AGT

Algebraic & Geometric Topology

Volume 25 (2025)

A new invariant of equivariant concordance and results on 2-bridge knots

Alessio Di Prisa
Giovanni Framba

msp

Algebraic & Geometric Topology **25**:2 (2025) 1117–1132
DOI: 10.2140/agt.2025.25.1117
Published: 16 May 2025

A new invariant of equivariant concordance and results on 2-bridge knots

ALESSIO DI PRISA
GIOVANNI FRAMBA

We study the equivariant concordance classes of 2-bridge knots and we prove that no 2-bridge knot is equivariantly slice. Finally, we introduce a new equivariant concordance invariant for strongly invertible knots. Using this invariant as an obstruction we strengthen the result on 2-bridge knots, proving that every 2-bridge knot has infinite order in the equivariant concordance group.

57K10; 57R85

1 Introduction

A strongly invertible knot is a pair (K,ρ), where $K \subseteq S^3$ is an oriented knot and $\rho \in \mathrm{Diffeo}^+(S^3)$ is an involution such that $\rho(K) = K$ and ρ reverses the orientation on K. By the resolution of the Smith conjecture [16] it is known that $\mathrm{Fix}(\rho)$ is an unknot which intersects K in two points. Sakuma [17] gave a well-defined notion of *equivariant connected sum* for strongly invertible knots by endowing them with a *direction*. Furthermore, Sakuma [17] studied strongly invertible knots up to *equivariant concordance* and introduced the *equivariant concordance group* $\widetilde{\mathcal{C}}$.

The equivariant concordance group is far from being understood. However, the first author proved in [7] that $\widetilde{\mathcal{C}}$ is not abelian, and several authors defined new invariants for equivariant concordance and obstructions for equivariant sliceness; see for example Boyle and Issa [2], Alfieri and Boyle [1], Dai, Hedden and Mallick [5], Dai, Hedden and Stoffregen [6] and Miller and Powell [15]. In particular, Boyle and Issa [2] defined the *butterfly link* associated with a directed strongly invertible knot and used it to define several equivariant concordance invariants.

We study the equivariant concordance of 2-bridge knots. In Proposition 3.2 we provide a formula to compute the *butterfly polynomial* [2] for a certain class of strongly invertible 2-bridge knots. Our initial goal was to prove Proposition 4.1 by combining the equivariant slice obstructions from the Kojima–Yamasaki *η-function* of Sakuma's link [17] and of the butterfly link [2]. This approach was inconclusive. However, using this formula we prove Corollary 3.3, which is a statement analogous to [17, Theorem II] in the case of the butterfly polynomial.

The main result of the paper is Theorem 5.10, which states the following:

Theorem *Let K be a directed strongly invertible knot and let $\widehat{L}_b(K)$ be its butterfly link endowed with the opposite semiorientation. If the Conway polynomial of $\widehat{L}_b(K)$ is nonzero then K is not equivariantly slice and has infinite order in $\widetilde{\mathcal{C}}$.*

Using this result we are able to prove Proposition 5.11, showing that every 2-bridge knot has infinite order in the equivariant concordance group.

Organisation of the paper

Section 2 contains a brief recap on the results on directed strongly invertible knots that we need later in the paper. In Section 3 we provide a formula for the butterfly polynomial [2, Definition 4.7] of 2-bridge knots. In Section 4 we prove that every 2-bridge knot is not equivariantly slice, using the nullity of the butterfly link as an obstruction to equivariant sliceness. Finally, in Section 5 we define a new equivariant concordance invariant for strongly invertible knots. We use this invariant to show that the equivariant concordance order of every 2-bridge knot is infinite.

2 Preliminaries

2.1 Directed strongly invertible knots

We briefly recall the notion of direction for a strongly invertible knot and of the equivariant concordance group. For the details see [17; 2].

Definition 2.1 A *direction* on a strongly invertible knot (K,ρ) is the choice of an oriented half-axis h, ie one of the two connected components of $\mathrm{Fix}(\rho)\setminus K$.

We call the triple (K,ρ,h) a *directed strongly invertible knot*. We write K instead of (K,ρ,h) when it is not strictly necessary to specify the choice of strong inversion and direction.

Definition 2.2 Let (K,ρ,h) be a directed strongly invertible knot. We define

- the *mirror* of (K,ρ,h) by $mK=(mK,\rho,h)$,
- the *axis-inverse* of (K,ρ,h) by $iK=(K,\rho,-h)$, where $-h$ is the direction given by the half-axis h with the opposite orientation,
- the *antipode* of (K,ρ,h) by $aK=(K,\rho,h')$, where h' is the direction given by the half-axis complementary to h. The orientation on h' is the one coherent with h.

Definition 2.3 We say that two directed strongly invertible knots (K_i,ρ_i,h_i), $i=0,1$, are *equivariantly concordant* if there exists a smooth properly embedded annulus $C\cong S^1\times I\subset S^3\times I$, invariant with respect to some involution ρ of $S^3\times I$ such that

- $\partial(S^3\times I,C)=(S^3,K_0)\sqcup -(S^3,K_1)$,

- ρ is in an extension of the strong inversion $\rho_0 \sqcup \rho_1$ on $S^3 \times 0 \sqcup S^3 \times 1$,
- the orientations of h_0 and $-h_1$ induce the same orientation on the annulus $\mathrm{Fix}(\rho)$, and h_0 and h_1 are contained in the same component of $\mathrm{Fix}(\rho) \setminus C$.

The *equivariant concordance group* is the set $\widetilde{\mathcal{C}}$ of equivalence classes of directed strongly invertible knots up to equivariant concordance, endowed with the operation induced by the *equivariant connected sum*, which we denote by $\widetilde{\#}$ (see [17; 2] for details).

Remark 2.4 It is easy to see that the mirror, axis-inverse and antipode induce involutive maps from the equivariant concordance group to itself. From the definition of equivariant connected sum we can easily deduce the following properties. Given two directed strongly invertible knots K and J, we have

- $m(K \,\widetilde{\#}\, J) = mK \,\widetilde{\#}\, mJ$,
- $i(K \,\widetilde{\#}\, J) = iJ \,\widetilde{\#}\, iK$,
- $a(K \,\widetilde{\#}\, J) = aJ \,\widetilde{\#}\, aK$.

Equivalently, we can say that m is an automorphism of $\widetilde{\mathcal{C}}$, while i and a are antiautomorphisms.

Remark 2.5 As a consequence, the equivariant concordance order of a directed strongly invertible knot (K, ρ, h) does not depend on the choice of a direction and it does not change when taking the mirror of the knot.

2.2 Butterfly links

Boyle and Issa [2, Definition 4.1] associated a directed strongly invertible knot (K, ρ, h) with a 2-component 2-periodic link (ie the involution ρ exchanges its components), called the *butterfly link* $L_b(K)$, defined as follows. Take an equivariant band B, parallel to the preferred half-axis h, which attaches to K at the two fixed points. Performing a band move on K along B produces a 2-component link. The link $L_b(K)$ is the one obtained from such a band move on K so that the linking number between its components is 0. Observe that $\partial B \setminus K$ consists of two arcs parallel to h, which we orient as h. The arcs lie in different components of $L_b(K)$ and the orientation on each component of $L_b(K)$ is the one induced from the orientation of the respective arc.

The following result can be proven easily by adapting the proof of [2, Proposition 7]. We report the proof because it will be useful for Proposition 5.4.

Proposition 2.6 *Let (K_i, ρ_i, h_i), $i = 0, 1$, be two equivariantly concordant directed strongly invertible knots. Then, $L_b(K_0)$ and $L_b(K_1)$ are equivariantly concordant (as 2-periodic links).*

Proof Let $C \subset S^3 \times I$ be an annulus between (K_0, ρ_0, h_0) and (K_1, ρ_1, h_1) that is invariant with respect to an extension $\rho\colon S^3 \times I \to S^3 \times I$ of $\rho_0 \sqcup \rho_1$. Let $A = \mathrm{Fix}(\rho)$ be the annulus of fixed points of ρ. Observe that $C \cap A = \alpha \cup \beta$, where α and β are two curves joining respectively the initial and final points of the half-axes of K_0 and K_1.

Now $A \setminus (\alpha \cup \beta)$ has two connected components: call D the component containing h_0 and h_1. Choose an equivariant tubular neighbourhood N of D and observe that $N \cap C$ is a D^1-subbundle of $N|_{\alpha\cup\beta}$. Consider two equivariant bands $B_i \subset S^3 \times \{i\}$, $i = 0, 1$, with B_i intersecting K_i and containing the half-axis h_i. We can choose B_i in such a way that $B_i \setminus K_i \subset N$; hence $B_0 \cup B_1 \cup C$ intersects N in a D^1-subbundle of $N|_{\partial D}$.

Choose B_0 so that the band move of K_0 along B_0 produces $L_b(K_0)$, and take B_1 so that the D^1-subbundle of $N|_{\partial D}$ given by $N \cap (B_0 \cup B_1 \cup C)$ extends to $E \subset N$, which is a D^1-bundle over D. Call L the 2-component link obtained from K_1 by the band move along B_1.

Now $E \cong D^1 \times D \cong D^1 \times D^1 \times D^1$, where $0 \times \partial D^1 \times D^1 = \alpha \cup \beta$. Then

$$C_b = (C \setminus D^1 \times \partial D^1 \times D^1) \cup \partial D^1 \times D^1 \times D^1$$

is an equivariant concordance between $L_b(K_0)$ and L. Since the linking number between components is a concordance invariant of 2-component links, we have $L = L_b(K_1)$. □

Corollary 2.7 *A directed strongly invertible knot is equivariantly slice if and only if its butterfly link is equivariantly slice (as a 2-periodic 2-component link).*

Proof Let K be a directed strongly invertible knot. Suppose that K is equivariantly slice, ie equivariantly concordant to the unknot. Then $L_b(K)$ is equivariantly concordant to the butterfly link of the unknot, which is easily seen to be a 2-component unlink, and hence $L_b(K)$ is equivariantly slice. Conversely, suppose that $L_b(K)$ bounds the disjoint union $D_1 \sqcup D_2$ of two disks in B^4 which is invariant under an extension of the involution of $L_b(K)$. Observe that the equivariant band move on K that gives $L_b(K)$ can be seen as an equivariant cobordism $C \subset S^3 \times [0, 1]$ of genus 0 (ie a pair of pants) between K and $L_b(K)$. Then $C \cup D_1 \cup D_2$ is an equivariant slice disk with boundary K. □

2.3 Strong inversions on 2-bridge knots

Let $K = K(p, q) \subseteq S^3$ be a 2-bridge knot. From [19] we know that we can write p/q as a continued fraction

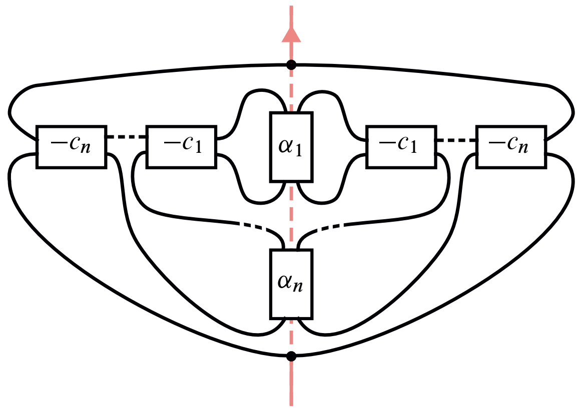
$$[a_1, \dots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

where $a_1, \dots, a_n$ and n are even nonzero integers.

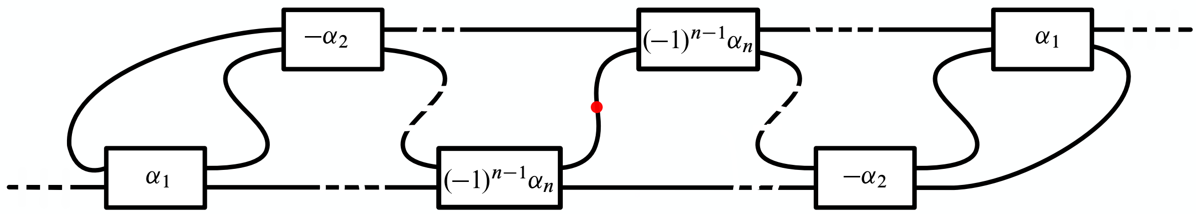
Recall that every 2-bridge knot is simple (see [8]). Sakuma [17] showed that a hyperbolic 2-bridge knot $K(p, q)$ admits exactly two inequivalent structures of strongly invertible knot. The first structure is given by the diagram

$$I_1\left(a_1, a_3, \dots, a_{n-1}; \tfrac{1}{2}a_2, \tfrac{1}{2}a_4, \dots, \tfrac{1}{2}a_n\right),$$

where the strong involution on a diagram $I_1(\alpha_1, \dots, \alpha_n; c_1 \dots, c_n)$ is given by the π-rotation around the vertical axis (see Figure 1).



$I_1(\alpha_1, \dots, \alpha_n; c_1 \dots, c_n)$



$I_2(\alpha_1, \alpha_2, \dots, \alpha_n)$

Figure 1: The two families of symmetric diagrams.

If $a_i = -a_{n-i+1}$ for all i, the second symmetry on $K(p,q)$ is represented by the diagram

$$I_2(a_1, a_2, \dots, a_{n/2}),$$

where the strong involution is the π-rotation around the central dot (see Figure 1, bottom). Otherwise it is given by

$$I_1\bigl(-a_n, -a_{n-2}, \dots, -a_2; -\tfrac{1}{2}a_{n-1}, \dots, -\tfrac{1}{2}a_3, -\tfrac{1}{2}a_1\bigr).$$

If $K(p,q)$ is a torus knot, it admits one only strong inversion, namely the one described by

$$I_1\bigl(a_1, a_3, \dots, a_{n-1}; \tfrac{1}{2}a_2, \tfrac{1}{2}a_4, \dots, \tfrac{1}{2}a_n\bigr).$$

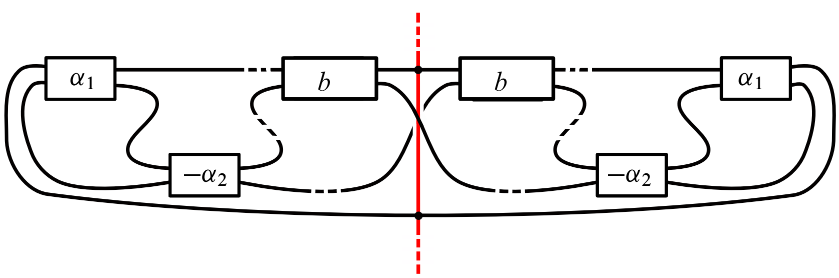


Figure 2: A diagram equivalent to the one in Figure 1, bottom.

Observe that the knot represented in Figure 1, bottom, can be equivariantly isotoped into the strongly invertible knot given in Figure 2, where $b = (-1)^{n-1}\alpha_n + 1$.

In the following, we will consider either $I_1(\alpha_1, \dots, \alpha_n; c_1 \dots, c_n)$ (Figure 1, top) or $I_2(\alpha_1, \dots, \alpha_n)$ (Figure 2) as a diagram for the directed strongly invertible knot $K = K(p, q)$, with the direction given by the oriented unbounded half-axis in the figures, unless the direction is otherwise specified. Here $n > 0$, $\alpha_1, \dots, \alpha_n \in 2\mathbb{Z} \setminus \{0\}$ and $c_1, \dots, c_n \in \mathbb{Z} \setminus \{0\}$.

3 η-function

Let $K \cup J$ be a 2-component link with linking number 0 between components. Kojima and Yamasaki [14] introduced the η-function, which is a topological concordance invariant for such links.

We briefly recall the construction of this invariant. Let X_K be the complement of K in S^3 and let $\widetilde{X}_K \to X_K$ be its infinite cyclic covering. Denote by t a generator of the deck transformation of $\widetilde{X}_K$. Recall that the Alexander module of K is $H_1(\widetilde{X}_K, \mathbb{Z})$ endowed with the $\mathbb{Z}[t, t^{-1}]$-module structure induced by the action of t. Now let l be the canonical longitude of J and let $\tilde{l}$ and $\tilde{J}$ be two nearby lifts of l and J to $\widetilde{X}_K$.

Since $\mathrm{lk}(K, J) = 0$, we know $\tilde{l}$ and $\tilde{J}$ are closed curves; hence they can be seen as classes in $H_1(\widetilde{X}_K, \mathbb{Z})$. Since the Alexander module is a torsion $\mathbb{Z}[t, t^{-1}]$-module, there exists a nonzero $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that $f(t) \cdot \tilde{l} = 0$, ie we can find a 2-chain Δ such that $\partial\Delta = f(t) \cdot \tilde{l}$. Then the η-function is defined as

$$\eta(K, J; t) = \frac{1}{f(t)} \sum_{n \in \mathbb{Z}} \#(\Delta \cap t^n \tilde{J}) \cdot t^n,$$

where $\#(\Delta \cap t^n \tilde{J})$ is the algebraic intersection.

One can check that η is well defined and has the following properties (see [14]):

(i) $\eta(K, J; t) = \eta(K, J; t^{-1})$.

(ii) $\eta(K, J; 1) = 0$.

(iii) η does not depend on the orientation of the link.

(iv) η is an invariant of topological concordance of links.

Observe that in general η does depend on the order of the components of the link, ie $\eta(K, J; t) \neq \eta(J, K; t)$.

In the following we will denote by $\mathbb{Z}\langle t \rangle \subseteq \mathbb{Z}[t, t^{-1}]$ the subgroup of Laurent polynomials satisfying properties (i) and (ii).

Boyle and Issa [2] defined the *butterfly polynomial* $\eta(L_b(K))$ of a directed strongly invertible knot K as the η-function of its butterfly link (since $L_b(K)$ is 2-periodic it does not depend on the order of its components) and showed that it induces a group homomorphism

$$\eta(L_b(-)) \colon \widetilde{\mathcal{C}} \to \mathbb{Q}(t).$$

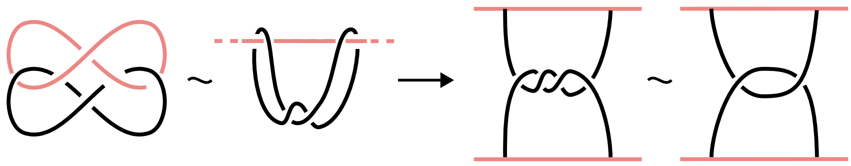


Figure 3: Fundamental domain of the Whitehead link.

Remark 3.1 Since the butterfly polynomial induces a homomorphism, every directed strongly invertible knot with nontrivial butterfly polynomial has infinite order in $\tilde{\mathcal{C}}$.

We now describe a formula to compute the butterfly polynomial of 2-bridge knots. To do so, we report a convenient algorithm [17; 20] to compute $\eta(K, J; t)$ when K is unknotted.

In this special case the infinite cyclic cover of X_K is diffeomorphic to $\mathbb{R} \times D^2$, and the η-function of L is simply

$$\eta(K, J; t) = \sum_{i \in \mathbb{Z}} \operatorname{lk}(\tilde{l}, t^i(\tilde{J}))t^i.$$

The algorithm consists of the following four steps.

(1) Start by noting that $\operatorname{lk}(\tilde{l}, t^i(\tilde{J}))t^i = \operatorname{lk}(\tilde{J}, t^i(\tilde{J}))t^i$ for $i \neq 0$, since $\tilde{l}$ is a nearby perturbation of $\tilde{J}$. Therefore, letting $r = \operatorname{lk}(\tilde{l}, \tilde{J})$, we get

$$\sum_{i \in \mathbb{Z}} \operatorname{lk}(\tilde{l}, t^i(\tilde{J}))t^i = \sum_{i \in \mathbb{Z} \setminus 0} \operatorname{lk}(\tilde{J}, t^i(\tilde{J}))t^i + r.$$

In the following steps we compute $\bar{\eta}(t) = \sum_{i \in \mathbb{Z} \setminus 0} \operatorname{lk}(\tilde{J}, t^i(\tilde{J}))t^i$. Since $\eta(1) = 0$, it is easy to retrieve $r = -\bar{\eta}(1)$.

(2) Draw a fundamental domain of the infinite cyclic cover $\widetilde{X}_K$ (see Figure 3). Assign a label and an orientation to each arc as follows:

- (i) The arc starting from the top right point has index 0 and is oriented downwards.
- (ii) Suppose an arc α is already labelled and oriented. Let A be the end point of α and B be the point opposite to A. Call β the strand that starts from B (by saying this we are orienting it). Define $\operatorname{index}(\beta)$ (the label on β) to be $\operatorname{index}(\alpha) + 1$ if B lies on the lower side of the domain or $\operatorname{index}(\alpha) - 1$ if B is on the upper side.

The labels we put on the strands keep track of which translate of $\tilde{J}$ in $\widetilde{X}_K$ they correspond to. A strand labelled by i is a portion of $t^i(\tilde{J})$. Hence a crossing where an arc labelled by i overcrosses an arc labelled by j corresponds to a crossing between $t^i(\tilde{J})$ and $t^j(\tilde{J})$ or, equivalently, between $\tilde{J}$ and $t^{i-j}(\tilde{J})$. This motivates the following step.

(3) Assign to each double point P a sign $\epsilon_P \in \{+, -\}$ and an integer $d_P \in \mathbb{Z}$ as follows. The sign ϵ_P is the sign of the crossing and d_P is the difference between the label on the overcrossing arc and the label of the undercrossing arc.

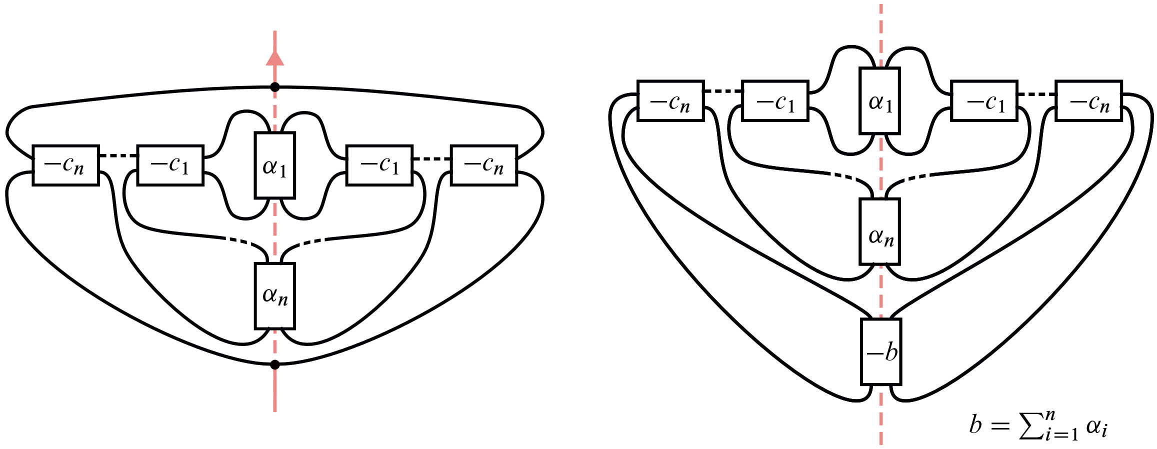


Figure 4: Strong inversion on a 2-bridge knot and construction of the butterfly link.

(4) Now let

$$\bar\eta(t) = \sum_P \epsilon_P t^{d_P} \quad \text{and} \quad r = -\bar\eta(1).$$

Then $\eta(t)$ is obtained as $\eta(t) = \bar\eta(t) + r$.

We will use this algorithm to prove Proposition 3.2.

Proposition 3.2 *Let $I_1(\alpha_1, \dots, \alpha_n; c_1, \dots, c_n)$ be a diagram for the directed strongly invertible knot $K = K(p,q)$. Then the butterfly polynomial of K is given by*

$$\eta_{L_b(K)}(t) = \sum_{i=1}^n c_i(t^{\sigma_i} + t^{-\sigma_i}) - 2\sum_{i=1}^n c_i,$$

where $\sigma_i = \frac{1}{2}\sum_{j=1}^i \alpha_j$.

Proof Figure 4 shows the construction of the butterfly link $L_b(K) = K_0 \sqcup K_1$ that starts from $I_1(\alpha_1, \dots, \alpha_n; c_1, \dots, c_n)$. We only add a box of $-2\sigma_n$ crossings, so that $\mathrm{lk}(K_0, K_1) = 0$. Observe that $L_b(K)$ is the *denominator closure* of the rational tangle T in Figure 5 (for details see [10, Figure 5]).

Using the procedure described in [11, Section 2], we can compute the coefficients of the continued fraction associated with this tangle, finding that they are

$$[-b, 2c_n, \alpha_n, \dots, 2c_1, \alpha_1].$$

The coefficients are all nonzero even integers; then, by [13, Exercise 2.1.14], we find that $L_b(K)$ is equivalent to the link represented by the diagram in Figure 5.

The diagram in Figure 5 has unknotted components, so we can compute the η-function of the link $L_b(K)$ drawing a fundamental domain (Figure 6) and applying the algorithm previously described.

Observe that we can assume $b \geq 0$. In fact it follows from [2, Proposition 11] that $\eta_{L_b(mK)}(t) = -\eta_{L_b(K)}(t)$.

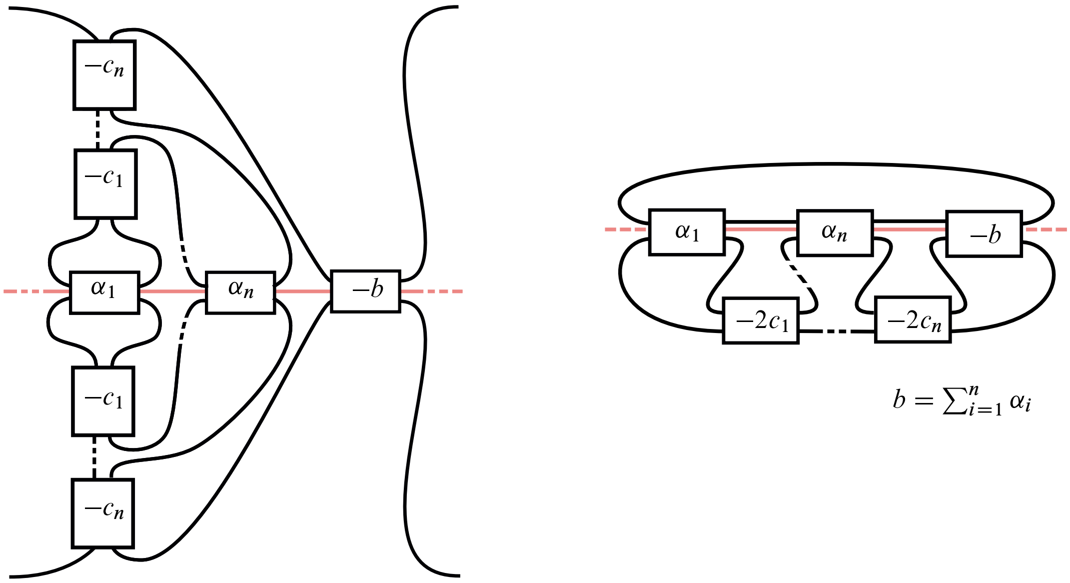


Figure 5: Left: rational tangle T. Right: canonical form of $L_b(K)$.

We start by labelling and orienting the arcs. Call γ the top-right arc, labelled 0. The arc γ runs across each $-2c_i$ box reaching the left side of the domain. If $\alpha_1 > 0$, then γ rises and ends in the upper horizontal bar. Then we will run across $\frac{1}{2}\alpha_1$ arcs riding from the lower bar to the upper bar, labelled with increasing indexes. If $\alpha_1 < 0$, then γ descends ending in the lower horizontal bar and the following $\frac{1}{2}\alpha_1$ arcs ride from the upper bar to the lower one with decreasing labels. This means that in both cases the last arc of this group is labelled by $\frac{1}{2}\alpha_1$.

This goes for all the groups of $\frac{1}{2}\alpha_i$ vertical arcs: in each group the labels increase or decrease by 1 and the arc entering the $-2c_i$ box, after the group of $\frac{1}{2}\alpha_i$ vertical arcs, is labelled by $\sum_{j=1}^{i} \frac{1}{2}\alpha_j = \sigma_i$.

The arc exiting the n^{th} box is labelled by $\sigma_n = \sum_{i=1}^{n} \frac{1}{2}\alpha_i$. The last group of vertical arcs consists of exactly $|\sigma_n|$ arcs oriented upwards or downwards depending on whether $\sigma_n < 0$ or $\sigma_n > 0$. It follows that the labels increase, or decrease, by 1 until they reach 0, at this point we meet the first arc, which is already oriented and labelled.

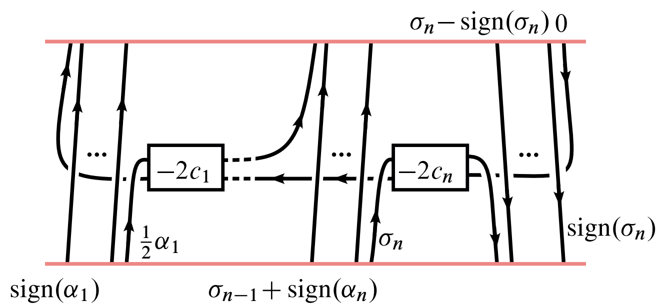


Figure 6: Fundamental domain with labelled oriented arcs.

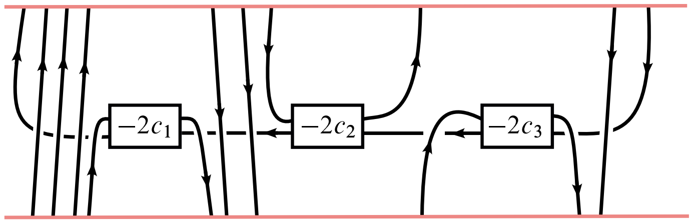


Figure 7: Example of fundamental domain.

At last, we count the crossings. We must count the crossings in the groups of vertical arcs and in the boxes for $i \in \{1, \dots, n\}$ and in the last group of $|\sigma_n|$ vertical arcs. For each $i \in \{1, \dots, n\}$ in the i^{th} box we find that the two strands run in opposite direction, one labelled 0 and one labelled σ_i. Hence we count

- $|c_i|$ crossings with $\epsilon = \operatorname{sign}(c_i)$ and $d = \sigma_i$,
- $|c_i|$ crossings with $\epsilon = \operatorname{sign}(c_i)$ and $d = -\sigma_i$.

Regarding the crossings in the vertical arcs, the idea is that they do not contribute inasmuch they simplify with each other. In fact the undercrossing arc is the same strand for each crossing and the labels of the overcrossing arcs start from one value and after increasing and decreasing they get back to the same value. This is due to the fact that the linking number between the components of the butterfly link is 0 and the two components only cross each other in the boxes of crossings that go from α_1 to α_n and in the box containing $-b$ crossings. Let us be more precise.

The $\frac{1}{2}|\alpha_i|$ crossings in the i^{th} group of vertical arcs contribute to the η-function with a polynomial that depends on the sign of α_i and of b in the following way:

$$f(\alpha_i, b) = \begin{cases} +t^{\sigma_{i-1}} \sum_{h=1}^{\alpha_i/2} t^h, & a_i > 0, \\ -t^{\sigma_{i-1}} \sum_{h=0}^{|\alpha_i|/2-1} t^{-h}, & a_i < 0. \end{cases}$$

Similarly, the crossings in the last group of vertical arcs contribute with

$$f(b) = -\sum_{h=1}^{\sigma_n} t^h.$$

See Figure 7 for an example. To prove that these two functions are correct we must examine the two possible cases.

(1) If $\alpha_i > 0$, then each crossing is positive, the undercrossing arc is labelled by 0 and the labels on the $\frac{1}{2}\alpha_i$ overcrossing arcs go from $\sigma_{i-1} + 1$ to $\sigma_{i-1} + \frac{1}{2}\alpha_i$.

(2) If $\alpha_i < 0$, then each crossing is negative, the undercrossing arc is labelled by 0 and the labels on the $\frac{1}{2}|\alpha_i|$ overcrossing arcs go from σ_{i-1} to $\sigma_{i-1} + 1 + \frac{1}{2}\alpha_i$.

The computation for the last group of vertical arcs works in the same way.

It follows that the count of the crossings on the vertical groups of arcs for $i \in \{1, \dots, n\}$ simplifies with the count of the crossings on the last group of vertical arcs:

$$\left(\sum_{i=1}^{n} f(\alpha_i, b)\right) + f(b) = 0.$$

This means that

$$\bar{\eta}(t) = \sum_{i=1}^{n} c_i (t^{\sigma_i} + t^{-\sigma_i}) \quad \text{and} \quad \bar{\eta}(1) = 2 \sum_{i=1}^{n} c_i;$$

hence $\eta(t) = \sum_{i=1}^{n} c_i (t^{\sigma_i} + t^{-\sigma_i}) - 2 \sum_{i=1}^{n} c_i$. □

In analogy with Sakuma's result [17, Theorem II], we can observe the following corollary to Proposition 3.2.

Corollary 3.3 *Every Laurent polynomial in $\mathbb{Z}\langle t \rangle$ is realised as butterfly polynomial of some directed strongly invertible knot.*

Proof Notice that the η-function of the butterfly link of the 2-bridge knot $K_n = I_1(2n; 1)$ is

$$\eta(t) = t^n + t^{-n} - 2,$$

and these form a set of generators for $\mathbb{Z}\langle t \rangle$. □

Remark 3.4 Using only the formula of Proposition 3.2 we are not able to deduce that no 2-bridge knot is equivariantly slice. As an example, the butterfly polynomial vanishes on the family of directed strongly invertible knots given by $I_1(2a, -2a, 2a, -2a; b, c, -b, d)$, with $a, b, c, d \in \mathbb{Z} \setminus \{0\}$.

4 No 2-bridge knot is equivariantly slice

In this section we prove the following two propositions.

Proposition 4.1 *The strongly invertible knot $K = I_1(\alpha_1, \dots, \alpha_n; c_1, \dots, c_n)$ is not equivariantly slice.*

Proposition 4.2 *The strongly invertible knot $K = I_2(\alpha_1, \dots, \alpha_n)$ is not equivariantly slice.*

To prove Proposition 4.1, we use the *nullity* of the butterfly link as an obstruction to equivariant sliceness. Let $L \subset S^3$ be a link, and denote by $\Sigma(L)$ the 2-fold cover of S^3 branched over L. Recall that the *nullity* of L is defined as

$$n(L) = 1 + \dim(H_1(\Sigma(L), \mathbb{Q}))$$

and that the nullity is an invariant for link concordance, as shown in [12].

Proof of Proposition 4.1 Consider on K the direction specified in Figure 4. Recall that the fraction associated with K is

$$p/q = [\alpha_1, 2c_1, \dots, \alpha_n, 2c_n].$$

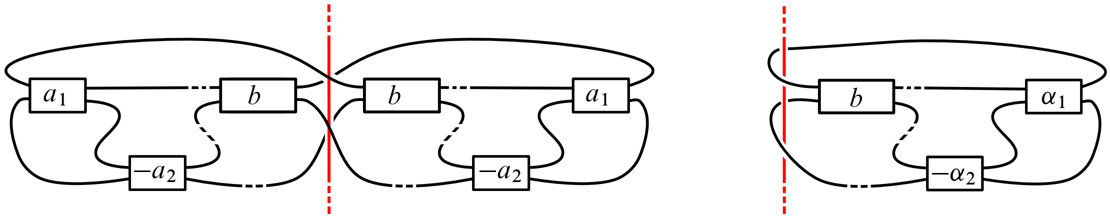


Figure 8: Left: butterfly link of $I_2(a_1, a_2, \dots, a_{n/2})$. Right: the quotient knot $\overline{K}$ and the quotient axis $\overline{A}$ (in red).

As shown in the proof of Proposition 3.2, observe that $L_b(K)$ is a 2-bridge link with continued fraction $\left[-\sum_{i=1}^{n} \alpha_i, 2c_n, \alpha_n, \dots, 2c_1, \alpha_1\right]$ (see Figure 5). It follows that the associated rational number is

$$p''/q'' = -\sum_{i=1}^{n} \alpha_i + q'/p',$$

where $p'/q' = [2c_n, \alpha_n, \dots, 2c_1, \alpha_1]$. It is a well-known fact (see [10, Theorem 4]) that $p = p'$ and q' is such that $q \cdot q' \equiv -1 \mod p$. It follows that $p''/q'' \in \mathbb{Q} \setminus \{0\}$. Since the 2-fold cover $\Sigma(L_b(K))$ of S^3 branched over $L_b(K)$ is a lens space and $p''/q'' \neq 0$, it is a rational homology S^3. In particular, the nullity of $L_b(K)$ is $n(L_b(K)) = 1$. Since the nullity of the 2-component unlink is easily seen to be 2, $L_b(K)$ is not concordant to the unlink. Therefore K is not equivariantly slice. □

Proof of Proposition 4.2 By [17, Proposition 2.3], if $\sum_{i \geq 0} \alpha_{2i+1} \neq 0$, then Sakuma's η-polynomial is nonvanishing for K, and hence K has infinite order in $\widetilde{\mathcal{C}}$.

Therefore, we can assume $\sum_{i \geq 0} \alpha_{2i+1} = 0$, and hence $n > 2$. In this case, we can see from Figure 8, left, that the butterfly link $L_b(K)$ of K is a 2-bridge link. In order to conclude the proof as in Proposition 4.1, we just have to show that $L_b(K)$ is not the unlink. Let $\overline{K}$ be the knot obtained by quotienting $(S^3, L_b(K))$ by the involution ρ, as depicted in Figure 8, right.

Suppose by contradiction that $L_b(K)$ is the unlink, and hence $\Sigma(L_b(K)) \cong S^1 \times S^2$. Observe that $\overline{K}$ is a 2-bridge knot with continued fraction $p/q = [\alpha_1, \dots, \alpha_n \pm 1]$.

We want to show that $\overline{K}$ is not an unknot. Observe that $\overline{K}$ is isotopic to the 2-bridge knot given by the continuous fraction $[-\alpha_n \mp 1, -\alpha_{n-1}, \dots, -\alpha_1]$. If $\overline{K}$ is unknotted, by [18] we know that $[-\alpha_n \mp 1, -\alpha_{n-1}, \dots, -\alpha_1] = 1/s$ for some $s \in \mathbb{Z}$. Then

$$\frac{\pm s + 1}{s} = [-\alpha_n, -\alpha_{n-1}, \dots, -\alpha_1].$$

Computing the continuous fraction term by term, one finds that

$$[\alpha_1, \dots, \alpha_n] = [\pm 2, \mp 2, \dots].$$

This proves that $\overline{K}$ cannot be an unknot, since these coefficients do not satisfy the constraint

$$\sum_{i \geq 0} \alpha_{2i+1} = 0.$$

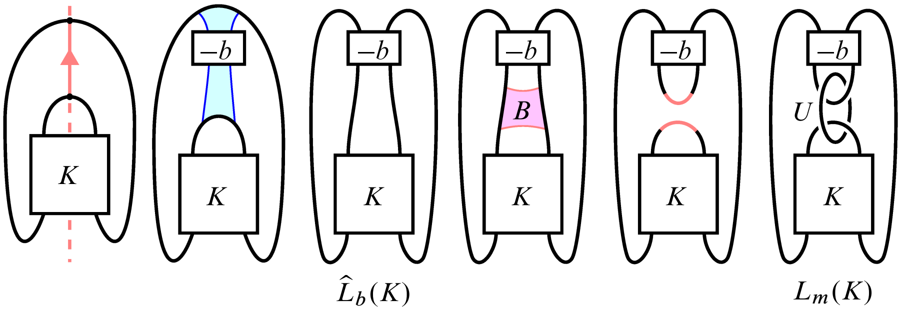


Figure 9: Construction of the moth link.

Now, since $L_b(K)$ is obtained from $\overline{K}$ by taking the double cover branched over the quotient of the axis $\overline{A}$, $\Sigma(L_b(K))$ is a double branched cover over $\Sigma(\overline{K})$. Since the transfer map (see [3, Chapter III, Section 2])

$$H_1\big(\Sigma(\overline{K}), \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \cong \mathbb{Z}/p\mathbb{Z} \hookrightarrow H_1\big(\Sigma(L_b(K)), \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \cong \mathbb{Z}\big[\tfrac{1}{2}\big]$$

is injective and $\mathbb{Z}\big[\frac{1}{2}\big]$ is torsion-free, we obtain a contradiction. It follows that $L_b(K)$ is a nontrivial 2-bridge link, $n(L_b(K)) = 1$ and hence that K is not equivariantly slice. □

5 A new invariant of equivariant concordance

Recall that a *semiorientation* on a link L is the choice of an orientation on each component of L, up to reversing the orientation on all components simultaneously.

Definition 5.1 Let K be a directed strongly invertible knot. We define $\widehat{L}_b(K)$ to be the 2-periodic, semioriented link obtained by endowing $L_b(K)$ with the opposite semiorientation.

Observe that $\widehat{L}_b(K)$ is obtained from K via a band move coherent with the (unique) semiorientation of K. Conversely, we can attach another equivariant band B to $\widehat{L}_b(K)$ obtaining again K (see Figure 9).

Definition 5.2 We define the *moth link of* K to be the link $L_m(K)$ given by the union of K and a meridian U of the core of the band B, as described in Figure 9. Observe that this meridian can be chosen so that $L_m(K)$ is a 2-component strongly invertible link.

Remark 5.3 Using the notation of [9], $L_m(K)$ is the *strong fusion* of the link $\widehat{L}_b(K)$ along the band B.

Proposition 5.4 *Let (K_0, ρ_0, h_0) and (K_1, ρ_1, h_1) be two equivariantly concordant directed strongly invertible knots. Then $L_m(K)$ is equivariantly concordant to $L_m(J)$.*

Proof Let $C \subset S^3 \times I$ be a concordance between (K_0, ρ_0, h_0) and (K_1, ρ_1, h_1) equivariant with respect to an extension $\rho\colon S^3 \times I \to S^3 \times I$ of $\rho_0 \sqcup \rho_1$. In the proof of Proposition 2.6 we found an equivariant embedding of $E \cong D^1 \times D^1 \times D^1$ in $S^3 \times I$ which intersects the concordance C in $D^1 \times \partial D^1 \times D^1$ and such that $C_b = (C \setminus E) \cup \partial D^1 \times D^1 \times D^1$ exhibits an equivariant concordance between $L_b(K_0)$ and $L_b(K_1)$.

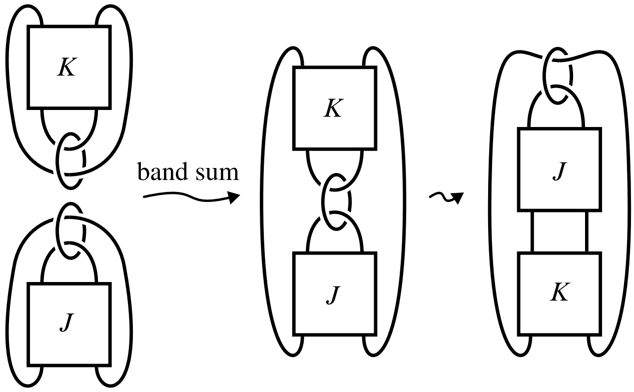


Figure 10: The band sum of $L_m(K)$ and $L_m(J)$ is $L_m(K \widetilde{\#} J)$.

Now let N be a tubular neighbourhood of $D^1 \times 0 \times D^1$ in $S^3 \times I$ and let c be the arc $0 \times 0 \times D^1$. We can take $\epsilon > 0$ small such that $E_\epsilon = D^1 \times \epsilon D^1 \times D^1 \subset N$. Then $(C_b \setminus E_\epsilon) \cup D^1 \times \partial(\epsilon D^1) \times D^1 \cup \partial N|_c$ is an equivariant concordance between $L_m(K_0)$ and $L_m(K_1)$. □

Definition 5.5 Let K be a directed strongly invertible knot. We define the *moth polynomial* of K as the η-function of $L_m(K) = K \cup U$, taken with respect to the component K, ie

$$\eta(L_m(K))(t) = \eta(K, U; t).$$

Proposition 5.6 *The moth polynomial induces a group homomorphism*

$$\eta(L_m(-)) \colon \widetilde{\mathcal{C}} \to \mathbb{Q}(t).$$

Proof Let K and J be two directed strongly invertible knots. By Proposition 5.4 if K and J are equivariantly concordant then $L_m(K)$ and $L_m(J)$ are concordant. Since the η-function is a concordance invariant, $\eta(L_m(K)) = \eta(L_m(J))$; therefore $\eta(L_m(-))$ is well defined. Next we have to show that $\eta(L_m(K \widetilde{\#} J)) = \eta(L_m(K)) + \eta(L_m(J))$. This follows by observing that $L_m(K \widetilde{\#} J)$ is obtained from $L_m(K)$ and $L_m(J)$ by a band sum, as shown in Figure 10 and using [4, Theorem 7.1]. □

We provide now a formula to compute the moth polynomial of a directed strongly invertible knot K from the Conway polynomial of K and $\widehat{L}_b(K)$.

Proposition 5.7 *The moth polynomial of a directed strongly invertible knot K can be computed by the formula*

$$\eta(L_m(K))(t) = \frac{\nabla_{\widehat{L}_b(K)}(z)}{z \nabla_K(z)},$$

where $\nabla_L(z)$ is the Conway polynomial of an oriented (or semioriented) link L and $z = i(2 - t - t^{-1})^{1/2}$.

The proposition above is an immediate consequence of Propositions 5.8 and 5.9.

Proposition 5.8 [9, Proposition 1] *Let L be a 2-component link and let b be a band with ends on different components of L. Denote by $L(b)$ the knot obtained by performing a band move on L along b and by $\widehat{L}(b)$ the strong fusion of L given by b. Then the Cochran invariant of $\widehat{L}(b)$ is given by*

$$\beta(\widehat{L}(b))(x) = \frac{-ix^{-1/2}\nabla_L(ix^{1/2})}{\nabla_{L(b)}(ix^{1/2})}.$$

Proposition 5.9 [4, Theorem 7.1] *The Kojima–Yamasaki η-function can be expanded in powers of $x = (1-t)(1-t^{-1})$ so that*

$$\eta(L)(t) = \beta(L)(x).$$

Since $\eta(L_m(-))$ is a homomorphism, Proposition 5.7 implies the following result.

Theorem 5.10 *Let K be a directed strongly invertible knot such that $\nabla_{\widehat{L}_b(K)}(z) \neq 0$. Then K is not equivariantly slice and has infinite order in $\widetilde{\mathcal{C}}$.*

As an immediate application of Theorem 5.10 we have the following refinement of the results in Section 4 on 2-bridge knots.

Proposition 5.11 *Every 2-bridge knot has infinite order in $\widetilde{\mathcal{C}}$, independently of the choice of strong inversion and direction.*

Proof First of all, by Remark 2.5 it is sufficient to show that a directed strongly invertible knot K of type $I_1(\alpha_1,\dots,\alpha_n,c_1,\dots,c_n)$ or $I_2(a_1,\dots,a_n)$, with the direction specified in Figures 1, top, and 2, has infinite order in $\widetilde{\mathcal{C}}$.

As proven in Propositions 4.1 and 4.2, either Sakuma's η-polynomial of K is nonzero, and hence K has infinite order, or $\Sigma(L_b(K)) = \Sigma(\widehat{L}_b(K))$ is a rational homology S^3.

Recall now that, for a link $L \subset S^3$, we have $|H_1(\Sigma(L),\mathbb{Z})| = |\Delta_L(-1)|$, where 0 means that the group is infinite. Since $H_1(\Sigma(\widehat{L}_b(K)),\mathbb{Z})$ is finite, we deduce that the Alexander polynomial of $\widehat{L}_b(K)$ is nonzero, and hence by Theorem 5.10 that K has infinite order in $\widetilde{\mathcal{C}}$. □

Acknowledgements We want to thank Christoph Lamm and Makoto Sakuma for pointing out the missing case in the previous version of the paper. We are also grateful to the referee for their helpful and kind comments.

The authors are members of the Italian National Group for Algebraic and Geometric Structures and their Applications (INdAM-GNSAGA).

References

[1] **A Alfieri**, **K Boyle**, *Strongly invertible knots, invariant surfaces, and the Atiyah–Singer signature theorem*, Michigan Math. J. 74 (2024) 1–17 MR Zbl

[2] **K Boyle**, **A Issa**, *Equivariant 4-genera of strongly invertible and periodic knots*, J. Topol. 15 (2022) 1635–1674 MR Zbl

[3] **G E Bredon**, *Introduction to compact transformation groups*, Pure and Appl. Math. 46, Academic Press, New York (1972) MR Zbl

[4] **T D Cochran**, *Geometric invariants of link cobordism*, Comment. Math. Helv. 60 (1985) 291–311 MR Zbl

[5] **I Dai**, **M Hedden**, **A Mallick**, *Corks, involutions, and Heegaard Floer homology*, J. Eur. Math. Soc. 25 (2023) 2319–2389 MR Zbl

[6] **I Dai**, **A Mallick**, **M Stoffregen**, *Equivariant knots and knot Floer homology*, J. Topol. 16 (2023) 1167–1236 MR Zbl

[7] **A Di Prisa**, *The equivariant concordance group is not abelian*, Bull. Lond. Math. Soc. 55 (2023) 502–507 MR Zbl

[8] **A Hatcher**, **W Thurston**, *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. 79 (1985) 225–246 MR Zbl

[9] **U Kaiser**, *Strong band sum and dichromatic invariants*, Manuscripta Math. 74 (1992) 237–251 MR Zbl

[10] **L H Kauffman**, **S Lambropoulou**, *On the classification of rational knots*, Enseign. Math. 49 (2003) 357–410 MR Zbl

[11] **L H Kauffman**, **S Lambropoulou**, *On the classification of rational tangles*, Adv. Appl. Math. 33 (2004) 199–237 MR Zbl

[12] **L H Kauffman**, **L R Taylor**, *Signature of links*, Trans. Amer. Math. Soc. 216 (1976) 351–365 MR Zbl

[13] **A Kawauchi**, *A survey of knot theory*, Birkhäuser, Basel (1996) MR Zbl

[14] **S Kojima**, **M Yamasaki**, *Some new invariants of links*, Invent. Math. 54 (1979) 213–228 MR Zbl

[15] **A N Miller**, **M Powell**, *Strongly invertible knots, equivariant slice genera, and an equivariant algebraic concordance group*, J. Lond. Math. Soc. 107 (2023) 2025–2053 MR Zbl

[16] **J W Morgan**, **H Bass** (editors), *The Smith conjecture*, Pure Appl. Math. 112, Academic, Orlando, FL (1984) MR Zbl

[17] **M Sakuma**, *On strongly invertible knots*, from "Algebraic and topological theories", Kinokuniya, Tokyo (1986) 176–196 MR Zbl

[18] **H Schubert**, *Knoten mit zwei Brücken*, Math. Z. 65 (1956) 133–170 MR Zbl

[19] **L Siebenmann**, *Exercices sur les nœuds rationnels*, preprint (1975)

[20] **M Snape**, *Homological invariants of strongly invertible knots*, PhD thesis, University of Glasgow (2018) Available at `https://theses.gla.ac.uk/39015/`

Scuola Normale Superiore di Pisa
Pisa, Italy

Università di Pisa
Pisa, Italy

alessio.diprisa@sns.it, giovanni.framba@phd.unipi.it

Received: 12 June 2023 Revised: 14 December 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2025 is US $760/year for the electronic version, and $1110/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 2 (pages 645–1264) 2025

An upper bound conjecture for the Yokota invariant 645

GIULIO BELLETTI

Arithmetic representations of mapping class groups 677

EDUARD LOOIJENGA

The geometry of subgroup embeddings and asymptotic cones 699

ANDY JARNEVIC

On k-invariants for (∞, n)-categories 721

YONATAN HARPAZ, JOOST NUITEN and MATAN PRASMA

Circular-orderability of 3-manifold groups 791

IDRISSA BA and ADAM CLAY

On the involutive Heegaard Floer homology of negative semidefinite plumbed 3-manifolds with $b_1 = 1$ 827

PETER K JOHNSON

Localization of a $\mathrm{KO}^*(\mathrm{pt})$-valued index and the orientability of the $\mathrm{Pin}^-(2)$ monopole moduli space 887

JIN MIYAZAWA

Verdier duality on conically smooth stratified spaces 919

MARCO VOLPE

Toward a topological description of Legendrian contact homology of unit conormal bundles 951

YUKIHIRO OKAMOTO

Enriched quasicategories and the templicial homotopy coherent nerve 1029

WENDY LOWEN and ARNE MERTENS

Closures of T-homogeneous braids are real algebraic 1075

BENJAMIN BODE

A new invariant of equivariant concordance and results on 2-bridge knots 1117

ALESSIO DI PRISA and GIOVANNI FRAMBA

Recipes to compute the algebraic K-theory of Hecke algebras of reductive p-adic groups 1133

ARTHUR BARTELS and WOLFGANG LÜCK

All known realizations of complete Lie algebras coincide 1155

YVES FÉLIX, MARIO FUENTES and ANICETO MURILLO

The space of nonextendable quasimorphisms 1169

MORIMICHI KAWASAKI, MITSUAKI KIMURA, SHUHEI MARUYAMA, TAKAHIRO MATSUSHITA and MASATO MIMURA

Synthetic approach to the Quillen model structure on topological spaces 1227

STERLING EBEL and KRZYSZTOF KAPULKIN