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Enriched quasicategories and the templicial homotopy coherent nerve

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We lay the foundations for a theory of quasicategories in a monoidal category $\mathcal{V}$ replacing Set, aimed at realising weak enrichment in the category $S\mathcal{V}$ of simplicial objects in $\mathcal{V}$. To accommodate noncartesian monoidal products, we make use of an ambient category $S_{\otimes}\mathcal{V}$ of templicial, or "tensor-simplicial", objects in $\mathcal{V}$, which are certain colax monoidal functors, following Leinster. Inspired by the description of the categorification functor due to Dugger and Spivak, we construct a templicial analogue of the homotopy coherent nerve functor which goes from $S\mathcal{V}$-enriched categories to $S_{\otimes}\mathcal{V}$. We show that an $S\mathcal{V}$-enriched category whose underlying simplicial category is locally Kan is turned into a quasicategory in $\mathcal{V}$ by this nerve functor.

18D20, 18N60; 18M05, 18N50

1 Introduction

1.A The main goal

The theory of $(\infty, 1)$-categories (or simply ∞-categories) is by now well established, with notable models including simplicial categories by Bergner [6], Segal categories by Hirschowitz and Simpson [20], complete Segal spaces by Rezk [37] and quasicategories by Joyal [23]. These models were all shown to be homotopically equivalent by the work of Bergner [7], Joyal [24; 25] and Lurie [28].

One may view ∞-categories as being "weakly enriched in spaces"; that is, between objects we have morphism spaces (usually formalised as simplicial sets) along with compositions that are only well defined and associative up to coherent homotopy. In analogy with ordinary enriched categories, one may thus

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conceive (weakly) enriched ∞-categories by replacing SSet by a suitable category $\mathcal{M}$ possessing some weak or higher structure, eg a monoidal model category or a monoidal ∞-category.

A general approach to enrichment is due to Gepner and Haugseng [16], who developed a theory of ∞-categories weakly enriched in a monoidal ∞-category. Their theory is built on Lurie's ∞-operads [29] and, as such, on the extensive framework of quasicategories.

Alternatively, one can consider enriched counterparts of each of the classical models for ∞-categories listed above. A particularly easy one is the "strict model" of simplicial categories, which one may replace by categories strictly enriched in a suitable monoidal model category $\mathcal{M}$. See the work of Berger and Moerdijk [5], Stanculescu [40] or Muro [35]. Further, in the case where $\mathcal{M}$ is cartesian (ie its monoidal structure is given by the cartesian product), Simpson [39] introduced $\mathcal{M}$-enriched Segal categories as certain simplicial objects in $\mathcal{M}$. Haugseng [19, Theorems 5.8 and 6.17] showed that both the strict model and Simpson's model are presentations of enriched ∞-categories in the sense of [16].

Building on Simpson's work and generalising Leinster's homotopy monoids [26], Bacard [3] defined $\mathcal{M}$-enriched Segal categories over a general monoidal model category $\mathcal{M}$ in order to encompass enriching categories of interest, like chain complexes over a commutative ring. Recently, an approach to complete dg-Segal spaces of a quite different flavour was put forward by Dimitriadis Bermejo [13], replacing the simplex category by a category of free dg-categories of finite type.

Finally, there are the particularly tangible quasicategories which have proven very successful, and whose theory has seen extensive development due to the work of Joyal, Lurie and many others. The main goal of the present paper is to lay the basic foundations for a concrete model of "enriched quasicategories", which stand to quasicategories as Bacard's enriched Segal categories stand to Segal categories. While the development of the homotopy theory of these objects is relegated to subsequent work, our constructions are motivated by homotopy-theoretic considerations, as we further explain.

For a suitable monoidal category $\mathcal{V}$, we define *quasicategories in* $\mathcal{V}$. Here, like the category Set of sets, $\mathcal{V}$ is a category not necessarily having any weak or higher structure. Instead, quasicategories in $\mathcal{V}$ should be viewed as being weakly enriched in the monoidal category $S\mathcal{V}$ of simplicial objects in $\mathcal{V}$ and, as such, they have a higher categorical nature. Here, we consider $S\mathcal{V}$ with the right-transferred model structure from SSet. This model structure exists for example if the monoidal unit I is a projective generator of $\mathcal{V}$, which goes back to Quillen [36, Section II.4]. The restriction to the case $\mathcal{M} = S\mathcal{V}$ allows us to keep our model tangible and elementary.

Like in the classical situation, quasicategories in $\mathcal{V}$ arise as a subclass of a larger category. We denote this category by $S_{\otimes}\mathcal{V}$ and call its objects *templicial* (short for *tensor-simplicial*) *objects* in $\mathcal{V}$. It is important to note that, while the hom-spaces are purported to be simplicial objects, templicial objects themselves are not. This change of perspective is necessary in order to make sense of basic constructions like the nerve, as we will explain in Section 1.B. Nonetheless, when $\mathcal{V} = \text{Set}$, both simplicial and templicial objects recover simplicial sets. Another case of particular interest is $\mathcal{V} = \mathrm{Mod}(k)$, the category of modules over a commutative ring k. Motivated by (noncommutative) algebraic geometry, where higher categorical structures like dg- and A_∞-categories play prominent roles as models for spaces, we focus on this case in subsequent work [27].

Classically, the Joyal model structure for quasicategories on SSet [23] and the model structure for simplicial categories by Bergner [6] are related by the homotopy coherent nerve functor, which is the right-adjoint in a Quillen equivalence. The construction of the homotopy coherent nerve goes back to Cordier [10] and in fact it was already shown by Cordier and Porter [11, Theorem 2.1] (though not with this terminology) that it preserves fibrant objects. Indeed, given a locally Kan simplicial category (that is, all its hom-objects are Kan complexes), its homotopy coherent nerve is a quasicategory. Taking this fact as a starting point, our main result is the following.

Theorem *There is a right-adjoint functor*

$$N^{\mathrm{hc}}_{\mathcal{V}} \colon \mathcal{V}\mathrm{Cat}_{\Delta} \to S_{\otimes}\mathcal{V}$$

from the category of small $S\mathcal{V}$-enriched categories to the category of templicial objects in $\mathcal{V}$ with the following properties:

(1) *If $\mathcal{V} = \mathrm{Set}$, then $N^{\mathrm{hc}}_{\mathcal{V}}$ recovers the classical homotopy coherent nerve.*

(2) *If $\mathcal{C}$ is a small $S\mathcal{V}$-enriched category whose underlying simplicial category is locally Kan, then $N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})$ is a quasicategory in $\mathcal{V}$.*

We call this functor the *templicial homotopy coherent nerve*. It is constructed in Section 4.B and the theorem is proven in Corollary 5.12. Some other enriched versions of the homotopy coherent nerve exist in the cartesian context. See the work of Gindi [17] and Moser, Rasekh and Rovelli [34]. We will investigate the relation of the latter nerve with ours in subsequent work [33].

The model structure on $S\mathcal{V}$-enriched categories generalises the one of Bergner on simplicial categories. We expect that a generalisation of Joyal's model structure on SSet exists for $S_{\otimes}\mathcal{V}$ (under suitable conditions on $\mathcal{V}$), having quasicategories in $\mathcal{V}$ as fibrant objects and making the templicial homotopy coherent nerve into a Quillen equivalence. The weak equivalences are likely reflected by the left-adjoint, which we call the *categorification functor*, for instance in the case of left transfer. This is work in progress. Note that, by [19], this would establish quasicategories in $\mathcal{V}$ as a model for ∞-categories enriched over $S\mathcal{V}$ in the sense of [16].

1.B Templicial objects and necklace categories

In order to define quasicategories in a monoidal category $\mathcal{V}$ and prove the theorem from Section 1.A, two larger categories play an important role: the category $S_{\otimes}\mathcal{V}$ of templicial objects and the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ of necklace categories. In this section, we will explain and motivate their occurrence, starting with the former.

Given a small category $\mathcal{C}$, recall that its nerve $N(\mathcal{C})$ is the simplicial set whose set of n-simplices is given by

$$N(\mathcal{C})_n = \coprod_{A_0,\dots,A_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(A_0, A_1) \times \cdots \times \mathcal{C}(A_{n-1}, A_n).$$

The inner face maps d_j for $0 < j < n$ are given by composing two consecutive morphisms in a sequence, and the degeneracy maps s_i for $0 \leq i \leq n$ are given by inserting an identity in the sequence. The outer face maps d_0 and d_n are defined by deleting respectively the first and last entry in a sequence. Now suppose $\mathcal{C}$ is enriched over the (possibly noncartesian) monoidal category $\mathcal{V}$. When defining the nerve of $\mathcal{C}$, a natural first attempt is to put

$$N(\mathcal{C})_n = \coprod_{A_0,\dots,A_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_{n-1}, A_n) \in \mathcal{V}$$

and try to make this into a simplicial object in $\mathcal{V}$. It is readily seen that we can define inner face morphisms and degeneracy morphisms in the same way. However, the same is not true for the outer face morphisms because in general there are no projections out of a tensor product, whence we cannot "project away" the factor $\mathcal{C}(A_0, A_1)$ or $\mathcal{C}(A_{n-1}, A_n)$ in the expression above. As a consequence, we do not obtain a simplicial object, but the above data can be organised into a colax monoidal functor

$$X : \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V},$$

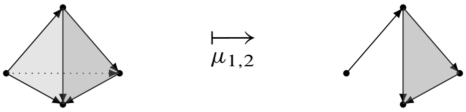
where $\mathbf{\Delta}_f$ is the monoidal category of finite intervals (see Section 1.D). Restricting from the usual simplex category $\mathbf{\Delta}$ to $\mathbf{\Delta}_f$, it follows that X no longer has any outer face maps, a loss which is compensated by the colax monoidal structure. It was shown by Leinster [26, Proposition 3.1.7] (also see Proposition 2.1 below) that, if $\mathcal{V}$ is cartesian, X may still be identified with a simplicial object in $\mathcal{V}$.

The philosophy of introducing coalgebraic structure in the noncartesian context is not uncommon. For example, Hopf algebras may be considered the group objects internal to $\mathrm{Mod}(k)$. Similarly, in their PhD thesis [1], Aguiar introduced graphs and categories internal to a monoidal category by means of bicomodules over comonoids. Such structure is invisible in a cartesian monoidal category because then every object has a unique comonoid structure. The same philosophy was applied by Bacard in their definition of $\mathcal{M}$-enriched Segal categories [3]. These are many-object versions of Leinster's homotopy monoids [26], based on the colax monoidal functors above.

Let us describe templicial objects in a little more detail. In a similar but nonequivalent way to [3], we define templicial objects as certain colax monoidal functors on $\mathbf{\Delta}_f$ with a discrete set of vertices. More precisely, a templicial object in $\mathcal{V}$ with vertex set S is a strongly unital, colax monoidal functor

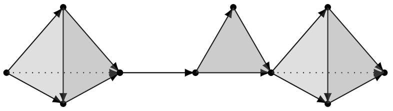
$$X : \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S,$$

where $\mathcal{V}\mathrm{Quiv}_S$ denotes the category of $\mathcal{V}$-enriched quivers with vertex set S. The colax monoidal structure equips X with quiver morphisms $\mu_{k,l} : X_{k+l} \to X_k \otimes_S X_l$ for all $k, l \geq 0$. For example, $\mu_{1,2}$ may be pictured as



Intuitively, $\mu_{k,l}$ involves pulling apart $(k+l)$-simplices into k-simplices attached to l-simplices at a vertex. We can thus no longer access outer faces of a simplex directly. The shift of focus to faces joint at a vertex naturally leads us to considering necklaces (see Section 3).

Necklaces were introduced by Baues [4] (under a different name) and popularised by Dugger and Spivak [14] in their description of the categorification functor. Roughly, a necklace is a sequence of simplices glued at the endpoints:



$\Delta^3 \vee \Delta^1 \vee \Delta^2 \vee \Delta^3$

Necklaces naturally occur in the interpretation of the comultiplications, with $\mu_{1,2}$ being parametrised by the necklace map

$$\nu_{1,2}\colon \Delta^1 \vee \Delta^2 \to \Delta^3$$

(see Notation 3.10). As such, they allow us to turn colax monoidal functors on $\mathbf{\Delta}_f$ into ordinary functors on the category $\mathcal{N}ec$ of necklaces, putting $X(\Delta^1 \vee \Delta^2) = X_1 \otimes_S X_2$ and $X(\nu_{1,2}) = \mu_{1,2}$ in the above example. Endowing the functor category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ with the Day convolution, we define a necklace category to be a category enriched in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. This allows us to realise $S_{\otimes}\mathcal{V}$ as a coreflective subcategory of the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ of necklace categories,

$$S_{\otimes}\mathcal{V} \hookrightarrow \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}. \tag{1}$$

This embedding will turn up as a crucial intermediate step in defining the templicial homotopy coherent nerve in Section 4. In the definition of quasicategories in $\mathcal{V}$ in Section 5, the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ also plays a fundamental role as the context in which we express the familiar lifting property with respect to inner horn inclusions.

1.C Overview of the paper

Next we give an overview of the contents of the paper. In Section 2, we formally introduce templicial objects and prove some basic properties. Starting in Section 2.A, we compare them to simplicial sets. In Section 2.B, we construct the templicial analogue of the classical nerve functor for small $\mathcal{V}$-enriched categories. For templicial objects which have nondegenerate simplices in an appropriate sense, in Section 2.C we prove a version of the Eilenberg–Zilber lemma. In general, the structure of a templicial object X is considerably richer than that of the underlying simplicial set $\widetilde{U}(X)$ (see Proposition 2.8 and

Remark 2.9). In particular, unlike in the classical case, simplices are no longer represented by morphisms from standard simplices (the "representation problem"; see Example 2.10).

This representation problem is solved in Section 3 with the introduction of necklace categories. In Section 3.A, we recall necklaces and give a combinatorial characterisation of their category $\mathcal{N}ec$. In Section 3.B, we define necklace categories and realise the category of templicial objects as a coreflective subcategory of the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ (Theorem 3.12). Finally, in Section 3.C, we observe that both the underlying simplicial set functor $\widetilde{U}$ and the templicial nerve $N_{\mathcal{V}}$ naturally factor through $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$.

In Section 4, we generalise the classical homotopy coherent nerve and the categorification functor (Definition 4.9). We follow the elegant approach from [14], which we recall in Section 4.A before presenting our enriched counterpart in Section 4.B. The key observation relating the two is a description of the categorification by means of a weighted colimit (Proposition 4.6).

Starting from the embedding (1), in order to construct the categorification, we construct a functor from necklace categories to $S\mathcal{V}$-enriched categories. Following [14], the categorification is simplified by using flanked flags in Section 4.C, in the presence of the nondegenerate simplices from Section 2.C. Finally, in Section 4.D, we show that the templicial homotopy coherent nerve reduces to the templicial nerve in the desired way. As usual, for a templicial object X, we naturally obtain a $\mathcal{V}$-enriched homotopy category $h_{\mathcal{V}}X$ as π_0 of the categorification. In general, the underlying simplicial set and underlying category functors do not commute with taking homotopy categories, as shown in Example 4.22.

In Section 5, we introduce the natural analogue of quasicategories in the templicial setting, which will remedy the aforementioned failure to commute. A quasicategory in $\mathcal{V}$ is defined as a templicial object satisfying a familiar lifting property with respect to inner horn inclusions (Definition 5.4). In contrast to the classical setup, this lifting property is considered in the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ rather than $S_{\otimes}\mathcal{V}$ because of the representation problem. The resulting notion is in general strictly stronger than requiring the underlying simplicial set $\widetilde{U}(X)$ to be a quasicategory. Nonetheless, when $\mathcal{V} = \mathrm{Set}$, we still recover ordinary quasicategories. We also show our main result (Corollary 5.12; see the theorem above). Finally, in Section 5.C, we show how the description of the homotopy category $h_{\mathcal{V}}X$ can be simplified when X is a quasicategory in $\mathcal{V}$. Moreover, the underlying category of the homotopy category corresponds to the homotopy category of the underlying ordinary quasicategory.

1.D Notation and conventions

(1) Throughout the text, we let $(\mathcal{V}, \otimes, I)$ be a fixed bicomplete, symmetric monoidal closed category (ie a Bénabou cosmos in the sense of Street [41]). Up to natural isomorphism, there is a unique colimit-preserving functor $F\colon \mathrm{Set} \to \mathcal{V}$ such that $F(\{*\}) = I$. This functor is left-adjoint to the forgetful functor $U = \mathcal{V}(I, -)\colon \mathcal{V} \to \mathrm{Set}$. Endowing Set with the cartesian monoidal structure, F is strong monoidal and U is lax monoidal. This notation will remain fixed as well.

(2) Let $(\mathcal{W}, \otimes, I)$ be an arbitrary monoidal category. Given a set S, we refer to a collection $Q = (Q(a,b))_{a,b\in S}$ with $Q(a,b) \in \mathcal{W}$ as a *$\mathcal{W}$-enriched quiver* with S its set of *vertices*. A *quiver morphism* $f\colon Q \to P$ is a collection $(f_{a,b})_{a,b\in S}$ of morphisms $f_{a,b}\colon Q(a,b) \to P(a,b)$ in $\mathcal{W}$. $\mathcal{W}$-enriched quivers with a fixed set of vertices S and morphisms between them form a category, which we denote by

$$\mathcal{W}\mathrm{Quiv}_S.$$

This category is monoidal with product $\otimes_S$ and unit I_S defined by

$$(Q \otimes_S P)(a,b) = \coprod_{c\in S} Q(a,c) \otimes P(c,b) \qquad \text{and} \qquad I_S(a,b) = \begin{cases} I & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

for all $Q, P \in \mathcal{W}\mathrm{Quiv}_S$ and $a, b \in S$.

(3) Let $f\colon S \to T$ be a map of sets. We have an induced lax monoidal functor $f^*\colon \mathcal{W}\mathrm{Quiv}_T \to \mathcal{W}\mathrm{Quiv}_S$ given by $f^*(Q)(a,b) = Q(f(a), f(b))$ for all $\mathcal{W}$-enriched quivers Q and $a, b \in S$. The functor f^* has a left-adjoint, which we denote by $f_!\colon \mathcal{W}\mathrm{Quiv}_S \to \mathcal{W}\mathrm{Quiv}_T$. It is given by

$$f_!(Q)(x,y) = \coprod_{\substack{a,b\in S\\ f(a)=x\\ f(b)=y}} Q(a,b)$$

for all $Q \in \mathcal{W}\mathrm{Quiv}_S$ and $x, y \in T$. As f^* is canonically lax monoidal, $f_!$ comes equipped with an induced colax monoidal structure.

(4) To relate $\mathcal{V}$-enriched and $S\mathcal{V}$-enriched categories to templicial objects (see Sections 2.B and 4.B), it will be more convenient for us to consider a $\mathcal{W}$-enriched category (or $\mathcal{W}$-category for short) as a pair $(\mathcal{C}, \mathrm{Ob}(\mathcal{C}))$ with $\mathrm{Ob}(\mathcal{C})$ its set of objects and $\mathcal{C}$ a monoid in $\mathcal{W}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$. Note that this convention implies that the composition in $\mathcal{C}$ is given by a collection of morphisms in $\mathcal{W}$, for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$,

$$m_{\mathcal{C}}\colon \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \to \mathcal{C}(A,C),$$

as opposed to the more conventional $\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \to \mathcal{C}(A,C)$. A $\mathcal{W}$-functor $\mathcal{C} \to \mathcal{D}$ is then a pair (H, f) with $f\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ a map of sets and $H\colon \mathcal{C} \to f^*(\mathcal{D})$ a morphism of monoids in $\mathcal{W}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$, where we used the lax structure of f^*. We denote the category of small $\mathcal{W}$-categories and $\mathcal{W}$-functors between them by

$$\mathcal{W}\mathrm{Cat}.$$

(5) We will make use of the simplex categories $\mathbf{\Delta}_{\mathrm{surj}} \subseteq \mathbf{\Delta}_f \subseteq \mathbf{\Delta}$, where:

- $\mathbf{\Delta}$ is the ordinary *simplex category*. Its objects are the posets $[n] = \{0, \dots, n\}$ with $n \geq 0$, and its morphisms are the order morphisms $[m] \to [n]$.
- $\mathbf{\Delta}_f$ is the category of *finite intervals*, which is the subcategory of $\mathbf{\Delta}$ consisting of all morphisms $f\colon [m] \to [n]$ that preserve the endpoints, that is, $f(0) = 0$ and $f(m) = n$.
- $\mathbf{\Delta}_{\mathrm{surj}}$ is the subcategory of $\mathbf{\Delta}$ of all surjective morphisms $[m] \twoheadrightarrow [n]$.

Unlike $\mathbf{\Delta}$, both $\mathbf{\Delta}_f$ and $\mathbf{\Delta}_{\text{surj}}$ carry a monoidal structure $(+,[0])$, which is given by identifying their respective top and bottom endpoints, as follows. For all $m,n \geq 0$,

$$[m]+[n]=[m+n].$$

For morphisms $f\colon [m]\to[m']$ and $g\colon [n]\to[n']$ in $\mathbf{\Delta}_f$ or $\mathbf{\Delta}_{\text{surj}}$,

$$(f+g)(i)=\begin{cases} f(i) & \text{if } i\leq m,\\ m'+g(i-m) & \text{if } i\geq m.\end{cases}$$

Note that, for any morphism $f\colon [m]\to[n]$ in $\mathbf{\Delta}_f$ and $k,l\geq 0$ such that $k+l=m$, there exist unique morphisms $f_1\colon [k]\to[p]$ and $f_2\colon [l]\to[q]$ in $\mathbf{\Delta}_f$ such that $f_1+f_2=f$.

There is a well-known monoidal equivalence between $\mathbf{\Delta}_f^{\text{op}}$ and the augmented simplex category $\mathbf{\Delta}_+$ (equipped with the join as monoidal product). This is known as Joyal's duality; see [21].

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2 Templicial objects

Aguiar [1] defined a graph internal to a monoidal category $\mathcal{W}$ as a pair (G_1,G_0) where G_0 is a comonoid in $\mathcal{W}$ and G_1 is a bicomodule over G_0. When $\mathcal{W}$ is cartesian monoidal, this recovers the usual notion of a graph internal to a category, namely a pair of morphisms $s,t\colon G_1 \rightrightarrows G_0$ expressing the source and target. Extending this philosophy to higher dimensions, we propose to define a *simplicial object internal to a monoidal category* $\mathcal{W}$ as a colax monoidal functor

$$X\colon \mathbf{\Delta}_f^{\text{op}}\to\mathcal{W},$$

where $\mathbf{\Delta}_f$ is the monoidal category of finite intervals (see Section 1.D). The restriction to $\mathbf{\Delta}_f$ precisely gets rid of the outer face maps, which are replaced by the colax monoidal structure. To justify this change, let us remark that the colax structure of X provides $X_0\in\mathcal{W}$ with the structure of a comonoid in $\mathcal{W}$ and X_1 with that of bicomodule over X_0. In other words, (X_1,X_0) is a graph internal to $\mathcal{W}$ in the sense of [1]. Moreover, it was shown by Leinster [26] (reappearing here as Proposition 2.1) that, if $\mathcal{W}$ is cartesian, then X recovers a simplicial object in $\mathcal{W}$.

Because enriched categories have a set of objects, we also equip such colax monoidal functors with a discrete set of vertices. Formally, we achieve this by putting $\mathcal{W} = \mathcal{V}\mathrm{Quiv}_S$ for a set S (see Section 1.D) and requiring the functor to be strongly unital. This leads to the definition of our main objects of study, templicial objects (Definition 2.3). We then define the natural analogue of the classical nerve functor in this context (Definition 2.11), taking $\mathcal{V}$-categories to templicial objects in $\mathcal{V}$. Finally, we show a generalisation of the Eilenberg–Zilber lemma (Lemma 2.19).

2.A Simplicial versus templicial objects

Let us make explicit what data are contained in a colax monoidal functor $X\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ for a monoidal category $\mathcal{W}$ and compare it to the data of a simplicial object. For general background on (co)lax monoidal functors, see eg [2]. Explicitly, such a functor X consists of a collection of objects $X_0, X_1, X_2, \dots$ of $\mathcal{W}$ along with

- inner face morphisms $d_j^X\colon X_n \to X_{n-1}$ for all $0 < j < n$,
- degeneracy morphisms $s_i^X\colon X_n \to X_{n+1}$ for all $0 \le i \le n$,
- comultiplication morphisms $\mu_{k,l}^X\colon X_{k+l} \to X_k \otimes X_l$ for all $k, l \ge 0$,
- a counit morphism $\epsilon^X\colon X_0 \to I$.

These data moreover must satisfy:

- **Simplicial identities** For all $i, j \ge 0$, whenever these equations are well defined,
$$d_i^X s_j^X = \begin{cases} s_{j-1}^X d_i^X & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1, \\ s_j^X d_{i-1}^X & \text{if } i > j+1, \end{cases}$$
$$d_i^X d_j^X = d_{j-1}^X d_i^X \quad \text{if } i < j, \qquad s_i^X s_j^X = s_j^X s_{i-1}^X \quad \text{if } i > j.$$
- **Naturality of μ^X** For all $k, l \ge 0$ and $0 < j < k+l+1$, $0 \le i \le k+l-1$,
$$\mu_{k,l}^X d_j^X \begin{cases} (d_j^X \otimes \mathrm{id}_{X_l})\mu_{k+1,l}^X & \text{if } j \le k, \\ (\mathrm{id}_{X_k} \otimes d_{j-k}^X)\mu_{k,l+1}^X & \text{if } j > k, \end{cases} \qquad \mu_{k,l}^X s_i^X = \begin{cases} (s_i^X \otimes \mathrm{id}_{X_l})\mu_{k-1,l}^X & \text{if } i < k, \\ (\mathrm{id}_{X_k} \otimes s_{i-k}^X)\mu_{k,l-1}^X & \text{if } i \ge k. \end{cases}$$
- **Coassociativity of μ^X** For all $r, s, t \ge 0$,
$$(\mathrm{id}_{X_r} \otimes \mu_{s,t}^X)\mu_{r,s+t}^X = (\mu_{r,s}^X \otimes \mathrm{id}_{X_t})\mu_{r+s,t}^X.$$
- **Counitality of μ^X with ϵ^X** For all $n \ge 0$,
$$(\mathrm{id}_{X_n} \otimes \epsilon^X)\mu_{n,0}^X = \mathrm{id}_{X_n} = (\epsilon^X \otimes \mathrm{id}_{X_n})\mu_{0,n}^X.$$

Note that, by the coassociativity, we have a well-defined morphism
$$\mu_{k_1,\dots,k_n}^X\colon X_{k_1+\dots+k_n} \to X_{k_1} \otimes \dots \otimes X_{k_n}$$
for all $n \ge 2$ and $k_1, \dots, k_n \ge 0$. Further, we will set $\mu_{k_1,\dots,k_n}^X$ to be the identity on X_{k_1} if $n = 1$, and to be the counit ϵ^X if $n = 0$.

Moreover, under these identifications, a monoidal natural transformation $\alpha\colon X \to Y$ between colax monoidal functors $X, Y\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ is equivalent to a collection of morphisms $(\alpha_n\colon X_n \to Y_n)_{n\geq 0}$ which satisfy:

- **Naturality of $\boldsymbol{\alpha}$** For all $0 < j < n$ and $0 \leq i \leq n$,

$$\alpha_{n-1} d_j^X = d_j^Y \alpha_n \quad \text{and} \quad \alpha_{n+1} s_i^X = s_i^Y \alpha_n.$$

- **Monoidality of $\boldsymbol{\alpha}$** For all $k, l \geq 0$,

$$\mu_{k,l}^Y \alpha_{k+l} = (\alpha_k \otimes \alpha_l)\mu_{k,l}^X \quad \text{and} \quad \epsilon^Y \alpha_0 = \epsilon^X.$$

Often we will drop the superscript X when it is clear from context.

We denote by $\mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}}, \mathcal{W})$ the category of colax monoidal functors $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ and monoidal natural transformations between them.

Proposition 2.1 [26, Proposition 3.1.7] *Let $\mathcal{W}$ be a cartesian monoidal category. There is an isomorphism of categories*

$$\mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}}, \mathcal{W}) \simeq S\mathcal{W}.$$

Remark 2.2 Suppose $\mathcal{W}$ is cartesian and let X be a simplicial object in $\mathcal{W}$. Its associated colax monoidal functor $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ has comultiplication morphisms given by

$$\mu_{k,l}\colon X_{k+l} \xrightarrow{(d_{k+1}\dots d_n, d_0 \dots d_0)} X_k \times X_l.$$

Conversely, suppose $X\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ is a colax monoidal functor. The outer face morphisms of the associated simplicial object are obtained by using the projections of the product

$$d_0\colon X_n \xrightarrow{\mu_{1,n-1}} X_1 \times X_{n-1} \xrightarrow{\pi_2} X_{n-1} \quad \text{and} \quad d_n\colon X_n \xrightarrow{\mu_{n-1,1}} X_{n-1} \times X_1 \xrightarrow{\pi_1} X_{n-1}.$$

If $\mathcal{W}$ is not cartesian, these projections are not available in general and the comultiplication μ of a colax monoidal functor can be considered as a replacement for the outer face morphisms in the monoidal context.

From now on, we will only consider such colax monoidal functors $X\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ with a discrete set of vertices. We formalise this by replacing $\mathcal{W}$ by a category $\mathcal{V}\mathrm{Quiv}_S$ of $\mathcal{V}$-enriched quivers (see Section 1.D) for some set S, and requiring that X be strongly unital.

An alternative but nonequivalent way to realise a set of vertices S consists in turning the monoidal category $\mathbf{\Delta}_f$ (which is a one-object bicategory) into a bicategory with object set S. This approach goes back to [28] and was used in [39; 3].

Definition 2.3 A *tensor-simplicial* or *templicial object* in $\mathcal{V}$ is a pair (X,S) with S a set and

$$X\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S$$

a colax monoidal functor which is strongly unital, ie its counit $\epsilon\colon X_0 \to I_S$ is an isomorphism. We call the elements of S the *vertices* of X. For $n>0$, an *n-simplex* of X is an element of the set $U(X_n(a,b)) \in \mathcal{V}$ for some $a,b \in S$.

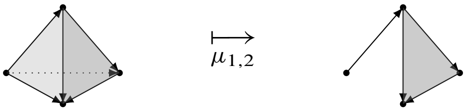
Let (X,S) and (Y,T) be templicial objects. A *templicial morphism* $(X,S)\to(Y,T)$ is a pair (α,f) with $f\colon S\to T$ a map of sets and $\alpha\colon f_!X \to Y$ a monoidal natural transformation between colax monoidal functors $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_T$. Here, we used the colax monoidal structure of $f_!$.

Sometimes we will denote a templicial object (X,S) or a templicial morphism (α,f) simply by X or α, respectively, assuming the underlying set or map of sets is clear.

Remark 2.4 Let (X,S) be a templicial object in $\mathcal{V}$ and consider $a,b\in S$. Then $X_n(a,b)\in\mathcal{V}$ should be interpreted as the *object of n-simplices of X with first vertex a and last vertex b*. Moreover, for all $k,l\geq 0$ and $a,b\in S$, the comultiplication morphism

$$(\mu^X_{k,l})_{a,b}\colon X_{k+l}(a,b) \to \coprod_{c\in S} X_k(a,c)\otimes X_l(c,b)$$

should be interpreted as taking a $(k+l)$-simplex from a to b and sending it to a k-simplex from a to some $c\in S$, along with an l-simplex from c to b, which are outer faces of the original $(k+l)$-simplex:



Unlike for simplicial objects, we thus no longer have direct access to the outer faces of a simplex, only to outer faces which are glued at a vertex.

Definition 2.5 Given maps of sets $f\colon S\to T$ and $g\colon T\to U$, there is a canonical monoidal natural isomorphism $(gf)_! \simeq g_!f_!$ between colax monoidal functors $\mathcal{V}\mathrm{Quiv}_S \to \mathcal{V}\mathrm{Quiv}_U$. Consequently, we can define the *composition* of two templicial morphisms $(\alpha,f)\colon(X,S)\to(Y,T)$ and $(\beta,g)\colon(Y,T)\to(Z,U)$ as the templicial morphism (γ,gf) with

$$\gamma\colon (gf)_!X \simeq g_!f_!X \xrightarrow{g_!\alpha} g_!Y \xrightarrow{\beta} Z.$$

Further, we have a canonical monoidal natural isomorphism $\varphi\colon (\mathrm{id}_S)_! \xrightarrow{\sim} \mathrm{id}_{\mathcal{V}\mathrm{Quiv}_S}$ for any set S, and the *identity* at (X,S) is defined as the templicial morphism $(\varphi X,\mathrm{id}_S)$. It is then easy to see that templicial objects in $\mathcal{V}$ and templicial morphisms between them form a category, which we denote by

$$S_\otimes\mathcal{V}.$$

Remark 2.6 A more abstract construction of the category $S_{\otimes}\mathcal{V}$ of templicial objects is as follows. Given a set S, consider the category $\Phi(S) = \mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}}, \mathcal{V}\mathrm{Quiv}_S)$. For a map of sets $f\colon S \to T$, let $\Phi(f)\colon \Phi(S) \to \Phi(T)$ be the functor given by postcomposition with $f_!$. Then Φ is not a functor since it does not preserve composition on the nose. But one can show that the isomorphisms $(gf)_! \simeq g_! f_!$ above do define a pseudofunctor $\Phi\colon \mathrm{Set} \to \mathrm{Cat}$. Taking the Grothendieck construction $\int \Phi$, we find $S_{\otimes}\mathcal{V}$ as the full subcategory spanned by the strongly unital colax functors $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S$.

Proposition 2.7 *There is an equivalence of categories*

$$S_{\times}\,\mathrm{Set} \simeq \mathrm{SSet}.$$

Proof Let K be a simplicial set. By Proposition 2.1, we may consider K as a colax monoidal functor $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathrm{Set}$ with comultiplication μ and counit ϵ. Then define, for all $n \geq 0$ and $a, b \in K_0$,

$$K_n(a,b) = \{\sigma \in K_n \mid d_1 \dots d_n(\sigma) = a,\ d_0 \dots d_0(\sigma) = b\}.$$

Given $f\colon [m] \to [n]$ in $\mathbf{\Delta}_f$, it follows from the simplicial identities that $K(f)\colon K_n \to K_m$ restricts to a map $K(f)_{a,b}\colon K_n(a,b) \to K_m(a,b)$. Moreover, it is clear that, for all $k, l \geq 0$ and $a, b \in K_0$, $\mu_{k,l}$ restricts to

$$\mu_{k,l}|_{K_{k+l}(a,b)}\colon K_{k+l}(a,b) \to \coprod_{c \in K_0} K_k(a,c) \times K_l(c,b)$$

and $K_0(a,a) = \{a\}$ if $a = b$, while $K_0(a,b) = \varnothing$ if $a \neq b$. Consequently, the functor

$$\varphi(K)\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathrm{Quiv}_{K_0}, \quad [n] \mapsto (K_n(a,b))_{a,b \in K_0},$$

is strongly unital and colax monoidal. Hence $(\varphi(K), K_0)$ is a templicial object.

Conversely, if (X, S) is a templicial object in Set, then we can define a simplicial set $\mathfrak{c}(X)$ by setting, for all $n \geq 0$,

$$\mathfrak{c}(X)_n = \coprod_{a,b \in S} X_n(a,b).$$

It is readily verified that the assignments $K \mapsto \varphi(K)$ and $X \mapsto \mathfrak{c}(X)$ can be extended to mutually inverse equivalences between SSet and $S_{\times}$ Set. □

As $F\colon \mathrm{Set} \to \mathcal{V}$ preserves colimits and is strong monoidal, postcomposition with F induces a functor

$$\tilde{F}\colon \mathrm{SSet} \simeq S_{\times}\,\mathrm{Set} \to S_{\otimes}\mathcal{V}.$$

More precisely, given a simplicial set K, $\tilde{F}(K)$ has vertex set K_0 and, for all $a, b \in K_0$ and $n \geq 0$,

$$\tilde{F}(K)_n(a,b) = F(K_n(a,b)),$$

where $K_n(a,b) = \{\sigma \in K_n \mid d_1 \dots d_n(\sigma) = a,\ d_0 \dots d_0(\sigma) = b\}$ is the set of n-simplices of K with first vertex a and last vertex b, as above.

Proposition 2.8 *The category of templicial objects* $S_{\otimes}\mathcal{V}$ *is cocomplete and the functor* $\widetilde{F}\colon \mathrm{SSet} \to S_{\otimes}\mathcal{V}$ *has a right-adjoint*

$$\widetilde{U}\colon S_{\otimes}\mathcal{V} \to \mathrm{SSet}.$$

Proof As $\mathcal{V}$ is cocomplete, so is $\mathcal{V}\mathrm{Quiv}_S$. It is readily verified that then also $\mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}}, \mathcal{V}\mathrm{Quiv}_S)$ is cocomplete with colimits given pointwise. Now consider a diagram $D\colon \mathcal{J} \to S_{\otimes}\mathcal{V}$, $j \mapsto (X^j, S^j)$, with $\mathcal{J}$ a small category. Let $S = \mathrm{colim}_{j\in\mathcal{J}} S^j$ in Set and write $\iota_j\colon S^j \to S$ for the canonical map. Then consider the colimit $X = \mathrm{colim}_{j\in\mathcal{J}}(\iota_j)_! X^j$ in $\mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}}, \mathcal{V}\mathrm{Quiv}_S)$. The counit of X is

$$\operatorname*{colim}_{j\in\mathcal{J}}(\iota_j)_!(X_0^j) \xrightarrow{\mathrm{colim}_j(\iota_j)_!(\epsilon_{X^j})} \operatorname*{colim}_{j\in\mathcal{J}}(\iota_j)_!(I_{S^j}) \xrightarrow{\sim} I_S,$$

which is an isomorphism since each ϵ_{X^j} is. Thus the pair (X, S) is a templicial object, which is easily seen to be the colimit of the diagram D in $S_{\otimes}\mathcal{V}$.

With the above description of the colimits in $S_{\otimes}\mathcal{V}$, it is clear that $\widetilde{F}$ preserves colimits and therefore has a right-adjoint $\widetilde{U}\colon S_{\otimes}\mathcal{V} \to \mathrm{SSet}$ given by $\widetilde{U}(X)_n = S_{\otimes}\mathcal{V}(\widetilde{F}(\Delta^n), X)$ for all templicial objects X and integers $n \geq 0$. □

Remark 2.9 Let us make the right-adjoint $\widetilde{U}$ a bit more explicit. Given a templicial object (X, S), an n-simplex of $\widetilde{U}(X)$ is a templicial morphism $\widetilde{F}(\Delta^n) \to X$, which is equivalent to a pair

$$((a_i)_{i=0}^n, (\alpha_{i,j})_{0\leq i<j\leq n})$$

with $a_i \in S$ and $\alpha_{i,j} \in U(X_{j-i}(a_i, a_j))$ such that, for all $0 \leq i < k < j \leq n$,

$$\mu_{k-i,j-k}(\alpha_{i,j}) = \alpha_{i,k} \otimes \alpha_{k,j}$$

in $U((X_{k-i} \otimes_S X_{j-k})(a_i, a_j))$. For example, $\widetilde{U}(X)_0$ recovers the set S and $\widetilde{U}(X)_1$ is given by the disjoint union of all sets $U(X_1(a, b))$ with $a, b \in S$.

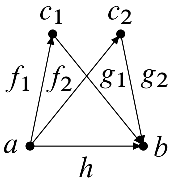
Unlike the case for simplicial objects $S\mathcal{V}$, not every n-simplex of a templicial object $(X, S) \in S_{\otimes}\mathcal{V}$ is uniquely represented by a morphism $\widetilde{F}(\Delta^n) \to X_n$. More precisely, the canonical map

$$\widetilde{U}(X)_n \to \coprod_{a,b\in S} U(X_n(a, b)), \quad (\alpha_{i,j})_{i,j} \mapsto \alpha_{0,n}, \tag{2}$$

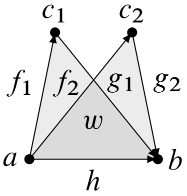
need not be injective or surjective for $n \geq 2$, as is shown in Example 2.10.

The lack of representation of simplices by morphisms makes templicial objects considerably harder to work with than ordinary simplicial sets. In an effort to resolve this issue, we extend $S_{\otimes}\mathcal{V}$ to a category of enriched categories $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ in Section 3.B.

Example 2.10 Let $\mathcal{V} = \mathrm{Ab}$ be the monoidal category of abelian groups with the tensor product as monoidal product and $\mathbb{Z}$ as monoidal unit. Consider the simplicial set $K = \partial\Delta^2 \amalg_{\Delta^1} \partial\Delta^2$:



We can extend $\widetilde{F}(K)$ to a templicial object X with a 2-simplex $w \in X_2(a,b)$ by setting $X_2(a,b) = \mathbb{Z}w \oplus F(K_2(a,b))$ and similarly adding the degeneracies of w. The inner face maps and comultiplication maps of X are uniquely determined by setting $d_1(w) = h$ and $\mu_{1,1}(w) = f_1 \otimes g_1 + f_2 \otimes g_2$:



Then w does not lie in the image of the map (2) for $n = 2$. Indeed, this would require a 2-simplex $(\alpha_{i,j})_{0 \le i < j \le 2}$ of $\widetilde{U}(X)$ with $\alpha_{0,2} = w$. But $\mu_{1,1}(w)$ is not a pure tensor while $\mu_{1,1}(\alpha_{0,2}) = \alpha_{0,1} \otimes \alpha_{1,2}$. Moreover, since $\mu_{1,1}(2s_0(h)) = 2s_0(a) \otimes h = s_0(a) \otimes 2h$, we have two distinct 2-simplices of $\widetilde{U}(X)_2$ which map to $2s_0(h) \in X_2(a,b)$.

2.B The templicial nerve

Given a monoidal category $\mathcal{W}$, there is a well-known equivalence (this goes back to Mac Lane [31, Section V.II])

$$\mathrm{Mon}(\mathcal{W}) \simeq \mathrm{StrMon}(\mathbf{\Delta}_+, \mathcal{W}) \tag{3}$$

between the categories of monoids in $\mathcal{W}$ and strong monoidal functors $\mathbf{\Delta}_+ \to \mathcal{W}$. Due to the monoidal equivalence $\mathbf{\Delta}_+ \simeq \mathbf{\Delta}_f^{\mathrm{op}}$, we may as well consider strong monoidal functors $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$.

Definition 2.11 Let $\mathcal{C}$ be a small $\mathcal{V}$-category, which we consider as a monoid $(\mathcal{C}, m_{\mathcal{C}}, u_{\mathcal{C}})$ in $\mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$. Applying (3) to the case $\mathcal{W} = \mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$, we obtain an associated strong monoidal functor, which we denote by

$$N_{\mathcal{V}}(\mathcal{C})\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}.$$

In particular, the pair $(N_{\mathcal{V}}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$ forms a templicial object, which we call the *templicial nerve* of the $\mathcal{V}$-category $\mathcal{C}$.

Explicitly, $N_{\mathcal{V}}(\mathcal{C})$ is given by taking the n-fold monoidal product of the $\mathcal{V}$-quiver $\mathcal{C}$,

$$N_{\mathcal{V}}(\mathcal{C})_n = \mathcal{C}^{\otimes n},$$

for all integers $n \geq 0$. Further, the inner face and degeneracy morphisms are

$$d_j = \mathrm{id}_{\mathcal{C}}^{\otimes j-1} \otimes_S m_{\mathcal{C}} \otimes_S \mathrm{id}_{\mathcal{C}}^{\otimes n-j-1} \colon \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes n-1}, \quad s_i = \mathrm{id}_{\mathcal{C}}^{\otimes i} \otimes_S u_{\mathcal{C}} \otimes_S \mathcal{C}^{\otimes n-i} \colon \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes n+1}$$

for all $0 \leq i \leq n$ and $0 < j < n$. Finally, the comultiplication morphisms and counit are given by the canonical isomorphisms

$$\mu_{k,l} \colon \mathcal{C}^{\otimes k+l} \xrightarrow{\sim} \mathcal{C}^{\otimes k} \otimes_S \mathcal{C}^{\otimes l} \quad \text{and} \quad \epsilon \colon \mathcal{C}^{\otimes 0} \xrightarrow{\sim} I_{\mathrm{Ob}(\mathcal{C})}$$

for any $k, l \geq 0$.

Recall the base change functor $f_! \colon \mathcal{V}\mathrm{Quiv}_S \to \mathcal{V}\mathrm{Quiv}_T$ and its right-adjoint $f^* \colon \mathcal{V}\mathrm{Quiv}_T \to \mathcal{V}\mathrm{Quiv}_S$ for a given map of sets $f \colon S \to T$ (see Section 1.D).

Lemma 2.12 *Let (X, S) be a templicial object, $\mathcal{C}$ a small $\mathcal{V}$-enriched category and $f \colon S \to \mathrm{Ob}(\mathcal{C})$ a map of sets. Then we have a bijection between monoidal natural transformations $f_! X \to N_{\mathcal{V}}(\mathcal{C})$ and quiver morphisms $H \colon X_1 \to f^*(\mathcal{C})$ such that the diagrams*

(4)

$$\begin{tikzcd}
X_1^{\otimes 2} \arrow[r, "H^{\otimes 2}"] & f^*(\mathcal{C})^{\otimes 2} \arrow[r] & f^*(\mathcal{C}^{\otimes 2}) \arrow[d, "f^*(\tilde{m}_{\mathcal{C}})"] \\
X_2 \arrow[u, "\mu_{1,1}"] \arrow[r, "d_1"'] & X_1 \arrow[r, "H"'] & f^*(\mathcal{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I_S \arrow[r] & f^*(I_{\mathrm{Ob}(\mathcal{C})}) \arrow[d, "f^*(u_{\mathcal{C}})"] \\
X_0 \arrow[ur, "\epsilon", "\sim"'] \arrow[r, "s_0"'] & X_1 \arrow[r, "H"'] & f^*(\mathcal{C})
\end{tikzcd}$$

commute.

Proof Given a monoidal natural transformation $\alpha \colon f_! X \to N_{\mathcal{V}}(\mathcal{C})$, define $H_\alpha \colon X_1 \to f^*(\mathcal{C})$ to be adjoint to $\alpha_1 \colon f_!(X_1) \to \mathcal{C}$. It follows from the monoidality of α that, for all $n \geq 0$, α_n is the composite

$$f_!(X_n) \xrightarrow{f_!(\mu_{1,\dots,1})} f_!(X_1^{\otimes n}) \to f_!(X_1)^{\otimes n} \xrightarrow{\alpha_1^{\otimes n}} \mathcal{C}^{\otimes n},$$

where we used the colax monoidal structure of $f_!$. So the assignment $\alpha \mapsto H_\alpha$ is injective. Moreover, it then follows from the naturality of α that H_α satisfies (4). Conversely, if $H \colon X_1 \to f^*(\mathcal{C})$ satisfies (4), then, defining α_1 as adjoint to H and α_n as above, it follows that $\alpha \colon f_! X \to N_{\mathcal{V}}(\mathcal{C})$ is a natural transformation. It is immediate that α is also monoidal. □

Proposition 2.13 *The assignment $\mathcal{C} \mapsto N_{\mathcal{V}}(\mathcal{C})$ of Definition 2.11 extends to a fully faithful functor $N_{\mathcal{V}} \colon \mathcal{V}\mathrm{Cat} \to S_{\otimes}\mathcal{V}$. The essential image of $N_{\mathcal{V}}$ consists of all templicial objects (X, S) for which $X \colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S$ is strong monoidal.*

Proof Let $\mathcal{C}$ and $\mathcal{D}$ be small $\mathcal{V}$-enriched categories, $f \colon \mathrm{Ob}(\mathcal{D}) \to \mathrm{Ob}(\mathcal{C})$ a map of sets and $H \colon \mathcal{D} \to f^*(\mathcal{C})$ a morphism in $\mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{D})}$. Then the diagrams (4) with $X = N_{\mathcal{V}}(\mathcal{D})$ precisely express that (H, f) is a $\mathcal{V}$-functor $\mathcal{D} \to \mathcal{C}$, and thus we have a bijection $\mathcal{V}\mathrm{Cat}(\mathcal{D}, \mathcal{C}) \simeq S_{\otimes}\mathcal{V}(N_{\mathcal{V}}(\mathcal{D}), N_{\mathcal{V}}(\mathcal{C}))$. More precisely, the templicial morphism $N_{\mathcal{V}}(H)$ corresponding to some $\mathcal{V}$-functor $H \colon \mathcal{D} \to \mathcal{C}$ is given by

$$N_{\mathcal{V}}(H)_n \colon f_!(\mathcal{D}^{\otimes n}) \to f_!(\mathcal{D})^{\otimes n} \xrightarrow{N_{\mathcal{V}}(H)_1^{\otimes n}} \mathcal{C}^{\otimes n}$$

for all $n \geq 0$, where $N_{\mathcal{V}}(H)_1\colon f_!(\mathcal{D}) \to \mathcal{C}$ is adjoint to $H\colon \mathcal{D} \to f^*(\mathcal{D})$. Thus clearly this defines a functor which is necessarily fully faithful. The characterisation of the essential image follows immediately from (3). □

Remark 2.14 When $(\mathcal{V}, \otimes, I) = (\mathrm{Set}, \times, \{*\})$, the templicial nerve functor $N_{\mathcal{V}}\colon \mathcal{V}\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ clearly recovers the classical nerve functor $N\colon \mathrm{Cat} \to \mathrm{SSet}$.

The adjunction $F \dashv U$ induces an adjunction between small categories and small $\mathcal{V}$-enriched categories, which we denote by

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\mathrm{Cat}. \arrow[l, bend left, "\mathcal{U}"] \end{tikzcd}$$

Proposition 2.15 *We have natural isomorphisms*

$$N_{\mathcal{V}} \circ \mathcal{F} \simeq \tilde{F} \circ N \quad \text{and} \quad \tilde{U} \circ N_{\mathcal{V}} \simeq N \circ \mathcal{U}.$$

Proof The first isomorphism follows from the construction of $N_{\mathcal{V}}$ and the fact that F is strong monoidal and preserves colimits. Further let $\mathcal{C}$ be a small $\mathcal{V}$-category and $n \geq 0$. Then we have isomorphisms, natural in $\mathcal{C}$ and n,

$$\begin{aligned} \tilde{U}(N_{\mathcal{V}}(\mathcal{C}))_n = S_{\otimes}\mathcal{V}(\tilde{F}(\Delta^n), N_{\mathcal{V}}(\mathcal{C})) &\simeq S_{\otimes}\mathcal{V}\big(N_{\mathcal{V}}(\mathcal{F}([n])), N_{\mathcal{V}}(\mathcal{C})\big) \\ &\simeq \mathcal{V}\mathrm{Cat}(\mathcal{F}([n]), \mathcal{C}) \simeq \mathrm{Cat}([n], \mathcal{U}(\mathcal{C})) \simeq N(\mathcal{U}(\mathcal{C}))_n, \end{aligned}$$

where we subsequently used the first isomorphism and Proposition 2.13. □

2.C The Eilenberg–Zilber lemma

Recall the classical Eilenberg–Zilber lemma for simplicial sets [15, (8.3)]. It states that, for any n-simplex x of a simplicial set K, there is a unique nondegenerate k-simplex y of K and a unique surjective map $\sigma\colon [n] \twoheadrightarrow [k]$ in $\mathbf{\Delta}$ such that $x = K(\sigma)(y)$. Equivalently, there exists a bijection

$$K_n \simeq \coprod_{\sigma\colon [n] \twoheadrightarrow [k] \text{ in } \mathbf{\Delta}_{\mathrm{surj}}} K_k^{\mathrm{nd}},$$

where $K_k^{\mathrm{nd}} \subseteq K_k$ denotes the subset of nondegenerate k-simplices of K, and $\mathbf{\Delta}_{\mathrm{surj}} \subseteq \mathbf{\Delta}$ is the subcategory of surjective maps (see Section 1.D).

The analogous statement for templicial objects (Lemma 2.19) also holds, but this requires an extra condition to ensure that they have a well-behaved notion of nondegenerate simplices. We will make use of this lemma when we reformulate the left-adjoint of the templicial homotopy coherent nerve in Section 4.C.

Definition 2.16 Consider a functor $X\colon \mathbf{\Delta}_{\mathrm{surj}}^{\mathrm{op}} \to \mathcal{W}$ with $\mathcal{W}$ a cocomplete category. For every integer $n \geq 0$, we let

$$X_n^{\mathrm{deg}} = \operatorname*{colim}_{\substack{\sigma\colon [n] \twoheadrightarrow [k] \text{ in } \mathbf{\Delta}_{\mathrm{surj}} \\ 0 \leq k < n}} X_k.$$

Note that we have a canonical morphism $X_n^{\mathrm{deg}} \to X_n$ in $\mathcal{W}$.

Let (X, S) be a templicial object and consider the restricted functor $X|_{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}}}\colon \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}} \to \mathcal{V}\mathrm{Quiv}_S$. For $n \geq 0$, we call X^{deg}_n the *quiver of degenerate n-simplices* of X. We say X *has nondegenerate simplices* if for every $n \geq 0$, the quiver morphism $X^{\mathrm{deg}}_n \to X_n$ is isomorphic to a coprojection

$$X^{\mathrm{deg}}_n \to X^{\mathrm{deg}}_n \amalg N$$

for some $N \in \mathcal{V}\mathrm{Quiv}_S$. In this case, we'll often denote N by X^{nd}_n. When considering an abstract templicial object that has nondegenerate simplices, we implicitly assume a choice for X^{nd}_n in each dimension has been made. Note that $X^{\mathrm{deg}}_0 = 0$ and so we always have $X^{\mathrm{nd}}_0 \simeq X_0$.

Example 2.17 Let (X, S) be a templicial object and suppose the underlying functor $X|_{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}}}\colon \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}} \to \mathcal{V}\mathrm{Quiv}_S$ is isomorphic to FZ for some $Z\colon \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}} \to \mathrm{Quiv}_S$ with $Z^{\mathrm{deg}}_n(a,b) \to Z_n(a,b)$ injective for all $a, b \in S$ and $n \geq 0$. Then X has nondegenerate simplices. Indeed, simply set

$$X^{\mathrm{nd}}_n(a,b) = F(Z_n(a,b) \setminus Z^{\mathrm{deg}}_n(a,b)).$$

In particular, for any simplicial set K, the templicial object $\widetilde{F}(K)$ has nondegenerate simplices.

Certainly not every templicial object has nondegenerate simplices, as the following example shows.

Example 2.18 Consider the monoidal category $\mathcal{V} = \mathrm{Ab}$ of abelian groups with the tensor product. Let $S = \{*\}$ be a singleton and define a functor $X\colon \mathbf{\Delta}^{\mathrm{op}}_f \to \mathrm{Ab}$ by setting $X_n = \mathbb{Z}$ for all $n \geq 0$ with

$$s_0\colon X_0 = \mathbb{Z} \xrightarrow{2\cdot} X_1 = \mathbb{Z}$$

and all other face and degeneracy maps given by the identity on $\mathbb{Z}$. Then X is a strongly unital, colax monoidal functor with comultiplication map $\mu_{k,l}$ for $k, l \geq 0$ given by

$$\mu_{k,l}\colon X_{k+l} = \mathbb{Z} \to X_k \otimes X_l \simeq \mathbb{Z}, \qquad z \mapsto \begin{cases} 2z & \text{if } k, l > 0, \\ z & \text{if } k = 0 \text{ or } l = 0. \end{cases}$$

We thus find a templicial abelian group (X, S) for which $X^{\mathrm{deg}}_1 \to X_1$ is given by the inclusion $2\mathbb{Z} \subseteq \mathbb{Z}$, which doesn't have a direct complement.

Lemma 2.19 *Let X be a templicial object and assume it has nondegenerate simplices. For any integer $n \geq 0$, we have an isomorphism of quivers*

$$X_n \simeq \coprod_{\sigma\colon [n] \twoheadrightarrow [k] \text{ in } \mathbf{\Delta}_{\mathrm{surj}}} X^{\mathrm{nd}}_k.$$

Proof By definition, $X_0 = X^{\mathrm{nd}}_0$. Take $n > 0$; then it follows by induction that

$$\begin{aligned} X_n &\simeq X^{\mathrm{nd}}_n \amalg X^{\mathrm{deg}}_n = X^{\mathrm{nd}}_n \amalg \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [k] \\ 0 \leq k < n}} X_k \simeq X^{\mathrm{nd}}_n \amalg \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [k] \\ 0 \leq k < n}} \coprod_{\sigma\colon [k] \twoheadrightarrow [l]} X^{\mathrm{nd}}_l \\ &\simeq X^{\mathrm{nd}}_n \amalg \coprod_{\substack{\sigma\colon [n] \twoheadrightarrow [l] \\ 0 \leq l < n}} \operatorname*{colim}_{\substack{[n] \overset{\sigma_1}{\twoheadrightarrow} [k] \overset{\sigma_2}{\twoheadrightarrow} [l] \\ \sigma = \sigma_2 \sigma_1}} X^{\mathrm{nd}}_l \simeq X^{\mathrm{nd}}_n \amalg \coprod_{\substack{\sigma\colon [n] \twoheadrightarrow [l] \\ 0 \leq l < n}} X^{\mathrm{nd}}_l. \end{aligned}$$

The last isomorphism is obtained by noting that the colimit on the left-hand side is taken over a category which has a terminal object given by the factorisation $[n] \overset{=}{\twoheadrightarrow} [n] \overset{\sigma}{\twoheadrightarrow} [l]$. □

3 Necklaces and necklace categories

Necklaces were first introduced by Baues [4] and popularised by Dugger and Spivak [14]. Their category $\mathcal{N}ec$ will play a crucial role in what follows. Morally a necklace is simply a sequence of simplices glued together at vertices. In view of Remark 2.4, necklaces appear naturally when applying the comultiplication morphism $\mu_{k,l}$ of a templicial object X. In this way, maps between necklaces parametrise the degeneracy and inner face morphisms of X, as well as its comultiplication morphisms. This change in perspective leads us to consider the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ of small categories enriched in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. We call these necklace categories and show in Section 3.B that we can recover $S_{\otimes}\mathcal{V}$ as a coreflective subcategory of $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ (see Theorem 3.12).

3.A Necklaces

We quickly recall the definition of a necklace and in Proposition 3.4 we give a combinatorial description of the category $\mathcal{N}ec$ of necklaces, which also appears in [18].

Definition 3.1 We denote by $\mathrm{SSet}_{*,*} = (\partial\Delta^1 \downarrow \mathrm{SSet})$ the category of *bipointed simplicial sets*. Its objects can be identified with tuples (K, a, b) where K is a simplicial set and $a, b \in K_0$ are called the *distinguished points* of K. We will also write $K_{a,b} = (K, a, b)$. A morphism $K_{a,b} \to L_{c,d}$ in $\mathrm{SSet}_{*,*}$ is a simplicial map $f : K \to L$ such that $f(a) = c$ and $f(b) = d$.

Let $K_{a,b}$ and $L_{c,d}$ be bipointed simplicial sets. The *wedge sum* $K \vee L$ of K and L is constructed by gluing K and L at the distinguished points b and c. More precisely, $K \vee L$ is given by the coequaliser

$$\begin{tikzcd} \Delta^0 \arrow[r, shift left, "b"] \arrow[r, shift right, "c"'] & K \amalg L \arrow[r, two heads] & K \vee L. \end{tikzcd}$$

We consider $K \vee L$ again as bipointed with distinguished points (a, d).

Remark 3.2 It is not difficult to verify that the wedge $\vee$ is a monoidal product on the category of bipointed simplicial sets $\mathrm{SSet}_{*,*}$ whose unit is given by Δ^0.

Definition 3.3 For any $n \geq 0$, we consider the standard simplex Δ^n as bipointed with distinguished points 0 and n. A *necklace* T is an iterated wedge of standard simplices. That is,

$$T = \Delta^{n_1} \vee \cdots \vee \Delta^{n_k} \in \mathrm{SSet}_{*,*}$$

for some $k \geq 0$ and $n_1, \dots, n_k > 0$ (if $k = 0$, then $T = \Delta^0$). We refer to the standard simplices $\Delta^{n_1}, \dots, \Delta^{n_k}$ as the *beads* of T. The distinguished points in every bead are called the *joints* of T.

We let $\mathcal{N}ec$ denote the full subcategory of $\mathrm{SSet}_{*,*}$ spanned by all necklaces. By construction, $(\mathcal{N}ec, \vee, \Delta^0)$ is again a monoidal category.

Proposition 3.4 *The category of necklaces $\mathcal{N}ec$ is equivalent to the category defined as follows*:

The objects are pairs (T, p) with $p \geq 0$ and $\{0 < p\} \subseteq T \subseteq [p]$. The morphisms $(T, p) \to (U, q)$ are morphisms $f \colon [p] \to [q]$ in $\mathbf{\Delta}_f$ such that $U \subseteq f(T)$, with compositions and identities defined as in $\mathbf{\Delta}_f$.

Moreover, under this equivalence, the wedge $\vee$ corresponds to

$$(T, p) \vee (U, q) = (T \cup (p + U), p + q),$$

where $p + U = \{p + u \mid u \in U\}$.

Proof We may identify a necklace $T = \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$ with $T = \{0 < n_1 < n_1 + n_2 < \cdots < p\} \subseteq [p]$, where $p = n_1 + \cdots + n_k$, and we will do so for the rest of this proof. Note that, under this identification, $[p]$ is the set of vertices and T is the set of joints of the necklace. Further, a necklace map $T \to U$ is completely determined on vertices, which is a morphism $[p] \to [q]$ in $\mathbf{\Delta}_f$. It remains to show that a morphism $f \colon [p] \to [q]$ in $\mathbf{\Delta}_f$ is the vertex map of a necklace map $T \to U$ if and only if $U \subseteq f(T)$.

Suppose f is the vertex map of some necklace map $T \to U$. Assume that there exists a $u \in U \setminus f(T)$. Then we may choose subsequent joints $t < t'$ in T such that $f(t) < u < f(t')$. Now the unique edge in T between t and t' must be sent to an edge in U between $f(t)$ and $f(t')$. But there is no such edge. Conversely, assume $U \subseteq f(T)$. We may write $T = \{0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = p\}$ and $U = \{0 = u_0 < u_1 < \cdots < u_{l-1} < u_l = q\}$. Then $f = f_1 + \cdots + f_k$ for some unique maps $f_i \colon [t_i - t_{i-1}] \to [f(t_i) - f(t_{i-1})]$ in $\mathbf{\Delta}_f$. Fixing $i \in \{1, \dots, k\}$, there is a unique $j \in \{1, \dots, l\}$ such that $u_{j-1} \leq f(t_{i-1}) \leq f(t_i) \leq u_j$. So we can extend f_i to an order morphism $[t_i - t_{i-1}] \to [u_j - u_{j-1}]$, which induces a simplicial map $\Delta^{t_i - t_{i-1}} \to \Delta^{u_j - u_{j-1}} \to U$. These maps combine to give a map of necklaces $T \to U$ whose vertex map is f.

Clearly, this correspondence is functorial and preserves the wedge sum $\vee$. □

Henceforth, we will identify $\mathcal{N}ec$ with the category described in Proposition 3.4. So we will also use the notation

$$T = \{0 = t_0 < t_1 < t_2 < \cdots < t_k = p\}$$

to refer to the necklace $\Delta^{t_1} \vee \Delta^{t_2 - t_1} \vee \cdots \vee \Delta^{p - t_{k-1}}$. We will often refer to a necklace (T, p) just by its underlying set of joints T.

Definition 3.5 Let $f \colon (T, p) \to (U, q)$ be a map of necklaces. We say f is *inert* if $p = q$ and $f = \mathrm{id}_{[p]}$. We say f is *active* if $f(T) = U$.

Remark 3.6 Every necklace map $f \colon (T, p) \to (U, q)$ can be uniquely factored as an active necklace map $(T, p) \to (f(T), q)$ followed by an inert necklace map $(f(T), q) \to (U, q)$. In fact, it is easy to see that the active and inert necklace maps form an (orthogonal) factorisation system on $\mathcal{N}ec$ in the sense of [9].

Remark 3.7 A simplex Δ^n, considered as a necklace with a single bead, is represented in $\mathcal{N}ec$ by the pair $(\{0 < n\}, n)$. On the other hand, the necklace $([n], n)$ represents the spine of Δ^n, that is, the union of the edges $0 \to 1 \to \cdots \to n$ in Δ^n.

More generally, for any necklace (T, p), we can consider $([p], p)$, which is the spine passing through all the vertices of T. Note that there is a unique inert necklace map $([p], p) \to (T, p)$ which represents the inclusion of the spine into T. Further, there is a unique order isomorphism $[k] \simeq T$, where k is the number of beads of T. Thus there is a unique active map $([k], k) \to (T, p)$, which is the inclusion of the spine passing through all the joints of T.

3.B Necklace categories

Consider the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ of functors $\mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$. As $\mathcal{N}ec^{\mathrm{op}}$ and $\mathcal{V}$ are both monoidal categories, we can endow $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ with the (nonsymmetric) monoidal structure given by Day convolution (see [12]). We denote the resulting monoidal category by $(\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}, \otimes_{\mathrm{Day}}, \underline{I})$.

Given two functors $X, Y : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$, their Day convolution $X \otimes_{\mathrm{Day}} Y$ is obtained by the left Kan extension of the composite

$$\mathcal{N}ec^{\mathrm{op}} \times \mathcal{N}ec^{\mathrm{op}} \xrightarrow{X \times Y} \mathcal{V} \times \mathcal{V} \xrightarrow{- \otimes -} \mathcal{V}$$

along $\vee : \mathcal{N}ec^{\mathrm{op}} \times \mathcal{N}ec^{\mathrm{op}} \to \mathcal{N}ec^{\mathrm{op}}$,

$$X \otimes_{\mathrm{Day}} Y = \mathrm{Lan}_{\vee}(X(-) \otimes Y(-)).$$

Further, the monoidal unit of $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ is given by the representable functor on the monoidal unit $\{0\}$ of $\mathcal{N}ec$. As $\{0\}$ is also the terminal object of $\mathcal{N}ec$, we find that $F(\mathcal{N}ec(-, \{0\})) \simeq \underline{I}$ is the constant functor on I, the monoidal unit of $\mathcal{V}$.

Definition 3.8 Consider the category

$$\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} = \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}\text{-}\mathrm{Cat}$$

of small categories enriched in the monoidal category $(\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}, \otimes_{\mathrm{Day}}, \underline{I})$. We call the objects of $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ *necklace categories* and its morphisms *necklace functors*.

If $\mathcal{V} = \mathrm{Set}$, we simply write $\mathrm{Cat}_{\mathcal{N}ec}$ for $\mathrm{Set}\,\mathrm{Cat}_{\mathcal{N}ec}$.

Construction 3.9 We construct a functor

$$(-)^{\mathrm{nec}} : S_{\otimes}\mathcal{V} \to \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$$

as follows. Let (X, S) be a templicial object. Define

$$X_T = X_{t_1} \otimes_S \cdots \otimes_S X_{n-t_{k-1}} \in \mathcal{V}\mathrm{Quiv}_S$$

for any necklace $T = \{0 = t_0 < t_1 < \cdots < t_k = p\}$. We will also write μ_T for the quiver morphism $\mu_{t_1, t_2 - t_1, \ldots, p - t_{k-1}} : X_p \to X_T$.

This extends to a functor $X_\bullet^{\mathrm{nec}}\colon \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S$ as follows. Take a necklace map $f\colon (T,p) \to (U,q)$ and write $U = \{0 = u_0 < u_1 < \cdots < u_l = q\}$.

- If f is inert, then $p = q$ and $U \subseteq T$. Then there exist unique necklaces $(T_i, u_i - u_{i-1})$ for $i \in \{1,\dots,l\}$ such that $T = T_1 \vee \cdots \vee T_l$. Now set
$$X(f)\colon X_U \xrightarrow{\mu_{T_1}\otimes\cdots\otimes\mu_{T_l}} X_T.$$
- If f is active, then there exist unique $f_i\colon [t_i - t_{i-1}] \to [f(t_i) - f(t_{i-1})]$ in $\mathbf{\Delta}_f$ for all $i \in \{1,\dots,k\}$ such that $f = f_1 + \cdots + f_k$. Now set
$$X(f)\colon X_U \simeq X_{f(t_1)} \otimes_S \cdots \otimes_S X_{q-f(t_{k-1})} \xrightarrow{X(f_1)\otimes\cdots\otimes X(f_k)} X_T,$$
where the isomorphism is induced by the strong unitality of X and the fact that $U = f(T)$.

It follows from the coassociativity of μ that $X_\bullet$ is functorial on inert morphisms, and from the functoriality of X that $X_\bullet$ is functorial on active morphisms. Then it follows from the naturality of μ that $X_\bullet$ is functorial on all morphisms.

If we fix vertices $a, b \in S$, then we obtain a functor
$$X_\bullet(a,b)\colon \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}, \quad T \mapsto X_T(a,b).$$

Now let X^{nec} denote the necklace category with S as its object set and $X_\bullet(a,b)$ as its hom-object for all $a, b \in S$. The composition $X_\bullet(a,b) \otimes_{\mathrm{Day}} X_\bullet(b,c) \to X_\bullet(a,c)$ for $a, b, c \in S$ is induced by the canonical morphism
$$X_T(a,b) \otimes X_U(b,c) \to X_{T\vee U}(a,c)$$
and the identities are given by the morphism $\underline{I} \to X_\bullet(a,a)$ for $a \in S$ induced by the isomorphism $I \simeq X_0(a,a) = X_{\{0\}}(a,a)$.

This clearly extends to a functor $(-)^{\mathrm{nec}}\colon S_\otimes\mathcal{V} \to \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$.

Notation 3.10 As in $\mathbf{\Delta}$, we distinguish some special maps in $\mathcal{N}ec$:

- For any $0 < j < n$, we write
$$\delta_j\colon \{0 < n-1\} \to \{0 < n\}$$
for the active necklace map whose underlying morphism in $\mathbf{\Delta}_f$ is the coface map $\delta_j\colon [n-1] \to [n]$, ie $\delta_j(i) = i$ if $i < j$ and $\delta_j(i) = j+1$ if $j \geq i$.
- For any $k, l > 0$, we write
$$\nu_{k,l}\colon \{0 < k < k+l\} \to \{0 < k+l\}$$
for the unique inert necklace map.

Construction 3.11 Let $\mathcal{C}$ be a necklace category with set of objects S. We construct a templicial object $(\mathcal{C}^{\mathrm{temp}}, S)$ as follows. For every necklace T, we have a $\mathcal{V}$-enriched quiver $\mathcal{C}_T = (\mathcal{C}_T(a,b))_{a,b\in S}$. Then the composition and identities of $\mathcal{C}$ induce quiver morphisms

$$m_{U,V}\colon \mathcal{C}_U \otimes_S \mathcal{C}_V \to \mathcal{C}_{U\vee V} \quad \text{and} \quad u\colon I_S \to \mathcal{C}_{\{0\}}$$

for all necklaces U and V. Set $\mathcal{C}^{\mathrm{temp}}_0 = I_S$ and $p_0 = u\colon \mathcal{C}^{\mathrm{temp}}_0 \to \mathcal{C}_{\{0\}}$. Now let $n > 0$. We inductively define an object $\mathcal{C}^{\mathrm{temp}}_n \in \mathcal{V}\mathrm{Quiv}_S$ along with morphisms p_n and $\mu_{k,l}$ as the limit of the diagram of solid arrows in $\mathcal{V}\mathrm{Quiv}_S$

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{temp}}_n \arrow[r, dashed, "(\mu_{k,l})_{k,l}"] \arrow[d, dashed, "p_n"'] & \prod\limits_{\substack{k,l>0\\ k+l=n}} \mathcal{C}^{\mathrm{temp}}_k \otimes_S \mathcal{C}^{\mathrm{temp}}_l \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\prod_{k,l} p_k\otimes p_l"] & \prod\limits_{\substack{r,s,t>0\\ r+s+t=n}} \mathcal{C}^{\mathrm{temp}}_r \otimes_S \mathcal{C}^{\mathrm{temp}}_s \otimes_S \mathcal{C}^{\mathrm{temp}}_t \\
\mathcal{C}_{\{0<n\}} \arrow[dr, "(\mathcal{C}(\nu_{k,l}))_{k,l}"'] & \prod\limits_{\substack{k,l>0\\ k+l=n}} \mathcal{C}_{\{0<k\}} \otimes_S \mathcal{C}_{\{0<l\}} \arrow[d, "\prod_{k,l} m_{\{0<k\},\{0<l\}}"] & \\
 & \prod\limits_{\substack{k,l>0\\ k+l=n}} \mathcal{C}_{\{0<k<k+l\}} &
\end{tikzcd} \tag{5}$$

where α and β are defined by

$$\pi_{r,s,t}\alpha = (\mathrm{id}_r \otimes \mu_{s,t})\pi_{r,s+t} \quad \text{and} \quad \pi_{r,s,t}\beta = (\mu_{r,s}\otimes \mathrm{id}_t)\pi_{r+s,t}.$$

For example, $\mathcal{C}^{\mathrm{temp}}_1 = \mathcal{C}_{\{0<1\}}$ with $p_1 = \mathrm{id}_{\mathcal{C}_{\{0,1\}}}$, and $\mathcal{C}^{\mathrm{temp}}_2$ is the pullback of $m_{\{0<1\},\{0<1\}}$ and $\mathcal{C}(\nu_{1,1})$. We further set $\mu_{0,n}$ and $\mu_{n,0}$ to be the left and right unit isomorphisms, respectively,

$$\mathcal{C}^{\mathrm{temp}}_n \xrightarrow{\sim} \mathcal{C}^{\mathrm{temp}}_0 \otimes_S \mathcal{C}^{\mathrm{temp}}_n, \quad \mathcal{C}^{\mathrm{temp}}_n \xrightarrow{\sim} \mathcal{C}^{\mathrm{temp}}_n \otimes_S \mathcal{C}^{\mathrm{temp}}_0.$$

Further, let $f\colon [m]\to[n]$ be a morphism in $\mathbf{\Delta}_f$. We define a quiver morphism $\mathcal{C}^{\mathrm{temp}}(f)\colon \mathcal{C}^{\mathrm{temp}}_n \to \mathcal{C}^{\mathrm{temp}}_m$ by induction on m. Set $\mathcal{C}^{\mathrm{temp}}(\mathrm{id}_{[0]})$ to be the identity on I_S. If $m > 0$, we let $\mathcal{C}^{\mathrm{temp}}(f)$ be the unique morphism satisfying, for all $k,l > 0$ with $k+l = m$,

$$\mu_{k,l}\mathcal{C}^{\mathrm{temp}}(f) = (\mathcal{C}^{\mathrm{temp}}(f_1)\otimes_S \mathcal{C}^{\mathrm{temp}}(f_2))\mu_{p,q} \quad \text{and} \quad p_m\mathcal{C}^{\mathrm{temp}}(f) = \mathcal{C}(f)p_n,$$

where $f_1\colon [k]\to[p]$ and $f_2\colon [l]\to[q]$ are unique in $\mathbf{\Delta}_f$ such that $f_1+f_2 = f$. (Note that, when $m=1$, the first condition is empty and $\mathcal{C}^{\mathrm{temp}}(f)$ is just $\mathcal{C}(f)p_n$.)

We have thus constructed a well-defined functor

$$\mathcal{C}^{\mathrm{temp}}\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S.$$

By construction, $\mathcal{C}^{\mathrm{temp}}$ is strongly unital and colax monoidal with comultiplication given by the morphisms $(\mu_{k,l})_{k,l\geq 0}$.

Theorem 3.12 *The functor* $(-)^{\mathrm{nec}}\colon S_{\otimes}\mathcal{V} \to \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ *is fully faithful and left-adjoint to a functor* $(-)^{\mathrm{temp}}\colon \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \to S_{\otimes}\mathcal{V}$ *which is given on objects by the assignment* $\mathcal{C}\mapsto\mathcal{C}^{\mathrm{temp}}$ *of Construction 3.11.*

Proof Let $\mathcal{C}$ be a necklace category and define a necklace functor $\varepsilon_{\mathcal{C}}\colon (\mathcal{C}^{\mathrm{temp}})^{\mathrm{nec}} \to \mathcal{C}$ by the quiver morphism

$$\varepsilon_{\mathcal{C}_T}\colon (\mathcal{C}^{\mathrm{temp}})_T \xrightarrow{p_{t_1}\otimes\cdots\otimes p_{p-t_{k-1}}} \mathcal{C}_{\{0<t_1\}}\otimes\cdots\otimes\mathcal{C}_{\{0<p-t_{k-1}\}} \xrightarrow{m_{\mathcal{C}}} \mathcal{C}_T$$

for any necklace $T=\{0=t_0<t_1<\cdots<t_k=p\}$.

Let (X,S) be a templicial object and $H\colon X^{\mathrm{nec}}\to\mathcal{C}$ a necklace functor with object map $f\colon S\to\mathrm{Ob}(\mathcal{C})$. We will construct a templicial morphism $(\alpha,f)\colon(X,S)\to(\mathcal{C}^{\mathrm{temp}},\mathrm{Ob}(\mathcal{C}))$ by induction. Let α_0 be the canonical quiver morphism $f_!(X_0)\simeq f_!(I_S)\to I_{\mathrm{Ob}(\mathcal{C})}=\mathcal{C}^{\mathrm{temp}}_0$. For $n>0$, let $\beta_n\colon f_!(X_n)=f_!(X_{\{0<n\}})\to\mathcal{C}_{\{0<n\}}$ be adjoint to $H_{\{0<n\}}\colon X_{\{0<n\}}\to f^*(\mathcal{C}_{\{0<n\}})$. By Construction 3.11, we have a unique morphism $\alpha_n\colon f_!(X_n)\to\mathcal{C}^{\mathrm{temp}}_n$ such that

$$p_n\alpha_n=\beta_n$$

and, for all $k,l>0$ with $k+l=n$, $\mu_{k,l}\circ\alpha_n$ is equal to the composite

$$f_!(X_n)\xrightarrow{f_!(\mu^X_{k,l})} f_!(X_k\otimes X_l)\to f_!(X_k)\otimes f_!(X_l)\xrightarrow{\alpha_k\otimes\alpha_l}\mathcal{C}^{\mathrm{temp}}_k\otimes\mathcal{C}^{\mathrm{temp}}_l,$$

where we used the colax monoidal structure of $f_!$. It follows that (α,f) is a well-defined templicial morphism which is unique such that $\varepsilon_{\mathcal{C}}\circ\alpha^{\mathrm{nec}}=H$. Hence, the assignment $\mathcal{C}\mapsto\mathcal{C}^{\mathrm{temp}}$ extends to a right-adjoint of $(-)^{\mathrm{nec}}$.

Now let X be a templicial object. Since the composition $X^{\mathrm{nec}}_T\otimes_S X^{\mathrm{nec}}_U\to X^{\mathrm{nec}}_{T\vee U}$ of X^{nec} is an isomorphism for all $T,U\in\mathcal{N}ec$, it follows from Construction 3.11 that $p_n\colon (X^{\mathrm{nec}})^{\mathrm{temp}}_n\to X_{\{0<n\}}=X_n$ is an isomorphism. We thus obtain a natural isomorphism $(-)^{\mathrm{temp}}\circ(-)^{\mathrm{nec}}\simeq\mathrm{id}_{S_\otimes\mathcal{V}}$, which shows that $(-)^{\mathrm{nec}}$ is fully faithful. □

3.C Some constructions revisited

We show that the functor $\tilde{U}\colon S_\otimes\mathcal{V}\to\mathrm{SSet}$ (Proposition 2.8) and the templicial nerve $N_{\mathcal{V}}\colon\mathcal{V}\mathrm{Cat}\to S_\otimes\mathcal{V}$ (Definition 2.11) factor through the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ of necklace categories.

Notation 3.13 By postcomposition, the adjunction $F\colon\mathrm{Set}\leftrightarrows\mathcal{V}:U$ induces an adjunction $F\colon\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}\leftrightarrows\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}:U$. Note that, as F is strong monoidal and preserves colimits, the induced functor $F\colon\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}\to\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ is strong monoidal as well. Therefore, we have an induced adjunction, which we denote by

$$\begin{tikzcd}
\mathrm{Cat}_{\mathcal{N}ec} \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}. \arrow[l, shift left, "\mathcal{U}"]
\end{tikzcd}$$

Proposition 3.14 *There is diagram of adjunctions*

$$\begin{tikzcd}
\mathrm{SSet} \arrow[r, shift left, "\tilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, hook, shift right, "(-)^{\mathrm{nec}}"'] \arrow[d, phantom, "\dashv"] & S_\otimes\mathcal{V} \arrow[l, shift left, "\tilde{U}"] \arrow[d, hook, shift right, "(-)^{\mathrm{nec}}"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{Cat}_{\mathcal{N}ec} \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] \arrow[u, shift right, "(-)^{\mathrm{temp}}"'] & \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \arrow[l, shift left, "\mathcal{U}"] \arrow[u, shift right, "(-)^{\mathrm{temp}}"']
\end{tikzcd}$$

which commutes in the sense that we have natural isomorphisms

$$(-)^{\mathrm{nec}} \circ \widetilde{F} \simeq \mathcal{F} \circ (-)^{\mathrm{nec}} \quad \textit{and} \quad \widetilde{U} \circ (-)^{\mathrm{temp}} \simeq (-)^{\mathrm{temp}} \circ \mathcal{U}.$$

In particular, we have a natural isomorphism

$$\widetilde{U} \simeq (-)^{\mathrm{temp}} \circ \mathcal{U} \circ (-)^{\mathrm{nec}}.$$

Proof It suffices to show the commutativity of the left-adjoints. But this immediately follows from the fact that $F : \mathrm{Set} \to \mathcal{V}$ is strong monoidal and preserves colimits. The final isomorphism $\widetilde{U} \simeq (-)^{\mathrm{temp}} \circ \mathcal{U} \circ (-)^{\mathrm{nec}}$ follows from the fact that $(-)^{\mathrm{nec}}$ is fully faithful. □

Lemma 3.15 *Let $\mathcal{C}$ be a finitely complete category. Let $f : A \to B$ be a morphism in $\mathcal{C}$ and $n \geq 2$. Then A is the limit of the diagram of solid arrows*

$$\begin{tikzcd}
A \arrow[r, dashed, "\Delta"] \arrow[d, dashed, "f"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\prod_{k,l} f"] & \prod\limits_{\substack{r,s,t>0\\r+s+t=n}} A \\
B \arrow[r, "\Delta"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} B &
\end{tikzcd}$$

where Δ is the diagonal morphism and α and β are defined by

$$\pi_{r,s,t}\alpha = \pi_{r+s,t} \quad \text{and} \quad \pi_{r,s,t}\beta = \pi_{r,s+t}$$

for all $r, s, t > 0$ with $r + s + t = n$.

Proof This is an easy verification. □

Let $\underline{(-)} : \mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ denote the diagonal functor associating to every object $V \in \mathcal{V}$ the constant functor on V. Then $\underline{(-)}$ is easily seen to be strong monoidal and thus it induces a functor

$$\underline{(-)} : \mathcal{V}\mathrm{Cat} \to \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}.$$

Proposition 3.16 *We have a natural isomorphism*

$$N_{\mathcal{V}} \simeq (-)^{\mathrm{temp}} \circ \underline{(-)}.$$

Proof Let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category and $n \geq 0$. Applying Lemma 3.15 to the n-fold composition $m_{\mathcal{C}}^{(n)} : \mathcal{C}^{\otimes n} \to \mathcal{C}$ as a morphism in $\mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$, it follows from Construction 3.11 that $\underline{\mathcal{C}}_n^{\mathrm{temp}} \simeq \mathcal{C}^{\otimes n}$ by induction on n. It quickly follows that this identification induces an isomorphism of templicial objects $\underline{\mathcal{C}}^{\mathrm{temp}} \simeq N_{\mathcal{V}}(\mathcal{C})$, which is clearly natural in $\mathcal{C}$. □

4 Enriching the homotopy coherent nerve

Let Cat_Δ denote the category of small simplicial categories, that is categories enriched in the cartesian monoidal category of simplicial sets $(\mathrm{SSet}, \times, \Delta^0)$. In this section we generalise the classical adjunction between the categorification functor $\mathfrak{C}\colon \mathrm{SSet} \to \mathrm{Cat}_\Delta$ and the homotopy coherent nerve $N^{\mathrm{hc}}\colon \mathrm{Cat}_\Delta \to \mathrm{SSet}$ to the templicial level, yielding an adjunction $\mathfrak{C}_\mathcal{V} \dashv N^{\mathrm{hc}}_\mathcal{V}$ which depends on $\mathcal{V}$ (Definition 4.9). We first recall in Section 4.A the homotopy coherent nerve due to Cordier [10]. It is most easily constructed as the formal right-adjoint to the categorification functor $\mathfrak{C}$, which has historically gone through several different equivalent descriptions. This goes back to Cordier and Porter [11] and a different definition is given in [28], which we outline below. Later, Dugger and Spivak [14] gave a very elegant and simple description of $\mathfrak{C}$ by means of necklaces. We will closely follow their approach and adapt it to the templicial setting in Section 4.C.

4.A The classical homotopy coherent nerve

We recall Cordier's homotopy coherent nerve. Further, we give a new expression for its left-adjoint (Proposition 4.6) which will allow us to generalise it more easily to the templicial setting in Section 4.B.

Notation 4.1 Given a necklace (T, p), consider the poset

$$\mathcal{P}_T = \{U \subseteq [p] \mid T \subseteq U\}$$

ordered by inclusion. Equivalently, it is the poset of inert necklace maps $U \hookrightarrow T$.

If $T = \{0 < p\}$ is a simplex, we also write $\mathcal{P}_T = \mathcal{P}_p$.

Remark 4.2 It is easy to see that the assignment $T \mapsto \mathcal{P}_T$ extends to a strong monoidal functor

$$\mathcal{P}\colon \mathcal{N}ec \to \mathrm{Cat},$$

where, for every necklace map $f\colon T \to U$, we have $\mathcal{P}(f)(V) = f(V)$ for all $V \in \mathcal{P}_T$. For necklaces T and U, the monoidal structure is given by

$$\mathcal{P}_T \times \mathcal{P}_U \to \mathcal{P}_{T\vee U}, \quad (V, W) \mapsto (V \vee W),$$

which is clearly an order isomorphism.

In [28, Section 1.1.5], a simplicial category $\mathfrak{C}[\Delta^n]$ is constructed as follows. Its objects are given by the set $[n]$ and, for all $i, j \in [n]$, we have

$$\mathfrak{C}[\Delta^n](i,j) = \begin{cases} N(\mathcal{P}_{j-i}) & \text{if } i \leq j, \\ \varnothing & i > j. \end{cases}$$

Note that $N(\mathcal{P}_{j-i}) \simeq (\Delta^1)^{\times j-i-1}$ if $i < j$ and $N(\mathcal{P}_{j-i}) \simeq \Delta^0$ if $i = j$. Further, given $i \leq j \leq k$ in $[n]$, the composition

$$m_{i,j,k}\colon \mathfrak{C}[\Delta^n](i,j) \times \mathfrak{C}[\Delta^n](j,k) \to \mathfrak{C}[\Delta^n](i,k)$$

is given by applying N to the order morphism

$$\mathcal{P}_{j-i} \times \mathcal{P}_{k-j} \simeq \mathcal{P}_{\{0<j-i<k-i\}} \hookrightarrow \mathcal{P}_{k-i}, \quad (T, U) \mapsto T \vee U.$$

Finally, the identities are given by the unique vertex of $\mathfrak{C}[\Delta^n](i,i) \simeq \Delta^0$ for $i \in [n]$.

It is now easy to see that the above construction extends to a functor

$$\mathfrak{C}[\Delta^{(-)}] \colon \mathbf{\Delta} \to \mathrm{Cat}_\Delta.$$

Then, by left Kan extension along the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathrm{SSet}$, the cosimplicial object $\mathfrak{C}[\Delta^{(-)}]$ induces an adjunction

$$\begin{tikzcd}\mathrm{SSet} \arrow[r, shift left=1.5, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_\Delta. \arrow[l, shift left=1.5, "N^{\mathrm{hc}}"]\end{tikzcd}$$

For all small simplicial categories $\mathcal{C}$ and $n \geq 0$,

$$N^{\mathrm{hc}}(\mathcal{C})_n \simeq \mathrm{Cat}_\Delta(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

Example 4.3 Given a small simplicial category $\mathcal{C}$, let us describe its homotopy coherent nerve in low dimensions:

- The vertices of $N^{\mathrm{hc}}(\mathcal{C})$ are given by the set of objects $\mathrm{Ob}(\mathcal{C})$.
- The edges of $N^{\mathrm{hc}}(\mathcal{C})$ are given by the morphisms of $\mathcal{C}$ (that is, vertices $f \in \mathcal{C}_0(A,B)$ for some $A, B \in \mathrm{Ob}(\mathcal{C})$).
- A 2-simplex of $N^{\mathrm{hc}}(\mathcal{C})$ is given by a (not necessarily commutative) diagram of morphisms in $\mathcal{C}$

$$\begin{tikzcd} & C \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\Uparrow \sigma"] & & B \end{tikzcd}$$

  along with an edge σ of $\mathcal{C}(A,B)$ from h to the composition $g \circ f$.

Proposition 4.4 [14, Proposition 3.7] *There is an isomorphism of simplicial sets*

$$\mathfrak{C}[T](0,p) \simeq N(\mathcal{P}_T)$$

that is natural in all necklaces $(T,p) \in \mathcal{N}ec$.

Proposition 4.5 [14, Proposition 4.3] *For every simplicial set K with vertices a and b, there is an isomorphism of simplicial sets*

$$\mathfrak{C}[K](a,b) \simeq \operatorname*{colim}_{\substack{T \to K_{a,b} \text{ in } \mathrm{SSet}_{*,*} \\ (T,p) \in \mathcal{N}ec}} \mathfrak{C}[T](0,p).$$

Recall that by Proposition 2.8 we may view a simplicial set K as a templicial set and thus, for any $a, b \in K_0$, we can apply Construction 3.9 to obtain a functor $K_\bullet(a,b) \colon \mathcal{N}ec^{\mathrm{op}} \to \mathrm{Set}$, $T \mapsto K_T(a,b)$. It follows from Yoneda's lemma that we have a canonical bijection, natural in $T \in \mathcal{N}ec$,

$$\mathrm{SSet}_{*,*}(T, K_{a,b}) \simeq K_T(a,b).$$

With this in mind, we introduce another description of $\mathfrak{C}$ by means of a weighted colimit. For background on weighted colimits, see [38, Definition 7.4.1], for example.

Proposition 4.6 *For any simplicial set K with vertices a and b, $\mathfrak{C}[K](a,b)$ is isomorphic to the weighted colimit in* SSet

$$\operatorname{colim}^{K_\bullet(a,b)} N\mathcal{P}_{(-)}$$

of $N\mathcal{P}_{(-)}\colon \mathcal{N}ec \to \mathrm{SSet}$ with weight $K_\bullet(a,b)\colon \mathcal{N}ec^{\mathrm{op}} \to \mathrm{Set}$.

Proof From Propositions 4.4 and 4.5, it is clear that $\mathfrak{C}[K](a,b)$ is given by the coequaliser in SSet

$$\coprod_{\substack{T\to U\to K_{a,b}\\ T,U\in\mathcal{N}ec}} N(\mathcal{P}_T) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{\substack{T\to K_{a,b}\\ T\in\mathcal{N}ec}} N(\mathcal{P}_T) \twoheadrightarrow \mathfrak{C}[K](a,b),$$

where α and β are given by respectively projecting onto $T\to K_{a,b}$ and applying $N\mathcal{P}_{(-)}$ to $T\to U$ for any $T\to U\to K_{a,b}$ in $\mathrm{SSet}_{*,*}$.

Since morphisms $T\to K_{a,b}$ in $\mathrm{SSet}_{*,*}$ with T a necklace correspond to elements of the set $K_T(a,b)$, we obtain a coequaliser diagram

$$\coprod_{T\to U \text{ in } \mathcal{N}ec} K_U(a,b)\times N(\mathcal{P}_T) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{T\in\mathcal{N}ec} K_T(a,b)\times N(\mathcal{P}_T) \twoheadrightarrow \mathfrak{C}[K](a,b),$$

where α and β are given by respectively applying $K_\bullet(a,b)$ and $N\mathcal{P}_{(-)}$ to $T\to U$ in $\mathcal{N}ec$. But this coequaliser is precisely the weighted colimit described in the statement. □

4.B The templicial homotopy coherent nerve

Consider the category $S\mathcal{V}$ of simplicial objects in $\mathcal{V}$, with the pointwise symmetric monoidal structure induced by that of $\mathcal{V}$. Note that its monoidal unit is the simplicial object $F(\Delta^0) = \underline{I}$ (the constant functor on I). Further, $S\mathcal{V}$ is canonically enriched and tensored over $\mathcal{V}$. We denote the enrichment over $\mathcal{V}$ by $[-,-]$. For every $V\in\mathcal{V}$ and $A\in S\mathcal{V}$, the tensoring $V\cdot A$ is given by the monoidal product $\underline{V}\otimes A$, where $\underline{V}$ denotes the constant functor on V.

We denote the category of small $S\mathcal{V}$-enriched categories by $\mathcal{V}\mathrm{Cat}_\Delta$. Note that the adjunction $F\colon \mathrm{Set} \leftrightarrows \mathcal{V}\colon U$ induces an adjunction $F\colon \mathrm{SSet}\leftrightarrows S\mathcal{V}\colon U$ by postcomposition, for which F is still strong monoidal. Hence, we have an induced adjunction between simplicial categories and $S\mathcal{V}$-categories, which we denote by

$$\begin{tikzcd} \mathrm{Cat}_\Delta \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\mathrm{Cat}_\Delta. \arrow[l, shift left, "\mathcal{U}"] \end{tikzcd}$$

From Proposition 4.6, it is easy to see that the adjunction $\mathfrak{C}\dashv N^{\mathrm{hc}}$ actually factors through the category $\mathrm{Cat}_{\mathcal{N}ec}$ of Definition 3.8. Moreover, it suggests that we can define the templicial categorification functor by means of a similar weighted colimit.

Construction 4.7 We construct an adjunction

$$\begin{tikzcd}\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}} \arrow[r, shift left, "\mathfrak{s}"] \arrow[r, phantom, "\perp"] & S\mathcal{V} \arrow[l, shift left, "\mathfrak{n}"]\end{tikzcd}$$

as follows. Given a functor $X : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$, consider the weighted colimit in $S\mathcal{V}$

$$\mathfrak{s}(X) = \operatorname{colim}^X FN\mathcal{P}_{(-)}$$

of the composite $\mathcal{N}ec \xrightarrow{\mathcal{P}_{(-)}} \mathrm{Cat} \xrightarrow{N} \mathrm{SSet} \xrightarrow{F} S\mathcal{V}$ with weight X. Explicitly, $\mathfrak{s}(X)$ may be realised as the coequaliser in $S\mathcal{V}$

$$\coprod_{f : T \to U \text{ in } \mathcal{N}ec} X_U \otimes FN(\mathcal{P}_T) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{T \in \mathcal{N}ec} X_T \otimes FN(\mathcal{P}_T) \twoheadrightarrow \mathfrak{s}(X), \tag{6}$$

where α and β are given by respectively applying X and $FN\mathcal{P}_{(-)}$ to a necklace morphism $f : T \to U$.

As a weighted colimit, $\mathfrak{s}(X)$ fits into a canonical bijection of sets

$$S\mathcal{V}(\mathfrak{s}(X), Y) \simeq \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}(X, [FN\mathcal{P}_{(-)}, Y])$$

which is natural in $Y \in S\mathcal{V}$. Hence, the assignment $X \mapsto \mathfrak{s}(X)$ extends to a functor $\mathfrak{s} : \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}} \to S\mathcal{V}$ which is left-adjoint to the functor

$$\mathfrak{n} : S\mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}, \quad Y \mapsto [FN\mathcal{P}_{(-)}, Y].$$

Proposition 4.8 *The functor* $\mathfrak{s} : \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}} \to S\mathcal{V}$ *of Construction 4.7 is strong monoidal.*

Proof For functors $X, Y : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$,

$$\begin{aligned}\mathfrak{s}(X \otimes_{\mathrm{Day}} Y) &= \operatorname*{colim}_{T \in \mathcal{N}ec}{}^{(X \otimes_{\mathrm{Day}} Y)(T)} FN\mathcal{P}_T \simeq \operatorname*{colim}_{U,V \in \mathcal{N}ec}{}^{X(U) \otimes Y(V)} FN\mathcal{P}_{U \vee V} \\ &\simeq \operatorname*{colim}_{U \in \mathcal{N}ec}{}^{X(U)} FN\mathcal{P}_U \otimes \operatorname*{colim}_{V \in \mathcal{N}ec}{}^{Y(V)} FN\mathcal{P}_V = \mathfrak{s}(X) \otimes \mathfrak{s}(Y),\end{aligned}$$

where we have used the presentation of $X \otimes_{\mathrm{Day}} Y$ as a left Kan extension and the strong monoidality of F, N and $\mathcal{P}_{(-)}$ (see Remark 4.2). Further, since $\underline{I} = F(\mathcal{N}ec(-, \{0\}))$,

$$\mathfrak{s}(\underline{I}) = \operatorname*{colim}_{T \in \mathcal{N}ec}{}^{\underline{I}} FN\mathcal{P}_T \simeq FN\mathcal{P}_{\{0\}} \simeq F(\Delta^0). \qquad \square$$

Definition 4.9 By virtue of Proposition 4.8, the adjunction $\mathfrak{s} \dashv \mathfrak{n}$ between $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ and $S\mathcal{V}$ induces an adjunction

$$\begin{tikzcd}\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \arrow[r, shift left, "\mathfrak{s}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\mathrm{Cat}_{\Delta} \arrow[l, shift left, "\mathfrak{n}"]\end{tikzcd}$$

We call the composite

$$\mathfrak{C}_{\mathcal{V}} : S_{\otimes}\mathcal{V} \xrightarrow{(-)^{\mathrm{nec}}} \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \xrightarrow{\mathfrak{s}} \mathcal{V}\mathrm{Cat}_{\Delta}$$

the *categorification functor*. It is left-adjoint to the composite

$$N_{\mathcal{V}}^{\mathrm{hc}} : \mathcal{V}\mathrm{Cat}_{\Delta} \xrightarrow{\mathfrak{n}} \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \xrightarrow{(-)^{\mathrm{temp}}} S_{\otimes}\mathcal{V},$$

which we call the *templicial homotopy coherent nerve*.

Remark 4.10 Suppose $\mathcal{V} = \mathrm{Set}$. Then the adjunction $\mathfrak{C}_{\mathcal{V}} \dashv N^{\mathrm{hc}}_{\mathcal{V}}$ reduces to the classical adjunction $\mathfrak{C} \dashv N^{\mathrm{hc}}$. Indeed, it suffices to note that $\mathfrak{C}_{\mathcal{V}}$ reduces to $\mathfrak{C}$, which follows from Proposition 4.6 and Construction 4.7.

Example 4.11 Let $\mathcal{C}$ be a small $S\mathcal{V}$-category. We describe the templicial object $N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})$ in low dimensions using Construction 3.11. Note the analogy with Example 4.3.

- The vertex set of $N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})$ is simply $\mathrm{Ob}(\mathcal{C})$.
- Further, for any $A, B \in \mathrm{Ob}(\mathcal{C})$, it follows from $N(\mathcal{P}_{\{0<1\}}) \simeq \Delta^0$ that

$$N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})_1(A,B) = \mathfrak{n}(\mathcal{C})_{\{0<1\}}(A,B) = [FN(\mathcal{P}_{\{0<1\}}), \mathcal{C}(A,B)] \simeq \mathcal{C}_0(A,B).$$

- In dimension 2, it follows from $N(\mathcal{P}_{\{0<2\}}) \simeq \Delta^1$ and $N(\mathcal{P}_{\{0<1<2\}}) \simeq \Delta^0$ that

$$\mathfrak{n}(\mathcal{C})_{\{0<2\}}(A,B) = [FN(\mathcal{P}_{\{0<2\}}), \mathcal{C}(A,B)] \simeq \mathcal{C}_1(A,B),$$
$$\mathfrak{n}(\mathcal{C})_{\{0<1<2\}}(A,B) = [FN(\mathcal{P}_{\{0<1<2\}}), \mathcal{C}(A,B)] \simeq \mathcal{C}_0(A,B).$$

Here, the morphism $\mathfrak{n}(\mathcal{C})_{\{0<2\}}(A,B) \to \mathfrak{n}(\mathcal{C})_{\{0<1<2\}}(A,B)$ is induced by the inert necklace map $\nu_{1,1}\colon \{0<1<2\} \hookrightarrow \{0<2\}$ and thus corresponds to the face map $d_0\colon \mathcal{C}_1(A,B) \to \mathcal{C}_0(A,B)$. It follows from (5) that we have a pullback diagram

$$\begin{tikzcd}
N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})_2(A,B) \arrow[r] \arrow[d] & \coprod_{C\in\mathrm{Ob}(\mathcal{C})} \mathcal{C}_0(A,C) \otimes \mathcal{C}_0(C,B) \arrow[d, "m_{0,0}"] \\
\mathcal{C}_1(A,B) \arrow[r, "d_0"'] & \mathcal{C}_0(A,B)
\end{tikzcd}$$

In particular, we see that the underlying set of the object $N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})_2(A,B)$ consists of pairs (σ,α) with $\alpha \in U\big(\coprod_{C\in\mathrm{Ob}(\mathcal{C})} \mathcal{C}_0(A,C)\otimes\mathcal{C}_0(C,B)\big)$ and $\sigma \in U(\mathcal{C}_1(A,B))$ an edge from $h = d_1(\sigma)$ to $m(\alpha)$.

Proposition 4.12 *There are canonical natural isomorphisms*

$$\mathfrak{C}_{\mathcal{V}} \circ \tilde{F} \simeq \mathcal{F} \circ \mathfrak{C} \quad \textit{and} \quad \tilde{U} \circ N^{\mathrm{hc}}_{\mathcal{V}} \simeq N^{\mathrm{hc}} \circ \mathcal{U}.$$

Proof As $\mathfrak{C}_{\mathcal{V}} \dashv N^{\mathrm{hc}}_{\mathcal{V}}$, $\mathfrak{C} \dashv N^{\mathrm{hc}}$, $\tilde{F} \dashv \tilde{U}$ and $\mathcal{F} \dashv \mathcal{U}$, it suffices to show the first natural isomorphism. Since $F\colon \mathrm{SSet} \to S\mathcal{V}$ preserves colimits and is strong monoidal, it is clear that

$$\operatorname*{colim}_{T\in\mathcal{N}ec}{}^{FX_T} FN\mathcal{P}_T \simeq F\big(\operatorname*{colim}_{T\in\mathcal{N}ec}{}^{X_T} N\mathcal{P}_T\big)$$

for any functor $X\colon \mathcal{N}ec^{\mathrm{op}} \to \mathrm{Set}$. It follows that we have a natural isomorphism $F\circ\mathfrak{s} \simeq \mathfrak{s}\circ F$ of functors $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}} \to S\mathcal{V}$, and thus also $\mathcal{F}\circ\mathfrak{s} \simeq \mathfrak{s}\circ\mathcal{F}$ of functors $\mathrm{Cat}_{\mathcal{N}ec} \to \mathcal{V}\mathrm{Cat}_{\Delta}$. Thus, by Proposition 3.14, $\mathcal{F}\circ\mathfrak{C} \simeq \mathfrak{C}_{\mathcal{V}}\circ\tilde{F}$ as well. □

4.C Simplification of the categorification functor

Following [14, Section 4], we give a simplified description of the categorification functor $\mathfrak{C}_{\mathcal{V}}\colon S_{\otimes}\mathcal{V} \to \mathcal{V}\mathrm{Cat}_{\Delta}$ of Definition 4.9. Let us first recall their approach.

Let (T, p) be a necklace and $n \geq 0$. A *flag of length* n on T is defined as an n-simplex of the nerve $N(\mathcal{P}_T)$. Explicitly, a flag of length n on T is a sequence of inclusions

$$\vec{T} = (T_0 \subseteq \cdots \subseteq T_n)$$

such that $T \subseteq T_0$ and $T_n \subseteq [p]$. We call a flag $\vec{T}$ on a necklace T *flanked* if $T = T_0$ and $T_n = [p]$.

Consider a simplicial set K with vertices a and b, and $T = \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$ a necklace. A map $T \to K_{a,b}$ in $\mathrm{SSet}_{*,*}$ is *totally nondegenerate* if, for every $i \in \{1, \dots, k\}$, the composite map in SSet

$$\Delta^{n_i} \hookrightarrow T \to K_{a,b}$$

represents a nondegenerate n_i-simplex of K.

As an immediate consequence of Proposition 4.5, an n-simplex of $\mathfrak{C}[K](a, b)$ consists of an equivalence class

$$[T, T \to K_{a,b}, \vec{T}] \tag{7}$$

of triples $(T, T \to K_{a,b}, \vec{T})$ where

- T is a necklace,
- $T \to K_{a,b}$ is a map in $\mathrm{SSet}_{*,*}$ (or equivalently an element of $K_T(a, b)$),
- $\vec{T}$ is a flag of length n on T.

The equivalence relation is generated by setting two triples $(T, T \to K_{a,b}, \vec{T})$ and $(U, U \to K_{a,b}, \vec{U})$ to be equivalent if there exists a map of necklaces $f : T \to U$ making the obvious diagram commute and such that $f(T_i) = U_i$ for all $0 \leq i \leq n$.

Then one can make the following reductions:

1. In every equivalence class (7), there exists a triple $(T, T \to K_{a,b}, \vec{T})$ such that $\vec{T}$ is flanked. Moreover, two such triples are equivalent if and only if they can be connected by a zigzag of morphisms of flagged necklaces in which every flag is flanked.
2. In every equivalence class (7), there exists a *unique* triple $(T, T \to K_{a,b}, \vec{T})$ such that $\vec{T}$ is flanked and $T \to K_{a,b}$ is totally nondegenerate. In other words, there is a bijection

   $$\mathfrak{C}[K]_n(a, b) \simeq \coprod_{\substack{T \in \mathcal{N}ec \\ \vec{T} \text{ flanked flag} \\ \text{of length } n}} K_T^{\mathrm{nd}}(a, b),$$

   where $K_T^{\mathrm{nd}}(a, b) \subseteq K_T(a, b)$ is the subset of totally nondegenerate maps $T \to K_{a,b}$.

The main ingredient in the first reduction is flankification. Given a necklace T with a flag $\vec{T} = (T_0 \subseteq \cdots \subseteq T_n)$, there is a unique order isomorphism $T_n \simeq [k]$, where k is the number of beads of T_n. For all $i \in [n]$, write T'_i for the image of T_i under this isomorphism, so that $T'_0 \subseteq \cdots \subseteq T'_n = [k]$. Further set $T' = T'_0$, so that the flag $\vec{T}'$ is flanked on T'. Then $(T', \vec{T}')$ is the *flankification* of $(T, \vec{T})$.

We now proceed to adapting these steps to the templicial setting. Generalising the first of the above reductions to templicial objects is fairly straightforward. This is done in Proposition 4.15. For the second reduction, we have to restrict to templicial objects that have nondegenerate simplices (Definition 2.16). Our definition of $\mathfrak{C}_{\mathcal{V}}[X]$ has the advantage that there is no reference to X in the indexing category of the colimit involved, which allows for more categorical and shorter proofs of the reduction steps.

Notation 4.13 Given an integer $n \geq 0$, let us write

$$\mathcal{N}ec^{\urcorner}[n]$$

for the category of pairs $(T, \vec{T})$ where T is a necklace and $\vec{T} = (T_0, \dots, T_n)$ is a flag of length n on T. A morphism $(T, \vec{T}) \to (U, \vec{U})$ in $\mathcal{N}ec^{\urcorner}[n]$ is a necklace map $f : T \to U$ such that $f(T_i) = U_i$ for all $i \in [n]$. Further, we let

$$\mathcal{N}ec^{\urcorner}_f[n]$$

denote the full subcategory of $\mathcal{N}ec^{\urcorner}[n]$ spanned by flagged necklaces whose flags are flanked. Note that a morphism in $\mathcal{N}ec^{\urcorner}_f[n]$ is necessarily active and surjective on vertices.

Lemma 4.14 *Let $n \geq 0$. Flankification extends to a functor $\mathcal{N}ec^{\urcorner}[n] \to \mathcal{N}ec^{\urcorner}_f[n]$ which is right-adjoint to the inclusion $\iota \colon \mathcal{N}ec^{\urcorner}_f[n] \hookrightarrow \mathcal{N}ec^{\urcorner}[n]$.*

Proof Write $\gamma(T, \vec{T})$ for the flankification of a flagged necklace $(T, \vec{T})$. If k is the number of beads of T, we obtain a morphism $\epsilon \colon \iota\gamma(T, \vec{T}) \to (T, \vec{T})$ in $\mathcal{N}ec^{\urcorner}[n]$ with underlying morphism $[k] \simeq T_n \hookrightarrow [p]$ in $\mathbf{\Delta}_f$. Given $(U, \vec{U}) \in \mathcal{N}ec^{\urcorner}_f[n]$ with (U, q) a necklace, and a morphism $f : \iota(U, \vec{U}) \to (T, \vec{T})$ in $\mathcal{N}ec^{\urcorner}[n]$, we have in particular that $T_n = f(U_n) = f([q])$. So the morphism $f : [q] \to [p]$ in $\mathbf{\Delta}_f$ factors uniquely as some $g \colon [q] \to [k]$ followed by $[k] \hookrightarrow [p]$. Moreover, g defines a morphism $(U, \vec{U}) \to \gamma(T, \vec{T})$ in $\mathcal{N}ec^{\urcorner}_f[n]$ such that $\epsilon \circ \iota(g)$. □

Proposition 4.15 *Let (X, S) be a templicial object and $a, b \in S$. Then, for every $n \geq 0$, we have a canonical isomorphism*

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) \simeq \operatorname*{colim}_{(T,\vec{T}) \in \mathcal{N}ec^{\urcorner}_f[n]} X_T(a,b).$$

Proof In view of (6), we have a coequaliser, for every integer $n \geq 0$,

$$\coprod_{\substack{f\colon T \to U \\ \vec{T} \text{ flag on } T \\ \text{of length } n}} X_U(a,b) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{\substack{T \in \mathcal{N}ec \\ \vec{T} \text{ flag on } T \\ \text{of length } n}} X_T(a,b) \twoheadrightarrow \mathfrak{C}_{\mathcal{V}}[X]_n(a,b),$$

where α is given by $X(f)$ and β is given by applying f to $\vec{T}$, for a necklace morphism $f : T \to U$. We thus have a canonical isomorphism

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) \simeq \operatorname*{colim}_{(T,\vec{T}) \in \mathcal{N}ec^{\rceil}[n]} X_T(a,b).$$

Now, as the inclusion $\mathcal{N}ec^{\rceil}_f[n] \hookrightarrow \mathcal{N}ec^{\rceil}[n]$ is a left-adjoint by Lemma 4.14, the corresponding functor between opposite categories is a right-adjoint and thus a final functor. Hence, the result follows. □

Remark 4.16 The simplicial structure of $\mathfrak{C}_{\mathcal{V}}[X](a,b) = \operatorname{colim}^{X_\bullet(a,b)} N\mathcal{P}_{(-)}$ is given by that of $N\mathcal{P}_T$, ie by deleting and copying terms in a flag, but the simplicial structure on $\operatorname{colim}_{(T,\vec{T}) \in \mathcal{N}ec^{\rceil}_f} X_T(a,b)$ is slightly more difficult. The degeneracy maps and inner face maps are still given by respectively copying and deleting terms in the flags. The outer face maps however are given by first deleting the term T_0 or T_n from a flag $(T_0, \dots, T_n)$ and then applying the flankification functor.

Let (X,S) be a templicial object with nondegenerate simplices and T a necklace, which we write as $\{0 = t_0 < t_1 < t_2 < \cdots < t_k = p\}$. Then let

$$X^{\mathrm{nd}}_T = X^{\mathrm{nd}}_{t_1} \otimes_S X^{\mathrm{nd}}_{t_2 - t_1} \otimes_S \cdots \otimes_S X^{\mathrm{nd}}_{p - t_{k-1}} \in \mathcal{V}\mathrm{Quiv}_S,$$

where X^{nd}_n denotes the quiver of nondegenerate simplices of Definition 2.16.

Proposition 4.17 *Let (X,S) be a templicial object with nondegenerate simplices. For all $n \geq 0$ and $a, b \in S$, we have an isomorphism in $\mathcal{V}$,*

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) \simeq \coprod_{\substack{T \in \mathcal{N}ec \\ \vec{T} \text{ flanked flag} \\ \text{of length } n}} X^{\mathrm{nd}}_T(a,b).$$

Proof By Proposition 4.15 and Lemma 2.19, we have an isomorphism

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) \simeq \operatorname*{colim}_{(T,\vec{T}) \in \mathcal{N}ec^{\rceil}_f[n]} \coprod_{\substack{f_i : [t_i - t_{i-1}] \twoheadrightarrow [n_i] \\ i \in \{1,\dots,k\}}} (X^{\mathrm{nd}}_{n_1} \otimes_S \cdots \otimes_S X^{\mathrm{nd}}_{n_k})(a,b),$$

where we've written $T = \{0 = t_0 < t_1 < \cdots < t_k = p\}$ for any $(T,\vec{T}) \in \mathcal{N}ec^{\rceil}_f[n]$. Now let $f : (T,p) \to (U,q)$ be an active necklace map whose underlying morphism $f : [p] \to [q]$ in $\mathbf{\Delta}_f$ is surjective. We can uniquely decompose $f = f_1 + \cdots + f_k$ with $f_i : [t_i - t_{i-1}] \twoheadrightarrow [n_i]$ in $\mathbf{\Delta}_{\mathrm{surj}}$ for all $i \in \{1, \dots, n\}$. Moreover, given a flag $\vec{T}$ of length n on T, there is a unique flanked flag $\vec{U} = (U_0, \dots, U_n)$ on U such that $f : T \to U$ lifts to a morphism $f : (T,\vec{T}) \to (U,\vec{U})$ in $\mathcal{N}ec^{\rceil}_f[n]$ (simply set $U_i = f(T_i)$). It follows that

$$\begin{aligned}
\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) &\simeq \operatorname*{colim}_{(T,\vec{T}) \in \mathcal{N}ec^{\rceil}_f[n]} \coprod_{(T,\vec{T}) \to (U,\vec{U}) \text{ in } \mathcal{N}ec^{\rceil}_f[n]} X^{\mathrm{nd}}_U(a,b) \\
&\simeq \coprod_{(U,\vec{U}) \in \mathcal{N}ec^{\rceil}_f[n]} \operatorname*{colim}_{(T,\vec{T}) \to (U,\vec{U}) \text{ in } \mathcal{N}ec^{\rceil}_f[n]} X^{\mathrm{nd}}_U(a,b) \simeq \coprod_{(U,\vec{U}) \in \mathcal{N}ec^{\rceil}_f[n]} X^{\mathrm{nd}}_U(a,b).
\end{aligned}$$

The last isomorphism is obtained by noting that the colimit on the left-hand side is indexed over the category $((\mathcal{N}ec_f^{\rceil}[n])_{/(U,\vec{U})})^{\mathrm{op}}$, which is connected, and the functor involved is constant on $X_U^{\mathrm{nd}}(a,b)$. □

4.D Comparison with the templicial nerve

Analogous to the classical homotopy coherent nerve, we show that the templicial homotopy coherent nerve $N_{\mathcal{V}}^{\mathrm{hc}}$ restricts to the templicial nerve $N_{\mathcal{V}}$ (see Section 2.B) when applied to ordinary $\mathcal{V}$-enriched categories.

Consider the $\mathcal{V}$-enriched left-adjoint $\pi_0\colon S\mathcal{V} \to \mathcal{V}$ to the functor $\underline{(-)}\colon \mathcal{V} \to S\mathcal{V}$ sending every object $V \in \mathcal{V}$ to the constant functor on V. Then, for any $Y \in S\mathcal{V}$, we have a reflexive coequaliser

(8)
$$\begin{tikzcd} Y_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y_0 \arrow[l, "s_0" description] \arrow[r, two heads] & \pi_0(Y). \end{tikzcd}$$

For example, if $\mathcal{V} = \mathrm{Set}$, then π_0 is the functor taking the set of connected components of a simplicial set.

As the monoidal product $\otimes$ of $\mathcal{V}$ preserves colimits in each variable, it follows that π_0 is strong monoidal and thus we have an induced adjunction

$$\begin{tikzcd} \mathcal{V}\mathrm{Cat}_{\Delta} \arrow[r, shift left=2, "\pi_0"] \arrow[r, phantom, "\perp"] & \mathcal{V}\mathrm{Cat}. \arrow[l, shift left=2, "\underline{(-)}"] \end{tikzcd}$$

Proposition 4.18 *We have a natural isomorphism*

$$N_{\mathcal{V}}^{\mathrm{hc}} \circ \underline{(-)} \simeq N_{\mathcal{V}}.$$

Proof By Proposition 3.16, $N_{\mathcal{V}} \simeq (-)^{\mathrm{temp}} \circ \underline{(-)}$ for $\underline{(-)}\colon \mathcal{V}\mathrm{Cat} \to \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$. Thus it suffices to show that we have an isomorphism $\mathfrak{n} \circ \underline{(-)} \simeq \underline{(-)}$ of functors $\mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. Take an object $A \in \mathcal{V}$ and write $[-,-]$ for the internal hom of $\mathcal{V}$. Since the simplicial set $N(\mathcal{P}_T)$ clearly only has one connected component, it follows from the fact that F preserves colimits that

$$[FN\mathcal{P}_T, \underline{A}] \simeq [\pi_0 FN\mathcal{P}_T, A] \simeq [F(\pi_0 N\mathcal{P}_T), A] \simeq [F(\{*\}), A] \simeq A$$

for all necklaces T. It follows that $\mathfrak{n}(\underline{A})$ is isomorphic to the constant functor on A. Clearly, this isomorphism is natural in A, as desired. □

Definition 4.19 It immediately follows from Proposition 4.18 that the templicial nerve $N_{\mathcal{V}}$ has a left-adjoint given by the composite

$$h_{\mathcal{V}} = \pi_0 \circ \mathfrak{C}_{\mathcal{V}}\colon S_{\otimes}\mathcal{V} \to \mathcal{V}\mathrm{Cat},$$

which we call the *homotopy category functor*.

Remark 4.20 By Remark 2.14, $h_{\mathcal{V}}$ necessarily recovers the classical homotopy category functor $h\colon \mathrm{SSet} \to \mathrm{Cat}$ when $\mathcal{V} = \mathrm{Set}$.

Corollary 4.21 *Let (X,S) be a templicial object with $a,b\in S$. Then we have a reflexive coequaliser*

$$\coprod_{\substack{T\in\mathcal{N}ec\\ T\neq\{0\}}} X_T(a,b) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{p>0} X_1^{\otimes p}(a,b) \twoheadrightarrow h_{\mathcal{V}}X(a,b)$$

with α and β induced by the unique active and inert necklace maps $([k],k)\to(T,p)$ and $([p],p)\to(T,p)$ of Remark 3.7, respectively, for any necklace (T,p) with k beads.

Proof It directly follows from Proposition 4.15 and (8) that, for all $a,b\in S$, we have the (reflexive) coequaliser

$$\operatorname*{colim}_{\substack{T\in\mathcal{N}ec_-\\ T\neq\{0\}}} X_T(a,b) \overset{\alpha'}{\underset{\beta'}{\rightrightarrows}} \operatorname*{colim}_{\substack{[p]\in\mathbf{\Delta}_{\mathrm{surj}}\\ p>0}} X_1^{\otimes p}(a,b) \twoheadrightarrow h_{\mathcal{V}}X(a,b), \tag{9}$$

where $\mathcal{N}ec_-$ denotes the subcategory of $\mathcal{N}ec$ consisting of all active necklace maps that are surjective on vertices, and α' and β' are defined similarly to α and β. Via the epimorphism $\coprod_T X_T(a,b)\twoheadrightarrow \operatorname{colim}_T X_T(a,b)$, we may replace the left-hand colimit by $\coprod_T X_T(a,b)$.

To show that we may also replace the right-hand colimit, observe that any surjective necklace map $f\colon([p],p)\twoheadrightarrow([q],q)$ with $q>0$ can be factored as an inert map $([p],p)\to(T,p)$ followed by some $\sigma\colon(T,p)\to([q],q)$ such that T has q beads and the unique active map $([q],q)\hookrightarrow(T,p)$ is a section of σ. Indeed, let $\sigma\colon[p]\twoheadrightarrow[q]$ be the underlying morphism of f in $\mathbf{\Delta}_f$. Then σ has a section δ in $\mathbf{\Delta}_f$. Now simply set $T=\delta([p])$. □

Recall the commutativity results for the templicial nerve (Proposition 2.15). While it follows immediately from Proposition 4.12 that $h_{\mathcal{V}}\circ\tilde{F}\simeq\mathcal{F}\circ h$, the homotopy category functors and forgetful functors do not commute in general, as the following example shows.

Example 4.22 Let $\mathcal{V}=\operatorname{Mod}(k)$ be the category of k-modules over with k an arbitrary unital commutative ring. Let $h_k=h_{\operatorname{Mod}(k)}$. Consider the templicial k-module $X=\tilde{F}(\partial\Delta^2)$. Then the hom-object $(h_kX)(0,2)\in\operatorname{Mod}(k)$ is isomorphic to

$$F(h(\partial\Delta^2)(0,2)) = F\Big(\Big\{\, {}_{0}\bullet\nearrow\overset{1}{\bullet}\searrow\bullet_{2},\ 0\bullet\!\longrightarrow\!\bullet 2 \,\Big\}\Big)\simeq k\oplus k.$$

On the other hand, note that each edge in $\tilde{U}(X)$ between two given vertices is uniquely determined by an element $a_i\in k$. So the set $h\tilde{U}(X)(0,2)$ consists of equivalence classes of sequences of edges $(a_1,\dots,a_n)$ from 0 to 2 in $\tilde{U}(X)$. One can check that

$$h\tilde{U}\tilde{F}(\partial\Delta^2)(0,2)\simeq U(k)\amalg_{U(0)}U(k),$$

which identifies a sequence $(a_1,\dots,a_n)$ with its product $a_n\cdots a_1$ in k. The two terms $U(k)$ correspond to paths either passing through the vertex 1 or not. Now the induced map $h\tilde{U}\tilde{F}(\partial\Delta^2)(0,2)\to U((h_kX)(0,2))$ on hom-sets corresponds to the canonical map

$$U(k)\amalg_{U(0)}U(k)\to U(k\oplus k),$$

which is certainly not a bijection if k is not the zero ring. Hence, the canonical functor

$$h\widetilde{U}(X) \to \mathcal{U}(h_k X)$$

is not an equivalence of categories.

In the next section, we will restrict to a special class of templicial objects, which we call quasicategories in $\mathcal{V}$. It turns out that, for a quasicategory X in $\mathcal{V}$, the canonical functor $h\widetilde{U}(X) \to \mathcal{U}(h_{\mathcal{V}} X)$ is always an isomorphism (under suitable hypotheses on $\mathcal{V}$); see Corollary 5.23 below.

5 Quasicategories in a monoidal category

Quasicategories are models for $(\infty, 1)$-categories first introduced by Joyal [22] as simplicial sets satisfying the weak Kan condition in the sense of Boardman and Vogt [8]. That is, a simplicial set X is a quasicategory if every simplicial map $\Lambda^n_j \to X$ from an inner horn can be extended to a map $\Delta^n \to X$ from the standard simplex. In [23], Joyal equips SSet with a model structure in which the fibrant objects are precisely the quasicategories. In this section, we introduce the natural analogue of quasicategories in the templicial context (Definition 5.4). However, in view of Example 2.10, we express the lifting condition in the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$, rather than $S_{\otimes}\mathcal{V}$. Nonetheless, we still recover classical quasicategories when $\mathcal{V} = \mathrm{Set}$ (Proposition 5.8). We continue in Section 5.B by showing our main result: that the templicial nerve produces quasicategories in $\mathcal{V}$ from locally Kan $S\mathcal{V}$-categories (Corollary 5.12). In Section 5.C, we discuss the homotopy category of a quasicategory in $\mathcal{V}$.

5.A Horn filling in necklaces

For integers $0 \leq j \leq n$, we denote by Λ^n_j the j^{th} horn of the n-simplex. That is, Λ^n_j is the union of all the faces of Δ^n, except the j^{th} face. In order to define quasicategories in $\mathcal{V}$, we wish to consider the usual horn lifting property in the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ via Construction 3.9. In this case, it is convenient to express the horn as a union of necklaces, rather than faces.

Proposition 5.1 *For all integers* $0 < j < n$,

$$(\Lambda^n_j)_\bullet(0,n) = \bigcup_{\substack{i=1 \\ i \neq j}}^{n-1} \delta_i(\Delta^{n-1})_\bullet(0,n) \cup \bigcup_{k=1}^{n-1} (\Delta^k \vee \Delta^{n-k})_\bullet(0,n)$$

as a subfunctor of $\Delta^n_\bullet(0,n)$ *in* $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}$.

Proof For all $0 < k, i < n$ with $i \neq j$, we have inclusions $\Delta^k \vee \Delta^{n-k} \subseteq \Lambda^n_j$ and $\delta_i(\Delta^{n-1}) \subseteq \Lambda^n_j$ in SSet. It follows that

$$\bigcup_{\substack{i=1 \\ i \neq j}}^{n-1} \delta_i(\Delta^{n-1})_\bullet(0,n) \cup \bigcup_{k=1}^{n-1} (\Delta^k \vee \Delta^{n-k})_\bullet(0,n) \subseteq (\Lambda^n_j)_\bullet(0,n).$$

Conversely, let $f\colon T \to (\Lambda^n_j)_{0,n}$ be a map in $\mathrm{SSet}_{*,*}$ with (T, p) a necklace. Suppose first that f is surjective on vertices. As the unique nondegenerate n-simplex of Δ^n is not contained in Λ^n_j, there must be some $k \in T$ such that $0 < f(k) < n$. Therefore, f factors through $\Delta^l \vee \Delta^{n-l}$ with $l = f(k)$. Now suppose that f is not surjective on vertices. Then f must factor through $\delta_i(\Delta^{n-1})$ for some $i \in [n] \setminus \{j\}$. As a map in $\mathrm{SSet}_{*,*}$, f always reaches the vertices 0 and n of Δ^n, and thus $0 < i < n$. □

Example 5.2 The outer horns aren't as well behaved in $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}$ as the inner horns. For example, Λ^2_0 is the pushout $\Delta^1 \amalg_{\{0\}} \Delta^1$ in SSet, but $(\Lambda^2_0)_\bullet(0,2)$ is isomorphic to just $\Delta^1_\bullet(0,1)$ as all maps $T \to (\Lambda^2_0)_{0,2}$ in $\mathrm{SSet}_{*,*}$ must factor through the edge $0 \to 2$ of Λ^2_0.

The following corollary expresses the advantage of working in the functor category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. While not every simplex of a templicial object is represented by a templicial morphism (see Example 2.10), it is represented by a morphism in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$.

Corollary 5.3 *Let (X, S) be a templicial object with $a, b \in S$.*

1. *Let (T, p) be a necklace. There is a bijective correspondence between morphisms $\widetilde{F}(T)_\bullet(0, p) \to X_\bullet(a, b)$ in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ and elements $\sigma \in U(X_T(a, b))$.*
2. *Let $0 < j < n$ be integers. There is a bijective correspondence between morphisms $\widetilde{F}(\Lambda^n_j)_\bullet(0, n) \to X_\bullet(a, b)$ in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ and elements*

$$x_k \in U((X_k \otimes_S X_{n-k})(a, b)) \quad \text{and} \quad y_i \in U(X_{n-1}(a, b))$$

for all $0 < k, i < n$ with $i \neq j$, which satisfy:

- *For all $0 < i < i' < n$ with $i \neq j \neq i'$,*

$$d_{i'-1}(y_i) = d_i(y_{i'}).$$

- *For all $0 < k < l < n$,*

$$(\mathrm{id}_{X_k} \otimes \mu_{l-k,n-l})(x_k) = (\mu_{k,l-k} \otimes \mathrm{id}_{X_{n-l}})(x_l).$$

- *For all $0 < k < n-1$ and $0 < i < n$ with $i \neq j$,*

$$\mu_{k,n-k-1}(y_i) = \begin{cases} (d_i \otimes \mathrm{id}_{X_{n-k-1}})(x_{k+1}) & \text{if } i \leq k, \\ (\mathrm{id}_{X_k} \otimes d_{i-k})(x_k) & \text{if } i > k. \end{cases}$$

Proof A morphism $F(T_\bullet(0, p)) \simeq \widetilde{F}(T)_\bullet(0, p) \to X_\bullet(a, b)$ is equivalent to a map $T_\bullet(0, p) \to U(X_\bullet(a, b))$ in $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}$, which corresponds to an element $\sigma \in U(X_T(a, b))$ by the Yoneda lemma. This shows (1). Statement (2) follows from Proposition 5.1. □

Definition 5.4 Let $Y : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$ be a functor. We say Y *lifts inner horns* if, for all $0 < j < n$, any lifting problem

$$\begin{tikzcd}
\widetilde{F}(\Lambda^n_j)_\bullet(0,n) \arrow[r] \arrow[d] & Y \\
\widetilde{F}(\Delta^n)_\bullet(0,n) \arrow[ur, dashed] &
\end{tikzcd}$$

has a solution in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. We say Y *lifts inner horns uniquely* if every such lifting problem has a unique solution in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$.

We call a templicial object (X, S) in $\mathcal{V}$ a *quasicategory in* $\mathcal{V}$ if the functor $X_\bullet(a,b)$ lifts inner horns for all $a, b \in S$. In this case, we will refer to the elements of S as the *objects* of X and to elements of $U(X_1(a,b))$ as the *morphisms* $a \to b$ in X.

Remark 5.5 Let $Y : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$ be a functor. By the adjunction $F \dashv U$, Y lifts inner horns in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ if and only if the composite $UY : \mathcal{N}ec^{\mathrm{op}} \to \mathrm{Set}$ lifts inner horns in $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}$.

As for ordinary quasicategories, there is an elementwise characterisation of quasicategories in $\mathcal{V}$, although it is bit more cumbersome to describe.

Proposition 5.6 *Let (X, S) be a templicial object. The following statements are equivalent*:

1. *X is a quasicategory in $\mathcal{V}$.*
2. *Let $a, b \in S$ and $0 < j < n$. For all collections of elements $(x_k)_{k=1}^{n-1}$, $(y_i)_{i=1,i\neq j}^{n-1}$ satisfying the conditions of Corollary 5.3(2), there is an element $z \in U(X_n(a,b))$ such that*

$$\mu_{k,n-k}(z) = x_k \quad \text{and} \quad d_i(z) = y_i$$

for all $0 < k, i < n$ with $i \neq j$.

Proof This immediately follows from Corollary 5.3. □

Remark 5.7 Note the similarities with the classical elementwise characterisation of a quasicategory. The elements y_i with $0 < i < n$, $i \neq j$ represent all inner faces of the horn Λ^n_j. They still have to satisfy the same conditions as usual. However, the two outer faces of the horn are replaced by the elements x_k with $0 < k < n$. The two new conditions of Corollary 5.3(2) merely express that these outer faces are glued to each other and to the inner faces in the appropriate way.

Indeed, when $\mathcal{V} = \mathrm{Set}$, we recover the classical notion of a quasicategory.

Proposition 5.8 *A simplicial set is a quasicategory if and only if it is a quasicategory in* Set (*in the sense of Definition 5.4*).

Proof Let X be a simplicial set, considered as a templicial set with X_0 its set of vertices. Then the assignment $(x_k)_{k=1}^{n-1} \mapsto (x^1_{n-1}, x^2_1)$ defines a bijection between the set of all collections of elements

$$(x_k = (x^1_k, x^2_k) \in X_k \times X_{n-k})_{k=1}^{n-1}$$

satisfying $(x_k^1, \mu_{l-k,n-l}(x_k^2)) = (\mu_{k,l-k}(x_k^1), x_l^2)$ for all $0<k<l<n$, and the set of all pairs $(y_n, y_0) \in X_{n-1}\times X_{n-1}$ satisfying $d_{n-1}(y_0) = d_0(y_n)$. It follows that condition (2) of Proposition 5.6 is equivalent to:

(2′) Let $0<j<n$. Consider elements $y_i \in X_{n-1}$ for all $0\le i\le n$ with $i\neq j$ which satisfy, for all $0\le i<i'\le n$ with $i\neq j\neq i'$,

$$d_{i'-1}(y_i) = d_i(y_{i'}).$$

Then there is an element $z\in X_n$ such that $d_i(z) = y_i$ for all $0\le i\le n$ with $i\neq j$.

But this precisely expresses that X is a quasicategory. □

5.B Nerves and quasicategories

We show that the earlier-defined templicial versions of classical nerves give examples of quasicategories in $\mathcal{V}$.

Lemma 5.9 *Let $\mathcal{C}$ be a necklace category with objects A and B. Consider the canonical morphism $\epsilon\colon \mathcal{C}_\bullet^{\mathrm{temp}}(A,B)\to\mathcal{C}_\bullet(A,B)$ induced by the counit of the adjunction $(-)^{\mathrm{nec}}\dashv(-)^{\mathrm{temp}}$. Given integers $0<j<n$, any lifting problem in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$*

$$\begin{tikzcd}
\widetilde{F}(\Lambda_j^n)_\bullet(0,n) \arrow[r] \arrow[d] & \mathcal{C}_\bullet^{\mathrm{temp}}(A,B) \arrow[d, "\epsilon"] \\
\widetilde{F}(\Delta^n)_\bullet(0,n) \arrow[r] \arrow[ur, dashed] & \mathcal{C}_\bullet(A,B)
\end{tikzcd}$$

has a unique solution.

Proof The top horizontal morphism corresponds to some collections of elements $(x_k)_{k=1}^{n-1}$ and $(y_i)_{i=1,i\neq j}^{n-1}$ with $x_k\in U((\mathcal{C}_k^{\mathrm{temp}}\otimes\mathcal{C}_{n-k}^{\mathrm{temp}})(a,b))$ and $y_i\in U(\mathcal{C}_{n-1}^{\mathrm{temp}}(a,b))$, satisfying the conditions of Corollary 5.3(2). Moreover, the bottom horizontal morphism corresponds to an element $z'\in U(\mathcal{C}_{\{0<n\}}(a,b))$ and the commutativity of the diagram comes down to the condition that $\mathcal{C}(\nu_{k,n-k})(z') = m_{\{0<k\},\{0<n-k\}}(p_k\otimes p_{n-k})(x_k)$ and $\mathcal{C}(\delta_i)(z') = p_{n-1}(y_i)$ for all $0<k,i<n$ with $i\neq j$.

Then, by the limit diagram (5), there exists a unique element $z\in U(\mathcal{C}_n^{\mathrm{temp}}(a,b))$ such that $\mu_{k,n-k}(z) = x_k$ for all $0<k<n$, and $p_n(z) = z'$. Again by (5), for all $0<k,i<n$ with $i\neq j$,

$$\begin{aligned}
\mu_{k,n-1-k}(d_i(z)) &= \begin{cases} (d_i\otimes \mathrm{id}_{\mathcal{C}_{n-k-1}^{\mathrm{temp}}})(\mu_{k+1,n-k}(z)) & \text{if } i\le k,\\ (\mathrm{id}_{\mathcal{C}_k^{\mathrm{temp}}}\otimes d_{i-k})(\mu_{k,n-k}(z)) & \text{if } i>k,\end{cases}\\
&= \mu_{k,n-1-k}(y_i),\\
p_{n-1}(d_i(z)) &= \mathcal{C}(\delta_i)p_n(z) = \mathcal{C}(\delta_i)(z') = p_{n-1}(y_i),
\end{aligned}$$

and thus $d_i(z) = y_i$. Hence, the element z determines the unique solution to the lifting problem. □

Proposition 5.10 *Let $\mathcal{C}$ be a necklace category with object set S. Suppose that, for all $A, B \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}_\bullet(A, B)$ lifts inner horns. Then $\mathcal{C}^{\mathrm{temp}}$ is a quasicategory in $\mathcal{V}$.*

Proof This is immediate from Lemma 5.9. □

Corollary 5.11 *For any small $\mathcal{V}$-category $\mathcal{C}$, the templicial object $N_{\mathcal{V}}(\mathcal{C})$ is a quasicategory in $\mathcal{V}$.*

Proof This follows from Propositions 3.16 and 5.10. □

Corollary 5.12 *Let $\mathcal{C}$ be small simplicial $\mathcal{V}$-category. Assume that, for all objects A and B of $\mathcal{C}$, the simplicial set $U(\mathcal{C}(A, B))$ is a Kan complex. Then the templicial object $N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})$ is a quasicategory in $\mathcal{V}$.*

Proof By Proposition 5.10, it suffices to check that, for all $A, B \in \mathrm{Ob}(\mathcal{C})$, the functor

$$\mathfrak{n}(\mathcal{C}(A, B))_\bullet = [FN\mathcal{P}_{(-)}, \mathcal{C}(A, B)] \colon \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$$

lifts inner horns in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. By the adjunction $\mathfrak{s} \dashv \mathfrak{n}$, this is equivalent to showing that, for all $0 < j < n$, every morphism $\mathfrak{s}(\widetilde{F}(\Lambda^n_j)_\bullet(0, n)) \to \mathcal{C}(A, B)$ in $S\mathcal{V}$ extends to $\mathfrak{s}(\widetilde{F}(\Delta^n)_\bullet(0, n))$. Now, by Proposition 4.12 and the adjunction $F \dashv U$, this is further equivalent to the lifting problem in SSet

$$\begin{tikzcd}
\mathfrak{C}[\Lambda^n_j](0, n) \arrow[r] \arrow[d] & U(\mathcal{C}(A, B)) \\
\mathfrak{C}[\Delta^n](0, n) \arrow[ur, dashed] &
\end{tikzcd}$$

This has a solution because $U(\mathcal{C}(A, B))$ is a Kan complex, as was shown in [28, Proposition 1.1.5.10] and is given in more detail in [30, Tag 00LH] (beware that, in the latter, the notation Path is used instead of $\mathfrak{C}$). □

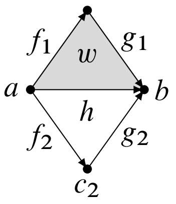
Corollary 5.13 *Let X be a quasicategory in $\mathcal{V}$. Then $\widetilde{U}(X)$ is a quasicategory.*

Proof This follows from Propositions 3.14, 5.8 and 5.10. □

The converse to Corollary 5.13 does not hold in general.

Example 5.14 Consider the over category $\mathcal{V} = \mathrm{Ab}/\mathbb{Z}$ of abelian groups A with a $\mathbb{Z}$-linear map $p \colon A \to \mathbb{Z}$. Then $\mathcal{V}$ is bicomplete and symmetric monoidal closed with monoidal unit given by $\mathrm{id}_{\mathbb{Z}} \colon \mathbb{Z} \to \mathbb{Z}$. The forgetful functor $U \colon \mathcal{V} \to \mathrm{Set}$ associates to every map $p \colon A \to \mathbb{Z}$ the set $\{a \in A \mid p(a) = 1\}$.

Now consider the simplicial set $\Delta^2 \amalg_{\{0,2\}} \Lambda^2_1$:



Set $X = \widetilde{F}(\Delta^2 \amalg_{\{0,2\}} \Lambda^2_1) \in S_{\otimes}\,\mathrm{Ab}$. We can promote X to a templicial object in $\mathcal{V}$ by equipping it with $\mathbb{Z}$-linear maps $p\colon X_n(x,y) \to \mathbb{Z}$ defined by

$$p(f_1) = p(f_2) = p(g_2) = p(h) = 1, \quad p(g_2) = 2 \quad \text{and} \quad p(w) = 1$$

Then, for example, $U(X_1(a,c_2)) = \{f_2\}$ but $U(X_1(c_2,b)) = \varnothing$. Consider the functor $\widetilde{U}\colon S_{\otimes}\mathcal{V} \to \mathrm{SSet}$ as induced by U above (not by $\mathrm{Ab} \to \mathrm{Set}$). Then $\widetilde{U}(X) \simeq \Delta^2 \amalg_{\{0\}} \Delta^1$, which is clearly a quasicategory. However, X is not a quasicategory in $\mathcal{V}$. To see this, consider the element

$$\alpha = f_2 \otimes g_2 - f_1 \otimes g_1 \in U((X_1 \otimes X_1)(a,b))$$

(note that, indeed, $(p \otimes p)(\alpha) = p(f_2)p(g_2) - p(f_1)p(g_1) = 1$). But there exists no element $\xi \in U(X_2(a,b))$ such that $\mu_{1,1}(\xi) = \alpha$.

We end this subsection by characterising the essential image of the templicial nerve functor $N_{\mathcal{V}}\colon \mathcal{V}\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ in terms of horn fillings.

Proposition 5.15 *Let $(X,S) \in S_{\otimes}\mathcal{V}$. Consider the following statements*:

(1) *(X,S) is isomorphic to the templicial nerve of a small $\mathcal{V}$-category.*

(2) *For all $a,b \in S$, $X_{\bullet}(a,b)$ lifts inner horns uniquely.*

Then (1) *implies* (2). *Moreover*, *if the functor $U\colon \mathcal{V} \to \mathrm{Set}$ is conservative*, *then* (1) *and* (2) *are equivalent.*

Proof Let $\mathcal{C}$ be a small $\mathcal{V}$-category. We wish to show that $N_{\mathcal{V}}(\mathcal{C})_{\bullet}(A,B)$ lifts inner horns uniquely for all $A, B \in \mathrm{Ob}(\mathcal{C})$. Since $N_{\mathcal{V}} \simeq (-)^{\mathrm{temp}} \circ \underline{(-)}$ (Proposition 3.16), it suffices by Lemma 5.9 to note that the lifting problem

$$\begin{tikzcd}
\widetilde{F}(\Lambda^n_j)_{\bullet}(0,n) \arrow[r] \arrow[d] & \underline{\mathcal{C}}(A,B) \\
\widetilde{F}(\Delta^n)_{\bullet}(0,n) \arrow[ur, dashed] &
\end{tikzcd}$$

has a unique solution for all $0 < j < n$, which is clear.

Assume that (2) holds and that U is conservative. By (3), it suffices to show that each comultiplication morphism $\mu_{k,n-k}$ with $0 < k < n$ is an isomorphism. Take $x_k \in U(X_k \otimes_S X_{n-k})$. By induction on n, we can define, for any $0 < l < n$ with $l \neq k$,

$$x_l = \begin{cases} (\mathrm{id}_{X_l} \otimes \mu^{-1}_{k-l,n-k})(\mu_{l,k-l} \otimes \mathrm{id}_{X_{n-k}})(x_k) & \text{if } l < k, \\ (\mu^{-1}_{k,l-k} \otimes \mathrm{id}_{X_{n-l}})(\mathrm{id}_{X_k} \otimes \mu_{l-k,n-l})(x_k) & \text{if } l > k. \end{cases}$$

Further set, for all $0 < i < n$ with $i \neq k$,

$$y_i = \begin{cases} \mu^{-1}_{k-1,n-k}(d_i \otimes \mathrm{id}_{X_{n-k}})(x_k) & \text{if } i < k, \\ \mu^{-1}_{k,n-k-1}(\mathrm{id}_{X_k} \otimes d_{i-k})(x_k) & \text{if } i > k. \end{cases}$$

It follows that the elements $(x_l)_{l=1}^{n-1}$ and $(y_i)_{i=1,i\neq k}^{n-1}$ satisfy the conditions of Corollary 5.3(2) and thus there is a unique element $z \in U(X_n(a,b))$ such that $\mu_{l,n-l}(z) = x_l$ and $d_i(z) = y_i$ for all $0 < l, i < n$ with $i \neq k$. In particular, $\mu_{k,n-k}(z) = x_k$. For any other $z' \in U(X_n(a,b))$ with $\mu_{k,n-k}(z') = x_k$, it follows from the definitions of the x_l and y_i that also $\mu_{l,n-l}(z') = x_l$ and $d_i(z') = y_i$ for all $0 < l, i < n$ with $i \neq k$. Thus $z' = z$ and hence the map

$$U(\mu_{k,n-k})\colon U(X_n(a,b)) \to U((X_k \otimes X_{n-k})(a,b))$$

is a bijection. As U is conservative, $\mu_{k,n-k}\colon X_n \to X_k \otimes X_{n-k}$ is an isomorphism of $\mathcal{V}$-enriched quivers. □

5.C Simplification of the homotopy category

We now turn our attention to the homotopy category $h_{\mathcal{V}}X$ when X is a quasicategory in $\mathcal{V}$. As is the case in the classical situation, this allows for a simpler description of $h_{\mathcal{V}}X$.

Construction 5.16 Let (X, S) be a templicial object and $a, b \in S$. We define an object $\mathrm{Hom}^L_X(a,b)_1 \in \mathcal{V}$ by the pullback

$$\begin{tikzcd}
\mathrm{Hom}^L_X(a,b)_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X_2(a,b) \arrow[d, "\mu^X_{1,1}"] \\
X_1(a,b) \arrow[r, "-\otimes s^X_0"'] & (X_1 \otimes_S X_1)(a,b)
\end{tikzcd}$$

Further, we let $d_1 = \pi_1$, $d_0 = d^X_1 \pi_2$ and we let $s_0\colon X_1(a,b) \to \mathrm{Hom}^L_X(a,b)_1$ be the unique morphism such that $\pi_1 s_0 = \mathrm{id}_{X_1(a,b)}$ and $\pi_2 s_0 = s^X_1$. We obtain a reflexive pair

$$\begin{tikzcd}
\mathrm{Hom}^L_X(a,b)_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_1(a,b) \arrow[l, "s_0" description]
\end{tikzcd}$$

and we define an object $h'_{\mathcal{V}}X(a,b)$ as the coequaliser of this pair,

$$\begin{tikzcd}
\mathrm{Hom}^L_X(a,b)_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_1(a,b) \arrow[r, two heads, "q"] & h'_{\mathcal{V}}X(a,b)
\end{tikzcd} \tag{10}$$

Remark 5.17 It is possible to extend Construction 5.16 to obtain a simplicial object $\mathrm{Hom}^L_X(a,b)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{V}$ which generalises the *left-pinched morphism space* of a simplicial set (as defined in [30, Tag 01KX]). In particular, $\mathrm{Hom}^L_X(a,b)_0 = X_1(a,b)$. Then the morphisms $d_0, d_1\colon \mathrm{Hom}^L_X(a,b)_1 \rightrightarrows X_1(a,b)$ and $s_0\colon X_1(a,b) \to \mathrm{Hom}^L_X(a,b)_1$ constitute the lowest-dimensional face and degeneracy morphisms of $\mathrm{Hom}^L_X(a,b)$. We will not go into them here however, and leave their investigation to later research.

Remark 5.18 As U preserves pullbacks, $U(\mathrm{Hom}^L_X(a,b)_1)$ is the set of all 2-simplices $\sigma \in \widetilde{U}(X)$ with $d_0(\sigma) = s_0(b)$ and $d_1 d_2(\sigma) = a$. In other words, it describes homotopies between two edges $a \to b$ in $\widetilde{U}(X)$.

Assuming that $\widetilde{U}(X)$ is a quasicategory and that U preserves reflexive coequalisers, it follows that we have an isomorphism

$$U(h'_{\mathcal{V}}X(a,b)) \simeq h\widetilde{U}(X)(a,b)$$

and the canonical morphism $X_1(a,b) \twoheadrightarrow h'_{\mathcal{V}}X(a,b)$ precisely takes the homotopy class $[f]$ in $h\widetilde{U}(X)$ of any $f \in U(X_1(a,b))$.

Lemma 5.19 *Assume that* $U\colon \mathcal{V} \to \mathrm{Set}$ *preserves reflexive coequalisers. Let* X *be a quasicategory in* $\mathcal{V}$ *with objects* a *and* b*. For any* $w, w' \in U(X_2(a,b))$,

$$(q\otimes q)\mu_{1,1}(w) = (q\otimes q)\mu_{1,1}(w') \implies q(d_1^X(w)) = q(d_1^X(w'))$$

in $h'_{\mathcal{V}}X(a,b)$.

Proof Let Q be the quiver given by $\mathrm{Hom}^L_X(a,b)_1$ for all objects a and b of X. Let $\sigma \in U((Q\otimes Q)(a,b))$ and $w, w' \in U(X_2(a,b))$ be such that $\mu_{1,1}(w) = (d_0\otimes d_0)(\sigma)$ and $\mu_{1,1}(w') = (d_1\otimes d_1)(\sigma)$. Then:

- Consider $x_1 = (d_1 \otimes s_0^X d_1)(\sigma) \in U((X_1\otimes X_2)(a,b))$, $x_2 = (\pi_2\otimes d_1)(\sigma) \in U((X_2\otimes X_1)(a,b))$ and $y_2 = w \in U(X_2(a,b))$. These define a morphism $\widetilde{F}(\Lambda^3_1)_\bullet(0,3) \to X_\bullet(a,b)$, which extends to an element $z \in U(X_3(a,b))$. Setting $w'' = d_1^X(z) \in U(X_2(a,b))$, we have $d_1^X(w'') = d_1^X(w)$.
- Similarly, consider $x_1 = (d_0\otimes\pi_2)(\sigma) \in U((X_1\otimes X_2)(a,b))$, $x_2 = w''\otimes s_0^X(b) \in U((X_2\otimes X_1)(a,b))$ and $y_2 = w'$. These define a morphism $\widetilde{F}(\Lambda^3_1)_\bullet(0,3) \to X_\bullet(a,b)$, which extends to an element $z \in U(X_3(a,b))$. Then set $\tau = d_1^X(z) \in U(X_2(a,b))$.

It follows that $\mu_{1,1}(\tau) = d_1^X(w)\otimes s_0^X(b)$ and $d_1^X(\tau) = d_1^X(w')$. Hence, $qd_1^X(w) = qd_1^X(w')$.

As the diagram (10) is a reflexive coequaliser, it is preserved by $-\otimes-$ in both variables simultaneously, so that we again have a reflexive coequaliser

$$\begin{tikzcd}(Q\otimes Q)(a,b) \arrow[r, shift left, "d_0\otimes d_0"] \arrow[r, shift right, "d_1\otimes d_1"'] & (X_1\otimes X_1)(a,b) \arrow[r, two heads, "q\otimes q"] & (h'_{\mathcal{V}}X\otimes h'_{\mathcal{V}}X)(a,b)\end{tikzcd}$$

Now assume that $(q\otimes q)\mu_{1,1}(w) = (q\otimes q)\mu_{1,1}(w')$. As U preserves reflexive coequalisers, there exist $\alpha_0,\dots,\alpha_n \in U((X_1\otimes X_1)(a,b))$ such that $\mu_{1,1}(w) = \alpha_0$, $\alpha_n = \mu_{1,1}(w')$ and, for all $i \in \{1,\dots,n\}$, there exists a $\sigma \in U((Q\otimes Q)(a,b))$ such that

$$\alpha_{i-1} = (d_0\otimes d_0)(\sigma) \text{ and } (d_1\otimes d_1)(\sigma) = \alpha_i \quad \text{or} \quad \alpha_{i-1} = (d_1\otimes d_1)(\sigma) \text{ and } (d_0\otimes d_0)(\sigma) = \alpha_i.$$

For every $0 < i < n$, α_i defines a horn $\widetilde{F}(\Lambda^2_1)_\bullet(0,2) \to X_\bullet(a,b)$, which we can extend to an element $w_i \in U(X_2(a,b))$ so that $\mu_{1,1}(w_i) = \alpha_i$. Thus, it follows by the previous that

$$qd_1(w) = qd_1(w_1) = \cdots = qd_1(w_{n-1}) = qd_1(w'). \qquad \square$$

Lemma 5.20 *Assume that* $U\colon \mathcal{V}\to \mathrm{Set}$ *is faithful. Let* $g\colon X\to Y$ *and* $f\colon X\to Z$ *be morphisms in* $\mathcal{V}$ *such that* g *is a regular epimorphism. Suppose that, for all* $x, y \in U(X)$,

$$g(x) = g(y) \implies f(x) = f(y).$$

Then there exists a unique morphism $h\colon Y\to Z$ *such that* $hg = f$.

Proof Denote the kernel pair $X\times_Y X \rightrightarrows X$ of g by π_1 and π_2. Since g is the coequaliser of this pair, it suffices to show that $f\pi_1 = f\pi_2$. As U is faithful, this is equivalent to showing that, for all $(x,y) \in U(X)\times_{U(Y)} U(X)$, we have $f(x) = f(y)$. But this is equivalent to the hypothesis on f and g. $\square$

Construction 5.21 Assume that $U\colon \mathcal{V} \to \mathrm{Set}$ is faithful and preserves and reflects reflexive coequalisers. Let (X, S) be a quasicategory in $\mathcal{V}$. We construct a $\mathcal{V}$-enriched category $h'_{\mathcal{V}}X$ whose hom-objects are given by $h'_{\mathcal{V}}X(a,b)$ of Construction 5.16. Let $h'_{\mathcal{V}}X$ denote the quiver given by $h'_{\mathcal{V}}X(a,b)$ for all $a, b \in S$, and let $q\colon X_1 \to h'_{\mathcal{V}}X$ denote the canonical quiver morphism.

First define $u\colon I_S \xrightarrow{s_0} X_1 \xrightarrow{q} h'_{\mathcal{V}}X$. Note that U also reflects regular epimorphisms (as they are the coequaliser of their kernel pair). Thus, as X is a quasicategory in $\mathcal{V}$, the comultiplication $\mu_{1,1}\colon X_2 \to X_1 \otimes_S X_1$ is a regular epimorphism. Further, q is a regular epimorphism by definition. Now $- \otimes -$ preserves reflexive coequalisers in each variable and thus also regular epimorphisms. It follows that $q^{\otimes 2} \circ \mu_{1,1}$ is a regular epimorphism as well. Using Lemmas 5.19 and 5.20, we have a unique quiver morphism $m\colon h'_{\mathcal{V}}X \otimes_S h'_{\mathcal{V}}X \to h'_{\mathcal{V}}X$ such that the following diagram commutes:

$$\begin{tikzcd}
X_2 \arrow[r, "\mu_{1,1}"] \arrow[dr, "d_1"'] & X_1^{\otimes 2} \arrow[r, "q^{\otimes 2}"] & (h'_{\mathcal{V}}X)^{\otimes 2} \arrow[d, "m"] \\
& X_1 \arrow[r, "q"'] & h'_{\mathcal{V}}X
\end{tikzcd}$$

Given a 2-simplex $(\alpha_{i,j})_{1 \leq i < j \leq 2}$ (see Remark 2.9) of $\widetilde{U}(X)$ with vertices a, b and c, we have $\mu_{1,1}(\alpha_{02}) = \alpha_{01} \otimes \alpha_{12}$ and thus $m(q(\alpha_{01}) \otimes q(\alpha_{02})) = q(d_1(\alpha_{02}))$. Therefore, the induced map

$$U(h'_{\mathcal{V}}X(a,b)) \times U(h'_{\mathcal{V}}X(b,c)) \to U(h'_{\mathcal{V}}X(a,b) \otimes h'_{\mathcal{V}}X(b,c)) \xrightarrow{U(m_{a,b,c})} U(h'_{\mathcal{V}}X(a,c))$$

coincides with the composition law of $h\widetilde{U}(X)$ under the isomorphisms supplied by Remark 5.18. The element $u_a = q(s_0(a))\colon I \to h'_{\mathcal{V}}X(a,a)$ is clearly the identity at a in $h\widetilde{U}(X)$. It then follows from the faithfulness of U that m is associative and unital with respect to u. So we obtain a $\mathcal{V}$-category $h'_{\mathcal{V}}X$.

Note that, by construction, we have an isomorphism of categories

$$\mathcal{U}(h'_{\mathcal{V}}X) \simeq h\widetilde{U}(X).$$

Proposition 5.22 *Assume that* $U\colon \mathcal{V} \to \mathrm{Set}$ *is faithful and preserves and reflects reflexive coequalisers. The assignment* $X \mapsto h'_{\mathcal{V}}X$ *of Construction 5.21 extends to a functor* $h'_{\mathcal{V}}$ *from the full subcategory of* $S_{\otimes}\mathcal{V}$ *spanned by all quasicategories in* $\mathcal{V}$ *to* $\mathcal{V}\mathrm{Cat}$, *which is left-adjoint to the templicial nerve functor* $N_{\mathcal{V}}$.

In particular, there exists a canonical isomorphism of $\mathcal{V}$*-enriched categories*

$$h_{\mathcal{V}}X \simeq h'_{\mathcal{V}}X$$

for every quasicategory X *in* $\mathcal{V}$.

Proof It follows from Construction 5.21 and Lemma 2.12 that we have a unique templicial morphism $\eta_X\colon X \to N_{\mathcal{V}}(h'_{\mathcal{V}}X)$ such that $\eta_{X_1}\colon X_1 \to h'_{\mathcal{V}}X$ is precisely q. We claim that η_X is the unit of an adjunction $h'_{\mathcal{V}} \dashv N_{\mathcal{V}}$.

Now let $\mathcal{C}$ be an arbitrary small $\mathcal{V}$-category and $(\zeta, f)\colon X \to N_{\mathcal{V}}(\mathcal{C})$ a templicial morphism. Then, by Lemma 2.12, $\zeta\colon f_!X \to N_{\mathcal{V}}(\mathcal{C})$ corresponds to a quiver morphism $H\colon X_1 \to f^*(\mathcal{C})$ such that the diagrams (4) commute. Letting Q denote the quiver given by $\mathrm{Hom}^L_X(a,b)_1$ for all objects a and b of X, we have a commutative diagram

$$\begin{tikzcd}
Q \arrow[r, "\pi_2"'] \arrow[rr, bend left, "d_0"] \arrow[d, "d_1=\pi_1"'] & X_2 \arrow[r, "d_1^X"'] \arrow[d, "\mu^X_{1,1}"] & X_1 \arrow[r, "H"] & f^*(\mathcal{C}) \\
X_1 \arrow[r, "-\otimes s_0^X"] \arrow[rr, bend right, "H\otimes u"'] & X_1^{\otimes 2} \arrow[r, "H^{\otimes 2}"] & f^*(\mathcal{C})^{\otimes 2} \arrow[r] & f^*(\mathcal{C}^{\otimes 2}) \arrow[u, "f^*(m_{\mathcal{C}})"']
\end{tikzcd}$$

It follows that $Hd_0 = Hd_1 \colon Q \to f^*(\mathcal{C})$ and thus there exists a unique quiver morphism $H' \colon h'_{\mathcal{V}}X \to f^*(\mathcal{C})$ such that $H'q = H$. By construction, H' defines a $\mathcal{V}$-functor $h'_{\mathcal{V}}X \to \mathcal{C}$, which is clearly unique, such that $N_{\mathcal{V}}(H) \circ \eta_X = (\zeta, f)$. □

Corollary 5.23 *Assume that* $U : \mathcal{V} \to \mathrm{Set}$ *is faithful and preserves and reflects reflexive coequalisers. Let* X *be a quasicategory in* $\mathcal{V}$*. The canonical functor*

$$h\widetilde{U}(X) \to \mathcal{U}(h_{\mathcal{V}}X)$$

is an isomorphism of categories.

Proof This is now an immediate consequence of Proposition 5.22 and the fact that $\mathcal{U}(h'_{\mathcal{V}}X) \simeq h\widetilde{U}(X)$. □

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