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Closures of T-homogeneous braids are real algebraic

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A link in S^3 is called real algebraic if it is the link of an isolated singularity of a polynomial map from $\mathbb{R}^4$ to $\mathbb{R}^2$. It is known that every real algebraic link is fibered and it is conjectured that the converse is also true. We prove this conjecture for a large family of fibered links, which includes closures of both T-homogeneous (and therefore also homogeneous) braids and braids that can be written as a product of the dual Garside element and a positive word in the Birman–Ko–Lee presentation. The proof offers a construction of the corresponding real polynomial maps, which can be written as semiholomorphic functions. We obtain information about their polynomial degrees.

57K10; 14P25, 32S55

1 Introduction

It is well known that the set of algebraic links, the links that arise as the links of isolated singularities of complex polynomials, consists of certain iterated cables of torus links; see Brauner [15; 16; 17]. In the last chapter of his seminal work [25], Milnor discusses the real analogue of this classification problem. Let $f:\mathbb{R}^4\to\mathbb{R}^2$ be a polynomial map with $f(0)=0$ whose Jacobian Df vanishes at the origin, but has full rank at all other points of some neighborhood of the origin. In this case, we say that the origin is an *isolated singularity* of f. Exactly as in the complex case, the intersection $f^{-1}(0)\cap S_\rho^3$ of the vanishing set of f and a 3-sphere of sufficiently small radius $\rho>0$ is a link, whose link type does not depend on ρ. We call this link the *link of the singularity* and denote it by L_f.

Analogous to the terminology from the complex setting, we say that a link L is *real algebraic* if $L=L_f$ for some polynomial f. In contrast to the set of algebraic links, the set of real algebraic links has not been classified yet. Note that a generic real polynomial in these dimensions does not have an isolated singularity. The difficulty of constructing polynomials with isolated singularities and a given link type has already been noted by Milnor [25].

While there are many differences between the complex and the real setting, the importance of (Milnor) fibrations can be found in both. In particular, Milnor [25] showed that every real algebraic link is fibered. It is an open conjecture by Benedetti and Shiota [6] that the set of real algebraic links and the set of fibered links are identical. In addition to the complex algebraic links, which are obviously also real algebraic, several families of fibered links have been proved to be real algebraic. However, compared with the set of all fibered links, these families are still comparatively small. Looijenga [23] proved that every fibered link that satisfies a certain "odd" symmetry is real algebraic. This includes the connected sum $K \# K$ of any fibered knot K with itself. Answering a question of Milnor [25], Perron [28] constructed a real polynomial map that realizes the figure-eight knot as a real algebraic knot. Later, Rudolph [32] came up with a different polynomial for the figure-eight knot and gave the example of certain unions of complex algebraic links (up to orientation or mirror reflections). These were then studied in more generality by Pichon [29], who proved that $L_f \cup \overline{L}_g$ is real algebraic if and only if it is fibered. Here the overline denotes a reversal of orientation and L_f and L_g are links of singularities of holomorphic polynomials f and g. The corresponding real polynomial map can be written as $f\bar{g}\colon \mathbb{R}^4 \cong \mathbb{C}^2 \to \mathbb{C} \cong \mathbb{R}^2$, where $\bar{g}$ is the complex conjugate of g. In earlier work [9], we proved that closures of braids of the form B^2 for any homogeneous braid B are real algebraic. Examples of real polynomial maps with isolated singularities have also been constructed by A'Campo [1] and Seade [35; 34]. Another result that can be interpreted as evidence for Benedetti and Shiota's conjecture is due to Kauffman and Neumann [22]. It states that if we consider real analytic maps with so-called tame singularities instead of polynomial maps, then the conjecture is true.

The main result of this paper is a construction of certain polynomials with isolated singularities, which allows us to prove that a large family of fibered links is real algebraic.

All links in this family are closures of so-called *P-fibered braids* or, equivalently, bindings of *braided open book decompositions*, which can be described in terms of certain diagrams called *Rampichini diagrams*; see Bode [12], Morton and Rampichini [26] and Rampichini [30]. We introduce an operation on Rampichini diagrams detailed in the later sections. We say that one P-fibered braid B_2 is obtained from another B_1 by the insertion of *inner loops* if their Rampichini diagrams are related by such operations. The family of fibered links for which we can prove real algebraicity can be characterized in terms of properties of their Rampichini diagrams.

Theorem 1.1 *Let B_1 be a P-fibered braid on n strands with an odd, pure Rampichini diagram. Let B_2 be a P-fibered braid that is obtained from B_1 by the insertion of inner loops. Then the closure of B_2 is real algebraic.*

Furthermore, the corresponding real polynomial map with an isolated singularity can be taken to be semiholomorphic (ie it can be written as a polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ in complex variables u, v and the complex conjugate $\bar{v}$) and of degree n with respect to the complex variable u.

The properties of being odd and pure will be defined at a later point (Definitions 3.5 and 2.6). It is not straightforward to decide if a given link satisfies the condition from Theorem 1.1. However, for large families of fibered links we prove that this is the case.

Theorem 1.2 *Let B be a T-homogeneous braid. Then the closure of B is real algebraic and the corresponding polynomial can be taken to be semiholomorphic.*

The family of T-homogeneous braids (see Definition 2.11) was introduced by Rudolph [33] as a generalization of the family of homogeneous braids (whose closures are known to be fibered; see Stallings [36]), so we immediately obtain the following corollary.

Corollary 1.3 *Let B be a homogeneous braid. Then the closure of B is real algebraic and the corresponding polynomial can be taken to be semiholomorphic.*

While previous constructions resulted in links that are unions of complex algebraic links (up to orientation or mirror reflection) (see Rudolph [32] and Pichon [29]) or in links that satisfy, in addition to their fiberedness, certain symmetry properties (being "odd" in the case of Looijenga [23], or the closure of the square of a braid in the case of Perron's [28] and Rudolph's figure-eight knot [32] as well as in our work [9; 13]), Theorem 1.2 and Corollary 1.3 have no such constraints. To be precise, together [33; 12] imply that every T-homogeneous braid is P-fibered. Combining this with [9; 13] shows that the closure of B^2 is real algebraic for every T-homogeneous braid B. Theorem 1.2 goes beyond that by stating that it is not necessary to square the braid. Since in general closures of T-homogeneous braids are neither odd nor 2-periodic nor are all of their components (mirror images of) complex algebraic knots, this yields a much larger family than previous constructions.

In the context of braided open books and Rampichini diagrams it is useful to represent braids in terms of the BKL-generators $a_{i,j}$ of Birman, Ko and Lee [8], also called band generators, instead of Artin generators σ_i. A word in BKL-generators is called BKL-positive if it does not include inverses of any of the generators $a_{i,j}$. Let $\delta = a_{1,2}a_{2,3}\dots a_{n-1,n} = \sigma_1\sigma_2\dots\sigma_{n-1}$ denote the dual Garside element [8].

Theorem 1.4 *Let $B = \delta P$, where P is a BKL-positive word. Then the closure of B is real algebraic.*

Banfield [4] showed that closures of braids of the form $\delta' P$, where $\delta' = a_{n-1,n}a_{n-2,n-1}\dots a_{1,2} = \sigma_{n-1}\sigma_{n-2}\dots\sigma_1$ and P is a BKL-positive braid, are fibered. The same arguments apply to δP. Likewise, [8] defines the dual Garside normal form using δ' as the dual Garside element. But the same arguments can be used to define a dual Garside normal form in terms of δ, which is for example used by Ito and Wiest [21]. This means that in the BKL-presentation every braid on n strands can be written as $\delta^k P$ for some $k \in \mathbb{Z}$, where $P = A_1 A_2 \dots A_m$ is BKL-positive. This representation becomes unique for every given braid once appropriate conditions on the factors A_i are imposed [8]. We then call $\delta^k P$ the dual Garside normal form of the braid. Since $\delta^{k-1} P$ is BKL-positive if $k \geq 1$, Theorem 1.4 immediately implies the following corollary.

Corollary 1.5 *Let B be a braid whose dual Garside normal form contains a positive power of the dual Garside element δ. Then the closure of B is real algebraic and the corresponding polynomial can be taken to be semiholomorphic.*

The remainder of the paper is structured as follows. Section 2 reviews the relevant concepts in the study of P-fibered braids and Rampichini diagrams. We also introduce the definitions of inner loops and pure Rampichini diagrams, which feature in Theorem 1.1. Section 3 introduces odd Rampichini diagrams and proves a result about corresponding (singular) P-fibered braids with a certain "odd" symmetry. The proof of Theorem 1.1 then follows a similar strategy as the proof of our main result in [11], which constructs weakly isolated singularities (as opposed to isolated singularities) for any link type. In comparison to the work in [11], the individual steps require much more care as the fibration properties and isolation of the singularity are very strong conditions. We first use trigonometric interpolation and approximation to find a loop in the space of monic, complex polynomials h_t of fixed degree, whose coefficients are finite Fourier series and whose roots form a singular braid B_{sing} that is related to the braids B_1 and B_2 and that has to satisfy the "odd" symmetry requirement (see Section 4). From this we construct a radially weighed homogeneous polynomial that is degenerate in the sense of Oka [27] or Araújo dos Santos, Bode and Sanchez Quiceno [2] and whose zeros form the cone over the singular braid B_{sing}. In Section 5 we show that trigonometric interpolation techniques can also be used to find a deformation of this polynomial that has an isolated singularity and whose link is obtained from the singular braid by an appropriate resolution of its singular crossings. In Section 6 the resulting link is shown to be the closure of the desired braid B_2, which concludes the proof of Theorem 1.1. Then Theorems 1.2 and 1.4 along with their corollaries are shown in Section 7.

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2 Background

2.1 Braids

A geometric braid on n strands is a loop in the configuration space of n distinct unmarked points in the complex plane. Via the fundamental theorem of algebra this space is identified with the space X_n of monic polynomials in one complex variable of degree n and with distinct roots. This way a braid parametrized as $(z_1(t), z_2(t), \dots, z_n(t))$, where t is going from 0 to 2π, corresponds to the loop $g_t(u) := \prod_{j=1}^{n}(u - z_j(t))$. We call an equivalence class of geometric braids under braid isotopies a braid. Braid isotopies correspond to homotopies of loops (with fixed basepoint) in X_n, so the braid group $\mathbb{B}_n$ on n strands can be defined to be the fundamental group of X_n. It is common to visualize geometric braids as the set of parametric curves $\bigcup_{j=1}^{n}(z_j(t), t)$ in $\mathbb{C} \times [0, 2\pi]$.

Definition 2.1 We say that a geometric braid B is *P-fibered* if the corresponding loop of polynomials g_t defines a fibration via $\arg(g)\colon (\mathbb{C} \times S^1) \setminus B \to S^1$, $g(u, \mathrm{e}^{\mathrm{i}t}) := g_t(u)$. We say that a braid is P-fibered if it can be represented by a P-fibered geometric braid.

The condition of being P-fibered has the following geometric interpretation in terms of the critical values $v_j(t)$ for $j = 1, 2, \dots, n-1$ of $g_t\colon \mathbb{C} \to \mathbb{C}$. Since the roots of g_t are simple, all of its critical values $v_j(t)$ are nonzero. A geometric braid is P-fibered if and only if, for every $j = 1, 2, \dots, n-1$, the derivative $\partial \arg(v_j(t))/\partial t$ never vanishes. Thus, as t increases from 0 to 2π, every critical value twists around $0 \in \mathbb{C}$

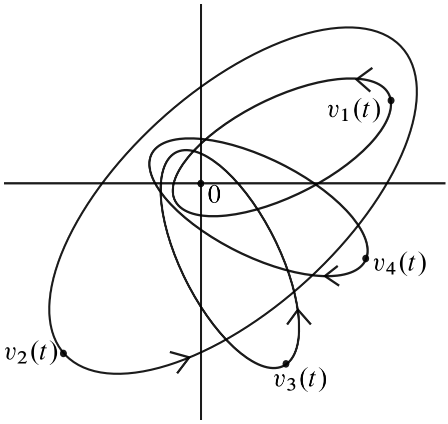

Figure 1: The motion in the complex plane of the critical values $v_j(t)$ of the loop of polynomials g_t corresponding to a P-fibered geometric braid.

in a fixed direction, either clockwise ($\partial\arg(v_j(t))/\partial t < 0$) or counterclockwise ($\partial\arg(v_j(t))/\partial t > 0$). This is illustrated in Figure 1.

The closure of a P-fibered braid B is a fibered link in the 3-sphere. Via this closing procedure (described in more detail in [12]), the braided surface $F_\varphi = \arg(g_t)^{-1}(\varphi)$ is identified with a fiber surface, whose boundary is the closure of B. A fibration of a link complement over a circle that comes from a P-fibered geometric braid and the corresponding loop of polynomials is also called a *braided open book decomposition* of S^3 [12].

Instead of the more common Artin presentation of the braid group $\mathbb{B}_n$, we will mostly work with the presentation of Birman, Ko and Lee [8], where the generators are $a_{i,j}$ for $1 \le i < j \le n$ and the relations are

$$a_{i,j}a_{k,m} = a_{k,m}a_{i,j} \qquad \text{if } (i-k)(i-m)(j-k)(j-m) > 0, \tag{1}$$

$$a_{i,j}a_{j,k} = a_{i,k}a_{i,j} = a_{j,k}a_{i,k} \qquad \text{for all } i, j, k \text{ with } 1 \le k < j < i \le n, \tag{2}$$

where we set $a_{j,i} = a_{i,j}$ for all $i, j \in \{1, 2, \dots, n\}$. A geometric realization of a BKL-generator (also called *band generator*) $a_{i,j}$ is depicted in Figure 2. The better-known Artin generator σ_i is equal to $a_{i,i+1}$.

We can associate in a straightforward way a *braided surface* (in the sense of Rudolph [31; 33]) or *banded surface* to a given BKL-word. Starting with n parallel disks, the braided surface is obtained by inserting a half-twisted band between the i^{th} and j^{th} disk for each generator $a_{i,j}$ in the given word, where the sign of the twist corresponds to the sign of the generator. Note that the topology of this surface does not change under the BKL-relations (1) and (2). However, it does change under the trivial group relation $a_{i,j}a_{i,j}^{-1} = a_{i,j}^{-1}a_{i,j} = e$.

In the construction of polynomials described in the later sections we will also encounter singular braids. These differ from usual (geometric) braids in that we allow a finite number of simple intersections (double

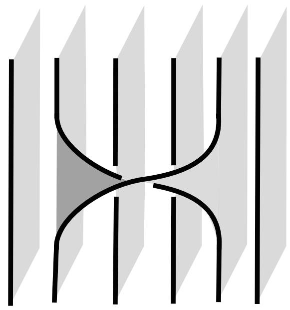

Figure 2: The band generator $a_{2,5}$ in $\mathbb{B}_6$.

points, or *singular crossings*) between the strands of a singular braid. There is now a rich theory of such singular braids [3; 7] and related variants, such as welded braids [20], but we will not really need any of these results apart from the definition itself.

2.2 Rampichini diagrams

In this subsection we review Rampichini diagrams and their connection to P-fibered braids. More details and proofs can be found in [12; 30].

Remark 2.2 Throughout this paper the expression $k \bmod n$ refers to the representative of k modulo n in $\{1,2,\ldots,n\}$. We use the cycle notation with arrows for permutations, ie $(1 \to 2)$ denotes the transposition that swaps 1 and 2. In diagrams, where there is little space and no chance of confusion with vectors, we also use $(1,2)$ for $(1 \to 2)$.

Let $g\colon \mathbb{C} \to \mathbb{C}$ be an element of X_n, that is, a monic polynomial of degree n with distinct roots. Since g is monic, it satisfies $\lim_{\rho\to\infty} \arg(g(\rho \mathrm{e}^{\mathrm{i}\chi})) = n\chi$. Thus, viewing the closed disk D as a compactification of $\mathbb{C}$, we can interpret g as a simple branched cover from the disk D to itself with $g(\mathrm{e}^{\mathrm{i}\chi}) = \mathrm{e}^{\mathrm{i}n\chi}$. The n^{th} roots of unity (labeled clockwise by 1 through n as in Figure 3(a)) are thus the n preimage points of $\arg(g) = 0$ on ∂D. They divide ∂D into n arcs A_i, where A_i connects i and $i+1 \bmod n$. The choice of which root of unity is labeled 1 is not important, but once we have made that choice, it should be the same for all polynomials.

Furthermore, we assume that the critical values v_j for $j = 1,2,\ldots,n-1$ of g have distinct arguments and are ordered so that $0 < \arg(v_1) < \arg(v_2) < \cdots < \arg(v_{n-1}) < 2\pi$. We denote the $n-1$ critical points of g by c_j for $j = 1,2,\ldots,n-1$, indexed so that $g(c_j) = v_j$. The map $\arg(g)$ induces a singular foliation on D, with the roots of g as elliptic singular points and the critical points as hyperbolic points (see Figure 3(a)).

The leaf of a critical point c_k has the shape of a cross, consisting of one line that connects two roots and one line that connects two points on ∂D, say on A_i and A_j, intersecting in c_k. Then we associate to a critical point c_k the transposition $\tau_k = (i \to j)$ (see Figure 3(b)). Since there is a one-to-one

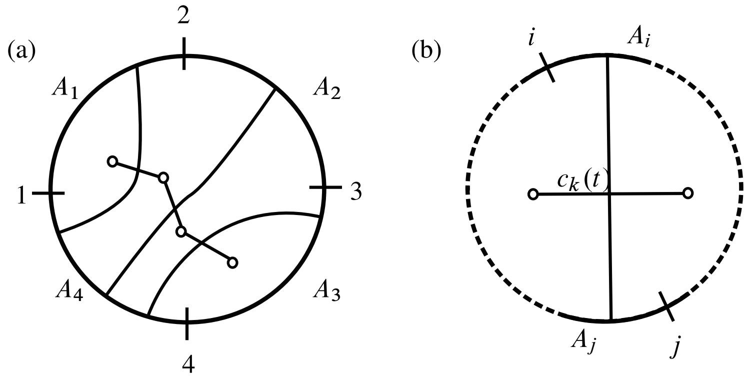

Figure 3: (a) The singular leaves of the singular foliation on D_t induced by $\arg(g_t)$ and the definition of the segments A_i. Small circles are roots of g_t. (b) The singular leaf containing $c_k(t)$. The transposition associated to $c_k(t)$ and $v_k(t) = g_t(c_k(t))$ is $(i \to j)$.

correspondence between critical points and critical values, we can also think of τ_k as a transposition associated with the critical value v_k. The ordered set of transpositions τ_k for $k = 1, 2, \dots, n-1$ is called the *cactus* of g and satisfies $\prod_{k=1}^{n-1} \tau_k = (1 \to 2 \to 3 \to \cdots \to n-1 \to n)$.

Polynomials like g above, that is, monic, of degree n, with distinct roots and critical values that have distinct arguments, are generic in the following sense. Consider the complex vector space $\mathbb{C}^n$ with the surjective map π that sends an n-tuple of complex numbers $(z_1, z_2, \dots, z_n)$ to the polynomial $\prod_{j=1}^{n}(z - z_j)$. The map π is a branched covering map and the preimage of X_n under π is the complement of $\frac{1}{2}n(n-1)$ complex hyperplanes in $\mathbb{C}^n$. If we also want to account for distinct critical values, it is better to use a different projection map π, namely the one that sends an n-tuple $(c_1, c_2, \dots, c_{n-1}, a_0)$ to the unique monic polynomial of degree n with critical points $c_1, c_2, \dots, c_{n-1}$ and constant term a_0. The preimage points of polynomials with nondistinct roots or nondistinct critical values correspond to the solutions of a polynomial equation [5], so the space of polynomials with distinct roots and distinct critical values X'_n is the image of a Zariski-open subset of $\mathbb{C}^n$, the complement of a number of algebraic hypersurfaces of complex dimension $n-1$, under π. Within X'_n the polynomials where the k distinct critical values have the same argument form real submanifolds of real dimension $k + 2(n-1-k) + 1$, where we have k dimensions corresponding to motions of one of the k critical values along the fixed half-line of constant argument, $2(n-1-k)$ dimensions for the unrestricted motion of the remaining $n-1-k$ critical values and one dimension for rotating the half-line of constant argument with k critical values on it. The set of polynomials like g above is therefore the complement of such manifolds of dimension at most $2n-3$ (for $k = 2$) in X'_n, which is itself the image of the complement of a number of complex hypersurfaces in $\mathbb{C}^n$ under the branched covering map π. In other words, the properties of g are generic.

Consider now a path g_t in X_n, with the parameter t going from 0 to 2π, whose endpoints satisfy the properties of g above. After a small homotopy of the path that does not change its endpoints, we may

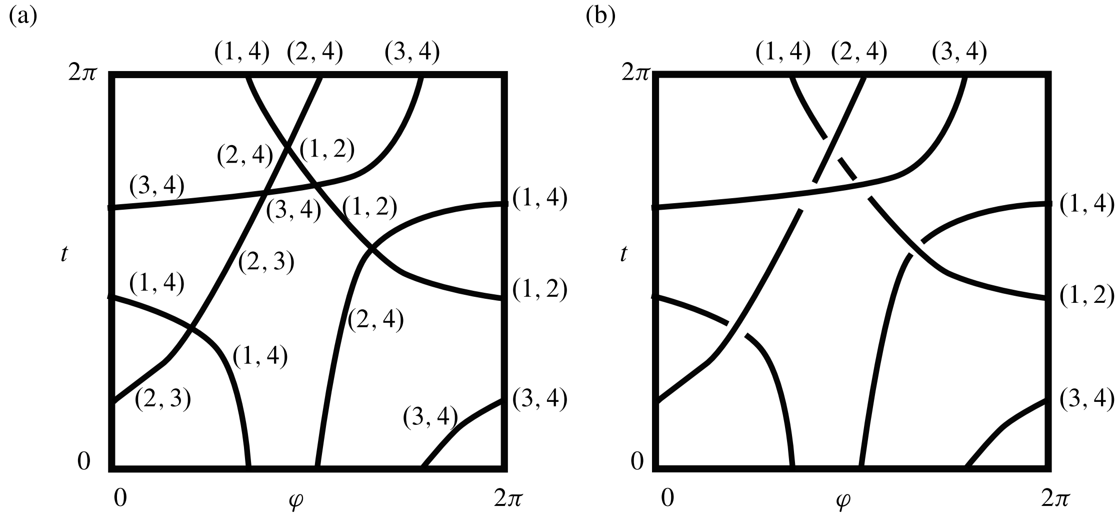

Figure 4: A Rampichini diagram. The transpositions at $t = 0$ are $\tau_1 = (1 \to 4)$, $\tau_2 = (2 \to 4)$ and $\tau_3 = (3 \to 4)$. A band word for the fiber $F_{\varphi=2\pi-\varepsilon}$ is given by $a_{3,4}a_{1,2}^{-1}a_{1,4}$.

assume that for all but finitely many values of $t \in [0, 2\pi]$ the arguments of the critical values $v_j(t)$ are distinct and at the remaining values of t there are only two critical values with the same argument (but different absolute values). For every value of t we order the critical values by their argument in $[0, 2\pi)$. Note that this means that the labeling of the critical values changes at the values of t where two critical values have the same argument or where one critical value crosses the line $\arg = 0$ (either from below or from above). We have a cactus $\tau_k(t)$ for $k = 1, 2, \ldots, n-1$, for every value of t for which the arguments of the critical values are distinct.

We store this information on the different cacti in a so-called *square diagram* as in Figure 4(a). It consists of a square containing curves that are labeled by transpositions in S_n, the symmetric group on n elements. The square represents a cylinder $S^1 \times [0, 2\pi]$ with coordinates (φ, t), both of which go from 0 to 2π, where φ is 2π-periodic. A point (φ, t) in the square lies on one of the curves in the diagram if and only if there is a critical value $v_j(t)$ of g_t with $\arg(v_j(t)) = \varphi$. Every point on a curve in the diagram thus corresponds to a critical value $v_j(t)$. Note that this means that every horizontal line ($t = \text{constant}$) has exactly $n-1$ intersections with the curves in the square, when counted with multiplicities. The label of a point on a curve (away from intersection points) is the transposition $\tau_j(t)$ associated to the corresponding critical value $v_j(t)$. At intersection points of curves in the square there is no well-defined transposition.

The labels in a square diagram can only change at the right edge ($\varphi = 2\pi$) of the square, where all labels are shifted by $-1 \bmod n$ or at intersection points between the different curves. An intersection point between two curves in the square at (φ_*, t_*) occurs if and only if $\arg(v_j(t_*)) = \arg(v_{j+1}(t_*)) = \varphi_*$ for some j, where the indexing of the critical values at $t = t_*$ is taken to be that at $t_* - \varepsilon$ for sufficiently small $\varepsilon > 0$. The labels τ_j and τ_{j+1} at the intersection point change as follows. If $|v_j(t)(t_*)| < |v_{j+1}(t_*)|$, then

$$(3) \qquad \tau_j(t_*+\varepsilon)=\tau_{j+1}(t_*-\varepsilon), \quad \tau_{j+1}(t_*+\varepsilon)=\tau_{j+1}(t_*-\varepsilon)\tau_j(t_*-\varepsilon)\tau_{j+1}(t_*-\varepsilon),$$

and if $|v_j(t)(t_*)|>|v_{j+1}(t_*)|$, then

$$(4) \qquad \tau_j(t_*+\varepsilon)=\tau_j(t_*-\varepsilon)\tau_{j+1}(t_*-\varepsilon)\tau_j(t_*-\varepsilon), \quad \tau_{j+1}(t_*+\varepsilon)=\tau_j(t_*-\varepsilon).$$

In other words, the indices of the labels are swapped and the label of the critical value with smaller absolute value is conjugated by the label of the other critical value. In Figure 4(a) we only label arcs of the curves with transpositions (as opposed to every single point), since it is understood that the labels do not change along the arcs.

If g_t is a loop, then its square diagram can be interpreted as a torus as the critical values at $t=0$ along with their labels match those at $t=2\pi$. The roots of g_t form a P-fibered geometric braid if and only if the curves in the corresponding square diagram are strictly monotone increasing or strictly monotone decreasing, ie they can be locally interpreted as graphs of strictly monotone increasing functions $t(\varphi)$ (corresponding to $\partial\arg(v_j(t))/\partial t>0$) or strictly monotone decreasing functions (corresponding to $\partial\arg(v_j(t))/\partial t<0$). In this case we call the square diagram a *Rampichini diagram*.

Instead of using transpositions as labels we may use band generators, where $(i\to j)$ is replaced by $a_{i,j}$ if the corresponding line in the Rampichini diagram is strictly monotone increasing and by $a_{i,j}^{-1}$ if the line is strictly monotone decreasing. This way, the rules (3) and (4) become the band relations (1) and (2).

Rampichini diagrams have the following nice properties. At every fixed value of t apart from those with nondistinct arguments of critical values, the labels $\tau_j(t)\in S_n$ at that height t, indexed with increasing φ, satisfy $\prod_{j=1}^n \tau_j(t)=(1\to 2\to 3\to\cdots\to n-1\to n)$, since we have a cactus at all such values of t. At every fixed value of φ (away from intersection points of curves in the square) the labels (thought of as BKL-generators) spell a band word for the fiber surface $F_\varphi=\arg(g_t)^{-1}(\varphi)$ when read from the bottom to the top (ie with increasing t).

Every P-fibered braid can be represented by a Rampichini diagram (after a small homotopy of g_t in X_n to ensure that the critical values are distinct) and, conversely, every Rampichini diagram describes a P-fibered braid [12; 30]. We say that a BKL-word w is a *P-fibered braid word* if it appears in a Rampichini diagram R as the band word for a fiber surface F_φ. In this case we also say that R is a Rampichini diagram for the P-fibered braid word w.

Remark 2.3 In [26; 12], the roots of unity on ∂D are labeled counterclockwise instead of clockwise. As pointed out in [12], this different convention means that a cactus $\{\tau_j\}_{j=1,2,\dots,n-1}$ is defined by the property $\prod_{j=1}^n \tau_j=(1\to n\to n-1\to\cdots\to 3\to 2)$ instead of $(1\to 2\to 3\to\cdots\to n-1\to n)$. Another consequence of this is that in [26; 12] the labels on the right edge of a Rampichini diagram increase by 1, while in this paper they decrease by 1 (modulo n). There are two reasons to prefer the convention that we use in this paper. Firstly, the definition of a cactus in this paper matches the one that is commonly used in the study of polynomials and branched coverings [19]. Secondly and more importantly, in contrast to what is claimed in [26; 12] the labels (i,j) in a Rampchini diagram coming from the counterclockwise convention do not directly correspond to the band generators $a_{i,j}$. Using the usual definition of the BKL-generators, where strands are labeled from left to right and bands are in front of all intermediate strands, the label (i,j) in a Rampichini diagram in [26] or [12] actually corresponds to $a_{n+1-i,n+1-j}$. This does not really affect any of the results in [26] or [12], but it is important to remember how extracting band words from a Rampichini diagram depends on the convention. For example, Morton and Rampichini [26] describe an algorithm based on their labeling convention that is meant to check if a given band word $B=\prod_{k=1}^{l}(a_{i_k,j_k})^{\varepsilon_k}$ represents a P-fibered braid. However, taking B as an input, the algorithm actually determines whether $\prod_{k=1}^{l}(a_{n+1-i_k,n+1-j_k})^{\varepsilon_k}$ is a P-fibered braid. Note that these are not only different braids, but have in general different closures; see Remark 2.10. Therefore, choosing the labels of the roots of unity on ∂D in order to define the transpositions in a Rampichini diagram is not simply a matter of taste, as was suggested in [12]. If we want the correspondence between labels in Rampichini diagrams and band generators, we need to label the roots of unity clockwise.

In order to keep Rampichini diagrams clear and less cluttered, we will from now on omit the labels apart from those at the edges of the square. Instead we will at each intersection point in the diagram mark one curve as the undercrossing curve by deleting the curve in a small neighborhood of the intersection (as is common in knot diagrams); see Figure 4(b). By choosing the curve whose critical value has the smaller absolute value to be the undercrossing strand, we can move between these two visual representations of Rampichini diagrams without losing any information, since at each crossing the label of the undercrossing curve is conjugated by the label of the overcrossing curve.

Remark 2.4 Originally, the decision to use crossings (as in knot diagrams) instead of labels in Rampichini diagrams was motivated by purely aesthetic aspects, as Rampichini diagrams with many intersection points needed many labels, which makes the visual representation rather confusing. However, it turns out that it is no coincidence that the resulting Figure 4(b) is a link diagram of a link in a thickened torus (ie a virtual link) with some labels on the edge of the square. We know that every P-fibered braid corresponds to a Rampichini diagram and vice versa. The critical values $v_j(t)$ of the corresponding polynomial g_t, which are known to be nonzero, form a closed (affine) braid

$$\bigcup_{t\in[0,2\pi]}\bigcup_{j=1}^{n}(v_j(t),\mathrm{e}^{\mathrm{i}t}) \tag{5}$$

in $(\mathbb{C}\setminus\{0\})\times S^1$, which is a thickened torus (see [10] and also recent work by Manturov and Nikonov [24]). It is this link that appears in a Rampichini diagram with our new crossing convention. The resulting virtual link is braided both in the t-direction and the φ-direction, ie transverse to all lines of constant t and all lines of constant φ. Any virtual Reidemeister move that preserves this braiding property corresponds to a homotopy of the loop in the space of critical values, which lifts to a homotopy of the loop in X_n [12]. In other words, the braided open book described by a Rampichini diagram does not change under isotopies of the corresponding virtual link as long as the link remains doubly braided throughout the isotopy. (In fact, it is sufficient if it remains braided in the t-direction throughout the isotopy and "doubly braided" at the end of the isotopy.) We thus have a 1-1-correspondence between Rampichini diagrams and links in a thickened torus that are "doubly braided" and labeled with certain transpositions. (Note that not every choice of labels on the boundary of the square diagram can be completed to a Rampichini diagram [26; 30].)

Definition 2.5 We call a Rampichini diagram R simple if the corresponding banded fiber surface consists of n disks connected by exactly $n-1$ bands. In other words, any horizontal line ($t=$ constant) has $n-1$ intersections with the curves in R (counted with multiplicity) and any vertical line ($\varphi=$ constant) also has $n-1$ intersections with the curves in R (counted with multiplicity).

Every fiber surface F_φ coming from a simple Rampichini diagram is a braided surface with n disks and $n-1$ bands. Thus they all have genus 0 and a connected boundary. Therefore, they are bounded by an unknot. The Rampichini diagram in Figure 4(a) is simple. It represents a braid on four strands, since every horizontal line has three intersection points with the curves in the square (counted with multiplicities), and every fiber surface F_φ has three bands, since each vertical line has three intersection points with the curves (again counted with multiplicities).

Let

(6) $$\widetilde{V}_n=\{(v_1,v_2,\dots,v_{n-1})\in(\mathbb{C}\setminus\{0\})^{n-1}: v_i\neq v_j \text{ if } i\neq j\}$$

and

(7) $$V_n=\widetilde{V}_n/S_{n-1},$$

where the action of the symmetric group S_{n-1} permutes the different v_i. The fundamental group of V_n is the affine braid group $\mathbb{B}_{n-1}^{\text{aff}}$, the subgroup of the braid group $\mathbb{B}_n$ where one of the strands is a vertical line, in this case $\{0\}\times[0,2\pi]$. This is discussed in more detail in [10]. The fundamental group of $\widetilde{V}_n$ is the intersection of the affine braid group and the pure braid group on n strands, both considered as subgroups of $\mathbb{B}_n$.

Let X_n' be the space of polynomials in X_n with distinct critical values. Then we may think of V_n as the space of sets of critical values of polynomials in X_n'. The map $\theta_n\colon X_n'\to V_n$ that sends a polynomial to its set of critical values maps a loop g_t in X_n' to a loop in V_n.

Definition 2.6 Let g_t be a loop in X_n' and R its Rampichini diagram. Then we say that R is pure if the loop $\theta_n(g_t)$ in V_n lifts to a loop in $\widetilde{V}_n$.

In other words, a Rampichini diagram is pure if the corresponding critical values form a pure braid in $(\mathbb{C}\setminus\{0\})\times S^1$. In this case we have two different ways to index the critical values. The first one is described above, where at every value of t we order the critical values by their arguments. This means that the indices are not constant along curves of critical values. They change at intersection points in the

Rampichini diagram and at the right edge of the square. The second one is given by the indices at $t=0$ and keeping the indices constant along each curve of critical values, ie the indexing comes from thinking of $\theta_n(g_t)$ as a loop in $\widetilde{V}_n$. In order to distinguish these two different orderings of critical values, we write $v(t)_j$ for the first ($0 \le \arg(v(t)_j) < \arg(v(t)_k) < 2\pi$ if and only if $j<k$) and $v_j(t)$ for the second ($j<k$ if and only if $0 \le \arg(v(0)_j) < \arg(v(0)_k) < 2\pi$). Note that the index of a transposition $\tau_j(t)$ always refers to the indexing of the first kind, ie $\tau_j(t)$ is the transposition associated with $v(t)_j$.

Critical values of loops of polynomials corresponding to P-fibered braids satisfy $\partial \arg(v_j(t))/\partial t \neq 0$, so every curve of critical values $v_j(t)$ has an associated sign $\varepsilon_j \in \{\pm 1\}$, which is equal to $\operatorname{sign}\big(\partial \arg(v_j(t))/\partial t\big)$ and which determines whether the corresponding curve in the Rampichini diagram is strictly monotone increasing or decreasing. We call ε_j the *sign of the critical value* $v_j(t)$ and say that a critical value v_j is positive/negative if ε_j is positive/negative.

The composition of two paths in V_n corresponds to a square diagram that is obtained by gluing the top edge of one square diagram along the bottom edge of the other square diagram. Since the paths have matching endpoints, the labels and endpoints in the respective square diagrams also match. Likewise, if the φ-coordinates and corresponding labels at the top edge of one square diagram match those at the bottom edge of another square diagram, we may glue the two diagrams along these edges to obtain a new diagram.

2.3 Inner loops

Let $(v_1, v_2, \dots, v_{n-1}) \in \widetilde{V}_n$ with $\arg(v_j) \neq \arg(v_k)$ if $j \neq k$. The v_i are not necessarily ordered by their arguments.

Definition 2.7 An inner loop is a loop γ_j (or γ_j^{-1}) in $\widetilde{V}_n$ based at $(v_1, v_2, \dots, v_{n-1})$, where every critical value except one is stationary and one critical value v_j moves in a counterclockwise (or clockwise) loop exactly once around the origin, such that whenever $\arg(v_j(t)) = \arg(v_i(t))$ with $j \neq i$ we have $|v_j(t)| < |v_i(t)|$. We call v_j the moving critical value of this inner loop.

Figure 5 shows the motion of the critical values during an inner loop. For a path γ in a topological space parametrized by t in $[0, 2\pi]$, we write $\gamma|_a^b$ for the segment of the path γ between $\gamma(a)$ and $\gamma(b)$.

Definition 2.8 Let g_t be a loop in X_n' whose roots form a P-fibered braid and let R be the corresponding Rampichini diagram, which we assume to be pure. Then $\theta_n(g_t)$ can be viewed as a loop in $\widetilde{V}_n$. Let $t_i \in [0, 2\pi]$ for $i = 1, 2, \dots, M$ be such that the arguments of the critical values of g_{t_i} are distinct for each fixed t_i and let $j(i) \in \{1, 2, \dots, n-1\}$ for $i = 1, 2, \dots, M$. Let $\varepsilon_{j(i)}$ be the sign of the critical value $v_{j(i)}(t)$. Let R' be the square diagram corresponding to the following loop in $\widetilde{V}_n$:

$$\theta_n(g_t)|_0^{t_1} \circ \gamma_{j(1)}^{\varepsilon_{j(1)}} \circ \theta_n(g_t)|_{t_1}^{t_2} \circ \gamma_{j(2)}^{\varepsilon_{j(2)}} \circ \cdots \circ \theta_n(g_t)|_{t_i}^{t_{i+1}} \circ \gamma_{j(i+1)}^{\varepsilon_{j(i+2)}} \circ \cdots \circ \gamma_{j(M)}^{\varepsilon_{j}(M)} \circ \theta_n(g_t)|_{t_M}^{2\pi}. \tag{8}$$

Then we say that R' is obtained from R by inserting inner loops.

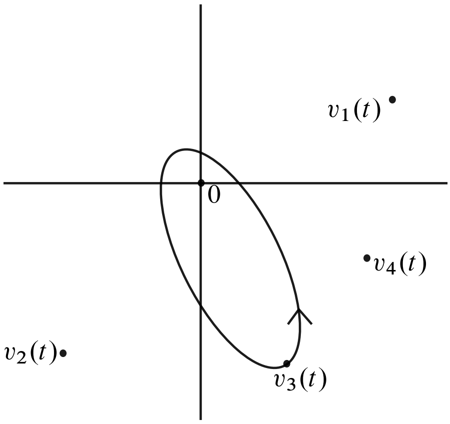

Figure 5: The inner loop γ_3.

First note that every loop in $\widetilde{V}_n$ corresponds to the critical values of some path in X_n [12], so R' is actually well defined as the square diagram of that path. (The path is not unique, but the choice of path does not matter.) For the construction of R' we may assume that the loop in (8) is rescaled so that it is parametrized by $t \in [0, 2\pi]$. The following lemma establishes that the corresponding path in X'_n is actually a loop and, moreover, R' is a Rampichini diagram.

Lemma 2.9 *Let R' be a square diagram that is obtained from a Rampichini diagram R by inserting inner loops. Then, after a small isotopy of the loop in (8), R' is a Rampichini diagram.*

Proof Figure 6 shows how the insertion of an inner loop affects a Rampichini diagram. It corresponds to the insertion of a line that is almost horizontal and that loops around the φ-coordinate exactly once. Since $|v_j(t)| < |v_i(t)|$ at all intersection points, the inserted line lies below all other curves in the diagram and thus its label must be conjugated at each crossing point in the diagram, while the other labels remain

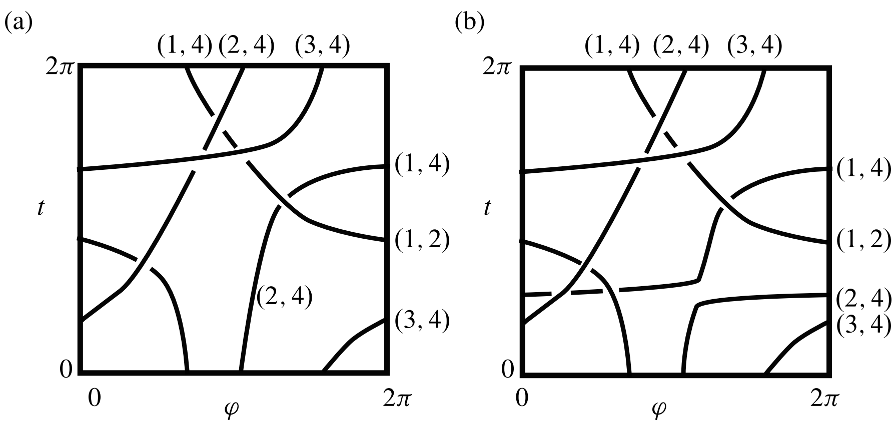

Figure 6: (a) A simple, pure Rampichini diagram. (b) The Rampichini diagram after the insertion of an inner loop.

unaffected. Since the sign of the inner loop $\gamma_j^{\varepsilon_j}$ matches the sign ε_j of the label of the moving critical value $v_j(t)$, the curves in R' are again strictly monotone after a small deformation of the curves (so that the critical values that are not moving during an inner loop become nonstationary). We now have to show that the labels $\tau_j(2\pi)$ at the top edge of the square diagram are identical to the labels $\tau_j(0)$ at the bottom edge. By the arguments in the proof of Theorem 5.6 in [12], this implies that the corresponding path in X_n' is a loop and hence R' is a Rampichini diagram.

We know that during an inner loop only the label of the moving critical value changes. We have to show that after the completion of the inner loop it is again the original label. Let $\tau_j(t_i)$ for $j=1,2,\dots,n-1$ be the transpositions in R associated with the critical values at the height $t=t_i$ at which an inner loop will be inserted, ordered by increasing value of φ. Let k be the index of the moving critical value for the inserted inner loop $\gamma_{j(i)}^{\varepsilon_{j(i)}}$, ie $v_{j(i)}(t_i)=v(t_i)_k$ in R. Let τ_k' denote its label after the inner loop.

By one of the defining properties of Rampichini diagrams, $\prod_{j=1}^{n-1}\tau_j(t_i)=(1\to 2\to 3\to\cdots\to n-1\to n)$. In fact, this equality is true for any height t in a square diagram, not just at $t=t_i$. We thus have

$$\prod_{j=1}^{n-1}\tau_j(t_i)=(1\to 2\to 3\to\cdots\to n-1\to n)=\left(\prod_{j=1}^{k-1}\tau_j(t_i)\right)\tau_k'\left(\prod_{j=k+1}^{n-1}\tau_j(t_i)\right), \tag{9}$$

where the left-hand side describes the cactus at $t=t_i-\varepsilon$ and the right-hand side the cactus at $t=t_i+\varepsilon$ for some small positive ε. This implies that $\tau_k(t_i)=\tau_k'$ after multiplying by $\left(\prod_{j=1}^{k-1}\tau_j(t_i)\right)^{-1}$ on the left and by $\left(\prod_{j=k+1}^{n-1}\tau_j(t_i)\right)^{-1}$ on the right. Thus the labels after an inserted inner loop are the same as the labels before the inner loop. Since R is a Rampichini diagram, the labels at its top edge are equal to the label at its bottom edge. It follows that the same holds for R', which proves the lemma. □

Remark 2.10 Figure 6(b) illustrates the arguments of Remark 2.3. It shows that the braid $a_{3,4}a_{2,4}a_{1,2}^{-1}a_{1,4}$, which closes to the Hopf link, is a P-fibered braid. If we had used the convention from [26] for labels on Rampichini diagrams, the labels on the right edge would read from the bottom to the top: $(1,2)$, $(1,3)$, $(3,4)$ and $(1,4)$. However, the closure of $a_{1,2}a_{1,3}a_{3,4}^{-1}a_{1,4}$ is a 2-component unlink and therefore not fibered. This is not a contradiction. We simply have to remember that labels (i,j) in the convention from [26] do not really correspond to band generators $a_{i,j}$, but to $a_{n+1-i,n+1-j}$. This means that some examples in [26] are not quite correct. Figure 10 in [26] for example displays a Rampichini diagram with their labeling convention, which is supposed to show that the braid $a_{3,4}a_{1,2}^{-1}a_{2,3}$ is P-fibered. Actually, it shows that $a_{1,2}a_{3,4}^{-1}a_{2,3}$ is P-fibered.

Instead of inserting inner loops into Rampichini diagrams, we can also insert them into a diagram as in Figure 7 corresponding to a trivial loop in X_n', where all curves in the diagram are vertical. We call such a square diagram *trivial*. Note that there are also nontrivial paths, whose square diagrams are trivial, since the only requirement is that the argument of the critical values do not change. Since there are no intersections of the curves with each other or with the right edge of the square, the labels in a trivial square diagram never change.

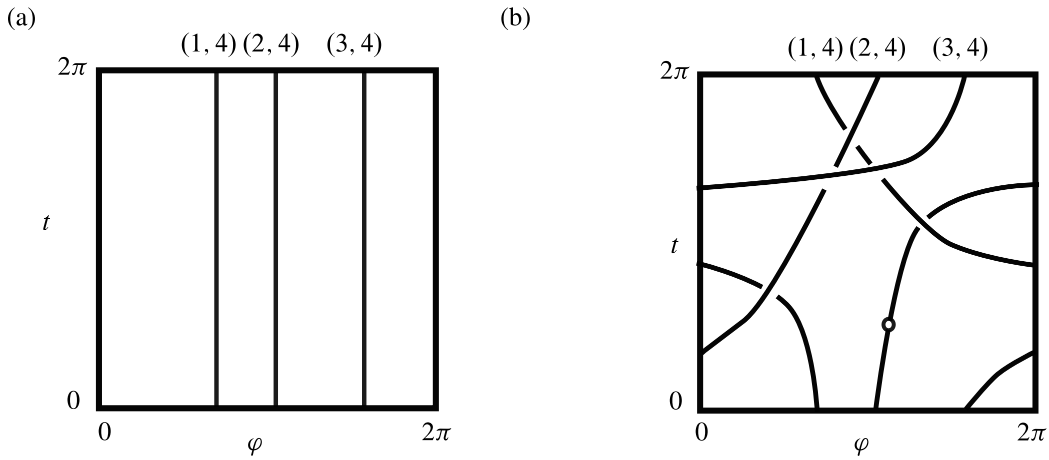

Figure 7: (a) A trivial square diagram corresponding to a constant loop in the space of polynomials. (b) A singular Rampichini diagram.

A trivial square diagram exists for any choice of cactus τ_j for $j = 1, 2, \dots, n-1$, which by definition satisfies $\prod_{i=1}^{n} \tau_i = (1 \to 2 \to 3 \to \cdots \to n-1 \to n)$, and signs $\varepsilon_j \in \{\pm 1\}$ for $j = 1, 2, \dots, n-1$. Inserting a finite number of inner loops $\gamma_{j(i)}^{\varepsilon_{j(i)}}$ for $i = 1, 2, \dots, M$ into the diagram results in a Rampichini diagram R' by the same arguments as in Lemma 2.9.

We can read off a band word for the resulting fiber surface $F_{2\pi-\varepsilon}$ of the corresponding braid from the Rampichini diagram R'. Note that every critical value $v_j(t)$ always contributes the same letter, which we call $a_j^{\varepsilon_j}$. Its sign is always equal to the sign ε_j of the critical value, while its associated transposition is $\tau_j^{\prod_{i=j+1}^{n+1} \tau_i}$. Here a^b denotes conjugation of a group element a by another group element b, that is, $a^b = b^{-1}ab$. The definition of $a_j^{\varepsilon_j}$ clearly depends on the choice of cactus $\{\tau_j\}_{j=1,2,\dots,n-1}$.

Definition 2.11 A braid B is called T-homogeneous if there is a choice of cactus $\{\tau_j\}_{j=1,2,\dots,n-1}$ and signs $\varepsilon_j \in \{\pm 1\}$ such that B can be represented by a word that only contains the letters $a_j^{\varepsilon_j}$ and that contains $a_j^{\varepsilon_j}$ for every $j = 1, 2, \dots, n-1$.

In some texts these braids are also called *strict*(*ly*) *T-homogeneous braids* to emphasize the second property, that every $a_j^{\varepsilon_j}$ appears in the word [33].

T-homogeneous braids are thus exactly those braids that can be obtained from a trivial square diagram as in Figure 7(a) by insertion of inner loops. Each inserted inner loop can be interpreted as a loop in $\widetilde{V}_n$ and a trivial square diagram corresponds to a constant loop in $\widetilde{V}_n$. Therefore every T-homogeneous braid has a pure Rampichini diagram.

Example 2.12 We may choose $\tau_j = (j \to n)$ for $j = 1, 2, \dots, n-1$ and obtain $a_j^{\varepsilon_j} = a_{j,j+1}^{\varepsilon_j} = \sigma_j^{\varepsilon_j}$, so T-homogeneous braids for this choice of cactus are exactly the *homogeneous braids* in Artin generators σ_j.

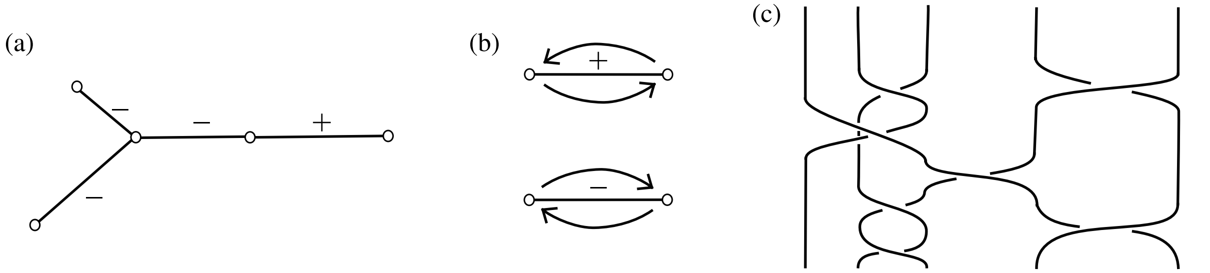

Figure 8: (a) An embedded tree T in the complex plane. The signs ε_e are drawn on each edge e. (b) The loop γ_e exchanges the roots on the endpoints of e as in the upper picture if $\varepsilon_e = +1$ and as in the lower picture if $\varepsilon_e = -1$. (c) A T-homogeneous braid with T as in (a).

Definition 2.11 describes the family of T-homogeneous braids in terms of their braid words. An alternative, equivalent definition explains the origin of the "T" in the name. Let T be an embedded tree in $\mathbb{C}$ with n vertices. We may assign to every edge e a sign $\varepsilon_e \in \{\pm 1\}$. We define a loop γ_e in X_n as follows. The basepoint of γ_e is the polynomial in X_n whose roots are the positions of the vertices of T. Throughout the loop all roots that do not correspond to endpoints of the edge e remain stationary, while the two endpoints of e exchange their position as in Figure 8(b), where the way in which they swap depends on the sign ε_e. A geometric braid B is T-homogeneous if its corresponding loop of polynomials g_t is the composition of loops of the form γ_e, where e runs through the set of edges of T, and for every edge e the corresponding loop γ_e appears in the factorization, ie $g_t = \prod_{i=1}^{l} \gamma_{e_i}$, where $\prod$ denotes composition of loops, and for every edge e there is an $i \in \{1, 2, \ldots, l\}$ with $e_i = e$. The two definitions are equivalent with different choices of an embedded tree T (up to planar isotopy) corresponding to different choices of a cactus $\{\tau_j\}_{j=1,2,\ldots,n-1}$. Homogeneous braids correspond to a path graph (or "linear graph").

Homogeneous braids are known to close to fibered links [36]. In fact, all T-homogeneous braids are known to close to bindings of totally braided open books [33], which is a property that is equivalent to being P-fibered [12].

Let B be a P-fibered geometric braid with corresponding loop of polynomials g_t and a pure Rampichini diagram R. Deforming its loop of critical values $(v_1(t), v_2(t), \ldots, v_{n-1}(t))$ in V_n lifts to a deformation of g_t and, as long as the deformation only changes the absolute values of the $v_j(t)$, the Rampichini diagram and the fiberedness property remain unchanged. Now consider such a deformation, where we only change one of the curves of critical values $v_j(t)$ so that it becomes 0 at some $t = \tau \in [0, 2\pi]$ and call the lifted loop $\tilde{g}_t$. Then $\tilde{g}_\tau$ has a double root and therefore the roots of $\tilde{g}_t$ do not form a braid, but a singular braid. Since the argument of the critical value at $t = \tau$ is not defined, we cannot associate to it a label or a transposition like we usually do in Rampichini diagrams. We define a *singular Rampichini diagram* $R_{\{(t_i, j(i))\}_{i=1,2,\ldots,M}}$ to be a Rampichini diagram R where, at a finite number of distinct values of $t = t_i$ for $i = 1, 2, \ldots, M$, one of the curves, corresponding to the critical value $v_{j(i)}(t_i)$, in R at height t is replaced by a small circle. The rest of the diagram is unchanged. This is displayed in Figure 7b), which can be obtained from the Rampichini diagram in Figure 6(a). The circles represent values of t at which a critical value becomes zero. While Rampichini diagrams visualize certain loops in V_n that lift to loops in X_n, singular Rampichini diagrams visualize certain loops in the space of $n-1$ distinct complex numbers (a space that is homeomorphic to X_{n-1}), which lift to loops in the space of monic polynomials of degree n, but not to loops in X_n.

Note that not every loop h_t in the space of monic polynomials of fixed degree n gives rise to a singular Rampichini diagram. Apart from the usual condition that $\partial\arg(v_j)(t)/\partial t \neq 0$ for all $v_j(t) \neq 0$, it is necessary that $\lim_{t\to t_i} \arg(v_{j(i)}(t))$ and $\lim_{t\to t_i} \partial\arg(v_{j(i)}(t))/\partial t$ are well defined, with the latter being nonzero. Furthermore, it was an assumption that there is only a finite number of singular crossings and no higher multiplicity than 2 for any root of h_t.

Since the roots of a loop of polynomials h_t corresponding to a singular Rampichini diagram form a singular braid, the level sets of $\arg(h_t)$ are not braided surfaces anymore. Therefore, the property of Rampichini diagrams that we can read off band words for the fibers is not true anymore.

3 Odd P-fibered braids

P-fibered braids play a big role in constructions of real algebraic links. If B is a P-fibered braid, then the closure of B^2 is real algebraic [9]. The corresponding semiholomorphic polynomial can be constructed explicitly as

$$f(u, r\mathrm{e}^{\mathrm{i}t}) = r^{2kn} g\left(\frac{u}{r^{2k}}, \mathrm{e}^{2\mathrm{i}t}\right), \tag{10}$$

where k is a sufficiently large integer, n is the number of strands and $g_t(u) = g(u,t)$ is the loop corresponding to the P-fibered geometric braid B parametrized in terms of trigonometric polynomials, so that the coefficients of g (as a polynomial in u) are polynomials in $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$. Note that f is radially weighted homogeneous, ie $f(\lambda^{2k}u, \lambda v) = \lambda^{2kn} f(u,v)$ for all $\lambda \in \mathbb{R}$. This construction requires 2-periodicity of a braid (ie B^2 instead of B), since this guarantees that f, which is a priori a polynomial in u, v, $\bar{v}$ and $\sqrt{v\bar{v}}$, is actually a polynomial in u, v and $\bar{v}$, since all terms with square roots come with an even exponent.

In [9] we discuss another symmetry, which will be essential here. Instead of the even symmetry of B^2, where all frequencies in the trigonometric parametrization of the roots are even and thus $g_{t+\pi} = g_t$, we require the odd symmetry $g_{t+\pi}(u) = -g_t(-u)$ for all $t \in [0,\pi)$ and all $u \in \mathbb{C}$. In this case, we say that g_t is *odd*.

Polynomials in $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$ with complex coefficients are also called finite Fourier series or complex trigonometric polynomials. We reserve the term trigonometric polynomial for finite Fourier series that are real functions. They can thus be expressed as $\mathbb{R}$-linear combinations of 1, $\cos(kt)$ and $\sin(kt)$, where k goes through a finite range of natural numbers.

We write $f^{(i)}$ for the i^{th} derivative of a function f. The C^k-norm is $|f|_k = \sup_{i\le k}\sup_{x\in\mathbb{R}} |f^{(i)}(x)|$.

A weaker version of the following result, where all components were required to have the same number of strands, was proved in [9, Lemma V.4].

Theorem 3.1 *Let B be a P-fibered geometric braid on n strands with corresponding loop of polynomials g_t. Assume that the coefficients of g_t are finite Fourier series. If n is odd and g_t is odd, then the closure of B is real algebraic, with the corresponding polynomial map given by*

$$f(u, re^{it}) = r^{kn} g\left(\frac{u}{r^k}, e^{it}\right), \tag{11}$$

where $g(u,t) = g_t(u)$ and k is a large odd integer.

Proof Equation (11) can be rewritten as

$$f(u,v) = \sqrt{v\bar{v}}^{kn} g\left(\frac{u}{\sqrt{v\bar{v}}^k}, \frac{v}{\sqrt{v\bar{v}}}\right). \tag{12}$$

As in the case of the even symmetry of B^2, choosing k sufficiently large clears the denominator and, since k, n and g_t are all odd, all monomials involving $\sqrt{v\bar{v}}$ come with an even exponent, so f is a polynomial in u, v and $\bar{v}$.

Note that f is radially weighted homogeneous, since $f(\lambda^k u, \lambda v) = \lambda^{kn} f(u,v)$. It then follows by the same arguments as in the proof of Theorem I.1 in [9] that the origin is an isolated singularity if and only if B is P-fibered. Furthermore, for each fixed r the zeros of $f|_{|v|=r}$ form the closed braid B in $\mathbb{C} \times rS^1$. By the same arguments as in [14] or [9], the link of the singularity is the closure of B. □

Note that the condition that coefficients of g_t should be finite Fourier series is not really a restriction. For an odd loop of polynomials g_t with $\deg_u(g_t)$ odd, the coefficient $a_j(t)$ of u^j is an odd function, ie $a_j(t+\pi) = -a_j(t)$, if j is even, and an even function, ie $a_j(t+\pi) = a_j(t)$, if j is odd. Odd/even trigonometric polynomials are dense in the space of odd/even 2π-periodic functions in the C^k-norm for any natural number k. Therefore, if B is a P-fibered geometric braid whose corresponding loop of polynomials g_t is odd, then an arbitrarily close approximation of B is an isotopic P-fibered braid whose corresponding loop of polynomials is odd and has coefficients that are finite Fourier series.

Definition 3.2 A cactus $\{\tau_j\}_{j=1,2,\dots,2n-2}$ with $\tau_j = (k(j) \to l(j))$ is called odd if it satisfies

$$\tau_{j+n-1} = (k(j) + n \pmod{2n-1} \to l(j) + n \pmod{2n-1}) \tag{13}$$

for all $j = 1, 2, \dots, n-1$.

Lemma 3.3 *Let $g\colon \mathbb{C} \to \mathbb{C}$, $g(u) = \prod_{j=1}^{2n-1}(u - z_j)$, be a monic polynomial of degree $2n-1$ with an odd cactus and critical values v_j for $j = 1, 2, \dots, 2n-2$ that satisfy*

$$0 < \arg(v_1) < \arg(v_2) < \cdots < \arg(v_{n-1}) < \pi < \arg(v_n) < \arg(v_{n+1}) < \cdots < \arg(v_{2n-2}) < 2\pi. \tag{14}$$

Then $\tilde{g}\colon \mathbb{C} \to \mathbb{C}$, $\tilde{g}(z) = \prod_{j=1}^{2n-1}(u + z_j)$, has the same cactus as g.

Proof By assumption, the critical values v_j for $j = 1, 2, \dots, n-1$ of g have an argument between 0 and π, while the critical values v_{j+n-1} for $j = 1, 2, \dots, n-1$ of g have an argument between π and 2π. Note that the critical values of $\tilde{g}$ are $\tilde{v}_{j+n-1} = -v_j$ for $j = 1, 2, \dots, 2n-2$, where the indices are taken mod $2n-2$. Hence the critical values $\tilde{v}_j$ for $j = 1, 2, \dots, n-1$ of $\tilde{g}$ have an argument between 0 and π and the critical values $\tilde{v}_{j+n-1}$ for $j = 1, 2, \dots, n-1$ have an argument between π and 2π.

The singular foliation of the disk induced by $\arg(\tilde{g})$ is exactly the singular foliation induced by $\arg(g)$ rotated by π. Thus, if $j \in \{1, 2, \dots, n-1\}$ and $\tau_j = (k(j) \to l(j))$, then $\arg(\tilde{v}_{j+n-1}) \in (\pi, 2\pi)$ with label $\tilde{\tau}_{j+n-1} = (k(j) + n \to l(j) + n) = \tau_{j+n-1}$, since g has an odd cactus.

Likewise, from $\tau_{j+n-1} = (k(j) + n \pmod{2n-1} \to l(j) + n \pmod{2n-1})$ for $j = 1, 2, \dots, n-1$, we get that $\tilde{v}_j$ for $j = 1, 2, \dots, n-1$ has the label $\tilde{\tau}_j = (k(j) \to l(j)) = \tau_j$. Thus $\tilde{g}$ has the same cactus as g. □

Lemma 3.4 *Let g be a polynomial in X'_{2n-1} with an odd cactus and critical values v_j for $j = 1, 2, \dots, 2n-2$ that satisfy the following property. For every $j \in \{1, 2, \dots, 2n-2\}$ there is a $k \in \{1, 2, \dots, 2n-2\}$ such that $\arg(v_k) = \arg(v_j) + \pi$. Then there is a path in X'_{2n-1} from g to $\tilde{g}$ whose square diagram is trivial, where $\tilde{g}$ is as in Lemma 3.3.*

Proof Riemann's uniqueness theorem states that (once we have fixed a labeling of the arcs A_j for $j = 1, 2, \dots, 2n-1$ on ∂D as we have done at the beginning of Section 2.2), for every cactus and set of critical values v_j, there is a unique monic polynomial with the given cactus and set of critical values v_j, up to translations in the complex plane $z \mapsto z + b$ [12].

In particular, we can translate the roots of g in parallel so that one of them becomes 0 without changing the cactus or the set of critical values. In other words, fixing one $k \in \{1, 2, \dots, 2n-1\}$, the path in X'_{2n-1} that corresponds to the path of polynomials with roots $z_j(t) = z_j - \frac{t}{2\pi} z_k$, with t going from 0 to 2π, has a trivial square diagram.

By Lemma 3.3, g and $\tilde{g}$ have the same cactus and, by the given symmetry of the critical values v_j of g, the critical values of $\tilde{g}$ have the same arguments as v_j for $j = 1, 2, \dots, 2n-2$. Therefore there is a path in V_{2n-1} that does not change the arguments of the critical values and goes from the set of critical values of g to the set of critical values of $\tilde{g}$. Then this path lifts to a unique path in X'_{2n-1} if we require that throughout the path the constant term of every polynomial is 0 [5]. By Riemann's uniqueness theorem, this path thus goes from the translated g to a translation of $\tilde{g}$. Since the arguments of the critical values do not change along the corresponding path in V_{2n-1}, the path in X'_{2n-1} has a trivial square diagram. The composition of the path that translates g and this path, followed by a translation that ends at $\tilde{g}$, is then the desired path in X'_{2n-1} with a trivial square diagram. □

We know from [12] that every Rampichini diagram arises from some loop g_t in X_n corresponding to a P-fibered braid.

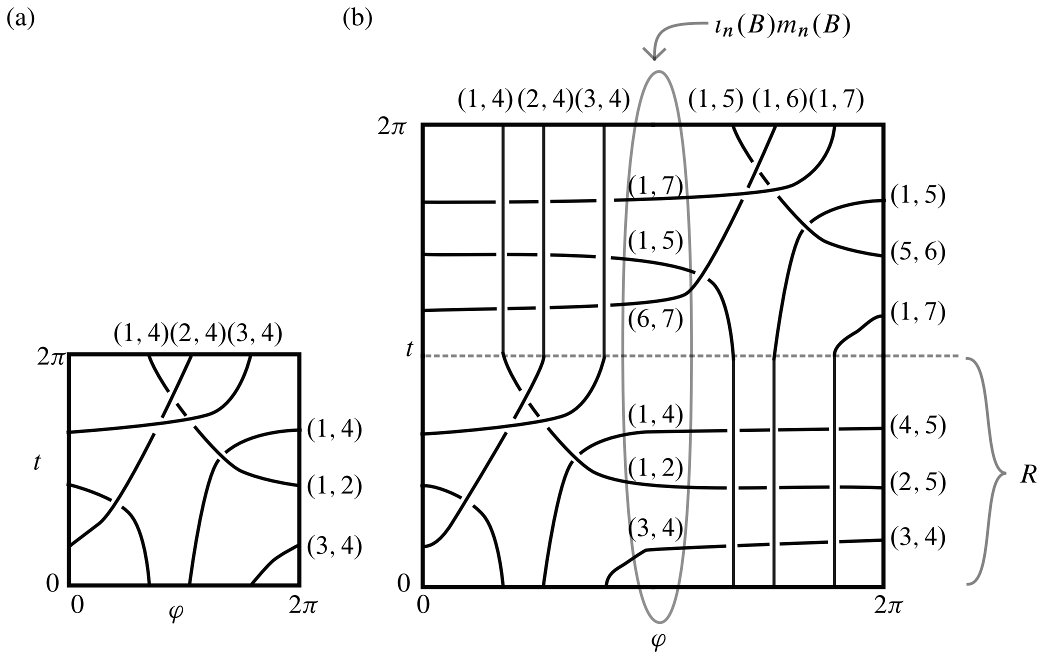

Figure 9: (a) A Rampichini diagram R. (b) The corresponding Rampichini diagram R'' with odd symmetry. Its lower half is the Rampichini diagram R'. The labels at $\varphi = \pi$ (interpreted as band generators) read from the bottom to the top spell $\iota_n(B)m_n(B)$.

Definition 3.5 We say that a Rampichini diagram is odd if there is a corresponding loop g_t in X_n that is odd.

We define $m_n : \mathbb{B}_n \to \mathbb{B}_{2n-1}$ to be the group homomorphism that sends a generator $a_{i,j}$ with $i, j \neq 1$ to $a_{i-n,j-n}$ and $a_{i,1} = a_{1,i}$ to $a_{1,i-n}$, where indices are understood modulo $2n-1$. Furthermore, we write $\iota_n : \mathbb{B}_n \to \mathbb{B}_{2n-1}$ for the inclusion which sends $a_{i,j}$ to $a_{i,j}$.

Lemma 3.6 *Let B be a P-fibered braid word on n strands. Then there is an odd Rampichini diagram for the P-fibered braid word $\iota_n(B)m_n(B)$.*

Proof Let R be the Rampichini diagram (see Figure 9(a)) whose labels at $\varphi = 2\pi - \varepsilon$ spell the word B. We can define a square diagram R' (see the lower half of Figure 9(b)) by gluing a trivial square diagram with $n-1$ vertical curves as in Figure 7(a) on the right edge of R and continuing all curves on the right edge of R towards the right edge of the new diagram R', crossing below the curves of the trivial diagram. The new cactus at $t = 0$ is given by $\tau_j(0) = (k(j) \to l(j))$, the original labels of R, for $j = 1, 2, \ldots, n-1$, and $\tau_{j+n-1} = (k(j) + n \pmod{2n-1} \to l(j) + n \pmod{2n-1})$ for $j = 1, 2, \ldots, n-1$. It is thus an odd cactus.

The square diagram R' corresponds to a loop in V_n and we claim that this loop lifts to a loop in X'_n. By the proof of Theorem 5.6 in [12], it is enough to check that the labels at the top edge of the square are the same as the labels at the bottom when we follow the rules of how labels change at crossings and at the right edge of the square. Since the vertical curves are the overcrossing strand in each crossing and since they never cross the right edge of the square, their labels do not change at all.

We call the curves in R and in R' that are strictly monotone increasing *positive curves* and the curves that are strictly monotone decreasing *negative curves*. We have to show that shifting the indices of every label of a positive curve on the right edge of R by $-1 \bmod n$ is equal to shifting the indices of the corresponding labels on the right edge of R' by $-1 \bmod 2n-1$. Likewise we have to show that the labels of the negative curves in R' at $\varphi = \pi$ are equal to the corresponding labels on the right edge of R. This implies that the labels in the left half of R' are equal to the corresponding labels in R.

If a label of a positive curve on the right edge of R is $(i \to j)$, then the corresponding label on the right edge of R' is given by the conjugate of $(i \to j)$ by $\prod_{k=1}^{n-1} \tau_{k+n-1} = (1 \to n+1 \to n+2 \to \cdots \to 2n-2 \to 2n-1)$. If both i and j are different from 1, the result of this conjugation is equal to $(i \to j)$, since $i, j \notin \{1, n+1, n+2, \ldots, 2n-1\}$. If one of them (say i) is equal to 1, then we obtain $(n+1 \to j)$. This is precisely what we wanted, since

$$(1-1 \to j-1) \bmod n = (n \to j-1) = (n+1-1 \to j-1) \bmod 2n-1. \tag{15}$$

If a label of a negative curve on the left edge of R is $(i \to j)$, then the corresponding label on the right edge of R is $(i+1 \to j+1)$ mod n, while it is $(i+1 \to j+1) \bmod 2n-1$ on the right edge of R'. The label of the same curve in R' at $\varphi = \pi$ after it has crossed all vertical curves is the conjugate of $(i+1 \to j+1) \bmod 2n-1$ by $\prod_{k=n-1}^{1} \tau_{k+n-1} = (1 \to 2n-1 \to 2n-2 \to \cdots \to n+1)$. If both i and j are different from n, this is simply $(i+1 \to j+1) \bmod 2n-1 = (i+1 \to j+1) \bmod n$, which is the desired label. If one of them (say j) is equal to n, then we obtain $(1 \to i+1) \bmod 2n-1 = (n+1 \to i+1) \bmod n$. Thus in all cases the labels in the left half of R' are the same as the corresponding labels in R. Since R is a Rampichini diagram, its labels at $t=0$ are equal to its labels at $t = 2\pi$ and thus R' corresponds to a loop g_t' in X_n'.

We may deform R' and the corresponding loop of polynomials g_t' in a neighborhood of $t=0$ such that the basepoint g_0' satisfies the property from Lemma 3.4: for every critical value $v_j'(0)$ of g_0' there is a critical value $v_k'(0)$ with $k \in \{1, 2, \ldots, 2n-2\}$ such that $\arg(v_j'(0)) = \arg(v_k'(0)) + \pi$.

Note that R' was constructed in such a way that g_0' has an odd cactus. By Lemma 3.4 there is a path in X_{2n-1}' from g_0' to $\tilde{g}_0'$ (the monic polynomial satisfying $\tilde{g}_0'(u) = -g_0'(-u)$) with trivial square diagram. The square diagram that is the composition of R' and this trivial square diagram can be interpreted as a square diagram of a path γ_t from g_0' to $\tilde{g}_0'$.

For a path γ_t in X_{2n-1}' given by $\prod_{j=1}^{n}(u - z_j(t))$ we denote by $\tilde{\gamma}_t$ the path given by $\prod_{j=1}^{n}(u + z_j(t))$. If γ_t is the path from g_0' to $\tilde{g}_0'$ described above, we can associate to $\tilde{\gamma}_t$ a square diagram (see the top half of Figure 9(b)). Its labels at the bottom edge match its labels at the top edge by the same arguments as above.

We now compose γ_t and $\tilde{\gamma}_t$, which is by construction an odd loop in X'_{2n-1}. Its corresponding square diagram R'' (see Figure 9(b)) is thus odd. It contains curves with vertical segments, but after a small isotopy that maintains oddness all curves are strictly monotone increasing or decreasing. Therefore, it is a Rampichini diagram.

We claim that the labels at $\varphi = \pi$ in this Rampichini diagram spell the word $\iota_n(B)m_n(B)$. We already know that the lower half of the diagram spells $\iota_n(B)$, since the lower left quadrant has the same labels as the original diagram R. By the same arguments, the upper right quadrant is the original diagram R except that all labels on the right edge have been shifted by n mod $2n-1$. It follows that the labels on the left edge in the upper half of R'' are the same as the labels on the left edge of R shifted by n mod $2n-1$. Therefore the labels in the upper half of R'' at $\varphi = \pi$ are the labels on the right edge of R shifted by $n-1 = -n$ mod $2n-1$, except if the corresponding label in R was of the form $a_{i,1}$, which is the only label that is affected by conjugation by $\prod_{j=1}^{n-1} \tau_j$. In this case the label on the right edge in the upper right quadrant is $a_{i+n,n+1}$. On the left edge of R'' it becomes $a_{n,i+n-1 \bmod 2n-1} = a_{n,i-n \bmod 2n-1}$ and after conjugation by $\prod_{j=1}^{n-1} \tau_j = (1 \to 2 \to 3 \to \cdots \to n-1 \to n)$ we have $a_{1,i-n \bmod 2n-1}$. Hence the labels in the top half at $\varphi = \pi$ spell $m_n(B)$. □

Note that if a knot K is a closure of a P-fibered braid B on n strands, then the closure of $\iota_n(B)m_n(B)$ is the connected sum $K \# K$. Following the construction above we obtain an odd loop of polynomials for the P-fibered braid $\iota_n(B)m_n(B)$ on $2n-1$ strands. By Theorem 3.1 we get a semiholomorphic polynomial with an isolated singularity at the origin and $K \# K$ as the link of that singularity. This proves the P-fibered/semiholomorphic version of a result by Looijenga [23].

Theorem 3.7 *Let K be a knot that is the closure of a P-fibered braid on n strands. Then $K \# K$ is real algebraic and the corresponding real polynomial can be taken to be semiholomorphic and of degree $2n-1$ with respect to the complex variable u.*

4 Trigonometric interpolation and approximation

The goal of this section is to prove that any singular Rampichini diagram that is obtained from a pure Rampichini diagram by inserting circles denoting singularities can be realized by a loop of polynomials h_t whose coefficients are polynomials in e^{it} and e^{-it}.

The following result follows immediately from a theorem by Deutsch [18], which is a generalization of work by Walsh [37].

Lemma 4.1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a 2π-periodic C^k-function. Then, for any chosen set of points z_j for $j = 1, 2, \dots, M$ and any $\varepsilon > 0$, there is a trigonometric polynomial f_{trig} with $f_{\mathrm{trig}}^{(i)}(z_j) = f^{(i)}(z_j)$ for all $j = 1, 2, \dots, M$ and $i = 1, 2, \dots, k$, and $|f - f_{\mathrm{trig}}|_k < \varepsilon$.*

Proof The proof is completely analogous to that of Corollary 3.2 in [18]. For every $j = 1, 2, \dots, M$ and $i = 1, 2, \dots, k$, we can define the operator $L_{i,j}(g) = g^{(i)}(z_j)$ on the space of 2π-periodic C^k-functions, which evaluates the i^{th} derivative of a function at the point z_j. Then every $L_{i,j}$ is a continuous linear operator, since

$$(16) \qquad |L_{i,j}(g)| = |g^{(i)}(z_j)| \le \sup_{x\in\mathbb{R}} |g^{(i)}(x)| \le \sum_{i=1}^{k} \sup_{x\in\mathbb{R}} |g^{(i)}(x)| = |g|_k.$$

The lemma then follows from [18, Theorem 0], since trigonometric polynomials are dense in the space of 2π-periodic functions with the C^k-norm. □

Applying this lemma to the real and imaginary part of a 2π-periodic C^k-function $f : \mathbb{R} \to \mathbb{C}$ results in the analogous result for finite Fourier series.

For a power series $f = \sum_{j=0}^{\infty} a_j t^j$, we call the smallest index j with $a_j \neq 0$ the *lowest order* of f.

Lemma 4.2 *Let B be a P-fibered geometric braid on n strands with a pure Rampichini diagram R. Let $v_j(t)$ for $j = 1, 2, \dots, n-1$ denote the critical values of the corresponding loop of polynomials. Let $\{(t_i, j(i))\}_{i=1,2,\dots,M}$ with $t_i \in [0, 2\pi]$, $t_i \neq t_j$ if $i \neq j$, and $j(i) \in \{1, 2, \dots, n-1\}$ be such that for every $i \in \{1, 2, \dots, M\}$ we have $\arg(v_{j(i)}(t_i)) \neq \arg(v_k(t_i))$ for all $k \neq j(i)$. Then there exists a loop h_t in the space of monic polynomials with fixed degree n such that all of its coefficients are finite Fourier series and h_t has an associated singular Rampichini diagram given by $R_{\{(t_i, j(i))\}_{i=1,2,\dots,M}}$.*

Proof Since B is a P-fibered geometric braid, we know that the critical values $v_j(t)$ of the corresponding loop of polynomials satisfy $\partial\arg(v_j(t))/\partial t \neq 0$ for all $t \in [0, 2\pi]$. This property does not change under small C^1-deformations of $v_j(t)$, ie there is an ε_0 such that, for all positive $\varepsilon < \varepsilon_0$ and any smooth function $F : [0, 2\pi] \to \mathbb{C}$ with $|F|_1 < \varepsilon$, we have $v_j(t) + F(f) \neq 0$ and $\partial\arg(v_j(t) + F(t))/\partial t \neq 0$ for all t. This can be seen from $\arg(v_j(t) + F(t)) = \operatorname{Im}\big(\operatorname{Log}(v_j(t) + F(t))\big)$ and thus

$$(17) \qquad \frac{\partial\arg(v_j + F)}{\partial t} = \frac{\operatorname{Re}(v_j + F)\,\partial\operatorname{Im}(v_j + F)/\partial t - \operatorname{Im}(v_j + F)\,\partial\operatorname{Re}(v_j + F)/\partial t}{(\operatorname{Re}(v_j + F))^2 + (\operatorname{Im}(v_j + F))^2}.$$

Let $\varepsilon < \varepsilon_0$ be such a small positive number. In particular, the above implies that $|v_j(t)| > \varepsilon$ for all j and t. Pick a small number $\delta > 0$ and define V_i as the open interval $(t_i - \delta, t_i + \delta)$. In particular, δ should be chosen so that in every V_i the argument of $v_{j(i)}(t)$ differs from the argument of any other critical value. Let $f_j : [0, 2\pi] \to \mathbb{R}$ be the smooth function defined to be $-\mathrm{e}^{-((t-t_i)/((t-t_i)^2-\delta^2))^2}$ on all V_i with $j(i) = j$ and everywhere else constant 0.

We may deform the critical values $v_j(t)$ without changing $\arg v_j(t)$ such that $|v_{j(i)}(t)|$ becomes small in V_i for all i. More precisely, we want that $|v_{j(i)}(t_i)|$ is a global minimum of $|v_{j(i)}(t)|$ (and in particular $|v_{j(i)}(t_i)| = |v_{j(i')}(t_{i'})|$ for all i and i' with $j(i) = j(i')$) and $|f_j v_{j(i)}(t)|_1 < \varepsilon/n$ for all t in any V_i with $j(i) = j$. Note that ε is the same as above and does not depend on i. This deformation of the critical values can be performed without changing $v_j(t)$ in any V_i with $j(i) \neq j$.

This deformation lifts to a homotopy of the loop of polynomials with distinct roots and fixed degree [12]. We can now approximate the critical points $c_j(t)$ for $j=1,2,\dots,n-1$ of the resulting polynomials g_t (the endpoint of the homotopy) and its constant term $a_0(t)$ by finite Fourier series using the C^2-norm, while interpolating the values of $c_j(t)$, $a_0(t)$ and their first and second derivatives at $t=t_i$ for $i=1,2,\dots,M$. (It is at this point that we use the fact that R is pure. Since $v_j(2\pi)=v_j(0)$, we have $c_j(2\pi)=c_j(0)$, which is necessary for a finite Fourier series.) This approximation and simultaneous interpolation exists by Lemma 4.1. By choosing the approximation close enough to the original $c_j(t)$ and $a_0(t)$, we can guarantee that there are still no argument-critical points of g_t. Since all critical values are smooth (in fact, analytic) functions of the critical points and the constant term, we can arrange that the critical values $w_j(t)$ for $j=1,2,\dots,n-1$ of g_t satisfy $|w_j(t)-v_j(t)|_1<\varepsilon/n$ on all V_i with $j(i)\neq j$. Furthermore, since all critical points and the constant term are parametrized by finite Fourier series, every coefficient of g_t is a finite Fourier series. It follows that the critical values $w_j(t)$ for $j=1,2,\dots,n-1$ of g_t are also parametrized by finite Fourier series and are nonzero. Moreover, the interpolation guarantees that we still have $|w_{j(i)}(t_i)|$ as a global minimum of $|w_{j(i)}(t)|$ and choosing the approximation sufficiently close gives $|f_j w_j(t)|_1<\varepsilon/n$. Note that this inequality is automatically satisfied outside of the V_i with $j(i)=j$, where $f_j\equiv 0$.

We claim that we can obtain the desired loop of polynomials h_t by only changing the constant term of g_t. More specifically, we are going to add $n-1$ finite Fourier series $A_1(t), A_2(t),\dots,A_{n-1}(t)$ to its constant term. These functions are defined successively as follows. We perform a simultaneous trigonometric interpolation and approximation of the function f_j and its first N derivatives, where N is some sufficiently large natural number. The resulting interpolating trigonometric polynomial F_j should agree with the values of f_j and its first N derivatives on all $t=t_i$. Then we define $A_j(t)=F_j(t)w_{j,j-1}(t)$, where $w_{k,j}(t)=w_k(t)+\sum_{i=1}^{j}A_i(t)$. Note that adding a term $A_j(t)$ to the constant term of a loop of polynomials shifts all critical values by $A_j(t)$, so $w_{k,j}(t)$ for $k=1,2,\dots,n-1$ are the critical values of $g_t+\sum_{i=1}^{j}A_i(t)$. We prove the following claim by induction on j.

Claim *If the approximation of f_j by F_j is sufficiently close for all $j=1,2,\dots,n-1$, then the following properties hold for all $j=1,2,\dots,n-1$:*

- $w_{k,j}(t)=0$ *if and only if* $k\le j$ *and* $t=t_i$ *for some* i *with* $j(i)=k$.
- $\arg(w_{k,j}(t))$ *and* $\partial\arg(w_{k,j}(t))/\partial t$ *have well-defined limits as* t *goes to* t_i *for all* i *with* $j(i)=k$, *with the latter limit being nonzero.*
- *The critical values* $w_{k,j}(t)$ *for* $k=1,2,\dots,n-1$ *have no argument-critical points, wherever* $w_{k,j}(t)\neq 0$.
- *For all* k *we have* $|w_{k,j}-v_k|_1<(j+1)\varepsilon/n$ *on all* V_i *with* $j(i)\neq k$.
- *For all* $k>j$ *we have* $|f_k w_{k,j}|_1<\varepsilon/n$ *on all* V_i.
- $|w_{k,j}(t_i)|$ *is a global minimum of* $|w_{k,j}(t)|$ *for all* i *with* $j(i)=k$.

Before we go into the proof of the claim, note that each A_j is indeed a finite Fourier series, since F_j is a trigonometric polynomial and w_j is a finite Fourier series for all $j = 1, 2, \dots, n-1$.

Base case $j = 1$ First we focus on $w_{1,1}$. The function $A_1(t)$ was constructed such that $w_{1,1}(t) = 0$ if and only if $t = t_i$ and $j(i) = 1$. This can be seen as follows. There is a neighborhood of $t = t_i$ with $j(i) = 1$ such that in that neighborhood any sufficiently close approximation of f_1 in the C^2-norm has positive second derivative. It follows that $t = t_i$ is the unique point where F_1 takes the value -1 in that neighborhood and thus $t = t_i$ is the only zero of $w_{1,1}$ in that neighborhood. We can guarantee that there are no zeros of $w_{1,1}$ outside of these neighborhoods, since their complement in $[0, 2\pi]$ is compact and we may choose the approximation arbitrarily close.

Furthermore, $\partial\arg(w_{1,1}(t))/\partial t \neq 0$ whenever $w_{1,1} \neq 0$, since in this case $\arg w_{1,1}(t) = \arg w_1(t)$. This equality also implies that $\arg(w_{1,1}(t))$ and $\partial\arg(w_{1,1}(t))/\partial t$ have well-defined limits as t goes to t_i with $j(i) = 1$, which are equal to $\arg(w_1(t_i))$ and $\partial\arg(w_1)/\partial t(t_i) \neq 0$, respectively.

We know that $w_1(t)$ differs from $v_1(t)$ on V_i with $j(i) \neq 1$ by at most ε/n in the C^1-norm. Since $f_1 \equiv 0$ on every such V_i, it follows that if F_1 is a sufficiently close approximation of f_1, it satisfies $|F_1(t)w_1(t)|_1 < \varepsilon/n$ on all such V_i. We thus obtain

$$|w_{1,1}(t) - v_1(t)|_1 \leq |F_1(t)w_1(t)|_1 + |w_1(t) - v_1(t)|_1 < \frac{2\varepsilon}{n} \tag{18}$$

on all V_i with $j(i) \neq 1$.

Now we turn our attention to $w_{k,1}$ with $k > 1$. We want to show that for any $k \neq 1$ the new critical values $w_{k,1}(t)$ are nonzero and satisfy $\partial w_{k,1}(t)/\partial t \neq 0$. We know that $|f_1 w_1|_1 < \varepsilon/n$ on all V_i (on those with $j(i) = 1$ by construction, on the others because $f_1 \equiv 0$). It follows that if F_1 is a sufficiently close approximation of f_1, it satisfies $|F_1(t)w_1(t)|_1 < \varepsilon/n$ on all V_i. Recall that in the V_i with $j(i) \neq k$, the critical value $w_k(t)$ with $k \neq 1$ differs from the original $v_k(t)$ by at most ε/n in the C^1-norm, so

$$|w_{k,1}(t) - v_k(t)|_1 \leq |F_1(t)w_1(t)|_1 + |w_k(t) - v_k(t)|_1 < \frac{2\varepsilon}{n}. \tag{19}$$

Thus $w_{k,1}(t)$ is nonzero and has no argument-critical points in V_i for all i with $j(i) \neq k$.

Outside of the V_i with $j(i) \neq k$ the same is true if $|F_1 w_1|_1$ is sufficiently small on $[0, 2\pi] \setminus \big(\bigcup_{i \mid j(i) \neq k} V_i\big)$, which can be arranged by choosing F_1 sufficiently close to f_1 on the compact set $[0, 2\pi] \setminus \big(\bigcup_{i \mid j(i) \neq k} V_i\big)$.

Obviously, $\arg(w_{k,1}(t))$ and $\partial\arg(w_{k,1}(t))/\partial t$ have well-defined limits as t approaches t_i with $j(i) = k > 1$, since $w_{k,1}$ with $k > 1$ has no zeros. Thus its argument and its derivative are well defined at $t = t_i$. Since there are no argument-critical points, this value of the derivative is nonzero. We have thus proved the first four items of the claim for $j = 1$.

Since for all i with $j(i) \neq 1$ we have $f_1 \equiv 0$ on all V_i, any sufficiently close approximation F_1 satisfies $|f_k w_{k,1}|_1 = |f_k(w_k + F_1 w_1)|_1 < \varepsilon/n$ on V_i for $k > 1$. Here we have used that $|f_k w_k|_1 < \varepsilon/n$ on V_i with $j(i) = k$ and $f_k \equiv 0$ everywhere else.

The sixth item is obviously true for $k=1$, since $w_{1,1}(t)=0$ if and only if $t=t_i$ by the first item. For $k>1$, note that by the interpolation property of F_1 we have

$$F_1(t_i)=F_1^{(1)}(t_i)=F_1^{(2)}(t_i)=\cdots=F_1^{(N)}(t_i)=0 \tag{20}$$

for all i with $j(i)>1$. Since $w_k(t)$ is given by a nonvanishing finite Fourier series, it is real-analytic and $|w_k(t)|$ is real-analytic, too. We want the natural number N, which determines to which order F_1 interpolates the derivatives of f_1 at $t=t_i$ with $j(i)>1$ to be larger than the lowest order of the power series of $|w_k(t)|-|w_k(t_i)|$ centered at $t=t_i$. (We are omitting the fact that this lowest order could depend on i and k. We simply choose N so that it is sufficient for all i and k.) Since $|w_k(t_i)|$ is a local minimum, this lowest order has a positive coefficient and by the interpolation property $|w_{k,1}(t)|-|w_{k,1}(t_i)|$ has the same lowest-order term. It follows that $|w_{k,1}(t_i)|$ is a local minimum and the only local minimum in a neighborhood. On the compact complement of these neighborhoods in $[0,2\pi]$ we can guarantee that $|w_{k,1}(t)|>|w_{k,1}(t_i)|$ by choosing a sufficiently close approximation F_1 of f_1. On $[0,2\pi]\backslash\big(\bigcup_{i\,|\,j(i)=1}V_i\big)$, where $f_1\equiv 0$, this is obvious. On V_i with $j(i)=1$ we may use that $|w_k(t)|=|v_k(t)|>\varepsilon$, so

$$\begin{aligned}|w_{k,1}(t)|=|w_k(t)+A_1(t)|\geq\big||w_k(t)|-|A_1(t)|\big|>\varepsilon-\frac{\varepsilon}{n}>\frac{\varepsilon}{n}\\ >|f_kw_k|_1\geq|f_k(t_i)w_k(t_i)|=|w_{k,1}(t_i)|.\end{aligned} \tag{21}$$

Here we have used that $|A_1(t)|=|F_1(t)w_1(t)|<\varepsilon/n$. Thus $|w_{k,1}(t_i)|$ with $j(i)=k$ are global minima of $|w_{k,1}(t)|$ for all k. This concludes the proof of the six items in the claim for the base case of $j=1$.

Induction We assume that the claim holds for some $j-1\in\{1,2,\dots,n-2\}$ and want to show that it then also holds for j.

By the same arguments as in the base case, we have $w_{j,j}(t)=0$ if and only if $t=t_i$ for i with $j(i)=j$. Away from these points we have $\arg(w_{j,j}(t))=\arg(w_{j,j-1}(t))$, so there are no argument-critical points of $w_{j,j}$ and $\arg(w_{j,j}(t))$ and $\partial\arg(w_{j,j}(t))/\partial t$ have well-defined limits, equal to $\arg(w_{j,j-1}(t_i))$ and $\partial\arg(w_{j,j-1})/\partial t(t_i)$, as t goes to t_i with $j(i)=j$.

For $w_{k,j}$ consider that, by the induction hypothesis, $|w_{k,j-1}-v_k|_1<j\varepsilon/n$ on all V_i with $j(i)\neq k$. The induction hypothesis also states that $|f_jw_{j,j-1}|_1<\varepsilon/n$ on all V_i. This means that on all V_i with $j(i)\neq k$ we have

$$|w_{k,j}-v_k|_1\leq|F_jw_{j,j-1}|_1+|w_{k,j-1}-v_k|_1<\frac{(j+1)\varepsilon}{n} \tag{22}$$

if F_j is sufficiently close to f_j. This implies that $w_{k,j}(t)$ is nonzero and has no argument-critical points in V_i with $j(i)\neq k$. Since for $k>j$ the critical value $w_{k,j-1}(t)$ is nonzero and has no argument-critical points, the same is true for $w_{k,j}(t)$ on the compact set $[0,2\pi]\setminus\big(\bigcup_{i\,|\,j(i)\neq k}V_i\big)$ if the approximation of f_j by F_j is sufficiently close. Since $w_{k,j}(t)$ with $k>j$ is nonvanishing, its argument and the derivative of its argument are well defined everywhere and naturally have a well-defined, nonzero limit at $t=t_i$. This proves the first four items of the claim for $w_{k,j}$ with $k\geq j$ and the fourth item for $w_{k,j}$ with $k<j$.

For the fifth item, consider that, on V_i with $j(i) \neq k$, the function f_k is constant zero, while on V_i with $j(i) = k > j$ the function f_j is constant zero. In either case we have that

$$|f_k w_{k,j}|_1 = |f_k(w_{k,j-1} + f_j w_{j,j-1})|_1 < \frac{\varepsilon}{n} \tag{23}$$

on V_i by the induction hypothesis.

The sixth item in the claim is trivial for all k with $k \leq j$, since in these cases we already know that $t = t_i$ with $j(i) = k$ are the only zeros of $w_{k,j}(t)$. For $k > j$ it is the same argument as in the base case ($j = 1$). The fact that F_j interpolates f_j and its first N derivatives guarantees that $t = t_i$ is a local minimum and, by choosing the approximation sufficiently close, we can show that these are global minima of $|w_{k,j}(t)|$.

What is left to show are the first three items of the claim for $w_{k,j}(t)$ with $k < j$. This is particularly subtle in neighborhoods of $t = t_i$ with $j(i) = k$, where $w_{k,j}$ has a zero.

Since $w_{k,j-1}$ for $k < j$ is a finite Fourier series, both the numerator and the denominator of $\partial \arg w_{k,j-1}/\partial t$, given by

$$\text{NUM} = \text{Re}(w_{k,j-1}(t))\frac{\partial \text{Im}(w_{k,j-1})}{\partial t}(t) - \text{Im}(w_{k,j-1}(t))\frac{\partial \text{Re}(w_{k,j-1})}{\partial t}(t) \tag{24}$$

and

$$\text{DEN} = \big(\text{Re}(w_{k,j-1}(t))\big)^2 + \big(\text{Im}(w_{k,j-1}(t))\big)^2, \tag{25}$$

are real analytic. As both of these go to zero as t approaches t_i with $j(i) = 1$, the derivative $\partial \arg w_{k,j-1}/\partial t$ is not defined at these points. However, by assumption it has a well-defined limit as t approaches t_i and, by the interpolation property of each F'_j for $j' < j$, it is equal to $\partial w_k/\partial t(t_i) \neq 0$.

Since both NUM and DEN are real-analytic, we may consider their power series at $t = t_i$. Let $\sum_m b_m(t - t_i)^m$ be the power series of DEN and let $m(i)$ denote the lowest index with $b_{m(i)} \neq 0$. (Since the limit of their quotient is well defined and nonzero, using the power series of DEN to define $m(i)$ would have the same result.) From the interpolation properties of F_i for $i = 1, 2, \dots, j$, we know that the lowest-order term of F_k is quadratic and the lowest order of F_i for $i \neq k$ is $N + 1 > 2$. Since $w_{k,k-1}(t)$ is nonzero for all t, we have that $m(i) = 4$. We know thus that for any $\delta' > 0$ there are neighborhoods $W_i \subset V_i$ of t_i on which |NUM| and |DEN| are both greater than $\delta'|(t - t_i)^m|$, where $m > m(i) = 4$.

Since we chose F_j so that $|F_j|_N < \varepsilon'$ outside of V_i with $j(i) = j$ for some arbitrarily small ε', we have $|F_j(t)| < |(t - t_i)^N|\varepsilon'$ and $|\partial F_j(t)/\partial t| < |(t - t_i)^{N-1}|\varepsilon'$ for all $t \in V_i$ with $j(i) \neq j$. It follows that, in $W_i \subset V_i$ with $j(i) \neq j$, we have $|w_{k,j}(t)| \geq \delta'|(t - t_i)^m| - |(t - t_i)^N|\varepsilon'' \max_{t \in W_i} |w_{j,j-1}(t)|$ with $N > m > m(i) = 4$, which has no zero in W_i apart from $t = t_i$ if ε'' is sufficiently small. Therefore, the t_i with $j(i) = k$ are the only zeros of $w_{k,j}$ in W_i when $j(i) \neq j$. Since these were also the only zeros of $w_{k,j-1}$ anywhere, a sufficiently close approximation F_j of f_j guarantees that there are no zeros of $w_{k,j}$ outside of the W_i with $j(i) \neq j$ either. As in the previous cases this follows from $f_j \equiv 0$ outside of V_i with $j(i) = j$, and $|w_{k,j}| \geq |v_k| - |w_{k,j} - v_k| > (n - j - 1)\varepsilon/n$ on V_i with $j(i) = j$.

Similarly, the modulus of the numerator $\mathrm{NUM}_{k,j}$ of $\partial \arg w_{k,,j}/\partial t$ satisfies

$$
\begin{aligned}
(26)\quad |\mathrm{NUM}_{k,j}| &= \left|\mathrm{Re}(w_{k,j-1}+A_j)\frac{\partial \mathrm{Im}(w_{k,j-1}+A_j)}{\partial t} - \mathrm{Im}(w_{k,j-1}+A_j)\frac{\partial \mathrm{Re}(w_{k,j-1}+A_j)}{\partial t}\right| \\
&= \left|\mathrm{NUM}_{k,j-1} + \mathrm{Re}(w_{k,j-1})\frac{\partial \mathrm{Im}(A_j)}{\partial t} - \mathrm{Im}(w_{k,j-1})\frac{\partial \mathrm{Re}(A_j)}{\partial t} + \mathrm{Re}(A_j)\frac{\partial \mathrm{Im}(w_{k,j-1})}{\partial t}\right. \\
&\qquad \left. - \mathrm{Im}(A_j)\frac{\partial \mathrm{Re}(w_{k,j-1})}{\partial t} + \mathrm{Re}(A_j)\frac{\partial \mathrm{Im}(A_j)}{\partial t} - \mathrm{Im}(A_j)\frac{\partial \mathrm{Re}(A_j)}{\partial t}\right| \\
&> \delta|(t-t_i)^m| - 2|(t-t_i)^N|\varepsilon' \max_{t\in W_i}\left|\frac{\partial w_{k,j-1}}{\partial t}(t)\right| \max_{t\in W_i}|w_{j,j-1}(t)| \\
&\qquad - 2|(t-t_i)^{N-1}|\varepsilon' \max_{t\in W_i}|w_{k,j-1}(t)| \max_{t\in W_i}|w_{j,j-1}(t)| \\
&\qquad - 2|(t-t_i)^N|\varepsilon' \max_{t\in W_i}|w_{k,j-1}(t)| \max_{t\in W_i}\left|\frac{\partial w_{j,j-1}}{\partial t}(t)\right| \\
&\qquad - 2|(t-t_i)^{2N-1}|\varepsilon'^2 \max_{t\in W_i}|w_{j,j-1}(t)|^2 \\
&\qquad - 4|(t-t_i)^{2N}|\varepsilon'\left|\frac{\partial w_{j,j-1}}{\partial t}(t)\right| \max_{t\in W_i}|w_{j,j-1}(t)| \\
&> 0
\end{aligned}
$$

in W_i with $j(i) \neq j$, where we use $A_j = F_j w_{j,j-1}$ and $|\mathrm{Re}(z)| \leq |z|$ for all $z \in \mathbb{C}$ (and likewise for $\mathrm{Im}(z)$). The last inequality holds because ε' can be chosen arbitrarily small and $N > m$.

Thus there are no argument-critical points of $w_{k,j}$ in W_i with $j(i) \neq j$. We already showed above that there are no argument-critical points of $w_{k,j}$ in V_i with $j(i) \neq k$. Since $w_{k,j-1}$ is nonzero and has no argument-critical points in the complement of these W_i and V_i, a sufficiently close approximation F_j of f_j guarantees that there are no argument-critical points of $w_{k,j}$ in the complement either.

Note that the interpolation property of F_j guarantees that $\arg(w_{k,j}(t))$ and $\partial \arg(w_{k,j}(t))/\partial t$ have well-defined limits as t goes to t_i with $j(i) = k$, which equal $\arg(w_{k,j-1}(t_i))$ and $\partial \arg(w_{k,j-1})/\partial t(t_i) \neq 0$, respectively. This concludes the proof of the claim by induction.

The desired loop of polynomials h_t is given by $g_t + \sum_{j=1}^{n-1} A_j(t)$ with critical values $w_{j,n-1}(t)$ for $j = 1, 2, \dots, n-1$. The first three items of the claim imply that the square diagram associated to h_t is a singular Rampichini diagram and, by construction, it is equal to $R_{\{(t_i, j(i))\}_{i=1,2,\dots,M}}$. □

Lemma 4.3 *Let B be a P-fibered braid on $n = 2n'-1$ strands with an odd, pure Rampichini diagram R. Suppose that the set $\{(t_i, j(i))\}_{i=1,2,\dots,M}$ with M even from Lemma 4.2 satisfies the following symmetry. For every $i = 1, 2, \dots, \frac{1}{2}M$, we have $t_{i+M/2} = t_i + \pi \mod 2\pi$ and $k\big(i+\frac{1}{2}M\big) = k(i) + n' - 1 \mod 2n'-2$, where $v_{j(i+M/2)}(t_{i+M/2}) = v(t_{i+M/2})_{k(i+M/2)}$ and $v_{j(i)}(t_i) = v(t_i)_{k(i)}$. Then h_t in Lemma 4.2 can be taken to be odd.*

Proof This lemma only requires small modifications to the proof of Lemma 4.2. Since R is odd, we may take g_t to be odd. Since the set $\{(t_i, j(i))\}_{i=1,2,\dots,M}$ has the symmetry above, the data set for the interpolation can be taken to have an odd symmetry. In other words, all functions f_j that are approximated can be taken to satisfy $f_j(x+\pi) = -f_j(x)$. Since the set of odd trigonometric polynomials is dense in the space of odd periodic functions, Lemma 4.1 guarantees that the interpolating function F_j can also be taken to be odd. Note that the critical values $w_j(t)$ of g_t are odd functions, so A_j is also odd. We obtain $h_{t+\pi}(u) = g_{t+\pi}(u) + \sum_{j=1}^{n-1} A_j(t+\pi) = -g_t(-u) - \sum_{j=1}^{n-1} A_j(t) = -h_t(-u)$. □

Lemma 4.4 *The functions $|w_{j,n-1}(t)|$ are real-analytic.*

Proof Since $w_{j,n-1}(t)$ is a finite Fourier series, its absolute value is real-analytic, except possibly at values of t, where $w_{j,n-1}(t) = 0$. By construction,

$$w_{j,n-1}(t) = (1+F_j(t))w_{j,j-1}(t) + \sum_{k>j} A_k(t) \tag{27}$$

and each $A_k(t)$ has a lowest order of at least 5, while the power series of $(1+F_j(t))^2|w_{j,j-1}(t)|^2$ has lowest order equal to 4. Thus the root of $|w_{j,n-1}(t)|^2$ at $t = t_i$ is exactly of order 4. We can thus write $|w_{j,n-1}(t)|^2 = (t-t_i)^4 W(t)$, where W is real-analytic with $W(t_i) > 0$. Then $|w_{j,n-1}(t)|$ is given by $(t-t_i)^2\sqrt{W(t)}$, which is a well-defined nonnegative function which squares to $|w_{j,n-1}(t)|^2$ and is real-analytic, since $W(t)$ is nonzero in a neighborhood of t_i. □

5 Isolated singularities

Let R be an odd, pure Rampichini diagram corresponding to a P-fibered braid B on n strands. By Lemma 4.2, we obtain a singular Rampichini diagram $R_{\{t_i, j(i)\}_{i=1,2,\dots,M}}$ from a loop h_t in the space of polynomials, whose coefficients are trigonometric polynomials, for any given set of t_i and $j(i)$. Furthermore, by Lemma 4.3 the new trigonometric loop h_t still is odd, satisfying $h_{t+\pi}(u) = -h_t(-u)$, if the data $(t_i, j(i))$ displays an odd symmetry. We write B_{sing} for the singular braid that is formed by the roots of h_t.

We can use the construction from Theorem 3.1 to obtain a semiholomorphic, radially weighted homogeneous polynomial

$$f(u, r\mathrm{e}^{\mathrm{i}t}) = r^{kn} h\left(\frac{u}{r^k}, \mathrm{e}^{\mathrm{i}t}\right), \tag{28}$$

where $h(u,t) = h_t(u)$, $n = \deg(h_t)$ and k is a sufficiently large odd integer. The fact that this is a polynomial (as opposed to a rational map or a function involving square roots) follows from the same reasoning as Theorem 3.1.

Note that the variety of f is the cone of the closed singular braid B_{sing}. Since the singular crossings are critical points, f does not have an isolated singularity (not even weakly isolated). It is a degenerate mixed function in the sense of Oka [27] and in the sense of [2]. In [27], Oka generalizes the definition of Newton polygons of complex polynomials to mixed functions and in [2] we investigate deformations of nondegenerate mixed functions and found that adding higher-order terms (above the boundary of the Newton polygon of the mixed function) does not change the link of a singularity. Since we are now dealing with a degenerate function, our situation is different. We claim that we can add a term of the form $r^m A(\mathrm{e}^{\mathrm{i}t})$, where m is a large even integer and A is an even polynomial in $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$, ie $A(t+\pi)=A(t)$, and construct in this way a polynomial with an isolated singularity. The function A is found via trigonometric interpolation.

Recall that the critical values of h_t are denoted by $w_{j,n-1}(t)$, whose zeros occur at $t=t_i$ for $i=1,2,\dots,M$, corresponding to singular crossings in B_{sing} and singularities in the singular Rampichini diagram. We assume that the conditions from Lemma 4.3 are satisfied, so M is even and the t_i are indexed so that $t_{i+M/2}=t_i+\pi \mod 2\pi$ for all $i=1,2,\dots,\frac{1}{2}M$. A singular crossing of B_{sing} corresponds to a circle in the singular Rampichini diagram at t_i on the curve corresponding to the critical value $w_{j(i),n-1}(t_i)$. This means that $w_{j,n-1}(t)=0$ if and only if $j=j(i)$ and $t=t_i$. By construction, $\arg(w_{j(i),n-1}(t))$ and $\partial\arg(w_{j(i),n-1})/\partial t$ have well-defined limits as t goes to t_i, which we denote by $\arg(w_{j(i),n-1}(t_i))$ and $\partial\arg(w_{j(i),n-1})/\partial t(t_i)$, respectively.

We will find A via a trigonometric interpolation that fixes the values of A and $\partial A/\partial t$ at each t_i for $i=1,2,\dots,M$. We can take a solution of this interpolation problem to be of degree M. Set $m=\max\{kn+1,M\}$, where k is the integer chosen in (28).

We define $A\colon \mathbb{R}\to\mathbb{C}$ to be the even finite Fourier series that solves

$$\arg(A(t_i))=\arg(w_{j(i),n-1}(t_i))+\pi,\qquad \frac{\partial|A|}{\partial t}(t_i)=0,\qquad \frac{\partial\arg(A)}{\partial t}=\frac{kn+m}{2m-kn}\frac{\partial\arg(w_{j(i),n-1})}{\partial t}(t_i) \tag{29}$$

for all $i=1,2,\dots,\frac{1}{2}M$. By the even symmetry of A, this also fixes the value of A and its derivative at t_i for $i=\frac{1}{2}M+1,\frac{1}{2}M+2,\dots,M$. It is implicit in the interpolation data that $A(t_i)\neq 0$ for all i.

Lemma 5.1 *Let $A\colon \mathbb{R}\to\mathbb{C}$ be an even finite Fourier series as above. Then $p=f+r^mA$ is a semiholomorphic polynomial with an isolated singularity.*

Proof The function f is a semiholomorphic polynomial. Since m is at least the degree of A, we can cancel the denominator of $r^mA(\mathrm{e}^{\mathrm{i}t})=(v\bar{v})^{m/2}A(v/\sqrt{v\bar{v}})$. Since A is even, all remaining factors of $\sqrt{v\bar{v}}$ come with an even exponent, so $f+r^mA$ is a polynomial in u, v and $\bar{v}$.

One advantage of working with semiholomorphic polynomials is that a critical point must satisfy $\partial p/\partial u=0$. Since $\partial p/\partial u=\partial f/\partial u$, this can only happen at critical points of f, considered as a family of complex polynomials, parametrized by v. Note that $p|_{v=0}=u^n$, so the origin is the only critical point with $v=0$. The u-coordinate of a point with $\partial f/\partial u=0$ is thus $r^k c_j(t)$, where $c_j(t)$ is a critical point of h_t. Thus all points where the derivative of p with respect to u vanishes are of the form $(r^k c_j(t), r\mathrm{e}^{\mathrm{i}t})$ with $p(r^k c_j(t), r\mathrm{e}^{\mathrm{i}t})=r^{kn}w_{j,n-1}(t)+r^mA(t)$. We now consider the real Jacobian matrix of p at such a point in coordinates $(\mathrm{Re}(u),\mathrm{Im}(u),r,t)=(\mathrm{Re}(u),\mathrm{Im}(u),|v|,\arg(v))$:

$$\begin{pmatrix} 0 & 0 & knr^{kn-1}\,\mathrm{Re}(w_{j,n-1}(t))+mr^{m-1}\,\mathrm{Re}(A(t)) & r^{kn}\,\partial\mathrm{Re}(w_{j,n-1}(t))/\partial t+r^m\,\partial\mathrm{Re}(A(t))/\partial t \\ 0 & 0 & knr^{kn-1}\,\mathrm{Im}(w_{j,n-1}(t))+mr^{m-1}\,\mathrm{Im}(A(t)) & r^{kn}\,\partial\mathrm{Im}(w_{j,n-1}(t))/\partial t+r^m\,\partial\mathrm{Im}(A(t))/\partial t \end{pmatrix}. \tag{30}$$

In order to show that the singularity at the origin is isolated, we show that for small positive values of r this matrix has full rank. For this we compute the determinant of the 2-by-2 matrix given by the nonzero entries above. The determinant is equal to

$$\begin{aligned}(31)\quad & knr^{2kn-1}\big(\mathrm{Re}(w_{j,n-1}(t))\,\mathrm{Im}(w'_{j,n-1}(t))-\mathrm{Im}(w_{j,n-1}(t))\,\mathrm{Re}(w'_{j,n-1}(t))\big)\\ &\quad+knr^{kn+m-1}\big(\mathrm{Re}(w_{j,n-1}(t))\,\mathrm{Im}(A'(t))-\mathrm{Im}(w_{j,n-1}(t))\,\mathrm{Re}(A'(t))\big)\\ &\quad+mr^{kn+m-1}\big(\mathrm{Re}(A(t))\,\mathrm{Im}(w'_{j,n-1}(t))-\mathrm{Im}(A(t))\,\mathrm{Re}(w'_{j,n-1}(t))\big)\\ &\quad+mr^{2m-1}\big(\mathrm{Re}(A(t))\,\mathrm{Im}(A'(t))-\mathrm{Im}(A(t))\,\mathrm{Re}(A'(t))\big),\end{aligned}$$

where the dash denotes the derivative with respect to t. By assumption we have that $m > kn$.

For a set of open neighborhoods U_i (to be determined later) of the t_i, the resulting expression goes to

$$(32)\quad \mathrm{Re}(w_{j,n-1}(t))\,\mathrm{Im}(w'_{j,n-1}(t))-\mathrm{Im}(w_{j,n-1}(t))\,\mathrm{Re}(w'_{j,n-1}(t))=|w_{j,n-1}(t)|^2\arg(w_{j,n-1})'(t)\neq 0$$

on $[0,2\pi]\setminus\big(\bigcup_{i\,|\,j(i)=j}U_i\big)$ as r goes to zero. Thus we obtain a nonzero determinant for all points $(r^k c_j(t), r\mathrm{e}^{\mathrm{i}t})$ with sufficiently small r except possibly for those with $j=j(i)$ and $t\in U_i$ for some i.

For every $i=1,2,\dots,M$ we define $R_1(t)=|A(t)|$ and $\delta_i(t)$ such that

$$(33)\quad A(t)=R_1(t)\mathrm{e}^{\mathrm{i}\left(\arg(w_{j(i),n-1}(t))+\delta_i(t)\right)}$$

on U_i. Furthermore, we write $\alpha_i(t)$ for $\arg(w_{j(i),n-1}(t))$ and $R_2(t)=|w_{j(i),n-1}(t)|$. This means that

$$(34)\quad \begin{aligned}\mathrm{Re}(A(t))&=R_1(t)\big(\cos\arg(w_{j(i),n-1}(t))\cos\delta_i(t)-\sin\arg(w_{j(i),n-1}(t))\sin\delta_i(t)\big),\\ \mathrm{Im}(A(t))&=R_1(t)\big(\cos\arg(w_{j(i),n-1}(t))\sin\delta_i(t)+\sin\arg(w_{j(i),n-1}(t))\cos\delta_i(t)\big).\end{aligned}$$

We now look at the individual terms in (31). First,

$$(35)\quad \mathrm{Re}(w_{j(i),n-1}(t))\,\mathrm{Im}(w'_{j(i),n-1}(t))-\mathrm{Im}(w_{j(i),n-1}(t))\,\mathrm{Re}(w'_{j(i),n-1}(t))=R_2(t)^2\alpha_i(t)',$$

which is 0 if and only if $t=t_i$.

Second,

$$\begin{aligned}(36)\quad & \mathrm{Re}(w_{j(i),n-1}(t))\,\mathrm{Im}(A'(t))-\mathrm{Im}(w_{j(i),n-1}(t))\,\mathrm{Re}(A(t))'\\ &=R_2(t)\big\{\cos\alpha_i(t)\big[R_1(t)\big(-\sin\alpha_i(t)\alpha_i(t)'\sin\delta_i(t)+\cos\alpha_i(t)\cos\delta_i(t)\,\delta_i(t)'\\ &\qquad+\cos\alpha_i(t)\,\alpha_i(t)'\cos\delta_i(t)-\sin\alpha_i(t)\sin\delta_i(t)\,\delta_i(t)'\big)\\ &\qquad+R_1'(t)(\cos\alpha_i(t)\sin\delta_i(t)+\sin\alpha_i(t)\cos\delta_i(t))\big]\\ &\quad-\sin\alpha_i(t)\,\big[R_1(t)\big(-\sin\alpha_i(t)\,\alpha_i(t)'\cos\delta_i(t)-\cos\alpha_i(t)\sin\delta_i(t)\,\delta_i(t)'\\ &\qquad-\cos\alpha_i(t)\,\alpha_i(t)'\sin\delta_i(t)-\sin\alpha_i(t)\cos\delta_i(t)\,\delta_i(t)'\big)\big]\\ &\qquad+R_1'(t)(\cos\alpha_i(t)\cos\delta_i(t)-\sin\alpha_i(t)\sin\delta_i(t))\big\}\\ &=R_2(t)(R_1(t)\cos\delta_i(t)\,(\delta_i(t)'+\alpha_i(t)')+R_1'\sin\delta_i(t)).\end{aligned}$$

Furthermore,

$$\begin{aligned}(37)\quad &\mathrm{Re}(A(t))\,\mathrm{Im}(w_{j(i),n-1}(t))' - \mathrm{Im}(A(t))\,\mathrm{Re}(w_{j(i),n-1}(t))' \\ &= R_1(t)\big[(\cos\alpha_i(t)\cos\delta_i(t) - \sin\alpha_i(t)\sin\delta_i(t))(R_2'(t)\sin\alpha_i(t) + R_2(t)\cos\alpha_i(t)\,\alpha_i(t)') \\ &\qquad - (\cos\alpha_i(t)\sin\delta_i(t) + \sin\alpha_i(t)\cos\delta_i(t))(R_2'(t)\cos\alpha_i(t) - R_2(t)\sin\alpha_i(t)\,\alpha_i(t)')\big] \\ &= R_1(t)R_2(t)\cos\delta_i(t)\alpha_i(t)' - R_1(t)R_2'(t)\sin\delta_i(t).\end{aligned}$$

Lastly,

$$(38)\qquad \mathrm{Re}(A(t))\,\mathrm{Im}(A(t))' - \mathrm{Im}(A(t))\,\mathrm{Re}(A(t))' = R_1(t)^2(\alpha_i(t)' + \delta_i(t)').$$

Thus (31) becomes

$$\begin{aligned}(39)\quad &\alpha_i(t)'(knr^{2kn-1}R_2(t)^2 + (kn+m)r^{kn+m-1}R_1(t)R_2(t)\cos\delta_i(t) + mr^{2m-1}R_1(t)^2) \\ &\qquad + \delta_i(t)'(knr^{kn+m-1}R_1(t)R_2(t)\cos\delta_i(t) + mr^{2m-1}R_1(t)^2) \\ &\qquad\qquad + \sin\delta_i(t)\,r^{kn+m-1}(knR_1'(t)R_2(t) - mR_1(t)R_2'(t)).\end{aligned}$$

Each term is a real analytic function of t (see Lemma 4.4) and thus can be locally written as a power series centered at t_i. Consider the lowest-order term of the series of $\sin\delta_i(t)\,(knR_1'(t)R_2(t) - mR_1(t)R_2'(t))$ and recall that 2 is the lowest order of R_2. Since 0 is the lowest order of R_1 and 1 is the lowest order of $\sin\delta_i$ (since $\delta_i(t_i)$ is either 0 or π, and $\delta_i'(t_i) \neq 0$ by the interpolation property), the lowest order of $\sin\delta_i\; mR_1R_2'$ is 2, while the lowest order of $\sin\delta_i\; knR_1'R_2$ is greater than 2. Likewise, the lowest orders of the series for $\alpha_i'(kn+m)R_1R_2\cos\delta_i$ and $\delta_i'knR_1R_2\cos\delta_i$ centered at t_i are also both 2.

Let r_2 denote the lowest-order coefficient of R_2. We compute the coefficient of $(t-t_i)^2$ for

$$\begin{aligned}(40)\quad &\alpha_i(t)'(kn+m)R_1(t)R_2(t)\cos\delta_i(t) + \delta_i(t)'knR_1(t)R_2(t)\cos\delta_i(t) \\ &\qquad + \sin\delta_i(t)\,(knR_1'(t)R_2(t) - mR_1(t)R_2(t)')\end{aligned}$$

and obtain

$$\begin{aligned}(41)\quad &\alpha_i'(t_i)(kn+m)r_2R_1(t_i)\cos\delta_i(t) + \delta_i'(t_i)knr_2R_1(t_i)\cos\delta_i(t) - \cos\delta_i(t)\delta_i'(t_i)mR_1(t_i)2r_2 \\ &\qquad = r_2R_1(t_i)\cos\delta_i(t_i)\,(\alpha_i'(t_i)(kn+m) + \delta_i'(t_i)(kn-2m)),\end{aligned}$$

which is 0, since $\delta_i'(t_i) = \alpha_i(t_i)(kn+m)/(2m-kn)$ by the interpolation property.

Therefore, the lowest order of (40) is at least 3. It follows that there is a neighborhood U_i of t_i where the absolute value of (40) is less than or equal to the absolute value of

$$(42)\qquad 2R_1(t)R_2(t)\sqrt{knm}\,\alpha_i(t)',$$

whose lowest order is 2. This implies that the absolute value of the determinant in (39) is at least

$$\begin{aligned}(43)\quad &|\alpha_i(t)'r^{2kn-1}(\sqrt{kn}R_2(t) - \sqrt{m}R_1(t)r^{m-kn})^2 + \delta_i(t)'mr^{2m-1}R_1(t)^2| \\ &\qquad = |\alpha_i(t)'|r^{2kn-1}(\sqrt{kn}R_2(t) - \sqrt{m}R_1(t)r^{m-kn})^2 + |\delta_i(t)'|mr^{2m-1}R_1(t)^2 \\ &\qquad \geq |\delta_i(t)'|mr^{2m-1}R_1(t)^2 > 0,\end{aligned}$$

since $r > 0$ and U_i can be chosen such that $R_1(t) > 0$ and $|\delta_i(t)'| > 0$. The first equality holds because $\alpha_i(t)'$ and $\delta_i(t)'$ have the same sign.

Thus there are no critical points $(r^k c_j(t), r\mathrm{e}^{\mathrm{i}t})$ of p with $t \in U_i$ and $r > 0$. Note that the U_i do not depend on r. Since $S^1 \setminus \left(\bigcup_{i=1}^M U_i\right)$ is compact, sufficiently small choices of $r > 0$ guarantee that $(r^k c_j(t), r\mathrm{e}^{\mathrm{i}t})$ with $t \in [0, 2\pi] \setminus \left(\bigcup_{i=1}^M U_i\right)$ is not a critical point of p either (see (32)). Therefore, the origin is an isolated singularity of p. □

6 The proof of Theorem 1.1

In order to finish the proof of Theorem 1.1 we have to determine the links of the constructed singularities via their Rampichini diagrams.

Lemma 6.1 *Let B be the vanishing set of $p|_{r=\varepsilon}\colon \mathbb{C} \times S^1 \to \mathbb{C}$, with $\varepsilon > 0$ sufficiently small. Then B is a closed geometric braid, whose braid isotopy class is P-fibered.*

A Rampichini diagram for B is obtained from $R_{\{(t_i, j(i))\}_{i=1,2,\dots,M}}$ by inserting inner loops $\gamma_{j(i)}^{\varepsilon_i}$ at $t = t_i$ for $i = 1, 2, \dots, \frac{1}{2}M$, where $\varepsilon_i = \operatorname{sign}(\partial \arg(w_{j(i),n-1})/\partial t(t_i))$, and removing the circles at $t = t_i$ for $i = \frac{1}{2}M + 1, \frac{1}{2}M + 2, \dots, M$.

Remark 6.2 We do not claim that B itself is a P-fibered *geometric* braid, as $p|_{r=\varepsilon}$ might have argument-critical points. However, B is braid isotopic to a P-fibered geometric braid.

Proof The function $p|_{r=\varepsilon}$ corresponds to a loop of monic complex polynomials of degree n with critical values $\varepsilon^{kn} w_{j,n-1}(t) + \varepsilon^m A(t)$.

We first have to show that no critical value is equal to 0, which implies that B is a closed braid. For this note that at $t = t_i$ we have $w_{j(i),n-1}(t_i) = 0$ and $A(t_i) \neq 0$, so $\varepsilon^{kn} w_{j(i),n-1}(t_i) + \varepsilon^m A(t_i) \neq 0$. Using the notation from the previous section we may write

$$\varepsilon^{kn} w_{j,n-1}(t) + \varepsilon^m A(t) = \mathrm{e}^{\mathrm{i}\alpha_i(t)} \varepsilon^{kn} (R_2(t) + R_1(t) \varepsilon^{m-kn} \mathrm{e}^{\mathrm{i}\delta_i(t)}). \tag{44}$$

Since $\delta_i(t_i) = 0$ and $\delta_i'(t_i) \neq 0$, there is a neighborhood U_i of t_i, where $\delta_i(t)$ is neither 0 nor π, such that (44) is nonzero for all t in U_i. The complement of these neighborhoods is a compact subset of S^1 on which R_2 is nonzero, so we can ensure that (44) is nonzero by choosing ε sufficiently small. This shows that no critical value is 0. Thus there are no double roots of $p|_{r=\varepsilon}$. It follows that B is a closed braid, since p is a semiholomorphic polynomial.

Now we consider the square diagram associated to the loop of polynomials $p|_{r=\varepsilon}$. Outside of the U_i the critical values approximate $w_{j,n-1}(t)$ arbitrarily well, so outside the U_i the square diagram looks exactly like the square diagram of h_t, which is a singular Rampichini diagram, all of whose singularities lie inside the U_i. Likewise, the critical values whose corresponding curves do not have a singularity in U_i are also arbitrarily well approximated by the critical values of $p|_{r=\varepsilon}$. Thus we only have to study what happens in a neighborhood of the singular points of the singular Rampichini diagram associated to h_t.

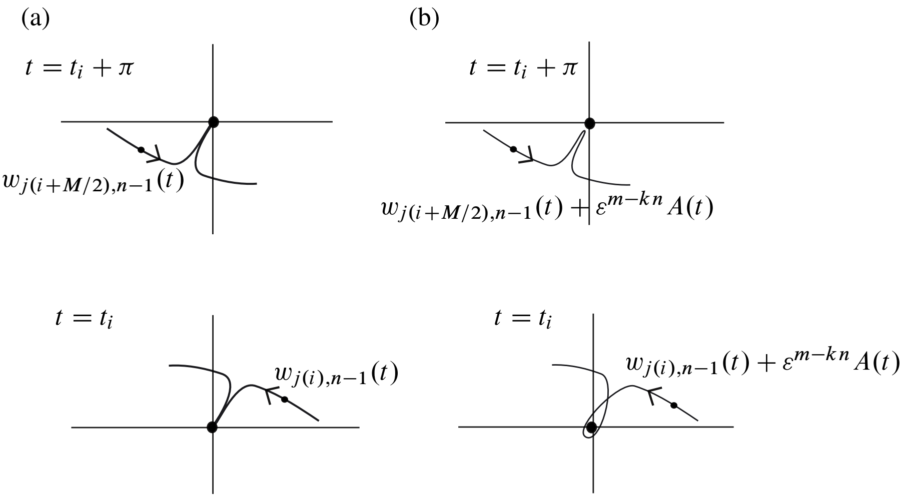

Figure 10: (a) The motion of the critical values $w_{j(i),n-1}(t)$ and $w_{j(i+M/2),n-1}(t)$ in neighborhoods of $t = t_i$ and $t = t_{i+M/2} = t_i + \pi$, respectively. (b) The motion (up to isotopy in $(\mathbb{C} \setminus \{0\}) \times S^1$) of the deformed critical values $w_{j(i),n-1}(t) + \varepsilon^{m-kn} A(t)$ and $w_{j(i+M/2),n-1}(t) + \varepsilon^{m-kn} A(t)$ in neighborhoods of $t = t_i$ and $t = t_{i+M/2} = t_i + \pi$, respectively.

There are two cases to consider. First, let $i \in \{1, 2, \dots, \frac{1}{2}M\}$. Equation (44) shows that $t = t_i$ is the unique value in U_i for which $\arg\big(\mathrm{e}^{\mathrm{i}\alpha_i(t)} \varepsilon^{kn} (R_2(t) + R_1(t)\varepsilon^{m-kn} \mathrm{e}^{\mathrm{i}\delta_i(t)})\big) = \alpha_i(t_i) + \pi$. At $t = t_i$ the derivative of $\arg\big(\mathrm{e}^{\mathrm{i}\alpha_i(t)} \varepsilon^{kn} (R_2(t) + R_1(t)\varepsilon^{m-kn} \mathrm{e}^{\mathrm{i}\delta_i(t)})\big)$ with respect to t is equal to $\delta_i'(t_i)$, which by construction has the same sign as $\alpha_i'(t_i)$.

Let $\tilde{\varepsilon}_i > 0$ and $U_i = (t_i - \tilde{\varepsilon}_i, t_i + \tilde{\varepsilon}_i)$. Note that we may choose $\tilde{\varepsilon}_i$ so small that both $\arg(w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i))$ and $\arg(w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i))$ are arbitrarily close to $\arg(w_{j(i),n-1}(t_i))$. Furthermore, we can assume that in U_i the critical value $\varepsilon^{kn} w_{j(i),n-1}(t) + \varepsilon^m A(t)$ has the smallest absolute value of all critical values. Then the argument of $\varepsilon^{kn} w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i) + \varepsilon^m A(t_i - \tilde{\varepsilon}_i)$ is arbitrarily close to the argument of $w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i)$ and the argument of $\varepsilon^{kn} w_{j(i),n-1}(t_i + \tilde{\varepsilon}_i) + \varepsilon^m A(t_i + \tilde{\varepsilon}_i)$ is arbitrarily close to the argument of $w_{j(i),n-1}(t_i + \tilde{\varepsilon}_i)$. Since the path $\varepsilon^{kn} w_{j(i),n-1}(t) + \varepsilon^m A(t)$ crosses the line $\varphi = \alpha_i(t_i) + \pi$ exactly once and since it has the smallest absolute value among all critical values, this path can be deformed into one without any argument-critical points and that winds around the origin exactly once (see Figure 10).

This deformation of the loop of critical values lifts to a deformation of the braid B [12]. By construction the direction of this path (clockwise (-1) or counterclockwise $(+1)$) is given by $\mathrm{sign}(\alpha_i'(t_i))$ and thus consistent with the monotonicity of the corresponding curve in the rest of the square diagram. Note that the deformation of the path taken by $\varepsilon^{kn} w_{j(i),n-1}(t) + r^m A(t)$ is an inner loop $\gamma_{j(i)}^{\mathrm{sign}(\alpha_i'(t_i))}$, since the moving critical value has the smallest absolute value throughout U_i.

Thus, after a deformation of the critical values, which lifts to a braid isotopy of B, there are no argument-critical points in U_i for $i = 1, 2, \dots, \frac{1}{2}M$, and by Lemma 2.9 the labels at $t = t_i - \tilde{\varepsilon}_i$ match the labels at $t = t_i + \tilde{\varepsilon}_i$.

Now we consider $i = \frac{1}{2}M + 1, \frac{1}{2}M + 2, \dots, M$. We know that at $t = t_i$ we have

$$\arg(\mathrm{e}^{\mathrm{i}\alpha_i(t)}\varepsilon^{kn}(R_2 + R_1\varepsilon^{m-kn}\mathrm{e}^{\mathrm{i}\delta_i(t)})) = \alpha_i(t_i). \tag{45}$$

Equation (44) shows that $t = t_i$ is the unique point in U_i where this is true and there is no point in U_i with $\arg(\mathrm{e}^{\mathrm{i}\alpha_i(t)}\varepsilon^{kn}(R_2 + R_1\varepsilon^{m-kn}\mathrm{e}^{\mathrm{i}\delta_i(t)})) = \alpha_i(t_i) + \pi$. It follows that the path taken by the critical value $\varepsilon^{kn}w_{j(i),n-1}(t) + \varepsilon^m A(t)$ in U_i does not twist around the origin. It has winding number zero and can thus be deformed to a curve without argument-critical points, connecting $\varepsilon^{kn}w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i) + \varepsilon^m A(t_i - \tilde{\varepsilon}_i)$ and $\varepsilon^{kn}w_{j(i),n-1}(t_i + \tilde{\varepsilon}_i) + \varepsilon^m A(t_i + \tilde{\varepsilon}_i)$ (see Figure 10).

Again this deformation of the loop of critical values lifts to a deformation of the polynomial and thus to a braid isotopy of B. In the U_i with $i \in \left\{\frac{1}{2}M + 1, \frac{1}{2}M + 2, \dots, M\right\}$ the deformed square diagram of $p|_{r=\varepsilon}$ thus looks exactly like the singular Rampichini diagram of h_t, except that the singularities have been removed (see Figure 10).

Therefore, the complete deformed square diagram of $p|_{r=\varepsilon}$ is obtained from the singular Rampichini diagram $R_{\{(t_i,j(i))\}_{i=1,2,\dots,M}}$ of h_t by inserting inner loops of the appropriate signs at the singularities at $t = t_i$ for $i = 1, 2, \dots, \frac{1}{2}M$, and simply removing the circles representing the singularities at $t = t_i$ for $i = \frac{1}{2}M + 1, \frac{1}{2}M + 2, \dots, M$.

Since there are no argument-critical points, the deformed diagram is a Rampichini diagram and B is P-fibered. □

By the same arguments as in [14, Theorem I.1] the link of the singularity is equivalent to the closure of B.

Note that this completes the proof of Theorem 1.1.

Proof of Theorem 1.1 Let B_2 be a P-fibered braid, whose Rampichini diagram R' can be obtained from an odd, pure Rampichini diagram R by the insertion of inner loops. Then there is a corresponding odd loop of polynomials h_t, parametrized by trigonometric polynomials, whose singular Rampichini diagram is R, but with singularities at $t = t_i$ for $i = 1, 2, \dots, \frac{1}{2}M$, the positions where we want to insert inner loops, and at $t = t_{i+M/2} = t_i + \pi$. We construct the functions f, A and p and by the previous lemmas p has an isolated singularity at the origin and the closure of B_2 is the link of the singularity. The polynomial p is semiholomorphic and its degree with respect to u is equal to n, the number of strands of B_2. □

Theorem 1.1 deals with P-fibered braids whose Rampichini diagrams are obtained from odd, pure Rampichini diagrams by inserting inner loops. The result remains true if we replace "odd" by "simple".

Theorem 6.3 *Let B_1 be a P-fibered braid on n strands with a simple, pure Rampichini diagram. Let B_2 be a P-fibered braid whose Rampichini diagram is obtained from that of B_1 by the insertion of inner loops. Then the closure of B_2 is real algebraic.*

Furthermore, the corresponding real polynomial map with an isolated singularity can be taken to be semiholomorphic (ie it can be written as a polynomial in complex variables u, v and the complex conjugate $\bar{v}$) and of degree $2n - 1$ with respect to the complex variable u.

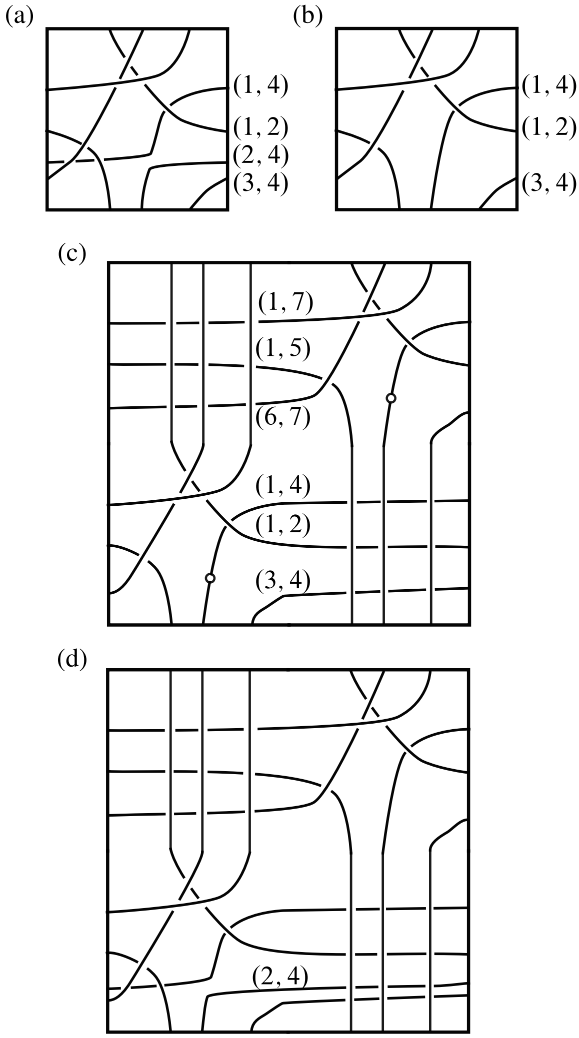

Figure 11: A sequence of Rampichini diagrams, where we have omitted several labels. (a) A Rampichini diagram that is obtained from a simple, pure Rampichini diagram by inserting an inner loop. (b) The corresponding simple, pure Rampichini diagram. (c) The singular Rampichini diagram with odd symmetry. (d) The final Rampichini diagram.

Proof Examples of Rampichini diagrams of B_2 and B_1 are shown in Figure 11(a) and (b), respectively. We want to show that B_2 can also be obtained from a P-fibered braid with odd, pure Rampichini diagram by the insertion of inner loops. As in Section 3 we can construct an odd Rampichini diagram R_{odd} for $\iota_n(B_1)m_n(B_1)$. Then the desired Rampichini diagram is obtained from R_{odd} by inserting inner loops into the lower left quadrant of R_{odd} (Figure 11(d)). We can thus construct an odd singular Rampichini diagram with singularities at $t = t_i \in (0, \pi)$ for $i = 1, 2, \dots, \frac{1}{2}M$ with $j(i) \in \{1, 2, \dots, n-1\}$ and $t = t_{i+M/2} = t_i + \pi$ for $i = 1, 2, \dots, \frac{1}{2}M$ with $j\left(i + \frac{1}{2}M\right) \in \{n, n+1, \dots, 2n-2\}$, corresponding to an odd loop of polynomials h_t as in the proof of Theorem 1.1 (see Figure 11(c)).

The braid that one obtains from this odd singular Rampichini diagram by inserting inner loops at the singularities with $t_i \in (0, \pi)$ and deleting the circles of the singularities with $t_i \in (\pi, 2\pi)$ is then $\iota_n(B_2)m_n(B_1)$. Since the Rampichini diagram of B_1 is simple, its closure is the unknot O, so the closure of $\iota_n(B_2)m_n(B_1)$ is a connected sum of the closure of B_2 and O and hence equal to the closure of B_2.

By Theorem 1.1, every P-fibered braid that can be obtained from an odd, pure Rampichini diagram closes to a real algebraic link. Thus the closure of B_2 is real algebraic. □

Example 6.4 Figure 11(b) shows a Rampichini diagram for the braid $B_1 = a_{3,4}a_{1,2}^{-1}a_{1,4}$, which closes to the unknot. We obtain a Rampichini diagram for the braid $B_2 = a_{3,4}a_{2,4}a_{1,2}^{-1}a_{1,4}$ in Figure 11(a) by inserting an inner loop. A braid word in BKL-generators for a braid with the same closure but with an odd, pure Rampichini diagram is then $\iota_n(B_2)m_n(B_1) = a_{3,4}a_{2,4}a_{1,2}^{-1}a_{1,4}a_{6,7}a_{1,5}^{-1}a_{1,7}$, which closes to the Hopf link. In fact, we could have inserted any number of inner loops into the Rampichini diagram of B_1, so that for example the closure of $a_{3,4}^{k_1}a_{2,4}^{k_2}a_{1,2}^{-k_3}a_{1,4}^{k_4}$ is real algebraic for every set of positive integers k_1, k_2, k_3, k_4. We obtain this braid word by inserting k_1 inner loops in the Rampichini diagram in Figure 11(b) into the curve in the bottom right corner, k_2 inner loops in the same position as in Figure 11(a), and k_3 and k_4 inner loops into the curves in Figure 11 next to the labels $(1,2)$ and $(1,4)$, respectively.

The proof of Theorem 1.1 is quite constructive. It describes explicitly the different approximations and interpolation steps that have to be performed in order to obtain the desired parametrization of the critical values. From the corresponding loop of polynomials h_t we obtain the function f as in (28) and the additional term A is again found via a very concrete interpolation process. However, it is still very challenging to produce examples of the resulting polynomials $p = f + A$.

The step that is difficult to perform in practice is the one that takes us from a parametrization of the critical values with labels (or, equivalently, from a Rampichini diagram) to the loop of polynomials h_t. For this it is necessary to solve a system of polynomial equations for every value of $t \in [0, 2\pi]$. This is very challenging even with the use of mathematical software. Recall that the different interpolation and approximation steps outlined in the previous sections were quite subtle. We can therefore not expect to obtain a loop of polynomials with the desired properties by simply solving the system of polynomials for a finite set of values of t in $[0, 2\pi]$ and then interpolating between these.

7 T-homogeneous braids and braids with positive Garside factor

We want to show that the family of T-homogeneous braid closures can be constructed as real algebraic links as in Theorems 1.1 and 6.3.

Definition 7.1 We say a braid word A in BKL-generators on n strands is a subword of a braid word B on n strands if deleting some letters (ie bands) from B results in A.

Proof of Theorem 1.2 We know that every T-homogeneous braid B on n strands is P-fibered and by definition B can be represented by a strictly homogeneous word w in a certain subset of BKL-generators, say $a_1, a_2, \dots, a_{n-1}$ with signs $\varepsilon_j \in \{\pm 1\}$ for $j = 1, 2, \dots, n$. We can choose a subword of w of length $n-1$ containing each $a_j^{\varepsilon_j}$. This subword is also T-homogeneous and hence P-fibered. Let R denote its Rampichini diagram, which is simple and pure.

It follows from the definition of T-homogeneity that the Rampichini diagram of B can be obtained from R by inserting inner loops. Theorem 6.3 then implies that the closure of B is real algebraic. □

Corollary 1.3 immediately follows, since homogeneous braids are T-homogeneous. By construction the corresponding polynomial is semiholomorphic. The degree of the polynomial with respect to the complex variable u is $2n-1$, where n is the number of strands of the homogeneous braid representative.

We proved in [13] that every link L is a sublink of a real algebraic link that consists of two nontrivially linked copies of L. Stallings showed that for every link L there is an unknot $O \in S^3 \setminus L$, nontrivially linked with L, such that $L \cup O$ is the closure of a homogeneous braid [36]. Combining this with Corollary 1.3 results in a proof of the following corollary.

Corollary 7.2 *For every link L in S^3 there is an unknot $O \in S^3 \setminus L$, nontrivially linked with L, such that $L \cup O$ is real algebraic.*

Now we prove Theorem 1.4. It states that any braid that is the product of a positive power of the dual Garside element δ and a BKL-positive word closes to a real algebraic link. As in the previous proof we show that this family of braids is obtained from a simple, pure Rampichini diagram by inserting inner loops.

Lemma 7.3 *For every positive generator $a_{i,j}$ there is a sequence of BKL-moves on the braid word $a_{1,n}a_{2,n}\dots a_{n-1,n}$ that results in a BKL-word whose last letter is $a_{i,j}$.*

Proof Assume $i < j$ and take the generator $a_{i,n}$ and move it towards $a_{j,n}$ conjugating the labels in between by $a_{i,n}$ at each step. After this the word looks like

$$a_{1,n}a_{2,n}\dots a_{i-1,n}a_{i,i+1}a_{i,i+2}\dots a_{i,j-1}a_{i,n}a_{j,n}a_{j+1,n}\dots a_{n-1,n}. \tag{46}$$

Then exchange $a_{i,n}$ and $a_{j,n}$, conjugating $a_{i,n}$ by $a_{j,n}$, to obtain

$$a_{1,n}a_{2,n}\dots a_{i-1,n}a_{i,i+1}a_{i,i+2}\dots a_{i,j-1}a_{j,n}a_{i,j}a_{j+1,n}\dots a_{n-1,n}. \tag{47}$$

Now move the generator $a_{i,j}$ all the way to the end of the word conjugating every generator in between by $a_{i,j}$. We thus obtain a word whose last letter is $a_{i,j}$. □

Note that such a sequence can be represented by a square diagram R, whose labels at the bottom edge spell $a_{1,n}a_{2,n}\dots a_{n-1,n}$ and whose labels at the top edge spell the final word of the sequence whose last letter is $a_{i,j}$. The lines are strictly monotone increasing and the words spelled by the labels at any horizontal slice of the diagram spell the intermediate BKL-word in the sequence.

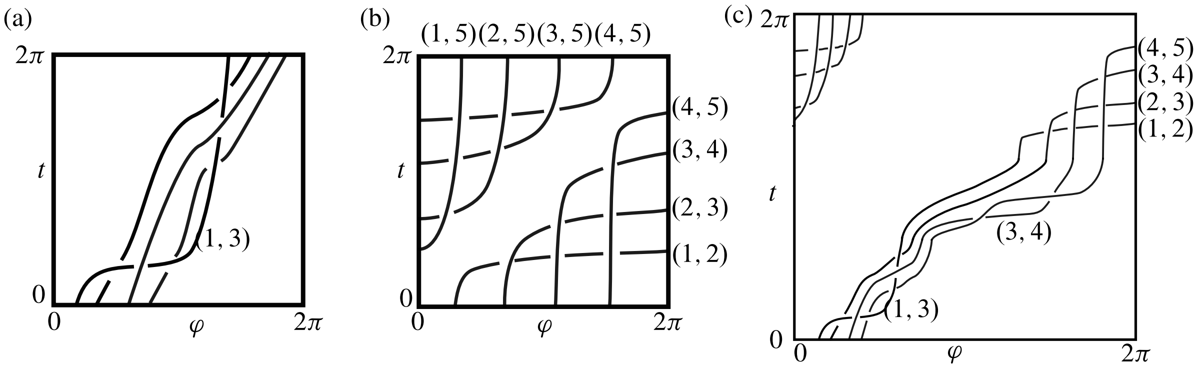

Figure 12: (a) $R'(a_{1,3})$ for $n = 5$. The transpositions at the top and bottom edge both read $(1,5)$, $(2,5)$, $(3,5)$, $(4,5)$ from left to right. (b) A Rampichini diagram for δ with $n = 5$. (c) A Rampichini diagram for δ built from $R'(a_{1,3})$, $R'(a_{3,4})$ and the diagram in (b). Inserting inner loops where the labels $(1,3)$ and $(3,4)$ are results in a Rampichini diagram for $a_{1,3}a_{3,4}\delta$.

Therefore, we can construct for every positive generator a square diagram $R'(a_{i,j})$, whose labels at the bottom and top edge both spell $a_{1,n}a_{2,n}\dots a_{n-1,n}$, whose curves do not intersect the right edge of the square ($\varphi = 2\pi$), and that for some value of t has the property that the label of the critical value with largest φ-coordinate is $a_{i,j}$. This diagram is obtained from R by composing it with the inverse of R, ie a square diagram that reverses the sequence of BKL-words from Lemma 7.3. Note that since all labels are positive, the inverse of R is still realized by strictly monotone increasing lines, see Figure 12(a). The reason why $R'(a_{i,j})$ is not a Rampichini diagram is that it is not 2π-periodic with respect to the t-coordinate. The endpoints at the top edge are strictly to the right of the starting points at the bottom edge.

Proof of Theorem 1.4 Let $B = \delta P$. Note that δ is a P-fibered braid with a simple, pure Rampichini diagram as in Figure 12(b). In particular, the labels at $t = 0$ read $a_{1,n}, a_{2,n}, \dots, a_{n-1,n}$ from left to right. We now have to show that every band in P can be obtained by inserting inner loops. Note that, since δ is a positive word, all curves in the Rampichini diagram are strictly monotone increasing.

Let $P = \prod_{k=1}^{M} a_{i_k,j_k}$. We glue the bottom edge of the square diagram $R'(a_{i_k,j_k})$ along the top edge of $R'(a_{i_{k-1},j_{k-1}})$ for all $k = 2, 3, \dots, M$. This results in a square diagram whose labels at the bottom edge and at the top edge are $a_{1,n}, a_{2,n}, \dots, a_{n-1,n}$. However the endpoints of the curves at the top edge of the square are strictly to the right of the endpoints of the curves at the bottom edge. We glue the top edge of this square diagram along the bottom edge of the Rampichini diagram of δ. In order to be able to perform these concatenations of square diagrams, we have to shift the curves in the upper square diagram to the right so that the endpoints of its curves on its bottom edge coincide with the endpoints of the lower square diagram on its top edge. Likewise we have to shift curves (in parallel) so that the endpoints at the top edge of the new diagram coincide with the endpoints on the bottom edge of $R'(a_{i_1,j_1})$. Note that none of these parallel shifts affect the crossing pattern or cactus at any value of t, nor do they change the strict monotonicity of the curves (see Figure 12).

The resulting square diagram is a simple, pure Rampichini diagram of δ and there are t_k for $k = 1, 2, \dots, M$ with $t_k > t_{k-1}$ such that $\tau_{n-1}(t_k) = a_{i_k, j_k}$. Thus inserting inner loops whose moving critical value corresponds to $\tau_{n-1}(t_k)$ results in a Rampichini diagram of $P\delta$.

It follows from Theorem 6.3 that the closure of $P\delta$, which is also the closure of B, is real algebraic. □

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