, X^a) := C_*^{\mathrm{dR}}(X^b) / i_*(C_*^{\mathrm{dR}}(X^a)).$$

Its homology is denoted by $H_*^{\mathrm{dR}}(X^b, X^a)$.

Next, we define a fiber product of de Rham chains.

Definition 2.8 Let (X, P_X) and (Y, P_Y) be differentiable spaces. Suppose that we have an oriented manifold M of dimension n and two smooth maps

$$f : (X, P_X) \to (M, P_M), \quad g : (Y, P_Y) \to (M, P_M^{\mathrm{reg}}) = M^{\mathrm{reg}}.$$

Then we define a linear map

$$C_{p+n}^{\mathrm{dR}}(X) \otimes C_{q+n}^{\mathrm{dR}}(Y) \to C_{p+q+n}^{\mathrm{dR}}(X \, _f\times_g Y), \quad x \otimes y \mapsto x \, _f\times_g y$$

by

$$x \, _f\times_g y := (-1)^{p|\eta|}[W, (\varphi \times \psi)|_W, (\omega \times \eta)|_W]$$

for $x = [U, \varphi, \omega] \in C_{p+n}^{\mathrm{dR}}(X)$ and $y = [V, \psi, \eta] \in C_{q+n}^{\mathrm{dR}}(Y)$. Here, $W := U \, _{f\circ\varphi}\times_{g\circ\psi} V$ is a fiber product over M.

It is straightforward to check the well-definedness of $x \, _f\times_g y$. It can also be checked that

$$\partial(x \, _f\times_g y) = (\partial x) \, _f\times_g y + (-1)^p x \, _f\times_g (\partial y)$$

for any $x \in C_{p+n}^{\mathrm{dR}}(X)$ and $y \in C_{q+n}^{\mathrm{dR}}(Y)$. When $M = \{0\}$, we simply write $x \, _f\times_g y$ as $x \times y$.

2.2.3 Collection of results about the de Rham chain complex In the rest of this section, let us summarize some basic results about the de Rham chain complex. The first result can be compared with the computation for M^{reg} in Example 2.6. Hereafter, $H^{\mathrm{sing}}(\cdot)$ denotes the singular homology with coefficients in $\mathbb{R}$.

Proposition 2.9 *For every oriented manifold M, there exists a canonical isomorphism*

$$\Psi_M : H_*^{\mathrm{sing}}(M) \to H_*^{\mathrm{dR}}(M)$$

such that, for any C^∞ map $f : M \to N$ between oriented manifolds, $\Psi_N \circ f_* = f_* \circ \Psi_M$.

For the details of the construction of Ψ_M, see [15, Section 4.7]. It is the composition of a natural isomorphism between $H_*^{\mathrm{sing}}(M)$ and $H_*^{\mathrm{sm}}(M)$ (the homology of smooth singular chains in M) and a canonical map from $H_*^{\mathrm{sm}}(M)$ to $H_*^{\mathrm{dR}}(M)$. For the proof that Ψ_M is an isomorphism, see [15, Section 5]. This result can be extended to relative homology groups for (M, N), where N is an open submanifold of M such that $N = f^{-1}((-\infty, a))$ for some approximately smooth map $f : M \to \mathbb{R}$.

Next, let $f, g : X \to Y$ be smooth maps between differentiable spaces. We say f is *homotopic* to g if there exists a smooth map $H : \mathbb{R} \times X \to Y$ such that $H(t, x) = f(x)$ for $t \leq 0$ and $H(t, x) = g(x)$ for $t \geq 1$. Then we have the following result. For the proof, see [15, Proposition 4.7].

Proposition 2.10 *For two smooth map $f, g : X \to Y$, if f is homotopic to g, then there exists a chain homotopy $K : C_*^{\mathrm{dR}}(X) \to C_{*+1}^{\mathrm{dR}}(Y)$ such that $\partial K + K\partial = f_* - g_*$. In particular, $f_* = g_* : H_*^{\mathrm{dR}}(X) \to H_*^{\mathrm{dR}}(Y)$.*

Remark 2.11 For three smooth maps $f, g, h\colon X \to Y$ such that f is homotopic to g and g is homotopic to h, we can ask whether f is homotopic to h. In fact, if the differentiable structure P_Y of Y satisfies the following condition, such transitivity holds (the proof is straightforward):

- For any $U \in \mathcal{U}$ and $(U_1, \varphi_1), (U_2, \varphi_2) \in P_Y$ such that $(U_i)_{i=1,2}$ is an open cover of U and $\varphi_1|_{U_1 \cap U_2} = \varphi_2|_{U_1 \cap U_2}$, $(U, \varphi) \in P_Y$ holds for $\varphi\colon U \to Y$ which maps $u \in U_i$ to $\varphi_i(u)$ for $i = 1, 2$.

All differentiable spaces appearing after Section 3 satisfy this condition. However, as mentioned in [15, Remark 4.4], it seems difficult in the general case to prove such transitivity.

The last one is a result about excisions.

Proposition 2.12 Let X be a differentiable space and $Y = f^{-1}((-\infty, a)) \subset X$ for some approximately smooth function $f\colon X \to \mathbb{R}$ and $a \in \mathbb{R}$. Suppose there is another approximately smooth function $g\colon X \to \mathbb{R}$ and $b_0 \in \mathbb{R}$ such that $g^{-1}((b_0, \infty)) \subset Y$. For every $b > b_0$, let $X^b := g^{-1}((-\infty, b))$ and $Y^b := (g|_Y)^{-1}((-\infty, b))$. Then the inclusion map of pairs $i\colon (X^b, Y^b) \to (X, Y)$ induces an isomorphism $i_*\colon C_*^{\mathrm{dR}}(X^b, Y^b) \to C_*^{\mathrm{dR}}(X, Y)$.

Proof We first prove the assertion when $g\colon X \to \mathbb{R}$ is a smooth map. For $b > b_0$, choose $\delta > 0$ and a smooth function $\kappa\colon \mathbb{R} \to [0, 1]$ such that $2\delta < b - b_0$ and $\kappa(b') = 1$ if $b' \leq b_0 + \delta$ while $\kappa(b') = 0$ if $b' \geq b - \delta$. Then we define a linear map

$$r\colon C_*^{\mathrm{dR}}(X) \to C_*^{\mathrm{dR}}(X^b), \quad [U, \varphi, \omega] \mapsto [U^b, \varphi|_{U^b}, (\kappa \circ g \circ \varphi) \cdot \omega|_{U^b}],$$

where $U^b := (g \circ \varphi)^{-1}((-\infty, b))$. This reduces to a map $\bar{r}\colon C_*^{\mathrm{dR}}(X, Y) \to C_*^{\mathrm{dR}}(X^b, Y^b)$. We claim that $\bar{r}$ is the inverse map of i_*. Indeed, for any $x = [U, \varphi, \omega] \in C_*^{\mathrm{dR}}(X)$, we have

$$x - i_* \circ \bar{r}(x) = [U, \varphi, \omega] - [U, \varphi, (\kappa \circ g \circ \varphi) \cdot \omega]$$
$$= [U_0, \varphi|_{U_0}, ((1 - \kappa) \circ g \circ \varphi) \cdot \omega|_{U_0}] \in C_*^{\mathrm{dR}}(Y) \quad \text{for } U_0 := (g \circ \varphi)^{-1}((b_0, \infty)).$$

Similarly, we can show that $x - \bar{r} \circ i_*(x) \in C_*^{\mathrm{dR}}(Y^b)$ for $x \in C_*^{\mathrm{dR}}(X^b)$.

In the general case, there exists a decreasing sequence $(g_j)_{j \geq 1}$ of smooth maps $g_j\colon X \to \mathbb{R}$ such that $g_j(x) \to g(x)$ as $j \to \infty$ for every $x \in X$. For $b > b_0$, let $X_j^b := g_j^{-1}((-\infty, b))$ and $Y_j^b := (g_j|_Y)^{-1}((-\infty, b))$. From [15, Corollary 4.12(i)], $\varinjlim_{j \to \infty} C_*^{\mathrm{dR}}(X_j^b, Y_j^b) \to C_*^{\mathrm{dR}}(X^b, Y^b)$, induced by inclusion maps, is an isomorphism. We have shown that $(i|_{(X_j^b, Y_j^b)})_*\colon C_*^{\mathrm{dR}}(X_j^b, Y_j^b) \to C_*^{\mathrm{dR}}(X, Y)$ is an isomorphism for every $j \geq 1$, so i_* is also an isomorphism. $\square$

3 Differentiable space of paths and operations from string topology

Throughout this paper, Q is a manifold of dimension n, and K is a compact submanifold of Q of codimension $d \geq 1$. In addition, both Q and K are required to have fixed orientations. The construction of $H_*^{\mathrm{string}}(Q, K)$ depends on the following auxiliary data:

(a) a complete Riemannian metric g on Q (we write $g(v, w) = \langle v, w \rangle_g$ and $\sqrt{g(v, v)} = |v|_g$);

(b) a constant $C_0 \geq 1$;

(c) a positive real number ε_0 for which the map

(3)
$$\{(x, v) \in (TK)^\perp \mid |v|_g < \varepsilon_0\} \to Q, \quad (x, v) \mapsto \exp_x(v),$$

is an open embedding;

(d) a C^∞ function $\mu \colon \left[0, \frac{3}{2}\right] \to [0, 1]$ such that $\mu(t) = t$ near $t = 0$, $\mu(t) = 1$ near $t = \frac{3}{2}$, and $0 \leq \mu'(t) \leq 1$ for every $t \in \left[0, \frac{3}{2}\right]$.

The independence of $H_*^{\text{string}}(Q, K)$ on these data up to isomorphism is proved in Section 4.5. Until then, these data are fixed, so $\langle v, w \rangle_g$ and $|v|_g$ are denoted by $\langle v, w \rangle$ and $|v|$, respectively.

We define $\mathscr{C}(K)$ to be the set of geodesics $\gamma \colon [0, T] \to Q$ with unit speed such that $\gamma(0), \gamma(T) \in K$ and $\gamma'(0) \in (T_{\gamma(0)}K)^\perp$ and $\gamma'(T) \in (T_{\gamma(T)}K)^\perp$. Such geodesics are called *binormal chords* of K. We also define, for $m \in \mathbb{Z}_{\geq 1}$,

$$\mathscr{L}_m(K) := \left\{ \sum_{k=1}^m \text{length } \gamma_m \;\middle|\; \gamma_1, \ldots, \gamma_m \in \mathscr{C}(K) \right\}, \quad \mathscr{L}(K) := \bigcup_{m=1}^\infty \mathscr{L}_m(K).$$

These are closed subsets of $\{a \in \mathbb{R} \mid a \geq 2\varepsilon_0\}$, since K is compact. Moreover, they are null sets with respect to Lebesgue measure. For the proof, see Lemma 3.10.

3.1 Differentiable space of paths

In this section, we introduce two differentiable spaces of paths, Σ_m^a and S_ε. Let $\Omega_K(Q)$ be the set of C^∞ paths $\gamma \colon [0, T] \to Q$ with $T > 0$ such that $\gamma(0), \gamma(T) \in K$ and $|\gamma'(t)| \leq C_0$ for any $t \in [0, T]$. For any C^∞ path $\gamma \colon [0, T] \to Q$, we define

$$\text{length } \gamma := \int_0^T |\gamma'(t)| \, dt.$$

For $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $m \in \mathbb{Z}_{\geq 1}$, we define Σ_m^a to be the subset of $\Omega_K(Q)^{\times m}$ which consists of $(\gamma_k \colon [0, T_k] \to Q)_{k=1,\ldots,m}$ satisfying *either* of the following two conditions:

- $\sum_{k=1}^m \text{length } \gamma_k < a$.
- $\min_{1 \leq k \leq m} \text{length } \gamma_k < \varepsilon_0$.

The differentiable structure on Σ_m^a is defined by

$$P_{\Sigma_m^a} := \{(U, \varphi) \mid U \in \mathcal{U} \text{ and } \varphi \colon U \to \Sigma_m^a \text{ is smooth}\}.$$

Here we say φ is smooth in the following sense: if we write $\varphi(u) = (\gamma_k^u \colon [0, T_k^u] \to Q)_{k=1,\ldots,m}$ for $u \in U$, then, for each $k \in \{1, \ldots, m\}$, the function $U \to \mathbb{R}_{>0}$, $u \mapsto T_k^u$, is of class C^∞ and

$$\{(u, t) \in U \times \mathbb{R} \mid 0 \leq t \leq T_k^u\} \to Q, \quad (u, t) \mapsto \gamma_k^u(t),$$

is a C^∞ map. As an exception, let us define $\Sigma_0^a := \{*\}$ if $a > 0$ and $\Sigma_0^0 := \varnothing$, together with the differentiable structure $P_{\Sigma_0^a} := \{(U, \varphi) \mid U \in \mathcal{U}, \varphi \colon U \to \Sigma_0^a\}$.

We consider the de Rham chain complex $(C_*^{\mathrm{dR}}(\Sigma_m^a), \partial)$ for $a \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 0}$. Lemma 2.7 implies that we may think of $C_*^{\mathrm{dR}}(\Sigma_m^a)$ as a linear subspace of $C_*^{\mathrm{dR}}(\Sigma_m^b)$ when $a \leq b$, since

$$\Sigma_m^b \to \mathbb{R}, \qquad (\gamma_l)_{l=1,\dots,m} \mapsto \text{length } \gamma_k \quad \text{for } k = 1, \dots, m,$$

$$\mathbb{R}^m \to \mathbb{R}, \qquad (a_k)_{k=1,\dots,m} \mapsto \min_{1 \leq k \leq m} a_k,$$

are approximately smooth functions. Thus the quotient complex $(C_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a), \partial)$ is defined.

Remark 3.1 When $a \leq m\varepsilon_0$, the condition that $\sum_{k=1}^m \text{length } \gamma_k < a$ implies that one of γ_k for $k = 1, \dots, m$ has length less than ε_0. Thus, $\Sigma_m^a = \Sigma_m^0$ if $a \leq m\varepsilon_0$. When $a = \infty$, $\Sigma_m^\infty = \Omega_K(Q)^{\times m}$, which will be used only in Section 6.

Next, we define another differentiable space of paths. For every $\varepsilon \in (0, \varepsilon_0]$, the open subset in Q

$$N_\varepsilon := \{\exp_x(v) \mid x \in K, \, v \in (T_x K)^\perp \text{ and } |v| < \tfrac{1}{2}\varepsilon\}$$

is a tubular neighborhood of K in Q. Then we define a set S_ε which consists of pairs of C^∞ paths $(\sigma_i : [0, \varepsilon_i] \to N_\varepsilon)_{i=1,2}$ satisfying:

- $0 < \varepsilon_i \leq \tfrac{1}{2}\varepsilon$ for $i = 1, 2$.
- $\sigma_1(\varepsilon_1), \sigma_2(0) \in K$ and $\sigma_1(0) = \sigma_2(\varepsilon_2)$.
- $|\sigma_i'(t)| \leq 1$ for $i = 1, 2$ and any $t \in [0, \varepsilon_i]$.

On this set, the evaluation map ev_0 is defined by

$$\mathrm{ev}_0 : S_\varepsilon \to N_\varepsilon, \quad (\sigma_1, \sigma_2), \mapsto \sigma_1(0).$$

The differentiable structure on S_ε is defined by

$$P_{S_\varepsilon} := \{(V, \psi) \mid V \in \mathcal{U}, \, \psi \text{ is a smooth map such that } \mathrm{ev}_0 \circ \psi : V \to N_\varepsilon \text{ is a submersion}\}.$$

Here we say ψ is smooth in the following sense: if we write $\psi(v) = (\sigma_i^v : [0, \varepsilon_i^v] \to N_\varepsilon)_{i=1,2}$ for $v \in V$, then, for $i \in \{1, 2\}$, the function $V \to \mathbb{R}_{>0}$, $v \mapsto \varepsilon_i^v$, is of class C^∞ and

$$\{(v, t) \in V \times \mathbb{R} \mid 0 \leq t \leq \varepsilon_i^v\} \to N_\varepsilon, \quad (v, t) \mapsto \sigma_i^v(t),$$

is a C^∞ map. Note that ev_0 is a smooth map from $(S_\varepsilon, P_{S_\varepsilon})$ to $(N_\varepsilon, P_{N_\varepsilon}^{\mathrm{reg}}) = N_\varepsilon^{\mathrm{reg}}$, defined in Example 2.3.

3.2 Homology groups

In this section, we examine the homology groups $H_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a)$ and $H_*^{\mathrm{dR}}(S_\varepsilon)$. The main results are Propositions 3.7 and 3.9. At the end, several additional results are proved.

3.2.1 Finite-dimensional approximation of Σ_m^a Let us fix $b_0 \in \mathbb{R}_{>0}$ and prepare some notation related to the Riemannian metric g. We note that there is a compact subset of Q which contains the images of all paths $\gamma \in \Omega_K(Q)$ with length $\gamma \leq b_0$, since K is compact and g is complete. For any two points

$q, q' \in Q$, let $d(q, q')$ be the distance between q and q'. Let us also fix $\varepsilon_g > 0$ such that, if q and q' in this compact set satisfy $d(q, q') < \varepsilon_g$, then there exists a unique geodesic path on $[0, 1]$ of length $d(q, q')$ from q to q'. We write this geodesic by $\overline{qq'} \colon [0, 1] \to Q$.

For every $a \in [0, b_0)$ and $m \in \mathbb{Z}_{\geq 1}$, let $\overline{\Sigma}_m^a$ be the subspace of Σ_m^a which consists of $(\gamma_k)_{k=1,\dots,m}$ satisfying $\sum_{k=1}^m \text{length } \gamma_k < b_0$. From [Proposition 2.12](#) about excision, the inclusion map $\iota \colon \overline{\Sigma}_m^b \to \Sigma_m^b$ induces an isomorphism

$$(4) \qquad \iota_* \colon H_*^{\mathrm{dR}}(\overline{\Sigma}_m^b, \overline{\Sigma}_m^a) \to H_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a)$$

for $a, b \in [0, b_0)$ with $a \leq b$. This means that we cut out a subset

$$\left\{ (\gamma_k)_{k=1,\dots,m} \mid \min_{1 \leq k \leq m} \text{length } \gamma_k < \varepsilon_0 \text{ and } \sum_{k=1}^m \text{length } \gamma_k \geq b_0 \right\}$$

to compute the homology group.

First, we approximate $\overline{\Sigma}_m^a$ for $a \in [0, b_0)$ by finite-dimensional manifolds. For every $\nu \in \mathbb{Z}_{\geq 1}$, let us define

$$(5) \qquad \overline{\Sigma}_m^a(\nu) := \left\{ (\gamma_k \colon [0, T_k] \to Q)_{k=1,\dots,m} \in \overline{\Sigma}_m^a \mid \max_{1 \leq k \leq m} T_k < C_0^{-1} \varepsilon_g \nu \right\},$$

so that $\bigcup_{\nu=1}^\infty \overline{\Sigma}_m^a(\nu) = \overline{\Sigma}_m^a$. Let us also define $B_m(\nu)$ to be a submanifold of $(Q^{\times(\nu+1)})^{\times m}$ which consists of $(q_k^l)_{k=1,\dots,m}^{l=0,\dots,\nu}$ satisfying:

- $q_k^0, q_k^\nu \in K$ for every $k = 1, \dots, m$.
- $\sum_{k=1}^m \sum_{l=0}^{\nu-1} d(q_k^l, q_k^{l+1}) < b_0$.
- $d(q_k^l, q_k^{l+1}) < \varepsilon_g$ for every $k = 1, \dots, m$ and $l = 0, \dots, \nu - 1$.

We then define $B_m^a(\nu)$ for $a < b_0$ to be the open submanifold of $B_m(\nu)$ which consists of $(q_k^l)_{k=1,\dots,m}^{l=0,\dots,\nu}$ satisfying *either* of the following two conditions:

- $\sum_{k=1}^m \sum_{l=0}^{\nu-1} d(q_k^l, q_k^{l+1}) < a$.
- $\min_{1 \leq k \leq m} \left(\sum_{l=0}^{\nu-1} d(q_k^l, q_k^{l+1}) \right) < \varepsilon_0$.

The differentiable structures on this manifold is $P_{B_m^a(\nu)}$ defined in [Example 2.3](#). For every $\nu \in Z_{\geq 1}$, there are two maps: $\iota_{\Sigma,\nu} \colon \overline{\Sigma}_m^a(\nu) \to \overline{\Sigma}_m^a(2\nu)$ is just the inclusion map, and $\iota_{B,\nu} \colon B_m^a(\nu) \to B_m^a(2\nu)$ is an embedding of a manifold which maps $(q_k^l)_{k=1,\dots,m}^{l=0,\dots,\nu} \in B_m^a(\nu)$ to $(\bar{q}_k^{l'})_{k=1,\dots,m}^{l'=0,\dots,2\nu} \in B_m^a(2\nu)$, where

$$\bar{q}_k^{l'} = \begin{cases} q_k^l & \text{if } l' \text{ is even and } l' = 2l, \\ \overline{q_k^l q_k^{l+1}}\left(\tfrac{1}{2}\right) & \text{if } l' \text{ is odd and } l' = 2l + 1. \end{cases}$$

In addition, we define two maps

$$f_\nu \colon \overline{\Sigma}_m^a(\nu) \to B_m^a(\nu), \quad g_\nu \colon B_m^a(\nu) \to \overline{\Sigma}_m^a(2\nu),$$

as follows: f_ν maps $(\gamma_k : [0, T_k] \to Q)_{k=1,\dots,m} \in \overline{\Sigma}_m^a(\nu)$ to $(\gamma_k((l/\nu)T_k))_{k=1,\dots,m}^{l=0,\dots,\nu} \in B_m^a(\nu)$. Note that for $l = 0, \dots, \nu - 1$ and $k = 1, \dots, m$,

$$d\left(\gamma_k\left(\frac{l}{\nu}T_k\right), \gamma_k\left(\frac{l+1}{\nu}T_k\right)\right) \leq \text{length } \gamma_k|_{[(l/\nu)T_k, ((l+1)/\nu)T_k]} \leq \frac{T_k}{\nu} \cdot \sup_{t \in [0, T_k]} |\gamma'(t)| < \varepsilon_g.$$

On the other hand, g_ν maps $(q_k^l)_{k=1,\dots,m}^{l=0,\dots,\nu} \in B_m^a(\nu)$ to

$$\left(\gamma_k : \left[0, \tfrac{3}{2}C_0^{-1}\varepsilon_g \nu\right] \to Q\right)_{k=1,\dots,m} \in \overline{\Sigma}_m^a(2\nu),$$

where, for $l = 0, \dots, \nu - 1$,

$$\gamma_k(t) := \overline{q_k^l q_k^{l+1}} \circ \chi\left(\frac{C_0}{\varepsilon_g} \cdot t - \frac{3}{2}l\right) \quad \text{if } \tfrac{3}{2}C_0^{-1}\varepsilon_g l \leq t \leq \tfrac{3}{2}C_0^{-1}\varepsilon_g(l+1).$$

Here $\chi : \left[0, \frac{3}{2}\right] \to [0, 1]$ is a C^∞ function such that $\chi(t) = 0$ near $t = 0$, $\chi(t) = \frac{1}{2}$ near $t = \frac{3}{4}$, $\chi(t) = 1$ near $t = \frac{3}{2}$, and $0 \leq \chi'(t) \leq 1$ for $t \in \left[0, \frac{3}{2}\right]$. Note that $|(\gamma_k)'(t)| \leq d(q_k^l, q_k^{l+1}) \cdot \sup_{t \in [0, 3/2]} |\chi'(t)| \cdot C_0/\varepsilon_g \leq C_0$. The next lemma shows that $B_m^a(\nu)$ approximates $\overline{\Sigma}_m^a(\nu)$ as $\nu \to \infty$. See [15, Lemma 6.3].

Lemma 3.2 *The following diagram commutes up to homotopy*:

$$
\begin{array}{ccc}
\overline{\Sigma}_m^a(\nu) & \xrightarrow{\iota_{\Sigma,\nu}} & \overline{\Sigma}_m^a(2\nu) \\
{\scriptstyle f_\nu}\downarrow & {\scriptstyle g_\nu}\nearrow & \downarrow{\scriptstyle f_{2\nu}} \\
B_m^a(\nu) & \xrightarrow{\iota_{B,\nu}} & B_m^a(2\nu)
\end{array}
$$

Proof The lower right triangle commutes in the strict sense. For the upper left triangle, we need to show that $\iota_{\Sigma,\nu}$ is homotopic to $g_\nu \circ f_\nu$.

We abbreviate $C_0^{-1} \cdot \varepsilon_g$ by c_0. For $(\gamma_k : [0, T_k] \to Q)_{k=1\dots,m} \in \overline{\Sigma}_m^a(\nu)$, let us define a path $\gamma_k^s : [0, c_0\nu] \to Q$ for $k = 1, \dots, m$ and $s \in [0, 1]$ by

$$\gamma_k^s(t) := \begin{cases} \overline{\gamma_k((l/\nu)T_k)\gamma_k(((l+s)/\nu)T_k)}(t - c_0 l) & \text{if } c_0 l \leq t \leq c_0(l+s), \\ \gamma_k\big((T_k/(c_0\nu))(t - c_0 l) + (l/\nu)T_k\big) & \text{if } c_0(l+s) \leq t \leq c_0(l+1), \end{cases} \quad \text{for } l = 0, \dots, l-1.$$

Then γ_k^0 is equal to $\gamma_k(T_k/(c_0\nu) \cdot)$ and γ_k^1 is a broken geodesic connecting $\big(\gamma_k((l/\nu)T_k)\big)^{l=0,\dots,\nu}$. We modify γ_k^s to a C^∞ path. For instance, we take a C^∞ function $\widetilde{\chi} : [0, 1] \times \left[0, \frac{3}{2}c_0\nu\right] \to [0, c_0\nu]$ satisfying $0 \leq \partial\widetilde{\chi}(s, t)/\partial t \leq 1$ and

$$\widetilde{\chi}(s, t) = \begin{cases} c_0 l & \text{on a neighborhood of } \{t = \frac{3}{2}c_0 l\}, \\ \frac{2}{3}\big(t - \frac{1}{4}c_0\big) & \text{on a neighborhood of } \{t = s + \frac{3}{2}c_0 l + \frac{1}{4}c_0\}. \end{cases}$$

Then we define $\widetilde{\gamma}_k^s := \gamma_k^s \circ \widetilde{\chi}(s, \cdot) : \left[0, \frac{3}{2}c_0\nu\right] \to Q$. If we take a C^∞ function $\kappa : \mathbb{R} \to [0, 1]$ such that $\kappa(t) = 0$ if $t \leq 0$ and $\kappa(t) = 1$ if $t \geq 1$, then we get a smooth map

$$H : \mathbb{R} \times \overline{\Sigma}_m^a(\nu) \to \overline{\Sigma}_m^a(2\nu), \quad (s, (\gamma_k)_{k=1,\dots,m}) \mapsto (\widetilde{\gamma}_k^{\kappa(s)})_{k=1,\dots,m}.$$

This gives a homotopy from $H(0, \cdot)$ to $H(1, \cdot)$. Moreover, $\iota_{\Sigma,\nu}$ is homotopic to $H(0, \cdot)$ since the paths of $\iota_{\Sigma,\nu}((\gamma_k)_{k=1,\dots,m})$ and those of $H(0, (\gamma_k)_{k=1,\dots,m})$ differ only by parametrizations, so the homotopy can be constructed by interpolating these parametrizations. For the same reason, $g_\nu \circ f_\nu$ is homotopic to $H(1, \cdot)$. Therefore, $\iota_{\Sigma,\nu}$ is homotopic to $g_\nu \circ f_\nu$. $\qquad\square$

From Lemma 3.2, it follows that, for any $a, b \in \mathbb{R}_{\geq 0}$ with $a \leq b < b_0$,

$$\varinjlim_{j\to\infty} (f_{2^j})_* : H_*^{\mathrm{dR}}(\overline{\Sigma}_m^b, \overline{\Sigma}_m^a) = \varinjlim_{j\to\infty} H_*^{\mathrm{dR}}(\overline{\Sigma}_m^b(2^j), \overline{\Sigma}_m^a(2^j)) \to \varinjlim_{j\to\infty} H_*^{\mathrm{dR}}(B_m^b(2^j), B_m^a(2^j))$$

is an isomorphism. Combining with (4), we get an isomorphism

$$(6) \qquad \left(\varinjlim_{j\to\infty} (f_{2^j})_* \right) \circ (\iota_*)^{-1} : H_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a) \to \varinjlim_{\nu\to\infty} H_*^{\mathrm{dR}}(B_m^b(\nu), B_m^a(\nu)).$$

Furthermore, from Proposition 2.9, $H_*^{\mathrm{sing}}(B_m^b(\nu), B_m^a(\nu)) \cong H_*^{\mathrm{dR}}(B_m^b(\nu), B_m^a(\nu))$.

3.2.2 Computation of homology by Morse theory Next, we examine the singular homology group $H_*^{\mathrm{sing}}(B_m^b(\nu), B_m^a(\nu))$ in terms of Morse theory. Fix $m \in \mathbb{Z}_{\geq 1}$ and $\nu \in \mathbb{Z}_{\geq 1}$. For $k \in \{1, \dots, m\}$ and $l \in \{0, \dots, \nu - 1\}$, we set

$$h_k^l : B_m(\nu) \to \mathbb{R}, \quad (q_k^l)_{k=1,\dots,m}^{l=0,\dots,\nu} \mapsto d(q_k^l, q_k^{l+1})^2.$$

For every $r > 0$, let us introduce

- a C^∞ function $\sigma_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}, z \mapsto \sqrt{z + r}$;
- a C^∞ function

$$L_r : B_m(\nu) \to \mathbb{R}, \quad q \mapsto \sum_{k=1}^{m} \sum_{l=0}^{\nu-1} \sigma_r \circ h_k^l(q);$$

- compact subsets of $B_m(\nu)$

$$Z_r := \{q \in L_r^{-1}([0, b_0]) \mid \sigma_r \circ h_k^l(q) \leq \varepsilon_g \text{ for every } k = 1, \dots, m \text{ and } l = 0, \dots, \nu - 1\},$$

$$Z_r^0 := \left\{ q \in Z_r \mid \min_{1 \leq k \leq m} \sum_{l=1}^{\nu-1} \sigma_r \circ h_k^l(q) \leq \varepsilon_0 \right\}.$$

The role of $\{\sigma_r\}_{r>0}$ is to approximate $\sqrt{z}$ by C^∞ functions. We define, for every $a \in [0, b_0)$ and $r > 0$,

$$Z_r^a := (L_r|_{Z_r})^{-1}([0, a)) \cup Z_r^0.$$

Then $Z_r^a \subset Z_{r'}^a$ if $0 < r' < r$. Furthermore, $\bigcup_{r>0} Z_r^a = B_m^a(\nu)$. Therefore, we have an isomorphism induced by the inclusion maps

$$\varinjlim_{r\to 0} H_*^{\mathrm{sing}}(Z_r^b, Z_r^a) \to H_*^{\mathrm{sing}}(B_m^b(\nu), B_m^a(\nu))$$

for $a, b \in \mathbb{R}_{\geq 0}$ with $a \leq b < b_0$. In order to apply Morse theory to L_r, we need to determine its critical points. The next lemma is fundamental. We omit the proof, but a similar result is proved, for instance in [17], when there is no boundary condition.

Lemma 3.3 *For $k_0 \in \{1, \ldots, m\}$ and $l_0 \in \{0, \ldots, \nu-1\}$, let $X_{k_0}^{l_0}$ be the gradient vector field of $h_{k_0}^{l_0}$. Define π^{l_0} to be the orthogonal projection from $TQ|_K$ to TK if $l_0 = 0$ or ν, and otherwise $\pi^{l_0} := \mathrm{id}_{TQ}$. Then each component of $X_{k_0}^{l_0} = (v_k^l)_{k=1,\ldots,m}^{l=0,\ldots,\nu}$ is determined by the following: If $(k,l) \neq (k_0, l_0), (k_0, l_0+1)$, then $v_k^l = 0$. Otherwise,*

$$v_{k_0}^{l_0} = \pi^{l_0}\left(-\frac{d}{dt}\Big|_{t=0} \overline{q_{k_0}^{l_0} q_{k_0}^{l_0+1}}\right), \quad v_{k_0}^{l_0+1} = \pi^{l_0+1}\left(\frac{d}{dt}\Big|_{t=1} \overline{q_{k_0}^{l_0} q_{k_0}^{l_0+1}}\right).$$

Proposition 3.4 *Suppose that $a \in [0, b_0) \setminus \mathscr{L}_m(K)$ and $r > 0$ is sufficiently small. Then $\boldsymbol{q} \in Z_r \setminus Z_r^0$ is a critical point of L_r with its value in $[0,a]$ if and only if there exist $\gamma_1, \ldots, \gamma_m \in \mathscr{C}(K)$ such that $(\gamma_k)_{k=1,\ldots,m} \in \overline{\Sigma}_m^a(\nu)$ and $\boldsymbol{q} = f_\nu((\gamma_k)_{k=1,\ldots,m})$.*

Proof From Lemma 3.3, $\boldsymbol{q} = (q_k^l)_{k=1,\ldots,m}^{l=0,\ldots,\nu} \in B_m(\nu)$ is a critical point of L_r if and only if the following conditions hold for every $k \in \{1, \ldots, m\}$:

(7)
$$(\sigma_r)'(h_k^0(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=0} \overline{q_k^0 q_k^1} \in (T_{q_k^0} K)^\perp \quad \text{and} \quad (\sigma_r)'(h_k^{\nu-1}(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=1} \overline{q_k^{\nu-1} q_k^\nu} \in (T_{q_k^\nu} K)^\perp,$$
$$(\sigma_r)'(h_k^{l-1}(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=1} \overline{q_k^{l-1} q_k^l} = (\sigma_r)'(h_k^l(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=0} \overline{q_k^l q_k^{l+1}} \quad \text{for every } l \in \{1, \ldots, \nu-1\}.$$

Comparing the norms of both sides in the second line of (7), we have, for $l \in \{1, \ldots, \nu-1\}$,

$$(\sigma_r)'(h_k^{l-1}(\boldsymbol{q})) \cdot \sqrt{h_k^{l-1}(\boldsymbol{q})} = (\sigma_r)'(h_k^l(\boldsymbol{q})) \cdot \sqrt{h_k^l(\boldsymbol{q})}.$$

Since $(\sigma_r)'(z)\sqrt{z}$ is a strictly increasing function of z, this equation means that $d(q_k^{l-1}, q_k^l) = d(q_k^l, q_k^{l+1})$ for $l \in \{1, \ldots, \nu-1\}$. Since $(\sigma_r)' > 0$, the conditions of (7) are equivalent to

(8)
$$\frac{d}{dt}\Big|_{t=0} \overline{q_k^0 q_k^1} \in (T_{q_k^0} K)^\perp \quad \text{and} \quad \frac{d}{dt}\Big|_{t=1} \overline{q_k^{\nu-1} q_k^\nu} \in (T_{q_k^\nu} K)^\perp,$$
$$\frac{d}{dt}\Big|_{t=1} \overline{q_k^{l-1} q_k^l} = \frac{d}{dt}\Big|_{t=0} \overline{q_k^l q_k^{l+1}} \quad \text{for every } l \in \{1, \ldots, \nu-1\}.$$

For every $\boldsymbol{q} \in B_m(\nu)$, let us define $T_{k,\boldsymbol{q}} := \sum_{l=0}^{\nu-1} d(q_k^l, q_k^{l+1})$ and a path $\gamma_{k,\boldsymbol{q}} : [0, T_{k,\boldsymbol{q}}] \to Q$ determined by

$$\gamma_{k,\boldsymbol{q}}(T_{k,\boldsymbol{q}} t) := \overline{q_k^l q_k^{l+1}}(\nu t - l) \quad \text{if } \frac{l}{\nu} \leq t \leq \frac{l+1}{\nu} \text{ for } l \in \{0, \ldots, \nu-1\}.$$

Note that, if $(\gamma_{k,\boldsymbol{q}})_{k=1,\ldots,m} \in \overline{\Sigma}_m^a(\nu)$, then $\boldsymbol{q} = f_\nu((\gamma_{k,\boldsymbol{q}})_{k=1,\ldots,m})$. We take $r > 0$ so small that $T_{k,\boldsymbol{q}} > 0$ for every $k = 1, \ldots, m$ and $\boldsymbol{q} \in Z_r \setminus Z_r^0$. Then, for $\boldsymbol{q} \in Z_r \setminus Z_r^0$, condition (8) is equivalent to $\gamma_{k,\boldsymbol{q}}$ being a binormal chord of K in Q, that is, $\gamma_{k,\boldsymbol{q}} \in \mathscr{C}(K)$. In addition,

$$\lim_{r \to 0} L_r(\boldsymbol{q}) = \sum_{k=1}^m \text{length } \gamma_{k,\boldsymbol{q}}$$

for every $\boldsymbol{q} \in Z_r$ uniformly. Recall that $\mathscr{L}_m(K)$ is a closed subset of $\mathbb{R}_{>0}$. Assuming that $a \notin \mathscr{L}_m(K)$ and $r > 0$ is sufficiently small, it follows that a critical point $\boldsymbol{q} \in Z_r \setminus Z_r^0$ of L_r satisfies $L_r(\boldsymbol{q}) \leq a$ if and only if the binormal chords $(\gamma_{k,\boldsymbol{q}})_{k=1,\ldots,m}$ satisfy $\sum_{k=1}^m \text{length } \gamma_{k,\boldsymbol{q}} < a$. □

Let Y_r be the gradient vector field of L_r. To prove the next lemma, which is rather technical, let us prepare a few computations. $X_{k,r}^l := ((\sigma_r)' \circ h_k^l) \cdot X_k^l$ is the gradient vector field of $\sigma_r \circ h_k^l$. For every

$k \in \{1, \ldots, m\}$, $l \in \{1, \ldots, \nu - 2\}$ and $\boldsymbol{q} \in B_m(\nu)$, we have

$$\langle Y_r, X_k^l \rangle(\boldsymbol{q}) = \left\langle (\sigma_r)'(h_k^{l-1}(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=1} \overline{q_k^{l-1}q_k^l}, -\frac{d}{dt}\Big|_{t=0} \overline{q_k^l q_k^{l+1}} \right\rangle$$

$$+ (\sigma_r)'(h_k^l(\boldsymbol{q})) \cdot \left| \frac{d}{dt}\Big|_{t=0} \overline{q_k^l q_k^{l+1}} \right|^2 + (\sigma_r)'(h_k^l(\boldsymbol{q})) \cdot \left| \frac{d}{dt}\Big|_{t=1} \overline{q_k^l q_k^{l+1}} \right|^2$$

$$+ \left\langle -(\sigma_r)'(h_k^{l+1}(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=0} \overline{q_k^{l+1}q_k^{l+2}}, \frac{d}{dt}\Big|_{t=1} \overline{q_k^l q_k^{l+1}} \right\rangle.$$

We abbreviate the increasing function $(\sigma_r)' \sqrt{z}$ by τ_r. Then, by the Cauchy–Schwarz inequality,

$$\langle Y_r, X_{k,r}^l \rangle(\boldsymbol{q}) \geq -\tau_r(h_k^{l-1}(\boldsymbol{q})) \cdot \tau(h_k^l(\boldsymbol{q})) + 2\big(\tau_r(h_k^l(\boldsymbol{q}))\big)^2 - \tau_r(h_k^{l+1}(\boldsymbol{q})) \cdot \tau_r(h_k^l(\boldsymbol{q}))$$

$$= \big(2\tau(h_k^l(\boldsymbol{q})) - \tau_r(h_k^{l-1}(\boldsymbol{q})) - \tau_r(h_k^{l+1}(\boldsymbol{q}))\big) \cdot \tau_r(h_k^l(\boldsymbol{q})).$$

Since σ_r and τ_r are increasing functions, we have, for $k_0 \in \{1, \ldots, m\}$ and $l_0 \in \{1, \ldots, \nu - 2\}$,

$$(9) \qquad \sigma_r \circ h_{k_0}^{l_0}(\boldsymbol{q}) = \max_{k,l} \sigma_r \circ h_k^l(\boldsymbol{q}) \implies \langle -Y_r, X_{k_0,r}^{l_0} \rangle(\boldsymbol{q}) \leq 0.$$

The same result holds when $l_0 = 0$ or $\nu - 1$. We also note that, for every $k_0 \in \{1, \ldots, m\}$,

$$(10) \qquad \left\langle -Y_r, \sum_{l=1}^{\nu-1} X_{k_0,r}^l \right\rangle(\boldsymbol{q}) \leq 0.$$

Lemma 3.5 *The trajectory of any point in Z_r (resp. Z_r^0) along $-Y_r$ never goes outside Z_r (resp. Z_r^0) at positive time.*

Proof Suppose that $\Gamma \colon [0, T] \to B_m(\nu)$ is a trajectory along $-Y_r$. Let us consider two continuous functions $f, g \colon [0, T] \to \mathbb{R}$ defined by

$$f(t) := \max_{k,l} \sigma_r \circ h_k^l(\Gamma(t)), \quad g(t) := \min_k \sum_{l=1}^{\nu-1} \sigma_r \circ h_k^l(\Gamma(t)).$$

To prove this lemma, it suffices to show that they are decreasing functions. Indeed, there exists a discrete subset $A \subset [0, T]$ such that f and g are differentiable at every $t \in [0, T] \setminus A$. Inequalities (9) and (10) imply that $f'(t) \leq 0$ and $g'(t) \leq 0$ for every $t \in [0, T] \setminus A$. Hence, f and g are decreasing on $[0, T]$. $\square$

We apply a general result from Morse theory.

Lemma 3.6 *Let B be a manifold and $L \colon B \to \mathbb{R}$ be a C^∞ function. For $a, b \in \mathbb{R}$ with $a \leq b$ and two compact subsets $Z, Z^0 \subset B$, suppose that there is no critical point of L in $(L|_Z)^{-1}([a, b]) \setminus Z^0$ and that the trajectory of any point in Z (resp. Z^0) along the negative gradient vector field of L never goes outside Z (resp. Z^0) at positive times. Let us define $Z^{a'} := (L|_Z)^{-1}((-\infty, a')) \cup Z^0$ for $a' \in \{a, b\}$. Then $H_*^{\mathrm{sing}}(Z^b, Z^a) = 0$.*

Proof The conditions on a, b, Z and Z^0 show that Z^b can be deformed into Z^a along the negative gradient flow of L. Therefore, we get a map from (Z^b, Z^a) to (Z^a, Z^a) which gives the inverse map of the inclusion map up to homotopy. $\square$

Combining the above results, we prove the first main proposition in this section.

Proposition 3.7 *If* $\mathscr{L}_m(K) \cap [a, b] = \varnothing$, *then* $H_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a) = 0$.

Proof From Proposition 3.4 and Lemma 3.5, we can apply Lemma 3.6 to show that, if $\mathscr{L}_m(K) \cap [a, b] = \varnothing$ and $r > 0$ is sufficiently small, then $H_*^{\mathrm{sing}}(Z_r^b, Z_r^a) = 0$, and thus

$$H_*^{\mathrm{sing}}(B_m^b(v), B_m^a(v)) \cong \varinjlim_{r \to 0} H_*^{\mathrm{sing}}(Z_r^b, Z_r^a) = 0.$$

From (6), it follows that $H_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a) \cong \varinjlim_{j \to \infty} H_*^{\mathrm{sing}}(B_m^b(2^j), B_m^a(2^j)) = 0$. □

3.2.3 $H_*^{\mathrm{dR}}(S_\varepsilon)$ and the evaluation map Next, we examine $H_*^{\mathrm{dR}}(S_\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0]$. Choose a Riemannian metric g' on N_{ε_0} for which K is a totally geodesic submanifold of N_{ε_0}. Then we can take a constant $C_1 \geq 1$ such that, for any $q, q' \in N_{\varepsilon_0/C_1}$ with $d(q, q') < \varepsilon_0/C_1$, there exists a unique shortest geodesic in N_{ε_0} with respect to g' from q to q'. Let us write this geodesic by $\widetilde{qq'} : [0, 1] \to N_{\varepsilon_0}$. In this subsection, exp denotes the exponential map with respect to g.

There is a smooth map between differentiable spaces $s_\varepsilon : N_\varepsilon^{\mathrm{reg}} \to S_\varepsilon$ which maps $\exp_x v \in N_\varepsilon$ for $x \in K$ and $v \in (T_x K)^\perp$ with $|v| < \frac{1}{2}\varepsilon$ to

$$s_\varepsilon(\exp_x v) := \left(\sigma_i^v : \left[0, \tfrac{1}{2}\varepsilon\right] \to N_\varepsilon \right)_{i=1,2},$$

where $\sigma_1^v(t) = \exp_x(((\varepsilon - 2t)/\varepsilon)v)$ and $\sigma_2^v(t) = \exp_x((2t/\varepsilon)v)$ for $t \in \left[0, \frac{1}{2}\varepsilon\right]$. This satisfies $\mathrm{ev}_0 \circ s_\varepsilon = \mathrm{id}_{N_\varepsilon}$. For $\varepsilon, \bar\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon \leq \bar\varepsilon$, let $i_{\varepsilon, \bar\varepsilon} : S_\varepsilon \to S_{\bar\varepsilon}$ denote the inclusion map.

Lemma 3.8 *There exists a constant* $C \geq C_1$ *such that, for any* $\varepsilon \in (0, \varepsilon_0/C]$, *the inclusion map* $i_{\varepsilon, C\varepsilon} : S_\varepsilon \to S_{C\varepsilon}$ *is homotopic to* $i_{\varepsilon, C\varepsilon} \circ s_\varepsilon \circ \mathrm{ev}_0$.

Proof We define a C^∞ map

$$G : \left\{ (q, q') \in N_{\varepsilon_0/C_1} \times N_{\varepsilon_0/C_1} \;\middle|\; d(q, q') < \frac{\varepsilon_0}{C_1} \right\} \times [0, 1] \to N_{\varepsilon_0}, \quad ((q, q'), s) \mapsto \widetilde{qq'}(s).$$

Then there is a constant $C \geq C_1$ such that

$$|d(G(\,\cdot\,, s))_{(q,q')}(v, v')|_g \leq \tfrac{1}{2}C(|v|_g + |v'|_g)$$

for every $s \in [0, 1]$ and $(v, v') \in T_q N_{\varepsilon_0/C} \times T_{q'} N_{\varepsilon_0/C}$ with $d(q, q') < \varepsilon_0/C_1$. For any $\varepsilon \in (0, \varepsilon_0/C]$ and $(\sigma_i : [0, \varepsilon_i] \to N_\varepsilon)_{i=1,2} \in S_\varepsilon$, we set $\bar\varepsilon := C\varepsilon$ and define $(\sigma_i^{(s)} : [0, \varepsilon_i^s] \to N_{\bar\varepsilon})_{i=1,2} \in S_{\bar\varepsilon}$ for $s \in \mathbb{R}$ as follows: Take $x \in K$ and $v \in T_x K$ such that $\sigma_1(0) = \exp_x v$ and $|v| < \frac{1}{2}\varepsilon$. Then we define

$$\varepsilon_i^s := \begin{cases} (1 - \kappa(s))\varepsilon_i + \kappa(s)\bar\varepsilon & \text{if } s \leq \frac{1}{3}, \\ \bar\varepsilon & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3}, \\ \left(1 - \kappa\left(s - \frac{2}{3}\right)\right)\bar\varepsilon + \kappa\left(s - \frac{2}{3}\right)\varepsilon & \text{if } s \leq \frac{1}{3}, \end{cases}$$

$$\sigma_i^{(s)}(t) := \begin{cases} \sigma_i(\varepsilon_i t/\varepsilon_i^s) & \text{if } s \leq \frac{1}{3}, \\ G\left(\sigma_i(\varepsilon_i t/\bar\varepsilon), \sigma_i^v(t/C), \kappa\left(s - \frac{1}{3}\right)\right) & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3}, \\ \sigma_i^v(\varepsilon t/\varepsilon_i^s) & \text{if } \frac{2}{3} \leq s. \end{cases}$$

Here, $\kappa \colon \mathbb{R} \to [0,1]$ is a C^∞ function such that $\kappa(s) = 0$ if $s \leq 0$ and $\kappa(s) = 1$ if $s \geq \frac{1}{3}$. Note that, when $\frac{1}{3} \leq s \leq \frac{2}{3}$,

$$|(\sigma_i^{(s)})'(t)|_g \leq \tfrac{1}{2} C \left(\frac{\varepsilon_i}{\bar\varepsilon} \sup |\sigma_i'|_g + \frac{1}{C} \sup |(\sigma_i^v)'|_g \right) \leq \tfrac{1}{2} C \left(\frac{\varepsilon_i}{\bar\varepsilon} + \frac{1}{C} \right) \leq 1.$$

Now the homotopy from $i_{\varepsilon,\bar\varepsilon}$ to $i_{\varepsilon,\bar\varepsilon} \circ s_\varepsilon \circ \mathrm{ev}_0$ is given by the map

$$\mathbb{R} \times S_\varepsilon \to S_{\bar\varepsilon} \colon (s, (\sigma_i)_{i=1,2}) \to (\sigma_i^{(s)}, \quad [0, \varepsilon_i^s] \mapsto N_{\bar\varepsilon})_{i=1,2}. \qquad \square$$

Proposition 3.9 *Let C be the constant of Lemma 3.8. For any $\varepsilon \in (0, \varepsilon_0/C]$ and $x \in H_*^{\mathrm{dR}}(S_\varepsilon)$,*

$$(\mathrm{ev}_0)_*(x) = 0 \in H_*^{\mathrm{dR}}(N_\varepsilon) \implies (i_{\varepsilon, C\varepsilon})_*(x) = 0 \in H_*^{\mathrm{dR}}(S_{C\varepsilon}).$$

Proof By Lemma 3.8, $(i_{\varepsilon,C\varepsilon})_*(x) = (i_{\varepsilon,C\varepsilon} \circ s_\varepsilon)_*((\mathrm{ev}_0)_*(x)) = 0$ if $(\mathrm{ev}_0)_*(x) = 0 \in H_*^{\mathrm{dR}}(N_\varepsilon)$. $\qquad \square$

3.2.4 Additional results Using the computations obtained in the former subsections, we prove several additional results. As before, b_0 is a fixed real number, and $a \in [0, b_0)$, which may belong to $\mathcal{L}(K)$.

Lemma 3.10 *There exists a manifold Z and a C^∞ function $f \colon Z \to \mathbb{R}$ such that $\mathcal{L}_1(K) \cap [0, a]$ is contained in the set of critical values of f.*

Proof We use the notation in the proof of Proposition 3.4 for $m = 1$. For every critical point $q \in Z_r \setminus Z_r^0$ of L_r, length $\gamma_{1,q} = f_r(L_r(q)) \in (0, f_r(b_0))$, where f_r is determined by $f_r(\nu \sigma_r(l)) = \nu \sqrt{l}$ for every $l \in \mathbb{R}_{>0}$. We choose $r > 0$ so that $a < f_r(b_0)$. Then, from the correspondence between binormal chords and critical points of L_r, $\mathcal{L}_m(K) \cap [0, a]$ is contained in the critical value of $f := f_r \circ L_r$ on $Z := Z_r \setminus Z_r^0$. $\qquad \square$

This proves that $\mathcal{L}(K) = \bigcup_{m=1}^\infty \mathcal{L}_m(K)$ is a null set. Indeed, for any $m \in \mathbb{Z}$, $\mathcal{L}_m(K) \cap [0, a]$ is contained in the set of critical values of the C^∞ function $Z^{\times m} \to \mathbb{R}$, $(x_1, \ldots, x_m) \mapsto f(x_1) + \cdots + f(x_m)$. By Sard's theorem, it is a null set in $\mathbb{R}$. Since b_0 was chosen arbitrarily, it follows that $\mathcal{L}_m(K)$ is a null set.

Next, recall that the definition of Σ_m^a depends on auxiliary data C_0 and ε_0.

Lemma 3.11 $H_*^{\mathrm{dR}}(\Sigma_m^a)$ *does not depend on the choice of C_0 and ε_0. More precisely, the following hold:*

- *If we write Σ_m^a as Σ_{m,C_0}^a to clarify the dependence on C_0, the inclusion map $\Sigma_{m,C_0}^a \to \Sigma_{m,C_0'}^a$ for $C_0 \leq C_0'$ induces an isomorphism on homology.*

- *If we write Σ_m^a as $\Sigma_{m,\varepsilon_0}^a$ to clarify the dependence on ε_0, the inclusion map $\Sigma_{m,\varepsilon_0'}^a \to \Sigma_{m,\varepsilon_0}^a$ for $\varepsilon_0' < \varepsilon_0$ induces an isomorphism on homology.*

Proof We define a smooth map $\Sigma_{m,C_0'}^a \to \Sigma_{m,C_0}^a$ which maps $(\gamma_k \colon [0, T_k] \to Q)_{k=1,\ldots,m}$ to

$$\left([0, C_0' T_k / C_0] \to Q, \quad t \mapsto \gamma_k(C_0 t / C_0')\right)_{k=1,\ldots,m}.$$

This gives the inverse map of the inclusion map up to homotopy. This proves the assertion for C_0.

To prove the assertion for ε_0, let us write Z_r^a by Z_{r,ε_0}^a. Then there is no critical point of $\sum_{l=0}^{\nu-1} \sigma_r \circ h_k^l$ in $Z_{r,\varepsilon_0}^a \setminus Z_{r,\varepsilon_0'}^a$ for every $k \in \{1, \ldots, m\}$. By deforming along the negative gradient vector field $-\sum_{l=0}^{\nu-1} X_{k,r}^l$ inductively on $k = 1, 2, \ldots, m$, we can see that $Z_{r,\varepsilon_0'}^a$ is a deformation retract of Z_{r,ε_0}^a. This implies that $H_*^{\mathrm{sing}}(Z_{r,\varepsilon_0}^a, Z_{r,\varepsilon_0'}^a) = 0$ and thus $H_*^{\mathrm{dR}}(\Sigma_{m,\varepsilon_0}^a, \Sigma_{m,\varepsilon_0'}^a) = 0$. $\qquad\square$

For the sake of discussions in Section 6, let us fix a topology on the set Σ_m^a for $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ as follows: $\Omega_K(Q)$ becomes a topological space such that the injection

$$\Omega_K(Q) \to C^\infty([0,1], Q) \times \mathbb{R}_{>0}, \quad (\gamma \colon [0,T] \to Q) \mapsto (\gamma(T^{-1}\cdot), T),$$

is a homeomorphism onto its image when $C^\infty([0,1], Q)$ is equipped with the C^∞-topology. We give Σ_m^a the restricted topology from $\Omega_K(Q)^{\times m}$. Then we can consider singular homology groups, such as $H_*^{\mathrm{sing}}(\Sigma_m^b, \Sigma_m^a)$ for $a \leq b$.

Suppose that $a, b \in \mathbb{R}_{\geq 0}$ and $a \leq b < b_0$. By the excision theorem, $\iota_* \colon H_*^{\mathrm{sing}}(\overline{\Sigma}_m^b, \overline{\Sigma}_m^a) \to H_*^{\mathrm{sing}}(\Sigma_m^b, \Sigma_m^a)$ is an isomorphism. All maps in Lemma 3.2 are continuous and the diagram commutes up to continuous homotopy. Therefore, we have an isomorphism

$$(11) \qquad \left(\varinjlim_{j \to \infty} (f_{2^j})_* \right) \circ (\iota_*)^{-1} \colon H_*^{\mathrm{sing}}(\Sigma_m^b, \Sigma_m^a) \to \varinjlim_{j \to \infty} H_*^{\mathrm{sing}}(B_m^b(2^j), B_m^a(2^j)).$$

3.3 Splitting and concatenating paths

For $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, we define an open subset of $\mathbb{R}^2$,

$$A_\varepsilon := \{(T, \tau) \mid T > 4\varepsilon \text{ and } 2\varepsilon < \tau < T - 2\varepsilon\}.$$

This becomes a differentiable space as a subspace of $(\mathbb{R}^2)^{\mathrm{reg}}$. For $a \in \mathbb{R}_{\geq 0}$, $m \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, m\}$, there are smooth maps

$$\mathrm{tl}_k \colon \Sigma_m^a \to \mathbb{R}, \quad (\gamma_l \colon [0, T_l] \to Q)_{l=1,\ldots,m} \mapsto T_k, \mathrm{pr}_T \colon A_\varepsilon \to \mathbb{R}^{\mathrm{reg}}, \quad (T, \tau) \mapsto T.$$

(Here tl stands for the time length.) These maps define a fiber product $\Sigma_m^a{}_{\mathrm{tl}_k}\times_{\mathrm{pr}_T} A_\varepsilon$ over $\mathbb{R}$, and the kth evaluation map ev_k is defined on it by

$$\mathrm{ev}_k \colon \Sigma_m^a{}_{\mathrm{tl}_k}\times_{\mathrm{pr}_T} A_\varepsilon \to Q \quad ((\gamma_l)_{l=1,\ldots,m}, (T, \tau)) \mapsto \gamma_k(\tau).$$

From ev_k and $\mathrm{ev}_0 \colon S_\varepsilon \to Q^{\mathrm{reg}}$, we obtain a fiber product over Q. We define a map on this fiber product

$$\mathrm{con}_k \colon (\Sigma_m^a{}_{\mathrm{tl}_k}\times_{\mathrm{pr}_T} A_\varepsilon){}_{\mathrm{ev}_k}\times_{\mathrm{ev}_0} S_\varepsilon \to \Sigma_{m+1}^{a+\varepsilon},$$

which maps $((\gamma_l)_{l=1,\ldots,m}, (T, \tau), (\sigma_i \colon [0, \varepsilon_i] \to N_\varepsilon)_{i=1,2})$ to $(\gamma_1, \ldots, \gamma_{k-1}, \widetilde{\gamma}_k^1, \widetilde{\gamma}_k^2, \gamma_{k+1}, \ldots, \gamma_m)$, where $\widetilde{\gamma}_k^i$ for $i = 1, 2$ are the paths

$$(12) \qquad \widetilde{\gamma}_k^1 \colon [0, \tau + 2\varepsilon_1] \to Q, \quad t \mapsto \begin{cases} \gamma_k(t) & \text{if } 0 \leq t \leq \tau - \varepsilon_1, \\ \gamma_k(\tau - \varepsilon_1 + \varepsilon_1\mu((t - \tau + \varepsilon_1)/\varepsilon_1)) & \text{if } \tau - \varepsilon_1 \leq t \leq \tau + \frac{1}{2}\varepsilon_1, \\ \sigma_1(\varepsilon_1 - \varepsilon_1\mu((\tau + 2\varepsilon_1 - t)/\varepsilon_1)) & \text{if } \tau + \frac{1}{2}\varepsilon_1 \leq t \leq \tau + 2\varepsilon_1, \end{cases}$$

$$\widetilde{\gamma}_k^2 \colon [0, T - \tau + 2\varepsilon_2] \to Q, \quad t \mapsto \begin{cases} \sigma_2(\varepsilon_2\mu(t/\varepsilon_2)) & \text{if } 0 \leq t \leq \frac{3}{2}\varepsilon_2, \\ \gamma_k(\tau + \varepsilon_2 - \varepsilon_2\mu((3\varepsilon_2 - t)/\varepsilon_2)) & \text{if } \frac{3}{2}\varepsilon_2 \leq t \leq 3\varepsilon_2, \\ \gamma_k(t + \tau - 2\varepsilon_2) & \text{if } 3\varepsilon_2 \leq t \leq T - \tau + 2\varepsilon_2. \end{cases}$$
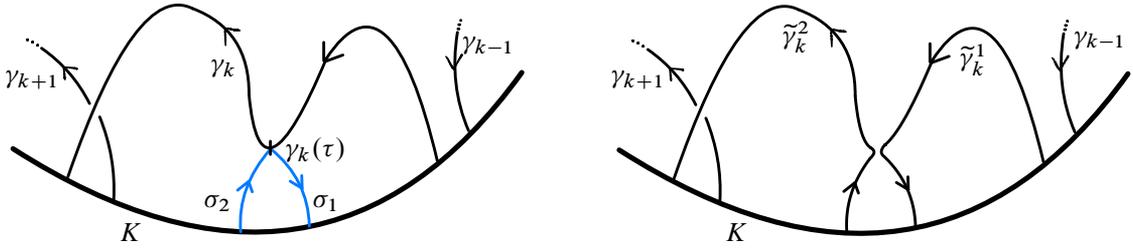
Figure 1: The process to define $\widetilde{\gamma}_k^1$ and $\widetilde{\gamma}_k^2$.

Here, $\mu \colon \left[0, \frac{3}{2}\right] \to [0, 1]$ is one of fixed data we have chosen in the beginning of Section 3.

This definition can be explained as follows (see Figure 1): We split the kth path $\gamma_k \colon [0, T] \to Q$ at $\tau \in (2\varepsilon, T - 2\varepsilon)$, where $\gamma_k(\tau) = \sigma_1(0)\, (= \sigma_2(\varepsilon_2)) \in N_\varepsilon$, and then concatenate $\gamma_k|_{[0,\tau]}$ (resp. $\gamma_k|_{[\tau,T]}$) with σ_1 (resp. σ_2). The reparametrizations via μ are necessary in order to modify them to C^∞ paths. Note that

$$\text{length } \widetilde{\gamma}_k^1 + \text{length } \widetilde{\gamma}_k^2 = \text{length } \gamma_k + \text{length } \sigma_1 + \text{length } \sigma_2 < \text{length } \gamma_k + \varepsilon.$$

The following lemma shows the cases where an element in the fiber product is mapped by con_k into Σ_{m+1}^0.

Lemma 3.12 *For* $((\gamma_l)_{l=1,\ldots,m}, (T, \tau), (\sigma_i)_{i=1,2}) \in (\Sigma_m^a \,_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T} A_\varepsilon)\, _{\mathrm{ev}_k}\!\times_{\mathrm{ev}_0} S_\varepsilon$, *we have*

$$\mathrm{con}_k((\gamma_l)_{l=1,\ldots,m}, (T, \tau), (\sigma_i)_{i=1,2}) \in \Sigma_{m+1}^0$$

if one of the following three conditions holds:

(i) $(\gamma_l)_{l=1,\ldots,m} \in \Sigma_m^0$.

(ii) $\tau < 4\varepsilon_0/(5C_0)$ *or* $T - 4\varepsilon_0/(5C_0) < \tau$.

(iii) γ_k *satisfies, for every* $\tau' \in (\gamma_k)^{-1}(N_\varepsilon)$, *that either* $\gamma_k|_{[0,\tau']}$ *or* $\gamma_k|_{[\tau',T]}$ *has length less than* $\frac{4}{5}\varepsilon_0$.

Proof As in the definition of con_k, let us write

$$\mathrm{con}_k((\gamma_l)_{l=1,\ldots,m}, (T, \tau), (\sigma_i)_{i=1,2}) = (\gamma_1, \ldots, \widetilde{\gamma}_k^1, \widetilde{\gamma}_k^2, \ldots, \gamma_m).$$

Under condition (i), length $\gamma_l < \varepsilon_0$ for some $l \in \{1, \ldots, m\}$. If $l \neq k$, the assertion is trivial. If $l = k$, either $\gamma_k|_{[0,\tau]}$ or $\gamma_k|_{[\tau,T]}$ has length less than $\frac{1}{2}\varepsilon_0$, and thus either $\widetilde{\gamma}_k^1$ or $\widetilde{\gamma}_k^2$ has length less than

$$\tfrac{1}{2}\varepsilon_0 + \max\{\text{length } \sigma_i \mid i = 1, 2\},$$

and this value is smaller than ε_0. Under condition (ii), if $\tau < 4\varepsilon_0/(5C_0)$ (resp. $T - 4\varepsilon_0/(5C_0) < \tau$), then length $\gamma_k|_{[0,\tau]} < \frac{4}{5}\varepsilon_0$ (resp. length $\gamma_k|_{[\tau,T]} < \frac{4}{5}\varepsilon_0$). Therefore, either $\widetilde{\gamma}_k^1$ or $\widetilde{\gamma}_k^2$ has length smaller than ε_0. The same result also holds under condition (iii). $\qquad\square$

3.4 Operation on de Rham chains

For $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, let us choose a C^∞ cutoff function $\rho_\varepsilon \colon A_\varepsilon \to [0, 1]$ such that

$$\rho_\varepsilon(T, \tau) = \begin{cases} 0 & \text{if } t \leq \frac{10}{3}\varepsilon \text{ or } T - \frac{10}{3}\varepsilon \leq \tau, \\ 1 & \text{if } \frac{11}{3}\varepsilon \leq \tau \leq T - \frac{11}{3}\varepsilon. \end{cases}$$

In particular, $\rho_\varepsilon(T, \tau) = 0$ if $T \leq 5\varepsilon$. We also choose a C^∞ function $\chi_\nu \colon A_\varepsilon \to [0, 1]$ for every $\nu \in \mathbb{Z}_{\geq 1}$ such that $\chi_\nu(T, \tau) = 1$ if $T \leq \nu$ and $\chi_\nu(T, \tau) = 0$ if $T \geq \nu + 1$. The support of $\chi_\nu \rho_\varepsilon$ is compact, so we obtain a de Rham chain

$$\alpha_{\varepsilon, \nu} := [A_\varepsilon, \mathrm{id}_{A_\varepsilon}, \chi_\nu \rho_\varepsilon] \in C_2^{\mathrm{dR}}(A_\varepsilon).$$

In addition, we define $\Sigma_m^a(\nu) := \bigcap_{k=1}^m (\mathrm{tl}_k)^{-1}([0, \nu))$, which is a subspace of Σ_m^a.

For $m \in \mathbb{Z}_{\geq 1}$, $k \in \{1, \dots, m\}$ and $\xi \in C_q^{\mathrm{dR}}(S_\varepsilon)$, we define a linear map

$$f_{k, \xi} \colon C_*^{\mathrm{dR}}(\Sigma_m^a) \to C_{*+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon})$$

such that, for $x \in C_*^{\mathrm{dR}}(\Sigma_m^a(\nu)) \subset C_*^{\mathrm{dR}}(\Sigma_m^a)$ with $\nu \in \mathbb{Z}_{\geq 1}$,

$$f_{k, \xi}(x) = (\mathrm{con}_k)_*((x \,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T} \alpha_{\varepsilon, \nu}) \,{}_{\mathrm{ev}_k}\!\times_{\mathrm{ev}_0} \xi).$$

This map is well defined since $x \,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T} (\alpha_{\varepsilon, \nu} - \alpha_{\varepsilon, \nu'}) = 0$ when $x \in C_*^{\mathrm{dR}}(\Sigma_m^a(\nu))$ and $\nu' \geq \nu$.

Returning to [Definition 2.8](#), we can describe the de Rham chain $f_{k, \xi}(x)$ explicitly. For $x = [U, \varphi, \omega] \in C_p^{\mathrm{dR}}(\Sigma_m^a)$, we write $\varphi(u) = (\gamma_l^u \colon [0, T_l^u] \to Q)_{l=1,\dots m}$ for $u \in U$. If we take $\nu > \sup_{u \in \mathrm{supp}\,\omega} T_k^u$, then $x \in C_p^{\mathrm{dR}}(\Sigma_m^a(\nu))$. First, we have $x \,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T} \alpha_{\varepsilon, \nu} = [\tilde{U}_k, \tilde{\varphi}_k, \tilde{\omega}_k]$, where

$$\tilde{U}_k := \{(u, \tau) \in U \times \mathbb{R} \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon\},$$
$$\tilde{\varphi}_k \colon \tilde{U}_k \to \Sigma_m^a \,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T} A_\varepsilon, \quad (u, \tau) \mapsto (\varphi(u), (T_k^u, \tau)),$$
$$\tilde{\omega}_k \in \Omega_c^*(\tilde{U}), \quad (\tilde{\omega}_k)_{(u, \tau)} := \rho_\varepsilon(T_k^u, \tau) \cdot \chi_\nu(T_k^u, \tau) \cdot \omega_u = \rho_\varepsilon(T_k^u, \tau) \cdot \omega_u.$$

Here, $\tilde{U}_k$ is oriented as an open submanifold of $U \times \mathbb{R}$. The last equality holds since $\chi_\nu(T_k^u, \tau) = 1$ for $u \in \mathrm{supp}\,\omega$. This shows the independence of $f_{k, \xi}(x)$ on the choice of χ_ν. For $\xi = [V, \psi, \eta] \in C_q^{\mathrm{dR}}(S_\varepsilon)$, we write $\psi(v) = (\sigma_i^v)_{i=1,2}$ for $v \in V$. Then $f_{k, \xi}(x) = (-1)^r [W_k, \Phi_k, \zeta_k]$, where

$$\begin{aligned}
&W_k := \{(u, \tau, v) \in U \times \mathbb{R} \times V \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon, \; \gamma_k^u(\tau) = \sigma_1^v(0)\}, \\
&\Phi_k \colon W_k \to \Sigma_{m+1}^{a+\varepsilon}, \quad (u, \tau, v) \mapsto \mathrm{con}_k(\varphi(u), (T_k^u, \tau), \psi(v)), \\
&\zeta_k \in \Omega_c^*(W_k), \quad (\zeta_k)_{(u, \tau, v)} := \rho_\varepsilon(T_k^u, \tau) \cdot (\omega_u \times \eta_v), \\
&r := (p + 1 - n)|\eta|.
\end{aligned}$$

(13)

Here W_k is oriented as a fiber product over Q of $\tilde{U}_k \to Q$, $(u, \tau) \mapsto \gamma_k^u(\tau)$, and $\mathrm{ev}_0 \circ \psi \colon V \to Q$.

Lemma 3.13 *For $x \in C_p^{\mathrm{dR}}(\Sigma_m^a)$ and $\xi \in C_q^{\mathrm{dR}}(S_\varepsilon)$,*

$$\partial \circ f_{k, \xi}(x) - f_{k, \xi} \circ \partial(x) - (-1)^{p+1-n} f_{k, \partial\xi}(x) \in C_{p+q-n}^{\mathrm{dR}}(\Sigma_m^0).$$

Proof Using the notation of [(13)](#) for $x = [U, \varphi, \omega] \in C_p^{\mathrm{dR}}(\Sigma_m^a)$ and $\xi = [V, \psi, \eta] \in C_q^{\mathrm{dR}}(S_\varepsilon)$, we have

$$\begin{aligned}
\partial \circ f_{k, \xi}(x) - f_{k, \xi} \circ \partial(x) - (-1)^{p+1-n} f_{k, \partial\xi}(x) &= (-1)^{p-1}(\mathrm{con}_k)_*((x \,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T} (\partial\alpha_{\varepsilon, \nu})) \,{}_{\mathrm{ev}_k}\!\times_{\mathrm{ev}_0} \xi) \\
&= (-1)^{(p-n)|\eta|+1}[W_k, \Phi_k, \theta_k],
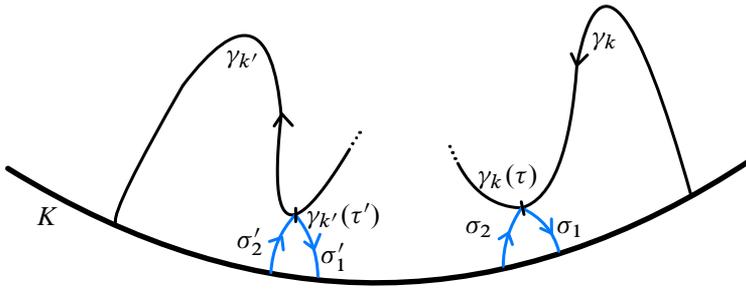\end{aligned}$$

Figure 2: The case where $(\gamma_l)_{l=1,\ldots,m}$ intersects both $(\sigma_i)_{i=1,2}$ and $(\sigma_i')_{i=1,2}$.

where $\theta_k \in \Omega_c^*(W_k)$ is defined by $(\theta_k)_{(u,\tau,v)} = \partial\rho_\varepsilon(T_k^u, \tau)/\partial\tau \cdot (\omega_u \times d\tau \times \eta_v)$. From the condition on ρ_ε, the support of θ lies in an open subset

$$\overline{W}_k := \{(u, \tau, v) \in W_k \mid \tau < 4\varepsilon \text{ or } T_k^u - 4\varepsilon < \tau\}.$$

Since $(\varphi(u), (T_k^u, \tau), \psi(v)) \in (\Sigma_m^a {}_{\mathrm{tl}_k}\times_{\mathrm{pr}_T} A_\varepsilon) {}_{\mathrm{ev}_k}\times_{\mathrm{ev}_0} S_\varepsilon$ for $(u, \tau, v) \in \overline{W}_k$ satisfies condition (ii) of Lemma 3.12, it follows that $\Phi_k(\overline{W}_k) \subset \Sigma_m^0$. Therefore,

$$[W_k, \Phi_k, \theta] = [\overline{W}_k, \Phi_k|_{\overline{W}_k}, \theta|_{\overline{W}_k}] \in C_{p+q-n}^{\mathrm{dR}}(\Sigma_m^0). \qquad \square$$

The next lemma is crucial to define chain complexes in Section 4.1. Before stating it, let us give an observation. Suppose that we have

$$(\gamma_l)_{l=1,\ldots,m} \in \Sigma_m^a, \quad (\sigma_i : [0, \varepsilon_i] \to N_\varepsilon)_{i=1,2}, \quad (\sigma_i' : [0, \varepsilon_i'] \to N_\varepsilon)_{i=1,2} \in S_\varepsilon,$$
$$k, k' \in \{1, \ldots, m\} \quad \text{with } k \leq k',$$

and there exist two points $\tau \in (2\varepsilon, T_k - 2\varepsilon)$ and $\tau' \in (2\varepsilon, T_{k'} - 2\varepsilon)$ such that

$$\gamma_k(\tau) = \sigma_1(0), \quad \gamma_{k'}(\tau') = \sigma_1'(0).$$

When $k = k'$, we additionally assume that $\tau + 2\varepsilon < \tau'$. (Figure 2 describes the situation we consider.) Then we can split γ_k at $t = \tau$ and $\gamma_{k'}$ at $t = \tau'$, and concatenate them with $(\sigma_i)_{i=1,2}$ and $(\sigma_i')_{i=1,2}$, respectively. Depending on which point we use first, there are two elements

$$\Phi := \mathrm{con}_k((\gamma_l)_{l=1,\ldots,m}, (T_k, \tau), (\sigma_i)_{i=1,2}) \in \Sigma_{m+1}^{a+\varepsilon},$$
$$\Phi' := \mathrm{con}_{k'}((\gamma_l)_{l=1,\ldots,m}, (T_{k'}, \tau'), (\sigma_i')_{i=1,2}) \in \Sigma_{m+1}^{a+\varepsilon}.$$

In either case, there remains another point which we have not yet used. When $k < k'$, we can split the $(k'+1)^{\mathrm{st}}$ path of Φ at $t = \tau'$ and the kth path of Φ' at $t = \tau$, and concatenate them with $(\sigma_i')_{i=1,2}$ and $(\sigma_i)_{i=1,2}$, respectively. When $k = k'$, we can split the $(k+1)^{\mathrm{st}}$ path of Φ at $t = \tau' - \tau + 2\varepsilon_2$ and the kth path of Φ' at $t = \tau$, and concatenate them with $(\sigma_i')_{i=1,2}$ and $(\sigma_i)_{i=1,2}$, respectively. After these two steps, we get the equations

(14)
$$\begin{cases} \mathrm{con}_{k'+1}(\Phi, (T_{k'}, \tau'), (\sigma_i')_{i=1,2}) = \mathrm{con}_k(\Phi', (T_k, \tau), (\sigma_i)_{i=1,2}) & \text{if } k < k', \\ \mathrm{con}_{k+1}(\Phi, (\widetilde{T}_k^2, \tau' - \tau + 2\varepsilon_2), (\sigma_i')_{i=1,2}) = \mathrm{con}_k(\Phi', (\widetilde{T}_k^1, \tau), (\sigma_i)_{i=1,2}) & \text{if } k = k'. \end{cases}$$

Here, $\widetilde{T}_k^2 := T_k - \tau + 2\varepsilon_2$ and $\widetilde{T}_k^1 := \tau' + 2\varepsilon_1'$. This observation leads us to the following lemma about de Rham chains.

Lemma 3.14 For $x \in C_p^{\mathrm{dR}}(\Sigma_m^a)$, $\xi \in C_q^{\mathrm{dR}}(S_\varepsilon)$ and $k, k' \in \{1, \ldots, m\}$ with $k \leq k'$, the following hold:

$$\begin{cases} f_{k'+1,\xi} \circ f_{k,\xi}(x) + (-1)^{q-n} f_{k,\xi} \circ f_{k',\xi}(x) = 0 & \text{if } k < k', \\ f_{k'+1,\xi} \circ f_{k,\xi}(x) + (-1)^{q-n} f_{k,\xi} \circ f_{k',\xi}(x) \in C_{p+2+2q-2n}^{\mathrm{dR}}(\Sigma_{m+2}^0) & \text{if } k = k'. \end{cases}$$

Proof We use the notation of (13) for $x = [U, \varphi, \omega]$ and $\xi = [V, \psi, \eta]$. For short, let us abbreviate for $(T, \tau, T', \tau') \in A_\varepsilon \times A_\varepsilon$

$$\rho(T, \tau, T', \tau) := \rho_\varepsilon(T, \tau) \cdot \rho_\varepsilon(T', \tau').$$

Case 1 We consider the case $k < k'$. We have $f_{k'+1,\xi} \circ f_{k,\xi}(x) = (-1)^s [W_{k,k'}, \Phi_{k,k'}, \zeta_{k,k'}]$ for

$$W_{k,k'} := \{(u, \tau, v, \tau', v') \mid (u, \tau, v) \in W_k, \, 2\varepsilon < \tau' < T_{k'}^u - 2\varepsilon, \, \gamma_{k'}^u(\tau') = \sigma_1^{v'}(0)\},$$

$$\Phi_{k,k'} \colon W_{k,k'} \to \Sigma_{m+2}^{a+2\varepsilon}, \quad (u, \tau, v, \tau', v') \mapsto \mathrm{con}_{k'+1}(\Phi_k(u, \tau, v), (T_{k'}^u, \tau'), \psi(v')),$$

$$\zeta_{k,k'} \in \Omega_c^*(W_{k,k'}), \quad (\zeta_{k,k'})_{(u,\tau,v,\tau',v')} := \rho(T_k^u, \tau, T_{k'}^u, \tau') \cdot (\omega_u \times \eta_v \times \eta_{v'}),$$

$$s := (q + 1 - n)|\eta|,$$

by substituting k and $[U, \varphi, \omega]$ in (13) with k' and $[W_k, \Phi_k, (-1)^r \zeta_k]$.

Similarly, $f_k \circ f_{k'}(x) = (-1)^s [W'_{k,k'}, \Phi'_{k,k'}, \zeta'_{k,k'}]$ for

$$W'_{k,k'} := \{(u, \tau', v', \tau, v) \mid (u, \tau', v') \in W_{k'}, \, 2\varepsilon < \tau < T_k^u - 2\varepsilon, \, \gamma_k^u(\tau) = \sigma_1^v(0)\},$$

$$\Phi'_{k,k'} \colon W'_{k,k'} \to \Sigma_{m+2}^{a+2\varepsilon}, \quad (u, \tau', v', \tau, v) \mapsto \mathrm{con}_k(\Phi_{k'}(u, \tau', v'), (T_k^u, \tau), \psi(v)),$$

$$\zeta'_{k,k'} \in \Omega_c^*(W'_{k,k'}), \quad (\zeta'_{k,k'})_{(u,\tau',v'\tau,v)} = \rho(T_{k'}^u, \tau', T_k^u, \tau) \cdot (\omega_u \times \eta_{v'} \times \eta_v).$$

We define a diffeomorphism

$$h \colon W_{k,k'} \to W'_{k,k'}, \quad (u, \tau, v, \tau', v') \mapsto (u, \tau', v', \tau, v),$$

which changes the sign of orientation by $(-1)^{(1+\dim V - n)^2}$. From (14), it follows that $\Phi'_{k,k'} \circ h = \Phi_{k,k'}$. Moreover, $h_*(\zeta_{k,k'}) = (-1)^{|\eta|^2} \zeta'_{k,k'}$. Combining these computations,

$$f_{k'+1,\xi} \circ f_{k,\xi}(x) = (-1)^{s+(1+\dim V - n + |\eta|)} [W'_{k,k'}, \Phi'_{k,k'}, \zeta'_{k,k'}] = (-1)^{q-n+1} f_{k,\xi} \circ f_{k',\xi}(x).$$

Case 2 We consider the case $k = k'$. We have $f_{k+1} \circ f_k(x) = (-1)^s [W_{k,k}, \Phi_{k,k}, \zeta_{k,k}]$ for

$$W_{k,k} := \{(u, \tau, v, \tau', v') \mid (u, \tau, v) \in W_k, \, 2\varepsilon < \tau' < \widetilde{T}_k^2(u, \tau, v) - 2\varepsilon, \, \gamma_k^u(\tau' + \tau - 2\varepsilon_2^v) = \sigma_2^{v'}(0)\},$$

$$\Phi_{k,k} \colon W_{k,k} \to \Sigma_{m+2}^{a+2\varepsilon}, \quad (u, \tau, v, \tau', v') \mapsto \mathrm{con}_{k+1}(\Phi_k(u, \tau, v), (\widetilde{T}_k^2(u, \tau, v), \tau'), \psi(v')),$$

$$\zeta_{k,k} \in \Omega_c^*(W_{k,k}), \quad (\zeta_{k,k})_{(u,\tau,v,\tau',v')} = \rho(T_k^u, \tau, \widetilde{T}_k^2(u, \tau, v), \tau') \cdot (\omega_u \times \eta_v \times \eta_{v'}),$$

where $\widetilde{T}_k^2(u, \tau, v) := T_k^u - \tau + 2\varepsilon_2^v$.

Similarly, $f_{k,\xi} \circ f_{k,\xi}(x) = (-1)^s [W'_{k,k}, \Phi'_{k,k}, \zeta'_{k,k}]$ for

$$W'_{k,k} := \{(u, \tau', v', \tau, v) \mid (u, \tau', v') \in W_k, 2\varepsilon < \tau < \widetilde{T}^1_k(\tau', v') - 2\varepsilon, \gamma^u_k(\tau) = \sigma^v_1(0)\},$$

$$\Phi'_{k,k} \colon W'_{k,k} \to \Sigma^{a+2\varepsilon}_{m+2}, \quad (u, \tau', v', \tau, v) \mapsto \mathrm{con}_k\big(\Phi_k(u, \tau', v'), (\widetilde{T}^1_k(\tau', v'), \tau), \psi(v)\big),$$

$$\zeta'_{k,k} \in \Omega^*_c(W'_{k,k}), \quad (\zeta'_{k,k})_{(u,\tau',v',\tau,v)} = \rho(T^u_k, \tau', \widetilde{T}^1_k(\tau', v'), \tau) \cdot (\omega_u \times \eta_{v'} \times \eta_v),$$

where $\widetilde{T}^1_k(\tau', v') := \tau' + 2\varepsilon^{v'}_1$. Since $\rho(T_k, \tau, \widetilde{T}^2_k(u, \tau, v), \tau') = 0$ for $\tau' \leq \frac{10}{3}\varepsilon$, we have $f_{k+1} \circ f_k(x) = (-1)^s [\overline{W}_{k,k}, \Phi_{k,k}, \zeta_{k,k}]$ for $\overline{W}_{k,k} := W_{k,k} \cap \{\tau' > 3\varepsilon\}$.

This time, we define a map

$$h \colon \overline{W}_{k,k} \to W'_{k,k}, \quad (u, \tau, v, \tau', v') \mapsto (u, \tau' + \tau - 2\varepsilon^v_2, v', \tau, v),$$

which is an open embedding and changes the sign of the orientation by $(-1)^{(1+\dim V - n)^2}$. From (14), it follows that $\Phi'_{k,k} \circ h = \Phi_{k,k}$. Indeed, if we set $\tau'_* := \tau' + \tau - 2\varepsilon^v_2$, then

$$\begin{aligned}
\Phi'_{k,k} \circ h(u, \tau, v, \tau', v') &= \mathrm{con}_k\big(\Phi_k(u, \tau'_*, v'), (\widetilde{T}^1_k(\tau'_*, v'), \tau), \psi(v)\big) \\
&= \mathrm{con}_{k+1}\big(\Phi_k(u, \tau, v), (\widetilde{T}^1_k(u, \tau, v), \tau'_* - \tau + 2\varepsilon^v_2), \psi(v')\big) \\
&= \Phi_{k,k}(u, \tau, v, \tau', v')
\end{aligned}$$

for every $(u, \tau, v, \tau', v') \in \overline{W}_{k,k}$. Therefore,

$$(-1)^{q-n+1} f_{k+1,\xi} \circ f_{k,\xi}(x) - f_{k,\xi} \circ f_{k,\xi}(x) = (-1)^s [W'_{k,k}, \Phi'_{k,k}, (-1)^{|\eta|^2} h_*(\zeta_{k,k}) - \zeta'_{k,k}].$$

For $(u, \tau', v', \tau, v) \in W'_{k,k}$,

$$\begin{aligned}
&(-1)^{|\eta|^2} (h_*(\zeta_{k,k}))_{(u,\tau',v',\tau,v)} - (\zeta'_{k,k})_{(u,\tau',v',\tau,v)} \\
&\quad = \big(\rho(T^u_k, \tau, \widetilde{T}^2_k(u, \tau, v), \tau' - \tau + 2\varepsilon^v_2) - \rho(T^u_k, \tau', \widetilde{T}^1_k(\tau', v'), \tau)\big) \cdot (\omega_u \times \eta_{v'} \times \eta_v).
\end{aligned}$$

If $\frac{11}{3}\varepsilon \leq \tau' \leq T^u_k - \frac{11}{3}\varepsilon$ and $\frac{11}{3}\varepsilon \leq \tau \leq \widetilde{T}^1_k(\tau', v') - \frac{11}{3}\varepsilon$, it can be checked that $\frac{11}{3}\varepsilon \leq \tau' - \tau + 2\varepsilon^v_2 \leq \widetilde{T}^2_k(u, \tau, v) - \frac{11}{3}\varepsilon$ and $\frac{11}{3} \leq \tau \leq T^u_k - \frac{11}{3}\varepsilon$ hold, and thus

$$\rho(T^u_k, \tau, \widetilde{T}^2_k(u, \tau, v), \tau' - \tau + 2\varepsilon^v_2) = 1 = \rho(T^u_k, \tau', \widetilde{T}^1_k(\tau', v'), \tau).$$

Therefore, $\mathrm{supp}((-1)^{|\eta|^2} h_*(\zeta_{k,k}) - \zeta'_{k,k})$ lies in $W^1_{k,k} \cup W^2_{k,k}$, where

$$W^1_{k,k} := \{(u, \tau', v', \tau, v) \in W'_{k,k} \mid \tau' < 4\varepsilon \text{ or } T^u_k - 4\varepsilon < \tau'\},$$

$$W^2_{k,k} := \{(u, \tau', v', \tau, v) \in W'_{k,k} \mid \tau < 4\varepsilon \text{ or } \widetilde{T}^1_k(\tau', v') - 4\varepsilon < \tau\}.$$

From Lemma 3.12, $\Phi_k(u, \tau', v') = \mathrm{con}_k(\varphi(u), (T^u_k, \tau'), \psi(v')) \in \Sigma^0_{m+1}$ for all $(u, \tau', v', \tau, v) \in W^1_{k,k}$. Then Lemma 3.12 is applied again to show that

$$\Phi'_{k,k}(u, \tau', v', \tau, v) = \mathrm{con}_k\big(\Phi_k(u, \tau', v'), (\widetilde{T}^1_k(\tau', v'), \tau), \psi(v)\big)$$

is an element of Σ_{m+2}^0 for every $(u, \tau', v', \tau, v) \in W_{k,k}^1 \cup W_{k,k}^2$. Indeed, we can apply case (i) of Lemma 3.12 for $(u, \tau', v', \tau, v) \in W_{k,k}^1$, and case (ii) for $(u, \tau', v', \tau, v) \in W_{k,k}^2$. As a consequence,

$$(-1)^{q-n+1} f_{k+1,\xi} \circ f_{k,\xi}(x) - f_{k,\xi} \circ f_{k,\xi}(x)$$
$$= [W_{k,k}^1 \cup W_{k,k}^2, \Phi'_{k,k}|_{W_{k,k}^1 \cup W_{k,k}^2}, ((-1)^{|\eta|^2} h_*(\zeta_{k,k}) - \zeta'_{k,k})|_{W_{k,k}^1 \cup W_{k,k}^2}] \in C_{p+2+2q-2n}^{\mathrm{dR}}(\Sigma_{m+2}^0). \qquad \square$$

In the definition of $f_{k,\xi}$ for $\xi \in C_q^{\mathrm{dR}}(S_\varepsilon)$, there is an ambiguity about the choice of $\rho_\varepsilon \colon A_\varepsilon \to [0, 1]$. Suppose that we choose another ρ'_ε, and define $\alpha'_{\varepsilon,v}$ and $f'_{k,\xi}$ in the same way as $\alpha_{\varepsilon,v}$ and $f_{k,\xi}$. Take an arbitrary chain $x \in C_p^{\mathrm{dR}}(\Sigma_m^a)$. Since $\mathrm{supp}(\rho'_\varepsilon - \rho_\varepsilon)$ lies in $\{(T, \tau) \in A_\varepsilon \mid \tau < 4\varepsilon \text{ or } T - 4\varepsilon < \tau\}$, $x \,_{\mathrm{tl}_k}\times_{\mathrm{pr}_T} (\alpha_{\varepsilon,v} - \alpha'_{\varepsilon,v})$ is a chain in the subspace

$$\{((\gamma_k)_{k=1,\ldots,m}, (T, \tau)) \in \Sigma_m^a \,_{\mathrm{tl}_k}\times_{\mathrm{pr}_T} A_\varepsilon \mid \tau < 4\varepsilon \text{ or } T - 4\varepsilon < \tau\}.$$

From Lemma 3.12, we have

$$f_{k,\xi}(x) - f'_{k,\xi}(x) = (\mathrm{con}_k)_* \big((x \,_{\mathrm{tl}_k}\times_{\mathrm{pr}_T} (\alpha_{\varepsilon,v} - \alpha'_{\varepsilon,v})) \,_{\mathrm{ev}_k}\times_{\mathrm{ev}_0} \xi\big) \in C_{p+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^0).$$

Therefore, for $a \in \mathbb{R}_{\geq 0}$, the induced map on the quotient spaces, which is denoted by the same symbol

$$f_{k,\xi} \colon C_*^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0) \to C_{*+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0),$$

is independent of the choice of ρ_ε. For this map, the equation

$$(15) \qquad \partial \circ f_{k,\xi} - f_{k,\xi} \circ \partial = (-1)^{p+1-n} f_{k,\partial\xi} \colon C_p^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0) \to C_{p+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0)$$

follows from Lemma 3.13, and the equation for $k' \geq k$,

$$(16) \qquad f_{k'+1,\xi} \circ f_{k,\xi} + (-1)^{q-n} f_{k,\xi} \circ f_{k',\xi} = 0 \colon C_p^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0) \to C_{p+2+2q-2n}^{\mathrm{dR}}(\Sigma_{m+2}^{a+2\varepsilon}, \Sigma_{m+2}^0),$$

follows from Lemma 3.14. When $\partial \xi = 0$, (15) implies that $f_{k,\xi}$ is a chain map shifting the degree by $1 + q - n$.

3.5 $[-1, 1]$- and $[-1, 1]^2$-modeled de Rham chains

In this section, we introduce two types of variants of de Rham chains. In this paper, they are necessary for only four kinds of differentiable spaces: Σ_m^a, $\Sigma_m^a \times \Sigma_{m'}^{a'}$, S_ε and (M, P_M) for a manifold M. Throughout this section, X stands for one of these differentiable spaces.

3.5.1 $[-1, 1]$-modeled de Rham chains We introduce chains in $\mathbb{R} \times X$. We define $\overline{P}_X$ as the set of tuples $(U, \varphi, (\tau_+, \tau_-))$ such that:

- $(U, \varphi) \in P_{\mathbb{R}^{\mathrm{reg}} \times X}$. If $X = S_\varepsilon$, we additionally require that $(\mathrm{id}_{\mathbb{R}} \times \mathrm{ev}_0) \circ \varphi \colon U \to \mathbb{R} \times Q$ is a submersion. Let us write $\varphi = (\varphi_{\mathbb{R}}, \varphi_X) \colon U \to \mathbb{R} \times X$ and $U_I := \varphi_{\mathbb{R}}^{-1}(I)$ for any subset $I \subset \mathbb{R}$.

- $\tau_+ \colon U_{\mathbb{R}_{\geq 1}} \to \mathbb{R}_{\geq 1} \times U_{\{1\}}$ and $\tau_- \colon U_{\mathbb{R}_{\leq -1}} \to \mathbb{R}_{\leq -1} \times U_{\{-1\}}$ are diffeomorphisms such that

$$\varphi \circ \tau_+^{-1} = i_{\mathbb{R}_{\geq 1}} \times \varphi_X|_{U_{\{1\}}}, \quad \varphi \circ \tau_-^{-1} = i_{\mathbb{R}_{\leq -1}} \times \varphi_X|_{U_{\{-1\}}}.$$

Here, $i_{\mathbb{R}_{\geq 1}}$ (resp. $i_{\mathbb{R}_{\leq -1}}$) is the inclusion map from $\mathbb{R}_{\geq 1}$ (resp. $\mathbb{R}_{\leq -1}$) to $\mathbb{R}$.

Remark 3.15 When $X = S_\varepsilon$, the condition that $(U, \varphi) \in P_{\mathbb{R}^{\mathrm{reg}} \times S_\varepsilon}$ implies only that the composition of $(\mathrm{id}_{\mathbb{R}} \times \mathrm{ev}_0) \circ \varphi \colon U \to \mathbb{R} \times Q$ with $\mathrm{pr}_{\mathbb{R}}$ (resp. pr_Q) is a submersion to $\mathbb{R}$ (resp. Q). The condition that $(\mathrm{id}_{\mathbb{R}} \times \mathrm{ev}_0) \circ \varphi$ itself is a submersion is necessary to define a fiber product of $[-1, 1]$-modeled de Rham chains in the latter subsection.

For $(U, \varphi, (\tau_+, \tau_-)) \in \overline{P}_X$, we define a linear subspace $\Omega_c^p(U, \varphi, (\tau_+, \tau_-))$ of $\Omega^p(U)$ which consists of p-forms ω on U such that $\operatorname{supp} \omega \cap U_{[-1,1]}$ is compact, $(\tau_+^{-1})^* \omega = 1 \times \omega|_{U_{\{1\}}}$ and $(\tau_-^{-1})^* \omega = 1 \times \omega|_{U_{\{-1\}}}$. We consider the graded $\mathbb{R}$-vector space

$$\overline{A}_*(X) := \bigoplus_{(U, \varphi, (\tau_+, \tau_-)) \in \overline{P}_X} \Omega_c^{\dim U - 1 - *}(U, \varphi, (\tau_+, \tau_-)).$$

For $\boldsymbol{U} = (U, \varphi, (\tau_+, \tau_-)) \in \overline{P}_X$ and $\omega \in \Omega_c^p(U, \varphi, (\tau_+, \tau_-))$, let $(U, \varphi, (\tau_+, \tau_-), \omega)$ denote the element of $\overline{A}_*(X)$ whose component for $\boldsymbol{V} \in \overline{P}_X$ is

$$(U, \varphi, (\tau_+, \tau_-), \omega)_{\boldsymbol{V}} = \begin{cases} \omega & \text{if } \boldsymbol{V} = \boldsymbol{U}, \\ 0 & \text{if } \boldsymbol{V} \neq \boldsymbol{U}. \end{cases}$$

We take the linear subspace $\overline{Z}_*(X)$ of $\overline{A}_*(X)$ generated by vectors

$$(V, \varphi \circ \pi, (\sigma_+, \sigma_-), \omega) - (U, \varphi, (\tau_+, \tau_-), \pi_! \omega)$$

for any submersion $\pi \colon V \to U$ such that

$$(\mathrm{id}_{\mathbb{R}_{\geq 1}} \times \pi|_{V_{\{1\}}}) \circ \sigma_+ = \tau_+ \circ \pi, \quad (\mathrm{id}_{\mathbb{R}_{\leq -1}} \times \pi|_{V_{\{-1\}}}) \circ \sigma_- = \tau_- \circ \pi.$$

We define the quotient vector space

$$\overline{C}_*^{\mathrm{dR}}(X) := \overline{A}_*(X) / \overline{Z}_*(X),$$

whose elements we call $[-1, 1]$-*modeled de Rham chains*. $[U, \varphi, (\tau_+, \tau_-), \omega]$ denotes the equivalence class of $(U, \varphi, (\tau_+, \tau_-), \omega)$. We define a degree -1 linear map $\partial \colon \overline{C}_*^{\mathrm{dR}}(X) \to \overline{C}_{*-1}^{\mathrm{dR}}(X)$ by

$$\partial[U, \varphi, (\tau_+, \tau_-), \omega] := (-1)^{|\omega|+1} [U, \varphi, (\tau_+, \tau_-), d\omega].$$

Obviously $\partial \circ \partial = 0$, and we obtain a chain complex $(\overline{C}_*^{\mathrm{dR}}(X), \partial)$. Its homology is denoted by $\overline{H}_*^{\mathrm{dR}}(X)$.

Naturally, there are three chain maps

(17) $\quad \overline{\iota} \colon C_*^{\mathrm{dR}}(X) \to \overline{C}_*^{\mathrm{dR}}(X), \quad [V, \psi, \omega] \mapsto (-1)^{\dim V} [\mathbb{R} \times V, \mathrm{id}_{\mathbb{R}} \times \psi, (\mathrm{id}_{\mathbb{R}_{\geq 1} \times V}, \mathrm{id}_{\mathbb{R}_{\leq -1} \times V}), 1 \times \omega],$

and

(18) $\quad \begin{aligned} e_+ &\colon \overline{C}_*^{\mathrm{dR}}(X) \to C_*^{\mathrm{dR}}(X), \quad [U, \varphi, (\tau_+, \tau_-), \omega] \mapsto (-1)^{\dim U - 1} [U_{\{1\}}, \varphi_X|_{U_{\{1\}}}, \omega|_{U_{\{1\}}}], \\ e_- &\colon \overline{C}_*^{\mathrm{dR}}(X) \to C_*^{\mathrm{dR}}(X), \quad [U, \varphi, (\tau_+, \tau_-), \omega] \mapsto (-1)^{\dim U - 1} [U_{\{-1\}}, \varphi_X|_{U_{\{-1\}}}, \omega|_{U_{\{-1\}}}]. \end{aligned}$

Here, $U_{\{1\}}$ and $U_{\{-1\}}$ are oriented so that τ_+ and τ_- preserve orientations. Clearly, $e_+ \circ \overline{\iota} = e_- \circ \overline{\iota} = \mathrm{id}_{C_*^{\mathrm{dR}}(X)}$. For $\overline{\iota} \circ e_+$ and $\overline{\iota} \circ e_-$, the next result holds.

Lemma 3.16 $\bar{\iota} \circ e_+$ and $\bar{\iota} \circ e_-$ are chain homotopic to the identity map $\mathrm{id}_{\overline{C}^{\mathrm{dR}}_*(X)}$.

Proof This assertion is essentially proved in [16, Lemma 4.8]. We should note that the result in the reference is proved for a specific differentiable space $\mathscr{L}_{k+1}(a)$ (a differentiable space of marked Moore loops in a manifold). However, even for a differentiable space X considered in this section, we can extend the definition of a chain

$$K\big([U, \varphi, (\tau_+, \tau_-), \omega]\big) := (-1)^{|\omega|+1}[\mathbb{R} \times U, \bar{\varphi}, (\bar{\tau}_+, \bar{\tau}_-), \bar{\omega}] \in \overline{C}^{\mathrm{dR}}_{*+1}(X)$$

for any $[U, \varphi, (\tau_+, \tau_-), \omega] \in \overline{C}^{\mathrm{dR}}_*(X)$, which appears in the proof of [16, Lemma 4.8]. Then $K \colon \overline{C}^{\mathrm{dR}}_*(X) \to \overline{C}^{\mathrm{dR}}_{*+1}(X)$ gives a chain homotopy from $\mathrm{id}_{\overline{C}^{\mathrm{dR}}_*(X)}$ to $\bar{\iota} \circ e_+$. The proof for $\bar{\iota} \circ e_-$ is completely parallel. $\square$

3.5.2 $[-1, 1]^2$-modeled de Rham chains

Let us define the smooth map

$$\iota \colon (\mathbb{R}^2)^{\mathrm{reg}} \times X \to (\mathbb{R}^2)^{\mathrm{reg}} \times X, \quad ((r_1, r_2), x) \mapsto ((r_2, r_1), x).$$

We often use the coordinate (r_1, r_2) of $\mathbb{R}^2$ to denote its subsets, for instance $\mathbb{R}_{\geq 1} \times \mathbb{R} = \{r_1 \geq 1\}$.

We introduce chains in $\mathbb{R}^2 \times X$. We define $\widehat{P}_X$ as the set of tuples $(U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-))$ such that:

- $(U, \varphi) \in P_{(\mathbb{R}^2)^{\mathrm{reg}} \times X}$. If $X = S_\varepsilon$, we additionally require that $(\mathrm{id}_{\mathbb{R}^2} \times \mathrm{ev}_0) \circ \varphi \colon U \to \mathbb{R}^2 \times Q$ is a submersion. Let us write $\varphi = ((\varphi^1_{\mathbb{R}}, \varphi^2_{\mathbb{R}}), \varphi_X) \colon U \to \mathbb{R}^2 \times X$ and $U_D := \{u \in U \mid (\varphi^1_{\mathbb{R}}(u), \varphi^2_{\mathbb{R}}(u)) \in D\}$ for any subset $D \subset \mathbb{R}^2$.

- τ^j_+ and τ^j_- for $j = 1, 2$ are diffeomorphisms such that

$$\varphi \circ (\tau^1_+)^{-1} = i_{\mathbb{R}_{\geq 1}} \times (\varphi^2_{\mathbb{R}} \times \varphi_X)|_{U_{\{r_1=1\}}}, \qquad \varphi \circ (\tau^1_-)^{-1} = i_{\mathbb{R}_{\leq -1}} \times (\varphi^2_{\mathbb{R}} \times \varphi_X)|_{U_{\{r_1=-1\}}},$$
$$\iota \circ \varphi \circ (\tau^2_+)^{-1} = i_{\mathbb{R}_{\geq 1}} \times (\varphi^1_{\mathbb{R}} \times \varphi_X)|_{U_{\{r_2=1\}}}, \quad \iota \circ \varphi \circ (\tau^2_-)^{-1} = i_{\mathbb{R}_{\leq -1}} \times (\varphi^1_{\mathbb{R}} \times \varphi_X)|_{U_{\{r_2=-1\}}}.$$

For $(U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-)) \in \widehat{P}_X$, we define the linear subspace $\Omega^p_c(U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-))$ of $\Omega^p(U)$ which consists of p-forms ω on U such that $\mathrm{supp}\,\omega \cap U_{[-1,1] \times [-1,1]}$ is compact and

$$((\tau^j_+)^{-1})^*\omega = 1 \times \omega|_{U_{\{r_j=1\}}}, \quad ((\tau^j_-)^{-1})^*\omega = 1 \times \omega|_{U_{\{r_j=-1\}}} \quad \text{for } j = 1, 2.$$

We consider the graded $\mathbb{R}$-vector space

$$\widehat{A}_*(X) := \bigoplus_{(U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-)) \in \widehat{P}_X} \Omega^{\dim U - 2 - *}_c(U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-)).$$

For $\boldsymbol{U} = (U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-)) \in \widehat{P}_X$ and $\omega \in \Omega^p_c(U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-))$, let

$$(U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega)$$

denote the element of $\widehat{A}_*(X)$ whose component for $\boldsymbol{V} \in \widehat{P}_X$ is

$$(U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega)_{\boldsymbol{V}} = \begin{cases} \omega & \text{if } \boldsymbol{V} = \boldsymbol{U}, \\ 0 & \text{if } \boldsymbol{V} \neq \boldsymbol{U}. \end{cases}$$

We take the linear subspace $\hat{Z}_*(X)$ of $\hat{A}_*(X)$ generated by vectors

$$(V, \varphi \circ \pi, (\sigma_+^1, \sigma_-^1), (\sigma_+^2, \sigma_-^2), \omega) - (U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \pi_! \omega)$$

for any submersion $\pi \colon V \to U$ such that, for $j = 1, 2$,

$$(\mathrm{id}_{\mathbb{R}_{\geq 1}} \times \pi|_{V_{\{r_j = 1\}}}) \circ \sigma_+^j = \tau_+^j \circ \pi, \quad (\mathrm{id}_{\mathbb{R}_{\leq -1}} \times \pi|_{V_{\{r_j = -1\}}}) \circ \sigma_-^j = \tau_-^j \circ \pi.$$

Now we define the quotient vector space

$$\hat{C}_*^{\mathrm{dR}}(X) := \hat{A}_*(X) / \hat{Z}_*(X),$$

whose elements we call $[-1, 1]^2$-*modeled de Rham chains*. $[U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega]$ denotes the equivalence class of $(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega)$. We define a degree -1 linear map $\partial \colon \hat{C}_*^{\mathrm{dR}}(X) \to \hat{C}_{*-1}^{\mathrm{dR}}(X)$ by

$$\partial[U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] := (-1)^{|\omega|+1}[U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), d\omega].$$

Obviously $\partial \circ \partial = 0$, and we obtain a chain complex $(\hat{C}_*^{\mathrm{dR}}(X), \partial)$. Its homology is denoted by $\hat{H}_*^{\mathrm{dR}}(X)$.

Naturally, there are six chain maps

(19) $$\hat{\imath}^1, \hat{\imath}^2 \colon \overline{C}_*^{\mathrm{dR}}(X) \to \hat{C}_*^{\mathrm{dR}}(X), \quad e_+^1, e_+^2, e_-^1, e_-^2 \colon \hat{C}_*^{\mathrm{dR}}(X) \to \overline{C}_*^{\mathrm{dR}}(X),$$

defined as follows: $\hat{\imath}^1$ and $\hat{\imath}^2$ map $x = [V, \psi, (\tau_+, \tau_-), \omega] \in \overline{C}_*^{\mathrm{dR}}(X)$ to

$$\hat{\imath}^1 x := (-1)^{\dim V - 1}[\mathbb{R} \times V, \mathrm{id}_{\mathbb{R}} \times \psi, (\mathrm{id}_{\mathbb{R}_{\geq 1} \times V}, \mathrm{id}_{\mathbb{R}_{\leq -1} \times V}), (\hat{\tau}_+, \hat{\tau}_-), 1 \times \omega],$$

$$\hat{\imath}^2 x := (-1)^{\dim V}[\mathbb{R} \times V, \iota \circ (\mathrm{id}_{\mathbb{R}} \times \psi), (\hat{\tau}_+, \hat{\tau}_-), (\mathrm{id}_{\mathbb{R}_{\geq 1} \times V}, \mathrm{id}_{\mathbb{R}_{\leq -1} \times V}), 1 \times \omega],$$

where $\hat{\tau}_+ \colon \mathbb{R} \times V_{\mathbb{R}_{\geq 1}} \to \mathbb{R}_{\geq 1} \times (\mathbb{R} \times V_{\{1\}})$ and $\hat{\tau}_- \colon \mathbb{R} \times V_{\mathbb{R}_{\leq -1}} \to \mathbb{R}_{\leq -1} \times (\mathbb{R} \times V_{\{-1\}})$ are determined by

$$\hat{\tau}_+(r', \tau_+^{-1}(r, u_+)) = (r, (r', u_+)) \quad \text{for } r' \in \mathbb{R} \text{ and } (r, u_+) \in \mathbb{R}_{\geq 1} \times V_{\{1\}},$$

$$\hat{\tau}_-(r', \tau_-^{-1}(r, u_-)) = (r, (r', u_-)) \quad \text{for } r' \in \mathbb{R} \text{ and } (r, u_-) \in \mathbb{R}_{\leq -1} \times V_{\{-1\}}.$$

In addition, e_+^j and e_-^j for $j = 1, 2$ map $y = [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] \in \hat{C}_*^{\mathrm{dR}}(X)$ to

$$e_+^1 y := (-1)^{\dim U}[U_{\{1\} \times \mathbb{R}}, (\varphi_{\mathbb{R}}^2, \varphi_X)|_{U_{\{1\} \times \mathbb{R}}}, (\tau_+^2|_{U_{\{1\} \times \mathbb{R}_{\geq 1}}}, \tau_-^2|_{U_{\{1\} \times \mathbb{R}_{\leq -1}}}), \omega|_{U_{\{1\} \times \mathbb{R}}}],$$

$$e_+^2 y := (-1)^{\dim U - 1}[U_{\mathbb{R} \times \{1\}}, (\varphi_{\mathbb{R}}^1, \varphi_X)|_{U_{\mathbb{R} \times \{1\}}}, (\tau_+^1|_{U_{\mathbb{R}_{\geq 1} \times \{1\}}}, \tau_-^1|_{U_{\mathbb{R}_{\leq -1} \times \{1\}}}), \omega|_{U_{\mathbb{R} \times \{1\}}}],$$

$$e_-^1 y := (-1)^{\dim U}[U_{\{-1\} \times \mathbb{R}}, (\varphi_{\mathbb{R}}^2, \varphi_X)|_{U_{\{-1\} \times \mathbb{R}}}, (\tau_+^2|_{U_{\{-1\} \times \mathbb{R}_{\geq 1}}}, \tau_-^2|_{U_{\{-1\} \times \mathbb{R}_{\leq -1}}}), \omega|_{U_{\{-1\} \times \mathbb{R}}}],$$

$$e_-^2 y := (-1)^{\dim U - 1}[U_{\mathbb{R} \times \{-1\}}, (\varphi_{\mathbb{R}}^1, \varphi_X)|_{U_{\mathbb{R} \times \{-1\}}}, (\tau_+^1|_{U_{\mathbb{R}_{\geq 1} \times \{-1\}}}, \tau_-^1|_{U_{\mathbb{R}_{\leq -1} \times \{-1\}}}), \omega|_{U_{\mathbb{R} \times \{-1\}}}].$$

Here, the orientations of U_D for $D = \{r_j = 1\}, \{r_j = -1\}$ for $j = 1, 2$ are determined so that τ_+^j and τ_-^j preserve orientations. The signs are chosen so that

(20) $$e_+ \circ e_+^1 = e_+ \circ e_+^2, \quad e_- \circ e_+^1 = e_+ \circ e_-^2, \quad e_+ \circ e_-^1 = e_- \circ e_+^2, \quad e_- \circ e_-^1 = e_- \circ e_-^2.$$

Clearly, $e_+^j \circ \hat{\imath}^j = e_-^j \circ \hat{\imath}^j = \mathrm{id}_{\overline{C}_*^{\mathrm{dR}}(X)}$ for $j = 1, 2$. For $\hat{\imath}^j \circ e_+^j$ and $\hat{\imath}^j \circ e_-^j$ for $j = 1, 2$, the next result holds.

Lemma 3.17 $\hat{\imath}^j \circ e_+^j$ and $\hat{\imath}^j \circ e_-^j$ for $j = 1, 2$ are chain homotopic to $\mathrm{id}_{\hat{C}_*^{\mathrm{dR}}(X)}$.

Proof We omit the detailed proof and refer to Lemma 3.16. For any $x = [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] \in \hat{C}_*^{\mathrm{dR}}(X)$, let us define diffeomorphisms, for $j = 1, 2$,

$$\tilde{\tau}_+^j \colon \mathbb{R} \times U_{\{r_j \geq 1\}} \to \mathbb{R}_{\geq 1} \times (\mathbb{R} \times U_{\{r_j = 1\}}), \quad \tilde{\tau}_-^j \colon \mathbb{R} \times U_{\{r_j \leq -1\}} \to \mathbb{R}_{\leq -1} \times (\mathbb{R} \times U_{\{r_j = -1\}})$$

such that $(\tilde{\tau}_+^j)^{-1}(r_j, (r, u)) = (r, (\tau_+^j)^{-1}(r_j, u))$ and $(\tilde{\tau}_-^j)^{-1}(r_j, (r, u)) = (r, (\tau_-^j)^{-1}(r_j, u))$. Referring to the proof of [16, Lemma 4.8], we can find $\bar{\varphi}^j$, $\tilde{\tau}_{\pm}^j$ and $\bar{\omega}^j$ for $j = 1, 2$ to define

$$K^1(x) := (-1)^{|\omega|+1}[\mathbb{R} \times U, \bar{\varphi}^1, (\tilde{\tau}_+^1, \tilde{\tau}_-^1), (\tilde{\tau}_+^2, \tilde{\tau}_-^2), \bar{\omega}^1],$$

$$K^2(x) := (-1)^{|\omega|+1}[\mathbb{R} \times U, \bar{\varphi}^2, (\tilde{\tau}_+^1, \tilde{\tau}_-^1), (\tilde{\tau}_+^2, \tilde{\tau}_-^2), \bar{\omega}^2]$$

such that $K^j \colon \hat{C}_*^{\mathrm{dR}}(X) \to \hat{C}_{*+1}^{\mathrm{dR}}(X)$ is a chain homotopy from $\mathrm{id}_{\hat{C}_*^{\mathrm{dR}}(X)}$ to $\hat{\imath}^j \circ e_+^j$ for $j = 1, 2$. The proof for $\hat{\imath}^j \circ e_-^j$ is completely parallel. □

3.5.3 Collection of analogies with ordinary de Rham chains As above, X and Y are chosen from the differentiable spaces Σ_m^a, $\Sigma_m^a \times \Sigma_{m'}^{a'}$, S_ε and (M, P_M) for a manifold M. Let $f \colon X \to Y$ be a smooth map. If $Y = S_\varepsilon$, we require that $X = S_{\varepsilon'}$ and $\mathrm{ev}_0 \circ f = \mathrm{ev}_0$. Then f induces chain maps

$$f_* \colon \overline{C}_*^{\mathrm{dR}}(X) \to \overline{C}_*^{\mathrm{dR}}(Y), \qquad\qquad [U, \varphi, (\tau_+, \tau_-), \omega] \mapsto [U, f \circ \varphi, (\tau_+, \tau_-), \omega],$$

$$f_* \colon \hat{C}_*^{\mathrm{dR}}(X) \to \hat{C}_*^{\mathrm{dR}}(Y), \quad [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] \mapsto [U, f \circ \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega].$$

If $X = \Sigma_m^a$ and $Y = \Sigma_m^b$ for $a \leq b$ and f is the inclusion map, then we claim that the above maps are injective. This can be proved in the same way as Lemma 2.7, so we omit the proof. As a consequence, we can define $\overline{C}_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a)$ and $\hat{C}_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a)$ as quotient complexes.

Next, let $(X, Y) = (\Sigma_m^a, \Sigma_{m'}^{a'})$ or $(\{0\}, S_\varepsilon)$. We identify $\{0\} \times S_\varepsilon$ with S_ε. Then a cross product $x \times y \in \overline{C}_{p+q}^{\mathrm{dR}}(X \times Y)$ is defined for $x = [U, \varphi, (\tau_+, \tau_-), \omega] \in \overline{C}_p^{\mathrm{dR}}(X)$ and $y = [V, \psi, (\sigma_+, \sigma_-), \eta] \in \overline{C}_q^{\mathrm{dR}}(Y)$ by

$$x \times y := (-1)^{p|\eta|}[W, \tilde{\varphi}, (\tilde{\tau}_+, \tilde{\tau}_-), \omega \times \eta].$$

Here, $W := U \, _{\varphi_{\mathbb{R}}} \times_{\psi_{\mathbb{R}}} V$ is a fiber product over $\mathbb{R}$ and $\tilde{\varphi}$, $\tilde{\tau}_+$ and $\tilde{\tau}_-$ are determined by

$$\tilde{\varphi} \colon W \to \mathbb{R} \times (X \times Y), \quad (u, v) \mapsto (\varphi_{\mathbb{R}}(u), \varphi_X(u), \psi_Y(v)),$$

$$\tilde{\tau}_+(u, v) = (r, (u_+, v_+)) \quad \text{for } (u, v) = ((\tau_+)^{-1}(r, u_+), (\sigma_+)^{-1}(r, v_+)) \in W_{\mathbb{R}_{\geq 1}},$$

$$\tilde{\tau}_-(u, v) = (r, (u_-, v_-)) \quad \text{for } (u, v) = ((\tau_-)^{-1}(r, u_-), (\sigma_-)^{-1}(r, v_-)) \in W_{\mathbb{R}_{\leq -1}}.$$

Similarly, a cross product $x \times y \in \hat{C}_{p+q}^{\mathrm{dR}}(X \times Y)$ is defined for $x = [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] \in \hat{C}_p^{\mathrm{dR}}(X)$ and $y = [V, \psi, (\sigma_+^1, \sigma_-^1), (\sigma_+^2, \sigma_-^2), \eta] \in \hat{C}_q^{\mathrm{dR}}(Y)$ by

$$x \times y := (-1)^{p|\eta|}[W, \tilde{\varphi}, (\tilde{\tau}_+^1, \tilde{\tau}_-^1), (\tilde{\tau}_+^2, \tilde{\tau}_-^2), \omega \times \eta].$$

Here $W := U \, _{(\varphi_{\mathbb{R}}^1, \varphi_{\mathbb{R}}^2)} \times_{(\psi_{\mathbb{R}}^1, \psi_{\mathbb{R}}^2)} V$ is a fiber product over $\mathbb{R}^2$ and $\tilde{\varphi}$, $\tilde{\tau}_+^j$ and $\tilde{\tau}_-^j$ for $j = 1, 2$ are determined by

$$\tilde{\varphi} \colon W \to \mathbb{R} \times (X \times Y), \quad (u, v) \mapsto (\varphi_{\mathbb{R}}(u), \varphi_X(u), \psi_Y(v)),$$

$$\tilde{\tau}_+^j(u, v) = (r, (u_+, v_+)) \quad \text{for } (u, v) = ((\tau_+^j)^{-1}(r, u_+), (\sigma_+^j)^{-1}(r, v_+)) \in W_{\{r_j = 1\}},$$

$$\tilde{\tau}_-^j(u, v) = (r, (u_-, v_-)) \quad \text{for } (u, v) = ((\tau_-^j)^{-1}(r, u_-), (\sigma_-^j)^{-1}(r, v_-)) \in W_{\{r_j = -1\}}.$$

The next result is analogous to Propositions 3.7 and 3.9. It follows immediately from the fact that $(e_+)_*\colon \overline{H}_*^{\mathrm{dR}}(X) \to H_*^{\mathrm{dR}}(X)$ is an isomorphism (see Lemma 3.16).

Proposition 3.18 *Let $a, b \in \mathbb{R}_{>0}$ with $a \le b$ and $\varepsilon \in (0, \varepsilon_0/C]$ for the constant C of Lemma 3.8. Then the following hold:*

- *If $\mathscr{L}_m(K) \cap [a, b] = \varnothing$, then $\overline{H}_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a) = 0$.*

- *For any $x \in \overline{H}_*^{\mathrm{dR}}(S_\varepsilon)$,*

$$(\mathrm{ev}_0)_* \circ (e_+)_*(x) = 0 \in H_*^{\mathrm{dR}}(N_\varepsilon) \implies (i_{\varepsilon, C\varepsilon})_*(x) = 0 \in \overline{H}_*^{\mathrm{dR}}(S_{C\varepsilon}).$$

3.5.4 Operations on $[-1, 1]$- and $[-1, 1]^2$-modeled de Rham chains In the rest of this section, let us define operators corresponding to $f_{k,\xi}$. We prefer to refer to the explicit description (13) of $f_{k,\xi}(x)$ rather than its original definition using fiber products of chains. Let $\varepsilon \in (0, \varepsilon_0/(5C_0)]$ and $\rho_\varepsilon\colon A_\varepsilon \to [0, 1]$ be the C^∞ function we have chosen in the beginning of Section 3.4

For $k = 1, \ldots, m$ and $\bar{\xi} \in \overline{C}_q^{\mathrm{dR}}(S_\varepsilon)$, we define a linear map

$$\bar{f}_{k,\bar{\xi}}\colon \overline{C}_*^{\mathrm{dR}}(\Sigma_m^a) \to \overline{C}_{*+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon})$$

as follows: Let

$$x = [U, \varphi, (\tau_+, \tau_-), \omega] \in \overline{C}_p^{\mathrm{dR}}(\Sigma_m^a), \quad \bar{\xi} = [V, \psi, (\sigma_+, \sigma_-), \eta] \in \overline{C}_q^{\mathrm{dR}}(S_\varepsilon),$$

and

$$\varphi(u) = (\varphi_{\mathbb{R}}(u), \varphi_\Sigma(u)) = (\varphi_{\mathbb{R}}(u), (\gamma_l^u\colon [0, T_l^u] \to Q)_{l=1,\ldots,m}) \in \mathbb{R} \times \Sigma_m^a,$$
$$\psi(v) = (\psi_{\mathbb{R}}(v), \psi_S(v)) = (\psi_{\mathbb{R}}(v), (\sigma_i^v)_{i=1,2}) \in \mathbb{R} \times S_\varepsilon$$

for every $u \in U$ and $v \in V$. Then we define $\bar{f}_{k,\bar{\xi}}(x) := (-1)^s[W_k, \Phi_k, (\tilde{\tau}_+, \tilde{\tau}_-), \zeta_k]$, where

$$W_k := \{(u, \tau, v) \in U \times \mathbb{R} \times V \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon, \ (\varphi_{\mathbb{R}}(u), \gamma_k^u(\tau)) = (\psi_{\mathbb{R}}(u), \sigma_1^v(0))\},$$

(21)
$$\Phi_k\colon W_k \to \mathbb{R} \times \Sigma_{m+1}^{a+\varepsilon}, \quad (u, \tau, v) \mapsto (\varphi_{\mathbb{R}}(u), \mathrm{con}_k(\varphi_\Sigma(u), (T_k^u, \tau), \psi_S(v))),$$
$$\zeta_k \in \Omega_c^*(W_k), \quad (\zeta_k)_{(u,\tau,v)} := \rho_\varepsilon(T_k^u, \tau) \cdot (\omega_u \times \eta_v),$$
$$s := (p + 1 - n)|\eta| + n + 1,$$

and

$$\tilde{\tau}_+(u, \tau, v) := (r, (u_+, \tau, v_+)) \in \mathbb{R} \times (W_k)_{\{r=1\}}$$

for $(u, \tau, v) = (\tau_+^{-1}(r, u_+), \tau, \sigma_+^{-1}(r, v_+)) \in (W_k)_{\{r \ge 1\}}$, and

$$\tilde{\tau}_-(u, \tau, v) := (r, (u_-, \tau, v_-)) \in \mathbb{R} \times (W_k)_{\{r=-1\}}$$

for $(u, \tau, v) = (\tau_-^{-1}(r, u_-), \tau, \sigma_-^{-1}(r, v_-)) \in (W_k)_{\{r \le -1\}}$. Here, W_k is oriented as a fiber product over $\mathbb{R} \times Q$ of the map

$$\{(u, \tau) \in U \times \mathbb{R} \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon\} \to \mathbb{R} \times Q, \quad (u, \tau) \mapsto (\varphi_{\mathbb{R}}(u), \gamma_k^u(\tau)),$$

and the submersion $(\mathrm{id}_{\mathbb{R}} \times \mathrm{ev}_0) \circ \psi \colon V \to \mathbb{R} \times Q$. It can be checked that $\bar{\imath}$ and e_+, e_- intertwine this operator and $f_{k,\xi}$ for $\xi \in C_q^{\mathrm{dR}}(S_\varepsilon)$. Namely,

$$\bar{\imath} \circ f_{k,\xi} = f_{k,\bar{\imath}\xi} \circ \bar{\imath}, \quad e_+ \circ \bar{f}_{k,\bar{\xi}} = f_{k,e_+\bar{\xi}} \circ e_+, \quad e_- \circ \bar{f}_{k,\bar{\xi}} = f_{k,e_-\bar{\xi}} \circ e_-.$$

Similar results as for $f_{k,\xi}$ are the following: $\bar{f}_{k,\bar{\xi}}$ induces a linear map

$$\bar{f}_{k,\bar{\xi}} \colon \overline{C}_*^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0) \to \overline{C}_{*+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0),$$

which is independent of ρ_ε. The next equations are variants of (15) and (16), and they follow from similar computations as Lemmas 3.13 and 3.14, so we omit the proof.

Proposition 3.19 For $k' \geq k$,

$$\partial \circ \bar{f}_{k,\bar{\xi}} - \bar{f}_{k,\bar{\xi}} \circ \partial = (-1)^{p+1-n} \bar{f}_{k,\partial\bar{\xi}} \colon \overline{C}_p^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0) \to \overline{C}_{p+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0),$$

$$\bar{f}_{k'+1,\bar{\xi}} \circ \bar{f}_{k,\bar{\xi}} + (-1)^{q-n} \bar{f}_{k,\bar{\xi}} \circ \bar{f}_{k',\bar{\xi}} = 0 \colon \overline{C}_p^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0) \to \overline{C}_{p+2+2q-2n}^{\mathrm{dR}}(\Sigma_{m+2}^{a+2\varepsilon}, \Sigma_{m+2}^0).$$

Next, for $k = 1, \ldots, m$ and $\hat{\xi} \in \hat{C}_q^{\mathrm{dR}}(S_\varepsilon)$, we define a linear map

$$\hat{f}_{k,\hat{\xi}} \colon \overline{C}_*^{\mathrm{dR}}(\Sigma_m^a) \to \overline{C}_{*+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon})$$

as follows: Let

$$x = [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2.\tau_-^2), \omega] \in \hat{C}_p^{\mathrm{dR}}(\Sigma_m^a), \quad \hat{\xi} = [V, \psi, (\sigma_+^1, \sigma_-^1), (\sigma_+^2, \sigma_-^2), \eta] \in \hat{C}_q^{\mathrm{dR}}(S_\varepsilon),$$

and

$$\varphi(u) = (\varphi_{\mathbb{R}^2}(u), \varphi_\Sigma(u)) = \big((\varphi_{\mathbb{R}}^1(u), \varphi_{\mathbb{R}}^2(u)), (\gamma_l^u \colon [0, T_l^u] \to Q)_{l=1,\ldots,m}\big) \in \mathbb{R}^2 \times \Sigma_m^a,$$

$$\psi(v) = (\psi_{\mathbb{R}^2}(v), \psi_S(v)) = \big((\psi_{\mathbb{R}}^1(v), \psi_{\mathbb{R}}^2(v)), (\sigma_i^v)_{i=1,2}\big) \in \mathbb{R}^2 \times S_\varepsilon$$

for every $u \in U$ and $v \in V$. Then we define $\hat{f}_{k,\hat{\xi}}(x) := (-1)^s [W_k, \Phi_k, (\tilde{\tau}_+^1, \tilde{\tau}_-^1), (\tilde{\tau}_+^2, \tilde{\tau}_-^2), \zeta_k]$, where

$$W_k := \{(u, \tau, v) \in U \times \mathbb{R} \times V \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon, (\varphi_{\mathbb{R}^2}(u), \gamma_k^u(\tau)) = (\psi_{\mathbb{R}^2}(u), \sigma_1^v(0))\},$$

$$\Phi_k \colon W_k \to \mathbb{R}^2 \times \Sigma_{m+1}^{a+\varepsilon}, \quad (u, \tau, v) \mapsto \big(\varphi_{\mathbb{R}^2}(u), \mathrm{con}_k(\varphi_\Sigma(u), (T_k^u, \tau), \psi_S(v))\big),$$

$$\zeta_k \in \Omega_c^*(W_k), \quad (\zeta_k)_{(u,\tau,v)} := \rho_\varepsilon(T_k^u, \tau) \cdot (\omega_u \times \eta_v),$$

$$s := (p + 1 - n)|\eta|,$$

and, for $j = 1, 2$,

$$\tilde{\tau}_+^j(u, \tau, v) := (r_j, (u_+^j, \tau, v_+^j)) \in \mathbb{R}_{\geq 1} \times (W_k)_{\{r_j = 1\}}$$

for $(u, \tau, v) = ((\tau_+^j)^{-1}(r_j, u_+^j), \tau, (\sigma^j)_+^{-1}(r_j, v_+^j)) \in (W_k)_{\{r_j \geq 1\}}$ and

$$\tilde{\tau}_-^j(u, \tau, v) := (r_j, (u_-^j, \tau, v_-^j)) \in \mathbb{R}_{\leq -1} \times (W_k)_{\{r_j = -1\}}$$

for $(u, \tau, v) = ((\tau_-^j)^{-1}(r_j, u_-^j), \tau, (\sigma^j)_-^{-1}(r_j, v_-^j)) \in (W_k)_{\{r_j \leq -1\}}$. Here, W_k is oriented as a fiber product over $\mathbb{R}^2 \times Q$ of the map

$$\{(u, \tau) \in U \times \mathbb{R} \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon\} \to \mathbb{R}^2 \times Q \colon (u, \tau) \to (\varphi_{\mathbb{R}^2}(u), \gamma_k^u(\tau))$$

and the submersion $(\mathrm{id}_{\mathbb{R}^2} \times \mathrm{ev}_0) \circ \psi \colon V \to \mathbb{R}^2 \times Q$. It can be checked that $\hat{\imath}^j$ and e_+^j and e_-^j for $j = 1, 2$ intertwine this operator and $\bar{f}_{k,\bar{\xi}}$ for $\bar{\xi} \in \overline{C}_q^{\mathrm{dR}}(S_\varepsilon)$.

Similar results as for $f_{k,\xi}$ are the following: $\hat{f}_{k,\hat{\xi}}$ induces a linear map

$$\hat{f}_{k,\hat{\xi}}\colon \hat{C}^{\mathrm{dR}}_*(\Sigma^a_m, \Sigma^0_m) \to \hat{C}^{\mathrm{dR}}_{*+1+q-n}(\Sigma^{a+\varepsilon}_{m+1}, \Sigma^0_{m+1}),$$

which is independent of ρ_ε. The next equations are variants of (15) and (16), and they follow from similar computations as Lemmas 3.13 and 3.14, so we omit the proof.

Proposition 3.20 *For $k' \geq k$,*

$$\partial \circ \hat{f}_{k,\hat{\xi}} - \hat{f}_{k,\hat{\xi}} \circ \partial = (-1)^{p+1-n} \hat{f}_{k,\partial\hat{\xi}}\colon \hat{C}^{\mathrm{dR}}_p(\Sigma^a_m, \Sigma^0_m) \to \hat{C}^{\mathrm{dR}}_{p+1+q-n}(\Sigma^{a+\varepsilon}_{m+1}, \Sigma^0_{m+1}),$$

$$\hat{f}_{k'+1,\hat{\xi}} \circ \hat{f}_{k,\hat{\xi}} + (-1)^{q-n} \hat{f}_{k,\hat{\xi}} \circ \hat{f}_{k',\hat{\xi}} = 0\colon \hat{C}^{\mathrm{dR}}_p(\Sigma^a_m, \Sigma^0_m) \to \hat{C}^{\mathrm{dR}}_{p+2+2q-2n}(\Sigma^{a+2\varepsilon}_{m+2}, \Sigma^0_{m+2}).$$

4 Construction of $H^{\mathrm{string}}_*(Q, K)$

4.1 Definition of chain complex

For $a \in \mathbb{R}_{>0}$ and $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, we define the graded $\mathbb{R}$-vector space

$$C^{<a}_*(\varepsilon) := \bigoplus_{m=0}^{\infty} C^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m).$$

If $m \geq 2a/\varepsilon_0$, then $a + m\varepsilon \leq m\left(\frac{1}{2}\varepsilon_0 + \varepsilon\right) \leq m\varepsilon_0$. In this case, $\Sigma^{a+m\varepsilon}_m = \Sigma^0_m$ by Remark 3.1. Therefore, the component for $m \in \mathbb{Z}_{\geq 0}$ vanishes if $m \geq 2a/\varepsilon_0$.

For each $m \in \mathbb{Z}_{\geq 0}$, we think of $C^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m)$ as a linear subspace of $C^{<a}_*(\varepsilon)$ in a natural way. For $\delta \in C^{\mathrm{dR}}_{n-d}(S_\varepsilon)$, we define a degree -1 linear map

$$D_\delta\colon C^{<a}_*(\varepsilon) \to C^{<a}_{*-1}(\varepsilon)$$

such that, for $x \in C^{\mathrm{dR}}_{p-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m)$,

$$D_\delta(x) = \partial x + \sum_{k=1}^{m} (-1)^{p+1+kd} f_{k,\delta}(x) \in C^{<a}_{p-1}(\varepsilon).$$

When $m = 0$, the right-hand side is just equal to ∂x.

Proposition 4.1 *If $\delta \in C^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ satisfies $\partial\delta = 0$, then $D_\delta \circ D_\delta = 0$.*

Proof Take an arbitrary $x \in C^{\mathrm{dR}}_{p-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m)$. Since $\partial \circ \partial = 0$,

$$D_\delta \circ D_\delta(x) = \sum_{k=1}^{m} \left((-1)^{p+1+kd} \partial \circ f_{k,\delta}(x) + (-1)^{p+kd} f_{k,\delta} \circ \partial(x)\right)$$

$$+ \sum_{k'=1}^{m+1} \sum_{k=1}^{m} (-1)^{(k+k')d-1} f_{k',\delta} \circ f_{k,\delta}(x).$$

Applying (15) for $q = n - d$ and $\xi = \delta$ for which $\partial \delta = 0$, we can see that the first summand is equal to 0. For the second summand, we apply (16) for $q = n - d$ and $\xi = \delta$. Then

$$\sum_{k'=1}^{m+1} \sum_{k=1}^{m} (-1)^{(k+k')d-1} f_{k',\delta} \circ f_{k,\delta}(x)$$
$$= \sum_{1 \le k \le k' \le m} ((-1)^{(k+k'+1)d-1} f_{k'+1,\delta} \circ f_{k,\delta}(x) + (-1)^{(k+k')d-1} f_{k,\delta} \circ f_{k',\delta}(x))$$
$$= 0.$$

This shows that $D_\delta \circ D_\delta(x) = 0$. □

In summary, for $a \in \mathbb{R}_{>0}$, $\varepsilon \in (0, \varepsilon_0/(5C_0)]$ and $\delta \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ with $\partial \delta = 0$, a chain complex $(C_*^{<a}(\varepsilon), D_\delta)$ is defined. Let $H_*^{<a}(\varepsilon, \delta)$ denote its homology.

The chain complex $(C_*^{<a}(\varepsilon), D_\delta)$ is filtered by subcomplexes $\{\mathscr{F}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ defined by

$$(22) \qquad \mathscr{F}_{\varepsilon,p}^{<a} := \bigoplus_{m \ge -p}^{\infty} C_{*-m(d-2)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

Let $E_{(\varepsilon,\delta)}^{<a} := (\{(E_{(\varepsilon,\delta)}^{<a})_{p,q}^r\}, \{(d_{(\varepsilon,\delta)}^{<a})_{p,q}^r\})$ be the spectral sequence determined by $\{\mathscr{F}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$. Note that $\mathscr{F}_{\varepsilon,p}^{<a} = 0$ for $p \le -2a/\varepsilon_0$ and $\mathscr{F}_{\varepsilon,p}^{<a} = \mathscr{F}_{\varepsilon,0}^{<a}$ for $p \ge 0$, and thus this spectral sequence converges to $H_*^{<a}(\varepsilon, \delta)$ in the sense of [21, Bounded Convergence 5.2.5]. The first page is given by

$$(E_{(\varepsilon,\delta)}^{<a})_{-m,q}^1 = \begin{cases} H_{(q-m)-m(d-2)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) = H_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) & \text{if } m \ge 0, \\ 0 & \text{if } m < 0. \end{cases}$$

Let us state an abstract lemma about morphisms in the category of spectral sequences. This result, which is a refinement of [21, Comparison Theorem 5.2.12], will be repeatedly used in the rest of this paper.

Lemma 4.2 *Let $E = (\{E_{p,q}^r\}, \{d_{p,q}^r\})$ and $E' = (\{E_{p,q}'^r\}, \{d_{p,q}'^r\})$ be bounded spectral sequences which converge to H_* and H_*', respectively, in the sense of [21, Bounded Convergence 5.2.5]. Let $f = \{f_{p,q}^r\}$ be a morphism from E to E' which is compatible with $\{h_n : H_n \to H_n'\}_{n \in \mathbb{Z}}$. Then the following assertion holds*:

- *Suppose that, for some $r_0 \ge 1$ and $n_0 \in \mathbb{Z}$, $f_{p,q}^{r_0}$ is an isomorphism if $p + q < n_0$, and a surjection if $p + q = n_0$. Then h_n is an isomorphism if $n < n_0$ and a surjection if $n = n_0$. In particular, if $f_{p,q}^{r_0}$ is an isomorphism for every $p, q \in \mathbb{Z}$ for some $r_0 \ge 1$, then h_n is an isomorphism for every $n \in \mathbb{Z}$.*

Proof Suppose that $\{f_{p,q}^r\}$ satisfy the condition of the assertion for $r_0 \ge 1$ and $n_0 \in \mathbb{Z}$. Note that, for any $r \ge 1$ and $p, q \in \mathbb{Z}$,

$$f_{p,q}^{r+1} : E_{p,q}^{r+1} \cong \operatorname{Ker} d_{p,q}^r / \operatorname{Im} d_{p+r,q-r+1}^r \to E_{p,q}'^{r+1} \cong \operatorname{Ker} d_{p,q}'^r / \operatorname{Im} d_{p+r,q-r+1}'^r$$

is induced by $\{f_{p,q}^r\}_{p,q}$. Therefore, by inductive arguments about $\{f_{p,q}^r\}$ on $r = r_0, r_0 + 1, \ldots$, we can prove that $f_{p,q}^\infty : E_{p,q}^\infty \to E_{p,q}'^\infty$ is an isomorphism if $p + q < n_0$, and a surjection if $p + q = n_0$. We omit the concluding argument about h_n, since it is parallel to [21, Comparison Theorem 5.2.12]. □

For $\varepsilon, \bar{\varepsilon} \in (0, \varepsilon_0/(5C_0)]$ with $\varepsilon \leq \bar{\varepsilon}$, let $j_{\varepsilon,\bar{\varepsilon}} \colon \Sigma_m^{a+m\varepsilon} \to \Sigma_m^{a+m\bar{\varepsilon}}$ for $m \in \mathbb{Z}_{\geq 0}$ be the inclusion maps. These maps induce a linear map

$$(j_{\varepsilon,\bar{\varepsilon}})_* \colon C_*^{<a}(\varepsilon) \to C_*^{<a}(\bar{\varepsilon}).$$

For any $\delta \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ with $\partial \delta = 0$, $(j_{\varepsilon,\bar{\varepsilon}})_*$ is a chain map from $(C_*^{<a}(\varepsilon), D_\delta)$ to $(C_*^{<a}(\bar{\varepsilon}), D_{(i_{\varepsilon,\bar{\varepsilon}})_*\delta})$ and preserves the filtrations $\{\mathcal{F}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{F}_{\bar{\varepsilon},p}^{<a}\}_{p\in\mathbb{Z}}$.

Lemma 4.3 *Suppose that $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$. Then the induced map on homology*

$$(j_{\varepsilon,\bar{\varepsilon}})_* \colon H_*^{<a}(\varepsilon, \delta) \to H_*^{<a}(\bar{\varepsilon}, (i_{\varepsilon,\bar{\varepsilon}})_*\delta)$$

is an isomorphism if $\bar{\varepsilon}$ satisfies $[a, a + (2a/\varepsilon_0)\bar{\varepsilon}] \cap \mathcal{L}(K) = \varnothing$.

Proof If $m \geq 2a/\varepsilon_0$, then $\Sigma_m^{a+m\bar{\varepsilon}} = \Sigma_m^{a+m\varepsilon} = \Sigma_m^0$. If $0 \leq m \leq 2a/\varepsilon_0$, then $[a+m\varepsilon, a+m\bar{\varepsilon}] \cap \mathcal{L}(K) = \varnothing$ from the condition on $\bar{\varepsilon}$. Thus, Proposition 3.7 is applied to show that $H_*^{\mathrm{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}, \Sigma_m^{a+m\varepsilon}) = 0$ for all $m \in \mathbb{Z}_{\geq 0}$. Therefore, the induced map on the $(-m, q)$-term for $m \geq 0$ of the first page

$$(j_{\varepsilon,\bar{\varepsilon}})_* \colon (E_{(\varepsilon,\delta)}^{<a})^1_{-m,q} = H_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) \to (E_{(\bar{\varepsilon},(i_{\varepsilon,\bar{\varepsilon}})_*\delta)}^{<a})^1_{-m,q} = H_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}, \Sigma_m^0)$$

is an isomorphism. Now the assertion follows from Lemma 4.2. $\qquad\square$

As we have seen in Example 2.6, $H_*^{\mathrm{dR}}(N_\varepsilon^{\mathrm{reg}}) \cong H_c^d(N_\varepsilon; \mathbb{R})$. Therefore, we can determine a unique homology class $\mathrm{Th}_\varepsilon \in H_{n-d}^{\mathrm{dR}}(N_\varepsilon^{\mathrm{reg}})$ which corresponds to the Thom class of $(TK)^\perp$ through the diffeomorphism $\{(x, v) \in (TK)^\perp \mid v < \varepsilon\} \to N_\varepsilon$, $(x, v) \mapsto \exp_x v$.

The above lemma leads us to define a set of data (ε, δ) as follows.

Definition 4.4 Let $C \geq 1$ be the constant of Lemma 3.8. We define $\mathcal{T}_a$ for every $a \in \mathbb{R}_{\geq 0} \setminus \mathcal{L}(K)$ to be the set of pairs (ε, δ) of $\varepsilon \in (0, \varepsilon_0/(5C^4)]$ and $\delta \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ such that:

(a) $[a, a + (2a/\varepsilon_0)\hat{\varepsilon}] \cap \mathcal{L}(K) = \varnothing$ for $\hat{\varepsilon} := C^3\varepsilon$.

(b) $\partial \delta = 0$ and $(\mathrm{ev}_0)_*[\delta] = \mathrm{Th}_\varepsilon \in H_{n-d}^{\mathrm{dR}}(N_\varepsilon^{\mathrm{reg}})$.

Let $a, b \in \mathbb{R}_{>0}$ with $a < b$. For $\varepsilon \in (0, \varepsilon_0/(5C_0)]$ and $\delta \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ with $\partial \delta = 0$, there exists a chain map $(I_\varepsilon^{a,b})_*$ from $(C_*^{<a}(\varepsilon), D_\delta)$ to $(C_*^{<b}(\varepsilon), D_\delta)$ induced by inclusion maps $I_\varepsilon^{a,b} \colon \Sigma_m^{a+m\varepsilon} \to \Sigma_m^{b+m\varepsilon}$ for all $m \in \mathbb{Z}_{\geq 0}$. We define the quotient complex

$$(C_*^{[a,b]}(\varepsilon) := C_*^{<b}(\varepsilon)/C_*^{<a}(\varepsilon), D_\delta).$$

Let $H_*^{[a,b]}(\varepsilon, \delta)$ denote its homology. Obviously, there exists a long exact sequence

$$(23) \qquad \cdots \to H_*^{<a}(\varepsilon, \delta) \xrightarrow{(I_\varepsilon^{a,b})_*} H_*^{<b}(\varepsilon, \delta) \to H_*^{[a,b]}(\varepsilon, \delta) \to H_{*-1}^{<a}(\varepsilon, \delta) \xrightarrow{(I_\varepsilon^{a,b})_*} \cdots.$$

The next result is a trivial computation from the spectral sequence.

Proposition 4.5 *For $a, b \in \mathbb{R} \setminus \mathscr{L}(K)$ with $a < b$ and $(\varepsilon, \delta) \in \mathscr{T}_a \cap \mathscr{T}_b$, the following hold:*

- *If $[a, b] \cap \mathscr{L}(K) = \varnothing$, then $H_*^{[a,b)}(\varepsilon, \delta) = 0$.*

- *If there exist $c \in \mathscr{L}(K)$ and $m_0 \in \mathbb{Z}_{\geq 1}$ such that $[a, b] \cap \mathscr{L}_m(K) = \{c\}$ if $m = m_0$ and $[a, b] \cap \mathscr{L}_m(K) = \varnothing$ otherwise, then*

$$H_*^{[a,b)}(\varepsilon, \delta) \cong H_{*-m_0(d-2)}^{\mathrm{dR}}(\Sigma_{m_0}^b, \Sigma_{m_0}^a).$$

Proof Let $E_{(\varepsilon,\delta)}^{[a,b)}$ be the spectral sequence determined by a filtration $\{\mathscr{F}_{\varepsilon,m}^{<b}/\mathscr{F}_{\varepsilon,m}^{<a}\}_{m\in\mathbb{Z}}$. We apply Proposition 3.7 to the first page. For the first case, $(E_{(\varepsilon,\delta)}^{[a,b)})_{p,q}^1 = 0$ for every $p, q \in \mathbb{Z}$, so the assertion is trivial. For the second case, $(E_{(\varepsilon,\delta)}^{[a,b)})_{p,q}^1 = 0$ for every $p \neq -m_0$, so all differentials are the zero map. Therefore, $H_q^{[a,b)}(\varepsilon, \delta) \cong (E_{(\varepsilon,\delta)}^{[a,b)})_{-m_0,q+m_0}^1$ and the assertion follows from

$$(E_{(\varepsilon,\delta)}^{[a,b)})_{-m_0,q+m_0}^1 = H_{q-m_0(d-2)}^{\mathrm{dR}}(\Sigma_{m_0}^{b+m_0\varepsilon}, \Sigma_{m_0}^{a+m_0\varepsilon}) \cong H_{q-m_0(d-2)}^{\mathrm{dR}}(\Sigma_{m_0}^b, \Sigma_{m_0}^a).$$

Here, the last isomorphism comes from Proposition 3.7. $\square$

4.2 Variants from $[-1, 1]$- and $[-1, 1]^2$-modeled de Rham chains

In Section 3.5, we introduced $[-1, 1]$-modeled and $[-1, 1]^2$-modeled de Rham chains. In this section, we define chain complexes as in Section 4.1 by using these types of chains. Their constructions and some computations are parallel to the former section, so we often omit proofs.

First, we deal with $[-1, 1]$-modeled chains. For $a \in \mathbb{R}_{>0}$ and $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, we consider the graded $\mathbb{R}$-vector space

$$\overline{C}_*^{<a}(\varepsilon) := \bigoplus_{m=0}^{\infty} \overline{C}_{*-m(d-2)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

For $\bar{\delta} \in \overline{C}_{n-d}^{\mathrm{dR}}(S_\varepsilon)$, we define a degree -1 map $\overline{D}_{\bar{\delta}} : \overline{C}_*^{<a}(\varepsilon) \to \overline{C}_{*-1}^{<a}(\varepsilon)$ by

$$\overline{D}_{\bar{\delta}}(x) := \partial x + \sum_{k=1}^{m} (-1)^{p+1+kd} \bar{f}_{k,\bar{\delta}}(x)$$

for $x \in \overline{C}_{p-m(d-2)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$.

Proposition 4.6 *If $\bar{\delta} \in \overline{C}_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ satisfies $\partial\bar{\delta} = 0$, then $\overline{D}_{\bar{\delta}} \circ \overline{D}_{\bar{\delta}} = 0$.*

This is analogous to Proposition 4.1 and can be deduced from the two equations of Proposition 3.19. From this proposition, for $\bar{\delta} \in \overline{C}_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ with $\partial\delta = 0$, we obtain a chain complex $(\overline{C}_*^{<a}(\varepsilon), \overline{D}_{\bar{\delta}})$. Let $\overline{H}_*^{<a}(\varepsilon, \bar{\delta})$ denote its homology.

Let us consider a relation to the chain complex defined in Section 4.1. The linear maps (18) for $X = \Sigma_m^{a+m\varepsilon}$

$$e_+, e_- : \overline{C}_*^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}) \to C_*^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}) \quad \text{for } m \in \mathbb{Z}_{\geq 0}$$

naturally induce linear maps

$$e_{\varepsilon,+}, e_{\varepsilon,-} : \overline{C}_*^{<a}(\varepsilon) \to C_*^{<a}(\varepsilon),$$

and these are chain maps from $(\overline{C}_*^{<a}(\varepsilon), \overline{D}_{\bar{\delta}})$ to $(C_*^{<a}(\varepsilon), D_{e_+\bar{\delta}})$ and $(C_*^{<a}(\varepsilon), D_{e_-\bar{\delta}})$, respectively.

We define a filtration $\{\overline{\overline{\mathscr{F}}}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ by

$$\overline{\overline{\mathscr{F}}}_{\varepsilon,p}^{<a} := \bigoplus_{m \geq -p} \overline{C}_{*-m(d-2)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

Let $\overline{E}_{(\varepsilon,\bar{\delta})}^{<a}$ be the spectral sequence determined by this filtration.

Lemma 4.7 $e_{\varepsilon,+}$ *and* $e_{\varepsilon,-}$ *are quasi-isomorphisms.*

Proof We prove this assertion for only $e_{\varepsilon,+}$. The proof for $e_{\varepsilon,-}$ is parallel. Since $e_{\varepsilon,+}$ preserves filtrations $\{\overline{\overline{\mathscr{F}}}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathscr{F}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$, this induces a map on the first page $(e_{\varepsilon,+})_*\colon (\overline{E}_{(\varepsilon,\bar{\delta})}^{<a})_{p,q}^1 \to (E_{(\varepsilon,e_+\bar{\delta})}^{<a})_{p,q}^1$. For $p = -m \leq 0$, this coincides with

$$(e_+)_*\colon \overline{H}_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) \to H_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

From [Lemma 3.16](), this map is an isomorphism. Now the assertion follows from [Lemma 4.2](). $\square$

For $\bar{\varepsilon}, \hat{\varepsilon} \in (0, \varepsilon_0/(5C_0)]$ with $\bar{\varepsilon} \leq \hat{\varepsilon}$, the linear map $(j_{\bar{\varepsilon},\hat{\varepsilon}})_*\colon \overline{C}_*^{<a}(\bar{\varepsilon}) \to \overline{C}_*^{<a}(\hat{\varepsilon})$, induced by the inclusion maps $j_{\bar{\varepsilon},\hat{\varepsilon}}\colon \Sigma_m^{a+m\bar{\varepsilon}} \to \Sigma_m^{a+m\hat{\varepsilon}}$ for all $m \in \mathbb{Z}_{\geq 0}$, is a chain map from $(\overline{C}_*^{<a}(\bar{\varepsilon}), \overline{D}_{\bar{\delta}})$ to $(\overline{C}_*^{<a}(\hat{\varepsilon}), \overline{D}_{(i_{\bar{\varepsilon},\hat{\varepsilon}})_*\bar{\delta}})$.

Lemma 4.8 *Suppose that* $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$. *Then the induced map on homology*

$$(j_{\bar{\varepsilon},\hat{\varepsilon}})_*\colon \overline{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}) \to \overline{H}_*^{<a}(\hat{\varepsilon}, (i_{\bar{\varepsilon},\hat{\varepsilon}})_*\bar{\delta})$$

is an isomorphism if $\hat{\varepsilon}$ *satisfies* $[a, a + (2a/\varepsilon_0)\hat{\varepsilon}] \cap \mathscr{L}(K) = \varnothing$.

Proof The proof is parallel to that of [Lemma 4.3](). The chain map $(j_{\bar{\varepsilon},\hat{\varepsilon}})_*$ preserves the filtrations $\{\overline{\overline{\mathscr{F}}}_{\bar{\varepsilon},p}^{<a}\}_{p \in \mathbb{Z}}$ and $\{\overline{\overline{\mathscr{F}}}_{\hat{\varepsilon},p}^{<a}\}_{p \in \mathbb{Z}}$. This induces an isomorphism on the first page since, for every $m \in \mathbb{Z}_{\geq 0}$,

$$\overline{H}_*(\Sigma_m^{a+m\hat{\varepsilon}}, \Sigma_m^{a+m\bar{\varepsilon}}) = 0$$

by [Proposition 3.18](). Now the assertion follows from [Lemma 4.2](). $\square$

The above lemma leads us to the following definition.

Definition 4.9 Let $C \geq 1$ be the constant of [Lemma 3.8](). We define $\overline{\overline{\mathscr{T}}}_a$ for $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$ to be the set of pairs $(\bar{\varepsilon}, \bar{\delta})$ of $\bar{\varepsilon} \in (0, \varepsilon_0/(5C^3)]$ and $\bar{\delta} \in \overline{C}_{n-d}^{\mathrm{dR}}(S_{\bar{\varepsilon}})$ such that:

(a) $[a, a + (2a/\varepsilon_0)\hat{\varepsilon}] \cap \mathscr{L}(K) = \varnothing$ for $\hat{\varepsilon} := C^2\bar{\varepsilon}$.

(b) $\partial\bar{\delta} = 0$ and $(\mathrm{ev}_0)_*[e_+\bar{\delta}] = \mathrm{Th}_{\bar{\varepsilon}} \in H_{n-d}^{\mathrm{dR}}(N_{\bar{\varepsilon}}^{\mathrm{reg}})$.

Next, we deal with $[-1, 1]^2$-modeled chains. For $a \in \mathbb{R}_{>0}$ and $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, we consider the graded $\mathbb{R}$-vector space

$$\widehat{C}_*^{<a}(\varepsilon) := \bigoplus_{m=0}^{\infty} \widehat{C}_{*-m(d-2)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

For $\hat{\delta} \in \hat{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$, we define a degree -1 map $\hat{D}_{\hat{\delta}} \colon \hat{C}^{<a}_*(\varepsilon) \to \hat{C}^{<a}_{*-1}(\varepsilon)$ by

$$\hat{D}_{\hat{\delta}}(x) := \partial x + \sum_{k=1}^{m} (-1)^{p+1+kd}\, \hat{f}_{k,\hat{\delta}}(x)$$

for $x \in \hat{C}^{\mathrm{dR}}_{p-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m)$.

Proposition 4.10 *If* $\hat{\delta} \in \hat{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ *satisfies* $\partial\hat{\delta} = 0$, *then* $\hat{D}_{\hat{\delta}} \circ \hat{D}_{\hat{\delta}} = 0$.

This is analogous to Proposition 4.1 and can be deduced from the two equations of Proposition 3.20. From this proposition, for $\hat{\delta} \in \hat{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ with $\partial\hat{\delta} = 0$, we obtain a chain complex $(\hat{C}^{<a}_*(\varepsilon), \hat{D}_{\hat{\delta}})$. Let $\hat{H}^{<a}_*(\varepsilon, \hat{\delta})$ denote its homology.

Let us consider a relation to the chain complex defined by $[-1, 1]$-modeled de Rham chains. For $j = 1, 2$, the linear maps of (19) for $X = \Sigma^{a+m\varepsilon}_m$,

$$e^j_+, e^j_- \colon \hat{C}^{\mathrm{dR}}_*(\Sigma^{a+m\varepsilon}_m) \to \overline{C}^{\mathrm{dR}}_*(\Sigma^{a+m\varepsilon}_m) \quad \text{for } m \in \mathbb{Z}_{\geq 0},$$

naturally induce linear maps

$$e^j_{\varepsilon,+}, e^j_{\varepsilon,-} \colon \hat{C}^{<a}_*(\varepsilon) \to \overline{C}^{<a}_*(\varepsilon),$$

and these are chain maps from $(\hat{C}^{<a}_*(\varepsilon), \hat{D}_{\hat{\delta}})$ to $(\overline{C}^{<a}_*(\varepsilon), \overline{D}_{e^j_+\hat{\delta}})$ and $(\overline{C}^{<a}_*(\varepsilon), \overline{D}_{e^j_-\hat{\delta}})$, respectively.

Lemma 4.11 $e^j_{\varepsilon,+}$ *and* $e^j_{\varepsilon,-}$ *for* $j = 1, 2$ *are quasi-isomorphisms.*

Proof The proof is parallel to that of Lemma 4.7. This time, we use the spectral sequence determined by a filtration $\{\widehat{\mathscr{F}}^a_{\varepsilon,p}\}_{p\in\mathbb{Z}}$, which is defined by

$$\widehat{\mathscr{F}}^a_{\varepsilon,p} := \bigoplus_{m \geq -p} \hat{C}^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m).$$

Then $e^j_{\varepsilon,+}$ and $e^j_{\varepsilon,-}$ for $j = 1, 2$ preserve the filtrations $\{\widehat{\mathscr{F}}^a_{\varepsilon,p}\}_{p\in\mathbb{Z}}$ and $\{\overline{\mathscr{F}}^a_{\varepsilon,p}\}_{p\in\mathbb{Z}}$. By Lemma 3.17, they induce an isomorphism on the first page. Now the assertion follows from Lemma 4.2. $\qquad\square$

4.3 The limit of $\varepsilon \to 0$

In this section, we define a transition map

$$k_{(\varepsilon',\delta'),(\varepsilon,\delta)} \colon H^{<a}_*(\varepsilon', \delta') \to H^{<a}_*(\varepsilon, \delta)$$

for every $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$ and $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathscr{T}_a$ with $\varepsilon' \leq \varepsilon$, by using $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathscr{T}}_a$ satisfying

$$(24) \qquad\qquad \varepsilon \leq \bar{\varepsilon}, \quad e_+\bar{\delta} = (i_{\varepsilon,\bar{\varepsilon}})_*\delta, \quad e_-\bar{\delta} = (i_{\varepsilon',\bar{\varepsilon}})_*\delta'.$$

In fact, $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}$ is an isomorphism. We also prove that $\big(\{H^{<a}_*(\varepsilon, \delta)\}_{(\varepsilon,\delta)\in\mathscr{T}_a}, \{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon'\leq\varepsilon}\big)$ forms an inverse system.

4.3.1 Construction of transition maps Let us first prove the existence of the above $(\bar{\varepsilon}, \bar{\delta})$.

Lemma 4.12 *For* $(\varepsilon, \delta), (\varepsilon', \delta) \in \mathscr{T}_a$ *with* $\varepsilon' \leq \varepsilon$, *there exists* $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathscr{T}}_a$ *satisfying* (24).

Proof Let us take $\bar{\varepsilon} := C\varepsilon$ for the constant C in Lemma 3.8, and rewrite $\delta_+ := (i_{\varepsilon,\bar{\varepsilon}})_*\delta$ and $\delta_- := (i_{\varepsilon',\bar{\varepsilon}})_*\delta'$ for short. Since

$$(\mathrm{ev}_0)_*[\delta - (i_{\varepsilon',\varepsilon})_*\delta'] = \mathrm{Th}_\varepsilon - (i_{\varepsilon',\varepsilon})_*\mathrm{Th}_{\varepsilon'} = 0 \in H^{\mathrm{dR}}_{n-d}(N_\varepsilon),$$

Proposition 3.9 shows that there exists $\theta \in C^{\mathrm{dR}}_{n-d+1}(S_{\bar{\varepsilon}})$ such that

$$\partial\theta = (i_{\varepsilon,\bar{\varepsilon}})_*(\delta - (i_{\varepsilon',\varepsilon})_*\delta') = \delta_+ - \delta_-.$$

Let $\kappa : \mathbb{R} \to [0,1]$ be a C^∞ function such that $\kappa(r) = 1$ if $1 \leq r$ and $\kappa(r) = 0$ if $r \leq -1$. We take chains $\beta_+, \beta_- \in \bar{C}^{\mathrm{dR}}_0(\{0\})$ defined by

$$\beta_+ := [\mathbb{R}, \mathrm{id}_\mathbb{R}, (\mathrm{id}_{\mathbb{R}_{\geq 1}}, \mathrm{id}_{\mathbb{R}_{\leq -1}}), \kappa], \quad \beta_- := [\mathbb{R}, \mathrm{id}_\mathbb{R}, (\mathrm{id}_{\mathbb{R}_{\geq 1}}, \mathrm{id}_{\mathbb{R}_{\leq -1}}), 1 - \kappa].$$

Now we define $\bar{\delta}$ by

$$\bar{\delta} := \beta_+ \times (\bar{\iota}\delta_+) + \beta_- \times (\bar{\iota}\delta_-) + (\partial\beta_+) \times (\bar{\iota}\theta) \in \bar{C}^{\mathrm{dR}}_{n-d}(S_{\bar{\varepsilon}}).$$

This satisfies condition (24). Moreover, $\partial\bar{\delta} = 0$ and $(\mathrm{ev}_0)_*[e_+\bar{\delta}] = (\mathrm{ev}_0)_*[\delta_+] = \mathrm{Th}_{\bar{\varepsilon}}$. Now it is clear that $(\bar{\varepsilon}, \bar{\delta}) = (C\varepsilon, \bar{\delta})$ satisfies the two conditions to be an element of $\overline{\mathcal{T}}_a$. $\qquad\square$

From Lemmas 4.3 and 4.7, we can define isomorphisms

$$f_{(\bar{\varepsilon},\bar{\delta}),+} := (j_{\varepsilon,\bar{\varepsilon}})^{-1}_* \circ (e_{\bar{\varepsilon},+})_* : \overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}) \to H^{<a}_*(\varepsilon, \delta),$$

$$f_{(\bar{\varepsilon},\bar{\delta}),-} := (j_{\varepsilon',\bar{\varepsilon}})^{-1}_* \circ (e_{\bar{\varepsilon},-})_* : \overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}) \to H^{<a}_*(\varepsilon', \delta')$$

such that the following diagrams commute:

$$
\begin{array}{ccc}
\overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}) & \xrightarrow{f_{(\bar{\varepsilon},\bar{\delta}),+}} & H^{<a}_*(\varepsilon, \delta) \\
{\scriptstyle (e_{\bar{\varepsilon},+})_*} \downarrow & & \downarrow {\scriptstyle (j_{\varepsilon,\bar{\varepsilon}})_*} \\
H^{<a}_*(\bar{\varepsilon}, e_+\bar{\delta}) & = & H^{<a}_*(\bar{\varepsilon}, (i_{\varepsilon,\bar{\varepsilon}})_*\delta)
\end{array}
\qquad
\begin{array}{ccc}
\overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}) & \xrightarrow{f_{(\bar{\varepsilon},\bar{\delta}),-}} & H^{<a}_*(\varepsilon', \delta') \\
{\scriptstyle (e_{\bar{\varepsilon},-})_*} \downarrow & & \downarrow {\scriptstyle (j_{\varepsilon',\bar{\varepsilon}})_*} \\
H^{<a}_*(\bar{\varepsilon}, e_-\bar{\delta}) & = & H^{<a}_*(\bar{\varepsilon}, (i_{\varepsilon',\bar{\varepsilon}})_*\delta')
\end{array}
$$

We define an isomorphism

$$k_{(\bar{\varepsilon},\bar{\delta})} := f_{(\bar{\varepsilon},\bar{\delta}),+} \circ (f_{(\bar{\varepsilon},\bar{\delta}),-})^{-1} : H^{<a}_*(\varepsilon', \delta') \to H^{<a}_*(\varepsilon, \delta).$$

Later, we will prove the independence on $(\bar{\varepsilon}, \bar{\delta})$ (Corollary 4.16), and this is the transition map $k_{(\varepsilon',\delta),(\varepsilon,\delta)}$ we need.

Lemma 4.13 When $(\varepsilon', \delta') = (\varepsilon, \delta) \in \mathcal{T}_a$, we may take $(\varepsilon, \bar{\iota}\delta) \in \overline{\mathcal{T}}_a$ as an element satisfying (24). In this case, we have

$$k_{(\varepsilon,\bar{\iota}\delta)} = \mathrm{id}_{H^{<a}_*(\varepsilon,\delta)}.$$

Proof To prove this assertion, let us introduce a chain map $\bar{\iota}_\varepsilon$ from $(C^{<a}_*(\varepsilon), D_\delta)$ to $(\overline{C}^{<a}_*(\varepsilon), \overline{D}_{\bar{\iota}\delta})$ induced by $\bar{\iota} : C^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m) \to \overline{C}^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m)$ (see (17)) for all $m \in \mathbb{Z}_{\geq 0}$. This satisfies $e_{\varepsilon,+} \circ \bar{\iota}_\varepsilon = \mathrm{id}_{C^{<a}_*(\varepsilon)} = e_{\varepsilon,-} \circ \bar{\iota}_\varepsilon$, and thus

(25) $$\qquad (e_{\varepsilon,+})_* = (\bar{\iota}_\varepsilon)^{-1}_* = (e_{\varepsilon,-})_* : \overline{H}^{<a}_*(\varepsilon, \bar{\iota}\delta) \to H^{<a}_*(\varepsilon, \delta).$$

Therefore, $k_{(\varepsilon,\bar{\iota}\delta)} = (e_{\varepsilon,+})_* \circ (e_{\varepsilon,-})^{-1}_* = \mathrm{id}_{H^{<a}_*(\varepsilon,\delta)}$. $\qquad\square$

4.3.2 Compositions Next, we think about compositions of maps in the set $\{k_{(\bar{\varepsilon},\bar{\delta})}\}_{(\bar{\varepsilon},\bar{\delta})\in\overline{\mathcal{T}}_a}$. For $(\varepsilon,\delta),(\varepsilon',\delta'),(\varepsilon'',\delta'')\in\mathcal{T}_a$ with $\varepsilon''\leq\varepsilon'\leq\varepsilon$, suppose that we have chosen $(\bar{\varepsilon},\bar{\delta}),(\bar{\varepsilon}',\bar{\delta}'),(\widetilde{\varepsilon},\widetilde{\delta})\in\overline{\mathcal{T}}_a$ satisfying

$$e_+\bar{\delta}=(i_{\varepsilon,\bar{\varepsilon}})_*\delta, \quad e_+\bar{\delta}'=(i_{\varepsilon',\bar{\varepsilon}'})_*\delta', \quad e_+\widetilde{\delta}=(i_{\varepsilon,\widetilde{\varepsilon}})_*\delta,$$
$$e_-\bar{\delta}=(i_{\varepsilon',\bar{\varepsilon}})_*\delta', \quad e_-\bar{\delta}'=(i_{\varepsilon'',\bar{\varepsilon}'})_*\delta'', \quad e_-\widetilde{\delta}=(i_{\varepsilon'',\widetilde{\varepsilon}})_*\delta''.$$

In this situation, let us first prove the following lemma.

Lemma 4.14 *There exists $\hat{\varepsilon}\in(0,\varepsilon_0/(5C_0)]$ and $\hat{\delta}\in\hat{C}^{\mathrm{dR}}_{n-d}(S_{\hat{\varepsilon}})$ such that $\partial\hat{\delta}=0$ and*

$$e^1_+\hat{\delta}=(i_{\widetilde{\varepsilon},\hat{\varepsilon}})_*\widetilde{\delta}, \quad e^1_-\hat{\delta}=(i_{\bar{\varepsilon}',\hat{\varepsilon}})_*\bar{\delta}', \quad e^2_+\hat{\delta}=(i_{\bar{\varepsilon},\hat{\varepsilon}})_*\bar{\delta}, \quad e^2_-\hat{\delta}=(i_{\varepsilon'',\hat{\varepsilon}})_*(\bar{\iota}\delta'').$$

Proof Let us take $\rho:=C\cdot\max\{\bar{\varepsilon},\bar{\varepsilon}',\widetilde{\varepsilon}\}$, $\hat{\varepsilon}:=C\rho$ and rewrite $\bar{\delta}_+:=(i_{\widetilde{\varepsilon},\rho})_*\widetilde{\delta}$ and $\bar{\delta}_-:=(i_{\bar{\varepsilon}',\rho})_*\bar{\delta}'$ for short. Since

$$(\mathrm{ev}_0)_*\circ(e_+)_*[(i_{\widetilde{\varepsilon},C^{-1}\rho})_*\widetilde{\delta}-(i_{\bar{\varepsilon}',C^{-1}\rho})_*\bar{\delta}']=(i_{\widetilde{\varepsilon},C^{-1}\rho})_*(\mathrm{Th}_{\widetilde{\varepsilon}})-(i_{\bar{\varepsilon}',C^{-1}\rho})_*(\mathrm{Th}_{\bar{\varepsilon}'})=0\in H^{\mathrm{dR}}_*(N_{C^{-1}\rho}),$$

Proposition 3.18 shows that there exists $\bar{\theta}_1\in\overline{C}^{\mathrm{dR}}_{n-d+1}(S_\rho)$ such that $\partial\bar{\theta}_1=\bar{\delta}_+-\bar{\delta}_-$. We define $\hat{\kappa}^1\colon\mathbb{R}\times\mathbb{R}\to[0,1]$, $(r_1,r_2)\mapsto\kappa(r_1)$, where κ is the function that appeared in the proof of Lemma 4.12. Then we take chains $\hat{\beta}_+,\hat{\beta}_-\in\hat{C}^{\mathrm{dR}}_0(\{0\})$ defined by

$$\hat{\beta}^1_+:=[\mathbb{R}^2,\mathrm{id}_{\mathbb{R}^2},(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-),\hat{\kappa}], \quad \hat{\beta}^1_-:=[\mathbb{R}^2,\mathrm{id}_{\mathbb{R}^2},(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-),1-\hat{\kappa}],$$

where $\tau^j_+=\mathrm{id}_{\{r_j\geq1\}}$ and $\tau^j_-=\mathrm{id}_{\{r_j\leq-1\}}$ for $j=1,2$. We define

$$\xi:=\hat{\beta}^1_+\times(\hat{\iota}^1\bar{\delta}_+)+\hat{\beta}^1_-\times(\hat{\iota}^1\bar{\delta}_-)+(\partial\hat{\beta}^1_+)\times(\hat{\iota}^1(\bar{\theta}_1-\bar{\iota}e_-\bar{\theta}_1))\in\hat{C}^{\mathrm{dR}}_{n-d}(S_\rho).$$

This chain satisfies $\partial\xi=0$ (note that $e_-(\partial\bar{\theta}_1)=e_-\bar{\delta}_+-e_-\bar{\delta}_-=0$). Moreover,

$$e^1_+\xi=(i_{\widetilde{\varepsilon},\rho})_*\widetilde{\delta}, \quad e^1_-\xi=(i_{\bar{\varepsilon}',\rho})_*\bar{\delta}', \quad e^2_-\xi=(i_{\varepsilon'',\rho})_*(\bar{\iota}\delta'')$$

hold. The former two equations are easy to check. The third equation can be checked as follows: since $e^2_-\circ\hat{\iota}^1=\bar{\iota}\circ e_-$, we have $e^2_-(\hat{\iota}^1(\bar{\theta}_1-\bar{\iota}e_-\bar{\theta}_1))=0$ and thus

$$e^2_-\xi=(e^2_-\hat{\beta}^1_+)\times\big(\bar{\iota}((i_{\varepsilon'',\rho})_*\delta'')\big)+(e^2_-\hat{\beta}^1_-)\times\big(\bar{\iota}((i_{\varepsilon'',\rho})_*\delta'')\big)=(i_{\varepsilon'',\rho})_*(\bar{\iota}\delta'').$$

Let us write $\underline{\delta}:=e^2_+\xi\in\overline{C}^{\mathrm{dR}}_{n-d}(S_\rho)$ and consider the difference of chains $\underline{\delta}-(i_{\bar{\varepsilon},\rho})_*\bar{\delta}$. We claim that there exists $\bar{\theta}_2\in\overline{C}^{\mathrm{dR}}_{n-d+1}(S_{\hat{\varepsilon}})$ such that

(26) $$\partial\bar{\theta}_2=(i_{\rho,\hat{\varepsilon}})_*\underline{\delta}-(i_{\bar{\varepsilon},\hat{\varepsilon}})_*\bar{\delta}, \quad e_+\bar{\theta}_2=e_-\bar{\theta}_2=0.$$

We prove this claim. Since $e_+\circ e^2_+=e_+\circ e^1_+$ and $e_-\circ e^2_+=e_+\circ e^1_-$, we have

$$(e_+)_*[\underline{\delta}-(i_{\bar{\varepsilon},\rho})_*\bar{\delta}]=(e_+\circ e^1_+)_*[\xi]-(i_{\varepsilon,\rho})_*[\delta]=0\in H^{\mathrm{dR}}_{n-d}(S_\rho),$$
$$(e_-)_*[\underline{\delta}-(i_{\bar{\varepsilon},\rho})_*\bar{\delta}]=(e_+\circ e^1_-)_*[\xi]-(i_{\varepsilon',\rho})_*[\delta']=0\in H^{\mathrm{dR}}_{n-d}(S_\rho).$$

From Lemma 3.16, there exists $\bar{\theta}'_2\in\overline{C}^{\mathrm{dR}}_{n-d+1}(S_\rho)$ such that $\partial\bar{\theta}'_2=\underline{\delta}-(i_{\bar{\varepsilon},\rho})_*\bar{\delta}$ and $\partial(e_+\bar{\theta}'_2)=\partial(e_-\bar{\theta}'_2)=0$. Since $(\mathrm{ev}_0)_*[e_+\bar{\theta}'_2]$ and $(\mathrm{ev}_0)_*[e_-\bar{\theta}'_2]$ are contained in $H^{\mathrm{dR}}_{n-d+1}(N_\rho)=\{0\}$, Proposition 3.18 shows that

there exist $\varphi_+, \varphi_- \in C^{\mathrm{dR}}_{n-d+2}(S_{\widehat{\varepsilon}})$ such that $\partial\varphi_+ = (i_{\rho,\widehat{\varepsilon}})_*(e+\bar{\theta}'_2)$ and $\partial\varphi_- = (i_{\rho,\widehat{\varepsilon}})_*(e-\bar{\theta}'_2)$. Then, by using β_+ and β_- in the proof of Lemma 4.12, we define

$$\bar{\theta}_2 := (i_{\rho,\widehat{\varepsilon}})_*\bar{\theta}'_2 - \partial(\beta_+ \times \bar{\imath}\varphi_+) - \partial(\beta_- \times \bar{\imath}\varphi_-) \in \overline{C}^{\mathrm{dR}}_{n-d+1}(S_{\widehat{\varepsilon}}),$$

and this chain satisfies (26).

We take $\widehat{\kappa}^2 \colon \mathbb{R}^2 \to [0,1]$, $(r_1, r_2) \mapsto \kappa(r_2)$, and

$$\widehat{\beta}^2_+ := [\mathbb{R}^2, \mathrm{id}_{\mathbb{R}^2}, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \widehat{\kappa}^2] \in \widehat{C}^{\mathrm{dR}}_0(\{0\}).$$

Finally, we define a chain

$$\widehat{\delta} := (i_{\rho,\widehat{\varepsilon}})_*\xi - \partial(\widehat{\beta}^2_+ \times \widehat{\imath}^2\bar{\theta}_2) \in \widehat{C}^{\mathrm{dR}}_{n-d}(S_{\widehat{\varepsilon}}).$$

This satisfies $\partial\widehat{\delta} = 0$ and the required four equations. $\qquad\square$

Lemma 4.14 is applied to prove the next proposition.

Proposition 4.15 $\qquad k_{(\bar{\varepsilon},\bar{\delta})} \circ k_{(\bar{\varepsilon}',\bar{\delta}')} = k_{(\widetilde{\varepsilon},\widetilde{\delta})} \colon H^{<a}_*(\varepsilon'', \delta'') \to H^{<a}_*(\varepsilon, \delta).$

Proof Let $(\widehat{\varepsilon}, \widehat{\delta})$ be the pair of Lemma 4.14. From Lemmas 4.8 and 4.11, we can define isomorphisms

$$f^1_{(\widehat{\varepsilon},\widehat{\delta}),+} := (j_{\bar{\varepsilon},\widehat{\varepsilon}})^{-1}_* \circ (e^1_{\widehat{\varepsilon},+})_* \colon \widehat{H}^{<a}_*(\widehat{\varepsilon}, \widehat{\delta}) \to \overline{H}^{<a}_*(\widetilde{\varepsilon}, \widetilde{\delta}),$$

$$f^1_{(\widehat{\varepsilon},\widehat{\delta}),-} := (j_{\bar{\varepsilon}',\widehat{\varepsilon}})^{-1}_* \circ (e^1_{\widehat{\varepsilon},-})_* \colon \widehat{H}^{<a}_*(\widehat{\varepsilon}, \widehat{\delta}) \to \overline{H}^{<a}_*(\bar{\varepsilon}', \bar{\delta}'),$$

$$f^2_{(\widehat{\varepsilon},\widehat{\delta}),+} := (j_{\bar{\varepsilon},\widehat{\varepsilon}})^{-1}_* \circ (e^2_{\widehat{\varepsilon},+})_* \colon \widehat{H}^{<a}_*(\widehat{\varepsilon}, \widehat{\delta}) \to \overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}).$$

From the definitions of $k_{(\bar{\varepsilon},\bar{\delta})}$, $k_{(\bar{\varepsilon}',\bar{\delta}')}$ and $k_{(\widetilde{\varepsilon},\widetilde{\delta})}$, it suffices to show that the following diagram commutes:
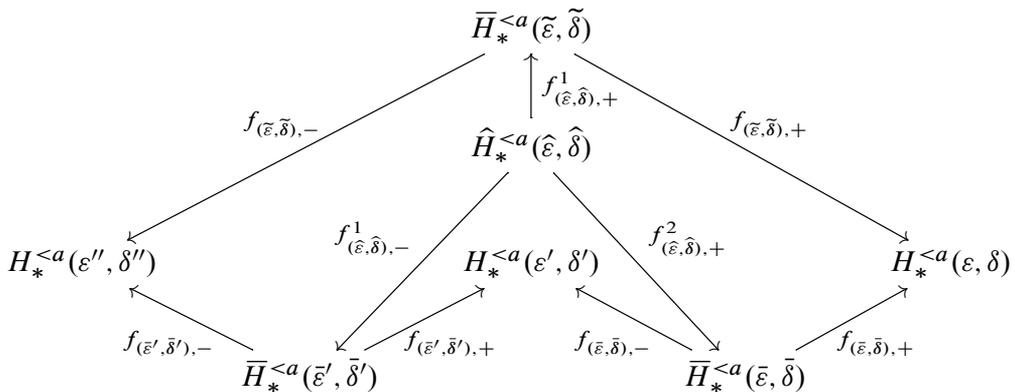
Note that all maps appearing in the diagram are isomorphisms. We need to prove the three equations

(27)
$$f_{(\widetilde{\varepsilon},\widetilde{\delta}),+} \circ f^1_{(\widehat{\varepsilon},\widehat{\delta}),+} = f_{(\bar{\varepsilon},\bar{\delta}),+} \circ f^2_{(\widehat{\varepsilon},\widehat{\delta}),+},$$
$$f_{(\bar{\varepsilon},\bar{\delta}),-} \circ f^2_{(\widehat{\varepsilon},\widehat{\delta}),+} = f_{(\bar{\varepsilon}',\bar{\delta}'),+} \circ f^1_{(\widehat{\varepsilon},\widehat{\delta}),-},$$
$$f_{(\widetilde{\varepsilon},\widetilde{\delta}),-} \circ f^1_{(\widehat{\varepsilon},\widehat{\delta}),+} = f_{(\bar{\varepsilon}',\bar{\delta}'),-} \circ f^1_{(\widehat{\varepsilon},\widehat{\delta}),-}.$$

Let us prove the first equation. Returning to the definition of $f_{(\widetilde{\varepsilon},\widetilde{\delta}),+}$ and $f^1_{(\widehat{\varepsilon},\widehat{\delta}),+}$,

$$
\begin{aligned}
f_{(\widetilde{\varepsilon},\widetilde{\delta}),+} \circ f^1_{(\widehat{\varepsilon},\widehat{\delta}),+} &= (j_{\varepsilon,\widetilde{\varepsilon}})^{-1}_* \circ (e_{\widetilde{\varepsilon},+})_* \circ (j_{\widetilde{\varepsilon},\widehat{\varepsilon}})^{-1}_* \circ (e^1_{\widehat{\varepsilon},+})_* \\
&= (j_{\varepsilon,\widehat{\varepsilon}})^{-1}_* \circ (e_{\widehat{\varepsilon},+} \circ e^1_{\widehat{\varepsilon},+})_* \\
&= (j_{\varepsilon,\widehat{\varepsilon}})^{-1}_* \circ (e_{\widehat{\varepsilon},+} \circ e^2_{\widehat{\varepsilon},+})_* \\
&= (j_{\varepsilon,\bar{\varepsilon}})^{-1}_* \circ (e_{\bar{\varepsilon},+})_* \circ (j_{\bar{\varepsilon},\widehat{\varepsilon}})^{-1}_* \circ (e^2_{\widehat{\varepsilon},+})_* \\
&= f_{(\bar{\varepsilon},\bar{\delta}),+} \circ f^2_{(\widehat{\varepsilon},\widehat{\delta}),+}.
\end{aligned}
$$

Here, the second and fourth equalities follow from the obvious equations

$$
(j_{\widetilde{\varepsilon},\widehat{\varepsilon}})_* \circ (e_{\widetilde{\varepsilon},+})_* = (e_{\widehat{\varepsilon},+})_* \circ (j_{\widetilde{\varepsilon},\widehat{\varepsilon}})_*, \quad (j_{\bar{\varepsilon},\widehat{\varepsilon}})_* \circ (e_{\bar{\varepsilon},+})_* = (e_{\widehat{\varepsilon},+})_* \circ (j_{\bar{\varepsilon},\widehat{\varepsilon}})_*.
$$

The third equality follows from

$$
e_{\widehat{\varepsilon},+} \circ e^1_{\widehat{\varepsilon},+} = e_{\widehat{\varepsilon},+} \circ e^2_{\widehat{\varepsilon},+},
$$

which comes from the relation $e_+ \circ e^1_+ = e_+ \circ e^2_+$ of (20). The second equation of (27) can be proved similarly by applying

$$
e_{\widehat{\varepsilon},-} \circ e^2_{\widehat{\varepsilon},+} = e_{\widehat{\varepsilon},+} \circ e^1_{\widehat{\varepsilon},-},
$$

which comes from the relation $e_- \circ e^2_+ = e_+ \circ e^1_-$ of (20). To prove the third equation of (27), there is one nontrivial matter: we need to apply

$$
(e_{\widehat{\varepsilon},-} \circ e^1_{\widehat{\varepsilon},+})_* = (e_{\widehat{\varepsilon},-} \circ e^1_{\widehat{\varepsilon},-})_* \colon \widehat{H}^{<a}_*(\widehat{\varepsilon},\widehat{\delta}) \to H^{<a}_*(\widehat{\varepsilon}, (i_{\varepsilon'',\widehat{\varepsilon}})_*\delta''),
$$

which does not follow from (20) directly. To check this equation, we consider the following diagram including $\overline{H}^{<a}_*(\widehat{\varepsilon}, (i_{\varepsilon'',\widehat{\varepsilon}})_*(\bar{\imath}\delta''))$:
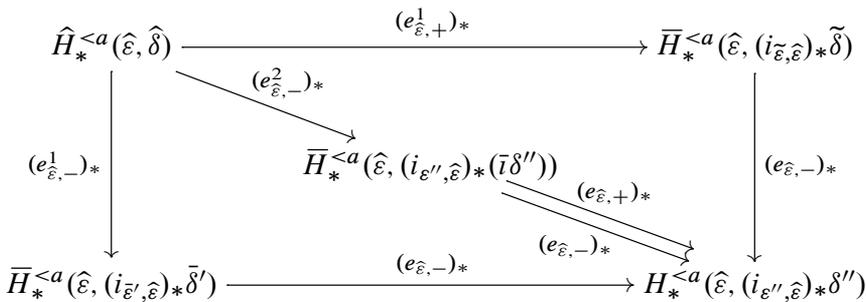
Then

$$
e_{\widehat{\varepsilon},-} \circ e^1_{\widehat{\varepsilon},+} = e_{\widehat{\varepsilon},+} \circ e^2_{\widehat{\varepsilon},-}, \quad e_{\widehat{\varepsilon},-} \circ e^1_{\widehat{\varepsilon},-} = e_{\widehat{\varepsilon},-} \circ e^2_{\widehat{\varepsilon},-}
$$

follow directly from the relations $e_- \circ e^1_+ = e_+ \circ e^2_-$ and $e_- \circ e^1_- = e_- \circ e^2_-$ of (20). If we rewrite $(i_{\varepsilon'',\widehat{\varepsilon}})_*\delta''$ by $\delta^\#$, equation (25) shows that

$$
(e_{\widehat{\varepsilon},+})_* = (e_{\widehat{\varepsilon},-})_* \colon \overline{H}^{<a}_*(\widehat{\varepsilon}, \bar{\imath}\delta^\#) \to H^{<a}_*(\widehat{\varepsilon}, \delta^\#).
$$

From the above diagram, we get $(e_{\widehat{\varepsilon},-})_* \circ (e^1_{\widehat{\varepsilon},+})_* = (e_{\widehat{\varepsilon},-})_* \circ (e^1_{\widehat{\varepsilon},-})_*.$ $\qquad\square$

Corollary 4.16 *For $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$, the isomorphism*

$$k_{(\bar{\varepsilon}, \bar{\delta})} \colon H_*^{<a}(\varepsilon', \delta') \to H_*^{<a}(\varepsilon, \delta)$$

does not depend on the choice of $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathcal{T}}_a$ satisfying (24).

Proof Let $(\bar{\varepsilon}, \bar{\delta})$ and $(\widetilde{\varepsilon}, \widetilde{\delta})$ be two arbitrary choices from $\overline{\mathcal{T}}_a$ satisfying (24). We apply Proposition 4.15 to the case where $(\varepsilon'', \delta'') = (\varepsilon', \delta')$ and $(\bar{\varepsilon}', \bar{\delta}') = (\varepsilon', \bar{\iota}\delta')$. By Lemma 4.13, $k_{(\varepsilon', \bar{\iota}\delta')}$ is equal to the identity map on $H_*^{<a}(\varepsilon', \delta')$, so we get

$$k_{(\bar{\varepsilon}, \bar{\delta})} \circ \mathrm{id}_{H_*^{<a}(\varepsilon', \delta')} = k_{(\widetilde{\varepsilon}, \widetilde{\delta})}. \qquad \square$$

From this result, we may rewrite $k_{(\bar{\varepsilon}, \bar{\delta})} \colon H_*^{<a}(\varepsilon', \delta') \to H_*^{<a}(\varepsilon, \delta)$ as $k_{(\varepsilon', \delta'),(\varepsilon, \delta)}$. The equations of Lemma 4.13 and Proposition 4.15 can be rewritten as

$$k_{(\varepsilon, \delta),(\varepsilon, \delta)} = \mathrm{id}_{H_*^{<a}(\varepsilon, \delta)}, \quad k_{(\varepsilon', \delta'),(\varepsilon, \delta)} \circ k_{(\varepsilon'', \delta''),(\varepsilon', \delta')} = k_{(\varepsilon'', \delta''),(\varepsilon, \delta)}.$$

Now, $\mathcal{T}_a$ becomes a directed set by the relation $(\varepsilon', \delta') \leq (\varepsilon, \delta)$ if and only if $\varepsilon' \leq \varepsilon$. Then we obtain an inverse system

$$\left(\{ H_*^{<a}(\varepsilon, \delta) \}_{(\varepsilon, \delta) \in \mathcal{T}_a}, \{ k_{(\varepsilon', \delta'),(\varepsilon, \delta)} \}_{\varepsilon' \leq \varepsilon} \right)$$

and its inverse limit

$$H_*^{<a}(Q, K) := \varprojlim_{\varepsilon \to 0} H_*^{<a}(\varepsilon, \delta)$$

is defined.

4.3.3 Spectral sequence Lastly, we extend the above work to spectral sequences. For $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$, we take $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathcal{T}}_a$ satisfying (24). Chain maps $(j_{\varepsilon, \bar{\varepsilon}})_*$, $e_{\bar{\varepsilon}+}$, $e_{\bar{\varepsilon}, -}$ and $(j_{\varepsilon', \bar{\varepsilon}})_*$ induce a zig zag of morphisms of spectral sequences

$$(28) \qquad E_{(\varepsilon', \delta')}^{<a} \xrightarrow{(j_{\varepsilon', \bar{\varepsilon}})_*} E_{(\bar{\varepsilon}, (i_{\varepsilon', \bar{\varepsilon}})_* \delta')}^{<a} \xleftarrow{(e_{\bar{\varepsilon}-})_*} \overline{E}_{(\bar{\varepsilon}, \bar{\delta})}^{<a} \xrightarrow{(e_{\bar{\varepsilon}, +})_*} E_{(\bar{\varepsilon}, (i_{\varepsilon, \bar{\varepsilon}})_* \delta)}^{<a} \xleftarrow{(j_{\varepsilon, \bar{\varepsilon}})_*} E_{(\varepsilon, \delta)}^{<a}.$$

All of them are isomorphisms. Let $k_{(\varepsilon', \delta'),(\varepsilon, \delta)} \colon E_{(\varepsilon', \delta')}^{<a} \to E_{(\varepsilon, \delta)}^{<a}$ denote the composition of these maps. (The independence on (ε, δ) can be proved in the same way as Corollary 4.16.)

Proposition 4.17 *There exists a spectral sequence $E^{<a} = \left(\{ (E^{<a})_{p,q}^r \}, \{ (d^{<a})_{p,q}^r \} \right)$ which converges to $H_*^{<a}(Q, K)$ in the sense of [21, Bounded Convergence 5.2.5] such that*

$$(E^{<a})_{-m,q}^1 = \begin{cases} H_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0 \end{cases}$$

Proof On the first page, the middle two maps of (28) have the form

$$H_{q+p(d-1)}(\Sigma_{-p}^{a-p\bar{\varepsilon}}, \Sigma_{-p}^0) \xleftarrow{(e_+)_*} \overline{H}_{q+p(d-1)}(\Sigma_{-p}^{a-p\bar{\varepsilon}}, \Sigma_{-p}^0) \xrightarrow{(e_-)_*} H_{q+p(d-1)}(\Sigma_{-p}^{a-p\bar{\varepsilon}}, \Sigma_{-p}^0).$$

Since $(e_-)_* = (\bar{\iota})_*^{-1} = (e_+)_*$, this composition is equal to the identity map. Therefore, $k_{(\varepsilon', \delta'),(\varepsilon, \delta)}$ is equal to $(j_{\varepsilon', \varepsilon})_* \colon H_{q+p(d-1)}(\Sigma_{-p}^{a-p\varepsilon'}, \Sigma_{-p}^0) \to H_{q+p(d-1)}(\Sigma_{-p}^{a-p\varepsilon}, \Sigma_{-p}^0)$ on the (p, q)-term with $p \leq 0$ of the first page.

By $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon' \le \varepsilon}$, we define $(E^{<a})^r_{p,q} := \varprojlim_{\varepsilon \to 0} (E^{<a}_{(\varepsilon,\delta)})^r_{p,q}$. Moreover,

$$(d^{<a})^r_{p,q} \colon (E^{<a})^r_{p,q} \to (E^{<a})^r_{p-r,q+r-1}$$

is defined to be the map induced by $\{(d^{<a}_{(\varepsilon,\delta)})^r_{p,q}\}_{(\varepsilon,\delta) \in \mathcal{T}_a}$. Then the (p,q)-term of the first page is given by

$$(E^{<a})^1_{p,q} = \varprojlim_{\varepsilon \to 0} H_{q+p(d-1)}(\Sigma^{a-p\varepsilon}_{-p}, \Sigma^0_{-p}) = H_{q+p(d-1)}(\Sigma^a_{-p}, \Sigma^0_{-p})$$

for $p \le 0$ and $(E^{<a})^1_{p,q} = 0$ for $p > 0$. Since $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon' \le \varepsilon}$ consists of isomorphisms, it is clear that $E^{<a}$ is a spectral sequence which converges to $H^{<a}_*(Q, K)$. $\qquad\square$

4.4 Definition of $H^{\text{string}}_*(Q, K)$

In this section, we define $I^{a,b} \colon H^{<a}_*(Q, K) \to H^{<b}_*(Q, K)$ for $a, b \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$ with $a \le b$ to get a direct system $\left(\{H^{<a}_*(Q, K)\}_{a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)}, \{I^{a,b}\}_{a \le b}\right)$. After defining $H^{\text{string}}_*(Q, K)$ as its direct limit, we will give a structure of a unital graded $\mathbb{R}$-algebra.

4.4.1 The limit of $a \to \infty$ Let a and b be the above real numbers. For any $(\varepsilon,\delta) \in \mathcal{T}_a \cap \mathcal{T}_b$, we have considered in Section 4.1 a linear map

$$(I^{a,b}_\varepsilon)_* \colon H^{<a}_*(\varepsilon,\delta) \to H^{<b}_*(\varepsilon,\delta),$$

which is induced by the inclusion maps $I^{a,b}_\varepsilon \colon \Sigma^{a+m\varepsilon}_m \to \Sigma^{b+m\varepsilon}_m$ for all $m \in \mathbb{Z}_{\ge 0}$.

Lemma 4.18 *Suppose that $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathcal{T}}_a \cap \overline{\mathcal{T}}_b$ satisfies* (24) *for $(\varepsilon,\delta), (\varepsilon',\delta) \in \mathcal{T}_a \cap \mathcal{T}_b$ with $\varepsilon' \le \varepsilon$. Then the following diagram commutes:*

$$
\begin{array}{ccc}
H^{<a}_*(\varepsilon,\delta) & \xrightarrow{(I^{a,b}_\varepsilon)_*} & H^{<b}_*(\varepsilon,\delta) \\[4pt]
{\scriptstyle k_{(\varepsilon',\delta'),(\varepsilon,\delta)}}\Big\uparrow & & \Big\uparrow{\scriptstyle k_{(\varepsilon',\delta'),(\varepsilon,\delta)}} \\[4pt]
H^{<a}_*(\varepsilon',\delta') & \xrightarrow[(I^{a,b}_{\varepsilon'})_*]{} & H^{<b}_*(\varepsilon',\delta')
\end{array}
$$

Proof $I^{a,b}_{\bar{\varepsilon}}$ induces a linear map $(I^{a,b}_{\bar{\varepsilon}})_* \colon \overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}) \to \overline{H}^{<b}_*(\bar{\varepsilon}, \bar{\delta})$. It suffices to show that

$$(I^{a,b}_\varepsilon)_* \circ f_{(\bar{\varepsilon},\bar{\delta}),+} = f_{(\bar{\varepsilon},\bar{\delta}),+} \circ (I^{a,b}_{\bar{\varepsilon}})_* \colon \overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}) \to H^{<b}_*(\varepsilon,\delta),$$

$$(I^{a,b}_{\varepsilon'})_* \circ f_{(\bar{\varepsilon},\bar{\delta}),-} = f_{(\bar{\varepsilon},\bar{\delta}),-} \circ (I^{a,b}_{\bar{\varepsilon}})_* \colon \overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}) \to H^{<b}_*(\varepsilon',\delta').$$

Let us check the first equation. Since $j_{\varepsilon,\bar{\varepsilon}} \circ I^{a,b}_\varepsilon = I^{a,b}_{\bar{\varepsilon}} \circ j_{\varepsilon,\bar{\varepsilon}} \colon \Sigma^{a+m\varepsilon}_m \to \Sigma^{b+m\bar{\varepsilon}}_m$,

$$
\begin{aligned}
(I^{a,b}_\varepsilon)_* \circ f_{(\bar{\varepsilon},\bar{\delta}),+} &= (I^{a,b}_\varepsilon)_* \circ (j_{\varepsilon,\bar{\varepsilon}})^{-1}_* \circ (e_{\bar{\varepsilon},+})_* \\
&= (j_{\varepsilon,\bar{\varepsilon}})^{-1}_* \circ (I^{a,b}_{\bar{\varepsilon}})_* \circ (e_{\bar{\varepsilon},+})_* \\
&= (j_{\varepsilon,\bar{\varepsilon}})^{-1}_* \circ (e_{\bar{\varepsilon},+})_* \circ (I^{a,b}_{\bar{\varepsilon}})_* \\
&= f_{(\bar{\varepsilon},\bar{\delta}),+} \circ (I^{a,b}_{\bar{\varepsilon}})_*.
\end{aligned}
$$

The second equation can be proved by replacing ε and $+$ in the above computation by ε' and $-$. $\qquad\square$

This lemma implies that, after taking the limits of $(\varepsilon, \delta) \in \mathcal{T}_a \cap \mathcal{T}_b$ as $\varepsilon \to 0$, we get a linear map

$$I^{a,b} := \lim_{\varepsilon \to 0} (I_\varepsilon^{a,b})_* \colon H_*^{<a}(Q, K) \to H_*^{<b}(Q, K).$$

Furthermore, for $b \geq a$ and $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a \cap \mathcal{T}_b$ with $\varepsilon' \leq \varepsilon$, $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}$ induces an isomorphism from $H_*^{[a,b)}(\varepsilon', \delta')$ to $H_*^{[a,b)}(\varepsilon, \delta)$. Thus, we can also define

$$H_*^{[a,b)}(Q, K) := \varprojlim_{\varepsilon \to 0} H_*^{[a,b)}(\varepsilon, \delta).$$

Note that a long exact sequence

(29) $$\cdots \to H_*^{<a}(Q, K) \xrightarrow{I^{a,b}} H_*^{<b}(Q, K) \to H_*^{[a,b)}(Q, K) \to H_{*-1}^{<a}(Q, K) \xrightarrow{I^{a,b}} \cdots$$

is induced by (23).

From the direct system $\left(\{H_*^{<a}(Q, K)\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}, \{I^{a,b}\}_{a \leq b} \right)$, we finally define a graded $\mathbb{R}$-vector space

$$H_*^{\text{string}}(Q, K) := \varinjlim_{a \to \infty} H_*^{<a}(Q, K).$$

4.4.2 Product structure Let us see that $H_*^{\text{string}}(Q, K)$ has the structure of a unital associative graded $\mathbb{R}$-algebra. For any $a, a' \in \mathbb{R}_{>0} \cup \{\infty\}$, $m, m' \in \mathbb{Z}_{\geq 0}$ and $\varepsilon \in \left(0, \frac{1}{5}\varepsilon_0\right]$, there is a map

$$\Pi \colon \Sigma_m^{a+m\varepsilon} \times \Sigma_{m'}^{a'+m'\varepsilon} \to \Sigma_{m+m'}^{(a+a')+(m+m')\varepsilon}, \quad ((\gamma_k)_{k=1,\ldots,m}, (\gamma'_l)_{l=1,\ldots,m'}) \mapsto (\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_{m'}).$$

When $m = 0$ or $m' = 0$, Π is identified with the identity map. We define a linear map

(30) $$C_p^{<a}(\varepsilon) \otimes C_q^{<a'}(\varepsilon) \to C_{p+q}^{<a+a'}(\varepsilon), \quad x \otimes y \mapsto x \star y,$$

such that, for $x \in C_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$ and $y \in C_{q-m'(d-2)}^{\text{dR}}(\Sigma_{m'}^{a'+m'\varepsilon}, \Sigma_{m'}^0)$,

$$x \star y = (-1)^{mqd} \Pi_*(x \times y).$$

We note that the associative relation $(x \star y) \star z = x \star (y \star z)$ holds. Suppose that $a, a', a + a' \notin \mathcal{L}(K)$. For any $(\varepsilon, \delta) \in \mathcal{T}_a \cap \mathcal{T}_{a'} \cap \mathcal{T}_{a+a'}$, the above map is compatible with the differential D_δ. Indeed, for $x \in C_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$ and $y \in C_{q-m'(d-2)}^{\text{dR}}(\Sigma_{m'}^{a'+m'\varepsilon}, \Sigma_{m'}^0)$,

$$(-1)^{mqd} D_\delta(x \star y) = \partial(\Pi_*(x \times y)) + \sum_{k=1}^{m+m'} (-1)^{p+q+1+kd} f_{k,\delta}(\Pi_*(x \times y))$$

$$= \Pi_*(\partial x \times y) + \sum_{k=1}^{m} (-1)^{p+q+1+kd+s_0} \Pi_*(f_{k,\delta}(x) \times y)$$

$$+ (-1)^{p-m(d-2)} \Pi_*(x \times \partial y) + \sum_{l=1}^{m'} (-1)^{p+q+1+(l+m)d} \Pi_*(x \times f_{l,\delta}(y))$$

$$= (-1)^{mqd}(D_\delta(x)) \star y + (-1)^{p+mqd} x \star (D_\delta(y)).$$

Here, $s_0 := (q - m'(d-2))(d+1)$. This computation shows that

$$D_\delta(x \star y) = (D_\delta(x)) \star y + (-1)^p x \star (D_\delta(y))$$

for $x \in C_p^{<a}(\varepsilon)$ and $y \in C_q^{<a'}(\varepsilon)$. Therefore, (30) induces a linear map on homology,

$$H_p^{<a}(\varepsilon, \delta) \otimes H_q^{<a'}(\varepsilon, \delta) \to H_{p+q}^{<a+a'}(\varepsilon, \delta),$$

for every $(\varepsilon, \delta) \in \mathcal{T}_a \cap \mathcal{T}_{a'} \cap \mathcal{T}_{a+a'}$.

Likewise, let us define $x \bar{\star} y := (-1)^{mqd} \Pi_*(x \times y) \in \overline{C}_{p+q}^{<a+a'}(\bar{\varepsilon})$ for $x \in \overline{C}_{p-m(d-2)}^{\mathrm{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}, \Sigma_m^0)$ and $y \in \overline{C}_{q-m'(d-2)}^{\mathrm{dR}}(\Sigma_{m'}^{a'+m'\bar{\varepsilon}}, \Sigma_{m'}^0)$. Then $e_{\bar{\varepsilon},+}$ and $e_{\bar{\varepsilon},-}$ intertwine the $\star$-operation and the $\bar{\star}$-operation. This shows that the $\star$-operation is compatible with $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon' \leq \varepsilon}$. Therefore, in the limit $\varepsilon \to 0$, a linear map

$$H_p^{<a}(Q, K) \otimes H_q^{<a'}(Q, K) \to H_{p+q}^{<a+a'}(Q, K)$$

is induced. The commutativity with $\{I^{a,b}\}_{a \leq b}$ naturally holds. As a result, we get an associative product structure on $H_*^{\mathrm{string}}(Q, K)$. The element $1 \in H_0^{\mathrm{string}}(Q, K)$, which comes from $1 \in \mathbb{R} = C_0^{\mathrm{dR}}(\Sigma_0^a, \Sigma_0^0) \subset C_0^{<a}(\varepsilon, \delta)$, is the unit of this graded algebra.

4.4.3 Explicit choice of (ε, δ) Lastly, let us introduce a condition on $(\varepsilon, \delta) \in \mathcal{T}_a$ so that δ can be written explicitly. The concrete computations in Sections 5 and 6 become easier by choosing (ε, δ) satisfying this condition.

Suppose that there exists a fixed trivialization $\mathbb{R}^d \times K \to (TK)^\perp$ of the normal bundle of K which preserves orientations and fiber metrics. For every $\varepsilon \leq \varepsilon_0$, let us write $\mathbb{O}_\varepsilon := \{w \in \mathbb{R}^d \mid |w| < \frac{1}{2}\varepsilon\}$. Composing with the exponential map (3), we obtain a diffeomorphism

$$h: \mathbb{O}_\varepsilon \times K \to N_\varepsilon,$$

which preserves orientations. In this case, we say $(\varepsilon, \delta) \in \mathcal{T}_a$ is *standard with respect to h* if $\delta \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ has the form

$$(31) \qquad\qquad\qquad\qquad \delta = [N_\varepsilon, \psi_\varepsilon, h_*(\nu_\varepsilon \times 1)]$$

satisfying:

- $\psi_\varepsilon: N_\varepsilon \to S_\varepsilon$, $v \to (\sigma_j^v)_{j=1,2}$, is defined by

$$\sigma_1^v: \left[0, \tfrac{1}{2}\varepsilon\right] \to N_\varepsilon, \quad t \mapsto h\left(\frac{\varepsilon - 2t}{\varepsilon}w, x\right), \quad \sigma_2^v: \left[0, \tfrac{1}{2}\varepsilon\right] \to N_\varepsilon, \quad t \mapsto h\left(\frac{2t}{\varepsilon}w, x\right),$$

  for $v = h(w, x) \in N_\varepsilon$.

- $h_*(\nu_\varepsilon \times 1) \in \Omega_c^{n-d}(N_\varepsilon)$ for some $\nu_\varepsilon \in \Omega_c^d(\mathbb{O}_\varepsilon)$ with $\int_{\mathbb{O}_\varepsilon} \nu_\varepsilon = 1$.

Suppose that $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathcal{T}}_a$ satisfies (24) for $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$ which are given as above. Then we say $(\bar{\varepsilon}, \bar{\delta})$ is *standard with respect to h* if $\bar{\varepsilon} = \varepsilon$ and $\bar{\delta} \in \overline{C}_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ has the form

$$(32) \qquad\qquad \bar{\delta} = (-1)^n [\mathbb{R} \times N_\varepsilon, \bar{\psi}_{\varepsilon',\varepsilon}, (\mathrm{id}_{\mathbb{R}_{\geq 1} \times N_\varepsilon}, \mathrm{id}_{\mathbb{R}_{\leq -1} \times N_\varepsilon}), \bar{\eta}_{\varepsilon',\varepsilon}]$$

such that, for some C^∞ function $\kappa \colon \mathbb{R} \to [0,1]$ with $\kappa(r) = 0$ if $r \le -1$ and $\kappa(r) = 1$ if $r \ge 1$, the following hold:

- $\bar{\psi}_{\varepsilon',\varepsilon}(r,v) = (r,(\sigma_j^{(r,v)})_{j=1,2}) \in \mathbb{R} \times S_\varepsilon$ is defined by

$$\sigma_1^{(r,v)} \colon \left[0, \tfrac{1}{2}\varepsilon_r\right] \to N_\varepsilon, \quad t \mapsto h\left(\frac{\varepsilon_r - 2t}{\varepsilon_r} w, x\right), \qquad \sigma_2^{(r,v)} \colon \left[0, \tfrac{1}{2}\varepsilon_r\right] \to N_\varepsilon, \quad t \mapsto h\left(\frac{2t}{\varepsilon_r} w, x\right),$$

  for $r \in \mathbb{R}$, $v = h(w,x) \in N_\varepsilon$ and $\varepsilon_r := \kappa(r)\varepsilon + (1 - \kappa(r))\varepsilon'$.

- $(\mathrm{id}_\mathbb{R} \times h)^* \bar{\eta}_{\varepsilon',\varepsilon} = \kappa \times (\nu_\varepsilon \times 1) + (1 - \kappa) \times (\nu_{\varepsilon'} \times 1) + (d\kappa) \times (\theta \times 1)$ for some $\theta \in \Omega_c^{d-1}(\mathbb{O}_\varepsilon)$ satisfying $d\theta = \nu_\varepsilon - \nu_{\varepsilon'}$.

In summary, in order to compute $H_*^{\mathrm{string}}(Q,K)$ when a trivialization h is given, we only need to deal with $(\varepsilon,\delta) \in \mathcal{T}_a$ and $(\varepsilon,\bar{\delta}) \in \overline{\mathcal{T}}_a$ which are standard with respect to h.

4.5 Invariance

In this section, we prove the invariance of $H_*^{\mathrm{string}}(Q,K)$ up to isomorphism under changing auxiliary data. More precisely, we consider the dependence of the construction on the following data (see the beginning of Section 3):

- (a) a complete Riemannian metric g on Q;
- (b) a constant $C_0 \ge 1$ which bounds the speed of all $\gamma \in \Omega_K(Q)$;
- (c) a real number $\varepsilon_0 > 0$ which is the diameter (in the direction of fibers) of a tubular neighborhood N_{ε_0} of K;
- (d) a C^∞ function $\mu \colon \left[0, \tfrac{3}{2}\right] \to [0,1]$ which is used to define con_k.

Notation Let X be an arbitrary notation, which we have defined in the former sections. As a rule, in this section, if its definition depends on some auxiliary data S, we rewrite X as X_S when discussing the dependence on S.

Independence on μ We choose a C^∞ family $\bar{\mu} := (\mu_r)_{r \in \mathbb{R}}$ such that each μ_r satisfies the same condition as μ, and $\mu_r = \mu_{-1}$ if $r \le -1$ while $\mu_r = \mu_1$ if $r \ge 1$. Then a map con_{k,μ_r} is defined for each $r \in \mathbb{R}$. For $(\bar{\varepsilon},\bar{\delta}) \in \mathcal{T}_a$, let us define $\bar{f}_{k,\bar{\delta},\bar{\mu}} \colon \overline{C}_*^{\mathrm{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}) \to \overline{C}_*^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\bar{\varepsilon}})$ by replacing con_k in the definition of Φ_k of (21) by $\mathrm{con}_{k,\mu_{r(u)}}$. We also replace $\bar{f}_{k,\bar{\delta}}$ in the definition of $\overline{D}_{\bar{\delta}}$ by $\bar{f}_{k,\bar{\delta},\bar{\mu}}$ to define a linear map

$$\overline{D}_{\bar{\delta},\bar{\mu}} \colon \overline{C}_*^{<a}(\bar{\varepsilon}) \to \overline{C}_{*-1}^{<a}(\bar{\varepsilon}).$$

This satisfies $\overline{D}_{\bar{\delta},\bar{\mu}} \circ \overline{D}_{\bar{\delta},\bar{\mu}} = 0$, so we get a chain complex $(\overline{C}_*^{<a}(\bar{\varepsilon}), \overline{D}_{\bar{\delta},\bar{\mu}})$. Let $\overline{H}_*^{<a}(\varepsilon,\bar{\delta},\bar{\mu})$ denote its homology group.

We rewrite $e_{\varepsilon,+}, e_{\varepsilon,-} \colon \overline{C}_*^{<a}(\varepsilon) \to C_*^{<a}(\varepsilon)$ by $e_{\varepsilon,\bar{\mu},+}$ and $e_{\varepsilon,\bar{\mu},-}$, respectively. They induce

$$(e_{\varepsilon,\bar{\mu},+})_* \colon \overline{H}_*^{<a}(\varepsilon,\bar{\delta},\bar{\mu}) \to H_*^{<a}(\varepsilon,e_+\delta)_{\mu_1}, \qquad (e_{\varepsilon,\bar{\mu},-})_* \colon \overline{H}_*^{<a}(\varepsilon,\bar{\delta},\bar{\mu}) \to H_*^{<a}(\varepsilon,e_-\delta)_{\mu_{-1}}.$$

We can prove, just as Lemma 4.7, that they are isomorphisms. When $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathcal{T}}_{[a,b)}$ satisfies (24), we define an isomorphism $k_{(\bar{\varepsilon}, \bar{\delta}, \bar{\mu})} : H_*^{<a}(\varepsilon', \delta')_{\mu_{-1}} \to H_*^{<a}(\varepsilon, \delta)_{\mu_1}$ by

$$k_{(\bar{\varepsilon}, \bar{\delta}, \bar{\mu})} := ((j_{\varepsilon, \bar{\varepsilon}})_*^{-1} \circ (e_{\bar{\varepsilon}, \bar{\mu}, +})_*) \circ ((j_{\varepsilon', \bar{\varepsilon}})_*^{-1} \circ (e_{\bar{\varepsilon}, \bar{\mu}, -})_*)^{-1}.$$

As in Proposition 4.15, the two triangles in the following diagram commute:

$$
\begin{array}{ccc}
H_*^{<a}(\varepsilon, \delta)_{\mu_{-1}} & \xrightarrow{\ k_{(\varepsilon, \bar{\imath}\delta, \bar{\mu})}\ } & H_*^{<a}(\varepsilon, \delta)_{\mu_1} \\[2pt]
{\scriptstyle k_{(\varepsilon', \delta'), (\varepsilon, \delta)}}\Big\uparrow \quad {}^{\nearrow\ k_{(\bar{\varepsilon}, \bar{\delta}, \bar{\mu})}} & & \Big\uparrow {\scriptstyle k_{(\varepsilon', \delta'), (\varepsilon, \delta)}} \\[2pt]
H_*^{<a}(\varepsilon', \delta')_{\mu_{-1}} & \xrightarrow[\ k_{(\varepsilon', \bar{\imath}\delta', \bar{\mu})}\ ]{} & H_*^{<a}(\varepsilon', \delta')_{\mu_1}
\end{array}
$$

Therefore, $\{k_{(\varepsilon, \bar{\imath}\delta, \bar{\mu})}\}_{(\varepsilon, \delta) \in \mathcal{T}_a}$ induces an isomorphism on the limits of $\varepsilon \to 0$,

$$k_{\bar{\mu}}^a : H_*^{<a}(Q, K)_{\mu_{-1}} \to H_*^{<a}(Q, K)_{\mu_1}.$$

It is easy to see, as in Lemma 4.18, that $\{k_{\bar{\mu}}^a\}_{a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)}$ commute with $\{I^{a,b}\}_{a \leq b}$, so we get an isomorphism from $H_*^{\mathrm{string}}(Q, K)_{\mu_{-1}}$ to $H_*^{\mathrm{string}}(Q, K)_{\mu_1}$. It is also possible to prove, in the same way as Corollary 4.16, that this isomorphism does not depend on the choice of $\bar{\mu}$.

Independence on ε_0 For $\varepsilon_0' \leq \varepsilon_0$, we consider the inclusion maps $j_{\varepsilon_0', \varepsilon_0} : \Sigma_{m, \varepsilon_0'}^{a+m\varepsilon} \to \Sigma_{m, \varepsilon_0}^{a+m\varepsilon}$ for all $m \in \mathbb{Z}_{\geq 0}$. They induce a chain map $(j_{\varepsilon_0', \varepsilon_0})_*$ from $(C_*^{<a}(\varepsilon)_{\varepsilon_0'}, D_\delta)$ to $(C_*^{<a}(\varepsilon)_{\varepsilon_0}, D_\delta)$ for every $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$ and $(\varepsilon, \delta) \in \mathcal{T}_{a, \varepsilon_0'} (\subset \mathcal{T}_{a, \varepsilon_0})$. This preserves filtrations $\{\mathscr{F}_{\varepsilon, p, \varepsilon_0}^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathscr{F}_{\varepsilon, p, \varepsilon_0'}^{<a}\}_{p \in \mathbb{Z}}$, so it induces an morphism of the spectral sequences. On the $(-m, q)$-term for $m \geq 0$ of the first page, it is equal to the map

$$(j_{\varepsilon_0', \varepsilon_0})_* : H_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_{m, \varepsilon_0'}^{a+m\varepsilon}, \Sigma_{m, \varepsilon_0'}^0) \to H_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_{m, \varepsilon_0}^{a+m\varepsilon}, \Sigma_{m, \varepsilon_0}^0),$$

which is an isomorphism by Lemma 3.11. Therefore, by Lemma 4.2, $(j_{\varepsilon_0', \varepsilon_0})_* : H_*^{<a}(\varepsilon, \delta)_{\varepsilon_0'} \to H_*^{<a}(\varepsilon, \delta)_{\varepsilon_0}$ is also an isomorphism.

We can prove its commutativity with $\{k_{(\varepsilon', \delta'), (\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon}$ as in Lemma 4.18, so we get an isomorphism on the limit of $\varepsilon \to 0$,

$$\mathfrak{J}_{\varepsilon_0', \varepsilon_0}^a : H_*^{<a}(Q, K)_{\varepsilon_0'} \to H_*^{<a}(Q, K)_{\varepsilon_0}.$$

It holds naturally that $\{\mathfrak{J}_{\varepsilon_0', \varepsilon_0}^a\}_{a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)}$ commutes with $\{I^{a,b}\}_{a \leq b}$. Therefore, in the limit $a \to \infty$, we get an isomorphism from $H_*^{\mathrm{string}}(Q, K)_{\varepsilon_0'}$ to $H_*^{\mathrm{string}}(Q, K)_{\varepsilon_0}$.

Independence on C_0 For $C_0' \geq C_0 \geq 1$, we consider the inclusion maps $j_{C_0, C_0'} : \Sigma_{m, C_0}^a \to \Sigma_{m, C_0'}^a$ for all $m \in \mathbb{Z}_{\geq 0}$. Parallel to the proof of the independence on ε_0, we apply Lemma 3.11 to show that an isomorphism

$$\mathfrak{J}_{C_0, C_0'}^a : H_*^{<a}(Q, K)_{C_0} \to H_*^{<a}(Q, K)_{C_0'}$$

is induced. It holds naturally that $\{\mathfrak{J}_{C_0, C_0'}^a\}_{a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)}$ commutes with $\{I^{a,b}\}_{a \leq b}$. Therefore, in the limit $a \to \infty$, we get an isomorphism from $H_*^{\mathrm{string}}(Q, K)_{C_0}$ to $H_*^{\mathrm{string}}(Q, K)_{C_0'}$.

Independence on g First, let us introduce an graded algebra $\check{H}^{\mathrm{string}}_*(Q, K)_g$ which is isomorphic to $H^{\mathrm{string}}_*(Q, K)_{(g, C_0, \varepsilon_0)}$, but whose definition does not depend on ε_0 and C_0. For every $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)_g$, we define the limit of $\varepsilon_0 \to 0$ and $C_0 \to \infty$,

$$\check{H}^{<a}_*(Q, K)_g := \varinjlim_{C_0 \to \infty} \varprojlim_{\varepsilon_0 \to 0} H^{<a}_*(Q, K)_{(g, C_0, \varepsilon_0)},$$

by the $\{\mathfrak{J}^a_{\varepsilon'_0, \varepsilon_0}\}_{\varepsilon'_0 \leq \varepsilon_0}$ and $\{\mathfrak{J}^a_{C_0, C'_0}\}_{C_0 \leq C'_0}$ defined above. Then $\{I^{a,b}\}_{a \leq b}$ induces a family of maps

$$\{\check{I}^{a,b}_g : \check{H}^{<a}_*(Q, K)_g \to \check{H}^{<b}_*(Q, K)_g\}_{a \leq b},$$

and we can take the limit as $a \to \infty$ to define $\check{H}^{\mathrm{string}}_*(Q, K)_g := \varinjlim_{a \to \infty} \check{H}^{<a}_*(Q, K)_g$.

Suppose that g and g' are complete Riemannian metrics on Q. For $a > 0$, there exists a compact subset Z_a which contains the images of all $\gamma \in \bigcup_{C_0 \geq 1} \Omega_K(Q)_{(g, C_0)}$ with $\mathrm{length}_g \gamma < a$ and the images of all $\gamma \in \bigcup_{C_0 \geq 1} \Omega_K(Q)_{(g', C_0)}$ with $\mathrm{length}_{g'} \gamma < a$. For any $a \in \mathbb{R}_{>0} \setminus (\mathscr{L}(K)_g \cup \mathscr{L}(K)_{g'})$, there exists a constant $c_a \geq 1$ such that $|\cdot|_{g'} \leq c_a |\cdot|_g$ and $|\cdot|_g \leq c_a |\cdot|_{g'}$ on Z_a. We may additionally assume that $a c_a \notin \mathscr{L}(K)_g \cup \mathscr{L}(K)_{g'}$.

Let $a \in \mathbb{R}_{>0} \setminus (\mathscr{L}(K)_g \cup \mathscr{L}(K)_{g'})$. For any $C_0 \geq 1$ and $\varepsilon_0 > 0$, we have the inclusion map

$$\mathrm{j}_{(C_0, \varepsilon_0)} : \Sigma^a_{m, (g, C_0, \varepsilon_0)} \to \Sigma^{a c_a}_{m, (g', C_0 c_a, \varepsilon_0 c_a)}.$$

In addition, let $S^a_{\varepsilon, g}$ be the subspace of $S_{\varepsilon, g}$ which consists of $(\sigma_i)_{i=1,2}$ satisfying $|(\sigma_i)'(t)|_g \leq c_a^{-1}$ for $i = 1, 2$. If ε is sufficiently small, we have the inclusion map

$$\mathrm{i}_\varepsilon : S^a_{\varepsilon, g} \to S_{\varepsilon c_a, g'}.$$

These maps induces a linear map on the homology groups

$$(\mathrm{j}_{(C_0, \varepsilon_0)})_* : H^{<a}_*(\varepsilon, \delta)_{(g, C_0, \varepsilon_0)} \to H^{<a c_a}_*(\varepsilon c_a, (\mathrm{i}_\varepsilon)_* \delta)_{(g', C_0 c_a, \varepsilon_0 c_a)}$$

for $(\varepsilon, \delta) \in \mathscr{T}_{a, (g, C_0, \varepsilon_0)}$ such that $\delta \in C^{\mathrm{dR}}_{n-d}(S^a_{\varepsilon, g})$. Its commutativity with $\{k_{(\varepsilon', \delta'), (\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon}$ can be proved as in Lemma 4.18. Let us write the induced map on the limits of $\varepsilon \to 0$ by

$$\mathfrak{J}^a_{(C_0, \varepsilon_0)} : H^{<a}_*(Q, K)_{(g, C_0, \varepsilon_0)} \to H^{<a c_a}_*(Q, K)_{(g', C_0 c_a, \varepsilon_0 c_a)}.$$

Moreover, it holds naturally that the maps of $\{\mathfrak{J}^a_{(C, \varepsilon_0)}\}_{C_0 \geq 1, \varepsilon_0 > 0}$ commute with those of $\{\mathfrak{J}^a_{\varepsilon'_0, \varepsilon_0}\}_{\varepsilon'_0 \leq \varepsilon_0}$ and $\{\mathfrak{J}^a_{C_0, C'_0}\}_{C_0 \leq C'_0}$, so we get a map on the limit of $\varepsilon_0 \to 0$ and $C_0 \to \infty$,

$$\mathfrak{J}^a := \varinjlim_{C_0 \to \infty} \varprojlim_{\varepsilon_0 \to 0} \mathfrak{J}^a_{(C_0, \varepsilon_0)} : \check{H}^{<a}_*(Q, K)_g \to \check{H}^{<c_a a}_*(Q, K)_{g'}.$$

Lastly, $\{\mathfrak{J}^a\}_{a \in \mathbb{R}_{>0} \setminus (\mathscr{L}(K)_g \cup \mathscr{L}(K)_{g'})}$ is compatible with $\{\check{I}^{a,b}_g\}_{a \leq b}$, so it induces a map on the limit of $a \to \infty$,

$$\mathfrak{J} : \check{H}^{\mathrm{string}}_*(Q, K)_g \to \check{H}^{\mathrm{string}}_*(Q, K)_{g'}.$$

If we exchange g and g', we can also define the map $(\mathfrak{J}')^a \colon \check{H}_*^{<a}(Q, K)_{g'} \to \check{H}_*^{<ac_a}(Q, K)_g$ for $a \in \mathbb{R}_{>0} \setminus (\mathscr{L}(K)_g \cup \mathscr{L}(K)_{g'})$. For $b := ac_a$, we have

$$(\mathfrak{J}')^b \circ \mathfrak{J}^a = \check{I}_g^{a,bc_b}, \quad \mathfrak{J}^b \circ (\mathfrak{J}')^a = \check{I}_{g'}^{a,bc_b}.$$

Therefore, $\varinjlim_{a \to \infty} (\mathfrak{J}')^a$ is the inverse map of $\mathfrak{J}$. This proves the independence on g.

This completes the proof of the independence on auxiliary data.

Finally, let us prove the invariance under changing the orientation of K. Suppose that $K = \bigsqcup_{\alpha \in A} K_\alpha$ for connected components $\{K_\alpha\}_{\alpha \in A}$. Then $N_\varepsilon = \bigsqcup_{\alpha \in A} N_{\varepsilon,\alpha}$ and $S_\varepsilon = \bigsqcup_{\alpha \in A} S_{\varepsilon,\alpha}$, where $N_{\varepsilon,\alpha}$ is a tubular neighborhood of K_α and $S_{\varepsilon,\alpha}$ consists of pairs of paths in $N_{\varepsilon,\alpha}$. In addition, for every $\alpha_1, \ldots, \alpha_{2m} \in A$, let $\Sigma_{m,(\alpha_1,\ldots,\alpha_{2m})}^a$ be the subspace of Σ_m^a consisting of $(\gamma_k \colon [0, T_k] \to Q)_{k=1,\ldots,m}$ such that $\gamma_k(0) \in K_{\alpha_{2k-1}}$ and $\gamma_k(T_k) \in K_{\alpha_{2k}}$ for $k = 1, \ldots, m$. Then $C_{n-d}^{\mathrm{dR}}(S_\varepsilon) = \bigoplus_{\alpha \in A} C_{n-d}^{\mathrm{dR}}(S_{\varepsilon,\alpha})$ and $C_*^{\mathrm{dR}}(\Sigma_m^a) = \bigoplus_{\alpha_1,\ldots,\alpha_{2m} \in A} C_*^{\mathrm{dR}}(\Sigma_{m,(\alpha_1,\ldots,\alpha_{2m})}^a)$.

For any subset $B \subset A$, let K_B be an oriented submanifold obtained from K by reversing the orientations of $\{K_\alpha\}_{\alpha \in B}$. For every $\delta = \sum_{\alpha \in A} \delta_\alpha \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ (where δ_α is a chain in $S_{\varepsilon,\alpha}$), let us write $\delta_B := \sum_{\alpha \in A \setminus B} \delta_\alpha - \sum_{\alpha \in B} \delta_\alpha$. By using the notation

$$s(\alpha_1, \ldots, \alpha_{2m}) := \#\{k \in \{1, \ldots, m\} \mid \alpha_{2k} \in B\},$$

we define a linear map $F_\varepsilon^a \colon C_*^{<a}(\varepsilon) \to C_*^{<a}(\varepsilon)$ such that $F_\varepsilon^a(x) = (-1)^{s(\alpha_1,\ldots,\alpha_{2m})} x$ for every $x \in C_{*-m(d-2)}^{\mathrm{dR}}(\Sigma_{m,(\alpha_1,\ldots,\alpha_{2m})}^a, \Sigma_{m,(\alpha_1,\ldots,\alpha_{2m})}^0)$ for $m \geq 1$, and $F_\varepsilon^a(x) = x$ for $x \in C_*^{\mathrm{dR}}(\Sigma_0^a, \Sigma_0^0)$. For every $(\varepsilon, \delta) \in \mathcal{T}_a$, (ε, δ_B) satisfies the conditions of [Definition 4.4](#) for (Q, K_B), and F_ε^a is a chain map from $(C_*^{<a}(\varepsilon), D_\delta)$ to $(C_*^{<a}(\varepsilon), D_{\delta_B})$. Moreover, F_ε^a is compatible with the $\star$-operation. By taking the limit of $\varepsilon \to 0$ and $a \to \infty$, we obtain an isomorphism of graded $\mathbb{R}$-algebras

$$F \colon H_*^{\mathrm{string}}(Q, K) \to H_*^{\mathrm{string}}(Q, K_B).$$

Remark 4.19 Similarly, one can prove the invariance of $H_*^{\mathrm{string}}(Q, K)$ under changing the orientation of Q.

Proposition 4.20 $H_*^{\mathrm{string}}(Q, K)$ *is invariant under a C^∞ isotopy of K.*

Proof For two oriented compact submanifolds K_0 and K_1 of Q, suppose that there exists a C^∞ family of embeddings $\{f_t \colon K_0 \to Q\}_{t \in [0,1]}$ such that f_0 is the inclusion map of K_0 and $f_1(K_0) = K_1$. Then this isotopy can be extended to an ambient isotopy $\{F_t\}_{t \in [0,1]}$ such that $F_0 = \mathrm{id}_Q$ and $F_1(K_0) = K_1$. Since F_1 is an isometry from $(Q, (F_1)^*g)$ to (Q, g), it naturally induces an isomorphism

$$H_*^{\mathrm{string}}(Q, K_0)_{(F_1)^*g} \to H_*^{\mathrm{string}}(Q, K_1)_g.$$

The assertion follows from the independence on the Riemannian metric on Q. $\qquad\square$

5 Examples

In this section we determine the algebraic structure of $H_*^{\text{string}}(Q, K)$ for two examples when $Q = \mathbb{R}^{2d-1}$ with $d \geq 2$. These examples are higher-dimensional generalizations of the Hopf link and the unlink in $\mathbb{R}^3$.

The manifold $Q = \mathbb{R}^{2d-1}$ has the standard orientation. Let us use the coordinate $(z_0, z_1, z_2) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1} = \mathbb{R}^{2d-1}$. The unit sphere $S^{d-1} \subset \mathbb{R}^d$ is oriented as the boundary of the unit ball. We consider three ways of embedding the unit sphere $S^{d-1} \subset \mathbb{R}^d$ into $\mathbb{R}^{2d-1}$:

$$S^{d-1} \subset \mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^{2d-1}, \quad (z_0, z_1) \mapsto (z_0, z_1, 0),$$
$$S^{d-1} \subset \mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}^{2d-1}, \quad (z_1, z_2) \mapsto (0, z_1 + 1, z_2),$$
$$S^{d-1} \subset \mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^{2d-1}, \quad (z_0, z_1) \mapsto (z_0, z_1, z_2^*),$$

for a fixed vector $z_2^* \in \mathbb{R}^{d-1} \setminus \{0\}$. Their images are written as K_0, K_1, K_2 in order. These submanifolds are oriented so that the diffeomorphisms $S^{d-1} \to K_i$ from the above maps change the sign of the orientation by $(-1)^{d-1}$.

As notation, given a set $\mathscr{S}$ and a map $\mathscr{S} \to \mathbb{Z}$, $s \mapsto |s|$, let $\mathscr{A}_*(\mathscr{S})$ denote the unital noncommutative graded $\mathbb{R}$-algebra freely generated by $\mathscr{S}$ such that $s \in \mathscr{A}_{|s|}(\mathscr{S})$ for every $s \in \mathscr{S}$.

5.1 Computation of $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_1)$

Let us define $\mathscr{A}_*^{\text{Hopf}} := \mathscr{A}_*(\mathscr{C} \cup \mathscr{D} \cup \mathscr{E})$ by the three sets

$$\mathscr{C} := \{c_{i,j}^0\}_{i \neq j} \cup \{c_{i,i}^1\}_i \cup \{c_{i,j}^1, \bar{c}_{i,j}^1\}_{i \neq j} \cup \{c_{i,j}^2\}_{i,j}, \quad \mathscr{D} := \{d_{i,i}^1\}_i \cup \{d_{i,j}^2\}_{i,j}, \quad \mathscr{E} := \{e_{i,i}^1\}_i \cup \{e_{i,j}^2\}_{i,j},$$

where i and j run over $\{0, 1\}$. The degree of each element is given by

$$|c_{i,j}^0| = d - 2, \qquad |c_{i,j}^1| = |\bar{c}_{i,j}^1| = 2d - 3 \quad \text{for } i \neq j,$$
$$|c_{i,i}^1| = 2d - 3, \quad |c_{i,j}^2| = 3d - 4, \quad |d_{i,i}^1| = 2d - 3, \quad |d_{i,j}^2| = 3d - 4, \quad |e_{i,i}^1| = 2d - 4, \quad |e_{i,j}^2| = 3d - 5.$$

We define a graded derivation $\partial : \mathscr{A}_*^{\text{Hopf}} \to \mathscr{A}_{*-1}^{\text{Hopf}}$ such that

$$\partial c_{i,j}^0 = 0, \quad \partial c_{i,i}^1 = \partial c_{i,j}^1 = \partial \bar{c}_{i,j}^1 = 0, \quad \partial c_{i,j}^2 = 0, \quad \partial d_{i,i}^1 = e_{i,i}^1, \quad \partial d_{i,j}^2 = e_{i,j}^2, \quad \partial e_{i,i}^1 = 0, \quad \partial e_{i,j}^2 = 0,$$

and they are extended by the Leibniz rule. We also define another graded derivation $F : \mathscr{A}_*^{\text{Hopf}} \to \mathscr{A}_{*-1}^{\text{Hopf}}$ such that

$$F c_{i,j}^0 = F c_{i,j}^1 = F \bar{c}_{i,j}^1 = 0 \quad \text{for } i \neq j,$$

$$\begin{aligned}
&F c_{0,0}^1 = (-1)^d e_{0,0}^1 + (-1)^d c_{0,1}^0 c_{1,0}^0, && F c_{1,1}^1 = (-1)^d e_{1,1}^1 + c_{1,0}^0 c_{0,1}^0, \\
&F c_{0,0}^2 = -e_{0,0}^2 - (\bar{c}_{0,1}^1 c_{1,0}^0 + (-1)^d c_{0,1}^1 c_{1,0}^0), && F c_{1,1}^2 = -e_{1,1}^2 - ((-1)^d \bar{c}_{1,0}^1 c_{0,1}^0 + c_{1,0}^1 c_{0,1}^0), \\
&F c_{0,1}^2 = -e_{0,1}^2, && F c_{1,0}^2 = -e_{1,0}^2, \\
&F d_{i,i}^1 = 0, && F d_{i,j}^2 = 0, \\
&F e_{i,i}^1 = 0, && F e_{i,j}^2 = 0,
\end{aligned}$$

and they are extended by the Leibniz rule. It is easy to see that $\partial \circ \partial = 0$, $\partial \circ F + F \circ \partial = 0$ and $F \circ F = 0$. Therefore, we obtain a differential graded $\mathbb{R}$-algebra $(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F)$. Note that the differential graded $\mathbb{R}$-algebra $(\mathscr{A}_*^{\mathrm{Hopf}}, \partial)$ is obtained from $(\mathscr{A}_*(\mathscr{C}), 0)$ by *stabilizations* (see [13, Definition 3.9]). Thus,

$$(33) \qquad (\mathscr{A}_*(\mathscr{C}), 0) \xrightarrow{\ i\ } (\mathscr{A}_*^{\mathrm{Hopf}}, \partial) \xleftarrow{\ \tau\ } (\mathscr{A}_*(\mathscr{C}), 0),$$

where i is the inclusion map and τ is the projection map, are quasi-isomorphisms. For the proof, see [13, Corollary 3.11].

The goal of this section is to prove the next theorem.

Theorem 5.1 *There exists an isomorphism of graded $\mathbb{R}$-algebras*

$$H_*(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F) \cong H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_1).$$

To compute $H_*^{\mathrm{string}}(Q, K_0 \cup K_1)$, we fix auxiliary data such that g is the standard Riemannian metric on $\mathbb{R}^{2d-1}$. The constant C_0 is required to be $C_0 > 3$. The other data, ε_0 and μ, are not specified. The proof is divided into three steps.

Step 1 We first observe de Rham chains in $C_*^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0)$. We define a map

$$\varphi \colon (K_0 \cup K_1)^2 \to \Omega_{K_0 \cup K_1}(\mathbb{R}^{2d-1})$$

such that each $(p, p') \in (K_0 \cup K_1)^2$ is mapped to a path of a segment

$$\varphi(p, p') \colon [0, 1] \to \mathbb{R}^{2d-1}, \quad t \mapsto (1-t)p + tp'.$$

We fix two points $p_0 := (0, 1, 0) \in K_0$ and $p_1 := (0, 0, 0) \in K_1$. Then we define de Rham chains

$$
\begin{aligned}
(34) \qquad
& x_{i,j}^0 := [\{(p_i, p_j)\}, \varphi|_{\{(p_i, p_j)\}}, 1] \in C_0^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0) && \text{if } i \neq j, \\
& x_{i,i}^1 := [\{p_i\} \times K_i, \varphi|_{\{p_i\} \times K_i}, 1] \in C_{d-1}^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0), \\
& x_{i,j}^1 := [\{p_i\} \times K_j, \varphi|_{\{p_i\} \times K_j}, 1] \in C_{d-1}^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0) && \text{if } i \neq j, \\
& \bar{x}_{i,j}^1 := [K_i \times \{p_j\}, \varphi|_{K_i \times \{p_j\}}, 1] \in C_{d-1}^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0) && \text{if } i \neq j, \\
& x_{i,j}^2 := \ [K_i \times K_j, \varphi|_{K_i \times K_j}, 1] \ \in C_{2d-2}^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0).
\end{aligned}
$$

Here, $a > 3$ and i and j run over $\{0, 1\}$. Obviously, they are cycle chains for ∂. We write the set of these chains as

$$\mathscr{X} := \{x_{i,j}^0\}_{i \neq j} \cup \{x_{i,i}^1\}_i \cup \{x_{i,j}^1, \bar{x}_{i,j}^1\}_{i \neq j} \cup \{x_{i,j}^2\}_{i,j}.$$

If we define a function $\mathfrak{l} \colon \mathscr{X} \to \mathbb{R}_{>0}$ by

$$\mathfrak{l}(x_{i,j}^0) = \mathfrak{l}(x_{i,j}^1) = \mathfrak{l}(\bar{x}_{i,j}^1) = 1 \quad \text{for } i \neq j, \qquad \mathfrak{l}(x_{i,i}^1) = \mathfrak{l}(x_{i,i}^2) = 2, \qquad \mathfrak{l}(x_{i,j}^2) = 3 \quad \text{for } i \neq j,$$

then each $x \in \mathscr{X}$ satisfies $x \in C_*^{\mathrm{dR}}(\Sigma_1^{\mathfrak{l}(x)+\varepsilon}, \Sigma_1^0)$ for any $\varepsilon > 0$.

For every $a \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 1}$, let us consider the manifold

$$B_m^a := \left\{ (q_1^0, q_1^1, \dots, q_m^0, q_m^1) \in (K_0 \cup K_1)^{2m} \ \Big| \ \sum_{k=1}^m |q_k^0 - q_k^1| < a \text{ or } \min_{1 \leq k \leq m} |q_k^0 - q_k^1| < \varepsilon_0 \right\}.$$

This is homotopy equivalent to Σ_m^a by two smooth maps

$$\pi_m \colon \Sigma_m^a \to B_m^a, \quad (\gamma_k \colon [0, T_k] \to \mathbb{R}^{2d-1})_{k=1,\ldots,m} \mapsto (\gamma_1(0), \gamma_1(T_1), \ldots, \gamma_m(0), \gamma_m(T_m)),$$

$$i_m \colon B_m^a \to \Sigma_m^a, \qquad (q_1^0, q_1^1, \ldots, q_m^0, q_m^1) \mapsto (\varphi(q_k^0, q_k^1))_{k=1,\ldots,m},$$

for which $\pi_m \circ i_m = \mathrm{id}_{B_m^a}$ and $i_m \circ \pi_m$ is homotopic to $\mathrm{id}_{\Sigma_m^a}$ (see Lemma 3.2).

Notation In this section, if N is a submanifold of a manifold M, then the inclusion map $N \to M$ is denoted by ι_N.

Lemma 5.2 *Let M be an oriented manifold with open submanifold N. Suppose that there exists an approximately smooth function $f \colon M \to \mathbb{R}$ such that $N = f^{-1}((-\infty, 0))$. In addition, assume that $H_*^{\mathrm{sing}}(M, N)$ has a finite dimension. Then there exists an isomorphism between $H_*^{\mathrm{sing}}(M, N)$ and $H_*^{\mathrm{dR}}(M, N)$ such that, for every closed oriented k-dimensional submanifold K of M, the fundamental class $[K] \in H_k^{\mathrm{sing}}(M, N)$ corresponds to $(-1)^{s(k)}[K, \iota_K, 1] \in H_k^{\mathrm{dR}}(M, N)$, where $s(k) := \frac{1}{2}(k - \dim M)(k - \dim M - 1)$.*

Proof We consider the correspondence through the isomorphisms

$$H_*^{\mathrm{sing}}(M, N) \cong H_{c,\mathrm{dR}}^{\dim M - *}(M, N) \cong H_*^{\mathrm{dR}}(M^{\mathrm{reg}}, N^{\mathrm{reg}}) \to H_*^{\mathrm{dR}}(M, N).$$

The first isomorphism is defined by Poincaré duality. The second isomorphism was given in Example 2.6. The last isomorphism is induced by $\mathrm{id}_M \colon M^{\mathrm{reg}} \to M$ [15, Proposition 5.2]. Let us identify the tubular neighborhood N_K of K with the normal bundle of K. Then $[K] \in H_k^{\mathrm{sing}}(M, N)$ corresponds to $[\eta] \in H_{c,\mathrm{dR}}^{\dim M - k}(M, N)$, where $\eta \in \Omega_c^{\dim M - k}(M)$ has its support in N_K and represents the Thom class of the normal bundle. Recalling Example 2.6, we can see that this cohomology class corresponds to $(-1)^{s(k)}[M, \mathrm{id}_M, \eta] = (-1)^{s(k)}[N_K, \mathrm{id}_{N_K}, \eta] \in H_k^{\mathrm{dR}}(M^{\mathrm{reg}}, N^{\mathrm{reg}})$. Let $\pi_{N_K} \colon N_K \to K$ be the bundle projection. Then, as a de Rham chain in $C_k^{\mathrm{dR}}(M, N)$, $[N_K, \iota_{N_K}, \eta]$ is homologous to $[N_K, \iota_K \circ \pi_{N_K}, \eta]$ since $\iota_{N_K} \colon N_K \to M$ is homotopic to $\iota_K \circ \pi_{N_K} \colon N_K \to K \subset M$. Now the assertion follows since

$$[N_K, \iota_K \circ \pi_{N_K}, \eta] = [K, \iota_K, (\pi_{N_K})_! \eta] = [K, \iota_K, 1] \in H_k^{\mathrm{dR}}(M, N). \qquad \square$$

Let us see that a basis of $H_*^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0)$ is given by $\mathscr{X}$ and the $\star$-operation. First, suppose that $m = 1$ and $a > 3$. Then $B_1^a = K \times K$. Through the isomorphism $H_*^{\mathrm{dR}}(B_1^a, B_1^0) \cong H_*^{\mathrm{sing}}(B_1^a, B_1^0)$ of Lemma 5.2, $\{(\pi_1)_*[x] \mid x \in \mathscr{X}\}$ corresponds to the set of singular homology classes

$$\{[\{(p_i, p_j)\}]\}_{i \neq j} \cup \{[\{p_i\} \times K_i]\}_i \cup \{[\{p_i\} \times K_j]\}_{i \neq j} \cup \{[K_i \times \{p_j\}]\}_{i \neq j} \cup \{[K_i \times K_j]\}_{i,j},$$

which is a basis of $H_*^{\mathrm{sing}}(K \times K, B_1^0)$. Therefore, $\{[x] \mid x \in \mathscr{X}\}$ is a basis of $H_*^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0)$ for $a > 3$.

For $a \in (0, 3]$, we consider the deformation along the negative gradient vector field of a C^∞ function $E_{i,j} \colon (K_i \cup K_j)^2 \to \mathbb{R} \colon (p, p') \to |p - p'|^2$. If $i = j$, then $\max E_{i,j} = 4$, $\min E_{i,j} = 0$, and the subset $\{(p, p') \in K_i \times K_i \mid |p - p'| < a\}$ for $a \leq 2$ has $E_{i,i}^{-1}(0) = \{(p, p') \in K_i \times K_i \mid p = p'\}$ (the diagonal) as a deformation retract. If $i \neq j$, then $\max E_{i,j} = 9$, $\min E_{i,j} = 1$, and the subset $\{(p, p') \in K_i \times K_j \mid |p - p'| < a\}$ for $a \leq 3$ has $E_{i,j}^{-1}(0) = (K_i \times \{p_j\}) \cup (\{p_i\} \times K_j)$ (a bouquet) as

a deformation retract. From these observations, we can see that $\{[x] \mid x \in \mathcal{X}, \mathfrak{l}(x) < a\}$ is a basis of $H_*^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0)$. In general, for any $m \in \mathbb{Z}_{\geq 1}$ and $a \in \mathbb{R}_{>0}$,

$$\text{(35)} \qquad \{[x_1 \star \cdots \star x_m] \mid x_1, \ldots, x_m \in \mathcal{X}, \mathfrak{l}(x_1) + \cdots + \mathfrak{l}(x_m) < a\}$$

is a basis of $H_*^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0)$.

We fix a trivialization $h \colon \mathbb{O}_{\varepsilon_0} \times (K_0 \cup K_1) \to N_{\varepsilon_0}$ such that $h(w, p_0) = p_0 + (0, w)$ and $h(w, p_1) = p_1 + (-w, 0)$ for every $w \in \mathbb{O}_{\varepsilon_0} \subset \mathbb{R}^d$. (The orientations of K_0 and K_1 are chosen so that this map preserves orientations.)

Suppose that $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K_0 \cup K_1)$ $(= \mathbb{R}_{>0} \setminus \mathbb{Z}_{\geq 1})$. In the following series of three lemmas, we will observe the chains $f_{1,\delta}(x)$ for each $x \in \mathcal{X}$. Hereafter, we assume that $(\varepsilon, \delta) \in \mathcal{T}_a$ is standard with respect to h (see Section 4.4.3).

Lemma 5.3 *For $i \neq j$,*

$$f_{1,\delta}(x_{i,j}^0) = 0, \quad f_{1,\delta}(x_{i,j}^1) = f_{1,\delta}(\bar{x}_{i,j}^1) = 0.$$

Proof For $(p, p') \in K_i \times K_j$ with $i \neq j$ such that either $p = p_i$ or $p' = p_j$, $\varphi(p, p')$ satisfies condition (iii) of Lemma 3.12 for $m = k = 1$. The equations follow from Lemma 3.12. $\square$

Lemma 5.4 *For $i \in \{0, 1\}$, there exist $y_{i,i}^1 \in C_1^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ and $y_{i,i}^2 \in C_d^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ such that*

$$\text{(36)} \qquad \begin{aligned} \partial y_{0,0}^1 &= f_{1,\delta}(x_{0,0}^1) - x_{0,1}^0 \star x_{1,0}^0, \\ \partial y_{1,1}^1 &= f_{1,\delta}(x_{1,1}^1) - (-1)^d x_{1,0}^0 \star x_{0,1}^0, \\ \partial y_{0,0}^2 &= f_{1,\delta}(x_{0,0}^2) - (\bar{x}_{0,1}^1 \star x_{1,0}^0 + (-1)^d x_{0,1}^1 \star x_{1,0}^0), \\ \partial y_{1,1}^2 &= f_{1,\delta}(x_{1,1}^2) - ((-1)^d \bar{x}_{1,0}^1 \star x_{0,1}^0 + x_{1,0}^1 \star x_{0,1}^0), \end{aligned}$$

and

$$\text{(37)} \qquad (f_{1,\delta} + (-1)^d f_{2,\delta})(y_{i,i}^1) = 0, \quad (f_{1,\delta} + (-1)^d f_{2,\delta})(y_{i,i}^2) = 0.$$

Proof We only show the existence of $y_{0,0}^2$ and $y_{0,0}^1$. Replacing (K_0, p_0) by (K_1, p_1), $y_{1,1}^2$ and $y_{1,1}^1$ are constructed in a parallel way, except for the difference of signs.

Since (ε, δ) is standard, δ has the form (31). Using the notation of (13), we can write $f_{1,\delta}(x_{0,0}^2) = [W_1, \Phi_1, \zeta_1]$, where

$$\begin{aligned} W_1 &= \{((p, p'), \tau, v) \in (K_0 \times K_0) \times \mathbb{R} \times N_\varepsilon \mid 2\varepsilon < \tau < 1 - 2\varepsilon, (1 - \tau)p + \tau p' = v\}, \\ \Phi_1 &\colon W_1 \to \Sigma_2^{2+2\varepsilon}, \quad ((p, p'), \tau, v) \mapsto \mathrm{con}_1(\varphi(p, p'), (1, \tau), \psi_\varepsilon(v)), \\ \zeta_1 &\in \Omega_c^d(W_1), \quad (\zeta_1)_{((p,p'),\tau,v)} = \rho_\varepsilon(1, \tau) \cdot (h_*(\nu_\varepsilon \times 1))_v. \end{aligned}$$

N_ε is a disjoint union of $h(\mathbb{O}_\varepsilon \times K_0)$ and $h(\mathbb{O}_\varepsilon \times K_1)$. Corresponding to this division, we define $W_i' := \{((p, p'), \tau, v) \in W_1 \mid v \in h(\mathbb{O}_\varepsilon \times K_i)\}$ for $i = 0, 1$. If $((p, p'), \tau, v) \in W_0'$, then τ satisfies condition (ii)
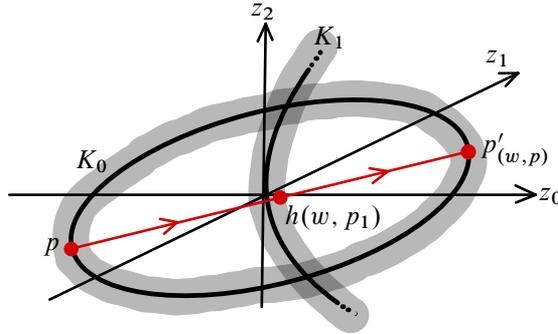
Figure 3: The path $\varphi(p, p'_{(w,p)})$. The gray region is the tubular neighborhood N_ε of $K_0 \cup K_1$.

of Lemma 3.12, so $[W'_0, \Phi_1, \zeta_1] = 0 \in C^{dR}_{d-1}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$. For $((p, p'), \tau, v) \in W'_1$, we have $\rho_\varepsilon(1, \tau) = 1$. Moreover, there is an diffeomorphism

$$I: \mathbb{O}_\varepsilon \times K_0 \to W'_1, \quad (w, p) \mapsto ((p, p'_{(w,p)}), |p - h(w, p_1)|, h(w, p_1)).$$

Here $p'_{(w,p)} \in K_0 \setminus \{p\}$ is determined by $h(w, p_1) \in \mathrm{Im}(\varphi(p, p'_{(w,p)}))$, as described in Figure 3. The diffeomorphism I preserves orientations and $I^*\zeta_1 = v_\varepsilon \times 1$.

Likewise, we consider an explicit description of $f_{1,\delta}(x^1_{0,0})$. Then

$$f_{1,\delta}(x^2_{0,0}) = [\mathbb{O}_\varepsilon \times K_0, \Phi_1 \circ I, v_\varepsilon \times 1] \in C^{dR}_{d-1}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0),$$
$$f_{1,\delta}(x^1_{0,0}) = [\mathbb{O}_\varepsilon \times \{p_0\}, \Phi_1 \circ I|_{\mathbb{O}_\varepsilon \times \{p_0\}}, v_\varepsilon \times 1] \in C^{dR}_0(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0).$$

Let us define $\widetilde{\Phi}_1: \mathbb{R} \times (\mathbb{O}_\varepsilon \times K_0) \to \Sigma_2^{2+2\varepsilon}$ as follows: Choose a C^∞ function $\kappa: \mathbb{R} \to [0, 1]$ such that $\kappa(s) = 0$ if $s \leq \frac{1}{2}$ and $\kappa(s) = 1$ if $s \geq 1$. For $s \geq \frac{1}{2}$, we define $\widetilde{\Phi}_1(s, (w, p)) := \Phi_1 \circ I(\kappa(s) \cdot w, p)$. Then the first path (resp. the second path) of $\widetilde{\Phi}_1\left(\frac{1}{2}, (w, p)\right)$ is equal to $\varphi(p, p_1)$ (resp. $\varphi(p_1, -p)$) up to a reparametrization, so we define $\widetilde{\Phi}(s, (w, p))$ for $s \leq \frac{1}{2}$ by interpolating the parametrizations so that $\widetilde{\Phi}(s, (w, p)) = (\varphi(p, p_1), \varphi(p_1, -p))$ for $s \leq 0$. Now we define

$$\tilde{y}^2_{0,0} := (-1)^d [\mathbb{R} \times (\mathbb{O}_\varepsilon \times K_0), \widetilde{\Phi}_1, \chi \times (v_\varepsilon \times 1)] \in C^{dR}_d(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0),$$
$$\tilde{y}^2_{0,0} := (-1)^d [\mathbb{R} \times (\mathbb{O}_\varepsilon \times \{p_0\}), \widetilde{\Phi}_1|_{\mathbb{R} \times (\mathbb{O}_\varepsilon \times \{p_0\})}, \chi \times (v_\varepsilon \times 1)] \in C^{dR}_1(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0),$$

where $\chi: \mathbb{R} \to [0, 1]$ is a C^∞ function with compact support such that $\chi \equiv 1$ on $[0, 1]$.

From the constructions of $\tilde{y}^2_{0,0}$ and $\tilde{y}^1_{0,0}$, we can compute their boundary chains as follows: Let us introduce two maps $\varphi_0: K_0 \to K_0 \times K_0: p \mapsto (p, -p)$ and $\tilde{i}_2: K_0 \times K_0 \to \Sigma_2^{2+2\varepsilon}, (p, p') \mapsto (\varphi(p, p_1), \varphi(p_1, p'))$. Then

$$\partial \tilde{y}^2_{0,0} = f_{1,\delta}(x^2_{0,0}) - (\tilde{i}_2)_*[\mathbb{O}_\varepsilon \times K_0, \varphi_0 \circ \mathrm{pr}_{K_0}, v_\varepsilon] = f_{1,\delta}(x^2_{0,0}) - (\tilde{i}_2)_*[\varphi_0(K_0), \iota_{\varphi_0(K_0)}, 1],$$
$$\partial \tilde{y}^1_{0,0} = f_{1,\delta}(x^1_{0,0}) - (\tilde{i}_2)_*[\mathbb{O}_\varepsilon \times \{p_0\}, \varphi_0 \circ \mathrm{pr}_{\{p_0\}}, v_\varepsilon \times 1] = f_{1,\delta}(x^1_{0,0}) - (\tilde{i}_2)_*[\{\varphi_0(p_0)\}, \iota_{\{\varphi_0(p_0)\}}, 1].$$

Here we used the condition that $\int_{\mathbb{O}_\varepsilon} v_\varepsilon = 1$. Moreover, we can check from the definition that $\widetilde{\Phi}_1(s, (w, p))$ satisfies condition (iii) of Lemma 3.12 for $m = 2$ and $k = 1, 2$. Therefore, $f_{k,\delta}(\tilde{y}^2_{0,0}) = 0$ and $f_{k,\delta}(\tilde{y}^1_{0,0}) = 0$ hold for $k = 1, 2$.

As homology classes in $H_*^{\mathrm{sing}}(K_0 \times K_0)$,

$$[\varphi_0(K_0)] = [K_0 \times \{p_0\}] + (-1)^d [\{p_0\} \times K_0] \in H_{d-1}^{\mathrm{sing}}(K_0 \times K_0),$$

$$[\{\varphi_0(p_0)\}] = [\{(p_0, p_0)\}] \in H_0^{\mathrm{sing}}(K_0 \times K_0).$$

Therefore, by Lemma 5.2, there exist $z_{0,0}^2 \in C_d^{\mathrm{dR}}(K_0 \times K_0)$ and $z_{0,0}^1 \in C_1^{\mathrm{dR}}(K_0 \times K_0)$ such that

$$\partial z_{0,0}^2 = [\varphi_0(K_0), \iota_{\varphi_0(K_0)}, 1] - \big([K_0 \times \{p_0\}, \iota_{K_0 \times \{p_0\}}, 1] + (-1)^d [\{p_0\} \times K_0, \iota_{\{p_0\} \times K_0}, 1]\big),$$

$$\partial z_{0,0}^1 = [\{\varphi_0(p_0)\}, \iota_{\{\varphi_0(p_0)\}}, 1] - [\{(p_0, p_0)\}, \iota_{\{(p_0, p_0)\}}, 1].$$

It is clear from the definition of each $x \in \mathscr{X}$ that

$$(\tilde{\imath}_2)_*[K_0 \times \{p_0\}, \iota_{K_0 \times \{p_0\}}, 1] = \bar{x}_{0,1}^1 \star x_{1,0}^0,$$

$$(\tilde{\imath}_2)_*[\{p_0\} \times K_0, \iota_{\{p_0\} \times K_0}, 1] = x_{0,1}^1 \star x_{1,0}^0, \quad (\tilde{\imath}_2)_*[\{(p_0, p_0)\}, \iota_{\{(p_0, p_0)\}}, 1] = x_{0,1}^0 \star x_{1,0}^0.$$

Therefore, $y_{0,0}^2 := \tilde{y}_{0,0}^2 + (\tilde{\imath}_2)_* z_{0,0}^2 \in C_d^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ and $y_{0,0}^1 := \tilde{y}_{0,0}^1 + (\tilde{\imath}_2)_* z_{0,0}^1 \in C_0^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ satisfy the first and the second equations of (36). Moreover, any path in the image of $\tilde{\imath}_2$ satisfies condition (iii) of Lemma 3.12 for $m = 2$ and $k = 1, 2$. Therefore, $f_{k,\delta}((\tilde{\imath}_2)_* z_{0,0}^2) = 0$ and $f_{k,\delta}((\tilde{\imath}_2)_* z_{0,0}^1) = 0$ hold for $k = 1, 2$, and thus $y_{0,0}^2$ and $y_{0,0}^1$ satisfy (37). $\qquad \square$

Lemma 5.5 *There exist* $y_{0,1}^2, y_{1,0}^2 \in C_{2d-1}^{\mathrm{dR}}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0)$ *such that*

$$(38) \qquad\qquad \partial y_{0,1}^2 = f_{1,\delta}(x_{0,1}^2), \quad \partial y_{1,0}^2 = f_{1,\delta}(x_{1,0}^2)$$

and

$$(39) \qquad\qquad (f_{1,\delta} + (-1)^d f_{2,\delta})(y_{0,1}^2) = 0, \quad (f_{1,\delta} + (-1)^d f_{2,\delta})(y_{1,0}^2) = 0.$$

Proof We only show the existence of $y_{0,1}^2$. Exchanging K_0 and K_1, $y_{1,0}^2$ is constructed in a parallel way.

Since (ε, δ) is standard, δ has the form (31). Using the notation of (13), we can write $f_{1,\delta}(x_{0,1}^2) = [W_1, \Phi_1, \zeta_1]$, where

$$W_1 = \{((p, p'), \tau, v) \in (K_0 \times K_1) \times \mathbb{R} \times N_\varepsilon \mid 2\varepsilon < \tau < 1 - 2\varepsilon, (1 - \tau)p + \tau p' = v\},$$

$$\Phi_1 : W_1 \to \Sigma_2^{3+2\varepsilon}, \quad ((p, p'), \tau, v) \mapsto \mathrm{con}_1(\varphi(p, p'), (1, \tau), \psi_\varepsilon(v)),$$

$$\zeta_1 \in \Omega_c^d(W_1), \quad (\zeta_1)_{((p,p'),\tau,v)} = \rho_\varepsilon(1, \tau) \cdot (h_*(v_\varepsilon \times 1))_v.$$

If $p \in K_0$ is sufficiently close to p_0, then $\varphi(p, p')$ satisfies condition (iii) of Lemma 3.12 for any $p' \in K_1$. Symmetrically, if $p' \in K_1$ is sufficiently close to p_1, then $\varphi(p, p')$ satisfies the same condition for any $p \in K_0$. See Figure 4. Therefore, for any bump function $b : K_0 \times K_1 \to \mathbb{R}$ whose support is localized near $(K_0 \times \{p_1\}) \cup (\{p_0\} \times K_1)$, we have $[W_1, \Phi_1, \zeta_1'] = 0$ for

$$\zeta_1' \in \Omega_c^d(W_1), \quad (\zeta_1')_{((p,p'),\tau,v)} = b(p, p') \cdot \rho_\varepsilon(1, \tau) \cdot (h_*(v_\varepsilon \times 1))_v.$$

We remark that $[W_1, \Phi_1, \zeta_1']$ is an explicit description (13) of $f_{1,\delta}([K_0 \times K_1, \varphi|_{K_0 \times K_1}, b])$.
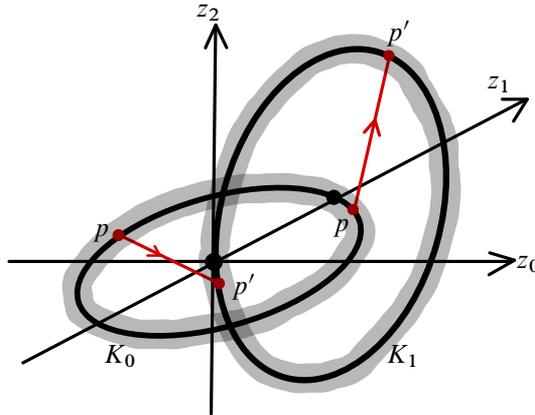
Figure 4: The path $\varphi(p, p')$ when p is close to p_0 or p' is close to p_1. The gray region is the tubular neighborhood N_ε of $K_0 \cup K_1$

Now we choose b so that it is constant to 1 on a neighborhood of $(K_0 \times \{p_1\}) \cup (\{p_0\} \times K_1)$. Then the above computation shows that

$$f_{1,\delta}(x_{0,1}^2) = f_{1,\delta}([K_0 \times K_1, \varphi|_{K_0 \times K_1}, 1-b]) + f_{1,\delta}([K_0 \times K_1, \varphi|_{K_0 \times K_1}, b])$$
$$= f_{1,\delta}([K_0' \times K_1', \varphi|_{K_0' \times K_1'}, 1-b|_{K_0' \times K_1'}]),$$

where K_i' for $i = 0, 1$ is the complement of a small closed ball containing p_i. Since $K_0' \times K_1'$ is contractible, there exists a map $R \colon \mathbb{R} \times K_0' \times K_1' \to K_0' \times K_1'$ such that $R(s, \cdot) = \mathrm{id}_{K_0' \times K_1'}$ for $s \geq 1$ and $R(s, \cdot)$ is constant to some point in $K_0' \times K_1'$ for $s \leq 0$. Using the function χ in the proof of Lemma 5.4, let us define a chain

$$\tilde{x}_{0,1}^2 := [\mathbb{R} \times (K_0' \times K_1'), \varphi \circ R, \chi \times (1-b|_{K_0' \times K_1'})] \in C_{2d-1}^{\mathrm{dR}}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0).$$

This chain satisfies

$$\partial \tilde{x}_{0,1}^2 = [K_0' \times K_1', \varphi|_{K_0' \times K_1'}, 1-b|_{K_0' \times K_1'}] + [\mathbb{R} \times (K_0' \times K_1'), \varphi \circ R, \chi \times db|_{K_0' \times K_1'}].$$

Note that $[K_0' \times K_1', \varphi \circ R(0, \cdot), 1-b|_{K_0' \times K_1'}] = 0$ since $\varphi \circ R(0, \cdot)$ is constant. The second chain of the right-hand side is mapped by $f_{1,\delta}$ to 0 since the support of db is localized near $(K_0 \times \{p_1\}) \cup (\{p_0\} \times K_1)$. Now we define $y_{0,1}^2 := f_{1,\delta}(\tilde{x}_{0,1}^2) \in C_{2d-1}^{\mathrm{dR}}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0)$. Then the first equation of (38) holds since $\partial y_{0,1}^2 = f_{1,\delta}(\partial \tilde{x}_{0,1}^2) = f_{1,\delta}(x_{0,1}^2)$. The first equation (39) follows from $(f_{1,\delta} + (-1)^d f_{2,\delta}) \circ f_{1,\delta} = 0$ (see the proof of Proposition 4.1). □

Step 2 We define a function $\mathfrak{l} \colon \mathscr{C} \cup \mathscr{D} \cup \mathscr{E} \to \mathbb{R}_{>0}$ by

$$\mathfrak{l}(c_{i,j}^0) = \mathfrak{l}(c_{i,j}^1) = \mathfrak{l}(\bar{c}_{i,j}^1) = 1 \quad \text{for } i \neq j, \qquad \mathfrak{l}(c_{i,i}^1) = 2, \qquad \mathfrak{l}(c_{i,j}^2) = \mathfrak{l}(d_{i,j}^2) = \mathfrak{l}(e_{i,j}^2) = \begin{cases} 2 & \text{if } i = j, \\ 3 & \text{if } i \neq j. \end{cases}$$

For every $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K_0 \cup K_1)$, let $\mathscr{A}_*^{<a}$ be an $\mathbb{R}$-subspace of $\mathscr{A}_*^{\mathrm{Hopf}}$ spanned by words of elements in $\mathscr{C} \cup \mathscr{D} \cup \mathscr{E}$ such that the sum of the values of $\mathfrak{l}$ is less than a. Then $(\partial + F)(\mathscr{A}_*^{<a}) \subset \mathscr{A}_*^{<a}$, so we get a subcomplex $(\mathscr{A}_*^{<a}, \partial + F)$.

We continue to use $(\varepsilon, \delta) \in \mathcal{T}_a$ which is standard with respect to h. The second step is to construct a chain map from $(\mathscr{A}_*^{<a}, \partial + F)$ to $(C_*^{<a}(\varepsilon), D_\delta)$, and prove that it is a quasi-isomorphism.

Let $y_{i,i}^1$ and $y_{i,j}^2$ be the chains of Lemmas 5.4 and 5.5, which depend on (ε, δ). We define a linear map $\Phi_{(\varepsilon,\delta)}^{<a}: \mathscr{A}_*^{<a} \to C_*^{<a}(\varepsilon)$ such that

(40)
$$\Phi_{(\varepsilon,\delta)}^{<a}(c_{i,j}^0) := x_{i,j}^0, \qquad \Phi_{(\varepsilon,\delta)}^{<a}(c_{i,j}^1) := x_{i,j}^0, \quad \Phi_{(\varepsilon,\delta)}^{<a}(\bar{c}_{i,j}^1) := \bar{x}_{i,j}^1 \quad \text{for } i \neq j,$$
$$\Phi_{(\varepsilon,\delta)}^{<a}(c_{i,i}^1) := x_{i,i}^1, \quad \Phi_{(\varepsilon,\delta)}^{<a}(c_{i,j}^2) := x_{i,j}^2, \quad \Phi_{(\varepsilon,\delta)}^{<a}(d_{i,i}^1) := y_{i,i}^1,$$
$$\Phi_{(\varepsilon,\delta)}^{<a}(d_{i,j}^2) := y_{i,j}^2, \quad \Phi_{(\varepsilon,\delta)}^{<a}(e_{i,i}^1) := \partial y_{i,i}^1, \quad \Phi_{(\varepsilon,\delta)}^{<a}(e_{i,j}^2) := \partial y_{i,j}^1,$$

and extend them naturally by the product map on $\mathscr{A}_*^{\mathrm{Hopf}}$ and the $\star$-operation.

Proposition 5.6 $\Phi_{(\varepsilon,\delta)}^{<a}$ *is a chain map from* $(\mathscr{A}_*^{<a}, \partial + F)$ *to* $(C_*^{<a}(\varepsilon), D_\delta)$.

Proof This follows immediately from the series of three lemmas in Step 1. For each $\xi \in \mathscr{C} \cup \mathscr{D} \cup \mathscr{E}$, $D_\delta \circ \Phi_{(\varepsilon,\delta)}^{<a}(\xi) = (\partial + F) \circ \Phi_{(\varepsilon,\delta)}^{<a}(\xi)$ is proved by

- Lemma 5.3 if $\xi = c_{i,j}^0, c_{i,j}^1, \bar{c}_{i,j}^1$ with $i \neq j$;
- the equations (36) if $\xi = c_{i,i}^1, c_{i,i}^2$;
- the equations (38) if $\xi = c_{i,j}^2$ with $i \neq j$;
- the equations (37) if $\xi = d_{i,i}^1, d_{i,i}^2, e_{i,i}^1, e_{i,i}^2$;
- the equations (39) if $\xi = d_{i,j}^2, e_{i,j}^2$ with $i \neq j$. $\square$

Therefore, we have a linear map on the homology groups
$$(\Phi_{(\varepsilon,\delta)}^{<a})_* : H_*(\mathscr{A}_*^{<a}, \partial + F) \to H_*^{<a}(\varepsilon, \delta).$$

Proposition 5.7 $(\Phi_{(\varepsilon,\delta)}^{<a})_*$ *is an isomorphism.*

Proof We introduce a function $\mathfrak{m}: \mathscr{C} \cup \mathscr{D} \cup \mathscr{E} \to \mathbb{Z}_{\geq 1}$ such that $\mathfrak{m}(\mathscr{C}) = \{1\}$ and $\mathfrak{m}(\mathscr{D}) = \mathfrak{m}(\mathscr{E}) = \{2\}$. For every $m \in \mathbb{Z}_{\geq 0}$, let $\mathscr{A}_*^{<a}(m)$ be an $\mathbb{R}$-subspace of $\mathscr{A}_*^{<a}$ generated by words of elements of $\mathscr{C} \cup \mathscr{D} \cup \mathscr{E}$ such that the sum of the values of $\mathfrak{m}$ is equal to m. (When $m = 0$, $\mathscr{A}_*^{<a}(0) := \mathbb{R} \cdot 1$.) Then the chain complex $(\mathscr{A}_*^{<a}, \partial + F)$ is filtered by subcomplexes $\{\mathscr{G}_p^{<a}\}_{p \in \mathbb{Z}}$ defined by $\mathscr{G}_p^{<a} := \bigoplus_{m \geq -p} \mathscr{A}_*^{<a}(m)$. Let us consider the spectral sequence determined by this filtration. The $(-m, q)$-term for $m \geq 0$ of its first page is given by
$$H_{q-m}(\mathscr{A}_*^{<a}(m), \partial) \cong H_{q-m}(\mathscr{A}_*^{<a}(m, \mathscr{C}), 0) = \mathscr{A}_{q-m}^{<a}(m, \mathscr{C}).$$

Here, $\mathscr{A}_*^{<a}(m, \mathscr{C}) := \mathscr{A}_*^{<a}(m) \cap \mathscr{A}_*(\mathscr{C})$ and the first isomorphism is induced by restricting the quasi-isomorphisms (33). $\Phi_{(\varepsilon,\delta)}^{<a}$ preserves the filtrations $\{\mathscr{G}_p^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathscr{F}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$. The induced map on the $-m^{\mathrm{th}}$ column for $m \geq 0$ of the first page has the form
$$(\Phi_{(\varepsilon,\delta)}^{<a})_* : \mathscr{A}_{*-m}^{<a}(m, \mathscr{C}) \to H_{*-m(d-1)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

This map is an isomorphism since the basis $\{c_1 \cdots c_m \mid c_1, \ldots, c_m \in \mathscr{C}, \mathfrak{l}(c_1) + \cdots + \mathfrak{l}(c_m) < a\}$ of $\mathcal{A}_{*-m}^{<a}(m, \mathscr{C})$ is mapped to the basis (35). In the $-m^{\text{th}}$ column for $m < 0$, $(\Phi_{(\varepsilon,\delta)}^{<a})_*$ is a map between the zero vector spaces. The proposition now follows from Lemma 4.2. $\qquad\square$

Step 3 The last step is to show that the family of maps

$$\{(\Phi_{(\varepsilon,\delta)}^{<a})_* \mid a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K), (\varepsilon, \delta) \in \mathcal{T}_a \text{ is standard with respect to } h\}$$

induces an isomorphism on the limit of $\varepsilon \to 0$ and $a \to \infty$.

Lemma 5.8 *For* $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ *with* $\varepsilon' \leq \varepsilon$ *which are standard with respect to* h,

$$k_{(\varepsilon',\delta'),(\varepsilon,\delta)} \circ (\Phi_{(\varepsilon',\delta')}^{<a})_* = (\Phi_{(\varepsilon,\delta)}^{<a})_*.$$

Proof We have defined $\Phi_{(\varepsilon,\delta)}^{<a}$ by the chains $\{y_{i,i}^1, y_{i,j}^2\}$, which depends on (ε, δ). As notation, let $\{(y_{i,i}^1)', (y_{i,j}^2)'\}$ be the corresponding chains constructed from (ε', δ'), by which $\Phi_{(\varepsilon',\delta')}^{<a}$ is defined.

We take $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathcal{T}}_a$ satisfying (24) for (ε, δ) and (ε', δ'). We may assume that it is standard with respect to h, and thus $\bar{\varepsilon} = \varepsilon$. As in Lemma 5.3, $\bar{f}_{1,\bar{\delta}}(\bar{\iota}(x)) = 0$ holds for $x = x_{i,j}^0, x_{i,j}^1, \bar{x}_{i,j}^1$ with $i \neq j$, where $\bar{\iota}$ is the map (17). We claim that there exist $[-1, 1]$-modeled chains $\bar{y}_{i,i}^1 \in \overline{C}_1^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$, $\bar{y}_{i,i}^2 \in \overline{C}_d^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ and $\bar{y}_{i,j}^2 \in \overline{C}_d^{\mathrm{dR}}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0)$ with $i \neq j$ which satisfy the following equations:

- the variants of the equations (36), (37), (38) and (39) determined by replacing $\{y_{i,i}^1, y_{i,j}^2, f_{k,\delta}, \star\}$ by $\{\bar{y}_{i,i}^1, \bar{y}_{i,j}^2, \bar{f}_{k,\bar{\delta}}, \overline{\star}\}$ and $x \in \mathscr{X}$ by $\bar{\iota}(x)$;
- $e_+ \bar{y}_{i,i}^1 = y_{i,i}^1$, $e_+ \bar{y}_{i,j}^2 = y_{i,j}^2$, $e_- \bar{y}_{i,i}^1 = (j_{\varepsilon',\varepsilon})_*(y_{i,i}^1)'$, and $e_- \bar{y}_{i,j}^2 = (j_{\varepsilon',\varepsilon})_*(y_{i,j}^2)'$.

This claim is proved by rewriting the proofs of Lemmas 5.4 and 5.5 for $[-1, 1]$-modeled chains. We omit the proof.

We define a linear map $\overline{\Phi}_\varepsilon^{<a} : \mathcal{A}_*^{<a} \to \overline{C}_*^{<a}(\varepsilon)$ as in (40) by replacing $x \in \mathscr{X}$ by $\bar{\iota}(x)$ and $\{y_{i,i}^1, y_{i,j}^2\}$ by $\{\bar{y}_{i,i}^1, \bar{y}_{i,j}^2\}$, and extend naturally by the product on $\mathcal{A}_*^{\mathrm{Hopf}}$ and the $\overline{\star}$-operation. The former equations about $\bar{y}_{i,i}^1$ and $\bar{y}_{i,j}^2$ ensure that $\overline{\Phi}_\varepsilon^{<a}$ is a chain map from $(\mathcal{A}_*^{<a}, \partial + F)$ to $(\overline{C}_*^{<a}(\varepsilon), \overline{D}_{\bar{\delta}})$, as in Proposition 5.6. The latter equations about $\bar{y}_{i,i}^1$ and $\bar{y}_{i,j}^2$ ensure the commutativity of the diagram

$\qquad\square$

Therefore, the family of maps $\{(\Phi_{(\varepsilon,\delta)}^{<a})_* \mid (\varepsilon, \delta) \in \mathscr{T}_a$ is standard with respect to $h\}$ induces an isomorphism on the limit of $\varepsilon \to 0$,

$$\Phi_*^{<a}: H_*(\mathscr{A}_*^{<a}, \partial + F) \to H_*^{<a}(\mathbb{R}^{2d-1}, K_0 \cup K_1)$$

for every $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K_0 \cup K_1)$. Moreover, $\{\Phi_*^{<a}\}_{a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K_0 \cup K_1)}$ is naturally compatible with $\{I^{a,b}\}_{a \le b}$ and the family of linear maps

$$\{H_*(\mathscr{A}_*^{<a}, \partial + F) \to H_*(\mathscr{A}_*^{<b}, \partial + F)\}_{a \le b}$$

which is induced by the inclusion map $\mathscr{A}_*^{<a} \to \mathscr{A}_*^{<b}$. Therefore, on the limit of $a \to \infty$, we obtain an isomorphism

$$H_*(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F) = \varinjlim_{a \to \infty} H_*(\mathscr{A}_*^{<a}, \partial + F) \to H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_1).$$

This finishes the proof of Theorem 5.1.

5.2 Computation of $H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_2)$

We define $\mathscr{A}_*^{\mathrm{unlink}} := \mathscr{A}_*(\mathscr{C}')$ by the set

$$\mathscr{C}' := \{c_{i,j}^0\}_{i \ne j} \cup \{c_{i,i}^1\}_i \cup \{c_{i,j}^1, \bar{c}_{i,j}^1\}_{i \ne j} \cup \{c_{i,j}^2\}_{i,j},$$

where i and j run over $\{0, 2\}$. The degree of each element is given by

$$|c_{i,j}^0| = d-2, \qquad |c_{i,j}^1| = |c_{i,j}^1| = |\bar{c}_{i,j}^1| = 2d-3 \quad \text{for } i \ne j, \qquad |c_{i,i}^1| = 2d-3, \qquad |c_{i,j}^2| = 3d-4.$$

(Obviously, there exists an isomorphism $\mathscr{A}_*(\mathscr{C}) \cong \mathscr{A}_*^{\mathrm{unlink}}$ as graded $\mathbb{R}$-algebras.) We define a graded derivation $\partial := 0: \mathscr{A}_*^{\mathrm{unlink}} \to \mathscr{A}_{*-1}^{\mathrm{unlink}}$. For a differential graded algebra $(\mathscr{A}_*^{\mathrm{unlink}}, \partial)$, we have the following result.

Theorem 5.9 *There exists an isomorphism of graded $\mathbb{R}$-algebras*

$$H_*(\mathscr{A}_*^{\mathrm{unlink}}, \partial) \cong H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_2).$$

To compute $H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_2)$, we fix auxiliary data such that g is the standard Riemannian metric on $\mathbb{R}^{2d-1}$. The constant C_0 is required to be $C_0 > \sqrt{|z_2^*|^2 + 4}$. The other data, ε_0 and μ, are not specified. The strategy of the proof is the same as for Theorem 5.1, but it is much more simple. We only give the outline of each step.

Step 1 We may assume that $|z_2^*| > 2$. Let us fix points $p_0 := (0, 1, 0) \in K_0$ and $p_2 := (0, 1, z_2^*) \in K_2$, and define submanifolds of $(K_0 \cup K_2)^2$,

$$K_{0,2} := \{(p, p') \in K_0 \times K_2 \mid p' = p + (0, 0, z_2^*)\}, \quad K_{2,0} := \{(p, p') \in K_2 \times K_0 \mid p' = p - (0, 0, z_2^*)\}.$$

Let $\varphi: (K_0 \cup K_2)^2 \to \Omega_{K_0 \cup K_2}(\mathbb{R}^{2d-1})$ be the map defined as in Section 5.1 by replacing K_1 by K_2. Then we define the set of chains

$$\mathscr{X}' := \{x_{i,j}^0\}_{i \ne j} \cup \{x_{i,i}^1\}_i \cup \{x_{i,j}^1, \bar{x}_{i,j}^1\}_{i \ne j} \cup \{x_{i,j}^2\}_{i,j},$$

where i and j run over $\{0, 2\}$, as follows:

$$
\begin{aligned}
x_{i,j}^0 &:= [\{(p_i, p_j)\}, \varphi|_{\{(p_i, p_j)\}}, 1] \in C_0^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0) & \text{if } (i \neq j), \\
x_{i,i}^1 &:= [\{p_i\} \times K_i, \varphi|_{\{p_i\} \times K_i}, 1] \in C_{d-1}^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0), \\
x_{i,j}^1 &:= [K_{i,j}, \varphi|_{K_{i,j}}, 1] \in C_{d-1}^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0) & \text{if } (i \neq j), \\
\bar{x}_{i,j}^1 &:= [K_i \times \{p_j\}, \varphi|_{K_i \times \{p_j\}}, 1] \in C_{d-1}^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0) & \text{if } (i \neq j), \\
x_{i,j}^2 &:= [K_i \times K_j, \varphi|_{K_i \times K_j}, 1] \in C_{2d-2}^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0).
\end{aligned}
$$

Here, $a > \sqrt{|z_2^*|^2 + 4}$. If we define $\mathfrak{l} \colon \mathscr{X}' \to \mathbb{R}_{>0}$ by

$$
\mathfrak{l}(x_{i,j}^0) = \mathfrak{l}(x_{i,j}^1) = |z_2^*| \quad \text{and} \quad \mathfrak{l}(\bar{x}_{i,j}^1) = \mathfrak{l}(x_{i,j}^2) = \sqrt{|z_2^*|^2 + 4} \quad \text{for } i \neq j, \qquad \mathfrak{l}(x_{i,i}^1) = \mathfrak{l}(x_{i,i}^2) = 2,
$$

then each $x \in \mathscr{X}'$ satisfies $x \in C_*^{\mathrm{dR}}(\Sigma_1^{\mathfrak{l}(x)+\varepsilon}, \Sigma_1^0)$ for any $\varepsilon > 0$. Furthermore, a basis of $H_*^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0)$ for $a \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 1}$ is given by the set of homology classes

$$
\{[x_1 \star \cdots \star x_m] \mid x_1, \ldots, x_m \in \mathscr{X}', \mathfrak{l}(x_1) + \cdots + \mathfrak{l}(x_m) < a\}.
$$

The reason this case is simpler is the following: for any $x \in \mathscr{X}'$ and $(\varepsilon, \delta) \in \mathscr{T}_a$ with $a > \mathfrak{l}(x)$, the equation

$$
f_{1,\delta}(x) = 0 \in C_*^{\mathrm{dR}}(\Sigma_2^{\mathfrak{l}(x)+2\varepsilon}, \Sigma_2^0)
$$

holds since the path $\varphi(p, p')$ satisfies condition (iii) of Lemma 3.12 for any $(p, p') \in (K_0 \cup K_2)^2$ (see Lemma 5.3).

Step 2 There exists a bijection $\mathscr{C}' \to \mathscr{X}'$ which maps $c_{i,j}^k$ to $x_{i,j}^k$ and $\bar{x}_{i,j}^1$ to $\bar{c}_{i,j}^1$ for $k \in \{0, 1, 2\}$ and $i, j \in \{0, 2\}$. Composing $\mathfrak{l} \colon \mathscr{X}' \to \mathbb{R}_{>0}$ with this bijection, a function $\mathfrak{l} \colon \mathscr{C}' \to \mathbb{R}_{>0}$ is defined. Similar to $\mathscr{A}_*^{\mathrm{Hopf}}$, $\mathscr{A}_*^{\mathrm{unlink}}$ is filtered by subcomplexes $(\mathscr{A}_*^{<a}, \partial)$ for all $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K_0 \cup K_2)$ which is defined by using $\mathfrak{l} \colon \mathscr{C}' \to \mathbb{R}_{>0}$.

Now $\Phi_\varepsilon^{<a} \colon \mathscr{A}_*^{<a} \to C_*^{<a}(\varepsilon)$ is defined so that $c \in \mathscr{C}'$ is mapped to $x \in \mathscr{X}'$ which corresponds to c via the above bijection, and extended naturally via the product map on $\mathscr{A}_*^{\mathrm{unlink}}$ and the $\star$-operation. It is clear in this case that $\Phi_\varepsilon^{<a}$ is a chain map from $(\mathscr{A}_*^{<a}, \partial = 0)$ to $(C_*^{<a}(\varepsilon), D_\delta)$. The fact that this map is a quasi-isomorphism is proved by a similar argument as in Proposition 5.7 about spectral sequences.

Step 3 We check that the family of maps $(\Phi_\varepsilon^{<a})_*$ induces an isomorphism on the limit of $\varepsilon \to 0$ and $a \to \infty$. This finishes the proof of Theorem 5.9.

5.3 A corollary and its potential application

The next result is a corollary from the above computations

Corollary 5.10 *As graded $\mathbb{R}$-algebras,*

$$
H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_1) \ncong H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_2).
$$

Proof From Theorems 5.1 and 5.9, it suffices to show that $H_*(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F)$ is not isomorphic to $\mathscr{A}_*^{\mathrm{unlink}}$ as a graded $\mathbb{R}$-algebra. Let us rewrite $(c_{0,1}^0, c_{1,0}^0, e_{0,0}^1, e_{1,1}^1)$ by (a_0, a_1, b_0, b_1) and $(c_{0,2}^0, c_{2,0}^0)$ by (a_0', a_1'). In addition, we define $C_0 := b_0 + a_0 a_1$ and $C_1 := b_1 + (-1)^d a_1 a_0$.

If $d = 2$, $H_0(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F)$ is the a priori noncommutative $\mathbb{R}$-algebra generated by $\{a_0, a_1, b_0, b_1\}$ modulo the ideal generated by $\{b_0, b_1, C_0, C_1\}$. This is isomorphic to the commutative algebra $\mathbb{R}[a_0, a_1]/(a_0 a_1)$. On the other hand, $\mathscr{A}_0^{\mathrm{unlink}}$ is the noncommutative algebra freely generated by $\{a_0', a_1'\}$. Therefore, $H_0(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F) \not\cong \mathscr{A}_0^{\mathrm{unlink}}$ as $\mathbb{R}$-algebras.

If $d \geq 3$, the lower-degree parts are isomorphic as vector spaces. Indeed, for $p \leq 2d - 5$,

$$H_p(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F) \cong \mathscr{A}_p^{\mathrm{unlink}} \cong \begin{cases} \mathbb{R} & \text{if } p = 0, \\ \mathbb{R}a_0 \oplus \mathbb{R}a_1 & \text{if } p = d - 2, \\ 0 & \text{otherwise.} \end{cases}$$

However, $H_{2d-4}(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F)$ is the $\mathbb{R}$-vector space spanned by $\{a_i a_j \mid i, j \in \{0, 1\}\} \cup \{b_0, b_1\}$ modulo the subspace generated by $\{b_0, b_1, C_0, C_1\}$, so its dimension is equal to 2. On the other hand, $\mathscr{A}_{2d-4}^{\mathrm{unlink}}$ is the $\mathbb{R}$-vector space spanned by $\{a_i' a_j' \mid i, j \in \{0, 1\}\}$, so its dimension is equal to 4. Therefore, $H_{2d-4}(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F) \not\cong \mathscr{A}_{2d-4}^{\mathrm{unlink}}$ as $\mathbb{R}$-vector spaces. $\square$

Let us see a potential application of this result. First we determine spin structures on unit conormal bundles

Proposition 5.11 *Let Q be an n-dimensional Riemannian manifold with a fixed spin structure. Then, for every submanifold K in Q, we can assign a spin structure on its unit conormal bundle Λ_K so that, if K is isotopic to K' as a submanifold in Q, then Λ_K is isotopic to $\Lambda_{K'}$ as a Legendrian submanifold with a spin structure.*

Proof Let us identify T^*Q with TQ via the Riemannian metric. We also identify Q with the zero section of TQ. Let L_K be the conormal bundle of K. Note that the tangent space of TQ at $(q, 0) \in Q$ is equal to $T_q Q \oplus T_q Q$, where the first component is the tangent space of the base space Q, and the second component is the tangent space of the fiber $T_q Q$. For every $q \in K$, $T_{(q,0)}(L_K) = T_q K \oplus (T_q K)^\perp$. Thus the vector bundle $T(L_K)|_K$ has a spin structure induced by $TQ|_K$. Since K is a deformation retract of L_K, this spin structure is extended to $T(L_K)$. By using the diffeomorphism $\mathbb{R}_{>0} \times \Lambda_K \to L_K \setminus K$, $(r, (q, p)) \mapsto (q, r \cdot p)$, we can determine a spin structure on $T(\Lambda_K)$ such that the spin structure on $\mathbb{R} \oplus T\Lambda_K$, induced by the inclusion map $\mathrm{Spin}(2n-1) \to \mathrm{Spin}(2n)$, is equal to the spin structure on $T(L_K)|_{\Lambda_K} \cong \mathbb{R} \oplus T\Lambda_K$. This spin structure on Λ_K for every submanifold K clearly satisfies the condition of this proposition. $\square$

Let us consider the unit conormal bundles of $K_0 \cup K_1$ and $K_0 \cup K_2$.

Proposition 5.12 *As a $(2d-2)$-dimensional submanifold of $UT^*\mathbb{R}^{2d-1}$ with the spin structure determined by Proposition 5.11, $\Lambda_{K_0 \cup K_1}$ is isotopic to $\Lambda_{K_0 \cup K_2}$.*

Proof For $s \in [0,1]$, we define $K_1^s := \{q + (0, 2s, 0) \in \mathbb{R}^{2d-1} \mid q \in K_1\}$. We also choose a C^∞ function $[0,1] \to [0,\pi]$, $s \mapsto \theta_s$, such that $\theta_0 = \theta_1 = 0$ and $\theta_{1/2} = \frac{\pi}{2}$, and define $R_s \in SO(2d-1)$ for $s \in [0,1]$ by $R_s(v_0, v_1, v_2) := ((\cos\theta_s)v_0 - (\sin\theta_s)v_2, v_1, (\sin\theta_s)v_0 + (\cos\theta_s)v_2)$ for every $(v_0, v_1, v_2) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1}$. We then define an isotopy $(\Lambda_s)_{s \in [0,1]}$ from $\Lambda_{K_1^0} = \Lambda_{K_1}$ to $\Lambda_{K_1^1}$ by

$$\Lambda_s := \{(q, p) \in UT^*\mathbb{R}^{2d-1} \mid q \in K_s, \, p \circ R_s|_{T_q K_s^1} = 0\}.$$

Λ_s intersects Λ_{K_0} if and only if $s = \frac{1}{2}$, and $\Lambda_{1/2} \cap \Lambda_{K_0} = \Lambda_{K_0} \cap UT^*_{p_0}\mathbb{R}^{2d-1}$, where $p_0 = (0, 1, 0) \in \mathbb{R}^{2d-1}$. We can slightly perturb $(\Lambda_s)_{s \in [0,1]}$ around $s = \frac{1}{2}$ to an isotopy $(\Lambda'_s)_{s \in [0,1]}$ such that Λ'_s does not intersect Λ_{K_0} for every $s \in [0,1]$. This isotopy is homotopic to an isotopy $(\Lambda_{K_1^s})_{s \in [0,1]}$, which preserves the spin structure of [Proposition 5.11](#). In addition, $K_0 \cup K_1^1$ is isotopic to $K_0 \cup K_2$ in $\mathbb{R}^{2d-1}$. Therefore, as a C^∞ submanifold with a spin structure, $\Lambda_{K_0} \cup \Lambda_{K_1}$ is isotopic to $\Lambda_{K_0} \cup \Lambda_{K_2}$. $\square$

If [Conjecture 1.4](#) in the introduction is true, then [Corollary 5.10](#) can be applied to show that the unit conormal bundle $\Lambda_{K_0 \cup K_1}$ is not isotopic to $\Lambda_{K_0 \cup K_2}$ as a Legendrian submanifold with a spin structure in $UT^*\mathbb{R}^{2d-1}$, though they are isotopic as C^∞ submanifolds with spin structures by the above proposition.

6 Cord algebra and $H_0^{string}(Q, K)$

Throughout this section, we consider the case where the codimension of K is 2 (ie $d = 2$) and the normal bundle $(TK)^\perp$ is trivial. The purpose is to show that $H_0^{string}(Q, K)$ is isomorphic to an isotopy invariant of K, called *cord algebra*.

6.1 Cord algebra and string homology

In this section, we refer to [6; 19] and give a definition of cord algebra and string homology. Note that, in this paper, their coefficients are reduced from the original $\mathbb{Z}[\pi_1(\partial N_{\varepsilon_0})]$ to $\mathbb{R}$.

We fix a frame of $(TK)^\perp$ to give an isomorphism $\mathbb{R}^2 \times K \cong (TK)^\perp$ of vector bundles over K which preserves their fiber metrics and orientations. Combining with the map [(3)](#), we obtain a diffeomorphism

$$h : \mathbb{O}_{\varepsilon_0} \times K \to N_{\varepsilon_0},$$

which preserves orientations. Here, $\mathbb{O}_\varepsilon = \{w \in \mathbb{R}^2 \mid |w| < \frac{1}{2}\varepsilon\}$ for every $\varepsilon \le \varepsilon_0$, as defined in [Section 4.4.3](#).

First, we define an $\mathbb{R}$-algebra $\mathrm{Cord}(Q, K; \mathbb{R})$. Its relation to the cord algebra defined in [6; 19] is discussed later in [Remark 6.2](#). Let us prepare some notation. We fix $w_0 \in \mathbb{O}_{\varepsilon_0} \setminus \{0\}$ and define a submanifold disjoint from K,

$$K' := \{h(w_0, x) \mid x \in K\} \subset N_{\varepsilon_0}.$$

For every $x \in K$, we define $c_x : [0,1] \to Q \setminus K$ to be the constant path at $h(w_0, x) \in K'$. We also define $m_x : [0,1] \to Q \setminus K$ to be a loop in a punctured disk $h((\mathbb{O}_{\varepsilon_0} \setminus \{0\}) \times \{x\}) \subset N_{\varepsilon_0} \setminus K$ based at $h(w_0, x) \in K'$, whose winding number around $h(0, x)$ is equal to 1. In addition, let $\pi_1(Q \setminus K, K')$ be the set of homotopy classes of continuous paths $\gamma : [0,1] \to Q \setminus K$ such that $\gamma(\{0, 1\}) \subset K'$.

Definition 6.1 Let $\mathscr{A}$ be the unital noncommutative $\mathbb{R}$-algebra freely generated by the set $\pi_1(Q \setminus K, K')$. We define the two-sided ideal $\mathscr{I}$ generated by the elements

$$[c_x], \quad [\gamma_1 \cdot \gamma_2] - [\gamma_1 \cdot m_x \cdot \gamma_2] - [\gamma_1][\gamma_2]$$

for all $x \in K$ and $\gamma_j : [0, 1] \to Q \setminus K$ for $j = 1, 2$ such that $\gamma_j(\{0, 1\}) \subset K'$ and $\gamma_1(1) = h(w_0, x) = \gamma_2(0)$. Then we define an $\mathbb{R}$-algebra $\mathrm{Cord}(Q, K; \mathbb{R}) := \mathscr{A}/\mathscr{I}$, and call it the *cord algebra* of (Q, K).

Remark 6.2 When K is 1-dimensional and connected (ie K is an oriented knot in a 3-manifold Q), we fix a basepoint $* \in K'$. A *cord* is a path $\gamma : [0, 1] \to Q$ such that $\gamma([0, 1]) \cap K = \varnothing$ and $\gamma(0), \gamma(1) \in K' \setminus \{*\}$. The notion of cord algebra (or *cord ring*) of knots was defined in [6; 18; 19], for instance. The most refined one is [6, Definition 2.6], which is defined as a noncommutative algebra over $\mathbb{Z}$ generated by the set of homotopy classes of cords and $\{\lambda^{\pm}, \mu^{\pm}\}$, modulo the relations about $\{\lambda^{\pm}, \mu^{\pm}\}$ and the "skein relations". If we replace both λ and μ by $1 \in \mathbb{Z}$ and tensor this $\mathbb{Z}$-algebra with $\mathbb{R}$, we obtain an $\mathbb{R}$-algebra isomorphic to $\mathrm{Cord}(Q, K; \mathbb{R})$. (The isomorphism is induced by a natural map from the set of homotopy classes of cords to $\pi_1(Q \setminus K, K')$.)

We should also note that, in [19, Definition 2.1], the cord algebra over $\mathbb{Z}[H_1(\partial N_{\varepsilon_0})]$ was defined when K is a connected codimension 2 submanifold of an arbitrary manifold and its normal bundle is oriented. In our setting, we have an isomorphism $(h^{-1})_* : H_1(\partial N_{\varepsilon_0}) \to H_1(S^1 \times K) \cong H_1(S^1) \oplus H_1(K)$. There exists a ring homomorphism $\varphi : \mathbb{Z}[H_1(\partial N_{\varepsilon_0})] \to \mathbb{R}$ determined by $\varphi(h_*([S^1])) = -1$ and $\varphi(h_*(c)) = 1$ for every $c \in H_1(K)$. If the base change of the cord algebra of [19, Definition 2.1] is done by φ, we obtain an $\mathbb{R}$-algebra isomorphic to $\mathrm{Cord}(Q, K; \mathbb{R})$.

Next, we refer to [6, Section 2.1] and define the string homology, which is simplified for our purpose. For $m \in \mathbb{Z}_{\geq 1}$, $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $p = 0, 1$, we define an $\mathbb{R}$-subspace $C_p^{\pitchfork}(m, a) \subset C_p^{\mathrm{sing}}(\Sigma_m^a)$ consisting of generic singular p-chains satisfying jet transversality conditions. (Recall that we have fixed a topology of Σ_m^a in Section 3.2.4.)

In the case of $p = 0$, $C_0^{\pitchfork}(m, a)$ is generated by $(\gamma_k : [0, T_k] \to Q)_{k=1,\ldots,m} \in \Sigma_m^a$ satisfying the following conditions:

(0a) $(\gamma_k)'(0), (\gamma_k)'(T_k) \notin TK$ for every $k \in \{1, \ldots, m\}$.

(0b) $\gamma_k(t) \notin K$ for every $k \in \{1, \ldots, m\}$ and $t \neq 0, T_k$.

In the case of $p = 1$, $C_1^{\pitchfork}(m, a)$ is generated by 1-parameter families of paths

$$[0, 1] \to \Sigma_m^a, \quad u \mapsto (\gamma_k^u : [0, T_k^u] \to Q)_{k=1,\ldots,m},$$

such that $[0, 1] \to \mathbb{R}_{>0}, u \mapsto T_k^u$, is a C^{∞} function and

$$\Gamma_k : \{(u, t) \mid 0 \leq u \leq 1, 0 \leq t \leq T_k^u\} \to Q$$

is a C^{∞} map for every $k \in \{1, \ldots, m\}$, and satisfies the following conditions:

(1a) $(\gamma_k^0)_{k=1,\dots,m}$ and $(\gamma_k^1)_{k=1,\dots,m}$ satisfy (0a)–(0b).

(1b) $(\gamma_k^u)'(0), (\gamma_k^u)'(T_k^u) \notin TK$ for every $u \in [0,1]$ and $\Gamma_k^{\mathrm{int}} := \Gamma_k|_{\{(u,t)|t\neq 0,T_k^u\}}$ is transverse to K for every $k \in \{1,\dots,m\}$.

(1c) If $(u_*, t_*), (u'_*, t'_*) \in \bigsqcup_{k=1}^m (\Gamma_k^{\mathrm{int}})^{-1}(K)$ are distinct points, then $u_* \neq u'_*$.

Note that condition (1b) implies that $(\Gamma_k^{\mathrm{int}})^{-1}(K)$ is a finite set. In addition, we define $C_p^{\pitchfork}(0,a) := C_p^{\mathrm{sing}}(\Sigma_0^a)$ for $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $p = 0,1$.

By (1a), the boundary operator of the singular chain complex $\partial^{\mathrm{sing}}: C_1^{\mathrm{sing}}(\Sigma_m^a) \to C_0^{\mathrm{sing}}(\Sigma_m^a)$ is restricted to the map

$$\partial^{\mathrm{sing}}: C_1^{\pitchfork}(m,a) \to C_0^{\pitchfork}(m,a), \quad (\gamma_k^u)_{k=1,\dots,m}^{u\in[0,1]} \mapsto (\gamma_k^1)_{k=1,\dots,m} - (\gamma_k^0)_{k=1,\dots,m}.$$

We also define a linear map $f_k^{\pitchfork}: C_1^{\pitchfork}(m,a) \to C_0^{\pitchfork}(m+1,a)$ for $m \in \mathbb{Z}_{\geq 1}$ such that, for any 1-chain $x = (\gamma_k^u: [0, T_k^u] \to Q)_{k=1,\dots,m}^{u\in[0,1]} \in C_1^{\pitchfork}(m,a)$,

$$f_k^{\pitchfork}(x) := \sum_{(u_*,t_*)\in(\Gamma_k^{\mathrm{int}})^{-1}(K)} \mathrm{sign}(u_*, t_*) \cdot (\gamma_1^{u_*} \dots, \gamma_{k-1}^{u_*}, \widehat{\gamma_k^{u_*}1}, \widehat{\gamma_k^{u_*}2}, \gamma_{k+1}^{u_*}, \dots, \gamma_m^{u_*}).$$

Here

$$\widehat{\gamma_k^{u_*}1} := \gamma_k^{u_*}|_{[0,t_*]}: [0, t_*] \to Q, \quad \widehat{\gamma_k^{u_*}2} := \gamma_k^{u_*}|_{[t_*, T_k^{u_*}]}(\cdot - t_*): [0, T_k^{u_*} - t_*] \to Q,$$

and $\mathrm{sign}(u_*, t_*) \in \{\pm 1\}$ is the orientation sign of the open embedding into $\mathbb{O}_{\varepsilon_0} \subset \mathbb{R}^2$

$$\Gamma_k^{\mathrm{fiber}} := \mathrm{pr}_{\mathbb{R}^2} \circ h^{-1} \circ \Gamma_k$$

defined on a small neighborhood of $(u_*, t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K) \subset (0,1) \times \mathbb{R}_{>0}$. For convenience, let us define, for $p \notin \{0,1\}$, $C_p^{\pitchfork}(m,a) := 0$, $\partial^{\mathrm{sing}} := 0: C_p^{\pitchfork}(m,a) \to C_{p-1}^{\pitchfork}(m,a)$ and $f_k^{\pitchfork} := 0: C_p^{\pitchfork}(m,a) \to C_{p-1}^{\pitchfork}(m+1,a)$.

For $a \in \mathbb{R}_{>0} \cup \{\infty\}$ and $m \in \mathbb{Z}_{\geq 0}$, we define the quotient vector space

$$C_*^{\pitchfork,<a}(m) := C_*^{\pitchfork}(m,a)/C_*^{\pitchfork}(m,0).$$

Then ∂^{sing} and $f_k^{\pitchfork}$ induce linear maps

$$\partial^{\mathrm{sing}}: C_*^{\pitchfork,<a}(m) \to C_{*-1}^{\pitchfork,<a}(m), \quad f_k^{\pitchfork}: C_*^{\pitchfork,<a}(m) \to C_{*-1}^{\pitchfork,<a}(m+1).$$

Now we define the graded $\mathbb{R}$-vector space

$$C_*^{\pitchfork,<a} := \bigoplus_{m=0}^{\infty} C_*^{\pitchfork,<a}(m).$$

For each $m \in \mathbb{Z}_{\geq 0}$, $C_*^{\pitchfork,<a}(m)$ is considered to be its linear subspace in a natural way. Then we define a degree -1 map $D^{\pitchfork}: C_*^{\pitchfork,<a} \to C_{*-1}^{\pitchfork,<a}$ by

$$(41) \qquad\qquad D^{\pitchfork}(x) := \partial^{\mathrm{sing}} x + \sum_{k=1}^m f_k^{\pitchfork}(x)$$
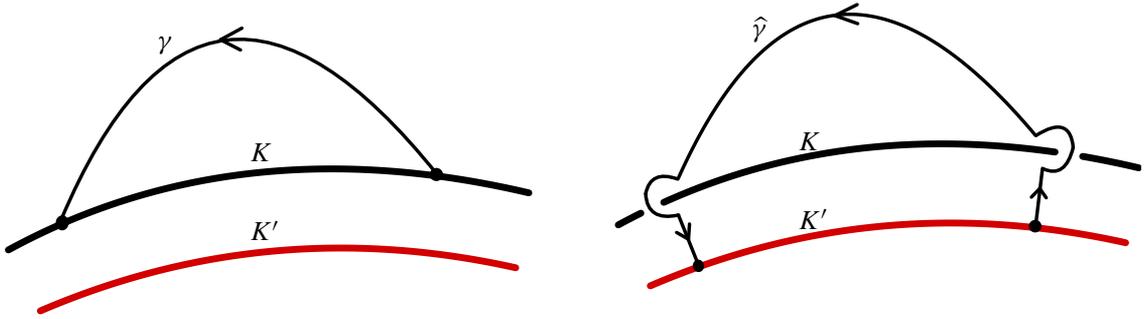
Figure 5: The picture of $\widehat{\gamma}$ defined for $(\gamma)_{k=1} \in \Sigma_1^\infty$.

for $x \in C_*^{\pitchfork, <a}(m)$. For $x \in C_*^{\pitchfork, <a}(0)$, the right-hand side is just equal to $\partial^{\mathrm{sing}} x$. Then we obtain a chain complex $(C_*^{\pitchfork, <a}, D^{\pitchfork})$. Let $H_*^{\pitchfork, <a}$ denote its homology group. In addition, for $a, b \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ with $a \leq b$, we define a linear map $J^{a,b}: H_*^{\pitchfork, <a} \to H_*^{\pitchfork, <b}$ induced by the inclusion maps $C_*^{\pitchfork}(m, a) \to C_*^{\pitchfork}(m, b)$ for all $m \in \mathbb{Z}_{\geq 0}$.

In this section, we call $H_*^{\pitchfork, <\infty}$ the *string homology* of (Q, K). Note that the direct limit $\varinjlim_{a \to \infty} H_*^{\pitchfork, <a}$ defined by $\{J^{a,b}\}_{a \leq b}$ is isomorphic to $H_0^{\pitchfork, <\infty}$. Furthermore, $H_0^{\pitchfork, <\infty}$ has a associative product structure induced by $\Pi: \Sigma_m^\infty \times \Sigma_{m'}^\infty \to \Sigma_{m+m'}^\infty$. Thus $H_0^{\pitchfork, <\infty}$ is a unital associative $\mathbb{R}$-algebra, whose unit comes from $* \in C_0^{\pitchfork, <\infty}(0)$.

The next proposition is essentially proved in [6, Proposition 2.9].

Proposition 6.3 *As an $\mathbb{R}$-algebra, $\mathrm{Cord}(Q, K; \mathbb{R})$ is isomorphic to $H_0^{\pitchfork, <\infty}$.*

Proof For every homotopy class $z \in \pi_1(Q \setminus K, K')$, we choose a C^∞ path γ which represents z. We then define a path $\bar{\gamma}$ as follows: for $x_0, x_1 \in K$ such that $\gamma(i) = h(w_0, x_i)$ for $i \in \{0, 1\}$,

$$\bar{\gamma}: [0, 3] \to Q, \qquad t \mapsto \begin{cases} h(t \cdot w_0, x_0) & \text{if } 0 \leq t \leq 1, \\ \gamma(t-1) & \text{if } 1 \leq t \leq 2, \\ h((3-t) \cdot w_0, x_1) & \text{if } 2 \leq t \leq 3. \end{cases}$$

We modify $\bar{\gamma}$ to $\widetilde{\gamma}$ by a reparametrization so that $(\widetilde{\gamma})_{k=1} \in \Sigma_1^\infty$ and satisfies (0a) and (0b). Then a homomorphism of unital $\mathbb{R}$-algebras

$$F: \mathscr{A} \to H_0^{\pitchfork, <\infty}$$

is defined so that $z = [\gamma]$ is mapped to $[(\widetilde{\gamma})_{k=1}]$. From the definition of $D^{\pitchfork}$, it can be checked that F is a well-defined surjection and maps the ideal $\mathscr{I}$ into 0. Therefore, we obtain a surjective homomorphism of unital $\mathbb{R}$-algebras $\bar{F}: \mathscr{A}/\mathscr{I} \to H_0^{\pitchfork, <\infty}$.

We prove that $\bar{F}$ is injective by describing its inverse map. For $(\gamma: [0, T] \to Q)_{k=1} \in \Sigma_1^\infty$ satisfying (0a) and (0b), let us define

$$\check{\gamma}: [0, 1] \to Q, \qquad t \mapsto \begin{cases} h((1-3t) \cdot w_0, \gamma(0)) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \gamma(3Tt - T) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ h((3t-2) \cdot w_0, \gamma(T)) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$
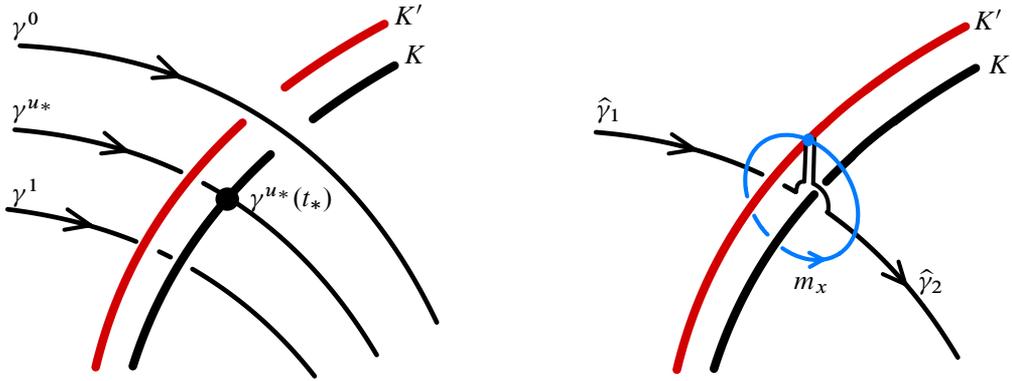
Figure 6: The left-hand side describes the 1-chain $(\gamma^u)_{k=1}^{u \in [0,1]}$ such that $\mathrm{sign}(u_*, t_*) = +1$ at $(u_*, t_*) \in (\Gamma_1^{\mathrm{int}})^{-1}(K)$. From the right-hand side, we can see that $[\hat{\gamma}^0] = [\hat{\gamma}_1 \cdot \hat{\gamma}_2]$ and $[\hat{\gamma}^1] = [\hat{\gamma}_1 \cdot m_x \cdot \hat{\gamma}_2]$ as elements of $\pi_1(Q \setminus K, K')$.

As described in Figure 5, we change $\check{\gamma}$ into $\hat{\gamma}$ by small perturbations inside N_{ε_0} around $t \in \{\frac{1}{3}, \frac{2}{3}\}$ so that $\hat{\gamma}$ does not intersect K. We then obtain a homotopy class $[\hat{\gamma}] \in \pi_1(Q \setminus K, K')$. Note that, for any $\hat{\gamma}^j$ for $j = 1, 2$ from two choices of perturbations, there exist $l_0, l_1 \in \{0, +1, -1\}$ such that

$$[\hat{\gamma}^2] = [(m_{\gamma(0)})^{l_0} \cdot \hat{\gamma}^1 \cdot (m_{\gamma(T)})^{l_1}] \in \pi_1(Q \setminus K, K').$$

Here, $(m_x)^1 := m_x$, $(m_x)^0 := c_x$, and $(m_x)^{-1}$ denotes the inverse path of m_x. (As a natural extension, $(m_x)^l$ for $l \in \mathbb{Z}$ is defined.) Thus, $[\hat{\gamma}^1] = [\hat{\gamma}^2]$ as an element of $\mathcal{A}/\mathcal{I}$. If $\gamma \in \Sigma_1^0$ (ie length $\gamma < \varepsilon_0$), then $[\hat{\gamma}] = [(m_x)^l] \in \pi_1(Q \setminus K, K')$ for some $x \in K$ and $l \in \mathbb{Z}$. In this case, $[\hat{\gamma}] = 0$ as an element of $\mathcal{A}/\mathcal{I}$. Therefore, we have a well-defined linear map

$$G : C_0^{\pitchfork, <\infty} \to \mathcal{A}/\mathcal{I}, \qquad \begin{cases} C^{\pitchfork, <\infty}(m) \ni (\gamma_k)_{k=1,\dots,m} \mapsto [\hat{\gamma}_1] \cdots [\hat{\gamma}_m] & \text{for } m \geq 1, \\ C^{\pitchfork, <\infty}(0) \ni 1 \mapsto \text{the unit}. \end{cases}$$

From the transversality condition (1b) together with (1c), it follows that $\mathrm{Im}\, D^{\pitchfork}$ is mapped into 0. Indeed, in a simple case, for $(\gamma^u)_{k=1}^{u \in [0,1]} \in C_1^{\pitchfork}(1, \infty)$ such that $(\Gamma_1^{\mathrm{int}})^{-1}(K) = \{(u_*, t_*)\}$, we can see that

$$G(D^{\pitchfork}(x)) = \begin{cases} [\hat{\gamma}_1 \cdot m_x \cdot \hat{\gamma}_2] - [\hat{\gamma}_1 \cdot \hat{\gamma}_2] + \mathrm{sign}(u_*, t_*)[\hat{\gamma}_1][\hat{\gamma}_2] & \text{if } \mathrm{sign}(u_*, t_*) = +1, \\ [\hat{\gamma}_1 \cdot \hat{\gamma}_2] - [\hat{\gamma}_1 \cdot m_x \cdot \hat{\gamma}_2] + \mathrm{sign}(u_*, t_*)[\hat{\gamma}_1][\hat{\gamma}_2] & \text{if } \mathrm{sign}(u_*, t_*) = -1, \end{cases}$$

for $x := \gamma^{u_*}(t_*)$, $\gamma_1 := \gamma^{u_*}|_{[0,t_*]}$ and $\gamma_2 := \gamma^{u_*}|_{[t_*, T^{u_*}]}(\cdot - t_*)$. Figure 6 describes the case where $\mathrm{sign}(u_*, t_*) = +1$. Thus $G\big(D^{\pitchfork}((\gamma^u)_{k=1}^{u \in [0,1]})\big) = 0 \in \mathcal{A}/\mathcal{I}$. Condition (1c) implies that the general case can be reduced to this simple case. Therefore, we obtain a well-defined linear map $\overline{G} : H_0^{\pitchfork, <\infty} \to \mathcal{A}/\mathcal{I}$. Finally, for any $[\gamma] \in \pi_1(Q \setminus K, K')$ such that $\gamma(i) = h(w_0, x_i)$ for $i \in \{0, 1\}$, there exist $l_0, l_1 \in \{0, +1, -1\}$ such that

$$\overline{G} \circ \overline{F}([\gamma]) = [(m_{x_0})^{l_0} \cdot \gamma \cdot (m_{x_1})^{l_1}],$$

which is equal to $[\gamma]$ in $\mathcal{A}/\mathcal{I}$. This implies that $\overline{F}$ is injective. □

6.2 Connecting string homology and $H_*^{\text{string}}(Q, K)$

The purpose of this section is to construct a linear map from $H_*^{\text{ᚠ},<a}$ to $H_*^{<a}(Q, K)$ for all $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$. On the limit of $a \to \infty$, an $\mathbb{R}$-algebra homomorphism from $H_*^{\text{ᚠ},<\infty}$ to $H_*^{\text{string}}(Q, K)$ is defined. Before constructing this map, we will prepare two things.

6.2.1 Preliminaries First, we define a map Ψ which associates de Rham chains with singular chains. Let $\kappa, \chi \colon \mathbb{R} \to [0, 1]$ be C^∞ functions such that:

- $\kappa(u) = 0$ if $u \leq 0$ and $\kappa(u) = 1$ if $u \geq 1$. In addition, $\kappa'(u) > 0$ if $0 < u < 1$.
- $\chi \colon \mathbb{R} \to [0, 1]$ has compact support and $\chi(s) = 1$ for every $s \in [0, 1]$.

For $p \in \mathbb{Z}$, a linear map

$$\Psi \colon C_p^{\text{ᚠ}}(m, a) \to C_p^{\text{dR}}(\Sigma_m^a)$$

is defined by

$$\begin{cases} \Psi((\gamma_k)_{k=1,\dots,m}) := [\{0\}, c_0, 1] & \text{if } p = 0, \\ \Psi((\gamma_k^u)_{k=1,\dots,m}^{u \in [0,1]}) := [\mathbb{R}, c_1, \chi] & \text{if } p = 1, \\ \Psi = 0 & \text{otherwise,} \end{cases}$$

where c_0 is the constant map to $(\gamma_k)_{k=1,\dots,m}$ and $c_1 \colon \mathbb{R} \to \Sigma_m^a$, $u \mapsto (\gamma_k^{\kappa(u)})_{k=1,\dots,m}$. These maps commute with boundary operators, namely

$$\partial \circ \Psi = \Psi \circ \partial^{\text{sing}} \colon C_p^{\text{ᚠ}}(m, a) \to C_{p-1}^{\text{dR}}(\Sigma_m^a).$$

Next, we define a filtration $\{C_p^{\text{ᚠ}}(m, a, \varepsilon)\}_{\varepsilon > 0}$ of $C_p^{\text{ᚠ}}(m, a)$. When $p = 0$, $C_0^{\text{ᚠ}}(m, a, \varepsilon)$ is an $\mathbb{R}$-subspace generated by $(\gamma_k \colon [0, T_k] \to Q)_{k=1,\dots,m} \in \Sigma_m^a$ satisfying (0a)–(0b) and the following condition:

(0c) There exists $\tau_0 \in (0, \varepsilon_0/(5C_0)]$ such that $\gamma_k([\tau_0, T_k - \tau_0]) \cap N_\varepsilon = \varnothing$ for every $k \in \{1, \dots, m\}$.

In the case of $p = 1$, $C_1^{\text{ᚠ}}(m, a, \varepsilon)$ is an $\mathbb{R}$-subspace generated by $(\gamma_k^u \colon [0, T_k^u] \to Q)_{k=1,\dots,m}^{u \in [0,1]}$ satisfying (1a)–(1c) and the following conditions:

(1d) $(\gamma_k^0)_{k=1,\dots,m}$ and $(\gamma_k^1)_{k=1,\dots,m}$ satisfy (0c).

(1e) There exist $\tau_0 \in (0, \varepsilon_0/(5C_0)]$ and an open neighborhood $U_{(u_*, t_*)}$ for each $(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)$ for $k = 1, \dots, m$ such that

$$(u_*, t_*) \in U_{(u_*, t_*)} \subset \{(u, t) \mid 0 < u < 1, \tau_0 < t < T_k^u - \tau_0\},$$

and the following hold:

(1e-1) $U_{(u_*, t_*)} \subset \{(u, t) \mid |t - t_*| < \tau_0\}$.

(1e-2) For any two distinct points $(u_*, t_*), (u'_*, t'_*) \in \bigsqcup_{k=1}^m (\Gamma_k^{\text{int}})^{-1}(K)$, the projections of the sets $U_{(u_*, t_*)}, U_{(u'_*, t'_*)} \subset [0, 1] \times \mathbb{R}_{>0}$ to $[0, 1]$,

$$\text{pr}_{[0,1]}(U_{(u_*, t_*)}), \text{pr}_{[0,1]}(U_{(u'_*, t'_*)}) \subset [0, 1],$$

are disjoint.

(1e-3) For every $k \in \{1, \ldots, m\}$,

$$(\Gamma_k|_{\{(u,t)|\tau_0 \leq t \leq T_k^u - \tau_0\}})^{-1}(N_\varepsilon) = \bigcup_{(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} U_{(u_*,t_*)}.$$

Moreover, for each $(u_*, t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)$,

$$\Gamma_k^{\mathrm{fiber}} \colon U_{(u_*,t_*)} \to \mathbb{O}_\varepsilon$$

is a diffeomorphism. (Recall that $\Gamma_k^{\mathrm{fiber}} = \mathrm{pr}_{\mathbb{R}^2} \circ h^{-1} \circ \Gamma_k$.)

In addition, we define $C_p^{\pitchfork}(m, a, \varepsilon) := C_p^{\pitchfork}(m, a)$ if $p \notin \{0, 1\}$ or $m = 0$.

Roughly speaking, (0c) means that $\gamma_k(t)$ is far from K by a distance at least $\frac{1}{2}\varepsilon$, except when t is close to $\{0, T_k\}$. Condition (1e) means that $\gamma_k^u(t)$ is far from K by a distance at least $\frac{1}{2}\varepsilon$, except when t is close to $\{0, T_k^u\}$ or when (u, t) is close to some point in $(\Gamma_k^{\mathrm{int}})^{-1}(K)$. Note that, when $0 < \varepsilon' \leq \varepsilon$, we have $C_*^{\pitchfork}(m, a, \varepsilon) \subset C_*^{\pitchfork}(m, a, \varepsilon')$, and

(42)
$$\bigcup_{\varepsilon > 0} C_*^{\pitchfork}(m, a, \varepsilon) = C_*^{\pitchfork}(m, a).$$

Moreover, ∂^{sing} and $f_k^{\pitchfork}$ for $k = 1, \ldots, m$ are restricted to linear maps

$$\partial^{\mathrm{sing}} \colon C_*^{\pitchfork}(m, a, \varepsilon) \to C_{*-1}^{\pitchfork}(m, a, \varepsilon), \quad f_k^{\pitchfork} \colon C_*^{\pitchfork}(m, a, \varepsilon) \to C_{*-1}^{\pitchfork}(m + 1, a, \varepsilon).$$

For every $\varepsilon > 0$, we define $C_*^{\pitchfork, <a}(m, \varepsilon) := C_*^{\pitchfork}(m, a, \varepsilon)/C_*^{\pitchfork}(m, 0, \varepsilon)$ and

$$C_*^{\pitchfork, <a}(\varepsilon) := \bigoplus_{m=0}^{\infty} C_*^{\pitchfork, <a}(m, \varepsilon).$$

A linear map $D_\varepsilon^{\pitchfork} \colon C_*^{\pitchfork, <a}(\varepsilon) \to C_{*-1}^{\pitchfork, <a}(\varepsilon)$ is defined by the same form as (41). Then we obtain a chain complex $(C_*^{\pitchfork, <a}(\varepsilon), D_\varepsilon^{\pitchfork})$. Let $H_*^{\pitchfork, <a}(\varepsilon)$ denote its homology. When $0 < \varepsilon' \leq \varepsilon$, the inclusion maps $C_*^{\pitchfork}(m, a, \varepsilon) \to C_*^{\pitchfork}(m, a, \varepsilon')$ for all $m \in \mathbb{Z}_{\geq 0}$ induce a linear map

$$l_{\varepsilon, \varepsilon'} \colon H_*^{\pitchfork, <a}(\varepsilon) \to H_*^{\pitchfork, <a}(\varepsilon'),$$

and we have a direct system $\left(\{H_*^{\pitchfork, <a}(\varepsilon)\}_{\varepsilon > 0}, \{l_{\varepsilon, \varepsilon'}\}_{\varepsilon \geq \varepsilon'}\right)$. From the relation (42), we have

$$\varinjlim_{\varepsilon \to 0} H_*^{\pitchfork, <a}(\varepsilon) = H_*^{\pitchfork, <a}.$$

6.2.2 Construction of the chain map With the above preparations, we consider the maps

$$C_p^{\pitchfork}(m, a, \varepsilon) \to C_p^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}), \quad x \mapsto \Psi(x), \quad \text{for } m \in \mathbb{Z}_{\geq 0}$$

for $\varepsilon \in (0, \varepsilon_0/(5C_0)]$. They induce a linear map from $C_*^{\pitchfork, <a}(\varepsilon)$ to $C_*^{<a}(\varepsilon)$, but this is not a chain map. In order to fill in the gap, we need to prove the following lemma for $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$.

Lemma 6.4 *Suppose that $(\varepsilon, \delta) \in \mathcal{T}_a$ is standard with respect to h. Then, for $m \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, m\}$, there exists a linear map*

$$o_{k,(\varepsilon,\delta)}: C_1^{\pitchfork}(m, a, \varepsilon) \to C_1^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon})$$

such that the following hold for any $x \in C_1^{\pitchfork}(m, a, \varepsilon)$:

(i) $\partial(o_{k,(\varepsilon,\delta)}(x)) - (f_{k,\delta} \circ \Psi(x) - \Psi \circ f_k^{\pitchfork}(x)) \in C_0^{\mathrm{dR}}(\Sigma_{m+1}^0)$.

(ii) $f_{l,\delta}(o_{k,(\varepsilon,\delta)}(x)) \in C_0^{\mathrm{dR}}(\Sigma_{m+2}^0)$ *for every* $l \in \{1, \ldots, m+1\}$.

Proof   It suffices to define $o_{k,(\varepsilon,\delta)}(x)$ for $x = (\gamma_l^u)_{l=1,\ldots,m}^{u \in [0,1]}$ satisfying (1a),…,(1e). The proof is divided into three steps: We define de Rham chains in the first two steps. In the last step, $o_{k,(\varepsilon,\delta)}(x)$ is defined as the sum of these chains and we check conditions (i) and (ii).

Step 1   From the definitions of Ψ and $f_k^{\pitchfork}$,

$$f_{k,\delta} \circ \Psi(x) - \Psi \circ f_k^{\pitchfork}(x) = f_{k,\delta}(\Psi(x)) - \sum_{(u_*, t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} \mathrm{sign}(u_*, t_*) \cdot [\{0\}, c_0(u_*, t_*), 1],$$

where $c_0(u_*, t_*)$ is a constant map to $(\gamma_1^{u_*}, \ldots, \widehat{\gamma_k^{u_* 1}}, \widehat{\gamma_k^{u_* 2}}, \ldots, \gamma_m^{u_*})$. Since (ε, δ) is standard, it has the form (31). Using the notation of (13), we can explicitly write $f_{k,\delta}(\Psi(x)) = [W_k, \Phi_k, \zeta_k]$, where

$$W_k = \{(u, \tau, v) \in \mathbb{R} \times \mathbb{R} \times N_\varepsilon \mid 2\varepsilon < \tau < T_k^{\mu(u)} - 2\varepsilon, \gamma_k^{\mu(u)}(\tau) = \sigma_1^v(0)\},$$

$$\Phi_k: W_k \to \Sigma_{m+1}^{a+(m+1)\varepsilon}, \quad (u, \tau, v) \mapsto \mathrm{con}_k((\gamma_l^{\mu(u)})_{l=1,\ldots,m}, (T_k^{\mu(u)}, \tau), \psi_\varepsilon(v)),$$

$$\zeta_k \in \Omega_c^2(W_k), \quad (\zeta_k)_{(u,\tau,v)} = \rho_\varepsilon(T_k^{\mu(u)}, \tau) \cdot \chi(u) \cdot (h_*(\nu_\varepsilon \times 1))_v.$$

Recall condition (1e-3) and note that $\sigma_1^v(0) = v$. We define $\overline{W}_k := W_k \cap \{\tau_0 < \tau < T_k^{\mu(u)} - \tau_0\}$. Then

$$\overline{W}_k \to \bigcup_{(u_*, t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} U_{(u_*, t_*)}, \quad (u, \tau, v) \mapsto (\mu(u), \tau),$$

is an orientation-preserving diffeomorphism. Moreover, $\rho_\varepsilon(T_k^{\mu(u)}, \tau) \cdot \chi(u) = 1$ for $(u, \tau, v) \in \overline{W}_k$. On the other hand, it follows from (1e-1) and Lemma 3.12 that $\Phi_k(u, \tau, v) \in \Sigma_{m+1}^0$ for $(u, \tau, v) \in W_k \setminus \overline{W}_k$. Therefore,

$$f_{k,\delta}(\Psi(x)) - \sum_{(u_*, t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} [U_{(u_*, t_*)}, \Phi'_{(u_*, t_*)}, \zeta'_{(u_*, t_*)}] = f_{k,\delta}(\Psi(x)) - [\overline{W}_k, \Phi_k|_{\overline{W}_k}, \zeta_k|_{\overline{W}_k}]$$

$$\in C_0^{\mathrm{dR}}(\Sigma_{m+1}^0),$$

where

$$\Phi'_{(u_*, t_*)}: U_{(u_*, t_*)} \to \Sigma_{m+1}^{a+(m+1)\varepsilon}, \quad (u, \tau) \mapsto \mathrm{con}_k((\gamma_l^u)_{l=1,\ldots,m}, (T_k^u, \tau), \psi_\varepsilon(\gamma_k^u(\tau))),$$

$$\zeta'_{(u_*, t_*)} = (\Gamma_k^{\mathrm{int}}|_{U_{(u_*, t_*)}})^*(h_*(\nu_\varepsilon \times 1)) \in \Omega_c^2(U_{(u_*, t_*)}).$$

As a result, $f_{k,\delta} \circ \Psi(x) - \Psi \circ f_k^{\pitchfork}(x)$ is equal to the chain

$$(43) \qquad \sum_{(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} \left([U_{(u_*,t_*)}, \Phi'_{(u_*,t_*)}, \zeta'_{(u_*,t_*)}] - \mathrm{sign}(u_*,t_*) \cdot [\{0\}, c_0(u_*,t_*), 1]\right)$$

modulo $C_0^{\mathrm{dR}}(\Sigma_{m+1}^0)$.

For each $(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)$, consider a diffeomorphism

$$\Gamma_k^{\mathrm{fiber}} = \mathrm{pr}_{\mathbb{R}^2} \circ h^{-1} \circ \Gamma_k^{\mathrm{int}}|_{U_{(u_*,t_*)}} \colon U_{(u_*,t_*)} \to \mathbb{O}_\varepsilon$$

and a scalar multiplication $m_s \colon \mathbb{O}_\varepsilon \to \mathbb{O}_\varepsilon$, $w \mapsto \kappa(s) \cdot w$, for $s \in \mathbb{R}$. We define a deformation retraction to $\{(u_*,t_*)\}$,

$$\mathbb{R} \times U_{(u_*,t_*)} \to U_{(u_*,t_*)}, \quad (s,(u,\tau)) \mapsto (u_s,\tau_s) := (\Gamma_k^{\mathrm{fiber}})^{-1} \circ m_s \circ (\Gamma_k^{\mathrm{fiber}})(u,\tau).$$

Now we define a map,

$$\widetilde{\Phi}'_{(u_*,t_*)} \colon \mathbb{R} \times U_{(u_*,t_*)} \to \Sigma_{m+1}^{a+(m+1)\varepsilon}, \quad (s,(u,\tau)) \mapsto \Phi'_{(u_*,t_*)}(u_s,\tau_s),$$

and the de Rham chain

$$o_k^1 := \sum_{(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} [\mathbb{R} \times U_{(u_*,t_*)}, \widetilde{\Phi}'_{(u_*,t_*)}, \chi \times \zeta'_{(u_*,t_*)}] \in C_1^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon}).$$

If $s \geq 1$, $\widetilde{\Phi}'_{(u_*,t_*)}(s,(u,\tau)) = \Phi'_{(u_*,t_*)}(u,\tau)$. If $s \leq 0$, $\widetilde{\Phi}'_{(u_*,t_*)}(s,(u,\tau))$ is constant to

$$(\gamma_1^{u_*}, \ldots, \widetilde{\gamma_k^{u_*}1}, \widetilde{\gamma_k^{u_*}2}, \ldots, \gamma_m^{u_*}) =: c_0'(u_*,t_*),$$

which is defined by (12) for $\gamma_k = \gamma_k^{u_*}$, $(T_k, \tau) = (T_k^{u_*}, t_*)$ and $\sigma_i \colon [0, \frac{1}{2}\varepsilon] \to \{\gamma_k^{u_*}(t_*)\} \subset Q$. Therefore, the boundary chain ∂o_k^1 is equal to

$$\sum_{(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} \left([U_{(u_*,t_*)}, \Phi'_{(u_*,t_*)}, \zeta'_{(u_*,t_*)}] - \left[\{0\}, c_0'(u_*,t_*), \int_{U_{(u_*,t_*)}} (\Gamma_k)^*(h_*(\nu_\varepsilon \times 1))\right]\right).$$

Since $\int_{\mathbb{O}_\varepsilon} \nu_\varepsilon = 1$, we can compute that

$$\int_{U_{(u_*,t_*)}} (\Gamma_k)^*(h_*(\nu_\varepsilon \times 1)) = \int_{U_{(u_*,t_*)}} (h^{-1} \circ \Gamma_k)^*(\nu_\varepsilon \times 1) = \int_{U_{(u_*,t_*)}} (\Gamma_k^{\mathrm{fiber}})^* \nu_\varepsilon = \mathrm{sign}(u_*,t_*) \in \{\pm 1\}.$$

Thus,

$$(44) \qquad \partial(o_k^1) = \sum_{(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} \left([U_{(u_*,t_*)}, \Phi'_{(u_*,t_*)}, \zeta'_{(u_*,t_*)}] - \mathrm{sign}(u_*,t_*) \cdot [\{0\}, c_0'(u_*,t_*), 1]\right).$$

Step 2 For each $(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)$, $c_0'(u_*,t_*)$ coincides with $c_0(u_*,t_*)$ up to reparametrizations of the kth and $(k+1)^{\mathrm{st}}$ paths. We define by interpolating parametrizations

$$c_1(u_*,t_*) \colon [0,1] \to \Sigma_{m+1}^{a+(m+1)\varepsilon}, \quad s \mapsto (\gamma_1^{u_*}, \ldots, \gamma_{k-1}^{u_*}, \widetilde{\gamma_k^{u_*}1} \circ \mu_s^1, \widetilde{\gamma_k^{u_*}2} \circ \mu_s^2, \gamma_{k+1}^{u_*}, \ldots, \gamma_m^{u_*}),$$

so that $c_1(u_*, t_*)(0) = c_0(u_*, t_*)$ and $c_1(u_*, t_*)(1) = c'_0(u_*, t_*)$. Then we obtain a chain

$$o^2_{(u_*, t_*)} := [\mathbb{R}, c_1(u_*, t_*) \circ \kappa, \chi] \in C_1^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon}),$$

which satisfies $\partial(o^2_{(u_*, t_*)}) = [\{0\}, c'_0(u_*, t_*), 1] - [\{0\}, c_0(u_*, t_*), 1]$.

Step 3 We define a chain

$$o_{k,(\varepsilon,\delta)}(x) := o^1_k + \sum_{(u_*, t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} \mathrm{sign}(u_*, t_*) \cdot o^2_{(u_*, t_*)}.$$

From (44), $\partial(o_{k,(\varepsilon,\delta)}(x))$ is equal to the chain of (43). Therefore, $o_{k,\delta}(x)$ satisfies condition (i). Condition (ii) can be checked as follows: From conditions (1e-2) and (1e-4), those paths in $\Phi'_{(u_*, t_*)}(s, (u, \tau))$ and $c_1(u_*, t_*)(s)$ satisfy condition (iii) of Lemma 3.12. Therefore, $f_{l,\delta}(o^1_k)$ and $f_{l,\delta}(o^2_{(u_*, t_*)})$ belongs to $C_0^{\mathrm{dR}}(\Sigma_{m+2}^0)$ for $l = 1, \ldots, m+1$. $\qquad \square$

For $(\varepsilon, \delta) \in \mathcal{T}_a$ which is standard, we define a linear map $\Phi^{<a}_{(\varepsilon,\delta)} \colon C_*^{\pitchfork, <a}(\varepsilon) \to C_*^{<a}(\varepsilon)$ such that, for $x \in C_p^{\pitchfork}(m, a, \varepsilon)$, the equivalence class $[x] \in C_p^{\pitchfork, <a}(m, \varepsilon)$ is mapped to

$$\Phi^{<a}_{(\varepsilon,\delta)}([x]) = \begin{cases} [\Psi(x)] & \text{if } p = 0, \\ [\Psi(x)] - \sum_{k=1}^{m} [o_{k,(\varepsilon,\delta)}(x)] & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The two properties of $o_{k,(\varepsilon,\delta)}$ show that $\Phi^{<a}_{(\varepsilon,\delta)}$ is a chain map from $(C_*^{\pitchfork, <a}(\varepsilon), D_\varepsilon^{\pitchfork})$ to $(C_*^{<a}(\varepsilon), D_\delta)$. Therefore, we obtain a linear map on homology,

$$(\Phi^{<a}_{(\varepsilon,\delta)})_* \colon H_*^{\pitchfork, <a}(\varepsilon) \to H_*^{<a}(\varepsilon, \delta).$$

6.2.3 Commutativity with transition maps We need to check the relation of $\Phi^{<a}_{(\varepsilon,\delta)}$ for $\{k_{(\varepsilon',\delta),(\varepsilon,\delta)}\}_{\varepsilon' \leq \varepsilon}$ and $\{l_{\varepsilon,\varepsilon'}\}_{\varepsilon \geq \varepsilon'}$.

Proposition 6.5 *For $(\varepsilon, \delta), (\varepsilon', \delta) \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$ which are standard with respect to h, the following diagram commutes:*

$$
\begin{array}{ccc}
H_*^{\pitchfork, <a}(\varepsilon) & \xrightarrow{(\Phi^{<a}_{(\varepsilon,\delta)})_*} & H_*^{<a}(\varepsilon, \delta) \\
l_{\varepsilon,\varepsilon'} \downarrow & & \uparrow k_{(\varepsilon',\delta'),(\varepsilon,\delta)} \\
H_*^{\pitchfork, <a}(\varepsilon') & \xrightarrow{(\Phi^{<a}_{(\varepsilon',\delta')})_*} & H_*^{<a}(\varepsilon', \delta)
\end{array}
$$

To prove this proposition, we return to the definition $k_{(\varepsilon',\delta'),(\varepsilon,\delta)} = k_{(\bar\varepsilon, \bar\delta)}$ by $(\bar\varepsilon, \bar\delta) \in \overline{\mathcal{T}}_a$ satisfying (24) for (ε, δ) and (ε', δ'). We require $(\bar\varepsilon, \bar\delta)$ to be standard with respect to h, and thus $\bar\varepsilon = \varepsilon$.

We set $\overline{\Psi} := \bar\iota \circ \Psi \colon C_*^{\pitchfork}(m, a) \to \overline{C}_*^{\mathrm{dR}}(\Sigma_m^a)$ for all $m \in \mathbb{Z}_{\geq 0}$. Again, the induced map from $C_*^{\pitchfork, <a}(\varepsilon)$ to $\overline{C}_*^{<a}(\varepsilon)$ is not a chain map. To fill in the gap, we need the following lemma.

Lemma 6.6 *For $m \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, m\}$, suppose that we have taken maps $o_{k,(\varepsilon,\delta)}$ and $o_{k,(\varepsilon',\delta')}$ of Lemma 6.4 for (ε, δ) and (ε', δ'). Then there exists a linear map*

$$\bar{o}_{k,(\varepsilon,\bar\delta)} \colon C_1^{\pitchfork}(m, a, \varepsilon) \to \overline{C}_1^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon})$$

such that the following hold for any $x \in C_1^{\pitchfork}(m, a, \varepsilon)$:

- $\partial(\bar{o}_{k,(\varepsilon,\bar{\delta})}) - (\bar{f}_{k,\bar{\delta}} \circ \overline{\Psi}(x) - \overline{\Psi} \circ f_k^{\pitchfork}(x)) \in \overline{C}_0^{\mathrm{dR}}(\Sigma_{m+1}^0)$.

- $\bar{f}_{l,\bar{\delta}}(\bar{o}_{k,(\varepsilon,\bar{\delta})}(x)) \in \overline{C}_0^{\mathrm{dR}}(\Sigma_{m+2}^0)$ for every $l \in \{1, \ldots, m+1\}$.

- $e_+(\bar{o}_{k,(\varepsilon,\bar{\delta})}(x)) = o_{k,(\varepsilon,\delta)}(x)$ and $e_-(\bar{o}_{k,(\varepsilon,\bar{\delta})}(x)) = (j_{\varepsilon',\varepsilon})_*(o_{k,(\varepsilon',\delta')}(x))$.

Proof We omit the detailed proof. Note that $\bar{\delta}$ has the form (32). For any $x = (\gamma_k)_{k=1,\ldots,m}$, we can compute explicitly that the chain $\bar{f}_{k,\bar{\delta}} \circ \overline{\Psi}(x) - \overline{\Psi} \circ f_k^{\pitchfork}(x)$ is equal to the sum of chains, for all $(u_*, t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)$,

$$[\mathbb{R} \times U_{(u_*,t_*)}, \overline{\Phi}'_{(u_*,t_*)}, (\mathrm{id}_{\mathbb{R}_{\geq 1}} \times U_{(u_*,t_*)}, \mathrm{id}_{\mathbb{R}_{\leq -1}} \times U_{(u_*,t_*)}), \bar{\zeta}'_{(u_*,t_*)}]$$
$$- \mathrm{sign}(u_*, t_*) \cdot [\mathbb{R}, c_0(u_*, t_*), (\mathrm{id}_{\mathbb{R}_{\geq 1}}, \mathrm{id}_{\mathbb{R}_{\leq -1}}), 1]$$

modulo $\overline{C}_0^{\mathrm{dR}}(\Sigma_{m+1}^0)$. Here $\overline{\Phi}'_{(u_*,t_*)} \colon \mathbb{R} \times U_{(u_*,t_*)} \to \mathbb{R} \times \Sigma_{m+1}^{a+(m+1)\varepsilon}$ is determined by

$$\overline{\Phi}'_{(u_*,t_*)}(r, (u, \tau)) := \left(r, \mathrm{con}_k\left((\gamma_l^u)_{l=1,\ldots,m}, (T_k^u, \tau), \bar{\psi}_{\varepsilon',\varepsilon}(r, \gamma_k^u(\tau))\right)\right)$$

and $(\bar{\zeta}'_{(u_*,t_*)}) := (\mathrm{id}_{\mathbb{R}} \times \Gamma_k^{\mathrm{int}}|_{U_{(u_*,t_*)}})^*(1 \times \bar{\eta}_{\varepsilon',\varepsilon}) \in \Omega^2(\mathbb{R} \times U_{(u_*,t_*)})$. The $[-1, 1]$-modeled chain $\bar{o}_{k,(\varepsilon,\bar{\delta})}$ is defined in a similar way as $o_{k,(\varepsilon,\delta)}$ in Lemma 6.4, and we can check that this chain satisfies the required three conditions. $\qquad \square$

Proof of Proposition 6.5 We define a linear map $\overline{\Phi}_{(\varepsilon,\bar{\delta})}^{<a} \colon C_*^{\pitchfork,<a}(\varepsilon) \to \overline{C}_*^{<a}(\varepsilon)$ such that, for $x \in C_p^{\pitchfork,<a}(m, a, \varepsilon)$,

$$\overline{\Phi}_{(\varepsilon,\bar{\delta})}^{<a}([x]) = \begin{cases} [\overline{\Psi}(x)] & \text{if } p = 0, \\ [\overline{\Psi}(x)] - \sum_{k=1}^m [\bar{o}_{k,(\varepsilon,\bar{\delta})}(x)] & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The first two properties of $\bar{o}_{k,(\varepsilon,\bar{\delta})}$ show that this is a chain map from $(C_*^{\pitchfork,<a}(\varepsilon), D_\varepsilon^{\pitchfork})$ to $(\overline{C}_*^{<a}(\varepsilon), \overline{D}_{\bar{\delta}})$. Therefore, we get a linear map on homology,

$$(\overline{\Phi}_{(\varepsilon,\bar{\delta})}^{<a})_* \colon H_*^{\pitchfork,<a}(\varepsilon) \to \overline{H}_*^{<a}(\varepsilon, \bar{\delta}).$$

The third property of $\bar{o}_{k,(\varepsilon,\bar{\delta})}$ implies that the following diagram commutes:

The proposition is now proved since $k_{(\varepsilon,\bar{\delta})} = ((j_{\varepsilon,\varepsilon})_*^{-1} \circ (e_{\varepsilon,+})_*) \circ ((j_{\varepsilon',\varepsilon})_*^{-1} \circ (e_{\varepsilon_-})_*)^{-1}$. $\qquad \square$

Let $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$. Proposition 6.5 shows that the family of maps

$$\{(\Phi_{(\varepsilon,\delta)}^{<a})_* \mid (\varepsilon, \delta) \in \mathcal{T}_a \text{ is standard with respect to } h\}$$

induces a linear map on the limits of $\varepsilon \to 0$,

$$\Phi^{<a} : H_*^{\pitchfork,<a} = \varinjlim_{\varepsilon \to 0} H_*^{\pitchfork,<a}(\varepsilon) \to H_*^{<a}(Q, K) = \varprojlim_{\varepsilon \to 0} H_*^{<a}(\varepsilon, \delta).$$

Naturally, those maps of $\{\Phi^{<a}\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}$ commutes with $\{I^{a,b} : H_*^{<a}(Q, K) \to H_*^{<b}(Q, K)\}_{a \le b}$ and $\{J^{a,b} : H_*^{\pitchfork,<a} \to H_*^{\pitchfork,<b}\}_{a \le b}$. Therefore, on the limit of $a \to \infty$, we have a linear map

$$\Phi : H_*^{\pitchfork,\infty} \to H_*^{\text{string}}(Q, K).$$

Moreover, it is straightforward to check that Φ is a homomorphism of unital $\mathbb{R}$-algebras.

6.3 Proof of isomorphism

In this section, we prove that for every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$, $\Phi^{<a}$ is an isomorphism in the 0th degree. As an immediate consequence, it is shown that the cord algebra of (Q, K) is isomorphic to $H_0^{\text{string}}(Q, K)$.

For each $m \in \mathbb{Z}_{\ge 0}$, let $\partial_m^{\text{sing}} : C_1^{\pitchfork,<a}(m) \to C_0^{\pitchfork,<a}(m)$ denote the singular boundary operator. We also write

(45)
$$\Psi_{0,m} : \operatorname{Coker} \partial_m^{\text{sing}} \to H_0^{\text{dR}}(\Sigma_m^a, \Sigma_m^0), \quad [x] \mapsto [\Psi(x)],$$
$$\Psi_{1,m} : \operatorname{Ker} \partial_m^{\text{sing}} \to H_1^{\text{dR}}(\Sigma_m^a, \Sigma_m^0), \quad x \mapsto [\Psi(x)].$$

Lemma 6.7 $\Psi_{0,m}$ *is an isomorphism and* $\Psi_{1,m}$ *is a surjection.*

Proof Naturally, there are two maps

$$\operatorname{Coker} \partial_m^{\text{sing}} \to H_0^{\text{sing}}(\Sigma_m^a, \Sigma_m^0), \quad \operatorname{Ker} \partial_m^{\text{sing}} \to H_1^{\text{sing}}(\Sigma_m^a, \Sigma_m^0).$$

induced by the inclusions $C_p^{\pitchfork}(m, a) \to C_p^{\text{sing}}(\Sigma_m^a)$ for $p = 0, 1$. The subset of Σ_m^a (resp. $C^0([0, 1], \Sigma_m^a)$) consisting of elements satisfying conditions (0a)–(0b) (resp. (1a)–(1c)) is open dense. This fact implies that the first map is an isomorphism and the second map is a surjection. Then we consider the following diagram for $p = 0, 1$:

$$
\begin{array}{ccccc}
K_{p,m} & \xrightarrow{\;\Psi_{p,m}\;} & H_p^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) & \xrightarrow{\;(6)\;} & \varinjlim_{j \to \infty} H_p^{\text{dR}}(B_m^a(2^j), B_m^0(2^j)) \\
\downarrow & & & & \uparrow \\
H_p^{\text{sing}}(\Sigma_m^a, \Sigma_m^0) & & \xrightarrow{\;\;\;\;\;(11)\;\;\;\;\;} & & \varinjlim_{j \to \infty} H_p^{\text{sing}}(B_m^a(2^j), B_m^0(2^j))
\end{array}
$$

Here, $K_{0,m} := \operatorname{Coker} \partial_m^{\text{sing}}$ and $K_{1,m} := \operatorname{Ker} \partial_m^{\text{sing}}$. The left vertical map is defined as above. The right vertical map is an isomorphism from Proposition 2.9. The horizontal maps are the isomorphisms of (6) and (11). The commutativity follows from the definition of the right vertical map. See [15, Section 4.7]. Now the assertion of the lemma follows from the diagram.   □

For the chain complexes $(C_*^{\pitchfork,<a}, D^{\pitchfork})$ and $(C_*^{\pitchfork,<a}(\varepsilon), D_\varepsilon^{\pitchfork})$, we define their filtrations $\{\mathcal{H}_p^{<a}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{H}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$ by

$$\mathcal{H}_p^{<a} := \bigoplus_{m\geq -p} C_*^{\pitchfork,<a}(m), \quad \mathcal{H}_{\varepsilon,p}^{<a} := \bigoplus_{m\geq -p} C_*^{\pitchfork,<a}(m,\varepsilon).$$

Let $E^{\pitchfork,<a}$ and $E_\varepsilon^{\pitchfork,<a}$ be the spectral sequences determined by $\{\mathcal{H}_p^{<a}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{H}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$, respectively. The $(-m,q)$-terms of their first pages are given by

$$(E^{\pitchfork,<a})^1_{-m,q} = \begin{cases} \text{Coker}\,\partial_m^{\text{sing}} & \text{if } q-m=0 \text{ and } m\geq 0, \\ \text{Ker}\,\partial_m^{\text{sing}} & \text{if } q-m=1 \text{ and } m\geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$(E_\varepsilon^{\pitchfork,<a})^1_{-m,q} = \begin{cases} \text{Coker}\,\partial_{\varepsilon,m}^{\text{sing}} & \text{if } q-m=0 \text{ and } m\geq 0, \\ \text{Ker}\,\partial_{\varepsilon,m}^{\text{sing}} & \text{if } q-m=1 \text{ and } m\geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\partial_{\varepsilon,m}^{\text{sing}}: C_1^{\pitchfork,<a}(m,\varepsilon) \to C_0^{\pitchfork,<a}(m,\varepsilon)$ also denotes the singular boundary operator. If $0 < \varepsilon' \leq \varepsilon$, there exists a morphism $l_{\varepsilon,\varepsilon'}: E_\varepsilon^{\pitchfork,<a} \to E_{\varepsilon'}^{\pitchfork,<a}$ induced by the inclusion maps $C_*^{\pitchfork}(m,a,\varepsilon) \to C_*^{\pitchfork}(m,a,\varepsilon')$ for all $m \in \mathbb{Z}_{\geq 0}$. Naturally, $\varinjlim_{\varepsilon\to 0} E_\varepsilon^{\pitchfork,<a} \cong E^{\pitchfork,<a}$.

For $(\varepsilon,\delta) \in \mathcal{T}_a$ which is standard with respect to h, the chain map $\Phi_{(\varepsilon,\delta)}^{<a}$ preserves the filtrations $\{\mathcal{H}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{F}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$, so it induces a morphism of spectral sequences

$$(\Phi_{(\varepsilon,\delta)}^{<a})_*: E_\varepsilon^{\pitchfork,<a} \to E_{(\varepsilon,\delta)}^{<a}.$$

Note that, on the $(-m,q)$-term for $m \geq 0$ of their first pages, this can be written as

(46)
$$(\Phi_{(\varepsilon,\delta)}^{<a})_*: \text{Coker}\,\partial_{\varepsilon,m}^{\text{sing}} \to H_0^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0), \qquad [x] \mapsto [\Psi(x)] \quad \text{if } q = m,$$
$$(\Phi_{(\varepsilon,\delta)}^{<a})_*: \text{Ker}\,\partial_{\varepsilon,m}^{\text{sing}} \to H_1^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0), \qquad x \mapsto [\Psi(x)] \quad \text{if } q = m+1.$$

Recall that we have defined $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}: E_{(\varepsilon',\delta')}^{<a} \to E_{(\varepsilon,\delta)}^{<a}$ by the composition of the maps of (28). The next result is a variant of Proposition 6.5 for spectral sequences.

Proposition 6.8 *The following diagram commutes*:

$$\begin{array}{ccc} E_\varepsilon^{\pitchfork,<a} & \xrightarrow{(\Phi_{(\varepsilon,\delta)}^{<a})_*} & E_{(\varepsilon,\delta)}^{<a} \\ {\scriptstyle l_{\varepsilon,\varepsilon'}}\Big\downarrow & & \Big\uparrow{\scriptstyle k_{(\varepsilon',\delta'),(\varepsilon,\delta)}} \\ E_{\varepsilon'}^{\pitchfork,<a} & \xrightarrow{(\Phi_{(\varepsilon',\delta')}^{<a})_*} & E_{(\varepsilon',\delta')}^{<a} \end{array}$$

This can be proved as Proposition 6.5 by taking $(\overline{\Phi}_{(\varepsilon,\bar\delta)}^{<a})_*: E_\varepsilon^{\pitchfork,<a} \to \overline{E}_{(\varepsilon,\bar\delta)}^{<a}$ into consideration. We omit the proof.

We use the spectral sequence $E^{<a}$ of Proposition 4.17. The above proposition and (46) immediately implies the existence of the following morphism of spectral sequences.

Proposition 6.9 *There exists a morphism of spectral sequences* $\Phi^{<a} : E^{\pitchfork,<a} \to E^{<a}$ *such that, on the* $(-m, q)$-*term for* $m \geq 0$ *of the first page,*

$$\Phi^{<a} = \Psi_{0,m} : (E^{\pitchfork,<a})^1_{-m,q} = \text{Coker } \partial^{\text{sing}}_m \to (E^{<a})^1_{-m,q} = H_0^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \quad \text{if } q = m,$$

$$\Phi^{<a} = \Psi_{1,m} : (E^{\pitchfork,<a})^1_{-m,q} = \text{ Ker } \partial^{\text{sing}}_m \to (E^{<a})^1_{-m,q} \; = H_1^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \quad \text{if } q = m+1.$$

This property of $\Phi^{<a} : E^{\pitchfork,<a} \to E^{<a}$ implies a result on the compatible map $\Phi^{<a} : H_p^{\pitchfork,<a} \to H_p^{<a}(Q, K)$.

Proposition 6.10 $\Phi^{<a} : H_p^{\pitchfork,<a} \to H_p^{<a}(Q, K)$ *is an isomorphism if* $p = 0$ *and a surjection if* $p = 1$.

Proof By Lemma 6.7 and Proposition 6.9, $\Phi^{<a} : (E^{\pitchfork,<a})^1_{p,q} \to (E^{<a})^1_{p,q}$ is an isomorphism if $p+q \leq 0$ and a surjection if $p+q = 1$. Since $E^{\pitchfork,<a}$ converges to $H_*^{\pitchfork,<a}$ and $E^{<a}$ converges to $H_*^{<a}(Q, K)$, we can apply Lemma 4.2 to prove the above assertion for $\Phi^{<a} : H_*^{\pitchfork,<a} \to H_*^{<a}(Q, K)$. $\square$

On their limits of $a \to \infty$, $\{\Phi^{<a}\}_{a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)}$ induces an isomorphism

$$\Phi : H_0^{\pitchfork,<\infty} \to H_0^{\text{string}}(Q, K).$$

Combining with Proposition 6.3, we finally obtain the following result.

Theorem 6.11 *As a unital* $\mathbb{R}$-*algebra,* $\text{Cord}(Q, K; \mathbb{R})$ *is isomorphic to* $H_0^{\text{string}}(Q, K)$.

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