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Morimichi Kawasaki
Mitsuaki Kimura
Shuhei Maruyama
Takahiro Matsushita
Masato Mimura

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The space of nonextendable quasimorphisms

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For a pair (G, N) of a group G with normal subgroup N, we consider the space of quasimorphisms and quasicocycles on N nonextendable to G. To treat this space, we establish the five-term exact sequence of cohomology relative to the bounded subcomplex. As an application, we study the spaces associated with the kernel of the (volume) flux homomorphism, the IA-automorphism group of a free group, and certain normal subgroups of Gromov-hyperbolic groups.

Furthermore, we employ this space to prove that the stable commutator length is equivalent to the stable mixed commutator length for certain pairs of a group and normal subgroup.

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1 Introduction

1.1 Motivations

A *quasimorphism* on a group G is a real-valued function $f : G \to \mathbb{R}$ on G satisfying

$$D(f) := \sup\{|f(xy) - f(x) - f(y)| : x, y \in G\} < \infty.$$

We call $D(f)$ the *defect* of the quasimorphism f. A quasimorphism f on G is said to be *homogeneous* if $f(x^n) = n \cdot f(x)$ for every $x \in G$ and every integer n. Let $\mathrm{Q}(G)$ denote the real vector space consisting of homogeneous quasimorphisms on G. The (homogeneous) quasimorphisms are closely related to the second bounded cohomology group $\mathrm{H}_b^2(G) = \mathrm{H}_b^2(G;\mathbb{R})$, and have been extensively studied in geometric group theory and symplectic geometry (see Calegari [26], Frigerio [40] and Polterovich and Rosen [88]).

In this paper, we consider a pair (G, N) of a group G with normal subgroup N. Let $i : N \to G$ be the inclusion map. In this setting, we can construct the following two real vector spaces:

- the space $\mathrm{Q}(N)^G$ of all *G-invariant homogeneous quasimorphisms on N*, where $f : N \to \mathbb{R}$ is said to be *G-invariant* if $f(gxg^{-1}) = f(x)$ for every $g \in G$ and every $x \in N$;
- the space $\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, where $\mathrm{H}^1(N)^G$ is the space of all G-invariant homomorphisms from N to $\mathbb{R}$ and i^* is the linear map from $\mathrm{Q}(G)$ to $\mathrm{Q}(N)$ induced by $i : N \hookrightarrow G$.

An element $f \in \mathrm{Q}(N)$ belongs to $i^*\mathrm{Q}(G)$ if and only if there exists $\hat{f} \in \mathrm{Q}(G)$ such that $\hat{f}|_N \equiv f$; in this case, we say that f is *extendable* to G. Since a homogeneous quasimorphism is conjugation-invariant (see Lemma 3.1), the space $i^*\mathrm{Q}(G)$ is contained in $\mathrm{Q}(N)^G$. The *extendability problem* asks whether there exists $f \in \mathrm{Q}(N)^G$ that is not extendable to G or, equivalently, whether the quotient vector space

$$\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$$

is nonzero. A stronger version of this problem asks whether the quotient space

$$\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$$

is nonzero. We have some reasons to take the quotient vector space over $\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, instead of one over $i^*\mathrm{Q}(G)$. Elements in $\mathrm{H}^1(N)^G$ seem "trivial" as quasimorphisms in $\mathrm{Q}(N)^G$; also, when we apply the Bavard duality theorem for stable mixed commutator lengths (see Theorem 7.1), precisely the elements in $\mathrm{H}^1(N)^G$ behave trivially. An example of a pair (G, N) such that $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$ is nonzero is provided by Shtern [90], and an example of a pair such that $\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ is nonzero is provided by the first and second authors [58]. Some of the authors generalize the result of [58] and provide an extrinsic application in [61] (see Theorem 1.3).

Here we show that, under a certain condition on $\Gamma = G/N$ and a mild condition on G, the quotient real vector space $\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ is finite-dimensional. For example, amenability of Γ and finite presentability of G suffice. We exhibit here two such examples: one corresponds to a surface group (Theorem 1.1), and the other to the fundamental group of a hyperbolic mapping torus (Theorem 1.2). We discuss this point in more detail in the latter part of this subsection. We remark that in Theorem 1.2 the group quotient $\Gamma = G/N$ is nonabelian solvable in general. The main novel point of these theorems is that we obtain nonzero finite-dimensionality of vector spaces associated with quasimorphisms: the (quotient) spaces of homogeneous quasimorphisms modulo genuine homomorphisms tend to be either zero- or infinite-dimensional for groups naturally appearing in geometric group theory. To be more precise, if a group admits hyperbolicity in a certain weak sense, then the space is infinite-dimensional (see Bestvina and Fujiwara [12]); the space vanishes for higher-rank lattices (see Burger and Monod [22; 23]). We also mention that there are some exceptions in the world of one-dimensional dynamics to the aforementioned tendency, such as certain Thompson-type groups; see Fournier-Facio and Lodha [36] and Calegari [26, Chapter 5].

Theorem 1.1 (nonzero finite-dimensionality in surface groups) *Let l be an integer greater than 1, $G=\pi_1(\Sigma_l)$ the surface group with genus l, and N the commutator subgroup $[\pi_1(\Sigma_l),\pi_1(\Sigma_l)]$ of $\pi_1(\Sigma_l)$. Then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = l(2l-1) \quad \textit{and} \quad \dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\big)=1.$$

For $l\in\mathbb{N}$, let $\mathrm{Mod}(\Sigma_l)$ be the mapping class group of the surface Σ_l and $s_l\colon \mathrm{Mod}(\Sigma_l)\to \mathrm{Sp}(2l,\mathbb{Z})$ the symplectic representation. For a mapping class $\psi\in\mathrm{Mod}(\Sigma_l)$, we take a diffeomorphism f representing ψ and let T_f denote the mapping torus of f. The fundamental group of T_f is isomorphic to the semidirect product $\pi_1(\Sigma_l)\rtimes_{f_*}\mathbb{Z}$ and surjects onto $\mathbb{Z}^{2l}\rtimes_{s_l(\psi)}\mathbb{Z}$ via the abelianization map $\pi_1(\Sigma_l)\to \mathrm{H}_1(\Sigma_l;\mathbb{Z})$. Note that the kernel of the surjection is equal to the commutator subgroup of $\pi_1(\Sigma_l)$.

Theorem 1.2 (nonzero finite-dimensionality in hyperbolic mapping tori) *Let l be an integer greater than 1, $\psi\in\mathrm{Mod}(\Sigma_l)$ a pseudo-Anosov element and f a diffeomorphism representing ψ. Let G be the fundamental group of the mapping torus T_f and N the kernel of the surjection $G\to\mathbb{Z}^{2l}\rtimes_{s_l(\psi)}\mathbb{Z}$. Then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim\mathrm{Ker}(I_{2l}-s_l(\psi)) + \dim\mathrm{Ker}\big(I_{\binom{2l}{2}}-\textstyle\bigwedge^2 s_l(\psi)\big)$$

and

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\big) = \dim\mathrm{Ker}(I_{2l}-s_l(\psi))+1.$$

Here, for $n\in\mathbb{N}$, I_n denotes the identity matrix of size n, and $\bigwedge^2 s_l(\psi)\colon \bigwedge^2\mathbb{R}^{2l}\to\bigwedge^2\mathbb{R}^{2l}$ is the map induced by $s_l(\psi)$.

In particular, if ψ is in the Torelli group (that is, $s_l(\psi)=I_{2l}$), then

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = 2l+\binom{2l}{2} \quad \textit{and} \quad \dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\big)=2l+1.$$

In Theorem 1.2, the pseudo-Anosov property for ψ is assumed to ensure hyperbolicity of G; see Theorem 4.8. In Theorems 4.5 and 4.11, we also obtain results analogous to Theorems 1.1 and 1.2 in the free group setting.

In study of quasimorphisms, it is often quite hard to obtain nonzero finite-dimensionality. For instance, if a group G can act nonelementarily in a certain good manner on a Gromov-hyperbolic geodesic space, then the dimension of $\mathrm{Q}(G)$ is of the cardinal of the continuum [12]; contrastingly, a higher-rank lattice G has zero $\mathrm{Q}(G)$ [22]. For a group G such that the dimension of $\mathrm{Q}(G)$ is of the cardinal of the continuum, understanding *all* quasimorphisms on G might have been considered an impossible subject. Our study of the space of nonextendable quasimorphisms might have some possibility of shedding light on this problem modulo "trivial or extendable" quasimorphisms.

Theorems 1.1 and 1.2 treat the case where G is a nonelementary Gromov-hyperbolic group and N a subgroup with solvable quotient. In this case, a result of Epstein and Fujiwara [34] implies that the dimension of $\mathrm{Q}(G)$ is the cardinal of the continuum. The kernel of the restriction $i^*:\mathrm{Q}(G)\to\mathrm{Q}(N)^G$ can be identified with $\mathrm{Q}(\Gamma)$. Since Γ is now finitely generated solvable, the dimension of $\mathrm{Q}(\Gamma)=\mathrm{H}^1(\Gamma)$ is finite. This implies that the dimension of $\mathrm{Q}(N)^G$ is also the cardinal of the continuum. Nevertheless, our results (Theorems 1.9 and 1.10) imply that the spaces $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))$ are always both finite-dimensional. Theorems 1.1 and 1.2 provide nonvanishing examples, and it might be an interesting problem to understand *all* quasimorphism classes in these examples.

We outline how we deduce finite-dimensionality of $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))$ under certain conditions in our results. Our main theorem, Theorem 1.5 (stated in Section 1.2), establishes the five-term exact sequence of the cohomology $\mathrm{H}^\bullet_{/b}$ associated with a short exact sequence of groups

$$1\to N\to G\to\Gamma\to 1.$$

Here, $\mathrm{H}^\bullet_{/b}$ relates the bounded cohomology $\mathrm{H}^\bullet_b$ with the ordinary cohomology $\mathrm{H}^\bullet$; see Section 1.2 for the precise definition of $\mathrm{H}^\bullet_{/b}$. In Theorems 1.9 and 1.10, we assume that Γ is *boundedly* 3*-acyclic*, meaning that $\mathrm{H}^2_b(\Gamma;\mathbb{R})=0$ and $\mathrm{H}^3_b(\Gamma;\mathbb{R})=0$ (Definition 1.8). Then the five-term exact sequence enables us to relate $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))$, respectively, to the ordinary second cohomology $\mathrm{H}^2(\Gamma)=\mathrm{H}^2(\Gamma;\mathbb{R})$ and $\mathrm{H}^2(G)=\mathrm{H}^2(G;\mathbb{R})$. Since second ordinary cohomology is finite-dimensional under certain mild conditions, we obtain the desired finite-dimensionality results. In this point of view, our main theorem (Theorem 1.5) might be regarded as filling in a missing piece between the bounded cohomology theory and the ordinary cohomology theory.

We also note that the extendability and nonextendability of invariant quasimorphisms themselves have applications. See Section 2.1 for an application to the stable (mixed) commutator lengths, and Section 2.3 for one to symplectic geometry. As a notable extrinsic application, we recall our following theorem:

Theorem 1.3 [61, Theorem 1.1] *Let Σ_l be a closed orientable surface whose genus l is at least two and Ω an area form on S. Let $\mathrm{Diff}_0(\Sigma_l,\Omega)$ denote the identity component of the group of diffeomorphisms of Σ_l that preserve Ω. Assume that a pair $f,g\in\mathrm{Diff}_0(\Sigma_l,\Omega)$ satisfies $fg=gf$. Let $\mathrm{Flux}_\Omega:\mathrm{Diff}_0(\Sigma_l,\Omega)\to\mathrm{H}^1(\Sigma_l;\mathbb{R})$ be the volume flux homomorphism and $\smile:\mathrm{H}^1(\Sigma_l;\mathbb{R})\times\mathrm{H}^1(\Sigma_l;\mathbb{R})\to\mathrm{H}^2(\Sigma_l;\mathbb{R})\cong\mathbb{R}$ the cup product. Then*

$$\mathrm{Flux}_\Omega(f)\smile\mathrm{Flux}_\Omega(g)=0.$$

The statement of Theorem 1.3 might not seem to have any relation to quasimorphisms. Nevertheless, the key to the proof is comparison between vanishing and nonvanishing of $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$, where $G=\mathrm{Flux}_\Omega^{-1}(\langle\mathrm{Flux}_\Omega(f),\mathrm{Flux}_\Omega(g)/k\rangle)$ for a sufficiently large integer k and $N=\mathrm{Ker}(\mathrm{Flux}_\Omega)$; see Section 2.3 for basic concepts around volume flux homomorphisms. (We discuss a related example in Example 7.15.)

1.2 Main theorem

To treat the spaces of nonextendable quasimorphisms, we establish the five-term exact sequence of group cohomology $\mathrm{H}^\bullet_{/b}$ relative to bounded cochain complexes. Throughout the paper, the coefficient module of the cohomology groups is the field $\mathbb{R}$ of real numbers unless otherwise specified. The main reason we are interested in this cohomology $\mathrm{H}^\bullet_{/b}$ is that $\mathrm{H}^1_{/b}(G)$ and $\mathrm{Q}(G)$ are isomorphic (see Remark 1.6).

Now we start the definition of $\mathrm{H}^\bullet_{/b}$. Let V be a left normed G-module, and $C^n(G;V)$ the space of functions from the n-fold direct product $G^{\times n}$ of G to V. The group cohomology is defined by the cohomology group of $C^n(G;V)$ with a certain differential (see Section 3 for the precise definition). Recall that the spaces $C^n_b(G;V)$ of the bounded functions form a subcomplex of $C^\bullet(G;V)$, and its cohomology group is the bounded cohomology group of G. We write $C^\bullet_{/b}(G;V)$ to indicate the quotient complex $C^\bullet(G;V)/C^\bullet_b(G;V)$, and write $\mathrm{H}^\bullet_{/b}(G;V)$ to mean its cohomology group.

Our main result is the five-term exact sequence of the cohomology $\mathrm{H}^\bullet_{/b}$. Before stating our main theorem, we first recall the five-term exact sequence of ordinary group cohomology.

Theorem 1.4 (see for example Brown [19, Corollary VII.6.4]) *Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence of groups and V a left $\mathbb{R}[\Gamma]$-module. Then there exists an exact sequence*

$$0 \to \mathrm{H}^1(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^1(G;V) \xrightarrow{i^*} \mathrm{H}^1(N;V)^G \xrightarrow{\tau} \mathrm{H}^2(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^2(G;V).$$

The following theorem is the main result in this paper:

Theorem 1.5 (main theorem) *Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence of groups and V a left Banach $\mathbb{R}[\Gamma]$-module equipped with a Γ-invariant norm $\|\cdot\|$. Then there exists an exact sequence*

$$0 \to \mathrm{H}^1_{/b}(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^1_{/b}(G;V) \xrightarrow{i^*} \mathrm{H}^1_{/b}(N;V)^G \xrightarrow{\tau_{/b}} \mathrm{H}^2_{/b}(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^2_{/b}(G;V). \tag{1-1}$$

Moreover, the exact sequence above is compatible with the five-term exact sequence of group cohomology; that is, the following diagram commutes:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{H}^1(\Gamma;V) & \xrightarrow{p^*} & \mathrm{H}^1(G;V) & \xrightarrow{i^*} & \mathrm{H}^1(N;V)^G & \xrightarrow{\tau} & \mathrm{H}^2(\Gamma;V) & \xrightarrow{p^*} & \mathrm{H}^2(G;V) \\
 & & \downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi_3} & & \downarrow{\scriptstyle \xi_4} & & \downarrow{\scriptstyle \xi_5} \\
0 & \longrightarrow & \mathrm{H}^1_{/b}(\Gamma;V) & \xrightarrow{p^*} & \mathrm{H}^1_{/b}(G;V) & \xrightarrow{i^*} & \mathrm{H}^1_{/b}(N;V)^G & \xrightarrow{\tau_{/b}} & \mathrm{H}^2_{/b}(\Gamma;V) & \xrightarrow{p^*} & \mathrm{H}^2_{/b}(G;V)
\end{array} \tag{1-2}$$

Here the ξ_i are the maps induced from the quotient map $C^\bullet \to C^\bullet_{/b}$.

Remark 1.6 By the definition of $\mathrm{H}^\bullet_{/b}$, there exists a natural map $\mathrm{Q}(G) \to \mathrm{H}^1_{/b}(G) = \mathrm{H}^1_{/b}(G;\mathbb{R})$, and it is known that this is an isomorphism (see the proof of [26, Theorem 2.50]). Then diagram (1-2) gives rise to

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{H}^1(\Gamma) & \xrightarrow{p^*} & \mathrm{H}^1(G) & \xrightarrow{i^*} & \mathrm{H}^1(N)^G & \xrightarrow{\tau} & \mathrm{H}^2(\Gamma) & \xrightarrow{p^*} & \mathrm{H}^2(G) \\
 & & \downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi_3} & & \downarrow{\scriptstyle \xi_4} & & \downarrow{\scriptstyle \xi_5} \\
0 & \longrightarrow & \mathrm{Q}(\Gamma) & \xrightarrow{p^*} & \mathrm{Q}(G) & \xrightarrow{i^*} & \mathrm{Q}(N)^G & \xrightarrow{\tau_{/b}} & \mathrm{H}^2_{/b}(\Gamma) & \xrightarrow{p^*} & \mathrm{H}^2_{/b}(G)
\end{array} \tag{1-3}$$

Note that the exactness of the sequence

$$0 \to \mathrm{Q}(\Gamma) \xrightarrow{p^*} \mathrm{Q}(G) \xrightarrow{i^*} \mathrm{Q}(N)^G$$

is well known (see [26, Remark 2.90]).

Remark 1.7 It is straightforward to show that the quotient space $\mathrm{H}^1_{/b}(N;V)^G/i^*\mathrm{H}^1_{/b}(G;V)$ is isomorphic to $\widehat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}/i^*\widehat{\mathrm{Q}}Z(G;V)$, where $\widehat{\mathrm{Q}}Z(G;V)$ and $\widehat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$ are the spaces of quasicocycles on G and G-quasiequivariant V-valued quasimorphisms on N, respectively (see Definition 6.1 and Section 8.2; see also Remark 6.4). In Section 8.2, we will apply Theorem 1.5 to the extension problem of G-quasiequivariant quasimorphisms on N to quasicocycles on G. We also mention that, for a group–subgroup pair (G,H) for H (*not* being normal in G, but) being hyperbolically embedded into G, certain extension theorems of quasicocycles have been obtained by Hull and Osin [49] and Frigerio, Pozzetti and Sisto [41].

This theorem provides several arguments to estimate the dimensions of the spaces $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ as follows. Here we recall the definition of *bounded k-acyclicity* of groups from Ivanov [53] and Moraschini and Raptis [82].

Definition 1.8 (bounded k-acyclicity) Let k be a positive integer. A group G is said to be *boundedly k-acyclic* if $\mathrm{H}^i_b(G) = 0$ for every positive integer i with $i \le k$.

We note that $\mathrm{H}^1_b(G) = 0$ for every group G. We recall properties and examples of boundedly k-acyclic groups in Theorem 3.6. In particular, we recall that amenable groups, such as abelian groups, are boundedly k-acyclic for all k (Theorem 3.5(5)).

Theorem 1.9 *If the quotient group $\Gamma = G/N$ is boundedly 3-acyclic, then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2(\Gamma).$$

Moreover, if G is Gromov-hyperbolic, then

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma).$$

In fact, the assumption of Gromov-hyperbolicity of G can be weakened to the surjectivity of the comparison map $c_G\colon \mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$; see Theorem 4.1.

On the space $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$, we also obtain the following:

Theorem 1.10 *If $\Gamma = G/N$ is boundedly 3-acyclic, then the map $p^* \circ (\xi_4)^{-1} \circ \tau_{/b}$ induces an isomorphism*

$$\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) \cong \mathrm{Im}(p^*) \cap \mathrm{Im}(c_G),$$

where $c_G\colon \mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is the comparison map. In particular, if Γ is boundedly 3-acyclic, then

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) \le \dim \mathrm{H}^2(G).$$

When $N = [G, G]$, we have a more precise calculation of $\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big)$ (see Corollary 4.3). As we mentioned in the previous subsection, there are many examples of a finitely presented group whose space of homogeneous quasimorphisms is infinite-dimensional, for instance any nonelementary Gromov-hyperbolic group [34]. Nevertheless, if we assume that $\Gamma = G/N$ is boundedly 3-acyclic, then we have the following two statements: the space $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ is finite-dimensional if G is finitely presented (following from Theorem 1.10); and the space $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ is finite-dimensional if Γ is finitely presented (following from Theorem 1.9).

There are several known conditions that guarantee $\mathrm{Q}(N)^G = i^*\mathrm{Q}(G)$, ie every G-invariant quasimorphism is extendable (see Malyutin [67], Ishida [50; 51] Shtern [90] and Kawasaki, Kimura, Matsushita and Mimura [60]). We say that a group homomorphism $p\colon G \to \Gamma$ *virtually splits* if there exist a subgroup Λ of finite index in Γ and a group homomorphism $s\colon \Lambda \to G$ such that $f \circ s(x) = x$ for every $x \in \Lambda$. The first, second, fourth and fifth authors showed that, if the group homomorphism $p\colon G \to \Gamma$ virtually splits, then $\mathrm{Q}(N)^G = i^*\mathrm{Q}(G)$ [60]. Thus the space $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$, which we consider in Theorem 1.9, can be seen as a space of obstructions to the existence of virtual splittings.

2 Other applications of the main theorem

In this section, we provide several other applications of our main theorem (Theorem 1.5); we also use its corollaries, Theorems 1.9 and 1.10. To conclude the section, we briefly describe the organization of the rest of the paper.

2.1 On equivalences of scl_G and $\mathrm{scl}_{G,N}$

As an application of the spaces of nonextendable quasimorphisms, we treat the equivalence problems of the stabilizations of usual and mixed commutator lengths. For two nonnegative-valued functions μ and ν on a group G, we say that μ and ν are *bi-Lipschitzly equivalent* (or *equivalent* in short) if there exist positive constants C_1 and C_2 such that $C_1\nu \le \mu \le C_2\nu$. By Theorem 1.10, $\mathrm{H}^2(G) = 0$ implies that $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) = 0$ if $\Gamma = G/N$ is boundedly 3-acyclic. We show that the condition $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) = 0$ implies that certain two stable word lengths related to commutators are bi-Lipschitzly equivalent.

Let G be a group and N a normal subgroup. A (G, N)*-commutator* is an element of G of the form $[g, x] = gxg^{-1}x^{-1}$ for some $g \in G$ and $x \in N$. Let $[G, N]$ be the group generated by the set of (G, N)-commutators. Then $[G, N]$ is a normal subgroup of G and is included in N. For an element x in $[G, N]$, the (G, N)*-commutator length* or the *mixed commutator length* of x is defined to be the minimum number n such that there exist n (G, N)-commutators $c_1, \dots, c_n$ such that $x = c_1 \cdots c_n$, and is denoted by $\mathrm{cl}_{G,N}(x)$. Then there exists a limit

$$\mathrm{scl}_{G,N}(x) := \lim_{n\to\infty} \frac{\mathrm{cl}_{G,N}(x^n)}{n}$$

and we call $\mathrm{scl}_{G,N}(x)$ the *stable* (G, N)*-commutator length* of x.

When $N = G$, $\mathrm{cl}_{G,G}(x)$ and $\mathrm{scl}_{G,G}(x)$ equal the commutator length and stable commutator length of x, respectively. We write $\mathrm{cl}_G(x)$ and $\mathrm{scl}_G(x)$ instead of $\mathrm{cl}_{G,G}(x)$ and $\mathrm{scl}_{G,G}(x)$. The commutator lengths and stable commutator lengths have a long history (see [26]), for instance in the study of theory of mapping class groups (see [32; 27; 10]) and diffeomorphism groups (see [21; 95; 96; 97; 17]). The celebrated Bavard duality theorem [6] describes the relationship between homogeneous quasimorphisms and the stable commutator length. In particular, for an element $x \in [G, G]$, $\mathrm{scl}_G(x)$ is nonzero if and only if there exists a homogeneous quasimorphism f on G with $f(x) \neq 0$.

In [58; 60], we construct a pair (G, N) such that scl_N and $\mathrm{scl}_{G,N}$ are not bi-Lipschitzly equivalent on $[N, N]$. Contrastingly, in several cases it is known that scl_G and $\mathrm{scl}_{G,N}$ are bi-Lipschitzly equivalent on $[G, N]$. For example, if the map $p\colon G \to \Gamma = G/N$ virtually splits, then scl_G and $\mathrm{scl}_{G,N}$ are bi-Lipschitzly equivalent on $[G, N]$. In this paper, the vanishing of $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ implies the equivalence of scl_G and $\mathrm{scl}_{G,N}$ as follows. We note that $\mathrm{H}^2(G) = 0$ implies $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) = 0$ by Theorem 1.10 when $\Gamma = G/N$ is boundedly 3-acyclic.

Theorem 2.1 *Assume that* $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. *Then*:

(1) scl_G *and* $\mathrm{scl}_{G,N}$ *are bi-Lipschitzly equivalent on* $[G, N]$.

(2) *If* $\Gamma = G/N$ *is amenable*, *then* $\mathrm{scl}_G(x) \leq \mathrm{scl}_{G,N}(x) \leq 2 \cdot \mathrm{scl}_G(x)$ *for all* $x \in [G, N]$.

(3) *If* $\Gamma = G/N$ *is solvable*, *then* $\mathrm{scl}_G(x) = \mathrm{scl}_{G,N}(x)$ *for all* $x \in [G, N]$.

Remark 2.2 Recently, several examples of nonamenable boundedly acyclic groups have been constructed (see [38; 37; 79; 78]; we also recall some of them in Theorem 3.6). However, our proof of Theorem 2.1(2) does *not* work if the assumption of amenability of Γ in (2) is replaced with bounded 3-acyclicity. Indeed, in our proof, we use the fact that $\mathrm{H}_b^2(G) \to \mathrm{H}_b^2(N)^G$ is *isometric*, which is deduced from the amenability of Γ [74, Proposition 8.6.6].

By Theorem 1.10, when G/N is boundedly 3-acyclic, $\mathrm{H}^2(G) = 0$ implies $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, and hence $\mathrm{scl}_{G,N}$ and scl_G are equivalent on $[G, N]$. There are plenty of examples of groups whose second cohomology groups vanish, as follows:

- The free groups F_n.
- Let l be a positive integer. Let N_l be the nonorientable closed surface with genus l, and set $G = \pi_1(N_l)$. Then, $G = \langle a_1, \dots, a_l \mid a_1^2 \cdots a_l^2 \rangle$ and $\mathrm{H}^2(G) = \mathrm{H}^2(N_l) = 0$.
- Let K be a knot in S^3. Then the knot group G of K is defined to be the fundamental group of the complement $S^3 \setminus K$. Since $S^3 \setminus K$ is an Eilenberg–Mac Lane space, we have $\mathrm{H}^2(G) = \mathrm{H}^2(S^3 \setminus K) = \widetilde{\mathrm{H}}_0(K) = 0$.
- The braid group B_n. Akita and Liu [1, Corollary 3.21] gave sufficient conditions on a labeled graph Γ for the real second cohomology group of the Artin group $A(\Gamma)$ to vanish.
- The Baumslag–Solitar groups $\mathrm{BS}(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$ with $m \neq n$ (see [13], for example).
- Free products of the above groups.

For other examples satisfying $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, see Example 4.13 and Corollaries 4.12, 4.14 and 4.16.

Finally, we discuss pairs (G, N) with nonequivalent scl_G and $\mathrm{scl}_{G,N}$. In [58], the first and second authors provided the first example of such (G, N) (Example 7.14); we obtain another example with smaller G in Example 7.15, which follows from [61]. These two examples may be seen as one example, coming from symplectic geometry. Unfortunately, in the present paper, we are unable to provide any new example from a different background. We remark that some of the authors [69] provided new examples after our work here; see the discussion below Problem 9.9. By Theorem 2.1, the vanishing of $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ implies the equivalence of scl_G and $\mathrm{scl}_{G,N}$. After this work, the authors proved that its converse holds if $N = [G, G]$ [59]. We discuss problems on this equivalence/nonequivalence in more detail in Section 9.2.

2.2 The case of IA-automorphism groups of free groups

Here we provide an example where $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ but Γ is not amenable. Our example comes from the automorphism group of a free group and the IA-automorphism group. The group of automorphisms of a group G is denoted by $\mathrm{Aut}(G)$. Let IA_n be the IA-automorphism group of the free group F_n, ie the kernel of the natural homomorphism $\mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$. Let $\mathrm{Aut}(F_n)_+$ denote the preimage of $\mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{Aut}(F_n)$. The following theorem will be proved in Section 8; see Theorems 8.9 and 8.17 for more general statements.

Theorem 2.3 (1) *For every* $n \geq 2$, *we have* $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}(F_n)} = i^*\mathrm{Q}(\mathrm{Aut}(F_n))$ *and* $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}_+(F_n)} = i^*\mathrm{Q}(\mathrm{Aut}_+(F_n))$.

(2) *For every* $n \geq 6$ *and every subgroup* G *of* $\mathrm{Aut}(F_n)$ *of finite index*, $\mathrm{Q}(N)^G = i^*\mathrm{Q}(G)$. *Here*, $N = \mathrm{IA}_n \cap G$.

Remark 2.4 (1) The bound $n \geq 6$ in Theorem 2.3(2) comes from Theorem 8.6(1), which treats an effective bound of the *Borel stable range* for second ordinary cohomology with the trivial real coefficients of SL_n.

In fact, by appealing to a recent result of Bader and Sauer [3], we are able to improve this bound to $n \geq 4$. We will see this in Theorem 8.17.

(2) Corollary 3.8 of [43] implies that $\mathrm{H}^2(\mathrm{Aut}(F_n)) = 0$ for $n \geq 5$. However, $\mathrm{H}^2(\Lambda)$ for a subgroup Λ of finite index in $\mathrm{Aut}(F_n)$ is mysterious in general. Even on H^1, only quite recently it has been proved that $\mathrm{H}^1(\Lambda) = 0$ if $n \geq 4$; the proof is based on Kazhdan's property (T) for $\mathrm{Aut}(F_n)$ for $n \geq 4$. See [55; 54; 86]. We refer to [8] for a comprehensive treatise on property (T). Contrastingly, by [71], there exists a subgroup Λ of finite index in $\mathrm{Aut}(F_3)$ such that $\mathrm{H}^1(\Lambda) \neq 0$.

(3) The same conclusions as in Theorem 2.3 hold if we replace $\mathrm{Aut}(F_n)$ and IA_n with $\mathrm{Out}(F_n)$ and $\overline{\mathrm{IA}}_n$, respectively. Here, $\overline{\mathrm{IA}}_n$ denotes the kernel of the natural map $\mathrm{Out}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$. Indeed, the proofs which will be presented in Section 8 work without any essential change.

(4) If $n \geq 3$ and G is a subgroup of $\mathrm{Aut}(F_n)$ of finite index, then the real vector space $i^*\mathrm{Q}(G)$ is infinite-dimensional. Indeed, we can apply [12] to the acylindrically hyperbolic group $\mathrm{Out}(F_n)$, whose amenable radical is trivial. Moreover, thanks to [39, Corollary 4.3], we may construct an infinite collection of homogeneous quasimorphisms on $\mathrm{Out}(F_n)$ which is linearly independent even when these quasimorphisms are restricted to $[\overline{\mathrm{IA}}_n \cap \overline{G}, \overline{\mathrm{IA}}_n \cap \overline{G}]$. Here $\overline{G}$ is the image of G under the natural projection $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$. Then consider the restriction of this collection on $\overline{G}$, and take the pullback of it under the projection $G \to \overline{G}$.

In fact, Corollary 1.2 of [9] treats quasicocycles into unitary representations. Then the following may be deduced in a similar manner to the above: Let G be a subgroup of $\mathrm{Aut}(F_n)$ of finite index with $n \geq 3$ and $\Gamma := G/(\mathrm{IA}_n \cap G)$. Let $(\pi, \mathcal{H})$ be a unitary Γ-representation, and $(\bar{\pi}, \mathcal{H})$ the pullback of it under the projection $G \to \Gamma$. Then the vector space $i^*\widehat{\mathrm{Q}}Z(G, \bar{\pi}, \mathcal{H})$ of the quasicocycles is infinite-dimensional. Furthermore, Corollary 1.2 of [9] and its proof can be employed to obtain the corresponding result in the setting where G is a subgroup of $\mathrm{Mod}(\Sigma_l)$ of finite index with $l \geq 3$ and $(\pi, \mathcal{H})$ is a unitary representation of $G/(\mathcal{I}(\Sigma_l) \cap G)$. Here $\mathcal{I}(\Sigma_l)$ denotes the Torelli group.

If (G, N) equals $(\mathrm{Mod}(\Sigma_l), \mathcal{I}(\Sigma_l))$ or its analogue for the setting of subgroups of finite index, then the situation is subtle. See Theorems 8.10 and 8.14 for our results. We remark that the question on the extendability of quasimorphisms might be open; see Problem 8.18.

2.3 Applications to volume flux homomorphisms

In Section 5, we will provide applications of Theorem 1.10 to diffeomorphism groups.

We study the problem to determine which cohomology class admits a bounded representative. Notably, the problem on (subgroups of) diffeomorphism groups is interesting and has been studied in view of characteristic classes of fiber bundles. However, the problem is often quite difficult and, in fact, there are only a few cohomology classes that are known to be bounded or not. Here we restrict our attention to the case of degree two cohomology classes. The best-known example is the Euler class of $\mathrm{Diff}_+(S^1)$, which has a bounded representative. The Godbillon–Vey class integrated along the fiber defines a cohomology class of $\mathrm{Diff}_+(S^1)$, which has no bounded representatives [91]. It was shown in [25] that the Euler class of $\mathrm{Diff}_0(\mathbb{R}^2)$ is unbounded. In the case of three-dimensional manifolds, the identity components of the diffeomorphism groups of several closed Seifert-fibered three-manifolds admit cohomology classes of degree two which do not have bounded representatives [68].

Let M be an m-dimensional manifold and Ω a volume form. Then we can define the flux homomorphism (on the universal covering) $\widetilde{\mathrm{Flux}}_\Omega \colon \widetilde{\mathrm{Diff}}_0(M, \Omega) \to \mathrm{H}^{m-1}(M)$, the flux group Γ_Ω, and the flux homomorphism $\mathrm{Flux}_\Omega \colon \mathrm{Diff}_0(M, \Omega) \to \mathrm{H}^{m-1}(M)/\Gamma_\Omega$; see Section 5 for the precise definitions.

As applications of Theorem 1.10, we have a few results related to the comparison maps $\mathrm{H}_b^2(\mathrm{Diff}_0(M, \Omega)) \to \mathrm{H}^2(\mathrm{Diff}_0(M, \Omega))$ and $\mathrm{H}_b^2(\widetilde{\mathrm{Diff}}_0(M, \Omega)) \to \mathrm{H}^2(\widetilde{\mathrm{Diff}}_0(M, \Omega))$.

It is known that the spaces $\mathrm{H}^2(\mathrm{Diff}_0(M,\Omega))$ and $\mathrm{H}^2(\widetilde{\mathrm{Diff}}_0(M,\Omega))$ can be very large (see for instance the following proposition in Kotschick and Morita [65]). Note that $\mathrm{H}^n(\mathbb{R}^m;\mathbb{R})$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}\big(\bigwedge^n_{\mathbb{Z}}(\mathbb{R}^m);\mathbb{R}\big)$.

Proposition 2.5 [65] *The homomorphisms*

$$\mathrm{Flux}^*_\Omega\colon \mathrm{H}^2(\mathrm{H}^{m-1}(M)/\Gamma_\Omega)\to \mathrm{H}^2(\mathrm{Diff}_0(M,\Omega)),\quad \widetilde{\mathrm{Flux}}^*_\Omega\colon \mathrm{H}^2(\mathrm{H}^{m-1}(M))\to \mathrm{H}^2(\widetilde{\mathrm{Diff}}_0(M,\Omega))$$

induced by the flux homomorphisms are injective.

As an application of Theorem 1.10, we have the following theorem:

Theorem 2.6 *Let (M,Ω) be an m-dimensional closed manifold with volume form Ω. Then the following hold*:

(1) *If $m=2$ and the genus of M is at least 2, then there exists at least one nontrivial element of $\mathrm{Im}(\mathrm{Flux}^*_\Omega)$ represented by a bounded 2-cochain.*
(2) *Otherwise, every nontrivial element of $\mathrm{Im}(\mathrm{Flux}^*_\Omega)$ and $\mathrm{Im}(\widetilde{\mathrm{Flux}}^*_\Omega)$ cannot be represented by a bounded 2-cochain.*

Note that in case (1) it is known that $\pi_1(\mathrm{Diff}_0(M,\Omega))=0$ (see for example [87, Section 7.2.B]); in particular, the flux group Γ_Ω is zero.

In the proof of Theorem 2.6(1), we essentially prove the nontriviality of the cohomology class $c_P\in\mathrm{Im}(\mathrm{Flux}^*_\Omega)$, called the *Py class*. In Section 9.1, we provide some observations on the Py class.

Organization of the paper

Section 3 collects preliminary facts. In Section 4, we first prove Theorems 1.9 and 1.10, assuming Theorem 1.5. Secondly, we show Theorems 1.1 and 1.2. In Section 5, we provide applications of Theorem 1.5 to the volume flux homomorphisms. Section 6 is devoted to the proof of Theorem 1.5. In Section 7, we prove Theorem 2.1. In Section 8, we prove Theorem 2.3, as well as Theorems 8.9 and 8.10 (and furthermore Theorems 8.17 and 8.14). In Section 9, we provide several open problems.

3 Preliminaries

Before proceeding to the main part of this section, we collect basic properties of quasimorphisms and state them as Lemmas 3.1 and 3.2; see [26, Sections 2.2 and 2.4] for more details. Lemma 3.1 follows from the equality $(ghg^{-1})^n=gh^ng^{-1}$ for every $g,h\in G$ and every $n\in\mathbb{Z}$.

Lemma 3.1 *A homogeneous quasimorphism is conjugation-invariant.*

In particular, the restriction of a homogeneous quasimorphism f of G to a normal subgroup N is G-invariant.

For a quasimorphism $f\colon G\to\mathbb{R}$, the Fekete lemma guarantees that the limit

$$\bar{f}(x)=\lim_{n\to\infty}\frac{f(x^n)}{n}$$

exists. The function $\bar{f}$ defined by the above equation is called the *homogenization of* f.

Lemma 3.2 (1) *$\bar{f}$ is a homogeneous quasimorphism.*

(2) *$|\bar{f}(x)-f(x)|\le D(f)$ for every $x\in G$.*

(3) [26, Lemma 2.58] *$D(\bar{f})\le 2D(f)$.*

In this section, we recall definitions and facts related to the ordinary and bounded cohomology of groups. For a more comprehensive introduction to this subject, we refer to [45; 26; 40].

Let V be a left $\mathbb{R}[G]$-module and $C^n(G;V)$ the vector space consisting of functions from the n-fold direct product $G^{\times n}$ to V. Let $\delta\colon C^n(G;V)\to C^{n+1}(G;V)$ be the $\mathbb{R}$-linear map defined by

$$(\delta f)(g_0,\ldots,g_n)=g_0\cdot f(g_1,\ldots,g_n)+\sum_{i=1}^{n}(-1)^i f(g_0,\ldots,g_{i-1}g_i,\ldots,g_n)+(-1)^{n+1}f(g_0,\ldots,g_{n-1}).$$

Then $\delta^2=0$ and its n^{th} cohomology is the *ordinary group cohomology* $\mathrm{H}^n(G;V)$.

Next, suppose that V is equipped with a G-invariant norm $\|\cdot\|$, ie $\|g\cdot v\|=\|v\|$ for every $g\in G$ and $v\in V$. Define $C_b^n(G;V)$ to be the subspace

$$C_b^n(G;V)=\Bigl\{f\colon G^{\times n}\to V \mid \sup_{(g_1,\ldots,g_n)\in G^{\times n}}\|f(g_1,\ldots,g_n)\|<\infty\Bigr\}$$

of $C^n(G;V)$. Then $C_b^\bullet(G;V)$ is a subcomplex of $C^\bullet(G;V)$, and we call the n^{th} cohomology of $C_b^\bullet(G;V)$ the *n^{th} bounded cohomology of* G, and denote it by $\mathrm{H}_b^n(G;V)$. The inclusion $C_b^\bullet(G;V)\to C^\bullet(G;V)$ induces a map $c_G\colon \mathrm{H}_b^\bullet(G;V)\to\mathrm{H}^\bullet(G;V)$, called the *comparison map*.

Let $\mathrm{H}_{/b}^\bullet(G;V)$ denote their relative cohomology, that is, the cohomology of the quotient complex $C_{/b}^\bullet(G;V)=C^\bullet(G;V)/C_b^\bullet(G;V)$. Then the short exact sequence of cochain complexes

$$0\to C_b^\bullet(G;V)\to C^\bullet(G;V)\to C_{/b}^\bullet(G;V)\to 0$$

induces the cohomology long exact sequence

$$\cdots\to\mathrm{H}_b^n(G;V)\xrightarrow{c_G}\mathrm{H}^n(G;V)\to\mathrm{H}_{/b}^n(G;V)\to\mathrm{H}_b^{n+1}(G;V)\to\cdots. \tag{3-1}$$

If we need to specify the G-representation ρ, we may use the symbols $\mathrm{H}^\bullet(G;\rho,V)$, $\mathrm{H}_b^\bullet(G;\rho,V)$ and $\mathrm{H}_{/b}^\bullet(G;\rho,V)$ instead of $\mathrm{H}^\bullet(G;V)$, $\mathrm{H}_b^\bullet(G;V)$ and $\mathrm{H}_{/b}^\bullet(G;V)$, respectively. Let $\mathbb{R}$ denote the field of real numbers equipped with the trivial G-action. In this case, we write $\mathrm{H}^n(G)$, $\mathrm{H}_b^n(G)$ and $\mathrm{H}_{/b}^n(G)$ instead of $\mathrm{H}^n(G;\mathbb{R})$, $\mathrm{H}_b^n(G;\mathbb{R})$ and $\mathrm{H}_{/b}^n(G;\mathbb{R})$, respectively.

Let N be a normal subgroup of G. Then G acts on N by conjugation, and hence G acts on $C^n(N;V)$. This G-action is described by

$$({}^g f)(x_1,\ldots,x_n)=g\cdot f(g^{-1}x_1g,\ldots,g^{-1}x_ng).$$

The action induces G-actions on $\mathrm{H}^n(N;V)$, $\mathrm{H}^n_b(N;V)$ and $\mathrm{H}^n_{/b}(N;V)$. When $N=G$, these G-actions on $\mathrm{H}^n(G;V)$, $\mathrm{H}^n_b(G;V)$ and $\mathrm{H}^n_{/b}(G;V)$ are trivial. Indeed, for $h\in G$, the conjugation by h on $C^\bullet(G;V)$ is homotoped to the identity map by the chain homotopy $\Phi_h\colon C^n(G;V)\to C^{n-1}(G;V)$ defined by

$$(\Phi_h(c))(g_1,\ldots,g_{n-1})=\sum_{i=0}^{n-1}(-1)^i c(g_1,\ldots,g_i,h,h^{-1}g_{i+1}h,\ldots,h^{-1}g_{n-1}h).$$

This chain homotopy induces ones between the conjugations by h and the identity maps on $C^\bullet_b(G;V)$ and $C^\bullet_{/b}(G;V)$. By definition, a cocycle $f\colon N\to V$ in $C^1_{/b}(N;V)$ defines a class of $\mathrm{H}^1_{/b}(N;V)^G$ if and only if the function ${}^g f-f\colon N\to V$ is bounded for every $g\in G$.

Until the end of Section 5, we consider the case of trivial real coefficients. Let $f\colon G\to\mathbb{R}$ be a homogeneous quasimorphism. Then f is considered as an element of $C^1(G)$, and its coboundary δf is

$$(\delta f)(x,y)=f(x)-f(xy)+f(y).$$

Since f is a quasimorphism, the coboundary δf is a bounded cocycle. Hence we obtain a map $\delta\colon \mathrm{Q}(G)\to\mathrm{H}^2_b(G)$ given by $f\mapsto[\delta f]$. Then the following lemma is well known:

Lemma 3.3 *The following sequence is exact*:

$$0\to\mathrm{H}^1(G)\to\mathrm{Q}(G)\xrightarrow{\delta}\mathrm{H}^2_b(G)\xrightarrow{c_G}\mathrm{H}^2(G).$$

Let $\varphi\colon G\to H$ be a group homomorphism. A *virtual section of* φ is a pair (Λ,x) consisting of a subgroup Λ of finite index in H and a group homomorphism $s\colon\Lambda\to G$ satisfying $\varphi(s(x))=x$ for every $x\in\Lambda$. The group homomorphism φ is said to *virtually split* if φ admits a virtual section. As mentioned at the end of the introduction, some of the authors showed the following proposition. For a further generalization of this result, see [61, Theorem 1.4].

Proposition 3.4 [60, Proposition 6.4] *If the projection $p\colon G\to\Gamma$ virtually splits, then the map $i^*\colon\mathrm{Q}(G)\to\mathrm{Q}(N)^G$ is surjective.*

In this paper, we often consider amenable groups and boundedly acyclic groups. Here we review basic properties related to them. First we collect those for amenable groups (see for example [40] for more details).

Theorem 3.5 (known results for amenable groups) (1) *Every finite group is amenable.*

(2) *Every abelian group is amenable.*

(3) *Every subgroup of an amenable group is amenable.*

(4) *Let $1\to N\to G\to\Gamma\to 1$ be an exact sequence of groups. Then G is amenable if and only if N and Γ are amenable.*

(5) *Every amenable group is boundedly k-acyclic for all $k\geq 1$.*

Secondly, we collect known results on bounded k-acyclicity for $k \geq 3$ due to various researchers; these results are not used in this paper, but it might be convenient to the reader to have some examples of boundedly 3-acyclic groups that are nonamenable. See also Remark 8.8 for one more example.

Theorem 3.6 (known results for boundedly acyclic groups) (1) [70] *Let $n \in \mathbb{N}$. Then the group $\mathrm{Homeo}_c(\mathbb{R}^n)$ of homeomorphisms on $\mathbb{R}^n$ with compact support is boundedly acyclic.*

(2) [75; 80] *For $n \geq 3$, every lattice in $\mathrm{SL}(n,\mathbb{R})$ is 3-boundedly acyclic.*

(3) [20] *Burger–Mozes groups* [24] *are 3-boundedly acyclic.*

(4) (see [82]) *Let $k \in \mathbb{N}$. Let $1 \to N \to G \to \Gamma \to 1$ be a short exact sequence of groups. Assume that N is boundedly k-acyclic. Then G is boundedly k-acyclic if and only if Γ is.*

(5) [38] *Every binate group* (see [38, Definition 3.1]) *is boundedly acyclic.*

(6) [37] *There exist continuum many nonisomorphic 5-generated nonamenable groups that are boundedly acyclic. There exists a finitely presented nonamenable group that is boundedly acyclic.*

(7) [78] *Thompson's group F is boundedly acyclic.*

(8) [78] *Let L be an arbitrary group. Let Γ be an infinite amenable group. Then the wreath product $L \wr \Gamma = \left(\bigoplus_\Gamma L\right) \rtimes \Gamma$ is boundedly acyclic.*

(9) [79] *For every integer n at least two, the identity component $\mathrm{Homeo}_0(S^n)$ of the group of orientation-preserving homeomorphisms of S^n is boundedly 3-acyclic. The group $\mathrm{Homeo}_0(S^3)$ is boundedly 4-acyclic.*

On (7), we remark that it is a major open problem whether Thompson's group F is amenable.

The seven-term exact sequence and the calculation of first cohomology mentioned below will be used in the proof of Theorem 4.11.

Theorem 3.7 (seven-term exact sequence; see [31] for example) *Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence. Then we have the exact sequence*

$$0 \to \mathrm{H}^1(\Gamma) \xrightarrow{p^*} \mathrm{H}^1(G) \xrightarrow{i^*} \mathrm{H}^1(N)^G \to \mathrm{H}^2(\Gamma) \to \mathrm{Ker}(i^*\colon \mathrm{H}^2(G) \to \mathrm{H}^2(N)) \xrightarrow{\rho} \mathrm{H}^1(\Gamma; \mathrm{H}^1(N)) \to \mathrm{H}^3(\Gamma).$$

Here $\mathrm{H}^1(N)$ is regarded as a left $\mathbb{R}[\Gamma]$-module by the Γ-action induced from the conjugation G-action on N.

Lemma 3.8 *For a left $\mathbb{R}[\mathbb{Z}]$-module V, let $\rho\colon \mathbb{Z} \to \mathrm{Aut}(V)$ be the representation. Then the first cohomology group $\mathrm{H}^1(\mathbb{Z}; V)$ is isomorphic to $V/\mathrm{Im}(\mathrm{id}_V - \rho(1))$.*

Proof By definition, the set $Z^1(\mathbb{Z}; V)$ of cocycles on $\mathbb{Z}$ with coefficients in V is equal to the set of crossed homomorphisms, that is,

$$\{h\colon \mathbb{Z} \to V \mid h(n+m) = \rho(n)(h(m)) + h(n) \text{ for every } n, m \in \mathbb{Z}\}.$$

Since every crossed homomorphism on $\mathbb{Z}$ is determined by its value on $1 \in \mathbb{Z}$, we have $Z^1(\mathbb{Z}; V) \cong V$. The set $B^1(\mathbb{Z}; V)$ of coboundaries on $\mathbb{Z}$ with coefficients in V is equal to

$$\{h\colon \mathbb{Z} \to V \mid h(1) = v - \rho(1)(v) \text{ for some } v \in V\}.$$

Hence, $B^1(\mathbb{Z}; V) \cong \mathrm{Im}(\mathrm{id}_V - \rho(1))$, and the lemma follows. □

4 The spaces of nonextendable quasimorphisms

The purpose of this section is to provide several applications of our main theorem (Theorem 1.5) to the spaces $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$. In Section 4.1 we prove Theorems 1.9 and 1.10 modulo Theorem 1.5, and in Sections 4.2–4.4 we provide several examples of pairs (G, N) such that the space $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ does not vanish (Theorems 1.1, 1.2 and 4.18).

Here we restate Theorems 1.9 and 1.10 for the convenience of the reader.

Theorem 1.9 *If the quotient group $\Gamma = G/N$ is boundedly 3-acyclic, then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2(\Gamma).$$

Moreover, if G is Gromov-hyperbolic, then

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma).$$

Theorem 1.10 *If $\Gamma = G/N$ is boundedly 3-acyclic, then the map $p^* \circ (\xi_4)^{-1} \circ \tau_{/b}$ induces an isomorphism*

$$\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) \cong \mathrm{Im}(p^*) \cap \mathrm{Im}(c_G),$$

where $c_G\colon \mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is the comparison map. In particular, if Γ is boundedly 3-acyclic, then

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) \le \dim \mathrm{H}^2(G).$$

4.1 Proofs of Theorems 1.9 and 1.10

The goal of this section is to prove Theorems 1.9 and 1.10 modulo Theorem 1.5.

First we prove Theorem 1.9. Recall that, if G is Gromov-hyperbolic, then the comparison map $\mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is surjective [73; 46; 84]. Hence, Theorem 1.9 follows from the following:

Theorem 4.1 *Let $1 \to N \to G \to \Gamma \to 1$ be an exact sequence of groups. Assume that Γ is boundedly 3-acyclic. Then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2(\Gamma).$$

Moreover, if the comparison map $c_G\colon \mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is surjective, then

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma).$$

Proof By Theorem 1.5, we have the exact sequence

$$\mathrm{Q}(G) \xrightarrow{i^*} \mathrm{Q}(N)^G \xrightarrow{\tau_{/b}} \mathrm{H}^2_{/b}(\Gamma).$$

Hence,

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2_{/b}(\Gamma).$$

Since Γ is boundedly 3-acyclic, the map $\xi_4\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2_{/b}(\Gamma)$ is an isomorphism by (3-1), and therefore

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2(\Gamma).$$

Next we show the latter assertion. Suppose that the comparison map $c_G\colon \mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is surjective. Then $\xi_5\colon \mathrm{H}^2(G) \to \mathrm{H}^2_{/b}(G)$ is the zero map. Since $\xi_4\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2_{/b}(\Gamma)$ is an isomorphism, the map $p^*\colon \mathrm{H}^2_{/b}(\Gamma) \to \mathrm{H}^2_{/b}(G)$ is also zero. Hence,

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2_{/b}(\Gamma) = \dim \mathrm{H}^2(\Gamma).$$ □

To prove Theorem 1.10, we use the following lemma in homological algebra:

Lemma 4.2 *For a commutative diagram of $\mathbb{R}$-vector spaces*

$$\begin{array}{ccccccc}
 & & & & & & C \\
 & & & & & & \big\downarrow{\scriptstyle c} \\
 & & B_2 & \xrightarrow{b_2} & B_3 & \xrightarrow{b_3} & B_4 \\
 & & \big\downarrow{\scriptstyle c_2} & & \cong\big\downarrow{\scriptstyle c_3} & & \big\downarrow{\scriptstyle c_4} \\
A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \xrightarrow{a_3} & A_4
\end{array}$$

where the rows and the last column are exact and c_3 is an isomorphism, the map $b_3 \circ c_3^{-1} \circ a_2$ induces an isomorphism

$$A_2/(\mathrm{Im}(a_1) + \mathrm{Im}(c_2)) \cong \mathrm{Im}(b_3) \cap \mathrm{Im}(c).$$

Because the proof of Lemma 4.2 is done by a standard diagram chase, we omit it.

Proof of Theorem 1.10 If $\Gamma = G/N$ is boundedly 3-acyclic, $\xi_4\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2_{/b}(\Gamma)$ is an isomorphism. Therefore, Theorem 1.10 follows by applying Lemma 4.2 to the commutative diagram (1-3). □

The following corollary of Theorem 1.10 will be used in the proof of Theorem 1.1:

Corollary 4.3 *Assume that N is contained in the commutator subgroup $[G, G]$ of G, and Γ is boundedly 3-acyclic. Then*

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) \le \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G.$$

Moreover, if the comparison map $\mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is surjective, then

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) = \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G.$$

Proof Since N is contained in the commutator subgroup of G, the map $i^*\colon \mathrm{H}^1(G) \to \mathrm{H}^1(N)^G$ is zero and hence $\dim \mathrm{Im}(p^*) = \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G$. Therefore, Theorem 1.10 implies the corollary. □

Remark 4.4 Let $1 \to N \to G \to \Gamma \to 1$ be an exact sequence and suppose that the group N is amenable. Then it is known that the map $\xi_3 \colon \mathrm{H}^1(N)^G \to \mathrm{Q}(N)^G$ in (1-3) is an isomorphism. Hence, Lemma 4.2 implies that the composite $\tau \circ \xi_3^{-1} \circ i^*$ induces an isomorphism

$$\mathrm{Q}(G)/(\mathrm{H}^1(G) + p^*\mathrm{Q}(\Gamma)) \cong \mathrm{Im}(\tau) \cap \mathrm{Im}(c_\Gamma).$$

This isomorphism was obtained in [62] in a different way and applied to study boundedness of characteristic classes of foliated bundles.

4.2 Proof of Theorem 1.1

The goal of this subsection is to prove Theorem 1.1 by using the results proved in the previous subsection. This theorem treats surface groups. Before proceeding to this case, we first prove the following theorem for free groups:

Theorem 4.5 (computations of dimensions for free groups) *For $n \geq 1$, set $G = F_n$ and $N = [F_n, F_n]$. Then*

$$\dim(\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)) = \tfrac{1}{2}n(n-1) \quad \textit{and} \quad \dim\big(\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) = 0.$$

Proof By Theorem 4.1,

$$\dim(\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(G/N) = \dim \mathrm{H}^2(\mathbb{Z}^n) = \tfrac{1}{2}n(n-1).$$

By Theorem 1.10, we obtain

$$\dim\big(\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) \leq \dim \mathrm{H}^2(G) = 0. \qquad \square$$

Next we show Theorem 1.1. In the proof, we need the precise description of the space $\mathrm{H}^1([F_n, F_n])^{F_n}$ of F_n-invariant homomorphisms on the commutator subgroup $[F_n, F_n]$ of the free group F_n. Throughout this subsection, we write $a_1, \ldots, a_n$ to mean the canonical basis of F_n.

Lemma 4.6 *Let i and j be integers such that $1 \leq i < j \leq n$. Then there exist F_n-invariant homomorphisms $\alpha_{i,j} \colon [F_n, F_n] \to \mathbb{R}$ such that, for $k, l \in \mathbb{Z}$ with $1 \leq k < l \leq n$,*

$$\alpha_{i,j}([a_k, a_l]) = \begin{cases} 1 & \text{if } (i,j) = (k,l), \\ 0 & \text{otherwise.} \end{cases} \tag{4-1}$$

Moreover, the $\alpha_{i,j}$ are a basis of $\mathrm{H}^1([F_n, F_n])^{F_n}$. In particular,

$$\dim \mathrm{H}^1([F_n, F_n])^{F_n} = \tfrac{1}{2}n(n-1).$$

Proof When $G = F_n$ and $N = [F_n, F_n]$, the five-term exact sequence (Theorem 1.4) implies that the dimension of $\mathrm{H}^1([F_n, F_n])^{F_n}$ is $\frac{1}{2}n(n-1)$. Hence it suffices to construct $\alpha_{i,j}$ satisfying (4-1).

We first consider the case $n = 2$. Since $\dim(\mathrm{H}^1([F_2, F_2])^{F_2}) = 1$, it suffices to show that there exists an F_2-invariant homomorphism $\alpha \colon [F_2, F_2] \to \mathbb{R}$ with $\alpha([a_1, a_2]) \neq 0$. Let $\varphi \colon [F_2, F_2] \to \mathbb{R}$ be a nontrivial F_2-invariant homomorphism. Then there exists a pair x and y of elements of F_2 such that $\varphi([x, y]) \neq 0$.

Let $f: F_2 \to F_2$ be the group homomorphism sending a_1 to x and a_2 to y. Since φ is F_2-invariant, we have

$$\varphi \circ f(gxg^{-1}) = \varphi(f(g)f(x)f(g)^{-1}) = \varphi \circ f(x)$$

for every $g \in F_2$ and every $x \in [F_2, F_2]$. Hence, $\varphi \circ (f|_{[F_2,F_2]}): [F_2, F_2] \to \mathbb{R}$ is an F_2-invariant homomorphism satisfying $\varphi \circ f([a_1, a_2]) \neq 0$. This completes the proof of the case $n = 2$.

Suppose that $n \geq 2$. Then, for $i, j \in \{1, \dots, n\}$ with $i < j$, define a homomorphism $q_{i,j}: F_n \to F_2$ which sends a_i to a_1, a_j to a_2, and a_k to the unit element of F_2 for $k \neq i, j$. Then $q_{i,j}$ induces a surjection $[F_n, F_n]$ to $[F_2, F_2]$, and induces a homomorphism $q_{i,j}^*: \mathrm{H}^1([F_2, F_2])^{F_2} \to \mathrm{H}^1([F_n, F_n])^{F_n}$. Set $\alpha_{i,j} = \alpha_{1,2} \circ q_{i,j}$. Then $\alpha_{i,j}$ clearly satisfies (4-1), and this completes the proof. □

Theorem 1.1 follows from Corollary 4.3 and the following proposition:

Proposition 4.7 *For $l \geq 1$,*

$$\dim \mathrm{H}^1\big([\pi_1(\Sigma_l), \pi_1(\Sigma_l)]\big)^{\pi_1(\Sigma_l)} = l(2l-1) - 1.$$

Proof Recall that $\pi_1(\Sigma_l)$ has the presentation

$$\langle a_1, \dots, a_{2l} \mid [a_1, a_2] \cdots [a_{2l-1}, a_{2l}] \rangle.$$

Let $f: F_{2l} \to \pi_1(\Sigma_l)$ be the natural epimorphism sending a_i to a_i, and K the kernel of f, ie K is the normal subgroup generated by $[a_1, a_2] \cdots [a_{2l-1}, a_{2l}]$ in F_{2l}. Then f induces an epimorphism $f|_{[F_{2l}, F_{2l}]}: [F_{2l}, F_{2l}] \to [\pi_1(\Sigma_l), \pi_1(\Sigma_l)]$ between their commutator subgroups, and its kernel coincides with K since K is contained in $[F_{2l}, F_{2l}]$. This means that, for a homomorphism $\varphi: [F_{2l}, F_{2l}] \to \mathbb{R}$, φ induces a homomorphism $\bar{\varphi}: [\pi_1(\Sigma_l), \pi_1(\Sigma_l)] \to \mathbb{R}$ if and only if

$$\varphi([a_1, a_2] \cdots [a_{2l-1}, a_{2l}]) = 0.$$

It is straightforward to show that φ is F_{2l}-invariant if and only if $\bar{\varphi}$ is $\pi_1(\Sigma_l)$-invariant. Hence the image of the monomorphism $\mathrm{H}^1\big([\pi_1(\Sigma_l), \pi_1(\Sigma_l)]\big)^{\pi_1(\Sigma_l)} \to \mathrm{H}^1([F_{2l}, F_{2l}])^{F_{2l}}$ is the subspace consisting of elements

$$\sum_{i<j} k_{ij} \alpha_{ij}$$

such that

$$k_{1,2} + k_{3,4} + \cdots + k_{2l-1,2l} = 0.$$

Since the dimension of $\mathrm{H}^1([F_{2l}, F_{2l}])^{F_{2l}}$ is $l(2l-1)$ (see Lemma 4.6), this completes the proof. □

Proof of Theorem 1.1 Since the abelianization $\Gamma = \pi_1(\Sigma_l)/[\pi_1(\Sigma_l), \pi_1(\Sigma_l)]$ of the surface group is isomorphic to $\mathbb{Z}^{2l}$, we have $\dim \mathrm{H}^2(\Gamma) = l(2l-1)$. Thus the first assertion follows from Theorem 4.1. Since the comparison map $\mathrm{H}_b^2(\pi_1(\Sigma_l)) \to \mathrm{H}^2(\pi_1(\Sigma_l))$ is surjective, we obtain

$$\dim\big(\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) = 1$$

by Corollary 4.3 and Proposition 4.7. □

4.3 Proof of Theorem 1.2 and a related example

To prove Theorem 1.2, we now recall some terminology of mapping class groups.

Let l be an integer at least 2 and Σ_l the oriented closed surface with genus l. The *mapping class group* $\mathrm{Mod}(\Sigma_l)$ *of* Σ_l is the group of isotopy classes of orientation-preserving diffeomorphisms on Σ_l. By considering the action on the first homology group, $\mathrm{Mod}(\Sigma_l)$ has a natural epimorphism $s_l\colon \mathrm{Mod}(\Sigma_l) \to \mathrm{Sp}(2l;\mathbb{Z})$, called the *symplectic representation*.

For $\psi \in \mathrm{Mod}(\Sigma_l)$, we take a diffeomorphism f that represents ψ. The mapping torus T_f is an orientable closed 3-manifold equipped with a natural fibration structure $\Sigma_l \to T_f \to S^1$. The following is known:

Theorem 4.8 [93] *A mapping class ψ is a pseudo-Anosov element if and only if the mapping torus T_f is a hyperbolic manifold.*

Set $\Gamma = \mathbb{Z}^{2l} \rtimes_{s_l(\psi)} \mathbb{Z}$ and

(4-2) $$\begin{aligned} G &= \pi_1(T_f) = \pi_1(\Sigma_l) \rtimes_{f_*} \mathbb{Z} \\ &= \langle a_1, \dots, a_{2l+1} \mid [a_1,a_2]\cdots[a_{2l-1},a_{2l}] = 1_G,\ a_{2l+1}\cdot a_i = (f_* a_i)\cdot a_{2l+1} \text{ for every } 1 \le i \le 2l \rangle, \end{aligned}$$

where $f_*\colon \pi_1(\Sigma_l) \to \pi_1(\Sigma_l)$ is the pushforward of f.

Lemma 4.9 (1) $\dim \mathrm{H}^2(\Gamma) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + \dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \bigwedge^2 s_l(\psi)\big)$.

(2) $\dim \mathrm{H}^2(G) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + 1$.

Proof Let T^{2l} be the $2l$-dimensional torus. By the natural inclusion $\mathrm{Sp}(2l;\mathbb{Z}) \to \mathrm{Homeo}(T^{2l})$, we regard the element $s_l(\psi)$ as a homeomorphism of T^{2l}. Let $M_{s_l(\psi)}$ be the mapping torus of $s_l(\psi) \in \mathrm{Homeo}(T^{2l})$. Since $M_{s_l(\psi)}$ is a $K(\Gamma, 1)$-manifold, we have $\dim \mathrm{H}^2(\Gamma) = \dim \mathrm{H}^2(M_{s_l(\psi)})$.

For the mapping torus $M_{s_l(\psi)}$, we have the long exact sequence

(4-3) $$\cdots \to \mathrm{H}^1(T^{2l}) \xrightarrow{\delta_1} \mathrm{H}^1(T^{2l}) \to \mathrm{H}^2(M_{s_l(\psi)}) \to \mathrm{H}^2(T^{2l}) \xrightarrow{\delta_2} \mathrm{H}^2(T^{2l}) \to \cdots,$$

where the map δ_n is given by

$$\mathrm{id}_{\mathrm{H}^n(T^{2l})} - s_l(\psi)^*\colon \mathrm{H}^n(T^{2l}) \to \mathrm{H}^n(T^{2l})$$

(see [48, Example 2.48]). This, together with (4-3) and the fact that $H^2(T^{2l}) \cong \bigwedge^2 H^1(T^{2l})$, implies that

$$\dim \mathrm{H}^2(\Gamma) = \dim \mathrm{H}^2(M_{s_l(\psi)}) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + \dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \textstyle\bigwedge^2 s_l(\psi)\big).$$

The computation of $\dim \mathrm{H}^2(G)$ is done in a similar manner. □

Let N be the kernel of the natural epimorphism $G \to \Gamma$. Note that N is isomorphic to the commutator subgroup of $\pi_1(\Sigma_l)$.

Lemma 4.10 $$\dim \mathrm{H}^1(N)^G \le \dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \textstyle\bigwedge^2 s_l(\psi)\big) - 1.$$

Proof We set $\mathrm{H} = \mathrm{H}_1(\Sigma_l;\mathbb{Z})$. Let $\iota\colon \mathrm{H}^1(N)^G \to \mathrm{Hom}\big(\bigwedge^2 \mathrm{H}, \mathbb{R}\big)$ be the map defined by

$$\iota(h)(q(x)\wedge q(y)) = h([x,y]),$$

where $x, y \in \pi_1(\Sigma_l)$ and $q\colon \pi_1(\Sigma_l) \to \mathrm{H}$ is the abelianization map. We claim that this map ι is well defined. To verify this, let $h \in \mathrm{H}^1(N)^G$. By commutator calculus, $[x_1x_2, y] = x_1[x_2,y]x_1^{-1}\cdot[x_1,y]$ for every $x_1, x_2, y \in \pi_1(\Sigma_l)$. Since h is G-invariant, this implies that

$$h([x_1x_2,y]) = h([x_1,y]) + h([x_2,y]).$$

In a similar manner to the above, we can see that

$$h([xz, yw]) = h([x,y])$$

for every $x, y \in \pi_1(\Sigma_l)$ and every $z, w \in N = [\pi_1(\Sigma_l), \pi_1(\Sigma_l)]$. Now it is straightforward to confirm that ι is well defined. Moreover, since N is normally generated by $\{[a_i,a_j]\}_{1\le i<j\le 2l}$ in G, the map ι is injective.

We set

$$\mathrm{Hom}\big(\textstyle\bigwedge^2 \mathrm{H},\mathbb{R}\big)^{\bigwedge^2 s_l(\psi)} = \big\{h \in \mathrm{Hom}\big(\bigwedge^2 \mathrm{H},\mathbb{R}\big) \mid h\circ \bigwedge^2 s_l(\psi) = h\big\}.$$

Then the image of ι is contained in $\mathrm{Hom}\big(\bigwedge^2 \mathrm{H},\mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$. Indeed, for $1 \le i < j \le 2l$ and $h \in \mathrm{H}^1(N)^G$,

$$\begin{aligned}\iota(h)\big(\textstyle\bigwedge^2 s_l(\psi)(q(a_i)\wedge q(a_j))\big) &= h([f_*a_i, f_*a_j])\\ &= h([a_{2l+1}\cdot a_i\cdot a_{2l+1}^{-1}, a_{2l+1}\cdot a_j\cdot a_{2l+1}^{-1}])\\ &= h([a_i,a_j]) = \iota(h)(q(a_i)\wedge q(a_j)),\end{aligned}$$

where the second equality comes from the relation in (4-2) and the third equality comes from the G-invariance of h.

Since $\dim \mathrm{Hom}\big(\bigwedge^2 \mathrm{H},\mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$ is equal to $\dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \bigwedge^2 s_l(\psi)\big)$, it suffices to show that the map

$$\iota\colon \mathrm{H}^1(N)^G \to \mathrm{Hom}\big(\textstyle\bigwedge^2 \mathrm{H},\mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$$

is not surjective. We set $v_1 = q(a_1)\wedge q(a_2) + \cdots + q(a_{2l-1})\wedge q(a_{2l}) \in \bigwedge^2 \mathrm{H}$. Then the map $\bigwedge^2 s_l(\psi)\colon \bigwedge^2 \mathrm{H} \to \bigwedge^2 \mathrm{H}$ preserves v_1. Hence, for a suitable basis containing v_1, the dual v_1^* is contained in $\mathrm{Hom}\big(\bigwedge^2 \mathrm{H},\mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$. However, v_1^* is not contained in the image of ι. Indeed, for every $h \in \mathrm{H}^1(N)^G$,

$$\iota(h)(v_1) = h([a_1,a_2]\cdots[a_{2l-1},a_{2l}]) = 0.$$

Hence the map $\iota\colon \mathrm{H}^1(N)^G \to \mathrm{Hom}\big(\bigwedge^2 \mathrm{H},\mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$ is not surjective, and the lemma follows. □

Proof of Theorem 1.2 The group G is Gromov-hyperbolic by Theorem 4.8 and Γ is amenable by Theorem 3.5(4). Hence, Theorem 1.9, together with Lemma 4.9(1), asserts that

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + \dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \textstyle\bigwedge^2 s_l(\psi)\big).$$

By Theorem 1.10 and Lemma 4.9(2), we obtain

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) \le \dim \mathrm{H}^2(G) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + 1.$$

On the other hand,

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) = \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G \ge \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + 1$$

by Corollary 4.3 and Lemmas 4.9(1) and 4.10. □

As we mentioned in the introduction, we obtain an analogue (Theorem 4.11) of Theorem 1.2 in the free group setting. For $n \in \mathbb{N}$, let $\mathrm{Aut}(F_n)$ be the automorphism group of F_n. Let $t_n\colon \mathrm{Aut}(F_n) \to \mathrm{GL}(n,\mathbb{Z})$ be the representation induced by the action of $\mathrm{Aut}(F_n)$ on the abelianization of F_n. Then the group $F_n \rtimes_\psi \mathbb{Z}$ naturally surjects onto $\mathbb{Z}^n \rtimes_{t_n(\psi)} \mathbb{Z}$ via the abelianization of F_n. We say that an automorphism ψ of F_n is *atoroidal* if it has no periodic conjugacy classes, that is, there does not exist a pair $(a,k) \in F_n \times \mathbb{Z}$ with $a \neq 1_{F_n}$ and $k \neq 0$ such that $\psi^k(a)$ is conjugate to a. Bestvina and Feighn [11] showed that $\psi \in \mathrm{Aut}(F_n)$ is atoroidal if and only if $F_n \rtimes_\psi \mathbb{Z}$ is Gromov-hyperbolic.

Theorem 4.11 (computations of dimensions for free-by-cyclic groups) *Let n be an integer greater than 1 and $\psi \in \mathrm{Aut}(F_n)$ an atoroidal automorphism. Set $G = F_n \rtimes_\psi \mathbb{Z}$ and let N be the kernel of the surjection $G \to \mathbb{Z}^n \rtimes_{t_n(\psi)} \mathbb{Z}$ defined via the abelianization map $F_n \to \mathbb{Z}^n$. Then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{Ker}(I_n - t_n(\psi)) + \dim \mathrm{Ker}\Bigl(I_{\binom{n}{2}} - \textstyle\bigwedge^2 t_n(\psi)\Bigr)$$

and

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) = \dim \mathrm{Ker}(I_n - t_n(\psi)),$$

where $\bigwedge^2 t_n(\psi)$ is the map induced by $t_n(\psi)$.

Proof Since G/N is isomorphic to $\Gamma = \mathbb{Z}^n \rtimes_{t_n(\psi)} \mathbb{Z}$, Theorem 1.9 implies that $\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma)$. Moreover,

$$\dim \mathrm{H}^2(\Gamma) = \dim \mathrm{Ker}(I_n - t_n(\psi)) + \dim \mathrm{Ker}\Bigl(I_{\binom{n}{2}} - \textstyle\bigwedge^2 t_n(\psi)\Bigr)$$

by the same argument as in the proof of Lemma 4.9(1). Hence the former statement holds.

We set $\mathrm{H} = \mathrm{H}_1(F_n;\mathbb{Z}) = F_n/[F_n,F_n]$. As in Lemma 4.10, we can define a monomorphism $\iota\colon \mathrm{H}^1(N)^G \to \mathrm{Hom}\bigl(\bigwedge \mathrm{H}, \mathbb{R}\bigr)^{\bigwedge^2 t_n(\psi)}$. Hence, together with Corollary 4.3, we obtain

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) \ge \dim \mathrm{Ker}(I_n - t_n(\psi)).$$

On the other hand,

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) \le \dim \mathrm{H}^2(G)$$

by Theorem 1.10. By the seven-term exact sequence (Theorem 3.7) applied to the short exact sequence $1 \to F_n \to G \to \mathbb{Z} \to 1$,

$$\mathrm{H}^2(G) \cong \mathrm{H}^1(\mathbb{Z};\mathrm{H}^1(F_n)).$$

By Lemma 3.8, $\mathrm{H}^1(\mathbb{Z};\mathrm{H}^1(F_n))$ is isomorphic to

$$\mathrm{H}^1(F_n)/\mathrm{Im}(\mathrm{id}_{\mathrm{H}^1(F_n)}-\psi^*),$$

where $\psi^*\colon \mathrm{H}^1(F_n)\to\mathrm{H}^1(F_n)$ is the pullback of ψ. Hence,

$$\dim\mathrm{H}^2(G)=\dim\mathrm{H}^1(\mathbb{Z};\mathrm{H}^1(F_n))=\dim\mathrm{Ker}(I_n-t_n(\psi))$$

and the theorem follows. □

4.4 Other examples

It follows from Theorem 1.10 that $\mathrm{H}^2(G)=0$ implies $\mathrm{Q}(N)^G=\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G)$, and we provide several examples of groups G with $\mathrm{H}^2(G)=0$ in Section 2.1.

As an application of [42, Theorem 2.4], we provide another example of a group G satisfying $\mathrm{Q}(N)^G=\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G)$.

Corollary 4.12 *Let L be a hyperbolic link in S^3 such that the number of the connected components of L is two. Let G be the link group of L (ie the fundamental group of the complement $S^3\setminus L$ of L) and N the commutator subgroup of G. Then $\mathrm{Q}(N)^G=\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G)$.*

Proof By Theorem 1.10, it suffices to show that the comparison map $c_G\colon \mathrm{H}^2_b(G)\to\mathrm{H}^2(G)$ is equal to zero. Theorem 2.4 of [42] gives $\mathrm{Im}(c_G)\neq\mathrm{H}^2(G)$. Since the number of the connected components of L is two, the second cohomology group $\mathrm{H}^2(G)$ is isomorphic to $\mathbb{R}$. Hence, $\mathrm{Im}(c_G)=0$. □

Here we provide other examples (G,N) such that $\mathrm{H}^2(G)\neq 0$ and $\mathrm{Q}(N)^G=\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G)$.

Example 4.13 Let $n\in\mathbb{N}$. For $i=1,2,\dots,n$, let H_i be a boundedly 2-acyclic group and assume that $\mathrm{H}^2(H_1)\neq 0$ (for example, we can take $H_1=\mathbb{Z}^2$). Set $G=H_1*H_2*\cdots*H_n$ and $N=[G,G]$. Then $\mathrm{H}^2(G)=\mathrm{H}^2(H_1)\oplus\mathrm{H}^2(H_2)\oplus\cdots\oplus\mathrm{H}^2(H_n)\neq 0$ but the comparison map $c_G\colon\mathrm{H}^2_b(G)\to\mathrm{H}^2(G)$ is the zero map. It follows from Theorem 1.10 that $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))=0$.

We considered free products in Example 4.13, but the comparison map in degree 2 is also trivial for graph products of amenable groups (such as RAAGs) and graphs of groups with amenable vertex groups (see [66, Example 4.7]). See [28, Corollary 5.4] for more cases in which the comparison map in degree 2 is trivial.

Corollary 4.14 *Let $E\to\Sigma_l$ be a nontrivial orientable circle bundle over a closed oriented surface of genus $l>1$. For the fundamental group $G=\pi_1(E)$ and its normal subgroup $N=[G,G]$,*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G))=l(2l-1)\quad\text{and}\quad\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\big)=0.$$

Remark 4.15

(1) The dimension of $\mathrm{Q}(G)$ (and hence the dimension of $\mathrm{Q}(N)^G$) is the cardinal of the continuum since G surjects onto the surface group $\pi_1(\Sigma_l)$.

(2) The dimension of $\mathrm{H}^2(G)$ is equal to $2l$. Indeed, $\mathrm{H}^2(G)$ is isomorphic to $\mathrm{H}^2(E)$ since E is a $K(G,1)$-space. Moreover, $\mathrm{H}^2(E)$ is isomorphic to $\mathrm{H}^1(\Sigma_l)$ by the Thom–Gysin sequence.

Proof of Corollary 4.14 Let n be the Euler number of the bundle $E \to \Sigma_l$. Note that n is nonzero since the bundle is nontrivial (see [40, Theorem 11.16]). Since the group G has a presentation

$$G = \pi_1(E) = \langle a_1, \dots, a_{2l+1} \mid [a_1, a_2] \cdots [a_{2l-1}, a_{2l}] = a_{2l+1}^{-n}, [a_i, a_{2l+1}] = 1_G \text{ for every } 1 \le i \le 2l \rangle,$$

the abelianization $\Gamma = G/N$ is isomorphic to $\mathbb{Z}^{2l} \times (\mathbb{Z}/n\mathbb{Z})$. Hence, $\dim \mathrm{H}^2(\Gamma) = l(2l-1)$. By the relation $[a_i, a_{2l+1}] = 1_G$ for each i and the fact that N is normally generated by $\{[a_i, a_j]\}_{1 \le i < j \le 2l+1}$ in G, we obtain $\dim \mathrm{H}^1(N)^G = l(2l+1) - 2l = l(2l-1)$ by an argument similar to the proof of Proposition 4.7. Hence Corollary 4.3 asserts that

$$\dim\big(\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) \le \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G = 0.$$

Since N is the commutator subgroup of G, the space $\mathrm{H}^1(N)^G$ injects into $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$. Hence,

$$\dim(\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)) \ge l(2l-1).$$

On the other hand, Theorem 4.1 asserts that

$$\dim(\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)) \le \mathrm{H}^2(\Gamma) = l(2l-1). \qquad \square$$

For elements $r_1, \dots, r_m \in G$, we write $\langle\langle r_1, \dots, r_m \rangle\rangle$ to mean the normal subgroup of G generated by $r_1, \dots, r_m$.

Corollary 4.16 *Let $r_1, \dots, r_m \in [F_n, [F_n, F_n]]$ and set*

$$G = F_n / \langle\langle r_1, \dots, r_m \rangle\rangle.$$

Then $\mathrm{Q}([G, G])^G = \mathrm{H}^1([G, G])^G + i^*\mathrm{Q}(G)$.

Proof Let q be the natural projection $F_n \to G$. Then the image of the monomorphism $q^* : \mathrm{H}^1([G, G])^G \to \mathrm{H}^1([F_n, F_n])^{F_n}$ is the space of F_n-invariant homomorphisms $f : [F_n, F_n] \to \mathbb{R}$ satisfying $f(r_1) = \cdots = f(r_m) = 0$. Since every F_n-invariant homomorphism of $[F_n, F_n]$ vanishes on $[F_n, [F_n, F_n]]$, we conclude that q^* is an isomorphism, and hence $\dim \mathrm{H}^1([G, G])^G = \frac{1}{2} n(n-1)$. Since $\Gamma = G/[G, G] = \mathbb{Z}^n$, we have $\dim \mathrm{H}^2(\Gamma) = \frac{1}{2} n(n-1)$. Hence Corollary 4.3 implies that $\mathrm{Q}([G, G])^G / (\mathrm{H}^1([G, G])^G + i^*\mathrm{Q}(G))$ is trivial. $\square$

Remark 4.17 Suppose that N is the commutator subgroup of G. As will be seen in Corollaries 6.20 and 7.11, the sum $\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ is actually a direct sum in this case, and the map $\mathrm{H}^1(N)^G \to \mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$ is an isomorphism. Hence, if G is a group as provided in Corollary 4.16 and N is the commutator subgroup of G, then the basis of $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$ is provided by the G-invariant homomorphism $\alpha'_{i,j} : N \to \mathbb{R}$ for $1 \le i < j \le n$, which is the homomorphism induced by $\alpha_{i,j} : [F_n, F_n] \to \mathbb{R}$ described in Lemma 4.6.

As an example of a pair (G, N) satisfying $\mathrm{Q}(N) \ne \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, we provide a certain family of one-relator groups. Recall that a *one-relator group* is a group isomorphic to $F_n / \langle\langle r \rangle\rangle$ for some positive integer n and an element r of F_n.

Theorem 4.18 *Let n and k be integers at least 2 and r an element of $[F_n, F_n] \setminus [F_n, [F_n, F_n]]$. Set $G = F_n/\langle\langle r^k \rangle\rangle$ and $N = [G, G]$. Then*

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) = 1.$$

Note that $r \in [F_n, F_n] \setminus [F_n, [F_n, F_n]]$ is equivalent to the existence of $f_0 \in \mathrm{H}^1([F_n, F_n])^{F_n}$ with $f_0(r) \neq 0$.

Proof of Theorem 4.18 By Newman's spelling theorem [85], every one-relator group with torsion is hyperbolic, and hence G is hyperbolic. Indeed, r does not belong to $\langle\langle r^k \rangle\rangle$ since $f_0(x)$ belongs to $k f_0(r)\mathbb{Z}$ for every element x of $\langle\langle r^k \rangle\rangle$. Since $\Gamma = G/N$ is abelian, Γ is boundedly 3-acyclic. By Corollary 4.3, it suffices to show

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) = \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G = 1.$$

Since $r^k \in [F_n, F_n]$, we have $\Gamma = \mathbb{Z}^n$, and $\dim \mathrm{H}^2(\Gamma) = \frac{1}{2}n(n-1)$. Hence, it only remains to show that

$$\dim \mathrm{H}^1(N)^G = \tfrac{1}{2}n(n-1) - 1. \tag{4-4}$$

Let $q\colon F_n \to G = F_n/\langle\langle r^k \rangle\rangle$ be the natural quotient. Then q induces a monomorphism $q^*\colon \mathrm{H}^1(N)^G \to \mathrm{H}^1([F_n, F_n])^{F_n}$. As in the proof of Proposition 4.7, it is straightforward to show that the image of $q^*\colon \mathrm{H}^1(N)^G \to \mathrm{H}^1([F_n, F_n])^{F_n}$ is the space of F_n-invariant homomorphisms $f\colon [F_n, F_n] \to \mathbb{R}$ such that $f(r) = 0$. Since there exists an element f_0 of $\mathrm{H}^1([F_n, F_n])^{F_n}$ with $f_0(r) \neq 0$, the codimension of the image of $q^*\colon \mathrm{H}^1(N)^G \to \mathrm{H}^1([F_n, F_n])^{F_n}$ is 1, which implies (4-4). □

After the authors completed this work, they obtained a generalization of Theorem 4.18; see [59, Theorem 11.15].

Remark 4.19 Let k be a positive integer. Here we construct a finitely presented group G satisfying

$$\dim\bigl(\mathrm{Q}([G, G])^G/(\mathrm{H}^1([G, G])^G + i^*\mathrm{Q}(G))\bigr) = k. \tag{4-5}$$

Let $F_{2k} = \langle a_1, \dots, a_{2k} \rangle$ be a free group and define G by

$$G = \langle a_1, \dots, a_{2k} \mid [a_1, a_2]^2, \dots, [a_{2k-1}, a_{2k}]^2 \rangle.$$

Set $H = \langle a_1, a_2 \mid [a_1, a_2]^2 \rangle$. Then G is the k-fold free product of H. Since H is a one-relator group with torsion, H is hyperbolic. Since a finite free product of hyperbolic groups is hyperbolic, G is hyperbolic. Hence the comparison map $\mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is surjective.

Let $q\colon F_{2k} \to G$ be the natural quotient. Then $q^*\colon \mathrm{H}^1([G, G])^G \to \mathrm{H}^1([F_{2k}, F_{2k}])^{F_{2k}}$ is a monomorphism whose image comprises the F_{2k}-invariant homomorphisms $\varphi\colon [F_{2k}, F_{2k}] \to \mathbb{R}$ such that $\varphi([a_{2i-1}, a_{2i}]) = 0$ for $i = 1, \dots, k$. Therefore Corollary 4.3 implies (4-5).

5 Cohomology classes induced by the flux homomorphism

First we review the definition of the (volume) flux homomorphism (see for instance [5]).

Let $\mathrm{Diff}(M,\Omega)$ denote the group of diffeomorphisms on an m-dimensional smooth manifold M which preserve a volume form Ω on M, $\mathrm{Diff}_0(M,\Omega)$ the identity component of $\mathrm{Diff}(M,\Omega)$, and $\widetilde{\mathrm{Diff}}_0(M,\Omega)$ the universal cover of $\mathrm{Diff}_0(M,\Omega)$. Then the (*volume*) *flux homomorphism* $\widetilde{\mathrm{Flux}}_\Omega\colon \widetilde{\mathrm{Diff}}_0(M,\Omega) \to \mathrm{H}^{m-1}(M)$ is defined by

$$\widetilde{\mathrm{Flux}}_\Omega([\{\psi^t\}_{t\in[0,1]}]) = \int_0^1 [\iota_{X_t}\Omega]\,dt,$$

where ι is the inner product, $\{\psi^t\}_{t\in[0,1]}$ is a path representing an element of $\widetilde{\mathrm{Diff}}_0(M,\Omega)$, and X_t is the time-dependent vector field generating the isotopy $\{\psi^t\}_{t\in[0,1]}$. The value $\widetilde{\mathrm{Flux}}_\Omega([\{\psi^t\}_{t\in[0,1]}])$ does not depend on the choice of the isotopy $\{\psi^t\}_{t\in[0,1]}$ and thus the map $\widetilde{\mathrm{Flux}}_\Omega$ is well defined. Moreover, $\widetilde{\mathrm{Flux}}_\Omega$ is a homomorphism. The image of $\pi_1(\mathrm{Diff}_0(M,\Omega))$ under $\widetilde{\mathrm{Flux}}_\Omega$ is called the *flux group* of the pair (M,Ω) and denoted by Γ_Ω. The flux homomorphism $\widetilde{\mathrm{Flux}}_\Omega$ descends to a homomorphism

$$\mathrm{Flux}_\Omega\colon \mathrm{Diff}_0(M,\Omega) \to \mathrm{H}^{m-1}(M)/\,\Gamma_\Omega.$$

These homomorphisms are fundamental objects in theory of diffeomorphism groups and have been extensively studied (see for example [63; 52]).

As we wrote in Section 2.3, Proposition 2.5 is essentially in [65]; we prove it for the reader's convenience.

Proof of Proposition 2.5 Suppose that the pair (G,N) of groups is $(\mathrm{Diff}_0(M,\Omega), \mathrm{Ker}(\mathrm{Flux}_\Omega))$ or $(\widetilde{\mathrm{Diff}}_0(M,\Omega), \mathrm{Ker}(\widetilde{\mathrm{Flux}}_\Omega))$. Since the kernels of the homomorphisms Flux_Ω and $\widetilde{\mathrm{Flux}}_\Omega$ are perfect (see [92; 4] and also [5, Theorems 4.3.1 and 5.1.3]), we have $\mathrm{H}^1(N)=0$. Hence this proposition follows from the five-term exact sequence (Theorem 1.4). □

To prove Theorem 2.6(1), we use Py's Calabi quasimorphism $f_P\colon \mathrm{Ker}(\mathrm{Flux}_\Omega)\to\mathbb{R}$, which was introduced in [89]. For an oriented closed surface whose genus l is at least 2 and a volume form Ω on M, Py constructed a $\mathrm{Diff}_0(M,\Omega)$-invariant homogeneous quasimorphism $f_P\colon \mathrm{Ker}(\mathrm{Flux}_\Omega)\to\mathbb{R}$ on $\mathrm{Ker}(\mathrm{Flux}_\Omega)$.

Proof of Theorem 2.6 First we prove (1). Suppose that Σ_l is an oriented closed surface whose genus l is at least 2, and let Ω be its volume form. Since in this case Γ_Ω is trivial (as mentioned just after Theorem 2.6), the two flux homomorphisms Flux_Ω and $\widetilde{\mathrm{Flux}}_\Omega$ coincide.

Set $G=\mathrm{Diff}_0(\Sigma_l,\Omega)$ and $N=\mathrm{Ker}(\mathrm{Flux}_\Omega)$. Since N is perfect [4, théorème II.6.1], we have $\mathrm{H}^1(N)=\mathrm{H}^1(N)^G=0$. Since G/N is abelian, Theorem 1.10 implies that

$$\mathrm{Q}(N)^G/\,i^*\mathrm{Q}(G)=\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\cong \mathrm{Im}(\mathrm{Flux}_\Omega^*)\cap\mathrm{Im}(c_G).$$

Since Py's Calabi quasimorphism f_P is not extendable to $G=\mathrm{Diff}_0(\Sigma_l,\omega)$ [58, Theorem 1.11], $\mathrm{Q}(N)^G/\,i^*\mathrm{Q}(G)$ is not trivial. Hence, $\mathrm{Flux}_\Omega^*\circ\xi_4^{-1}\circ\tau_{/b}([f_P])\in\mathrm{Im}(\mathrm{Flux}_\omega^*)\cap\mathrm{Im}(c_G)$ is nonzero.

Now we show (2). Suppose that $m=2$. The case that M is a 2-sphere is clear since $\mathrm{H}^1(M)=0$ and hence the flux homomorphisms are trivial. The case of M a torus follows from the fact that both Flux_Ω and $\widetilde{\mathrm{Flux}}_\Omega$ have section homomorphisms: hence, by Proposition 3.4, $\mathrm{Im}(\mathrm{Flux}_\Omega^*)\cap\mathrm{Im}(c_G)\cong\mathrm{Q}(N)^G/\,i^*\mathrm{Q}(G)=0$.

Suppose that $m\geq 3$. Then Proposition 5.1 below implies that Flux_Ω has a section homomorphism. Hence, by Proposition 3.4, $\mathrm{Im}(\mathrm{Flux}_\Omega^*)\cap\mathrm{Im}(c_G)\cong\mathrm{Q}(N)^G/\,i^*\mathrm{Q}(G)=0$. □

Proposition 5.1 [35, Proposition 6.1] *Let m be an integer at least 3, M an m-dimensional differential manifold, and Ω a volume form on M. Then there exists a section homomorphism of the reduced flux homomorphism $\mathrm{Flux}_\Omega\colon \mathrm{Diff}_0(M,\Omega) \to \mathrm{H}^{m-1}(M,\Omega)/\Gamma_\Omega$. In addition, there exists a section homomorphism of $\widetilde{\mathrm{Flux}}_\Omega\colon \widetilde{\mathrm{Diff}}_0(M,\Omega) \to \mathrm{H}^{m-1}(M,\Omega)$.*

The idea of Theorem 2.6 is also useful in (higher-dimensional) symplectic geometry. For notions in symplectic geometry, see for example [5; 88]. For a symplectic manifold (M,ω), let $\mathrm{Ham}(M,\omega)$ denote the group of Hamiltonian diffeomorphisms with compact support. For an exact symplectic manifold (M,ω), let $\mathrm{Cal}_\omega\colon \mathrm{Ham}(M,\omega) \to \mathbb{R}$ denote the Calabi homomorphism. We note that the map Cal_ω^* is injective, where $\mathrm{Cal}_\omega^*\colon \mathrm{H}^2(\mathbb{R};\mathbb{R}) \to \mathrm{H}^2(\mathrm{Ham}(M,\omega);\mathbb{R})$ is the homomorphism induced by Cal_ω. Here $\mathrm{H}^2(\mathbb{R};\mathbb{R})$ denotes the group cohomology of $\mathbb{R}$ (as a discrete group) with trivial real coefficients and is isomorphic to $\mathrm{Hom}_\mathbb{Z}\left(\bigwedge_\mathbb{Z}^2(\mathbb{R});\mathbb{R}\right)$ due to the discussion before Proposition 2.5. Indeed, because $\mathrm{Ker}(\mathrm{Cal}_\omega)$ is perfect [4], we can prove the injectivity of Cal_ω^* similarly to the proof of Proposition 2.5. Then we have the following theorem:

Theorem 5.2 *For an exact symplectic manifold (M,ω), every nontrivial element of $\mathrm{Im}(\mathrm{Cal}_\omega^*)$ cannot be represented by a bounded 2-cochain.*

Note that $\mathrm{Cal}_\omega\colon \mathrm{Ham}(M,\omega) \to \mathbb{R}$ has a section homomorphism. Indeed, for a (time-independent) Hamiltonian function whose integral over M is 1 and its Hamiltonian flow $\{\phi^t\}_{t\in\mathbb{R}}$, the homomorphism $t \mapsto \phi^t$ is a section of the Calabi homomorphism Cal_ω. Hence, the proof of Theorem 5.2 is similar to Theorem 2.6.

6 Proof of the main theorem

The goal in this section is to prove Theorem 1.5, which is the five-term exact sequence of the cohomology of groups relative to the bounded subcomplex; we restate it for the convenience of the reader:

Theorem 1.5 (main theorem) *Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence of groups and V a left Banach $\mathbb{R}[\Gamma]$-module equipped with a Γ-invariant norm $\|\cdot\|$. Then there exists an exact sequence*

$$0 \to \mathrm{H}^1_{/b}(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^1_{/b}(G;V) \xrightarrow{i^*} \mathrm{H}^1_{/b}(N;V)^G \xrightarrow{\tau_{/b}} \mathrm{H}^2_{/b}(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^2_{/b}(G;V).$$

Moreover, the exact sequence above is compatible with the five-term exact sequence of group cohomology; that is, the following diagram commutes:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{H}^1(\Gamma;V) & \xrightarrow{p^*} & \mathrm{H}^1(G;V) & \xrightarrow{i^*} & \mathrm{H}^1(N;V)^G & \xrightarrow{\tau} & \mathrm{H}^2(\Gamma;V) & \xrightarrow{p^*} & \mathrm{H}^2(G;V) \\
 & & \downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi_3} & & \downarrow{\scriptstyle \xi_4} & & \downarrow{\scriptstyle \xi_5} \\
0 & \longrightarrow & \mathrm{H}^1_{/b}(\Gamma;V) & \xrightarrow{p^*} & \mathrm{H}^1_{/b}(G;V) & \xrightarrow{i^*} & \mathrm{H}^1_{/b}(N;V)^G & \xrightarrow{\tau_{/b}} & \mathrm{H}^2_{/b}(\Gamma;V) & \xrightarrow{p^*} & \mathrm{H}^2_{/b}(G;V)
\end{array}$$

Here the ξ_i are the maps induced from the quotient map $C^\bullet \to C^\bullet_{/b}$.

Notation Throughout this section, V denotes a Banach space equipped with the norm $\|\cdot\|$ and an isometric G-action whose restriction to N is trivial. For a nonnegative real number $D \geq 0$ and $v, w \in V$, the symbol $v \approx_D w$ means that $\|v - w\| \leq D$. For functions $f, g\colon S \to V$ on a set S, the symbol $f \approx_D g$ means that $f(s) \approx_D g(s)$ for every $s \in S$.

6.1 N-quasicocycle

To define the map $\tau_{/b}\colon \mathrm{H}^1_{/b}(N; V)^G \to \mathrm{H}^2_{/b}(\Gamma; V)$ in Theorem 1.5, it is convenient to introduce the notion of an N-quasicocycle. First, we recall the definition of quasicocycles.

Definition 6.1 Let G be a group and V a left $\mathbb{R}[G]$-module with a G-invariant norm $\|\cdot\|$. A function $F\colon G \to V$ is called a *quasicocycle* if there exists a nonnegative number D such that

$$F(g_1 g_2) \approx_D F(g_1) + g_1 \cdot F(g_2)$$

for every $g_1, g_2 \in G$. The smallest such D is called the *defect* of F and denoted by $D(F)$. Let $\widehat{\mathrm{Q}}Z(G; V)$ denote the $\mathbb{R}$-vector space of all quasicocycles on G.

Remark 6.2 If we need to specify the G-representation ρ, we write $\widehat{\mathrm{Q}}Z(G; \rho, V)$ instead of $\widehat{\mathrm{Q}}Z(G; V)$.

We introduce the concept of N-quasicocycles, which is a generalization of the concept of partial quasimorphisms introduced in [33] (see also [81; 56; 64; 18; 60]).

Definition 6.3 Let N be a normal subgroup of G. A function $F\colon G \to V$ is called an *N-quasicocycle* if there exists a nonnegative number D'' such that

$$F(ng) \approx_{D''} F(n) + F(g) \quad \text{and} \quad F(gn) \approx_{D''} F(g) + g \cdot F(n) \tag{6-1}$$

for every $g \in G$ and $n \in N$. The smallest such D'' is called the *defect* of the N-quasicocycle F and denoted by $D''(F)$. Let $\widehat{\mathrm{Q}}Z_N(G; V)$ denote the $\mathbb{R}$-vector space of all N-quasicocycles on G.

If the G-action on V is trivial, then a quasicocycle is also called a *V-valued quasimorphism*. In this case, we use the symbol $\widehat{\mathrm{Q}}(G; V)$ instead of $\widehat{\mathrm{Q}}Z(G; V)$ to denote the space of V-valued quasimorphisms. A V-valued quasimorphism F is said to be *homogeneous* if $F(g^k) = k \cdot F(g)$ for every $g \in G$ and every $k \in \mathbb{Z}$. The homogenization of V-valued quasimorphisms is well defined, as in the case of ($\mathbb{R}$-valued) quasimorphisms. We write $\mathrm{Q}(G; V)$ for the space of V-valued homogeneous quasimorphisms.

Recall that in our setting the restriction of the G-action on V to N is always trivial. Then a left G-action on $\mathrm{Q}(N; V)$ is defined by

$$({}^g f)(n) = g \cdot f(g^{-1} n g)$$

for every $g \in G$ and every $n \in N$. We call an element of $\mathrm{Q}(N; V)^G$ a *G-equivariant V-valued homogeneous quasimorphism.*

Remark 6.4 An element $f \in \mathrm{Q}(N;V)$ belongs to $\mathrm{Q}(N;V)^G$ if and only if

$$g \cdot f(n) = f(gng^{-1})$$

for every $g \in G$ and every $n \in N$. This is why we call an element of $\mathrm{Q}(N;V)^G$ G-equivariant.

Remark 6.5 The isomorphism $\mathrm{H}^1_{/b}(N;V) \to \mathrm{Q}(N;V)$ given by the homogenization is compatible with the G-actions. In particular, this isomorphism induces an isomorphism $\mathrm{H}^1_{/b}(N;V)^G \to \mathrm{Q}(N;V)^G$.

The elements of $\mathrm{Q}(N;V)^G = \mathrm{H}^1_{/b}(N;V)^G$ are G-invariant (as cohomology classes). However, respecting the condition $g \cdot f(n) = f(gng^{-1})$ for $f \in \mathrm{Q}(N;V)^G$, we call the elements of $\mathrm{Q}(N;V)^G$ *G-equivariant V-valued homogeneous quasimorphisms*.

Lemma 6.6 *Let N be a normal subgroup of G and V a left $\mathbb{R}[G]$-module. Assume that the induced N-action on V is trivial. Then, for an N-quasicocycle $F \in \widehat{\mathrm{Q}}Z_N(G;V)$, there exists a bounded cochain $b \in C^1_b(G;V)$ such that the restriction $(F+b)|_N$ is in* $\mathrm{Q}(N;V)^G$.

Proof By the definition of N-quasicocycles, the restriction $F|_N \colon N \to V$ is a quasimorphism. Let $\bar{F}|_N$ be the homogenization of $F|_N$. Then the map

$$b' = \bar{F}|_N - F|_N \colon N \to V$$

is bounded. Define $b \colon G \to V$ by

$$b(g) = \begin{cases} b'(g) & \text{if } g \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Then the map b is also bounded. Set $\Phi = F + b$; then $\Phi|_N = (F+b)|_N = \bar{F}|_N$. Since Φ is an N-quasicocycle, we have

$$({}^g\Phi)(n) = g \cdot \Phi(g^{-1}ng) \approx_{D''(\Phi)} \Phi(g \cdot g^{-1}ng) - \Phi(g) = \Phi(ng) - \Phi(g) \approx_{D''(\Phi)} \Phi(n)$$

for $g \in G$ and $n \in N$. Hence the difference ${}^g\Phi - \Phi$ is in $C^1_b(N;V)$. Since $({}^g\Phi)|_N$ and $\Phi|_N$ are homogeneous quasimorphisms, we have ${}^g\Phi|_N - \Phi|_N = 0$, and this implies that the element $\Phi|_N = (F+b)|_N$ belongs to $\mathrm{Q}(N;V)^G$. □

If V is the trivial G-module $\mathbb{R}$, then N-quasicocycles are also called N-quasimorphisms (this word was first introduced in [57]). In this case, Lemma 6.6 reads as follows:

Corollary 6.7 *Let N be a normal subgroup of G. For an N-quasimorphism $F \in \widehat{\mathrm{Q}}_N(G)$, there exists a bounded cochain $b \in C^1_b(G)$ such that the restriction $(F+b)|_N$ is in* $\mathrm{Q}(N)^G$.

6.2 The map $\tau_{/b}$

Now we proceed to the proof of Theorem 1.5. The goal in this subsection is to construct the map $\tau_{/b} \colon \mathrm{H}^1_{/b}(N)^G \to \mathrm{H}^2_{/b}(G)$. Here we only present the proofs in the case where the coefficient module V

is the trivial module $\mathbb{R}$. When $V \neq \mathbb{R}$, the proofs work without any essential change (see Remarks 6.5, 6.8, 6.14 and 6.16).

First we define the map $\tau_{/b}\colon \mathrm{H}^1_{/b}(N)^G \to \mathrm{H}^2_{/b}(\Gamma)$. Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be a group extension. As a special case of Remark 6.5, we have isomorphisms $\mathrm{H}^1_{/b}(N) \to \mathrm{Q}(N)$ and $\mathrm{H}^1_{/b}(N)^G \to \mathrm{Q}(N)^G$. By using these, we identify $\mathrm{H}^1_{/b}(N)$ and $\mathrm{H}^1_{/b}(N)^G$ with $\mathrm{Q}(N)$ and $\mathrm{Q}(N)^G$, respectively.

Let $\overline{\mathrm{Q}}_N(G) = \overline{\mathrm{Q}}_N(G;\mathbb{R})$ be the $\mathbb{R}$-vector space of all N-quasimorphisms whose restrictions to N are homogeneous quasimorphisms on N; that is,

$$\overline{\mathrm{Q}}_N(G) = \{F\colon G \to \mathbb{R} \mid F \text{ is an } N\text{-quasimorphism such that } F|_N \in \mathrm{Q}(N)^G\} \subset \widehat{\mathrm{Q}}_N(G).$$

By definition, the restriction of the domain defines a map

$$i^*\colon \overline{\mathrm{Q}}_N(G) \to \mathrm{Q}(N)^G.$$

Remark 6.8 When the G-action on V is nontrivial, we need to replace the space $\overline{\mathrm{Q}}_N(G)$ by

$$\overline{\mathrm{Q}}Z^1_N(G;V) = \{F\colon G \to V \mid F \text{ is an } N\text{-quasicocycle such that } F|_N \in \mathrm{Q}(N;V)^G\}.$$

Lemma 6.9 *The map $i^*\colon \overline{\mathrm{Q}}_N(G) \to \mathrm{Q}(N)^G$ is surjective.*

Proof Let $s\colon \Gamma \to G$ be a set-theoretic section of p satisfying $s(1_\Gamma) = 1_G$. For $f \in \mathrm{Q}(N)^G$, define a map $F_{f,s}\colon G \to \mathbb{R}$ by

$$F_{f,s}(g) = f(g \cdot sp(g)^{-1})$$

for $g \in G$. Then $F_{f,s}|_N = f$ since $sp(n) = 1_G$ for every $n \in N$. Moreover, $F_{f,s}$ is an N-quasimorphism. Indeed,

$$F_{f,s}(ng) = f(ng \cdot sp(ng)^{-1}) = f(ng \cdot sp(g)^{-1}) \approx_{D(f)} f(n) + f(g \cdot sp(g)^{-1}) = F_{f,s}(n) + F_{f,s}(g)$$

and

$$\begin{aligned} F_{f,s}(gn) = F_{f,s}(gng^{-1}g) &\approx_{D(f)} F_{f,s}(gng^{-1}) + F_{f,s}(g) \\ &= f(gng^{-1}) + F_{f,s}(g) = f(n) + F_{f,s}(g) = F_{f,s}(n) + F_{f,s}(g) \end{aligned}$$

by the definition of quasimorphisms and the G-invariance of f. This means $i^*(F_{f,s}) = f$, and hence the map i^* is surjective. □

Lemma 6.10 *For $F \in \overline{\mathrm{Q}}_N(G)$ and $g_i, g_i' \in G$ satisfying $p(g_i) = p(g_i') \in \Gamma$,*

$$\delta F(g_1, g_2) \approx_{4D''(F)} \delta F(g_1', g_2').$$

Proof By the assumption, there exist $n_1, n_2 \in N$ satisfying $g_1' = n_1 g_1$ and $g_2' = g_2 n_2$. Therefore,

$$\begin{aligned} \delta F(g_1', g_2') &= F(g_2 n_2) - F(n_1 g_1 g_2 n_2) + F(n_1 g_1) \\ &\approx_{4D''(F)} F(g_2) + F(n_2) - (F(n_1) + F(g_1 g_2) + F(n_2)) + F(n_1) + F(g_1) \\ &= \delta F(g_1, g_2). \end{aligned}$$ □

For $F \in \overline{\mathrm{Q}}_N(G)$ and a section $s\colon \Gamma \to G$ of p, we set $\alpha_{F,s} = s^*\delta F \in C^2(\Gamma)$. By Lemma 6.10, the element $[\alpha_{F,s}] \in C^2_{/b}(\Gamma) = C^2(\Gamma)/C^2_b(\Gamma)$ is independent of the choice of the section s. Therefore we set $\alpha_F = [\alpha_{F,s}] \in C^2_{/b}(\Gamma)$.

Lemma 6.11 *The cochain α_F is a cocycle on $C^{\bullet}_{/b}(\Gamma)$.*

Proof It suffices to show that the coboundary $\delta\alpha_{F,s}$ belongs to $C^3_b(\Gamma)$. For $f, g, h \in \Gamma$,

$$\begin{aligned}\delta\alpha_{F,s}(f,g,h) &= \delta F(s(g),s(h)) - \delta F(s(fg),s(h)) + \delta F(s(f),s(gh)) - \delta F(s(f),s(g))\\ &\approx_{8D''(F)} \delta F(s(g),s(h)) - \delta F(s(f)s(g),s(h)) + \delta F(s(f),s(g)s(h)) - \delta F(s(f),s(g))\\ &= \delta(\delta F)(s(f),s(g),s(h)) = 0\end{aligned}$$

by Lemma 6.10. □

By Lemmas 6.9 and 6.11, we obtain a map

$$\overline{\mathrm{Q}}_N(G) \to \mathrm{H}^2_{/b}(\Gamma);\ F \mapsto [\alpha_F]. \tag{6-2}$$

Lemma 6.12 *The cohomology class $[\alpha_F] \in \mathrm{H}^2_{/b}(\Gamma)$ depends only on the restriction $F|_N$.*

Proof Let $s\colon \Gamma \to G$ be a section of p and Φ an element of $\overline{\mathrm{Q}}_N(G)$ satisfying $\Phi|_N = F|_N$. Then, for every $g, h \in \Gamma$,

$$\begin{aligned}(\alpha_{F,s}-\alpha_{\Phi,s})(g,h) &= \delta F(s(g),s(h)) - \delta\Phi(s(g),s(h))\\ &= F(s(h)) - F(s(g)s(h)) + F(s(g)) - \big(\Phi(s(h)) - \Phi(s(g)s(h)) + \Phi(s(g))\big)\\ &= \delta(F\circ s)(g,h) - \delta(\Phi\circ s)(g,h) + F(s(gh)) - F(s(g)s(h)) - \big(\Phi(s(gh)) - \Phi(s(g)s(h))\big).\end{aligned}$$

Since F and Φ are N-quasimorphisms, we have

$$\begin{aligned}F(s(gh)) - F(s(g)s(h)) &\approx_{D''(F)} F(s(gh)s(h)^{-1}s(g)^{-1}),\\ \Phi(s(gh)) - \Phi(s(g)s(h)) &\approx_{D''(\Phi)} \Phi(s(gh)s(h)^{-1}s(g)^{-1}).\end{aligned}$$

Together with $F(s(gh)s(h)^{-1}s(g)^{-1}) = \Phi(s(gh)s(h)^{-1}s(g)^{-1})$, we have

$$\alpha_{F,s} - \alpha_{\Phi,s} \approx_{D''(F)+D''(\Phi)} \delta(F\circ s - \Phi\circ s),$$

and this implies $[\alpha_F] = [\alpha_\Phi] \in \mathrm{H}^2_{/b}(\Gamma)$. □

By Lemma 6.12, the map defined in (6-2) descends to a map $\tau_{/b}\colon \mathrm{Q}(N)^G \to \mathrm{H}^2_{/b}(\Gamma)$; that is, $\tau_{/b}$ is defined by

$$\tau_{/b}(f) = [\alpha_F],$$

where F is an element of $\overline{\mathrm{Q}}_N(G)$ satisfying $F|_N = f$. Under the isomorphism $\mathrm{Q}(N)^G \cong \mathrm{H}^1_{/b}(N)^G$, we obtain the map

$$\tau_{/b}\colon \mathrm{H}^1_{/b}(N)^G \to \mathrm{H}^2_{/b}(\Gamma).$$

6.3 Proof of the exactness

Now we proceed to the proof of the exactness of the sequence

$$0 \to \mathrm{H}^1_{/b}(\Gamma) \xrightarrow{p^*} \mathrm{H}^1_{/b}(G) \xrightarrow{i^*} \mathrm{Q}(N)^G \xrightarrow{\tau_{/b}} \mathrm{H}^2_{/b}(\Gamma) \xrightarrow{p^*} \mathrm{H}^2_{/b}(G), \tag{6-3}$$

where we identify $\mathrm{Q}(N)^G$ with $\mathrm{H}^1_{/b}(N)^G$.

Lemma 6.13 *The sequence (6-3) is exact at $\mathrm{H}^1_{/b}(\Gamma)$ and $\mathrm{H}^1_{/b}(G)$.*

Remark 6.14 In the case of trivial real coefficients, this proposition is well known (see [26]). Indeed, the spaces $\mathrm{H}^1_{/b}(\Gamma)$ and $\mathrm{H}^1_{/b}(G)$ are isomorphic to $\mathrm{Q}(\Gamma)$ and $\mathrm{Q}(G)$, respectively, and exactness of the sequence

$$0 \to \mathrm{Q}(\Gamma) \to \mathrm{Q}(G) \to \mathrm{Q}(N)^G$$

follows from the homogeneity of the elements of $\mathrm{Q}(\Gamma)$. However, in general, the spaces $\mathrm{H}^1_{/b}(\Gamma; V)$ and $\mathrm{H}^1_{/b}(G; V)$ are not isomorphic to the spaces of V-valued homogeneous quasimorphisms $\mathrm{Q}(\Gamma; V)$ and $\mathrm{Q}(G; V)$, respectively. Therefore, we present a proof of Lemma 6.13 which can be modified to the case of nontrivial coefficients without any essential change.

Proof of Lemma 6.13 We first show the exactness at $\mathrm{H}^1_{/b}(\Gamma)$. Let $a \in \mathrm{H}^1_{/b}(\Gamma)$ and suppose $p^*a = 0$. Let $f \in C^1(\Gamma)$ be a representative of a. Since $p^*a = 0$ in $\mathrm{H}^1_{/b}(G)$, there exists $c \in \mathbb{R} \cong C^0(\Gamma)$ such that $p^*f - \delta c = p^*f$ is bounded. Since p is surjective, f is bounded, and hence $a = 0$. This implies the exactness at $\mathrm{H}^1_{/b}(\Gamma)$.

Next we prove the exactness at $\mathrm{H}^1_{/b}(G)$. Since the map $p \circ i$ is zero, the composite $i^* \circ p^*$ is also zero. For $a \in \mathrm{H}^1_{/b}(G)$ satisfying $i^*a = 0$, it follows from Lemma 6.6 that there exists a representative $f \in C^1(G)$ of a satisfying $f|_N = 0$. For a section $s\colon \Gamma \to G$ of p, set $f_s = s^*f\colon \Gamma \to \mathbb{R}$. Then f_s is a quasimorphism on Γ. Indeed, since f is a quasimorphism on G,

$$\begin{aligned} f_s(g_1g_2) = f(s(g_1g_2)) &= f(s(g_1g_2)s(g_2)^{-1}s(g_1)^{-1}s(g_1)s(g_2)) \\ &\approx_{D(f)} f(s(g_1g_2)s(g_2)^{-1}s(g_1)^{-1}) + f(s(g_1)s(g_2)) = f(s(g_1)s(g_2)) \\ &\approx_{D(f)} f(s(g_2)) + f(s(g_1)) = f_s(g_2) + f_s(g_1) \end{aligned}$$

by the triviality $f|_N = 0$. Hence the cochain f_s is a cocycle on $C^1_{/b}(\Gamma)$; let $a_s \in \mathrm{H}^1_{/b}(\Gamma)$ denote the relative cohomology class represented by f_s. For $g \in G$,

$$p^*f_s(g) = f(sp(g)) = f(sp(g)g^{-1}g) \approx_{D(f)} f(sp(g)g^{-1}) + f(g) = f(g).$$

Therefore, the cochain p^*f_s is equal to f as relative cochains on G, and this implies $p^*a_s = a$. □

Lemma 6.15 *The sequence (6-3) is exact at $\mathrm{Q}(N)^G$.*

Proof Note that every $\mu \in \mathrm{Q}(G)$ is an N-quasimorphism. Hence, by the definition of $\tau_{/b}\colon \mathrm{Q}(N)^G \to \mathrm{H}^2_{/b}(\Gamma)$,

$$\tau_{/b}(i^*(\mu)) = [\alpha_\mu] = [[\alpha_{\mu,s}]]$$

for a section $s\colon \Gamma \to G$. Since μ is a quasimorphism, $\alpha_{\mu,s} = s^*\delta\mu$ is an element of $C^2_b(\Gamma)$ and hence $\alpha_\mu = [\alpha_{\mu,s}] \in C^2_{/b}(\Gamma)$ is zero. This implies $\tau_{/b} \circ i^* = 0$.

Suppose that $f \in \mathrm{Q}(N)^G$ satisfies $\tau_{/b}(f) = 0$. By Lemma 6.9, we obtain $F \in \overline{\mathrm{Q}}_N(G)$ satisfying $F|_N = f$. Let $s\colon \Gamma \to G$ be a section of p. The triviality of $[\alpha_F] = \tau_{/b}(f) = 0$ implies that there exist $\beta \in C^1(\Gamma)$ and $b \in C^2_b(\Gamma)$ satisfying

$$\alpha_{F,s} - \delta\beta = b.$$

For $g_i \in G$,

$$\delta F(g_1, g_2) \approx_{4D''(F)} \delta F(sp(g_1), sp(g_2)) = \alpha_{F,s}(p(g_1), p(g_2))$$

by Lemma 6.10. Hence,

$$\delta(F - p^*\beta)(g_1, g_2) \approx_{4D''(F)} (\alpha_{F,s} - \delta\beta)(p(g_1), p(g_2)) = p^*b(g_1, g_2). \tag{6-4}$$

Since the cochain b is bounded, the function $F - p^*\beta$ is a quasimorphism. By the fact that the restriction $(F - p^*\beta)|_N$ is equal to the homogeneous quasimorphism f, the homogenization $\overline{F - p^*\beta}$ of $F - p^*\beta$ satisfies $i^*(\overline{F - p^*\beta}) = f$. This implies $\operatorname{Ker}\tau_{/b} \subset \operatorname{Im} i^*$. □

Remark 6.16 The last step of the proof of Lemma 6.15 uses the homogenization of a quasimorphism. In the case where the G-action on V is nontrivial (in particular, where we cannot use the homogenization), we replace the argument of the last step with the following: we have shown that the cochain $F - p^*\beta$ is a cocycle in $C^1_{/b}(G)$ by (6-4). Since $F|_N = f$, the restriction $(F - p^*\beta + \beta(1_\Gamma))|_N$ is equal to f. Therefore, $i^*\big([F - p^*\beta + \beta(1_\Gamma)]\big) = f$, and this implies $\operatorname{Ker}\tau_{/b} \subset \operatorname{Im} i^*$.

Lemma 6.17 *The sequence (6-3) is exact at* $\mathrm{H}^2_{/b}(\Gamma)$.

Proof For $f \in \mathrm{Q}(N)^G$, we have $F \in \overline{\mathrm{Q}}_N(G)$ satisfying $F|_N = f$ by Lemma 6.9. Then a representative of $p^*(\tau_{/b}(f)) \in \mathrm{H}^2_{/b}(G)$ is given by $p^*\alpha_{F,s} \in C^2(G)$ for some section $s\colon \Gamma \to G$ of p. For $g_i \in G$,

$$p^*\alpha_{F,s}(g_1, g_2) = s^*\delta F(p(g_1), p(g_2)) = \delta F(sp(g_1), sp(g_2)) \approx_{4D''(F)} \delta F(g_1, g_2)$$

by Lemma 6.10. This implies $p^*(\tau_{/b}(f)) = 0$.

For $a \in \mathrm{H}^2_{/b}(\Gamma)$ satisfying $p^*a = 0$, let $\alpha \in C^2(\Gamma)$ be a representative of a. We can assume that the cochain satisfies

$$\alpha(1_\Gamma, 1_\Gamma) = 0. \tag{6-5}$$

Indeed, if $\alpha(1_\Gamma, 1_\Gamma) = c \in \mathbb{R}$, then the cochain $\alpha - c$ satisfies (6-5) and is also a representative of a since the constant function c is bounded. Note that the cocycle condition of $C^\bullet_{/b}(\Gamma)$ implies that there exists a nonnegative constant D such that

$$\delta\alpha \approx_D 0.$$

Hence, for $\gamma_1, \gamma_2 \in \Gamma$,

$$0 \approx_D \delta\alpha(\gamma_1, 1_\Gamma, \gamma_2) = \alpha(1_\Gamma, \gamma_2) - \alpha(\gamma_1, 1_\Gamma).$$

In particular,

$$\alpha(1_\Gamma, \gamma) \approx_D \alpha(1_\Gamma, 1_\Gamma) = 0 \quad \text{and} \quad \alpha(\gamma, 1_\Gamma) \approx_D \alpha(1_\Gamma, 1_\Gamma) = 0 \tag{6-6}$$

for every $\gamma \in \Gamma$. The equality $p^* a = 0$ implies that there exists $\beta \in C^1(G)$ and a nonnegative constant D' satisfying

$$p^*\alpha - \delta\beta \approx_{D'} 0. \tag{6-7}$$

Define a cochain $\zeta \colon G \to \mathbb{R}$ by

$$\zeta(g) = \beta(g) - \alpha(p(g), 1_\Gamma); \tag{6-8}$$

then it is an N-quasimorphism. Indeed, by using $p(n) = 1_\Gamma$, we have

$$\delta\zeta(n, g) = \delta\beta(n, g) - \big(\alpha(p(g), 1_\Gamma) - \alpha(p(g), 1_\Gamma) + \alpha(1_\Gamma, 1_\Gamma)\big) \approx_D (\delta\beta - p^*\alpha)(g, n) \approx_{D'} 0$$

and

$$\delta\zeta(g, n) = \delta\beta(g, n) - \big(\alpha(1_\Gamma, 1_\Gamma) - \alpha(p(g), 1_\Gamma) + \alpha(p(g), 1_\Gamma)\big) \approx_D (\delta\beta - p^*\alpha)(g, n) \approx_{D'} 0$$

by (6-6) and (6-7). By Lemma 6.6, there exists a bounded cochain $b \in C_b^1(G)$ such that the restriction $(\zeta + b)|_N$ is in $\mathrm{Q}(N)^\Gamma$. Set $\Phi = \zeta + b \in \overline{\mathrm{Q}}_N(G)$; then a representative of $\tau_{/b}(\Phi|_N)$ is given by $\alpha_{\Phi,s}$ for some section $s \colon \Gamma \to G$ of p. For $g_1, g_2 \in \Gamma$,

$$(\alpha_{\Phi,s} - \alpha)(g_1, g_2) = (\delta\Phi - p^*\alpha)(s(g_1), s(g_2)) \approx_{D'} (\delta\Phi - \delta\beta)(s(g_1), s(g_2))$$

by (6-7). By (6-8),

$$(\Phi - \beta)(g) = (\zeta + b - \beta)(g) = b(g) - \alpha(p(g), 1_\Gamma).$$

Together with (6-6) and the boundedness of b, the cochain $\Phi - \beta \colon G \to \mathbb{R}$ is bounded. Hence the cochain $\alpha_{\Phi,s} - \alpha$ is also bounded, and this implies $a = [\alpha_\Phi] = \tau_{/b}(\Phi|_N)$. □

Proof of Theorem 1.5 The exactness is obtained from Lemmas 6.13, 6.15 and 6.17. Commutativity of the first, second and fourth squares is obtained from the cochain level calculations. The commutativity of the third square follows from the definition of the map $\tau_{/b}$ and Proposition 6.18 below. □

Proposition 6.18 [83, Proposition 1.6.6] *Let $1 \to N \to G \to \Gamma \to 1$ be an exact sequence and V an Γ-module. For a G-invariant homomorphism $f \in \mathrm{H}^1(N; V)^G$, there exists a map $F \colon G \to V$ such that the restriction $F|_N$ is equal to f and the coboundary δF descends to a group two cocycle $\alpha_F \in C^2(\Gamma; V)$; that is, $p^*\alpha_F = \delta F$. Then the map $\tau \colon \mathrm{H}^1(N; V)^G \to \mathrm{H}^2(\Gamma; V)$ in the five-term exact sequence of group cohomology is obtained by $\tau(f) = [\alpha_F]$.*

We conclude this section with the following applications of Theorem 1.5 to the extendability of G-invariant homomorphisms:

Proposition 6.19 *Let $\Gamma = G/N$. Assume that $\mathrm{H}_b^2(\Gamma) = 0$ and $f: N \to \mathbb{R}$ is a G-invariant homomorphism on N. If f is extended to a quasimorphism on G, then f is extended to a homomorphism on G.*

Proof Note that the assumption $\mathrm{H}_b^2(\Gamma) = 0$ implies that the map $\mathrm{H}^2(\Gamma) \to \mathrm{H}_{/b}^2(\Gamma)$ is injective. By diagram chasing on (1-3), the proposition holds. □

This proposition immediately implies the following corollary:

Corollary 6.20 *Let $\Gamma = G/N$. Assume that $\mathrm{H}_b^2(\Gamma) = 0$ and N is a subgroup of $[G, G]$. Then every nonzero G-invariant homomorphism $f: N \to \mathbb{R}$ cannot be extended to G as a quasimorphism. Namely, $\mathrm{H}^1(N)^G \cap i^*\mathrm{Q}(G) = 0$.*

Proof Assume that a homomorphism $f: N \to \mathbb{R}$ can be extended to G as a quasimorphism. Then Proposition 6.19 implies that there exists a homomorphism $f': G \to \mathbb{R}$ with $f'|_N = f$. Since f' vanishes on $[G, G]$, we have $f = f'|_N = 0$. □

Without the assumption $\mathrm{H}_b^2(\Gamma) = 0$, there exists a G-invariant homomorphism which is extendable to G as a quasimorphism but is *not* extendable to G as a (genuine) homomorphism. To see this, let $G = \widetilde{\mathrm{Homeo}}_+(S^1)$ and $N = \pi_1(\mathrm{Homeo}_+(S^1))$. Then $\Gamma = \mathrm{Homeo}_+(S^1)$ and hence $\mathrm{H}_b^2(\Gamma) \cong \mathbb{R} \neq 0$. Poincaré's rotation number ρ: $\widetilde{\mathrm{Homeo}}_+(S^1) \to \mathbb{R}$ is an extension of the homomorphism $\pi_1(\mathrm{Homeo}_+(S^1)) \cong \mathbb{Z} \hookrightarrow \mathbb{R}$. However, this homomorphism cannot be extendable to $\widetilde{\mathrm{Homeo}}_+(S^1)$ as a homomorphism since $\widetilde{\mathrm{Homeo}}_+(S^1)$ is perfect.

7 Proof of equivalences of scl_G and $\mathrm{scl}_{G,N}$

The goal of this section is to prove Theorem 2.1. Here we recall its precise statement.

Theorem 2.1 *Assume that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. Then:*

(1) scl_G *and* $\mathrm{scl}_{G,N}$ *are bi-Lipschitzly equivalent on* $[G, N]$.

(2) *If $\Gamma = G/N$ is amenable, then $\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x) \le 2 \cdot \mathrm{scl}_G(x)$ for all $x \in [G, N]$.*

(3) *If $\Gamma = G/N$ is solvable, then $\mathrm{scl}_G(x) = \mathrm{scl}_{G,N}(x)$ for all $x \in [G, N]$.*

In this section, in order to specify the domain of a quasimorphism, we use the symbols D_G and D_N to denote the defect of a quasimorphism on G and N, respectively. The main tool in this section is the Bavard duality theorem for $\mathrm{scl}_{G,N}$, which was proved by the first, second, fourth and fifth authors:

Theorem 7.1 (Bavard duality theorem for stable mixed commutator lengths [60]) *Let N be a normal subgroup of a group G. Then, for every $x \in [G,N]$,*

$$\mathrm{scl}_{G,N}(x) = \frac{1}{2} \sup_{f \in \mathrm{Q}(N)^G - \mathrm{H}^1(N)^G} \frac{|f(x)|}{D_N(f)}.$$

Here we set the supremum in the right-hand side of the above equality to be zero if $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G$.

This theorem yields the following criterion to show the equivalence of $\mathrm{scl}_{G,N}$ and scl_G:

Proposition 7.2 *Let C be a real number such that, for every $f \in \mathrm{Q}(N)^G$, there exists $f' \in \mathrm{Q}(G)$ satisfying $f'|_N - f \in \mathrm{H}^1(N)^G$ and $D_G(f') \le C \cdot D_N(f)$. Then, for every $x \in [G,N]$,*

$$\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x) \le C \cdot \mathrm{scl}_G(x).$$

The existence of a C as in the assumption of Proposition 7.2 is equivalent to saying that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. See Section 7.1 for details.

Proof Let $x \in [G,N]$. It is clear that $\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x)$. Let $\varepsilon > 0$. Then Theorem 7.1 implies that there exists $f \in \mathrm{Q}(N)^G$ such that

$$\mathrm{scl}_{G,N}(x) - \varepsilon \le \frac{f(x)}{2D_N(f)}.$$

By assumption, there exists $f' \in \mathrm{Q}(G)$ such that $f'' = f'|_N - f \in \mathrm{H}^1(N)^G$ and $D_G(f') \le C \cdot D_N(f)$. Since f'' is a G-invariant homomorphism and $x \in [G,N]$, we have $f''(x) = 0$ and hence $f'(x) = f(x)$. Hence,

$$\mathrm{scl}_{G,N}(x) - \varepsilon \le \frac{f(x)}{2D_N(f)} \le C \cdot \frac{f'(x)}{2D_G(f')} \le C \cdot \mathrm{scl}_G(x).$$

Here we use the original Bavard duality theorem [6] to prove the last inequality. Since ε is an arbitrary number, we complete the proof. □

In the proofs of Theorem 2.1(2)–(3), we use the following corollary of Proposition 7.2:

Corollary 7.3 *Assume that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ and that there exists $C \ge 1$ such that $f' \in \mathrm{Q}(G)$ implies that $D_G(f') \le C \cdot D_N(f'|_N)$. Then, for every $x \in [G,N]$,*

$$\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x) \le C \cdot \mathrm{scl}_G(x).$$

Proof Let $f \in \mathrm{Q}(N)^G$. Then, by the assumption that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, there exists $f' \in \mathrm{Q}(G)$ such that $f'|_N - f$ is a G-invariant homomorphism. Note that $D_N(f'|_N) = D_N(f)$. Indeed, for every $a, b \in N$,

$$f(ab) - f(a) - f(b) = f'(ab) - f'(a) - f'(b)$$

since $f'|_N - f$ is a homomorphism. Hence, $C \cdot D_N(f) = C \cdot D_N(f'|_N) \ge D_G(f')$. Hence, Proposition 7.2 implies that

$$\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x) \le C \cdot \mathrm{scl}_G(x)$$

for every $x \in [G,N]$. □

concept	defect	definition	vector space
quasimorphism on G	D	$f(g_1g_2)\approx_D f(g_1)+f(g_2)$	$\widehat{\mathrm{Q}}(G)$
G-quasi-invariant quasimorphism on N	D, D'	$f(x_1x_2)\approx_D f(x_1)+f(x_2)$, $f(gxg^{-1})\approx_{D'} f(x)$	$\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}$
N-quasimorphism on G	D''	$f(gx)\approx_{D''} f(g)+f(x)$, $f(xg)\approx_{D''} f(x)+f(g)$	$\widehat{\mathrm{Q}}_N(G)$

Table 1: The concepts and symbols on quasimorphisms.

7.1 Proof of Theorem 2.1(1)

The main difficulty in the proof of Theorem 2.1 is distilled in Theorem 7.4, stated below. Note that the defect D_N defines a seminorm on $\mathrm{Q}(N)^G$, and its kernel is $\mathrm{H}^1(N)^G$.

Theorem 7.4 *The normed space* $(\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G, D_N)$ *is a Banach space.*

To show this theorem, we recall some concepts introduced in [60]. Let $\widehat{\mathrm{Q}}_N(G)=\widehat{\mathrm{Q}}_N(G;\mathbb{R})$ denote the $\mathbb{R}$-vector space of N-quasimorphisms (see Definition 6.3 and Table 1). We call $f\in\widehat{\mathrm{Q}}_N(G)$ an *N-homomorphism* if $D''(f)=0$, and let $\mathrm{H}^1_N(G)$ denote the space of N-homomorphisms on G. It is clear that the defect D'' is a seminorm on $\widehat{\mathrm{Q}}_N(G)$, and, in fact, the norm space $\widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$ is complete:

Proposition 7.5 [60, Corollary 3.6] *The normed space* $(\widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G), D'')$ *is a Banach space.*

A quasimorphism $f\colon N\to\mathbb{R}$ is said to be *G-quasi-invariant* if the number

$$D'(f)=\sup_{g\in G,x\in N}|f(gxg^{-1})-f(x)|$$

is finite. Let $\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}$ denote the space of G-quasi-invariant quasimorphisms on N. The function D_N+D', which assigns $D_N(f)+D'(f)$ to $f\in\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}$, defines a seminorm on $\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}$. For an N-quasimorphism f on G, the restriction $f|_N$ is a G-quasi-invariant quasimorphism [60, Lemma 2.3]. Conversely, for every G-quasi-invariant quasimorphism f on N, there exists an N-quasimorphism $f'\colon G\to\mathbb{R}$ satisfying $f'|_N=f$ [60, Proposition 2.4]. We summarize the concepts and symbols on quasimorphisms in Table 1.

Lemma 7.6 *The normed space* $(\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G, D_N+D')$ *is a Banach space.*

Proof In what follows, we will define bounded operators

$$A\colon\widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)\to\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G,\quad B\colon\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G\to\widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$$

such that $A\circ B$ is the identity of $\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G$. First, we define A by the restriction, ie $A(f)=f|_N$. Then the operator norm of A is at most 3 since $D_N\le D''$ and $D'\le 2D''$. Indeed, $D_N\le D''$ follows by definition, and $D'\le 2D''$ follows from the estimate

$$f(gxg^{-1})+f(g)\approx_{D''(f)} f(gx)\approx_{D''(f)} f(g)+f(x)$$

for $g\in G$ and $x\in N$.

Let S be a subset of G such that $1_G \in S$ and the map

$$S \times N \to G, \quad (s,x) \mapsto sx,$$

is bijective. For an $f \in \hat{\mathrm{Q}}(N)^{\mathrm{Q}G}$, define a function $B(f)\colon G \to \mathbb{R}$ by $B(f)(sx) = f(x)$ for $s \in S$ and $x \in N$. Then $B(f)$ is an N-quasimorphism on G satisfying $D''(B(f)) \le D_N(f) + D'(f)$. Hence, the map B induces a bounded operator $\hat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G \to \hat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$ whose operator norm is at most 1, and we conclude that $\hat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G$ is isomorphic to $B(\hat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G)$. Proposition 7.5 implies that $\hat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$ is a Banach space. Therefore, it suffices to show that $B(\hat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G)$ is a closed subset of $\hat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$, which follows from the well-known Lemma 7.7, given below. □

Lemma 7.7 *Let X be a topological subspace of a Hausdorff space Y. If X is a retract of Y, then X is a closed subset of Y.*

Proof Let $r\colon Y \to X$ be a retraction of the inclusion map $i\colon X \to Y$. Since $X = \{y \in Y \mid i \circ r(y) = y\}$ and Y is a Hausdorff space, we conclude that X is a closed subset of Y. □

Proof of Theorem 7.4 For $n \in \mathbb{Z}$ and $x \in N$, define a function $\alpha_{n,x}\colon \hat{\mathrm{Q}}(N)^{\mathrm{Q}G} \to \mathbb{R}$ by

$$\alpha_{n,x}(f) = f(x^n) - n \cdot f(x).$$

Since $|\alpha_{n,x}(f)| \le (n-1)D_N(f)$, we conclude that $\alpha_{n,x}$ is bounded with respect to the norm $D_N + D'$, and hence $\alpha_{n,x}$ induces a bounded operator $\bar{\alpha}_{n,x}\colon \hat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G \to \mathbb{R}$. Since

$$\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G = \bigcap_{n\in\mathbb{Z},x\in N} \mathrm{Ker}(\bar{\alpha}_{n,x}),$$

the space $\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G$ is a closed subspace of the Banach space $\hat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G$ (see Lemma 7.6). Since $D' = 0$ on $\mathrm{Q}(N)^G$ [60, Lemma 2.1], $(\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G, D_N)$ is a Banach space. □

Proof of Theorem 2.1(1) It is clear that $\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x)$ for every $x \in [G,N]$. Hence, it suffices to show that there exists $C > 1$ such that, for every $x \in [G,N]$, we have $\mathrm{scl}_{G,N}(x) \le C \cdot \mathrm{scl}_G(x)$.

It follows from Theorem 7.4 that $(\mathrm{Q}(G)/\mathrm{H}^1(G), D_G)$ and $(\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G, D_N)$ are Banach spaces. Let $T\colon \mathrm{Q}(G)/\mathrm{H}^1(G) \to \mathrm{Q}(N)^G/\mathrm{H}^1(N)^G$ be the bounded operator induced by the restriction $\mathrm{Q}(G) \to \mathrm{Q}(N)^G$. Let X be the kernel of T. Then T induces a bounded operator

$$\bar{T}\colon (\mathrm{Q}(G)/\mathrm{H}^1(G))/X \to \mathrm{Q}(N)^G/\mathrm{H}^1(N)^G.$$

The assumption $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ implies that the map T is surjective, and hence $\bar{T}$ is a bijective bounded operator. By the open mapping theorem, the inverse $S = \bar{T}^{-1}$ is a bounded operator, and we set $C = \|S\| + 1$, where $\|S\|$ denotes the operator norm of S. Then, for every $[f] \in \mathrm{Q}(N)^G/\mathrm{H}^1(N)^G$, there exists $f' \in \mathrm{Q}(G)$ such that $D_G(f') \le C \cdot D_N(f)$ and $f'|_N - f \in \mathrm{H}^1(N)^G$. Hence, Proposition 7.2 implies that

$$\mathrm{scl}_G \le \mathrm{scl}_{G,N} \le C \cdot \mathrm{scl}_G$$

on $[G,N]$. □

7.2 Proof of Theorem 2.1(2)

Here we recall the definition of the seminorm on $\mathrm{H}_b^n(G)$. The space $C_b^n(G;\mathbb{R})$ of bounded n-cochains on G is a Banach space with respect to the ∞-norm

$$\|\varphi\|_\infty = \sup\{|\varphi(x_1,\ldots,x_n)| : x_1,\ldots,x_n \in G\}.$$

Since $\mathrm{H}_b^n(G) = \mathrm{H}_b^n(G;\mathbb{R})$ is a subquotient of $C_b^n(G;\mathbb{R})$, it has a seminorm induced by the norm on $C_b^n(G;\mathbb{R})$. Namely, for $\alpha \in \mathrm{H}_b^n(G)$, the seminorm $\|\alpha\|$ on $\mathrm{H}_b^n(G)$ is defined by

$$\|\alpha\| = \inf\{\|\varphi\|_\infty \mid \varphi \text{ is a bounded } n\text{-cocycle on } G \text{ satisfying } [\varphi]=\alpha\}.$$

Theorem 7.8 (see [74, Proposition 8.6.6]) *If $\Gamma = G/N$ is amenable, then the map $\mathrm{H}_b^2(G) \to \mathrm{H}_b^2(N)^G$ is an isometric isomorphism.*

We recall the following estimate of the defect of the homogenization:

Lemma 7.9 [26, Lemma 2.58] *Let $\delta\colon \mathrm{Q}(G) \to \mathrm{H}_b^2(G)$ be the natural map. Then*

$$\|[\delta f]\| \le D_G(f) \le 2\cdot \|[\delta f]\|.$$

Proof of Theorem 2.1(2) Suppose that $\Gamma = G/N$ is amenable and $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. Let $f \in \mathrm{Q}(G)$. By Corollary 7.3, it suffices to show that $2D_N(f|_N) \ge D_G(f)$ for every $f \in \mathrm{Q}(G)$. This follows from

$$2D_N(f|_N) \ge 2\|[\delta f|_N]\| = 2\|[\delta f]\| \ge D_G(f),$$

where the equality and the inequalities are deduced from Theorem 7.8 and Lemma 7.9, respectively. □

7.3 Proof of Theorem 2.1(3)

Lemma 7.10 *Let $f\colon N \to \mathbb{R}$ be an extendable homogeneous quasimorphism on N. Then, for each $a, b \in G$ satisfying $[a,b] \in N$,*

$$|f([a,b])| \le D_N(f).$$

Proof We first prove

$$[a^n,b] = a^{n-1}[a,b]a^{-(n-1)} \cdot a^{(n-2)}[a,b]a^{-(n-2)} \cdots [a,b]. \tag{7-1}$$

Indeed,

$$[a^n,b] = a^nba^{-n}b^{-1} = a^{n-1}\cdot aba^{-1}b^{-1}\cdot a^{-(n-1)}\cdot a^{n-1}ba^{-(n-1)}b^{-1} = a^{n-1}[a,b]a^{-(n-1)}\cdot[a^{n-1},b].$$

By induction on n, (7-1) follows. Since f is G-invariant, we have

$$f([a^n,b]) \approx_{(n-1)D_N(f)} f(a^{n-1}[a,b]a^{-(n-1)}) + \cdots + f([a,b]) = n\cdot f([a,b]).$$

Therefore,

$$|f([a^n,b])| \geq n \cdot \big(|f([a,b])| - D_N(f)\big).$$

Suppose that $|f([a,b])| > D_N(f)$. Then the right side of the above inequality can be unbounded with respect to n. However, since f is extendable, the left side of the above inequality is bounded. This is a contradiction. □

In Corollary 6.20, we provide a condition that ensures a G-invariant homomorphism $f\colon N \to \mathbb{R}$ cannot be extended to G as a quasimorphism. Here we present another condition:

Corollary 7.11 *Let $f\colon N \to \mathbb{R}$ be a G-invariant homomorphism and assume that N is generated by simple commutators of G. If f is nonzero, then f is not extendable.*

Proof If f is extendable, then Lemma 7.10 implies that $f(c) = 0$ for every simple commutator c of G contained in N. Since N is generated by simple commutators of G, this means that $f = 0$. □

Lemma 7.12 *Let f be a homogeneous quasimorphism on G, and assume that $\Gamma = G/N$ is solvable. Then $D_G(f) = D_N(f|_N)$.*

Proof We first assume that Γ is abelian. It is known that $D_G(f) = \sup_{a,b\in G} |f([a,b])|$ (see [26, Lemma 2.24]). Applying Lemma 7.10 to $f|_N$, we have

$$D_G(f) = \sup_{a,b\in G} |f([a,b])| \leq D_N(f|_N) \leq D_G(f),$$

and, in particular, $D_G(f) = D_N(f|_N)$.

Next we consider the general case. Let $G^{(n)}$ denote the n^{th} derived subgroup of G. Then there exists a positive integer n such that $G^{(n)} \subset N$ since Γ is solvable. By the previous paragraph,

$$D_G(f) = D_{G^{(1)}}(f|_{G^{(1)}}) = \cdots = D_{G^{(n)}}(f|_{G^{(n)}}) \leq D_N(f|_N) \leq D_G(f). \qquad \square$$

Proof of Theorem 2.1(3) Combine Lemma 7.12 and Corollary 7.3. □

Next we provide some applications of Theorem 2.1(3):

Corollary 7.13 *If one of the following conditions holds, then $\mathrm{scl}_G = \mathrm{scl}_{G,N}$ on $[G,N]$ (here $\Gamma = G/N$):*

(1) *Γ is a finite solvable group.*

(2) *Γ is a finitely generated abelian group whose rank is at most 1.*

Proof This clearly follows from Proposition 3.4 and Theorem 2.1(3). □

In Section 9.2, we propose several problems on the coincidence and equivalence of scl_G and $\mathrm{scl}_{G,N}$.

7.4 Examples with nonequivalent scl_G and $\mathrm{scl}_{G,N}$

To conclude this section, we provide some examples of group pairs (G,N) for which scl_G and $\mathrm{scl}_{G,N}$ *fail* to be bi-Lipschitzly equivalent on $[G,N]$.

Example 7.14 Let l be an integer at least 2, and Ω be an area form of Σ_l. In this case, the flux group Γ_Ω is known to be trivial; thus we have the volume flux homomorphism $\mathrm{Flux}_\Omega\colon \mathrm{Diff}_0(\Sigma,\Omega)\to \mathrm{H}^1(\Sigma_l)$. In [58], the authors proved that, for the pair

$$(G,N)=(\mathrm{Diff}_0(\Sigma_l,\Omega),\mathrm{Ker}(\mathrm{Flux}_\Omega)),$$

scl_G and $\mathrm{scl}_{G,N}$ are *not* bi-Lipschitzly equivalent on $[G,N]$. More precisely, we found an element γ in $[G,N]$ such that

$$\mathrm{scl}_G(\gamma)=0 \quad \text{but} \quad \mathrm{scl}_{G,N}(\gamma)>0.$$

Example 7.15 We can provide the following example, which is related to Example 7.14 with smaller G, from results in [61]. We stick to the setting of Example 7.14. Take an arbitrary pair (v,w) with $v,w\in \mathrm{H}^1(\Sigma_l)$ that satisfies

$$v\smile w\neq 0. \tag{7-2}$$

Here, recall from Theorem 1.3 that $\smile\colon \mathrm{H}^1(\Sigma_l)\times \mathrm{H}^1(\Sigma_l)\to \mathrm{H}^2(\Sigma_l)\cong\mathbb{R}$ denotes the cup product. Then, from results in [61], we can deduce the following: there exists a positive integer k_0, depending only on w and the area of Σ_l, such that, for every $k\geq k_0$, if we set

$$\Lambda_k=\langle v,w/k\rangle,$$

namely the subgroup of $\mathrm{H}^1(\Sigma_l)$ generated by v and w/k, and

$$(G,N)=(\mathrm{Flux}_\Omega^{-1}(\Lambda_k),\mathrm{Ker}(\mathrm{Flux}_\Omega)),$$

then scl_G and $\mathrm{scl}_{G,N}$ are *not* bi-Lipschitzly equivalent on $[G,N]$. To see this, by following arguments in [61, Section 4], we construct a sequence $(\gamma_m)_{m\in\mathbb{N}}$ in $[G,N]$. Then [61, Proposition 4.6], together with Bavard's duality theorem, implies that

$$\sup_{m\in\mathbb{N}} \mathrm{scl}_G(\gamma_m)\leq \tfrac{3}{2}.$$

Contrastingly, Proposition 4.7(3) in [61], together with Theorem 7.1, implies that

$$\liminf_{m\to\infty}\frac{\mathrm{scl}_{G,N}(\gamma_m)}{m}\geq \frac{1}{2k\cdot D_N(f_P)}|\mathfrak{b}_I(v,w)|>0.$$

Here $\mathfrak{b}_I(v,w)=\langle v\smile w,[\Sigma_l]\rangle_{\Sigma_l}\in\mathbb{R}$ is the intersection number of v and w, where $[\Sigma_l]$ is the fundamental class of Σ_l and $\langle\,\cdot\,,\cdot\,\rangle_{\Sigma_l}\colon \mathrm{H}^2(\Sigma_l)\times \mathrm{H}_2(\Sigma_l)\to\mathbb{R}$ denotes the Kronecker pairing of Σ_l. The map $f_P\colon N\to\mathbb{R}$ is Py's Calabi quasimorphism (recall Section 5; see also [61, Sections 2.4 and 2.5]). We also note that v, w and f_P here correspond to $\bar{v}$, $\bar{w}$ and μ_P in [61], respectively.

8 Aut(F_n) and Mod(Σ_l)

8.1 Proof of Theorem 2.3

An *IA-automorphism* of a group G is an automorphism f on G which acts as the identity on the abelianization $\mathrm{H}_1(G;\mathbb{Z})$ of G. We write IA_n to indicate the group of IA-automorphisms on F_n. Then we have exact sequences

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n,\mathbb{Z}) \to 1, \quad 1 \to \mathrm{IA}_n \to \mathrm{Aut}_+(F_n) \to \mathrm{SL}(n,\mathbb{Z}) \to 1.$$

Theorem 2.3(1) claims that $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}(F_n)} = i^*\mathrm{Q}(\mathrm{Aut}(F_n))$ and $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}_+(F_n)} = i^*\mathrm{Q}(\mathrm{Aut}_+(F_n))$. To show it, we use the following facts, which can be derived from the computation of the second integral homology $\mathrm{H}_2(\mathrm{SL}(n,\mathbb{Z});\mathbb{Z})$:

Theorem 8.1 (see [72]) *For $n \geq 3$, $\mathrm{H}^2(\mathrm{SL}(n,\mathbb{Z})) = 0$ and $\mathrm{H}^2(\mathrm{GL}(n,\mathbb{Z})) = 0$.*

The following is obtained from [77, Corollary 1.4; 75, Theorem 1.2]:

Theorem 8.2 *Let n be an integer at least 3 and Γ_0 a subgroup of finite index in $\mathrm{SL}(n,\mathbb{Z})$. Then $\mathrm{H}^3_b(\Gamma_0) = 0$.*

The following theorem is a special case of [74, Proposition 8.6.2]:

Theorem 8.3 *Let N be a subgroup of finite index in G and V a Banach G-module; then the restriction $\mathrm{H}^n_b(G;V) \to \mathrm{H}^n_b(N;V)$ is injective for every $n \geq 0$.*

Now we proceed to the proof of Theorem 2.3(1). First we show the following lemma:

Lemma 8.4 *Let n be an integer at least 3 and Γ_0 a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$. Then $\mathrm{H}^3_b(\Gamma_0) = 0$.*

Proof Since $\Gamma_0 \cap \mathrm{SL}(n,\mathbb{Z})$ is a subgroup of finite index in $\mathrm{SL}(n,\mathbb{Z})$, we have $\mathrm{H}^3_b(\Gamma_0 \cap \mathrm{SL}(n,\mathbb{Z})) = 0$ by Theorem 8.2. Since $\Gamma_0 \cap \mathrm{SL}(n,\mathbb{Z})$ is a subgroup of finite index in Γ_0, we obtain $\mathrm{H}^3_b(\Gamma_0) = 0$ by Theorem 8.3. □

Proof of Theorem 2.3(1) Suppose that $n = 2$. Then $\mathrm{GL}(n,\mathbb{Z})$ and $\mathrm{SL}(n,\mathbb{Z})$ have a subgroup of finite index which is isomorphic to a free group. Therefore this case is proved by Proposition 3.4. So suppose that $n > 2$. Let Γ be either $\mathrm{GL}(n,\mathbb{Z})$ or $\mathrm{SL}(n,\mathbb{Z})$. By Theorem 8.1, Lemma 8.4 and the cohomology long exact sequence, we have $\mathrm{H}^2_{/b}(\Gamma) = 0$. Hence, Theorem 1.5 implies that $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}(F_n)}/i^*\mathrm{Q}(\mathrm{Aut}(F_n)) = 0$ and $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}_+(F_n)}/i^*\mathrm{Q}(\mathrm{Aut}_+(F_n)) = 0$. □

Next we prove Theorem 2.3(2). First we recall the following application of the transfer:

Lemma 8.5 (see [19, Proposition III.10.4]) *Let N be a normal subgroup of finite index in G, V a real G-module and q a positive integer. Then the restriction induces an isomorphism $\mathrm{H}^q(G;V) \xrightarrow{\cong} \mathrm{H}^q(N;V)^{\Gamma}$, where $\Gamma = G/N$.*

In the proof of Theorem 2.3(2), we furthermore use the following theorem, due to Borel [14; 15; 16] and Hain [47] (and Tshishiku [94]):

Theorem 8.6 (1) *For every $n \geq 6$ and every subgroup Γ_0 of finite index in $\mathrm{GL}(n,\mathbb{Z})$, $\mathrm{H}^2(\Gamma_0) = 0$.*

(2) *For every $l \geq 3$ and every subgroup Γ_0 of finite index in $\mathrm{Sp}(2l,\mathbb{Z})$, the inclusion map $\Gamma_0 \hookrightarrow \mathrm{Sp}(2l,\mathbb{Z})$ induces an isomorphism of cohomology $\mathrm{H}^2(\mathrm{Sp}(2l,\mathbb{Z})) \cong \mathrm{H}^2(\Gamma_0)$. In particular, the cohomology $\mathrm{H}^2(\Gamma_0)$ is isomorphic to $\mathbb{R}$.*

For the convenience of the reader, we describe the deduction of Theorem 8.6 from the work of Borel, Hain and Tshishiku.

Proof First we discuss (2). This is stated in [47, Theorem 3.2]; see [94] for the complete proof.

Next we treat (1). Let $\Lambda = \Gamma_0 \cap \mathrm{SL}(n,\mathbb{Z})$. Then Λ is a subgroup of finite index both in Γ_0 and in $\mathrm{SL}(n,\mathbb{Z})$. By Lemma 8.5, the restriction $\mathrm{H}^2(\Gamma_0) \to \mathrm{H}^2(\Lambda)$ is injective. Hence, to prove (1), it suffices to show that $\mathrm{H}^2(\Lambda) = 0$. In what follows, we sketch the deduction of $\mathrm{H}^2(\Lambda) = 0$ from the work of Borel; see also the discussion in the introduction of [94].

We appeal to Borel's theorem [15, Theorem 1], with $G = \mathrm{SL}_n$, $\Gamma = \Lambda$ and r the trivial complex representation. (See also [14, Theorem 11.1].) Then there exists a natural homomorphism $\mathrm{H}^q(\mathfrak{g},\mathfrak{k};\mathbb{C})^{\Lambda} \to \mathrm{H}^q(\Lambda;\mathbb{C})$ and, if $q \leq \min\{c(\mathrm{SL}_n), \mathrm{rank}_{\mathbb{R}}(\mathrm{SL}(n,\mathbb{R})) - 1\}$, then this map is an isomorphism. Here $\mathrm{H}^q(\mathfrak{g},\mathfrak{k};\mathbb{C})^{\Lambda}$ is a Lie algebraic cohomology ($\mathfrak{g}$ and $\mathfrak{k}$ stand for the Lie algebras of $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n)$, respectively); it is known that $\mathrm{H}^2(\mathfrak{g},\mathfrak{k};\mathbb{C})^{\Lambda} = 0$; see [14, 11.4]. For the definition of the constant $c(G) = c(G,0)$ for G being a connected semisimple group defined over $\mathbb{Q}$, see [14, 7.1]. We remark that, for the trivial complex representation r, $c(G,r) = (c(G,0) =) c(G)$. Since SL_n is of type A_{n-1}, the constant $c(\mathrm{SL}_n)$ equals $\lfloor \frac{1}{2}(n-2) \rfloor$; see [15] (and [14, 9.1]). Here, $\lfloor \cdot \rfloor$ denotes the floor function. The number $\mathrm{rank}_{\mathbb{R}}(\mathrm{SL}(n,\mathbb{R}))$ means the real rank of $\mathrm{SL}(n,\mathbb{R})$; it equals $n-1$. Since $n \geq 6$, we hence have $\min\{c(\mathrm{SL}_n), \mathrm{rank}_{\mathbb{R}}(\mathrm{SL}(n,\mathbb{R})) - 1\} \geq 2$. Therefore, $\mathrm{H}^2(\Lambda;\mathbb{C}) = 0$. This immediately implies that $\mathrm{H}^2(\Lambda) = 0$, as desired. (Borel [16] considered a better constant $C(G)$ than $c(G)$ in general, but $C(\mathrm{SL}_n) = \lfloor \frac{1}{2}(n-2) \rfloor$; see [16; 94].) □

Remark 8.7 In the proof of Theorem 2.3(2), we only use Theorem 8.6(1). We will use Theorem 8.6(2) in the proofs of claims in the next subsection.

concept	defect	definition	vector space
quasicocycle on G	D	$F(g_1g_2) \approx_D F(g_1) + g_1 \cdot F(g_2)$	$\hat{\mathrm{Q}}Z(G;V)$
G-quasiequivalent quasimorphism on N	D, D'	$f(x_1x_2) \approx_D f(x_1) + f(x_2),\ f(gxg^{-1}) \approx_{D'} g \cdot f(x)$	$\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$
N-quasicocycle on G	D''	$F(gx) \approx_{D''} F(g) + g \cdot F(x),\ F(xg) \approx_{D''} F(x) + F(g)$	$\hat{\mathrm{Q}}Z_N(G;V)$

Table 2: The concepts and symbols on quasicocycles.

Proof of Theorem 2.3(2) Let n be an integer at least 6. Let G be a group of finite index in $\mathrm{Aut}(F_n)$. Set $N = G \cap \mathrm{IA}_n$ and $\Gamma = G/N$. Then we have an exact sequence

$$1 \to N \to G \to \Gamma \to 1$$

and Γ is a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$. By Lemma 8.4 and Theorem 8.6(1), the second relative cohomology group $\mathrm{H}^2_{/b}(\Gamma)$ is trivial. Therefore, by Theorem 1.5, $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G) = 0$. □

Remark 8.8 By [80, Theorem 1.4] and Theorem 8.2, for every $n \geq 3$, every subgroup of finite index in $\mathrm{SL}(n,\mathbb{Z})$ is boundedly 3-acyclic.

8.2 Quasicocycle analogues of Theorem 2.3: finite-dimensional unitary coefficients

To state our next result, we need some notation. In Section 7.1, we introduced the notion of G-quasiequivariant quasimorphism. Let V be an $\mathbb{R}[G]$-module whose G-action on V is trivial at N. The G-quasi-invariance can be extended to the V-valued quasimorphisms as the G-quasiequivariance. Recall from Remark 6.4 that a V-valued quasimorphism $f: N \to V$ is G-equivariant if $f(gxg^{-1}) - g \cdot f(x) = 0$. A V-valued quasimorphism $f: N \to V$ is said to be G*-quasiequivariant* if the number

$$D'(f) = \sup_{g \in G, x \in N} \|f(gxg^{-1}) - g \cdot f(x)\|$$

is finite. Let $\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$ denote the $\mathbb{R}$-vector space of all G-quasiequivariant V-valued quasimorphisms. Let $F: G \to V$ be a quasicocycle; then the restriction $F|_N$ belongs to $\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$ by definition. It is straightforward to show that $\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}/i^*\hat{\mathrm{Q}}Z(G;V)$ is isomorphic to $\mathrm{Q}(N;V)^G/i^*\mathrm{H}^1_{/b}(G;V) = \mathrm{H}^1_{/b}(N;V)^G/i^*\mathrm{H}^1_{/b}(G;V)$. We summarize the concepts and symbols on quasicocycles in Table 2.

Our main results in this section are the following two theorems. In fact, in Section 8.3, we will deduce further generalizations (Theorems 8.17 and 8.14) of Theorems 8.9 and 8.10, respectively, from a recent result of Bader and Sauer [3].

Theorem 8.9 (result for $\mathrm{Aut}(F_n)$) *Let n be an integer at least 6 and G a subgroup of finite index in $\mathrm{Aut}(F_n)$. Then, for every finite-dimensional unitary representation π of Γ,*

$$\hat{\mathrm{Q}}(N;\mathcal{H})^{\mathrm{Q}G} = i^*\hat{\mathrm{Q}}Z(G;\bar{\pi},\mathcal{H}).$$

Here $(\bar{\pi},\mathcal{H})$ is the pullback to G of the representation $(\pi,\mathcal{H})$ of Γ.

Theorem 8.10 (result for $\mathrm{Mod}(\Sigma_l)$) *Let l be an integer at least 3 and G a subgroup of finite index in $\mathrm{Mod}(\Sigma_l)$. Set $N = G \cap \mathcal{I}(\Sigma_l)$ and $\Gamma = G/N$. Let $(\pi, \mathcal{H})$ be a finite-dimensional unitary Γ-representation such that $\pi \not\supset 1$, ie $\mathcal{H}^{\pi(\Gamma)} = 0$. Then*

$$\widehat{\mathrm{Q}}(N; \mathcal{H})^{\mathrm{Q}G} = i^* \widehat{\mathrm{Q}}Z(G; \bar{\pi}, \mathcal{H}).$$

Here $\bar{\pi}$ is the pullback of π by the quotient homomorphism $G \to \Gamma$.

Before proceeding to the proofs of Theorems 8.9 and 8.10, we mention some known results we need in the proofs. The following theorem is well known (see [7, Corollaries 4.C.16 and 4.B.6]):

Theorem 8.11 *Let Γ_0 be a subgroup of finite index in $\mathrm{GL}(n, \mathbb{Z})$ for $n \geq 3$ or $\mathrm{Sp}(2l, \mathbb{Z})$ for $l \geq 3$, and $(\pi, \mathcal{H})$ a finite-dimensional unitary Γ_0-representation. Then $\Gamma_0(\pi) := \mathrm{Ker}(\pi : \Gamma_0 \to \mathcal{U}(\mathcal{H}))$ is a subgroup of finite index in Γ_0, where $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$.*

Theorem 8.12 [77, Corollary 1.6] *Let l be an integer at least 2 and Γ_0 a subgroup of finite index in $\mathrm{Sp}(2l, \mathbb{Z})$. Let $(\pi, \mathcal{H})$ be a unitary Γ_0-representation with $\mathcal{H}$ separable and $\pi \not\supset 1$. Then $\mathrm{H}_b^3(\Gamma_0; \pi, \mathcal{H}) = 0$.*

Corollary 8.13 (1) *Let n be an integer at least 6 and Γ_0 a subgroup of finite index in $\mathrm{GL}(n, \mathbb{Z})$. Let $(\pi, \mathcal{H})$ be a finite-dimensional unitary Γ_0-representation. Then $\mathrm{H}^2(\Gamma_0; \pi, \mathcal{H}) = 0$.*

(2) *Let l be an integer at least 3, Γ_0 a subgroup of finite index in $\mathrm{Sp}(2l, \mathbb{Z})$, and $(\pi, \mathcal{H})$ a finite-dimensional unitary Γ_0-representation such that $\pi \not\supset 1$. Then $\mathrm{H}^2(\Gamma_0; \pi, \mathcal{H}) = 0$.*

Proof We first prove (2). Set $\Gamma_0(\pi) = \mathrm{Ker}(\pi)$. Then Theorem 8.11 implies that $\Gamma_0(\pi)$ is of finite index in Γ_0. Hence, by Lemma 8.5, $\mathrm{H}^2(\Gamma_0; \pi, \mathcal{H}) \cong \mathrm{H}^2(\Gamma_0(\pi); \mathcal{H})^{\Gamma_0/\Gamma_0(\pi)}$.

We now show the following claims:

Claim *The conjugation action by Γ_0 on the cohomology $\mathrm{H}^2(\Gamma_0(\pi))$ is trivial.*

Proof By Theorems 8.11 and 8.6(2), the inclusion $i : \Gamma_0(\pi) \hookrightarrow \Gamma_0$ induces an isomorphism $i^* : \mathrm{H}^2(\Gamma_0) \cong \mathrm{H}^2(\Gamma_0(\pi))$. Lemma 8.5 implies that the map i in fact induces an isomorphism $\mathrm{H}^2(\Gamma_0) \cong \mathrm{H}^2(\Gamma_0(\pi))^{\Gamma_0}$. Therefore, the Γ_0-action on $\mathrm{H}^2(\Gamma_0(\pi))$ is trivial. ◁

Claim *There exists a canonical isomorphism $\mathrm{H}^2(\Gamma_0(\pi); \mathcal{H}) \cong \mathcal{H}$, and this induces an isomorphism $\mathrm{H}^2(\Gamma_0(\pi); \mathcal{H})^{\Gamma_0/\Gamma_0(\pi)} \cong \mathcal{H}^{\Gamma_0/\Gamma_0(\pi)}$.*

Proof By Theorem 8.6(2), the cohomology $\mathrm{H}^2(\Gamma_0(\pi))$ is isomorphic to $\mathbb{R}$; hence, by the universal coefficient theorem, the cohomology $\mathrm{H}^2(\Gamma_0(\pi); \mathcal{H})$ is isomorphic to $\mathcal{H}$ (here, note that $\mathcal{H}$ is a trivial $\mathbb{R}[\Gamma_0(\pi)]$-module of finite dimension). In what follows, we exhibit a concrete isomorphism. For $\alpha \in \mathcal{H}$, we define a cochain $c_\alpha \in C^2(\Gamma_0(\pi); \mathcal{H})$ by

$$c_\alpha(\gamma_1, \gamma_2) = c(\gamma_1, \gamma_2) \cdot \alpha \in \mathcal{H},$$

where $c \in C^2(\Gamma_0(\pi))$ is a cocycle whose cohomology class corresponds to $1 \in \mathbb{R}$ under the isomorphism $\mathrm{H}^2(\Gamma_0(\pi)) \cong \mathbb{R}$. This cochain c_α is a cocycle since the $\Gamma_0(\pi)$-action on $\mathcal{H}$ is trivial. Then the map sending α to $[c_\alpha]$ gives rise to an isomorphism $\mathcal{H} \xrightarrow{\cong} \mathrm{H}^2(\Gamma_0(\pi);\mathcal{H})$. For $\gamma \in \Gamma_0$ and $\gamma_1, \gamma_2 \in \Gamma_0(\pi)$,

$$({}^\gamma c_\alpha)(\gamma_1,\gamma_2) = \pi(\gamma)\cdot c_\alpha(\gamma^{-1}\gamma_1\gamma, \gamma^{-1}\gamma_2\gamma) = \pi(\gamma)\cdot(({}^\gamma c)(\gamma_1,\gamma_2)\cdot\alpha) = ({}^\gamma c)(\gamma_1,\gamma_2)\cdot(\pi(\gamma)\cdot\alpha).$$

Moreover, by the claim above, there exists a cochain $b \in C^1(\Gamma_0(\pi))$ satisfying ${}^\gamma c = c + \delta b$. Hence,

$$({}^\gamma c_\alpha)(\gamma_1,\gamma_2) = ({}^\gamma c)(\gamma_1,\gamma_2)\cdot(\pi(\gamma)\cdot\alpha) = (c+\delta b)(\gamma_1,\gamma_2)\cdot(\pi(\gamma)\cdot\alpha) = (c+\delta b)_{\pi(\gamma)\cdot\alpha}(\gamma_1,\gamma_2).$$

Therefore the cohomology class ${}^\gamma[c_\alpha]$ corresponds to the element $\pi(\gamma)\cdot\alpha$ under the isomorphism, and this implies the claim. ◁

By the claims above and the assumption that π does not contain the trivial representation, we have $\mathrm{H}^2(\Gamma_0;\pi,\mathcal{H}) = 0$. This completes the proof of (2).

We can deduce (1) by the same arguments as above with Lemma 8.5 and Theorems 8.6 and 8.11. □

Proof of Theorem 8.9 Let n be an integer at least 6. Let G be a group of finite index in $\mathrm{Aut}(F_n)$. Set $N = G \cap \mathrm{IA}_n$ and $\Gamma = G/N$. Then we have an exact sequence

$$1 \to N \to G \to \Gamma \to 1$$

and Γ is a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$. Let $(\pi,\mathcal{H})$ be a finite-dimensional unitary Γ-representation. Set $\Gamma(\pi) = \mathrm{Ker}(\pi)$. By Theorem 8.11, $\Gamma(\pi)$ is a normal subgroup of finite index in Γ. By using Lemma 8.4, we can show that $\mathrm{H}^3_b(\Gamma(\pi);\mathcal{H}) = 0$. Here, we use [74, Corollary 8.2.10]; see also the proof of Theorem 8.17 in the next subsection. Together with Theorem 8.3, we obtain $\mathrm{H}^3_b(\Gamma;\pi,\mathcal{H}) = 0$. Hence, by Corollary 8.13(1), $\mathrm{H}^2_{/b}(\Gamma;\pi,\mathcal{H}) = 0$. Therefore, the quotient $\mathrm{H}^1_{/b}(N;\mathcal{H})/i^*\mathrm{H}^1_{/b}(G;\pi,\mathcal{H})$ is trivial by Theorem 1.5. Since $\mathrm{H}^1_{/b}(N;\mathcal{H})/i^*\mathrm{H}^1_{/b}(G;\pi,\mathcal{H})$ is isomorphic to $\widehat{\mathrm{Q}}(N;\mathcal{H})^{\mathrm{Q}G}/i^*\widehat{\mathrm{Q}}Z(G;\pi,\mathcal{H})$, this completes the proof. □

Proof of Theorem 8.10 Let l be an integer at least 3. Let G be a subgroup of finite index in $\mathrm{Mod}(\Sigma_l)$. Set $N = G \cap \mathcal{I}(\Sigma_l)$ and $\Gamma = G/N$. Let $(\pi,\mathcal{H})$ be a finite-dimensional unitary Γ-representation not containing the trivial representation. Then Theorem 8.12 and Corollary 8.13(2) imply that the second relative cohomology group $\mathrm{H}^2_{/b}(\Gamma;\pi,\mathcal{H})$ is trivial. Hence, by arguments similar to those in the proof of Theorem 8.9, we obtain the theorem. □

8.3 Quasicocycle analogues of Theorem 2.3: including infinite-dimensional unitary coefficients

After the submitted version of this paper was completed, work of Bader and Sauer [3] on vanishing of higher group cohomology with unitary coefficients for higher-rank Lie groups and their lattices has come out. This work enables us to improve Theorems 8.10 and 8.9 to the following Theorems 8.14 and 8.17, respectively. For the reader's convenience, we add this subsection to state and deduce these two theorems. We are grateful to one of the referees for informing us of [3]. We start with the following strengthening of Theorem 8.10:

Theorem 8.14 (stronger result for $\mathrm{Mod}(\Sigma_l)$) *Fix an integer l at least 3. Then, for every subgroup G of finite index in $\mathrm{Mod}(\Sigma_l)$ and every unitary representation $(\pi, \mathcal{H})$ of Γ such that $\pi \not\supset 1$,*

$$\widehat{\mathrm{Q}}(N; \mathcal{H})^{\mathrm{Q}G} = i^* \widehat{\mathrm{Q}}Z(G; \bar{\pi}, \mathcal{H}).$$

Here we set $N = G \cap \mathcal{I}(\Sigma_l)$ and $\Gamma = G/N$; the representation $(\bar{\pi}, \mathcal{H})$ of G is the pullback of the representation $(\pi, \mathcal{H})$ of Γ.

For the proof of Theorem 8.14, we employ the following definitions and theorem:

Definition 8.15 Let H be a locally compact second countable group. Let k be a positive integer.

(1) The group H is said to be *strongly k-Kazhdan* [30] or, alternatively, to have *property* $[\mathrm{T}_k]$ [2; 3] if, for every (strongly continuous) unitary H-representation $(\pi, \mathcal{H})$ and every positive integer i with $i \leq k$, the continuous cohomology $\mathrm{H}_c^i(H; \pi, \mathcal{H})$ vanishes.

(2) [2; 3] The group H is said to have *property* (T_k) if, for every (strongly continuous) unitary H-representation $(\pi, \mathcal{H})$ *with* $\pi \not\supset 1$ and every positive integer i with $i \leq k$, the continuous cohomology $\mathrm{H}_c^i(H; \pi, \mathcal{H})$ vanishes.

The celebrated Delorme–Guichardet theorem states that property $[\mathrm{T}_1]$ (the strong 1-Kazhdan property) and property (T_1) (for a locally compact second countable group) are both equivalent to Kazhdan's property (T); see [8] for details. By definition, for positive integers k_1 and k_2 with $k_1 \geq k_2$, property (T_{k_1}) implies property (T_{k_2}), and property $[\mathrm{T}_{k_1}]$ implies property $[\mathrm{T}_{k_2}]$ (we also remark that, if Λ is a discrete group, then, for a unitary Λ-representation $(\pi, \mathcal{H})$, continuous cohomology $\mathrm{H}_c^\bullet(\Lambda; \pi, \mathcal{H})$ coincides with $\mathrm{H}^\bullet(\Lambda; \pi, \mathcal{H})$). The main motivation in [30] comes from their theorem [30, Theorem 1.2], which states that a finitely presented group with property $[\mathrm{T}_2]$ is Frobenius stable. For applications of property (T_2) to the Frobenius stability, see [2].

Theorem 8.16 [3, Theorem B and Appendix A] *Let H be a connected semisimple Lie group with a finite center. Then H has property $(\mathrm{T}_{r_0(H)-1})$ (as a topological group), where $r_0(H)$ is the invariant given in* [3, Appendix A]. *Let Λ be an irreducible lattice in H. Then Λ has property (T_{r-1}), where $r = \min\{r_0(H), \mathrm{rank}_{\mathbb{R}}(H)\}$.*

Here, both for $H = \mathrm{SL}(n+1, \mathbb{R})$ with n at least 2 and for $H = \mathrm{Sp}(2n, \mathbb{R})$ with n at least 3, we have $r_0(H) = n = \mathrm{rank}_{\mathbb{R}}(H)$.

We note that $\mathrm{H}^2(\mathrm{Sp}(2n, \mathbb{Z})) = \mathbb{R}$ for every integer n at least 2; see [14]. In particular, $\mathrm{Sp}(2(n+1), \mathbb{Z})$ with $n \geq 2$ *fails* to have property $[\mathrm{T}_n]$, whereas Theorem 8.16 ensures property (T_n) for this group.

Proof of Theorem 8.14 using Theorem 8.16 First we prove the theorem under the additional assumption that $\mathcal{H}$ is separable. We appeal to Theorem 8.16 by setting $H = \mathrm{Sp}(2l, \mathbb{R})$ and $\Lambda = \Gamma$ (recall that l is assumed to be at least 3). Then Γ has property (T_2). Therefore, $\mathrm{H}^2(\Gamma; \pi, \mathcal{H}) = 0$. Since $\pi \not\supset 1$, Theorem 8.12 (Monod's theorem) shows that $\mathrm{H}_b^3(\Gamma; \pi, \mathcal{H}) = 0$. Now Theorem 1.5, together with the exact sequence (3-1) and Remark 1.7, ends our proof for the case where $\mathcal{H}$ is separable.

Now we treat the general case; here, we use the following trick to reduce to the separable case. Suppose that

$$\hat{\mathrm{Q}}(N;\mathscr{H})^{\mathrm{Q}G} \setminus i^*\hat{\mathrm{Q}}Z(G;\bar{\pi},\mathscr{H}) \neq \varnothing.$$

Take some f from the left-hand side. Let $\mathscr{K}$ be the closure of the linear span of $\{\pi(\gamma)f(x) \mid \gamma\in\Gamma,\ x\in N\}$ in $\mathscr{H}$. Then $\mathscr{K}$ is a closed $\pi(\Gamma)$-invariant subspace in $\mathscr{H}$ that is *separable* (note that Γ and N are both countable). Hence, we can view $\mathscr{K}$ as a unitary Γ-subrepresentation space of $\mathscr{H}$; we write $(\sigma,\mathscr{K})$ for this representation. We define a unitary G-representation $(\bar{\sigma},\mathscr{K})$ by taking the pullback of σ. Thus, we can view f as an element in $\hat{\mathrm{Q}}(N;\mathscr{K})^{\mathrm{Q}G}$, where we view $(\bar{\sigma},\mathscr{K})$ as an $\mathbb{R}[G]$-module. Since $\mathscr{K}$ is separable and $\sigma \not\supset 1$, the first paragraph of this proof implies that

$$\hat{\mathrm{Q}}(N;\mathscr{K})^{\mathrm{Q}G} = i^*\hat{\mathrm{Q}}Z(G;\bar{\sigma},\mathscr{K}).$$

Hence, we can find an element $F\in\hat{\mathrm{Q}}Z(G;\bar{\sigma},\mathscr{K})$ such that $f=i^*F$. However, since F may be seen as an element in $\hat{\mathrm{Q}}Z(G;\bar{\pi},\mathscr{H})$, we have $f=i^*F\in i^*\hat{\mathrm{Q}}Z(G;\bar{\pi},\mathscr{H})$, a contradiction. ◻

Next we state the following strengthening of Theorem 8.9:

Theorem 8.17 (stronger result for $\mathrm{Aut}(F_n)$) *Fix an integer n at least 4. Then, for every subgroup G of finite index in $\mathrm{Aut}(F_n)$ and every unitary representation $(\pi,\mathscr{H})$ of Γ such that $\mathscr{H}^\Gamma$ is finite-dimensional,*

$$\hat{\mathrm{Q}}(N;\mathscr{H})^{\mathrm{Q}G} = i^*\hat{\mathrm{Q}}Z(G;\bar{\pi},\mathscr{H}).$$

Here we set $N=G\cap\mathrm{IA}_n$ and $\Gamma=G/N$; the representation $(\bar{\pi},\mathscr{H})$ of G is the pullback of the representation $(\pi,\mathscr{H})$ of Γ.

In particular, the assumptions of $n\geq 6$ in Theorems 2.3(2) and 8.9 can both be weakened to $n\geq 4$.

Proof of Theorem 8.17 using Theorem 8.16 By employing the same trick as in the final part of the proof of Theorem 8.14, we may assume that $\mathscr{H}$ is separable throughout this proof. By Theorem 1.5 and the exact sequence (3-1), it suffices to prove that $\mathrm{H}^2(\Gamma;\pi,\mathscr{H})=0$ and $\mathrm{H}^3_b(\Gamma;\pi,\mathscr{H})=0$. Note that Γ is a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$. First we prove that $\mathrm{H}^3_b(\Gamma;\pi,\mathscr{H})=0$. Decompose the Γ-representation space $\mathscr{H}$ as $\mathscr{H}=\mathscr{H}^\Gamma\oplus(\mathscr{H}^\Gamma)^\perp$, where $(\mathscr{H}^\Gamma)^\perp$ is the orthogonal complement of $\mathscr{H}^\Gamma$ in $\mathscr{H}$. Then the restriction of π on $\mathscr{H}^\Gamma$ is trivial, and it is finite-dimensional by assumption; the restriction π^{orth} of π on $(\mathscr{H}^\Gamma)^\perp$ does not admit a nonzero Γ-invariant vector. Now we claim that $\mathrm{H}^3_b(\Gamma;\pi^{\mathrm{inv}},\mathscr{H}^\Gamma)=0$. Indeed, by Lemma 8.4, $\mathrm{H}^3_b(\Gamma)=0$. Since we assume that $\mathscr{H}^\Gamma$ is finite-dimensional, we can decompose $\mathscr{H}^\Gamma$ as a finite direct sum of the one-dimensional trivial module; then [74, Corollary 8.2.10] ends the proof of the claim above. We also claim that $\mathrm{H}^3_b(\Gamma;\pi^{\mathrm{orth}},(\mathscr{H}^\Gamma)^\perp)=0$. Indeed, this follows from another theorem of Monod [76, Theorem 2]; see [77, Corollary 1.6] for a more general statement and Theorem 8.3. Again by [74, Corollary 8.2.10],

$$\mathrm{H}^3_b(\Gamma;\pi,\mathscr{H}) \cong \mathrm{H}^3_b(\Gamma;\pi^{\mathrm{inv}},\mathscr{H}^\Gamma)\oplus\mathrm{H}^3_b(\Gamma;\pi^{\mathrm{orth}},(\mathscr{H}^\Gamma)^\perp);$$

hence $\mathrm{H}^3_b(\Gamma;\pi,\mathscr{H})=0$, as desired.

Next we will show that $\mathrm{H}^2(\Gamma;\pi,\mathcal{H})=0$ by appealing to Theorem 8.16; the proof of this part works without the assumption that $\mathcal{H}^\Gamma$ is finite-dimensional. Let $\Gamma_0=\Gamma\cap\mathrm{SL}(n,\mathbb{Z})$, which is a subgroup of finite index both of Γ and of $\mathrm{SL}(n,\mathbb{Z})$. First, apply Theorem 8.16 with $H=\mathrm{SL}(n,\mathbb{R})$ and $\Lambda=\mathrm{SL}(n,\mathbb{Z})$ (recall that n is assumed to be at least 4). Then $\mathrm{SL}(n,\mathbb{Z})$ has property (T_2). Together with Theorem 8.1 and the universal coefficient theorem, we conclude that $\mathrm{SL}(n,\mathbb{Z})$ has property $[\mathrm{T}_2]$. By [30, Proposition 4.4], property $[\mathrm{T}_2]$ passes to a subgroup of finite index. (Strictly speaking, it follows that property $[\mathrm{T}_2]$ passes to a normal subgroup of finite index; then we employ [19, Proposition III.10.4] to have the full heredity. Or, alternatively, we may apply the Shapiro lemma.) Hence, Γ_0 also has property $[\mathrm{T}_2]$. By Lemma 8.5, Γ has property $[\mathrm{T}_2]$ as well, and thus $\mathrm{H}^2(\Gamma;\pi,\mathcal{H})=0$. □

As we explained in the proof above, the main result of [3] in particular improves the Borel stable range for second ordinary cohomology with the trivial coefficients of SL_n from $n\geq 6$ to $n\geq 4$. We also note that the assumption of the finite-dimensionality of $\mathcal{H}^\Gamma$ in Theorem 8.17 is used to deduce $\mathrm{H}_b^3(\Gamma;\pi^{\mathrm{inv}},\mathcal{H}^\Gamma)=0$ from $\mathrm{H}_b^3(\Gamma)=0$. As we mentioned in the proof, for ordinary cohomology we do not need this finite-dimensionality assumption thanks to the universal coefficient theorem. For a Banach space V, the following question might be of interest: (for a fixed q, say 2 or 3) when does the vanishing of $\mathrm{H}_b^q(G)$ imply that of $\mathrm{H}_b^q(G;V)$ for a group G, where V is viewed as an $\mathbb{R}[G]$-module with the trivial action? For results in this direction, we refer the reader to [44; 82, Proposition 2.39].

The counterpart of Theorem 8.14 in the case of the trivial real coefficients is an open problem.

Problem 8.18 *Let G be a subgroup of finite index in $\mathrm{Mod}(\Sigma_l)$. Set $N=G\cap\mathcal{I}(\Sigma_l)$ and $\Gamma=G/N$. Then does $\mathrm{Q}(N)^G=i^*\mathrm{Q}(G)$ hold?*

Cochran, Harvey and Horn [29] constructed $\mathrm{Mod}(\Sigma)$-invariant quasimorphisms on $\mathcal{I}(\Sigma)$ for a surface Σ with at least one boundary component. The problem asking whether their quasimorphisms are extendable may be of special interest.

8.4 Extension theorem of quasicocycles

As an appendix to this section, we present the following extension theorem of quasicocycles. In this subsection we treat a general theory, and the topic has no specific relation to $\mathrm{Aut}(F_n)$ or $\mathrm{Mod}(\Sigma_l)$. Recall that every G-quasi-invariant quasimorphism on N is extendable to G if the projection $G\to G/N$ virtually splits (Proposition 3.4). This can be generalized as follows:

Theorem 8.19 *Let $1\to N\to G\xrightarrow{p}\Gamma\to 1$ be an exact sequence and V an $\mathbb{R}[\Gamma]$-module with a Γ-invariant norm $\|\cdot\|$. Assume that the exact sequence virtually splits. Then, for every V-valued G-quasiequivariant quasimorphism $f\in\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$, there exists a quasicocycle $F\in\hat{\mathrm{Q}}Z(G;V)$ such that $F|_N=f$ and $D(F)\leq D(f)+3D'(f)$.*

The proof is parallel to that of [60, Proposition 6.4] (Proposition 3.4 above). For the sake of completeness, we include the proof; see [loc. cit.] for more details.

Proof Let (s,Λ) be a virtual section of $p\colon G\to\Gamma$ (see Section 2). Let B be a finite subset of Γ such that the map $\Lambda\times B\to\Gamma$, $(\lambda,b)\mapsto\lambda b$, is bijective. Let $s'\colon B\to G$ be a map satisfying $p\circ s'(b)=b$ for every $b\in B$. Define a map $t\colon\Gamma\to G$ by setting $t(\lambda b)=s(\lambda)s'(b)$. Note that t is a (set-theoretic) section of p. Given $f\in\widehat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$, define a function $F\colon G\to V$ by

$$F(g)=\frac{1}{\#B}\sum_{b\in B}f\bigl(g\cdot t(b\cdot p(g))^{-1}\cdot t(b)\bigr).$$

Then $F|_N=f$. Moreover, for $g_1,g_2\in G$, by using that $f(h_1h_2)\approx_{D'(f)}p(h_1)\cdot f(h_2h_1)$ and $f(h_1h_2)\approx_{D'(f)}p(h_2)^{-1}\cdot f(h_2h_1)$ for every $h_1,h_2\in G$ with $h_1h_2\in N$, we have

$$\begin{aligned}F(g_1g_2)&=\frac{1}{\#B}\sum_{b\in B}f(g_1g_2\cdot t(b\cdot p(g_1g_2))^{-1}t(b))\\&\approx_{D'(f)}\frac{1}{\#B}\sum_{b\in B}b^{-1}\cdot f\bigl(t(b)\cdot g_1g_2\cdot t(b\cdot p(g_1g_2))^{-1}\bigr)\\&=\frac{1}{\#B}\sum_{b\in B}b^{-1}\cdot f\bigl(t(b)\cdot g_1\cdot t(b\cdot p(g_1))^{-1}\cdot t(b\cdot p(g_1))\cdot g_2\cdot t(b\cdot p(g_1g_2))^{-1}\bigr)\\&\approx_{D(f)}\frac{1}{\#B}\sum_{b\in B}b^{-1}\cdot\bigl(f\bigl(t(b)\cdot g_1\cdot t(b\cdot p(g_1))^{-1}\bigr)+f\bigl(t(b\cdot p(g_1))\cdot g_2\cdot t(b\cdot p(g_1g_2))^{-1}\bigr)\bigr)\\&\approx_{2D'(f)}\frac{1}{\#B}\sum_{b\in B}f(g_1\cdot t(b\cdot p(g_1))^{-1}\cdot t(b))+\frac{1}{\#B}\sum_{b\in B}p(g_1)\cdot f\bigl(g_2\cdot t(b\cdot p(g_1g_2))^{-1}\cdot t(b\cdot p(g_1))\bigr)\\&=F(g_1)+g_1\cdot\biggl(\frac{1}{\#B}\sum_{b\in B}f\bigl(g_2\cdot t\bigl((b\cdot p(g_1))\cdot p(g_2)\bigr)^{-1}\cdot t(b\cdot p(g_1))\bigr)\biggr).\end{aligned}$$

By the arguments in the proof of [60, Proposition 6.4],

$$\frac{1}{\#B}\sum_{b\in B}f\bigl(g_2\cdot t\bigl((b\cdot p(g_1))\cdot p(g_2)\bigr)^{-1}\cdot t(b\cdot p(g_1))\bigr)=F(g_2).$$

Therefore, $F(g_1g_2)\approx_{D(f)+3D'(f)}F(g_1)+g_1\cdot F(g_2)$. □

9 Open problems

9.1 Mystery of the Py class

Let Σ_l be a closed connected orientable surface whose genus l is at least 2 and Ω a volume form on Σ_l. Recall that Py [89] constructed a Calabi quasimorphism f_P on $\mathrm{Ker}(\mathrm{Flux}_\Omega)$ which is $\mathrm{Diff}_0(\Sigma_l,\Omega)$-invariant, and the first and second authors showed that f_P is not extendable to $\mathrm{Diff}_0(\Sigma_l,\Omega)$ (recall Section 5 and Example 7.15). We define $\bar{c}_P\in\mathrm{H}^2(\mathrm{H}^1(\Sigma_l))$ and $c_P\in\mathrm{H}^2(\mathrm{Diff}_0(\Sigma_l,\Omega))$ by $\bar{c}_P=\xi_4^{-1}\circ\tau_{/b}(f_P)$ and $c_P=\mathrm{Flux}_\Omega^*(\bar{c}_P)$, respectively. We call c_P the *Py class*. Note that we essentially proved the nontriviality of the Py class in the proof of Theorem 2.6(1).

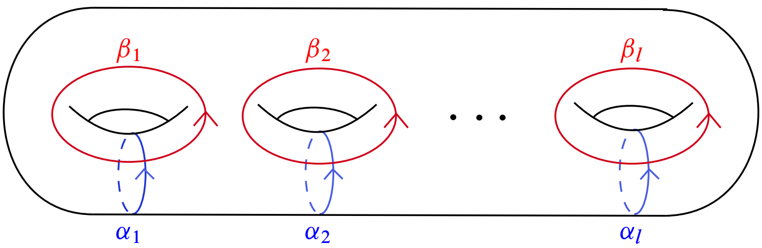

Figure 1: $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l : [0,1] \to \Sigma_l$.

When we constructed the class $\bar{c}_P \in \mathrm{H}^2(\mathrm{H}^1(\Sigma_l))$, we used the morphism $\xi_4 : \mathrm{H}^2(\mathrm{H}^1(\Sigma_l)) \to \mathrm{H}^2_{/b}(\mathrm{H}^1(\Sigma_l))$. Here we apply the exact sequence

$$1 \to \mathrm{Ker}(\mathrm{Flux}_\Omega) \to \mathrm{Diff}_0(\Sigma_l, \Omega) \xrightarrow{\mathrm{Flux}_\Omega} \mathrm{H}^1(\Sigma_l) \to 1$$

to diagram (1-3). Since the bounded cohomology groups of an amenable group are zero, the map ξ_4 is an isomorphism and we have the inverse $\xi_4^{-1} : \mathrm{H}^2_{/b}(\mathrm{H}^1(\Sigma_l)) \to \mathrm{H}^2(\mathrm{H}^1(\Sigma_l))$. Because the vanishing of the bounded cohomology of amenable groups is shown by a transcendental method, we do not have a precise description of the map ξ_4^{-1}.

Remark 9.1 If we fix a (right-)invariant mean m on the amenable group Γ, then we have the following description of the map ξ_4^{-1}. For a cocycle $[c] \in C^2_{/b}(\Gamma)$, a cocycle $f \in C^2(\Gamma)$ representing the class $\xi_4^{-1}([[c]])$ can be given by

$$f(\gamma_1, \gamma_2) = c(\gamma_1, \gamma_2) - m(\delta c(\cdot, \gamma_1, \gamma_2)).$$

However, we have the following observations on the Py class. Here we consider $\mathrm{H}^1(\Sigma_l)$ as a symplectic vector space by the intersection form.

Theorem 9.2 *Let Σ_l be a closed connected orientable surface whose genus l is at least 2 and Ω a volume form on Σ_l. For a subgroup Λ of $\mathrm{H}^1(\Sigma_l)$, let $\iota_\Lambda : \Lambda \to \mathrm{H}^1(\Sigma_l)$ be the inclusion map.*

(1) *Let v and w be elements in $\mathrm{H}^1(\Sigma_l)$ with $v \smile w \neq 0$. Then there exists a positive integer k_0 such that, for every integer k at least k_0, for the subgroup $\Lambda = \langle v, w/k \rangle$ of $\mathrm{H}^1(\Sigma_l)$, we have $\iota_\Lambda^* \bar{c}_P \neq 0$. Here $\smile$ denotes the cup product.*

(2) *If a subgroup Λ of $\mathrm{H}^1(\Sigma_l)$ is contained in a linear subspace $\langle [\alpha_1]^*, \dots, [\alpha_l]^* \rangle$ or $\langle [\beta_1]^*, \dots, [\beta_l]^* \rangle$, then $\iota_\Lambda^* \bar{c}_P = 0$, where $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l$ are the curves described in Figure 1.*

To prove Theorem 9.2, we use the following observation.

Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence of groups such that Γ is amenable. For a subgroup Γ^0 of Γ, $1 \to N \xrightarrow{i} p^{-1}(\Gamma^0) \xrightarrow{p} \Gamma^0 \to 1$ is also an exact sequence and Γ^0 is also amenable (Theorem 3.5(3)).

Then, by Theorem 1.5, we have the commuting diagrams

(9-1)
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{H}^1(\Gamma) & \xrightarrow{p^*} & \mathrm{H}^1(G) & \xrightarrow{i^*} & \mathrm{H}^1(N)^G & \xrightarrow{\tau} & \mathrm{H}^2(\Gamma) & \xrightarrow{p^*} & \mathrm{H}^2(G) \\ & & \downarrow{\xi_1} & & \downarrow{\xi_2} & & \downarrow{\xi_3} & & \downarrow{\xi_4} & & \downarrow{\xi_5} \\ 0 & \longrightarrow & \mathrm{Q}(\Gamma) & \xrightarrow{p^*} & \mathrm{Q}(G) & \xrightarrow{i^*} & \mathrm{Q}(N)^G & \xrightarrow{\tau_{/b}} & \mathrm{H}^2_{/b}(\Gamma) & \xrightarrow{p^*} & \mathrm{H}^2_{/b}(G) \end{array}$$

(9-2)
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{H}^1(\Gamma^0) & \xrightarrow{p^*} & \mathrm{H}^1(p^{-1}(\Gamma^0)) & \xrightarrow{i^*} & \mathrm{H}^1(N)^{p^{-1}(\Gamma^0)} & \xrightarrow{\tau^0} & \mathrm{H}^2(\Gamma^0) & \xrightarrow{p^*} & \mathrm{H}^2(p^{-1}(\Gamma^0)) \\ & & \downarrow{\xi_1^0} & & \downarrow{\xi_2^0} & & \downarrow{\xi_3^0} & & \downarrow{\xi_4^0} & & \downarrow{\xi_5^0} \\ 0 & \longrightarrow & \mathrm{Q}(\Gamma^0) & \xrightarrow{p^*} & \mathrm{Q}(p^{-1}(\Gamma^0)) & \xrightarrow{i^*} & \mathrm{Q}(N)^{p^{-1}(\Gamma^0)} & \xrightarrow{\tau^0_{/b}} & \mathrm{H}^2_{/b}(\Gamma^0) & \xrightarrow{p^*} & \mathrm{H}^2_{/b}(p^{-1}(\Gamma^0)) \end{array}$$

Since Γ and Γ^0 are boundedly 3-acyclic (Theorem 3.5(5)), $\xi_4\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2_{/b}(\Gamma)$ and $\xi_4^0\colon \mathrm{H}^2(\Gamma^0) \to \mathrm{H}^2_{/b}(\Gamma^0)$ are isomorphisms. The following lemma can be deduced from the definitions of $\tau_{/b}$ and $\tau^0_{/b}$:

Lemma 9.3 *We have*
$$(\xi_4^0)^{-1} \circ \tau^0_{/b} \circ I_1^* = I_2^* \circ (\xi_4)^{-1} \circ \tau_{/b},$$
where $I_1^*\colon \mathrm{Q}(N)^G \to \mathrm{Q}(N)^{p^{-1}(\Gamma^0)}$ and $I_2^*\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2(\Gamma^0)$ are the homomorphisms induced from the inclusion $I\colon \Gamma^0 \to \Gamma$.

We employ the following theorem, which is related to Example 7.15, in order to prove Theorem 9.2:

Theorem 9.4 [61, Theorems 1.6 and 1.10] *Let Σ_l be a closed connected orientable surface whose genus l is at least 2 and Ω a volume form on Σ_l. Let Λ be a subgroup of $\mathrm{H}^1(\Sigma_l)$ and set $G = \mathrm{Flux}^{-1}(\Lambda)$ and $N = \mathrm{Ker}(\mathrm{Flux}_\Omega)$. Then:*

(1) *Let v and w be elements in $\mathrm{H}^1(\Sigma_l)$ with $v \smile w \neq 0$. Then there exists a positive integer k_0 such that, for every integer k at least k_0, for $\Lambda = \langle v, w/k \rangle$, $[f_P]$ is a nontrivial element of $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$.*

(2) *If Λ is contained in a linear subspace $\langle [\alpha_1]^*, \dots, [\alpha_l]^* \rangle$ or $\langle [\beta_1]^*, \dots, [\beta_l]^* \rangle$, then $[f_P]$ is the trivial element of $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$, where $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l$ are the curves described in Figure 1.*

On (2), see also [61, Remark 4.8].

Proof of Theorem 9.2 Set $\Gamma = \mathrm{H}^1(\Sigma_l)$, $\Gamma^0 = \Lambda$ and $G = \mathrm{Flux}_\Omega^{-1}(\Lambda)$. We use the notation in the diagrams (9-1) and (9-2).

First, to prove (1), suppose that the dimension of Λ is larger than l. Then, since $[f_P]$ is a nontrivial element of $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$, by Theorem 1.10, $(\xi_4^0)^{-1} \circ \tau^0_{/b} \circ I_1^*(f_P)$ is also a nontrivial element of $\mathrm{H}^2(\Gamma^0) = \mathrm{H}^2(\Lambda)$. Hence, by Lemma 9.3, $\iota_\Lambda^* \bar{c}_P = I_2^* \circ (\xi_4)^{-1} \circ \tau_{/b}(f_P) = (\xi_4^0)^{-1} \circ \tau^0_{/b} \circ I_1^*(f_P)$ is also a nontrivial element of $\mathrm{H}^2(\Gamma^0) = \mathrm{H}^2(\Lambda)$.

Next, to prove (2), suppose that Λ is contained in linear subspaces $\langle[\alpha_1]^*, \dots, [\alpha_l]^*\rangle$ or $\langle[\beta_1]^*, \dots, [\beta_l]^*\rangle$. Then, since $[f_P]$ is the trivial element of $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$, by Theorem 1.10 and Proposition 2.5, $(\xi_4^0)^{-1} \circ \tau_{/b}^0 \circ I_1^*(f_P)$ is also the trivial element of $\mathrm{H}^2(\Gamma^0) = \mathrm{H}^2(\Lambda)$. Hence, by Lemma 9.3, $\iota_\Lambda^* \bar{c}_P = I_2^* \circ (\xi_4)^{-1} \circ \tau_{/b}(f_P) = (\xi_4^0)^{-1} \circ \tau_{/b}^0 \circ I_1^*(f_P)$ is also the trivial element of $\mathrm{H}^2(\Gamma^0) = \mathrm{H}^2(\Lambda)$. □

Finally, we pose the following problems on the Py class:

Problem 9.5 *Give a precise description of a cochain representing* $\bar{c}_P \in \mathrm{H}^2(\mathrm{H}^1(\Sigma_l))$ *and a bounded cochain representing* $c_P \in \mathrm{H}^2(\mathrm{Diff}_0(M, \Omega))$.

Problem 9.6 *Let* Σ_l *be a closed connected orientable surface whose genus* l *is at least* 2 *and* Ω *a volume form on* Σ_l. *Is the vector space* $\mathrm{Im}(\mathrm{Flux}_\Omega^*) \cap \mathrm{Im}(c_{\mathrm{Diff}_0(\Sigma_l,\Omega)})$ *spanned by* c_P?

By Theorem 1.10, Problem 9.6 is rephrased as follows.

Problem 9.7 *Let* Σ_l *be a closed connected orientable surface whose genus* l *is at least* 2 *and* Ω *a volume form on* Σ_l. *Is the vector space* $\mathrm{Q}(\mathrm{Ker}(\mathrm{Flux}_\Omega))^{\mathrm{Diff}_0(\Sigma_l,\Omega)}/i^*\mathrm{Q}(\mathrm{Diff}_0(\Sigma_l, \Omega))$ *spanned by* $[f_P]$?

9.2 Problems on equivalences and coincidences of scl_G and $\mathrm{scl}_{G,N}$

By Theorem 2.1, $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ implies that scl_G and $\mathrm{scl}_{G,N}$ are equivalent on $[G, N]$. Moreover, if N is the commutator subgroup of G and $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, then scl_G and $\mathrm{scl}_{G,N}$ coincide on $[G, N]$. Since $\mathrm{H}^2(G) = 0$ implies $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ (Theorem 1.10), there are several examples of pairs (G, N) such that $\mathrm{scl}_{G,N}$ and scl_G are equivalent (see Section 2.1). In Section 3, we provided several examples of groups G with $\mathrm{Q}(N)^G \neq \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ (see Theorems 1.1, 1.2 and 4.18), but we were unable to determine whether scl_G and $\mathrm{scl}_{G,N}$ are equivalent on $[G, N]$ in these examples. Hence, the example of $G = \mathrm{Diff}(\Sigma_l, \omega)$ with $l \geq 2$ and $N = [G, G]$ raised by [58] (see also [61]) has remained essentially the only known example where scl_G and $\mathrm{scl}_{G,N}$ are not equivalent on $[G, N]$. In fact, this is the only example where scl_G and $\mathrm{scl}_{G,N}$ do not coincide on $[G, N]$. Here we provide several problems on equivalences and coincidences of scl_G and $\mathrm{scl}_{G,N}$.

Problem 9.8 *Is it true that* $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ *implies that* $\mathrm{scl}_G = \mathrm{scl}_{G,N}$ *on* $[G, N]$?

Problem 9.9 *Find a pair* (G, N) *such that* G *is finitely generated and* scl_G *and* $\mathrm{scl}_{G,N}$ *are not equivalent. In particular, are* $\mathrm{scl}_{\pi_1(\Sigma_l)}$ *and* $\mathrm{scl}_{\pi_1(\Sigma_l),[\pi_1(\Sigma_l),\pi_1(\Sigma_l)]}$ *equivalent on* $[\pi_1(\Sigma_l), [\pi_1(\Sigma_l), \pi_1(\Sigma_l)]]$ *for* $l \geq 2$?

After the current work, Problem 9.9 was solved by some of the authors [69]: for $l \geq 2$, $\mathrm{scl}_{\pi_1(\Sigma_l)}$ and $\mathrm{scl}_{\pi_1(\Sigma_l),[\pi_1(\Sigma_l),\pi_1(\Sigma_l)]}$ are not equivalent. Moreover, the authors proved in [59] that scl_G and $\mathrm{scl}_{G,[G,G]}$ are not equivalent if $\mathrm{Q}([G, G])^G \neq \mathrm{H}^1([G, G])^G + i^*\mathrm{Q}(G)$.

We also pose the following problem. Let B_n be the n^{th} braid group and P_n the n^{th} pure braid group.

Problem 9.10 *For* $n \geq 3$, *does* $\mathrm{scl}_{B_n} = \mathrm{scl}_{B_n,[P_n,P_n]}$ *hold on* $[B_n, [P_n, P_n]]$?

In light of the following proposition, we can regard Problem 9.10 as a special case of Problem 9.8:

Proposition 9.11 *For $n \geq 2$, let $G = B_n$ and $N = [P_n, P_n]$. Then $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. In particular, $\mathrm{scl}_G(x) \leq \mathrm{scl}_{G,N}(x) \leq 2 \cdot \mathrm{scl}_G(x)$ for all $x \in [G, N]$.*

Proof Consider the exact sequence

$$1 \to P_n/[P_n, P_n] \to B_n/[P_n, P_n] \to \mathfrak{S}_n \to 1,$$

where $\mathfrak{S}_n$ is the symmetric group. By Theorem 3.5(1)–(2), $\mathfrak{S}_n$ and $P_n/[P_n, P_n]$ are amenable. Hence, Theorem 3.5(4) implies that $B_n/[P_n, P_n]$ is also amenable. As pointed out in Section 2.1, the second cohomology of the braid group B_n vanishes. Hence, Theorem 1.10 implies that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. The equivalence between scl_{B_n} and $\mathrm{scl}_{B_n,[P_n,P_n]}$ follows from Theorem 2.1(2). □

As another special case of Problem 9.8, we provide the following problem:

Problem 9.12 *For $n \geq 2$, does $\mathrm{scl}_{\mathrm{Aut}(F_n)} = \mathrm{scl}_{\mathrm{Aut}(F_n),\mathrm{IA}_n}$ hold on $[\mathrm{Aut}(F_n), \mathrm{IA}_n]$?*

Even the following weaker variant of Problem 9.12 seems open. We note that Theorem 2.1(2) does *not* apply to the setting of Theorem 2.3.

Problem 9.13 *Let $n \geq 3$. Find an explicit real constant $C \geq 1$ such that $\mathrm{scl}_{\mathrm{Aut}(F_n),\mathrm{IA}_n} \leq C \cdot \mathrm{scl}_{\mathrm{Aut}(F_n)}$ on $[\mathrm{Aut}(F_n), \mathrm{IA}_n]$.*

In [60], the first, second, fourth and fifth authors considered the equivalence problem between cl_G and $\mathrm{cl}_{G,N}$. We provide the following problem:

Problem 9.14 *Is it true that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ implies the bi-Lipschitz equivalence of cl_G and $\mathrm{cl}_{G,N}$ on $[G, N]$?*

We note that Theorem 2.1(1) states that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ implies the bi-Lipschitz equivalence of scl_G and $\mathrm{scl}_{G,N}$. To the best knowledge of the authors, Problem 9.14, even for the case where $1 \to N \to G \to \Gamma \to 1$ virtually splits, might be open in general.

In light of Proposition 9.11 and Theorem 2.3, we can regard the following problem as special cases of Problem 9.14:

Problem 9.15 *For (G, N) either $(B_n, [P_n, P_n])$ with $n \geq 3$ or $(\mathrm{Aut}(F_n), \mathrm{IA}_n)$ with $n \geq 2$, are cl_G and $\mathrm{cl}_{G,N}$ equivalent on $[G, N]$?*

Note that cl_G and $\mathrm{cl}_{G,N}$ are bi-Lipschitzly equivalent when $(G, N) = (B_n, P_n \cap [B_n, B_n] = [P_n, B_n])$ [60].

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MKa: *Department of Mathematics, Faculty of Science, Hokkaido University*
Sapporo, Japan

MKi: *Department of Mathematics, Osaka Dental University*
Hirakata, Japan

SM: *School of Mathematics and Physics, College of Science and Engineering, Kanazawa University*
Kanazawa, Japan

TM: *Department of Mathematical Sciences, Faculty of Science, Shinshu University*
Matsumoto, Japan

MM: *Mathematical Institute, Tohoku University*
Sendai, Japan

kawasaki@math.sci.hokudai.ac.jp, kimura-m@cc.osaka-dent.ac.jp,
smaruyama@se.kanazawa-u.ac.jp, matsushita@shinshu-u.ac.jp,
m.masato.mimura.m@tohoku.ac.jp

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