Synthetic approach to the Quillen model structure on topological spaces

STERLING EBEL

KRZYSZTOF KAPULKIN

Synthetic approach to the Quillen model structure on topological spaces

STERLING EBEL

KRZYSZTOF KAPULKIN

We provide an axiomatic treatment of Quillen's construction of the model structure on topological spaces to make it applicable to a wider range of settings, including Δ-generated spaces and pseudotopological spaces. We use this axiomatization to construct a model structure on the category of locales.

Introduction

The construction of Quillen's model structure on the category Top of topological spaces is a cornerstone result of modern homotopy theory [20, Section II.3]. The importance of many other model structures, like the Kan–Quillen model structure on simplicial sets or the Thomason model structure on the category of small categories [25], is justified by their equivalence with Quillen's structure on topological spaces. Numerous accounts of it have been given since, including by Hovey [11], Hirschhorn [9; 10] and May and Ponto [16, Section 17.2], further underscoring its significance.

Although the category of (all) topological spaces is perhaps most commonly studied, quite often one works with either subcategories of topological spaces (eg compactly generated, weakly Hausdorff spaces or Δ-generated spaces) or larger categories containing Top as a subcategory (eg pseudotopological spaces). The former are natural choices for a convenient category of spaces, ie a subcategory of Top with nice categorical properties, including cartesian closure. The latter, being a common generalization of both graphs and spaces, finds applications in topological data analysis when quantifying to what extent a space can be recovered by sampling only finitely many of its points; see Rieser [22; 21].

The purpose of this paper is to axiomatize the requirements on various categories of interest to make a more modern version of Quillen's argument work. Such a modern treatment differs from Quillen's argument in a variety of ways, including: isolating the pushout–product construction, utilizing characterization of the class of weak equivalences in terms of an "up to homotopy" lifting property, and identifying the key role of the subcategory of topological cubes. Our axiomatization first introduces the notion of a *category with intervals* (Definition 1.1), a framework in which one can speak of homotopies, defined using the topological interval, and their basic properties. We then introduce the framework of a *Q-structure*

(Definition 2.2), which is the structure on a category that permits Quillen's construction (Theorem 2.6). Moreover, in a *good* Q-structure, weak equivalences can be defined in terms of isomorphisms on the sets of connected components and all higher homotopy groups (Theorem 3.12).

Several examples of good Q-structures are identified and give rise to model structures on topological spaces (recovering the model structure Quillen originally constructed [20]), compactly generated, weakly Hausdorff spaces (recovering the model structure of Hovey [11]), Δ-generated spaces (recovering the model structure of Haraguchi [6]), sober spaces (Section 4), pseudotopological spaces (Section 5, recovering the model structure of Rieser [22]), and locales (Section 6). This in particular settles an open question of constructing a model structure on the category of locales.

We begin in Section 1 by introducing the notion of a category with intervals (Definition 1.1) and developing basic homotopy theory therein. We then proceed to introduce (good) Q-structures in Section 2 and define cofibrations and fibrations. We conclude our proof of the existence of a model structure in Section 3 before turning our attention to examples of Q-structures in Sections 4, 5 and 6.

Acknowledgements

1   Basic notions in a nice category of spaces

In this section, we introduce the notion of a category with intervals (Definition 1.1) and show that it is sufficient to develop basics of the theory of homotopies.

Throughout the paper, we write Top for the category of (all) topological spaces and continuous maps. For notational convenience, the unit interval $[0, 1]$ will be denoted by I.

Let $\mathcal{C}$ denote a bicomplete category, to be thought of as a "nice" category of spaces. As such, we will refer to objects in $\mathcal{C}$ as spaces, and morphisms in $\mathcal{C}$ as (continuous) maps. Let $\varnothing$ and $*$ denote the initial and terminal object of $\mathcal{C}$, respectively. Let $\square$ be the full subcategory of Top consisting of all n-cubes and their boundaries for $n \geq 0$. Let $\square_{\leq 2}$ denote the full subcategory of $\square$ whose objects are $*$, I, I^2, $\varnothing$, ∂I and ∂I^2.

Let $e_k \colon * \to I$ be the appropriate endpoint inclusion for $k \in \{0, 1\}$. Denote the left, top, right and bottom edge inclusions $I \hookrightarrow I^2$ by $\partial_L = e_0 \times \mathrm{id}_I$, $\partial_T = \mathrm{id}_I \times e_1$, $\partial_R = e_1 \times \mathrm{id}_I$ and $\partial_B = \mathrm{id}_I \times e_0$, respectively. By embedding, we shall mean a faithful functor throughout this paper.

Definition 1.1 A *category with intervals*[1] is a bicomplete category $\mathcal{C}$ along with an embedding $\iota: \square_{\leq 2} \hookrightarrow \mathcal{C}$ such that:

(S1) $\iota \varnothing$ is initial and $\iota *$ is terminal.

(S2) $\iota I^2 \cong \iota I \times \iota I$, where the product is taken in $\mathcal{C}$.

(S3) ι preserves a pushout of the form

$$
\begin{array}{ccc}
* & \xrightarrow{\ e_0\ } & I \\
{\scriptstyle e_1}\big\downarrow & & \big\downarrow{\scriptstyle b} \\
I & \xrightarrow[\ a\ ]{} & I
\end{array}
$$

such that $a(0) = 0$ and $b(1) = 1$, and $X \times -$ preserves this pushout for all $X \in \mathcal{C}$.

(S4) $\varnothing \times X \cong \varnothing$ for all $X \in \mathcal{C}$.

The axioms given above are fairly rigid in structure; as such, it is difficult to give a "noncanonical" example. We give several examples in Sections 4, 5 and 6, which will moreover be of good Q-structure (see Definition 2.2). All of these examples are given by restricting a functor $\mathsf{Top} \to \mathcal{C}$ to $\square$. We are unaware of any examples of categories which admit two distinct embeddings of $\square$ defining two distinct categories with intervals.

We require that the pushout in (S3) be of maps in $\square_{\leq 2}$ to use relationships internal to $\square_{\leq 2}$ to develop a basic theory of homotopies. Necessarily, a and b are injective, and hence homeomorphisms onto their image. Unless the distinction is necessary, we will identify elements in $\square_{\leq 2}$ with their image in $\mathcal{C}$. For the rest of this section, fix an embedding ι making $(\mathcal{C}, \iota)$ into a category with intervals.

Homotopies

In this subsection, working in an arbitrary category with intervals (Definition 1.1), we develop a notion of homotopy, leading to the proof (Theorem 1.13) that any such category carries a natural structure of a 2-category.

Definition 1.2 Let X and Y be spaces and $f, g: X \to Y$ be maps. A *homotopy* from f to g is given by a map $H: X \times I \to Y$ such that the following commutes:

$$
\begin{array}{ccccc}
X & \xrightarrow{\ e_0\ } & X \times I & \xleftarrow{\ e_1\ } & X \\
& {\scriptstyle f}\searrow & \ \ \big\downarrow{\scriptstyle H} & \swarrow{\scriptstyle g} & \\
& & Y & &
\end{array}
$$

If there exists a homotopy from f to g, then f *is homotopic to* g, denoted by $f \sim g$ or $H: f \sim g$.

In particular, we will be using $X \times I$ as a cylinder object for X, in which case our notion of homotopy is that of a *left homotopy*. In Section 2, we will assume further that $- \times I$ admits a right adjoint $(-)^I$, giving path spaces X^I; this gives an equivalent notion of *right homotopy*, which we will not explicitly use.

---

[1]Note that [17] introduces the notion of a *site with interval*. Our notion differs from theirs, as it is strongly based on the properties of the topological interval, whereas that of [17] is abstract and specific to the category of sheaves.

Let $f, g\colon X \to Y$, $u\colon W \to X$ and $v\colon Y \to X$ be maps, and H a homotopy from f to g. For a space A, let $\pi_A\colon A \times I \to A$ denote the projection. We will follow these notational conventions for homotopies:

(1) $Hu = H(u \times \mathrm{id}_I)$, which gives a homotopy from fu to gu.

(2) $\mathrm{const}_f = f\pi_X = \pi_Y(f \times \mathrm{id}_I)$.

Homotopies are subject to the following identities, which can easily be verified:

(1) $(u \times \mathrm{id}_I)e_k = e_k u$.

(2) $\pi_X e_k = \mathrm{id}_X$.

(3) $v \circ \mathrm{const}_f \circ (u \times \mathrm{id}_X) = \mathrm{const}_{vfu}$.

Definition 1.3   Let $f, g\colon X \to Y$ be maps, and suppose $i\colon A \to X$ is a map such that $fi = gi$. A *homotopy relative to A* is a homotopy $H\colon X \times I \to Y$ from f to g such that the following commutes:

$$
\begin{array}{ccc}
A \times I & \xrightarrow{\ \pi_A\ } & A \\
{\scriptstyle i \times \mathrm{id}_I}\big\downarrow & & \big\downarrow{\scriptstyle fi=gi} \\
X \times I & \xrightarrow[\ H\ ]{} & Y
\end{array}
$$

Explicitly, $Hi = \mathrm{const}_{fi}$. If such a homotopy exists, it is denoted by $f \sim g$ rel A.

Definition 1.4   Let $f, g, h\colon X \to Y$ be maps. Given a homotopy H from f to g and a homotopy K from g to h, their *track composite* $H \cdot K$ is the induced map in the pushout

$$
\begin{array}{ccc}
X & \xrightarrow{\ e_0\ } & X \times I \\
{\scriptstyle e_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_X \times b} \quad K \\
X \times I & \xrightarrow[\ \mathrm{id}_X \times a\ ]{} & X \times I \\
& H & \quad H \cdot K \quad Y
\end{array}
$$

Let $-\mathrm{id}_I\colon I \to I$ be the map $t \mapsto 1 - t$. The *inverse homotopy* of H, denoted by $-H$, is $H(\mathrm{id}_X \times (-\mathrm{id}_I))$:

$$
\begin{array}{ccc}
X \times I & \xrightarrow{\ \mathrm{id}_X \times (-\mathrm{id}_I)\ } & X \times I \\
& {\scriptstyle -H} \searrow & \big\downarrow{\scriptstyle H} \\
& & Y
\end{array}
$$

Since $(\mathrm{id}_X \times a)e_0 = e_0$ and $(\mathrm{id}_X \times b)e_1 = e_1$ by (S3), $H \cdot K$ is a homotopy from f to h. Similarly, $-H$ gives a homotopy from g to f. Given maps $q\colon W \to X$ and $r\colon Y \to Z$, we have $q(H \cdot K)r = qHr \cdot qKr$. Thus, if $fi = gi = hi$ for $i\colon A \to X$, then $(H \cdot K)i = Hi \cdot Ki = \mathrm{const}_{fi}$, and similarly $-Hi = \mathrm{const}_{fi}$, showing that composition and inverse homotopies preserve relativity.

Definition 1.5   Let $H, K\colon X \times I \to Y$ be homotopies from f to g. Then H and K are *homotopic rel endpoints* if there is a map $\alpha\colon X \times I^2 \to Y$ such that $\alpha\partial_L = H$, $\alpha\partial_T = \mathrm{const}_g$, $\alpha\partial_R = K$ and $\alpha\partial_B = \mathrm{const}_f$.

Analogous comments to the above hold for homotopies rel endpoints. Thus, constant homotopies, inverse homotopies and track composites show respectively that (relative) homotopy is reflexive, symmetric and transitive. In particular, we obtain the following:

Corollary 1.6 *Homotopy defines an equivalence relation on $\mathcal{C}(X, Y)$ for every pair of objects $X, Y \in \mathcal{C}$. Similarly, for any fixed pair of maps $f, g \colon X \to Y$, homotopy rel endpoints defines an equivalence relation on the set of homotopies from f to g.* □

Lemma 1.7 *Let H and H' be homotopies from f to g, and K and K' be homotopies from g to h. If $H \sim H'$ rel endpoints and $K \sim K'$ rel endpoints, then $H \cdot K \sim H' \cdot K'$ rel endpoints. That is, track composition is well defined up to homotopy rel endpoints.*

Proof By the assumptions, we may choose a map $\alpha \colon X \times I^2 \to Y$ such that $\alpha \partial_L = H$, $\alpha \partial_T = \mathrm{const}_g$, $\alpha \partial_R = H'$ and $\alpha \partial_B = \mathrm{const}_f$; likewise, we may choose $\beta \colon X \times I^2 \to Y$ with $\beta \partial_L = K$, $\beta \partial_T = \mathrm{const}_h$, $\beta \partial_R = K'$ and $\beta \partial_B = \mathrm{const}_g$. Then the induced map in the pushout

gives a homotopy rel endpoints from $H \cdot K$ to $H' \cdot K'$. □

Lemma 1.8 *For any homotopy H from f to g, $H \cdot \mathrm{const}_g \sim H$ rel endpoints and $\mathrm{const}_f \cdot H \sim H$ rel endpoints.*

Proof Let $\Gamma \colon I^2 \to I$ be given by $\Gamma(s, t) = st + (1-s)a^{-1}\big(\min(t, a(1))\big)$. Then the map $\alpha = H\Gamma$ gives the required homotopy. To see this, note that $\Gamma(e_0 \times a) = \mathrm{id}_I$ and $\Gamma(e_0 \times b) = e_1$, so $\alpha \partial_L = H \cdot \mathrm{const}_g$, and clearly $\alpha \partial_R = H$, $\alpha \partial_B = \mathrm{const}_f$ and $\alpha \partial_T = \mathrm{const}_g$. □

Lemma 1.9 *For any homotopy H from f to g, $H \cdot (-H) \sim \mathrm{const}_f$ rel endpoints and $-H \cdot H \sim \mathrm{const}_g$ rel endpoints.*

Proof Let $\gamma \colon I \to I$ be given by $\gamma(t) = a^{-1}(t)$ for $t \le a(1)$ and $\gamma(t) = 1 - b^{-1}(t)$ for $t \ge b(0)$. Define $\Gamma \colon I^2 \to I$ by
$$\Gamma(s, t) = \begin{cases} \min(a^{-1}(t), 1 - s) & \text{if } t \le u(1), \\ \min(1 - b^{-1}(t), 1 - s) & \text{if } t \ge v(0). \end{cases}$$
Then Γ is continuous by the pasting lemma, $\Gamma \partial_L = \gamma$ and $\Gamma \partial_T = \Gamma \partial_B = \Gamma \partial_R = 0$. Note that $\gamma a = \mathrm{id}_I$ and $\gamma b = -\mathrm{id}_I$, so $H(\mathrm{id}_X \times \gamma a) = H$ and $H(\mathrm{id}_X \times \gamma b) = -H$; hence, $H(\mathrm{id}_X \times \gamma) = H \cdot (-H)$. Thus, $H(\mathrm{id}_X \times \Gamma) \colon X \times I^2 \to Y$ gives a homotopy from $H \cdot (-H)$ to const_f rel endpoints. The other part is analogous. □

Lemma 1.10 *For any homotopies H from f_1 to f_2, J from f_2 to f_3, and K from f_3 to f_4, we have $(H \cdot J) \cdot K \sim H \cdot (J \cdot K)$ rel endpoints. That is, track composition of homotopies is associative up to homotopy rel endpoints.*

Proof Let $\Omega = a(1) = b(0)$. As before, note that a^{-1} is defined and continuous on $[0, \Omega]$, as is b^{-1} on $[\Omega, 1]$. Let $\gamma \colon I \to I$ be given by

$$\gamma(t) = \begin{cases} b^{-1}(t) & \text{if } t \in [b\Omega, 1], \\ aba^{-1}b^{-1}(t) & \text{if } t \in [\Omega, b\Omega], \\ a(t) & \text{if } t \in [0, \Omega]. \end{cases}$$

By the pasting lemma, γ is continuous, and $\gamma a = aa$, $\gamma ba = ab$ and $\gamma bb = b$. Let $\Gamma \colon I^2 \to I$ be defined by $\Gamma(s, t) = (1 - s)t + s\gamma(t)$. Then $\Gamma \partial_B = e_0$, $\Gamma \partial_T = e_1$, $\Gamma \partial_L = \mathrm{id}_I$ and $\Gamma \partial_R = \gamma$. Let $\alpha = ((H \cdot J) \cdot K) \circ (\mathrm{id}_X \times \Gamma)$. Clearly $\alpha \partial_L = (H \cdot J) \cdot K$, $\alpha \partial_B = \mathrm{const}_{f_1}$ and $\alpha \partial_T = \mathrm{const}_{f_4}$. Moreover, $(\alpha \partial_R)a = ((H \cdot J) \cdot K)aa = H$, and similarly $(\alpha \partial_R)ba = J$ and $(\alpha \partial_R)bb = K$. Thus, both $\alpha \partial_R$ and $H \cdot (J \cdot K)$ fit in the following diagram, so by uniqueness they are equal, giving the required homotopy:

Lemma 1.11 *Let $f, g_0, g_1, h \colon X \to Y$ be maps and $A \colon f \sim g_0$, $B \colon g_0 \sim h$, $C \colon f \sim g_1$ and $D \colon g_1 \sim h$ be homotopies. If there is a map $\alpha \colon X \times I^2 \to Y$ with $\alpha \partial_B = A$, $\alpha \partial_R = B$, $\alpha \partial_L = C$ and $\alpha \partial_T = D$, then $A \cdot B \sim C \cdot D$ rel endpoints.*

Proof Let $\gamma_0, \gamma_1 \colon I \to I^2$ be given by

$$\gamma_0(x) = \begin{cases} (a^{-1}(x), 0) & \text{if } x \leq a(1), \\ (1, b^{-1}(x)) & \text{if } x \geq b(0), \end{cases} \quad \text{and} \quad \gamma_1(x) = \begin{cases} (0, a^{-1}(x)) & \text{if } x \leq a(1), \\ (b^{-1}(x), 1) & \text{if } x \geq b(0). \end{cases}$$

Then $\gamma_0 a = \partial_B$, $\gamma_0 b = \partial_R$, $\gamma_1 a = \partial_L$ and $\gamma_1 b = \partial_T$. Let $\Gamma \colon I^2 \to I^2$ be given by $\Gamma(s, t) = s\gamma_1(t) + (1 - s)\gamma_0(t)$. Since $\Gamma \partial_B = e_0 \times e_0$ and $\Gamma \partial_T = e_1 \times e_1$, $\alpha(\mathrm{id}_X \times \Gamma)$ gives a homotopy rel endpoints from $A \cdot B$ to $C \cdot D$. $\square$

Lemma 1.12 [27, Lemma 1] *Let $u, v \colon X \to Y$ and $f, g \colon Y \to Z$ be maps. Given a homotopy H from u to v and a homotopy K from f to g, $fH \cdot Kv \sim Ku \cdot gH$ rel endpoints.*

Proof Let $\alpha \colon X \times I \times I \to Z$ be $K(H \times \mathrm{id}_I)$. Then $\alpha \partial_L = K(H \times \mathrm{id}_I)(e_0 \times \mathrm{id}_I) = K(He_0) = Ku$, and similarly $\alpha \partial_T = gH$, $\alpha \partial_R = Kv$ and $\alpha \partial_B = fH$. The result follows from Lemma 1.11. $\square$

Theorem 1.13 *If $\mathcal{C}$ satisfies the assumptions in Definition 1.1, then $\mathcal{C}$ admits a 2-category structure, where each pair of spaces A and B in $\mathcal{C}$ are assigned the groupoid $\mathcal{C}(A, B)$ whose objects are maps $A \to B$ and morphisms are homotopy classes of homotopies rel endpoints.*

Proof The composition in $\mathcal{C}(A, B)$ is given by track composition as in Definition 1.2. Lemmas 1.7, 1.8, 1.9 and 1.10 assert that this is well defined and associative, that each map $f: A \to B$ has an identity homotopy const_f, and that each class of homotopies $[H]$ has an inverse $[-H]$. Thus, $\mathcal{C}(A, B)$ is a groupoid for every pair of spaces A and B. For each space A, the functor $I_A: [0] \to \mathcal{C}(A, A)$ maps the unique morphism in $[0]$ to $[\mathrm{const}_{\mathrm{id}_A} : \mathrm{id}_A \sim \mathrm{id}_A]$. The composition functors $c_{A,B,C}: \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ act on objects by sending a pair of maps $(f: A \to B, g: B \to C)$ to $gf: A \to C$ and, by Lemma 1.12, there is a canonical, well-defined choice for a pair of homotopy classes $([H: u \sim v], [K: f \sim g])$, namely mapping them to $[fH \cdot Kv] = [Ku \cdot gH]$. $\qquad\square$

We now turn our attention to defining homotopy equivalences.

Definition 1.14 A map $f: X \to Y$ is a *homotopy equivalence* if there exists $g: Y \to X$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$. If $gf = \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$ rel X, then g is a *deformation retraction*.

Definition 1.15 Let ho $\mathcal{C}$ denote the category whose objects are spaces in $\mathcal{C}$ and whose maps are homotopy classes of continuous maps. By Lemma 1.12, composition of homotopy classes is well defined. An isomorphism in this category is a class of homotopy equivalences. There is a canonical quotient functor $\Pi: \mathcal{C} \to \mathrm{ho}\,\mathcal{C}$, which maps $f: X \to Y$ to $[f]: X \to Y$.

We record two lemmas about homotopy equivalences for future use.

Lemma 1.16 *A map $f: X \to Y$ is a homotopy equivalence if and only if Πf is an isomorphism in* ho $\mathcal{C}$.

Proof If f has a homotopy inverse $g: Y \to X$, then, since $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$, we have $[g][f] = [\mathrm{id}_X]$ and $[f][g] = [\mathrm{id}_Y]$, so $\Pi f = [f]$ is an isomorphism. Conversely, if $\Pi f = [f]$ is an isomorphism, choose a representative $g: Y \to X$ for $[f]^{-1}$. Then $[gf] = [g][f] = [\mathrm{id}_X]$, so $gf \sim \mathrm{id}_X$, and likewise $fg \sim \mathrm{id}_Y$, so f is a homotopy equivalence. $\qquad\square$

Lemma 1.17 *The following properties hold for homotopy equivalences*:

(1) *Homotopy equivalences are closed under 2-out-of-6.*

(2) *A retract of a homotopy equivalence in the arrow category is again a homotopy equivalence.*

(3) *If $f: X \to Y$ and $f': X' \to Y'$ are homotopy equivalences, then so is $f \times f'$.*

(4) *Assuming $-\times I$ preserves coproducts, then, if $\{f_k\}_{k \in K}$ is a family of homotopy equivalences, so is $\coprod_{k \in K} f_k$.*

Proof (1) and (2) are immediate from Lemma 1.16, since isomorphisms satisfy these in any category and homotopy equivalences are precisely the maps inverted by Π.

For (3), choose homotopy inverses $g: Y \to X$ and $g': Y' \to X'$, as well as homotopies H from gf to id_X and H' from $g'f'$ for $\mathrm{id}_{X'}$. Then a homotopy from $gf \times g'f' = (g \times g')(f \times f')$ to $\mathrm{id}_{X \times X'}$ is given by (H, H'). Similar reasoning gives a homotopy from $(f \times f')(g \times g')$ to $\mathrm{id}_{Y \times Y'}$, so $f \times f'$ is a homotopy equivalence.

Lastly, for (4), given families $\{g_k\}_{k\in K}$ of homotopy inverses with $\{H_k\}_{k\in K}$ and $\{J_k\}_{k\in K}$ homotopies from $g_k f_k$ and $f_k g_k$ to the respective identities, $\coprod H_k$ and $\coprod J_k$ give homotopies from $\coprod g_k \circ \coprod f_k$ and $\coprod f_k \circ \coprod g_k$ to the respective identities since $\left(\coprod \mathrm{dom}(f_k)\right) \times I \cong \coprod(\mathrm{dom}(f_k) \times I)$. $\qquad\square$

Pushout products

In this final subsection, we collect the requisite background on the Leibniz construction. Specifically, we recall the definitions of pushout product (Definition 1.18) and pullback power (Definition 1.20) and prove their basic properties in the category of topological spaces.

Definition 1.18 Let $f\colon X \to Y$ and $g\colon A \to B$ be maps in $\mathcal{C}$. Their *pushout product*, denoted by $f \mathbin{\hat{\times}} g$, is the factorization of $f \times g\colon X \times A \to Y \times B$ through the pushout in the following diagram:

$$
\begin{array}{ccc}
X \times A & \xrightarrow{\;f \times \mathrm{id}_A\;} & Y \times A \\
{\scriptstyle \mathrm{id}_X \times g}\big\downarrow & & \big\downarrow \\
X \times B & \xrightarrow{\qquad\qquad} & P \\
& \searrow{\scriptstyle f \times \mathrm{id}_A} & \;\;\searrow{\scriptstyle f \mathbin{\hat{\times}} g} \\
& & Y \times B
\end{array}
$$

with $\mathrm{id}_X \times g$ on the right.

Note that, since $\times$ is symmetric, $f \mathbin{\hat{\times}} g \cong g \mathbin{\hat{\times}} f$, and that $\hat{\times}$ is associative up to isomorphism.

Lemma 1.19 *If $f\colon A \to B$ and $f'\colon A' \to B'$ are isomorphic in the arrow category, as are $g\colon C \to D$ and $g'\colon C' \to D'$, then $f \mathbin{\hat{\times}} g$ and $f' \mathbin{\hat{\times}} g'$ are isomorphic in the arrow category.*

Proof This is a standard diagram chase, given that naturally isomorphic diagrams induce an isomorphism between their colimits which commutes with the colimit legs. $\qquad\square$

A functor $\otimes\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is *divisible on the right* if, for every $B \in \mathcal{B}$, the functor $- \otimes B\colon \mathcal{A} \to \mathcal{C}$ admits a right adjoint. When the functor $- \times A\colon \mathcal{C} \to \mathcal{C}$ admits a right adjoint, A is an *exponentiable* space. The right adjoint will be denoted by $(-)^A\colon \mathcal{C} \to \mathcal{C}$, and its action on a morphism $f\colon X \to Y$ will be denoted by $f_*\colon X^A \to Y^A$. A map $g\colon A \to B$ between exponentiable spaces induces a natural transformation $g^*\colon (-)^B \Rightarrow (-)^A$.

Definition 1.20 Let $f\colon X \to Y$ and $g\colon A \to B$ be maps in $\mathcal{C}$ with A and B both exponentiable. Their *pullback power*, denoted by $f \mathbin{\triangleright} g$, is the factorization of $f_* g^* = g^* f_*$ in the following diagram:

$$
\begin{array}{ccc}
X^B & & \\
& \searrow{\scriptstyle f_*} & \\
{\scriptstyle g^*}\big\downarrow \;\;{\scriptstyle f \triangleright g} & & \\
& P & \longrightarrow Y^B \\
& \big\downarrow & \big\downarrow{\scriptstyle g^*} \\
& X^A & \xrightarrow{\;f_*\;} Y^A
\end{array}
$$

Note that, unlike the pushout product, in general $f \mathbin{\triangleright} g \neq g \mathbin{\triangleright} f$.

The following statement holds in general for *closed* monoidal products in a given category. Since we will not be taking $\mathcal{C}$ to be cartesian closed, we need to ensure that we are only exponentiating spaces that we have assumed or shown to be exponentiable.

Lemma 1.21 *Let $i : A \to B$ be a map between exponentiable spaces. Then $f \mathbin{\hat{\times}} i$ has the left lifting property with respect to g if and only if f has the left lifting property with respect to $g \triangleright i$*

Proof This follows from [13, Proposition D.1.18], since the functor $- \times - : \mathcal{C} \times \mathcal{C}_{\exp} \to \mathcal{C}$ is divisible on the right, where $\mathcal{C}_{\exp}$ denotes the full subcategory of $\mathcal{C}$ consisting of exponentiable spaces. $\square$

Lemma 1.22 *Let A be a space and $!_A : \varnothing \to A$ the unique map. For any map $f : X \to Y$ and space A, we have $!_A \mathbin{\hat{\times}} f \cong \mathrm{id}_A \times f$.*

Proof This is immediate from (S4). $\square$

Lemma 1.23 *The following identities hold for the pushout product in the arrow category $\mathsf{Top}^{\to}$, where $!_X : \varnothing \to X$ and $i : \partial I \to I$ is the inclusion of the endpoints*:

(1) $e_0 \mathbin{\hat{\times}} e_0 \cong !_I \mathbin{\hat{\times}} e_0$.

(2) $i \mathbin{\hat{\times}} e_0 \cong !_I \mathbin{\hat{\times}} e_0$.

Proof For (1), it follows from Lemma 1.22 that we must find an automorphism $\phi : I \times I \to I \times I$ that restricts to an isomorphism $I \times \{0\} \cong I \times \{0\} \cup \{0\} \times I$, ie such that the following commutes:

$$
\begin{array}{ccc}
I \times \{0\} & \xrightarrow{\;\phi|_{I \times \{0\}}\;} & I \times \{0\} \cup \{0\} \times I \\
\downarrow & & \downarrow \\
I \times I & \xrightarrow[\phi]{} & I \times I
\end{array}
$$

One such ϕ is given by the composite $g f$, where $g(x, y) = \left(\frac{1}{2}(1 + x), y\right)$ and

$$
f(x, y) = \begin{cases}
(x, 2y) & \text{if } y \le \frac{1}{2}x, \\
(2(x - y), x) & \text{if } \frac{1}{2}x \le y \le x, \\
(2(x - y), y) & \text{if } x \le y \le 2x, \\
(-y, 2x) & \text{if } y \ge 2x.
\end{cases}
$$

The proof of (2) is similar. $\square$

Note that, since the pushout product is well defined up to isomorphism in the arrow category by Lemma 1.19, the isomorphism in (1) holds regardless of what pushout we take for $I \amalg_* I$, even though we chose a specific pushout for the calculation in (1). In particular, since we have taken I for the pushout $I \amalg_* I$ in $\mathcal{C}$, and $\square_{\le 2}$ to be a full subcategory, we may use (1) in any category with intervals.

2 Q-structures: fibrations and cofibrations

In this and the next section, we will show that the axioms of a Q-structure given in Definition 2.2 are sufficient for $\mathcal{C}$ to admit a model structure defined analogously to the standard model structure on Top. If

$\mathcal{C}$ satisfies the additional requirements for being a good Q-structure, given in (Q7) and (Q8), then the weak equivalences of the model structure are precisely the ones which induce isomorphisms on path-connected components and all homotopy groups, as defined in Definition 3.7.

We begin by introducing (relative) cell complexes.

Definition 2.1 Let $\mathcal{J}$ be a class of maps. A $\mathcal{J}$-*cell complex* is a transfinite composition of pushouts of coproducts of maps in $\mathcal{J}$, as in the following diagram:

$$
\begin{array}{ccc}
\coprod \mathrm{dom}(j_\alpha) & \xrightarrow{\;\coprod j_\alpha\;} & \coprod \mathrm{cod}(j_\alpha) \\
\downarrow & & \downarrow{\scriptstyle p_1} \\
X_0 \xrightarrow{\quad} & X_1 \xrightarrow{\qquad\qquad} & X_2 \xrightarrow{\qquad} \cdots \xrightarrow{\qquad} \mathrm{colim}\, X_k \\
& \uparrow & \uparrow{\scriptstyle p_2} \\
\coprod \mathrm{dom}(j_\beta) & \xrightarrow[\;\coprod j_\beta\;]{} & \coprod \mathrm{cod}(j_\beta)
\end{array}
$$

The collection of all $\mathcal{J}$-cell complexes will be denoted by $\mathrm{cell}(\mathcal{J})$, and $\mathrm{cof}(\mathcal{J})$ will denote the closure of $\mathrm{cell}(\mathcal{J})$ under retracts. Let $\mathrm{rlp}(\mathcal{J})$ denote the class of maps which have the right lifting property against $\mathcal{J}$. A standard proof shows that $\mathrm{rlp}(\mathcal{J}) = \mathrm{rlp}(\mathrm{cof}(\mathcal{J}))$ [9, Proposition 10.3.2].

Notation 2.1 Let $\mathcal{I} = \{i_n\colon \partial I^n \hookrightarrow I^n\}_{n\geq 0}$ and $\mathcal{J} = \{j_n\colon I^n \times \{0\} \hookrightarrow I^n \times I\}_{n\geq 0}$ be the subspace inclusions in Top.

As a slight abuse of notation, let $e_k\colon * \hookrightarrow \partial I$ for $k = 0, 1$ be the endpoint inclusions.

Definition 2.2 Let $\iota\colon \square \to \mathcal{C}$ be an embedding of $\square$ into a bicomplete category $\mathcal{C}$. The pair $(\mathcal{C}, \iota)$ is a *Q-structure* if the following hold:

(Q1) $\square_{\leq 2} \hookrightarrow \square \xrightarrow{\;\iota\;} \mathcal{C}$ makes $\mathcal{C}$ a category with intervals.

(Q2) $X \times \partial I$ is a coproduct $X \amalg X$ in $\mathcal{C}$, with inclusions $\mathrm{id}_X \times e_k$ for all $X \in \mathcal{C}$.

(Q3) $\mathcal{I} \,\hat{\times}\, \mathcal{I} \subseteq \mathrm{cof}(\mathcal{I})$ and $\mathcal{I} \,\hat{\times}\, \mathcal{J} \subseteq \mathrm{cof}(\mathcal{J})$.

(Q4) $\mathcal{J} \subseteq \mathrm{cof}(\mathcal{I})$ and $\{\varnothing \to \partial I^n\} \subseteq \mathrm{cof}(\mathcal{I})$.

(Q5) ∂I^n and I^n are exponentiable in $\mathcal{C}$ for all $n \geq 0$.

(Q6) ∂I^n and I^n are small relative to maps in $\mathrm{cell}(\mathcal{I})$ and $\mathrm{cell}(\mathcal{J})$, respectively, for all $n \geq 0$.

The pair $(\mathcal{C}, \iota)$ is a *good Q-structure* if, in addition:

(Q7) I^n is a product $\prod_{1 \leq k \leq n} I$ for all positive integers n.

(Q8) There are pushouts, for all $n \geq 1$, of the form

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{\;e_1\;} & \partial I^n \times I \\
{\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \rho} \\
* & \xrightarrow[\;c\;]{} & I^n
\end{array}
\qquad\qquad
\begin{array}{ccc}
\partial I^{n-1} & \xhookrightarrow{\;i_n\;} & I^{n-1} \\
{\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \sigma} \\
* & \xrightarrow[\;(*)\;]{} & \partial I^n
\end{array}
$$

in $\mathcal{C}$ such that $\rho e_0 = i_n$.

The axioms listed above are key properties used to construct the Quillen model structure on Top, restated for an arbitrary $\mathcal{C}$. We will use the axioms as follows: (Q1) allows us to make use of the homotopy theory developed in Section 1; (Q2) is used in lifting homotopies along certain maps, with prescribed endpoints; (Q3) and (Q5) allow us to obtain a weak version of the pushout product axiom; lastly, (Q4) and (Q6) are used in applying the small object argument. For good Q-structures, (Q7) will be used to define homotopy groups (see Definition 3.7) and (Q8) are some familiar quotient identities from Top, namely $S^{n-1} \times I / S^{n-1} \times \{0\} \cong D^n$ and $D^n / S^{n-1} \cong S^n$.

It follows from (Q5) that $- \times I^n$, as a left adjoint, preserves colimits for all $n \geq 0$. (Q6) is sufficient for $\mathcal{C}$ to admit a small object argument on $\mathcal{I}$ and $\mathcal{J}$, which generates two weak factorization systems in $\mathcal{C}$. For $\mathcal{K}$ equal to either $\mathcal{I}$ or $\mathcal{J}$, one has for the left class $\mathrm{cof}(\mathcal{K})$ and for the right class $\mathrm{rlp}(\mathcal{K})$ of the respective weak factorization system.

In Sections 4, 5 and 6, we discuss several examples of good Q-structures. We do not know of any examples of Q-structures that are not good; however, we do not expect the axioms (Q1)–(Q6) to imply (Q7)–(Q8), since the latter require the existence of certain pushouts squares which are otherwise not assumed to exist. We chose to isolate the axioms (Q1)–(Q6) to underscore the fact that they are the only ones required to establish a model structure, while (Q7)–(Q8) are used to characterize the class of weak equivalences.

We can now define the classes of maps that will form a model structure on $\mathcal{C}$.

Definition 2.3 (1) A *cofibration* is a map in $\mathrm{cof}(\mathcal{I})$.

(2) A *fibration* is a map in $\mathrm{rlp}(\mathcal{J})$.

(3) A *trivial cofibration* is a map in $\mathrm{cof}(\mathcal{J})$.

(4) A *trivial fibration* is a map in $\mathrm{rlp}(\mathcal{I})$.

Using the small object argument, we obtain the expected factorizations:

Lemma 2.4 *Every map in $\mathcal{C}$ can be factored as*

(1) *a cofibration followed by a trivial fibration, and*

(2) *a trivial cofibration followed by a fibration.* □

Definition 2.5 Let $f : X \to Y \in \mathcal{C}$ and n a nonnegative integer. Then:

(1) f is *n-compressible* if any square of the form

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{\ u\ } & X \\
{\scriptstyle i_n}\big\uparrow & & \big\downarrow{\scriptstyle f} \\
I^n & \xrightarrow{\ v\ } & Y
\end{array}
$$

admits a diagonal map $h : I^n \to X$ such that $h i_n = u$ and $f h \sim v$ rel ∂I^n.

(2) f is *n-connected* if it is k-compressible for all $k \leq n$.

(3) f is a *weak equivalence* if it is n-connected for all nonnegative integers.

A map f has the *weak right lifting property* against $i : A \to B$ if it has the lifting property as in (1) against i: given maps such that $fu = vi$, there is a filler h satisfying $hi = u$ and $fh \sim v$ rel A.

Theorem 2.6 *Any Q-structure* $(\mathcal{C}, \iota)$ *induces a model structure on* $\mathcal{C}$ *whose cofibrations and fibrations are as defined in Definition 2.3 and weak equivalences are as defined in Definition 2.5.*

We will defer the proof of this to Section 3.

Definition 2.7 A space X is *fibrant* if the map $X \to *$ is a fibration. Dually, X is *cofibrant* if the map $\varnothing \to X$ is a cofibration.

Lemma 2.8 *Every space is fibrant.*

Proof Given the square

$$
\begin{array}{ccc}
I^n & \xrightarrow{\ f\ } & X \\
{\scriptstyle j_n}\downarrow & & \downarrow{\scriptstyle !} \\
I^n \times I & \xrightarrow{\ !\ } & *
\end{array}
$$

the map $f\pi_{I^n}$ gives a lift. $\square$

Lemma 2.9 *Every trivial fibration is both a fibration and a weak equivalence.*

Proof Clearly, if f has the right lifting property with respect to $\mathcal{I}$, it is a weak equivalence, since every square of the form

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{\ u\ } & X \\
{\scriptstyle i_n}\uparrow & & \downarrow{\scriptstyle f} \\
I^n & \xrightarrow{\ v\ } & Y
\end{array}
$$

admits a strict lift, not just one up to homotopy on the lower triangle. Moreover, by (Q4), since f has the right lifting property with respect to all maps in $\mathrm{cof}(\mathcal{I})$, it has the right lifting property with respect to $\mathcal{J}$, and thus is a fibration. $\square$

Proposition 2.10 *The following identities hold in* $\mathcal{C}$:

(1) $\mathrm{cof}(\mathcal{I}) \mathbin{\hat{\times}} \mathcal{I} \subseteq \mathrm{cof}(\mathcal{I})$.

(2) $\mathrm{cof}(\mathcal{I}) \mathbin{\hat{\times}} \mathcal{J} \subseteq \mathrm{cof}(\mathcal{J})$.

(3) $\mathcal{I} \mathbin{\hat{\times}} \mathrm{cof}(\mathcal{J}) \subseteq \mathrm{cof}(\mathcal{J})$.

Proof Since $\mathcal{I} \mathbin{\hat{\times}} \mathcal{I} \subseteq \mathrm{cof}(\mathcal{I})$ by (Q3), $\mathcal{I} \mathbin{\hat{\times}} \mathcal{I}$ has the left lifting property with respect to $\mathrm{rlp}(\mathcal{I})$, so $\mathcal{I}$ has the left lifting property with respect to $\mathrm{rlp}(\mathcal{I}) \triangleright \mathcal{I}$ by Lemma 1.21. Thus, $\mathrm{cof}(\mathcal{I})$ has the left lifting property with respect to $\mathrm{rlp}(\mathcal{I}) \triangleright \mathcal{I}$, and hence $\mathrm{cof}(\mathcal{I}) \mathbin{\hat{\times}} \mathcal{I} \subseteq \mathrm{cof}(\mathcal{I})$ by Lemma 1.21 again. The other parts are analogous. $\square$

For closed monoidal products $\otimes$, we can obtain the stronger result that $\mathrm{cof}(\mathcal{I}) \mathbin{\hat{\otimes}} \mathrm{cof}(\mathcal{J}) \subseteq \mathrm{cof}(\mathcal{K})$ whenever $\mathcal{I} \mathbin{\hat{\otimes}} \mathcal{J} \subseteq \mathcal{K}$. In this case, maps in $\mathrm{cof}(\mathcal{I})$ might not be between exponentiable spaces, so we cannot apply

the same reasoning again. Thus, some care will be needed when applying Proposition 2.10 to ensure that at least one map is in either $\mathcal{I}$ or $\mathcal{J}$.

Definition 2.11   Let $[-, -]$ denote the induced map from the pushout

$$
\begin{array}{ccc}
* & \xrightarrow{\ e_0\ } & I \\
{\scriptstyle e_1}\downarrow & & \downarrow \\
I & \xrightarrow{\quad} & I \\
\end{array}
$$
∂_T, ∂_L, $[\partial_L,\partial_T]$, I^2
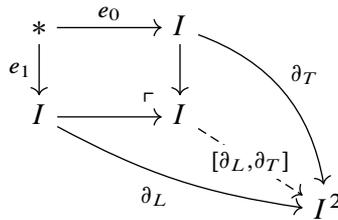
The *open box inclusion* is the induced map in the pushout

$$
\begin{array}{ccc}
* & \xrightarrow{\ e_0\ } & I \\
{\scriptstyle e_1}\downarrow & & \downarrow \\
I & \xrightarrow{\quad} & I \\
\end{array}
$$
$-\partial_R$, $[[\partial_L,\partial_T],-\partial_R]$, $[\partial_L,\partial_T]$, I^2
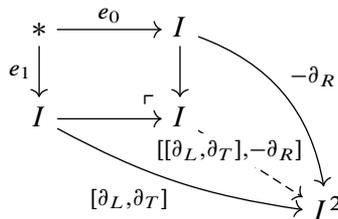
By Lemmas 1.19 and 1.23, we recognize the top pushout as $[\partial_L, \partial_T] \cong e_0 \,\hat{\times}\, e_0 \cong \,!_I \,\hat{\times}\, e_0 \cong \mathrm{id}_I \times e_0 = j_1$, so, by the same reasoning, $[[\partial_L, \partial_T], -\partial_R] \cong j_1$ in the arrow category. When we wish to make the distinction, we will use $\sqcap$ for I in the latter pushout and inc$: \sqcap \hookrightarrow I^2$ for $[[\partial_L, \partial_T], -\partial_R]$. Geometrically, it should be thought of as the left, top, and right edges of the square, which we will use to "fill out" homotopies. By Proposition 2.10, if $i: A \rightarrowtail X$ is a cofibration, $i \,\hat{\times}\, \mathrm{inc}$ is a trivial cofibration. To verify the equality of two maps from $\sqcap$ in this context, one of which is a composition including inc, it suffices to check that they agree after precomposing with b, aa and ab (see Definition 1.1).

Lemma 2.12   *Let $f: X \to Y$ be a homotopy equivalence. Then there is a map $g: Y \to X$ along with homotopies H from id_X to gf and K from id_Y to fg and a map $\alpha: X \times I^2 \to Y$ such that $\alpha\partial_L = fH$, $\alpha\partial_T = \mathrm{const}_{fgf}$, $\alpha\partial_R = Kf$ and $\alpha\partial_B = \mathrm{const}_f$.*

Proof   This is the statement that any equivalence can be promoted to an adjoint equivalence, applied to the 2-category $\mathcal{C}$ as given in Theorem 1.13.   □

Theorem 2.13   *Homotopy equivalences are weak equivalences.*

Proof   Let $f: X \to Y$ be a homotopy equivalence and g, H, K and α be as in the statement of Lemma 2.12. Suppose that, for a nonnegative integer n, the following diagram commutes:

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{\ u\ } & X \\
{\scriptstyle i_n}\uparrow\downarrow & & \downarrow{\scriptstyle f} \\
I^n & \xrightarrow{\ v\ } & Y \\
\end{array}
$$

By Proposition 2.10, $i_n \,\hat{\times}\, e_1 \in \mathrm{cof}(\mathcal{J})$, so there is a lift $J: I^n \times I \to X$ since X is fibrant:

$$
\begin{array}{ccc}
\partial I^n \times I \amalg_{\partial I^n \times \{1\}} I^n \times \{1\} & \xrightarrow{\;[Hu, g\,v]\;} & X \\[2pt]
{\scriptstyle i_n \hat{\times} e_1}\downarrow & \nearrow^{\!\!J} & \\[2pt]
I^n \times I & &
\end{array}
$$

Let $w = Je_0 : I^n \to X$. Since $He_0 = \mathrm{id}_X$, we have $w i_n = Je_0 i_n = J(i_n \times \mathrm{id}_I)e_0 = Hue_0 = u$. By the lifting triangle above, $fJi_n = fHu$, and, by the initial square, $Kvi_n = Kfu$. Taking the convention for $\mathrm{inc}: \sqcap \hookrightarrow I^2$ as in Definition 2.11,

$$
\alpha u \circ \mathrm{inc} \circ b = \alpha u \circ -\partial_R = -Kfu = -Kvi_n,
$$

$$
\alpha u \circ \mathrm{inc} \circ aa = \alpha u \partial_L = fHu = fJi_n,
$$

$$
\alpha u \circ \mathrm{inc} \circ ab = \alpha u \partial_T = \mathrm{const}_{fgfu} = \mathrm{const}_{fgv}i_n;
$$

hence, the top map in the diagram

$$
\begin{array}{ccc}
\partial I^n \times I^2 \amalg_{\partial I^n \times \sqcap} I^n \times \sqcap & \xrightarrow{\;[\alpha u, [[fJ, \mathrm{const}_{fgv}], -Kv]]\;} & Y \\[2pt]
{\scriptstyle i_n \hat{\times} \mathrm{inc}}\downarrow & \nearrow^{\!\!\tilde{\alpha}} & \\[2pt]
I^n \times I^2 & &
\end{array}
$$

is defined, so it admits a lift $\tilde{\alpha}: I^n \times I^2 \to Y$ by Proposition 2.10. The composite $\tilde{\alpha}\partial_B$ gives a homotopy from fw to v rel ∂I^n, since $\tilde{\alpha}\partial_B i_n = \tilde{\alpha}(i_n \times \mathrm{id}_{I^2})\partial_B = \alpha u \partial_B = \mathrm{const}_{fu} = \mathrm{const}_{vi_n}$. $\square$

3 Quillen model structure

In this section, we put all the pieces together and prove Theorem 2.6, while also characterizing weak equivalences in terms of homotopy groups (Theorem 3.12), which are introduced in Definition 3.7.

Lemma 3.1 *For a weak equivalence $f: X \to Y$ and a cofibration $i: A \rightarrowtail B$, if the square*

$$
\begin{array}{ccc}
A & \xrightarrow{\;u\;} & X \\[2pt]
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\[2pt]
B & \xrightarrow{\;v\;} & Y
\end{array}
$$

commutes, then there is a filler $h: B \to X$ such that $hi = u$ and $fh \sim v$ rel A.

Proof This is analogous to the proof that the left class of a weak factorization system is closed under coproducts, pushouts, transfinite composition and retracts, except special care is needed to check that relativity is preserved in each part. To get the induced homotopy on the colimits, one needs to use relativity and cocontinuity of $- \times I$. We will only give the proof for pushouts. Suppose that the square

$$
\begin{array}{ccc}
A & \xrightarrow{\;\alpha\;} & A' \\[2pt]
{\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle i'} \\[2pt]
B & \xrightarrow{\;\beta\;} & B'
\end{array}
$$

is a pushout and that f has the weak right lifting property (see Definition 2.5) against i. We can combine this square with

$$
\begin{array}{ccc}
A' & \xrightarrow{u} & X \\
i' \downarrow & & \downarrow f \\
B' & \xrightarrow{v} & Y
\end{array}
$$

to get a diagonal filler $h\colon B \to X$ with $hi = u\alpha$ and a homotopy $H\colon B \times I \to Y$ from fh to $v\beta$ rel A. This induces the map $h'\colon B' \to X$ and homotopy $H'\colon B' \times I \to Y$, by cocontinuity of $-\times I$, in

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
i \downarrow & & \downarrow i' \\
B & \xrightarrow{\beta} & B'
\end{array}
\qquad u,\ h',\ h \to X
$$

$$
\begin{array}{ccc}
A \times I & \xrightarrow{\alpha \times \mathrm{id}_I} & A' \times I \\
i \times \mathrm{id}_I \downarrow & & \downarrow i' \times \mathrm{id}_I \\
B \times I & \xrightarrow{\beta \times \mathrm{id}_I} & B' \times I
\end{array}
\qquad \mathrm{const}_{fu},\ H',\ H \to Y
$$

since $Hi = \mathrm{const}_{fhi} = \mathrm{const}_{fu\alpha} = \mathrm{const}_{fu}\alpha$ by relativity of H. By commutativity, $H'i' = \mathrm{const}_{fu}$, so H' is relative to A', and one can verify through the pushout on the right that $H'e_0 = fh'$ and $H'e_1 = v$. $\square$

Definition 3.2 Let $\mathrm{ho}\,\mathcal{C}_{\mathrm{cof}}$ denote the full subcategory of $\mathrm{ho}\,\mathcal{C}$ (Definition 1.15) consisting of cofibrant spaces.

Theorem 3.3 *A map $f\colon X \to Y$ is a weak equivalence if and only if*

$$(*) \qquad\qquad f_*\colon [A, X] \to [A, Y]$$

is a bijection for all cofibrant spaces A.

Proof Suppose first that f is a weak equivalence, and let A be a cofibrant space. By the above lemma, for any $[\beta] \in [A, Y]$, there is a filler $\alpha\colon A \to X$ in the diagram

$$
\begin{array}{ccc}
\varnothing & \xrightarrow{\ !\ } & X \\
! \downarrow & & \downarrow f \\
A & \xrightarrow{\beta} & Y
\end{array}
$$

such that $f\alpha \sim \beta$; hence, f_* is surjective. If $f_*[\alpha] = f_*[\beta]$ for $[\alpha], [\beta] \in [A, X]$, then there is a homotopy $H\colon A \times I \to Y$ from $f\alpha$ to $f\beta$. Since A is cofibrant, $!_A \mathbin{\widehat{\times}} i_1 = \mathrm{id}_A \times i_1 \colon A \times \partial I \to A \times I$ is a cofibration by Proposition 2.10 and Lemma 1.22. By (Q2), $A \times \partial I$ is a coproduct $A \amalg A$, so the square

$$
\begin{array}{ccc}
A \times \partial I & \xrightarrow{\alpha + \beta} & X \\
!_A \widehat{\times} i_1 \downarrow & & \downarrow f \\
A \times I & \xrightarrow{H} & Y
\end{array}
$$

admits a filler $K\colon A \times I \to X$ such that the upper triangle commutes, so K is a homotopy from α to β; hence, f_* is injective.

Suppose instead that $f_* \colon [A, X] \to [A, Y]$ is a bijection for all cofibrant spaces A. Using Lemma 2.4, factor $\varnothing \to X$ as a cofibration followed by a trivial fibration, which is a weak equivalence and a fibration by Lemma 2.9: $\varnothing \rightarrowtail \widetilde{X} \xrightarrow{\sim}\!\!\!\twoheadrightarrow X$. Similarly, factor $\varnothing \to Y$ into $\varnothing \rightarrowtail \widetilde{Y} \xrightarrow{\sim}\!\!\!\twoheadrightarrow Y$. Then there is a lift $\tilde{f} \colon \widetilde{X} \to \widetilde{Y}$ in the diagram

$$
\begin{array}{ccc}
\varnothing & \rightarrowtail & \widetilde{Y} \\
\downarrow & \nearrow^{\tilde{f}} & \downarrow\wr \\
\widetilde{X} & \longrightarrow & Y
\end{array}
$$

where the bottom arrow is the composite $\widetilde{X} \xrightarrow{\sim}\!\!\!\twoheadrightarrow X \xrightarrow{f} Y$. Since we have already shown that weak equivalences satisfy $(*)$, $\tilde{f}$ does as well, by the 2-out-of-3 property for isomorphisms, since the following diagram commutes:

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\;\sim\;}\!\!\!\twoheadrightarrow & X \\
\tilde{f}\downarrow & & \downarrow f \\
\widetilde{Y} & \xrightarrow{\;\sim\;}\!\!\!\twoheadrightarrow & Y
\end{array}
$$

Then $[\tilde{f}]_* \colon \operatorname{ho} \mathcal{C}_{\mathrm{cof}}(-, \widetilde{X}) \Rightarrow \operatorname{ho} \mathcal{C}_{\mathrm{cof}}(-, \widetilde{Y})$ is a natural isomorphism, so it follows from the Yoneda lemma that $[\tilde{f}]$ is an isomorphism in $\operatorname{ho} \mathcal{C}_{\mathrm{cof}}$, and hence a homotopy equivalence and a weak equivalence by Theorem 2.13. Suppose now that the following diagram commutes:

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{\;u\;} & X \\
i_n\downarrow & & \downarrow f \\
I^n & \xrightarrow[v]{} & Y
\end{array}
$$

Since ∂I^n is cofibrant by (Q4), we may choose lifts for the following squares:

$$
\begin{array}{ccc}
\varnothing & \longrightarrow & \widetilde{X} \\
\downarrow & \nearrow^{\tilde{u}} & \downarrow\wr \\
\partial I^n & \xrightarrow[u]{} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
\partial I^n & \xrightarrow{\tilde{f}\tilde{u}} & \widetilde{Y} \\
i_n\downarrow & \nearrow^{\tilde{v}} & \downarrow\wr \\
I^n & \xrightarrow[v]{} & Y
\end{array}
$$

Then we obtain a map $\tilde{h} \colon I^n \to \widetilde{X}$ such that $\tilde{h}i_n = \tilde{u}$ and $\tilde{f} \circ \tilde{h} \sim \tilde{v}$ rel ∂I^n in the following diagram:

$$
\begin{array}{ccccc}
\partial I^n & \xrightarrow{\tilde{u}} & \widetilde{X} & \longrightarrow\!\!\!\!\!\twoheadrightarrow & X \\
i_n\downarrow & & \downarrow\tilde{f} & & \downarrow f \\
I^n & \xrightarrow[\tilde{v}]{} & \widetilde{Y} & \longrightarrow\!\!\!\!\!\twoheadrightarrow & Y
\end{array}
$$

Composing $\tilde{h}$ with $\widetilde{X} \xrightarrow{\sim}\!\!\!\twoheadrightarrow X$ gives the required map. $\square$

Proposition 3.4 *Weak equivalences satisfy the 2-out-of-3 property.*

Proof This is immediate from Theorem 3.3 and the 2-out-of-3 property for isomorphisms. $\square$

Theorem 3.5 *A map $f \colon X \to Y$ is a trivial fibration if and only if f is a weak equivalence and a fibration.*

Proof The forward direction is given in Lemma 2.9. Suppose instead that f is a weak equivalence and a fibration. Given a square of the form

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{\ u\ } & X \\
{\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f} \\
I^n & \xrightarrow{\ v\ } & Y
\end{array}
$$

choose diagonal filler $w\colon I^n \to X$ such that $w i_n = u$ and a homotopy $H\colon I^n \times I \to Y$ from fw to v relative to ∂I^n. Since $i_n \mathbin{\hat{\times}} j_0 \in \mathrm{cof}(\mathcal{J})$ by Proposition 2.10, we may choose a lift $\tilde{H}\colon I^n \times I \to X$ for the following square:

$$
\begin{array}{ccc}
\partial I^n \times I \amalg_{\partial I^n \times \{0\}} I^n \times \{0\} & \xrightarrow{[\mathrm{const}_u, w]} & X \\
{\scriptstyle i_n \hat{\times} j_0}\downarrow \quad\quad \nearrow{\scriptstyle \tilde{H}} & & \downarrow{\scriptstyle f} \\
I^n \times I & \xrightarrow{\quad H \quad} & Y
\end{array}
$$

Commutativity of this square comes from $H e_0 = fw$ and H being relative to ∂I^n. Let $\tilde{w} = \tilde{H} e_1$. Then $\tilde{w} i_n = \tilde{H} i_n e_1 = \mathrm{const}_u e_1 = u$ and $f \tilde{w} = H e_1 = v$, so $\tilde{w}$ is the required lift. $\qquad\square$

Theorem 3.6 *A map $f\colon X \to Y$ is a trivial cofibration if and only if f is a weak equivalence and a cofibration.*

Proof It follows from (Q4) that $\mathrm{cof}(\mathcal{J}) \subseteq \mathrm{cof}(\mathcal{I})$, so every trivial cofibration is a cofibration. Suppose f is a trivial cofibration. Since X is fibrant, we obtain a lift $p\colon Y \to X$ in the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \mathrm{id}_X\ } & X \\
{\scriptstyle f}\downarrow \quad \nearrow{\scriptstyle p} & & \downarrow{\scriptstyle !} \\
Y & \xrightarrow{\quad !\quad} & *
\end{array}
$$

such that $pf = \mathrm{id}_X$. We show that $fp \sim \mathrm{id}_Y$, from which it will follow from Theorem 2.13 that f is a weak equivalence. Since $f \mathbin{\hat{\times}} i_1 \in \mathrm{cof}(\mathcal{J})$ by Proposition 2.10 and $Y \times \partial I$ is a coproduct $Y \amalg Y$ by (Q2), we obtain a lift $H\colon Y \times I \to Y$ in

$$
\begin{array}{ccc}
X \times I \amalg_{X \times \partial I} (Y \times \partial I) & \xrightarrow{[\mathrm{const}_f, \mathrm{id}_Y + fp]} & Y \\
{\scriptstyle f \hat{\times} i_1}\downarrow \quad\quad \nearrow{\scriptstyle H} & & \downarrow{\scriptstyle !} \\
Y \times I & \xrightarrow{\quad !\quad} & *
\end{array}
$$

By commutativity, H is a homotopy from fp to id_Y.

The converse follows immediately by the retract argument. $\qquad\square$

We are now ready to show that the three classes given in Definitions 2.3 and 2.5 define a model structure on $\mathcal{C}$.

Proof of Theorem 2.6 By (Q6), $\mathcal{C}$ admits two weak factorization systems. The first, cofibrantly generated by $\mathcal{I}$, has cofibrations for its left class by definition and, by Theorem 3.5, the intersection of fibrations

and weak equivalences for its right class. The second, cofibrantly generated by $\mathcal{J}$, has, by Theorem 3.6, the intersection of cofibrations and weak equivalences for its left class and fibrations for its right class by definition. Moreover, by Proposition 3.4, weak equivalences satisfy the 2-out-of-3 property.   □

Homotopy groups in a good Q-structure

For the remainder of this section, we assume that $(\mathcal{C}, \iota)$ is a good Q-structure. For a space $X \in \mathcal{C}$, a *point* x_0 in X is a map $x_0 \colon * \to X$; if there is a map $* \to X$, then X *admits points*. We will denote the composite $X \to * \xrightarrow{y_0} Y$ by const_{y_0} or just y_0. Let $\mathcal{C}_*$ be the slice category under $*$: its objects are pairs $(X, x_0 \colon * \to X)$ and a morphism $f \colon (X, x_0) \to (Y, y_0)$ is a based map $f \colon X \to Y$ such that $f(x_0) = y_0$. Given two maps $f, g \colon (X, x_0) \to (Y, y_0)$, a homotopy between them in this category is a homotopy H from f to g relative to x_0. As in Section 1, this defines an equivalence relation on $\mathcal{C}_*((X, x_0), (Y, y_0))$.

Recall that, in (Q7), there is a point $(*) \colon * \to \partial I^n$ in the second pushout, which we will treat as a distinguished basepoint of ∂I^n to define homotopy groups. Such a point is given for all $n \geq 1$, though we will not make the distinction between $(*)$ for different n.

Definition 3.7 Let $n \geq 0$ and $x_0 \colon * \to X$ a point in X. Let $\pi_n(X, x_0) = \mathcal{C}_*\big((\partial I^{n+1}, (*)), (X, x_0)\big)/\sim$ denote the set of based maps $(\partial I^{n+1}, (*)) \to (X, x_0)$ quotiented by homotopy rel $(*)$. By the pushout in (Q8), this is equivalently the set of all maps $f \colon I^n \to X$ such that $f i_n = \mathrm{const}_{x_0}$, quotiented by homotopy relative to ∂I^n. For $n \geq 1$, this set admits a group operation, with which it is the nth *homotopy group* of X at x_0. The operation on $\pi_n(X, x_0)$ is defined by $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$, where $\alpha \cdot \beta$ is the induced map in

$$
\begin{array}{ccc}
I^{n-1} & \xrightarrow{\ e_0\ } & I^n \\
{\scriptstyle e_1}\downarrow & {\scriptstyle \mathrm{id}_{I^{n-1}} \times v}\downarrow & \ \ \searrow^{\beta} \\
I^n & \xrightarrow[\ \mathrm{id}_{I^{n-1}} \times u\ ]{} & I^n \\
& {}_{\alpha}\searrow & {}^{\alpha \cdot \beta}\searrow \\
& & X
\end{array}
$$

since $\alpha e_1 = \beta e_0 = \mathrm{const}_{x_0}$.

Note that we are implicitly using (Q7) and (S3) in defining the group operation above to recognize $I^n \cong I^{n-1} \times I$. We will continue to make this identification without further reference. As in the topological case, $\pi_0(X, x_0)$ will not form a group, but rather acts as the set of path-connected components.

Theorem 3.8 *The operation on $\pi_n(X, x_0)$ defined above makes it a group for $n \geq 1$.*

Proof The identity is const_{x_0} by Lemma 1.8, associativity comes from Lemma 1.10, and the inverse of $[\alpha]$ is $[-\alpha]$ by Lemma 1.9.   □

Lemma 3.9 *Let X be a space and $i \colon A \rightarrowtail X$ a cofibration. Suppose $f, g, h \colon X \to Y$ are continuous maps such that $f i = g i$. Given homotopies H from f to h and K from g to h, if $H i \sim K i$ rel endpoints then there is a homotopy from f to g that is relative to A.*

Proof Choose a map $\alpha \colon A \times I^2 \to Y$ such that $\alpha \partial_L = Hi$, $\alpha \partial_T = \mathrm{const}_{hi}$, $\alpha \partial_R = Ki$ and $\alpha \partial_B = \mathrm{const}_{fi}$. By Proposition 2.10, $i \mathbin{\hat{\times}} \mathrm{inc}$ is in $\mathrm{cof}(\mathcal{J})$, where $\mathrm{inc} \colon \sqcap \to I^2$ is as in Definition 2.11. Since

$$\alpha \circ \mathrm{inc} \circ b = \alpha \circ -\partial_R = -Ki, \quad \alpha \circ \mathrm{inc} \circ aa = \alpha \partial_L = Hi, \quad \alpha \circ \mathrm{inc} \circ ab = \alpha \partial_T = \mathrm{const}_{hi},$$

the top map is defined, so there is a lift $\beta \colon X \times I^2 \to Y$ in

$$
\begin{array}{ccc}
A \times I^2 \amalg_{A \times \sqcap} X \times \sqcap & \xrightarrow{\;[\alpha, [[Hi, \mathrm{const}_{hi}], -Ki]]\;} & Y \\
{\scriptstyle i \hat{\times} \mathrm{inc}} \big\downarrow & \nearrow^{\;\beta} & \\
X \times I^2 & &
\end{array}
$$

The composite $\beta \partial_B$ gives a homotopy from f to g that is relative to A, since $\beta \partial_B i = \beta(i \times \mathrm{id}_{I^2}) \partial_B = \alpha \partial_B = \mathrm{const}_{fi}$. $\qquad \square$

Lemma 3.10 *Suppose $(\mathcal{C}, \iota)$ is a good Q-structure and let $n \geq 1$. Then a map $u \colon \partial I^n \to X$ is homotopic to x rel $(*)$ if and only if there is a map $\tilde{u} \colon I^n \to X$ such that $\tilde{u} i_n = u$.*

Proof Suppose u is homotopic to the point $x = u(*)$ rel $(*)$. Then there is a homotopy $H \colon \partial I^n \times I \to X$ such that $He_0 = u$ and $He_1 = \mathrm{const}_x$. Thus, by the left pushout in (Q8), H factors through I^n to a map $\tilde{u} \colon I^n \to X$ such that $\tilde{u} i_n = He_0 = u$. Conversely, suppose that u admits an extension $\tilde{u} \colon I^n \to X$. Let $H \colon I^n \times I \to I^n$ be $H(s, t) = (1-t)s + t \cdot (*)$, the straight line homotopy from id_{I^n} to $(*)$, which satisfies $H \circ (*) = \mathrm{const}_{(*)}$. Then $\tilde{H} = \tilde{u} H i_n$ satisfies $\tilde{H} e_0 = u$, $\tilde{H} e_1 = u(*)$ and $\tilde{H}(*) = \mathrm{const}_{u(*)} = \mathrm{const}_x$, so $\tilde{H}$ is the required homotopy relative to $(*)$. $\qquad \square$

Lemma 3.11 *Suppose $(\mathcal{C}, \iota)$ is a good Q-structure and that X admits points. Let $f \colon X \to Y$ be continuous and n a positive integer. Then:*

(1) *f is 0-compressible if and only if $\pi_0 f$ is surjective.*

(2) *f is n-compressible if and only if, for all points x in X, $\pi_k f \colon \pi_k(X, x) \to \pi_k(Y, f(x))$ is injective for $k = n-1$ and surjective for $k = n$.*

Proof For (1), since $\partial I^0 = \varnothing$ and $I^0 \cong *$ by (S1), the diagram in Definition 2.5 always admits a diagonal filler if and only if, for any point y in Y, we may choose a point x in X such that fx is homotopic to y, that is, $\pi_0 f$ is surjective. For (2), suppose first that f is n-compressible. Let $x \in X$ and fix $[\beta] \in \pi_n(Y, f(x))$. Regarding β as a map $I^n \to Y$ such that $\beta i_n = f(x)$, we get a diagonal filler $\alpha \colon I^n \to X$ for the square

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{\;x\;} & X \\
{\scriptstyle i_n} \big\downarrow & & \big\downarrow {\scriptstyle f} \\
I^n & \xrightarrow[\;\beta\;]{} & Y
\end{array}
$$

where $\alpha i_n = x$, so $[\alpha] \in \pi_n(X, x)$, and $f\alpha \sim \beta$ rel ∂I^n, so $\pi_n f([\alpha]) = [\beta]$. Thus, $\pi_n f$ is surjective. For injectivity, when $n-1 = 0$, if $\pi_0 f[x] = [f(x)] = [f(x)'] = \pi_0 f[x']$ in $\pi_0 Y$ for $[x], [x'] \in \pi_0 X$, then

choosing a homotopy p from $f(x)$ to $f(x)'$ induces a diagonal filler in

$$
\begin{array}{ccc}
\partial I & \xrightarrow{[x,x']} & X \\
{\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle f} \\
I & \xrightarrow[p]{} & Y
\end{array}
$$

since ∂I is a coproduct $* \amalg *$ by (Q2), which gives a homotopy from x to x'. When $n - 1 > 0$, if $[f\alpha] = [\mathrm{const}_{f(x)}]$ for $[\alpha] \in \pi_{n-1}(X, x)$, choose an extension $\widetilde{f\alpha}\colon I^{n-1} \to Y$ given by Lemma 3.10, so there is a diagonal filler in

$$
\begin{array}{ccc}
\partial I^{n-1} & \xrightarrow{\alpha} & X \\
{\scriptstyle i_{n-1}}\downarrow & & \downarrow{\scriptstyle f} \\
I^{n-1} & \xrightarrow[\widetilde{f\alpha}]{} & Y
\end{array}
$$

from which, by Lemma 3.10, $[\alpha] = 0$; hence, $\pi_k f$ is injective. Conversely, suppose that π_k is injective for $k = n - 1$ and surjective for $k = n$. Given the commutative square

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{u} & X \\
{\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f} \\
I^n & \xrightarrow[v]{} & Y
\end{array}
$$

let $x = u(*)$. By commutativity and Lemma 3.10, fu is homotopic to $\mathrm{const}_{f(x)}$ rel $(*)$, which, by injectivity of $\pi_{n-1} f$, gives that u is homotopic to const_x, say through a homotopy $H\colon \partial I^n \times I \to X$ rel $(*)$. Since $fu = vi_n$ and Y is fibrant, by Proposition 2.10 there is a lift in the diagram

$$
\begin{array}{ccc}
\partial I^n \times I \cup I^n \times \{0\} & \xrightarrow{[fH,v]} & Y \\
{\scriptstyle i_n \,\widehat{\times}\, j_0}\downarrow & \nearrow{\scriptstyle J} & \\
I^n \times I & &
\end{array}
$$

where the left map is in $\mathrm{cof}(\mathcal{J})$ by Proposition 2.10. Let $\tilde{v} = Je_1$. Since $\tilde{v}i_n = Ji_ne_1 = fHe_1 = \mathrm{const}_{f(x)}$, we can regard $\tilde{v}$ as a map on ∂I^{n+1} by the right pushout in (Q8), so $[\tilde{v}] \in \pi_n(Y, f(x))$. Since $\pi_n f$ is surjective, there is a $[\widetilde{w}] \in \pi_n(X, x)$ such that there is a homotopy $K\colon I^n \times I \to Y$ from $f\widetilde{w}$ to $\tilde{v}$ rel ∂I^n. Regarding $\widetilde{w}$ as a map on I^n, we obtain a lift in the diagram

$$
\begin{array}{ccc}
\partial I^n \times I \cup I^n \times \{1\} & \xrightarrow{[H,\widetilde{w}]} & Y \\
\downarrow & \nearrow{\scriptstyle G} & \\
I^n \times I & &
\end{array}
$$

since $He_1 = \mathrm{const}_x$. Let $w = Ge_0$, so $wi_n = Gi_ne_0 = He_0 = u$, and $vi_n = fu = fwi_n$. Since $J\,\boldsymbol{\cdot}{-}K$ is a homotopy from v to $f\widetilde{w}$, fG is a homotopy from fw to $f\widetilde{w}$ and

$$(J\,\boldsymbol{\cdot}{-}K)i_n = Ji_n\,\boldsymbol{\cdot}{-}Ki_n = fH \,\boldsymbol{\cdot}\, \mathrm{const}_{f(x)} \sim fH = fGi_n \ \text{ rel endpoints},$$

we may apply Lemma 3.9 to get that $fw \sim v$ rel ∂I^n; hence, f is n-connected. $\qquad\square$

Thus, we may conclude that, in a good Q-structure $(\mathcal{C}, \iota)$, the definition of weak equivalences given in Definition 2.5 is equivalent to the classical definition given in terms of homotopy groups.

Theorem 3.12 *In a good Q-structure, if a space X admits points, then a map $f : X \to Y$ is n-connected if and only if, for all points x in X, $\pi_k f : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism for $k < n$ and a surjection for $k = n$. In particular, if X admits points, then a map $f : X \to Y$ is a weak equivalence if and only if $\pi_n f : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for every nonnegative integer n and every point x in X.* □

4 Example: subcategories of topological spaces

We begin by recovering the usual Quillen model structure on the category Top of topological spaces. From this, we will also be able to verify that the result holds for several subcategories of Top, as well as for pseudotopological spaces in PsTop and locales in Loc.

Theorem 4.1 *The category* Top *of topological spaces carries a model structure whose weak equivalences are the weak homotopy equivalences, fibrations are the Serre fibrations, and cofibrations are the Serre cofibrations.*

To prove the theorem, we will appeal to Theorem 2.6 with the obvious inclusion $\iota : \square \hookrightarrow$ Top. We begin however with a technical lemma, which verifies the only tricky axiom, (Q3):

Lemma 4.2 *For n and m nonnegative integers, $i_n \,\hat{\times}\, i_m \cong i_{n+m}$ and $i_n \,\hat{\times}\, j_m \cong j_{n+m}$ in the arrow category* $\mathsf{Top}^{\to}$. *Thus, in Notation 2.1, $\mathcal{I} \,\hat{\times}\, \mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I} \,\hat{\times}\, \mathcal{J} \subseteq \mathcal{J}$.*

Proof In Top, we may take the pushout product of two inclusions $A \hookrightarrow B$ and $C \hookrightarrow D$ to be the inclusion $A \times D \cup B \times C \hookrightarrow B \times D$. Thus, $i_n \,\hat{\times}\, i_m$ is the inclusion $\partial I^n \times I^m \cup I^n \times \partial I^m \hookrightarrow I^{n+m}$. Since $\partial I^n \times I^m$ is the subspace of I^{n+m} where at least one of the first n coordinates is either 0 or 1, and likewise for $I^n \times \partial I^m$ with the last m coordinates, we have $\partial I^n \times I^m \cup I^n \times \partial I^m \cong \partial I^{n+m}$. Thus, $i_n \,\hat{\times}\, i_m \cong i_{n+m}$. Similarly, $i_n \,\hat{\times}\, j_m$ is the inclusion $\partial I^n \times I^m \times I \cup I^n \times I^m \times \{0\} \hookrightarrow I^{n+m} \times I$. But $\partial I^n \times I \cup I^n \times \{0\} \cong I^n$, so $i_n \,\hat{\times}\, j_0 \cong j_0$ and thus, since $- \times I^m$ preserves colimits, $i_n \,\hat{\times}\, j_m \cong (I^n \times I^m \hookrightarrow I^{n+m} \times I) \cong j_{n+m}$. □

Proof of Theorem 4.1 Having fixed the inclusion $\iota : \square \hookrightarrow$ Top, (S1), (S2) and (S4) hold, and (S3) is given by letting $a(t) = \frac{1}{2}t$ and $b(t) = \frac{1}{2}(1 + t)$. Moreover, (Q2) and (Q7) are obvious, (Q5) holds since ∂I^n and I^n are locally compact Hausdorff, and (Q6) is well known (see for instance [11, Proposition 2.4.2]). For (Q3), we then use Lemma 4.2.

(Q4) follows from the given pushouts

$$
\begin{array}{ccc}
\partial I^n & \xhookrightarrow{\;\;i_n\;\;} & I^n \\
{\scriptstyle i_n}\big\downarrow & {\scriptstyle \ulcorner} & \big\downarrow \\
I^n & \xhookrightarrow[\;\in \text{cell}(\mathcal{I})\;]{} & \partial I^{n+1} \xhookrightarrow{\;i_{n+1}\;} I^{n+1}
\end{array}
\qquad
\begin{array}{ccc}
\partial I^{n-1} & \xhookrightarrow{\;i_{n-1}\;} & I^{n-1} \\
\big\downarrow & {\scriptstyle \ulcorner} & \big\downarrow \\
\varnothing \xhookrightarrow{\;i_0\;} * & \xrightarrow[\;\in \text{cell}(\mathcal{I})\;]{} & \partial I^n
\end{array}
$$

Lastly, (Q8) is precisely the statement that, under the identifications $\partial I^n \times I / \partial I^n \times \{1\} \cong I^n$ and $I^{n-1}/\partial I^{n-1} \cong \partial I^n$, if we take ρ and σ to be the quotient maps, then any continuous map that is constant

on fibers of the quotient factors through the quotient. By the description of the classes of maps, it is clear this is the classic Quillen model structure.   □

In a (co)reflective subcategory of Top, much of the work has already been done. In particular, if $\mathcal{C}$ is a (co)reflective subcategory of Top containing the CW complexes, then we may take $\iota\colon \square \hookrightarrow \mathcal{C}$ to be the canonical inclusion. In this case, the image of ι is naturally isomorphic to the image of $\square$ under the (co)reflection; hence, we may take each of the above pushouts to be the same as in Top. All that remains is to check that $X \times -$ preserves the pushout in (S3), as well as the conditions of (Q5) and (Q6). We introduce four (co)reflective subcategories which admit good Q-structures.

Let X be a topological space. Recall that a closed subset $F \subseteq X$ is *irreducible* if it cannot be written as the union of two proper closed subsets of F. That is, if $F = F_1 \cup F_2$ and F_1 and F_2 are closed, then either $F_1 = F$ or $F_2 = F$.

Definition 4.3 We take the following subcategories to all be full:

(1) Let cgHaus be the subcategory of compactly generated Hausdorff spaces.

(2) Let CGWH be the subcategory of compactly generated, weakly Hausdorff spaces.

(3) Let DTop be the subcategory of Δ-generated spaces. A Δ-generated space X is a topological space which has the final topology with respect to all maps from simplices $\Delta^n \to X$. There is an obvious functor Δ-ify$\colon$ Top $\to$ DTop, which simply refines the topology on a space X to be the finest topology for which all maps $\Delta^n \to X$ are continuous. This forms an adjunction $\iota \dashv \Delta$-ify, making DTop a coreflective subcategory of Top [26, Corollary 1.4].

(4) Let Sob be the subcategory of sober spaces. A sober space X is a topological space such that every irreducible closed subset of X is the closure of exactly one point. In particular, every Hausdorff space is sober. There is a soberification functor sober$\colon$ Top $\to$ Sob, which is a left adjoint to the inclusion $\iota\colon$ Sob $\hookrightarrow$ Top, making Sob a reflective subcategory of Top.

There is a chain of (co)reflections relating cgHaus and CGWH to Top, so the comments above apply to them. The categories cgHaus, CGWH and DTop are cartesian closed (see for instance [23; 24; 26], respectively). In Sob, the exponentiable spaces are precisely the locally compact spaces [12, Theorem VII.4.12]. Thus, in each of these categories, (Q5) is satisfied. A standard proof (such as the one in [11]) shows that (Q6) is also satisfied in cgHaus and CGWH, and DTop is locally presentable [5, Theorem 3.7] and hence admits the small object argument.

Most of the model structures on the above categories are well known (see for instance the proof in [10]). The only model structure that we are unaware of an existing reference for is on Sob, so we will carefully check the remaining axiom (Q6) there.

Lemma 4.4 *Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$. Then every map in* cell$(\mathcal{K})$ *is a closed T_1 inclusion; that is, if $f\colon X \to Y \in$ cell$(\mathcal{K})$, then f is a closed map such that every point in $Y - fX$ is closed. Moreover, $Y - fX$ is Hausdorff in the subspace topology.*

Proof The first part is given in [11, Lemma 2.4.5]. For the second part, notice that Y/fX is a cell complex and hence Hausdorff, so the result follows since $Y - fX \cong Y/fX - [fX]$ as subspaces. □

Lemma 4.5 *Coproducts of sober spaces, as formed in* Top, *are sober.*

Proof Let $\coprod X_\alpha$ be a coproduct in Top, where each X_α is sober. Clearly, any irreducible closed subset must be contained within exactly one component of $\coprod X_\alpha$, say X_β; hence, it is the closure of a single point since X_β is sober. □

Lemma 4.6 *Suppose X is a space and $A, B \subseteq X$ are subsets such that X is the disjoint union of A and B as sets, A is closed in X, and every point of B is closed in X. If A is sober and B is Hausdorff under the subspace topology, then X is sober.*

Proof Let $F \subseteq X$ be an irreducible closed subset of X. Suppose F has at least two distinct points of B, say x and y. Then, since B is Hausdorff, there exist disjoint sets $U \subseteq B$ and $V \subseteq B$ that are open neighborhoods of x and y, respectively, in B. Since A is closed, B is open, so U and V are open in X. Then $F = (F - U) \cup (F - V)$ gives F as a union of two nonempty closed subsets of X, contradicting irreducibility. Thus, F has at most one point of B. If F intersected both A and B, then it would contain exactly one point of B, say x, and, since $\{x\}$ is closed in X, we would have $F = (F \cap A) \cup \{x\}$, contradicting irreducibility. Thus, F is either a single point of B or an irreducible closed subset of A; in either case, F is the closure of exactly one point since A is sober. □

Corollary 4.7 *Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$, and let $\{k_\alpha : S_\alpha \to K_\alpha\}$ be a collection of maps in $\mathcal{K}$. If X is sober and the following is a pushout in* Top, *then Y is sober*:

$$
\begin{array}{ccc}
\coprod_\alpha S_\alpha & \xrightarrow{\coprod k_\alpha} & \coprod_\alpha K_\alpha \\
\downarrow & & \downarrow \\
X & \xrightarrow{\quad f \quad} & Y
\end{array}
$$

Proof This follows from Lemmas 4.4 and 4.6, given that Y is the disjoint union of fX and $Y - fX$ as sets. □

In particular, this means that any time cells are attached to a sober space in Sob in a single step, we may take the resulting relative cell complex, as formed in Top, to be the pushout. In fact, this then holds for transfinite composites of such maps as well:

Lemma 4.8 *Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X = \operatorname{colim} X_\alpha$ be a transfinite composite of pushouts of coproducts of maps in $\mathcal{K}$, formed in* Top. *If X_0 is sober, then so is X.*

Proof Again, this holds by Lemmas 4.4 and 4.6. □

Corollary 4.9 *In* Sob, *compact Hausdorff spaces are small relative to maps in* $\operatorname{cell}(\mathcal{I})$ *and* $\operatorname{cell}(\mathcal{J})$.

Proof By the above lemmas, transfinite composites of pushouts of maps in $\mathcal{I}$ and $\mathcal{J}$ are formed the same in Sob as in Top. Thus, the maps in $\mathrm{cell}(\mathcal{I}_{\mathsf{Sob}})$ and $\mathrm{cell}(\mathcal{J}_{\mathsf{Sob}})$ are precisely those in $\mathrm{cell}(\mathcal{I}_{\mathsf{Top}})$ and $\mathrm{cell}(\mathcal{J}_{\mathsf{Top}})$ whose domain is sober. The result follows from the usual statement in Top. $\square$

It is also easily verified that (S3), (Q7) and (Q8) hold in each of the categories in Definition 4.3, making each a good Q-structure with the canonical inclusions. Thus, we may apply Theorems 2.6 and 3.12 for each of the categories to get the following:

Theorem 4.10 *Each of the categories of*

- *compactly generated Hausdorff spaces* cgHaus,
- *compactly generated, weakly Hausdorff spaces* CGWH,
- Δ-*generated spaces* DTop,
- *sober spaces* Sob

admits a model structure by taking the weak homotopy equivalences, Serre fibrations and relative cell complexes to be the weak equivalences, fibrations and cofibrations. $\square$

Many of the model structures above are known to be Quillen equivalent to the standard model structure on Top. The only remaining one to our knowledge is the model structure on Sob, which we easily verify is Quillen equivalent as well:

Lemma 4.11 *A map $f : A \to B$ in* Sob *is a weak equivalence if and only if ιf is in* Top.

Proof We have characterized weak equivalences in both categories by maps from sober spaces. The result follows from Sob being full. $\square$

Theorem 4.12 *The adjunction* sober $\dashv \iota$ *is a Quillen equivalence.*

Proof That it is a Quillen adjunction holds by construction, since $\mathrm{sober}(\mathcal{I}_{\mathsf{Top}}) \cong \mathcal{I}_{\mathsf{Sob}}$, and likewise with $\mathcal{J}$; hence, sober preserves (trivial) cofibrations since it preserves colimits. Thus, let $X \in$ Top be cofibrant, which is sober as a Hausdorff space, and $Y \in$ Sob be any sober space. By the following diagram and Lemma 4.11, sober $X \to Y$ is a weak equivalence if and only if $X \to \iota Y$ is:

$$\begin{array}{ccc} \iota \circ \mathrm{sober}\, X & \longrightarrow & \iota Y \\ {\scriptstyle\cong}\big\uparrow & \nearrow & \\ X & & \end{array}$$

$\square$

5 Example: pseudotopological spaces

Pseudotopological spaces are a generalization of topological spaces, and a more specialized type of convergence spaces. Rather than describing the structure in terms of open sets, one takes convergence of ultrafilters to be foundational. The category of pseudotopological spaces PsTop has several nice properties

which Top lacks, most notably the fact that PsTop is quasitopos [7]. In fact, [8, Theorem 3.12] showed that PsTop is the cartesian closed topological hull of the category of pretopological spaces PreTop (also known as Čech closure spaces) in the category of convergence spaces Conv. In particular, since DiGraph and Graph both fully embed into PreTop [2, Proposition 1.56], the existence of a model structure on PsTop could be applicable to the homotopy theory of graphs and Čech closure spaces. We now properly introduce pseudotopological spaces and give a brief recollection of the necessary lemmas.

Definition 5.1 Let X be a set, $\mathbb{F}X$ be the set of filters on X, and βX be the set of ultrafilters on X. For each $x \in X$, let $x_\bullet = \{A \subseteq X \mid x \in A\}$ be the principal ultrafilter at x. A *pseudotopology* on X is a function $\lim_\xi : \beta X \to \mathcal{P}X$ such that $x \in \lim_\xi x_\bullet$ for all $x \in X$. A *pseudotopological space* is a pair $(X, \lim_\xi)$ such that $\lim_\xi$ is a pseudotopology on X.

Given a pseudotopology, we can define the limit of any filter $\mathcal{F}$ on X,

$$\lim_\xi \mathcal{F} = \bigcap_{\mathcal{U} \in \beta \mathcal{F}} \lim_\xi \mathcal{U},$$

where $\beta \mathcal{F} = \{\mathcal{U} \in \beta X \mid \mathcal{F} \subseteq \mathcal{U}\}$. By the ultrafilter lemma, $\beta \mathcal{F}$ is nonempty unless $\mathcal{F} = \mathcal{P}X$, in which case the empty intersection is understood to be X (that is, $\mathcal{P}X$ converges to every element of X).

Definition 5.2 Given pseudotopologies $\lim_\xi$ on X and $\lim_\zeta$ on Y, a function $f : X \to Y$ is *continuous* if $f(\lim_\xi \mathcal{F}) \subseteq \lim_\zeta f_* \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}X$, where $f_* \mathcal{F} = \{B \subseteq Y \mid \text{there is } A \in \mathcal{F} \text{ such that } fA \subseteq B\}$ is the pushforward of $\mathcal{F}$. The category of pseudotopological spaces and continuous functions will be denoted by PsTop.

Definition 5.3 Given two pseudotopologies $\lim_\xi$ and $\lim_\zeta$ on X, $\lim_\xi$ is *finer* than $\lim_\zeta$ (and $\lim_\zeta$ is *coarser* than $\lim_\xi$), written $\zeta \leq \xi$, if $\lim_\xi \mathcal{U} \subseteq \lim_\zeta \mathcal{U}$ for all $\mathcal{U} \in \beta X$. Equivalently, $\lim_\xi$ is finer than $\lim_\zeta$ if and only if $\mathrm{id}_X : (X, \lim_\xi) \to (X, \lim_\zeta)$ is continuous.

Lemma 5.4 [4, Section III.6] *For any set X, the collection of pseudotopologies Ξ on X is a complete lattice.*

Proof Let $\mathcal{U}$ be an ultrafilter on X, and $\Gamma \subseteq \Xi$. Then an infimum is given by $\lim_{\bigwedge \Gamma} \mathcal{U} = \bigcup_{\xi \in \Gamma} \lim_\xi \mathcal{U}$ and a supremum is given by $\lim_{\bigvee \Gamma} \mathcal{U} = \bigcap_{\xi \in \Gamma} \lim_\xi \mathcal{U}$. $\qquad\square$

The existence of a finest and a coarsest pseudotopology implies the existence of discrete and indiscrete functors from Set to PsTop. Indeed, these form the usual "discrete $\dashv$ forgetful $\dashv$ indiscrete" adjunctions, so any (co)limit in PsTop has, as its underlying set, the (co)limit set. Specifically, for the discrete pseudotopology, only the principal ultrafilter $x_\bullet$ converges to x, and, for the indiscrete pseudotopology, every ultrafilter converges to every point. Moreover, for any set of maps $\{f_k : X_k \to Y\}_{k \in K}$, we may take

the *final pseudotopology* on Y, which is the finest pseudotopology for which each f_k is continuous [4, Section IV.3]. Dually, we may take the *initial pseudotopology* with respect to a set of maps with common domain, which is the coarsest pseudotopology for which each map is continuous.

Corollary 5.5 PsTop *is (co)complete.*

Proof Take the (co)limit of the underlying sets, and then take the initial (final) pseudotopology under the (co)limit cone. □

For any pair of pseudotopological spaces X and Y, there is a natural pseudotopology on $\mathsf{PsTop}(X, Y)$. We do not describe it here, instead referring the reader to [8, Theorem 3.7]:

Proposition 5.6 [8, Theorem 3.7] *The category* PsTop *is cartesian closed.* □

Every topology τ on X induces a pseudotopology, by taking $\lim_\tau \mathcal{U} = \{x \in X \mid \mathcal{N}(x) \subseteq \mathcal{U}\}$; that is, one takes the usual limits of the filter. Given a topology σ on Y, a function $f \colon X \to Y$ is continuous with respect to the topologies if and only if it is continuous with respect to the induced pseudotopologies. Thus, we get a full embedding $\iota \colon \mathsf{Top} \hookrightarrow \mathsf{PsTop}$. In the other direction, given any pseudotopology, we can take its topological modification.

Definition 5.7 Let $(X, \lim_\xi)$ be a pseudotopological space. A set $O \subseteq X$ is *open* if $\lim_\xi \mathcal{U} \cap O \neq \varnothing$ implies $O \in \mathcal{U}$. A set is *closed* if its complement is open.

One can verify that the collection of open sets $\tau\xi$ forms a topology, which is the *topological modification* of $\lim_\xi$; the topological modification of a space $(X, \lim_\xi)$ is $(X, \tau\xi)$. One can verify that this process preserves continuity, so this defines a functor $\tau \colon \mathsf{PsTop} \to \mathsf{Top}$ which is left adjoint to ι. This makes Top a reflective subcategory of PsTop. In particular, since the underlying sets and set functions have not changed, the unit is the identity map $\mathrm{id}_X \colon (X, \lim_\xi) \to (X, \tau\xi)$. Through this, we are able to verify most of the required axioms from the required identities holding in Top through a few lemmas.

Definition 5.8 [4, Definition III.3.1, Proposition IV.2.14] Let $(X, \lim_\xi)$ be a pseudotopological space and $i \colon A \hookrightarrow X$ a set inclusion. The *subspace pseudotopology*, denoted by $\lim_{\xi|_A}$, is the initial pseudotopology under i. Explicitly, this is given by $x \in \lim_{\xi|_A} \mathcal{F}$ if and only if $x \in \lim_\xi i_*\mathcal{F}$.

Lemma 5.9 [4, Proposition IV.2.15] *Let* $f \colon (X, \lim_\xi) \to (Y, \lim_\zeta)$ *be continuous and* $A \subset X$ *and* $B \subset Y$ *such that* $fA \subseteq B$. *Then* $f|_A \colon (A, \lim_{\xi|_A}) \to (B, \lim_{\zeta|_B})$ *is continuous.* □

Lemma 5.10 [4, Proposition V.4.24] *For any psuedotopological space* $(X, \lim_\xi)$ *and any set inclusion* $i \colon A \hookrightarrow X$, *the topological modification of the subspace pseudotopology,* $\tau(\xi|_A)$, *is finer than the subspace topology under the topological modification,* $(\tau\xi)|_A$. *That is,* $\mathrm{id}_A \colon (A, \lim_{\tau(\xi|_A)}) \to (A, \lim_{(\tau\xi)|_A})$ *is continuous.* □

Definition 5.11 Two collections of subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}X$ *mesh*, written $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \varnothing$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The *grill* of $\mathcal{A}$ is

$$\mathcal{A}^{\#} = \{B \subseteq X \mid A \cap B \neq \varnothing \text{ for all } A \in \mathcal{A}\}.$$

Definition 5.12 Let $(X, \lim_{\xi})$ be a pseudotopological space. For $\mathcal{A} \subseteq \mathcal{P}X$, the *adherence* of $\mathcal{A}$ is

$$\mathrm{adh}_{\xi} \mathcal{A} = \bigcup_{\mathcal{F}\#\mathcal{A}} \lim_{\xi} \mathcal{F},$$

where the union is taken over all filters $\mathcal{F}$ that mesh with $\mathcal{A}$. Given two subsets $A, B \subseteq X$, A is ξ-*compact at* B if, for every filter $\mathcal{F}$, $A \in \mathcal{F}^{\#}$ implies $\mathrm{adh}_{\xi} \mathcal{F} \cap B \neq \varnothing$. A is *compact* if A is ξ-compact at A, and *relatively compact* if A is ξ-compact at X.

Note that, if A is ξ-compact at B and $B \subseteq C$, then A is ξ-compact at C. Moreover, if A and B are ξ-compact at C, then $A \cup B$ is ξ-compact at C. A topological space is compact in the above sense if and only if it is compact in the usual sense.

Lemma 5.13 [4, Proposition IX.1.26] *Let* $f : (X, \lim_{\xi}) \to (Y, \lim_{\zeta})$ *be continuous. For* $A, B \subseteq X$, *if* A *is* ξ-*compact at* B, *then* fA *is* ζ-*compact at* fB. □

Proposition 5.14 [4, Corollary IX.1.18] *Let* X *be a compact pseudotopological space and* Y *a Hausdorff topological space. If* $f : X \to Y$ *is a continuous bijection, then* f *is a homeomorphism.* □

Corollary 5.15 *Suppose*

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
g \downarrow & & \downarrow i \\
C & \xrightarrow{j} & P
\end{array}
$$

is a pushout diagram in Top, *where* B *and* C *are compact and* P *is Hausdorff. Then* P *is the pushout in* PsTop.

Proof Let $\lim_{\xi}$ be the colimit pseudotopology on P, and τ the colimit topology. By Lemma 5.13, iB and jC are ξ-compact at P, so $P = iB \cup jC$ is compact. The result follows from Proposition 5.14, since, by the adjunction $\tau \dashv \iota$, the map $\mathrm{id}_P : (P, \lim_{\xi}) \to (P, \lim_{\tau})$ is continuous. □

We are now ready to show that PsTop carries a model structure. Note that this is identical to the model structure defined in [22], but constructed in a different manner.

Theorem 5.16 *There is a model structure on* PsTop *whose cofibrations are maps in* $\mathrm{cof}(\mathcal{I})$, *fibrations are maps in* $\mathrm{rlp}(\mathcal{J})$, *and weak equivalences are maps which have the weak right lifting property against* $\mathcal{I}$ *or, equivalently by* Theorem 3.12, *maps which induce isomorphisms on all homotopy groups.*

Proof We are now ready to verify the axioms of Definition 2.2. That (S1) and (S4) hold is obvious, (S2) follows from ι being a right adjoint, and (S3) by Corollary 5.15 and Proposition 5.6. Thus, $(\mathcal{C}, \iota)$ is a category with intervals. For (Q2), note that $* \amalg *$ is given the finest pseudotopology for which the

inclusions $* \hookrightarrow * \amalg *$ are continuous. Since these are always continuous, $* \amalg *$ must have the discrete topology; hence, $* \amalg * \cong \partial I$. The general case follows from Proposition 5.6. Next, (Q3) and (Q4) follow from Lemma 4.2 and Corollary 5.15, and (Q5) by Proposition 5.6. The verification of (Q6) is done in [22, Theorem 5.16]. Lastly, (Q7) follows from ι preserving limits, and (Q8) by Corollary 5.15. $\square$

Of course, since only limits of topological spaces are again topological in PsTop, not every cofibration in Top is necessarily a cofibration in PsTop. In fact, this need not hold for cofibrant cell complexes even.

Next we will verify that this model structure is Quillen equivalent to the standard model structure on Top, via the adjunction $\tau \dashv \iota$. We will first check that it is a Quillen adjunction as well as some intermediate lemmas.

Proposition 5.17 *The adjunction $\tau \dashv \iota$ is a Quillen adjunction.*

Proof Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$, and $\mathcal{K}_{\mathsf{Top}}$ and $\mathcal{K}_{\mathsf{PsTop}}$ be the respective generating (acyclic) cofibrations. Clearly, $\tau \mathcal{K}_{\mathsf{PsTop}} = \mathcal{K}_{\mathsf{Top}}$. Since τ is a left adjoint, it preserves colimits, so $\tau(\mathrm{cell}(\mathcal{K}_{\mathsf{PsTop}})) \subseteq \mathrm{cell}(\mathcal{K}_{\mathsf{Top}})$. Since functors preserve retracts in the arrow category, τ preserves (trivial) cofibrations. $\square$

Lemma 5.18 *A map $f : X \to Y$ between topological spaces is a weak equivalence in PsTop if and only if it is a weak equivalence in Top.*

Proof Since we have characterized weak equivalences by maps from topological spaces via lifting properties (and, for homotopies, ι preserves products), this follows from Top being a full subcategory of PsTop. $\square$

Let $\varnothing \rightarrowtail A$ be a cofibrant cell complex in PsTop, and fix a presentation of A as a transfinite composite, $A = \mathrm{colim}\, A_\alpha$. For each cell attached to A, there is a *characteristic map*, denoted by $\Phi_k : I^{n_k} \to A$, which is the composite $I^{n_k} \hookrightarrow \coprod_k I^{n_k} \to A_\alpha \to A$. A subset $S \subseteq A$ is a (finite) subcomplex if it can be written as a (finite) union, $S = \bigcup_k \Phi_k I^{n_k}$.

Lemma 5.19 *Let $(X, \lim_\xi)$ be a cofibrant cell complex in PsTop, and S a finite subcomplex of X. Then the subspace pseudotopology on S agrees with the subspace topology on τX.*

Proof Let $S = \bigcup_{k=1}^{N} \Phi_k I^{n_k}$ and let $\lim_{\xi|_S}$ be the subspace pseudotopology under $\lim_\xi$ and $\lim_{\tau(\xi)|_S}$ be the subspace topology under $\tau \xi$. By adjunction, $\mathrm{id}_S : (S, \lim_{\xi|_S}) \to (S, \lim_{\tau(\xi|_S)})$ is continuous, and, by Lemma 5.10, $\mathrm{id}_S : (S, \lim_{\tau(\xi|_S)}) \to (S, \lim_{(\tau\xi)|_S})$ is as well; thus their composite is continuous. Since τX is a cell complex in Top, it is Hausdorff, and since the restricted maps $\Phi_k : I^{n_k} \to (S, \lim_{\xi|_S})$ are continuous by Lemma 5.9, their image is compact in S. The result follows from Proposition 5.14: since S is a finite union of sets compact in $\lim_{\xi|_S}$, S itself is compact in $\lim_{\xi|_S}$, and $\lim_{(\tau\xi)|_S}$ is Hausdorff as a subspace of τX. $\square$

Theorem 5.20 *The adjunction $\tau \dashv \iota$ is a Quillen equivalence between the standard model structure on Top and the model structure on PsTop given in Theorem 5.16.*

Proof By [11, Corollary 1.3.16], since ι preserves weak equivalences by Lemma 5.18, it suffices to check that $\mathrm{id}_X\colon X \to \iota\tau X = \tau X$ is a weak equivalence for cofibrant X. The general case will follow from considering just $\mathcal{I}$-cell complexes. Suppose X is a cofibrant $\mathcal{I}$-cell complex and that the following diagram commutes:

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{\;u\;} & X \\
\scriptstyle{i_n}\big\downarrow & & \big\downarrow\scriptstyle{\mathrm{id}_X} \\
I^n & \xrightarrow{\;v\;} & \tau X
\end{array}
$$

Since τX is a cell complex in Top and vI^n is a compact subset, it is contained in a finite subcomplex, say S. By Lemma 5.19, the subspace topology under τX and subspace pseudotopology under X agree, so the inclusion maps $S \hookrightarrow X$ and $S \hookrightarrow \tau X$ are both continuous. Factoring v through S, we get the following diagram, which gives a strict lift:

$$
\begin{array}{ccc}
\partial I^n & \xrightarrow{\quad\quad u \quad\quad} & X \\
\scriptstyle{i_n}\big\downarrow & \;\;S\;\; & \big\downarrow\scriptstyle{\mathrm{id}_X} \\
I^n & \xrightarrow[\quad\quad v \quad\quad]{} & \tau X
\end{array}
$$

$\square$

Of course, the approach in this paper is not suitable for every topological category. For instance, in the category PreTop of pretopological spaces [2; 21] (also referred to as Čech closure spaces), which is reflective in PsTop and contains Top as a reflective subcategory, most of the axioms assumed are suitable, except for the exponentiability requirement. In particular, [15, Theorem 3.4] showed that a pretopological space X is exponentiable if and only if every point in X has a smallest neighborhood, which of course means I is not exponentiable. Similarly, the only exponentiable T_0 uniform spaces are the discrete spaces [18, Corollary 3.4]; hence, Theorem 2.6 is not applicable to the category of uniform spaces either.

6 Example: locales

As the main application of Theorem 2.6, we will show that Loc admits a model structure. As with PsTop, we will first recall some basic theory about locales before verifying the assumptions of Definition 2.2.

Definition 6.1 A *frame* is a complete lattice $(L, \leq)$ which satisfies the infinitary distributive law

$$
a \wedge \left(\bigvee_{k \in K} x_k \right) = \bigvee_{k \in K} (a \wedge x_k).
$$

Given two frames L and M, a *frame homomorphism* is a monotone function $f\colon L \to M$ such that $f\left(\bigvee K\right) = \bigvee fK$ for all $K \subseteq L$ and $f\left(\bigwedge_{k=1}^n x_k\right) = \bigwedge_{k=1}^n f(x_k)$ for all $\{x_k\}_{k=1}^n \subseteq L$.

The category of frames and frame homomorphisms is denoted by Frm.

Definition 6.2 The category of locales, denoted by Loc, is the opposite of the category of frames, Frm$^{\mathrm{op}}$.

Therefore, locales are frames, but a locale morphism $L \to M$ is a frame homomorphism $f : M \to L$. Since frames are complete lattices, there is a right adjoint $g : L \to M$ going in the localic direction; such a map is a *localic map*, which preserves all meets. When necessary, we distinguish between the left and right adjoint by writing f^L and f^R. Since a Galois connection $f : L \rightleftarrows M : g$ satisfies $fgf = f$ and $gfg = g$, f is injective if and only if g is surjective, and f is surjective if and only if g is injective.

Let $\Omega : \mathsf{Top} \to \mathsf{Loc}$ denote the functor taking a space X to its topology regarded as a frame, and a continuous map $f : X \to Y$ to a frame homomorphism $f^{-1} : \Omega Y \to \Omega X$. Restricting Ω to Sob, this becomes a full embedding which admits a right adjoint.

Proposition 6.3 [12, Corollary II.1.7] *There is an adjunction $\Omega : \mathsf{Sob} \rightleftarrows \mathsf{Loc} : \mathrm{pt}$. In this, Ω is a full embedding of Sob into Loc, making Sob a coreflective subcategory of Loc.* $\square$

We will take the restriction of Ω to $\square$ to be the inclusion ι.

Notation 6.1 We write $\mathcal{I}_{\mathsf{Loc}}$ and $\mathcal{J}_{\mathsf{Loc}}$ for the images of $\mathcal{I}$ and $\mathcal{J}$ under Ω, respectively.

A locale L is *spatial* if it is in the essential image of Ω, that is, if $L \cong \Omega X$ for some sober space X. As a left adjoint, Ω preserves colimits and, in particular, the pushouts described in the section on the model structure on Top.

For a frame L, a *prenucleus* is an order-preserving map $k_0 : L \to L$ such that $x \le k_0(x)$ and $k_0(x) \wedge y \le k_0(x \wedge y)$ for all $x, y \in L$. A *nucleus* is a closure operator $k : L \to L$ that preserves binary meets. Any prenucleus k_0 generates a nucleus with the same fixed points as k_0 by letting $\mathrm{Fix}(k_0) \subseteq L$ be the set of fixed points of k_0 and defining $k(x) = \bigwedge \{ y \in \mathrm{Fix}(k_0) \mid x \le y \}$. We will now fix a construction of the necessary limits and colimits in Frm and Loc.

Limits in Frm are formed as in Set, with the natural pointwise order on products: $(x_k) \le (y_k)$ if and only if $x_k \le y_k$ for all k. We use a description of binary coproducts in Frm as given by Banaschewski [1] and Chen [3]. For a frame L and $S \subseteq L$, let $\downarrow S = \{ x \in L \mid x \le s \text{ for some } s \in S \}$. Define the downset functor $\mathfrak{D} : \mathsf{Frm} \to \mathsf{Frm}$ by $\mathfrak{D} X = \{ U \subseteq \mathcal{P} X \mid U \text{ downward closed} \}$ as the set of downsets (ordered by inclusion, with intersection for meets and unions for joins) and $\mathfrak{D} f(U) = \downarrow f(U)$. To distinguish downsets from other sets, we will denote them by $\downarrow S$ from here on. Given two frames L_1 and L_2, let $L_1 \times L_2$ be their product. Define the following prenuclei on $\mathfrak{D}(L_1 \times L_2)$:

$$\sigma_0(U) = \{ \textstyle\bigvee D \mid D \subseteq U \text{ up-directed} \},$$
$$\pi_1(U) = \{ (\textstyle\bigvee X, y) \mid X \subseteq L_1, X \times \{y\} \subseteq U \},$$
$$\widehat{\pi}_2(U) = \{ (x, \textstyle\bigvee Y) \mid Y \subseteq L_2 \text{ finite}, \{x\} \times Y \subseteq U \}.$$

Let σ be the associated nucleus of σ_0 and let $\pi = \sigma \circ \pi_1 \circ \widehat{\pi}_2$. Then the set $\mathrm{Fix}(\pi)$ of fixed elements, also called π-saturated elements, constitutes the coproduct of L_1 and L_2 in Frm, denoted by $L_1 \otimes L_2$. For a downset $\downarrow S$ to be π-saturated means that, if $x \in L_1$ is such that $\{(x, y_\alpha)\} \subset \downarrow S$, then $(x, \bigvee y_\alpha) \in \downarrow S$, and likewise for joins in L_1. The coproduct inclusions $\iota_i : L_i \to L_1 \otimes L_2$ are given by $\iota_1(x) = \{(a, b) \mid a \le x\} \cup \bar{n}$,

where $\bar{n} = \{(a, b) \mid a = 0 \text{ or } b = 0\}$, and likewise for ι_2. The elements of the form $\iota_1(x) \cap \iota_2(y) = \downarrow(x, y) \cup \bar{n}$ are denoted by $x \otimes y$. In particular, due to the possibility of an empty index set for π-saturation, $(x, \bigvee \varnothing) = (x, 0) \in \downarrow S$, and similarly $(0, y) \in \downarrow S$ for all $\downarrow S \in L_1 \otimes L_2$, so $\bar{n}$ is the least element of $L_1 \otimes L_2$.

For arbitrary $\downarrow S \in L_1 \otimes L_2$, we have the identity $\downarrow S = \bigvee_{a \otimes b \le \downarrow S} a \otimes b = \bigcup_{a \otimes b \le \downarrow S} a \otimes b$; moreover, for $\{x_\alpha\} \subseteq L_1$ and $y \in L_2$, we have $\bigvee(x_\alpha \otimes y) = (\bigvee x_\alpha) \otimes y$ [19, Proposition IV.5.2]. Given a frame homomorphism $f : L_2 \to M$, the map $\mathrm{id}_{L_1} \otimes f : L_1 \otimes L_2 \to L_1 \otimes M$ satisfies

$$(\mathrm{id}_{L_1} \otimes f)(1 \otimes x) = (\mathrm{id}_{L_1} \otimes f)\iota_2(x) = \iota_2 \circ f(x) = (1 \otimes f(x)),$$

and likewise for ι_1; thus, $(\mathrm{id}_{L_1} \otimes f)(a \otimes b) = a \otimes f(b)$ and so $(\mathrm{id}_{L_1} \otimes f)(\downarrow S) = \bigvee_{a \otimes b \le \downarrow S} a \otimes f(b)$.

Lastly, we may form the pushout of a span $B \xleftarrow{f} A \xrightarrow{g} C$ in Loc by taking the underlying set P to be the pullback of the left adjoints in Frm, $P = \{(b, c) \in B \times C \mid f^L(b) = g^L(c)\}$. In Frm, the projection maps are the obvious projections, so the pushout legs in Loc are the right adjoints, namely $i_B : B \to P$ given by $i_B(x) = \bigvee\{(b, c) \in P \mid b = i_B^L(b, c) \le x\}$, and analogously for i_C. Note that, since pullbacks of injections (surjections) are again injections (surjections) in Frm since they are in Set, the analogous statement holds for pushouts in Loc.

Proposition 6.4 [19, Corollary IV.4.3.5] *The categories* Frm *and* Loc *are bicomplete.* □

The coproduct in Frm distributes over arbitrary products or, equivalently, the product in Loc distributes over arbitrary coproducts.

Proposition 6.5 [14, Proposition I.5.2] *Let L be a frame and $\{M_k\}_{k \in K}$ be a collection of frames. Then $L \otimes \prod_{k \in K} M_k \cong \prod_{k \in K}(L \otimes M_k)$ is a product with projections $\mathrm{id}_L \otimes p_k$, where p_k are the projections from $\prod_{k \in K} M_k$.* □

Proposition 6.6 [12, Proposition II.2.13] *If X and Y are sober spaces with X locally compact, then $\Omega(X \times Y) \cong \Omega X \otimes \Omega Y$, where $\otimes$ is the product in* Loc. □

The initial object in Frm is the poset $T = \{0 < 1\}$: since every frame homomorphism preserves 0 and 1, this uniquely determines a morphism out of T. Terminal objects in Frm are trivial posets $*$: the constant map from any frame is clearly a frame homomorphism. Note that, since $0 = 1$ in $*$, there are no maps out of $*$, except to singletons. Considering the dual then, any singleton is an initial object in Loc, and T is the terminal object in Loc. From this, it is clear that $\Omega\varnothing$ is initial and $\Omega*$ is terminal. In particular, since there are no maps into $\Omega\varnothing$ except the identity, any product with $\Omega\varnothing$ is again $\Omega\varnothing$; hence, $\Omega\varnothing$ satisfies (S4).

In a frame, a is *way below* b, written $a \ll b$, if, whenever $b = \bigvee K$, there is a finite subset $K' \subseteq K$ such that $a \le \bigvee K'$. A frame is *continuous* or *locally compact* if $a = \bigvee\{x \in L \mid x \ll a\}$ for all $a \in L$; a locale is locally compact if it is locally compact as a frame. Note that, if X is a locally compact Hausdorff space, then ΩX is locally compact. Johnstone characterizes the exponentiable locales as the locally compact locales, as described in the following proposition:

Proposition 6.7 [12, Theorem VII.4.11] *A locale is exponentiable in* Loc *if and only if it is locally compact.*   □

Hence, the images of ∂I^n and I^n under Ω are exponentiable in Loc. Let us now verify that $\mathcal{I}_{\mathsf{Loc}}$ and $\mathcal{J}_{\mathsf{Loc}}$ admit the small object argument.

Lemma 6.8 *Let $X_0 \to X = \operatorname{colim} X_n$ be an ω-transfinite composite of injective localic maps. Denote the colimit inclusions by $i_n \colon X_n \hookrightarrow X$. A map $f \colon A \to X$ factors through X_N if and only if $fA \subseteq i_N X_N$.*

Proof The forward direction is clear. Suppose conversely that $fA \subseteq i_N X_N$ for some positive integer N, so for each $a \in A$ there is $x_a \in X_N$ with $f(a) = i_N(x_a)$. Define $\bar{f} \colon A \to X_N$ by $\bar{f}(a) = x_a$. Clearly, $i_N \bar{f}(a) = f(a)$, so, since i_N is injective and both f and i_N are localic maps, $\bar{f}$ is a localic map as well by [19, IV.1.1(1)].   □

In the following, we write an element of $\coprod \Omega X_\alpha$ as a union $\bigcup U_\alpha$, where U_α is an open subset of X_α.

Proposition 6.9 *Let K be a compact Hausdorff space. Then ΩK is small relative to transfinite composites of pushouts of coproducts of maps in $\mathcal{I}_{\mathsf{Loc}}$ and $\mathcal{J}_{\mathsf{Loc}}$ in* Loc.

Proof Let $X_0 \to X = \operatorname{colim} X_n$ be an ω-transfinite composite of pushouts of coproducts of maps in $\mathcal{K}_{\mathsf{Loc}}$, where $\mathcal{K}_{\mathsf{Loc}} = \{\Omega i_n \colon \Omega S_n \to \Omega C_n\}_{n \geq 0}$ is either $\mathcal{I}_{\mathsf{Loc}}$ or $\mathcal{J}_{\mathsf{Loc}}$. Note that, when $X_0 \cong \Omega *$, any map $f \colon \Omega K \to X$ factors through some X_n since it does in Sob and Ω preserves colimits. We can reduce the general case to this as follows. Let $Y_0 = \Omega *$, and inductively define Y_{n+1} as the pushout

$$
\begin{array}{ccccc}
\coprod \Omega S_{n_\alpha} & \xrightarrow{\;\sigma\;} & X_n & \xrightarrow{\;p_n\;} & Y_n \\
{\scriptstyle \coprod \Omega i_{n_\alpha}}\downarrow & & \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\
\coprod \Omega C_{n_\alpha} & \longrightarrow & X_{n+1} & \dashrightarrow[p_{n+1}] & Y_{n+1}
\end{array}
$$

where the left square is the defining pushout for X_{n+1} and both the outer and right squares are pushouts. Explicitly, since $\left(\coprod \Omega i_{n_\alpha}\right)^L \left(\bigcup U_\alpha\right) = \left(\coprod i_{n_\alpha}\right)^{-1}\left(\bigcup U_\alpha\right) = \bigcup (U_\alpha \cap S_{n_\alpha})$ for $\bigcup U_\alpha \subseteq \bigcup C_{n_\alpha}$, we obtain

$$
X_{n+1} = \left\{\left(x, \bigcup U_\alpha\right) \in X_n \times \coprod \Omega C_{n_\alpha} \;\middle|\; \sigma^L(x) = \bigcup (U_\alpha \cap S_{n_\alpha})\right\},
$$

$$
Y_{n+1} = \left\{\left(y, x, \bigcup U_\alpha\right) \in Y_n \times X_{n+1} \;\middle|\; p_n^L(y) = i_X^L\left(x, \bigcup U_\alpha\right) = x\right\}.
$$

For $x \in X_n$, let $\mathfrak{m}(x) = \sigma^L(x) \cup \bigcup(C_{n_\alpha} - S_{n_\alpha})$, which is open since each i_{n_α} is a closed map. Then $i_X(x) = \bigvee\{(\tilde{x}, \bigcup V_\alpha) \in X_{n+1} \mid \tilde{x} \leq x\}$; clearly $(x, \mathfrak{m}(x))$ is the maximum within this set, so $i_X(x) = (x, \mathfrak{m}(x))$. Similarly, $i_Y(y) = \left(y, p_n^L(y), \mathfrak{m}(p_n^L(y))\right)$. Repeating this process for all n, we get an induced map $\hat{p} \colon X \to Y = \operatorname{colim} Y_n$ in the diagram

$$
\begin{array}{ccccccccc}
X_0 & \hookrightarrow & X_1 & \hookrightarrow & X_2 & \hookrightarrow & \cdots & \hookrightarrow & X \\
{\scriptstyle p_0}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & & & \downarrow{\scriptstyle \hat{p}} \\
Y_0 & \hookrightarrow & Y_1 & \hookrightarrow & Y_2 & \hookrightarrow & \cdots & \hookrightarrow & Y
\end{array}
$$

By [Lemma 6.8](), a map $f : \Omega K \to X$ factors through X_N if and only if $f(\Omega K)$ is contained in X_N, identified with its inclusion into X. Suppose there was a map $f : \Omega K \to X$ that did not factor through any X_n. Necessarily, then, $f(\Omega K)$ intersects the image of $X_{n+1} - i_X X_n$ for arbitrarily large n; we will show that this holds for Y_n as well.

If $(x, \bigcup U_\alpha) \in f(\Omega K) \cap (X_{n+1} - i_X X_n)$, then necessarily $\bigcup U_\alpha \ne \mathfrak{m}(x)$. Since $\sigma^L(x) = \bigcup(U_\alpha \cap S_{n_\alpha})$, we have $\bigcup(U_\alpha \cap S_{n_\alpha}) = \mathfrak{m}(x) \cap \bigcup S_{n_\alpha}$; hence, there must be a point $t \in (\mathfrak{m}(x) - U_\alpha) \cap (\bigcup(C_{n_\alpha} - S_{n_\alpha}))$. In particular, $t \in \mathfrak{m}(x')$ for any $x' \in X_n$ since $\bigcup(C_{n_\alpha} - S_{n_\alpha}) \subseteq \mathfrak{m}(x')$. Let

$$\left(\tilde{y}, \tilde{x}, \bigcup V_\alpha \right) = p_{n+1}\left(x, \bigcup U_\alpha \right) = \bigvee \left\{ \left(y', x', \bigcup V'_\alpha \right) \in Y_{n+1} \mid \left(x', \bigcup V'_\alpha \right) \le \left(x, \bigcup U_\alpha \right) \right\}.$$

Then $\bigcup V_\alpha \subseteq \bigcup U_\alpha$, so $t \notin V_\alpha$. But, by the description of i_Y, if $(\tilde{y}, \tilde{x}, \bigcup V_\alpha) = i_Y(y)$ for some $y \in Y_n$, then $\bigcup V_\alpha = \mathfrak{m}(p_n^L(y))$, which cannot be true since $t \in \mathfrak{m}(p_n^L(y))$. Thus, $p_{n+1}(x, \bigcup U_\alpha) \notin i_Y Y_n$.

Hence, $\hat{p} f(\Omega K)$ intersects $Y_{n+1} - i_Y Y_n$ for arbitrarily large n too, which is a contradiction since $\hat{p} f$ must factor through some Y_N. Thus, no such f exists, so all maps $f : \Omega K \to X$ factor through some X_n. $\square$

The only remaining condition to check is that products with L preserve the pushout in (S3) for any locale L, which we will verify the dual statement for in Frm. This however requires a few lemmas. Recall that, as in the beginning of [Section 4](), $a(t) = \frac{1}{2}t$ and $b(t) = \frac{1}{2}(t + 1)$ are pushout legs $I \to I$ such that a restricts to a homeomorphism $[0, 1) \to [0, \frac{1}{2})$ and b restricts to a homeomorphism $(0, 1] \to (\frac{1}{2}, 1]$.

Lemma 6.10 [3, Proposition 2.2] *If N is a continuous frame and $a \otimes b \le \pi(U)$ in $L \otimes N$, then, for any $c \ll b$, we have $(a, c) \in \pi_1 \hat{\pi}_2(U)$.* $\square$

Lemma 6.11 *Let L be a frame and $\downarrow S \in L \otimes \Omega I$. Then the following hold:*

(1) $\mathfrak{D}(\mathrm{id}_L \times a^{-1})(\downarrow S) = (\mathrm{id}_L \times a^{-1})_*(\downarrow S) = \{(x, a^{-1}U) \mid (x, U) \in \downarrow S\}$. *That is, the image of $\downarrow S$ under $\mathrm{id}_L \times a^{-1}$ is a downset.*

(2) $(\mathrm{id}_L \times a^{-1})_*(\downarrow S) = \hat{\pi}_2((\mathrm{id}_L \times a^{-1})_*(\downarrow S)) = \pi_1((\mathrm{id}_L \times a^{-1})_*(\downarrow S))$. *In particular, $(\mathrm{id}_L \times a^{-1})_*(\downarrow S)$ is fixed under $\pi_1 \hat{\pi}_2$.*

(3) $\mathrm{id}_L \otimes a^{-1}(\downarrow S) \le \pi \circ (\mathrm{id}_L \times a^{-1})_*(\downarrow S)$.

Analogous statements hold for b^{-1} in place of a^{-1}.

Proof Since $\downarrow S \in L \otimes \Omega I$, it is fixed under π; hence, if $\{(x, U_\alpha)\} \subseteq \downarrow S$, then $(x, \bigcup U_\alpha) \in \downarrow S$. For (1) then, suppose $(x, U) \in \mathfrak{D}(\mathrm{id}_L \times a^{-1})(\downarrow S) = \downarrow[(\mathrm{id}_L \times a^{-1})(\downarrow S)]$. Then there is $(y, V) \in \downarrow S$ such that $(x, U) \le (y, a^{-1}V)$, so $x \le y$ and $U \subseteq a^{-1}V$. If $1 \notin U$, then aU is open, $aU \subseteq aa^{-1}V \subseteq V$ and $a^{-1}aU = U$, so $(x, aU) \in \downarrow S$ satisfies $(\mathrm{id}_L \times a^{-1})(x, aU) = (x, U)$. Otherwise, if $1 \in U$ and hence $\frac{1}{2} \in V$, then $W = aU \cup ([\frac{1}{2}, 1] \cap V)$ is open and satisfies $a^{-1}W = U$ and $W \subseteq V$, so $(x, W) \in \downarrow S$ satisfies $(\mathrm{id}_L \times a^{-1})(x, W) = (x, U)$. Thus, the direction $\subseteq$ holds, and the other inclusion is clear, so they are equal.

For the remainder of the proof, let $\mathcal{S} = (\mathrm{id}_L \times a^{-1})_*(\downarrow S)$. For (2), suppose first that $(x, V) \in \hat{\pi}_2 \mathcal{S}$. Then there must be a finite subset $\{a^{-1}U_n\}_{n=1}^N \subseteq \Omega I$ such that $\{x\} \times \{a^{-1}U_n\}_{n=1}^N \subseteq \mathcal{S}$ and $\bigcup_{n=1}^N a^{-1}U_n = V$.

Since each $(x, a^{-1}U_n)$ is in $\mathcal{S}$, $(x, U_n) \in \downarrow S$, so, since $\downarrow S$ is π-saturated, $\left(x, \bigcup_{n=1}^{N} U_n\right) \in \downarrow S$ and $(x, V) = \left(x, a^{-1}\left(\bigcup_{n=1}^{N} U_n\right)\right) \in \mathcal{S}$. Thus, $\hat{\pi}_2 \mathcal{S} \subseteq \mathcal{S}$. Similarly, if $\left(\bigvee x_\alpha, a^{-1}U\right) \in \pi_1 \mathcal{S}$ for some $\{(x_\alpha, a^{-1}U)\} \subseteq \mathcal{S}$, then $(x_\alpha, U) \in \downarrow S$ for each α; hence, by π-saturation, $\left(\bigvee x_\alpha, U\right) \in \downarrow S$ and $\left(\bigvee x_\alpha, a^{-1}U\right) \in \mathcal{S}$. Thus, $\pi_1 \mathcal{S} \subseteq \mathcal{S}$. Since the reverse inclusions are clear, the equalities hold.

Lastly, for (3), for each $(x, U) \in \downarrow S$, we have $(x, a^{-1}U) \in \mathcal{S}$, so also in $\pi\mathcal{S}$, so $\mathrm{id}_L \otimes a^{-1}(x, U) = x \otimes a^{-1}U \leq \pi\mathcal{S}$. Since this holds for all such (x, U),

$$\mathrm{id}_L \otimes a^{-1}(\downarrow S) = \bigvee_{x \otimes U \leq \downarrow S} x \otimes a^{-1}U \leq \pi\mathcal{S}. \qquad \square$$

Lemma 6.12 *Let $f \colon M \to N$ be a frame homomorphism, and suppose $\{n_k\}_{k \in K} \subseteq N$ is such that every $n \in N$ is of the form $n = \bigvee_{k \in K'} n_k$ for some $K' \subseteq K$. If $\{n_k\}_{k \in K}$ is in the image of f, then f is surjective.*

Proof Suppose that $f(m_k) = n_k$ for each $k \in K$. Fix $n \in N$, and let $K' \subseteq K$ be such that $n = \bigvee_{k \in K'} n_k$. Then $\bigvee_{k \in K'} m_k$ satisfies $f\left(\bigvee_{k \in K'} m_k\right) = \bigvee_{k \in K'} f(m_k) = \bigvee_{k \in K'} n_k = n$. $\qquad \square$

We will identify $L \otimes \{0 < 1\}$ with L and write χ_k for $\mathrm{id}_L \otimes i_k \colon L \otimes \Omega I \to L$ for $k = 0, 1$. For $x \in L$ and $U \in \Omega I$, we have $\chi_k(x \otimes U) = x$ if $k \in U$ and $\chi_k(x \otimes U) = 0$ if $k \notin U$; for a general $\downarrow S \in L \otimes \Omega I$, $\chi_k(\downarrow S) = \bigvee \{x \in L \mid \text{there is } U \in \Omega I \text{ with } k \in U \text{ and } x \otimes U \leq \downarrow S\}$.

Proposition 6.13 *For any frame L, the following is a pullback square in Frm:*

$$
\begin{array}{ccc}
L \otimes \Omega I & \xrightarrow{\mathrm{id}_L \otimes a^{-1}} & L \otimes \Omega I \\
{\scriptstyle \mathrm{id}_L \otimes b^{-1}} \downarrow & & \downarrow {\scriptstyle \chi_1} \\
L \otimes \Omega I & \xrightarrow[\chi_0]{} & L
\end{array}
$$

Proof Let P be the canonical pullback of the square (constructed as a subset of the product); since limits in Frm have the corresponding limit in Set for their underlying set, we may take

$$P = \{(\downarrow S, \downarrow T) \subseteq (L \otimes \Omega I)^2 \mid \chi_1(\downarrow S) = \chi_0(\downarrow T)\}$$

along with the usual projections pr_1 and pr_2. Let $\phi \colon L \otimes \Omega I \to P$ be the induced map, which is given by $\phi(\downarrow S) = (\mathrm{id}_L \otimes a^{-1}(\downarrow S), \mathrm{id}_L \otimes b^{-1}(\downarrow S))$. We will show that ϕ is a bijection, and hence an isomorphism.

By [Lemma 6.12](), it suffices to show every $(x \otimes U, y \otimes V)$ is in the image of ϕ for surjectivity, since every element of P can be written as a join of such elements. Indeed, let $(\downarrow S, \downarrow T) \in P$. For each $x \otimes U \leq \downarrow S$, if $1 \notin U$, then let $t_{x \otimes U} = 0$, so $(x \otimes U, t_{x \otimes U}) \in P$. Otherwise, if $1 \in U$, then let $\mathcal{T} = \{y \otimes V \leq \downarrow T \mid 0 \in V\}$, so $((x \wedge y) \otimes U, (x \wedge y) \otimes V) \in P$ for each $y \otimes V \in \mathcal{T}$. Since $\chi_0(\downarrow T) = \chi_1(\downarrow S)$ and this holds for all such $y \otimes V$,

$$
\begin{aligned}
\bigvee_{y \otimes V \in \mathcal{T}} ((x \wedge y) \otimes U, (x \wedge y) \otimes V) &= \left(\left(\bigvee_{y \otimes V \in \mathcal{T}} (x \wedge y)\right) \otimes U, t_{x \otimes U}\right) \\
&= \left(\left(x \wedge \bigvee_{y \otimes V \in \mathcal{T}} y\right) \otimes U, t_{x \otimes U}\right) \\
&= ((x \wedge \chi_1(\downarrow S)) \otimes U, t_{x \otimes U}) = (x \otimes U, t_{x \otimes U}) \leq (\downarrow S, \downarrow T),
\end{aligned}
$$

where $t_{x \otimes U} = \bigvee_{y \otimes V \in \mathcal{T}} ((x \wedge y) \otimes V)$; the first equality holds since $\bigvee(x_\alpha \otimes y) = (\bigvee x_\alpha) \otimes y$, the second by distributivity, and the last since $x \leq \chi_1(\downarrow S)$. Thus, whether $1 \in U$ or not, $(x \otimes U, t_{x \otimes U})$ can be written as a join of pairs of tensor elements, and hence so can

$$\bigvee_{x \otimes U \leq \downarrow S} (x \otimes U, t_{x \otimes U}) = \left(\bigvee_{x \otimes U \leq \downarrow S} x \otimes U, t_{\downarrow S}\right) = (\downarrow S, t_{\downarrow S}) \in P,$$

where $t_{\downarrow S} = \bigvee_{x \otimes U \leq \downarrow S} t_{x \otimes U} \leq \downarrow T$. Analogous reasoning for $\downarrow T$ shows $(\downarrow S, \downarrow T) = (\downarrow S, t_{\downarrow S}) \vee (s_{\downarrow T}, \downarrow T)$ can be written as a join of elements of the form $(x \otimes U, y \otimes V)$.

For surjectivity then, fix an element $(x \otimes U, y \otimes V) \in P$. Suppose first that $1 \in U$. If $x = 0$, then $\chi_1(x \otimes U) = 0 = \chi_0(y \otimes V)$; thus, either $0 \notin V$ or $y = 0$. If $y = 0$, then $\phi(0) = (0,0) = (x \otimes U, y \otimes V)$, so suppose not. Then, since $0 \notin V$, bV is open in I and $a^{-1}bV$ is empty, so $y \otimes bV$ satisfies

$$\phi(y \otimes bV) = (y \otimes a^{-1}bV, y \otimes b^{-1}bV) = (0, y \otimes V) = (x \otimes U, y \otimes V).$$

Suppose now that $x \neq 0$. Then $x = \chi_1(x \otimes U) = \chi_0(y \otimes V)$; necessarily then $0 \in V$ and $y = x$. Then $C = aU \cup bV$ is open in I and satisfies $a^{-1}C = U$ and $b^{-1}C = V$, so $\phi(x \otimes C) = (x \otimes U, y \otimes V)$. Suppose now that $1 \notin U$, so aU is open in I and $b^{-1}aU$ is empty. If $y = 0$, then $\phi(x \otimes aU) = (x \otimes a^{-1}aU, 0) = (x \otimes U, y \otimes V)$. If $y \neq 0$, then, since $\chi_0(y \otimes V) = \chi_1(x \otimes U) = 0$, it must be that $0 \notin V$. Thus, aU and bV are both open in I and $a^{-1}bV$ is empty, so

$$\phi((x \otimes aU) \vee (y \otimes bV)) = \phi(x \otimes aU) \vee \phi(y \otimes bV) = (x \otimes U, 0) \vee (0, y \otimes V) = (x \otimes U, y \otimes V).$$

Thus, in any case we see that $(x \otimes U, y \otimes V)$ is in the image of ϕ; hence, ϕ is surjective.

For injectivity, let $\downarrow S \neq \downarrow T$ in $L \otimes \Omega I$. Without loss of generality, assume there is $(x, U) \in \downarrow S - \downarrow T$. Suppose that $(x \otimes a^{-1}U, x \otimes b^{-1}U) \leq \phi(\downarrow T)$. Then $x \otimes a^{-1}U \leq (\mathrm{id}_L \otimes a^{-1})(\downarrow T) \leq \pi \circ (\mathrm{id}_L \times a^{-1})_*(\downarrow T)$. For each $t_a \in a^{-1}U$, by local compactness we may choose an open interval I_{t_a} (or half-open interval $[0, \alpha)$ or $(\beta, 1]$ if $t_a = 0, 1$ is in $a^{-1}U$), not containing 1 unless $t_a = 1$, such that $I_{t_a} \ll a^{-1}U$. Thus, by Lemmas 6.10 and 6.11, $(x, I_{t_a}) \in (\mathrm{id}_L \times a^{-1})_*(\downarrow T)$. Similarly, for each $t_b \in b^{-1}U$, there is an interval J_{t_b}, not containing 0 unless $t_b = 0$, such that $(x, J_{t_b}) \in (\mathrm{id}_L \times a^{-1})_*(\downarrow T)$. If $1 \in a^{-1}U$, and hence $0 \in b^{-1}U$, then we may choose some neighborhood $J_{1/2} \subseteq U$ of $\frac{1}{2}$ such that $(x, J_{1/2}) \in \downarrow T$, $a^{-1}J_{1/2} = I_1$ and $b^{-1}J_{1/2} = J_0$; otherwise let $J_{1/2} = \varnothing$. For all $t_a \neq 1$ in $a^{-1}U$, $aI_{t_a} \subseteq U$ is open since $1 \notin I_{t_a}$, and $(x, aI_{t_a}) \in \downarrow T$ since aI_{t_a} is the least open set V such that $a^{-1}V = I_{t_a}$; likewise, for each $t_b \neq 0$, $bJ_{t_b} \subseteq U$ is open and $(x, bJ_{t_b}) \in \downarrow T$. Thus, since a and b are jointly surjective, each $t \in U$ is in some aI_{t_a}, bJ_{t_b} or $J_{1/2}$, so their union is U. Since $\downarrow T$ is π-saturated,

$$\left(x, \bigvee_{t_a \in a^{-1}U} aI_{t_a} \bigvee_{t_b \in b^{-1}U} bJ_{t_b} \vee J_{1/2}\right) = (x, U) \in \downarrow T,$$

which contradicts our assumption. Thus, $(x \otimes a^{-1}U, x \otimes b^{-1}U) \not\leq \phi(\downarrow T)$, but $(x \otimes a^{-1}U, x \otimes b^{-1}U) \leq \phi(\downarrow S)$ and hence $\phi(\downarrow T) \neq \phi(\downarrow S)$. $\qquad\square$

We are now ready to verify that (Loc, Ω) carries a model structure.

Theorem 6.14 *There is a model structure on* Loc *whose cofibrations are maps in* $\mathrm{cof}(\mathcal{I}_{\mathsf{Loc}})$ *(see* [Notation 6.1](#)*), fibrations are maps in* $\mathrm{rlp}(\mathcal{J}_{\mathsf{Loc}})$ *(see idem), and weak equivalences are maps which have the weak right lifting property against* $\mathcal{I}_{\mathsf{Loc}}$ *or, equivalently by* [Theorem 3.12](#)*, maps which induce isomorphisms on all homotopy groups.*

Proof We verify that (Loc, Ω) is a good Q-structure. By the description of Ω, (S1) and (S4) are clear, (S2) and (Q7) hold by [Proposition 6.6](#), and (S3) holds by [Proposition 6.13](#); thus, (Loc, Ω) is a category with intervals. By [Proposition 6.5](#), (Q2) holds, (Q5) follows from [Proposition 6.7](#), and (Q6) by [Proposition 6.9](#). Lastly, (Q3), (Q4) and (Q8) hold since Ω preserves colimits. □

Since Ω preserves colimits and the generating (acyclic) cofibrations, we have an immediate corollary:

Proposition 6.15 *The adjunction* $\Omega \dashv \mathrm{pt}$ *is a Quillen adjunction between the Quillen model structures on* Sob *and* Loc.

We will show that $\Omega \dashv \mathrm{pt}$ is in fact a Quillen equivalence. Note that every object in Loc is fibrant and, for any $X \in \mathsf{Sob}$, $X \to \mathrm{pt}\,\Omega X$ is an isomorphism as Ω is fully faithful. Thus, by [11, Corollary 1.3.16(c)], it is necessary and sufficient to show that pt reflects weak equivalences. The following lemma will be useful:

Lemma 6.16 *Let* $i : A \to X$ *be a map between sober spaces and* $f, g : \Omega X \to L$ *be maps in* Loc. *Then* $f \sim g$ *rel* ΩA *in* Loc *if and only if* $\hat{f} \sim \hat{g}$ *rel* A *in* Sob, *where* $\hat{f}, \hat{g} : X \to \mathrm{pt}\,L$ *are the adjunct maps.*

Proof Recall from [Proposition 6.6](#) that the canonical map $\Omega(X \times I) \to \Omega X \otimes \Omega I$ is an isomorphism as I is locally compact, where $\otimes$ is again the product in Loc. By this and adjointness, a map $K : X \times I \to \mathrm{pt}\,L$ makes the two diagrams

$$
\begin{array}{ccccc}
X & \xrightarrow{e_0} & X \times I & \xleftarrow{e_1} & X \\
& {\hat{f}}\searrow & \downarrow{K} & \swarrow{\hat{g}} & \\
& & \mathrm{pt}\,L & &
\end{array}
\qquad
\begin{array}{ccc}
A \times I & \xrightarrow{\pi_A} & A \\
{i \times \mathrm{id}_I}\downarrow & & \downarrow{\hat{f}i = \hat{g}i} \\
X \times I & \xrightarrow{K} & \mathrm{pt}\,L
\end{array}
$$

commute in Sob if and only if $\Omega X \otimes \Omega I \cong \Omega(X \times I) \xrightarrow{\hat{K}} L$ makes the following two diagrams commute in Loc:

$$
\begin{array}{ccccc}
\Omega X & \xrightarrow{e_0} & \Omega X \otimes \Omega I & \xleftarrow{e_1} & \Omega X \\
& {f}\searrow & \downarrow{\hat{K}} & \swarrow{g} & \\
& & L & &
\end{array}
\qquad
\begin{array}{ccc}
\Omega A \otimes \Omega I & \xrightarrow{\pi_{\Omega A}} & \Omega A \\
{\Omega i \otimes \mathrm{id}_{\Omega I}}\downarrow & & \downarrow{f \Omega i = g \Omega i} \\
\Omega X \otimes \Omega I & \xrightarrow{\hat{K}} & L
\end{array}
\qquad \square
$$

Theorem 6.17 *The adjunction* $\Omega \dashv \mathrm{pt}$ *is a Quillen equivalence between the Quillen model structures on* Sob *and* Loc.

Proof Suppose that $f : X \to Y \in \mathsf{Loc}$ is a map such that $\mathrm{pt}\,f$ is a weak equivalence. Given the square on the left, we obtain the square on the right by adjointness:

$$\begin{array}{ccc} \Omega\partial I^n & \xrightarrow{\;u\;} & X \\ {\scriptstyle\Omega i_n}\downarrow & & \downarrow{\scriptstyle f} \\ \Omega I^n & \xrightarrow{\;v\;} & Y \end{array} \qquad\qquad \begin{array}{ccc} \partial I^n & \xrightarrow{\;\hat{u}\;} & \mathrm{pt}\,X \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle \mathrm{pt}\,f} \\ I^n & \xrightarrow{\;\hat{v}\;} & \mathrm{pt}\,Y \end{array}$$

As pt f is a weak equivalence, we obtain a map $h\colon I^n \to \mathrm{pt}\,X$ with $hi_n = \hat{u}$ and a homotopy $K\colon \mathrm{pt}\,f \circ h \sim \hat{v}$ rel ∂I^n. Then, by adjointness, $\hat{h} \circ \Omega i_n = u$, and, by [Lemma 6.16](), $\hat{K}$ is a homotopy $f\hat{h} \sim v$ rel $\Omega(\partial I^n)$. Thus, $\hat{h}$ is the required filler in the original diagram; hence, f is a weak equivalence. $\qquad\square$

In fact, recognizing that $\mathrm{rlp}(\mathcal{I}_{\mathsf{Loc}}) = \mathrm{rlp}(\Omega(\mathcal{I}_{\mathsf{Sob}})) = \mathrm{pt}^{-1}(\mathrm{rlp}(\mathcal{I}_{\mathsf{Sob}}))$ by adjunction, the weak equivalences and fibrations in Loc are preserved and reflected by pt. Thus, the model structure on Loc is precisely the transferred model structure from Sob.

Corollary 6.18 *The Quillen model structure on* Loc *is the right transferred model structure from* Sob *along* $\Omega \dashv \mathrm{pt}$. $\qquad\square$

With this Quillen equivalence, we can also compare localic homotopy groups to their homotopy groups under pt. Since every locale is fibrant, [[11](), Corollary 1.3.16(b)] implies that the counit $\Omega\,\mathrm{pt}\,L \to L$ is a weak equivalence for every $L \in \mathsf{Loc}$. Thus, we get the following:

Proposition 6.19 *If (X, x_0) is a pointed sober space, then $\pi_n^{\mathsf{Sob}}(X, x_0) \cong \pi_n^{\mathsf{Loc}}(\Omega X, \Omega x_0)$ for any $n \geq 0$. In particular, for any pointed locale (L, l_0) and $n \geq 0$, $\pi_n^{\mathsf{Sob}}(\mathrm{pt}\,L, \mathrm{pt}\,l_0) \cong \pi_n^{\mathsf{Loc}}(L, l_0)$ since $\Omega\,\mathrm{pt}\,L \to L$ is a weak equivalence.*

Proof This follows from [Lemma 6.16](), as two maps $f, g\colon I^n \to X \cong \mathrm{pt}\,\Omega X$ are homotopic rel ∂I^n if and only if $\hat{f}, \hat{g}\colon \Omega I^n \to \Omega X$ are homotopic rel $\Omega(\partial I^n)$. $\qquad\square$

References

[1] **B Banaschewski**, *Bourbaki's fixpoint lemma reconsidered*, Comment. Math. Univ. Carolin. 33 (1992) 303–309 MR Zbl

[2] **P Bubenik**, **N Milićević**, *Homotopy, homology, and persistent homology using closure spaces*, J. Appl. Comput. Topol. 8 (2024) 579–641 MR Zbl

[3] **X D Chen**, *On binary coproducts of frames*, Comment. Math. Univ. Carolin. 33 (1992) 699–712 MR Zbl

[4] **S Dolecki**, **F Mynard**, *Convergence foundations of topology*, World Scientific, Hackensack, NJ (2016) MR Zbl

[5] **L Fajstrup**, **J Rosický**, *A convenient category for directed homotopy*, Theory Appl. Categ. 21 (2008) 7–20 MR Zbl

[6] **T Haraguchi**, *On model structure for coreflective subcategories of a model category*, Math. J. Okayama Univ. 57 (2015) 79–84 MR Zbl

[7] **H Herrlich**, *Topological improvements of categories of structured sets*, Topology Appl. 27 (1987) 145–155 MR Zbl

[8] **H Herrlich**, **E Lowen-Colebunders**, **F Schwarz**, *Improving Top: PrTop and PsTop*, from "Category theory at work" (H Herrlich, H-E Porst, editors), Res. Exp. Math. 18, Heldermann, Berlin (1991) 21–34 MR Zbl

[9]   **P S Hirschhorn**, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, Amer. Math. Soc., Providence, RI (2003) MR Zbl

[10] **P S Hirschhorn**, *The Quillen model category of topological spaces*, Expo. Math. 37 (2019) 2–24 MR Zbl

[11] **M Hovey**, *Model categories*, Mathematical Surveys and Monographs 63, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[12] **P T Johnstone**, *Stone spaces*, Cambridge Studies in Advanced Mathematics 3, Cambridge Univ. Press (1982) MR Zbl

[13] **A Joyal**, *The theory of quasi-categories and its applications*, lecture notes (2008) Available at `https://ncatlab.org/nlab/files/JoyalTheoryOfQuasiCategories.pdf`

[14] **A Joyal**, **M Tierney**, *An extension of the Galois theory of Grothendieck*, Mem. Amer. Math. Soc. 309, Amer. Math. Soc., Providence, RI (1984) MR Zbl

[15] **E Lowen-Colebunders**, **G Sonck**, *Exponential objects and Cartesian closedness in the construct Prtop*, Appl. Categ. Structures 1 (1993) 345–360 MR Zbl

[16] **J P May**, **K Ponto**, *More concise algebraic topology*: *localization*, *completion*, *and model categories*, Univ. of Chicago Press (2012) MR Zbl

[17] **F Morel**, **V Voevodsky**, $\mathbb{A}^1$-*homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999) 45–143 MR Zbl

[18] **S B Niefield**, *Cartesianness: topological spaces, uniform spaces, and affine schemes*, J. Pure Appl. Algebra 23 (1982) 147–167 MR Zbl

[19] **J Picado**, **A Pultr**, *Frames and locales: topology without points*, Springer (2012) MR Zbl

[20] **D G Quillen**, *Homotopical algebra*, Lecture Notes in Math. 43, Springer (1967) MR Zbl

[21] **A Rieser**, *Čech closure spaces: a unified framework for discrete and continuous homotopy*, Topology Appl. 296 (2021) art. id. 107613 MR Zbl

[22] **A Rieser**, *Cofibration and model category structures for discrete and continuous homotopy*, preprint (2022) arXiv 2209.13510

[23] **N E Steenrod**, *A convenient category of topological spaces*, Michigan Math. J. 14 (1967) 133–152 MR Zbl

[24] **N P Strickland**, *The category of cgwh spaces*, preprint (2009) Available at `https://ncatlab.org/nlab/files/StricklandCGHWSpaces.pdf`

[25] **R W Thomason**, *Cat as a closed model category*, Cahiers Topologie Géom. Différentielle 21 (1980) 305–324 MR Zbl

[26] **R M Vogt**, *Convenient categories of topological spaces for homotopy theory*, Arch. Math. (Basel) 22 (1971) 545–555 MR Zbl

[27] **R M Vogt**, *A note on homotopy equivalences*, Proc. Amer. Math. Soc. 32 (1972) 627–629 MR Zbl

Department of Mathematics, University of Western Ontario
London, ON, Canada

Department of Mathematics, University of Western Ontario
London, ON, Canada

sebel@uwo.ca,   kkapulki@uwo.ca

See inside back cover or msp.org/agt for submission instructions.