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**Recipes to compute the algebraic  $K$ -theory of  
Hecke algebras of reductive  $p$ -adic groups**

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# Recipes to compute the algebraic $K$ -theory of Hecke algebras of reductive $p$ -adic groups

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We compute the algebraic  $K$ -theory of the Hecke algebra of a reductive  $p$ -adic group  $G$  using the fact that the Farrell–Jones conjecture is known in this context. The main tools will be the properties of the associated Bruhat–Tits building and an equivariant Atiyah–Hirzebruch spectral sequence. In particular, the projective class group can be written as the colimit of the projective class groups of the compact open subgroups of  $G$ .

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## 1 Introduction

We begin with stating the main theorem of this paper; explanation will follow:

**Main Theorem 1.1** *Let  $G$  be a td-group which is modulo a normal compact subgroup a subgroup of a reductive  $p$ -adic group. Let  $R$  be a uniformly regular ring with  $\mathbb{Q} \subseteq R$ . Choose a model  $E_{\text{Cop}}(G)$  for the classifying space for proper smooth  $G$ -actions. Let  $\mathcal{I} \subseteq \text{Cop}$  be the set of isotropy groups of points in  $E_{\text{Cop}}(G)$ .*

Then:

- (i) *The map induced by the projection  $E_{\text{Cop}}(G) \rightarrow G/G$  induces for every  $n \in \mathbb{Z}$  an isomorphism*

$$H_n^G(E_{\text{Cop}}(G); \mathbf{K}_R) \rightarrow H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R)).$$

- (ii) *There is a (strongly convergent) spectral sequence*

$$E_{p,q}^2 = SH_p^{G,\mathcal{I}}(E_{\text{Cop}}(G); \overline{K_q(\mathcal{H}(?; R))}) \Rightarrow K_{p+q}(\mathcal{H}(G; R)),$$

*whose  $E^2$ -term is concentrated in the first quadrant.*

- (iii) *The canonical map induced by the various inclusions  $K \subseteq G$ ,*

$$\text{colim}_{K \in \text{Sub}_{\mathcal{I}}(G)} K_0(\mathcal{H}(K; R)) \rightarrow K_0(\mathcal{H}(G; R)),$$

*can be identified with the isomorphism appearing in assertion (i) in degree  $n = 0$  and hence is bijective.*

- (iv) *We have  $K_n(\mathcal{H}(G; R)) = 0$  for  $n \leq -1$ .*

We proved assertion (i) of Theorem 1.1 in [2, Corollary 1.8]. So this paper deals with implications of it concerning computations of the algebraic  $K$ -groups  $K_n(\mathcal{H}(G))$  of the Hecke algebra of  $G$ .

A  $td$ -group  $G$  is a locally compact, second countable, totally disconnected topological Hausdorff group. It is *modulo a normal compact subgroup a subgroup of a reductive  $p$ -adic group* if it contains a (not necessarily open) normal compact subgroup  $K$  such that  $G/K$  is isomorphic to a subgroup of some reductive  $p$ -adic group.

A ring is called *uniformly regular* if it is Noetherian and there exists a natural number  $l$  such that any finitely generated  $R$ -module admits a resolution by projective  $R$ -modules of length at most  $l$ . We write  $\mathbb{Q} \subseteq R$  if, for any integer  $n$ , the element  $n \cdot 1_R$  is a unit in  $R$ . Examples of uniformly regular rings  $R$  with  $\mathbb{Q} \subseteq R$  are fields of characteristic zero.

We denote by  $\mathcal{H}(G; R)$  the *Hecke algebra* consisting of locally constant functions  $s: G \rightarrow R$  with compact support, where the additive structure comes from the additive structure of  $R$  and the multiplicative structure from the convolution product. Note that  $\mathcal{H}(G; R)$  is a ring without unit.

We denote by  $E_{\text{Cop}}(G)$  a model for the *classifying space for proper smooth  $G$ -actions*, ie a  $G$ -CW-complex, whose isotropy groups are all compact open subgroups of  $G$  and for which  $E_{\text{Cop}}(G)^H$  is weakly contractible for any compact open subgroup  $H \subseteq G$ . Two such models are  $G$ -homotopy equivalent. Hence  $H_n^G(E_{\text{Cop}}(G); \mathbf{K}_R)$  is independent of the choice of a model. If  $G$  is a reductive  $p$ -adic group with compact center, then its Bruhat–Tits building is a model for  $E_{\text{Cop}}(G)$ . If the center is not compact, one has to pass to the extended Bruhat–Tits building.

We will construct a *smooth  $G$ -homology theory*  $H_*^G(-; \mathbf{K}_R)$  in Section 3. It assigns to a smooth  $G$ -CW-pair  $(X, A)$  a collection of abelian groups  $\mathcal{H}_n^G(X, A; \mathbf{K}_R)$  for  $n \in \mathbb{Z}$  that satisfies the expected axioms, ie long exact sequence of a pair,  $G$ -homotopy invariance, excision, and the disjoint union axiom. Moreover, for every open subgroup  $U \subseteq G$  and  $n \in \mathbb{Z}$ , we have

$$(1.2) \quad H_n^G(G/U; \mathbf{K}_R) \cong K_n(\mathcal{H}(U; R)).$$

Let  $\mathcal{F}$  be a collection of open subgroups of  $G$  which is closed under conjugation. Examples are the set  $\mathcal{C}_{\text{op}}$  of compact open subgroups of  $G$  and the set  $\mathcal{I}$  of isotropy groups of points of some model for  $E_{\text{Cop}}(G)$ . The subgroup category  $\text{Sub}_{\mathcal{F}}(G)$  appearing in Theorem 1.1(iii) has  $\mathcal{F}$  as set of objects and will be described in detail in Section 2.A.

The abelian groups  $SH_p^{G, \mathcal{F}}(E_{\mathcal{F}}(G); \overline{K_q(\mathcal{H}(?; R))})$  appearing in Theorem 1.1(ii) will be defined for the covariant functor  $\overline{K_q(\mathcal{H}(?; R))}: \text{Sub}_{\mathcal{F}}(G) \rightarrow \mathbb{Z}\text{-Mod}$ , whose value at  $U \in \mathcal{F}$  is  $K_n(\mathcal{H}(U; R))$ , in Section 2.B. They are closely related to the *Bredon homology groups*  $BH_p^{G, \mathcal{F}}(E_{\mathcal{F}}(G); K_q(\mathcal{H}(?; R)))$ .

The proof of Theorem 1.1 will be given in Section 4.

The relevance of the Hecke algebra  $\mathcal{H}(G; R)$  is that the category of nondegenerate modules over it is isomorphic to the category of smooth  $G$ -representations with coefficients in  $R$ ; see for instance

Bernstein [5] or Garrett [13]. Hence, in particular, its projective class group  $K_0(\mathcal{H}(G; R))$  is important. The various inclusions  $K \rightarrow G$  for  $K \in \mathcal{C}_{\text{op}}$  induce a map

$$(1.3) \quad \bigoplus_{K \in \mathcal{C}_{\text{op}}} K_0(\mathcal{H}(K; R)) \rightarrow K_0(\mathcal{H}(G; R)),$$

which factors over the canonical epimorphism  $\bigoplus_{K \in \mathcal{C}_{\text{op}}} K_0(\mathcal{H}(K; R)) \rightarrow \text{colim}_{K \in \text{Sub}_{\mathcal{I}}(G)} K_0(\mathcal{H}(K; R))$  to the isomorphism appearing in Theorem 1.1(iii) and is hence surjective. Dat [10] has shown that the map (1.3) is rationally surjective for  $G$  a reductive  $p$ -adic group and  $R = \mathbb{C}$ . In particular, the cokernel of it is a torsion group. Dat [9, Conjecture 1.11] conjectured that this cokernel is  $\tilde{w}_G$ -torsion. Here  $\tilde{w}_G$  is a certain multiple of the order of the Weyl group of  $G$ . Dat [9, Proposition 1.13] proved this conjecture for  $G = \text{GL}_n(F)$  for a  $p$ -adic field  $F$  of characteristic zero and asked about the integral version — see the comment following [9, Proposition 1.10] — which is now proven by Theorem 1.1(iii).

The computations simplify considerably in the case of a reductive  $p$ -adic group thanks to the associated (extended) Bruhat–Tits building; see Sections 5 and 7. As an illustration we analyze the projective class groups of the Hecke algebras of  $\text{SL}_n(F)$ ,  $\text{PGL}_n(F)$  and  $\text{GL}_n(F)$  in Section 6.

One of our main tools will be the *smooth equivariant Atiyah–Hirzebruch spectra sequence*, which we will establish and examine in Section 2.

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## 2 The smooth equivariant Hirzebruch spectral sequence

Throughout this section we fix a set  $\mathcal{F}$  of open subgroups of  $G$  which is closed under conjugation. Our main examples for  $\mathcal{F}$  are the family  $\mathcal{O}p$  of all open subgroups and the family  $\mathcal{C}op$  of all compact open subgroups. An  $\mathcal{F}$ - $G$ -CW-complex  $X$  is a  $G$ -CW-complex  $X$  such that, for every  $x \in X$ , its isotropy group  $G_x$  belongs to  $\mathcal{F}$ . A smooth  $G$ -CW-complex is the same as an  $\mathcal{O}p$ -CW-complex and a proper smooth  $G$ -CW-complex is the same as a  $\mathcal{C}op$ -CW-complex. Let  $\mathcal{H}_*^G$  be a smooth  $G$ -homology theory.

The main result of this section is:

**Theorem 2.1** Consider a pair  $(X, A)$  of  $\mathcal{F}$ - $G$ -CW-complexes and a smooth  $G$ -homology theory  $\mathcal{H}_*^G$ . Then there is an equivariant Atiyah–Hirzebruch spectral sequence converging to  $\mathcal{H}_{p+q}^G(X, A)$ , whose  $E^2$ -term is given by

$$E_{p,q}^2 = BH_p^{G,\mathcal{F}}(X, A; \mathcal{H}_q^G)$$

for the Bredon homology  $BH_p^{G,\mathcal{F}}(X, A; \mathcal{H}_q^G)$  of  $(X, A)$  with coefficients in the covariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $\mathcal{H}_q^G$  that sends  $G/H$  to  $\mathcal{H}_q^G(G/H)$ .

The remainder of this section is devoted to the definition of the Bredon homology, the construction of the equivariant Atiyah–Hirzebruch spectral sequence, and some general calculations concerning the  $E^2$ -term. Convergence means that there is an ascending filtration  $F_{l,m-l}\mathcal{H}_m^G(X, A)$  for  $l = 0, 1, 2, \dots$  of  $\mathcal{H}_m^G(X, A)$  such that  $F_{p,q}\mathcal{H}_{p+q}^G(X, A)/F_{p-1,q+1}\mathcal{H}_{p+q}^G(X, A) \cong E_{p,q}^\infty$  for  $E_{p,q}^\infty = \text{colim}_{r \rightarrow \infty} E_{p,q}^r$ .

### 2.A The smooth orbit category and the smooth subgroup category

The  $\mathcal{F}$ -orbit category  $\text{Or}_{\mathcal{F}}(G)$  has as objects homogeneous  $G$ -spaces  $G/H$  with  $H \in \mathcal{F}$ . Morphisms from  $G/H$  to  $G/K$  are  $G$ -maps  $G/H \rightarrow G/K$ . We will put no topology on  $\text{Or}_{\mathcal{F}}(G)$ . For any  $G$ -map  $f: G/H \rightarrow G/K$  of smooth homogeneous spaces, there is an element  $g \in G$  such that  $gHg^{-1} \subseteq K$  and  $f$  is the  $G$ -map  $R_{g^{-1}}: G/H \rightarrow G/K$  sending  $g'H$  to  $g'g^{-1}K$ . Given two elements  $g_0, g_1 \in G$  such that  $g_iHg_i^{-1} \subseteq K$  for  $i = 0, 1$ , we have  $R_{g_0^{-1}} = R_{g_1^{-1}}$  if and only if  $g_1g_0^{-1} \in K$ . We get a bijection

$$(2.2) \quad K \backslash \{g \in G \mid gHg^{-1} \subseteq K\} \xrightarrow{\cong} \text{map}_G(G/H, G/K), \quad g \mapsto R_{g^{-1}}.$$

The  $\mathcal{F}$ -subgroup category  $\text{Sub}_{\mathcal{F}}(G)$  has  $\mathcal{F}$  as the set of objects. For  $H, K \in \mathcal{F}$ , denote by  $\text{conhom}_G(H, K)$  the set of group homomorphisms  $f: H \rightarrow K$  for which there exists an element  $g \in G$  with  $gHg^{-1} \subseteq K$  such that  $f$  is given by conjugation with  $g$ , ie  $f = c(g): H \rightarrow K, h \mapsto ghg^{-1}$ . Note that  $c(g) = c(g')$  holds for two elements  $g, g' \in G$  with  $gHg^{-1} \subseteq K$  and  $g'Hg'^{-1} \subseteq K$  if and only if  $g^{-1}g'$  lies in the centralizer  $C_G H = \{g \in G \mid gh = hg \text{ for all } h \in H\}$  of  $H$  in  $G$ . The group of inner automorphisms  $\text{Inn}(K)$  of  $K$  acts on  $\text{conhom}_G(H, K)$  from the left by composition. Define the set of morphisms

$$\text{mor}_{\text{Sub}_{\text{cop}}(G)}(H, K) := \text{Inn}(K) \backslash \text{conhom}_G(H, K).$$

There is an obvious bijection

$$(2.3) \quad K \backslash \{g \in G \mid gHg^{-1} \subseteq K\} / C_G H \xrightarrow{\cong} \text{Inn}(K) \backslash \text{conhom}_G(H, K), \quad KgC_G H \mapsto [c(g)],$$

where  $[c(g)] \in \text{Inn}(K) \backslash \text{conhom}_G(H, K)$  is the class represented by the element  $c(g): H \rightarrow K, h \mapsto ghg^{-1}$ , in  $\text{conhom}_G(H, K)$  and  $K$  acts from the left and  $C_G H$  from the right on  $\{g \in G \mid gHg^{-1} \subseteq K\}$  by the multiplication in  $G$ .

Let

$$(2.4) \quad P: \text{Or}_{\mathcal{F}}(G) \rightarrow \text{Sub}_{\mathcal{F}}(G)$$

be the canonical projection which sends an object  $G/H$  to  $H$  and is given on morphisms by the obvious projection under the identifications (2.2) and (2.3).

### 2.B Cellular chain complexes and Bredon homology

Given an  $\mathcal{F}$ - $G$ -CW-complex  $X$ , we obtain a contravariant  $\text{Or}_{\mathcal{F}}(G)$ -space  $O_X : \text{Or}_{\mathcal{F}}(G) \rightarrow \text{Spaces}$  by sending  $G/H$  to  $\text{map}_G(G/H, X) = X^H$ . We get a contravariant  $\text{Sub}_{\mathcal{F}}(G)$ -space  $S_X : \text{Sub}_{\mathcal{F}}(G) \rightarrow \text{Spaces}$  by sending  $H$  to  $C_G H \backslash \text{map}_G(G/H, X) = C_G H \backslash X^H$ . A morphism  $H \rightarrow K$  given by an element  $g \in G$  satisfying  $gHg^{-1} \subseteq K$  is sent to the map  $C_G K \backslash X^K \rightarrow C_G H \backslash X^H$  induced by the map  $X^K \rightarrow X^H, x \mapsto g^{-1}x$ .

Given a pair  $(Y, A)$  with a filtration  $A = Y_{-1} \subseteq Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots \subseteq Y$  with  $Y = \text{colim}_{n \rightarrow \infty} Y_n$ , we associate to it a  $\mathbb{Z}$ -chain complex  $C_*^c(Y, A)$ , whose  $n^{\text{th}}$  chain module is the singular homology  $H_n^{\text{sing}}(Y_n, Y_{n-1})$  of the pair  $(Y_n, Y_{n-1})$  (with coefficients in  $\mathbb{Z}$ ) and whose  $n^{\text{th}}$  differential is given by the composite

$$H_n^{\text{sing}}(Y_n, Y_{n-1}) \xrightarrow{\partial_n} H_{n-1}^{\text{sing}}(Y_{n-1}) \xrightarrow{H_{n-1}^{\text{sing}}(i_{n-1})} H_{n-1}^{\text{sing}}(Y_{n-1}, Y_{n-2})$$

for  $\partial_n$  the boundary operator of the pair  $(Y_n, Y_{n-1})$  and  $i_{n-1} : Y_{n-1} = (Y_{n-1}, \emptyset) \rightarrow (Y_{n-1}, Y_{n-2})$  the inclusion.

Given a pair of  $\mathcal{F}$ - $G$ -CW-complexes  $(X, A)$ , the filtration by its skeletons induces filtrations on  $X^H$  and  $C_G H \backslash X^H$  for every  $H \in \mathcal{F}$ . We get a contravariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -chain complex  $C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) : \text{Or}_{\mathcal{F}}(G) \rightarrow \mathbb{Z}\text{-Ch}$  and a contravariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -chain complex  $C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A) : \text{Sub}_{\mathcal{F}}(G) \rightarrow \mathbb{Z}\text{-Ch}$  by putting

$$\begin{aligned} C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A)(G/H) &:= C_*^c(O_X(G/H), O_A(G/H)) = C_*^c(X^H, A^H), \\ C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A)(H) &:= C_*^c(S_X(X)(H), S_A(H)) = C_*^c(C_G H \backslash X^H, C_G H \backslash A^H). \end{aligned}$$

Choose a  $G$ -pushout

$$(2.5) \quad \begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_{n-1} \end{array}$$

It induces, for every closed subgroup  $H \subseteq G$ , pushouts

$$\begin{array}{ccc} \coprod_{i \in I_n} (G/H_i)^H \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} (q_i^n)^H} & X_{n-1}^H \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} (G/H_i)^H \times D^n & \xrightarrow{\coprod_{i \in I_n} (Q_i^n)^H} & X_{n-1}^H \end{array}$$

and

$$\begin{array}{ccc} \coprod_{i \in I_n} C_G H \backslash (G/H_i)^H \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} C_G H \backslash (q_i^n)^H} & C_G H \backslash X_{n-1}^H \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} C_G H \backslash (G/H_i)^H \times D^n & \xrightarrow{\coprod_{i \in I_n} C_G H \backslash (Q_i^n)^H} & C_G H \backslash X_{n-1}^H \end{array}$$

Note that  $(G/H_i)^H$  agrees with  $\text{mor}_{\text{Or}_{\mathcal{F}}(G)}(G/H, G/H_i) = \text{map}_G(G/H, G/H_i)$ . In the sequel, for a set  $S$  we denote by  $\mathbb{Z}S$  the free  $\mathbb{Z}$ -module with basis  $S$ . Since singular homology satisfies the disjoint union axiom, homotopy invariance and excision, we obtain an isomorphism of contravariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -modules

$$(2.6) \quad \bigoplus_{i \in I_n} \mathbb{Z}\text{mor}_{\text{Or}_{\mathcal{F}}(G)}(?, G/H_i) \xrightarrow{\cong} C_n^{\text{Or}_{\mathcal{F}}(G)}(X, A),$$

where  $\mathbb{Z}\text{mor}_{\text{Or}_{\mathcal{F}}(G)}(?, G/H_i)$  is the free  $\mathbb{Z}\text{Or}(G)$ -module based at the object  $G/H_i$  [14, Example 9.8 on page 164], and analogously an isomorphism of contravariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -modules

$$(2.7) \quad \bigoplus_{i \in I_n} \mathbb{Z}\text{mor}_{\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)}(?, H_i) \xrightarrow{\cong} C_n^{\text{Sub}_{\mathcal{F}}(G)}(X, A).$$

If  $P_*C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A)$  is the  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -chain complex obtained by induction with  $P: \text{Or}_{\mathcal{F}}(G) \rightarrow \text{Sub}_{\mathcal{F}}(G)$  from  $C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A)$  — see [14, Example 9.15 on page 166] — we conclude from (2.6) and (2.7) that the canonical map of  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -chain complexes

$$(2.8) \quad P_*C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \xrightarrow{\cong} C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A)$$

is an isomorphism.

For a covariant  $\mathbb{Z}\text{Or}(G)$ -module  $M$ , we get from the tensor product over  $\text{Or}_{\mathcal{F}}(G)$  — see [14, 9.13 on page 166] — a  $\mathbb{Z}$ -chain complex  $C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{F}}(G)} M$ .

**Definition 2.9** (Bredon homology) We define the  $n^{\text{th}}$  Bredon homology to be the  $\mathbb{Z}$ -module

$$BH_n^{G, \mathcal{F}}(X, A; M) = H_n(C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{F}}(G)} M).$$

Given a covariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -module  $N$ , define analogously

$$SH_n^{G, \mathcal{F}}(X, A; N) = H_n(C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)} N).$$

Given a covariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -module  $N$ , define the covariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $P^*N$  to be  $N \circ P$ . We get from the adjunction of [14, 9.22 on page 169] and (2.8) a natural isomorphism of  $\mathbb{Z}$ -chain complexes

$$(2.10) \quad C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)} N \xrightarrow{\cong} C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{F}}(G)} P^*N$$

and hence a natural isomorphism of  $\mathbb{Z}$ -modules

$$(2.11) \quad BH_n^{G, \mathcal{F}}(X, A; P^*N) \xrightarrow{\cong} SH_n^{G, \mathcal{F}}(X, A; N).$$

Let  $(X, A)$  be a pair of  $\mathcal{F}$ -CW-complexes. Denote by  $\mathcal{I}$  the set of isotropy groups of points in  $X$ . Let  $M$  be a covariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module and  $N$  be a covariant  $\text{Sub}_{\mathcal{F}}(G)$ -module. Denote by  $M|_{\mathcal{I}}$  and  $N|_{\mathcal{I}}$  their restrictions to  $\text{Or}_{\mathcal{I}}(G)$  and  $\text{Sub}_{\mathcal{I}}(G)$ . Then one easily checks using [11, Lemma 1.9] that there are canonical isomorphisms

$$(2.12) \quad BH_n^{G, \mathcal{I}}(X, A; M|_{\mathcal{I}}) \cong BH_n^{G, \mathcal{F}}(X, A; M),$$

$$(2.13) \quad SH_n^{G, \mathcal{I}}(X, A; N|_{\mathcal{I}}) \cong BH_n^{G, \mathcal{F}}(X, A; N).$$

### 2.C The construction of the equivariant Atiyah–Hirzebruch spectral sequence

**Proof of Theorem 2.1** Since  $(X, A)$  comes with the skeletal filtration, there is by a general construction a spectral sequence

$$E_{p,q}^r, \quad d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$$

converging to  $\mathcal{H}_{p+q}^G(X, A)$ , whose  $E_1$ -term is given by

$$E_{p,q}^1 = \mathcal{H}_{p+q}^G(X_p, X_{p-1}),$$

and the first differential is the composite

$$d_{p,q}^1: E_{p,q}^1 = \mathcal{H}_{p+q}^G(X_p, X_{p-1}) \rightarrow \mathcal{H}_{p+q-1}^G(X_{p-1}) \rightarrow \mathcal{H}_{p+q-1}^G(X_{p-1}, X_{p-2}) = E_{p-1,q}^1,$$

where the first map is the boundary operator of the pair  $(X_p, X_{p-1})$  and the second is induced by the inclusion. The elementary construction is explained for trivial  $G$  for instance in [18, 15.6 on page 339]. The construction carries directly over to the equivariant setting.

The straightforward proof of the identification of  $E_{p,q}^2$  with  $BH_p^{G,\mathcal{F}}(X, A; \mathcal{H}_q)$  is left to the reader.  $\square$

### 2.D Passing to the subgroup category

**Condition 2.14** ( $\text{Sub}|_{\mathcal{F}}$ ) Let  $\mathcal{H}_*^G(-)$  be a smooth  $G$ -homology theory. Then  $\mathcal{H}_*^G(-)$  satisfies the condition ( $\text{Sub}|_{\mathcal{F}}$ ) if, for any  $H \in \mathcal{F}$  and  $g \in C_G H$ , the  $G$ -map  $R_{g^{-1}}: G/H \rightarrow G/H$  sending  $g'H$  to  $g'g^{-1}H$  induces the identity on  $\mathcal{H}_q^G(G/H)$ , ie  $\mathcal{H}_q^G(R_{g^{-1}}) = \text{id}_{\mathcal{H}_q^G(G/H)}$ .

**Remark 2.15** Suppose that the  $G$ -homology theory  $\mathcal{H}_*^G$  satisfies the condition ( $\text{Sub}|_{\mathcal{F}}$ ). Then the covariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $\mathcal{H}_q^G$  sending  $G/H$  with  $H \in \mathcal{F}$  to  $\mathcal{H}_q^G(G/H)$  defines a covariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -module  $\overline{\mathcal{H}_q^G}: \text{Sub}_{\mathcal{F}}(G) \rightarrow \mathbb{Z}\text{-Mod}$  uniquely determined by  $\mathcal{H}_q^G = \overline{\mathcal{H}_q^G} \circ P$  for the projection  $P: \text{Or}_{\mathcal{F}}(G) \rightarrow \text{Sub}_{\mathcal{F}}(G)$ . Moreover, we obtain from (2.11), for every pair  $(X, A)$  of  $\mathcal{F}$ - $G$ -CW-complexes, natural isomorphisms

$$BH_n^{G,\mathcal{F}}(X, A; \mathcal{H}_q^G(-)) \cong SH_n^{G,\mathcal{F}}(X, A; \overline{\mathcal{H}_q^G(-)}).$$

Note that the right-hand side is often easier to compute than the left-hand side. One big advantage of  $\text{Sub}(G)$  in comparison with  $\text{Or}(G)$  is that, for a finite subgroup  $H \subseteq G$ , the set of automorphisms of  $H$  is the group  $N_G H/H \cdot C_G H$ , which is finite, whereas the set of automorphisms of  $G/H$  in  $\text{Or}(G)$  for a finite group  $H$  is the group  $N_G H/H$ , which is not necessarily finite. This is a key ingredient in the construction of an equivariant Chern character for discrete groups  $G$  and proper  $G$ -CW-complexes in [15; 16].

If  $G$  is abelian,  $\text{Sub}_{\mathcal{F}}(G)$  reduces to the poset of open subgroups of  $G$  ordered by inclusion.

## 2.E The connective case

**Theorem 2.16** (i) Suppose that  $\mathcal{H}_q^G(G/H) = 0$  for every  $H \in \mathcal{F}$  and  $q \in \mathbb{Z}$  with  $q < 0$ . Then we get, for every pair  $(X, A)$  of  $\mathcal{F}$ - $G$ -CW-complexes and every  $q \in \mathbb{Z}$  with  $q < 0$ ,

$$\mathcal{H}_q^G(X, A) = 0.$$

(ii) Choose a model  $E_{\text{Cop}}(G)$  for the classifying space of smooth proper  $G$ -actions. Let  $\mathcal{I}$  be the set of isotropy groups of points in  $E_{\text{Cop}}(G)$ . Suppose that  $\mathcal{H}_q^G(G/H) = 0$  for every open  $H \in \mathcal{I}$  and  $q \in \mathbb{Z}$  with  $q < 0$ .

(a) Then, for every  $q < 0$ , we have  $\mathcal{H}_q^G(E_{\text{Cop}}(G)) = 0$ , the edge homomorphism induces an isomorphism

$$BH_0^G(E_{\text{Cop}}(G); \mathcal{H}_q^G(-)) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{F}}(G))$$

and the canonical map

$$\text{colim}_{G/H \in \text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G(G/H) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{F}}(G))$$

is bijective.

(b) Suppose additionally that  $\mathcal{H}_*^G$  satisfies condition  $(\text{Sub}_{\mathcal{I}})$ ; see Condition 2.14. Then the edge homomorphism induces an isomorphism

$$SH_0^G(E_{\text{Cop}}(G); \overline{\mathcal{H}_q^G(-)}) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{F}}(G))$$

and the canonical map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{I}}(G)} \overline{\mathcal{H}_0^G(H)} \xrightarrow{\cong} \mathcal{H}_0^G(E_{\text{Cop}}(G))$$

is bijective.

**Proof** (i) This follows directly from the smooth equivariant Atiyah–Hirzebruch spectral sequence of Theorem 2.1.

(a) We get  $\mathcal{H}_q^G(E_{\text{Cop}}(G)) = 0$  for  $q < 0$  from assertion (i).

We get from the smooth equivariant Atiyah–Hirzebruch spectral sequence of Theorem 2.1 an isomorphism

$$BH_0^{G, \mathcal{I}}(E_{\text{Cop}}(G); \mathcal{H}_0^G) = H_0(C_*^{\text{Or}_{\mathcal{I}}(G)}(E_{\text{Cop}}(G)) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\text{Cop}}(G))$$

since  $E_{p,q}^2 = BH_0^{G, \mathcal{I}}(E_{\text{Cop}}(G); \mathcal{H}_q^G) = 0$  is valid for  $p, q \in \mathbb{Z}$  if  $p < 0$  or  $q < 0$ . Since the  $\mathbb{Z}\text{Or}_{\mathcal{I}}(G)$ -module  $C_n^{\text{Or}_{\mathcal{I}}(G)}(E_{\text{Cop}}(G))$  is free in the sense of [14, 9.16 on page 167] for  $n \geq 0$  by (2.6) and  $E_{\text{Cop}}(G)^H$  is weakly contractible for  $H \in \mathcal{I}$ , the  $\mathbb{Z}\text{Or}_{\mathcal{I}}(G)$ -chain complex  $C_*^{\text{Or}_{\mathcal{I}}(G)}(E_{\text{Cop}}(G))$  is a projective  $\mathbb{Z}\text{Or}_{\mathcal{I}}(G)$ -resolution of the constant contravariant  $\mathbb{Z}\text{Or}_{\mathcal{I}}(G)$ -module  $\mathbb{Z}$ , whose value is  $\mathbb{Z}$  at each object and which assigns to any morphism  $\text{id}_{\mathbb{Z}}$ . Since  $-\otimes_{\mathbb{Z}\text{Or}_{\mathcal{I}}(G)} \mathcal{H}_q^G$  is right exact by [14, 9.23 on page 169], we get an isomorphism

$$H_0(C_*^{\text{Or}_{\mathcal{I}}(G)}(E_{\text{Cop}}(G)) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G) \cong \mathbb{Z} \otimes_{\mathbb{Z}\text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G.$$

We conclude from the adjunction appearing in [14, 9.21 on page 169] and the universal property of the colimit that there is a canonical isomorphism

$$\operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G(G/H) \cong \mathbb{Z} \otimes_{\mathbb{Z}\operatorname{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G.$$

This finishes the proof of assertion (a).

(b) This follows from assertion (a), since we get from  $(\operatorname{Sub}_{\mathcal{I}})$  a canonical isomorphism

$$\operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G(G/H) \xrightarrow{\cong} \operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{I}}(G)} \overline{\mathcal{H}}_q^G(H)$$

for the covariant  $\mathbb{Z}\operatorname{Sub}_{\mathcal{I}}(G)$ -module  $\overline{\mathcal{H}}_q^G$  determined by the covariant  $\mathbb{Z}\operatorname{Or}_{\mathcal{I}}(G)$ -module  $\mathcal{H}_q^G$ ; see Remark 2.15. □

## 2.F The first differential

Let  $X$  be an  $\mathcal{F}$ - $G$ -CW-complex. Suppose that  $X_0 = \coprod_{j \in J} G/V_j$  and that  $X_1$  is given by the  $G$ -pushout

$$(2.17) \quad \begin{array}{ccc} \coprod_{i \in I} G/U_i \times S^0 & \xrightarrow{\coprod_{i \in I_n} q_i} & X_0 \\ \downarrow & & \downarrow \\ \coprod_{i \in I} G/U_i \times D^1 & \xrightarrow{\coprod_{i \in I} \mathcal{Q}_i} & X_1 \end{array}$$

We want to figure out the map of  $\mathbb{Z}\operatorname{Or}_{\mathcal{F}}(G)$ -modules  $\gamma$  making the diagram

$$\begin{array}{ccc} \bigoplus_{i \in I} \mathbb{Z}\operatorname{mor}_{\operatorname{Or}_{\mathcal{F}}(G)}(?, G/U_i) & \xrightarrow{\gamma} & \bigoplus_{j \in J} \mathbb{Z}\operatorname{mor}_{\operatorname{Or}_{\mathcal{F}}(G)}(?, G/V_j) \\ \cong \downarrow & & \downarrow \cong \\ C_1^{\operatorname{Or}_{\mathcal{F}}(G)}(X) & \xrightarrow{c_n} & C_0^{\operatorname{Or}_{\mathcal{F}}(G)}(X) \end{array}$$

commute, where the vertical isomorphisms come from the isomorphisms (2.6). In order to describe  $\gamma$ , we have to define for each  $i \in I$  and  $j \in J$  a map of  $\mathbb{Z}\operatorname{Or}(G)$ -modules

$$\gamma_{i,j} : \mathbb{Z}\operatorname{mor}_{\operatorname{Or}_{\mathcal{F}}(G)}(?, G/H_i) \rightarrow \mathbb{Z}\operatorname{mor}_{\mathbb{Z}\operatorname{Or}_{\mathcal{F}}(G)}(?, G/K_j)$$

such that  $\{j \in I_{n-1} \mid \gamma_{i,j} \neq 0\}$  is finite for every  $i \in I_n$ . Note that  $\gamma_{i,j}$  is determined by the image of  $\operatorname{id}_{G/H_i}$ . Hence we need to specify for  $i \in I$  and  $j \in J$  an element

$$(2.18) \quad \bar{\gamma}_{i,j} \in \mathbb{Z}\operatorname{mor}_{\operatorname{Or}_{\mathcal{F}}(G)}(G/U_i, G/V_j) = \mathbb{Z}\operatorname{map}_G(G/U_i, G/V_j).$$

For each  $i \in I$ , there are two elements  $j_-(i)$  and  $j_+(i)$  in  $J$  such that the image of  $G/H_i \times \{\pm 1\}$  under the map  $q_i$  appearing in (2.17) is the summand  $G/K_{j_{\pm}(i)}$  belonging to  $j_{\pm}(i)$  of  $\coprod_{j \in I_0} G/K_j$  if we write  $S^0 = \{-1, 1\}$ . Denote by  $(q_i^1)_{\pm 1} : G/H_i \rightarrow G/K_{j_{\pm}(i)}$  the restriction of  $q_i^1$  to  $G/H_i \times \{\pm 1\}$ . We leave the elementary proof of the next lemma to the reader.

**Lemma 2.19** In  $\mathbb{Z}\text{map}_G(G/H_i, G/K_j)$ ,

$$\bar{\gamma}_{i,j} = \begin{cases} \pm[(q_i^1)_{\pm 1}] & \text{if } j = j_{\pm}(i) \text{ and } j_{-}(i) \neq j_{+}(i), \\ [(q_i^1)_{+1}] - [(q_i^1)_{-1}] & \text{if } j = j_{-}(i) = j_{+}(i), \\ 0 & \text{if } j \notin \{j_{-}(i), j_{+}(i)\}. \end{cases}$$

**Remark 2.20** This implies, for the  $\mathbb{Z}$ -chain complex  $C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\text{Or}_{\mathcal{F}}(G)} M$  for a covariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $M$ , that its first differential agrees with the  $\mathbb{Z}$ -homomorphism

$$\alpha = (\alpha_{i,j})_{i \in I, j \in J}: \bigoplus_{i \in I} M(G/U_i) \rightarrow \bigoplus_{j \in J} M(G/V_j),$$

where the  $\mathbb{Z}$ -homomorphisms  $\alpha_{i,j}: M(G/U_i) \rightarrow M(G/V_j)$  are given, in the notation of Lemma 2.19, by

$$\alpha_{i,j} = \begin{cases} \pm M((q_i^1)_{\pm}) & \text{if } j = j_{\pm}(i) \text{ and } j_{-}(i) \neq j_{+}(i), \\ M((q_i^1)_{+1}) - M((q_i^1)_{-1}) & \text{if } j = j_{-}(i) = j_{+}(i), \\ 0 & \text{if } j \notin \{j_{-}(i), j_{+}(i)\}. \end{cases}$$

Note that the cokernel of  $\alpha$  is  $BH_0^{G, \mathcal{F}}(X; M)$ .

We get a computation of the first differential of  $C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)} N$  for a covariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -module  $N$  from the isomorphism (2.10). Explicitly, the first differential is given by

$$\beta = (\beta_{i,j})_{i \in I, j \in J}: \bigoplus_{i \in I_n} N(U_i) \rightarrow \bigoplus_{j \in I_{n-1}} N(V_j),$$

where the  $\mathbb{Z}$ -homomorphisms  $\beta_{i,j}: N(G/U_i) \rightarrow N(G/V_j)$  are given as follows. Choose for the map  $(q_i)_{\pm}: G/U_i \rightarrow G/V_j$  an element  $(g_i)_{\pm}$  with  $(q_i)_{\pm}(eU_i) = (g_i)_{\pm}^{-1}V_j$ . Let  $[c(g_i)_{\pm}]: U_i \rightarrow V_j$  be the morphism in  $\text{Sub}_{\mathcal{F}}(G)$  represented by  $c(g_i)_{\pm}: U_i \rightarrow V_j$  sending  $u$  to  $gug^{-1}$ . Then

$$\beta_{i,j} = \begin{cases} \pm N([c(g_i)_{\pm}]) & \text{if } j = j_{\pm}(i) \text{ and } j_{-}(i) \neq j_{+}(i), \\ N([c(g_i)_{+}]) - N([c(g_i)_{-}]) & \text{if } j = j_{-}(i) = j_{+}(i), \\ 0 & \text{if } j \notin \{j_{-}(i), j_{+}(i)\}. \end{cases}$$

Note that the cokernel of  $\beta$  is  $SH_0^{G, \mathcal{F}}(X; N)$ .

### 3 A brief review of the Farrell–Jones conjecture for the algebraic $K$ -theory of Hecke algebras

In this section we give a review of the Farrell–Jones conjecture for the algebraic  $K$ -theory of Hecke algebras. Further information can be found in [4; 2].

Let  $R$  be a (not necessarily commutative) associative unital ring with  $\mathbb{Q} \subseteq R$ . Let  $G$  be a td-group. Let  $\mathcal{H}(G; R)$  be the associated Hecke algebra.

One can construct a covariant functor

$$K_R: \text{Or}_{\mathcal{O}_p}(G) \rightarrow \text{Spectra}$$

such that  $\pi_n(\mathbf{K}_R(Q'/U')) \cong K_n(\mathcal{H}(U; R))$  for any  $n \in \mathbb{Z}$  and open subgroup  $U \subseteq Q$ . Associated to it is a smooth  $G$ -homology theory  $H_*^G(-; \mathbf{K}_R)$  such that

$$(3.1) \quad H_n^G(G/U; \mathbf{K}_R) \cong K_n(\mathcal{H}(U; R))$$

for every  $n \in \mathbb{Z}$  and every open subgroup  $U \subseteq Q$ .

The next result follows from [2, Corollary 1.8].

**Theorem 3.2** *Let  $G$  be a td-group which is modulo a normal compact subgroup a subgroup of a reductive  $p$ -adic group. Let  $R$  be a uniformly regular ring with  $\mathbb{Q} \subseteq R$ .*

*Then the map induced by the projection  $E_{\text{cop}}(G) \rightarrow G/G$  induces for every  $n \in \mathbb{Z}$  an isomorphism*

$$H_n^G(E_{\text{cop}}(G); \mathbf{K}_R) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R)).$$

## 4 Proof of Theorem 1.1

**Proof of Theorem 1.1** (i) This is exactly Theorem 3.2.

(ii) Since an open group homomorphism  $U \rightarrow V$  between two td-groups induces a ring homomorphism  $\mathcal{H}(U; R) \rightarrow \mathcal{H}(V; R)$  between the Hecke algebras and hence a homomorphism  $K_n(\mathcal{H}(U; R)) \rightarrow K_n(\mathcal{H}(V; R))$ , and inner automorphisms of a td-group  $U$  induce the identity on  $K_n(\mathcal{H}(U; R))$ , we get a covariant  $\mathbb{Z}\text{Sub}_{\text{com}}(G)$ -module  $K_n(\mathcal{H}(?: R))$  whose value at  $U$  is  $K_n(\mathcal{H}(U; R))$ . Since the isomorphism (3.1) is natural, we get an isomorphism of covariant  $\mathbb{Z}\text{Or}_{\text{op}}(G)$ -modules

$$P^* K_n(\mathcal{H}(?: R)) \xrightarrow{\cong} \pi_n(\mathbf{K}_R)$$

for the projection  $P: \text{Or}_{\text{op}}(G) \rightarrow \text{Sub}_{\text{op}}(G)$  of (2.4). So the smooth equivariant Atiyah–Hirzebruch spectral sequence applied to the smooth homology theory  $H_*^G(-; \mathbf{K}_R)$  takes, for an  $\mathcal{F}$ - $G$ -CW-complex  $X$ , the form

$$(4.1) \quad E_{p,q}^2 = SH_q^{G,\mathcal{F}}(X; K_q(\mathcal{H}(?: R))) \Rightarrow H_{p+q}^G(X; \mathbf{K}_R).$$

Now assertion (ii) follows from the special case  $X = E_{\text{cop}}(G)$  and assertion (i).

(iii)–(iv) As  $K_q(\mathcal{H}(K; R))$  vanishes for every compact td-group  $K$  and every  $q \leq -1$  [4, Lemma 8.1], assertions (iii)–(iv) follow from Theorem 2.16 applied in the case  $X = E_{\text{cop}}(G)$  and from assertion (i).  $\square$

## 5 The main recipe for the computation of the projective class group

Throughout this section,  $G$  will be a td-group and  $R$  a uniformly regular ring with  $\mathbb{Q} \subseteq R$ , eg a field of characteristic zero. We will assume that the assembly map  $H_n^G(E_{\text{cop}}(G); \mathbf{K}_R) \rightarrow H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R))$  is bijective for all  $n \in \mathbb{Z}$ . This is known to be true for subgroups of reductive  $p$ -adic groups by Theorem 3.2.

### 5.A The general case

Let  $X$  be an abstract simplicial complex with a simplicial  $G$ -action such that all isotropy groups are compact open, the  $G$ -action is cellular, and  $|X|^K$  is nonempty and connected for every compact open subgroup  $K$  of  $G$ .

We can choose a subset  $V$  of the set of vertices of  $X$  such that the  $G$ -orbit through any vertex in  $X$  meets  $V$  in precisely one element. Fix a total ordering on  $V$ . Let  $E$  be the subset of  $V \times V$  consisting of those pairs  $(v, w)$  such that  $v \leq w$  and there exists  $g \in G$  for which  $v$  and  $gw$  satisfy  $v \neq gw$  and span an edge  $[v, gw]$  in  $X$ . For  $(v, w) \in E$ , define  $\overline{F(v, w)}$  to be the subset of  $G_v \backslash G / G_w$  consisting of elements  $x$  for which  $v$  and  $gw$  satisfy  $v \neq gw$  and span an edge  $[v, gw]$  in  $X$  for some (and hence all) representative  $g$  of  $x$ . Choose a subset  $F(v, w)$  of  $G$  such that the projection  $G \rightarrow G_v \backslash G / G_w$  induces a bijection  $F(v, w) \rightarrow \overline{F(v, w)}$ .

Then, for every edge of  $X$ , the  $G$ -orbit through it meets the set  $\{[v, gv] \mid (v, w) \in E, g \in F(v, w)\}$  in precisely one element. Moreover, the 0-skeleton of  $|X|$  is given by  $|X|_0 = \coprod_{u \in V} G/G_u$  and  $|X|_1$  is given by the  $G$ -pushout

$$\begin{CD} \coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} G/(G_v \cap G_{gw}) \times S^0 @>\coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} q_{(v,w),g}>> |X|_0 \\ @VVV @VVV \\ \coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} G/(G_v \cap G_{gw}) \times D^1 @>>> |X|_1 \end{CD}$$

where  $q_{(v,w),g}: G/(G_v \cap G_{gw}) \times S^0 \rightarrow |X|_0 = \coprod_{u \in V} G/G_u$  is defined as follows. Write  $S^0 = \{-1, 1\}$ . The restriction of  $q_{(v,w),g}$  to  $G/(G_v \cap G_{gw}) \times \{-1\}$  lands in the summand  $G/G_v$  and is given by the canonical projection. The restriction of  $q_{(v,w),g}$  to  $G/(G_v \cap G_{gw}) \times \{1\}$  lands in the summand  $G/G_w$  and is given by the  $G$ -map  $R_{g^{-1}}: G/(G_v \cap G_{gw}) \rightarrow G/G_w$  sending  $z(G_v \cap G_{gw})$  to  $zgG_w$ .

Next we define a map

$$\beta = (\beta_{(v,w),g,u}): \bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(G_v \cap G_{gw}; R)) \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(G_u; R)).$$

If  $u = v$ , then  $\beta_{(v,w),g,v}: K_0(\mathcal{H}(G_v \cap G_{gw}; R)) \rightarrow K_0(\mathcal{H}(G_v; R))$  is the map induced by the inclusion  $G_v \cap G_{gw} \rightarrow G_v$  multiplied with  $-1$ . If  $u = w$ , then  $\beta_{(v,w),g,w}: K_0(\mathcal{H}(G_v \cap G_{gw}; R)) \rightarrow K_0(\mathcal{H}(G_w; R))$  is the map induced by the group homomorphism  $G_v \cap G_{gw} \rightarrow G_w$  sending  $z$  to  $g^{-1}zg$ . If  $u \notin \{v, w\}$ , then  $\beta_{(v,w),g,u} = 0$ .

**Lemma 5.1** *The cokernel of  $\beta$  is isomorphic to  $K_0(\mathcal{H}(G; R))$ .*

**Proof** We conclude from Remark 2.20 that the cokernel of  $\beta$  is  $SH_0^{G, \text{Cop}}(X; \overline{K_0^G(-)})$ . The up-to- $G$ -homotopy unique  $G$ -map  $f: X \rightarrow E_{\text{Cop}}(G)$  induces for every compact open subgroup  $K \subset G$  a 1-connected map  $f^K: |X|^K \rightarrow E_{\text{Cop}}(G)^K$ . This implies that the map  $SH_0^{G, \text{Cop}}(X; \overline{K_0^G(-)}) \rightarrow SH_0^{G; \text{Cop}}(E_{\text{Cop}}(G); \overline{K_0^G(-)})$  induced by  $f$  is an isomorphism; see [14, Proposition 23(iii) on page 35].

Theorem 2.16(b) implies  $SH_0^G(E_{\text{cop}}(G); \overline{K_0^G(-)}) \cong H_0^G(E_{\text{cop}}(G); \mathbf{K}_R)$ . Since by assumption we have  $H_0^G(E_{\text{cop}}(G); \mathbf{K}_R) \cong K_0(\mathcal{H}(G; R))$ , Lemma 5.1 follows.  $\square$

**Remark 5.2** Suppose additionally that  $X$  possesses a strict fundamental domain  $\Delta$ , ie a simplicial subcomplex  $\Delta$  that contains exactly one simplex from each orbit for the  $G$ -action on the set of simplices of  $X$ . Then one can take  $V$  to be the set of vertices of  $\Delta$  and, for  $(v, w) \in E$ , the set  $F(v, w)$  to be  $\{e\}$ . Moreover,  $\beta$  reduces to the map

$$\beta = (\beta_{(v,w,u)}): \bigoplus_{(v,w) \in E} K_0(\mathcal{H}(G_v \cap G_w; R)) \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(G_u; R)),$$

where  $\beta_{(v,w,u)}$  is the map induced by the inclusion  $G_v \cap G_w \rightarrow G_v$  multiplied with  $-1$  for  $u = v$ , the map induced by the inclusion  $G_v \cap G_w \rightarrow G_w$  for  $u = w$ , and zero for  $u \notin \{v, w\}$ . Note that  $E$  is the subset of  $V \times V$  consisting of elements  $(v, w)$  for which  $v < w$  holds and  $v$  and  $w$  span an edge  $[v, w]$  in  $\Delta$ .

### 5.B A variation

Consider a central extension  $1 \rightarrow \tilde{C} \rightarrow \tilde{G} \xrightarrow{\text{pr}} G \rightarrow 1$  of td-groups together with a group homomorphism  $\mu: \tilde{G} \rightarrow \mathbb{Z}$  such that  $\tilde{C} \cap \tilde{M}$  is compact for  $\tilde{M} := \ker(\mu)$ . We still consider the abstract simplicial complex  $X$  of Section 5.A coming with a simplicial  $G$ -action such that all isotropy groups are compact open, and  $|X|^K$  is nonempty and connected for every compact open subgroup  $K$  of  $G$ . Furthermore, we will assume that the assembly map  $H_n^{\tilde{G}}(E_{\text{cop}}(\tilde{G}); \mathbf{K}_R) \rightarrow H_n^{\tilde{G}}(\tilde{G}/\tilde{G}; \mathbf{K}_R) = K_0(\mathcal{H}(\tilde{G}; R))$  is bijective for all  $n \in \mathbb{Z}$ .

If  $\tilde{C}$  is compact, then we can consider  $X$  as a  $\tilde{G}$ -CW-complex by restricting the  $G$ -action with pr and Section 5.A applies. Hence we will assume that  $\tilde{C}$  is not compact, or, equivalently, that  $\tilde{C}$  is not contained in the kernel  $\tilde{M} := \ker(\mu)$ . Then the index  $m := [\mathbb{Z} : \mu(C)]$  is a natural number  $m \geq 1$ . We fix an element  $\tilde{c} \in \tilde{C}$  with  $\mu(\tilde{c}) = m$ . In the sequel, we choose for every  $g \in G$  an element  $\tilde{g}$  in  $\tilde{G}$  satisfying  $\text{pr}(\tilde{g}) = g$  and denote, for an open subgroup  $U \subseteq G$ , by  $\tilde{U} \subseteq \tilde{G}$  its preimage under pr:  $\tilde{G} \rightarrow G$ . Let

$$\gamma: \bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M}; R)) \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(\tilde{G}_u \cap \tilde{M}; R))$$

be the map whose component for  $(v, w) \in E$ ,  $g \in F(v, w)$  and  $u \in V$  is the map

$$(5.3) \quad \gamma_{(v,w),g,u}: K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M}; R)) \rightarrow K_0(\mathcal{H}(\tilde{G}_u \cap \tilde{M}; R))$$

defined next. If  $u = v$ , it is the map coming from the inclusion  $\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M} \rightarrow \tilde{G}_v \cap \tilde{M}$  multiplied with  $-1$ . If  $u = w$ , it is the map coming from the group homomorphism  $\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M} \rightarrow \tilde{G}_w \cap \tilde{M}$  sending  $x$  to  $\tilde{g}x\tilde{g}^{-1}$ . If  $u \notin \{v, w\}$ , it is trivial. Note that this definition is independent of the choice of  $\tilde{g} \in \tilde{G}$  satisfying  $\text{pr}(\tilde{g}) = g$  for  $g \in F(v, w)$ .

**Lemma 5.4** *The cokernel of  $\gamma$  is  $K_0(\mathcal{H}(\tilde{G}; R))$ .*

**Proof** Note that  $|X| \times \mathbb{R}$  carries the  $G \times \mathbb{Z}$ -CW-complex structure coming from the product of the  $G$ -CW-complex structure on  $|X|$  and the standard free  $\mathbb{Z}$ -CW-structure on  $\mathbb{R}$ . Since the  $\mathbb{Z}$ -CW-complex  $\mathbb{R}$

has precisely one equivariant 1-cell and one equivariant 0-cell, the set of equivariant 0-cells of the  $G \times \mathbb{Z}$ -CW-complex  $|X| \times \mathbb{R}$  can be identified with the set  $V$  and the set of equivariant 1-cells can be identified with the disjoint union of  $V$  and the set  $\coprod_{(v,w) \in E} F(v, w)$ . Now the 0-skeleton of  $|X| \times \mathbb{R}$  is given by the disjoint union  $\coprod_{u \in V} \tilde{G}/\tilde{G}_u \times \mathbb{Z}$  and the 1-skeleton of  $|X| \times \mathbb{R}$  is given by the  $G \times \mathbb{Z}$ -pushout

$$(5.5) \quad \begin{array}{ccc} \coprod_{v \in V} \tilde{G}/\tilde{G}_v \times \mathbb{Z} \times S^0 \amalg \coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} \tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \times S^0 & \xrightarrow{\tilde{q}} & \coprod_{u \in V} \tilde{G}/\tilde{G}_u \times \mathbb{Z} \\ \downarrow & & \downarrow \\ \coprod_{v \in V} \tilde{G}/\tilde{G}_v \times \mathbb{Z} \times D^1 \amalg \coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} \tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \times S^0 & \longrightarrow & (|X| \times \mathbb{R})_1 \end{array}$$

where  $\tilde{q}$  is given as follows. Write  $S^0 = \{-1, 1\}$ . Fix  $u \in V$ . The restriction of  $\tilde{q}$  to the summand  $\tilde{G}/\tilde{G}_v \times \mathbb{Z} \times \{\epsilon\}$  lands in the summand  $\tilde{G}/\tilde{G}_v \times \mathbb{Z}$  and is given by  $\text{id}$  for  $\epsilon = -1$  and by  $\text{id} \times \text{sh}_1$  for  $\epsilon = 1$ , where  $\text{sh}_a: \mathbb{Z} \rightarrow \mathbb{Z}$  sends  $b$  to  $a + b$  for  $a, b \in \mathbb{Z}$ . Fix  $(v, w) \in E$  and  $g \in F(v, w)$ . The restriction of  $\tilde{q}$  to the summand  $\tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \times \{-1\}$  belonging to  $(v, w)$  and  $g$  lands in the summand for  $u = v$  and is the canonical projection  $\tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \rightarrow \tilde{G}/\tilde{G}_v \times \mathbb{Z}$ . The restriction of  $\tilde{q}$  to the summand  $\tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \times \{1\}$  belonging to  $(v, w)$  and  $g$  lands in the summand for  $u = w$  and is the map  $R_{\tilde{g}^{-1}} \times \text{id}_{\mathbb{Z}}: \tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \rightarrow \tilde{G}/\tilde{G}_w \times \mathbb{Z}$ , where  $R_{\tilde{g}^{-1}}$  sends  $\tilde{z}(\tilde{G}_v \cap \tilde{G}_{gw})$  to  $\tilde{z}\tilde{g}^{-1}\tilde{G}_w$ .

We have the group homomorphism

$$\iota := \text{pr} \times \mu: \tilde{G} \rightarrow G \times \mathbb{Z}.$$

Its kernel is  $\tilde{C} \cap \tilde{M}$ . Its image has finite index in  $G \times \mathbb{Z}$ , which agrees with the index  $m$  of the image of  $\mu$  in  $\mathbb{Z}$ .

We are interested in the  $\tilde{G}$ -CW-complex  $\iota^*(|X| \times \mathbb{R})$  obtained by restriction with  $\iota$  from the  $G \times \mathbb{Z}$ -CW-complex  $|X| \times \mathbb{R}$ . So we have to analyze how the  $G \times \mathbb{Z}$ -cells in  $\iota^*(|X| \times \mathbb{R})$  viewed as  $\tilde{G}$ -spaces decompose as disjoint unions of  $\tilde{G}$ -cells. Consider any open subgroup  $U \subseteq G$ . Then we obtain a  $\tilde{G}$ -homeomorphism

$$\alpha(U): \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) \xrightarrow{\cong} \iota^*(G/U \times \mathbb{Z})$$

by sending the element  $\tilde{z}(\tilde{U} \cap \tilde{M})$  in the  $p^{\text{th}}$  summand to  $(\text{pr}(\tilde{z})U, \mu(\tilde{z}) + p)$ . Next we have to analyze the naturality properties of  $\alpha(U)$ . The diagram, for  $a \in \mathbb{Z}$ ,

$$\begin{array}{ccc} \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) & \xrightarrow{\alpha(U)} & \iota^*(G/U \times \mathbb{Z}) \\ \hat{\pi}^a \downarrow & & \downarrow \text{id} \times \text{sh}_a \\ \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) & \xrightarrow{\alpha(U)} & \iota^*(G/U \times \mathbb{Z}) \end{array}$$

commutes, where  $\hat{\pi}$  sends the summand for  $p = 0, \dots, m - 2$  by the identity to the summand for  $p + 1$  and sends the summand for  $p = m - 1$  to the summand for  $p = 0$  by the map  $R_{\tilde{c}}: \tilde{G}/(\tilde{U} \cap \tilde{M}) \rightarrow \tilde{G}/(\tilde{U} \cap \tilde{M})$  for  $\tilde{c} \in \tilde{C}$  satisfying  $\mu(\tilde{c}) = m$ . Note for the sequel that the endomorphism  $\pi_n(\mathbf{K}_R(R_{\tilde{c}}))$

of  $\pi_n(K_R(\tilde{G}/\tilde{U} \cap \tilde{M})) = K_0(\mathcal{H}(\tilde{U} \cap \tilde{M}))$  is the identity, since conjugation with  $\tilde{c}$  induces the identity on  $\tilde{U} \cap \tilde{M}$ .

Consider two open subgroups  $U$  and  $V$  of  $G$  and an element  $g \in G$  with  $gUg^{-1} \subseteq V$ . Then we get well-defined  $\tilde{G}$ -maps  $R_{\tilde{g}-1}: \tilde{G}/(\tilde{U} \cap \tilde{M}) \rightarrow \tilde{G}/(\tilde{V} \cap \tilde{M})$  sending  $\tilde{z}(\tilde{U} \cap \tilde{M})$  to  $\tilde{z}\tilde{g}^{-1}(\tilde{V} \cap \tilde{M})$  and  $R_{g-1} \times \text{id}: \iota^*(G/U \times \mathbb{Z}) \rightarrow \iota^*(G/V \times \mathbb{Z})$  sending  $(zU, n)$  to  $(zg^{-1}V, n)$  and the following diagram commutes:

$$\begin{CD} \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) @>{\alpha(U)}>> \iota^*(G/U \times \mathbb{Z}) \\ @V{\coprod_{p=0}^{m-1} R_{\tilde{g}-1}}VV @VV{R_{g-1} \times \text{sh}_{\mu(\tilde{g}-1)}}V \\ \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{V} \cap \tilde{M}) @>{\alpha(V)}>> \iota^*(G/V \times \mathbb{Z}) \end{CD}$$

In particular the following diagram commutes:

$$\begin{CD} \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) @>{\alpha(U)}>> \iota^*(G/U \times \mathbb{Z}) \\ @V{\pi^{\mu(\tilde{g})} \circ (\coprod_{p=0}^{m-1} R_{\tilde{g}-1)} }VV @VV{R_{g-1} \times \text{id}}V \\ \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{V} \cap \tilde{M}) @>{\alpha(V)}>> \iota^*(G/V \times \mathbb{Z}) \end{CD}$$

Now we obtain from the  $G \times \mathbb{Z}$ -pushout (5.5) by applying restriction with  $\iota$  and the maps  $\alpha_U$  above a  $\tilde{G}$ -pushout describing how the 1-skeleton of the  $\tilde{G}$ -CW-complex  $\iota^*(|X| \times \mathbb{R})$  is obtained from its 0-skeleton and explicit descriptions of the attaching maps.

In the sequel  $A^m$  stands for the  $m$ -fold direct sum of copies of  $A$  for an abelian group  $A$  and  $\pi: A^m \rightarrow A^m$  denotes the permutation map sending  $(a_1, a_2, \dots, a_m)$  to  $(a_m, a_1, \dots, a_{m-1})$  and  $\text{aug}: A^m \rightarrow A$  denotes the augmentation map sending  $(a_1, \dots, a_m)$  to  $a_1 + \dots + a_m$ .

Let  $\delta$  be the map given by the direct sum

$$\delta = \bigoplus_{v \in V} \delta_v: \bigoplus_{v \in V} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{M}; R))^m \rightarrow \bigoplus_{v \in V} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{M}; R))^m,$$

where  $\delta_v: K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{M}; R))^m \rightarrow K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{M}; R))^m$  is  $\pi - \text{id}$ . Let

$$\epsilon: \bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M}; R))^m \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(\tilde{G}_u \cap \tilde{M}; R))^m$$

be the map given by the components  $\epsilon_{(v,w),g,u}$  defined as follows. For  $u = v$ , the map  $\epsilon_{(v,w),g,v}$  is the  $m$ -fold direct sum  $\gamma_{(v,w),g,v}^m$  of the maps  $\gamma_{(v,w),g,v}$  defined in (5.3). For  $u = w$ , we put

$$\epsilon_{(v,w),g,w}: K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M}; R))^m \xrightarrow{\gamma_{(v,w),g,w}^m} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{M}; R))^m \xrightarrow{\pi^{\mu(\tilde{g})}} K_0(\mathcal{H}(\tilde{G}_w \cap \tilde{M}; R))^m.$$

Since  $\pi^m = \text{id}$ , the map  $\pi^{\mu(\tilde{g})}$  depends only on  $\bar{\mu}(g)$ , where  $\bar{\mu}: G \rightarrow \mathbb{Z}/m$  sends  $g$  to the image of  $\tilde{g}$  under the projection  $\mathbb{Z} \rightarrow \mathbb{Z}/m$  for any choice of an element  $\tilde{g} \in \tilde{G}$  with  $\text{pr}(\tilde{g}) = g$ .

The cokernel of the map

$$\delta \oplus \epsilon: \left( \bigoplus_{v \in V} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{M}; R))^m \right) \oplus \left( \bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M}; R))^m \right) \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(\tilde{G}_u \cap \tilde{M}; R))^m$$

is  $K_0(\mathcal{H}(\tilde{G}; R))$  because of Theorem 2.16(b) and Remark 2.20, by the same argument as appears in the proof of Lemma 5.1, since  $(\iota^*(|X| \times \mathbb{R}))^K$  is connected for every compact open subgroup  $K$  of  $\tilde{G}$ . It does not matter that  $\iota^*(|X| \times \mathbb{R})$  is a  $\tilde{G}$ -CW-complex but not a simplicial complex, since in the description of  $\beta_{i,j}$  appearing in Remark 2.20 the case  $j_i(+) = j_-(i)$  never occurs.

We can identify  $\bigoplus_{v \in V} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{M}; R))$  and the cokernel of  $\delta$ , since we have the exact sequence  $A^m \xrightarrow{\pi - \text{id}} A^m \xrightarrow{\alpha} A \rightarrow 0$  for every abelian group  $A$ . The cokernel of  $\delta \oplus \epsilon$  is isomorphic to the cokernel of the composite of  $\epsilon$  with the map

$$\bigoplus_{v \in V} \text{aug}: \bigoplus_{v \in V} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{M}; R))^m \rightarrow \bigoplus_{v \in V} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{M}; R)) = \text{cok}(\delta).$$

For every  $(v, w) \in E$ ,  $g \in F(v, w)$  and  $u \in V$ , the diagram

$$\begin{CD} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M}; R))^m @>\epsilon_{(v,w),u}>> K_0(\mathcal{H}(\tilde{G}_u \cap \tilde{M}; R))^m \\ @V \text{aug} VV @VV \text{aug} V \\ K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M}; R)) @>\gamma_{(v,w),u}>> K_0(\mathcal{H}(\tilde{G}_u \cap \tilde{M}; R)) \end{CD}$$

commutes, since  $\alpha \circ \pi = \alpha$ . □

## 6 The projective class group of the Hecke algebras of $SL_n(F)$ , $PGL_n(F)$ and $GL_n(F)$

Next we apply the recipes of Section 5 to some prominent reductive  $p$ -adic groups  $G$  as an illustration. For the remainder of this section  $R$  is a uniformly regular ring with  $\mathbb{Q} \subseteq R$ .

Note that, for a reductive  $p$ -adic group  $G$ , the assembly map  $H_n^G(E_{\text{cop}}(G); \mathbf{K}_R) \rightarrow H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R))$  is bijective for all  $n \in \mathbb{Z}$  by Theorem 3.2. Moreover, the Bruhat–Tits building  $X$  of  $G$  or of  $G/\text{cent}(G)$  can serve as the desired simplicial complex  $X$  appearing in Section 5. The original construction of the Bruhat–Tits building can be found in [8]. For more information about buildings, we refer to [1; 6; 7; 17]. The space  $X$  carries a CAT(0)-metric, which is invariant under the action of  $G$  or  $G/\text{cent}(G)$ ; see [6, Theorem 10A.4 on page 344]. Hence  $|X|^H$  is contractible for any compact open subgroup  $H$  of  $G$  or  $G/\text{cent}(G)$ , since  $X^H$  is a convex nonempty subset of  $X$  and hence contractible by [6, Corollary II.2.8 on page 179]. Therefore the geometric realization of the Bruhat–Tits building  $X$  is (after possibly subdividing to achieve a cellular action) a model for  $E_{\text{cop}}(G)$  or  $E_{\text{cop}}(G/\text{cent}(G))$ .

### 6.A $SL_n(F)$

We begin with computing  $K_0(\mathcal{H}(SL_n(F); R))$ , where  $F$  is a non-Archimedean local field with valuation  $v: F \rightarrow \mathbb{Z} \cup \{\infty\}$ . The following claims about the Bruhat–Tits building  $X$  for  $SL_n(F)$  (and later about  $X'$ ) can all be verified from the description of  $X$  in [1, Section 6.9].

For  $l = 0, \dots, n - 1$ , let  $U_l^S$  be the compact open subgroup of  $SL_n(F)$  consisting of all matrices  $(a_{ij})$  in  $SL_n(F)$  satisfying  $v(a_{i,j}) \geq -1$  for  $1 \leq i \leq n - l < j \leq n$ ,  $v(a_{i,j}) \geq 1$  for  $1 \leq j \leq n - l < i \leq n$  and  $v(a_{i,j}) \geq 0$  for all other  $i$  and  $j$ . In particular,  $U_0^S = SL_n(\mathcal{O})$ , where  $\mathcal{O} = \{z \in F \mid v \geq 0\}$ . The intersection of the  $U_l^S$  is the Iwahori subgroup  $I^S$  of  $SL_n(F)$ . It is given by those matrices  $A$  in  $SL_n(F)$  for which  $v(a_{i,j}) \geq 1$  for  $i > j$  and  $v(a_{i,j}) \geq 0$  for  $i \leq j$ .

The  $(n-1)$ -simplex  $\Delta$  can be chosen with an ordering on its vertices such that the isotropy group of its  $l^{\text{th}}$  vertex  $v_l$  is  $U_l^S$ . The isotropy group of a face  $\sigma$  of  $\Delta$  is the intersection of the isotropy groups of the vertices of  $\sigma$ . In particular, the isotropy group of  $\Delta$  is the Iwahori subgroup  $I^S$  of  $SL_n(F)$ . Consider the map

$$d^{SL_n(F)}: \bigoplus_{0 \leq i < j \leq n-1} K_0(\mathcal{H}(U_i^S \cap U_j^S; R)) \rightarrow \bigoplus_{0 \leq l \leq n-1} K_0(\mathcal{H}(U_l^S; R))$$

for which the component  $d_{i < j, l}^{SL_n(F)}: K_0(\mathcal{H}(U_i^S \cap U_j^S; R)) \rightarrow K_0(\mathcal{H}(U_l^S; R))$  is given by  $-K_0(\mathcal{H}(f_{i < j}^i; R))$  if  $l = i$ , by  $K_0(\mathcal{H}(f_{i < j}^j; R))$  if  $l = j$ , and zero if  $l \notin \{i, j\}$ , where  $f_{i < j}^k: U_i^S \cap U_j^S \rightarrow U_k^S$  is the inclusion for  $k = i, j$ .

Then the cokernel of  $d^{SL_n(F)}$  is  $K_0(\mathcal{H}(SL_n(F); R))$  by Lemma 5.1 and Remark 5.2.

### 6.B $PGL_n(F)$

Next we compute  $K_0(\mathcal{H}(PGL_n(F); R))$ . The action of  $SL_n(F)$  on  $X$  extends to an action of  $GL_n(F)$ . This action factors through the canonical projection  $\text{pr}: GL_n(F) \rightarrow PGL_n(F)$  to an action of  $PGL_n(F)$ . These actions are still simplicial, but no longer cellular. Let

$$\hat{h} := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ \zeta & & & 1 \end{pmatrix} \in GL_n(F),$$

where we choose a uniformizer  $\zeta \in F$ , ie an element in  $F$  satisfying  $v(\zeta) = 1$ . Obviously  $\hat{h}^n$  is the diagonal matrix  $\zeta \cdot I_n$ , all of whose diagonal entries are  $\zeta$ , and hence is central in  $GL_n(F)$ . Define  $h \in PGL_n(F)$  by  $h = \text{pr}(\hat{h})$ . Then  $h v_l = v_{l+1}$  for  $l = 0, \dots, n - 2$  and  $h v_{n-1} = v_0$  and  $h^n$  is the unit in  $PGL_n(F)$ . In particular, the action of  $PGL_n(F)$  is transitive on the vertices of  $X$ . To obtain a cellular action,  $X$  can be subdivided to  $X'$  as follows. The  $(n-2)$ -skeleton of  $X$  is unchanged, while the  $(n-1)$ -simplices of  $X$  are replaced in  $X'$  with cones on their boundary. More formally, the vertices of  $X'$  are the vertices of  $X$  and the barycenters  $b_\sigma$  of  $(n-1)$ -simplices  $\sigma$  of  $X$ . A set  $S$  of vertices of  $X'$  is a simplex of  $X'$  if and only if either  $S$  is a  $k$ -simplex of  $X$  and  $k < n - 1$  or  $S$  contains exactly one barycenter  $b_\sigma$  and

all  $v \in S \setminus \{b_\sigma\}$  are vertices of  $\sigma$  (in the simplicial structure of  $X$ ). The action of  $\mathrm{PGL}_n(F)$  on  $X'$  is then cellular and is transitive on  $(n-1)$ -simplices of  $X'$ . There are two orbits of vertices, represented by  $v_0$  and  $b_\Delta$ . Let  $k := \lfloor \frac{1}{2}n \rfloor$ . There are  $k + 1$  orbits of 1-simplices, represented by  $\{v_0, v_1\}, \dots, \{v_0, v_k\}$  and  $\{v_0, b_\Delta\}$ . Next we describe some isotropy groups.

For an open subgroup  $W \subseteq \mathrm{PGL}_n(F)$ , let  $\widetilde{W}$  be its preimage under the projection  $\mathrm{pr}: \mathrm{GL}_n(F) \rightarrow \mathrm{PGL}_n(F)$ . For  $l = 0, \dots, n - 1$ , let  $U_l^G$  be the compact open subgroup of  $\mathrm{GL}_n(F)$  given by  $\hat{h}^l \mathrm{GL}_n(\mathcal{O}) \hat{h}^{-l} = \widetilde{\mathrm{PGL}_n(F)}_{v_l} = \widetilde{\mathrm{PGL}_n(F)}_{h_l v_0}$ . In particular,  $U_0^G = \mathrm{GL}_n(\mathcal{O})$ . Note that

$$U_l^G \cap \mathrm{SL}_n(F) = (\hat{h}^l \mathrm{GL}_n(\mathcal{O}) \hat{h}^{-l}) \cap \mathrm{SL}_n(F) = \hat{h}^l \mathrm{SL}_n(\mathcal{O}) \hat{h}^{-l} = U_l^S.$$

The intersection of the  $U_l^G$  is the Iwahori subgroup  $I^G$  of  $\mathrm{GL}_n(F)$ . Let  $U_l^P$  be the image of  $U_l^G$  in  $\mathrm{PGL}_n(F)$ . This is the isotropy group of the vertex  $v_l$  for the action of  $\mathrm{PGL}_n(F)$ . The Iwahori subgroup  $I^P$  of  $\mathrm{PGL}_n(F)$  is the image of  $I^G$  under  $\mathrm{pr}$ . It is the pointwise isotropy subgroup for  $\Delta$ . Let  $H$  be the subgroup generated by the image of  $h$  in  $\mathrm{PGL}_n(F)$ . It is a cyclic subgroup of order  $n$  that cyclically permutes the vertices of  $\Delta$ . This subgroup normalizes  $I^P$  and the isotropy group of  $b_\Delta$  is the product  $HI^P$ . Recall that  $v_l = h^l v_0$  and hence  $U_l^P = h^l U_0 P h^{-l}$ .

Write  $i_H: I^P \rightarrow HI^P$ ,  $i_0: I^P \rightarrow U_0^P$ ,  $c_0: U_0^P \cap U_l^P \rightarrow U_0^P$  for the inclusions and define  $c_l: U_0^P \cap U_l^P \rightarrow U_0^P$  by  $z \mapsto h^{-l} z h^l$ . Let

$$d^{\mathrm{PGL}_n(F)}: K_0(\mathcal{H}(I^P; R)) \oplus \bigoplus_{l=1}^k K_0(\mathcal{H}(U_0^P \cap U_l^P; R)) \rightarrow K_0(\mathcal{H}(HI^P; R)) \oplus K_0(\mathcal{H}(U_0^P; R))$$

be the map that is  $K_0(i_H) \times -K_0(i_0)$  on  $K_0(\mathcal{H}(I^P; R))$  and  $0 \times (K_0(c_l) - K_0(c_0))$  on  $K_0(\mathcal{H}(U_0^P \cap U_l^P; R))$ . The cokernel of the homomorphism  $d^{\mathrm{PGL}_n(F)}$  agrees with  $SH_0^{\mathrm{PGL}_n(F)}(X'; K_0(\mathcal{H}(?; R)))$  by Lemma 5.1 if, using the notation of Section 5.A, we put  $E = \{v_0, b_\Delta\}$  with  $v_0 < b_\Delta$ ,  $F(v_0, v_0) = \{h, h^2, \dots, h^k\}$  and  $F(v_0, b_\Delta) = \{e\}$ .

### 6.C $\mathrm{GL}_n(F)$

Next we compute  $K_0(\mathcal{H}(\mathrm{GL}_n(F); R))$ . Note that  $\mathrm{GL}_n(F)$  has a noncompact center. Hence Section 5.A does not apply and we have to pass to the setting of Section 5.B using the short exact sequence  $1 \rightarrow C = \mathrm{cent}(\mathrm{GL}_n(F)) \rightarrow \mathrm{GL}_n(F) \xrightarrow{\mathrm{pr}} \mathrm{PGL}_n(F) \rightarrow 1$ , the discussion in Section 6.B and Lemma 5.4.

Let  $\widetilde{M}$  be the kernel of the composite  $\mu: \mathrm{GL}_n(F) \xrightarrow{\det} F^\times \xrightarrow{\nu} \mathbb{Z}$ . Let  $\widehat{H} \subseteq \mathrm{GL}_n(F)$  be the infinite cyclic subgroup generated by the element  $\hat{h}$ . Note that  $\widetilde{M} \cap C$  consists of those diagonal matrices whose entries on the diagonal are all the same and are sent to 0 under  $\nu$ . We conclude  $(\mathrm{GL}_n(\mathcal{O}) \cdot C) \cap \widetilde{M} = \mathrm{GL}_n(\mathcal{O})$  from  $C \cap \widetilde{M} \subseteq \mathrm{GL}_n(\mathcal{O}) \subseteq \widetilde{M}$ . Recall that, for  $W \subseteq \mathrm{PGL}_n(F)$ , we denote by  $\widetilde{W}$  its preimage under  $\mathrm{pr}: \mathrm{GL}_n(F) \rightarrow \mathrm{PGL}_n(F)$ . Since  $\mathrm{pr}(U_l^G) = U_l^P$ , we get, for  $l = 0, \dots, n - 1$ ,

$$\widetilde{U}_l^P \cap \widetilde{M} = (U_l^G \cdot C) \cap \widetilde{M} = (\hat{h}^l \mathrm{GL}_n(\mathcal{O}) \hat{h}^{-l} \cdot C) \cap \widetilde{M} = \hat{h}^l ((\mathrm{GL}_n(\mathcal{O}) \cdot C) \cap \widetilde{M}) \hat{h}^{-l} = \hat{h}^l \mathrm{GL}_n(\mathcal{O}) \hat{h}^{-l} = U_l^G.$$

Now one easily checks  $\widetilde{I}^P \cap \widetilde{M} = I^G$ . Finally we show  $\widetilde{HI}^P \cap \widetilde{M} = I^G$ . We get  $I^G \subseteq \widetilde{HI}^P \cap \widetilde{M}$  from  $\widetilde{I}^P \cap \widetilde{M} = I^G$ . Consider an element  $A \in \widetilde{HI}^P \cap \widetilde{M}$ . We can find an integer  $b$ , an element  $B \in I^G$ , and an

element  $D \in C$  such that  $A = \hat{h}^b B D$  and  $v(A) = 0$ . From  $I^G \subseteq \tilde{M}$ , we conclude  $\hat{h}^b D \in \tilde{M}$ . Since  $\mu(D)$  is divisible by  $n$  and  $\mu(\hat{h}) = 1$  holds,  $b$  is divisible by  $n$ . This implies  $\hat{h}^b \in C$  and hence  $\hat{h}^b D \in C \cap \tilde{M}$ . As  $(C \cap \tilde{M})I^G = I^G$ , we conclude  $A \in I^G$ . Hence  $\widetilde{HT}^P \cap \tilde{M} = I^G$ .

Let  $\tilde{i}_0: I^G \rightarrow U_0^G$  and  $\tilde{c}_0: U_0^G \cap U_i^G \rightarrow U_0^G$  be the inclusions and let  $\tilde{c}_l: U_0^G \cap U_l^G \rightarrow U_0^G$  be the map sending  $\tilde{z}$  to  $\hat{h}^{-l} \tilde{z} \hat{h}^l$ . Let

$$\tilde{d}^{\text{GL}_n(F)}: K_0(\mathcal{H}(I^G; R)) \oplus \bigoplus_{l=1}^k K_0(\mathcal{H}(U_0^G \cap U_l^G; R)) \rightarrow K_0(\mathcal{H}(I^G; R)) \oplus K_0(\mathcal{H}(U_0^G; R))$$

be the map that is  $\text{id}_{K_0(I^G)} \times -K_0(\tilde{i}_0)$  on  $K_0(\mathcal{H}(I^G; R))$  and  $0 \times (K_0(\tilde{c}_l) - K_0(\tilde{c}_0))$  on  $K_0(\mathcal{H}(U_0^G \cap U_l^G; R))$ . The cokernel of the map  $\tilde{d}^{\text{GL}_n(F)}$  is  $K_0(\mathcal{H}(\text{GL}_n(F); R))$  by Lemma 5.4. Let

$$\tilde{d}^{\text{GL}_n(F)}: \bigoplus_{l=1}^k K_0(\mathcal{H}(U_0^G \cap U_l^G; R)) \rightarrow K_0(\mathcal{H}(U_0^G; R))$$

be the map which is given by  $K_0(\tilde{c}_l) - K_0(\tilde{c}_0)$  on  $K_0(\mathcal{H}(U_0^G \cap U_l^G; R))$ . Since  $\tilde{d}^{\text{GL}_n(F)}$  has the same cokernel as  $\tilde{d}^{\text{GL}_n(F)}$ , the cokernel of  $\tilde{d}^{\text{GL}_n(F)}$  is  $K_0(\mathcal{H}(\text{GL}_n(F); R))$ .

## 7 Homotopy colimits

### 7.A The Farrell–Jones assembly map as a map of homotopy colimits

Next we want to extend the considerations of Section 6 to the higher  $K$ -groups. For this purpose and the proofs appearing in [2], it is worthwhile to write down the assembly map in terms of homotopy colimits. The projections  $G/U \rightarrow G/G$  for  $U$  compact open in  $G$  induce a map

$$(7.1) \quad \text{hocolim}_{G/U \in \text{Or}_{\text{cop}}(G)} \mathbf{K}_R(G/U) \rightarrow \mathbf{K}_R(G/G) \simeq \mathbf{K}(\mathcal{H}(G; R)).$$

This map can be identified after applying  $\pi_n$  with the assembly map appearing in Theorems 1.1(i) and 3.2. This follows from [11, Section 5].

### 7.B Simplifying the source of the Farrell–Jones assembly map

Let  $X$  be an abstract simplicial complex with simplicial  $G$ -action such that the isotropy group of each vertex is compact open and the  $G$ -action is cellular. Furthermore we assume that  $|X|^K$  is weakly contractible for any compact open subgroup of  $G$ . Then  $|X|$  is a model for  $E_{\text{cop}}(G)$ .

Let  $C$  be a collection of simplices of  $X$  that contains at least one simplex from each orbit of the action of  $G$  on the set of simplices of  $X$ . Define a category  $\mathcal{C}(C)$  as follows. Its objects are the simplices from  $C$ . A morphism  $gG_\sigma: \sigma \rightarrow \tau$  is an element  $gG_\sigma \in G/G_\sigma$  satisfying  $g\sigma \subseteq \tau$ . The composite of  $gG_\sigma: \sigma \rightarrow \tau$  with  $hG_\tau: \tau \rightarrow \rho$  is  $hgG_\sigma: \sigma \rightarrow \rho$ . Define a functor

$$(7.2) \quad \iota_C: \mathcal{C}(C)^{\text{op}} \rightarrow \text{Or}_{\text{cop}}(G)$$

by sending an object  $\sigma$  to  $G/G_\sigma$  and a morphism  $gG_\sigma: \sigma \rightarrow \tau$  to  $R_g: G/G_\tau \rightarrow G/G_\sigma$ ,  $g'G_\tau \mapsto g'gG_\sigma$ .

**Lemma 7.3** Under the assumptions above, the map induced by the functor  $\iota_C$ ,

$$\operatorname{hocolim}_{\sigma \in \mathcal{C}(C)^{\text{op}}} \mathbf{K}_R(G/G_\sigma) \xrightarrow{\simeq} \operatorname{hocolim}_{G/U \in \text{Or}_{\mathcal{C}\text{op}}(G)} \mathbf{K}_R(G/U),$$

is a weak homotopy equivalence.

**Proof** We want to apply the criterion [12, 9.4]. So we have to show that the geometric realization of the nerve of the category  $G/K \downarrow \iota_C$  is a contractible space for every object  $G/K$  in  $\text{Or}_{\mathcal{C}\text{op}}(G)$ . An object in  $G/K \downarrow \iota_C$  is a pair  $(\sigma, u)$  consisting of an element  $\sigma \in C$  and a  $G$ -map  $u: G/K \rightarrow G/G_\sigma$ . A morphism  $(\sigma, u) \rightarrow (\tau, v)$  in  $G/K \downarrow \iota_C$  is given by a morphism  $gG_\tau: \tau \rightarrow \sigma$  in  $\mathcal{C}(C)$  such that the  $G$ -map  $R_g: G/G_\sigma \rightarrow G/G_\tau$  sending  $zG_\sigma$  to  $zgG_\tau$  satisfies  $v \circ R_g = u$ .

Let  $\mathcal{P}(X^K)$  be the poset given by the simplices of  $X^K$  ordered by inclusion. Then we get an equivalence of categories

$$F: \mathcal{P}(X^K)^{\text{op}} \xrightarrow{\cong} G/K \downarrow \iota_C$$

as follows. It sends a simplex  $\sigma$  to the object  $(\sigma, \text{pr}_\sigma: G/K \rightarrow G/G_\sigma)$  for the canonical projection  $\text{pr}_\sigma$ . A morphism  $\sigma \rightarrow \tau$  in  $\mathcal{P}(X^K)^{\text{op}}$  is sent to the morphism  $(\sigma, \text{pr}_\sigma) \rightarrow (\tau, \text{pr}_\tau)$  in  $G/K \downarrow \iota_C$  which is given by the morphism  $eG_\tau: \tau \rightarrow \sigma$  in  $\mathcal{C}(C)$ .

Consider an object  $(\sigma, u)$  in  $G/K \downarrow \iota_C$ . We want to show that it is isomorphic to an object in the image of  $F$ . Choose  $g \in G$  such that  $g^{-1}Kg \subseteq G_\sigma$  and let  $u$  be the  $G$ -map  $R_g: G/K \rightarrow G/G_\sigma$  sending  $zK$  to  $zgG_\sigma$ . Then  $K \subseteq G_{g\sigma}$  and we can consider the object  $F(g\sigma) = (g\sigma, \text{pr}_{g\sigma})$  for the projection  $\text{pr}_{g\sigma}: G/K \rightarrow G_{g\sigma}$ . Now the isomorphism  $gG_\sigma: \sigma \rightarrow g\sigma$  in  $\mathcal{C}(C)$  induces an isomorphism  $F(g\sigma) \xrightarrow{\cong} (\sigma, u)$  in  $G/K \downarrow \iota_C$ .

Obviously  $F$  is faithful. It remains to show that  $F$  is full. Fix two objects  $\sigma$  and  $\tau$  in  $\mathcal{P}(X^K)$ . Consider a morphism  $f: F(\sigma) = (\sigma, \text{pr}_\sigma) \rightarrow F(\tau) = (\tau, \text{pr}_\tau)$  in  $G/K \downarrow \iota_C$ . It is given by a morphism  $gG_\tau: \tau \rightarrow \sigma$  in  $\mathcal{C}(C)$  such that the composite of  $R_g: G/G_\sigma \rightarrow G/G_\tau$  with  $\text{pr}_\sigma$  is  $\text{pr}_\tau$ . This implies  $gG_\tau = G_\tau$  and hence  $g \in G_\tau$ . Since  $g\tau \subseteq \sigma$  by the definition of a morphism in  $\mathcal{C}(C)$ , we get  $\tau \subseteq \sigma$ . Hence  $f$  is the image of the morphism  $\sigma \rightarrow \tau$  under  $F$ . This shows that  $F$  is full.

Hence it remains to show that geometric realization of the nerve of  $\mathcal{P}(X^K)^{\text{op}}$  is contractible. Since this is the barycentric subdivision of  $|X|^K$ , this follows from the assumptions. □

Suppose additionally that  $X$  admits a strict fundamental domain  $\Delta$ , ie a simplicial subcomplex  $\Delta$  that contains exactly one simplex from each orbit for the  $G$ -action on the set of simplices of  $X$ . Then we can take for  $C$  the simplices from  $\Delta$ . In this case  $\mathcal{C}(C)$  can be identified with the poset  $\mathcal{P}(\Delta)$  of simplices of  $\Delta$ . Recall that, for any open subgroup  $U$  of  $G$ , there is an explicit weak homotopy equivalence  $\mathbf{K}(\mathcal{H}(U; R)) \xrightarrow{\cong} \mathbf{K}_R(G/U)$ , where the source is the  $K$ -theory spectrum  $\mathbf{K}(\mathcal{H}(U; R))$  of the Hecke algebra  $\mathcal{H}(U; R)$ ; see [3, (5.6) and Remark 6.7]. Lemma 7.3 implies:

**Theorem 7.4** Let  $X$  be an abstract simplicial complex with a simplicial  $G$ -action such that the isotropy group of each vertex is compact open, the  $G$ -action is cellular, and  $|X|^K$  is weakly contractible for every compact open subgroup  $K$  of  $G$ . Let  $\Delta$  be a strict fundamental domain.

Then the assembly map

$$(7.5) \quad \operatorname{hocolim}_{\sigma \in \mathcal{P}(\Delta)^{\text{op}}} \mathbf{K}(\mathcal{H}(G_\sigma; R)) \rightarrow \operatorname{hocolim}_{G/U \in \text{Or}_{\text{cop}}(G)} \mathbf{K}_R(G/U)$$

induced by the functor  $\mathcal{P}(\Delta)^{\text{op}} \rightarrow \text{Or}_{\text{cop}}(G)$  sending a simplex  $\sigma$  to  $G_\sigma$  is a weak homotopy equivalence.

**Example 7.6** ( $\text{SL}_n(F)$ ) Let  $X$  be the Bruhat–Tits building for  $\text{SL}_n(F)$ . Then the canonical  $\text{SL}_n(F)$  action on  $X$  is cellular. We will use again the notation introduced in Section 6. The  $(n-1)$ -simplex  $\Delta$ , viewed as a subcomplex of  $X$ , is a strict fundamental domain. Applying this in the case  $n = 2$  yields the homotopy pushout diagram

$$\begin{array}{ccc} \mathbf{K}(\mathcal{H}(I^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_1^S; R)) \\ \downarrow & & \downarrow \\ \mathbf{K}(\mathcal{H}(U_0^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(\text{SL}_2(F); R)) \end{array}$$

For the  $K$ -groups this yields a Mayer–Vietoris sequence, infinite to the left,

$$(7.7) \quad \begin{aligned} \cdots \rightarrow K_n(\mathcal{H}(I^S; R)) &\rightarrow K_n(\mathcal{H}(U_1^S; R)) \oplus K_n(\mathcal{H}(U_0^S; R)) \rightarrow K_n(\mathcal{H}(\text{SL}_2(F); R)) \\ &\rightarrow K_{n-1}(\mathcal{H}(I^S; R)) \rightarrow K_{n-1}(\mathcal{H}(U_1^S; R)) \oplus K_{n-1}(\mathcal{H}(U_0^S; R)) \rightarrow \cdots \\ &\cdots \rightarrow K_0(\mathcal{H}(I^S; R)) \rightarrow K_0(\mathcal{H}(U_1^S; R)) \oplus K_0(\mathcal{H}(U_0^S; R)) \rightarrow K_0(\mathcal{H}(\text{SL}_2(F); R)) \rightarrow 0 \end{aligned}$$

and  $K_n(\mathcal{H}(\text{SL}_2(F); R)) = 0$  for  $n \leq -1$ .

For  $n = 3$  we obtain the homotopy pushout diagram

$$\begin{array}{ccccc} & & \mathbf{K}(\mathcal{H}(U_{12}^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_2^S; R)) \\ & \nearrow & \downarrow & & \nearrow \\ \mathbf{K}(\mathcal{H}(I^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_{02}^S; R)) & & \\ \downarrow & & \downarrow & & \downarrow \\ & & \mathbf{K}(\mathcal{H}(U_1^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(\text{SL}_3(F); R)) \\ \downarrow & \nearrow & \downarrow & & \nearrow \\ \mathbf{K}(\mathcal{H}(U_{01}^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_0^S; R)) & & \end{array}$$

where we abbreviated  $U_{ij}^S := U_i^S \cap U_j^S$ . In general, for  $\text{SL}_n(F)$  we obtain a homotopy pushout diagram whose shape is an  $n$ -cube.

To such an  $n$ -cube there is assigned a spectral sequence concentrated in the region for  $p \geq 0$  and  $0 \leq q \leq n - 1$ , which corresponds to the spectral sequence appearing in Theorem 1.1(ii).

## 8 Allowing central characters and actions on the coefficients

So far we have only considered the standard Hecke algebra  $\mathcal{H}(G; R)$ . There are more general Hecke algebras  $\mathcal{H}(G; R, \rho, \omega)$  [4], and all the discussions of this paper carry over to them in the obvious way.

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
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