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An upper bound conjecture for the Yokota invariant

GIULIO BELLETTI

We conjecture an upper bound on the growth of the Yokota invariant of polyhedral graphs, extending a previous result on the growth of the $6j$-symbol. Using Barrett's Fourier transform we are able to prove this conjecture in a large family of examples. As a consequence of this result, we prove the Turaev–Viro volume conjecture for a new infinite family of hyperbolic manifolds.

57K16, 57K32

1 Introduction

In [7] Chen and Yang proposed and provided extensive computations for the following conjecture, relating the hyperbolic volume of a manifold to its Turaev–Viro invariants TV_r (see [22, page 869] for the original definition):

Conjecture 1 (the Turaev–Viro volume conjecture) *Let M be a hyperbolic 3-manifold, either closed with cusps, or compact with geodesic boundary. Then as r varies along the odd natural numbers,*

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(\mathrm{TV}_r(M, e^{2\pi i/r})) = \mathrm{Vol}(M). \tag{1-1}$$

This conjecture has been verified for the complements of the Borromean rings and of the figure-eight knot by Detcherry, Kalfagianni and Yang [12], all the hyperbolic Dehn surgeries on the figure-eight knot (for integral surgeries by Ohtsuki [19] and later for rational surgeries by Wong and Yang [26]), and all complements of fundamental shadow links by Belletti, Detcherry, Kalfagianni and Yang [5].

A useful tool introduced in [5] to study the asymptotic behavior of quantum invariants such as TV_r is a sharp upper bound on the growth of the $6j$-symbol, which is the basic building block in their definition.

Such an upper bound can be used to prove very quickly the volume conjecture for complements of fundamental shadow links.

The upper bound just mentioned can be interpreted as an upper bound for the Yokota invariant Y_r, which is an invariant of embedded graphs (see Definition 2.7). Indeed, the square of the $6j$-symbol is also the Yokota invariant of the tetrahedral graph; thus it is natural to ask if an upper bound analogous to that of [5] holds for any polyhedral graph (which is to say, any graph which is the 1-skeleton of a hyperbolic polyhedron). We propose the following:

Conjecture 2 (the upper bound conjecture) *Let $r > 2$ be odd. If Γ is a polyhedral graph and* col *is any r-admissible coloring of its edges* (see Definition 2.3), *then*

$$\frac{\pi}{r}\log|Y_r(\Gamma,\mathrm{col})| \leq \sup_P \mathrm{Vol}(P) + O_{r\to\infty}\left(\frac{\log(r)}{r}\right),$$

where P varies among all proper generalized hyperbolic polyhedra with 1-skeleton Γ (see Definition 3.2; these are hyperbolic polyhedra with possibly hyperideal vertices) *and* $\mathrm{Vol}(P)$ *is the hyperbolic volume of P.*

Moreover, the inequality is sharp, with equality attained by the sequence of colorings giving the color $\frac{1}{2}(r-2\pm 1)$ to each edge (the sign is chosen so that the colors are even).

We are able to prove the upper bound conjecture for a large family of examples:

Theorem 4.9 *The upper bound conjecture is verified for any planar graph obtained from the tetrahedron by applying any sequence of the following two moves:*

- *blowing up a trivalent vertex (see Figure 1), or*
- *triangulating a triangular face (see Figure 2).*

The upper bound conjecture naturally leads to the question of what is the supremum of all volumes of polyhedra sharing the same 1-skeleton. This is answered by Belletti [4, Theorem 4.2]:

Theorem 1.1 *For any polyhedral graph Γ,*

$$\sup_P \mathrm{Vol}(P) = \mathrm{Vol}(\overline{\Gamma}),$$

where P varies among all proper generalized hyperbolic polyhedra with 1-skeleton Γ and $\overline{\Gamma}$ is the rectification of Γ.

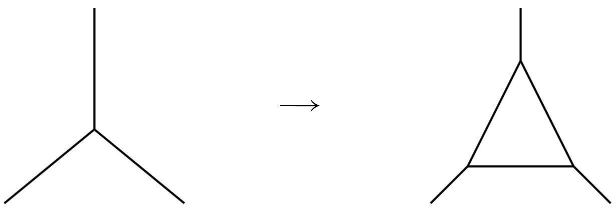

Figure 1: Truncating a vertex.

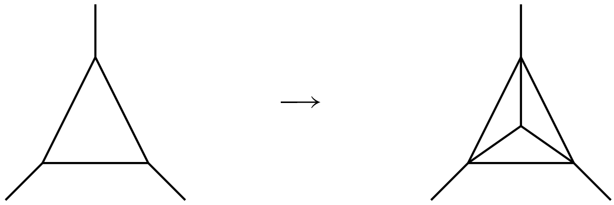

Figure 2: Triangulating a face.

The rectification of a graph is defined in [4, Section 3.4] (see also Remark 3.9); for our purposes it suffices to say that $\overline{\Gamma}$ is the polyhedron with 1-skeleton Γ with every edge tangent to $\partial\mathbb{H}^3$ in the Klein model of hyperbolic space (and hence which has dihedral angle 0 at each edge). This polyhedron can be canonically truncated to give an ideal right-angled hyperbolic polyhedron, and hence it makes sense to speak of $\mathrm{Vol}(\overline{\Gamma})$ as the volume of the truncation.

As an application of Theorem 4.9, we prove in Theorem 5.6 that the Turaev–Viro volume conjecture holds for a new infinite family of cusped manifolds. These are complements of certain links in $S^3 \#^g (S^1 \times S^2)$; their hyperbolic structure is obtained by gluing right-angled octahedra.

In Section 2 we set the notation, give the basic properties of the Kauffman bracket and define the Yokota invariant. In Section 3 we discuss previous volume conjectures for polyhedra and state the upper bound conjecture. In Section 4 we introduce the Fourier transform of Barrett, and use it to prove Theorem 4.9. Section 5 contains the proof of the Turaev–Viro volume conjecture for a new family of manifolds. Finally in an appendix we propose numerical evidence for a related volume conjecture for polyhedra.

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2 The Kauffman bracket and the Yokota invariant

2.1 The Kauffman bracket

Throughout the rest of the paper $r \geq 3$ is an odd integer and $q = e^{2\pi i/r}$. All the definitions we give in this section are standard; the only notable difference is that in some papers (eg [2]) the graphs are colored with half-integer colors, while here we use integers.

For an integer $n \geq 0$, the *quantum integer* $[n]$ is defined as $(q^n - q^{-n})/(q - q^{-1}) = \sin(2\pi n/r)/\sin(2\pi/r)$, and the quantum factorial $[n]!$ is $\prod_{i=1}^{n}[i]$ (with the convention that $[0]! = 1$). Furthermore, we denote by I_r the set of all even nonnegative integers at most equal to $r-2$.

Remark 2.1 Because of the choice of root of unity q, we need to work with the SO(3) version of the quantum invariants, rather than the SU(2) version. This essentially amounts to using only even numbers as colors; a brief overview of how these invariants are related can be found for example in [12, Section 2]. Because of this, some terms in the upcoming formulas appear redundant; we still include them to keep the notation uniform with other papers dealing with the SU(2) version.

Definition 2.2 We say that a triple $(a,b,c) \in I_r^3$ is *r-admissible* if

- $a,b,c \leq r-2$,
- $a+b+c$ is even and $a+b+c \leq 2r-4$,
- $a \leq b+c$, $b \leq a+c$ and $c \leq a+b$.

We say that a 6-tuple $(n_1,n_2,n_3,n_4,n_5,n_6)$ of elements in I_r is *r-admissible* if the four triples (n_1,n_2,n_3), (n_1,n_5,n_6), (n_2,n_4,n_6) and (n_3,n_4,n_5) are r-admissible.

For $n \in \mathbb{N}$ define

$$\Delta_n = (-1)^{n+1}[n+1]. \tag{2-1}$$

For an r-admissible triple (a,b,c) we can define

$$\Theta(a,b,c) = (-1)^{(a+b+c)/2} \frac{\left[\frac{1}{2}(a+b+c)+1\right]!}{\left[\frac{1}{2}(a+b-c)\right]!\left[\frac{1}{2}(a-b+c)\right]!\left[\frac{1}{2}(-a+b+c)\right]!} \tag{2-2}$$

and $\Delta(a,b,c) := \Theta(a,b,c)^{-1/2}$. Notice that the number inside the square root is real; by convention we take the positive square root of a positive number, and the square root with positive imaginary part of a negative number.

Definition 2.3 An *r-admissible coloring* for a tetrahedron T is an assignment of an r-admissible 6-tuple $(n_1,n_2,n_3,n_4,n_5,n_6) \in I_r^6$ to the set of edges of T, as shown in Figure 3. More generally, we say that an r-admissible coloring for a trivalent graph $\Gamma \subseteq S^3$ is an assignment of elements of I_r to the edges of Γ such that the colors at each vertex form an r-admissible triple. Even more generally we say that an

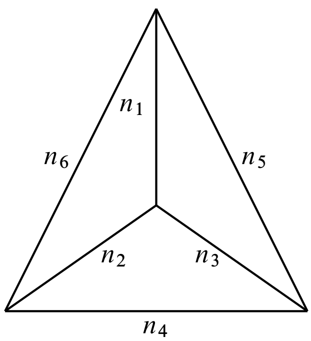


Figure 3: An r-admissible coloring for a tetrahedron.

assignment of elements of I_r to edges of a (not necessarily trivalent) graph is a *coloring*, and a graph Γ together with its coloring col is a *colored graph* (Γ, col).

If v is a trivalent vertex of a graph whose incident edges are colored by an r-admissible triple (a,b,c), we write for short $\Theta(v)$ and $\Delta(v)$ instead of $\Theta(a,b,c)$ and $\Delta(a,b,c)$.

Moreover, for an r-admissible 6-tuple $(n_1,n_2,n_3,n_4,n_5,n_6)$ we can define its $6j$-symbol as usual as

$$\begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix} = \prod_{i=1}^{4} \Delta(v_i) \sum_{z=\max T_i}^{\min Q_j} \frac{(-1)^z [z+1]!}{\prod_{i=1}^{4}[z-T_i]! \prod_{j=1}^{3}[Q_j - z]!}, \tag{2-3}$$

where

- $v_1 = (n_1,n_2,n_3)$, $v_2 = (n_1,n_5,n_6)$, $v_3 = (n_2,n_4,n_6)$ and $v_4 = (n_3,n_4,n_5)$,
- $T_1 = \frac{1}{2}(n_1+n_2+n_3)$, $T_2 = \frac{1}{2}(n_1+n_5+n_6)$, $T_3 = \frac{1}{2}(n_2+n_4+n_6)$ and $T_4 = \frac{1}{2}(n_3+n_4+n_5)$,
- $Q_1 = \frac{1}{2}(n_1+n_2+n_4+n_5)$, $Q_2 = \frac{1}{2}(n_1+n_3+n_4+n_6)$ and $Q_3 = \frac{1}{2}(n_2+n_3+n_5+n_6)$.

By convention we define the $6j$-symbol of a non-r-admissible tuple to be equal to 0.

The *Kauffman bracket* is an invariant of *trivalent framed graphs*; before defining the Kauffman bracket we recall the definition of framed graphs:

Definition 2.4 A *framed graph* $\Gamma \subseteq S^3$ is a graph in S^3 together with a 2-dimensional oriented thickening, considered up to isotopy. More precisely, a framed graph Γ is a pair (G, F) with G an embedded graph in S^3 and F an embedded orientable surface containing G as a deformation retract. As is usual for framed links, we draw planar diagrams of framed graphs with over and undercrossing information, and such that the "thickness" of the surface always lies flat on the projection plane.

Definition 2.5 The *Kauffman bracket* is the unique map

$$\langle \cdot \rangle \colon \{\text{colored trivalent framed graphs in } S^3\} \to \mathbb{C}$$

satisfying the following properties:

(i) If Γ is the planar circle colored with $n \in I_r$ then $\langle \Gamma \rangle = \Delta_n$.

(ii) If Γ is a theta graph (see Figure 4) colored with the r-admissible triple $(a,b,c) \in I_r^3$ then $\langle \Gamma \rangle = 1$.

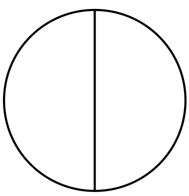
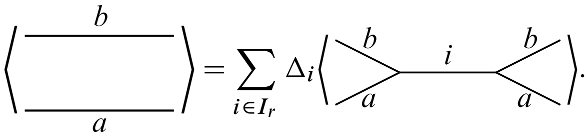


Figure 4: A theta graph.

(iii) If Γ is a tetrahedron colored with the r-admissible 6-tuple $(n_1, \dots, n_6) \in I_r^6$ then

$$\langle\Gamma\rangle = \begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix}.$$

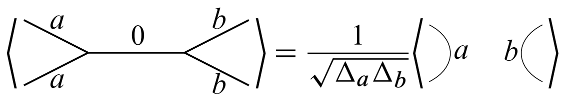
(iv) The *fusion rule*:

(2-4)


(v) If Γ has a bridge (that is to say, an edge that disconnects the graph if removed) colored with $i \neq 0$, then $\langle\Gamma\rangle = 0$.

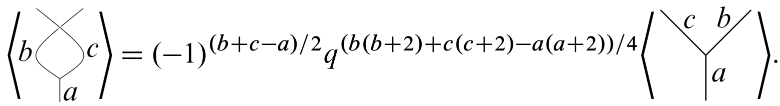
(vi) If at some vertex of Γ the colors do not form an r-admissible triple, then $\langle\Gamma\rangle = 0$.

(vii) If Γ is colored with an r-admissible coloring such that the color of an edge e is equal to 0, then

(2-5)


and $\langle\Gamma\rangle = (1/\sqrt{\Delta_a\Delta_b})\langle\Gamma'\rangle$, where Γ' is Γ with e removed, and a and b are the colors of the edges that share a vertex with e (notice that since the coloring is r-admissible, two edges sharing the same vertex with e will have the same color).

(viii) The undoing of a crossing:



(ix) If Γ is the disjoint union of Γ_1 and Γ_2, then $\langle\Gamma\rangle = \langle\Gamma_1\rangle\langle\Gamma_2\rangle$.

It is absolutely not clear from the definition that such a map exists; a proof is in [16, Chapter 9]. However, it is straightforward to see that (i)–(ix) are enough to calculate $\langle\Gamma\rangle$. Taking any planar diagram of Γ, apply a fusion rule near each crossing, and then undo the crossing using (viii); therefore we only need to calculate $\langle\cdot\rangle$ on planar graphs. For a planar graph, repeated applications of the fusion rule create a bridge, and (v), (vii) and (ix) allow one to compute $\langle\Gamma\rangle$ from the Kauffman bracket of two graphs, each with fewer vertices.

Remark 2.6 There are a few different normalizations of the Kauffman bracket in the literature. Here we use the *unitary normalization*; it should be noted that [16] uses a different one, however the results there apply to the unitary normalization with little modification.

In what follows sometimes we will color the edges of Γ with linear combinations of colors; the Kauffman bracket can be extended linearly to this context. In particular, we will use *Kirby's color* $\Omega := \sum_{i\in I_r} \Delta_i i$.

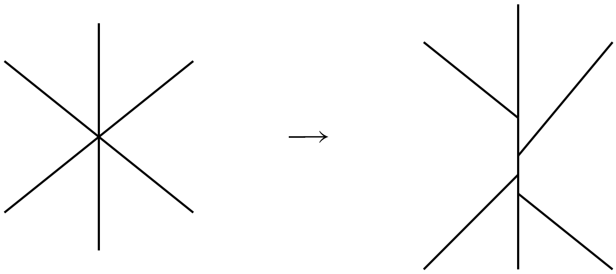


Figure 5: Desingularization of a vertex of valence 6

2.2 The definition of the Yokota invariant from the Kauffman bracket

In this subsection we give an overview of the Yokota invariant, which generalizes the Kauffman bracket invariant of trivalent graphs to graphs with vertices of any valence; it was first introduced in [27].

Suppose $\Gamma \subseteq S^3$ is a framed graph with vertices of valence at least 3; as before $r > 2$ is odd and $q = e^{2\pi i/r}$.

For a vertex v of Γ, we can take a small ball B containing v, and replace $\Gamma \cap B$ with a trivalent planar tree in B having the same endpoints in $\partial B \cap \Gamma$ (see Figure 5). We call this procedure a *desingularization* of Γ at v. Notice that if v has valence greater than 3, then this procedure is not unique; however, any desingularization is related to any other via a sequence of Whitehead moves (see Figure 6). This fact can be most easily seen by thinking about the dual graph: the vertex corresponds to a polygon and a desingularization corresponds to a choice of enough diagonals to triangulate the polygon. Then a Whitehead move acts on the dual as a diagonal flip, and clearly diagonal flips are enough to go from any choice of diagonals to any other.

We say that the trivalent graph Γ' is a desingularization of Γ if it is obtained from Γ by desingularization of each vertex of valence at least 4.

Definition 2.7 Let (Γ, col) be a framed graph in S^3 colored with elements of I_r. Let Γ' be a desingularization of Γ. Call $e'_1, \dots, e'_k$ the edges of Γ' that were added by the desingularization. If $k > 0$, then the *Yokota invariant* of (Γ, col) is

$$Y_r(\Gamma, \mathrm{col}) := \sum_{\mathrm{col}' \in I_r^k} \left(\prod_{i=1}^{k} \Delta_{\mathrm{col}'(e'_i)} \right) |\langle \Gamma', \mathrm{col} \cup \mathrm{col}' \rangle|^2,$$

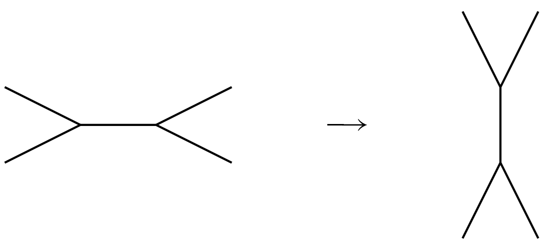


Figure 6: A Whitehead move.

with col' coloring the edges $e'_1, \dots, e'_k$. If instead $k = 0$ (ie $\Gamma = \Gamma'$, ie Γ is trivalent) then $Y_r(\Gamma, \mathrm{col}) = |\langle \Gamma, \mathrm{col} \rangle|^2$.

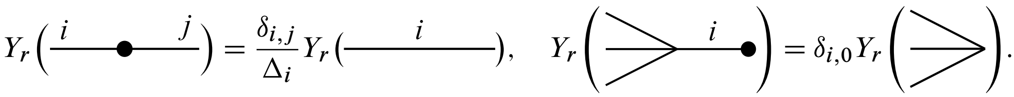
As we did with the Kauffman bracket, we extend linearly the Yokota invariant to linear combinations of colors. Notice that in this case, even if Γ is trivalent, we may get $Y_r(\Gamma, \mathrm{col}) \neq |\langle \Gamma, \mathrm{col} \rangle|^2$.

Remark 2.8 We stress the fact that we are using the unitary normalization for the Kauffman bracket. If we instead used the Kauffman normalization $\langle \cdot \rangle_K$ of [16], the definition of the Yokota invariant of (Γ, col) would be

$$Y_r(\Gamma, \mathrm{col}) := \sum_{\mathrm{col}' \in I_r^k} \frac{\prod_{i=1}^k \Delta_{\mathrm{col}'(e'_i)}}{\prod_{v \text{ vertex of } \Gamma} \Theta(v)} |\langle \Gamma', \mathrm{col} \cup \mathrm{col}' \rangle_K|^2.$$

Proposition 2.9 [27] *The Yokota invariant does not depend on the choice of desingularization.*

We can easily extend the Yokota invariant to graphs with 1-valent and 2-valent vertices as well, via the following formulas:



We normalize the invariant so that it is equal to 1 for the graph with a single vertex and no edges.

Now we give three important properties of the Yokota invariant, all easy consequences of the definitions:

Proposition 2.10 (1) *The Yokota invariant does not depend on the framing of Γ.*

(2) *If an edge e of Γ is colored with the Kirby color Ω, and Γ' is obtained from Γ via a Whitehead move on the edge e (coloring the edge that replaces e with Ω and keeping every other color the same) then $Y_r(\Gamma, \mathrm{col}) = Y_r(\Gamma', \mathrm{col})$.*

(3) *If Γ is a vertex sum of Γ_1 and Γ_2 along trivalent vertices $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$ (see Figure 7), then $Y_r(\Gamma, \mathrm{col}) = Y_r(\Gamma_1, \mathrm{col}_1) Y_r(\Gamma_2, \mathrm{col}_2)$, where col_1 and col_2 are the restrictions of col to Γ_1 and Γ_2, respectively.*

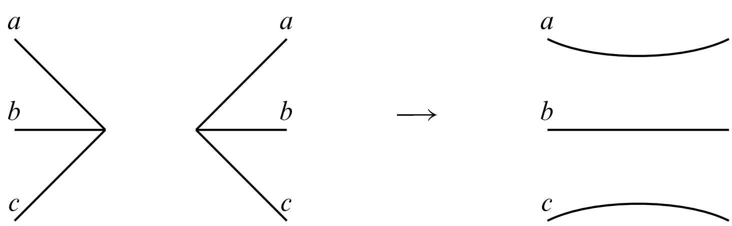


Figure 7: A vertex sum of two trivalent vertices

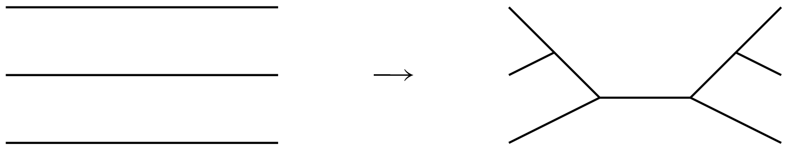


Figure 8: Applying the fusion rule to three edges arising from a vertex sum.

Proof Part (1) holds because $\langle\Gamma\rangle$ depends on the framing of Γ only up to a factor of q^a; thus when taking squared norms this becomes 1. Part (2) is essentially the fact that the Yokota invariant is well defined: since both e and the corresponding edge in Γ' are colored with Ω, both sides of the equality are equal to the Yokota invariant of the graph obtained by collapsing e to a point.

Part (3) follows from the analogous property for the Kauffman bracket; this is obtained via two applications of the fusion rule and one application of the bridge rule of Definition 2.5(v); see Figure 8. $\square$

It is very important that the vertex sum in Proposition 2.10(3) is done between trivalent vertices; the assertion is false in general.

Remark 2.11 The Kauffman bracket (and hence, the Yokota invariant) can also be defined in the much larger setting of framed trivalent graphs in closed oriented 3-manifolds (see for example [16; 18]); since we will not need such a generality that carries some more technical details, we will restrict ourselves to the S^3 case.

3 Volume conjecture for polyhedra

3.1 The volume conjecture for polyhedra

Costantino first conjectured in [8] that the growth of the $6j$-symbol is given by the volume of a hyperbolic tetrahedron. A volume conjecture for trivalent graphs (and their Kauffman bracket invariant) was proposed in [24] and later refined in [10] to the case of planar trivalent graphs and hyperbolic polyhedra with trivalent vertices. The conjecture of [10] evaluates the invariant at the first root of unity $q = e^{\pi i/r}$; the downside of this choice is that they have to consider poles of the Kauffman bracket, instead of its values directly. Recently Murakami and Kolpakov [17] proposed a volume conjecture for polyhedra at the second root of unity $q = e^{2\pi i/r}$, but only stated it for simple polyhedra without hyperideal vertices (see Remark 3.6 and Definition 3.2); remarkably this conjecture directly involves the value of the Kauffman bracket. Here we propose Conjecture 3, which is an extension of Kolpakov and Murakami's volume conjecture to a very general setting, and then propose Conjecture 4, which concerns an upper bound for the Yokota invariant of polyhedral graphs.

Geometric background Recall the projective model for hyperbolic space $\mathbb{H}^3 \subseteq \mathbb{R}^3 \subseteq \mathbb{RP}^3$ where $\mathbb{H}^3$ is the unit ball of $\mathbb{R}^3$; for the basic definitions see for example [1]. Notice that for convenience we have picked an affine chart $\mathbb{R}^3 \subseteq \mathbb{RP}^3$, so that it always make sense to speak of segments between two

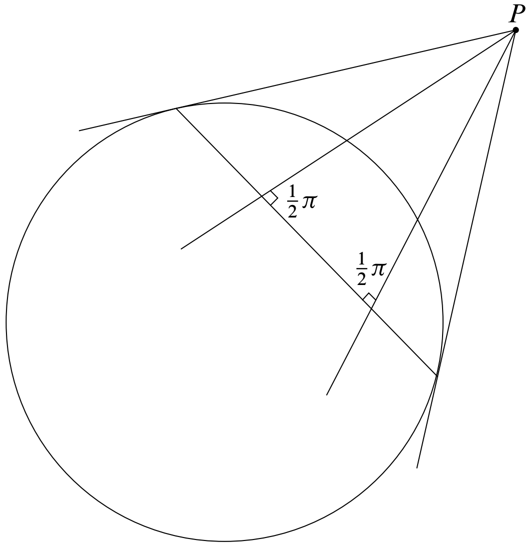


Figure 9: The dual of a point P.

points, half-spaces, et cetera; this choice is inconsequential, up to isometry. It should be mentioned that isometries, in this model, correspond to projective transformations that preserve the unit sphere.

The space $\mathbb{RP}^3$ has a duality that comes from the Minkowski scalar product on $\mathbb{R}^{3,1}$; using this we can associate to a point p lying in $\mathbb{R}^3 \setminus \overline{\mathbb{H}^3}$ a plane $\Pi_p \subseteq \mathbb{H}^3$, called the *polar plane* of p, such that all lines passing through $\mathbb{H}^3$ and p are orthogonal to Π_p (see Figure 9 for a 2-dimensional picture). If $p \in \mathbb{R}^3 \setminus \overline{\mathbb{H}^3}$, denote by $H_p \subseteq \mathbb{H}^3$ the half-space delimited by the polar plane Π_p on the other side of p; in other words, H_p contains $0 \in \mathbb{R}^3$. If the line from p to p' passes through $\mathbb{H}^3$, then Π_p and $\Pi_{p'}$ are disjoint [1, Lemma 4]. In particular, if the segment from p to p' intersects $\mathbb{H}^3$, then $\Pi_p \subseteq H_{p'}$ and $\Pi_{p'} \subseteq H_p$; if however the segment does not intersect $\mathbb{H}^3$, but the half-line from p to p' does, then $H_p \subseteq H_{p'}$. If p gets pushed away from $\mathbb{H}^3$, then Π_p gets pushed closer to the origin of $\mathbb{R}^3$.

Definition 3.1 A *projective polyhedron* in $\mathbb{RP}^3$ is a convex polyhedron in some affine chart of $\mathbb{RP}^3$. Alternatively, it is the closure of a connected component of the complement of finitely many planes in $\mathbb{RP}^3$ that does not contain any projective line.

Definition 3.2 Following [15, Definition 4.7]:

- We say that a projective polyhedron $P \subseteq \mathbb{R}^3 \subseteq \mathbb{RP}^3$ is a *generalized hyperbolic polyhedron* if each edge of P intersects $\mathbb{H}^3$.
- A vertex of a generalized hyperbolic polyhedron is *real* if it lies in $\mathbb{H}^3$, *ideal* if it lies in $\partial\mathbb{H}^3$ and *hyperideal* otherwise.
- A generalized hyperbolic polyhedron P is *proper* if for each hyperideal vertex v of P *the interior* of the polar half-space H_v contains all the other real vertices of P (see Figure 10).

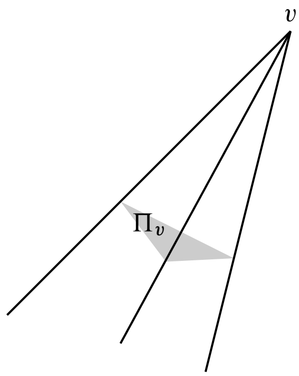


Figure 10: A proper vertex.

- We define the *truncation* of a generalized hyperbolic polyhedron P at a hyperideal vertex v to be the intersection of P with H_v; similarly the *truncation* of P is the truncation at every hyperideal vertex, that is to say $P \cap \left(\bigcap_{v \text{ hyperideal}} H_v\right)$. We say that the *volume* of P is the volume of its truncation. Notice that the volume of a nonempty generalized hyperbolic polyhedron could be 0 if the truncation is empty.

In the remainder of the paper we simply say *proper polyhedra* for proper generalized hyperbolic polyhedra.

When it has positive volume, the truncation of a generalized hyperbolic polyhedron P is itself a polyhedron; some of its faces are the truncations of the faces of P, while the others are the intersection of P with some truncating plane; we call such faces *truncation faces*. If an edge of the truncation of P lies in a truncation face we say that the edge arises from the truncation.

Remark 3.3 For proper polyhedra the dihedral angles at the edges arising from the truncation are $\frac{1}{2}\pi$.

Remark 3.4 If Γ can be embedded as the 1-skeleton of a projective polyhedron, then it is 3-connected (that is to say, it cannot be disconnected by removing two nonadjacent vertices). Furthermore, any 3-connected planar graph can be embedded as the 1-skeleton of a proper polyhedron [20]. If a planar graph is 3-connected, then it admits a unique embedding in S^2 (up to isotopies of S^2 and mirror symmetry) [13, Corollary 3.4]. Hence when in the following we consider a planar graph, it is always going to be 3-connected and embedded in S^2. In particular, it will make sense to talk about the dual of Γ, denoted by Γ^*. The graph Γ^* is the 1-skeleton of the cellular decomposition of S^2 dual to that of Γ.

Remark 3.5 It is important not to mix up the 1-skeleton of a projective polyhedron with the 1-skeleton of its truncation. In what follows, whenever we refer to 1-skeletons we always refer to those of projective polyhedra (and not their truncation) unless specified.

We propose the following formulation of the volume conjecture for polyhedra, generalizing the previously mentioned versions:

Conjecture 3 (the volume conjecture for polyhedra) *Let P be a proper polyhedron with dihedral angles $\alpha_1, \dots, \alpha_m$ at the edges $e_1, \dots, e_m$, and 1-skeleton Γ. Let col_r be a sequence of r-admissible colorings of the edges $e_1, \dots, e_m$ of Γ such that*

$$2\pi \lim_{r \to +\infty} \frac{\mathrm{col}_r(e_i)}{r} = \pi - \alpha_i.$$

Then

$$\lim_{r \to +\infty} \frac{\pi}{r} \log |Y_r(\Gamma, \mathrm{col}_r)| = \mathrm{Vol}(P).$$

Remark 3.6 In the case where P is a simple polyhedron in $\mathbb{H}^3$ (ie a compact polyhedron with only trivalent vertices) this conjecture is the same as the volume conjecture of Kolpakov and Murakami [17].

Conjecture 3 was verified in [6] for tetrahedra with at least one hyperideal vertex; we provide some further supporting numerical evidence for Conjecture 3 for some pyramids in the appendix, and prove it for a large family of examples in Proposition 4.8 and Remark 4.10 (however, only for a single sequence of colors).

Remark 3.7 Conjecture 3 would imply that Conjecture 2 is verified when restricted to colors which correspond to hyperbolic polyhedra.

3.2 The upper bound conjecture

In [5] the authors proved an upper bound on the growth of the $6j$-symbol. When stated in terms of the Yokota invariant of the tetrahedral graph T, the result is the following:

Theorem 3.8 *For any r and any r-admissible coloring* col *of the graph T, we have*

$$\frac{\pi}{r} \log |Y_r(T, \mathrm{col})| \le v_8 + O\left(\frac{\log(r)}{r}\right),$$

where $v_8 \sim 3.66$ is the volume of the regular ideal right-angled octahedron. Furthermore, this inequality is sharp, with the upper bound achieved at the 6-tuple $\left(\frac{1}{2}(r-2\pm 1), \dots, \frac{1}{2}(r-2\pm 1)\right)$ with the signs chosen so that $\frac{1}{2}(r-2\pm 1)$ is even.

It is natural to ask if a similar upper bound holds for other graphs. The reason the quantity v_8 is involved in the statement of Theorem 3.8 is that it is the upper bound of the volume of all proper tetrahedra. In [4] the author proved that, given a polyhedral graph Γ, the upper bound of the volume of all proper polyhedra with 1-skeleton Γ is equal to the volume of the rectification of Γ, denoted by $\overline{\Gamma}$ (see Remark 3.9). In light of this, we reword Conjecture 2 as follows:

Conjecture 4 *If Γ is a polyhedral graph and* col *is any r-admissible coloring of its edges, then*

$$\frac{\pi}{r} \log |Y_r(\Gamma, \mathrm{col})| \le \mathrm{Vol}(\overline{\Gamma}) + O\left(\frac{\log(r)}{r}\right).$$

Moreover, the inequality is sharp, with equality attained by the sequence of colorings giving the color $\frac{1}{2}(r-2\pm 1)$ to each edge (the sign is chosen so that the colors are even).

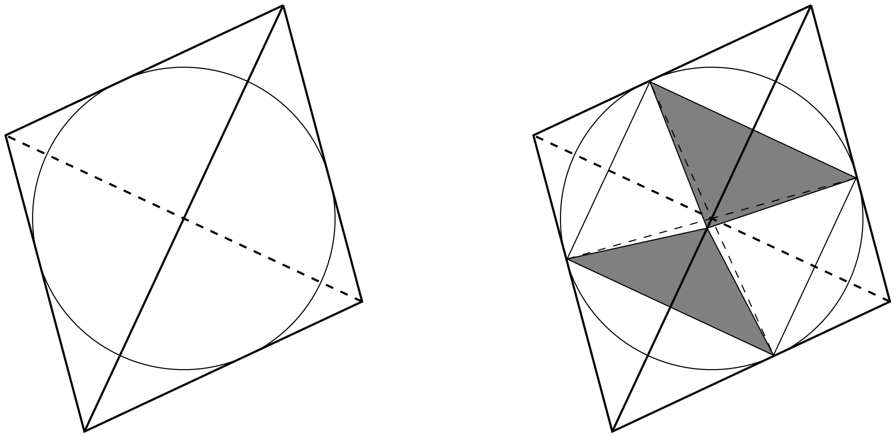


Figure 11: The rectification of a tetrahedron (left) and its truncation (right), the ideal right-angled octahedron. The gray faces arise from the truncation of the top and bottom vertices.

Remark 3.9 The rectification of Γ is defined as the unique projective polyhedron with 1-skeleton Γ and with every edge tangent to $\partial\mathbb{H}^3$ (see Figure 11). While $\overline{\Gamma}$ is not a proper (or even generalized hyperbolic) polyhedron, we can still speak of its truncation and its volume; for more details see [4, Section 3.4].

Remark 3.10 It would be natural to ask whether a similar upper bound would work for nonpolyhedral graphs; however in this case it is unclear what would be the geometric object to replace $\overline{\Gamma}$.

Theorem 4.9 *Conjecture 4 is verified for any planar graph obtained from the tetrahedron by applying any sequence of the following two moves*:

- *blowing up a trivalent vertex* (*see Figure 1*), *or*
- *triangulating a triangular face* (*see Figure 2*).

This theorem will be proven in Section 4.

4 The Fourier transform

In this section we prove Theorem 4.9. The first main tool used is Theorem 3.8.

The second is the Fourier transform introduced in [2] by Barrett. We describe it here in a slightly different context and notation.

Let $H \subseteq S^3$ be the 0-framed Hopf link as in Figure 12. For $i, j \in I_r$ we denote by $H(i, j) \in \mathbb{C}$ the value of the Kauffman bracket of the Hopf link colored with i, j; applying the relation of [18, Figure 22] and an easy induction on j shows that

$$H(i,j) = (-1)^{i+j}[(i+1)(j+1)] = (-1)^{i+j}\frac{\sin(2\pi(i+1)(j+1)/r)}{\sin(2\pi/r)}.$$

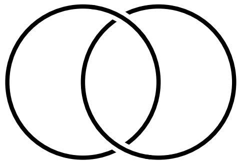


Figure 12: The 0-framed Hopf link.

Furthermore define

$$N := \frac{r}{4\sin^2(2\pi/r)} = \langle U, \Omega\rangle = \sum_{i\in I_r} \Delta_i^2,$$

where U is the 0-framed unknot in S^3 colored with $\Omega := \sum_{i\in I_r} \Delta_i i$; see [18, page 185].

Remark 4.1 Once again, we are using the SO(3) version of the invariants evaluated at $q = e^{2\pi i/r}$. However, the Fourier transform and its properties hold with any choice of primitive $2r^{\text{th}}$ root of unity, or any choice of primitive $4r^{\text{th}}$ root of unity for the SU(2) case; the proofs work verbatim in every other case.

Definition 4.2 The *Fourier transform* of $Y_r(\Gamma, \mathrm{col})$ is the invariant of the colored graph (Γ, col') given by the formula

$$\mathcal{F}_r(\Gamma, \mathrm{col}') = \sum_{\mathrm{col}\text{ coloring of }\Gamma} Y_r(\Gamma, \mathrm{col}) H(\mathrm{col}, \mathrm{col}'),$$

where

$$H(\mathrm{col}, \mathrm{col}') := \prod_{e\text{ edge of }\Gamma} H(\mathrm{col}(e), \mathrm{col}'(e^*)).$$

The following proposition was first noticed by Barrett in [2, Section 5]; a concise proof was later given in [3, Theorem 1]. For the sake of completeness, we include a detailed proof.

Proposition 4.3 *If Γ is a planar framed graph, Γ^* is its planar dual and col' is a coloring of the edges of Γ^*, then*

$$Y_r(\Gamma^*, \mathrm{col}') = N^{-g} \sum_{\mathrm{col}\text{ coloring of }\Gamma} Y_r(\Gamma, \mathrm{col}) H(\mathrm{col}, \mathrm{col}') = N^{-g} \mathcal{F}_r(\Gamma, \mathrm{col}'),$$

where g is the genus of a regular neighborhood of Γ.

Proof The proof is entirely diagrammatic; when we display an equality between (linear combinations of) diagrams, we mean that they have the same Kauffman bracket. Throughout the proof we will liberally add Ω-colored 0-framed unknots that are unlinked from anything else; this will generate an ambiguity of a power of N that we will account for at the end.

Step 1 Calculate $Y_r(\Gamma, \mathrm{col})$ as the Kauffman bracket of a certain framed colored link $L(\Gamma, \mathrm{col})$.

The colored link $L(\Gamma, \mathrm{col})$ is obtained from (Γ, col) as in Figure 13. Every vertex is replaced by a circle colored with Ω, and every edge is replaced by a circle colored with the same color as the edge, wrapping

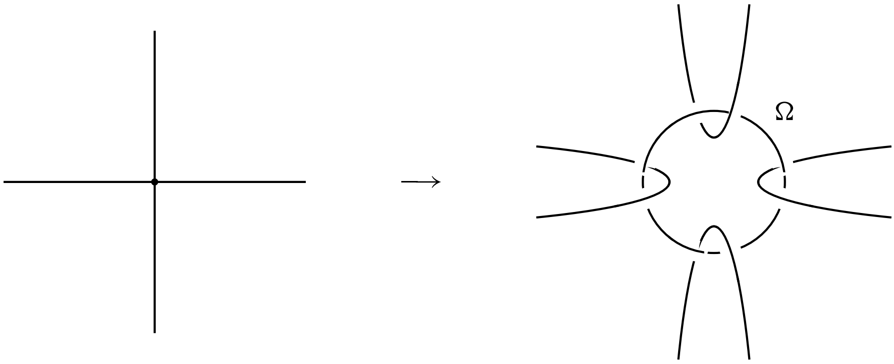
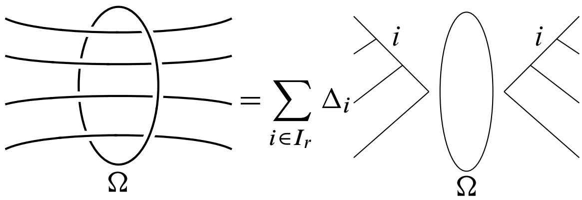


Figure 13: Obtaining $L(\Gamma, \mathrm{col})$ using the *chainmail rule*. Each circle has the same color as its corresponding edge, and it has two consecutive overcrossings and two consecutive undercrossings.

once around each of the two circles corresponding to its vertices in a minimally twisted way (ie the circle has two consecutive overcrossings and two consecutive undercrossings). Notice that the link itself only depends on Γ; only its coloring depends on col. Furthermore we can define the framing to be the blackboard framing of the diagram we just constructed.

The fact that $\langle L(\Gamma, \mathrm{col})\rangle = Y_r(\Gamma, \mathrm{col})$ can be shown by using the definition of Y_r after applying the following identity to L:

(4-1)



This holds for any number of strands; it is obtained by repeated application of the fusion rule followed by the well-known fact (see [18, Lemma 6]) that if a diagram contains the portion depicted in Figure 14 it is equal to 0 unless $i = 0$.

When passing from Γ to $L(\Gamma, \mathrm{col})$ we still speak of edges and vertices of $L(\Gamma, \mathrm{col})$: we mean the circles corresponding to edges and vertices of Γ, respectively. Slightly more improperly, we speak of faces of $L(\Gamma, \mathrm{col})$, by which we mean the portions of the plane delimited by edges of Γ. To do this, until the start of Step 3, we fix the diagram of $L(\Gamma, \mathrm{col})$ that we just created.

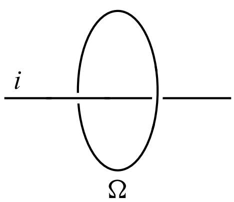


Figure 14

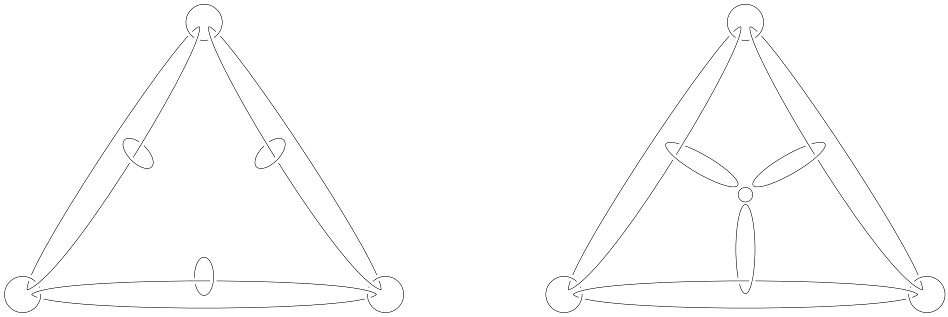


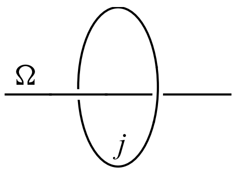
Figure 15: Stretching edges towards the center and adding an extra component.

Step 2 For a given coloring col' of Γ, calculate $\mathcal{F}_r(\Gamma, \mathrm{col}')$ as the Kauffman bracket of a link $\widehat{L}(\Gamma, \mathrm{col}')$.

The Fourier transform is given by the formula

$$\mathcal{F}_r(\Gamma, \mathrm{col}') = \sum_{\mathrm{col} \text{ coloring of } \Gamma} \langle L(\Gamma, \mathrm{col}) \rangle H(\mathrm{col}, \mathrm{col}').$$

We wish to express this formula as the bracket of a single colored link; to do so we use the following relationship (which can be proven via a particular case of the vertex sum formula from (3) after we introduce extra edges colored with 0):

(4-2) $$\sum_{i \in I_r} H(i, j) \underset{i}{\overline{\hspace{2cm}}} =$$ 

Therefore $\mathcal{F}_r(\Gamma, \mathrm{col}') = \langle \widehat{L}(\Gamma, \mathrm{col}') \rangle$, where $\widehat{L}(\Gamma, \mathrm{col}')$ is the colored link obtained from $L(\Gamma, \mathrm{col})$ by changing the color of each edge e of $L(\Gamma, \mathrm{col})$ to Ω and by adding a meridional circle around it colored with $\mathrm{col}'(e)$. We call the meridional circles added via this process the *transverse* circles; they will correspond to edges of Γ^*. Notice that this step only added circles, and did not otherwise change the link diagram we created in Step 1 (not even via planar isotopy).

Step 3 Manipulate $\widehat{L}(\Gamma, \mathrm{col}')$ to obtain $L(\Gamma^*, \mathrm{col}')$.

Take a face F of $\widehat{L}(\Gamma, \mathrm{col}')$, stretch the circles transverse to its edges so that they are close to the center of F and add an unknot U colored with Ω at the center of F (see Figure 15). Handleslide this new unknotted component along all the edges of F (see Figure 16); the result is that U gets linked to each transverse circle and remains unlinked from any edge or vertex of Γ as in Figure 17. Because the edges are colored with Ω this procedure does not change the Kauffman bracket (see for example [18, Corollary, page 181]). The circle U will correspond to a vertex in Γ^*. Repeat this procedure for every face of $\widehat{L}(\Gamma, \mathrm{col}')$; notice also that the procedure we just carried out only changes the link diagram in the portion of the plane corresponding to F.

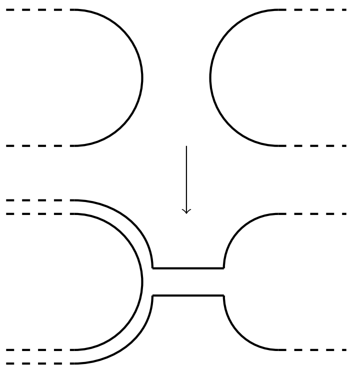


Figure 16: Handleslide between two different components of $L(\Gamma, \mathrm{col})$.

Now apply (4-1) to each circle corresponding to a vertex of Γ and each circle corresponding to a vertex of Γ^*. The result (see Figure 18) is going to be four connected graphs (and several unlinked unknots that for now we ignore), which lie in parallel planes and are therefore unlinked from each other. Two of these give $Y_r(\Gamma^*, \mathrm{col}')$ and two of these give $Y_r(\Gamma, \Omega)$ (where we still denote by Ω the coloring of Γ with color Ω on each edge).

Step 4 Prove that

$$Y_r(\Gamma, \Omega) = N^g.$$

To do this, recall that the Yokota invariant does not change when performing a Whitehead move on an edge colored with Ω; see Proposition 2.10(2). Further recall that a sequence of Whitehead moves can change any trivalent graph into any other trivalent graph with the same number of vertices; this is because

- two trivalent graphs with the same number of vertices also have the same number of faces,
- their duals are triangulations with the same number of vertices,
- their duals can be changed into one another via "edge flips" (see [14]),
- edge flips are dual to Whitehead moves,
- two planar graphs with isotopic duals are themselves isotopic by [25, Theorem 11].

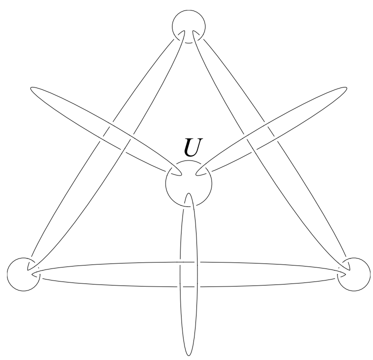


Figure 17: The central component U gets linked by handleslides.

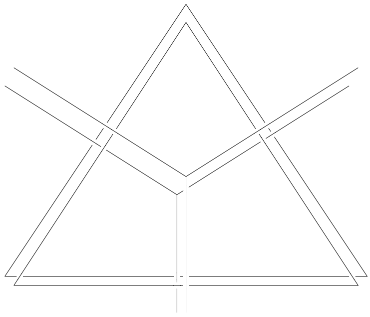


Figure 18: After applying (4-1), we get four unlinked graphs.

Therefore, we can desingularize and then apply Whitehead moves to Γ until it becomes a "bicycle" graph as in Figure 19, with some circles connected linearly by segments; since desingularizing and performing a Whitehead move do not change the genus of the regular neighborhood, there are g circles. Because of the bridge rule (v), the Kauffman bracket is 0 unless the color of every connecting edge is 0, and therefore

$$Y_r(\Gamma,\Omega) = \left(\sum_{i_1,\dots,i_g\in I_r} \Delta_{i_1}^2\cdots\Delta_{i_g}^2\right) = N^g.$$

Step 5 Account for the extra N factors.

At the beginning we added an unknot for each vertex of Γ, and then for each face. However when we applied the inverse of the chainmail relation we removed the exact same number of components; therefore there is no additional N factor. □

Proposition 4.4 *For any coloring* col *of a planar graph* Γ,

$$\frac{\pi}{r}\log|Y_r(\Gamma,\mathrm{col})| \le \max_{\mathrm{col}'}\frac{\pi}{r}\log|Y_r(\Gamma^*,\mathrm{col}')| + O\left(\frac{\log r}{r}\right),$$

where the maximum is taken over all r-admissible colorings of the dual graph Γ^*.

Proof Let $\mathrm{col}_{\max}$ be an r-admissible coloring of Γ^* such that $|Y_r(\Gamma^*,\mathrm{col}_{\max})|$ is maximum.

By Proposition 4.3,

$$\begin{aligned}\frac{\pi}{r}\log|Y_r(\Gamma,\mathrm{col})| &= \frac{\pi}{r}\log\Big|\sum_{\mathrm{col}'} H(\mathrm{col},\mathrm{col}')Y_r(\Gamma^*,\mathrm{col}')\Big| \\ &\le \frac{\pi}{r}\log\sum_{\mathrm{col}'}|H(\mathrm{col},\mathrm{col}')Y_r(\Gamma^*,\mathrm{col}_{\max})| = \frac{\pi}{r}\log|Y_r(\Gamma^*,\mathrm{col}_{\max})| + O\left(\frac{\log r}{r}\right),\end{aligned}$$

where the last equality stems from the fact that $\sum_{\mathrm{col}'} H(\mathrm{col},\mathrm{col}')$ grows polynomially in r. □

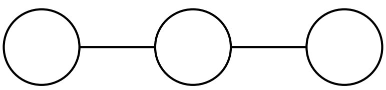


Figure 19: The bicycle with three wheels.

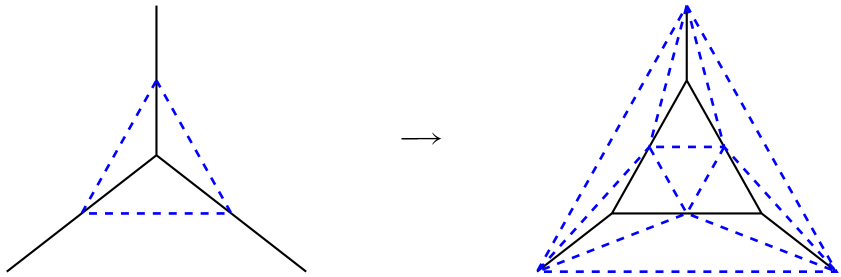


Figure 20: The 1-skeleton of the rectification is outlined with dashed lines; a blow-up of a vertex corresponds to gluing an octahedron to its truncation face.

Corollary 4.5 *Conjecture 4 is true for Γ if and only if it is true for Γ^*.*

Proof Corollary 4.6 of [4] states that $\mathrm{Vol}(\overline{\Gamma}) = \mathrm{Vol}(\overline{\Gamma^*})$; this and Proposition 4.4 imply the result. □

We now turn to the proof of Theorem 4.9. This will use a few intermediate propositions which we now state and prove.

We first calculate the volume of the rectification of the graphs at hand:

Proposition 4.6 *If Γ is obtained from the tetrahedron by a sequence of g blow-ups of vertices or triangulations of triangular faces, then*

$$\mathrm{Vol}(\overline{\Gamma}) = (g+1)v_8.$$

Proof The case of $g=0$ is well known and appears in [23, Theorem 4.2]. Take now any Γ obtained from Γ' by a blow-up of a vertex v and consider P the truncated rectification of Γ'. The vertex v corresponds to a truncation face of P: this face is an ideal triangle. Given an octahedron, we can glue it to P by identifying any of its faces to the face corresponding to v (since they are triangular faces the result does not depend on any choice). Notice that the gluing is done along an ideal triangular face, and along right dihedral angles. It is immediate to see that this gluing gives the truncation of $\overline{\Gamma}$: it has the correct 1-skeleton (see Figure 20) and it is right-angled. Therefore, by blowing up a vertex, the maximum volume grows by v_8. Dually, triangulating a triangular face makes the maximum volume grow by v_8 as well. □

Next, we prove the upper bound:

Proposition 4.7 *If Γ is obtained from the tetrahedron by a sequence of g blow-ups of vertices or triangulations of triangular faces, and* col *is any r-admissible coloring, then*

$$\frac{\pi}{r}\log|Y_r(\Gamma,\mathrm{col})| \le (g+1)v_8 + O\left(\frac{\log(r)}{r}\right).$$

Proof The base case $g=0$ is Theorem 3.8.

If Γ is obtained from Γ' as a blow-up of a single vertex, then

$$Y_r(\Gamma,\mathrm{col}) = Y_r(\Gamma',\mathrm{col}_1)Y_r(T,\mathrm{col}_2),$$

where T is a tetrahedron, and col_1 and col_2 are the colorings induced by col on Γ' and T, respectively. Therefore $Y_r(\Gamma,\mathrm{col}) \le Y_r(\Gamma',\mathrm{col}_1)Y_r(T,\mathrm{col}_2)$, and by induction

$$\frac{\pi}{r}\log|Y_r(\Gamma,\mathrm{col})| \le (g+1)v_8 + O\left(\frac{\log(r)}{r}\right).$$

By Proposition 4.4, this inequality also holds if Γ is obtained from Γ' by triangulating a single triangular face. □

Finally we prove the sharpness of the upper bound:

Proposition 4.8 *If Γ is obtained from the tetrahedron by a sequence of g blow-ups of vertices or triangulations of triangular faces, and* $\mathrm{col} = \left(\frac{1}{2}(r-2\pm 1),\dots,\frac{1}{2}(r-2\pm 1)\right)$ *— where the signs are chosen so that $r-2\pm 1$ is a multiple of 4 — then*

$$\lim_{r\to+\infty}\frac{\pi}{r}\log(Y_r(\Gamma,\mathrm{col})) = (g+1)v_8.$$

Proof The proof is by induction; the base case is Theorem 3.8. Suppose Γ is obtained from the tetrahedron by g blow-ups and triangulations, and at least one blow-up. Then Γ is a vertex sum of Γ_1 and Γ_2, with both graphs obtained from the tetrahedron via g_1 and g_2 blow-ups or triangulations, respectively, and $g_1+g_2 = g-1$. Since $Y_r(\Gamma,\mathrm{col}) = Y_r(\Gamma_1,\mathrm{col}_1)Y_r(\Gamma_2,\mathrm{col}_2)$ — with col_1 and col_2 the colorings induced by col on Γ_1 and Γ_2, respectively — we have

$$\begin{aligned}\lim_{r\to+\infty}\frac{\pi}{r}\log(Y_r(\Gamma,\mathrm{col})) &= \lim_{r\to+\infty}\frac{\pi}{r}\log(Y_r(\Gamma_1,\mathrm{col}_1)Y_r(\Gamma_2,\mathrm{col}_2))\\ &= (g_1+1+g_2+1)v_8 = (g+1)v_8.\end{aligned}$$

We need to deal with the case of Γ being obtained via g triangulations. In this case, Γ^* is obtained from the tetrahedron via g blow-ups. Apply the Fourier transform to $Y_r(\Gamma^*,\mathrm{col}')$:

$$Y_r(\Gamma,\mathrm{col}) = \sum_{\mathrm{col}'} H(\mathrm{col},\mathrm{col}')Y_r(\Gamma^*,\mathrm{col}').$$

However, since col is constantly $\frac{1}{2}(r-2\pm 1)$ and even, we have

$$\begin{aligned}H\left(\tfrac{1}{2}(r-2\pm 1), j\right) &= (-1)^j\frac{\sin\left((2\pi/r)\cdot\frac{1}{2}(r\pm 1)(j+1)\right)}{\sin(2\pi/r)}\\ &= (-1)^j\frac{\sin(\pi(j+1)\pm\pi(j+1)/r)}{\sin(2\pi/r)} = -\frac{\sin(\pm\pi(j+1)/r)}{\sin(2\pi/r)},\end{aligned}$$

which has $\mp$ sign since $0\le j\le r-1$. Moreover, since Γ^* is a trivalent graph, $Y_r(\Gamma^*,\mathrm{col}') = |\langle\Gamma^*,\mathrm{col}'\rangle|^2$ is nonnegative for every coloring; therefore $Y_r(\Gamma,\mathrm{col})$ is a sum with constant sign of $Y_r(\Gamma^*,\mathrm{col}')$ over all possible colorings. This shows that $Y_r(\Gamma,\mathrm{col})$ grows as the maximum growth of $Y_r(\Gamma^*,\mathrm{col}')$ over all colorings, which is $(g+1)v_8$. □

Putting Propositions 4.6, 4.7 and 4.8 together, we obtain the following theorem:

Theorem 4.9 *If Γ is obtained from the tetrahedron by a sequence of blow-ups of vertices or triangulations of triangular faces, then Conjecture 4 is verified.*

Remark 4.10 Proposition 4.8 actually proves Conjecture 3 for a large family of polyhedra (albeit for a single sequence of colors each) since the volume of a polyhedron with internal angles 0 is the volume of its rectification (notice how $(2\pi/r)\cdot\frac{1}{2}(r\pm 1-2)\to\pi$ as $r\to+\infty$).

5 The Turaev–Viro volume conjecture

In this section we apply Theorem 4.9 to prove the Turaev–Viro volume conjecture for an infinite family of examples.

Recall (for example from [18, Section 4.2]) that the Reshetikhin–Turaev invariant of a colored framed link L in a manifold M is defined as

$$\mathrm{RT}_r(M, L, \mathrm{col}) = \eta\kappa^{\sigma(L')}\langle L\sqcup L', \mathrm{col}\cup\Omega\rangle,$$

where

- $L'\subseteq S^3$ is a framed link giving M via Dehn surgery,
- $L\sqcup L'$ is the disjoint union of L' and L viewed as a subset of S^3 (if need be, after isotoping L to be disjoint from L'),
- the components of L' are all colored with Ω,
- $\sigma(L')$ is the signature of the linking matrix of $L'\subseteq S^3$,
- $\eta=\langle U,\Omega\rangle^{-1}=(A^2-A^{-2})/\sqrt{-2r}$, and
- $\kappa=\langle U_+,\Omega\rangle$, where U_+ is the unknot with framing equal to $+1$.

Proposition 5.1 *Let $\Gamma\subseteq S^3$ be a graph obtained from the tetrahedron by a sequence of $g-1$ blow-ups of vertices or triangulations of triangular faces as in the hypothesis of Theorem 4.9; let $e_1,\dots,e_k$ be its edges, and denote by h the number of vertices of Γ. Then there is a k-component link $L=L_1\sqcup\dots\sqcup L_k$ in $S^3\,\#^{h-1}\,(S^1\times S^2)$ such that for any coloring $\mathrm{col}\in I_r^k$ (seen both as a coloring of Γ and as a coloring of L) we have*

$$Y_r(\Gamma, \mathrm{col}) = \mathrm{RT}_r(S^3\,\#^{h-1}\,(S^1\times S^2), L, \mathrm{col}).$$

Proof We have seen in the proof of Proposition 4.3 (specifically, in Step 1) that there is a way to associate to any (Γ,col) a colored framed link $L(\Gamma,\mathrm{col})$ in S^3 such that $Y_r(\Gamma,\mathrm{col})=\langle L(\Gamma,\mathrm{col})\rangle$. The link $L(\Gamma,\mathrm{col})$ is a link with $k+h$ components; k of these components are in bijection with the edges of Γ and are colored with the corresponding color of col. The other h are unknotted components in bijection with the vertices of Γ and are colored with Ω. The link $L(\Gamma,\mathrm{col})$ — without its coloring — almost satisfies the requirements we desire; however it has one too many components.

We now want to remove a component from $L(\Gamma, \mathrm{col})$; we do this so that the end result of the proposition is a link in $S^3 \#^{h-1} (S^1 \times S^2)$ rather than a link in $S^3 \#^h (S^1 \times S^2)$.

Pick an equatorial S^2 in S^3 and isotope $L(\Gamma, \mathrm{col})$ so that all its Ω-colored components lie flat on it, and all other components intersect the S^2 twice; the fact that this can be done is evident from the construction of $L(\Gamma, \mathrm{col})$. Each Ω-colored component will bound a disk that contains the intersection of its edges with S^2; every intersection lands inside one of these disks. Pick a component of $L(\Gamma, \mathrm{col})$ colored with Ω: it is possible to handleslide it along each other Ω-colored component without modifying the Kauffman bracket (by [18, Corollary, page 181]). Once one such handleslide is performed, the new curve will bound a disk that contains the intersection points of both families of curves. Repeating this procedure and handlesliding the chosen component over all others will make it bound a disk containing all transverse intersection points of $L(\Gamma, \mathrm{col})$ with the plane, thus making it unlinked from everything; therefore $\langle L(\Gamma, \mathrm{col})\rangle = \langle U\rangle\langle L(\Gamma, \mathrm{col})'\rangle = \eta^{-1}\langle L(\Gamma, \mathrm{col})'\rangle$, where U is an unknotted unlinked component colored with Ω and $L(\Gamma, \mathrm{col})'$ is the remaining part of the colored link. By the definition of the Reshetikhin–Turaev invariant of links

$$\langle L(\Gamma, \mathrm{col})'\rangle = \eta\, \mathrm{RT}_r(S^3 \#^{h-1} (S^1 \times S^2), L, \mathrm{col}),$$

where L is the link obtained from $L(\Gamma, \mathrm{col})'$ by doing a 0-framed Dehn surgery on the components of $L(\Gamma, \mathrm{col})$ colored with Ω. Notice that L only depends on Γ and not on the coloring. $\square$

Definition 5.2 We denote the link constructed in Proposition 5.1 by $K(\Gamma)$ — notice that this is a link rather than a colored link. The next several propositions explore the geometric properties of $K(\Gamma)$, culminating in proving the Turaev–Viro volume conjecture for it.

Proposition 5.3 *Let $\Gamma \subseteq S^3$ be a graph obtained from the tetrahedron by a sequence of $g-1$ blow-ups of vertices or triangulations of triangular faces; suppose Γ has k edges and h vertices. Then $L := K(\Gamma)$ is hyperbolic, and the hyperbolic structure on its complement is obtained by gluing $4g$ right-angled hyperbolic ideal octahedra.*

Proof Let $\overline{\Gamma}$ be the rectification of Γ, and let P be its truncation. We have seen in the proof of Theorem 4.9 that P can be obtained by gluing g right-angled hyperbolic octahedra. Take two copies of P and glue them along each corresponding truncation face. This gives a manifold homeomorphic to a handlebody of genus $h-1$ with some annuli removed from the boundary (corresponding to the ideal vertices of P); the decomposition into octahedra makes it into a finite-volume manifold M with geodesic boundary. Take the double of M along the geodesic boundary: this gives a manifold N which is homeomorphic to $S^3 \#^{h-1} (S^1 \times S^2) \setminus L$.

To see this, take an octahedron O and truncate a small link of each of its vertices. This truncation can be seen as the basic building block of the fundamental shadow links (see Figure 21 and [11, Proposition 3.33]): each truncated vertex corresponds to an arc, four of the faces of the octahedron correspond to the discs and the remaining four faces correspond to the regions of the spheres delimited by the arcs.

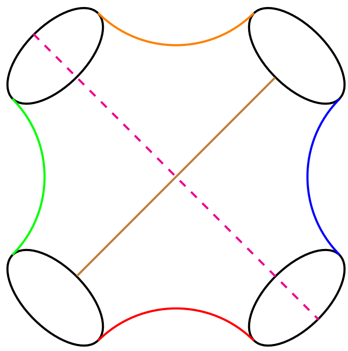


Figure 21: The building block: a ball with 4 disks in its boundary, and six arcs connecting them.

The polyhedron P is obtained by gluing octahedra together following a certain pattern; we can glue the building blocks in the same pattern. The result of this gluing is a ball with h discs on its boundary and some arcs connecting the discs. If we take the double of this ball along the discs, we obtain a genus $h-1$ handlebody with a link in its boundary. The link $L(\Gamma, \mathrm{col})$ corresponds to the link on the boundary of the handlebody plus $h-1$ components corresponding to the boundary of the gluing disks (after pushing them out of the handlebody slightly).

Doubling this handlebody is equivalent to performing 0-surgery on each of these $h-1$ components in S^3; therefore by doing this we obtain $S^3 \#^{h-1} (S^1 \times S^2)$ as ambient manifold and the link in the boundary gives L. □

Proposition 5.4 *Let Γ be a graph obtained from the tetrahedron by a sequence of $g-1$ blow-ups of vertices or triangulations of triangular faces; let t be the maximal number of disjoint triangular faces in the truncation of $\overline{\Gamma}$. Let $L := K(\Gamma)$, and E_L be its complement. Then E_L contains at most $t + 2g - 2$ disjoint geodesic thrice-punctured spheres.*

Proof The reasoning in this proof is similar to the proof of [9, Proposition 3.4].

Let P be the truncation of $\overline{\Gamma}$; we have seen in the proof of Proposition 5.3 that E_L is obtained by doubling P along the truncation faces (to obtain a hyperbolic manifold with geodesic boundary H) and doubling again along the geodesic boundary.

The truncation faces of P can be colored with black and the remaining with white; this way two faces of the same color never share an edge.

The proof of Proposition 5.3 shows that E_L decomposes into octahedra; take O an octahedron in this decomposition, and let S be any thrice-punctured sphere.

Claim *$S \cap O$ is either the empty set or a facet of O.*

We will prove the claim later; for now let us see how this concludes the proof.

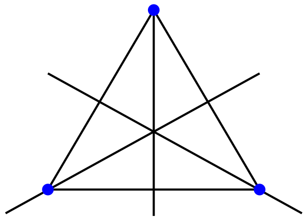


Figure 22: The six geodesics in a thrice punctured sphere cutting it into triangles.

Let $\mathcal{S}$ be a set of disjoint thrice-punctured spheres. This determines a set of disjoint ideal triangles in each of the four copies of P that make up E_L; some of them are in the boundary of a polyhedron while some of them are properly embedded. Each polyhedron contains exactly $g-1$ disjoint properly embedded geodesic triangles (the ones that decompose P into octahedra). These glue up to give $2g-2$ disjoint thrice-punctured spheres in E_L. Furthermore, a disjoint collection $T_1, \dots, T_t$ of triangles in ∂P induces a set of disjoint thrice-punctured spheres. Therefore there are at most $2g-2+t$ disjoint thrice-punctured spheres in E_L.

Proof of the claim We first look at $S \cap O$ as a subset of S. It must be a convex region of S delimited by geodesics. Since S contains exactly six maximal embedded geodesics (since it contains no closed geodesics and maximal embedded geodesics are determined by the cusp in which they end) the possible configurations are easy to list. Figure 22 shows the six geodesics cutting S into triangles; the possibilities for $S \cap O$ can be obtained by looking at all the possible ways to glue these triangles to obtain a convex set. The convex subsets of S obtained by gluing triangle regions are

(1) a triangle with one ideal vertex (a single triangle region),

(2) a triangle with two ideal vertices (gluing two triangle regions without an ideal vertex in common),

(3) a square with one ideal vertex and two right angles (gluing two triangle regions with an ideal vertex in common),

(4) a triangle with two ideal vertices and a right angle (gluing a triangle region to the triangle in (2)),

(5) a square with two ideal vertices (gluing two triangles in (2) along a geodesic),

(6) a bigon with one ideal point in its interior (gluing all triangle regions sharing an ideal vertex),

(7) a bigon with one ideal point in its boundary (gluing two triangle regions that have all the edges on the same geodesics),

(8) a region with three ideal points (obtained in several possible ways).

Every other possible way of gluing together the triangle regions of Figure 22 does not give a convex subset.

On the other hand, $S \cap O$ as a subset of O must coincide with the intersection of O with a plane $\Pi \subseteq \mathbb{H}^3$; therefore it cannot be either a bigon with an ideal point in its interior (6) or a bigon with an ideal point

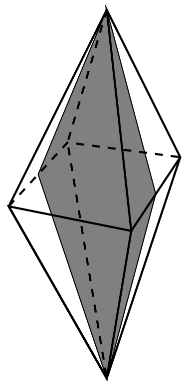


Figure 23: A square arising as the intersection of a thrice-punctured sphere and an octahedron of E_L.

in its boundary (7), since these regions cannot be realized as a hyperbolic polygon in $\mathbb{H}^3$ (hence, neither in O). Moreover, $\Pi \cap O$ cannot be a triangle with one or two ideal vertices (this excludes (1), (2) and (4)), nor can it be a square with one ideal vertex and two right angles (3), since none of these configurations can be realized as intersections of a plane with O. The remaining possibilities are that $S \cap O$ is a region with three ideal points (8), a square with two ideal vertices ((5), see Figure 23), or a facet of dimension 0 or 1. However by construction O is glued to at least three octahedra which are different from O and each other; therefore the case of a square with two ideal vertices is impossible since the intersection of S with these octahedra must also be a square with two ideal vertices, which would contradict the fact that S is a thrice-punctured sphere. Finally there are no properly embedded totally geodesic surfaces with exactly three ideal points in O; therefore the only possibility is that it is a face of O. To sum up, the only possible cases are that $S \cap O$ (when nonempty) is a vertex, an edge or a face of O. Therefore $S \cap O$ must be a facet of O. □

Remark 5.5 If M is the exterior of a fundamental shadow link with volume $2nv_8$, then it contains exactly $2n$ disjoint geodesic thrice-punctured spheres. This can be used to show that some of the exterior of the links provided by Proposition 5.1 are not homeomorphic to exteriors of fundamental shadow links; the simplest such example is the link associated to the graph shown in Figure 24. An easy check shows that the truncation of $\overline{\Gamma}$ contains at most six disjoint triangular faces: they correspond to the truncation faces of the three vertices on the left half of the picture, and to the three triangular faces on the right half of the picture. This means that (by Proposition 5.4) E_L contains at most 10 thrice-punctured spheres and has volume $12v_8$; on the other hand a fundamental shadow link complement with the same volume as E_L must contain 12 such spheres.

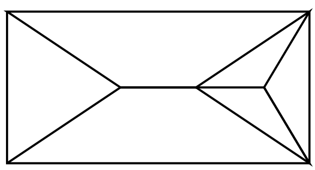


Figure 24: A graph whose link is not a fundamental shadow link.

More generally, if Γ is obtained from the tetrahedron through at least one triangulation and at least one blow-up, then the associated link exterior is not diffeomorphic to the exterior of a fundamental shadow link (and there is at least one such manifold of volume $4nv_8$ for each $n>1$).

Theorem 5.6 *Let $\Gamma \subseteq S^3$ be a graph obtained from the tetrahedron by a sequence of $g-1$ blow-ups of vertices or triangulations of triangular faces; suppose Γ has k edges and h vertices. Take the k-component link $L := K(\Gamma) \subseteq S^3 \#^{h-1} (S^1 \times S^2)$. Then the Turaev–Viro volume conjecture (Conjecture 1) holds for the exterior of L.*

Proof Theorem 4.9 implies that for any choice of r-admissible coloring col,

$$\frac{\pi}{r}\log|\mathrm{RT}_r(S^3 \#^{h-1} (S^1\times S^2), L, \mathrm{col})| = \frac{\pi}{r}\log|Y_r(\Gamma,\mathrm{col})| \le g v_8 + O\left(\frac{\log(r)}{r}\right).$$

The equality is a consequence of Proposition 5.1; the subsequent inequality is the content of Theorem 4.9. Furthermore if we denote by c the coloring $\left(\frac{1}{2}(r\pm 1),\dots,\frac{1}{2}(r\pm 1)\right)$ (where the sign is chosen so that the color is always even), we have

$$\frac{\pi}{r}\log|\mathrm{RT}_r(S^3 \#^{h-1} (S^1\times S^2), L, c)| = \frac{\pi}{r}\log|Y_r(\Gamma,c)| = g v_8 + O\left(\frac{\log(r)}{r}\right)$$

because of Proposition 4.8.

If E_L is the exterior of L, then

$$\mathrm{TV}_r(E_L) = \sum_{\mathrm{col}\in I_r^k} |\mathrm{RT}_r(S^3 \#^{h-1} (S^1\times S^2), L, \mathrm{col})|^2$$

by [5, Proposition 5.3], and $\mathrm{Vol}(E_L) = 4g v_8$ by Proposition 5.3.

This implies the result since

$$\lim_{r\to\infty} \frac{2\pi}{r}\log(\mathrm{TV}_r(E_L)) = 4g v_8,$$

because the sum in the formula for $\mathrm{TV}_r(E_L)$ has polynomially many terms all with the same sign. □

Remark 5.7 There is an overlap between Theorem 5.6 and [5, Theorem 1.1]. Some links of Theorem 5.6 are also fundamental shadow links (FSL); namely, those links corresponding to graphs obtained from the tetrahedron by blow-ups. However as we have seen in Remark 5.5 (infinitely) many others are not.

Appendix Numerical evidence for Conjecture 3

Supporting evidence for Conjecture 3 in the case of simple polyhedra can be found in [17]. In this appendix we show numerical computations supporting the conjecture for the square and pentagonal pyramids; all the calculations are performed with Mathematica. The notebook is available on GitHub at https://github.com/Giulio451/UpperBound; all calculations were performed on a Dell XPS 13 laptop.

The ideal regular square pyramid By Bao and Bonahon [1, Theorem 1] there is a unique square pyramid such that the angles at the base are $\frac{1}{4}\pi$ and the vertical angles are $\frac{1}{2}\pi$. Such a pyramid is ideal and

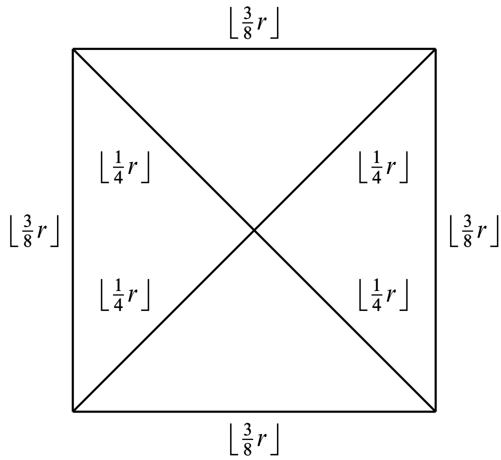


Figure 25: The coloring of a square pyramid associated to the ideal regular pyramid.

is maximally symmetric; it is decomposed into two ideal tetrahedra with angles $\frac{1}{4}\pi$, $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$; hence its hyperbolic volume is equal to $4\Lambda\left(\frac{1}{4}\pi\right) = \frac{1}{2}v_8 \sim 1.83193$ (where Λ is the Lobachevsky function). Consider the coloring of Figure 25; it converges to the angles of the ideal pyramid in the sense of Conjecture 3.

Its Yokota invariant can be calculated by desingularizing the 4-valent vertex and by using the vertex sum formula; it is given by

$$\sum_{k \in I_r} \Delta_k \begin{vmatrix} \lfloor \frac{1}{4}r \rfloor & \lfloor \frac{1}{4}r \rfloor & k \\ \lfloor \frac{3}{8}r \rfloor & \lfloor \frac{3}{8}r \rfloor & \lfloor \frac{3}{8}r \rfloor \end{vmatrix}^4,$$

where $\lfloor x \rfloor$ is the floor of x. The growth is shown in Figure 26.

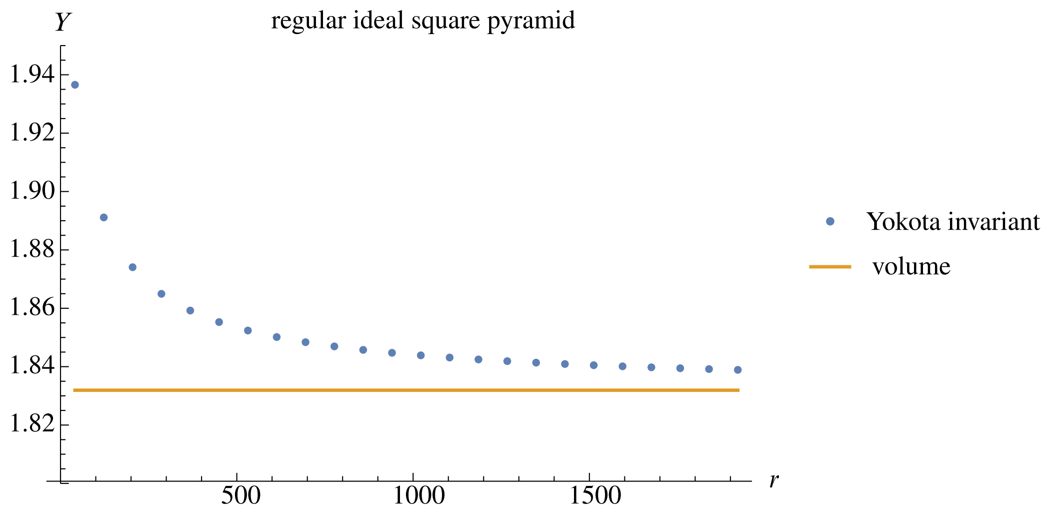


Figure 26

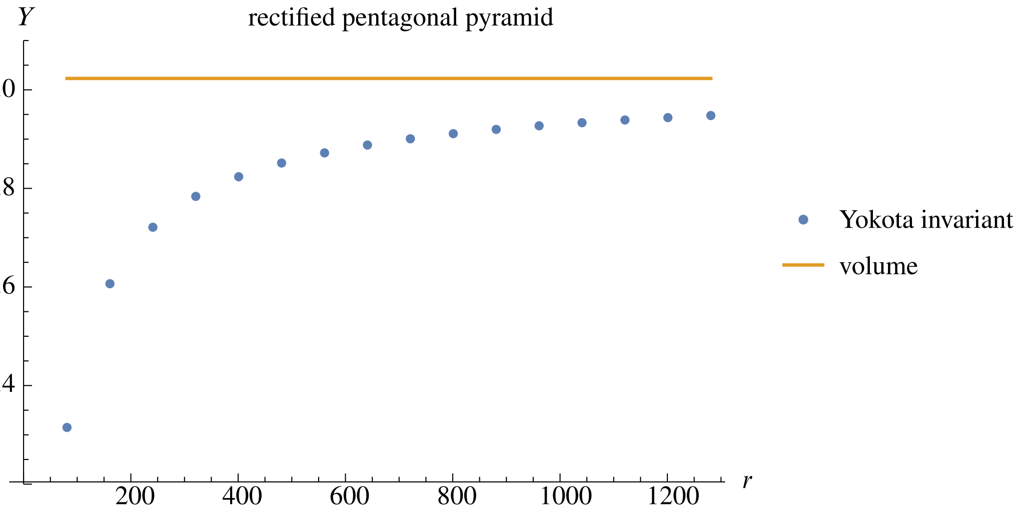


Figure 27

The 0-angled squared pyramid Because of [4, Theorem 4.2], the square pyramid with every dihedral angle equal to 0 exists and attains the maximum volume of any square pyramid (it is in fact the rectified pyramid). Its truncation is the right-angled ideal square antiprism. The volume of a right-angled ideal antiprism with n-gonal face is given by [21, page 151]

$$2n\left(\Lambda\left(\frac{\pi}{4}+\frac{\pi}{2n}\right)+\Lambda\left(\frac{\pi}{4}-\frac{\pi}{2n}\right)\right),$$

and for $n=4$ this gives ~ 6.02305.

Color the pyramid with $\lfloor \frac{1}{2}r \rfloor$ at every vertex; this coloring converges to the angles of the rectified pyramid. Its Yokota invariant is given by

$$\sum_{k\in I_r} \Delta_k \begin{vmatrix} \lfloor \frac{1}{2}r \rfloor & \lfloor \frac{1}{2}r \rfloor & k \\ \lfloor \frac{1}{2}r \rfloor & \lfloor \frac{1}{2}r \rfloor & \lfloor \frac{1}{2}r \rfloor \end{vmatrix}^4,$$

and its growth is shown in Figure 27.

The ideal regular pentagonal pyramid As before there is a unique ideal pentagonal pyramid with vertical angles $\frac{3}{5}\pi$ and base angles $\frac{1}{5}\pi$; this pyramid is maximally symmetric. We can decompose it into three ideal tetrahedra, two with dihedral angles $\frac{1}{5}\pi$, $\frac{1}{5}\pi$ and $\frac{3}{5}\pi$ and the remaining with dihedral angles $\frac{1}{5}\pi$, $\frac{2}{5}\pi$ and $\frac{2}{5}\pi$. Its volume then is

$$5\Lambda\left(\tfrac{1}{5}\pi\right)+2\Lambda\left(\tfrac{2}{5}\pi\right)+\Lambda\left(\tfrac{3}{5}\pi\right)\sim 2.49339.$$

Consider the coloring in Figure 28, converging to the angles of the ideal pyramid. Its Yokota invariant can be calculated (by desingularization and the vertex sum formula) as

$$\sum_{k,j\in I_r} \Delta_k\Delta_j \left|\left(\begin{vmatrix} \lfloor \frac{2}{5}r \rfloor & \lfloor \frac{2}{5}r \rfloor & k \\ \lfloor \frac{1}{5}r \rfloor & \lfloor \frac{1}{5}r \rfloor & \lfloor \frac{1}{5}r \rfloor \end{vmatrix} \begin{vmatrix} \lfloor \frac{2}{5}r \rfloor & \lfloor \frac{2}{5}r \rfloor & j \\ \lfloor \frac{1}{5}r \rfloor & \lfloor \frac{1}{5}r \rfloor & \lfloor \frac{1}{5}r \rfloor \end{vmatrix} \begin{vmatrix} \lfloor \frac{2}{5}r \rfloor & k & j \\ \lfloor \frac{1}{5}r \rfloor & \lfloor \frac{1}{5}r \rfloor & \lfloor \frac{1}{5}r \rfloor \end{vmatrix}\right)\right|^2,$$

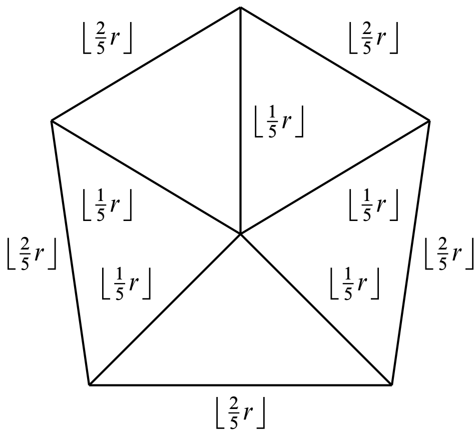


Figure 28: The coloring of the pentagonal pyramid corresponding to an ideal regular pyramid.

and its growth is shown in Figure 29.

The 0-angled pentagonal pyramid The truncation of the rectified pentagonal pyramid is the pentagonal antiprism, whose volume is ~ 8.13789, and the corresponding Yokota invariant is

$$\sum_{k,j \in I_r} \Delta_k \Delta_j \left| \left(\begin{vmatrix} \lfloor \frac{1}{4} r \rfloor & \lfloor \frac{1}{4} r \rfloor & k \\ \lfloor \frac{1}{4} r \rfloor & \lfloor \frac{1}{4} r \rfloor & \lfloor \frac{1}{4} r \rfloor \end{vmatrix} \begin{vmatrix} \lfloor \frac{1}{4} r \rfloor & \lfloor \frac{1}{4} r \rfloor & j \\ \lfloor \frac{1}{4} r \rfloor & \lfloor \frac{1}{4} r \rfloor & \lfloor \frac{1}{4} r \rfloor \end{vmatrix} \begin{vmatrix} \lfloor \frac{1}{4} r \rfloor & k & j \\ \lfloor \frac{1}{4} r \rfloor & \lfloor \frac{1}{4} r \rfloor & \lfloor \frac{1}{4} r \rfloor \end{vmatrix} \right) \right|^2.$$

Because of the greater range of the sum, it is considerably slower to compute than the other examples; we were only able to arrive to level $r = 321$, and the Yokota invariant is within 4% of the volume, as can be seen in Figure 30. However this is similar to the error (at level 321) in the previous examples.

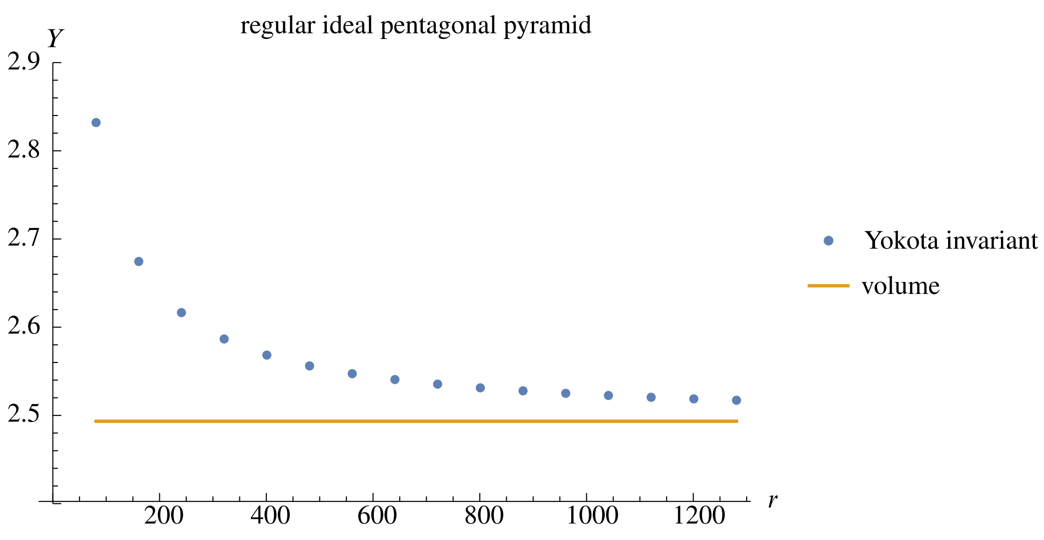


Figure 29

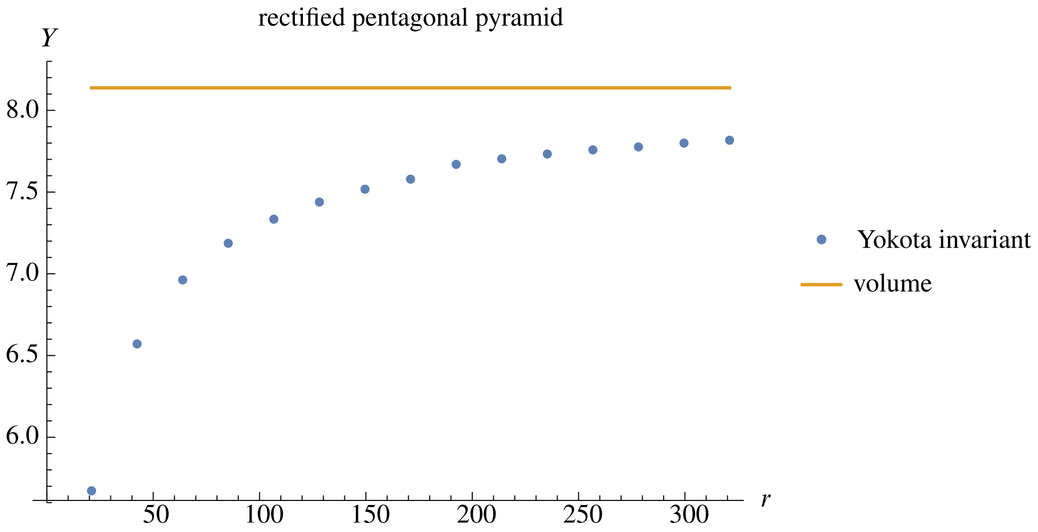


Figure 30

References

[1] **X Bao**, **F Bonahon**, *Hyperideal polyhedra in hyperbolic 3-space*, Bull. Soc. Math. France 130 (2002) 457–491 MR Zbl

[2] **J W Barrett**, *Geometrical measurements in three-dimensional quantum gravity*, Int. J. Modern Phys. A 18 (2003) 97–113 MR Zbl

[3] **J W Barrett**, **J Faria Martins**, **J M García-Islas**, *Observables in the Turaev–Viro and Crane–Yetter models*, J. Math. Phys. 48 (2007) art. id. 093508 MR Zbl

[4] **G Belletti**, *The maximum volume of hyperbolic polyhedra*, Trans. Amer. Math. Soc. 374 (2021) 1125–1153 MR Zbl

[5] **G Belletti**, **R Detcherry**, **E Kalfagianni**, **T Yang**, *Growth of quantum $6j$-symbols and applications to the volume conjecture*, J. Differential Geom. 120 (2022) 199–229 MR Zbl

[6] **Q Chen**, **J Murakami**, *Asymptotics of quantum $6j$ symbols*, J. Differential Geom. 123 (2023) 1–20 MR Zbl

[7] **Q Chen**, **T Yang**, *Volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants*, Quantum Topol. 9 (2018) 419–460 MR Zbl

[8] **F Costantino**, *$6j$-symbols, hyperbolic structures and the volume conjecture*, Geom. Topol. 11 (2007) 1831–1854 MR Zbl

[9] **F Costantino**, **R Frigerio**, **B Martelli**, **C Petronio**, *Triangulations of 3-manifolds, hyperbolic relative handlebodies, and Dehn filling*, Comment. Math. Helv. 82 (2007) 903–933 MR Zbl

[10] **F Costantino**, **F Guéritaud**, **R van der Veen**, *On the volume conjecture for polyhedra*, Geom. Dedicata 179 (2015) 385–409 MR Zbl

[11] **F Costantino**, **D Thurston**, *3-manifolds efficiently bound 4-manifolds*, J. Topol. 1 (2008) 703–745 MR Zbl

[12] **R Detcherry**, **E Kalfagianni**, **T Yang**, *Turaev–Viro invariants, colored Jones polynomials, and volume*, Quantum Topol. 9 (2018) 775–813 MR Zbl

[13] **H Fleischner**, *The uniquely embeddable planar graphs*, Discrete Math. 4 (1973) 347–358 MR Zbl

[14] **A Hatcher**, *On triangulations of surfaces*, Topology Appl. 40 (1991) 189–194 MR Zbl

[15] **C D Hodgson**, **I Rivin**, *A characterization of compact convex polyhedra in hyperbolic 3-space*, Invent. Math. 111 (1993) 77–111 MR Zbl

[16] **L H Kauffman**, **S L Lins**, *Temperley–Lieb recoupling theory and invariants of 3-manifolds*, Ann. of Math. Stud. 134, Princeton Univ. Press (1994) MR Zbl

[17] **A Kolpakov**, **J Murakami**, *Combinatorial decompositions, Kirillov–Reshetikhin invariants, and the volume conjecture for hyperbolic polyhedra*, Exp. Math. 27 (2018) 193–207 MR Zbl

[18] **W B R Lickorish**, *The skein method for three-manifold invariants*, J. Knot Theory Ramifications 2 (1993) 171–194 MR Zbl

[19] **T Ohtsuki**, *On the asymptotic expansion of the quantum* $\mathrm{SU}(2)$ *invariant at* $q = \exp(4\pi\sqrt{-1}/N)$ *for closed hyperbolic 3-manifolds obtained by integral surgery along the figure-eight knot*, Algebr. Geom. Topol. 18 (2018) 4187–4274 MR Zbl

[20] **E Steinitz**, *Polyeder und Raumeinteilungen*, from "Geometries" (F Klein, W Meyer, editors), Encycl. Math. Wiss. 3, Teubner, Leipzig (1922) 1–139

[21] **W P Thurston**, *The geometry and topology of three-manifolds*, lecture notes, Princeton University (1979) Available at `https://url.msp.org/gt3m`

[22] **V G Turaev**, **O Y Viro**, *State sum invariants of 3-manifolds and quantum* $6j$*-symbols*, Topology 31 (1992) 865–902 MR Zbl

[23] **A Ushijima**, *A volume formula for generalised hyperbolic tetrahedra*, from "Non-Euclidean geometries", Math. Appl. (N.Y.) 581, Springer (2006) 249–265 MR Zbl

[24] **R van der Veen**, *The volume conjecture for augmented knotted trivalent graphs*, Algebr. Geom. Topol. 9 (2009) 691–722 MR Zbl

[25] **H Whitney**, *Non-separable and planar graphs*, Proc. Nat. Acad. Sci. USA 17 (1931) 125–127 Zbl

[26] **K H Wong**, **T Yang**, *On the volume conjecture for hyperbolic Dehn-filled 3-manifolds along the figure-eight knot*, preprint (2020) arXiv 2003.10053

[27] **Y Yokota**, *Topological invariants of graphs in 3-space*, Topology 35 (1996) 77–87 MR Zbl

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Arithmetic representations of mapping class groups

EDUARD LOOIJENGA

Let S be a closed oriented surface and G a finite group of orientation-preserving automorphisms of S whose orbit space has genus at least two. There is a natural group homomorphism from the G-centralizer in $\mathrm{Diff}^+(S)$ to the G-centralizer in $\mathrm{Sp}(H_1(S))$. We give a sufficient condition for its image to be a subgroup of finite index.

57K20, 57M12; 11E39

1 Introduction and statement of the main result

Let S be a closed connected oriented surface of genus ≥ 2. The group $\mathrm{Diff}^+(S)$ of orientation-preserving diffeomorphisms of S acts on $H_1(S)$ via its connected component group $\pi_0(\mathrm{Diff}^+(S))$, known as the *mapping class group* of S, and it is a classical fact that the image of this representation is the full symplectic group $\mathrm{Sp}(H_1(S))$ of integral linear transformations which preserve the intersection form on $H_1(S)$. This paper concerns an equivariant version, where it is assumed that we are given a finite subgroup $G \subset \mathrm{Diff}^+(S)$. The centralizer $\mathrm{Diff}^+(S)^G$ of G in $\mathrm{Diff}^+(S)$ lands under the above symplectic representation in $\mathrm{Sp}(H_1(S))^G$ and the question we address here is how much smaller the image is. Besides its intrinsic interest, the answer has consequences for understanding the mapping class group of the G-orbit space of S. We shall regard the latter as an orbifold surface and denote it by S_G; the regular orbits then define an open subset $S_G^\circ \subset S_G$ with finite complement. This punctured surface S_G° has negative Euler characteristic. The image of $\mathrm{Diff}^+(S)^G$ in the mapping class group of the punctured surface S_G° is of finite index and thus makes $\mathrm{Sp}(H_1(S))$ a "virtual representation" of that mapping class group.

The work of Putman and Wieland [6] relates our question to the Ivanov conjecture as follows. Let us say that the G-action on S has the *Putman–Wieland property* if $\mathrm{Diff}^+(S)^G$ has no finite orbit in $H_1(S) \smallsetminus \{0\}$. These authors prove that if that property holds for a given genus h of S_G (no matter what S and G are), then every finite-index subgroup of a mapping class group of a connected oriented surface of finite type of genus $> h$ has zero first Betti number. The first part of our main result is about that property.

Theorem 1.1 *Let $S \to S_G$ be a G-cover as above.*

(i) *If this cover is trivial over a compact genus-one subsurface of S_G° with connected boundary, then the action of $\mathrm{Diff}^+(S)^G$ on $H_1(S)$ has no nonzero finite orbits.*

(ii) *If this cover is trivial over a compact genus-two subsurface of S_G° with connected boundary, then the image of $\mathrm{Diff}^+(S)^G$ in $\mathrm{Sp}(H_1(S))^G$ is of finite index.*

We will also obtain an arithmeticity property in the setting of (i). See the discussion and the end of this introduction as well as Remark 5.6.

Remark 1.2 In either case the cover over the complement of such a subsurface of S_G must be connected (for the subsurface has a connected boundary and S is connected). Since this complement has a boundary component over which the covering is trivial, we may contract that component (and each of the components lying over it) to obtain a G-covering $S' \to S'_G$, where the genus of S'_G is now one (resp. two) less than that of S_G. As this covering represents all the topological input, we may paraphrase our main theorem as saying that the Putman–Wieland property (resp. the arithmeticity property) holds after a "stabilization" by taking a connected sum of the base orbifold with a closed surface of genus one (resp. two).

We will prove this theorem under the apparently weaker assumption that there exists a closed one-dimensional submanifold nonempty $A \subset S_G^\circ$, so a disjoint union of say $k \geq 1$ embedded circles (with $k = 1$ in case (i) and $k = 2$ in case (ii)) such that $S \to S_G$ is trivial over A and connected over $S_G \smallsetminus A$. This looks as if this is a more general result, because it is easy to find in the respective cases such an A inside the postulated subsurface with the property that its complement is connected. But we will see that this generalization is only apparent.

There is also a useful geometric interpretation for this last formulation: given such an A, then we can obtain the G-covering $S' \to S'_G$ as above by regarding $S_G \smallsetminus A$ as a punctured surface (so with two punctures for each component of A) and letting $S'_G \supset S_G \smallsetminus A$ be the closed orbifold obtained by filling in these punctures as nonorbifold points. Our assumptions say that S'_G is a closed connected surface (the genus drop is the number of connected components of A) and that the given G-covering $S \to S_G$ arises from a G-covering $S' \to S'_G$ with, for each component of A, an identification of the fibers of this covering over the two associated points (as principal G-sets). If we give S_G a complex structure and thus turn it into a smooth complex-projective curve with an orbifold structure, then an algebraic geometer might be tempted to regard this orbifold curve as being in its moduli space near the Deligne–Mumford stratum where the orbifold acquires k nodes, but for which the G-covering stays irreducible and does not ramify over the nodes. The covering $S' \to S'_G$ then appears as the normalization of such a degeneration. No algebraic geometry is used in the proof, though, for the topological part of this paper uses methods that directly generalize those of Looijenga [3].

Let us compare the above theorem with the work of Grunewald, Larsen, Lubotzky and Malestein [1], whose main motivation was to construct, via the virtual isomorphism mentioned above, new arithmetic quotients of the mapping class group of S_G. They assume that G acts freely so that $S_G = S_G^\circ$ and impose another, more technical condition, which in our setup translates into requiring that we are in the context of (i) and demanding that the covering $S' \to S'_G$ is of "handlebody type", in the sense that it extends to a handlebody that has S'_G as boundary. They prove that the image of $\mathrm{Diff}^+(S)^G$ in each simple factor of $\mathrm{Sp}(H^1(S, \mathbb{Q}))^G$ is arithmetic. Our approach differs from theirs in several aspects, but mostly in our direct and relatively simple way of constructing G-equivariant mapping classes.

When speaking of arithmetic subgroups of $\mathrm{Sp}(H^1(S,\mathbb{Q}))^G$, it is of course tacitly understood that the latter can be regarded as the group of rational points of an algebraic group defined over $\mathbb{Q}$. Let us make this explicit.

Denote by $X(\mathbb{Q}G)$ the set of irreducible characters of $\mathbb{Q}G$ and choose for every $\chi\in X(\mathbb{Q}G)$ a representing irreducible (left) $\mathbb{Q}G$-module V_χ. We also fix on every V_χ a G-invariant inner product $s_\chi\colon V_\chi\times V_\chi\to\mathbb{Q}$ (which can be obtained as the G-average of an arbitrary inner product). This exhibits V_χ as a self-dual $\mathbb{Q}G$-module. Then $\mathrm{End}_{\mathbb{Q}G}(V_\chi)$ is a skew field which is of finite $\mathbb{Q}$-dimension. We denote its opposite by D_χ (meaning that the underling $\mathbb{Q}$-vector space is $\mathrm{End}_{\mathbb{Q}G}(V_\chi)$, but that composition is taken in opposite order) so that V_χ is now a right D_χ-module. Adopting as a convention that D_χ used as superscript (resp. subscript) indicates that we are dealing with right (resp. left) D_χ-module endomorphisms, then the natural map

$$\mathbb{Q}G\cong\prod_{\chi\in X(\mathbb{Q}G)}\mathrm{End}^{D_\chi}(V_\chi)$$

is an isomorphism of $\mathbb{Q}$-algebras. This is in fact the Wedderburn decomposition of $\mathbb{Q}G$, as each factor is a minimal two-sided ideal.

The group algebra $\mathbb{Q}G$ comes with an anti-involution $r\mapsto r^\dagger$ which takes each basis element e_g for $g\in G$ to the basis element $e_{g^{-1}}$. This identifies $\mathbb{Q}G$ with its opposite. Since V_χ is self-dual, in the above decomposition the involution leaves each factor $\mathrm{End}^{D_\chi}(V_\chi)$ invariant and induces one in the skew-field D_χ: the involution on $\mathrm{End}^{D_\chi}(V_\chi)$ is given by taking the s_χ-adjoint, given by $s_\chi(\sigma v,v')=s_\chi(v,\sigma^\dagger v')$. Since we have D_χ acting on V_χ on the right, this means that $s_\chi(v\lambda,v')=s_\chi(v,v'\lambda^\dagger)$. (We note in passing that any other G-invariant inner product s'_χ on V_χ is of the form $s_\chi(v\lambda,v')$ for some nonzero λ with $\lambda^\dagger=\lambda$; the associated anti-involution of D_χ is then a conjugate of $\dagger$.) The center of D_χ, which we denote by L_χ, is a number field, and the fixed-point set of $\dagger$ in L_χ is a subfield $K_\chi\subset L_\chi$ with $[L_\chi:K_\chi]\le 2$.

For a finitely generated $\mathbb{Q}G$-module H, denote by $H[\chi]$ the associated χ-isogeny space $\mathrm{Hom}_{\mathbb{Q}G}(V_\chi,H)$. The right D_χ-module structure on V_χ determines a left D_χ-module structure on $H[\chi]$ and the *isotypical decomposition* of H is the assertion that the natural map

$$\bigoplus_{\chi\in X(\mathbb{Q}G)}V_\chi\otimes_{D_\chi}H[\chi]\to H,\quad v\otimes_{D_\chi}u\in V_\chi\otimes_{D_\chi}H[\chi]\mapsto u(v),$$

is an isomorphism of $\mathbb{Q}G$-modules. So the χ*-isotypical subspace* of H, ie the image H_χ of $V_\chi\otimes_{D_\chi}H[\chi]$ in V, has the structure of a K_χ-vector space.

Assume now that H comes equipped with a nondegenerate G-invariant symplectic form $(a,b)\in H\times H\mapsto a\cdot b\in\mathbb{Q}$. Then the isotypical decomposition of H is symplectic, so that we also have a decomposition $\mathrm{Sp}(H)^G=\prod_{\chi\in X(\mathbb{Q}G)}\mathrm{Sp}(H_\chi)^G$. Every factor $\mathrm{Sp}(H_\chi)^G$ can be understood with the help of the skew-hermitian form

$$(f,f')\in H[\chi]\times H[\chi]\mapsto\langle f,f'\rangle_\chi\in D_\chi,$$

which is characterized by the property that for all $v, v' \in V_\chi$,

$$f(v)\cdot f'(v') = s_\chi(v\langle f, f'\rangle_\chi, v')$$

(*skew-hermitian* means that the form is D_χ-linear in the first variable and $\langle f', f\rangle_\chi = -\langle f, f'\rangle_\chi^\dagger$). Indeed, for fixed f and f', the map $(v,v') \in V_\chi \times V\chi \mapsto f(v)\cdot f(v') \in \mathbb{Q}$ is a bilinear form on V_χ. Since $s_\chi\colon V_\chi \times V_\chi \to \mathbb{Q}$ is nondegenerate, there exists a unique $\mathbb{Q}$-linear endomorphism σ of V_χ such that $f(v)\cdot f(v') = s_\chi(\sigma(v), v')$. The G-invariance of both bilinear forms implies that σ is G-equivariant, ie is an element of $\mathrm{End}_{\mathbb{Q}G}(V_\chi)$. We prefer to regard it as an element of its opposite D_χ, so that $f(v)\cdot f(v') = s_\chi(v\sigma, v')$. This σ is evidently $\mathbb{Q}$-linear in both f and f', and that is why we denote it by $\langle f, f'\rangle_\chi \in D_\chi$. It is then a little exercise to check that $\langle\,,\rangle_\chi$ is skew-hermitian. This form is nondegenerate in an obvious sense. The group of automorphisms $H[\chi]$ that preserve this form is a generalized unitary group, and therefore written as $U(H[\chi])$.

Any element of $\mathrm{Sp}(H_\chi)^G$ acts via the isomorphism $H_\chi \cong V_\chi \otimes_{D_\chi} H[\chi]$ as an element of the form $1_{V_\chi} \otimes u$ with $u \in U(H[\chi])$, and this identifies $\mathrm{Sp}(H_\chi)^G$ with $U(H[\chi])$. The group $U(H[\chi])$ is the group of K_χ-points of a reductive algebraic group defined over K_χ, whereas $\mathrm{Sp}(H_\chi)^G$ is the group of $\mathbb{Q}$-points of an algebraic group defined over $\mathbb{Q}$. Indeed, the latter is obtained from the former by the restriction of scalars $K_\chi|\mathbb{Q}$.

Theorem 1.1 then amounts to the assertion that the image of $\mathrm{Diff}^+(S)^G$ in the product of unitary groups $\prod_{\chi\in X(\mathbb{Q}G)} U(H_1(S;\mathbb{Q})[\chi])$ is arithmetic. We use this decomposition to prove the theorem, since we first prove arithmeticity for a single factor. This leads to a somewhat stronger result, for we show that in the setting of (i) of our main theorem (so when $S \to S_G$ is trivial over a genus-one subsurface) the image of $\mathrm{Diff}^+(S)^G$ in $U(H_1(S;\mathbb{Q})[\chi])$ is almost always arithmetic (see Remark 5.6).

Structure Of the four sections, only the last one is topological, but in order to put the constructions given there to work, we need a considerable amount of algebra and that explains the nature of the preceding sections.

Section 2 collects useful (and essentially known) algebraic proprieties of constructs that we encounter in the symplectic representation theory of a finite group over $\mathbb{Q}$. So there is little or no claim of originality here, although it was (for us) a bit of an effort to extract this material from the literature. In Section 3 we introduce and study what we might regard as the basic symplectic module associated to an irreducible $\mathbb{Q}G$-module, where G is a finite group. The main result is Proposition 3.1, which states an arithmeticity property and also lists the (few) cases for which this arithmetic group has real rank ≤ 1. This prepares us for stating and proving the arithmeticity criterion Theorem 4.2 in Section 4, which furnishes the main algebraic input for Section 5. As mentioned, this last section is essentially topological: we there construct sufficiently many G-equivariant mapping classes to ensure that we can apply said theorem to obtain our main theorem, Theorem 1.1.

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2 Brief review of special unitary groups

The Albert classification In this subsection, D is a skew field of finite dimension over $\mathbb{Q}$ endowed with an anti-involution $\dagger$. We assume that the involution $\dagger$ is positive in the sense that $\lambda \in D \mapsto \mathrm{tr}_{D/\mathbb{Q}}(\lambda\lambda^\dagger)$ is a positive-definite form. We remark that this is so for the cases that matter here, for the given anti-involution on $\mathbb{Q}G$ is evidently positive: for $r \in \mathbb{Q}G$, the $\mathbb{Q}$-trace of $rr^\dagger$ is $|G|$ times the coefficient of e_1 in $rr^\dagger$, and hence is positive definite. The same is then true for its Wedderburn factors $\mathrm{End}^{D_\chi}(V_\chi)$ and their associated skew fields D_χ.

We denote the center of D by L (so it is a number field) and denote by D_+ (resp. K) the $\dagger$-invariant part of D (resp. L). Albert's classification of such pairs $(D, \dagger)$ — see for example [5, Chapter IV, Theorem 2] — then tells us that K is totally real, so that $\mathbb{R} \otimes_{\mathbb{Q}} D = \prod_\sigma \mathbb{R} \otimes_\sigma D$, where σ runs over the distinct field embeddings $\sigma\colon K \hookrightarrow \mathbb{R}$, and that there are essentially four cases:

- (I) $D = L = K$ so that $\mathbb{R} \otimes_\sigma D = \mathbb{R}$ for each σ,
- (II) $L = K$ and for each σ there exists an isomorphism $\mathbb{R} \otimes_\sigma D \cong \mathrm{End}_{\mathbb{R}}(\mathbb{R}^2)$ which sends $\dagger$ to taking the transpose (so $[D : L] = 4$),
- (III) $L = K$ and for each σ there exists an isomorphism $\mathbb{R} \otimes_\sigma D \cong \mathbb{K}$, where $\mathbb{K}$ denotes the Hamilton quaternions, which sends $\dagger$ to quaternion conjugation (so $[D : L] = 4$),
- (IV) L is a purely imaginary extension of K (in other words, L is a CM field) and for each σ there exists an isomorphism $\mathbb{R} \otimes_\sigma D \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}^d)$, which takes $\mathbb{R} \otimes_\sigma L$ to $\mathbb{C}$ (so $[D : L] = d^2$) and sends $\dagger$ to taking the conjugate transpose.

Let M be a left D-module of finite rank. We write $M^\dagger$ for M endowed with the structure of a right D-module via the rule $a\lambda := \lambda^\dagger a$ for $a \in M$ and $\lambda \in D$. So if M' is another left D-module, then $M^\dagger \otimes_D M'$ is defined. It is a K-vector space with the property that $a \otimes_D \lambda a' = (\lambda^\dagger a) \otimes_D a'$ for all $\lambda \in D$, $a \in M$ and $a' \in M'$. In particular, we have in $M^\dagger \otimes_D M$ a K-linear involution defined by $(a \otimes_D b)' = b \otimes_D a$. We denote its fixed-point set by $u(M) \subset M^\dagger \otimes_D M$. As a $\mathbb{Q}$-subspace of $M^\dagger \otimes_D M$, it is spanned by the symmetric tensors $a \otimes_D a$.

Isotropic transvections and Eichler transformations Let $(a,b) \in M \times M \mapsto \langle a,b\rangle \in D$ be a skew-hermitian form on M. We denote its radical by M_o, so that the form descends to a nondegenerate one on $\overline{M} := M/M_o$. We define the associated unitary group $U(M)$ as the group of D-linear automorphisms of M that preserve the form and act as the identity on M_o. It is the group of K-points of an algebraic group defined over K. If the form is nondegenerate ($M_o = 0$), then $U(M)$ is what is called in [2, Section 5.2B] a *classical unitary group*.

If $c \in M$ is *isotropic* (meaning that $\langle c,c\rangle = 0$), then we have the associated *isotropic transvection* $T_c = T(c \otimes_D c) \in U(M)$ defined by $x \in M \mapsto x + \langle x,c\rangle c$ (so $c \otimes_D c$ is here understood as an element of $M^\dagger \otimes_D M$). It "generates" an abelian unipotent subgroup of $U(M)$ defined by

$$\lambda \in D_+ \mapsto T(c \otimes_D \lambda c) \in U(M), \quad T(c \otimes_D \lambda c)(x) = x + \langle x, \lambda c\rangle c.$$

Isotropic transvections are particular cases of Eichler transformations. These are defined as follows: Let $c \in M$ be isotropic, $a \in M$ perpendicular to c and $\lambda \in D$ such that $\lambda - \lambda^\dagger = \langle a,a\rangle$ (equivalently, $\lambda - \frac{1}{2}\langle a,a\rangle \in D_+$). Then the associated *Eichler transformation* is

$$E(c,a,\lambda)\colon x \in M \mapsto x + \langle x,a\rangle c + \langle x,c\rangle a + \langle x,c\rangle \lambda c \in M.$$

It is a D-linear transformation which preserves the form. When $\lambda = \frac{1}{2}\langle a,a\rangle$, we shall write $E(c,a)$ instead. Since $T(c \otimes_D c) = E\big(\frac{1}{2}c, c\big)$, isotropic transvections are Eichler transformations, as asserted.

One checks that each Eichler transformation lies in $U(M)$ as defined above. In fact, $t \in K \mapsto E(tc,a,\lambda) = E(c,ta,t^2\lambda)$ is a closed one-parameter subgroup of $U(M)$ whose infinitesimal generator is represented by $a \otimes_D c + c \otimes_D a \in u(M)$ — or rather by its image in $u(M)/u(M_o)$, for if both a and c lie in M_o, then we get the identity. By a general property of algebraic groups [8, Corollary 2.2.7], such subgroups then generate a closed algebraic K-subgroup of $U(M)$. Following [2], we denote that group by $\mathrm{EU}(M)$.

We note the commutator identity

$$[E(c,a_1,\lambda_1), E(c,a_2,\lambda_2)] = T(c \otimes \lambda c) \quad \text{with } \lambda = \langle a_1,a_2\rangle + \langle a_1,a_2\rangle^\dagger. \tag{1}$$

It follows that if we fix c, but let a and λ vary (subject to the conditions above, so with $a \in c^\perp$), then the $E(c,a,\lambda)$ generate a unipotent group that appears as an extension of the vector group $c^\perp/(M_o + Dc)$ by the abelian subgroup of $U(M)$ defined by the $T(c \otimes_D \lambda c)$.

The group $\mathrm{EU}(M)$ is already generated by the isotropic transvections: When $\dagger$ is nontrivial this follows from [2, (6.3.1)]. The remaining case is the one we labeled (I). This is when $D = L = K$ and $U(M)$ is a symplectic group over K, but then there is no issue, because every $a \in M$ is isotropic, and then $E(c,a) = T_{a+c}T_a^{-1}T_c^{-1}$.

Unipotent radical and Levi quotient If the form is nondegenerate (ie $M_o = \{0\}$), then $\mathrm{EU}(M)$ is a K-form of a classical semisimple algebraic group, and hence has finite center. To be precise, it is a group of symplectic type in cases (I) and (II), of orthogonal type in case (III) and of special linear type in case (IV).

If M_o is possibly nonzero, then per convention, the elements of $U(M)$ act trivially on M_o. The natural map $U(M) \to U(\overline{M})$ is evidently onto. Its kernel consists of the transformations that act trivially on both M_o and M/M_o, and is therefore the unipotent radical $R_u(U(M))$ of $U(M)$ — recall that $U(\overline{M})$ is reductive. We have an exact sequence

$$1 \to R_u(U(M)) \to U(M) \to U(\overline{M}) \to 1.$$

The elements of $R_u(U(M))$ are the Eichler transformations $E(c,a)$ with $c \in M_o$ and $a \in M$ arbitrary. In this case $E(c,a)$ only depends on the image of $c \otimes_{\mathbb{Z}} a$ in $M_o^\dagger \otimes_D \overline{M})$, so that the resulting map

$$M_0^\dagger \otimes_D \overline{M} \to R_u(U(M))$$

is an isomorphism. So $R_u(U(M))$ is a vector group over K (ie a K-vector space regarded as the group of K-points of a K-algebraic group of additive type). Since $\mathrm{EU}(\overline{M})$ is a normal semisimple subgroup of $U(M)$, it has the same unipotent radical: $R_u(\mathrm{EU}(M)) = R_u(U(M))$.

The relation between EU(M) and $U(M)$ Since in what follows the notion of the real rank of an algebraic group shows up, let us begin with reviewing this concept briefly.

Let $\mathcal{G}$ be a reductive algebraic group. Suppose first that $\mathcal{G}$ is defined over $\mathbb{R}$. Then the real rank $\mathrm{rk}_{\mathbb{R}}(\mathcal{G})$ of $\mathcal{G}$ is by definition the dimension of a Cartan subgroup of $\mathcal{G}$ defined over $\mathbb{R}$. For example, if $\mathcal{G}$ is the orthogonal group of a nondegenerate quadratic form over $\mathbb{R}$, then its real rank is the Witt index of this form: the dimension of a maximal isotropic subspace defined over $\mathbb{R}$. If $\mathcal{G}$ is defined over a number field k, then we restrict scalars à la Weil so that $\mathrm{Res}_{k|\mathbb{Q}}\,\mathcal{G}$ is a group defined over $\mathbb{Q}$. We then regard $\mathrm{Res}_{k|\mathbb{Q}}\,\mathcal{G}$ (by base change) as a group over $\mathbb{R}$, and define the real rank of $\mathcal{G}$ to be the real rank of the latter. Concretely, if $\sigma_1, \dots, \sigma_r$ are the real embeddings of k in $\mathbb{R}$ and $\tau_1, \bar{\tau}_1, \dots, \tau_s, \bar{\tau}_s$ are the remaining distinct (complex) embeddings (they come in complex conjugate pairs), then the definition comes down to

$$\mathrm{rk}_{\mathbb{R}}(\mathcal{G}) = \sum_{i=1}^{r} \mathrm{rk}_{\mathbb{R}}(\mathcal{G}_{\sigma_i}) + \sum_{i=1}^{s} \mathrm{rk}_{\mathbb{C}}(\mathcal{G}_{\tau_i})$$

(this is also the sum over all the archimedean valuations of k, taking as general term the real rank of the corresponding completion of $\mathcal{G}(k)$). The Dirichlet unit theorem often gives lower bounds for the rank. For example, if the skew field D is as in the Albert classification, the group of units $D^\times$ is a reductive group defined over K, and its group of real points and its real rank are then as follows, putting $e := [K : \mathbb{Q}]$:

(I) $\mathrm{Res}_{K|\mathbb{Q}}\, D^\times(\mathbb{R})$ is open in $(\mathbb{R}^\times)^e$; the real rank of $D^\times$ is e,

(II) $\mathrm{Res}_{K|\mathbb{Q}}\, D^\times(\mathbb{R})$ is open in $\mathrm{GL}_2(\mathbb{R})^e$; the real rank of $D^\times$ is $2e$,

(III) $\mathrm{Res}_{K|\mathbb{Q}}\, D^\times(\mathbb{R})$ is open in $(\mathbb{K}^\times)^e$; the real rank of $D^\times$ is e,

(IV) $\mathrm{Res}_{K|\mathbb{Q}}\, D^\times(\mathbb{R})$ is open in $\mathrm{GL}_d(\mathbb{C})^e$; the real rank of $D^\times$ is de.

So $D^\times$ has real rank ≥ 2, unless D equals $\mathbb{Q}$ (I), is a definite quaternion algebra with center $\mathbb{Q}$ (III) or is an imaginary quadratic extension of $\mathbb{Q}$.

It is clear that $\mathrm{EU}(M)$ is a normal subgroup of $U(M)$. We already noticed that it is closed in $U(M)$, and hence the quotient $U(M)/\mathrm{EU}(M)$ is also an algebraic group. Note however that if $\overline{M}$ has no nonzero isotropic vectors, then $\mathrm{EU}(\overline{M})$ is trivial. We mention for future reference a consequence of a theorem of G E Wall [10]:

Lemma 2.1 *If M is nondegenerate and contains a nonzero isotropic vector, then $U(M)/\mathrm{EU}(M)$ is anisotropic (all its real forms are compact). So $\mathrm{EU}(M)$ and $U(M)$ have the same real rank, and any arithmetic subgroup of $U(M)$ will have finite image in $U(M)/\mathrm{EU}(M)$.*

Proof Theorem 1 of [10] identifies $U(M)/\mathrm{EU}(M)$ as a quotient of $D^\times$ by a normal subgroup which contains $D^\times \cap D_+$. It is easy to check that in all four cases (I)–(IV) in the Albert classification such a quotient must be anisotropic. □

Wall's result is more specific and tells us that $U(M)/\mathrm{EU}(M)$ is often an anisotropic torus. But this need not be so when $\dim_D M = 2$.

Lemma 2.2 *If M is nondegenerate isotropic, then the K-algebraic group $\mathrm{EU}(M)$ is almost simple (by which we mean that $\mathrm{EU}(M)$ is perfect and every proper normal subgroup is contained in its center) unless $D = K$ and $M \cong K^4$ is endowed with a nondegenerate symmetric form which admits an isotropic plane defined over K.*

Proof This follows from [2, Theorem 6.3.16 combined with Theorem 6.3.15]. □

The excepted case is genuine, for in that case $M \cong M_1 \otimes_K M_2$ as modules endowed with K-forms, where M_i is a two-dimensional K-vector space endowed with a nondegenerate symplectic form. The resulting map $\mathrm{SL}(M_1)\times\mathrm{SL}(M_2) \to \mathrm{GL}(M)$ has image $\mathrm{EU}(M) \cong O(M)$ and its kernel has $(-1,-1)$ as its unique nonidentity element.

Remark 2.3 The *reduced norm* is the homomorphism $N : U(M) \to L^\times$ characterized by the following property: if $T \in U(M)$, then for some (or equivalently, every) real embedding $\sigma : K \hookrightarrow \mathbb{R}$, the D-linear T induces a linear transformation of the $(\mathbb{R}\otimes_\sigma L)$-vector space $\mathbb{R} \otimes_K M$ whose determinant is $1 \otimes_\sigma N(T)$. The kernel of N, usually denoted by $\mathrm{SU}(M)$, contains $\mathrm{EU}(M)$ and is often equal to it. But in our context this group does not show up in a natural manner.

3 The hyperbolic module attached to a finite group

A hermitian extension Our discussion of symplectic $\mathbb{Q}G$-modules also applies to orthogonal $\mathbb{Q}G$-modules. One such module is $\mathbb{Q}G$ itself (regarded as a left module). It comes indeed with a G-invariant pairing $\mathbb{Q}G \times \mathbb{Q}G \to \mathbb{Q}$, the *trace form*, which assigns to the pair (r_1, r_2) the trace of $r_1 r_2^\dagger$ considered as an endomorphism of $\mathbb{Q}G$ as a $\mathbb{Q}$-vector space (this is simply $|G|$ times the coefficient of e_1). This pairing is symmetric and nondegenerate.

The Wedderburn decomposition $\mathbb{Q}G \cong \prod_\chi \mathrm{End}^{D_\chi}(V_\chi)$ is also the isotypical decomposition, for the K_χ-linear map

$$V_\chi \otimes_{D_\chi} \mathrm{Hom}^{D_\chi}(V_\chi, D_\chi) \to \mathrm{End}^{D_\chi}(V_\chi), \quad v_1 \otimes_{D_\chi} f : v \in V_\chi \mapsto v_1 f(v),$$

which assigns to $v_1 \otimes_{D_\chi} f$ the endomorphism $v \in V_\chi \mapsto v_1 f(v)$ is well defined and is an isomorphism of left $\mathbb{Q}G$-modules (and also as right $\mathbb{Q}G$-modules). This also shows that $\mathbb{Q}G[\chi] \cong \mathrm{Hom}^{D_\chi}(V_\chi, D_\chi)$ as a right D_χ-module.

We claim that $\mathrm{Hom}^{D_\chi}(V_\chi, D_\chi) \cong V_\chi^\dagger$ as left D_χ-modules. This is based on a hermitian extension of s_χ to $V_\chi^\dagger$: if we follow the same recipe as in the introduction for a symplectic representation, then we find that there is a hermitian form $h_\chi : V_\chi^\dagger \times V_\chi^\dagger \to D_\chi$ characterized by the property that for all $v, v' \in V_\chi$ and $f, f' \in V_\chi^\dagger$,

$$s_\chi(v, f')s_\chi(v', f) = s_\chi(vh_\chi(f, f'), v').$$

This formula implies that h_χ is G-invariant (we let G act on the right of $V_\chi^\dagger$). By taking $v = f = f'$, we also see that $h_\chi(f, f) = s_\chi(f, f)$, so that h_χ is a hermitian extension of s_χ. For every $v \in V_\chi^\dagger$, the expression $h_\chi(v, -)$ yields an element of $\mathrm{Hom}^{D_\chi}(V_\chi, D_\chi)$ and defines the stated isomorphism.

Isotropic transvections Henceforth we write R for the integral group ring $\mathbb{Z}G$. Let M be a finitely generated (left) R-module, free over $\mathbb{Z}$, and let $(a, b) \in M \times M \mapsto a \cdot b \in \mathbb{Z}$ be a nondegenerate (but not necessarily unimodular) G-equivariant symplectic form. We extend this in the standard manner to a form

$$(a, b) \in M \times M \mapsto \langle a, b\rangle := \sum_{g \in G} (g^{-1}a \cdot b)e_g = \sum_{g \in G} (a \cdot gb)e_g \in R.$$

This form is skew-hermitian: it is R-linear in the first variable and $\langle a, b\rangle = -\langle b, a\rangle^\dagger$. A $\mathbb{Z}$-linear automorphism of M is G-equivariant and preserves the symplectic form if and only if it is an R-module automorphism which preserves this skew-hermitian form. We denote the group of such automorphisms by $U(M)$.

Let R_+ stand for the fixed-point set of $\dagger$ in R; this is an additive subgroup of R. If $a \in M$ is *R-isotropic* in the sense that $\langle a, a\rangle = 0$, then for every $r \in R_+$ the isotropic transvection

$$T_a(r) : x \in M \mapsto x + \langle x, a\rangle ra \in M \tag{2}$$

lies in $U(M)$ and $r \in R_+ \mapsto T_a(r)$ is a homomorphism from (the additively written) R_+ to (the multiplicatively written) $U(M)$. Since $T_a(r)$ only depends on $a \otimes ra \in M^\dagger \otimes_R M$, we also denote this transformation by $T(a \otimes ra)$.

The basic hyperbolic module Let A be a (not necessarily commutative) unital ring with unit endowed with an anti-involution $\dagger$. The *basic hyperbolic A-module* $\mathcal{H}^2(A)$ is the free left A-module of rank two (whose generators we denote by $\boldsymbol{e}$ and $\boldsymbol{f}$) endowed with the skew-hermitian form defined by

$\langle \boldsymbol{e}, \boldsymbol{e}\rangle = \langle \boldsymbol{f}, \boldsymbol{f}\rangle = 0$ and $\langle \boldsymbol{e}, \boldsymbol{f}\rangle = 1$. It can be regarded as the A-form of the standard symplectic module $\mathbb{Z}^2$. In vector notation:

$$\left\langle \begin{pmatrix} a' \\ a'' \end{pmatrix}, \begin{pmatrix} b' \\ b'' \end{pmatrix} \right\rangle = a'b''^{\dagger} - a''b'^{\dagger}. \tag{3}$$

It is unimodular in the sense that $a \in \mathcal{H}^2(A) \mapsto \langle -, a\rangle \in \mathrm{Hom}_A(\mathcal{H}^2(A), A)$ is an antilinear isomorphism. We will write $U_2(A)$ and $\mathrm{EU}_2(A))$ for $U(\mathcal{H}^2(A))$ and $\mathrm{EU}(\mathcal{H}^2(A))$, respectively. The latter contains $\mathrm{SL}_2(\mathbb{Z})$ in an obvious manner. Let $A_+ \subset A$ be the set of $\dagger$-invariant elements. One verifies that $T_{\boldsymbol{e}} : x \in \mathcal{H}^2(A) \mapsto x + \langle x, \boldsymbol{e}\rangle \boldsymbol{e}$ and the similarly defined $T_{\boldsymbol{f}}$ have the matrix form

$$T_{\boldsymbol{e}}(a) = \begin{pmatrix} 1 & -\rho_a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T_{\boldsymbol{f}}(a) = \begin{pmatrix} 1 & 1 \\ \rho_a & 1 \end{pmatrix},$$

respectively, where ρ_a stands for right multiplication with a.

The group $\Gamma(G)$ We take $A = R(= \mathbb{Z}G)$. The elements of the form $r + r^{\dagger}$ with $r \in R$ make up a subgroup R_{++} of R_+ such that R_+/R_{++} is a finite-dimensional $\mathbb{F}_2$-vector space. Let $\Gamma_+(G)$ and $\Gamma_-(G)$ denote the subgroups of $\mathrm{EU}_2(R)$ generated by $T_{\boldsymbol{e}}(R_{++})$ and $T_{\boldsymbol{f}}(R_{++})$, respectively, and let $\Gamma_o(G) \subset \mathrm{EU}_2(R)$ stand for the subgroup generated by $\Gamma_+(G)$ and $\mathrm{SL}_2(\mathbb{Z})$. Since $\Gamma_-(G)$ is a $\mathrm{SL}_2(\mathbb{Z})$-conjugate of $\Gamma_+(G)$, the group $\Gamma_o(G)$ contains $T_{\boldsymbol{f}}(R_{++})$. The right (inverse) action of G on $\mathcal{H}^2(R)$ defines an embedding of G in $U_2(R)$. One checks that

$$\rho_g T_{\boldsymbol{e}}(r) \rho_{g^{\dagger}} = T_{\boldsymbol{e}}(grg^{-1}),$$

and so G normalizes $\Gamma_o(G)$. We put $\Gamma(G) := \Gamma_o(G).G$.

Arithmetic nature of $\Gamma(G)$ The notion of a basic hyperbolic module generalizes in a straightforward manner to $\mathcal{H}^2(V_\chi^{\dagger})$, the skew-hermitian form being given by

$$\left\langle \begin{pmatrix} v' \\ v'' \end{pmatrix}, \begin{pmatrix} w' \\ w'' \end{pmatrix} \right\rangle = h_\chi(v', w'') - h_\chi(v'', w'). \tag{4}$$

So we have defined $U_2(V_\chi^{\dagger})$; it is the group of K_χ-points of a reductive algebraic group defined over K_χ. If we write an element of $U_2(V_\chi^{\dagger})$ in block form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, with A, B, C and D in $\mathrm{End}_{D_\chi}(V_\chi^{\dagger})$, then the subgroup defined by $C = 0$ is parabolic. Its unipotent radical is given by requiring that in addition A and D are the identity. The corresponding subgroup is then the vector group $\left(\begin{smallmatrix} 1 & B \\ 0 & 1 \end{smallmatrix}\right)$ for which B is hermitian relative to h_χ. In other words, h_χ identifies B with an element of $u(V_\chi^{\dagger})$ — that is, a symmetric element of $V_\chi \otimes_{D_\chi} V_\chi^{\dagger}$ — and hence defines an isotropic transvection. In particular, this is also the unipotent radical of the corresponding subgroup of $\mathrm{EU}_2(V_\chi^{\dagger})$. An opposite parabolic subgroup is defined by $B = 0$ and has a similar description of its unipotent radical. Let us denote these unipotent radicals by $\mathcal{R}_+(U_2(V_\chi^{\dagger}))$ and $\mathcal{R}_-(U_2(V_\chi^{\dagger}))$, respectively.

We run into this when we consider the isotypical decomposition of $\mathcal{H}^2(\mathbb{Q}G)$. The isogeny space $\mathcal{H}^2(\mathbb{Q}G)[\chi] = \mathrm{Hom}_{\mathbb{Q}G}(V_\chi, \mathcal{H}^2(\mathbb{Q}G))$ is then a left D_χ-module that is naturally identified with $\mathcal{H}^2(V_\chi^\dagger)$. This gives rise to a decomposition

$$U_2(\mathbb{Q}G) = \prod_{\chi\in X(\mathbb{Q}G)} U_2(V_\chi^\dagger).$$

The image of $\Gamma_\pm(G)$ in $U_2(V_\chi^\dagger)$ clearly lands in $\mathcal{R}_\pm(U_2(V_\chi^\dagger))$. Since $\mathrm{End}_{D_\chi}(V_\chi^\dagger) = \mathrm{End}^{D_\chi}(V_\chi)$ is a Wedderburn factor of $\mathbb{Q}G$, it follows that the image of R in $\mathrm{End}_{D_\chi}(V_\chi^\dagger)$ is an order (a lattice that is also a unital algebra). This is compatible with the anti-involutions, and hence the image of R_{++} in $\mathrm{End}_{D_\chi}(V_\chi^\dagger)$ is a lattice in the subspace of hermitian matrices. So the image of $\Gamma_\pm(G)$ in $U_2(V_\chi^\dagger)$ is a lattice in $\mathcal{R}_\pm(U_2(V_\chi^\dagger))$.

It is clear that $\Gamma_o(G)$ maps to $\mathrm{EU}_2(V_\chi^\dagger)$.

Proposition 3.1 *The image of $\Gamma(G)$ in $U_2(V_\chi^\dagger)$ is an arithmetic subgroup. The real rank of $U_2(V_\chi^\dagger)$ is ≥ 2 unless $\dim_{D_\chi} V_\chi = 1$. In that last case, where we can assume that $V_\chi^\dagger = D_\chi$ with G acting on the right and mapping to its group of units, one of the following holds:*

- (i) $D_\chi = \mathbb{Q}$ *and G maps to μ_2,*
- (iia) D_χ *is the Gaussian field $\mathbb{Q}(\sqrt{-1})$ and G maps onto μ_4,*
- (iib) D_χ *is the Eisenstein field $\mathbb{Q}(\sqrt{-3})$ and G maps onto μ_3 or μ_6,*
- (iiia) $D_\chi \cong \mathbb{Q} + \mathbb{Q}\boldsymbol{i} + \mathbb{Q}\boldsymbol{j} + \mathbb{Q}\boldsymbol{k}$ *and G maps onto its group of units (a binary tetrahedral group of order 24) or onto the quaternion group of order 8, or*
- (iiib) $D_\chi \cong \mathbb{Q} + \mathbb{Q}\sqrt{3}\boldsymbol{i} + \mathbb{Q}\boldsymbol{j} + \mathbb{Q}\sqrt{3}\boldsymbol{k}$ *and G maps onto the binary dihedral group of order 12.*

In all cases, $\Gamma(G)$ acts $\mathbb{Q}$-irreducibly in $\mathcal{H}^2(V_\chi^\dagger)$.

Remark 3.2 It is well known that the quaternion group appearing in Proposition 3.1(iiia) is realized as the Galois group of a torus ramified at four points (the covering surface has genus three). This example is like a Swiss army knife for illustrating (and refuting) statements in complex dynamics, which is why that community refers to it as the *eierlegende Wollmilchsau*. We do not know whether its appearance here is just a coincidence.

For the proof we need:

Theorem 3.3 (Raghunathan [7], Venkataramana [9]) *Let $\mathcal{G}$ be an almost simple simply connected $\mathbb{Q}$-algebraic group of real rank ≥ 2. Let $\mathcal{R}_-$ and $\mathcal{R}_+$ be $\mathbb{Q}$-subgroups that contain the unipotent radicals of opposite $\mathbb{Q}$-parabolic subgroups of $\mathcal{G}$. Then for any pair of lattices $\Gamma_+ \subset \mathcal{R}_+(\mathbb{Q})$ and $\Gamma_- \subset \mathcal{R}_-(\mathbb{Q})$, the subgroup of $\mathcal{G}(\mathbb{Q})$ generated by their union $\Gamma_+ \cup \Gamma_-$ is a congruence subgroup of $\mathcal{G}(\mathbb{Q})$.*

Proof We first prove the arithmeticity property of $\Gamma_o(G)$ in $\mathrm{EU}_2(V_\chi^\dagger)$. Let us first observe that the group $\mathrm{EU}_2(V_\chi^\dagger)$ is almost simple by Lemma 2.2. Indeed, this can only fail if $D_\chi = K_\chi$ and the form is symmetric (with $\dim_{K_\chi} V_\chi^\dagger = 2$), and this is clearly not the case.

If the real rank of $\mathrm{EU}_2(V_\chi^\dagger)$ is ≥ 2, then the theorem of Raghunathan and Venkataramana applies and we conclude that $\Gamma_o(\chi)$ is an arithmetic subgroup of $\mathrm{EU}_2(V_\chi^\dagger)$. It is then also easy to see that $\Gamma(G)$ acts $\mathbb{Q}$-irreducibly in $\mathcal{H}^2(V_\chi^\dagger)$.

Since the real rank of $\mathrm{EU}_2(V_\chi^\dagger)$ is $\geq \dim_{D_\chi} V_\chi^\dagger$, it remains to treat the case when $\dim_{D_\chi} V_\chi^\dagger = 1$. In other words, we can assume that $V_\chi^\dagger = D_\chi$. The real rank of $\mathrm{EU}_2(D_\chi)$ is then still ≥ 2 most of the time. As we saw above, the exceptions are the cases for which $K_\chi = \mathbb{Q}$ and D_χ is either $\mathbb{Q}$, a definite quaternion algebra over $\mathbb{Q}$ with center $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$. Since D_χ is also an irreducible $\mathbb{Q}G$-module, we have a homomorphism $\rho\colon G \to D_\chi^\times$ whose image contains a $\mathbb{Q}$-basis of D_χ. In particular, D is generated over $\mathbb{Q}$ by its units. So if D_χ is an imaginary quadratic extension of $\mathbb{Q}$, then D_χ is either $\mathbb{Q}$, the Gaussian field or the Eisenstein field. In the definite quaternion case, $D_\chi^\times(\mathbb{R})$ is the group of unit quaternions, and hence $\rho(G)$ is one of the subgroups classified by Klein: this group must be binary tetrahedral, binary octahedral, binary icosahedral or binary dihedral (of order $4n$). In these cases $\rho(\mathbb{Q}G) \cap \mathbb{R}$ equals $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{5})$ or $\mathbb{Q}(\cos(\pi/n))$, respectively. Since we want this intersection to be $\mathbb{Q}$, only the two groups listed have that property.

Note that in each of these exceptional cases, $D_{\chi,+} = K_\chi = \mathbb{Q}$. The isotropic subspaces in $\mathcal{H}^2(D_\chi)$ are defined over $\mathbb{Q}$, and hence $\mathrm{EU}_2(D_\chi) \cong \mathrm{SL}_2(\mathbb{Q})$. The group $\Gamma_o(\chi)$ is then a copy of $\mathrm{SL}_2(\mathbb{Z})$. So $\Gamma_o(G)$ is arithmetic in $\mathrm{EU}_2(V_\chi^\dagger)$. In view of Lemma 2.1 this also implies the arithmeticity of $\Gamma(G)$ in $U_2(V_\chi^\dagger)$. The actions of G (on the right) and $\mathrm{SL}_2(\mathbb{Z})$ (on the left) on $\mathcal{H}^2(V_\chi^\dagger)$ commute and make $\mathcal{H}^2(V_\chi^\dagger)$ an exterior tensor product of irreducible $\mathbb{Q}$-representations: it is the right $\mathbb{Q}G$-module $V_\chi^\dagger$ tensored with the tautological representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Q}^2$ (which is absolutely irreducible). Hence $\mathcal{H}^2(V_\chi^\dagger)$ is irreducible as a representation of $\mathrm{SL}_2(\mathbb{Z}) \times G$. This implies that $\mathcal{H}^2(V_\chi^\dagger)$ is irreducible as a $\Gamma(G)$-module. $\square$

4 An arithmeticity criterion

In this section we fix a rational character $\chi \in X(\mathbb{Q}G)$. We therefore suppress the subscript χ and write D for D_χ and V for V_χ.

Proposition 3.1 tells us that $\Gamma(G) \subset U_2(V^\dagger)$ is an arithmetic subgroup which acts $\mathbb{Q}$-irreducibly on $\mathcal{H}^2(V^\dagger)$ and that, with a few exceptions, the group $U_2(V^\dagger)$ is of real rank ≥ 2.

Eichler transformations revisited Let M be (left) D-module of finite rank endowed with a nondegenerate skew-hermitian form $\langle -,- \rangle\colon M \times M \to D$. Given a D-submodule $N \subset M$, we denote by $U_M(N)$ the subgroup of the group of transformations that act trivially on $N^\perp$. This group preserves N and acts trivially on its radical $N_o = N \cap N^\perp$. Hence "restriction to N" defines a homomorphism $U_M(N) \to U(N)$. This homomorphism is easily shown to be onto. Its kernel consists of the unitary transformations that act trivially on $N + N^\perp$, and one checks that this is the image of $u(N_o)$ under T. We saw that the homomorphism $U(N) \to U(\overline{N})$ is also onto, and we identified its kernel with the vector group $N_o^\dagger \otimes \overline{N}$. So the Levi quotient of $U_M(N)$ is $U(\overline{N})$ and its unipotent radical $R_u(U_M(N))$ is an extension of vector groups:

$$1 \to u(N_o) \xrightarrow{T} R_u(U_M(N)) \to N_o^\dagger \otimes_D \overline{N} \to 0.$$

As is clear from (1), this extension is usually nontrivial. If N_o is spanned by a single element c, then we can write this sequence as

$$0 \to c \otimes D_+ c \xrightarrow{T} R_u(U_M(N)) \to c \otimes \overline{N} \to 0. \tag{5}$$

Any element of $R_u(U_M(N))$ is an Eichler transformation $E(c,a,\lambda)$ whose image in $c \otimes \overline{N}$ is $c \otimes \bar{a}$ (where $\bar{a} \in \overline{N}$ is the image of a). We will often use the following lemma:

Lemma 4.1 *Let $\Gamma \subset U_M(N)$ be a discrete subgroup whose image in $U(\overline{N})$ is arithmetic and which acts $\mathbb{Q}$-irreducibly in $\overline{N}$. If $\Gamma \cap R_u(U_M(N))$ contains an Eichler transformation $E(c,a,\lambda)$ with $a \in N \smallsetminus D_+c$, then $\Gamma \cap U_M(N)$ is arithmetic in $U_M(N)$.*

Proof We are given that in the exact sequence of algebraic groups

$$1 \to R_u(U_M(N)) \to U_M(N) \to U(\overline{N}) \to 1,$$

the image $\overline{\Gamma}$ of Γ in $U(\overline{N})$ is arithmetic. Hence for Γ to be arithmetic, it suffices that $\Gamma \cap R_u(\mathrm{EU}_M(N))$ be a lattice. For this we turn to the exact sequence (5). The Eichler transformation $E(c,a,\lambda)$ has image $c \otimes \bar{a}$ in $c \otimes \overline{N}$, and this image is nonzero by assumption. The image of the Γ-conjugacy class of $E(c,a,\lambda)$ in $c \otimes \overline{N}$ is equal to $c \otimes \overline{\Gamma}\bar{a}$. Since our assumptions also imply that $\overline{\Gamma}$ acts $\mathbb{Q}$-irreducibly in $\overline{N}$, it follows that the image of $\Gamma \cap R_u(U_M(N))$ in $c \otimes \overline{N}$ is a lattice in $c \otimes \overline{N}$.

Next observe that if $E(c,a_1,\lambda_1)$ and $E(c,a_2,\lambda_2)$ lie in $\Gamma \cap R_u(U_M(N))$, then so does their commutator, which by the identity (1) is $T(c \otimes \lambda c)$ with $\lambda = \langle a_1, a_2\rangle + \langle a_1, a_2\rangle^\dagger$. Since the $\langle a_1, a_2\rangle$ generate a lattice in D, it follows that the λ generate a lattice in D_+. In other words, the preimage of $\Gamma \cap R_u(U_M(N))$ in $c \otimes D_+c$ is also a lattice. Hence $\Gamma \cap R_u(U_M(N))$ is a lattice. $\square$

Hyperbolic submodules If $j : \mathcal{H}^2(V^\dagger) \hookrightarrow M$ is an embedding of hermitian D-modules, then M is the orthogonal direct sum of the image of j and its perp (for $\mathcal{H}^2(V^\dagger)$ is nondegenerate), so that j gives rise to an injective homomorphism of groups $j_* : U_2(V^\dagger) \hookrightarrow U(M)$. Let us refer to such an embedding as a *$V^\dagger$-hyperbolic summand* in M.

The following criterion for arithmeticity will be central to our argument:

Theorem 4.2 *Let M be a nondegenerate skew-hermitian D-module of finite rank, and*

$$a : V^\dagger \hookrightarrow M, \quad \{b : V^\dagger \hookrightarrow M\}_{b \in \mathcal{B}}$$

a collection of D-linear embeddings (with $\mathcal{B}$ finite and nonempty) whose images span M over D and are such that for each $b \in \mathcal{B}$, the pair (a,b) defines a hyperbolic summand of M. If $\dim_D V^\dagger = 1$ **and** *$\dim_D M > 2$, assume in addition that there exist $b_1, b_2 \in \mathcal{B}$ for which $b_1(V^\dagger)$ and $b_2(V^\dagger)$ are perpendicular.*

Then the subgroup Γ of $U(M)$ generated by $\{(a,b)_*\Gamma(G)\}_{b\in\mathcal{B}}$ is an arithmetic subgroup of $U(M)$ which acts $\mathbb{Q}$-irreducibly in M.

The proof will be by induction on $\dim_D M$. As may be inferred from the statement of the theorem, the case when $\dim_D V^\dagger = 1$ is a bit more delicate. Indeed, the first induction step then requires special care and so we do that case first. Once we have dealt with it, we indicate how to modify the arguments in order to obtain a proof of the unrestricted version of Theorem 4.2.

Let us say that a D-subspace $N \subset M$ is Γ-*arithmetic* if $\Gamma \cap U_M(N)$ is arithmetic in $U_M(N)$ and acts $\mathbb{Q}$-irreducibly in $\overline{N}$ (the last property is a consequence of the first if the real rank of $U_M(N)$ is ≥ 2).

The case $\dim_D V^\dagger = 1$ We then identify $V^\dagger$ with D and a and each $b \in \mathcal{B}$ with the image of $1 \in D$ under these embeddings, so that $\langle a,b\rangle = 1$ for all $b \in \mathcal{B}$. Note that $\{a\} \cup \mathcal{B} \subset M$ consists of isotropic elements and generates M over D. We write $\Gamma(a,b)$ for the image of $\Gamma(G)$ under (a,b), so that Γ is generated by $\{\Gamma(a,b)\}_{b\in\mathcal{B}}$. As any $b \in \mathcal{B}$ lies in $\Gamma(a,b)a$, it follows that $\{a\} \cup \mathcal{B} \subset \Gamma a$.

By Proposition 3.1, $Da + Db$ is Γ-arithmetic for every $b \in \mathcal{B}$. We therefore assume that M is not of the form $Da + Db$. So there exist $b_1, b_2 \in \mathcal{B}$ with $b_2 \notin Da + Db_1$ such that $\langle b_1, b_2\rangle = 0$.

Lemma 4.3 *Put $N := Da + Db_1$. Then $N' := N + Db_2$ is Γ-arithmetic.*

Proof We verify that the assumptions of Lemma 4.1 are satisfied by $\Gamma \cap U_M(N')$. It is clear that the radical of N' is spanned by $c := b_2 - b_1$, so that $\overline{N}'$ is the isomorphic image of N. We know that N is Γ-arithmetic and so Γ has arithmetic image in $U(\overline{N}')$. Since T_{b_1} and T_{b_2} lie in Γ, so does $T_{b_2}T_{b_1}^{-1}$. We check that

$$T_{b_2}T_{b_1}^{-1}(x) = x - \langle x, b_1\rangle + \langle x, b_1 + c\rangle(b_1 + c) = E(c, b_1, 1)(x).$$

So the image of $\Gamma \cap R_u(U_M(N'))$ in $c \otimes \overline{N}'$ contains $c \otimes b_1$. Now apply Lemma 4.1. □

From this point onward the argument will be inductive. The union of Lemmas 4.4 and 4.5 will establish the theorem in the case $\dim_D V^\dagger = 1$.

Lemma 4.4 *Let $N \subsetneq M$ be a D-subspace which contains a, b_1 and b_2, and whose radical is of D-dimension one. If N is Γ-arithmetic, then there exists a $b \in \mathcal{B}$ such that $N' := N + Db$ is nondegenerate and Γ-arithmetic, and the real rank of $U(N')$ is ≥ 2.*

Proof Let $c \in N$ span the radical of N. Since M is nondegenerate and D-spanned by $\{a\} \cup \mathcal{B}$, there must exist a $b \in \mathcal{B}$ such that c is not in the radical of $N' := N + Db$. Then it is easily seen that N' is nondegenerate, so that $U_M(N') \cong U(N')$. In the case $N = Da + Db_1 + Db_2$ (where we can take $c = b_2 - b_1$), one checks that N is the perpendicular sum of two copies of $\mathcal{H}^2(D)$. Otherwise, N' contains such a sum. This implies that $U(N')$ has real rank ≥ 2.

The $U(N')$-stabilizer $U(N')_c$ of c is equal to $U_M(N)$ and hence contains $\Gamma \cap U(N')_c$ as an arithmetic subgroup. Observe that $c' := T_b(c) = c + \langle c, b\rangle b$ is another isotropic element with $\langle c', c\rangle = \langle c, b\rangle\langle b, c\rangle \neq 0$, and so $Dc + Dc' \cong \mathcal{H}^2(D)$. The two $U(N')$-stabilizers of Dc and Dc' are opposite parabolic subgroups of $U(N')$ whose unipotent radicals are contained in $U(N')_c$ and $U(N')_{c'}$, respectively. Since $U(N')_{c'}$ is a Γ-conjugate of $U(N')_c$, it follows that $\Gamma \cap U(N')_{c'}$ is an arithmetic subgroup of $U(N')_{c'}$. We have thus satisfied the hypotheses of Theorem 3.3 and we conclude that $\Gamma \cap U(N')$ is arithmetic in $U(N')$. The fact that $\Gamma \cap U(N')$ acts $\mathbb{Q}$-irreducibly in N' follows from the fact that $\Gamma \cap U(N)$ has this property in N, for the $(\Gamma \cap U(N'))$-translates of N span N' over $\mathbb{Q}$, but do not decompose N'. □

Lemma 4.5 *Let $N \subsetneq M$ be a proper nondegenerate D-subspace of dimension ≥ 4 and contain a, b_1 and b_2. If N is Γ-arithmetic, then so is $N' := N + Db'$ for every $b' \in \mathcal{B} \smallsetminus N$.*

Proof when N' is degenerate We verify that the assumptions of Lemma 4.1 are satisfied by $\Gamma \cap U_M(N')$. The radical of N' is necessarily spanned by an element of the form $c := b' - b$, where $b \in N$ is characterized by the property that $\langle x, b\rangle = \langle x, b'\rangle$ for all $x \in N$. So N maps isomorphically onto $N'/Dc = \overline{N}'$. In particular, the natural map $U(N) \to U(\overline{N}')$ is an isomorphism, and hence $\Gamma \cap U_M(N')$ maps onto an arithmetic subgroup of $U(\overline{N}')$.

Let n be a positive integer such that $T_b^n \in \Gamma$. Then

$$T_{b'}^n T_b^{-n}(x) = x - n\langle x, b\rangle b + n\langle x, b'\rangle b' = x + n\langle x, b\rangle c + n\langle x, c\rangle b + n\langle x, c\rangle c = E(c, nb, n)(x).$$

So $E(c, nb, n) \in \Gamma \cap R_u(U_M(N'))$ and this element has image $c \otimes nb$ in $c \otimes \overline{N}'$. It then follows from Lemma 4.1 that N' is Γ-arithmetic. □

Proof when N' is nondegenerate Then $N_1' := N' \cap b'^{\perp}$ is degenerate with radical spanned by b'. We first prove that N_1' is Γ-arithmetic by verifying the assumptions of Lemma 4.1. The subspace $N_1 := N \cap b'^{\perp}$ supplements b' in N_1'. It is therefore nondegenerate and maps isomorphically onto $\overline{N}_1' = N_1'/Db'$. This enables us to regard $U(N_1)$ as a subgroup of $U_M(N_1')$ that acts trivially on both b' and its orthogonal projection in N.

Since $\Gamma \cap U_M(N)$ is arithmetic in $U_M(N)$, its subgroup $\Gamma \cap U(N_1)$ is arithmetic in $U(N_1)$ and has arithmetic image in $U(\overline{N}_1')$. We show that $E(b', c') \in \Gamma$ for some $c' \in N_1' \smallsetminus Db'$. Then Lemma 4.1 will imply that N_1' is Γ-arithmetic.

For this we recall that $c := b_2 - b_1$ is perpendicular to b_1 and has nonzero image in $Db_1^{\perp}/Db_1$. Let $\gamma \in \Gamma(a, b')\Gamma(a, b_1) \subset \Gamma(N')$ take b_1 to b'. Since c is perpendicular to b_1, $c' := \gamma(c) \in N'$ is perpendicular to $\gamma(b_1) = b'$ (so lies in N_1') and has nonzero image $[c']$ in $\overline{N}_1' \cong N_1$. Since $E(b', c') = E(c', b')^{-1}$ is a Γ-conjugate of $E(c, b_1)^{-1}$, it lies in Γ.

Now $U(N')_{b'} = U(N' \cap b'^{\perp})$ and so $\Gamma \cap U(N')_{b'}$ is arithmetic in $U(N')_{b'}$. As $a \in \Gamma b'$, the same is true for $\Gamma \cap U(N')_a$. Since a and b' span a copy of $\mathcal{H}^2(D)$, their $U(N')$-stabilizers contain the unipotent radicals of opposite parabolic subgroups of $U(N')$. The real rank of $U(N')$ is ≥ 2, so Theorem 3.3 applies and tells us that N' is Γ-arithmetic. □

The case when $\dim_D V^\dagger > 1$ The same scheme works when $\dim_D V^\dagger > 1$. The difference is that we deal with larger hyperbolic packets, to wit, the images of hyperbolic embeddings $(a, b)\colon \mathcal{H}^2(V^\dagger) \hookrightarrow M$. The essential difference is that we start off in a better position, since we begin with a $V^\dagger$-hyperbolic embedding $j : \mathcal{H}^2(V^\dagger) \hookrightarrow M$ and we already know that its image N is Γ-arithmetic and that $U_M(N) \cong U(N)$ has real rank ≥ 2.

5 Finding liftable mapping classes

In this section the G-covering $S \to S_G$ is as in the introduction and $A \subset S_G^\circ$ is a nonempty closed one-submanifold such that the covering is trivial over A and is connected over $S \smallsetminus A$. We also choose a connected component α of A, so that $A' := A \smallsetminus \alpha$ might be empty. We orient α and regard it as the oriented image of an embedding of the circle in S_G°. We will see that this gives rise to enough copies of $\Gamma(G)$ in the representation of the G-equivariant mapping classes as to satisfy the hypotheses of our arithmeticity criterion, Theorem 4.2.

We denote by $S_G(\alpha)$ the singular surface obtained from S_G by contracting α to a point (that we will denote by ∞). Its topological normalization is a closed connected surface, denoted by $\widehat{S}_G(\alpha)$, whose genus is one less than that of S_G. The surface $\widehat{S}_G(\alpha)$ comes with two points over ∞ and the orientation of α enables us to tell them apart: we let p_- be "to the left" of α and p_+ be "to the right" of α. If we regard A' also as a submanifold of $\widehat{S}_G^\circ(\alpha)$, then the surjection $\widehat{S}_G^\circ(\alpha) \to S_G^\circ(\alpha)$ defines a map from the set $\Pi(\widehat{S}_G^\circ(\alpha) \smallsetminus A'; p_-, p_+)$ of path homotopy classes in $\widehat{S}_G^\circ(\alpha) \smallsetminus A'$ from p_- to p_+ to the fundamental group $\pi_1(S_G^\circ(\alpha), \infty)$. This map is injective. We do the same (in a G-equivariant manner) for the preimage of $S_G \smallsetminus \alpha$ in S, and thus get G-covers $S(\alpha) \to S_G(\alpha)$ and $\widehat{S}^\circ(\alpha) \to \widehat{S}_G(\alpha)$, and G-orbits $P_\infty \subset S(\alpha)^\circ$ and $P_\pm \subset \widehat{S}^\circ(\alpha)$, so that we end up with the diagram below (in which the vertical maps are G-coverings):

$$\begin{tikzcd}
\pi^{-1}\alpha \arrow[rr, two heads] \arrow[dr, hook] \arrow[dd, two heads] & & P_\infty \arrow[dr, hook] \arrow[dd, two heads] & & P_- \cup P_+ \arrow[ll, two heads] \arrow[dr, hook] \arrow[dd, two heads] & \\
& S \arrow[rr, two heads] \arrow[dd, two heads] & & S(\alpha) \arrow[dd, two heads] & & \widehat{S}(\alpha) \arrow[ll, two heads] \arrow[dd] \\
\alpha \arrow[rr, two heads] \arrow[dr, hook] & & \{\infty\} \arrow[dr, hook] & & \{p_-, p_+\} \arrow[ll, two heads] \arrow[dr, hook] & \\
& S_G \arrow[rr, two heads] & & S_G(\alpha) & & \widehat{S}_G(\alpha) \arrow[ll, two heads]
\end{tikzcd}$$

This construction comes with G-equivariant bijections $P_- \cong P_\infty \cong P_+$. Our assumption on the covering $S \to S_G$ amounts to the properties that $\widehat{S}(\alpha)$ is connected and stays so if we remove the preimage of A', and the three G-orbits P_∞ and $P_\pm$ are regular. So the choice of a point in P_∞ (which is equivalent to the choice of a lift $\tilde{\alpha}$ of α) identifies these G-sets with G (on which G acts by left translation). In particular, we thus identify $\mathrm{Iso}_G(P_-, P_+) \cong \mathrm{Aut}_G(P_\infty)$ with G (where $g \in G$ acts on G by right translation over g^{-1}).

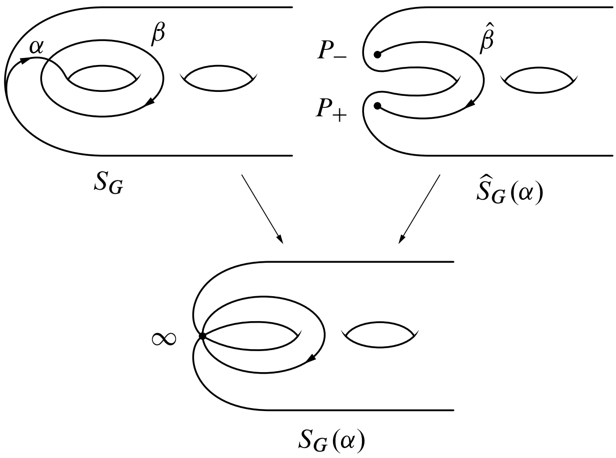

Figure 1: The surface S_G, its quotient $S_G(\alpha)$ and the normalization $\widehat{S}_G(\alpha)$.

For any path in $\widehat{S}_G^\circ(\alpha) \smallsetminus A'$ from p_- to p_+, the G covering is trivial over it, so that we have an associated G-bijection $P_- \cong P_+$. Since the G-covering over $\widehat{S}_G^\circ(\alpha) \smallsetminus A'$ is connected, the resulting map

$$\Pi(\widehat{S}_G^\circ(\alpha) \smallsetminus A'; p_-, p_+) \to \mathrm{Iso}_G(P_-, P_+) \cong \mathrm{Aut}_G(P_\infty)$$

is onto by standard covering theory. We will say that an element of $\Pi(\widehat{S}_G^\circ(\alpha) \smallsetminus A'; p_-, p_+)$ is *G-trivial* if its image in $\mathrm{Aut}_G(P_\infty)$ under the above map is the identity. Such elements make up a coset for the kernel of the natural homomorphism $\pi_1(\widehat{S}_G^\circ(\alpha) \smallsetminus A', p_-) \to G$.

Lemma 5.1 *Every element of $\Pi(\widehat{S}_G^\circ(\alpha) \smallsetminus A'; p_-, p_+)$ is representable by some arc (ie some embedded unit interval) in $\widehat{S}_G^\circ(\alpha) \smallsetminus A'$ from p_- to p_+. We can arrange that this arc lifts to an embedding of the circle $\mathbb{R}/\mathbb{Z}$ in S_G° which meets α in a single point with intersection number one. In particular, every element of $\mathrm{Aut}_G(P)$ is realized by the monodromy along an embedded circle β which does not meet A', and meets α in one point only and does so transversally with intersection number one.*

Proof We first represent the homotopy class by an immersion of the unit interval with only transverse self-intersections. That number of self-intersections is finite and if this number is positive, we lower it by moving the last point of self-intersection towards p_+ and then slide the path over p_+. By iterating this procedure we obtain a representative which is an embedding. It is clear that we can make this arc lift to an embedded circle. The second assertion then follows. □

We choose a lift $\tilde\alpha$ of α and write $a \in H_1(S)$ for its homology class. Since Ra is an isotropic sublattice of $H_1(S)$, we have that $\langle a, a\rangle = 0$. We let $I \subset H_1(S)$ be the homology supported by the preimage of A'; this is a free R-submodule (where as before, $R = \mathbb{Z}G$) with a generator for every connected component of A'. It is clear that $Ra + I$ is isotropic. We saw that the lift $\tilde\alpha$ identifies $\mathrm{Aut}_G(P)$ with the group G with G acting on itself by right translations. Lemma 5.1 above shows that all such elements are obtained from a loop of the type described there. From that lemma we also derive:

Corollary 5.2 *Let $b \in I^{\perp}$ be such that $a \cdot b = 1$ and $ga \cdot b = 0$ for $g \in G \smallsetminus \{1\}$. Then some $b' \in b + Ra$ can be represented by a lift of an embedded circle β in $S_G^{\circ} \smallsetminus A'$ which meets α transversally at a unique point* (*and for which necessarily $\alpha \cdot \beta = 1$*)*. In particular, b' is R-isotropic and the R-linear map defined by*

$$\mathcal{H}^2(R) \to I^{\perp}, \qquad \boldsymbol{e} \mapsto a, \quad \boldsymbol{f} \mapsto b',$$

is an embedding of skew-hermitian R-modules whose orthogonal complement supplements its image; *we obtain a basic hyperbolic summand of the R-module $H_1(S)$.*

Proof It is not difficult to see that the homology class b is representable by a map from the circle to S° which meets the preimage of α exactly once (and hence in a point of $\tilde{\alpha}$) and does not meet the preimage of A'. We apply Lemma 5.1 to its image in $S_G^{\circ} \smallsetminus A'$, or rather to the resulting arc in $\widehat{S}_G^{\circ}(\alpha) \smallsetminus A'$ which connects p_- with p_+; this then produces an embedding β of the circle in $S_G^{\circ} \smallsetminus A'$ which meets α only once and with intersection number one over which the G-covering is trivial. The lift $\tilde{\beta}$ of β which meets $\tilde{\alpha}$ defines a homology class b' which differs from b by a class supported by the preimage of α, that is, an element of Ra.

The proof of the last paragraph is straightforward. $\square$

Let us call an ordered pair (a, b) in $H_1(S)$ *R-hyperbolic* if $\langle a, a\rangle = \langle b, b\rangle = 0$ and $\langle a, b\rangle = 1$. Such a pair defines a basic hyperbolic summand $Ra + Rb \subset H_1(S)$ and gives rise to an embedding of $\Gamma(G)$ in $\mathrm{Sp}(H_1(S))^G$. We shall denote the latter's image by $\Gamma(a, b)$ and the image of $\Gamma_o(G)$ by $\Gamma_o(a, b)$. We write $\mathcal{B}_a$ for the set of $b \in H_1(S)$ for which the pair (a, b) is R-hyperbolic, and $\Gamma_o(a)$ and $\Gamma(a)$ for the subgroups of $\mathrm{Sp}(H_1(S))^G$ generated by its subgroups $\Gamma_o(a, b)$ and $\Gamma(a, b)$, respectively, with $b \in \mathcal{B}_a$.

Fix a basepoint p on α and write $\tilde{p}$ for its preimage in $\tilde{\alpha}$.

Let $h \in G$ and regard h as an element of $\mathrm{Aut}_G(P)$. Let β be as in Lemma 5.1, which we may (and will) assume to meet α in p such that the lift $\tilde{\beta}$ of β (as an arc) begins in $\tilde{p}$ ends in $h\tilde{p}$. Let S_G^{β} be a closed regular neighborhood of $\alpha \cup \beta$ in $S_G^{\circ} \smallsetminus A'$. This is a compact genus-one surface whose boundary ∂S_G^{β} is connected. The homotopy class of this boundary (with its natural orientation) is in the free homotopy class of the commutator $[\beta]^{-1}[\alpha]^{-1}[\beta][\alpha]$ — we write path composition functorially, so the order of travel is read from right to left. This commutator has trivial image in G (since $[\alpha]$ has), and so the G-covering $S \to S_G$ is trivial over ∂S_G^{β}. The preimage of ∂S_G^{β} in S is the boundary of the preimage S^{β} of S_G^{β} and the Dehn twist along ∂S_G^{β} lifts in a G-equivariant manner to a multi-Dehn twist D^{β} along that boundary. The following lemma generalizes one of the constructions given in [3] for the case when G is cyclic (in that paper they are depicted as Figures 2 and 3).

Lemma 5.3 *The multi-Dehn twist D^{β} acts on $H_1(S)$ as $T_a(2 - e_h - e_h^{\dagger})$* (where we use formula (2), noting that $2 - e_h - e_h^{\dagger} \in R_+$).

Proof The lift of the commutator $[\beta]^{-1}[\alpha]^{-1}[\beta][\alpha]$ that passes through $\tilde{p}$ first traverses the embedded circle $\tilde{\alpha}$, then traverses $\tilde{\beta}$, then traverses the circle $h\tilde{\alpha}$ in the opposite direction and then returns via the inverse of $\tilde{\beta}$ to $\tilde{p}$. So the homology class of this lift of the commutator (and hence of the corresponding lift of ∂S_G^β) is $a - ha$. If we replace $\tilde{p}$ by $g\tilde{p}$ with $g \in G$, then this replaces a by ga and h by ghg^{-1}, so that the corresponding class is $ga - gha = g(1-h)a$. By a standard formula, the resulting action on $H_1(S)$ is given by

$$\begin{aligned} D_*^\beta(x) &= x + \sum_{g\in G}(x\cdot g(1-h)a)g(1-h)a \\ &= x + \sum_{g\in G}((x\cdot ga)ga - (x\cdot gha)ga - (x\cdot ga)gha + (x\cdot gha)gha) \\ &= x + 2\langle x,a\rangle a - \langle x,ha\rangle a - \langle x,a\rangle ha = T_a(2-e_h-e_h^\dagger)(x). \end{aligned}$$ □

Proposition 5.4 *Let $b \in H_1(S)$ be such that (a,b) is an R-hyperbolic pair. Then the image of* $\mathrm{Diff}^+(S)^G \to \mathrm{Sp}(H_1(S))^G$ *contains* $\Gamma(a,b)$.

Proof We first show this for $\Gamma_0(a,b)$. Let β be as in Corollary 5.2 (and thus represent an element of $b+Ra$). The diffeomorphisms of S_G with support in the interior of S_G^β have, as their image in the mapping class group of S_G, a centrally extended copy of $\mathrm{SL}(2,\mathbb{Z})$ with the central subgroup generated by the Dehn twist along the boundary of S_G^β. This Dehn twist acts trivially on $H_1(S_G^\beta)$. These diffeomorphisms lift to diffeomorphisms of S with support in S^β with the central subgroup acting trivially on $H_1(S)$. We thus obtain in the image of $\mathrm{Diff}^+(S)^G \to \mathrm{Sp}(H_1(S))^G$ a copy of $\mathrm{SL}_2(\mathbb{Z})$.

The multi-Dehn twist associated to α acts on $H_1(S)$ as $x \mapsto x + \sum_{g\in G}(x\cdot ga)ga = x + \langle x,a\rangle a = T_a(1)(x)$. By Lemma 5.3 the image of $\mathrm{Diff}^+(S)^G \to \mathrm{Sp}(H_1(S))^G$ also contains the transvections $T_a(2-e_h-e_h^\dagger)$ for all $h \in G$. Hence that image contains all of $T_a(R_{++})$. This proves that the image of $\mathrm{Diff}^+(S)^G \to \mathrm{Sp}(H_1(S))^G$ contains $\Gamma_o(a,b)$.

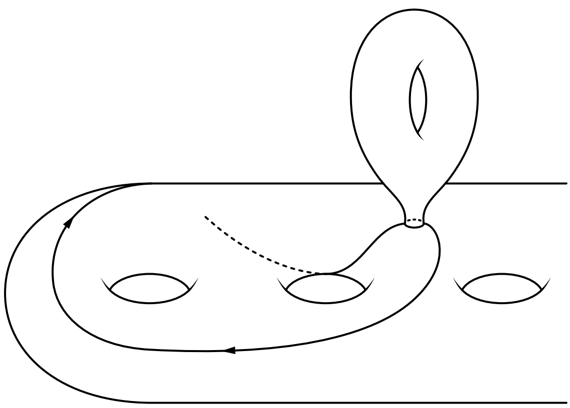

Figure 2: Point pushing (or rather, "small circle pushing") the genus-one surface S^β along S_G/S_G^β.

So it remains to show that the image of G in $\Gamma(a,b)$ is realized by $\mathrm{Diff}^+(S)^G$. For this we use mapping classes of *push type*. Consider the smooth surface S_G/S_G^β that is obtained as a quotient of S_G by contracting S_G^β to a point (that we shall call q). If we do the same for the connected components of S^β in G we get a G-cover $S/\!\!/S^\beta \to S_G/S_G^\beta$ fitting in the commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "G"'] & S/\!\!/S^\beta \arrow[d, "G"] \\ S_G \arrow[r] & S_G/S_G^\beta \end{tikzcd}$$

The covering on the right does not branch over q, and so its preimage Q in $S/\!\!/S^\beta$ is a regular G-orbit. For every $g \in G$ there is a closed loop γ of S_G/S_G^β based at q which avoids branch points and induces in Q right multiplication by g. The corresponding point-pushing map on S_G/S_G^β (chosen to fix branch points) lifts to a G-equivariant diffeomorphism φ of $S/\!\!/S^\beta$ that extends this permutation.

Such a point-pushing map is isotopic to the identity on S_G/S_G^β, and hence the same is true for its lift φ. In particular, φ acts trivially on $H_1(S/\!\!/S^\beta)$. It is not difficult to see that the point-pushing map and its lift φ can be "lifted" to S and S_G by "small circle pushing". Since $H_1(S\smallsetminus S^\beta, \partial(S\smallsetminus S^\beta)) \to H^1(S/\!\!/S^\beta)$ is an isomorphism, the action on $H_1(S\smallsetminus S^\beta, \partial(S\smallsetminus S^\beta))$ will be trivial. Clearly the components of S_β will be permuted according to the right action of g, and thus $g \in \Gamma(a,b)$ is realized in the image of $\mathrm{Diff}^+(S)^G$. □

Part (ii) of the corollary below establishes the Putman–Wieland property of Theorem 1.1.

Corollary 5.5 (hyperbolic generation) *The following properties hold*:

(i) *The subset* $\{a\}\cup\mathcal{B}_a$ *of* $H_1(S;\mathbb{Q})$ *spans the latter over* $\mathbb{Q}G$.

(ii) *The subgroup* $\Gamma_o(a)$ *of* $\mathrm{Sp}(H_1(S))$ *generated by the subgroups* $\Gamma_o(a,b)$ *with* $b\in\mathcal{B}_a$ (*and hence* $\mathrm{Diff}^+(S)^G$) *has no nonzero finite orbit in* $H_1(S)$.

Proof Let $c \in H_1(S;\mathbb{Q})$ be perpendicular to $\{a\}\cup\mathcal{B}_a$. We prove that c is then perpendicular to every $x \in H_1(S)$; since the intersection form is nondegenerate, this will imply that $c=0$ and hence that the $\mathbb{Z}G$-submodule of $H_1(S)$ generated by $\{a\}\cup\mathcal{B}_a$ is of finite index. To this end, let $b\in\mathcal{B}_a$. Then $x' := (1+\langle x,a\rangle)b + x$ has the property that $\langle a,x'\rangle = 1$. By Corollary 5.2 (a,x'') is a hyperbolic pair for some $x'' \in x' + Ra$, so that $0=\langle x'',c\rangle = \langle x,c\rangle$.

For (ii) it suffices to show that for every finite-index subgroup $\Gamma\subset\Gamma_o(a)$, the fixed part $H_1(S)^\Gamma$ is trivial. Note that $H_1(S)^{\Gamma_o(a,b)}$ is the perp of $Ra+Rb$ in $H_1(S)$ with respect to the intersection pairing. The $\Gamma_o(a,b)$-invariant part of $H_1(S)$ is not changed if we replace $\Gamma_o(a,b)$ by the finite-index subgroup $\Gamma\cap\Gamma_o(a,b)$, and hence $H_1(S)^\Gamma$ is perpendicular to $Ra+Rb$. As this is true for all $b\in\mathcal{B}_a$ and $\{a\}\cup\mathcal{B}_a$ generates $H_1(S)$ as an R-module, it follows that $H_1(S)^\Gamma$ must be trivial. □

We can now finish the proof of our main theorem.

Proof of Theorem 1.1 Let us denote the image of $\mathrm{Diff}^+(M)^G$ in $\mathrm{Sp}(H_1(S;\mathbb{Q}))^G$ by Γ and the image of the latter in the factor $U(H_1(S)[\chi])$ of $\mathrm{Sp}(H_1(S;\mathbb{Q}))^G = \prod_\chi U(H_1(S)[\chi])$ by Γ_χ. By combining Corollary 5.5 with Theorem 4.2, we see that under the assumptions of (ii), Γ_χ is an arithmetic subgroup of $U(H_1(S)[\chi])$.

Note that $\Gamma^\chi := \Gamma \cap \mathrm{EU}(H_1(S)[\chi])$ is a normal subgroup of Γ_χ. It remains to see that Γ^χ is of finite index in Γ_χ. Since $\mathrm{EU}(H_1(S)[\chi])$ is almost simple and of real rank ≥ 2, it follows from a general result of Margulis [4, Assertion (A), Chapter VIII] that this is the case unless Γ^χ meets $\mathrm{EU}(H_1(S)[\chi])$ in the center. But Proposition 5.4 shows that Γ^χ contains a subgroup isomorphic to $\Gamma_o(\chi)$, and so this last possibility does not occur. □

Remark 5.6 This argument shows that if we are in the setting of (i) (triviality of the cover over a genus-one subsurface of S_G), then Γ_χ is an arithmetic subgroup of $U(H_1(S)[\chi])$, unless $V_\chi^\dagger \cong D_\chi$ and the image of G in V_χ is of the type given in Proposition 3.1. Denote that image by G_χ and let G^χ stand for the kernel of $G \to G_\chi$. Since $H_1(S)[\chi]$ already arises on the G_χ-cover $S_{G^\chi} \to S_G$ in the sense that $H_1(S)[\chi] = H_1(S_{G^\chi})[\chi]$, we may for the arithmeticity question just as well focus on this intermediate cover.

In Proposition 3.1(i)–(ii), that is, when D_χ equals $\mathbb{Q}$, the Gaussian field or the Eisenstein field, then G_χ is a group of roots of unity and hence cyclic. When the genus of S_G is at least three, we can always find a closed subsurface of genus two over which the covering $S_{G^\chi} \to S_G$ is trivial, and so Γ_χ is then arithmetic. In the remaining cases, G_χ is a particular kind of Kleinian group. It might well be that a G_χ-cover is then also trivial over the complement of a genus-two subsurface of the quotient surface when the genus of the latter is ≥ 3. If true, then it would follow that Γ would always be arithmetic if the genus of S_G is at least three and the covering is trivial over a genus-one subsurface of S_G.

References

[1] **F Grunewald**, **M Larsen**, **A Lubotzky**, **J Malestein**, *Arithmetic quotients of the mapping class group*, Geom. Funct. Anal. 25 (2015) 1493–1542 MR Zbl

[2] **A J Hahn**, **O T O'Meara**, *The classical groups and K-theory*, Grundl. Math. Wissen. 291, Springer (1989) MR Zbl

[3] **E Looijenga**, *Prym representations of mapping class groups*, Geom. Dedicata 64 (1997) 69–83 MR Zbl

[4] **G A Margulis**, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Math. 17, Springer (1991) MR Zbl

[5] **D Mumford**, *Abelian varieties*, Tata Inst. Fund. Res. Stud. Math. 5, Oxford Univ. Press (1970) MR Zbl

[6] **A Putman**, **B Wieland**, *Abelian quotients of subgroups of the mappings class group and higher Prym representations*, J. Lond. Math. Soc. 88 (2013) 79–96 MR Zbl

[7] **M S Raghunathan**, *A note on generators for arithmetic subgroups of algebraic groups*, Pacific J. Math. 152 (1992) 365–373 MR Zbl

[8] **T A Springer**, *Linear algebraic groups*, 2nd edition, Progr. Math. 9, Birkhäuser, Boston, MA (1998) MR Zbl

[9] **T N Venkataramana**, *On systems of generators of arithmetic subgroups of higher rank groups*, Pacific J. Math. 166 (1994) 193–212 MR Zbl

[10] **G E Wall**, *The structure of a unitary factor group*, Inst. Hautes Études Sci. Publ. Math. 1 (1959) 23 MR Zbl

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The geometry of subgroup embeddings and asymptotic cones

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Given a finitely generated subgroup H of a finitely generated group G and a nonprincipal ultrafilter ω, we consider a natural subspace, $\mathrm{Cone}^{\omega}_{G}(H)$, of the asymptotic cone of G corresponding to H. Informally, this subspace consists of the points of the asymptotic cone of G represented by elements of the ultrapower H^{ω}. We show that the connectedness and convexity of $\mathrm{Cone}^{\omega}_{G}(H)$ detect natural properties of the embedding of H in G. We begin by defining a generalization of the distortion function and show that this function determines whether $\mathrm{Cone}^{\omega}_{G}(H)$ is connected. We then show that whether H is strongly quasiconvex in G is detected by a natural convexity property of $\mathrm{Cone}^{\omega}_{G}(H)$ in the asymptotic cone of G.

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1 Introduction

The asymptotic cone of a group G is a metric space which captures certain aspects of the coarse geometry of G. Roughly speaking, the asymptotic cone is how the group looks from infinitely far away, and is constructed by taking a certain limit of scaled-down copies of the group viewed as a metric space. The roots of asymptotic cones come from a paper of Gromov proving that finitely generated groups of polynomial growth are nilpotent [8]. Van den Dries and Wilkie added nonstandard analysis to the construction in this paper, formally introducing asymptotic cones [4]. Since then, several other standard algebraic and geometric properties of groups have been shown to have natural parallels in their asymptotic cones. For instance, a finitely generated group is virtually abelian if and only if all of its asymptotic cones are quasi-isometric to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ (see Gromov [9]), and a finitely generated group is hyperbolic if and only if all of its asymptotic cones are $\mathbb{R}$-trees [9].

Given a group G and an ultrafilter ω, we will denote the asymptotic cone of G with respect to ω by $\mathrm{Cone}^{\omega}(G)$. Our goal here is to study the way that geometric properties of embeddings of subgroups in groups can be detected using asymptotic cones. In order to accomplish this, we define a natural subspace of $\mathrm{Cone}^{\omega}(G)$ corresponding to a subgroup H. Essentially, points in the asymptotic cone of a group G can be represented by certain elements of the ultrapower G^{ω}. We denote by $\mathrm{Cone}^{\omega}_{G}(H)$ the subspace of $\mathrm{Cone}^{\omega}(G)$ consisting of points with a representative from H^{ω}. For the formal definition of this subspace, see Definition 4.10.

The first property of $\mathrm{Cone}^{\omega}_{G}(H)$ we study is connectedness. We show that whether $\mathrm{Cone}^{\omega}_{G}(H)$ is connected is closely related to a generalization of the distortion function of H in G.

Definition 1.1 Let H be a subgroup of a group G, with $G = \langle X \rangle$ and $H = \langle Y \rangle$ where X and Y are finite sets. The *distortion function* of H in G with respect to X and Y is defined by the formula

$$\Delta_{H,Y}^{G,X}(n) = \max\{|h|_Y : h \in H,\, |h|_X \le n\},$$

where $|h|_Y$ denotes the word length of h with respect to the generating set Y. A subgroup H of a group G is called *undistorted* if $\Delta_{H,Y}^{G,X}$ is bounded from above by a linear function.

We consider distortion up to the following equivalence relation:

Definition 1.2 For nondecreasing functions $f, g\colon \mathbb{N} \to \mathbb{N}$, we write that $f \preceq g$ if there exists a constant C such that $f(n) \le Cg(Cn)$ for all $n \in \mathbb{N}$. We write $f \sim g$ if $f \preceq g$ and $g \preceq f$.

Under this equivalence, distortion is independent of the choice of the finite generating set. We denote by Δ_H^G the distortion function of H in G for some choice of the finite generating set X.

Definition 1.3 Assume that X is a finite generating set for a group G, and H is a subgroup of G such that X contains a generating set for H. We define the *generalized distortion function* $\mu_H^{G,X}(m,n)\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by the formula

$$\mu_H^{G,X}(m,n) = \max\{|h|_{Y_m} : h \in H,\, |h|_X \le n\} = \Delta_{H,Y_m}^{G,X}(n),$$

where $Y_m = \{h \in H : |h|_X \le m\}$.

We consider generalized distortion functions up to the following equivalence:

Definition 1.4 Given two functions $f, g\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ which are nonincreasing in the first variable and nondecreasing in the second variable, we write $f \preceq g$ if there exists a constant $C \in \mathbb{N}$ such that

$$f(Cm, n) \le Cg(m, Cn) + C$$

for all $m, n \in \mathbb{N}$, and we say that $f \cong g$ if $f \preceq g$ and $g \preceq f$.

Under this equivalence, $\mu_H^{G,X}(n)$ is independent of the choice of the finite generating set X of G, so we use μ_H^G to mean $\mu_H^{G,X}$ where X is some finite generating set of G. For example, if H is undistorted in G, then

$$\mu_H^G(m,n) \cong \frac{n}{m}.$$

We show that the generalized distortion function determines whether $\mathrm{Cone}_G^\omega(H)$ is connected. Specifically, we prove the following result, which also shows that for such a subspace, connectedness is equivalent to path connectedness.

Definition 1.5 We say that a function $f\colon \mathbb{R}^{\ge 1} \times \mathbb{R}^{\ge 0} \to \mathbb{R}$ is *homogeneous* if $f(r,s) = g(s/r)$ for some function $g\colon \mathbb{R}^{\ge 0} \to \mathbb{N}$.

Theorem 1.6 (Theorem 4.13) *For any finitely generated group G and any subgroup H, the following conditions are equivalent*:

(i) *H is finitely generated and $\mu_H^G(m,n)$ is bounded from above by a homogeneous function.*

(ii) *$\mathrm{Cone}_G^\omega(H)$ is path connected for all nonprincipal ultrafilters ω.*

(iii) *$\mathrm{Cone}_G^\omega(H)$ is connected for all nonprincipal ultrafilters ω.*

This theorem enables us to relate the ordinary distortion function to the connectedness of $\mathrm{Cone}_G^\omega(H)$, and to construct pairs $H \le G$ such that $\mathrm{Cone}_G^\omega(H)$ is disconnected, but the distortion of H in G is small. Consider the following properties of a finitely generated subgroup H of a finitely generated group G:

(a) H is undistorted in G.

(b) $\mathrm{Cone}_G^\omega(H)$ is connected for all nonprincipal ultrafilters ω.

(c) Δ_H^G is bounded by a polynomial function.

The following theorem collects the relationship between these three properties:

Theorem 1.7 (Theorem 4.19) *For any finitely generated subgroup H of a finitely generated group G, the following implications hold*:

$$(a) \implies (b) \implies (c)$$

Further, the missing implications do not hold. Specifically:

(i) *For any $k \in \mathbb{N}$, there exists a finitely generated group G and a finitely generated subgroup H of G such that $\Delta_H^G(n) \sim n^k$ and $\mathrm{Cone}_G^\omega(H)$ is connected for any nonprincipal ultrafilter ω.*

(ii) *For any real number $\epsilon > 0$, there exists a finitely generated group G with a finitely generated subgroup H such that $\Delta_H^G(n) \preceq n^{1+\epsilon}$ but $\mathrm{Cone}_G^\omega(H)$ is disconnected for some nonprincipal ultrafilter ω.*

Next, we show that the property of a subgroup being strongly quasiconvex, introduced independently by Tran and Genevois [7; 17], can be detected by a natural property of the embedding of $\mathrm{Cone}_G^\omega(H)$ in $\mathrm{Cone}^\omega(G)$.

Definition 1.8 A subgroup H of a group G with finite generating set X is said to be *quasiconvex* if there exists a number M such that any geodesic in the Cayley graph $\Gamma(G,X)$ connecting two points in H is contained in the M neighborhood of H. H is said to be *strongly quasiconvex* if for all real numbers $\lambda \ge 1$ and $C \ge 0$ there exists a constant $N(\lambda,C)$ such that any (λ,C)-quasigeodesic in $\Gamma(G,X)$ connecting two points in H is entirely contained in the N neighborhood of H.

In general, quasiconvexity is not independent of the choice of the finite generating set of G. For instance, in the group $\mathbb{Z}\times\mathbb{Z} = \langle a\rangle \times \langle b\rangle$, the subgroup $\langle ab\rangle$ is not quasiconvex with respect to the generating set $\langle a,b\rangle$, but is quasiconvex with respect to the generating set $\langle ab,a\rangle$. In the case where G is hyperbolic, quasiconvexity is independent of the choice of the finite generating set.

We have the following relationship between these properties of a subgroup H of a finitely generated group G:

$$\text{strongly quasiconvex} \implies \text{quasiconvex} \implies \text{finitely generated and undistorted.}$$

None of the reverse implications hold. To see this again consider $G = \mathbb{Z} \times \mathbb{Z} = \langle a \rangle \times \langle b \rangle$. The subgroup $\langle ab \rangle$ is undistorted but not quasiconvex, and the subgroup $\langle a \rangle$ is quasiconvex but not strongly quasiconvex. However, in the case when G is hyperbolic, all of these properties are in fact equivalent.

Strong quasiconvexity is a generalization of quasiconvexity that is preserved under quasi-isometry in general. Tran [17] characterized strongly quasiconvex subgroups based on a certain divergence function, and showed that they satisfy many properties of quasiconvex subgroups of hyperbolic groups. Specifically, any strongly quasiconvex subgroup is undistorted, has finite index in its commensurator, and the intersection of any two strongly quasiconvex subgroups is strongly quasiconvex. Examples of strongly quasiconvex subgroups include peripheral subgroups of relatively hyperbolic groups and hyperbolically embedded subgroups of finitely generated groups.

We show that the property of being strongly quasiconvex is equivalent to a natural property of the embedding of $\mathrm{Cone}^{\omega}_{G}(H)$ in $\mathrm{Cone}^{\omega}(G)$.

Definition 1.9 We say that a subspace T of a metric space S is *strongly convex* if any simple path in S starting and ending in T is entirely contained in T.

Theorem 1.10 (Theorem 5.12) *Let H be a finitely generated subgroup of a finitely generated group G. H is strongly quasiconvex in G if and only if $\mathrm{Cone}^{\omega}_{G}(H)$ is strongly convex in $\mathrm{Cone}^{\omega}(G)$ for all nonprincipal ultrafilters ω.*

This characterization gives useful information about the structure of the asymptotic cones of groups with strongly quasiconvex subgroups. For instance:

Theorem 1.11 (Theorem 5.13) *If G is a finitely generated group containing an infinite, infinite-index strongly quasiconvex subgroup H, then all asymptotic cones of G contain a cut point.*

A precursor to Theorems 1.10 and 1.11 can be found in [2], where Behrstock showed that any asymptotic cone of a mapping class group contains an isometrically embedded copy of an $\mathbb{R}$-tree, and that this $\mathbb{R}$-tree is strongly convex in the asymptotic cone. This is then used to deduce that any asymptotic cone of a mapping class group contains a cut point. I would like to thank Jason Behrstock for pointing out this connection.

Combining Theorem 1.11 with a result of Druțu and Sapir [6] gives the following result:

Corollary 1.12 (Corollary 5.15) *If G is a finitely generated group containing an infinite, infinite-index strongly quasiconvex subgroup, then G does not satisfy a law.*

This result can be applied to show, for instance, that solvable groups and groups satisfying the law $x^n = 1$ for some $n \in \mathbb{N}$ cannot have infinite, infinite-index strongly quasiconvex subgroups.

Organization Section 2 covers some necessary background on asymptotic cones and establishes our notation. Section 3 establishes some basic properties of the generalized distortion function and formulates a relationship between the generalized distortion function and the distortion function. Section 4 contains the proof of Theorems 1.6 and 1.7. Finally, Section 5 contains the proof of Theorems 1.10 and 1.11.

2 Background

In this section, we provide some background and fix our notation for asymptotic cones.

Recall that given an ultrafilter ω and any bounded sequence of real numbers, (r_i), $\lim^{\omega}(r_i)$ exists and is unique.

Now let (S,d) be a metric space, and let c_i be an unbounded strictly increasing sequence of positive real numbers. Denote by d_i the metric on S defined by $d_i(x,y)=d(x,y)/c_i$. We call the sequence (c_i) the *scaling sequence*.

Definition 2.1 Given a metric space (S,d), a scaling sequence (c_i), and an infinite sequence of points $z=(s_i)$ in S, denote by $S_z^{\mathbb{N}}$ the set of infinite sequences (t_i) in S such that $d_i(s_i,t_i)$ is bounded. The sequence (s_i) is called the *observation point*.

Definition 2.2 Given $(x_i),(y_i)\in S_z^{\mathbb{N}}$, let $d^*((x_i),(y_i))=\lim^{\omega} d_i(x_i,y_i)$.

Note that this is a bounded sequence so the limit exists. However, in general d^* will not be a metric, as there can be different sequences (x_i) and (y_i) such that $d^*((x_i),(y_i))=0$.

Definition 2.3 We will denote by $\mathrm{Cone}_z^{\omega}((d_i),S)$ the metric space that results from quotienting the pseudometric d^* by the equivalence relation $(x_i)\sim(y_i)$ if $d^*((x_i),(y_i))=0$. We will denote the resultant metric by d_S^{ω}. When the choice of the basepoint or the scaling sequence is clear, we will simply write $\mathrm{Cone}^{\omega}(S)$. We will denote the equivalence class of (x_i) by $(x_i)^{\omega}$, so $d_S^{\omega}((x_i)^{\omega},(y_i)^{\omega})=d^*((x_i),(y_i))$.

Definition 2.4 A map f between two metric spaces (S,d_S) and (T,d_T) is called a (λ,C)-*quasi-isometric embedding* if for all $s,t\in S$,

$$\frac{d_S(s,t)}{\lambda}-C\le d_T(f(s),f(t))\le\lambda d_S(s,t)+C.$$

Here f is called ϵ-*quasisurjective* if for all $t\in T$, there exists an $s\in S$ such that $d_T(f(s),t)\le\epsilon$. A map f is called a (λ,C,ϵ)-*quasi-isometry* if f is a (λ,C)-quasi-isometric embedding, and is ϵ-quasisurjective. When we don't care about the quasi-isometry constants, we will simply call f a quasi-isometry and say that S and T are quasi-isometric.

Definition 2.5 Let S be a metric space. A path $p\colon[0,l]\to S$ is called a (λ,C)-*quasigeodesic* if p is a (λ,C)-quasi-isometric embedding.

Definition 2.6 Given a pointed metric space (S, x) and (λ, C)-quasigeodesic paths $p_i : [0, l_i] \to S$ such that the sequence l_i/c_i is bounded and $(p_i(0)) \in S_z^{\mathbb{N}}$, let $L = \lim^\omega l_i/c_i$. If $L \neq 0$, define the *ω-limit* of the paths p_i, written

$$p = \lim{}^\omega(p_i) \colon [0, L] \to \mathrm{Cone}^\omega(S),$$

by $p(x) = (p_i(x(l_i/L)))^\omega$. If $L = 0$, define $p = \lim^\omega(p_i) \colon \{0\} \to \mathrm{Cone}^\omega(S)$ by $p(0) = (p_i(0))^\omega$.

Definition 2.7 A geodesic in $\mathrm{Cone}^\omega(S)$ is called a *limit geodesic* if it is an ω-limit of geodesic paths.

Note that the limit of geodesics is a geodesic in the asymptotic cone. Thus if S is a geodesic metric space, then so is $\mathrm{Cone}^\omega(S)$.

A finitely generated group G can be considered as a metric space using the word metric arising from any finite generating set X. Given an ultrafilter ω, we will denote the asymptotic cone of G with respect to ω by $\mathrm{Cone}^\omega(G)$ where we assume all scaling sequences are $c_i = i$ unless otherwise specified, and the observation point will always be $(e)^\omega$. Note that G is $\left(1, 0, \frac{1}{2}\right)$-quasi-isometric to its Cayley graph $\Gamma(G, X)$, and so its asymptotic cone is isometric to the asymptotic cone of $\Gamma(G, X)$. This is a geodesic space, and so $\mathrm{Cone}^\omega(G)$ is a geodesic space.

The asymptotic cone of G depends on the choice of a finite generating set X, an ultrafilter ω, and the choice of a scaling sequence (d_i). Note that changing the generating set of a group gives a quasi-isometric Cayley graph, and so will give a bi-Lipschitz asymptotic cone. In general, however, the other choices can matter, and a group can have many different asymptotic cones. For instance, Thomas and Velickovic exhibited a group such that one of its asymptotic cones is an $\mathbb{R}$-tree, and another is not simply connected [16]. These two choices turn out to be closely related. Specifically, given any scaling sequence (c_i) such that the sizes of the sets $S_r = \{i : c_i \in [r, r+1)\}$ are bounded, and any ultrafilter ω, there exists an ultrafilter ω' such that $\mathrm{Cone}^\omega((c_i), G) = \mathrm{Cone}^{\omega'}((i), G)$ [14]. This justifies our choice to take all scaling sequences as $c_i = i$ unless otherwise specified.

Definition 2.8 We say that a metric space S is transitive if for any two points $s, t \in S$ there exists an isometry $\phi \colon S \to S$ such that $\phi(s) = t$.

Recall that for any group G, $\mathrm{Cone}^\omega(G)$ is a transitive space, and that any asymptotic cone is complete.

3 The generalized distortion function

We begin by defining a variant of distortion that will help us calculate generalized distortion in a variety of groups.

Definition 3.1 Let H be an infinite subgroup of a group G and let Y and X be finite generating sets of H and G, respectively. Define the *lower distortion function* of H in G, written $\nabla_{H,Y}^{G,X}(n)$, by the formula

$$\nabla_{H,Y}^{G,X}(n) = \min\{|h|_Y : |h|_X > n, h \in H\}.$$

We consider lower distortion up to the same equivalence as distortion, and denote by ∇_H^G the function $\nabla_{H,Y}^{G,X}$ for some choices of the finite generating sets X and Y.

Example 3.2 For $p \in \mathbb{N}$ with $p \geq 2$, let $G = \mathrm{BS}(1,p) = \langle a, b : b^{-1}ab = a^p \rangle$, and let $H = \langle a \rangle$. Note that $a^{p^n} = b^{-n}ab^n$, and so $\Delta_H^G(n) \succeq p^n$. In fact, $\Delta_H^G \sim p^n$ [9]. Next, note that if $k < p^n$, then we can write $k = \sum_{i=0}^{n-1} c_i p^i$, with $0 \leq c_i < p$. This in turn means that we can write $a^k = \prod_{i=0}^{n-1} b^{-i} a^{c_i} b^i = \left(\prod_{i=0}^{n-1} a^{c_i} b^{-1}\right) b^{n-1}$. This implies that $|a^k|_X \leq n + n(p) = n(p+1)$. Thus $\nabla_H^G(n) \succeq p^n$.

Example 3.3 Let G be the discrete Heisenberg group, ie the group of all upper triangular integer matrices with ones along the diagonal, and let H be the center of this group, ie the subgroup of all matrices of the form

$$\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } c \in \mathbb{Z}.$$

Let X be the generating set for the group G given by $G = \langle x, y, z \rangle$ where

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $Y = \{z\}$, a generating set for H. Note that $x^n y^n x^{-n} y^{-n} = z^{n^2}$. Now let m be a natural number such that $(n-1)^2 < m < n^2$. We know that $|z^{n^2}|_X \leq 4n$. Thus

$$|z^m|_X \leq 4n + (n^2 - (n-1)^2) = 4n + 2n - 1 \leq 6n.$$

Therefore if $m \leq n^2$, then $|z^m|_X \leq 6n$, and so $\nabla_H^G(n) \succeq n^2$.

Now we will show that if $|h|_X \leq n$, then $|h|_Y \leq n^2$. Let $f : G \to \mathbb{N}$ and $k : G \to \mathbb{N}$ be the functions given by

$$f\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = |a| \quad \text{and} \quad k\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = |b|,$$

respectively. We have that

$$f(gx) \leq f(g) + 1, \quad f(gy) = f(g), \quad \text{and} \quad f(gz) = f(g),$$

and thus if $|g|_X \leq n$, then $f(g) \leq n$. Similarly

$$k(gx) = k(g), \quad k(gy) \leq f(g) + k(g), \quad \text{and} \quad k(gz) \leq k(g) + 1.$$

Thus if $|g|_X \leq n$, then $k(g) \leq n^2$. If $h \in H$, then $|h|_Y = k(h)$, and so if $|h|_X \leq n$, then $|h|_Y \leq n^2$. Thus $\Delta_H^G(n) \preceq n^2$.

Example 3.4 Let $G = \langle a, b, c : [a,b] = 1, [a,c] = 1, c^{-1}bc = b^2 \rangle \cong \mathbb{Z} \times \mathrm{BS}(1,2)$, and let $H = \langle a, b \rangle \cong \mathbb{Z} \times \mathbb{Z}$. Let $X = \{a, b, c\}$. Note that $|b^{2^n}|_X \leq 2n + 1$, so $\Delta_H^G(n) \succeq 2^n$, but $|a^n|_X = n$, and so $\nabla_H^G(n) \preceq n$. Thus $\Delta_H^G \not\sim \nabla_H^G$.

Note that if f_1, f_2, g_1, and g_2 are strictly increasing functions such that $f_1(n) \sim f_2(n)$ and $g_1(n) \sim g_2(n)$, then $f_1(n)/g_1(m) \cong f_2(n)/g_2(m)$. Thus:

Proposition 3.5 *For a finitely generated infinite subgroup H of a finitely generated group G,*

$$\frac{\Delta_H^G(n)}{\Delta_H^G(m)} \preceq \mu_H^G(m,n) \preceq \frac{\Delta_H^G(n)}{\nabla_H^G(m)}. \tag{1}$$

Proof First choose a finite generating set X for G containing a generating set Y for H. Fix $n \in \mathbb{N}$ and let h be an element of H such that $|h|_X \leq n$ and $|h|_Y = \Delta_{H,Y}^{G,X}(n)$. By definition, if $k \in Y_m$ then $|k|_X \leq m$, and so $|k|_Y \leq \Delta_{H,Y}^{G,X}(m)$. Thus $|h|_{Y_m} \geq \lceil \Delta_{H,Y}^{G,X}(n)/\Delta_{H,Y}^{G,X}(m) \rceil$, and we obtain the first inequality in (1). For the next inequality, note that if $|h|_X \leq n$, then $|h|_Y \leq \Delta_{H,Y}^{G,X}(n)$. Thus we can write h as a product of at most $\lceil \Delta_{H,Y}^{G,X}(n)/(\nabla_{H,Y}^{G,X}(m)-1) \rceil$ elements of length less than or equal to $\nabla_{H,Y}^{G,X}(m)-1$ with respect to Y. Note that if h is an element of H such that $|h|_Y < \nabla_{H,Y}^{G,X}(m)$, then by the definition of $\nabla_{H,Y}^{G,X}$, $|h|_X \leq m$, and $h \in Y_m$. This gives the second inequality in (1). □

Definition 3.6 We call an infinite subgroup H of a group G *uniformly distorted* if $\Delta_H^G \sim \nabla_H^G$.

Combining the previous observations gives the following corollary:

Corollary 3.7 *If H is an infinite uniformly distorted finitely generated subgroup of a finitely generated group G, then $\mu_H^G(m,n) \cong \Delta_H^G(n)/\Delta_H^G(m) \cong \Delta_H^G(n)/\nabla_H^G(m)$.*

Example 3.8 Example 3.2 showed that if $G = \mathrm{BS}(1,p) = \langle a, b : b^{-1}ab = a^p \rangle$ and $H = \langle a \rangle$, then H is uniformly distorted in G, so we can apply Corollary 3.7 to get that $\mu_H^G(m,n) \cong p^{n-m}$.

Example 3.9 Example 3.3 showed that if G is the discrete Heisenberg group and H is the center of G, then H is uniformly distorted in G and we have from Corollary 3.7 that $\mu_H^G(m,n) \cong (n/m)^2$.

We conclude with an example demonstrating that for a group G with finite generating set X containing a generating set for a subgroup H, $\mu_H^{G,X}(n-1,n)$ can be very large.

Example 3.10 Let H be a finitely generated subgroup of a finitely generated group G such that the membership problem is undecidable, and let X be a finite generating set for G containing a generating set of H. The existence of such subgroups was demonstrated independently by Mihailova and Rips [11; 15]. Gromov [9] showed that the distortion function of H in G is bounded by a computable function if and only if the membership problem is solvable. Note that

$$\Delta_{H,Y}^{G,X}(n) = \mu_H^{G,X}(1,n) \leq \mu_H^{G,X}(1,2)\mu_H^{G,X}(2,3)\cdots\mu_H^{G,X}(n-1,n).$$

Thus, if $\mu_H^{G,X}(n-1,n)$ is bounded by a computable function, then so is $\Delta_{H,Y}^{G,X}(n)$, a contradiction. Thus $\mu_H^{G,X}(n-1,n)$ is not bounded by any computable function.

4 Connectedness in asymptotic cones

We begin by defining an analog of the generalized distortion function for the case of a metric space S.

Definition 4.1 Given a metric space S, a real number $r > 0$, and two points $s, t \in S$, an *r-path* connecting s and t is a sequence of points $s = s_0, s_1, \dots, s_k = t$ with $d_S(s_i, s_{i+1}) \le r$ for all $0 \le i < k$. We call k the *length* of the r-path. We say a metric space S is *r-connected* if for any two points $s, t \in S$ there exists an r-path connecting s and t. If (S, s) is a pointed r-connected metric space, and t is in S, let $|t|_r$ be the length of a shortest r-path connecting s and t.

Definition 4.2 Let (S, s) be a proper r-connected pointed metric space. Define $\nu_S(m, n)\colon \mathbb{R}^{\ge r} \times \mathbb{R}^{\ge 0} \to \mathbb{N}$ to be $\max\{|t|_m : d_S(s, t) \le n\}$.

Lemma 4.3 *The value ν_S is well defined, ie for all real numbers $m \ge r, n$ there exists a constant $K \in \mathbb{R}$ such that for any point $t \in S$ with $d(s, t) \le n$, $|t|_m \le K$.*

Proof Fix $n \in \mathbb{R}^{\ge 0}$, and let B be the closed ball centered at s of radius n. As B is compact, it can be covered by some finite number p of open balls of radius m. Let $s_1, \dots, s_p$ be the centers of these balls. As S is r-connected, for each s_i there exists a sequence of points

$$s = s_{0,i}, s_{1,i}, \dots, s_{K_i,i} = s_i$$

with $d_S(s_{j,i}, s_{j+1,i}) \le m$ for all $0 \le i < K_i$. Let $K = \max\{K_i : 1 \le i \le p\}$. Any point in B is within m of some s_i, and so $\nu_S(m, n) \le K + 1$. $\square$

If H is a finitely generated subgroup of a finitely generated group G, and X is a finite generating set of G containing a generating set for H, then H is 1-connected and proper with respect to the word metric induced by X. It is clear in this case that μ_H^G is the restriction of ν_H to $\mathbb{N} \times \mathbb{N}$, where we consider H with the word metric induced from G.

Definition 4.4 Given two functions $f, g\colon \mathbb{R}^{\ge r} \times \mathbb{R}^{\ge 0} \to \mathbb{R}$ which are nonincreasing in the first variable, and nondecreasing in the second variable, we write $f \preceq_\nu g$ if there exists a constant $C \in \mathbb{R}$ such that $f(Cm, n) \le Cg(m, Cn)$ for all $m, n \in \mathbb{R}^{\ge 0}$ with $m \ge r$, and we say that $f \cong_\nu g$ if $f \preceq_\nu g$ and $g \preceq_\nu f$.

Essentially, ν measures how far away S is from being a geodesic metric space. For instance, if S is geodesic, then $\nu_S(m, n) = \lceil n/m \rceil$.

Lemma 4.5 *If (S, s) and (T, t) are proper r-connected pointed metric spaces, and f is a (λ, C, ϵ)-quasi-isometry between S and T such that $f(s) = t$, then, $\nu_S \cong_\nu \nu_T$.*

Proof First, fix $n \in \mathbb{R}^{\ge 0}$ and $m \in \mathbb{R}^{\ge r}$, and let $y \in S$ with $d_S(s, y) \le n$. This means $d_T(t, f(y)) \le \lambda n + C$. Let $K = \nu_T(m, \lambda n + C)$. There exist $K + 1$ points $y_0, y_1, \dots, y_K$ such that $t = y_0, y_1, \dots, y_K = f(y)$ with $d_T(y_i, y_{i+1}) \le m$. By quasisurjectivity, for each i there exists a $y_i' \in S$ such that $d_T(f(y_i'), y_i) \le \epsilon$.

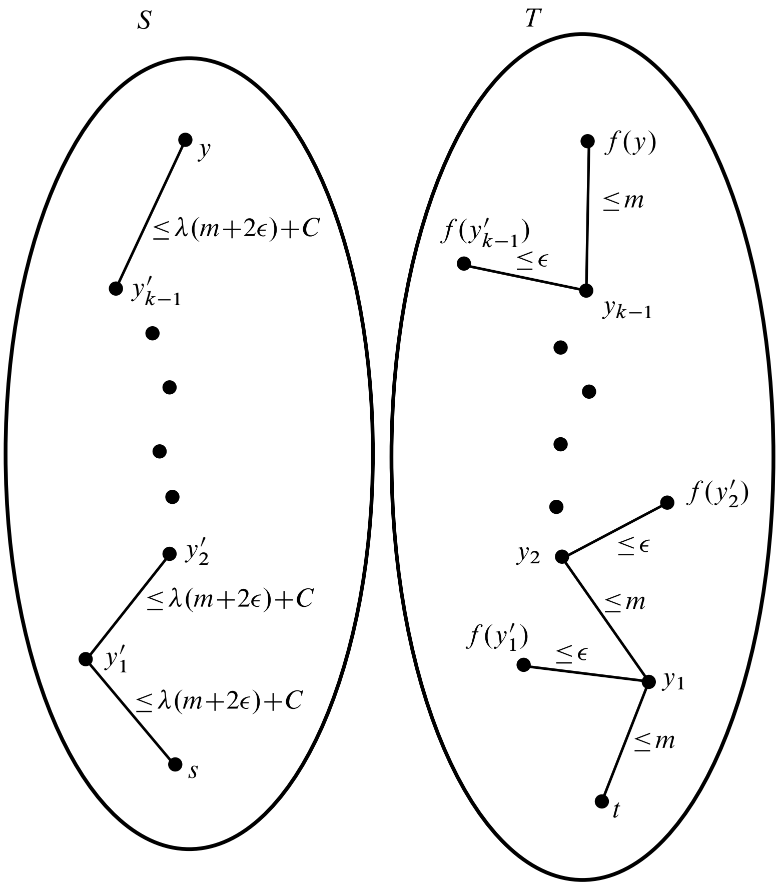

Figure 1: Lemma 4.5.

Thus $d_T(f(y'_i), f(y'_{i+1})) \leq m + 2\epsilon$, and so $d_S(y'_i, y'_{i+1}) \leq \lambda(m + 2\epsilon) + C \leq \lambda' m$ for some fixed λ' as $m \geq r$. Note that we can choose y'_0 to be s, and y'_K to be y. Thus $\nu_S(\lambda' m, n) \leq \nu_T(m, \lambda n + C)$. If $\lambda n + C \leq m$, we have that $\nu_T(m, \lambda n + C) = 1$, so we can assume that $\lambda n + C$ is greater than r as well, and hence $\nu_S(\lambda' m, n) \leq \nu_T(m, \lambda'' n)$ for some fixed λ''. By symmetry, $\nu_T \preceq_\nu \nu_S$, and so $\nu_T \cong_\nu \nu_S$. □

Definition 4.6 We call a metric space S *asymptotically transitive* if $\mathrm{Cone}^\omega(S)$ is transitive for all ultrafilters ω.

Theorem 4.7 *Let r be a positive number and let (S, s) be an asymptotically transitive proper r-connected pointed metric space. The following are equivalent*:

(i) *There exists a function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that for all $m \geq r$ and $n \geq 0$, $\nu_S(m, n) \leq f(n/m)$.*

(ii) *There exists a constant K such that $\nu_S(i, 4i) \leq K$ for all real numbers $i \geq r$.*

(iii) $\mathrm{Cone}^\omega(S)$ *is path connected for all nonprincipal ultrafilters ω.*

(iv) $\mathrm{Cone}^\omega(S)$ *is connected for all nonprincipal ultrafilters ω.*

Note that the implication (i) $\implies$ (ii) is clear, simply by letting $K = f(4)$. The implication (iii) $\implies$ (iv) is also immediate.

To show that (ii) implies (iii) we will need the following lemma:

Lemma 4.8 *Let $r \in \mathbb{R}^{\geq 0}$. If (S, s) is an asymptotically transitive proper r-connected pointed metric space and there exists a constant K such that $\nu_S(i, 4i) \leq K$ for all real numbers $i \geq r$, then for any points $p = (y_i)^\omega, q = (z_i)^\omega \in \mathrm{Cone}^\omega(S)$, there exist $K+1$ points $p = p_0, p_1, p_2, \dots, p_K = q$ in $\mathrm{Cone}^\omega(S)$ such that $d_S^\omega(p_i, p_{i+1}) \leq \frac{1}{2} d_S^\omega(p, q)$.*

This lemma is reminiscent of a lemma in [13] used to prove that any group satisfying a quadratic isoperimetric inequality has a simply connected asymptotic cone. There Papasoglu used the isoperimetric inequality to build sequences of loops to fill a loop in the asymptotic cone. This is very similar to the approach we will use to prove that (ii) implies (iii). Similar ideas can also be found in [3; 10; 14].

Proof If $(y_i)^\omega = (z_i)^\omega$, the result is trivial, so let $(y_i)^\omega$ and $(z_i)^\omega$ be points in $\mathrm{Cone}^\omega(S)$ such that $d_S^\omega((y_i)^\omega, (z_i)^\omega) = C > 0$. Note that by the transitivity of $\mathrm{Cone}^\omega(S)$, we can assume that $(y_i)^\omega = (s)^\omega$. This means in particular that $d_S(s, z_i) \leq 2Ci$ ω-almost surely. Note that $\frac{1}{2}Ci \geq r$ ω-almost surely, and hence $\nu_S\left(\frac{1}{2}Ci, 2Ci\right) \leq K$ ω-almost surely. It follows that there exist points $s = y_{i,0}, y_{i,1}, \dots, y_{i,K} = z_i$ with $d_S(y_{i,j}, y_{i,j+1}) \leq \frac{1}{2}Ci$ for all $0 \leq j \leq K-1$ ω-almost surely. Now define $p_j = (y_{i,j})^\omega$. Note that $d_S^\omega(p_j, p_{j+1}) = \lim^\omega d_S(y_{i,j}, y_{i,j+1})/i \leq \frac{1}{2}C$, and so we have our desired $p_0, \dots, p_K$. $\square$

We will also need the following lemma in order to prove that (iv) implies (i):

Lemma 4.9 *If S is a connected metric space, then for any real number $r > 0$, S is r-connected.*

Proof For a fixed $r > 0$, and fixed $p \in S$, consider the set C of points q such that there exists a finite sequence of points $p = p_0, p_1, \dots, p_K = q$ with $d(p_i, p_{i+1}) \leq r$. If $x \in C$, then clearly $B_r(x) \subset C$, and so C is open. Similarly, if $x \notin C$, then $B_r(x) \subset S \setminus C$, so C is closed. Hence C is open, closed, and nonempty, so $C = S$, as desired. $\square$

Proof of Theorem 4.7 We begin by proving (ii) implies (iii).

Let $p, q \in \mathrm{Cone}^\omega(S)$, and let $C = d_S^\omega(p, q)$. We will define a uniformly continuous function f from numbers of the form a/K^n with $a, n \in \mathbb{N}$ and $a \leq K^n$ to the asymptotic cone such that $f(0) = p$ and $f(1) = q$. Note that this is sufficient, since asymptotic cones are complete, and these numbers are dense in the interval $[0, 1]$.

We will define the function inductively as follows. First define $f(0) = p$ and $f(1) = q$. Then fix $n \in \mathbb{N}$, and assume we've defined f on all numbers of the form a/K^n in such a way that for all $s \in \mathbb{N} \cup \{0\}$ with $s < K^n$,

$$d_S^\omega\left(f\left(\frac{s}{K^n}\right), f\left(\frac{s+1}{K^n}\right)\right) \leq \frac{C}{2^n}.$$

Now let $t = (Kl + b)/K^{n+1}$ where $1 \le b < K$ and $l \in \mathbb{N} \cup \{0\}$ with $l \le K^{n-1}$. According to Lemma 4.8, there exist points $p_0, p_1, \dots, p_K$ such that

$$f\left(\frac{l}{K^n}\right) = p_0, p_1, \dots, p_K = f\left(\frac{l+1}{K^n}\right),$$

and

$$d_S^\omega(p_i, p_{i+1}) \le \tfrac{1}{2} d_S^\omega\left(f\left(\frac{l}{K^n}\right), f\left(\frac{l+1}{K^n}\right)\right) \le \frac{C}{2^{n+1}}.$$

Let $f(t) = p_b$. It is straightforward to verify that f is uniformly continuous.

We will now show that (iv) implies (i) by contradiction. Assume that $\mathrm{Cone}^\omega(S)$ is connected, and that $\nu_S(m, n)$ is not bounded by any homogeneous function. Hence there exists a $c \in \mathbb{R}^{>0}$ such that $\nu_S(n, cn)$ is not bounded. Let n_i be a sequence of natural numbers such that $\nu_S(n_i, cn_i) \ge i$. Let ω be an ultrafilter containing $\{n_i : i \in \mathbb{N}\}$. Consider a sequence of points $t_i \in S$ such that $d_S(s, t_i) \le ci$, and $|t_i|_i = \nu_S(i, ci)$. According to Lemma 4.9, we can pick points $(s)^\omega = p_0, p_1, \dots, p_k = (t_i)^\omega$ in $\mathrm{Cone}^\omega(S)$ such that $d_S^\omega(p_i, p_{i+1}) \le \frac{1}{2}$. Let $p_j = (t_{i,j})^\omega$. We have that $d_S(t_{i,j}, t_{i,j+1}) \le i$ ω-almost surely, so $\nu_S(i, ci) = |t_i|_i \le k$ ω-almost surely. On the other hand if $j > k$, then $\nu_S(n_j, cn_j) > k$. However,

$$\{n_j : j > k\} = \{n_j : j \in \mathbb{N}\} \cap \{n : n > n_k\} \in \omega,$$

a contradiction. $\square$

We now want to study how distortion of groups relates to connectedness in asymptotic cones. We begin by defining a natural subspace of the asymptotic cone of G corresponding to H:

Definition 4.10 Let T be a subspace of a metric space S. Denote by $\mathrm{Cone}_S^\omega(T)$ the set of all points in $\mathrm{Cone}^\omega(S)$ with a representative $(t_i)^\omega$ with each component in T.

Lemma 4.11 *For all subspaces $T \subset S$, $\mathrm{Cone}_S^\omega(T)$ is closed in $\mathrm{Cone}^\omega(S)$.*

Proof Note that $\mathrm{Cone}_S^\omega(T) = \mathrm{Cone}^\omega(T)$, where we consider T under the induced metric from S. Since asymptotic cones are complete, this is a complete metric space. A complete subspace of a complete metric space is closed, and so $\mathrm{Cone}_S^\omega(T)$ is closed in $\mathrm{Cone}^\omega(S)$. $\square$

Note that we can think about a subgroup H of a group G as a subspace of the metric space we get by considering the word metric on G.

Lemma 4.12 *If H is a subgroup of a finitely generated group G such that $\mathrm{Cone}_G^\omega(H)$ is connected for all ultrafilters ω, then H is finitely generated.*

Proof Let H be a subgroup of a finitely generated group G, and let X be a finite generating set for G. We call an element h of H *reducible* if there exists a constant $k \in \mathbb{N}$ and k elements of H, $h_1, h_2, \dots, h_k$, with $|h_i|_X < |h|_X$ for all $0 \le i \le k$ such that $h = h_1 h_2 \cdots h_k$. We call an element $h \in H$ *irreducible* if it is not reducible. We can assume that there exists no i such that all elements $h \in H$ with $|h|_X \ge i$ are reducible, as this would imply that H is finitely generated. Thus we can find a sequence (h_i) of irreducible elements of H such that $|h_i|_X > |h_{i-1}|_X$ for all i. Fix an ultrafilter ω and consider the asymptotic cone $\mathrm{Cone}^\omega_G(H)$ with respect to ω and the scaling sequence $(|h_i|_X)$. Assume this asymptotic cone is connected. As $(h_i)^\omega \in \mathrm{Cone}^\omega_G(H)$, there exist points $(e)^\omega = p_0, p_1, \dots, p_k = (h_i)^\omega$ with $d(p_i, p_{i+1}) \le \frac{1}{4}$ for all $0 \le i < k$. Let $p_j = (h_{i,j})^\omega$. We have that $|h_{i,j}^{-1} h_{i,j+1}|_X \le \frac{1}{2}|h_i|_X$ ω-almost surely. Finally, note that $h_i = h_{i,k} = h_{1,i}(h_{i,1}^{-1}h_{i,2})\cdots(h_{i,k-1}^{-1}h_{i,k})$. This, however, implies that h_i is ω-almost surely reducible, a contradiction. □

We can apply Lemma 4.8 to a subgroup H of a finitely generated group G, where H is given the word metric induced from G. In this case, the relationship between ν_H and μ^G_H combined with Theorem 4.7 gives the following theorem:

Theorem 4.13 *The following are equivalent for a subgroup H of a finitely generated group G:*

(i) *H is finitely generated and there exists a constant K such that $\mu^G_H(i, 4i) \le K$ for all i.*

(ii) *H is finitely generated and there exists a function f such that $\mu^G_H(m, n) \le f(n/m)$.*

(iii) *$\mathrm{Cone}^\omega_G(H)$ is path connected for all ultrafilters ω.*

(iv) *$\mathrm{Cone}^\omega_G(H)$ is connected for all ultrafilters ω.*

Example 4.14 We have previously seen that if $G = \mathrm{BS}(1, p) = \langle a, b : b^{-1}ab = a^p\rangle$ and $H = \langle a\rangle$, then $\mu^G_H(m, n) \cong p^{n-m}$. Thus $\mu^G_H(i, 2i)$ is unbounded, and we can conclude from Theorem 4.13 that there exists an ultrafilter ω such that $\mathrm{Cone}^\omega_G(H)$ is disconnected. In fact, $\mathrm{Cone}^\omega_G(H)$ is disconnected for all ultrafilters ω. This follows from the proof of Theorem 4.7 and observing that for all $c, n \in \mathbb{N}$ the set of $k \in \mathbb{N}$ such that $\mu^{G,\{a,b\}}_{H,\{a\}}(k, ck) \le n$ is finite.

Example 4.15 If G is the discrete Heisenberg group and H is the center of G, then we have seen in a previous example that $\mu^G_H(m, n) \cong n^2/m^2$, and so $\mu^G_H(i, 4i)$ is bounded and $\mathrm{Cone}^\omega_G(H)$ is connected for all ultrafilters ω.

We now want to relate the connectedness of $\mathrm{Cone}^\omega_G(H)$ to the distortion of H in G. In order to do this, we need a couple preliminary results. The first of these is due to Olshansky:

Theorem 4.16 [12] *For any group H, and any function $\ell\colon H \to \mathbb{N}$ satisfying*

(i) *for all $h \in H$, $\ell(h) = 0$ if and only if $h = 1$,*

(ii) *$\ell(h) = \ell(h^{-1})$ for all $h \in H$,*

(iii) *$\ell(gh) \le \ell(g) + \ell(h)$ for all $g, h \in H$,*

(iv) *there exists a constant a such that $|\{h \in H : \ell(h) \le n\}| \le a^n$,*

there exists a group $G = \langle X\rangle$ with $|X| < \infty$, an embedding ϕ of H in G, and a constant C such that for all $h \in H$,

$$\frac{|\phi(h)|_X}{C} \le \ell(h) \le C|\phi(h)|_X.$$

Definition 4.17 A function $f:\mathbb{R}^{\geq 1}\to\mathbb{R}$ is called *superlinear* if for all $k\in\mathbb{R}$ the set $\{x : f(x)\leq kx\}$ is bounded, and f is called *sublinear* if for all $k\in\mathbb{R}$ the set $\{x : f(x)\geq kx\}$ is bounded.

Lemma 4.18 *Let $f:\mathbb{R}^{\geq 1}\to\mathbb{R}$ be an increasing sublinear function with $f(r)\leq r$ for all real numbers $r\geq 1$. There exists a function $\ell:\mathbb{R}^{\geq 1}\to\mathbb{R}^{\geq 1}$ such that*

(i) *for all $m,n\in\mathbb{N}$, $\ell(m)+\ell(n)\geq \ell(m+n)$,*

(ii) *for all $n\in\mathbb{N}$, $\ell(n)\geq f(n)$,*

(iii) *for all $k\in\mathbb{N}$, there exists a $p_k\in\mathbb{R}$ such that ℓ is constant on the interval $[p_k,kp_k]$.*

Proof We will define p_k and ℓ by induction on k. First let $p_1=1$ and let $\ell(1)=1$. Assume we have defined p_k and $\ell(n)$ for $n\leq kp_k$ in a way that satisfies (i)–(iii). Let p_{k+1} be the least real number such that for all $r\in\mathbb{R}$, if $r\geq (k+1)p_{k+1}$, then $f(r)\leq r/(k+1)!$. For $s\in\mathbb{R}$, if $kp_k<s\leq p_{k+1}$, define $\ell(s)=s/k!$. For $s\in\mathbb{R}$, if $p_{k+1}\leq s\leq (k+1)p_{k+1}$, define $\ell(s)=p_{k+1}/k!$. By definition, $\ell((k+1)p_{k+1})=p_{k+1}/k!=(k+1)p_{k+1}/(k+1)!$.

We will now show that ℓ satisfies (i)–(iii). First, fix $r\in\mathbb{R}^{\geq 1}$, and let $k\in\mathbb{N}$ such that $kp_k\leq r\leq (k+1)p_{k+1}$. If $kp_k<r<p_{k+1}$, then $\ell(r)=r/k!$, and if $s<r$, then $\ell(s)\geq s/k!$. Thus, if $p+q=r$, then $\ell(p)+\ell(q)\geq p/k!+q/k!=r/k!=\ell(r)$. If $p_{k+1}<r\leq (k+1)p_{k+1}$, then $\ell(r)=\ell(p_{k+1})$, and (i) follows immediately as ℓ is increasing. For $s\in\mathbb{R}$, if $kp_k\leq s\leq p_{k+1}$, then $\ell(s)=s/k!>f(s)$ by definition. If $p_{k+1}\leq s\leq (k+1)p_{k+1}$, then $\ell(s)=\ell((k+1)p_{k+1})=(k+1)p_{k+1}/(k+1)!\geq f((k+1)p_{k+1})\geq f(s)$, so ℓ satisfies (ii). It is clear that this definition of ℓ satisfies (iii). □

We are now ready to relate the connectedness of $\mathrm{Cone}^{\omega}_{G}(H)$ to the distortion of H in G:

Theorem 4.19 *If H is a finitely generated subgroup of a finitely generated group G, then the following implications hold*:

(i) *If $\Delta^{G}_{H}(n)$ is linear, then $\mathrm{Cone}^{\omega}_{G}(H)$ is connected for all ultrafilters ω.*

(ii) *If $\mathrm{Cone}^{\omega}_{G}(H)$ is connected for all ultrafilters ω, then $\Delta^{G}_{H}(n)\preceq f$ for some polynomial f.*

(iii) *For every increasing superlinear function $\phi:\mathbb{N}\to\mathbb{N}$ there exists a group G with a subgroup H such that $\mathrm{Cone}^{\omega}_{G}(H)$ is disconnected for some ultrafilter ω, but $\Delta^{G}_{H}(n)\preceq\phi$.*

(iv) *For all $k\in\mathbb{N}$, there exists a group G with a subgroup H such that $\mathrm{Cone}^{\omega}_{G}(H)$ is connected for all ultrafilters ω, and $\Delta^{G}_{H}\sim n^k$.*

Proof (i) If H is a subgroup of G, then we can define a continuous function ρ from $\mathrm{Cone}^{\omega}(H)$ to $\mathrm{Cone}^{\omega}_{G}(H)$ by $\rho((h_i)^{\omega})=(h_i)^{\omega}$. For all $h\in H$, $|h|_X\leq C|h|_Y$ for some fixed constant C, so ρ is well defined. Assume $(h_i)^{\omega}\in\mathrm{Cone}^{\omega}_{G}(H)$. This means that there exists B such that for all $i\in\mathbb{N}$, $|h_i|_X/i\leq B$. Since distortion is linear, there exists D such that $|h_i|_Y/i\leq D(|h_i|_X/i)\leq DB$. Thus ρ is surjective, and $\mathrm{Cone}^{\omega}_{G}(H)$ is connected, as $\mathrm{Cone}^{\omega}_{G}(H)$ is connected.

(ii) Assume that $\mathrm{Cone}_G^\omega(H)$ is connected in $\mathrm{Cone}^\omega(G)$, and hence that $\mu_H^G(i, 2i)$ is bounded by some constant K for all i. By induction, $\Delta_H^G(2^n) = \mu_H^G(1, 2^n) \leq K^n$ for all $n \in \mathbb{N}$.

Now let $n \in \mathbb{N}$, and let $m \in \mathbb{R}$ such that $2^{m-1} \leq n < 2^m$. We have that

$$\Delta_H^G(n) \leq \Delta_H^G(2^m) \leq K^m = (2^m)^{\log_2 K} \leq (2n)^{\log_2 K}.$$

Thus $\Delta_H^G(n) \preceq n^{\log_2 K}$.

(iii) Let ϕ be a superlinear increasing function $\mathbb{N} \to \mathbb{N}$. Then ϕ can be extended to an invertible increasing superlinear function from $\mathbb{R}^{\geq 1}$ to $\mathbb{R}$. We can now apply Lemma 4.18 to ϕ^{-1} to get a function ℓ which is always larger than ϕ^{-1}. We can then restrict ℓ to the natural numbers and take ceilings to get a function from $\mathbb{N}$ to $\mathbb{N}$. We can extend this to a function from $\mathbb{Z}$ to $\mathbb{Z}$ by defining $\ell(0) = 0$ and $\ell(-z) = \ell(z)$ for $z < 0$. As $\ell \geq \phi^{-1}$, we have that $\phi(\ell(n)) \geq n$. If ϕ is subexponential, then this ℓ now satisfies all of the conditions of Theorem 4.16, and hence there exists a group $G = \langle X \rangle$, a constant C, and an embedding $\psi\colon \mathbb{Z} \to G$ such that

$$\frac{\ell(n)}{C} \leq |\psi(n)|_X \leq C\ell(n).$$

Now note that if $|\psi(n)|_X \leq m$, then $\ell(n) \leq C|\psi(n)|_X \leq Cm$, and so $n < \phi(\ell(n)) \leq \phi(Cm)$. Hence, distortion is bounded by ϕ. On the other hand, $\ell(p_k) = \ell(p_k + 1) = \cdots = \ell(kp_k)$ implies that $C|\psi(q)|_X > \ell(p_k)$ for all $p_k \leq q \leq kp_k$ while $|\psi(kp_k)|_X \leq C\ell(p_k)$, and so $\mu_H^G(\ell(p_k)/C, C\ell(p_k)) \geq k$. By Theorem 4.13, $\mathrm{Cone}_G^\omega(H)$ is disconnected for some ultrafilter ω.

Note that if ϕ is superexponential, then Theorem 4.19(ii) shows that $\mathrm{Cone}_G^\omega(H)$ is not connected for all ultrafilters ω.

(iv) We will use the same method as in (iii).

Fix $k \in \mathbb{N}$, and for $z \in \mathbb{Z}$ let $\ell(z) = \lceil |z|^{1/k} \rceil$. Let G be a group with finite generating set X and ψ an embedding of $\mathbb{Z}$ into G such that

$$\frac{\ell(z)}{C} \leq |\psi(z)|_X \leq C\ell(z).$$

Note that if $|\psi(z)|_X \leq m$, then $|z|^{1/k} \leq \lceil |z|^{1/k} \rceil = \ell(z) \leq C|\psi(z)|_X \leq Cm$, which implies that $|z| \leq C^k m^k$. Thus $\Delta_H^G(m) \preceq m^k$. Now note that $\ell(m^k) = m$, so $|\psi(m^k)|_X \leq Cm$, which implies $\Delta_H^G(Cm) \geq m^k$. Thus $\Delta_H^G(m) \sim m^k$. The above calculations show that if $|\psi(z)|_X \leq 4i$, then $|z| \leq 4^k C^K i^k$. Further, if $|z| \leq (i/C)^K$ then $|\psi(z)|_X \leq C\ell(z) \leq i$. Thus $\mu_H^G(i, 4i) \leq 4^k C^{2k}$, and so by Theorem 4.13 we have that $\mathrm{Cone}_G^\omega(H)$ is connected. $\square$

5 Convexity in asymptotic cones

Definition 5.1 A subspace T of a metric space S is called *Morse* if for all constants λ and C there exists a constant M such that any (λ, C)-quasigeodesic connecting points in T is contained in the M neighborhood of T.

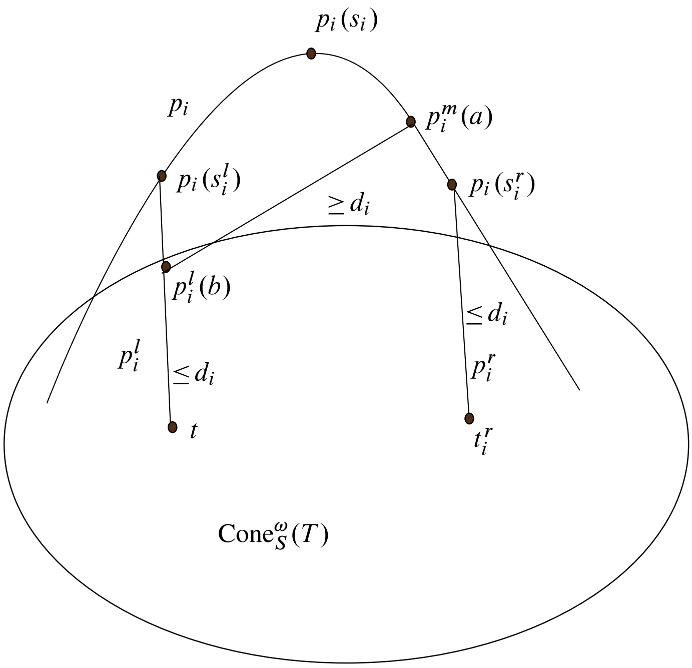

Figure 2: Theorem 5.3.

Definition 5.2 We say a subset T of a metric space S is *strongly convex* if every simple path starting and ending in T is entirely contained in T.

Theorem 5.3 *Let T be a closed subspace of a geodesic metric space S. Assume that $\mathrm{Cone}_S^\omega(T)$ is strongly convex in $\mathrm{Cone}^\omega(S)$ for all ultrafilters ω, and for any two points t_1 and t_2 in $\mathrm{Cone}_S^\omega(T)$ there exists an isometry ϕ of $\mathrm{Cone}^\omega(S)$ fixing $\mathrm{Cone}_S^\omega(T)$ such that $\phi(t_1) = t_2$. Then T is Morse.*

Proof Assume T is not Morse. This means that there exist constants $\lambda \geq 1$ and $C \geq 0$ such that for all $i \in \mathbb{N}$ there exists a (λ, C)-quasigeodesic $p_i : [0, k_i] \to S$ parametrized by length, and $s_i \in [0, k_i]$ with $p_i(0)$ and $p_i(k_i)$ in T and $d_S(p_i(s_i), T) \geq i$. For all i let

$$d_i = \sup\{d_S(p_i(s), T) : s \in [0, k_i]\}. \tag{2}$$

We can choose our paths p_i to make the sequence (d_i) increasing with all $d_i > C$. For each i, let s_i be a point in $[0, k_i]$ such that $d_S(p_i(s_i), T) = d_i$ (such a point exists as paths are compact). Let $s_i^l = \max\{s_i - 3\lambda d_i, 0\}$, and similarly let $s_i^r = \min\{s_i + 3\lambda d_i, k_i\}$. By (2), $d_S(p_i(s_i^l), T)$ and $d_S(p_i(s_i^r), T)$ are less than or equal to d_i. Let $d_S(p_i(s_i^l), T) = k_i^l$, and $d_S(p_i(s_i^r), T) = k_i^r$. Let t_i^l be a point in T such that $d_S(p_i(s_i^l), t_i^l) = k_i^l$, and let $p_i^l : [0, k_i^l] \to \Gamma(G)$ be a geodesic from t_i^l to $p_i(s_i^l)$. Note that by assumption we can take $t_i^l = t$, where t is some fixed point in T, by taking an isometry fixing T sending t_i^l to t. Similarly, let $p_i^r : [0, k_i^r]$ be a geodesic from $p_i(s_i^r)$ to a point $t_i^r \in T$ such that $d_S(t_i^r, p_i(s_i^r)) = k_i^r$. Denote by $p_i^m : [s_i^l, s_i^r] \to S$ the segment of p_i from $p_i(s_i^l)$ to $p_i(s_i^r)$.

We will need the following lemma:

Lemma 5.4 (i) *For all $i \in \mathbb{N}$, if $s_i^l \neq 0$, $a \in [s_i, s_i^r]$, and $b \in [0, k_i^l]$, then $d_S(p_i^m(a), p_i^l(b)) \geq d_i$.*

(ii) *For all $i \in \mathbb{N}$, if $s_i^r \neq k_i$, $a \in [s_i^l, s_i]$, and $b \in [0, k_i^r]$, then $d_S(p_i^m(a), p_i^r(b)) \geq d_i$.*

Proof First, if $s_i^l \neq 0$, then $s_i^l = s_i - 3\lambda d_i$. Now note that

$$d_S(p_i^m(a), p_i^m(s_i^l)) \geq \frac{3\lambda d_i}{\lambda} - C = 3d_i - C > 3d_i - d_i = 2d_i,$$

as p_i is a (λ, C)-geodesic, and we assumed that $d_i > C$. Thus, as $d_S(p_i^l(b), p_i^m(x_i^l)) \leq d_i$, we have $d_S(p_i^m(a), p_i^l(b)) \geq d_i$. The second claim follows similarly. □

We return to the proof of Theorem 5.3.

Fix an ultrafilter ω, and consider the asymptotic cone of S with respect to ω and the scaling sequence d_i. By construction, $d_S(t, p_i^l(k_i^l)) \leq d_i$, and so $(p_i^l(k_i^l))^\omega \in \mathrm{Cone}^\omega((d_i), G)$. Since $|s_i^l - s_i^r| \leq 6\lambda d_i$, we have that $d_S(p_i(s_i^l), p_i(s_i^r)) \leq 6\lambda^2 d_i + C$, and so since $(p_i(s_i^l))^\omega \in \mathrm{Cone}^\omega((d_i), S)$, we have that $(p_i(s_i^r))^\omega \in \mathrm{Cone}^\omega((d_i), S)$. Since $d_S(p_i(s_i^r), p_i^r(k_i^r)) = d(p_i^r(0), p_i^r(k_i^r)) \leq d_i$, we have that $(p_i^r(k_i^r))^\omega \in \mathrm{Cone}^\omega((d_i), S)$. Thus we can define

$$k^l = \lim{}^\omega \frac{k_i^l}{d_i}, \quad s^l = \lim{}^\omega \frac{s_i^l}{d_i}, \quad s^r = \lim{}^\omega \frac{s_i^r}{d_i}, \quad \text{and} \quad k^r = \lim{}^\omega \frac{k_i^r}{d_i},$$

and we can define $p^l : [0, k^l] \to \mathrm{Cone}^\omega((d_i), S)$ as $\lim^\omega(p_i^l)$, $p^m : [s^l, s^r] \to \mathrm{Cone}^\omega((d_i), S)$ as $\lim^\omega(p_i^m)$, and $p^r : [0, k^r]$ as $\lim^\omega(p_i^r)$. We have that p^l and p^r are geodesics, and p^m is a $(\lambda, 0)$-quasigeodesic, and hence all are simple.

Now we have three simple paths, p^l, p^m and p^r, such that $p^l(0)$ and $p^r(k^r)$ are in $\mathrm{Cone}_S^\omega((d_i), T)$, and p^l and p^r both intersect p^m. Unfortunately, the concatenation of these three paths may not be simple, as p^l and p^r could intersect p^m more than once. To deal with this case, we need the following lemma:

Lemma 5.5 *Let $s = \lim^\omega s_i/d_i$.*

(i) *If $a \in [0, k^l]$ and $b \in [s^l, s^r]$ with $p^l(a) = p^m(b)$, then $b \leq s$.*

(ii) *if $a \in [0, k^r]$ and $b \in [s^l, s^r]$ with $p^r(a) = p^m(b)$, then $b \geq s$.*

Proof Note that if $\{i : k_i^l = 0\} \in \omega$, then p^l is a trivial path, and the result is clear. Otherwise $\{i : k_i^l \neq 0\} \in \omega$. In this case we can use Lemma 5.4 to say that if $(b_i)^\omega$ is on p^l and $(a_i)^\omega$ is on p^m after s, then $d_S^\omega((b_i)^\omega, (a_i)^\omega) \geq \lim^\omega d_i/d_i \geq 1$. The proof of (ii) follows similarly. □

Thus we can form a simple path which starts and ends in $\mathrm{Cone}_S^\omega((d_i), T)$ as follows. Let

$$p = \max\{t \in [s^l, s^r] : \exists a \in [0, k^l] \text{ such that } p^l(a) = p^m(t)\},$$

and let

$$q = \min\{t \in [s^l, s^r] : \exists a \in [0, k^r] \text{ such that } p^r(a) = p^m(t)\}.$$

We obtain a simple path by following p^l up to $p^m(p)$, then following p^m up to $p^m(q)$, and finally following p^r back to $p^r(k^r)$. This path contains $p^m(s)$ by Lemma 5.5. Finally, as $p^m(s) = (p_i^m(s_i))^\omega$,

$$d_S^\omega(p^m(s), \mathrm{Cone}_S^\omega((d_i)T)) = \lim^\omega \frac{d_S(p_i^m(s_i), \mathrm{Cone}_S^\omega((d_i), T))}{d_i} = \lim^\omega \frac{d_i}{d_i} = 1.$$

Thus we have a simple path starting and ending in $\mathrm{Cone}_S^\omega(T)$ that is not entirely contained in $\mathrm{Cone}_S^\omega(T)$. □

In order to prove a partial converse of this statement we will need the following results from Druțu, Mozes, and Sapir [5]. Note that an error was found in this paper [1], but none of the following lemmas were affected.

Lemma 5.6 [5, Lemma 2.3] *Let S be a geodesic metric space, ω an ultrafilter, and B a closed subset of $\mathrm{Cone}^\omega(S)$. If x and y are in the same connected component of $\mathrm{Cone}^\omega(S) \setminus B$, then there exists a sequence of paths $(p_i)_{i=1}^n$ such that each path is a limit geodesic in X, and the concatenation of the paths p_i is a simple path from x to y.*

Definition 5.7 A path is called C-bi-Lipschitz if it is a $(C, 0)$-quasigeodesic.

Lemma 5.8 [5, Lemma 2.5] *In the same setting as Lemma 5.6, let p be a simple path in $\mathrm{Cone}^\omega(S)$ which is a concatenation of limit geodesics. For all δ there exists a constant C and a C-bi-Lipschitz path p' such that the Hausdorff distance between p and p' is less than δ, and p' is also a concatenation of limit geodesics connecting the same points.*

Lemma 5.9 [5, Lemma 2.6] *Let p be a C-bi-Lipschitz path in $\mathrm{Cone}^\omega(S)$ which is a concatenation of limit geodesics. There exists a constant C' and a sequence of paths (p_n) in S such that each p_n is C'-bi-Lipschitz, and $\lim^\omega(p_n) = p$.*

Theorem 5.10 *If T is a Morse subspace of a metric space S, then $\mathrm{Cone}_S^\omega(T)$ is strongly convex in $\mathrm{Cone}^\omega(S)$.*

Proof Let p be a simple path in $\mathrm{Cone}^\omega(S)$ starting and ending in $\mathrm{Cone}_S^\omega(T)$ but not entirely contained in $\mathrm{Cone}_S^\omega(T)$. As $\mathrm{Cone}_S^\omega(T)$ is closed, there is a subpath p' of p which starts and ends in $\mathrm{Cone}_S^\omega(T)$, but no interior point of p' is in $\mathrm{Cone}_S^\omega(T)$. Let x be the initial point of p and let y be the terminal point of p. Let x' and y' be points on p' such that

$$\max\{d_S^\omega(x, x'), d_S^\omega(y, y')\} < \tfrac{1}{2} d_S^\omega(x, y),$$

and let p^l and p^r be limit geodesics from x to x' and from y' to y, respectively. Let p^m be a concatenation of limit geodesics connecting x' to y' avoiding $\mathrm{Cone}_S^\omega(T)$. Such a path exists by Lemma 5.6 as $\mathrm{Cone}_S^\omega(T)$ is closed. The concatenation of p^l, p^m, and p^r may not be simple, so we let a be the first point of p^l

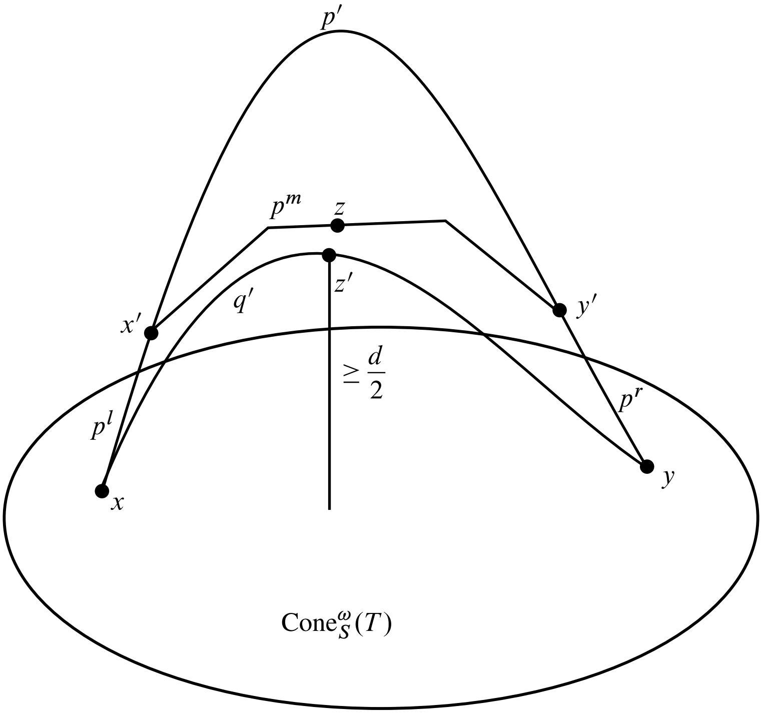

Figure 3: Lemma 5.9.

on p^m, and b be the last point of p^r on p^m. By the choice of x' and y', p^l does not intersect p^r, so we can obtain a simple path by following p^l from x to a, p^m from a to b, and p^r from b to y. Call this concatenation q.

Let z be a point on q such that $d^{\omega}_S(z, \mathrm{Cone}^{\omega}_S(T)) = d > 0$. Using Lemma 5.8, we can find a path q' such that q' is a C-bi-Lipschitz path which is a concatenation of limit geodesics, and the Hausdorff distance between q and q' is less than $\frac{1}{2}d$. Thus there is a point z' on q' such that $d^{\omega}_S(z, z') \leq \frac{1}{2}d$, so $d^{\omega}_S(z', \mathrm{Cone}^{\omega}_S(T)) \geq \frac{1}{2}d$.

Finally we can apply Lemma 5.9 to this new path q' to get that $q' = \lim^{\omega}(q_n)$ with each q_n being a C'-bi-Lipschitz path starting and ending in T. Thus, as T is Morse, each path is in some fixed neighborhood of T. This implies that $q = \lim^{\omega}(q_n)$ is entirely contained in $\mathrm{Cone}^{\omega}_S(T)$, a contradiction.

Thus, if T is Morse in S, then $\mathrm{Cone}^{\omega}_S(T)$ is strongly convex in $\mathrm{Cone}^{\omega}(S)$. □

Definition 5.11 A subgroup H of a group G with finite generating set X is called *strongly quasiconvex* if it is Morse as a subspace of the Cayley graph G with respect to X.

Note that if H is a subgroup of G, then for any two points $(h_i)^{\omega}$ and $(k_i)^{\omega}$ in $\mathrm{Cone}^{\omega}_G(H)$ there exists an isometry of $\mathrm{Cone}^{\omega}(G)$ fixing $\mathrm{Cone}^{\omega}_G(H)$ which sends $(h_i)^{\omega}$ to $(k_i)^{\omega}$. Thus we can combine the previous two results to give:

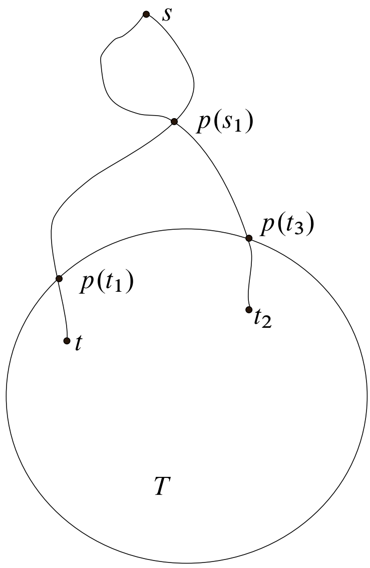

Figure 4: Theorem 5.13.

Theorem 5.12 *A subgroup H of a group G is strongly quasiconvex if and only if $\mathrm{Cone}^{\omega}_{G}(H)$ is strongly convex in $\mathrm{Cone}^{\omega}(G)$ for all ultrafilters ω.*

We conclude by proving a large class of groups cannot contain infinite, infinite-index strongly quasiconvex subgroups.

Theorem 5.13 *If a path connected metric space S contains a proper closed strongly convex subspace T consisting of more than one point, then S contains a cut point.*

Proof Let $s \in S \setminus T$, and let $t \in T$. Let $p\colon [0,l] \to S$ be a simple path connecting s and t. Let $t_1 = \min\{a \in [0,l] : p(a) \in T\}$. This is well defined as T is closed. We will show that $p(t_1)$ is a cut point. Let $t_2 \neq p(t_1)$ be a point in T. If $p(t_1)$ is not a cut point, then there exists a path $p'\colon [0,k]$ connecting s and t_2 such that $p(t_1)$ is not on p'. Let $t_3 = \min\{a \in [0,k] : p'(a) \in T\}$. Let $s_1 = \max\{a \in [0,t_1] : p(s_1) \in p'\}$. Create a simple path by following p from t_1 to s_1 and then following p' from s_1 to t_2. This is a simple path connecting two points of T that is not entirely contained in T, a contradiction. $\square$

Theorem 5.14 (Sapir and Druțu [6]) *If G is a nonvirtually cyclic group satisfying a law, then no asymptotic cone of G contains a cut point.*

If H is an infinite, infinite-index subgroup of a finitely generated group G, then it is easy to see that $\mathrm{Cone}^{\omega}_{G}(H)$ is a proper subspace of $\mathrm{Cone}^{\omega}(G)$ that consists of more than one point. Thus we can combine the previous two results to get the following corollary:

Corollary 5.15 *If G is a finitely generated group containing a nondegenerate strongly quasiconvex subgroup H, then G does not satisfy a law.*

References

[1] **T Aougab**, **M G Durham**, **S J Taylor**, *Pulling back stability with applications to* $\mathrm{Out}(F_n)$ *and relatively hyperbolic groups*, J. Lond. Math. Soc. 96 (2017) 565–583 MR Zbl

[2] **J A Behrstock**, *Asymptotic geometry of the mapping class group and Teichmüller space*, Geom. Topol. 10 (2006) 1523–1578 MR Zbl

[3] **G R Conner**, **C Kent**, *Local topological properties of asymptotic cones of groups*, Algebr. Geom. Topol. 14 (2014) 1413–1439 MR Zbl

[4] **L van den Dries**, **A J Wilkie**, *Gromov's theorem on groups of polynomial growth and elementary logic*, J. Algebra 89 (1984) 349–374 MR Zbl

[5] **C Druțu**, **S Mozes**, **M Sapir**, *Divergence in lattices in semisimple Lie groups and graphs of groups*, Trans. Amer. Math. Soc. 362 (2010) 2451–2505 MR Zbl

[6] **C Druțu**, **M Sapir**, *Tree-graded spaces and asymptotic cones of groups*, Topology 44 (2005) 959–1058 MR Zbl

[7] **A Genevois**, *Hyperbolicities in* CAT(0) *cube complexes*, Enseign. Math. 65 (2019) 33–100 MR Zbl

[8] **M Gromov**, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. 53 (1981) 53–73 MR Zbl

[9] **M Gromov**, *Asymptotic invariants of infinite groups*, from "Geometric group theory, II", Lond. Math. Soc. Lect. Note Ser. 182, Cambridge Univ. Press (1993) 1–295 MR Zbl

[10] **C Kent**, *Asymptotic cones of* HNN *extensions and amalgamated products*, Algebr. Geom. Topol. 14 (2014) 551–595 MR Zbl

[11] **K A Mihailova**, *The occurrence problem for direct products of groups*, Mat. Sb. 70(112) (1966) 241–251 MR Zbl In Russian.

[12] **A Y Olshansky**, *On the distortion of subgroups of finitely presented groups*, Mat. Sb. 188 (1997) 51–98 MR Zbl In Russian; translated in Sb. Math. 188 (1997) 1617–1664

[13] **P Papasoglu**, *On the asymptotic cone of groups satisfying a quadratic isoperimetric inequality*, J. Differential Geom. 44 (1996) 789–806 MR Zbl

[14] **T R Riley**, *Higher connectedness of asymptotic cones*, Topology 42 (2003) 1289–1352 MR Zbl

[15] **E Rips**, *Subgroups of small cancellation groups*, Bull. Lond. Math. Soc. 14 (1982) 45–47 MR Zbl

[16] **S Thomas**, **B Velickovic**, *Asymptotic cones of finitely generated groups*, Bull. Lond. Math. Soc. 32 (2000) 203–208 MR Zbl

[17] **H C Tran**, *On strongly quasiconvex subgroups*, Geom. Topol. 23 (2019) 1173–1235 MR Zbl

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On k-invariants for (∞, n)-categories

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Every (∞, n)-category can be approximated by its tower of homotopy (m, n)-categories. In this paper, we prove that the successive stages of this tower are classified by k-invariants, analogously to the classical Postnikov system for spaces. Our proof relies on an abstract analysis of Postnikov-type systems equipped with k-invariants, and also yields a construction of k-invariants for algebras over ∞-operads and enriched ∞-categories.

18N65, 55S45

1 Introduction

The weak homotopy type of a topological space can be conveniently studied using its Postnikov tower

$$X \to \cdots \to \tau_{\leq a} X \to \tau_{\leq a-1} X \to \cdots \to \tau_{\leq 0} X = \pi_0(X).$$

The Postnikov tower allows one (theoretically) to reconstruct X from algebraic and cohomological data. Indeed, the lowest stages of this tower encode the path components of X and its fundamental groupoid. For the higher stages, the passage from $\tau_{\leq a-1} X$ to $\tau_{\leq a} X$ is completely determined by a cohomology class

$$k_a \in \mathrm{H}^{a+1}(\tau_{\leq a-1} X, \pi_a(X)).$$

Indeed, given a map $f : Y \to \tau_{\leq a-1} X$, there exists a lift

$$\begin{tikzcd} & \tau_{\leq a} X \arrow[d] \\ Y \arrow[ur, dotted] \arrow[r, "f"'] & \tau_{\leq a-1} X \end{tikzcd} \tag{1-1}$$

if and only if the cohomology class $f^* k_a$ vanishes on Y. In this case, the i^{th} homotopy group of the space of lifts (1-1) can be identified (noncanonically) with the $(a-i)^{\text{th}}$ cohomology group of Y with coefficients in $f^*\pi_a(X)$. Here it should be noted that the homotopy groups $\pi_a(X)$ typically form a *local system of abelian groups*.

The purpose of this paper is to describe an analogue of the Postnikov tower for (∞, n)-categories. More precisely, every (∞, n)-category $\mathcal{C}$ admits a tower of homotopy (m, n)-categories, as shown by Lurie [24, Section 3.5] (see Section 6)

$$\mathcal{C} \to \cdots \to \mathrm{ho}_{(m,n)} \mathcal{C} \to \mathrm{ho}_{(m-1,n)} \mathcal{C} \to \cdots \to \mathrm{ho}_{(n,n)} \mathcal{C}.$$

Our main result asserts that there are again cohomology classes which control the passage from the homotopy (m, n)-category to the homotopy $(m+1, n)$-category:

Theorem 1.1 (informal) *For each $a \geq 2$, the extension $\mathrm{ho}_{(n+a,n)} \mathcal{C} \to \mathrm{ho}_{(n+a-1,n)} \mathcal{C}$ is classified by a k-invariant*

$$k_a \in \mathrm{H}^{a+1}(\mathrm{ho}_{(n+a-1,n)} \mathcal{C}, \pi_a(\mathcal{C})),$$

*where $\pi_a(\mathcal{C})$ is a **local system of abelian groups** on the (∞, n)-category $\mathrm{ho}_{(n+1,n)} \mathcal{C}$.*

In the case of $(\infty, 1)$-categories, these k-invariants have also been constructed explicitly in terms of simplicial categories by Dwyer, Kan, and Smith [7]. For $n > 1$, the above result is stated (without proof) and used by Lurie in [24]. In [14], we have used this result as part of an obstruction-theoretic proof of the fact that adjunctions in $(\infty, 2)$-categories are uniquely determined at the level of the homotopy 2-category (see also the work of Riehl and Verity [29]).

To make Theorem 1.1 more precise, let us recall that for any local system of abelian groups $\mathcal{A}$ on a space X, there exist Eilenberg–MacLane spaces $\mathrm{K}(\mathcal{A}, a) \to \tau_{\leq 1} X$, defined in the homotopy category $\mathrm{ho}(\mathcal{S}_{/\tau_{\leq 1} X})$ by the following universal property: for every map $f : Y \to \tau_{\leq 1} X$, there is a natural bijection

$$\mathrm{H}^a(Y, f^*\mathcal{A}) \cong \pi_0 \operatorname{Map}_{/\tau_{\leq 1}(X)}(Y, \mathrm{K}(\mathcal{A}, a)).$$

In fact, the Eilenberg–MacLane spaces $\mathrm{K}(\mathcal{A}, a)$ are related by equivalences

$$\mathrm{K}(\mathcal{A}, a) \xrightarrow{\sim} \Omega_{/\tau_{\leq 1} X} \mathrm{K}(\mathcal{A}, a+1),$$

where $\Omega_{/\tau_{\leq 1} X} \mathrm{K}(\mathcal{A}, a+1)$ computes the fiberwise loop space of $\mathrm{K}(\mathcal{A}, a+1)$ over $\tau_{\leq 1} X$ (at the basepoints given by the canonical section classifying the zero cohomology class). In other words, these

Eilenberg–MacLane spaces can be organized into a parametrized spectrum $\mathrm{H}\mathcal{A}$ over $\tau_{\leq 1}X$ such that $\mathrm{K}(\mathcal{A}, a) \simeq \Omega^\infty(\Sigma^a \mathrm{H}\mathcal{A})$ (see work of May and Sigurdsson [27]). From an ∞-categorical perspective, this parametrized spectrum can also be described more precisely as follows (Ando, Blumberg, Gepner, Hopkins, and Rezk [1]): the local system $\mathcal{A}$ determines a functor of ∞-categories $\mathrm{H}\mathcal{A}\colon \tau_{\leq 1}X \to \mathrm{Ab} \to \mathrm{Sp}$ sending each $x \in \tau_{\leq 1}X$ to the Eilenberg–MacLane spectrum of the abelian group $\mathcal{A}_x$. By the Grothendieck construction, such an ∞-functor to spectra can equivalently be viewed as a spectrum object in spaces over $\tau_{\leq 1}X$.

In these terms, the k-invariants can be interpreted as maps that fit into commuting squares for $a \geq 2$:

$$\begin{tikzcd}
\tau_{\leq a}X \arrow[r] \arrow[d] & \tau_{\leq 1}X \arrow[d, "0"] \\
\tau_{\leq a-1}X \arrow[r, "k_a"'] & \Omega^\infty(\Sigma^{a+1}\mathrm{H}\pi_a(X))
\end{tikzcd}$$

Here the right vertical map classifies the zero cohomology class. In fact, this square is homotopy Cartesian, which implies that the space of sections (1-1) is homotopy equivalent to the space of null-homotopies of f^*k_a.

Our more precise version of Theorem 1.1 is then the following:

Theorem 1.2 (Theorem 6.3) *For any (∞, n)-category $\mathcal{C}$ and $a \geq 2$, there is a parametrized spectrum object $\mathrm{H}\pi_a(\mathcal{C})$ internal to (∞, n)-categories, whose base object is $\mathrm{ho}_{(n+1,n)}\,\mathcal{C}$, so that there is a pullback square of (∞, n)-categories*

$$\begin{tikzcd}
\mathrm{ho}_{(n+a,n)}\,\mathcal{C} \arrow[r] \arrow[d] & \mathrm{ho}_{(n+1,n)}\,\mathcal{C} \arrow[d, "0"] \\
\mathrm{ho}_{(n+a-1,n)}\,\mathcal{C} \arrow[r, "k_a"'] & \Omega^\infty(\Sigma^{a+1}\mathrm{H}\pi_a(\mathcal{C})).
\end{tikzcd} \tag{1-2}$$

Furthermore, we prove that the parametrized spectrum $\mathrm{H}\pi_a(\mathcal{C})$ can indeed be thought of as an Eilenberg–MacLane spectrum: it is contained in the heart of a certain t-structure on the ∞-category of parametrized spectrum objects over $\mathrm{ho}_{(n+1,n)}\,\mathcal{C}$ (Corollary 6.17). This heart consists of local systems of abelian groups on the (∞, n)-category $\mathrm{ho}_{(n+1,n)}\,\mathcal{C}$, as defined (somewhat informally) by Lurie in [24] (see Definition 6.13 and Remark 6.15).

To prove Theorem 1.2, the main idea will be to proceed by induction on the categorical dimension n. More precisely, the structure of the Postnikov tower, together with its k-invariants, can be axiomatized in terms of "Postnikov structures". We prove that a (functorial) Postnikov structure on a symmetric monoidal ∞-category $\mathcal{V}$ that is compatible with the tensor product gives rise to a Postnikov structure on the ∞-category $\mathrm{Cat}(\mathcal{V})$ of $\mathcal{V}$-enriched ∞-categories (Theorem 5.18). Furthermore, the resulting Postnikov structure on $\mathrm{Cat}(\mathcal{V})$ respects the natural symmetric monoidal structure on $\mathrm{Cat}(\mathcal{V})$ inherited from $\mathcal{V}$. This can be used to proceed inductively.

More generally, this argument can also be used to provide k-invariants for Postnikov towers of algebras over ∞-operads (see Proposition 4.14 and Example 4.24). These k-invariants typically take values in certain André–Quillen cohomology groups, and have also been considered (in specific cases) by Goerss and Hopkins [10], Basterra and Mandell [4] and Lurie [26].

Outline

Let us now give an outline of this paper: In Section 2, we recall the definition of the tangent bundle of an ∞-category and the related theory of "square zero extensions". Furthermore, we discuss the "square zero" monoidal structure on the tangent bundle of a symmetric monoidal presentable ∞-category $\mathcal{V}$, which is useful to describe tangent bundles to categories of algebras. This square zero monoidal structure is particularly simple when $\mathcal{V}$ is already stable; we discuss this case in a bit more detail in Section 3.

In Section 4, we give an abstract axiomatization of towers of square zero extensions, which we call *Postnikov structures*, as well as multiplicative refinements thereof. In particular, we show how multiplicative Postnikov structures induce (multiplicative) Postnikov structures for algebras over ∞-operads. As the basis of our inductive proof, we show that the Postnikov tower of spaces is part of a multiplicative (functorial) Postnikov structure. Section 5 contains our main result, Theorem 5.18: we show that multiplicative Postnikov structures induce multiplicative Postnikov structures at the level of enriched ∞-categories.

In Section 6, we apply this result inductively to prove that the homotopy (m,n)-categories of an (∞,n)-category are part of a multiplicative Postnikov structure (Theorem 6.3); in particular, this provides the required pullback squares (1-2). Finally, we discuss how the tangent bundle of (∞,n)-categories carries a (family of) t-structures, whose heart consists of the category of local systems of abelian groups on (∞,n)-categories (Definition 6.13). The parametrized spectra $\mathrm{H}\pi_a(\mathcal{C})$ appearing in (1-2) then appear as the Eilenberg–MacLane spectra associated to such local systems.

Conventions

We will make use of the language of ∞-categories, ie quasicategories, and ∞-operads, following the standard references by Lurie [23; 26]; we will not distinguish between an ordinary category and its nerve. Furthermore, we will refer to symmetric monoidal ∞-categories as *SM ∞-categories*. Recall that SM ∞-categories and (lax) SM functors form (full) subcategories of the ∞-category of ∞-operads, which we will denote by

$$\mathrm{SMCat} \hookrightarrow \mathrm{SMCat}^{\mathrm{lax}} \hookrightarrow \mathrm{Op}_{\infty}.$$

A presentable SM ∞-category is a presentable ∞-category equipped with a closed symmetric monoidal structure, ie an object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Given an ∞-operad $\mathcal{O}$, ie a map $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$, and a collection of objects S in the underlying ∞-category $\mathcal{O}_{\langle 1\rangle}$ that is closed under equivalences, the *full suboperad* of $\mathcal{O}$ on S is the full subcategory of $\mathcal{O}^{\otimes}$ spanned by all objects of the form $x_1 \oplus \cdots \oplus x_n$ with all $x_i \in S$ (see [26, Section 2.2.1]).

Let $f\colon \mathcal{C} \to \mathcal{D}$ be an SM functor and let W be the class of maps in $\mathcal{C}$ that are sent to equivalences by f. We will say that f is a *monoidal localization* if it defines an initial object in full subcategory of $\mathrm{CAlg}(\mathrm{Cat})_{\mathcal{C}/}$ on those symmetric monoidal functors $g\colon \mathcal{C} \to \mathcal{E}$ sending W to equivalences. If $\mathcal{C}$ is an SM ∞-category and $f\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ is a (non-SM) localization such that W is closed under tensor products with objects in $\mathcal{C}$, then f admits a unique lift to a monoidal localization of SM ∞-categories [26, Proposition 4.1.7.4].

If $\mathcal{C}$ and $\mathcal{D}$ are SM ∞-categories, let us define a *reflective monoidal localization* to be an adjoint pair $L\colon \mathcal{C}^{\otimes} \rightleftarrows \mathcal{D}^{\otimes}\colon R$ in the homotopy 2-category of ∞-operads such that $\epsilon\colon LR \to \mathrm{id}_{\mathcal{D}}$ is a natural equivalence. Note that a reflective monoidal localization is determined uniquely by any one of the two maps L and R (Riehl and Verity [29]). If (L, R) is a reflective monoidal localization, then the (a priori only *lax* SM) left adjoint L is a monoidal localization in the sense above (Lurie [26, Corollary 7.3.2.12], Haugseng [18, Theorem 4.6]). Conversely, if L is a monoidal localization which admits a (fully faithful) right adjoint at the level of the underlying ∞-categories, then it determines a reflective monoidal localization [26, Corollary 7.3.2.7].

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2 Tangent bundles of ∞-categories

The purpose of this section is to recall some elements of the cotangent complex formalism described by Lurie [26, Section 7.3]. In particular, we will recall the definition of the tangent bundle of an ∞-category $\mathcal{C}$ and the notion of a square zero extension. To motivate this terminology, we show in Section 2.2 that the tangent bundle inherits a "square zero" monoidal structure from $\mathcal{V}$. In Section 2.3, we introduce the notion of a "t-orientation" on the tangent bundle, allowing one to make sense of connective (and discrete) objects in its fibers. The tangent bundle of stable (or more generally, additive) ∞-categories has a particularly simple structure, which we discuss in more detail in Section 3.

2.1 Recollections on tangent bundles and square zero extensions

Let $\mathcal{V}$ be an ∞-category with finite limits. Following Lurie [26, Definition 7.3.1.9], we define the *tangent bundle* of $\mathcal{V}$ to be the ∞-category

$$\mathcal{T}\mathcal{V} = \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$$

of excisive functors $F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ from the ∞-category of finite pointed spaces, ie those functors sending pushout squares to pullback squares. The ∞-category $\mathcal{T}\mathcal{V}$ comes with functors

$$\pi = \mathrm{ev}_*\colon \mathcal{T}\mathcal{V} \to \mathcal{V}, \quad \Omega^\infty = \mathrm{ev}_{S^0}\colon \mathcal{T}\mathcal{V} \to \mathcal{V}$$

taking the base, or the (parametrized) infinite loop space object underlying such a parametrized spectrum, respectively. The functor π is a Cartesian fibration and admits both a left and a right adjoint, both taking the constant excisive functor on an object in $\mathcal{V}$. We refer to the fiber of π at an object $X \in \mathcal{V}$ as the *tangent ∞-category* $\mathcal{T}_X\mathcal{V}$ of $\mathcal{V}$ at X. The diagram

$$\begin{tikzcd} \mathcal{T}\mathcal{V} \arrow[rr, "E\mapsto[E(S^0)\to E(*)]"] \arrow[dr, "\pi"'] & & \mathrm{Fun}(\Delta^1, \mathcal{V}) \arrow[dl, "\mathrm{codom}"] \\ & \mathcal{V} & \end{tikzcd}$$

then exhibits each fiber $\mathcal{T}_X\mathcal{V}$ as the stabilization $\mathrm{Sp}(\mathcal{V}_{/X})$ of the over-category $\mathcal{V}_{/X}$ [26, Section 7.3.1]. When $\mathcal{V}$ is presentable, $\mathcal{T}\mathcal{V}$ and each of the fibers $\mathcal{T}_X\mathcal{V}$ are presentable as well and the functor Ω^∞ admits a left adjoint Σ_+^∞.

Definition 2.1 Let $\mathcal{V}$ be a presentable ∞-category. Then the inclusion $\mathcal{T}\mathcal{V} \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ admits a left adjoint, which we will denote by $X \mapsto X^{\mathrm{exc}}$. We will say that a map $X \to Y$ in $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is a *$\mathcal{T}\mathcal{V}$-local equivalence* if the map $X^{\mathrm{exc}} \to Y^{\mathrm{exc}}$ is an equivalence.

Example 2.2 The tangent bundle $\mathcal{T}\mathcal{S}$ can be thought of as the ∞-category of parametrized spectra (with varying base space). Note that $\mathcal{T}\mathcal{S}$ is in some sense the universal tangent bundle. Indeed, using the tensor product on presentable ∞-categories [26, Section 4.8.1] (with unit $\mathcal{S}$, exhibiting that all presentable ∞-categories are tensored over $\mathcal{S}$), we have that

$$\mathcal{T}\mathcal{V} \simeq \mathcal{T}\mathcal{S} \otimes \mathcal{V}.$$

Indeed, using [26, Proposition 4.8.1.17], the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ on the excisive functors coincides under restriction along the Yoneda embedding with the full subcategory of $\mathrm{Fun}^{\mathrm{R}}(\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})^{\mathrm{op}}, \mathcal{V})$ of right adjoint functors that factor over the localization $(-)^{\mathrm{exc}}\colon \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \to \mathcal{T}\mathcal{S}$.

Remark 2.3 For any $S \in \mathcal{S}_*^{\mathrm{fin}}$ and $C \in \mathcal{V}$, let $h_S \otimes C = \mathrm{Map}(S, -) \otimes C$ be the corresponding corepresentable functor, ie the left Kan extension of $C\colon * \to \mathcal{V}$ along $S\colon * \to \mathcal{S}_*^{\mathrm{fin}}$. Note that $F \in \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is excisive if and only if it is a local object with respect to the set of maps

$$(h_{S_1} \amalg_{h_{S_3}} h_{S_2}) \otimes C_\alpha \to (h_{S_0} \otimes C_\alpha) \tag{2-1}$$

for any set of generators $\{C_\alpha\}$ of $\mathcal{V}$ and any pushout square in $\mathcal{S}_*^{\mathrm{fin}}$

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d] & S_1 \arrow[d] \\ S_2 \arrow[r] & S_3 \end{tikzcd}$$

In particular, the $\mathcal{T}\mathcal{V}$-local equivalences are strongly generated by this set of maps [23, Proposition 5.5.4.15].

Remark 2.4 For any presentable ∞-category $\mathcal{V}$, the description of the generating $\mathcal{T}\mathcal{V}$-local equivalences from Remark 2.3 shows that evaluation at $* \in \mathcal{S}_*^{\mathrm{fin}}$ sends $\mathcal{T}\mathcal{V}$-local equivalences to equivalences in $\mathcal{V}$. It follows that there is a commuting diagram

$$\begin{tikzcd} \mathcal{T}\mathcal{V} \arrow[r, hook] \arrow[dr] & \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \arrow[r, "(-)^{\mathrm{exc}}"] \arrow[d] & \mathcal{T}\mathcal{V} \arrow[dl] \\ & \mathcal{V} & \end{tikzcd}$$

The vertical functors are Cartesian (and co-Cartesian) fibrations, with right adjoint sections taking the constant $\mathcal{S}_*^{\mathrm{fin}}$-diagram. In particular, an arrow in $\mathcal{T}\mathcal{V}$ or $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is Cartesian if and only if it is the pullback of a map between constant diagrams. It follows that the (right adjoint) inclusion $\mathcal{T}\mathcal{V} \hookrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ preserves Cartesian arrows. When $\mathcal{V}$ is compactly generated, or more generally differentiable [26, Definition 6.1.1.6] (see Lemma 6.5), the functor $(-)^{\mathrm{exc}}$ preserves Cartesian arrows by [26, Theorem 6.1.1.10].

Let $\mathcal{V}$ be an ∞-category with finite limits and $B \in \mathcal{V}$ an object. For every $E \in \mathcal{T}_B\mathcal{V}$, there is a natural map $\Omega^\infty(E) \to B$, arising from the map of finite pointed spaces $S^0 \to *$. For every map $X \to B$, we denote by

$$H^0_{\mathrm{Q}}(X; E) = \pi_0 \operatorname{Map}_{/B}(X, \Omega^\infty(E))$$

the set of homotopy classes of lifts $\eta\colon X \to \Omega^\infty(E)$. Since $\Omega^\infty(E)$ is a grouplike $\mathbb{E}_\infty$-monoid over B by Proposition 2.28, this forms an abelian group; its unit is the zero map $0\colon X \to B \to \Omega^\infty(E)$ induced by the map of finite pointed spaces $* \to S^0$. More generally, we will refer to the groups $H^n_{\mathrm{Q}}(X; E) = H^0_{\mathrm{Q}}(X; \Sigma^n E)$ as the n^{th} *Quillen cohomology* groups of X with coefficients in E. Given a section $\eta\colon X \to \Omega^\infty(E)$, we will say that the pullback square

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & B \arrow[d, "0"] \\ X \arrow[r, "\eta"'] & \Omega^\infty(E) \end{tikzcd} \tag{2-2}$$

exhibits $\widetilde{X}$ as a *square zero extension* of X [26, Definition 7.4.1.6]. When η is homotopic to

$$0\colon X \to B \to \Omega^\infty(E),$$

we will refer to $\widetilde{X} \simeq X \times_B \Omega^{\infty+1}(E)$ as the *trivial square zero extension*.

Remark 2.5 The above definition of a square zero extension looks slightly more general than the one appearing in [26, Definition 7.4.1.6], where it is assumed that $B = X$. However, note that there is a natural map $p\colon X \to B$ (induced by the projection $\Omega^\infty(E) \to B$); pulling back the parametrized spectrum E along p, one can also realize $\widetilde{X}$ as the square zero extension of X classified by the canonical map $\eta'\colon X \to \Omega^\infty(p^*E)$.

2.2 Monoidal structure on the tangent bundle

Our next goal will be to construct a (closed) symmetric monoidal structure on the tangent bundle $\mathcal{T}\mathcal{V}$ of a presentable SM ∞-category. To this end, let us recall that if $\mathcal{V}$ is an SM ∞-category and $\mathcal{I}$ is an ∞-category, then $\mathrm{Fun}(\mathcal{I},\mathcal{V})$ can be endowed with a levelwise tensor product, as follows:

Construction 2.6 [26, Remark 2.1.3.4] Let $\mathcal{V}$ be an SM ∞-category, encoded by a co-Cartesian fibration of ∞-operads $\mathcal{V}^{\otimes} \to \mathrm{Fin}_*$. If $\mathcal{I}$ is another ∞-category, let us consider the map

$$\mathrm{Fun}(\mathcal{I},\mathcal{V})^{\otimes_{\mathrm{lev}}} := \mathrm{Fun}(\mathcal{I},\mathcal{V}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{I},\mathrm{Fin}_*)} \mathrm{Fin}_* \to \mathrm{Fin}_*.$$

This is again a co-Cartesian fibration of ∞-operads [26, Remark 2.1.3.4], which endows the functor category $\mathrm{Fun}(\mathcal{I},\mathcal{V})$ with a symmetric monoidal structure that we will refer to as the *levelwise tensor product*. For every $f\colon \mathcal{I} \to \mathcal{J}$, the restriction functor $f^*\colon \mathrm{Fun}(\mathcal{J},\mathcal{V}) \to \mathrm{Fun}(\mathcal{I},\mathcal{V})$ has the natural structure of a symmetric monoidal functor because the induced map $f^*\colon \mathrm{Fun}(\mathcal{J},\mathcal{V})^{\otimes_{\mathrm{lev}}} \to \mathrm{Fun}(\mathcal{I},\mathcal{V})^{\otimes_{\mathrm{lev}}}$ preserves co-Cartesian arrows over Fin_*. On the other hand, every SM functor $\mathcal{V} \to \mathcal{W}$ induces an SM functor $\mathrm{Fun}(\mathcal{I},\mathcal{V})^{\otimes_{\mathrm{lev}}} \to \mathrm{Fun}(\mathcal{I},\mathcal{W})^{\otimes_{\mathrm{lev}}}$ by postcomposition.

For future reference, let us mention two alternative descriptions of the levelwise tensor product:

Remark 2.7 The levelwise tensor product is adjoint to the *Boardman–Vogt tensor product*. Indeed, we can view $\mathcal{I}$ as an ∞-operad via the functor $\mathcal{I} \to * \to \mathrm{Fin}_*$, where the second functor is the inclusion of the object $\langle 1\rangle$. For any ∞-operad $\mathcal{O}$, recall that the ∞-category of ∞-operad maps $\mathcal{O}^{\otimes} \otimes_{\mathrm{BV}} \mathcal{I} \to \mathcal{V}^{\otimes}$ is then equivalent to the ∞-category $\mathrm{BiFunc}(\mathcal{O}^{\otimes},\mathcal{I};\mathcal{V}^{\otimes})$ of (dotted) bifunctors of ∞-operads [26, Definition 2.2.5.3]

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{I} \arrow[rrr, dotted] \arrow[d] & & & \mathcal{V}^{\otimes} \arrow[d] \\
\mathrm{Fin}_* \times * \arrow[r, "\mathrm{id}\times\{\langle 1\rangle\}"] & \mathrm{Fin}_* \times \mathrm{Fin}_* \arrow[rr, "\wedge"] & & \mathrm{Fin}_*
\end{tikzcd}$$

Since the bottom horizontal composite can simply be identified with the identity functor on Fin_*, the ∞-category $\mathrm{BiFunc}(\mathcal{O}^{\otimes},\mathcal{I};\mathcal{V}^{\otimes})$ is equivalent to the ∞-category of functors $f\colon \mathcal{O}^{\otimes} \times \mathcal{I} \to \mathcal{V}^{\otimes}$ relative to Fin_* with the following equivalent properties:

(a) For each inert map $\alpha\colon x \to y$ in $\mathcal{O}^{\otimes}$ and each equivalence $\beta\colon i \to j$ in $\mathcal{I}$, $f(\alpha,\beta)$ is an inert map in $\mathcal{V}^{\otimes}$.

(b) For each inert map $\alpha\colon x \to y$ in $\mathcal{O}^{\otimes}$ and each object $i \in \mathcal{I}$, $f(\alpha,\mathrm{id}_i)$ is an inert map in $\mathcal{V}^{\otimes}$.

These conditions are indeed equivalent since $f(\alpha,\beta) \simeq f(\mathrm{id}_y,\beta) \circ f(\alpha,\mathrm{id}_i)$, where $f(\mathrm{id}_y,\beta)$ is an equivalence. The ∞-category of functors f satisfying condition (b) is in turn equivalent to the ∞-category of ∞-operad maps $\mathcal{O}^{\otimes} \to \mathrm{Fun}(\mathcal{I},\mathcal{V})^{\otimes_{\mathrm{lev}}}$. Consequently, we have natural equivalences

$$\mathrm{Alg}_{\mathcal{O}\otimes_{\mathrm{BV}}\mathcal{I}}(\mathcal{V}^{\otimes}) \simeq \mathrm{BiFunc}(\mathcal{O}^{\otimes},\mathcal{I};\mathcal{V}^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{I},\mathcal{V})^{\otimes_{\mathrm{lev}}}).$$

Let us point out that by symmetry of the Boardman–Vogt tensor product, we also have that

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}) \simeq \mathrm{Alg}_{\mathcal{O} \otimes_{\mathrm{BV}} \mathcal{I}}(\mathcal{V}^{\otimes}) \simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}^{\otimes})).$$

Remark 2.8 If $\mathcal{I}$ has coproducts, then the levelwise tensor product can be identified with the *Day convolution product*. Indeed, let $\mathcal{I}^{\amalg}$ be the corresponding co-Cartesian ∞-operad [26, Definition 2.4.3.7] and let us write $\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{Day}}} \to \mathrm{Fin}_*$ for the ∞-operad obtained from $\mathcal{I}^{\amalg}$ and $\mathcal{V}^{\otimes}$ by Day convolution [26, Definition 2.2.6.1]. By [26, Proposition 2.2.6.16], this is a co-Cartesian fibration of ∞-operads that endows $\mathrm{Fun}(\mathcal{I}, \mathcal{V})$ with a (closed) SM structure. For any ∞-operad $\mathcal{O}$, we then have equivalences of ∞-categories of maps of ∞-operads (ie algebras)

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{Day}}}) \simeq \mathrm{Alg}_{\mathcal{O} \times \mathcal{I}^{\amalg}}(\mathcal{V}^{\otimes}) \simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}^{\otimes})) \simeq \mathrm{Alg}_{\mathcal{O} \otimes_{\mathrm{BV}} \mathcal{I}}(\mathcal{V}^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}).$$

Here $\mathcal{O} \times \mathcal{I}^{\amalg}$ is the product of ∞-operads, given explicitly by $\mathcal{O}^{\otimes} \times_{\mathrm{Fin}_*} \mathcal{I}^{\amalg} \to \mathrm{Fin}_*$. The first equivalence then follows from the universal property of the Day convolution [26, Definition 2.2.6.1], the second from [26, Theorem 2.4.3.18] and the last two equivalences follow from the relation between the levelwise tensor product and the Boardman–Vogt tensor product (which is symmetric).

Lemma 2.9 *Let $\mathcal{V}$ be a presentable SM ∞-category and $f : \mathcal{I} \to \mathcal{J}$ a functor between ∞-categories with finite coproducts that preserves finite coproducts. Then the SM functor $f^* : \mathrm{Fun}(\mathcal{J}, \mathcal{V})^{\otimes_{\mathrm{lev}}} \to \mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$ admits a symmetric monoidal left adjoint $f_!$.*

Proof Remark 2.8 identifies the lax SM functor $f^* : \mathrm{Fun}(\mathcal{J}, \mathcal{V})^{\otimes_{\mathrm{lev}}} \to \mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$ (which happens to be strong SM) with the lax SM functor $f^* : \mathrm{Fun}(\mathcal{J}, \mathcal{V})^{\otimes_{\mathrm{Day}}} \to \mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{Day}}}$ arising from naturality of the Day convolution product. The latter admits an SM left adjoint $f_!$ (given by left Kan extension) by [22, Remark 3.31]. □

Proposition 2.10 *Let $\mathcal{V}$ be a presentable SM ∞-category and endow $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ with the levelwise tensor product $\otimes_{\mathrm{lev}}$. Then the localization of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ at the $\mathcal{TV}$-local equivalences is monoidal. In particular:*

- *The localization functor $(-)^{\mathrm{exc}} : \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \to \mathcal{TV}$ has a unique lift to an SM functor between SM ∞-categories with domain given by $(\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}), \otimes_{\mathrm{lev}})$.*
- *The closed SM structure on $\mathcal{TV}$ is given by $X \otimes Y = (X \otimes_{\mathrm{lev}} Y)^{\mathrm{exc}}$.*

Proof By [26, Proposition 4.1.7.4], it suffices to verify that $X \otimes_{\mathrm{lev}} Y \to X \otimes_{\mathrm{lev}} Y'$ is a $\mathcal{TV}$-local equivalence for every $X : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ and every $\mathcal{TV}$-local equivalence $Y \to Y'$. Since the $\mathcal{TV}$-local equivalences are closed under colimits and $\otimes_{\mathrm{lev}}$ preserves colimits in each variable, we may assume that $Y \to Y'$ is a generating local equivalence of the form (2-1) and that $X = h_T \otimes D$. Since the tensoring $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \times \mathcal{V} \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is monoidal (for the levelwise tensor product), there are equivalences

$$X \otimes_{\mathrm{lev}} Y' := (h_T \otimes D) \otimes_{\mathrm{lev}} (h_{S_0} \otimes C) \simeq (h_T \times h_{S_0}) \otimes (C \otimes D) \simeq (h_{T \vee S_0}) \otimes (C \otimes D).$$

The last equivalence uses that the copresheaf $h_T \times h_{S_0} = \mathrm{Map}(T, -) \times \mathrm{Map}(S_0, -)$ (valued in spaces) is corepresentable by the coproduct $T \vee S_0$ in $\mathcal{S}_*^{\mathrm{fin}}$. Similarly, we have that

$$\begin{aligned} X \otimes_{\mathrm{lev}} Y &:= (h_T \otimes D) \otimes_{\mathrm{lev}} ((h_{S_1} \amalg_{h_{S_3}} h_{S_2}) \otimes C) \\ &\simeq (h_T \times (h_{S_1} \amalg_{h_{S_3}} h_{S_2})) \otimes (D \otimes C) \simeq (h_{T\vee S_1} \amalg_{h_{T\vee S_3}} h_{T\vee S_2}) \otimes (D \otimes C), \end{aligned}$$

where the last equivalence uses that $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ is Cartesian closed and that $h_T \times h_{S_i} = h_{T\vee S_i}$. It therefore suffices to show that the map

$$(h_{T\vee S_1} \amalg_{h_{T\vee S_3}} h_{T\vee S_2}) \otimes (D \otimes C) \to h_{T\vee S_0} \otimes (D \otimes C)$$

is a $\mathcal{TV}$-local equivalence. This is obvious since

$$\begin{tikzcd} T \vee S_0 \arrow[r] \arrow[d] & T \vee S_1 \arrow[d] \\ T \vee S_2 \arrow[r] & T \vee S_3 \end{tikzcd}$$

is a pushout square in $\mathcal{S}_*^{\mathrm{fin}}$. ◻

Lemma 2.11 *Let $\mathcal{V}$ be a presentable SM ∞-category and endow $\mathcal{TV}$ with the closed symmetric monoidal structure from Proposition 2.10. Then*:

(1) *The functor $\pi : \mathcal{TV} \to \mathcal{V}$ admits a natural symmetric monoidal structure.*

(2) *The induced oplax symmetric monoidal structure on the left adjoint to π* [26, Corollary 7.3.2.7] *is strong monoidal. Consequently*, *$\mathcal{TV}$ is tensored over $\mathcal{V}$ via the formula*

$$C \otimes X = (C \otimes_{\mathrm{lev}} X(-))^{\mathrm{exc}}.$$

(3) *$\Omega^\infty : \mathcal{TV} \to \mathcal{V}$ has a natural lax symmetric monoidal structure.*

Remark 2.12 The lax monoidal structure on Ω^∞ induces an oplax symmetric monoidal structure on $\Sigma^\infty_+ : \mathcal{V} \to \mathcal{TV}$ [19]. This does not make Σ^∞_+ a strong monoidal functor. For example, taking $\mathcal{V} = \mathcal{S}$, we have that $\Sigma^\infty_+(X) \in \mathrm{Sp}(\mathcal{S}_{/X})$ corresponds to the constant parametrized spectrum over X with fiber given by the sphere spectrum $\mathbb{S}$. Unraveling the definitions (eg using equivalence (2-3)), one then sees that $\Sigma^\infty_+(X) \otimes \Sigma^\infty_+(Y)$ corresponds to the constant parametrized spectrum over $X \times Y$ with fiber $\mathbb{S} \vee \mathbb{S}$, while $\Sigma^\infty_+(X \times Y)$ has fiber $\mathbb{S}$.

Proof Let $t : * \to \mathcal{S}_*^{\mathrm{fin}}$ be the inclusion of the initial (and also terminal) object. By Construction 2.6 and Lemma 2.9, restriction and left Kan extension along t yield an adjoint pair of SM functors

$$\mathrm{cst} = t_! : \mathcal{V} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) : t^* = \mathrm{ev}_*,$$

where the left adjoint takes the constant diagram and the right adjoint evaluates at $*$.

For (1), we then note that ev_* is itself a left adjoint and sends $\mathcal{T}\mathcal{V}$-local equivalences to equivalences in $\mathcal{V}$, since the domain and codomain of the generating $\mathcal{T}\mathcal{V}$-local equivalences (2-1) are both sent to C_α. It follows that $\pi\colon \mathcal{T}\mathcal{V} \to \mathcal{V}$ is symmetric monoidal for $\otimes$ as well. For (2), one simply notes that the SM functor $\mathrm{cst}\colon \mathcal{V} \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ already takes values in $\mathcal{T}\mathcal{V} \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$. For (3), note that Ω^∞ is the composite of the lax symmetric monoidal inclusion $\mathcal{T}\mathcal{V} \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ and the symmetric monoidal functor $\mathrm{ev}_{S^0}\colon \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \to \mathcal{V}$ (for the levelwise tensor product on the domain). □

For any functor $X\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$, there is a canonical (counit) map $X(*) \to X$, where we consider $X(*) \in \mathcal{V}$ as a constant diagram.

Lemma 2.13 *Let $X, Y\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ be two functors. Then the pushout-product map*

$$\psi(X,Y)\colon X(*) \otimes_{\mathrm{lev}} Y \amalg_{X(*)\otimes_{\mathrm{lev}} Y(*)} X \otimes_{\mathrm{lev}} Y(*) \to X \otimes_{\mathrm{lev}} Y$$

is a $\mathcal{T}\mathcal{V}$-local equivalence.

Proof Suppose that $X = \mathrm{colim}\, X_i$ for some diagram of functors X_i. Since evaluation and taking the constant diagram preserve colimits, we can identify the pushout-product map $\psi(X,Y)$ with the colimit $\mathrm{colim}_i\, \psi(X_i, Y)$ in the arrow category of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$. As $\mathcal{T}\mathcal{V}$-local equivalences are stable under colimits, we can therefore reduce to the case where $X = h_S \otimes C$ and $Y = h_T \otimes D$ are corepresentables.

Using that the constant diagram on $X(*)$ is given by $h_* \otimes X(*)$, the pushout-product map can then be identified with

$$h_T \otimes (C \otimes D) \amalg_{h_* \otimes (C\otimes D)} h_S \otimes (C \otimes D) \to (h_S \otimes C) \otimes_{\mathrm{lev}} (h_T \otimes D).$$

As in the proof of Proposition 2.10, the codomain can be identified with $h_{S\vee T} \otimes (C \otimes D)$. The above map is then a $\mathcal{T}\mathcal{V}$-local equivalence because

$$\begin{tikzcd} S \vee T \arrow[r] \arrow[d] & S \vee * \arrow[d] \\ * \vee T \arrow[r] & * \vee * \end{tikzcd}$$

is a co-Cartesian square (see Remark 2.3). □

The above lemma can be described somewhat informally as follows: we can identify an object of $\mathcal{T}\mathcal{V}$ with a tuple consisting of $C \in \mathcal{V}$ and $E \in \mathrm{Sp}(\mathcal{V}_{/C})$. Using the tensoring of $\mathcal{T}\mathcal{V}$ over $\mathcal{V}$ from Lemma 2.11, we then have an equivalence

$$(C, E) \otimes (D, F) \simeq (C \otimes D, (C \otimes F) \oplus (E \otimes D)), \tag{2-3}$$

where the direct sum is taken in the fiber $\mathcal{T}_{C\otimes D}\mathcal{V}$. This justifies the following terminology:

Definition 2.14 Let $\mathcal{V}$ be a presentable SM ∞-category. The *square zero tensor product* on $\mathcal{T}\mathcal{V}$ is the symmetric monoidal structure provided by Proposition 2.10.

For any SM left adjoint $f\colon \mathcal{V} \to \mathcal{W}$, postcomposition with f defines an SM left adjoint $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{W})$ that descends to a natural SM left adjoint $\mathcal{T}(f)\colon \mathcal{T}\mathcal{V} \to \mathcal{T}\mathcal{W}$ between localizations.

Remark 2.15 Let $\varnothing$ be the initial object of $\mathcal{V}$. Since $\{\varnothing\} \hookrightarrow \mathcal{V}$ is stable under the binary tensor product of $\mathcal{V}$ and $\pi\colon \mathcal{T}\mathcal{V} \to \mathcal{V}$ is symmetric monoidal, the full subcategory $\mathcal{T}_\varnothing\mathcal{V} = \mathcal{T}\mathcal{V} \times_{\mathcal{V}} \{\varnothing\} \hookrightarrow \mathcal{T}\mathcal{V}$ inherits a nonunital SM structure from $\mathcal{T}\mathcal{V}$. Lemma 2.13 shows that for all $E, F \in \mathcal{T}_\varnothing\mathcal{V}$, the tensor product $E \otimes F$ is the zero object in $\mathcal{T}_\varnothing\mathcal{V}$.

Example 2.16 Let $\mathcal{V}$ be a cartesian closed presentable ∞-category. In this case, the levelwise monoidal structure on $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ induced by the cartesian product on $\mathcal{V}$ is simply the cartesian monoidal structure. Since the (reflective) full subcategory $\mathcal{T}\mathcal{V} \hookrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is closed under the cartesian product, the induced square zero monoidal structure on $\mathcal{T}\mathcal{V}$ is simply the cartesian product as well.

Proposition 2.17 *Let $\mathcal{V}$ be a presentable SM ∞-category and let $\mathcal{O}$ be an ∞-operad. Then there is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \simeq \mathcal{T}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})),$$

where $\mathcal{T}\mathcal{V}$ is endowed with the square zero monoidal structure.

Proof The fully faithful functor $\mathcal{T}\mathcal{V} \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is lax symmetric monoidal and hence realizes $\mathcal{T}\mathcal{V}^\otimes$ as a full suboperad of the ∞-operad $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})^\otimes$. The ∞-category of $\mathcal{O}$-algebras in $\mathcal{T}\mathcal{V}$ then embeds as the full subcategory of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}))$ whose underlying functors are excisive. Using Remark 2.7 together with the commutativity of the Boardman–Vogt tensor product [26, Proposition 2.2.5.13], we obtain an equivalence

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \arrow[d, phantom, "\cap" description, "\rotatebox{90}{$\subseteq$}"] & & \mathcal{T}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})) \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}), \otimes_{\mathrm{lev}}) \arrow[rr, "\sim"] \arrow[dr] & & \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})) \arrow[dl] \\
& \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) &
\end{tikzcd}$$

of ∞-categories over $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$, where the diagonal functors are induced by forgetting algebra structures. In particular, this equivalence identifies the full subcategory $\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ on the left-hand side with the full subcategory on the right spanned by diagrams of $\mathcal{O}$-algebras in $\mathcal{V}$ whose underlying diagrams are excisive. But this is the same as diagrams $\mathcal{S}_*^{\mathrm{fin}} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ that are themselves excisive, because the forgetful functor from $\mathcal{O}$-algebras to $\mathcal{V}$ detects limits [26, Corollary 3.2.2.4]. We conclude that the horizontal equivalence above identifies $\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ with $\mathcal{T}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}))$, so the desired result follows. $\square$

The following result provides a symmetric monoidal refinement of Example 2.2:

Proposition 2.18 *Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ and consider the commuting square in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\eta"] \arrow[d, "\mathrm{cst}"'] & \mathcal{V} \arrow[d, "\mathrm{cst}"] \\ \mathcal{T}\mathcal{S} \arrow[r, "\mathcal{T}(\eta)"] & \mathcal{T}\mathcal{V} \end{tikzcd}$$

where the vertical functors are the SM left adjoints to the projection functors and the horizontal functors are induced by the map η from the initial presentable SM ∞-category. This is a pushout square in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

The proof requires some results about the tensor product of presentable ∞-categories [26, Section 4.8.1]. Let us recall that there is a sub-∞-operad $\mathrm{Pr}^{\mathrm{L},\otimes} \subseteq \mathrm{Cat}^{\mathrm{big},\times}$ of the cartesian operad of big ∞-categories, whose objects are presentable ∞-categories and $\mathrm{Map}_{\mathrm{Pr}^{\mathrm{L},\otimes}}(\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{D})$ is the union of path components of $\mathrm{Map}_{\mathrm{Cat}^{\mathrm{big}}}(\mathcal{C}_1 \times \dots \times \mathcal{C}_n, \mathcal{D})$ spanned by the functors preserving colimits in each variable. Then the ∞-operad $\mathrm{Pr}^{\mathrm{L},\otimes}$ describes a (closed) symmetric monoidal structure on Pr^{L} [26, Proposition 4.8.1.15].

In the proof below, let us refer to functors $g\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ preserving colimits in each variable simply as *bifunctors* and let us say that such a bifunctor g is initial if it defines an initial object in the ∞-category of presentable ∞-categories (with left adjoints between them) equipped with a bifunctor from $\mathcal{C}_1 \times \mathcal{C}_2$. An initial bifunctor $g\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ exhibits $\mathcal{D}$ as the tensor product of $\mathcal{C}_1$ and $\mathcal{C}_2$ in Pr^{L}.

Lemma 2.19 *Let $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{D}$ be presentable ∞-categories, $g\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ a bifunctor and consider the functor*

$$\Psi(g)\colon \mathcal{D} \xrightarrow{h} \mathcal{P}(\mathcal{D}) \xrightarrow{g^*} \mathcal{P}(\mathcal{C}_1 \times \mathcal{C}_2).$$

Then $\Psi(g)$ takes values in the full subcategory of right adjoint functors

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2) \subseteq \mathrm{Fun}(\mathcal{C}_1^{\mathrm{op}}, C_2) \subseteq \mathrm{Fun}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{P}(\mathcal{C}_2)) \simeq \mathcal{P}(\mathcal{C}_1 \times \mathcal{C}_2)$$

and g is an initial bifunctor if and only if $\Psi(g)\colon \mathcal{D} \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$ is an equivalence.

Proof This follows from the proof of [26, Proposition 4.8.1.17]. Indeed, the argument in [loc. cit.] shows that $\Psi(g)$ is in fact a right adjoint functor with values in the full subcategory $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$ and that the assignment $g \mapsto \Psi(g)$ determines a natural equivalence of spaces

$$\mathrm{Map}_{\mathrm{Pr}^{\mathrm{L},\otimes}}(\mathcal{C}_1, C_2; \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Pr}^{\mathrm{R}}}(\mathcal{D}, \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)) \simeq \mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2), \mathcal{D}).$$

In particular (as concluded in [loc. cit.]), it follows that the presentable ∞-category $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$ corepresents bifunctors, ie $\mathcal{C}_1 \otimes \mathcal{C}_2 \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$. This immediately implies that g is an initial bifunctor if and only if $\Psi(g)\colon \mathcal{D} \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$ is an equivalence. □

Proof of Proposition 2.18 Since $\mathcal{S}$ is the initial object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ and coproducts of $\mathbb{E}_\infty$-algebras are given by the tensor product in the underlying ∞-category [26, Proposition 3.2.4.7], it will suffice to verify that the SM left adjoint functor $F\colon \mathcal{T}\mathcal{S} \otimes \mathcal{V} \to \mathcal{T}\mathcal{V}$ induced by the commuting square is an equivalence. To verify this, we need to show that the underlying functor (forgetting SM structures) is an equivalence.

To this end, note that the proof of [26, Proposition 3.2.4.7] implies that F can be identified with the composite functor

$$F\colon \mathcal{T}\mathcal{S}\otimes\mathcal{V}\xrightarrow{\mathcal{T}(\eta)\otimes\mathrm{cst}}\mathcal{T}\mathcal{V}\otimes\mathcal{T}\mathcal{V}\xrightarrow{\otimes}\mathcal{T}\mathcal{V}.$$

The corresponding bifunctor is therefore given by

$$f\colon \mathcal{T}\mathcal{S}\times\mathcal{V}\xrightarrow{\mathcal{T}(\eta)\times\mathrm{cst}}\mathcal{T}\mathcal{V}\times\mathcal{T}\mathcal{V}\xrightarrow{\otimes}\mathcal{T}\mathcal{V}.$$

To see that F is an equivalence, we need to show that the bifunctor f satisfies the condition of Lemma 2.19, ie that $\Psi(f)\colon\mathcal{T}\mathcal{V}\to\mathrm{Fun}^{\mathrm{R}}(\mathcal{T}\mathcal{S}^{\mathrm{op}},\mathcal{V})$ is an equivalence. To identify the codomain of $\Psi(f)$, consider the functor $h^{\mathrm{exc}}\colon\mathcal{S}_*^{\mathrm{fin,op}}\hookrightarrow\mathcal{P}(\mathcal{S}_*^{\mathrm{fin,op}})\to\mathcal{T}\mathcal{S}$ given by the Yoneda embedding followed by the localization from Remark 2.3. The universal properties of the Yoneda embedding and this localization imply that restriction along h^{exc} induces an equivalence

$$(h^{\mathrm{exc}})^*\colon\mathrm{Fun}^{\mathrm{R}}(\mathcal{T}\mathcal{S}^{\mathrm{op}},\mathcal{V})\xrightarrow{\sim}\mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}},\mathcal{V})=\mathcal{T}\mathcal{V}.$$

Using this equivalence, $\Psi(f)$ can be identified with the functor $\Psi(f)\colon\mathcal{T}\mathcal{V}\to\mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}},\mathcal{V})$ sending $X\in\mathcal{T}\mathcal{V}$ to the functor $\mathcal{S}_*^{\mathrm{fin}}\to\mathcal{V}$ classifying the correspondence

$$\mathcal{S}_*^{\mathrm{fin}}\times\mathcal{V}\to\mathcal{S},\quad (S,v)\mapsto\mathrm{Map}_{\mathcal{T}\mathcal{V}}((h_S\otimes 1_{\mathcal{V}})^{\mathrm{exc}}\otimes\mathrm{cst}(v),X).$$

Here we used that $T(\eta)\colon\mathcal{T}\mathcal{S}\to\mathcal{T}\mathcal{V}$ sends h_S^{exc} to the excisive approximation of $(h_S\otimes 1_{\mathcal{V}})$. By Proposition 2.10, the tensor product $(h_S\otimes 1_{\mathcal{V}})^{\mathrm{exc}}\otimes\mathrm{cst}(v)$ in $\mathcal{T}\mathcal{V}$ is naturally equivalent to $(h_S\otimes v)^{\mathrm{exc}}\in\mathcal{T}\mathcal{V}$. This object has the universal property that

$$\mathrm{Map}_{\mathcal{T}\mathcal{V}}((h_S\otimes v)^{\mathrm{exc}},X)\simeq\mathrm{Map}_{\mathcal{V}}(v,X(S)).$$

It follows that $\Psi(f)$ can simply be identified with the identity on $\mathcal{T}\mathcal{V}$. In particular, it is an equivalence, so that Lemma 2.19 shows that f is an initial bifunctor and F is an equivalence, as desired. □

2.3 t-orientations on tangent categories

In later sections, we will consider various examples of tangent bundles whose fibers are stable categories with a natural "connective part". Let us axiomatize this situation as follows:

Definition 2.20 Let $p\colon\mathcal{E}\to\mathcal{B}$ be a *stable Cartesian fibration*, ie a Cartesian fibration such that each fiber $\mathcal{E}_X$ is stable and each arrow $f\colon X\to Y$ in $\mathcal{B}$ induces an exact functor $f^*\colon\mathcal{E}_Y\to\mathcal{E}_X$. A *$t$-orientation* on $p\colon\mathcal{E}\to\mathcal{B}$ is a tuple of full subcategories $(\mathcal{E}^{\geq 0},\mathcal{E}^{\leq 0})$ of $\mathcal{E}$ such that:

(1) For each p-Cartesian arrow $E\to F$ in $\mathcal{E}$ with $F\in\mathcal{E}^{\leq 0}$, we have that $E\in\mathcal{E}^{\leq 0}$.

(2) For every $X\in\mathcal{B}$, the tuple

$$(\mathcal{E}^{\geq 0}\cap\mathcal{E}_X,\ \mathcal{E}^{\leq 0}\cap\mathcal{E}_X)$$

defines a t-structure on the stable ∞-category $\mathcal{E}_X$.

In this case, we will refer to $\mathcal{E}^{\heartsuit}=\mathcal{E}^{\geq 0}\cap\mathcal{E}^{\leq 0}$ as the *heart* of the t-orientation.

Example 2.21 Let $\pi\colon \mathcal{T}\mathcal{V} \to \mathcal{V}$ be the tangent bundle of a presentable ∞-category. Then each $\mathcal{T}_X\mathcal{V}$ carries a t-structure such that $\mathcal{T}_X^{\leq -1}\mathcal{V}$ is the full subcategory of $E \in \mathcal{T}_X\mathcal{V}$ such that $\Omega^\infty(E) \simeq X$ is the terminal object in $\mathcal{V}_{/X}$ [26, Proposition 1.4.3.4]. Since such objects are stable under base change along a map $X' \to X$ in the base, it follows that $\mathcal{T}\mathcal{V}$ comes with a *canonical t-orientation* in which $\mathcal{T}^{\leq -1}\mathcal{V}$ consists of those E such that $\Omega^\infty(E) \simeq \pi(E)$.

Condition (1) of Definition 2.20 is equivalent to $\mathcal{E}^{\leq 0} \to \mathcal{B}$ being a Cartesian fibration and the inclusion $\mathcal{E}^{\leq 0} \hookrightarrow \mathcal{E}$ preserving Cartesian edges.

Lemma 2.22 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a stable Cartesian fibration with a t-orientation $(\mathcal{E}^{\geq 0}, \mathcal{E}^{\leq 0})$. Then:*

(1) *The restriction of the projection p to each of the three subcategories $\mathcal{E}^{\geq 0}$, $\mathcal{E}^{\leq 0}$ and $\mathcal{E}^{\heartsuit}$ is a Cartesian fibration.*

(2) *There exists a commuting square of adjunctions over $\mathcal{B}$, ie in $\mathrm{Cat}_\infty/\mathcal{B}$, of the form*

$$\begin{tikzcd} \mathcal{E}^{\heartsuit} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] \arrow[d, hook, shift left] \arrow[d, phantom, "\dashv"] & \mathcal{E}^{\leq 0} \arrow[l, shift left, "\tau_{\geq 0}"] \arrow[d, hook, shift left] \arrow[d, phantom, "\dashv"] \\ \mathcal{E}^{\geq 0} \arrow[u, shift left, "\tau_{\leq 0}"] \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "\tau_{\geq 0}"] \arrow[u, shift left, "\tau_{\leq 0}"] \end{tikzcd}$$

Furthermore, all right adjoint functors preserve Cartesian edges.

In particular, $\mathcal{E}^{\heartsuit} \to \mathcal{B}$ is a Cartesian fibration whose fibers are (ordinary) abelian categories.

Proof For each $X \in \mathcal{B}$, the fiber $\mathcal{E}_X$ comes equipped with a t-structure. In particular, for each X there are coreflective localizations [26, Proposition 1.2.1.5],

$$\begin{tikzcd} \mathcal{E}_X^{\geq 0} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E}_X \arrow[l, shift left, "\tau_{\geq 0}"] \end{tikzcd}, \quad \begin{tikzcd} \mathcal{E}_X^{\heartsuit} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E}_X^{\leq 0} \arrow[l, shift left, "\tau_{\geq 0}"] \end{tikzcd}. \tag{2-4}$$

The functors $\tau_{\geq 0}$ realize their codomain as the localization of the domain at the (-1)-coconnective morphisms, ie those morphisms whose cofiber in $\mathcal{E}_X$ is contained in $\mathcal{E}_X^{\leq 0}$. By condition (1) from Definition 2.20, each morphism $f\colon X \to Y$ in $\mathcal{E}$ induces a left t-exact functor $f^*\colon \mathcal{E}_Y \to \mathcal{E}_X$ between the fibers. It follows that the (-1)-coconnective morphisms in (each fiber of) $\mathcal{E}$ and $\mathcal{E}^{\leq 0}$ are stable under the functors f^*. Let us pass to a universe $\mathcal{U}$ such that $\mathcal{E}$ and $\mathcal{B}$ are $\mathcal{U}$-small and write $\mathcal{X}^{\geq 0}$ (resp. $\mathcal{X}^{\heartsuit}$) for the $\mathcal{U}$-small ∞-category obtained from $\mathcal{E}$ (resp. $\mathcal{E}^{\leq 0}$) by localizing at the (-1)-coconnective arrows in each fiber. We can then apply [21, Proposition 2.1.4] in the ($\mathcal{U}$-small) setting where the marked arrows in $\mathcal{B}$ are just the equivalences to obtain maps of Cartesian fibrations (preserving Cartesian arrows)

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\tau_{\geq 0}"] \arrow[dr, "\pi"'] & & \mathcal{X}^{\geq 0} \arrow[dl] \\ & \mathcal{B} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^{\leq 0} \arrow[rr, "\tau_{\geq 0}"] \arrow[dr, "\pi"'] & & \mathcal{X}^{\heartsuit} \arrow[dl] \\ & \mathcal{B} & \end{tikzcd} \tag{2-5}$$

By [loc. cit.], on the fiber over an object $X \in \mathcal{B}$ these maps can be identified with the localization functors from (2-4). In particular, it follows from [26, Proposition 7.3.2.6] that the localizations from (2-5) both admit a left adjoint over $\mathcal{B}$. These left adjoints are (fiberwise) fully faithful and identify $\mathcal{X}^{\geq 0}$ and $\mathcal{X}^{\heartsuit}$ with the full subcategories $\mathcal{E}^{\geq 0}$ and $\mathcal{E}^{\heartsuit}$, respectively. In particular, this shows that the projections from $\mathcal{E}^{\geq 0}$ and $\mathcal{E}^{\heartsuit}$ to $\mathcal{B}$ are Cartesian fibrations, proving (1). Furthermore, the functors from (2-5) provide the horizontal right adjoints (relative to $\mathcal{B}$) in (2). Finally, the inclusions $\mathcal{E}^{\heartsuit} \to \mathcal{E}^{\geq 0}$ and $\mathcal{E}^{\leq 0} \to \mathcal{E}$ admit left adjoints over $\mathcal{B}$ by [26, Proposition 7.3.2.6]. $\square$

Let us now specialize to the case of the tangent bundle.

Definition 2.23 Let $\mathcal{V}$ be an SM ∞-category with finite limits. A t-orientation on $\mathcal{T}\mathcal{V}$ is *monoidal* if $\mathcal{T}^{\geq 0}\mathcal{V}$ is closed under the square-zero tensor product and contains the unit.

Example 2.24 Consider the full subcategories of excisive functors $F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$

$$\mathcal{T}^{\geq 0}\mathcal{S} \subseteq \mathcal{T}\mathcal{S}, \qquad \mathcal{T}^{\leq 0}\mathcal{S} \subseteq \mathcal{T}\mathcal{S}$$

such that for every n, the map $F(S^n) \to F(*)$ has n-connected, (resp. n-truncated) fibers. This defines a t-orientation on $\mathcal{T}\mathcal{S}$, whose restriction to each fiber $\mathcal{T}_X\mathcal{S} \simeq \mathrm{Fun}(X, \mathrm{Sp})$ consists of diagrams of connective, (resp. coconnective) spectra. Furthermore, this t-orientation is monoidal (the square zero monoidal structure simply being the Cartesian product by Example 2.16). In particular, the heart $\mathcal{T}^{\heartsuit}\mathcal{S}$ can be identified with the ∞-category of *local systems of abelian groups*. The inclusion $\mathcal{T}^{\heartsuit}\mathcal{S} \subseteq \mathcal{T}\mathcal{S}$ sends a local system of abelian groups $\mathcal{A}$ to the corresponding parametrized Eilenberg–MacLane spectrum $\mathrm{H}\mathcal{A}$.

Let $\mathcal{V}$ be an SM ∞-category with finite limits and suppose that $\mathcal{T}\mathcal{V}$ carries a monoidal t-orientation. If $\mathcal{O}$ is an ∞-operad, we can use Proposition 2.17 to identify the Cartesian fibration $\pi\colon \mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ with $\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$. Using this identification, consider the two full subcategories

$$\mathcal{T}^{\geq 0}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) = \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}^{\geq 0}\mathcal{V}), \qquad \mathcal{T}^{\leq 0}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) = \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}^{\leq 0}\mathcal{V}),$$

where we view $\mathcal{T}^{\geq 0}\mathcal{V}$ and $\mathcal{T}^{\leq 0}\mathcal{V}$ as full suboperads of $\mathcal{T}\mathcal{V}$. In other words, these are the full subcategories of $\mathcal{O}$-algebras in $\mathcal{T}\mathcal{V}$ whose underlying objects (for every color $x \in \mathcal{O}$) are 0-connective (resp. 0-coconnective) in $\mathcal{T}\mathcal{V}$.

Proposition 2.25 *These two full subcategories $\mathcal{T}^{\geq 0}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ and $\mathcal{T}^{\leq 0}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ define a monoidal t-orientation on $\mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$. For every color $x \in \mathcal{O}$, the forgetful functor $x^*\colon \mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \to \mathcal{T}\mathcal{V}$ is t-exact, ie it preserves both 0-connective and 0-coconnective objects.*

Proof First, to see that the t-orientation is monoidal, note that the full subcategory $\mathcal{T}^{\geq 0}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}^{\geq 0}\mathcal{V}) \subseteq \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ is closed under tensor products, since evaluation on the set of colors detects tensor products of algebras.

To verify condition (1) of Definition 2.20, notice that a morphism in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ is π-Cartesian if and only if for every color $x \in \mathcal{O}$, its image under $x^* \colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \to \mathcal{T}\mathcal{V}$ is a Cartesian arrow [26, Corollary 3.2.2.3]. This immediately implies that for every Cartesian arrow in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ whose codomain is contained in $\mathcal{T}^{\leq 0}\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$, the domain is contained in $\mathcal{T}^{\leq 0}\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ as well.

For condition (2), consider the adjoint pair $\mathcal{T}^{\geq 0}\mathcal{V} \rightleftarrows \mathcal{T}\mathcal{V}$ from Lemma 2.22. Since the inclusion $\mathcal{T}^{\geq 0}\mathcal{V} \to \mathcal{T}\mathcal{V}$ is symmetric monoidal, its right adjoint $\tau_{\geq 0}$ inherits a lax symmetric monoidal structure [26, Corollary 7.3.2.7]. We therefore obtain an adjoint pair at the level of $\mathcal{O}$-algebras which is natural with respect to restriction along maps of ∞-operads $\mathcal{O} \to \mathcal{O}'$ (see [26, Remark 7.3.2.13]). In particular, both adjoints commute with the forgetful functor for each color $x \in \mathcal{O}$

$$\begin{tikzcd}
\mathcal{T}^{\geq 0}\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}^{\geq 0}\mathcal{V}) \arrow[r, hook, shift left] \arrow[d, "x^*"'] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \arrow[l, shift left] \arrow[d, "x^*"] \\
\mathcal{T}^{\geq 0}\mathcal{V} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{T}\mathcal{V} \arrow[l, shift left]
\end{tikzcd}$$

Since the unit of the adjoint pair $\mathcal{T}^{\geq 0}\mathcal{V} \rightleftarrows \mathcal{T}\mathcal{V}$ is an equivalence and its counit maps to an equivalence in $\mathcal{V}$ by Lemma 2.22, the induced adjunction on $\mathcal{O}$-algebras restricts to an adjunction between the fibers over an $\mathcal{O}$-algebra A

$$\begin{tikzcd}
\mathcal{T}_A^{\geq 0}\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[r, hook, shift left] \arrow[d, "x^*"'] \arrow[r, phantom, "\perp"] & \mathcal{T}_A\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[l, shift left, "\tau_{\geq 0}"] \arrow[d, "x^*"] \\
\mathcal{T}_{x^*A}^{\geq 0}\mathcal{V} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{T}_{x^*A}\mathcal{V} \arrow[l, shift left, "\tau_{\geq 0}"]
\end{tikzcd}$$

The left and right adjoint both commute with the forgetful functors and the unit of the adjunction is an equivalence. In particular, it follows that an object $E \in \mathcal{T}_A\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ is

(a) contained in $\mathcal{T}_A^{\geq 0}\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ if and only if $\tau_{\geq 0}(E) \simeq E$,

(b) contained in $\mathcal{T}_A^{\leq -1}\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ if and only if for every color $x \in \mathcal{O}$, $x^*E \in \mathcal{T}_{x^*A}\mathcal{V}$ is (-1)-coconnective, ie $\tau_{\geq 0}(x^*E) \simeq 0$; in turn, this is equivalent to $\tau_{\geq 0}(E) \simeq 0$ in $\mathcal{T}_A\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$.

By [26, Proposition 1.2.1.16], the subcategories $\mathcal{T}_A^{\geq 0}\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ and $\mathcal{T}_A^{\leq 0}\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ then determine a t-structure on $\mathcal{T}_A\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ if and only if the essential image of

$$\tau_{\geq 0} \colon \mathcal{T}_A\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \to \mathcal{T}_A\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$$

is closed under extensions. Since this functor is idempotent, (a) identifies its essential image with $\mathcal{T}_A^{\geq 0}\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$, which is closed under extensions because the forgetful functors x^* (which detect connectivity) preserve extensions and each $\mathcal{T}_{x^*A}^{\geq 0}\mathcal{V}$ is closed under extensions. □

In the remainder of this section, we will show that for a large class of presentable ∞-categories $\mathcal{V}$, the connective objects for the canonical t-orientation on $\mathcal{T}\mathcal{V}$ (Example 2.21) admit a simpler combinatorial description than that of an excisive functor. To this end, let us start by recalling that every $E \in \mathcal{T}_X\mathcal{V}$ defines a reduced excisive functor $E\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}_{/X}$. Restricting E to the full subcategory of finite pointed sets, we obtain a very special Γ-space object in $\mathcal{V}_{/X}$ in the sense of Segal, whose underlying object is $\Omega^\infty(E)$. Indeed, for any two finite pointed sets S, T, the pushout square of finite pointed spaces (in fact, sets)

$$\begin{tikzcd} S \vee T \arrow[r] \arrow[d] & * \vee T \arrow[d] \\ S \vee * \arrow[r] & * \end{tikzcd}$$

induces an equivalence $E(S \vee T) \to E(S) \times E(T)$, from which the grouplike Segal conditions follow. In other words, $\Omega^\infty(E)$ has the structure of a grouplike $\mathbb{E}_\infty$-monoid in the sense of [8].

Conversely, in the presence of loop space machinery, every grouplike $\mathbb{E}_\infty$-monoid arises from a spectrum. For later purposes, let us make this slightly more precise: suppose that $\mathcal{V}$ is a presentable ∞-category and let

$$\mathrm{Grp}(\mathcal{V}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V}), \quad \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}) \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})$$

denote the ∞-categories of grouplike monoids, (resp. grouplike $\mathbb{E}_\infty$-monoids) in $\mathcal{V}$. Both arise as full (reflective) subcategories of diagrams satisfying the grouplike Segal conditions [23, Definition 7.2.2.1; 26, Section 2.4.2, Definition 5.2.6.2; 8]. In addition, there is an adjoint pair

$$B\colon \mathrm{Grp}(\mathcal{V}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{V}_* :\Omega \tag{2-6}$$

where the left adjoint sends a grouplike monoid to its bar construction and the right adjoint sends a pointed object in $\mathcal{V}$ to its loop space (endowed with the group structure coming from the usual cogroup structure $S^1 \to S^1 \vee S^1$).

Definition 2.26 Let $\mathcal{V}$ be a presentable ∞-category. We will say that $\mathcal{V}$ *has loop space machinery* if it satisfies the following conditions:

(1) The Cartesian product $\mathcal{V} \times \mathcal{V} \xrightarrow{\times} \mathcal{V}$ preserves geometric realizations.

(2) The unit of the adjunction (2-6) is an equivalence.

We will say that $\mathcal{V}$ has *parametrized loop space machinery* if each slice ∞-category $\mathcal{V}_{/X}$ has loop space machinery.

Example 2.27 Note that $\mathcal{V}$ has loop space machinery if and only if $\mathcal{V}_{*/}$ has loop space machinery. Using this, one readily sees that all ∞-toposes and stable presentable ∞-categories have parametrized loop space machinery. More generally, a prestable presentable ∞-category (ie the connective part of a t-structure on a stable ∞-category [25, Section C.1]) has parametrized loop space machinery. If $\mathcal{V}$ has (parametrized) loop space machinery and $U\colon \mathcal{W} \to \mathcal{V}$ is a right adjoint functor preserving sifted colimits and detecting equivalences (in particular, it is monadic), then $\mathcal{W}$ has (parametrized) loop space machinery.

Recall that a simplicial object $\Delta^{\mathrm{op}} \to \mathcal{D}$ in some ∞-category $\mathcal{D}$ is said to be *n-skeletal* if it is left Kan extended from $\Delta^{\mathrm{op}}_{\leq n} \subseteq \Delta^{\mathrm{op}}$.

Proposition 2.28 *Let $\mathcal{V}$ be a presentable ∞-category with loop space machinery and consider the adjoint pair*

$$\boldsymbol{B}\colon \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Sp}(\mathcal{V}) = \mathrm{Exc}_{\mathrm{red}}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V}) : \Omega^\infty$$

whose right adjoint restricts a reduced excisive functor along the inclusion $i\colon \mathrm{Fin}_* \to \mathcal{S}^{\mathrm{fin}}_*$. Then the left adjoint $\boldsymbol{B}$ is fully faithful and a functor $F\colon \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{V}$ lies in its essential image (in particular, it will be reduced excisive) if and only if it satisfies the following two conditions:

1. *Its restriction to Fin_* satisfies the grouplike Segal conditions.*
2. *It preserves all **finite geometric realizations**, ie colimits of simplicial diagrams that are n-skeletal for some n.*

Before turning to the proof of Proposition 2.28, let us mention some consequences:

Definition 2.29 For a presentable ∞-category $\mathcal{V}$, let us say that a functor $A\colon \mathrm{Fin}_* \to \mathcal{V}$ is a *Segal $\mathbb{E}_\infty$-groupoid* if for any two finite pointed sets $S, T \in \mathrm{Fin}_*$, the square

$$\begin{tikzcd} A(S \vee T) \arrow[r] \arrow[d] & A(* \vee T) \arrow[d] \\ A(S \vee *) \arrow[r] & A(*) \end{tikzcd}$$

is cartesian. We will write $\mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})$ for the full subcategory on the Segal $\mathbb{E}_\infty$-groupoids.

A Segal $\mathbb{E}_\infty$-groupoid in $\mathcal{V}$ with $A(*) = X$ is equivalent to a grouplike $\mathbb{E}_\infty$-monoid in $\mathcal{V}_{/X}$.

Corollary 2.30 *Let $\mathcal{V}$ be a presentable ∞-category with parametrized loop space machinery. Then the following hold*:

1. *There is a relative adjoint pair*

$$\begin{tikzcd} \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \arrow[rr, shift left, "\boldsymbol{B}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "\mathrm{ev}_*"'] & & \mathcal{T}\mathcal{V} = \mathrm{Exc}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V}) \arrow[ll, shift left, "\Omega^\infty"] \arrow[dl, "\pi"] \\ & \mathcal{V} & \end{tikzcd}$$

   whose right adjoint restricts an excisive functor along the inclusion $i\colon \mathrm{Fin}_* \to \mathcal{S}^{\mathrm{fin}}_*$.
2. *The left adjoint $\boldsymbol{B}$ is fully faithful and a functor $F\colon \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{V}$ lies in its essential image (in particular, it will be excisive) if and only if it preserves finite geometric realizations and i^*F is a Segal $\mathbb{E}_\infty$-groupoid.*
3. *The connective part $\mathcal{T}^{\geq 0}\mathcal{V}$ of the canonical t-orientation (Example 2.21) on $\mathcal{T}\mathcal{V}$ coincides with the essential image of $\boldsymbol{B}$.*

Proof Each excisive functor $E\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ can also be considered as a reduced excisive functor with values in $\mathcal{V}_{/E(*)}$. The restriction to Fin_* then defines a grouplike $\mathbb{E}_\infty$-monoid in $\mathcal{V}_{/E(*)}$, or equivalently, an $\mathbb{E}_\infty$-groupoid in $\mathcal{V}$. It follows that there is a well-defined functor $\Omega^\infty\colon \mathcal{T}\mathcal{V} \to \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})$ compatible with the projections to $\mathcal{V}$. For each $X \in \mathcal{V}$, the induced functor between fibers admits a fully faithful left adjoint by Proposition 2.28 (applied to $\mathcal{V}_{/X}$).

For (1), we now note that the projections ev_* and π are both Cartesian fibrations, so that Ω^∞ admits a relative left adjoint $\boldsymbol{B}$ [26, Proposition 7.3.2.6]. For (2), note that $\boldsymbol{B}$ is given fiberwise by the fully faithful left adjoint from Proposition 2.28. Since π and ev_* are also co-Cartesian fibrations, this implies that $\boldsymbol{B}$ is fully faithful (by [23, Proposition 2.4.4.2]) and that its essential image is as asserted in (2).

For (3), the proof of [26, Proposition 1.4.3.4] shows that it suffices to verify that the essential image of $\boldsymbol{B}\colon \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X}) \hookrightarrow \mathcal{T}_X\mathcal{V}$ is closed under extensions. For this, we just need to verify that the additive presentable ∞-category $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X})$ satisfies the following condition [25, Proposition C.1.2.2]: for each map $Y \to \Sigma Z$ in $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X})$ to a suspension with fiber $F \to Y$, the natural map $0 \amalg_F Y \to \Sigma Z$ from the cofiber is an equivalence. To see this, consider the following diagram in $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X})$:

$$\begin{tikzcd} \cdots\ F\oplus Z\oplus Z \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & F\oplus Z \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & F \arrow[r] \arrow[d] & Y \arrow[d] \\ \cdots\ Z\oplus Z \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Z \arrow[r, shift left=1] \arrow[r, shift right=1] & 0 \arrow[r] & \Sigma Z \end{tikzcd}$$

Here the bottom row is the standard augmented simplicial object that computes ΣZ as a geometric realization of coproducts (by restricting along the cofinal functor $(\Delta_{/\Lambda_0^2})^{\mathrm{op}} \to \Lambda_0^2$ and taking the left Kan extension along the left fibration $(\Delta_{/\Lambda_0^2})^{\mathrm{op}} \to \Delta^{\mathrm{op}}$). The top row is obtained from the bottom row by base change along $Y \to \Sigma Z$ and each of the left vertical maps can be identified with the evident projection onto a summand. However, note that the simplicial structure of the top row is not just the direct sum of the bottom row and the constant diagram on F.

Since the forgetful functor $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X}) \to \mathcal{V}_{/X}$ detects geometric realizations and $\mathcal{V}$ has parametrized loop space machinery (so that the fiber product $\times_{\Sigma Z}$ preserves geometric realizations), the top row is then a colimit diagram as well. The canonical map $0 \amalg_F Y \to \Sigma Z$ is then an equivalence, since it can be identified with the geometric realization of the natural equivalence of simplicial objects

$$0 \amalg_F (F \oplus Z^{\oplus\bullet}) \to Z^{\oplus\bullet}. \qquad \square$$

Corollary 2.31 *Let $\mathcal{V}$ be a presentable SM ∞-category with parametrized loop space machinery. Then there is a commuting square of presentable SM ∞-categories and symmetric monoidal left adjoint functors*

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \arrow[r, "i_!"] \arrow[d] & \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \arrow[d, "(-)^{\mathrm{exc}}"] \\ \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \arrow[r, "\boldsymbol{B}"] & \mathcal{T}\mathcal{V} \end{tikzcd}$$

where the top ∞-categories come equipped with the levelwise monoidal structure and the vertical functors are monoidal localizations. In particular, the canonical t-orientation (Example 2.21) is monoidal.

Proof Corollary 2.30 already provides the desired square of presentable ∞-categories and left adjoints without monoidal structures. Here the functors $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \to \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})$ and $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \to \mathcal{T}\mathcal{V}$ are the localizations whose right adjoints are the evident inclusions of the full subcategories of Segal $\mathbb{E}_\infty$-groupoids and excisive functors. Since $\boldsymbol{B}$ is a fully faithful functor, the localization $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \to \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})$ precisely inverts the class W of maps that are sent to equivalences by $(-)^{\mathrm{exc}} \circ i$.

To refine this commuting square to a commuting square of SM functors, observe that Fin_* and $\mathcal{S}_*^{\mathrm{fin}}$ both admit finite coproducts (given by wedge sums) and that the inclusion $i\colon \mathrm{Fin}_* \hookrightarrow \mathcal{S}_*^{\mathrm{fin}}$ preserves coproducts. Lemma 2.9 now implies that $i_!\colon \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})^{\otimes_{\mathrm{lev}}} \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$ admits a natural SM structure (adjoint to the SM structure on i^*). The functor $(-)^{\mathrm{exc}}$ is an SM localization by Proposition 2.10.

Since $(-)^{\mathrm{exc}} \circ i$ is monoidal, the class W of arrows in $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})$ is closed under the tensor product with an object. It follows that the localization $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \to \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})$ is a symmetric monoidal localization [26, Proposition 4.1.7.4]; the functor $\boldsymbol{B}\colon \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \to \mathcal{T}\mathcal{V}$ then has a unique SM structure making the square commute.

For the conclusion about the canonical t-orientation being monoidal, note that $\mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \hookrightarrow \mathcal{T}\mathcal{V}$ is a fully faithful symmetric monoidal functor whose essential image coincides with $\mathcal{T}^{\geq 0}\mathcal{V}$ by Corollary 2.30. This implies that $\mathcal{T}^{\geq 0}\mathcal{V}$ contains the monoidal unit and is closed under the tensor product, as desired. $\square$

Let us now turn to the proof of Proposition 2.28, which requires some preliminaries.

Lemma 2.32 *Let $i\colon \mathrm{Fin}_* \to \mathcal{S}_*^{\mathrm{fin}}$ be the natural fully faithful inclusion. Then restriction and left Kan extension define an adjoint pair*

$$i_!\colon \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}} \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) :i^*$$

whose left adjoint is fully faithful. The essential image of $i_!$ consists exactly of those functors $F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ that preserve finite geometric realizations.

Proof Note that $i_!$ is fully faithful because i is. To identify the essential image, let us factor the Yoneda embedding as

$$\mathrm{Fin}_* \xrightarrow{i} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{j} \mathcal{P}(\mathrm{Fin}_*),$$

where j sends $T \in \mathcal{S}_*^{\mathrm{fin}}$ to $\mathrm{Map}_{\mathcal{S}_*^{\mathrm{fin}}}(i(-), T)$. Note that for each finite pointed set $S \in \mathrm{Fin}_*$, the functor $\mathrm{Map}_{\mathcal{S}_*^{\mathrm{fin}}}(i(S), -)$ preserves all finite geometric realizations in $\mathcal{S}_*^{\mathrm{fin}}$, since it sends $T \mapsto T^{\times |S|-1}$. Consequently, j preserves finite geometric realizations as well. Since every finite pointed space is the geometric realization of some n-skeletal simplicial diagram in Fin_* and the Yoneda embedding is fully faithful on Fin_*, it follows that j is fully faithful.

We then have a sequence of adjunctions given by restriction and left Kan extension

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \arrow[r, hook, shift left, "i_!"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \arrow[l, shift left, "i^*"] \arrow[r, hook, shift left, "j_!"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{P}(\mathrm{Fin}_*), \mathcal{V}) \arrow[l, shift left, "j^*"] \end{tikzcd}$$

where the left adjoints are fully faithful. By [23, Lemma 5.1.5.5], the essential image of $j_! i_!$ coincides with those functors $\mathcal{P}(\mathrm{Fin}_*) \to \mathcal{V}$ preserving all colimits. Consequently, the essential image of $i_!$ consists of those functors whose left Kan extension along j defines a colimit-preserving functor $\mathcal{P}(\mathrm{Fin}_*) \to \mathcal{V}$.

Since j preserves finite geometric realizations, it follows that any functor in the image of $i_!$ preserves finite geometric realizations. Conversely, given $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ preserving finite geometric realizations, we have to verify that the counit map

$$i_! i^* F(T) \to F(T)$$

is a natural equivalence for $T \in \mathcal{S}_*^{\mathrm{fin}}$. The domain and codomain both preserve finite geometric realizations in T. Since each T is the realization of a finite simplicial diagram in Fin_*, we can reduce to the case where $T \in \mathrm{Fin}_*$. But F and $i_! i^* F$ agree on finite pointed sets by construction. □

Recall that S^1 arises as the geometric realization of the 1-skeletal (finite) pointed simplicial set

$$N_\bullet(\Delta^1/\partial\Delta^1) \colon \Delta^{\mathrm{op}} \to \mathrm{Fin}_*,$$

given explicitly in simplicial degree n by the finite pointed set $\langle n \rangle$ with $n+1$ elements [31, page 295]. For every $S \in \mathcal{S}_*^{\mathrm{fin}}$, the levelwise smash product $N_\bullet(\Delta^1/\partial\Delta^1) \wedge S$ then determines a simplicial diagram in $\mathcal{S}_*^{\mathrm{fin}}$, given in degree n by the n-fold wedge sum $\langle n \rangle \wedge S = S^{\vee n}$.

Lemma 2.33 *Suppose that $\mathcal{V}$ is a presentable ∞-category with loop space machinery and that $A : \mathrm{Fin}_* \to \mathcal{V}$ satisfies the grouplike Segal conditions. Let $F = i_! A : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ be its image under the left adjoint from Lemma 2.32. For each $S \in \mathcal{S}_*^{\mathrm{fin}}$, the simplicial diagram*

$$F(N_\bullet(\Delta^1/\partial\Delta^1) \wedge S) \colon \Delta^{\mathrm{op}} \to \mathcal{V}$$

endows $F(S)$ with the structure of a grouplike monoid in the sense of [23, Definition 7.2.2.1].

Proof Consider the functor $Q : \mathcal{S}_*^{\mathrm{fin}} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$ sending S to $F(N_\bullet(\Delta^1/\partial\Delta^1) \wedge S)$. We have to show that Q takes values in the full subcategory $\mathrm{Grp}(\mathcal{V}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$ of simplicial objects satisfying the grouplike Segal conditions.

Observe that the full subcategory $\mathrm{Grp}(\mathcal{V}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$ of simplicial objects X satisfying the grouplike Segal conditions (ie the grouplike Segal maps $X(n) \to X(1)^{\times n}$ are equivalences) is stable under geometric realizations: for every simplicial diagram $X_\bullet$ in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$, the grouplike Segal maps $|X_\bullet(n)| \to |X_\bullet(1)|^{\times n}$ are equivalent to the geometric realizations of simplicial diagram of Segal maps $X_\bullet(n) \to X_\bullet(1)^{\times n}$. On the other hand, the functor Q preserves finite geometric realizations since F preserves finite geometric realizations (Lemma 2.32). Since every object in $\mathcal{S}_*^{\mathrm{fin}}$ is the geometric realization of a k-skeletal simplicial diagram of finite pointed sets, it thus suffices to show that $F(N_\bullet(\Delta^1/\partial\Delta^1) \wedge S)$ is a grouplike monoid when S is a finite pointed set.

When $S = \langle m \rangle$, the simplicial object $F(N_\bullet(\Delta^1/\partial\Delta^1) \wedge \langle m \rangle)$ can be identified explicitly as follows: it is obtained from $A((-) \wedge \langle m \rangle) : \mathrm{Fin}_* \to \mathcal{V}$ by restricting along the functor $N_\bullet(\Delta^1/\partial\Delta^1) \colon \Delta^{\mathrm{op}} \to \mathrm{Fin}_*$ from [31, page 295]. Since A satisfies the grouplike Segal conditions, $A((-) \wedge \langle m \rangle) \simeq A(-)^{\times m}$ satisfies the grouplike Segal conditions as well. The simplicial object obtained by restriction then satisfies the grouplike Segal conditions as well (as asserted somewhat implicitly in [loc. cit.]; see in particular Proposition 1.5 there). $\square$

Proof of Proposition 2.28 Consider the adjoint pair $(i_!, i^*)$ from Lemma 2.32. We claim that for every $A\colon \mathrm{Fin}_* \to \mathcal{V}$ satisfying the grouplike Segal conditions, the functor $F := i_!(A)\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ is reduced excisive. Assuming this, the adjoint pair $(i_!, i^*)$ simply restricts to an adjoint pair between spectra and grouplike $\mathbb{E}_\infty$-monoids, ie $\boldsymbol{B} = i_!$ and $\Omega^\infty = i^*$. The characterization of the essential image of $\boldsymbol{B}$ then follows from Lemma 2.32.

To verify the claim, note that $F(*) \simeq A(*) \simeq *$, so F is reduced. Since $* \in \mathcal{S}_*^{\mathrm{fin}}$ is the initial object, there is a canonical lift $\widetilde{F}\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}_*$ such that postcomposing with the forgetful functor $\mathcal{V}_* \to \mathcal{V}$ yields F: indeed, $\widetilde{F}$ is simply given by the functor sending S to the pointed object $* \simeq F(*) \to F(S)$ in $\mathcal{V}$. Since the forgetful functor $\mathcal{V}_* \to \mathcal{V}$ preserves limits, the functor F is excisive if and only if $\widetilde{F}$ is excisive.

To see that $\widetilde{F}$ is excisive, it suffices to verify that for every $S \in \mathcal{S}_*^{\mathrm{fin}}$, the natural map

$$\widetilde{F}(S) \to \Omega \widetilde{F}(\Sigma S) \tag{2-7}$$

is an equivalence [26, Proposition 1.4.2.13]. Using that $\Sigma S = S^1 \wedge S$ is the geometric realization of the 1-skeletal simplicial diagram $\mathrm{N}_\bullet(\Delta^1/\partial\Delta^1) \wedge S$ and that F (and hence $\widetilde{F}$) preserves finite geometric realizations, we have that $\widetilde{F}(\Sigma S)$ is the bar construction of the group object from Lemma 2.33. The map (2-7) can then be identified with the map underlying the canonical map of grouplike monoids $F(S) \to \Omega B(F(S))$, which is an equivalence because $\mathcal{V}$ has loop space machinery. $\square$

3 Tangent bundles of stable ∞-categories

The purpose of this section is to spell out the various definitions from Section 2 in the case where $\mathcal{V}$ is a stable or additive presentable ∞-category, for which the tangent bundle has a much simpler description.

3.1 Trivializing the tangent bundle

Let $\mathcal{V}$ be a pointed ∞-category with finite limits and consider the full subcategory $\mathrm{Ret} \subseteq \mathcal{S}_*^{\mathrm{fin}}$ on $*$ and S^0. Then Ret is equivalent to the retract category [23, Definition 4.4.5.2] and there are functors

$$\begin{tikzcd} \mathcal{T}\mathcal{V} \arrow[r, "G"] \arrow[dr, "\mathrm{ev}_*"'] & \mathrm{Fun}(\mathrm{Ret}, \mathcal{V}) \arrow[r, "\mathrm{fib}"] \arrow[d, "\mathrm{ev}_*"] & \mathcal{V} \times \mathcal{V} \arrow[dl, "\pi_1"] \\ & \mathcal{V} & \end{tikzcd} \tag{3-1}$$

Here the first horizontal functor is given by restriction and fib sends a retract diagram $X \to Y \to X$ to the tuple $(X, Y \times_X *)$. The first functor exhibits $\mathcal{T}\mathcal{V}$ as the fiberwise stabilization of $\mathrm{Fun}(\mathrm{Ret}, \mathcal{V})$: for every $X \in \mathcal{V}$ the induced functor $\mathcal{T}_X\mathcal{V} \to \mathrm{Fun}(\mathrm{Ret}, \mathcal{V})_X \simeq (\mathcal{V}_{/X})_*$ on fibers over X exhibits its domain as the stabilization of its target.

Now suppose that $\mathcal{V}$ is an additive ∞-category. Then the functor fib is an equivalence, with inverse sending (X,Y) to $X \to X \oplus Y \to X$ (see eg [5, Lemma 1.5.12]). In this case, we therefore obtain an equivalence

$$\begin{tikzcd} \mathcal{T}\mathcal{V} \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & \mathcal{V} \times \mathrm{Sp}(\mathcal{V}) \arrow[dl, "\pi_1"] \\ & \mathcal{V} & \end{tikzcd}$$

between $\mathcal{T}\mathcal{V}$ and the fiberwise stabilization of $\mathcal{V} \times \mathcal{V}$ over $\mathcal{V}$. For stable $\mathcal{V}$, the situation is even simpler:

Lemma 3.1 *If $\mathcal{V}$ is a stable ∞-category, then both G and* fib *are equivalences, so that there is an equivalence $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathcal{V}$ such that $\pi(X,Y) \simeq X$ and $\Omega^\infty(X,Y) \simeq X \oplus Y$.*

Proof The functor fib is an equivalence since $\mathcal{V}$ is additive, so that the fibers of $\mathrm{Fun}(\mathrm{Ret}, \mathcal{V})$ are equivalent to $\mathcal{V}$ and hence already stable, which in turn implies that the functor G exhibiting the fiberwise stabilization is an equivalence (see [16, Corollary 2.2.5] for a similar argument). $\square$

Lemma 3.2 *Let $\mathcal{V}$ be an additive presentable ∞-category and let $\Sigma^\infty \colon \mathcal{V} \to \mathrm{Sp}(\mathcal{V})$ be the left adjoint functor exhibiting $\mathrm{Sp}(\mathcal{V})$ as the stabilization of $\mathcal{V}$. Then the following induced square of tangent categories is Cartesian*:

$$\begin{tikzcd} \mathcal{T}\mathcal{V} \arrow[r, "\mathcal{T}(\Sigma^\infty)"] \arrow[d, "\pi"'] & \mathcal{T}(\mathrm{Sp}(\mathcal{V})) \arrow[d, "\pi"] \\ \mathcal{V} \arrow[r, "\Sigma^\infty"] & \mathrm{Sp}(\mathcal{V}) \end{tikzcd}$$

Proof The left adjoint functor $\Sigma^\infty \colon \mathcal{V} \to \mathrm{Sp}(\mathcal{V})$ commutes with the functor fib because one can identify $Y \times_X 0 \simeq Y \amalg_X 0$ for a retract diagram $X \to Y \to X$. This implies that the functor $\mathcal{T}(\Sigma^\infty)$ is obtained from the functor $\Sigma^\infty \times \Sigma^\infty_+ \colon \mathcal{V} \times \mathcal{V} \to \mathrm{Sp}(\mathcal{V}) \times \mathrm{Sp}(\mathcal{V})$ by stabilizing the second factor, which readily implies the result. $\square$

3.2 Square zero monoidal structure

If $\mathcal{V}$ is a stable presentable SM ∞-category, then the square zero monoidal structure on $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathcal{V}$ (Definition 2.14) can be made more explicit using the following:

Definition 3.3 Let $\mathcal{D}$ be a presentable SM ∞-category. We will say that an object $D \in \mathcal{D}$ is *square zero* if the canonical map $\varnothing \to D \otimes D$ from the initial object is an equivalence and denote by $\mathrm{SqZ}(\mathcal{D}) \subseteq \mathcal{D}$ the full subcategory on the square zero objects. Note that every SM left adjoint $F \colon \mathcal{D} \to \mathcal{D}'$ restricts to a natural map $F \colon \mathrm{SqZ}(\mathcal{D}) \to \mathrm{SqZ}(\mathcal{D}')$.

Recall that the ∞-category of *$\mathcal{V}$-linear SM ∞-categories* is given by the ∞-category $\mathrm{CAlg}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}}) \simeq \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathcal{V}/}$ of presentable SM ∞-categories $\mathcal{D}$ equipped with a symmetric monoidal left adjoint functor $\mathcal{V} \to \mathcal{D}$.

Definition 3.4 Let $\mathcal{W}$ be a $\mathcal{V}$-linear SM ∞-category together with a square zero object $M \in \mathcal{W}$. We say that this exhibits $\mathcal{W}$ as the *free $\mathcal{V}$-algebra on a square zero object* if for each $\mathcal{D} \in \mathrm{CAlg}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$, evaluation at M defines a natural equivalence

$$\mathrm{ev}_M : \mathrm{Fun}_{\mathcal{V}}^{\otimes}(\mathcal{W}, \mathcal{D}) \to \mathrm{SqZ}(\mathcal{D}).$$

Remark 3.5 Consider a pushout square in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r] \arrow[d] & \mathcal{V}_2 \arrow[d] \\ \mathcal{W}_1 \arrow[r, "f"'] & \mathcal{W}_2 \end{tikzcd}$$

If $M \in \mathcal{W}_1$ exhibits $\mathcal{W}_1$ as the free $\mathcal{V}_1$-algebra on a square zero object, then $f(M)$ exhibits $\mathcal{W}_2 \simeq \mathcal{V}_2 \otimes_{\mathcal{V}_1} \mathcal{W}_1$ as the free $\mathcal{V}_2$-algebra on a square zero object: indeed, the evaluation at $f(M)$ factors as two equivalences:

$$\mathrm{ev}_{f(M)} : \mathrm{Fun}_{\mathcal{V}_2}^{\otimes}(\mathcal{V}_2 \otimes_{\mathcal{V}_1} \mathcal{W}_1, \mathcal{D}) \xrightarrow[\sim]{f^*} \mathrm{Fun}_{\mathcal{V}_1}^{\otimes}(\mathcal{W}_1, \mathcal{D}) \xrightarrow[\sim]{\mathrm{ev}_M} \mathrm{SqZ}(\mathcal{D}).$$

Proposition 3.6 *There exists a free $\mathcal{S}$-algebra $\mathcal{S}[\epsilon]$ on a square zero object. Furthermore, the functor*

$$\{A, M\} \to \mathcal{S}[\epsilon]$$

that sends A to the monoidal unit and M to the (universal) square zero object, exhibits $\mathcal{S}[\epsilon]$ as the free presentable ∞-category on the two-element set $\{A, M\}$.

In particular, the tensor product functor $\otimes : \mathcal{S}[\epsilon] \times \mathcal{S}[\epsilon] \to \mathcal{S}[\epsilon]$ is the unique functor preserving colimits in each variable given on generating objects by $A \otimes A = A$, $A \otimes M = M \otimes A = M$ and $M \otimes M = \varnothing$ is the initial object.

Proof First, let $\mathrm{Fin}^{\mathrm{bij}}$ be the category of finite sets and bijections, with monoidal structure given by disjoint union. By [26, Proposition 2.2.4.9], the inclusion of the 1-element set $\{\underline{1}\} : * \to \mathrm{Fin}^{\mathrm{bij}}$ exhibits $\mathrm{Fin}^{\mathrm{bij}}$ as the free symmetric monoidal ∞-category on $*$. By [26, Corollary 4.8.1.12] (and the fact that $\mathrm{Fin}^{\mathrm{bij}} \simeq \mathrm{Fin}^{\mathrm{bij,op}}$), the ∞-category $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ of symmetric sequences admits a unique closed symmetric monoidal structure such that the Yoneda embedding

$$\mathrm{Fin}^{\mathrm{bij}} \xrightarrow{h} \mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$$

admits a symmetric monoidal structure. In particular, the (co)representable h_0 on the empty set is the monoidal unit and the universal property of $(\mathrm{Fin}^{\mathrm{bij}}, \amalg)$ and [26, Proposition 4.8.1.10] imply that the map

$\{h_1\}\colon * \to \mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ exhibits $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ as the free presentable SM ∞-category on $*$. Finally, [26, Remark 4.8.1.13] asserts that the resulting symmetric monoidal structure on $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ is in fact given by Day convolution.

Let us now denote by $\mathcal{S}[\epsilon]$ the (reflective) localization of symmetric sequences at the set of maps $\varnothing \to h_n$ from the initial object, for all $n \geq 2$. Then $\mathcal{S}[\epsilon] \subseteq \mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ is the full subcategory of symmetric sequences X such that $X(n) \simeq *$ for all $n \geq 2$. In particular, the functor $\{A, M\} \to \mathcal{S}[\epsilon]$ sending $A \mapsto h_0$ and $M \mapsto h_1$ exhibits $\mathcal{S}[\epsilon]$ as the free presentable ∞-category on $\{A, M\}$.

For any $m \geq 0$ and $n \geq 2$, the map $\varnothing \otimes h_m \to h_n \otimes h_m$ is equivalent to the map $\varnothing \to h_{n+m}$, so (by the same argument as in Proposition 2.10) this exhibits $\mathcal{S}[\epsilon]$ as a symmetric monoidal localization of $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$. By the universal property of symmetric monoidal localizations, the square zero object $h_1 \in \mathcal{S}[\epsilon]$ then realizes $\mathcal{S}[\epsilon]$ as the free presentable SM ∞-category on a square zero object. □

Corollary 3.7 *For every presentable SM ∞-category $\mathcal{V}$, there exists a free $\mathcal{V}$-algebra $\mathcal{V}[\epsilon]$ on a square zero object.*

Proof Proposition 3.6 provides the existence of the free $\mathcal{S}$-algebra on a square zero object $\mathcal{S}[\epsilon]$. By Remark 3.5, $\mathcal{V} \otimes_{\mathcal{S}} \mathcal{S}[\epsilon]$ then provides the free $\mathcal{V}$-algebra on a square zero object. □

Remark 3.8 Let $\mathcal{V}$ be a presentable SM ∞-category. Then the free $\mathcal{V}$-algebra $\mathcal{V}[\epsilon]$ on a square zero object can also be described in terms of a variant of the Day convolution product applicable to *promonoidal* ∞-categories, as developed in recent work of Nardin and Shah [28]. More precisely, one can check that the 2-colored operad $\mathcal{M}\mathrm{Com}$ for commutative algebras and modules is such a promonoidal (∞-)category. Since the underlying category of $\mathcal{M}\mathrm{Com}$ is simply the set $\{A, M\}$, this endows $\mathrm{Fun}(\{A, M\}, \mathcal{V})$ with a Day convolution product which has the property that

$$(h_A \otimes C) \otimes (h_A \otimes D) = h_A \otimes (C \otimes D),$$
$$(h_A \otimes C) \otimes (h_M \otimes D) = h_M \otimes (C \otimes D),$$
$$(h_M \otimes C) \otimes (h_M \otimes D) = \varnothing.$$

In particular, the square zero object $h_M \otimes 1_{\mathcal{V}}$ induces a symmetric monoidal functor from $\mathcal{V}[\epsilon]$ to this Day convolution, which is easily seen to be an equivalence. The universal property of the Day convolution therefore implies that for any ∞-operad $\mathcal{O}$, there is a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}[\epsilon]) \simeq \mathrm{Alg}_{\mathcal{O} \times \mathcal{M}\mathrm{Com}}(\mathcal{V}).$$

The ∞-operad $\mathcal{M}\mathcal{O} = \mathcal{O} \times \mathcal{M}\mathrm{Com}$ is the ∞-operad for $\mathcal{O}$-algebras and (operadic) modules over them [16; 20]. Combining this with Propositions 2.17 and 3.10 below, one finds that $\mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{M}\mathcal{O}}(\mathcal{V})$ for every stable presentable SM ∞-category $\mathcal{V}$ (see also [4; 26; 30]).

We will now relate the free $\mathcal{V}$-algebra on a square zero object to $\mathcal{T}\mathcal{V}$:

Construction 3.9 Let $\mathcal{V}$ be a stable presentable SM ∞-category and consider the cofiber sequence of excisive functors

$$h_* \otimes 1_{\mathcal{V}} \xrightarrow{i \otimes \mathrm{id}} h_{S^0} \otimes 1_{\mathcal{V}} \to M_{\mathcal{V}},$$

where $i \otimes \mathrm{id}$ is the canonical map of corepresentables induced by $* \to S^0$. Lemma 2.13 shows that the pushout-product of $h_* \otimes 1_{\mathcal{V}} \to h_{S^0} \otimes 1_{\mathcal{V}}$ with itself in $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V})$ is a $\mathcal{T}\mathcal{V}$-local equivalence. Consequently, the pushout-product map in the monoidal localization $\mathcal{T}\mathcal{V}$ becomes an equivalence. Since the cofiber of a pushout-product map is the tensor product of the cofibers (see eg [12, Theorem 6.2] for a proof at the level of stable monoidal derivators), it follows that $M_{\mathcal{V}} \otimes M_{\mathcal{V}} \simeq 0$ in $\mathcal{T}\mathcal{V}$.

Proposition 3.10 *Let $\mathcal{V}$ be a stable presentable SM ∞-category and consider $\mathcal{T}\mathcal{V}$ as a $\mathcal{V}$-linear SM ∞-category via the left adjoint $\mathcal{V} \to \mathcal{T}\mathcal{V}$ to the projection. Then the square zero object $M_{\mathcal{V}} \in \mathcal{T}\mathcal{V}$ exhibits $\mathcal{T}\mathcal{V}$ as the free $\mathcal{V}$-algebra on a square zero object.*

Proof Since $\mathcal{V}$ is a stable presentable SM ∞-category, the canonical SM left adjoint $\mathcal{S} \to \mathcal{V}$ factors canonically over spectra [26, Corollary 4.8.2.19]. This gives rise to the following diagram in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\begin{tikzcd}
\mathcal{S} \arrow[r] \arrow[dd] & \mathrm{Sp} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\
 & \mathrm{Sp}[\epsilon] \arrow[r] \arrow[d, "\phi"'] & \mathcal{V}[\epsilon] \arrow[d, "\phi_{\mathcal{V}}"] \\
\mathcal{T}\mathcal{S} \arrow[r] & \mathcal{T}\,\mathrm{Sp} \arrow[r] & \mathcal{T}\mathcal{V}
\end{tikzcd}$$

Here each composite vertical functor is the left adjoint to the projection (ie taking constant $\mathcal{S}^{\mathrm{fin}}_*$-diagrams). For $\mathcal{V}$ and the ∞-category of spectra, this left adjoint factors over the free algebra on a square zero object: ϕ is the functor classifying the square zero object $M_{\mathrm{Sp}} \in \mathcal{T}\,\mathrm{Sp}$ and $\phi_{\mathcal{V}}$ classifies $M_{\mathcal{V}}$. Since the functor $\mathcal{T}\,\mathrm{Sp} \to \mathcal{T}\mathcal{V}$ is a monoidal left adjoint, it sends M_{Sp} to $M_{\mathcal{V}}$ so that the diagram commutes.

Now notice that by Proposition 2.18, the total square and the left rectangle are both pushout squares in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. On the other hand, Remark 3.5 shows that the top right square is co-Cartesian. It therefore follows that the bottom right square is co-Cartesian as well. Consequently, $\phi_{\mathcal{V}}$ is an equivalence as soon as ϕ is an equivalence, so we can reduce to the case $\mathcal{V} = \mathrm{Sp}$. In this case, let us consider the composite functor

$$\{A, M\} \to \mathcal{S}[\epsilon] \to \mathrm{Sp}[\epsilon]$$

sending A to the monoidal unit and M to the universal square zero object. Proposition 3.6 asserts that the first functor exhibits $\mathcal{S}[\epsilon]$ as the free presentable ∞-category generated by $\{A, M\}$ and Remark 3.5 and [26, Proposition 4.8.2.18] imply that the second functor exhibits $\mathrm{Sp}[\epsilon]$ as the stabilization of $\mathcal{S}[\epsilon]$. The composite therefore exhibits $\mathrm{Sp}[\epsilon]$ as the free stable presentable ∞-category generated by $\{A, M\}$.

Now observe that by construction the monoidal functor

$$\phi\colon \mathrm{Fun}(\{A, M\}, \mathrm{Sp}) \simeq \mathrm{Sp}[\epsilon] \to \mathcal{T}\,\mathrm{Sp}$$

is given on generators by $\phi(h_A) = 1_{\mathcal{T}\,\mathrm{Sp}} = h_* \otimes 1_{\mathrm{Sp}}$ and $\phi(h_M) = M_{\mathrm{Sp}} = \mathrm{cof}(h_* \otimes 1_{\mathrm{Sp}} \to h_{S^0} \otimes 1_{\mathrm{Sp}})$. It follows that the right adjoint to ϕ is given by the composite functor $\mathcal{T}\,\mathrm{Sp} \to \mathrm{Fun}(\mathrm{Ret}, \mathrm{Sp}) \to \mathrm{Sp} \times \mathrm{Sp}$ appearing (3-1), which is an equivalence since Sp is stable. We conclude that ϕ is an equivalence, as desired. □

Remark 3.11 If $\mathcal{V}$ is an additive presentable SM ∞-category, its stabilization $\mathrm{Sp}(\mathcal{V})$ carries an induced symmetric monoidal structure and $\Sigma^\infty\colon \mathcal{V} \to \mathrm{Sp}(\mathcal{V})$ is a symmetric monoidal functor [8, Theorem 5.1]. The pullback square of Lemma 3.2 then becomes a pullback square of SM ∞-categories. Proposition 3.10 then provides an explicit description of the square zero monoidal structure on $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$, given informally by the formula

$$(C, E) \otimes_{\mathcal{T}\mathcal{V}} (D, F) \simeq \big(C \otimes_{\mathcal{V}} D, (\Sigma^\infty C \otimes_{\mathrm{Sp}(\mathcal{V})} F) \oplus (E \otimes_{\mathrm{Sp}(\mathcal{V})} \Sigma^\infty D)\big). \tag{3-2}$$

Example 3.12 Let $\mathcal{V}$ be a stable presentable SM ∞-category and suppose that $\mathcal{O}$ is a monochromatic ∞-operad in arity ≥ 1, ie $\mathcal{O}^{\otimes}_{\langle 0\rangle} = \varnothing$ and $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ is a category with (up to equivalence) one object. This implies that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ is pointed, ie the terminal algebra 0 is also the initial algebra (by [26, Proposition 3.1.3.13]).

For any $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$, $\mathcal{T}_A \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ can be identified with the ∞-category of operadic A-modules (see [16, Corollary 1.0.5] or [26, Theorem 7.3.4.13]). Alternatively, Remark 3.8 identifies $\mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{M}\mathcal{O}}(\mathcal{V})$ with the ∞-category of $\mathcal{O}$-algebras and modules over them.

Now, given such an A-module E, the $\mathcal{O}$-algebra $\Omega^\infty(E)$ can be identified with the split square zero extension $A \oplus E$. For any section $\eta\colon A \to A \oplus E$, we then obtain pullback squares of the form

$$\begin{tikzcd}
\Omega^\infty(0, E[-1]) \arrow[r] \arrow[d] & A_\eta \arrow[r] \arrow[d] & \Omega^\infty(A, E) \arrow[d] \\
\Omega^\infty(0,0) = 0 \arrow[r] & A \arrow[r, "\eta"] & \Omega^\infty(A, E)
\end{tikzcd}$$

Here the map $0 \to A$ is the initial map of $\mathcal{O}$-algebras and total pullback arises as the image under Ω^∞ of the pullback in $\mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{M}\mathcal{O}}(\mathcal{V})$ of the map $(A, 0) \to (A, E)$ along the initial map $(0,0) \to (A, E)$. In particular, $\Omega^\infty(0, E[-1])$ is the image of an $\mathcal{O}$-algebra $(0, E[-1])$ under the nonunital lax symmetric monoidal functor

$$\Omega^\infty\colon \mathcal{T}_0\mathcal{V} = \mathcal{T}\mathcal{V} \times_{\mathcal{V}} \{0\} \hookrightarrow \mathcal{T}\mathcal{V} \to \mathcal{V},$$

where the first functor is the inclusion of the nonunital SM sub-∞-category from Remark 2.15. We have seen there that the tensor product on $\mathcal{T}_0\mathcal{V}$ is null-homotopic, so that each operation in $\mathcal{O}$ of arity ≥ 2 acts on $(0, E[-1])$ by a null-homotopic map. Consequently, the resulting map $A_\eta \to A$ indeed behaves like a square zero extension in the sense of algebra: its fiber $\Omega^\infty(0, E[-1])$ is an $\mathcal{O}$-algebra on which all operations in $\mathcal{O}$ of arity ≥ 2 act by null-homotopic maps (see [26, Proposition 7.4.1.14]).

3.3 t-orientations

Let us conclude with some remarks about t-orientations on tangent bundles of additive and monoidal ∞-categories.

Example 3.13 Let $\mathcal{V}$ be an additive presentable ∞-category, so that $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ (Lemma 3.2). Then any t-structure on $\mathrm{Sp}(\mathcal{V})$ determines a t-orientation on $\mathcal{T}\mathcal{V}$. Now suppose that $\mathcal{V}$ is furthermore symmetric monoidal and recall that the square zero tensor product on $\mathcal{T}\mathcal{V}$ can be identified with the tensor product on $\mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ given by Remark 3.11. From this description, one sees that a t-structure on $\mathrm{Sp}(\mathcal{V})$ determines a *monoidal* t-orientation on $\mathcal{T}\mathcal{V}$ if and only if $\mathrm{Sp}(\mathcal{V})^{\geq 0}$ is closed under taking the tensor product in $\mathrm{Sp}(\mathcal{V})$ with objects of the form $\Sigma^\infty(X)$, for $X \in \mathcal{V}$.

Example 3.14 Suppose that $\mathcal{V}$ is an additive presentable ∞-category and consider the canonical t-orientation on $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ (Example 2.21). A tuple (C, E) is then contained in $\mathcal{T}^{\leq -1}\mathcal{V}$ if and only if $\Omega^\infty(E) = 0$ in $\mathcal{V}$. The proof of [26, Proposition 1.4.3.4] shows that $(C, E) \in \mathcal{T}^{\geq 0}\mathcal{V}$ if and only if E is contained in the smallest subcategory of $\mathrm{Sp}(\mathcal{V})$ which is closed under colimits and extensions and contains all $\Sigma^\infty(X)$ for $X \in \mathcal{V}$. If $\mathcal{V}$ is furthermore closed SM, then Example 3.13 shows that the canonical t-orientation is monoidal.

When $\mathcal{V}$ is stable, the canonical t-orientation simply produces the trivial t-structure ($\mathcal{T}^{\geq 0}\mathcal{V} = \mathcal{T}\mathcal{V}$). If $\mathcal{V}$ is prestable [25, Definition C.1.2.1], the canonical t-orientation has $\mathcal{T}^{\geq 0}\mathcal{V} \simeq \mathcal{V} \times \mathcal{V}$ under the equivalence $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ [25, Proposition C.1.2.2].

Example 3.15 Suppose that $\mathcal{V}$ is a prestable SM ∞-category and $\mathcal{O}$ an ∞-operad. Endowing $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ with its canonical monoidal t-orientation and applying Proposition 2.25, we obtain a t-orientation on $\mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$, and hence a t-structure on $\mathcal{T}_A\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ for any $\mathcal{O}$-algebra A. Under the identification $\mathcal{T}_A\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Mod}_A(\mathrm{Sp}(\mathcal{V}))$ from Example 3.12, this is simply the t-structure whose connective part is given by $\mathrm{Mod}_A(\mathcal{V}) \subseteq \mathrm{Mod}_A(\mathrm{Sp}(\mathcal{V}))$.

4 Postnikov structures

The goal of this section is to give an axiomatic description of a decomposition of an object in a nice ∞-category, together with the data of "k-invariants", analogous to the Postnikov tower of a space.

Definition 4.1 Let $\mathcal{V}$ be an ∞-category with finite limits. A *Postnikov structure* on an object X in $\mathcal{V}$ consists of the following data:

(1) An infinite tower
$$X \to \cdots \to X_a \to \cdots \to X_1$$
of objects $X_a \in \mathcal{V}$ under X for $a \geq 1$, exhibiting X as the limit of $\{X_a\}_{a \geq 1}$.

(2) For each $a \geq 2$, an object $K_a \colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ in $\mathcal{T}\mathcal{V}$ together with a Cartesian square

$$\begin{tikzcd} X_a \arrow[r] \arrow[d] & \pi(K_a) \arrow[d, "0"] \\ X_{a-1} \arrow[r, "k_a"'] & \Omega^\infty K_a \end{tikzcd}$$

exhibiting $X_a \to X_{a-1}$ as a square zero extension (see (2-2)).

The convention to start at $a = 1$ is rather arbitrary, and in various cases it can be more natural to start at $a = 0$.

Warning 4.2 The notion of a Postnikov structure on an object X is a priori unrelated to the tower of truncations of X, ie its underlying tower need not be given by the Postnikov tower of X in the sense of [23, Definition 5.5.6.23]. For example, Theorem 6.3 yields a Postnikov structure on an (∞, n)-category $\mathcal{C}$ whose underlying tower consists of the homotopy categories $\mathrm{ho}_{(n+a,n)}(\mathcal{C})$ and not on its truncations $\tau_{\leq n+a}(\mathcal{C})$ in $\mathrm{Cat}_{(\infty,n)}$.

Warning 4.2 notwithstanding, we will see that a good source of Postnikov structures is given by the usual Postnikov tower together with its k-invariants:

Example 4.3 The motivating example of a Postnikov structure is the usual Postnikov tower of a space X, together with the data of its k-invariants. In this case, $X_a = \tau_{\leq a} X$ and the K_a are given by the (suspended) parametrized Eilenberg–MacLane spectra $K_a = \Sigma^{a+1} \mathrm{H}\pi_a(X)$ over $\tau_{\leq 1} X$. We will come back to this in Example/Proposition 4.15.

To study naturality of Postnikov structures, it will be convenient to organize the data of an object X equipped with a Postnikov structure into a single diagram $T \colon \mathcal{E} \to \mathcal{V}$. To this end, let us start by recalling the following definition:

Definition 4.4 Let $\phi \colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We will denote by $M(\phi)$ the domain of the co-Cartesian fibration classified by $\phi \colon \Delta^1 \to \mathrm{Cat}_\infty$. By [23, Lemma 3.2.3.3], $M(\phi)$ can be identified with the *mapping simplex* [23, Section 3.2.2], ie it can be identified with the pushout of ∞-categories

$$M(\phi) := \mathcal{D} \amalg_{\{1\} \times \mathcal{C}} \Delta^1 \times \mathcal{C}.$$

Using the co-Cartesian fibration $M(\phi) \to \Delta^1$, one can understand $M(\phi)$ as follows: an object of $M(\phi)$ is either an object of $\mathcal{C}$ or an object of $\mathcal{D}$, and for each $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$ we have

$$\begin{aligned} \mathrm{Map}_{M(\phi)}(c, c') &= \mathrm{Map}_{\mathcal{C}}(c, c'), & \mathrm{Map}_{M(\phi)}(c, d) &= \mathrm{Map}_{\mathcal{D}}(\phi(c), d), \\ \mathrm{Map}_{M(\phi)}(d, d') &= \mathrm{Map}_{\mathcal{D}}(d, d'), & \mathrm{Map}_{M(\phi)}(d, c) &= \varnothing, \end{aligned}$$

with the evident composition. Let us write $\phi_* \colon \Delta^1 \times \mathcal{C} \to M(\phi)$ for the natural map into the pushout, sending $(0, c)$ to c and $(1, c)$ to $\phi(c)$.

Construction 4.5 For any integer a, let $\kappa_a\colon \{a \to (a-1)\} \to \mathcal{S}_*^{\mathrm{fin}}$ be the functor sending the walking arrow $a \to (a-1)$ to $* \to S^0$ and let $\mathcal{E}_a = M(\kappa_\alpha)$ be its mapping simplex. As in Definition 4.4, we will identify the objects of $\mathcal{E}_a$ with the objects of $\mathcal{S}_*^{\mathrm{fin}}$, together with two additional objects a, $a-1$. The functor $\sigma_{a*}\colon \Delta^1 \times \{a \to a-1\} \to \mathcal{E}_a$ is then given explicitly by

$$\sigma_a(0,a) = a, \quad \sigma_a(0,a-1) = a-1, \quad \sigma_a(1,a) = * \quad \sigma_a(1,a-1) = S^0.$$

For any integer m, let us then define $\mathcal{E}_{\geq m}$ as the pushout of ∞-categories

$$\begin{tikzcd}
\mathbb{Z}^{\mathrm{op}}_{\geq m} \arrow[r] \arrow[d] & \left[\mathcal{E}_{m+1} \amalg_{\{m+1\}} \mathcal{E}_{m+2} \amalg_{\{m+2\}} \mathcal{E}_{m+3} \amalg \cdots\right] \arrow[d] \\
(\mathbb{Z}^{\mathrm{op}}_{\geq m})^{\triangleleft} \arrow[r] & \mathcal{E}_{\geq m}
\end{tikzcd}$$

where the left vertical functor is the usual inclusion into the cone and the top horizontal functor sends each map $a \to (a-1)$ in $\mathbb{Z}^{\mathrm{op}}_{\geq m}$ to the corresponding nondegenerate arrow in $\mathcal{E}_a$. Given $T\colon \mathcal{E}_{\geq m} \to \mathcal{V}$, we then observe that:

- The restriction of T to $\mathcal{S}_*^{\mathrm{fin}} \subseteq \mathcal{E}_a$ corresponds to K_a.
- The restriction of T along $\sigma_{a*}\colon \Delta^1 \times \{a \to a-1\} \subseteq \mathcal{E}_a$ corresponds to the square

$$\begin{tikzcd}
X_a \arrow[r] \arrow[d] & K_a(*) = \pi(K_a) \arrow[d] \\
X_{a-1} \arrow[r, "k_a"'] & K_a(S^0) = \Omega^\infty(K_a)
\end{tikzcd}$$

- The restriction of T to $(\mathbb{Z}^{\mathrm{op}}_{\geq m})^{\triangleleft} \subseteq \mathcal{E}$ encodes the tower $X \to \cdots \to X_{m+1} \to X_m$.

By default, we will take $\mathcal{E} = \mathcal{E}_{\geq 1}$.

Definition 4.6 We define the *∞-category of objects equipped with a Postnikov structure* to be the full subcategory

$$\mathrm{PoStr}(\mathcal{V}) \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{V})$$

of diagrams T for which (a) the restriction to each $\mathcal{S}_*^{\mathrm{fin}} \subseteq \mathcal{E}_a$ is excisive, (b) the restriction along each σ_{a*} is a Cartesian square and (c) the restriction to $(\mathbb{Z}^{\mathrm{op}}_{\geq 1})^{\triangleleft}$ is a limit cone.

Remark 4.7 The conditions determining $\mathrm{PoStr}(\mathcal{V})$ inside $\mathrm{Fun}(\mathcal{E}, \mathcal{V})$ assert that certain designated cone diagrams $\mathcal{I}_\alpha^{\triangleleft} \to \mathcal{E}$, with $\mathcal{I}_\alpha$ contractible (either a span or $\mathbb{Z}^{\mathrm{op}}_{\geq 1}$), are sent to limit cones. In particular, $\mathrm{PoStr}(\mathcal{V}) \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ is closed under limits.

Evaluating at the cone point of the tower $\infty \in (\mathbb{Z}^{\mathrm{op}}_{\geq 1})^{\triangleleft} \subseteq \mathcal{E}$ determines a limit-preserving functor $\mathrm{ev}_\infty\colon \mathrm{PoStr}(\mathcal{V}) \to \mathcal{V}$.

Definition 4.8 A *Postnikov structure* on an ∞-category $\mathcal{V}$ is defined to be a section $\Phi\colon \mathcal{V} \to \mathrm{PoStr}(\mathcal{V})$ of the functor $\mathrm{ev}_\infty\colon \mathrm{PoStr}(\mathcal{V}) \to \mathcal{V}$.

Warning 4.9 Note the distinction between a Postnikov structure on an *object in* an ∞-category $\mathcal{V}$ (Definition 4.1) and a Postnikov structure *on* an ∞-category $\mathcal{V}$: the former is a single diagram in $\mathcal{V}$, while the latter is a family of diagrams depending functorially on $X \in \mathcal{V}$. This should not cause any confusion, since it is always clear from the context if we are dealing with a functor on $\mathcal{V}$.

Definition 4.10 Let $\mathcal{V}$ be an SM ∞-category and endow $\mathrm{Fun}(\mathcal{E}, \mathcal{V})$ with the levelwise tensor product. We define the ∞*-operad of objects equipped with a Postnikov structure* to be the full suboperad

$$\mathrm{PoStr}(\mathcal{V})^{\otimes} \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$$

spanned by the objects from Definition 4.8. A *multiplicative Postnikov structure* on $\mathcal{V}$ is a section of the map $\mathrm{ev}_\infty\colon \mathrm{PoStr}(\mathcal{V})^{\otimes} \to \mathcal{V}^{\otimes}$ in the ∞-category of ∞-operads.

Using that $\mathrm{PoStr}(\mathcal{V})^{\otimes}$ is a full suboperad of $\mathrm{Fun}(\mathcal{E}, \mathcal{V})^{\otimes}$, Definition 4.10 can be rephrased as follows: a multiplicative Postnikov structure on $\mathcal{V}$ is a lax symmetric monoidal section $\Phi\colon \mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ of $\mathrm{ev}_\infty\colon \mathrm{Fun}(\mathcal{E}, \mathcal{V}) \to \mathcal{V}$ with the property that the underlying functor of Φ is a Postnikov structure (Definition 4.8).

Remark 4.11 In general, the ∞-operad $\mathrm{PoStr}(\mathcal{V})^{\otimes}$ need not be an SM ∞-category.

Remark 4.12 Suppose that $\Phi\colon \mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ is a multiplicative Postnikov structure. Restricting to the copy of $\mathcal{S}_*^{\mathrm{fin}} \subseteq \mathcal{E}_m$ in level m, one obtains a lax monoidal functor $K_m(\Phi)\colon \mathcal{V} \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ taking values in $\mathcal{T}\mathcal{V} \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$. Since $\mathcal{T}\mathcal{V}$ is a monoidal localization of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$, each $K_m(\Phi)$ defines a lax monoidal functor $\mathcal{V} \to \mathcal{T}\mathcal{V}$ to the tangent bundle, equipped with the square zero monoidal structure (Definition 2.14).

Example 4.13 Suppose that the monoidal structure on $\mathcal{V}$ is given by the Cartesian product. Then the levelwise monoidal structure on $\mathrm{Fun}(\mathcal{E}, \mathcal{V})$ is the Cartesian monoidal structure as well. Consequently (see [26, Section 2.4.1]), strong symmetric monoidal functors $\mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ simply correspond to product preserving functors $\mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$, ie to functors $\mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ each of whose components $\mathcal{V} \to \mathcal{V}$ are product preserving. In particular, any Postnikov structure $\Phi\colon \mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ on $\mathcal{V}$ whose individual components $\Phi_i : \mathcal{V} \to \mathcal{V}$ are product preserving canonically refines to a multiplicative Postnikov structure.

The main point of multiplicative Postnikov structures is that they induce such structures on categories of algebras:

Proposition 4.14 *Let $\mathcal{O}$ be an ∞-operad and let $\mathcal{V}$ be a symmetric monoidal ∞-category equipped with a multiplicative Postnikov structure $\Phi\colon \mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$. Then the induced map*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \xrightarrow{\Phi_*} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{E}, \mathcal{V})) \simeq \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}))$$

is also a multiplicative Postnikov structure.

Proof First, note that we can view $\mathcal{E}$ as an ∞-operad (with only unary operations), so that

$$\mathrm{Fun}(\mathcal{E}, \mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{E}}(\mathcal{V}).$$

By Remark 2.7, the symmetry of the Boardman–Vogt tensor product of ∞-operads [26, Proposition 2.2.5.13] then induces a commuting diagram of symmetric monoidal ∞-categories

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}\otimes_{\mathrm{BV}}\mathcal{E}}(V) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{E}, \mathcal{V})) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_\infty)"'] & \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})) \simeq \mathrm{Alg}_{\mathcal{E}\otimes_{\mathrm{BV}}\mathcal{O}}(V) \arrow[d, "\mathrm{ev}_\infty"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[r, equal] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})
\end{tikzcd}$$

in which the horizontal arrows are equivalences. It follows that $\Phi_* = \mathrm{Alg}_{\mathcal{O}}(\Phi)$ defines a lax symmetric monoidal section of ev_∞. To see that Φ_* takes values in the full sub-∞-category $\mathrm{PoStr}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})) \subseteq \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}))$, consider the commuting diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[r, "\Phi_*"] \arrow[d] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{E}, \mathcal{V})) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})) \arrow[d] \\
\mathrm{Fun}(\mathcal{O}_{\langle 1\rangle}, \mathcal{V}) \arrow[r, "\Phi_*"'] & \mathrm{Fun}(\mathcal{O}_{\langle 1\rangle}, \mathrm{Fun}(\mathcal{E}, \mathcal{V})) \arrow[r, "\simeq"'] & \mathrm{Fun}(\mathcal{E}, \mathrm{Fun}(\mathcal{O}_{\langle 1\rangle}, \mathcal{V}))
\end{tikzcd}$$

where $\mathcal{O}_{\langle 1\rangle}$ is the underlying ∞-category of $\mathcal{O}$ [26, Remark 2.1.1.25]. Since the vertical functors preserve limits and detect equivalences, the top horizontal composite defines a Postnikov structure if and only if the bottom horizontal composite does (since an $\mathcal{E}$-diagram is a Postnikov structure if it sends certain subdiagrams to limit diagrams). But for the bottom horizontal composite this is clear, since limits are computed pointwise. $\square$

4.1 Examples

Together with Proposition 4.14, the main sources of examples of multiplicative Postnikov structures are the following:

Example/Proposition 4.15 *Let $\mathcal{S}$ be the ∞-category of spaces. Then the Postnikov tower*

$$X \to \cdots \to \tau_{\leq 2}(X) \to \tau_{\leq 1}(X),$$

together with its k-invariants, gives rise to a multiplicative Postnikov structure on $(\mathcal{S}, \times)$.

Proof Since we consider $\mathcal{S}$ with the Cartesian monoidal structure, Example 4.13 shows that it suffices to construct the Postnikov structure without its lax monoidal structure, and only check at the end that the individual components are product preserving. Now the underlying Postnikov structure can be produced at the level of simplicial sets (and is classical, see [6; 11]). Indeed, for every Kan complex X, let us make the following definitions:

(a) Let $P_a(X) = \operatorname{cosk}_{a+1}(X)$ be the $(a+1)$-coskeleton and note that there is a canonical weak equivalence $P_1(X) \to \mathrm{N}(\Pi_1(X))$ to the nerve of the fundamental groupoid.

(b) For every $a \geq 2$, there is a functor $\pi_a(X) \colon \Pi_1(X) \to \mathrm{Ab}$ sending a vertex x of X to the corresponding homotopy group. Let us recall that this homotopy group can be presented as quotient of the set of maps of pointed simplicial sets $(\operatorname{sk}_a \Delta^{a+1}, \{0\}) \to (X, x)$ by pointed homotopy.

(c) For every pointed simplicial set S, taking the free reduced $\pi_a(X)$-module on its simplices yields a simplicial $\Pi_1(X)$-set $\pi_a(X) \otimes S \colon \Pi_1(X) \to \mathrm{sSet}$. Taking $S = \Delta^n/\operatorname{sk}_{n-1}\Delta^n$, this gives the functor sending each vertex x of X to the classical (minimal) model for the Eilenberg–MacLane space $K(\pi_a(X,x), n)$. Recall that the latter is characterized up to *isomorphism* by the following universal property: there is a natural bijection between the set of maps of simplicial sets $T \to K(\pi_a(X,x), n)$ and the set of n-cocycles in the normalized cochain complex of T with coefficients in $\pi_a(X,x)$.

(d) Recall that there is a classifying space functor $(-)_{h\Pi_1(X)}$ from $\operatorname{Fun}(\Pi_1(X), \mathrm{sSet})$ to simplicial sets, given by the following explicit point-set model for the homotopy colimit: $Y_{h\Pi_1(X)}$ has n-simplices given by tuples of $x_0 \to \cdots \to x_n$ in $\Pi_1(X)$ and an n-simplex of $Y(x_0)$. In particular, $(*)_{h\Pi_1(X)} = \mathrm{N}(\Pi_1(X))$ is the nerve of the fundamental groupoid.

(e) Let $\mathrm{sSet}_*^{\mathrm{fin}}$ denote the full subcategory of pointed simplicial sets whose image in the ∞-category $\mathcal{S}_*$ of pointed spaces is finite. We then define $K_{X,a} \colon \mathrm{sSet}_*^{\mathrm{fin}} \to \mathrm{sSet}$ by

$$K_{X,a}(T) = [\pi_a(X) \otimes (T \wedge S^{a+1})]_{h\Pi_1(X)},$$

where $S^{a+1} = \Delta^{a+1}/\operatorname{sk}_a \Delta^{a+1}$.

(f) By [6, 1.2(vi)], there is a natural map of simplicial sets for each $a \geq 2$

$$k_a \colon P_{a-1}(X) \to K_{X,a}(S^0) = [K(\pi_a(X), a+1)]_{h\Pi_1(X)}.$$

Explicitly, this map is given as follows. The simplicial set $K_{X,a}(S^0)$ is $(a+1)$-coskeletal and the map $K_{X,a}(S^0) \to \mathrm{N}(\Pi_1(X))$ induces an isomorphism on a-skeleta. The map k_a then coincides with $P_{a-1}(X) \to P_1(X) \to \mathrm{N}(\Pi_1(X))$ on the a-skeleton, and sends an $(a+1)$-simplex of $P_{a-1}(X)$, ie a map $\sigma \colon \operatorname{sk}_a \Delta^{a+1} \to X$, to the associated element in $\pi_a(X, \sigma(0))$ (see point (b)).

By construction, the map k_a is trivial on all $(a+1)$-simplices in $P_{a-1}(X)$ that arise as the image of an $(a+1)$-simplex in $P_a(X)$, so that there is a commuting square

$$\begin{tikzcd} P_a(X) \arrow[r] \arrow[d] & K_{X,a}(*) \arrow[d, "0"] \\ P_{a-1}(X) \arrow[r, "k_a"'] & K_{X,a}(S^0) \end{tikzcd} \tag{4-1}$$

For any Kan complex X, the functor $K_{X,a}$ preserves weak equivalences of simplicial sets and hence determines a functor of ∞-categories $K_{X,a} \colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$. It is straightforward to verify the conditions of Proposition 2.28, which imply that $K_{X,a}$ is excisive because $\mathcal{S}$ admits loopspace machinery (Example 2.27).

Furthermore, the square (4-1) defines a pullback square in the ∞-category $\mathcal{S}$ by [6, Lemma 2.3] and the sequence $X \to \cdots \to P_a(X) \to P_{a-1}(X) \to \cdots$ is a homotopy limit sequence. It follows that the above construction defines, for every Kan complex X, a simplicial model for a Postnikov structure on the underlying object in the ∞-category $\mathcal{S}$.

All of the above data is strictly functorial in maps of Kan complexes and sends weak equivalences to pointwise weak equivalences of simplicial sets. It therefore defines a section

$$\begin{tikzcd} \mathcal{S} \arrow[r, dotted] & \mathrm{PoStr}(\mathcal{S}) \arrow[l, bend right, "\mathrm{ev}_\infty"'] \end{tikzcd}$$

on ∞-categorical localizations, as desired. To verify that the individual components of this tower are product preserving we note that:

(1) For each $a \geq 1$ the Postnikov piece functor $P_a(X)$ is product preserving. Indeed, on the level of Kan complexes it is given by $\mathrm{cosk}_a(-)$, which is product preserving on the nose.

(2) For each $a \geq 1$ and $T \in \mathcal{S}_*^{\mathrm{fin}}$, the functor

$$X \mapsto K_{X,a}(T) = (\pi_a(X) \otimes (T \wedge S^{a+1}))_{h\Pi_1(X)}$$

is product preserving. Indeed, this follows from the fact that:

- $\Pi_1(-)$ is product preserving.
- Taking a^{th} homotopy groups is product preserving when considered as a functor from pointed spaces to abelian groups. In other words, the map $\pi_a(X \times Y, (x, y)) \to \pi_a(X, x) \times \pi_a(Y, y)$ is an isomorphism.
- For a fixed finite set I the functor $A \mapsto A \otimes I = A^I$ from abelian groups to sets is product preserving.
- Products in spaces commute with homotopy quotients in each variable separately. Indeed, for two diagrams of simplicial sets $X : \mathcal{G} \to \mathrm{sSet}$ and $Y : \mathcal{H} \to \mathrm{sSet}$ indexed by groupoids, the map $(X \times Y)_{h(\mathcal{G}\times\mathcal{H})} \to X_{h\mathcal{G}} \times Y_{h\mathcal{H}}$ is an isomorphism by the explicit formula from (d).

It follows that the Postnikov structure is multiplicative. $\square$

Example 4.16 The multiplicative Postnikov structure of Example/Proposition 4.15 is not just lax symmetric monoidal, but strongly symmetric monoidal, as mentioned in its construction: it is a product preserving functor $\Phi \colon \mathcal{S} \to \mathrm{Fun}(\mathcal{E}, \mathcal{S})$. It follows that for any small ∞-category with finite products $\mathcal{T}$, the ∞-category $\mathrm{Fun}^\times(\mathcal{T}, \mathcal{S})$ of product preserving functors $\mathcal{T} \to \mathcal{S}$ comes with a Postnikov structure

$$\mathrm{Fun}^\times(\mathcal{T}, \mathcal{S}) \xrightarrow{\Phi_*} \mathrm{Fun}^\times(\mathcal{T}, \mathrm{Fun}(\mathcal{E}, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{E}, \mathrm{Fun}^\times(\mathcal{T}, \mathcal{S})).$$

For every $A \in \mathrm{Fun}^\times(\mathcal{T}, \mathcal{S})$, this provides a refinement of the tower $A \to \cdots \to \tau_{\leq 2} A \to \tau_{\leq 1} A$ of truncations of A. In particular, when $\mathcal{T}$ is an algebraic theory, this shows that $\mathcal{T}$-algebras over in $\mathcal{S}$ have Postnikov towers equipped with k-invariants (see [10] for algebras over simplicial operads).

Example 4.17 Let $\mathcal{X}$ be an ∞-topos in which Postnikov towers converge [23, Definition 5.5.6.23], ie $\mathcal{X} \to \lim_n \tau_{\leq n}\mathcal{X}$ is the limit of its full subcategories of truncated objects (this implies that $\mathcal{X}$ is hypercomplete). In this case, there exists a reflective localization $L\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \underset{\leftarrow}{\overset{\rightarrow}{\perp}} \mathcal{X} : i$ such that L is left exact and preserves (limits of) Postnikov towers. We then obtain a Postnikov structure on $\mathcal{X}$

$$\mathcal{X} \xrightarrow{i} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\Phi_*} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E}, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{E}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})) \xrightarrow{L_*} \mathrm{Fun}(\mathcal{E}, \mathcal{X}).$$

Indeed, this sends every object $X \in \mathcal{X}$ to the Postnikov structure of the presheaf $i(X)$ (applying Example/Proposition 4.15 pointwise in $\mathcal{C}$), and then applies L to the resulting diagram of presheaves. Since L is left exact and preserves Postnikov towers, the resulting $\mathcal{E}$-diagram in $\mathcal{X}$ is indeed a Postnikov structure.

Observation 4.18 The proof of Example/Proposition 4.15 admits the following modification: let $\mathcal{S}^{\pi-\mathrm{ab}} \subseteq \mathcal{S}$ be the full subcategory consisting of those spaces X such that each homotopy group $\pi_1(X,x)$ is abelian and acts trivially on the higher $\pi_n(X,x)$. Then there exists a multiplicative Postnikov structure

$$\mathcal{S}^{\pi-\mathrm{ab}} \to \mathrm{PoStr}(\mathcal{S}^{\pi-\mathrm{ab}}) \subseteq \mathrm{PoStr}(\mathcal{S})$$

whose value on a space X is the Postnikov structure $X \to \cdots \to \tau_{\leq 1}X \to \pi_0(X)$ including the zeroth stage. Furthermore, the k-invariants are given by maps

$$k_a\colon \tau_{\leq a-1}X \to \Omega^\infty(K_a(X)),$$

where $K_a(X)$ is the parametrized spectrum over $\pi_0(X)$ whose fiber over $x \in \pi_0(X)$ denotes the suspended Eilenberg–MacLane spectrum $H(\pi_a(X,x))[a+1]$. Indeed, this follows from the fact that the category of simplicial sets with homotopy type in $\mathcal{S}^{\pi-\mathrm{ab}}$ is closed under coskeleta and products, together with the fact that the local system of homotopy groups from (b) arises as the pullback of a local system along the map $\Pi_1(X) \to \pi_0(X)$.

Example 4.19 Let $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}^{\pi-\mathrm{ab}})$ be the ∞-category of $\mathbb{E}_\infty$-spaces whose underlying space has trivial actions of π_1. Proposition 4.14 and Observation 4.18 imply that the Postnikov tower

$$A \to \cdots \to \tau_{\leq 1}A \to \tau_{\leq 0}A$$

is part of a multiplicative Postnikov structure Φ^{ab} on $(\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}^{\pi-\mathrm{ab}}), \times)$.

Let A be a grouplike $\mathbb{E}_\infty$-space. Then A is in particular contained in $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}^{\pi-\mathrm{ab}})$. The corresponding Postnikov structure $\Phi^{\mathrm{ab}}_A\colon \mathcal{E} \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}^{\pi-\mathrm{ab}})$ has the property that $\pi_0(\Phi^{\mathrm{ab}}_A)$ is the constant diagram with value $\pi_0(A)$. In particular, Φ^{ab}_A takes values in the full subcategory of grouplike $\mathbb{E}_\infty$-monoids. It follows that the multiplicative Postnikov structure Φ^{ab} restricts to a multiplicative Postnikov structure on grouplike $\mathbb{E}_\infty$-monoids, which fits into a commuting square

$$\begin{tikzcd}
\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \arrow[r, "\Phi^{\mathrm{ab}}"] \arrow[d, "\text{forget}"'] & \mathrm{PoStr}(\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})) \arrow[d, "\text{forget}"] \\
\mathcal{S}^{\pi-\mathrm{ab}} \arrow[r, "\text{Observation 4.18}"] & \mathrm{PoStr}(\mathcal{S})
\end{tikzcd}$$

The Postnikov structure on the category of grouplike $\mathbb{E}_\infty$-spaces (or equivalently, connective spectra) from Example 4.19 admits a generalization to more general *complete Grothendieck prestable ∞-categories* [25, Definitions C.1.2.12, C.1.4.2].

Remark 4.20 Recall that a presentable ∞-category $\mathcal{V}$ is a complete Grothendieck prestable ∞-category if and only if the left adjoint $\mathcal{V} \to \mathrm{Sp}(\mathcal{V})$ to its stabilization is fully faithful and exhibits $\mathcal{V} \simeq \mathrm{Sp}(\mathcal{V})^{\geq 0}$ as the connective part of a left complete t-structure on $\mathrm{Sp}(\mathcal{V})$ with the property that the coconnective part $\mathrm{Sp}(\mathcal{V})^{\leq 0} \subseteq \mathrm{Sp}(\mathcal{V})$ is closed under filtered colimits (see [25, Proposition C.1.4.1] and its proof). In particular, this implies that $\mathcal{V}$ is an additive ∞-category and that the full subcategory of 0-truncated objects $\tau_{\leq 0}\mathcal{V}$ is an abelian category, equivalent to the heart $\mathrm{Sp}(\mathcal{V})^{\heartsuit}$. As usual, we will write $\pi_a X \in \mathrm{Sp}(\mathcal{V})^{\heartsuit}$ for the homotopy groups with respect to the t-structure. Finally, we will say that a map $f\colon A \to B$ in $\mathcal{V}$ is *a-connective* if its cofiber $\mathrm{cof}(f) \in \mathcal{V} \subseteq \mathrm{Sp}(\mathcal{V})$ is $(a+1)$-connective (in other words, the $(a+1)$-fold suspension of an object in $\mathcal{V}$).

For later purposes, let us record the following properties of prestable ∞-categories:

Remark 4.21 Consider a square $F\colon \Delta^1 \times \Delta^1 \to \mathcal{V}$ in a prestable ∞-category in which all maps induce isomorphisms on $\tau_{\leq 0}$. Then the square is Cartesian if and only if it is co-Cartesian in $\mathcal{V}$. Indeed, the condition that all maps induce isomorphisms in $\tau_{\leq 0}\mathcal{V} \simeq \mathrm{Sp}(\mathcal{V})^{\heartsuit}$ implies that the square is Cartesian in $\mathcal{V} \simeq \mathrm{Sp}(\mathcal{V})^{\geq 0}$ if and only if it is Cartesian in $\mathrm{Sp}(\mathcal{V})$, and likewise for being co-Cartesian. Since pullback and pushout squares in $\mathrm{Sp}(\mathcal{V})$ coincide, the result follows.

Lemma 4.22 *Let $\mathcal{V}$ be an SM prestable ∞-category such that the tensor product preserves finite colimits in each variable and let $n \geq 0$ and $a \geq 1$. For each $1 \leq i \leq n$, suppose we have an a-connective map $f_i\colon A_i \to B_i$ and a 1-connective map $g_i\colon A_i \to A_i'$, and let $B_i' = A_i' \amalg_{A_i} B_i$ be their pushout. For the induced square*

$$\begin{array}{ccc} \bigotimes_{i=1}^n A_i & \longrightarrow & \bigotimes_{i=1}^n A_i' \\ \downarrow & & \downarrow \\ \bigotimes_{i=1}^n B_i & \longrightarrow & \bigotimes_{i=1}^n B_i' \end{array}$$

the natural map $Q \to \bigotimes_{i=1}^n B_i'$ from the pushout is $(a+2)$-connective.

Proof We proceed by induction on n, the case $n = 1$ being evident. The map i_n can be factored as

$$i_n\colon Q_n \xrightarrow{\theta} B_1' \otimes Q_{n-1} \xrightarrow{B_1' \otimes i_{n-1}} \bigotimes_{i=1}^{n} B_i',$$

where the map θ arises as the colimit of the following natural transformation of spans:

$$\begin{array}{ccccc} \bigotimes_{i=1}^n A_i' & \longleftarrow & \bigotimes_{i=1}^n A_i & \longrightarrow & \bigotimes_{i=1}^n B_i \\ \downarrow & & \downarrow & & \downarrow \\ B_1' \otimes \bigotimes_{i=2}^n A_i' & \longleftarrow & B_1' \otimes \bigotimes_{i=2}^n A_i & \longrightarrow & B_1' \otimes \bigotimes_{i=2}^n B_i \end{array}$$

Using that $(a+3)$-connective objects are closed under extensions in $\mathcal{V}$, it suffices to verify that θ and $B_1' \otimes i_{n-1}$ are both $(a+2)$-connective. For $B_1' \otimes i_{n-1}$, this follows by inductive hypothesis. To show that $\mathrm{cof}(\theta)$ is $(a+3)$-connective, we can use the equivalence $\mathrm{cof}(A_1 \to B_1') \simeq \mathrm{cof}(f_1) \oplus \mathrm{cof}(g_1)$ to identify the cofiber of the above natural transformation with

$$\mathrm{cof}(f_1) \otimes \bigotimes_{i=2}^{n} A_i' \leftarrow (\mathrm{cof}(f_1) \oplus \mathrm{cof}(g_1)) \otimes \bigotimes_{i=2}^{n} A_i \to \mathrm{cof}(g_1) \otimes \bigotimes_{i=2}^{n} B_i .$$

The maps are given by projections in the first factor and by tensor products of g_i and f_i in the other factors. Taking the pushout of the above diagram, one therefore finds that

$$\mathrm{cof}(\theta) \simeq \Big[\mathrm{cof}(f_1) \otimes \mathrm{cof}\Big(\bigotimes_{i=2}^{n} g_i\Big)\Big] \oplus \Big[\mathrm{cof}(g_1) \otimes \mathrm{cof}\Big(\bigotimes_{i=2}^{n} f_i\Big)\Big].$$

It follows from [26, HA, Lemma 7.4.1.30] and our connectivity assumptions on the maps f_i and g_i that both summands are $(a+3)$-connective, so that θ is $(a+2)$-connective. $\square$

Example/Proposition 4.23 *Let $\mathcal{V}$ be a complete Grothendieck prestable ∞-category and let us write $\mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V}) \subseteq \mathrm{PoStr}(\mathcal{V})$ for the full sub-∞-category of objects equipped with a Postnikov structure (indexed over all $a \geq 0$) with the following properties:*

(a) *For each $a \geq 0$, the map $X \to X_a$ exhibits $X_a \simeq \tau_{\leq a} X$ as the a-truncation of X.*

(b) *For each $a \geq 1$, $\pi(K_a)$ is 0-truncated and all maps in the pullback square*

$$\begin{tikzcd} X_a \arrow[r] \arrow[d] & \pi(K_a) \arrow[d, "0"] \\ X_{a-1} \arrow[r, "k_a"] & \Omega^\infty K_a \end{tikzcd} \tag{4-2}$$

induce isomorphisms on 0-truncations.

(c) *For each $a \geq 1$, the object $K_a \in \mathcal{T}\mathcal{V}$ is contained in the connective part for the canonical t-orientation (Example 3.14).*

Then the map $\mathrm{ev}_\infty \colon \mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V}) \to \mathcal{V}$ is an equivalence. If $\mathcal{V}$ is furthermore closed SM, then ev_∞ refines to an equivalence $\mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V})^\otimes \to \mathcal{V}^\otimes$ between $\mathcal{V}^\otimes$ and the full suboperad of $\mathrm{PoStr}(\mathcal{V})^\otimes$ spanned by the objects equipped with Postnikov structures satisfying the above properties.

In particular, each object $A \in \mathcal{V}$ comes with a unique Postnikov structure satisfying the above three conditions, and the resulting Postnikov structure on $\mathcal{V}$ carries a unique multiplicative structure if $\mathcal{V}$ is symmetric monoidal. Let us point out that by Remark 4.21, the square (4-2) is also a pushout square. Together with (a), this implies that for each $a \geq 1$, the square (4-2) can be identified with the (co-)Cartesian square

$$\begin{tikzcd} \tau_{\leq a} A \arrow[r] \arrow[d] & \pi_0(A) \arrow[d] \\ \tau_{\leq a-1} A \arrow[r, "k_a"] & \pi_0(A) \oplus \Sigma^{a+1} \pi_a(A) \end{tikzcd} \tag{4-3}$$

since the cofiber of the left (and hence right) vertical map is $\pi_a(X)[a+1]$ and the right vertical map is the inclusion of a summand (since it admits a retraction). Taking algebras, we then obtain the following:

Example 4.24 Let $\mathcal{V}$ be a complete SM Grothendieck prestable ∞-category and let $\mathcal{O}$ be an ∞-operad. For example, one can take $\mathcal{V} = \mathrm{Sp}^{\geq 0}$ to be the ∞-category of connective spectra with the smash product. Combining Proposition 4.14 and Example/Proposition 4.23, we find that the Postnikov tower $A \to \cdots \to \tau_{\leq 1} A \to \tau_{\leq 0} A$ of an $\mathcal{O}$-algebra in $\mathcal{V}$ is part of a (multiplicative) Postnikov structure on $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$.

By Example 3.12, this means that each stage of the Postnikov tower fits into a pullback square of $\mathcal{O}$-algebras (4-3), where $\pi_0(A) \oplus \Sigma^{a+1}\pi_a(A)$ is the trivial square zero extension of $\pi_0(A)$ by the operadic module $\Sigma^{a+1}\pi_a(A)$. By specializing to $\mathcal{O} = \mathbb{E}_n$, this recovers [26, Corollary 7.4.1.28].

The remainder of this section is devoted to a proof of Example/Proposition 4.23. To avoid repetition, let us prove the claim for a symmetric monoidal $\mathcal{V}$; the much simpler nonmonoidal case can be proven in the same way, removing all references to the monoidal structure. Our proof will proceed by induction, where the inductive step relies on an analysis of the ∞-operad of pullback squares (4-2). To this end, let us introduce some auxiliary categories:

Construction 4.25 For each $a \geq 1$, let us denote by

$$\mathcal{E}_a := M(\kappa_a), \quad \mathcal{E}_a^{\mathrm{cn}} := M(\kappa'_a),$$

the mapping simplices (Definition 4.4) of the functors $\kappa_a \colon \{a \to (a-1)\} \to \mathcal{S}_*^{\mathrm{fin}}$, as in Construction 4.5, and $\kappa'_a \colon \{a \to (a-1)\} \to \mathrm{Fin}_*$ sending $a \mapsto *$ and $(a-1) \mapsto S^0$. Note that $\mathcal{E}_a^{\mathrm{cn}}$ is an ordinary category, since it is the unstraightening of a diagram of ordinary categories. In particular, Definition 4.4 provides a full description of $\mathcal{E}_a^{\mathrm{cn}}$, without need of specifying further homotopy coherences.

Now consider the chain of functors

$$a \colon * \xrightarrow{0} \Delta^2 \xhookrightarrow{j} \mathcal{E}_a^{\mathrm{cn}} \xrightarrow{\tilde{i}} \mathcal{E}_a, \tag{4-4}$$

where j is the inclusion of the full subcategory $\{a \to (a-1) \to *\}$ in $\mathcal{E}_a^{\mathrm{cn}}$ (using the description from Definition 4.4) and $\tilde{i}$ is the cobase change of the inclusion $i \colon \mathrm{Fin}_* \hookrightarrow \mathcal{S}_*^{\mathrm{fin}}$.

Definition 4.26 Let us denote by

$$\mathrm{Ext}_a^{\otimes} \hookrightarrow \mathrm{Fun}(\mathcal{E}_a, \mathcal{V})^{\otimes_{\mathrm{lev}}}, \quad \mathrm{Ext}_a^{\mathrm{cn},\otimes} \hookrightarrow \mathrm{Fun}(\mathcal{E}_a^{\mathrm{cn}}, \mathcal{V})^{\otimes_{\mathrm{lev}}}, \quad \mathrm{Trun}_a^{\otimes} \hookrightarrow \mathrm{Fun}(\Delta^2, \mathcal{V})^{\otimes_{\mathrm{lev}}},$$

the three full sub-∞-operads defined as follows:

(1) $\mathrm{Trun}_a^{\otimes}$ is spanned by sequences $T_a \to T_{a-1} \to T_*$ with $T_a \in \tau_{\leq a}\mathcal{V}$ and exhibiting $T_{a-1} \simeq \tau_{\leq a-1} T_a$ and $T_* \simeq \tau_{\leq 0} T_a$.

(2) $\mathrm{Ext}_a^{\mathrm{cn},\otimes}$ is spanned by the diagrams $T\colon \mathcal{E}_a^{\mathrm{cn}} \to \mathcal{V}$ such that

  (a) the restriction to $\{a \to (a-1) \to *\}$ is contained in $\mathrm{Trun}_a^{\otimes}$,

  (b) the restriction along $\kappa_{a*}\colon \Delta^1 \times \{a, a-1\} \to \mathcal{E}_a^{\mathrm{cn}}$ (Definition 4.4) is a pullback square in which all maps induce isomorphisms on 0-truncations,

  (c′) the restriction to Fin_* defines an $\mathbb{E}_\infty$-groupoid object (Definition 2.29).

(3) $\mathrm{Ext}_a^{\otimes}$ is spanned by the diagrams $T\colon \mathcal{E}_a \to \mathcal{V}$ satisfying conditions (a) and (b) above, as well as

  (c) the restriction to $\mathcal{S}_*^{\mathrm{fin}}$ defines an object in $\mathcal{T}^{\geq 0}\mathcal{V} \subseteq \mathcal{T}\mathcal{V} = \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$.

Lemma 4.27 *Let $\mathcal{V}$ be a complete Grothendieck prestable ∞-category and $T\colon \mathcal{E}_a^{\mathrm{cn}} \to \mathcal{V}$ a diagram. Then the following are equivalent*:

(1) *T is contained in* $\mathrm{Ext}_a^{\mathrm{cn}}$.

(2) *T is left Kan extended from its restriction to $\{a \to (a-1) \to *\}$, and this restriction is contained in* Trun_a.

Proof Recall that $j : \Delta^2 \hookrightarrow \mathcal{E}_a^{\mathrm{cn}}$ denotes the inclusion of the full subcategory on $a, (a-1)$ and $*$. For any diagram $F\colon \Delta^2 \to \mathcal{V}$ of the form $F(a) \to F(a-1) \to F(*)$ and a finite pointed set S with basepoint s_0, the left Kan extension $j_!F(S)$ can be computed as the pushout

$$\begin{tikzcd} \bigoplus_{s\in S} F(a) \arrow[r] \arrow[d] & F(a) \arrow[d] \\ F(*) \oplus \bigoplus_{s\in S\setminus\{s_0\}} F(a-1) \arrow[r] & j_!F(S) \end{tikzcd}$$

Here the vertical functor is given by $F(a) \to F(*)$ on the summand labeled by the basepoint of S and by $F(a) \to F(a-1)$ on the summand labeled by each other point of S. Indeed, the above colimit coincides with the colimit of

$$\Delta^2 \times_{\mathcal{E}_a^{\mathrm{cn}}} (\mathcal{E}_a^{\mathrm{cn}})_{/S} \to \Delta^2 \xrightarrow{F} \mathcal{V},$$

where one can use the explicit description of the (ordinary) category $\mathcal{E}_a^{\mathrm{cn}}$ to identify the comma category. Using that $\mathcal{V}$ is a prestable (and in particular additive) ∞-category, this implies that

$$j_!F(S) = F(*) \oplus \bigoplus_{s\in S\setminus\{s_0\}} \mathrm{cof}(F(a) \to F(a-1)). \tag{4-5}$$

This formula shows that the restriction $j_!F|_{\mathrm{Fin}_*}$ is a Segal $\mathbb{E}_\infty$-groupoid and that the square

$$\begin{tikzcd} j_!F(a) \arrow[r] \arrow[d] & j_!F(*) \arrow[d] \\ j_!F(a-1) \arrow[r] & j_!F(S^0) \end{tikzcd}$$

is co-Cartesian, and hence also Cartesian since $\mathcal{V}$ is prestable. It follows that (2) implies (1). For the converse, if $T \in \mathcal{E}_a^{\mathrm{cn}}$, then the natural map $\epsilon\colon j_!j^*T \to T$ is an equivalence at the objects $a, (a-1)$ and $*$ because j is fully faithful. In light of Remark 4.21, the map $F(a) \amalg_{F(a-1)} F(*) \to F(S^0)$ is an equivalence so that ϵ is also an equivalence at S^0. Since both $j_!j^*F$ and F restrict to Segal $\mathbb{E}_\infty$-groupoids on Fin_*, it follows that ϵ is also an equivalence at all other $S \in \mathrm{Fin}_*$. ∎

Lemma 4.28 *Let $\mathcal{V}$ be a complete Grothendieck prestable ∞-category with a closed SM structure. Then restriction along the maps (4-4) induces equivalences of ∞-operads*

$$\mathrm{ev}_{(0,a)}\colon \mathrm{Ext}_a^{\otimes} \xrightarrow{\sim} \mathrm{Ext}_a^{\mathrm{cn},\otimes} \xrightarrow{\sim} \mathrm{Trun}_a^{\otimes} \xrightarrow{\sim} (\tau_{\leq a}\mathcal{V})^{\otimes}.$$

Proof Restriction along the functors in (4-4) defines SM functors between the ∞-categories of $\mathcal{V}$-valued diagrams, with the levelwise tensor product, which preserve the full sub-∞-operads from Definition 4.26. We will check that each of the restriction functors is an equivalence.

Step 1 Using that $\tilde{i}\colon \mathcal{E}_a^{\mathrm{cn}} \to \mathcal{E}_a$ is the pushout of the inclusion $i\colon \mathrm{Fin}_* \to \mathcal{S}_*^{\mathrm{fin}}$, it follows that there is a pullback square of ∞-operads

$$\begin{tikzcd} \mathrm{Ext}_a^{\otimes} \arrow[r, "\tilde{i}^*"] \arrow[d] & \mathrm{Ext}_a^{\mathrm{cn},\otimes} \arrow[d] \\ (\mathcal{T}^{\geq 0}\mathcal{V})^{\otimes} \arrow[r, "i^* = \Omega^\infty"] & \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})^{\otimes} \end{tikzcd}$$

Here the ∞-operads in the bottom row are full sub-∞-operads of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$ and $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})^{\otimes_{\mathrm{lev}}}$, respectively. Corollary 2.31 implies that these bottom two ∞-operads are in fact SM ∞-categories and that Ω^∞ is an SM equivalence between them. Consequently, $\tilde{i}^*$ is an equivalence of ∞-operads as well.

Step 2 Let $j\colon \Delta^2 \hookrightarrow \mathcal{E}_a^{\mathrm{cn}}$ denote the inclusion of the full subcategory $\{a \to (a-1) \to *\}$. To see that $j^*\colon \mathrm{Ext}_a^{\mathrm{cn},\otimes} \to \mathrm{Trun}_a^{\otimes}$ is an equivalence of ∞-operads, we will show that it is essentially surjective and fully faithful, ie it induces equivalences on spaces of multimorphisms [2, Proposition 7.17]. Essential surjectivity follows from Lemma 4.27: indeed, each object $F \in \mathrm{Trun}_a$ arises as the restriction of its left Kan extension $j_! F \in \mathrm{Ext}_a^{\mathrm{cn}}$.

To check that j^* is fully faithful, let $T_1, \dots, T_n$ and T_0 be objects in $\mathrm{Ext}_a^{\mathrm{cn}}$, and let us abbreviate $X_i = T_i(a)$ and $Y_i = T_i(a-1)$. The condition that $T_i \in \mathrm{Ext}_a^{\mathrm{cn}}$ then implies that X_i is a-truncated and that

$$Y_i \simeq \tau_{\leq a-1} X_i, \quad T_i(*) = \pi_0 X_i, \quad \mathrm{cof}(T_i(a) \to T_i(a-1)) \simeq \Sigma^{a+1}\pi_a X_i. \tag{4-6}$$

We now need to show that restriction along j induces an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{E}_a^{\mathrm{cn}}, \mathcal{V})}(T_1 \otimes_{\mathrm{lev}} \cdots \otimes_{\mathrm{lev}} T_n, T_0) \to \mathrm{Map}_{\mathrm{Fun}(\Delta^2, \mathcal{V})}(j^*T_1 \otimes_{\mathrm{lev}} \cdots \otimes_{\mathrm{lev}} j^*T_n, j^*T_0). \tag{4-7}$$

By adjunction, the map (4-7) is obtained by applying $\mathrm{Map}_{\mathrm{Fun}(\mathcal{E}_a^{\mathrm{cn}}, \mathcal{V})}(-, T_0)$ to the counit map

$$\epsilon\colon j_! j^*(T_1 \otimes_{\mathrm{lev}} \cdots \otimes_{\mathrm{lev}} T_n) \to T_1 \otimes_{\mathrm{lev}} \cdots \otimes_{\mathrm{lev}} T_n.$$

We claim that ϵ is given pointwise by an $(a+2)$-connective map in $\mathcal{V}$. This implies that (4-7) is an equivalence, because $(a+2)$-connective maps induce equivalences on $(a+1)$-truncations and T_0 takes values in $(a+1)$-truncated objects, by equations (4-5) and (4-6).

It thus remains to verify that each component of the natural transformation ϵ is $(a+2)$-connective. This is clear for the components of the natural transformation ϵ at the objects a, $(a-1)$ and $*$ in $\mathcal{E}_a^{\mathrm{cn}}$, where the counit is an equivalence (since j is fully faithful). We will prove by induction on $k \geq 0$ that the component of ϵ at the finite pointed set $\langle k \rangle$ with $k+1$ elements is $(a+2)$-connective. The case $k = 0$ has already been treated, and for $k \geq 1$ consider the following commuting diagram:

$$\begin{tikzcd}
X_1 \otimes \cdots \otimes X_n \arrow[r] \arrow[d] & j_! j^*(T_1 \otimes \cdots \otimes T_n)(\langle k-1 \rangle) \arrow[r, "\epsilon"] \arrow[d] & T_1(\langle k-1 \rangle) \otimes \cdots \otimes T_n(\langle k-1 \rangle) \arrow[d] \\
Y_1 \otimes \cdots \otimes Y_n \arrow[r] & j_! j^*(T_1 \otimes \cdots \otimes T_n)(\langle k \rangle) \arrow[r, "\epsilon"] & T_1(\langle k \rangle) \otimes \cdots \otimes T_n(\langle k \rangle)
\end{tikzcd}$$

Formula (4-5) shows that the left square is a pushout, so that the pushout Q for the right cospan is equivalent to the pushout for the total cospan. Lemma 4.22 then implies that the natural map

$$Q \to T_1(\langle k \rangle) \otimes \cdots \otimes T_n(\langle k \rangle)$$

is $(a+2)$-connective. On the other hand, the map $j_! j^*(T_1 \otimes_{\mathrm{lev}} \cdots \otimes_{\mathrm{lev}} T_n)(\langle k \rangle) \to Q$ is the pushout of the counit map ϵ at $\langle k-1 \rangle$, which was $(a+2)$-connective by inductive hypothesis. We conclude that ϵ is a natural transformation given at each object by an $(a+2)$-connective map, as desired.

Step 3 Finally, let us show that $\mathrm{ev}_a \colon \mathrm{Trun}_a^{\otimes} \to (\tau_{\leq a}\mathcal{V})^{\otimes}$ is an equivalence of ∞-operads. The objects of Trun_a are simply given by sequences $\sigma = [X \to \tau_{\leq a-1} X \to \tau_{\leq 0} X]$ with $X \in \tau_{\leq a}\mathcal{V}$. In particular, ev_a is essentially surjective. To see that it is a fully faithful map of ∞-operads, ie that each

$$\mathrm{Map}_{\mathrm{Trun}_a^{\otimes}}(\sigma_1, \ldots, \sigma_n; \sigma_0) \to \mathrm{Map}_{(\tau_{\leq a}\mathcal{V})^{\otimes}}(\mathrm{ev}_a(\sigma_1), \ldots, \mathrm{ev}_a(\sigma_n); \mathrm{ev}_a(\sigma_0))$$

is an equivalence, it suffices to verify the following: for each diagram in $\tau_{\leq a}\mathcal{V}$ of the form

$$\begin{tikzcd}
X_1 \otimes \cdots \otimes X_n \arrow[r] \arrow[d] & (\tau_{\leq a-1} X_1) \otimes \cdots \otimes (\tau_{\leq a-1} X_n) \arrow[r] \arrow[d, dotted] & (\tau_{\leq 0} X_1) \otimes \cdots \otimes (\tau_{\leq 0} X_n) \arrow[d, dotted] \\
X_0 \arrow[r] & \tau_{\leq a-1} X_0 \arrow[r] & \tau_{\leq 0} X_0
\end{tikzcd}$$

there exists a contractible space of dotted extensions, as indicated. This follows from the fact that the first horizontal map is a-connective and the second is 1-connective. $\square$

Proof of Example/Proposition 4.23 Let us inductively define a tower of ∞-operads $\mathrm{PoStr}^{\mathrm{cn}}_{\leq a}(\mathcal{V})^{\otimes}$ by setting $\mathrm{PoStr}^{\mathrm{cn}}_{\leq 0}(\mathcal{V})^{\otimes} = (\tau_{\leq 0}\mathcal{V})^{\otimes}$ and taking pullbacks

$$\begin{tikzcd}
\mathrm{PoStr}^{\mathrm{cn}}_{\leq a}(\mathcal{V})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Ext}_a^{\otimes} \arrow[r, "\mathrm{ev}_a"] \arrow[d, "\mathrm{ev}_{a-1}"] & (\tau_{\leq a}\mathcal{V})^{\otimes} \\
\mathrm{PoStr}^{\mathrm{cn}}_{\leq a-1}(\mathcal{V})^{\otimes} \arrow[r, "\mathrm{ev}_{a-1}"] & (\tau_{\leq a-1}\mathcal{V})^{\otimes}
\end{tikzcd} \tag{4-8}$$

Each $\mathrm{ev}_a \colon \mathrm{PoStr}^{\mathrm{cn}}_{\leq a}(\mathcal{V})^{\otimes} \to (\tau_{\leq a}\mathcal{V})^{\otimes}$ is an equivalence of ∞-operads: by inductive hypothesis the first top horizontal arrow in (4-8) will be an equivalence, and the second map is an equivalence by Lemma 4.28. Furthermore, Step 3 of the proof of Lemma 4.28 shows that the map of ∞-operads

$(\tau_{\leq a}\mathcal{V})^{\otimes} \simeq \mathrm{Ext}_a^{\otimes} \to (\tau_{\leq a-1}\mathcal{V})^{\otimes}$ is given by the localization $\tau_{\leq a-1}\colon \tau_{\leq a}\mathcal{V} \to \tau_{\leq a-1}\mathcal{V}$ with its canonical SM structure. We thus obtain a natural diagram

$$\begin{tikzcd}
\mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V})^{\otimes} \arrow[r] \arrow[d, "\mathrm{ev}_\infty"'] & \cdots \arrow[r] & \mathrm{PoStr}^{\mathrm{cn}}_{\leq 2}(\mathcal{V})^{\otimes} \arrow[r] \arrow[d, "\mathrm{ev}_2", "\sim"'] & \mathrm{PoStr}^{\mathrm{cn}}_{\leq 1}(\mathcal{V})^{\otimes} \arrow[r] \arrow[d, "\mathrm{ev}_1", "\sim"'] & \mathrm{PoStr}^{\mathrm{cn}}_{\leq 0}(\mathcal{V})^{\otimes} \arrow[d, "\mathrm{ev}_0", "\sim"'] \\
\mathcal{V}^{\otimes} \arrow[r] & \cdots \arrow[r] & (\tau_{\leq 2}\mathcal{V})^{\otimes} \arrow[r, "\tau_{\leq 1}"] & (\tau_{\leq 1}\mathcal{V})^{\otimes} \arrow[r, "\tau_{\leq 0}"] & (\tau_{\leq 0}\mathcal{V})^{\otimes}
\end{tikzcd}$$

Since $\mathcal{V}$ was a complete Grothendieck prestable ∞-category (so that Postnikov towers are convergent), the bottom row exhibits $\mathcal{V}^{\otimes}$ as the limit of the $(\tau_{\leq a}\mathcal{V})^{\otimes}$. Using this and unraveling the definitions (see Construction 4.5), we then have an equivalence $\mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V})^{\otimes} \simeq \mathcal{V}^{\otimes} \times_{\lim_a (\tau_{\leq a}\mathcal{V})^{\otimes}} \lim_a \mathrm{PoStr}^{\mathrm{cn}}_{\leq a}(\mathcal{V})^{\otimes}$. Since this is the pullback of a span consisting of two equivalences, we conclude that $\mathrm{ev}_\infty\colon \mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V})^{\otimes} \to \mathcal{V}^{\otimes}$ is an equivalence, as desired. □

5 Postnikov structures on enriched categories

In the previous section we have seen how multiplicative Postnikov structures give rise to multiplicative Postnikov structures on ∞-categories of algebras over operads (Proposition 4.14). The purpose of this section is to prove that similarly, a multiplicative Postnikov structure on a symmetric monoidal ∞-category $\mathcal{V}$ induces a multiplicative Postnikov structure on the ∞-category of $\mathcal{V}$-enriched ∞-categories.

5.1 Recollections on enriched ∞-categories

Let us briefly recall some elements of the theory of enriched ∞-categories developed by Gepner and Haugseng [9].

Definition 5.1 For each space X, let us write $\mathcal{O}_X$ for the universal $(X \times X)$-colored (symmetric) ∞-operad receiving a map from $\Delta_X^{\mathrm{op}} \to \Delta^{\mathrm{op}} \to \mathrm{Fin}_*$, where Δ_X^{op} is the generalized nonsymmetric ∞-operad from [9, Definition 4.1.1]. By [9, Corollaries 3.7.8, 4.2.8], one can model $\mathcal{O}_X$ explicitly by the symmetrization of the simplicial operad from [9, Definition 4.2.4].

When the space X is a point, one recovers the associative operad $\mathcal{O}_* = \mathbb{E}_1$. The operads $\mathcal{O}_X$ depend functorially on the space X, so that we obtain a functor

$$\mathcal{O}_{(-)}\colon \mathcal{S} \to (\mathrm{Op}_\infty)_{/\mathbb{E}_1} \to \mathrm{Op}_\infty.$$

If $\mathcal{V}$ is a monoidal category, then an $\mathcal{O}_X$-algebra in $\mathcal{V}$ can be informally described as follows: an algebra consists of objects $\mathrm{Map}(x, y) \in \mathcal{V}$, depending functorially on $(x, y) \in X \times X$, together with composition operations satisfying obvious associativity conditions.

Definition 5.2 We will refer to the ∞-category $\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V})$ as the ∞-category of *$\mathcal{V}$-enriched categorical algebras with space of objects X*. These ∞-categories depend (contravariantly) functorially on X and we define the ∞-category of *categorical algebras*

$$\mathrm{Ob}\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) = \int_{X\in\mathcal{S}} \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V}) \to \mathcal{S} \tag{5-1}$$

to be the domain of the corresponding Cartesian fibration [9, Definition 4.3.1]. If $\mathcal{V}$ is a presentable monoidal ∞-category, then $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is presentable as well [9, Proposition 4.3.5].

For later purposes, we will mainly be interested in a refinement of this construction for *symmetric* monoidal $\mathcal{V}$.

Proposition 5.3 *Let $\mathcal{S}^\times \to \mathrm{Fin}_*$ denote the Cartesian ∞-operad associated to the ∞-category of spaces. Then there exists a natural functor*

$$\mathrm{Alg}_{\mathrm{Cat}}\colon \mathrm{SMCat}_\infty^{\mathrm{lax}} \to \mathrm{SMCat}_{\infty/\mathcal{S}^\times}^{\mathrm{lax,big}}$$

that sends each SM ∞-category $\mathcal{V}$ to the ∞-category $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ of categorical algebras, together with an SM structure such that the tensor product of categorical algebras with spaces of objects X and Y is a categorical algebra with space of objects $X\times Y$.

Let us point out that the results from [9] only provide functoriality of $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ with respect to (strong) SM functors in $\mathcal{V}$. Since the proof of Proposition 5.3 is rather technical, we will postpone it to the appendix and instead record two further consequences (which are also proven in the appendix). First, note that Proposition 5.3 asserts in particular that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ inherits a symmetric monoidal structure from $\mathcal{V}$, whose underlying tensor product functor can be identified as follows:

Lemma 5.4 *Let $\mathcal{V}$ be an SM ∞-category. Then the tensor product*

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "\otimes"] \arrow[d] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[d] \\ \mathcal{S}\times\mathcal{S} \arrow[r, "\times"] & \mathcal{S} \end{tikzcd}$$

arises as the unstraightening of the natural transformation of functors $\mathcal{S}^{\mathrm{op}}\times\mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}$ given at (X,Y) by

$$\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V}) \times \mathrm{Alg}_{\mathcal{O}_Y}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V}) \times \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V}) \xrightarrow{\otimes} \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V}), \tag{5-2}$$

where the first functor restricts along the maps $\mathcal{O}_X \leftarrow \mathcal{O}_{X\times Y} \to \mathcal{O}_Y$ and the second functor arises from the SM structure on algebras in $\mathcal{V}$ [26, Example 3.2.4.4].

Informally, this means that given two categorical algebras $\mathbb{C}$, $\mathbb{D}$ with spaces of objects X, Y, their tensor product $\mathbb{C}\otimes\mathbb{D}$ has space of objects $X\times Y$ and mapping objects

$$\mathrm{Map}_{\mathbb{C}\otimes\mathbb{D}}((x_0,y_0),(x_1,y_1)) = \mathrm{Map}_{\mathbb{C}}(x_0,x_1)\otimes\mathrm{Map}_{\mathbb{D}}(y_0,y_1).$$

In particular, the unit is given by the categorical algebra $[0]_{1_{\mathcal{V}}}$ with a single object $*$ and with $1_{\mathcal{V}}$ as endomorphisms.

Remark 5.5 If $\mathcal{V}$ is an SM ∞-category, then the natural map (5-2) can also be identified with the composite map $\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V}) \times \mathrm{Alg}_{\mathcal{O}_Y}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V} \times \mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V})$, where the first map is the "exterior product" from [9, Propositions 3.6.14, 4.3.11] and the second map is the image under $\mathrm{Alg}_{\mathcal{O}_{X\times Y}}(-)$ of the lax monoidal functor $\otimes_{\mathcal{V}} \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. Consequently, the functor $\otimes \colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ from Proposition 5.3 is naturally equivalent to the tensor product functor from [9, Corollary 4.3.13]. In particular, we find that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is a presentable SM ∞-category if $\mathcal{V}$ is a presentable SM ∞-category, ie the monoidal structure is *closed* [9, Corollary 4.3.16].

Proposition 5.6 *For each ∞-category $\mathcal{I}$, there is a commuting square depending functorially on $\mathcal{I}$*

$$\begin{tikzcd} \mathrm{SMCat}^{\mathrm{lax}} \arrow[r, "\mathrm{Alg}_{\mathrm{Cat}}"] \arrow[d, "\mathrm{Fun}(\mathcal{I},-)"'] & \mathrm{SMCat}^{\mathrm{lax,big}}_{/\mathcal{S}^\times} \arrow[d, "\mathrm{Fun}(\mathcal{I},-)\times_{\mathrm{Fun}(\mathcal{I},\mathcal{S}^\times)}\mathcal{S}^\times"] \\ \mathrm{SMCat}^{\mathrm{lax}} \arrow[r, "\mathrm{Alg}_{\mathrm{Cat}}"'] & \mathrm{SMCat}^{\mathrm{lax,big}}_{/\mathcal{S}^\times} \end{tikzcd}$$

where the vertical functors use the levelwise tensor product from Construction 2.6.

In other words, for each ∞-category $\mathcal{I}$ there is a natural monoidal equivalence

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{I}, \mathcal{V})) \simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{I},\mathcal{S})} \mathcal{S}.$$

When $\mathcal{I}$ is weakly contractible, the constant diagram functor $\mathcal{S} \to \mathrm{Fun}(\mathcal{I}, \mathcal{S})$ is fully faithful, so that we can rephrase this as follows: there is a natural (SM) fully faithful embedding $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{I}, \mathcal{V})) \hookrightarrow \mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}))$ whose essential image consists of $\mathcal{I}$-diagrams of categorical algebras whose underlying diagram of objects is constant.

For any presentable monoidal ∞-category $\mathcal{V}$, we then define the ∞-category of *$\mathcal{V}$-enriched ∞-categories* $\mathrm{Cat}(\mathcal{V})$ to be the full subcategory $\mathrm{Cat}(\mathcal{V}) \subseteq \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ of *complete* categorical algebras. More precisely, there is a functor

$$J[-] \colon \Delta \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$$

sending $[n]$ to the categorical algebra with object set $\{0, \ldots, n\}$, all mapping objects being $1_{\mathcal{V}}$ and all compositions being equivalences. We will abbreviate $J = J[1]$. Every categorical algebra $\mathbb{C}$ then defines a simplicial space

$$\Delta^{\mathrm{op}} \to \mathcal{S}, \quad [n] \mapsto \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})}(J[n], \mathbb{C}).$$

This simplicial space is a Segal groupoid [9, Corollary 5.2.7] and $\mathbb{C}$ is defined to be complete if this Segal groupoid is essentially constant. The above Segal space only depends on the *underlying space-valued categorical algebra*, ie the categorical algebra in $\mathcal{S}$ obtained by applying the lax monoidal functor $\mathrm{Map}(1_{\mathcal{V}}, -)\colon \mathcal{V} \to \mathcal{S}$ to all mapping objects [9, Proposition 5.1.11]. Furthermore, the space $\mathrm{Map}(J[n], \mathbb{C}) \subseteq \mathrm{Map}([n]_{1_{\mathcal{V}}}, \mathbb{C})$ is a union of path components in the space of n-composable sequences of arrows in $\mathbb{C}$ [9, Proposition 5.1.17].

The inclusion of $\mathcal{V}$-enriched ∞-categories into categorical algebras is part of an adjoint pair

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \underset{\hookleftarrow}{\overset{(-)^\wedge}{\underset{\perp}{\longrightarrow}}} \mathrm{Cat}(\mathcal{V})$$

whose left adjoint is called *completion* [9, Theorem 5.6.6]. When $\mathcal{V}$ is presentable symmetric monoidal, this is a symmetric monoidal localization [9, Proposition 5.7.14] (using Remark 5.5). Finally, let us recall that the completion functor realizes $\mathrm{Cat}(\mathcal{V})$ as the localization of $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ at the *Dwyer–Kan (DK) equivalences*, ie the fully faithful and essentially surjective functors in the following sense:

Definition 5.7 Let $f : \mathbb{C} \to \mathbb{D}$ be a map of categorical algebras.

(1) f is *fully faithful* if for every two objects $x, y \in \mathrm{Ob}(\mathbb{C})$, the map

$$f : \mathrm{Map}_{\mathbb{C}}(x, y) \to \mathrm{Map}_{\mathbb{D}}(f(x), f(y))$$

is an equivalence in $\mathcal{V}$. Equivalently, f is a Cartesian arrow for the Cartesian fibration (5-1).

(2) f is *essentially surjective* if the map

$$\mathrm{Map}(\{0\}, \mathbb{C}) \times_{\mathrm{Map}(\{0\},\mathbb{D})} \mathrm{Map}(J, \mathbb{D}) \to \mathrm{Map}(\{1\}, \mathbb{D})$$

is surjective on π_0. Here the mapping spaces are taken in the ∞-category $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$.

(3) f is an *isofibration* if the induced map

$$\mathrm{Map}(J, \mathbb{C}) \to \mathrm{Map}(J, \mathbb{D}) \times_{\mathrm{Map}(\{1\},\mathbb{D})} \mathrm{Map}(\{1\}, \mathbb{C})$$

is surjective on π_0.

Remark 5.8 Let $F : \mathcal{V} \to \mathcal{W}$ be an SM left adjoint functor between presentable SM ∞-categories, with (lax SM) right adjoint G. Then $\mathrm{Alg}_{\mathrm{Cat}}(G)\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{W}) \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ preserves underlying space-valued categorical algebras. Indeed, this follows from the equivalence of lax SM functors $\mathrm{Map}_{\mathcal{W}}(1_{\mathcal{W}}, -) \simeq \mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, G(-))$, which is right adjoint to the equivalence of SM functors between $\mathcal{S} \xrightarrow{1_{\mathcal{V}}\otimes -} \mathcal{V} \xrightarrow{F} \mathcal{W}$ and $1_{\mathcal{W}} \otimes - : \mathcal{S} \to \mathcal{W}$, where $1_{\mathcal{W}} \otimes -$ denotes the unique SM functor preserving colimits (and likewise for $\mathcal{V}$). In particular, the right adjoint $\mathrm{Alg}_{\mathrm{Cat}}(G)\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{W}) \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ detects completeness of categorical algebras, as well as essential surjectivity and being an isofibration for maps between these.

5.2 The cube and tower lemmas

Throughout, let $\mathcal{V}$ be a monoidal ∞-category. The purpose of this section is to record two kinds of ("homotopy") limits of categorical algebras that are preserved by the completion functor $(-)^\wedge : \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \to \mathrm{Cat}(\mathcal{V})$.

The results and arguments are very analogous to the usual way of computing homotopy limits of categories in terms of the canonical model structure on categories.

Lemma 5.9 *Consider a commutative square of categorical algebras*

$$\begin{tikzcd} \mathbb{C}' \arrow[r, "g'"] \arrow[d, "p"'] & \mathbb{D}' \arrow[d, "q"] \\ \mathbb{C} \arrow[r, "g"'] & \mathbb{D} \end{tikzcd} \tag{5-3}$$

such that g' is essentially surjective, g is fully faithful and p is an isofibration. Then

$$\mathrm{Map}(\{0\}, \mathbb{C}') \underset{\mathrm{Map}(\{0\},\mathbb{D}')}{\times} \mathrm{Map}(J, \mathbb{D}') \to \mathrm{Map}(\{0\}, \mathbb{C}) \underset{\mathrm{Map}(\{0\},\mathbb{D})}{\times} \mathrm{Map}(J, \mathbb{D}) \underset{\mathrm{Map}(\{1\},\mathbb{D})}{\times} \mathrm{Map}(\{1\}, \mathbb{D}')$$

is surjective on path components.

Informally, this means that for any object $d \in \mathbb{D}'$, each lift-up-to-equivalence of $q(d)$ to $\mathbb{C}$ refines to a lift-up-to-equivalence of d to $\mathbb{C}'$.

Proof By [9, Proposition 5.1.11], we may as well assume that $\mathcal{V} = \mathcal{S}$. Explicitly, suppose that we are given objects $c \in \mathbb{C}$ and $d' \in \mathbb{D}'$ together with an equivalence $\alpha\colon g(c) \xrightarrow{\sim} d = q(d')$ in $\mathbb{D}$, that is, a map from J. We then need to find an object $c' \in \mathbb{C}'$ lying over c and an equivalence $\alpha'\colon g'(c') \xrightarrow{\sim} d'$ in $\mathbb{D}'$ lying over α.

To begin, g' being essentially surjective provides an object $t' \in \mathbb{C}'$ and an equivalence $\gamma'\colon g'(t') \xrightarrow{\sim} d'$ in $\mathbb{D}'$. One can then complete $q(\gamma')$ and α to a commutative triangle

$$\begin{tikzcd} & q(g'(t')) \arrow[dr, "q(\gamma')"] & \\ g(c) \arrow[ur, "\delta"] \arrow[rr, "\alpha"] & & d \end{tikzcd} \tag{5-4}$$

in $\mathbb{D}$ for some equivalence $\delta\colon g(c) \to q(g'(t'))$. Using the commuting square (5-3), we can identify $q(g'(t')) \simeq g(p(t'))$ in the space of objects of $\mathbb{D}$. Because g is fully faithful, every map from $g(c) \to g(p(t'))$ lifts to a unique map $c \to p(t')$ in $\mathbb{C}$, so that there exists an equivalence $\varepsilon\colon c \to p(t')$ lying over δ. Since p is an isofibration we may lift ε to an equivalence $\varepsilon'\colon c' \to t'$ for some $c' \in \mathbb{C}'$ lying over c. We may then complete $g'(\varepsilon')$ and γ' to a commutative diagram

$$\begin{tikzcd} & g'(t') \arrow[dr, "\gamma'"] & \\ g'(c') \arrow[ur, "g'(\varepsilon')"] \arrow[rr, "\alpha'"] & & d' \end{tikzcd} \tag{5-5}$$

for some equivalence $\alpha'\colon g'(c') \to d'$ in $\mathbb{D}'$, since $\mathrm{Map}(J[-], \mathbb{C})$ is a Segal groupoid object. Because the image of triangle (5-5) under $q\colon \mathbb{D}' \to \mathbb{D}$ agrees with triangle (5-4) on the inner horn, it follows that $q(\alpha')$ is homotopic to α. This yields the desired data of c' and $\alpha'\colon g'(c') \to d'$ so that the proof is complete. □

Lemma 5.10 (cube lemma) *Consider a map of Cartesian squares in* $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$

$$\begin{tikzcd} \mathbb{P} \arrow[r] \arrow[d] & \mathbb{C}' \arrow[d, "p"] & & \mathbb{Q} \arrow[r] \arrow[d] & \mathbb{D}' \arrow[d, "q"] \\ \mathbb{C}'' \arrow[r, "h"'] & \mathbb{C} \arrow[rr, phantom, "\xRightarrow{\begin{pmatrix} f & g' \\ g'' & g \end{pmatrix}}"] & & \mathbb{D}'' \arrow[r, "k"'] & \mathbb{D} \end{tikzcd} \tag{5-6}$$

such that p *is an isofibration. If the components* $g\colon \mathbb{C} \to \mathbb{D}$, $g'\colon \mathbb{C}' \to \mathbb{D}'$ *and* $g''\colon \mathbb{C}'' \to \mathbb{D}''$ *are Dwyer–Kan equivalences, then the same holds for* $f\colon \mathbb{P} \to \mathbb{Q}$.

Corollary 5.11 *The completion functor* $(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \to \mathrm{Cat}(\mathcal{V})$ *sends pullback squares with one leg being an isofibration to pullback squares.*

Proof Apply Lemma 5.10 to the case where the maps g, g' and g'' exhibit $\mathbb{D}, \mathbb{D}'$ and $\mathbb{D}''$ as the completions of $\mathbb{C}, \mathbb{C}'$ and $\mathbb{C}''$ respectively (in this case $\mathbb{Q} \simeq \mathbb{D}' \times_{\mathbb{D}} \mathbb{D}''$ is automatically complete). □

Proof of Lemma 5.10 To show that f is fully faithful, let $x, y \in \mathbb{P}$ be two objects, and consider the induced map of squares

$$\begin{tikzcd} \mathrm{Map}_{\mathbb{P}}(x,y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{C}'}(x,y) \arrow[d, "p_*"] & & \mathrm{Map}_{\mathbb{Q}}(x,y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{D}'}(x,y) \arrow[d, "q_*"] \\ \mathrm{Map}_{\mathbb{C}''}(x,y) \arrow[r] & \mathrm{Map}_{\mathbb{C}}(x,y) \arrow[rr, phantom, "\Longrightarrow"] & & \mathrm{Map}_{\mathbb{D}''}(x,y) \arrow[r] & \mathrm{Map}_{\mathbb{D}}(x,y) \end{tikzcd}$$

Both squares are Cartesian in $\mathcal{V}$ and by assumption the three maps associated to g, g' and g'' are equivalences, so that the map $f_*\colon \mathrm{Map}_{\mathbb{P}}(x,y) \to \mathrm{Map}_{\mathbb{Q}}(x,y)$ is an equivalence as well.

Let us now show that f is essentially surjective as a map of categorical algebras. Essential surjectivity is detected on the level of the underlying space-valued categorical algebras [9, Proposition 5.1.11]. We may hence assume that $\mathcal{V} = \mathcal{S}$. Let $y \in \mathbb{Q}$ be an object and let $d' \in \mathbb{D}'$, $d \in \mathbb{D}$, $d'' \in \mathbb{D}''$ be its images. Since $g''\colon \mathbb{C}'' \to \mathbb{D}''$ is essentially surjective there exists an object $c'' \in \mathbb{C}''$ and an equivalence $\alpha''\colon g''(c'') \xrightarrow{\sim} d''$ in $\mathbb{D}''$. Let $c := h(c'') \in \mathbb{C}$. Applying Lemma 5.9 to the image

$$\alpha\colon g(c) \simeq k(g''(c'')) \xrightarrow{\sim} k(d'') \simeq d \simeq q(d')$$

of α'' in $\mathbb{D}$ we deduce the existence of an object $c' \in \mathbb{C}'$ lying over c and an equivalence $\alpha'\colon g'(c') \to d'$ in $\mathbb{D}'$ lying over α. The compatible triple (c, c', c'') now determines an object $x \in \mathbb{P}$ while the compatible triple $(\alpha, \alpha', \alpha'')$ determines an equivalence $g(x) \xrightarrow{\sim} y$ in $\mathbb{Q}$. □

Lemma 5.12 (tower lemma) *Consider a natural transformation between limit cones in* $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$

$$\begin{tikzcd} \mathbb{P} \arrow[r] \arrow[d, "f"'] & \cdots \arrow[r] & \mathbb{C}_2 \arrow[r, "p_2"] \arrow[d, "g_2"'] & \mathbb{C}_1 \arrow[r, "p_1"] \arrow[d, "g_1"'] & \mathbb{C}_0 \arrow[d, "g_0"] \\ \mathbb{Q} \arrow[r] & \cdots \arrow[r] & \mathbb{D}_2 \arrow[r, "q_2"'] & \mathbb{D}_1 \arrow[r, "q_1"'] & \mathbb{D}_0 \end{tikzcd}$$

Suppose that all p_i for $i \geq 1$ are isofibrations and all g_i for $i \geq 0$ are Dwyer–Kan equivalences. Then f is a Dwyer–Kan equivalence as well.

Corollary 5.13 *The completion functor* $(-)^\wedge : \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \to \mathrm{Cat}(\mathcal{V})$ *sends limits of towers of isofibrations to limits.*

Proof Apply Lemma 5.12 to the case where the maps g_i exhibit $\mathbb{D}_i$ as the completion of $\mathbb{C}_i$ (in which case $\mathbb{Q}$ is automatically complete). □

Proof of Lemma 5.12 To show that f is fully faithful, let $x, y \in \mathbb{P}$ be two objects, and consider the induced map of towers

$$\begin{tikzcd} \mathrm{Map}_{\mathbb{P}}(x,y) \arrow[r] \arrow[d, "f_*"'] & \cdots \arrow[r] & \mathrm{Map}_{\mathbb{C}_1}(x,y) \arrow[r] \arrow[d, "(g_1)_*"'] & \mathrm{Map}_{\mathbb{C}_0}(x,y) \arrow[d, "(g_0)_*"] \\ \mathrm{Map}_{\mathbb{Q}}(x,y) \arrow[r] & \cdots \arrow[r] & \mathrm{Map}_{\mathbb{D}_1}(x,y) \arrow[r] & \mathrm{Map}_{\mathbb{D}_0}(x,y) \end{tikzcd}$$

Then both towers are limit towers in $\mathcal{V}$ and by assumption the $(g_i)_*$ are equivalences, so that the map $f_* : \mathrm{Map}_{\mathbb{P}}(x,y) \to \mathrm{Map}_{\mathbb{Q}}(x,y)$ is an equivalence as well.

Let us now show that f is essentially surjective as a map of categorical algebras. We may again assume that $\mathcal{V} = \mathcal{S}$ [9, Proposition 5.1.11]. Let $y \in \mathbb{Q}$ be an object and let $d_i \in \mathbb{D}_i$ be its images. Since $g_0 : \mathbb{C}_0 \to \mathbb{D}_0$ is essentially surjective, there exists an object $c_0 \in \mathbb{C}_0$ and an equivalence $\alpha_0 : g_0(c_0) \xrightarrow{\sim} d_0$ in $\mathbb{D}_0$. Applying Lemma 5.9 to α_0 and $d_1 \in \mathbb{D}_1$, we deduce the existence of an object $c_1 \in \mathbb{C}_1$ lying over c_0 and an equivalence $\alpha_1 : g_1(c_1) \to d_1$ in $\mathbb{D}_1$ lying over α_1. Proceeding inductively, we obtain compatible sequences of objects $c_i \in \mathbb{C}_i$ and equivalences $\alpha_i : g_i(c_i) \to d_i$. These determine an object $x \in \mathbb{P}$ and an equivalence $g(x) \xrightarrow{\sim} y$ in $\mathbb{Q}$. □

5.3 Postnikov structures on enriched ∞-categories

We now turn to our main result, providing Postnikov structures on $\mathcal{V}$-enriched ∞-categories from (certain) multiplicative Postnikov structures on $\mathcal{V}$.

Definition 5.14 Let $\mathcal{V}$ be an SM ∞-category. We will say that a map $f : X \to Y$ in $\mathcal{V}$ is an *external π_0-isomorphism* if the induced map of spaces $\mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, X) \to \mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, Y)$ induces an isomorphism on π_0.

We will say that a Postnikov structure on an object $T : \mathcal{E} \to \mathcal{V}$ is *externally π_0-constant* if it sends every map in $\mathcal{E}$ to an external π_0-isomorphism in $\mathcal{V}$. A Postnikov structure $\Phi : \mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ on $\mathcal{V}$ is externally π_0-constant if it sends each object $X \in \mathcal{V}$ to an externally π_0-constant Postnikov structure on X.

Remark 5.15 If $T \in \mathrm{PoStr}(\mathcal{V})$ is externally π_0-constant then each $K_a \in \mathcal{T}\mathcal{V}$ has the property that the induced parametrized spectrum $E_a = \mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, K_a) \in \mathcal{T}\mathcal{S}$ is 0-connected (or 1-connective), ie each fiber is a 1-connective spectrum. Indeed, for each $n \geq 0$, the map $E_a(S^n) = \Omega^{\infty-n}(E_a) \to \pi(E_a)$ induces an isomorphism on π_0 by assumption and a surjection on π_1 since it admits a section, so that its fibers are all connected. It follows that the fiber $\Omega^{\infty-n}(E_a)_x$ is the connected delooping of $\Omega^{\infty-n+1}(E_a)_x$ for each $n \geq 0$ and $E_{a,x}$ is a 1-connective spectrum.

Example 5.16 The usual Postnikov structure on spaces (Example/Proposition 4.15) is externally π_0-constant: for every space X, the resulting Postnikov structure is even constant after applying $\tau_{\leq 1}$. For more general ∞-toposes (Example 4.17), the Postnikov structure is typically *not* externally π_0-constant: even though all maps induce isomorphisms on π_0-sheaves, on global sections they typically do not induce bijections on π_0. For example, for any finite CW-complex X, abelian group A and $n \geq 2$, the map of constant sheaves $\mathcal{K}(A,n) \to \tau_{\leq n-1}\mathcal{K}(A,n) \simeq *$ in $\mathrm{Sh}_\infty(X)$ induces an isomorphism on 1-truncations, but at the level of π_0 of the global sections we obtain $\mathrm{H}^n(X;A) \to *$, which need not be an isomorphism.

Example 5.17 The canonical Postnikov structure on $\mathrm{Sp}^{\geq 0}$ is externally π_0-constant. Indeed, for each $E \in \mathrm{Sp}^{\geq 0}$, the image of its Postnikov structure under $\mathrm{Map}_{\mathrm{Sp}^{\geq 0}}(\mathbb{S}, -)$ is simply the Postnikov structure on the space $\Omega^\infty(E)$, but extended down to dimension 0, see Example 4.19. All spaces appearing in the Postnikov structure for $\Omega^\infty(E)$ have isomorphic π_0.

More generally, let $\mathcal{V}$ be a stable, presentable SM ∞-category with a left complete t-structure such that the connective part $\mathcal{V}^{\geq 0}$ is closed under finite tensor products. If the mapping spectrum functor

$$\mathrm{Map}(1_{\mathcal{V}}, -)\colon \mathcal{V} \to \mathrm{Sp}$$

sends $\mathcal{V}^{\geq 0}$ to $\mathrm{Sp}^{\geq 0}$, then the Postnikov structure on $\mathcal{V}^{\geq 0}$ from Example/Proposition 4.23 is externally π_0-constant. This is notably the case when $\mathcal{V} = \mathrm{Mod}_R$ with R a connective ring spectrum (with the usual t-structure).

Theorem 5.18 *Let $\mathcal{V}$ be an SM ∞-category equipped with a multiplicative Postnikov structure*

$$\Phi\colon \mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V}).$$

If Φ is externally π_0-constant, then the composite

$$\mathrm{Cat}(\mathcal{V}) \subseteq \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \xrightarrow{\Phi_*} \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{E}, \mathcal{V})) \to \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \xrightarrow{(-)^\wedge} \mathrm{Fun}(\mathcal{E}, \mathrm{Cat}(\mathcal{V}))$$

defines a multiplicative Postnikov structure Φ_{Cat} on $\mathrm{Cat}(\mathcal{V})$. Furthermore, this Postnikov structure Φ_{Cat} is itself externally π_0-constant.

Slightly informally (ie up to Dwyer–Kan equivalence), the Postnikov structure $\Phi_{\mathrm{Cat}}(\mathbb{C})$ of a $\mathcal{V}$-enriched ∞-category $\mathbb{C}$ is obtained by applying Φ to all mapping objects in $\mathbb{C}$. To see that this still yields a Postnikov structure after completion, we will make use of the cube and tower lemmas (Lemmas 5.10 and 5.12), for which we will need Φ to be externally π_0-constant.

Theorem 5.18 can be applied repeatedly:

Definition 5.19 Let $\mathcal{V}$ be a presentable SM ∞-category. Then the presentable SM ∞-category of *$\mathcal{V}$-enriched (∞, n)-categories* is defined inductively as

$$\mathrm{Cat}_n(\mathcal{V}) := \mathrm{Cat}(\mathrm{Cat}_{n-1}(\mathcal{V})).$$

For later purposes, let us record the following:

Lemma 5.20 *Let $\mathcal{V}$ and $\mathcal{W}$ be presentable SM ∞-categories and $L\colon \mathcal{V} \rightleftarrows \mathcal{W} :\iota$ a reflective (symmetric) monoidal localization. This induces a reflective monoidal localization of presentable SM ∞-categories*

$$\mathrm{Cat}_n(L)\colon \mathrm{Cat}_n(\mathcal{V}) \rightleftarrows \mathrm{Cat}_n(\mathcal{W})\colon \mathrm{Cat}_n(\iota).$$

The essential image of ι_* consists of those $\mathcal{V}$-enriched (∞, n)-categories whose mapping objects are (in the essential image of) $\mathcal{W}$-enriched $(\infty, n-1)$-categories.

Proof An inductive application of [9, Corollary 5.7.12, Proposition 5.7.16] provides the presentable SM structure on $\mathrm{Cat}_n(\mathcal{V})$ and $\mathrm{Cat}_n(\mathcal{W})$. The induced reflective monoidal localization arises from an inductive application of [9, Proposition 5.7.18]. □

An inductive application of Theorem 5.18 then immediately yields the following:

Corollary 5.21 *Let $\mathcal{V}$ be a presentable SM ∞-category equipped with a multiplicative, externally π_0-constant Postnikov structure $\Phi\colon \mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$. Then there is an induced multiplicative Postnikov structure*

$$\Phi_{\mathrm{Cat}_n}\colon \mathrm{Cat}_n(\mathcal{V})^{\otimes} \to \mathrm{PoStr}(\mathrm{Cat}_n(\mathcal{V}))^{\otimes}$$

on the ∞-category of $\mathcal{V}$-enriched (∞, n)-categories. Explicitly, Φ_{Cat_n} is given (up to n-categorical Dwyer–Kan equivalence) by applying Φ to objects of n-morphisms.

We will now turn to the proof of Theorem 5.18. Our strategy will be to first prove a version of Theorem 5.18 for categorical algebras and then descend to enriched ∞-categories. The case of categorical algebras follows readily from the following observation:

Lemma 5.22 *Let $\mathcal{I}$ be a weakly contractible ∞-category, $\mathcal{I}^{\triangleleft}$ its cone and $\mathcal{V}$ a monoidal ∞-category with $\mathcal{I}$-indexed limits. Consider the natural functor*

$$\phi\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{V})) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{S})} \mathcal{S} \to \mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})),$$

where the first functor is the natural equivalence from Proposition 5.6. If $\mathbb{C}$ is a categorical algebra in $\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{V})$ whose mapping objects belong to the full subcategory of limit cones, then

$$\phi(\mathbb{C})\colon \mathcal{I}^{\triangleleft} \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$$

is a limit cone as well.

Proof By naturality, the equivalences of Proposition 5.6 fit into a commuting square where the horizontal functors restrict along $\mathcal{J} \hookrightarrow \mathcal{J}^{\triangleleft}$

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathcal{V})) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{J}, \mathcal{V})) \arrow[d, "\simeq"] \\
\mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathcal{S})} \mathcal{S} \arrow[r] & \mathrm{Fun}(\mathcal{J}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{J}, \mathcal{S})} \mathcal{S}
\end{tikzcd}$$

This induces a commuting square between the right adjoints of the two horizontal functors. The top right adjoint is a fully faithful embedding whose essential image consists precisely of categorical algebras enriched over limit cones. To compute the bottom right adjoint, consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathcal{S}) \arrow[d] & \mathcal{S} \arrow[l] \arrow[d, equal] \\
\mathrm{Fun}(\mathcal{J}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \arrow[r] & \mathrm{Fun}(\mathcal{J}, \mathcal{S}) & \mathcal{S} \arrow[l]
\end{tikzcd}$$

where the vertical functors restrict along $\mathcal{J} \to \mathcal{J}^{\triangleleft}$. The horizontal functors then commute with the right adjoints to the restriction functors as well: for the left square, this uses that the forgetful functor $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \to \mathcal{S}$ preserves limits, and for the right square, this uses that $\mathcal{J}$ is contractible, so that constant $\mathcal{J}^{\triangleleft}$-diagrams are limit cones. The right adjoint to $\mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathcal{S})} \mathcal{S} \to \mathrm{Fun}(\mathcal{J}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{J}, \mathcal{S})} \mathcal{S}$ is then the fiber product of the three right adjoint functors [26, Corollary 4.7.4.18]. In particular, the projection onto the first factor commutes with these right adjoints. It follows that the composite ϕ intertwines the right adjoints to restriction along $\mathcal{J} \hookrightarrow \mathcal{J}^{\triangleleft}$, which yields the result. □

Proof of Theorem 5.18 We have to verify that Φ_{Cat} is a lax symmetric monoidal section of the (lax) symmetric monoidal functor $\mathrm{ev}_\infty \colon \mathrm{Fun}(\mathcal{E}, \mathrm{Cat}(\mathcal{V})) \to \mathcal{V}$, and that it takes values in the full subcategory $\mathrm{PoStr}(\mathrm{Cat}(\mathcal{V})) \subseteq \mathrm{Fun}(\mathcal{E}, \mathrm{Cat}(\mathcal{V}))$ of Postnikov structures.

For the first assertion, consider the commuting diagram

$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{V}) \arrow[r, hook] \arrow[d, equal] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "\Phi_*"] \arrow[d, equal] & \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{E}, V)) \arrow[r, "\varphi"] \arrow[d, "\mathrm{ev}_\infty"'] & \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \arrow[r, "(-)^\wedge"] \arrow[d, "\mathrm{ev}_\infty"] & \mathrm{Fun}(\mathcal{E}, \mathrm{Cat}(\mathcal{V})) \arrow[d, "\mathrm{ev}_\infty"] \\
\mathrm{Cat}(\mathcal{V}) \arrow[r, hook] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, equal] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, equal] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "(-)^\wedge"'] & \mathrm{Cat}(\mathcal{V})
\end{tikzcd}$$

All arrows are lax SM functors. For the first and last horizontal arrows (in both rows), this follows from [9, Proposition 5.7.14] and for Φ_*, this follows from Proposition 5.3. The functor φ is the composite of the SM equivalence from Proposition 5.6 and the SM projection

$$\mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{S})} \mathcal{S} \to \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})).$$

Furthermore, the second square commutes since Φ_* is a multiplicative Postnikov structure and all other squares commute by naturality in the ∞-category $\mathcal{E}$. This provides a lax SM section of ev_∞ because the composition of the bottom row is naturally equivalent to the identity via the (lax SM) counit of the monoidal adjunction $(-)^\wedge \colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \rightleftarrows \mathrm{Cat}(\mathcal{V}) : \iota$ [9, Proposition 5.7.14].

For the second assertion, it suffices to verify that for a $\mathcal{V}$-enriched category $\mathbb{C}$, its image

$$\mathbb{T}^\wedge := \Phi_{\mathrm{Cat}}(\mathbb{C}) \colon \mathcal{E} \to \mathrm{Cat}(\mathcal{V})$$

defines a Postnikov structure in $\mathrm{Cat}(\mathcal{V})$. By construction, $\mathbb{T}^\wedge$ is the levelwise completion of the diagram which applies the multiplicative Postnikov structure Φ to all mapping objects

$$\mathbb{T} := \phi(\Phi_*(\mathbb{C})) \colon \mathcal{E} \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}).$$

For each of the cone diagrams $\mathcal{J}_\alpha^{\lhd} \to \mathcal{E}$ from Remark 4.7 (with $\mathcal{J}_\alpha$ contractible), $\mathbb{T}|_{\mathcal{J}^{\lhd}}$ is a limit cone in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ by Lemma 5.22. Consequently, $\mathbb{T}$ defines a Postnikov structure in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$. We have to show that these cones remain limit cones upon applying the completion functor objectwise. This will follow from Corollaries 5.11 and 5.13 once we verify that $\mathbb{T}$ sends every arrow in $\mathcal{E}$ to an isofibration.

To this end, consider a map $\mathbb{T}(i) \to \mathbb{T}(j)$ of categorical algebras induced by $i \to j$ in $\mathcal{E}$. By construction, f induces the identity on spaces of objects and for every two objects $x, y \in \mathbb{T}(i)$, the map

$$\mathrm{Map}(1_{\mathcal{V}}, \mathrm{Map}_{\mathbb{T}(i)}(x, y)) \to \mathrm{Map}(1_{\mathcal{V}}, \mathrm{Map}_{\mathbb{T}(j)}(x, y))$$

is a π_0-isomorphism since the Postnikov structure Φ is π_0-constant (Definition 5.14). The condition that $\mathbb{T}(i) \to \mathbb{T}(j)$ is an isofibration is determined at the level of the underlying $\mathcal{S}$-enriched categorical algebras, so we may as well assume that $\mathcal{V} = \mathcal{S}$. For every object x in $\mathbb{T}(i)$ and every arrow $\alpha \colon x \to y$ in $\mathbb{T}(j)$, we then have a lift of α to a map $\tilde{\alpha} \colon x \to y$ in $\mathbb{T}(i)$ (note that $\mathbb{T}(i) \to \mathbb{T}(j)$ is the identity on objects), which is furthermore unique up to homotopy. If α is an equivalence, then $\tilde{\alpha}$ is an equivalence by [9, Proposition 5.1.15]: indeed, using that such lifts of arrows to $\mathbb{T}(i)$ are unique up to homotopy, a homotopy inverse of $\tilde{\alpha}$ is provided by a lift of the homotopy inverse of α to $\mathbb{T}(i)$. We conclude that $\mathbb{T}(i) \to \mathbb{T}(j)$ is an isofibration.

Finally, we have to verify that $\mathbb{T}^\wedge$ is a π_0-constant Postnikov structure. The monoidal unit of $\mathrm{Cat}(\mathcal{V})$ is the completion $1_{\mathrm{Cat}(\mathcal{V})} = [0]^\wedge_{1_{\mathcal{V}}}$ of the unit object of $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, given by the operad map $\mathcal{O}_* \simeq \mathrm{Ass} \to \mathcal{V}^\otimes$ encoding the unit (ie initial) associative algebra $1_{\mathcal{V}}$ in $\mathcal{V}$. In particular, the functor

$$\mathrm{Map}_{\mathrm{Cat}(\mathcal{V})}(1_{\mathrm{Cat}(\mathcal{V})}, -) \colon \mathrm{Cat}(\mathcal{V}) \to \mathcal{S}$$

is equivalent to the functor sending a $\mathcal{V}$-enriched ∞-category to its space of objects. To see that $\mathbb{T}^\wedge$ is π_0-constant, it therefore suffices to verify that the diagram of object spaces

$$\mathrm{Ob}(\mathbb{T}^\wedge) \colon \mathcal{E} \xrightarrow{\mathbb{T}^\wedge} \mathrm{Cat}(\mathcal{V}) \xrightarrow{\mathrm{Ob}} \mathcal{S}$$

sends each map $i \to j$ in $\mathcal{E}$ to a map with 0-connected fibers (in particular, a π_0-isomorphism). Let us pick an object $x \in \mathbb{T}^\wedge(j)$ and verify that the fiber $\mathrm{Ob}(\mathbb{T}^\wedge(i))_x$ is connected. The object x determines a map $x \colon [0]_{1_{\mathcal{V}}} \to 1_{\mathrm{Cat}(\mathcal{V})} \to \mathbb{T}^\wedge(j)$. Since the functor $\mathbb{T}(j) \to \mathbb{T}^\wedge(j)$ induces a π_0-surjection on objects by [9, Theorem 5.6.2], this composite map factors over $\mathbb{T}(j)$. Taking pullbacks along these maps, we obtain a commuting square of categorical algebras

$$\begin{tikzcd}
\mathbb{T}(i) \times_{\mathbb{T}(j)} [0]_{1_{\mathcal{V}}} \arrow[r] \arrow[d] & \mathbb{T}^\wedge(i) \times_{\mathbb{T}^\wedge(j)} 1_{\mathrm{Cat}(\mathcal{V})} \arrow[d] \\
{[0]_{1_{\mathcal{V}}}} \arrow[r] & 1_{\mathrm{Cat}(\mathcal{V})}
\end{tikzcd}$$

Since $\mathbb{T}(i) \to \mathbb{T}(j)$ was an isofibration, the top and bottom horizontal maps are Dwyer–Kan equivalences (cube lemma, ie Lemma 5.10) and the left vertical map is an isofibration. Lemma 5.9, together with the fact that the right column consists of *complete* categorical algebras, then implies that $\mathbb{T}(i) \times_{\mathbb{T}(j)} [0]_{1_{\mathcal{V}}} \to \mathbb{T}^\wedge(i) \times_{\mathbb{T}^\wedge(j)} [0]_{1_{\mathcal{V}}}$ induces a π_0-surjection on spaces of objects. Since $\mathbb{T}(i) \to \mathbb{T}(j)$ is constant on objects, the domain of this map has a contractible space of objects. Consequently, $\mathrm{Ob}(\mathbb{T}^\wedge(i) \times_{\mathbb{T}^\wedge(j)} [0]_{1_{\mathcal{V}}}) \simeq \mathrm{Ob}(\mathbb{T}^\wedge(i))_x$ is connected, as desired. □

6 Local systems on (∞, n)-categories

In this section, we spell out the contents of Theorem 5.18 and Corollary 5.21 in the setting of (∞, n)-categories. There are many equivalent models for the ∞-category of (∞, n)-categories, one of which is the ∞-category $\mathrm{Cat}_n(\mathcal{S})$ of $\mathcal{S}$-enriched (∞, n)-categories of Definition 5.19 [17, Corollary 7.21]. This model is particularly well-adapted to definitions that proceed by induction on mapping objects, such as the following:

Definition 6.1 An $(m, 0)$-category is defined to be an m-truncated space. For any $0 \leq n \leq m$, an (∞, n)-category $\mathcal{C}$ is called an (m, n)-*category* if each mapping $(\infty, n-1)$-category is an $(m-1, n-1)$-category.

In light of [9, Corollary 6.1.10], this coincides with [9, Definition 6.1.1].

Lemma 6.2 *The fully faithful inclusion $\iota\colon \mathrm{Cat}_{(m,n)} \subseteq \mathrm{Cat}_{(\infty,n)}$ of the full subcategory of (m, n)-categories admits a left adjoint sending an (∞, n)-category $\mathcal{C}$ to its* ***homotopy (m, n)-category*** $\mathrm{ho}_{(m,n)}\,\mathcal{C}$. *The adjoint pair $\mathrm{ho}_{(m,n)}\colon \mathrm{Cat}_{(\infty,n)} \rightleftarrows \mathrm{Cat}_{(m,n)} :\iota$ has the canonical structure of a reflective monoidal localization.*

Proof Consider the reflective localization $\tau_{\leq m-n}\colon \mathcal{S} \rightleftarrows \tau_{\leq m-n}\mathcal{S} :\iota$. Since truncation preserve products, this is a reflective monoidal localization. Consequently, it induces a reflective monoidal localization $\mathrm{Cat}_n(\tau_{\leq m-n})\colon \mathrm{Cat}_n(\mathcal{S}) \rightleftarrows \mathrm{Cat}_n(\tau_{\leq m-n}\mathcal{S}) :\mathrm{Cat}_n(\iota)$ by Lemma 5.20. Note that (by induction) the essential image of $\mathrm{Cat}_n(\iota)$ is precisely the full subcategory $\mathrm{Cat}_{(m,n)} \subseteq \mathrm{Cat}_{(\infty,n)}$. It follows that there is a left adjoint $\mathrm{ho}_{(m,n)}\colon \mathrm{Cat}_{(\infty,n)} \to \mathrm{Cat}_{(m,n)}$ (equivalent to $\mathrm{Cat}_n(\tau_{\leq m-n})$), which has the canonical structure of a monoidal localization. □

The universal properties of the homotopy (m, n)-categories imply that they fit into a tower

(6-1) $$\mathcal{C} \to \cdots \to \mathrm{ho}_{(m,n)}(\mathcal{C}) \to \mathrm{ho}_{(m-1,n)}(\mathcal{C}) \to \cdots \to \mathrm{ho}_{(n+1,n)}(\mathcal{C}).$$

By induction on n, one sees that this tower is convergent. Indeed, using that $\mathrm{Cat}_{(\infty,n+1)} \subseteq \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Cat}_{(\infty,n)})$ preserves limits, this follows from the fact that:

(a) At the level of spaces of objects, the tower induces isomorphisms on π_0 so that $\mathcal{C} \to \lim_m \mathrm{ho}_{(m,n+1)}(\mathcal{C})$ is essentially surjective.

(b) The map $\mathrm{Map}_{\mathcal{C}}(c,d) \to \lim_m \mathrm{Map}_{\mathrm{ho}_{(m,n+1)}(\mathcal{C})}(c,d) \simeq \lim_m \mathrm{ho}_{(m,n)}(\mathrm{Map}_{\mathcal{C}}(c,d))$ is an equivalence for each $c, d \in \mathcal{C}$ by inductive hypothesis.

Theorem 5.18 and Corollary 5.21 then yield the following more precise statement of Theorem 1.1:

Theorem 6.3 *For each $n \geq 1$, the tower of natural transformations (6-1) refines to a multiplicative Postnikov structure on $\mathrm{Cat}_{(\infty,n)}$.*

Proof By Lemma 6.2, the reflective monoidal localization

$$\mathrm{ho}_{(m,n)}\colon \mathrm{Cat}_{(\infty,n)} = \mathrm{Cat}_n(\mathcal{S}) \underset{\mathrm{Cat}_n(\iota)}{\overset{\mathrm{Cat}_n(\tau_{\leq m-n})}{\rightleftarrows}} \mathrm{Cat}_n(\mathcal{S}_{\leq m-n}) = \mathrm{Cat}_{(m,n)} :\iota$$

arises from the reflective monoidal localization $\tau_{\leq m-n}\colon \mathcal{S} \underset{\perp}{\rightleftarrows} \mathcal{S}_{\leq m-n} :\iota$ via Lemma 5.20. Now let $\Phi\colon \mathcal{S} \to \mathrm{Fun}(\mathcal{E},\mathcal{S})$ be the multiplicative, π_0-constant Postnikov structure on spaces refining the classical Postnikov tower (Example/Proposition 4.15). By Theorem 5.18 and Corollary 5.21, this induces a multiplicative Postnikov structure Φ_{Cat_n} on $\mathrm{Cat}_{(\infty,n)}$.

We can view Φ_{Cat_n} as a diagram $\mathcal{E} \to \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{Cat}_{(\infty,n)}, \mathrm{Cat}_{(\infty,n)})$ of (lax symmetric monoidal) endofunctors of $\mathrm{Cat}_{(\infty,n)}$, by adjunction with the Boardman–Vogt tensor product (see Remark 2.7). Forgetting about the k-invariants, the underlying tower of Φ_{Cat_n} is given by the tower of functors

$$\mathrm{id} \to \cdots \to \mathrm{Cat}_n(\tau_{\leq a}) \to \mathrm{Cat}_n(\tau_{\leq a-1}) \to \cdots \to \mathrm{Cat}_n(\tau_{\leq 1}).$$

Lemma 6.2 identifies this with the natural tower of homotopy (m,n)-categories (6-1), as desired. □

In other words, for each (∞,n)-category $\mathcal{C}$ and $a \geq 2$, there exists a natural parametrized spectrum object

$$\mathrm{H}\pi_a(\mathcal{C}) \in \mathcal{T}_{\mathrm{ho}_{(n+1,n)}(\mathcal{C})}(\mathrm{Cat}_{(\infty,n)})$$

and a pullback square of (∞,n)-categories

$$\begin{tikzcd}
\mathrm{ho}_{(n+a,n)}\,\mathcal{C} \arrow[r] \arrow[d] & \mathrm{ho}_{(n+1,n)}\,\mathcal{C} \arrow[d, "0"] \\
\mathrm{ho}_{(n+a-1,n)}\,\mathcal{C} \arrow[r, "k_a"'] & \Omega^\infty(\Sigma^{a+1}\mathrm{H}\pi_a(\mathcal{C}))
\end{tikzcd}$$

The proof of Theorem 6.3 is not completely satisfying because these parametrized spectra $\mathrm{H}\pi_a(\mathcal{C})$ are defined somewhat implicitly. In the remainder of this section, we will explain how (as the notation suggests) the parametrized spectra $\mathrm{H}\pi_a(\mathcal{C})$ can be considered as the Eilenberg–MacLane spectra associated to *local systems of abelian groups* on $\mathrm{ho}_{(n+1,n)}(\mathcal{C})$, as considered in [24].

6.1 Tangent bundle of enriched ∞-categories

Our first goal will be to construct a t-orientation (Definition 2.20) on the tangent bundle to $\mathcal{V}$-enriched ∞-categories, using a version of Proposition 2.25. To this end, we will need a description of the tangent bundle to enriched ∞-categories along the lines of Proposition 2.17:

Theorem 6.4 *Let $\mathcal{V}$ be a differentiable presentable SM ∞-category such that $1_{\mathcal{V}}$ is compact. Then there exists a natural equivalence of SM ∞-categories*

$$\begin{tikzcd} \mathrm{Cat}(\mathcal{T}\mathcal{V}) \arrow[rr, "\mathcal{L}", "\sim"'] \arrow[dr, "\mathrm{Cat}(\pi_{\mathcal{V}})"'] & & \mathcal{T}\,\mathrm{Cat}(\mathcal{V}) \arrow[dl, "\pi"] \\ & \mathrm{Cat}(\mathcal{V}) & \end{tikzcd} \tag{6-2}$$

where $\mathrm{Cat}(\pi_{\mathcal{V}})$ is induced by the (monoidal) tangent projection $\pi_{\mathcal{V}}\colon \mathcal{T}\mathcal{V} \to \mathcal{V}$ and π is the tangent projection for $\mathrm{Cat}(\mathcal{V})$.

Recall from [26, Definition 6.1.1.6] that a presentable ∞-category $\mathcal{V}$ is differentiable if the sequential colimit functor $\mathrm{colim}\colon \mathrm{Fun}(\mathbb{N}, \mathcal{V}) \to \mathcal{V}$ is left exact. In particular, any compactly generated ∞-category is differentiable. To apply Theorem 6.4 inductively, let us record the following observation:

Lemma 6.5 *Let $\mathcal{V}$ be a presentable monoidal ∞-category which is differentiable and such that $1_{\mathcal{V}}$ is compact. Then $\mathrm{Cat}(\mathcal{V})$ is differentiable and $1_{\mathrm{Cat}(\mathcal{V})}$ (ie the image of the categorical algebra with one object with endomorphism algebra $1_{\mathcal{V}}$) is compact as well.*

Proof By [26, Remark 4.1.8.9], there exists a monoidal model category $\boldsymbol{V}$ presenting $\mathcal{V}$ of the following form: one can construct a simplicial monoid $A \in \mathrm{Alg}(\mathrm{sSet})$ and take $\boldsymbol{V} = \mathrm{sSet}_{/A}$, with monoidal structure given by $(X \to A) \otimes (Y \to A) = (X \times Y \to A \times A \to A)$ and model structure given by a left Bousfield localization of the covariant model structure. In particular, $\boldsymbol{V}$ is simplicial and combinatorial, its cofibrations are the monomorphisms, all objects are cofibrant and weak equivalences are stable under filtered colimits. Then $\mathrm{Cat}(\mathcal{V})$ arises from the model category $\mathrm{Cat}^{\mathrm{strict}}(\boldsymbol{V})$ on $\boldsymbol{V}$-enriched categories [17] and it then follows from [13, Corollary 3.1.12] that $\mathrm{Cat}(\mathcal{V})$ is again differentiable.

To see that $1_{\mathrm{Cat}(\mathcal{V})}$ is compact, note that the corepresentable functor $\mathrm{Map}(1_{\mathrm{Cat}(\mathcal{V})}, -)$ can be identified with the functor taking spaces of objects. This functor decomposes as

$$\mathrm{Cat}(\mathcal{V}) \to \mathrm{Cat}(\mathcal{S}) \simeq \mathrm{Cat}_\infty \xrightarrow{\mathrm{Core}} \mathcal{S},$$

where the first functor is induced by the lax monoidal functor $\mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, -)\colon \mathcal{V} \to \mathcal{S}$ and the second functor takes the core (or maximal sub-∞-groupoid). Taking cores preserves filtered colimits (as is easily checked using quasicategories). To see that $\mathrm{Cat}(\mathcal{V}) \to \mathrm{Cat}(\mathcal{S})$ preserves filtered colimits, we use model categories. Since $\boldsymbol{V}$ is simplicial and monoidal, there is a monoidal Quillen pair $1_{\boldsymbol{V}} \otimes - \colon \mathrm{sSet} \rightleftarrows \boldsymbol{V} : \mathrm{Map}_{\boldsymbol{V}}(1_{\boldsymbol{V}}, -)$.

Applying these functors on mapping objects yields a Quillen pair on enriched categories. In light of [23, Proposition 5.3.1.16], it now suffices to verify that the right Quillen functor $U:\mathrm{Cat}^{\mathrm{strict}}(\boldsymbol{V})\to\mathrm{Cat}^{\mathrm{strict}}(\mathrm{sSet})$ preserves filtered homotopy colimits.

Let $\mathbb{C}_\bullet\colon \mathcal{I}\to\mathrm{Cat}^{\mathrm{strict}}(\boldsymbol{V})$ be a projectively cofibrant filtered diagram with colimit $\mathbb{C}_\infty$. To see that the natural map $\operatorname{hocolim} U(\mathbb{C}_\bullet)\to U(\mathbb{C}_\infty)$ is a weak equivalence, note that the Dwyer–Kan equivalences of simplicial categories are closed under filtered colimits. Consequently, the homotopy colimit can simply be computed by the ordinary colimit and it suffices to verify that $\operatorname{colim} U(\mathbb{C}_\bullet)\to U(\mathbb{C}_\infty)$ is a Dwyer–Kan equivalence. At the level of objects, the map is simply an isomorphism and for each $c,d\in\operatorname{colim}U(\mathbb{C}_\bullet)$ arising from some $U(\mathbb{C}_i)$, the induced map on mapping spaces is given by

$$\operatorname{colim}_{j\in\mathcal{I}_{i/}}\mathrm{Map}_{\boldsymbol{V}}(1_{\boldsymbol{V}},\mathbb{C}_j(c,d))\to\mathrm{Map}_{\boldsymbol{V}}(1_{\boldsymbol{V}},\operatorname{colim}_{j\in\mathcal{I}_{i/}}\mathbb{C}_j(c,d)).$$

This map is a weak equivalence of simplicial sets because $\mathrm{Map}_{\boldsymbol{V}}(1_{\boldsymbol{V}},-)$ preserves filtered homotopy colimits and because weak equivalences in both sSet and $\boldsymbol{V}$ are closed under filtered colimits. □

The proof of Theorem 6.4 requires a few preliminary observations:

Proposition 6.6 *Let $\mathcal{V}$ be a presentable monoidal ∞-category and $\pi_{\mathcal{V}}\colon\mathcal{T}\mathcal{V}\to\mathcal{V}$ its tangent projection. Then the inclusions of complete objects and the completion functors fit into pullback squares*

$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{T}\mathcal{V}) \arrow[r, hook] \arrow[d, "\mathrm{Cat}(\pi_{\mathcal{V}})"'] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \arrow[r, "(-)^\wedge"] \arrow[d, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"'] & \mathrm{Cat}(\mathcal{T}\mathcal{V}) \arrow[d, "\mathrm{Cat}(\pi_{\mathcal{V}})"] \\
\mathrm{Cat}(\mathcal{V}) \arrow[r, hook] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "(-)^\wedge"] & \mathrm{Cat}(\mathcal{V})
\end{tikzcd}$$

in which the vertical functors are all Cartesian and co-Cartesian fibrations.

Proof Recall from Lemma 2.11 that the monoidal functor $\pi_{\mathcal{V}}\colon\mathcal{T}\mathcal{V}\to\mathcal{V}$ has a (strong) monoidal fully faithful *left* adjoint $\mathrm{cst}\colon\mathcal{V}\to\mathcal{T}\mathcal{V}$ taking constant diagrams, and that cst is also a monoidal fully faithful *right* adjoint to $\pi_{\mathcal{V}}$. Considering $\pi_{\mathcal{V}}$ as a right adjoint functor, Remark 5.8 then implies that $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})\colon\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})\to\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ preserves and detects completeness, so that the left square commutes and is Cartesian. On the other hand, considering $\pi_{\mathcal{V}}$ as a monoidal left adjoint, we find that the right square commutes: taking right adjoints, this comes down to $\mathrm{cst}\colon\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})\to\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ preserving complete objects.

Next, the two monoidal adjunctions $(\mathrm{cst},\pi_{\mathcal{V}})$ and $(\pi_{\mathcal{V}},\mathrm{cst})$ induce adjunctions

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, hook, shift left=1.5, "\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}), \arrow[l, shift left=1.5, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \arrow[r, shift left=1.5, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[l, hook', shift left=1.5, "\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})"]
\end{tikzcd}$$

in which $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})$ is fully faithful. This implies that $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$ preserves limits and colimits and by [5, Lemma 2.6.1], it is both a Cartesian and co-Cartesian fibration. Since the left square was a pullback, this implies that $\mathrm{Cat}(\pi_{\mathcal{V}})$ is a Cartesian and co-Cartesian fibration as well and that the inclusion $\mathrm{Cat}(\mathcal{T}\mathcal{V}) \hookrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ preserves Cartesian and co-Cartesian arrows.

It remains to verify that the right square is Cartesian. To see this, we claim that the following conditions are equivalent for a map $\alpha\colon \mathbb{C} \to \mathbb{D}$ in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$:

(a) α is a DK-equivalence.

(b) α is an $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-co-Cartesian lift of a DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$.

(c) α is an $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-Cartesian lift of a DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$.

Assuming this, it follows that $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$ classifies a diagram $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \to \mathrm{Cat}^{\mathrm{L}}$ sending each DK-equivalence to an adjoint equivalence: over a fixed DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, the $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-Cartesian and $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-co-Cartesian arrows coincide, so that the proof of [23, Proposition 5.2.2.8] shows that the unit and counit of the induced adjunction are equivalences. Since the DK-equivalences in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ are precisely the $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-(co)Cartesian lifts of DK-equivalences in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, it then follows from [21, Proposition 2.1.4] that the right square is a Cartesian square.

To see the claim, let us write $\alpha_0\colon \mathbb{C}_0 \to \mathbb{D}_0$ for the image of α in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ and let $\alpha_0'\colon \mathbb{C}_0' \to \mathbb{D}_0'$ be the image of α_0 under $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})$. The unit and the counit of the adjoint pairs above then determine a commuting diagram in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$:

$$\begin{tikzcd}
\mathbb{C}_0' \arrow[r, "\epsilon"] \arrow[d, "\alpha_0'"'] & \mathbb{C} \arrow[r, "\eta"] \arrow[d, "\alpha"] & \mathbb{C}_0' \arrow[d, "\alpha_0'"] \\
\mathbb{D}_0' \arrow[r, "\epsilon"] & \mathbb{D} \arrow[r, "\eta"] & \mathbb{D}_0'
\end{tikzcd}$$

Since $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})$ and $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$ both preserve DK-equivalences (having right adjoints preserving complete objects), α_0' is a DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ if and only if α_0 is a DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$. Furthermore, the left square is a pushout if and only if α is $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-co-Cartesian and the right square is a pullback if and only if α is $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-Cartesian, by [5, Lemma 2.6.1] and its opposite.

Using this, (b) implies (a) because DK-equivalences are stable under pushout. Conversely, if α is a DK-equivalence, then α_0' is a DK-equivalence as well. Consequently, $\mathbb{D}_0' \amalg_{\mathbb{C}_0'} \mathbb{C} \to \mathbb{D}$ is both a DK-equivalence and an equivalence on spaces of objects (since its image in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is an equivalence), and hence an equivalence.

Dually, (c) is equivalent to α_0' being a DK-equivalence and the right square being a pullback. Since fully faithful maps are stable under pullback, this implies that α is fully faithful. Since essential surjectivity is detected on the underlying space-valued categorical algebra, which in turn is determined by the underlying $\mathcal{V}$-enriched categorical algebra, we find that α is essentially surjective as well. Conversely, if α is a DK-equivalence, then α_0' is a DK-equivalence as well. Consequently, the map $\mathbb{C} \to \mathbb{C}_0' \times_{\mathbb{D}_0'} \mathbb{D}$ is fully faithful and an equivalence on spaces of objects, and is hence an equivalence. ◻

Lemma 6.7 *Let $\mathcal{V}$ be a presentable SM ∞-category and let $\mathcal{T}\mathcal{V}$ be its tangent bundle with the square zero monoidal structure. Then there is a natural SM equivalence*

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \simeq \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(V) \times_{\mathcal{T}\mathcal{S}} \mathcal{S}$$

between $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ and the full subcategory of excisive functors $\mathbb{C}_\bullet\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ which are constant at the level of objects.

Proof Recall the equivalence

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})) \simeq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})} \mathcal{S}$$

from Proposition 5.6. By Lemma 5.22, this restricts to an equivalence on full subcategories of excisive functors. □

Lemma 6.8 *Let $\mathcal{V}$ be a presentable monoidal ∞-category. Then there exists a natural relative adjunction*

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \arrow[rr, hook, shift left=1.5ex, "\varphi"] \arrow[rr, phantom, "\perp"] \arrow[dr, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"'] & & \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[ll, shift left=1.5ex, "\psi"] \arrow[dl, "\pi"] \\ & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) & \end{tikzcd}$$

Proof Using Lemma 6.7, we define φ as the projection onto the first factor

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \simeq \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times_{\mathcal{T}\mathcal{S}} \mathcal{S} \arrow[r, "\varphi"] \arrow[d, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"'] & \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(V) \arrow[d, "\pi"] \\ \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times_{\mathcal{S}} \mathcal{S} \arrow[r, "\sim"] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \end{tikzcd} \tag{6-3}$$

Here the square commutes by naturality of the equivalence from Proposition 5.6 with respect to restriction along $\{*\} \hookrightarrow \mathcal{S}_*^{\mathrm{fin}}$. Note that φ is the base change along the Cartesian fibration $\mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \to \mathcal{T}\mathcal{S}$ of the fully faithful functor $\mathrm{cst}\colon \mathcal{S} \to \mathcal{T}\mathcal{S}$ that is left adjoint to the tangent projection. The opposite of [23, Corollary 5.2.7.11] then implies that φ is fully faithful and admits a right adjoint ψ. This right adjoint can be described as follows: given $\mathbb{C}_\bullet\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, $\psi(\mathbb{C}_\bullet)\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ sends each space T to the full subcategorical algebra of $\mathbb{C}_T$ obtained by restricting the objects along the canonical map $\mathrm{Ob}(\mathbb{C}_*) \to \mathrm{Ob}(\mathbb{C}_T)$. In other words, the counit map $\psi(\mathbb{C}_T) \to \mathbb{C}_T$ is a Cartesian lift of the map of spaces $\mathrm{Ob}(\mathbb{C}_*) \to \mathrm{Ob}(\mathbb{C}_T)$. It follows from this description that ψ commutes with evaluation at $*$ as well, so that ϕ and ψ form a relative adjunction. □

Proof of Theorem 6.4 To define the functor $\mathcal{L}$, consider the composition

$$\mathcal{T}((-)^\wedge) \circ \varphi\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \hookrightarrow \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \to \mathcal{T}\,\mathrm{Cat}(\mathcal{V}).$$

Here φ is the functor from Lemma 6.8 and the last square arises from the adjoint pair

$$(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \underset{\perp}{\rightleftarrows} \mathrm{Cat}(\mathcal{V}) :\iota$$

by taking tangent bundles. Note that φ sends DK-equivalences in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ to $\mathcal{S}_*^{\mathrm{fin}}$-diagrams of DK-equivalences in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, which in turn are sent to equivalences by $\mathcal{T}((-)^\wedge)$. Consequently, the above composite induces a functor $\mathcal{L}\colon \mathrm{Cat}(\mathcal{T}\mathcal{V}) \to \mathcal{T}\,\mathrm{Cat}(\mathcal{V})$ from the localization at the DK-equivalences.

Since $(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \to \mathrm{Cat}(\mathcal{T}\mathcal{V})$ is a monoidal localization, $\mathcal{L}$ inherits a natural SM structure from the composite $\mathcal{T}((-)^\wedge) \circ \varphi\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$. Here $\mathcal{T}((-)^\wedge)$ inherits its SM structure from $(-)^\wedge$, and φ corresponds under the SM equivalence of Proposition 5.6 to the projection (6-3), which is an SM functor.

To show that $\mathcal{L}$ is an equivalence, observe that (since $(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \to \mathrm{Cat}(\mathcal{T}\mathcal{V})$ has a section), $\mathcal{L}$ coincides with the top horizontal composite

(6-4)
$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{T}\mathcal{V}) \arrow[r, hook] \arrow[d] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \arrow[r, hook, "\varphi"] \arrow[d] & \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "\mathcal{T}((-)^\wedge)"] \arrow[d] & \mathcal{T}\,\mathrm{Cat}(\mathcal{V}) \arrow[d] \\
\mathrm{Cat}(\mathcal{V}) \arrow[r, hook] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, equal] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "(-)^\wedge"] & \mathrm{Cat}(\mathcal{V})
\end{tikzcd}$$

Here all vertical functors are co-Cartesian fibrations, where we use Proposition 6.6 for the left two. The first two horizontal functors preserve co-Cartesian arrows by Proposition 6.6 and by Lemma 6.8 and [26, Proposition 7.3.2.6]. The last functor $\mathcal{T}((-)^\wedge)$ preserves co-Cartesian arrows for formal reasons: for any adjoint pair $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$, the induced adjoint pair $\mathcal{T}F\colon \mathcal{T}\mathcal{C} \rightleftarrows \mathcal{T}\mathcal{D} :\mathcal{T}G$ covering F and G has left adjoint $\mathcal{T}F$ preserving co-Cartesian arrows and right adjoint $\mathcal{T}G$ preserving Cartesian arrows.

Having proven that $\mathcal{L}$ is a map between co-Cartesian fibrations preserving co-Cartesian arrows, it suffices to verify that $\mathcal{L}$ induces an equivalence on fibers. Let us therefore fix a $\mathcal{V}$-enriched category $\mathbb{C} \in \mathrm{Cat}(\mathcal{V})$ and let us write $X = \mathrm{Ob}(\mathbb{C})$ for its space of objects. Since the left square in (6-4) was Cartesian, it suffices to verify that φ and completion induce an equivalence $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \times_{\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})} \{\mathbb{C}\} \to \mathcal{T}_{\mathbb{C}}\,\mathrm{Cat}(\mathcal{V})$. This follows essentially from [13, Section 3.1].

Indeed, note that the equivalence from Lemma 6.7 induces an equivalence on fibers

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \times_{\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})} \{\mathbb{C}\} \simeq \mathcal{T}_{\mathbb{C}}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times_{\mathcal{T}_X \mathcal{S}} \{X\}.$$

Recall that under the equivalence of Lemma 6.7, the functor φ was simply given by projection onto the first factor. It will therefore suffice to verify that the composite

(6-5)
$$\mathcal{T}_{\mathbb{C}}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times_{\mathcal{T}_X \mathcal{S}} \{X\} \underset{\psi}{\overset{\varphi=\pi_1}{\rightleftarrows}} \mathcal{T}_{\mathbb{C}}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \underset{\mathcal{T}_{\mathbb{C}}(\iota)}{\overset{\mathcal{T}_{\mathbb{C}}((-)^\wedge)}{\rightleftarrows}} \mathcal{T}_{\mathbb{C}}\,\mathrm{Cat}(\mathcal{V})$$

is an equivalence. As indicated, this composite admits a right adjoint: ψ is the right adjoint from Lemma 6.8, restricting the space of objects to X, and $\mathcal{T}(\iota)$ is induced by the canonical inclusion

$$\iota\colon \mathrm{Cat}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}).$$

Now note that the above diagram arises upon stabilization from the following diagram of adjunctions between ∞-categories of retractive objects over $\mathbb{C}$:

$$F\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})_{\mathbb{C}/\!/\mathbb{C}} \times_{\mathcal{S}_{X/\!/X}} \{X\} \underset{\psi'}{\overset{\pi_1}{\rightleftarrows}} \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})_{\mathbb{C}/\!/\mathbb{C}} \underset{\iota}{\overset{(-)^\wedge}{\rightleftarrows}} \mathrm{Cat}(\mathcal{V})_{\mathbb{C}/\!/\mathbb{C}} :G.$$

Here the right adjoint ψ' exists by (the opposite of) [23, Corollary 5.2.7.11]. Explicitly, it takes the full subcategorical algebra with objects X, ie the counit map $\psi'(\mathbb{D}) \to \mathbb{D}$ is a Cartesian lift of the map of spaces $X = \mathrm{Ob}(\mathbb{C}) \to \mathrm{Ob}(\mathbb{D})$ (see Definition 5.7). Upon stabilization, this induces the right adjoint pair in (6-5) by definition. For the left adjoint pair, note that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})_{\mathbb{C}/\!/\mathbb{C}} \times_{\mathcal{S}_{X/\!/X}} \{X\}$ is a fiber product of pointed ∞-categories along left exact functors, and that stabilization preserves such fiber products. Consequently, the projection onto the first factor induces the functor ϕ, and since stabilization sends an adjoint pair of left exact functors to an adjunction between stable ∞-categories, its right adjoint ψ' induces the functor ψ at the stable level.

We now follow the same proof as [13, Proposition 3.1.9]: applying (the ∞-categorical analogue of) [15, Corollary 2.39], it suffices to verify that the unit and counit of (F, G) become equivalences upon taking loop spaces. For the unit map, let $\mathbb{C} \to \mathbb{D} \to \mathbb{C}$ be a retract diagram of categorical algebras with spaces of objects X. Then $\mathbb{D} \to GF(\mathbb{D})$ is the natural map obtained by decomposing $\mathbb{D} \to \mathbb{D}^\wedge$ (essentially uniquely) into a map that is the identity on objects, followed by a fully faithful map. Since $\mathbb{D} \to \mathbb{D}^\wedge$ is itself fully faithful, the unit is itself already an equivalence.

For the counit, let $\mathbb{C} \to \mathbb{D} \to \mathbb{C}$ be a retract diagram of $\mathcal{V}$-enriched categories. Then the counit map $\epsilon_{\mathbb{D}}\colon FG(\mathbb{D}) = \psi'(\mathbb{D})^\wedge \to \mathbb{D}$ is the natural map from the completion of the full subcategorical algebra $\psi'(\mathbb{D}) \to \mathbb{D}$ with objects X. Since $\psi'(\mathbb{D}) \to \mathbb{D}$ is fully faithful, $\epsilon_{\mathbb{D}}$ is fully faithful as well. Consequently, the base change $FG(\mathbb{D}) \times_{\mathbb{D}} \mathbb{C} \to \mathbb{C}$ is fully faithful as well. This map has a canonical section (since we are working in $\mathrm{Cat}(\mathcal{V})_{\mathbb{C}/\!/\mathbb{C}}$) and is hence also essentially surjective. Since all categorical algebras involved were complete, it follows that $FG(\mathbb{D}) \times_{\mathbb{D}} \mathbb{C} \simeq \mathbb{C}$ is the zero object in $\mathrm{Cat}(\mathcal{V})_{\mathbb{C}/\!/\mathbb{C}}$. Using this, the looping of the counit map $\Omega_{/\mathbb{C}}(\epsilon_{\mathbb{D}})\colon \mathbb{C} \times_{FG(\mathbb{D})} \mathbb{C} \to \mathbb{C} \times_{\mathbb{D}} \mathbb{C}$ can be identified with

$$\mathbb{C} \times_{FG(\mathbb{D})} \mathbb{C} \to \mathbb{C} \times_{FG(\mathbb{D})} FG(\mathbb{D}) \times_{\mathbb{D}} \mathbb{C},$$

which is the base change of an equivalence. It follows that the counit is an equivalence upon taking loop space objects, so that (6-5) is indeed an equivalence. $\square$

Proposition 6.9 *Let $\mathcal{V}$ be presentable monoidal ∞-category and suppose that $\mathcal{T}\mathcal{V}$ carries a monoidal t-orientation. Then the full subcategories $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\geq 0}\mathcal{V})$ and $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$ define a t-orientation on the stable Cartesian fibration $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$.*

Proof Theorem 6.4 and Proposition 6.6 imply that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \to \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is a stable Cartesian fibration, being the base change of such. For a fixed space of objects, the restrictions

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\geq 0}\mathcal{V}) \times_{\mathcal{S}} \{X\} \simeq \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{T}^{\geq 0}\mathcal{V}), \quad \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V}) \times_{\mathcal{S}} \{X\} \simeq \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{T}^{\leq 0}\mathcal{V}) \tag{6-6}$$

coincide with the t-orientation on $\mathcal{O}_X$-algebras from Proposition 2.25. In particular, this implies that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\geq 0}\mathcal{V})$ and $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$ restrict to a t-structure on the fiber over a fixed categorical algebra $\mathbb{C}$.

For condition (1) of Definition 2.20, let $f\colon \mathbb{C} \to \mathbb{D}$ be a map in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, $\mathbb{E} \in \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$ an object living over $\mathbb{D}$ and $f^*\mathbb{E} \to \mathbb{E}$ the Cartesian lift of f. To see that $f^*\mathbb{E} \in \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$, factor f as a map

$g\colon \mathbb{C} \to \mathbb{D}'$ which is the identity on objects, followed by a fully faithful map $h\colon \mathbb{D}' \to \mathbb{D}$. Then $h^*(\mathbb{E}) \to \mathbb{E}$ is fully faithful; in particular, if all mapping objects of $\mathbb{E}$ are contained in $\mathcal{T}^{\leq 0}\mathcal{V}$, the same holds for $h^*(\mathbb{E})$. The map $f^*(\mathbb{E}) \to h^*(\mathbb{E})$ is then a Cartesian arrow in $\mathrm{Alg}_{\mathcal{O}_{\mathrm{Ob}(\mathbb{C})}}(\mathcal{T}\mathcal{V})$, so that Proposition 2.25 implies that $f^*(\mathbb{E}) \in \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$. □

Corollary 6.10 *Let $\mathcal{V}$ be a differentiable presentable SM ∞-category such that $1_{\mathcal{V}}$ is compact and suppose that $\mathcal{T}\mathcal{V}$ carries a monoidal t-orientation. Under the equivalence $\mathcal{T}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}\mathcal{V})$ from Theorem 6.4, the full subcategories*

$$\mathcal{T}^{\geq 0}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}^{\geq 0}\mathcal{V}), \quad \mathcal{T}^{\leq 0}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}^{\leq 0}\mathcal{V})$$

then determine a monoidal t-orientation of the tangent bundle $\mathcal{T}\,\mathrm{Cat}(\mathcal{V})$, with heart $\mathcal{T}^{\heartsuit}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}^{\heartsuit}\mathcal{V})$.

Proof Using the left pullback square from Proposition 6.6, this is simply the base change of the t-orientation from Proposition 6.9. It is a monoidal t-structure because $\mathrm{Cat}(-)$ preserves symmetric monoidal functors and fully faithful functors, so that $\mathcal{T}^{\geq 0}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}^{\geq 0}\mathcal{V}) \subseteq \mathcal{T}\,\mathrm{Cat}(\mathcal{V})$ is closed under the tensor product. □

Remark 6.11 In the setting of Corollary 6.10, let $\mathbb{C}$ be $\mathcal{V}$-enriched category together with a π_0-surjection of spaces $X \to \mathrm{Ob}(\mathbb{C})$. Let $\mathbb{C}_X \hookrightarrow \mathbb{C}$ be the induced fully faithful functor, which realizes $\mathbb{C}$ as the completion of $\mathbb{C}_X$. Theorem 6.4 and Proposition 6.6 then provide equivalences of stable ∞-categories

$$\mathcal{T}_{\mathbb{C}}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \times_{\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})} \{\mathbb{C}_X\} \simeq \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{T}\mathcal{V}) \times_{\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V})} \{\mathbb{C}_X\}.$$

In the presence of a t-orientation, the proof of Proposition 6.9 (see (6-6)) shows that this identifies $\mathcal{T}^{\heartsuit}_{\mathbb{C}}\,\mathrm{Cat}(\mathcal{V})$ with the fiber $\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{T}^{\heartsuit}\mathcal{V}) \times_{\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V})} \{\mathbb{C}_X\}$.

6.2 Local systems of abelian groups on (∞, n)-categories

Applying Corollary 6.10 inductively, starting with the t-structure on parametrized spectra of Example 2.24, we obtain:

Corollary 6.12 *Let $\mathcal{C}$ be an (∞, n)-category. Then the tangent ∞-category $\mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}_{(\infty,n)}$ carries a t-structure, in which an object E is (co)connective if and only if for any two objects $x, y \in \mathcal{C}$, the functor*

$$\mathrm{Map}_{(-)}(x, y)\colon \mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}_{(\infty,n)} \to \mathcal{T}_{\mathrm{Map}_{\mathcal{C}}(x,y)}\,\mathrm{Cat}_{(\infty,n-1)}$$

sends E to a (co)connective object.

Applying this inductively, one finds the following inductive description of the heart of the t-orientation on $\mathcal{T}\,\mathrm{Cat}_{(\infty,n)}$:

Definition 6.13 (see [24, Definition 3.5.10]) The ∞-category of *local systems of abelian groups on* $(\infty,0)$*-categories* is defined to be the domain of the Cartesian fibration

$$\mathrm{Loc}_{(\infty,0)} \to \mathcal{S}$$

classified by the functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending a space X to the category of local systems $\mathrm{Fun}(\Pi_1(X), \mathrm{Ab})$. This carries a symmetric monoidal structure given by the Cartesian product. For $n \geq 1$, we define the symmetric monoidal ∞-category of *local systems of abelian groups on* (∞, n)*-categories* to be the domain of the Cartesian fibration

$$\mathrm{Loc}_{(\infty,n)} = \mathrm{Cat}(\mathrm{Loc}_{(\infty,n-1)}) \to \mathrm{Cat}(\mathrm{Cat}_{(\infty,n-1)}) = \mathrm{Cat}_{(\infty,n)}\,.$$

Note that $\mathrm{Loc}_{(\infty,n)}$ inherits a symmetric monoidal structure from $\mathrm{Loc}_{(\infty,n-1)}$, such that the projection to $\mathrm{Cat}_{(\infty,n)}$ is symmetric monoidal.

For each $\mathcal{C}$, let us denote the fiber of $\mathrm{Loc}_{(\infty,n)}$ over $\mathcal{C}$ by $\mathrm{Loc}_{(\infty,n)}(\mathcal{C})$ and refer to it as the *abelian category of local systems on* $\mathcal{C}$. Note that $\mathrm{Loc}_{(\infty,n)}(\mathcal{C})$ is indeed an (ordinary) abelian category by the following immediate consequence of Corollary 6.10:

Corollary 6.14 *There are equivalences of ∞-categories over* $\mathrm{Cat}_{(\infty,n)}$

$$\mathrm{Loc}_{(\infty,n)} \simeq \mathrm{Cat}(\mathrm{Loc}_{(\infty,n-1)}) \simeq \mathrm{Cat}(\mathcal{T}^{\heartsuit}\,\mathrm{Cat}_{(\infty,n-1)}) \simeq \mathcal{T}^{\heartsuit}\,\mathrm{Cat}_{(\infty,n)}\,.$$

Given an (∞, n)-category $\mathcal{C}$, Remark 6.11 now implies that a local system $\mathcal{A}$ on $\mathcal{C}$ is given by the datum of map of ∞-operads

$$\begin{tikzcd} & & \mathrm{Loc}^{\otimes}_{(\infty,n-1)} \arrow[d] \\ \mathcal{O}_X \arrow[r] \arrow[urr, dotted, "\mathcal{A}"] & \mathcal{O}_{\mathrm{Ob}(\mathcal{C})} \arrow[r, "\mathcal{C}"'] & \mathrm{Cat}^{\otimes}_{(\infty,n-1)} \end{tikzcd}$$

for any choice of π_0-surjection of spaces $X \to \mathrm{Ob}(\mathcal{C})$.

Remark 6.15 For the canonical choice $X = \mathrm{Ob}(\mathcal{C})$, the datum of a local system $\mathcal{A}$ on an (∞, n)-category $\mathcal{C}$ can therefore be described informally as follows:

(0) For each $x, y \in \mathcal{C}$, a local system $\mathcal{A}_{x,y}$ over the $(\infty, n-1)$-category of maps $\mathcal{C}(x, y)$.

(i) a map of local systems for each triple $x, y, z \in \mathcal{C}$ and a map of abelian groups for each $x \in \mathcal{C}$

$$m_{x,y,z}\colon p_1^*\mathcal{A}_{y,z} \times p_0^*\mathcal{A}_{x,y} \to c^*\mathcal{A}_{x,z}, \quad u_x\colon \mathbb{Z} \to e^*\mathcal{A}_{x,x}.$$

Here $c\colon \mathcal{C}(y,z) \times \mathcal{C}(x,y) \to \mathcal{C}(x,z)$ is the composition,

$$p_0\colon \mathcal{C}(y,z) \times \mathcal{C}(x,y) \to \mathcal{C}(y,z) \quad \text{and} \quad p_1\colon \mathcal{C}(y,z) \times \mathcal{C}(x,y) \to \mathcal{C}(x,y)$$

are the projections and $e\colon * \to \mathcal{C}(x,x)$ is the unit.

(ii) An associativity condition for each quadruple $w, x, y, z \in \mathcal{C}$ and left and right unitality conditions for each tuple $x, y \in \mathcal{C}$, given by the commutativity of the diagrams

$$\begin{tikzcd}
& p_2^*\mathcal{A}_{z,w} \times p_1^*\mathcal{A}_{y,z} \times p_0^*\mathcal{A}_{x,y} \arrow[dl, "m_{x,z,w}\circ(\mathrm{id}\times m_{x,y,z})"'] \arrow[dr, "m_{x,y,w}\circ(m_{y,z,w}\times\mathrm{id})"] & \\
(c\circ(\mathrm{id}\times c))^*\mathcal{A}_{x,w} \arrow[rr, "\alpha^*", "\cong"'] & & (c\circ(c\times\mathrm{id}))^*\mathcal{A}_{x,w}
\end{tikzcd}$$

where α^* arises from the associator α of $\mathcal{C}$ by naturality of base change, and

$$\begin{tikzcd}
& \mathcal{A}_{x,y} \arrow[dl, "m_{x,x,y}\circ(\mathrm{id}\times u_x)"'] \arrow[d, equal] \arrow[dr, "m_{y,y,x}\circ(u_y\times\mathrm{id})"] & \\
(c\circ(\mathrm{id}\times e))^*\mathcal{A}_{x,y} \arrow[r, "\rho^*", "\cong"'] & \mathcal{A}_{x,y} & (c\circ(e\times\mathrm{id}))^*\mathcal{A}_{x,y} \arrow[l, "\lambda^*"', "\cong"]
\end{tikzcd}$$

where the bottom maps arise from the left and right unit equivalences λ and ρ.

There are no higher coherences because the local systems over each $\mathcal{C}(x, y)$ form an ordinary 1-category. Definition 6.13 therefore gives a precise formulation of the (informal) definition of local systems on (∞, n)-categories appearing in [24, Definition 3.5.10].

Remark 6.16 Taking $X \to \mathrm{Ob}(\mathcal{C})$ to be a π_0-surjection from a *set*, the datum of a local system over $\mathcal{C}$ can also be identified with a section of the map of operads $\mathcal{L}_{\mathcal{C},X} := \mathrm{Loc}^{\otimes}_{(\infty,n-1)} \times_{\mathrm{Cat}^{\otimes}_{(\infty,n-1)}} \mathcal{O}_X \to \mathcal{O}_X$. Now note that $\mathrm{Loc}^{\otimes}_{(\infty,n-1)} \to \mathrm{Cat}^{\otimes}_{(\infty,n-1)}$ induces maps on mapping spaces with *discrete fibers*. Since X is a set, both $\mathcal{O}_X$ and $\mathcal{L}_{\mathcal{C},X}$ are therefore *ordinary* operads and a section $\mathcal{O}_X \to \mathcal{L}_{\mathcal{C},X}$ is given by choosing images of objects and multimorphisms satisfying a certain associativity condition, but *no higher coherences*. In this situation, the informal description of a local system from Remark 6.15, for objects taken in the set X, is *exactly* the data of a local system on $\mathcal{C}$.

The inductive construction of the Postnikov structure in Theorem 6.3 shows that all parametrized spectra appearing in it are contained in the heart of the t-structure on $\mathcal{T}\,\mathrm{Cat}_{(\infty,n)}$. We therefore obtain the following result (which appears without proof as [24, Claim 3.5.18]):

Corollary 6.17 *For every (∞, n)-category $\mathcal{C}$, the parametrized spectrum $\mathrm{H}\pi_a(\mathcal{C})$ of Theorem 6.3 is the Eilenberg–MacLane spectrum associated to*

$$\pi_a(\mathcal{C}) \in \mathrm{Loc}_{(\infty,n)}(\mathcal{C}) = \mathcal{T}^{\heartsuit}_{\mathrm{ho}_{(n+1,n)}(\mathcal{C})} \mathrm{Cat}_{(\infty,n)}.$$

In terms of Remark 6.15, it is the Eilenberg–MacLane spectrum of the local system of abelian groups on $\mathrm{ho}_{(n+1,n)}\,\mathcal{C}$ given inductively by $\pi_a(\mathcal{C})_{x,y} = \pi_a \mathrm{Map}_{\mathcal{C}}(x, y)$, for any $x, y \in \mathcal{C}$.

Appendix Symmetric monoidal structure on categorical algebras

In this appendix, we provide the proofs of Lemma 5.4 and Propositions 5.3 and 5.6 about the symmetric monoidal structure on $\mathrm{Cat}_{\mathrm{Alg}}(\mathcal{V})$. The key ingredient of these proofs will be Construction A.3: given an ∞-category $\mathcal{C}$ with finite products, this produces a diagram $F_\Psi\colon \mathcal{D} \to \mathrm{SMCat}^{\mathrm{lax}}_{/\mathcal{C}^\times}$ of SM ∞-categories over $\mathcal{C}^\times$ from the data of a (suitable) diagram $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{SMCat}^{\mathrm{lax}}_\infty$. Let us start with a preliminary observation:

Definition A.1 For an ∞-category $\mathcal{D}$, let us write

$$\text{co-Cart}(\mathcal{D})^{\mathrm{lax}} \subseteq \mathrm{Cat}_{\infty/\mathcal{D}}, \quad \mathrm{Cart}(\mathcal{D})^{\mathrm{opl}} \subseteq \mathrm{Cat}_{\infty/\mathcal{D}}$$

for the full subcategories spanned by the co-Cartesian and Cartesian fibrations, respectively. Furthermore, let us denote by $\mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}}_\infty)^{\mathrm{strong}} \hookrightarrow \mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}}_\infty)$ the wide subcategory whose morphisms $\mu\colon F \to G$ are natural transformations such that for each $d \in \mathcal{D}$, the map $\mu_d\colon F(d) \to G(d)$ is a strong (as opposed to lax) SM functor.

Lemma A.2 *For any ∞-category $\mathcal{D}$, there is a natural (wide) subcategory inclusion*

$$\mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}})^{\mathrm{strong}} \hookrightarrow \mathrm{CAlg}(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}}),$$

where we take commutative algebras with respect to the fiber product over $\mathcal{D}$.

Proof We use unstraightening to identify both categories with (nonfull) subcategories of $\mathrm{Cat}_{\infty/\mathcal{D}^{\mathrm{op}} \times \mathrm{Fin}_*}$ and then show that one is naturally included in the other. First, note that $\mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}})^{\mathrm{strong}}$ is a subcategory of $\mathrm{Fun}(\mathcal{D}, \mathrm{Cat}_{\infty/\mathrm{Fin}_*})$. By [19, Corollary 2.3.4], unstraightening to a Cartesian fibration over $\mathcal{D}$ then provides an equivalence between $\mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}})^{\mathrm{strong}}$ and the following subcategory of $\mathrm{Cat}_{\infty/\mathcal{D}^{\mathrm{op}} \times \mathrm{Fin}_*}$:

(1) Objects are maps $p = (p_1, p_2)\colon \mathcal{E} \to \mathcal{D}^{\mathrm{op}} \times \mathrm{Fin}_*$ such that p_1 is a Cartesian fibration, p_2 is a co-Cartesian fibration, p_1 sends p_2-co-Cartesian arrows to equivalences and p_2 sends p_1-Cartesian arrows to equivalences. Furthermore, for each $d \in \mathcal{D}$, the fiber $\mathcal{E}_d \to \mathrm{Fin}_*$ is an SM ∞-category and for each $\alpha\colon d \to d'$ in $\mathcal{D}$, the change of fiber functor $\alpha^*\colon \mathcal{E}_{d'} \to \mathcal{E}_d$ preserves p_2-co-Cartesian lifts of inert morphisms in Fin_*.

(2) Morphisms are commuting triangles

(A-1)
$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "(p_1,p_2)=p"'] & & \mathcal{E}' \arrow[dl, "q=(q_1,q_2)"] \\ & \mathcal{D}^{\mathrm{op}} \times \mathrm{Fin}_* & \end{tikzcd}$$

such that f sends all p_1-Cartesian arrows to q_1-Cartesian arrows and all p_2-co-Cartesian arrows to q_2-co-Cartesian arrows.

Similarly, we can view $\mathrm{CAlg}(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}})$ as a subcategory of $\mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Cat}_{\infty/\mathcal{D}^{\mathrm{op}}})$. By [19, Corollary 2.3.4], unstraightening to a co-Cartesian fibration over Fin_* then provides an equivalence between $\mathrm{CAlg}(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}})$ and the following subcategory of $\mathrm{Cat}_{\infty/\mathcal{D}^{\mathrm{op}}\times\mathrm{Fin}_*}$:

(1′) Objects are maps $p = (p_1, p_2)\colon \mathcal{E} \to \mathcal{D}^{\mathrm{op}} \times \mathrm{Fin}_*$ such that p_1 is a Cartesian fibration, p_2 is a co-Cartesian fibration, p_1 sends p_2-co-Cartesian arrows to equivalences and p_2 sends p_1-Cartesian arrows to equivalences. Furthermore, for each $\langle n\rangle$ in Fin_*, the Segal maps induce an equivalence $\mathcal{E}_{\langle n\rangle} \simeq \mathcal{E}_{\langle 1\rangle} \times_{\mathcal{D}^{\mathrm{op}}} \cdots \times_{\mathcal{D}^{\mathrm{op}}} \mathcal{E}_{\langle 1\rangle}$ of Cartesian fibrations over $\mathcal{D}^{\mathrm{op}}$.

(2′) Morphisms are commuting triangles (A-1) such that f sends all p_2-co-Cartesian arrows to q_2-co-Cartesian arrows.

Notice that conditions (1) and (1′) are equivalent. Indeed, consider the Segal map

$$g\colon \mathcal{E}_{\langle n\rangle} \simeq \mathcal{E}_{\langle 1\rangle} \times_{\mathcal{D}^{\mathrm{op}}} \cdots \times_{\mathcal{D}^{\mathrm{op}}} \mathcal{E}_{\langle 1\rangle}$$

between categories over $\mathcal{D}^{\mathrm{op}}$. Then g preserves Cartesian arrows over $\mathcal{D}^{\mathrm{op}}$ if and only if for each $\alpha : d \to d'$ in $\mathcal{D}$, the change of fiber functor $\alpha^*\colon \mathcal{E}_{d'} \to \mathcal{E}_d$ preserves co-Cartesian lifts of inert morphisms in Fin_*. When this is the case, the Segal map is an equivalence if and only if it induces an equivalence between the fibers over each $d \in \mathcal{D}^{\mathrm{op}}$, ie if and only if each $\mathcal{E}_d$ is an SM ∞-category. We therefore obtain two subcategories with the same objects, while on morphisms the condition (2) is clearly stronger than (2′). This yields the desired wide subcategory inclusion. □

Construction A.3 Let $\mathcal{C}$ be an ∞-category with finite products, $\mathcal{D}$ an ∞-category and consider a functor $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{SMCat}^{\mathrm{lax}}$ that sends each arrow in $\mathcal{C}^{\mathrm{op}}$ to a strong SM functor. We will construct from Ψ a natural functor $F_\Psi\colon \mathcal{D} \to \mathrm{SMCat}^{\mathrm{lax}}_{/\mathcal{C}^\times}$, where $F_\Psi(d) \to \mathcal{C}^\times$ is an SM functor whose underlying functor is the Cartesian fibration classified by $\Psi(-, d)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

To do this, note that by adjunction and Lemma A.2, we obtain a natural functor

$$\Psi\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}}_\infty)^{\mathrm{strong}} \to \mathrm{CAlg}(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}}).$$

By [26, Theorem 2.4.3.18], this defines a map of ∞-operads

$$(\mathcal{C}^{\mathrm{op}})^{\amalg} \to \mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl},\times} \simeq \text{co-Cart}(\mathcal{D})^{\mathrm{lax},\times}$$

from the co-Cartesian ∞-operad $(\mathcal{C}^{\mathrm{op}})^{\amalg}$ to the Cartesian operad $\text{co-Cart}(\mathcal{D})^{\mathrm{lax},\times}$. Here the equivalence of Cartesian operads arises from the equivalence sending a Cartesian fibration $\mathcal{E} \to \mathcal{D}^{\mathrm{op}}$ to the opposite co-Cartesian fibration $\mathcal{E}^{\mathrm{op}} \to \mathcal{D}$.

Since the target of the above map is a Cartesian ∞-operad, this is uniquely determined by an $(\mathcal{C}^{\mathrm{op}})^{\amalg}$-monoid object $(\mathcal{C}^{\mathrm{op}})^{\amalg} \to \text{co-Cart}(\mathcal{D})^{\mathrm{lax}}$. The unstraightening of this functor determines a functor

$$p = (p_1, p_2)\colon \mathcal{X}^{\circ,\otimes}_\Psi \to (\mathcal{C}^{\mathrm{op}})^{\amalg} \times \mathcal{D} \tag{A-2}$$

with the following properties:

(1) $p_1\colon \mathcal{X}^{\circ,\otimes}_{\Psi} \to (\mathcal{C}^{\mathrm{op}})^{\amalg}$ is a co-Cartesian fibration and p_2 sends p_1-co-Cartesian arrows to equivalences.

(2) For each $\langle n\rangle \in \mathrm{Fin}_*$, the n inert maps $\sigma_i\colon \langle n\rangle \to \langle 1\rangle$ induce an equivalence

$$\mathcal{X}^{\circ,\otimes}_{\Psi,\langle n\rangle} \to \mathcal{X}^{\circ,\otimes}_{\Psi,\langle 1\rangle} \times_{\mathcal{D}} \cdots \times_{\mathcal{D}} \mathcal{X}^{\circ,\otimes}_{\Psi,\langle 1\rangle}.$$

(3) For each $\langle n\rangle \in \mathrm{Fin}_*$, the map between fibers over $\langle n\rangle$

$$p\colon \mathcal{X}^{\circ,\otimes}_{\Psi,\langle n\rangle} \to (\mathcal{C}^{\mathrm{op}})^{\amalg}_{\langle n\rangle} \times \mathcal{D} \simeq (\mathcal{C}^{\mathrm{op}})^{\times n} \times \mathcal{D}$$

is a co-Cartesian fibration, classified by the functor sending $(c_1,\dots,c_n,d)$ to $\Psi(c_1,d)^{\mathrm{op}}\times\cdots\times\Psi(c_n,d)^{\mathrm{op}}$.

Here (2) is equivalent to $(\mathcal{C}^{\mathrm{op}})^{\amalg} \to \text{co-Cart}(\mathcal{D})^{\mathrm{lax}}$ being a monoid object, after which (3) is equivalent to the fact that the underlying functor $\mathcal{C}^{\mathrm{op}} \to \text{co-Cart}(\mathcal{D})^{\mathrm{lax}}$ corresponds to $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Cat}$ under unstraightening over $\mathcal{D}$.

In particular, these conditions imply that for each $d \in \mathcal{D}$, the map between fibers $p_1\colon p_2^{-1}(d) \to (\mathcal{C}^{\mathrm{op}})^{\amalg}$ is a co-Cartesian fibration of ∞-operads and that each map $d \to d'$ induces a map of ∞-operads $p_2^{-1}(d) \to p_2^{-1}(d')$ over $(\mathcal{C}^{\mathrm{op}})^{\amalg}$. Unraveling the definitions, the co-Cartesian fibration

$$p_1\colon p_2^{-1}(d) \to (\mathcal{C}^{\mathrm{op}})^{\amalg}$$

arises as the co-Cartesian unstraightening of the functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}, \quad c \mapsto \Psi(c,d)^{\mathrm{op}},$$

with lax monoidal structure maps given by

$$\text{(A-3)} \qquad \Psi(c,d)^{\mathrm{op}} \times \Psi(c',d)^{\mathrm{op}} \to \Psi(c\times c',d)^{\mathrm{op}} \times \Psi(c\times c',d)^{\mathrm{op}} \to \Psi(c\times c',d)^{\mathrm{op}}.$$

Here the first map restricts along the maps $c \leftarrow c\times c' \to c'$ and the second map uses the SM structure on $\Psi(c\times c',d)^{\mathrm{op}}$. Similarly, unwinding the construction shows that the co-Cartesian fibration $p_1^{-1}(c_1,\dots,c_n) \to \mathcal{D}$ is classified by the functor sending d to $\Psi(c_1,d)^{\mathrm{op}}\times\cdots\times\Psi(c_n,d)^{\mathrm{op}}$.

Postcomposing with the map $(\mathcal{C}^{\mathrm{op}})^{\amalg} \to \mathrm{Fin}_*$, one can view (A-2) as a map of co-Cartesian fibrations over Fin_*. Let us take the induced map of fiberwise opposite co-Cartesian fibrations, ie the co-Cartesian fibrations classifying the Fin_*-diagram of opposite ∞-categories [3]. This yields a diagram of the form

$$\begin{tikzcd} \mathcal{X}^{\otimes}_{\Psi} \arrow[rr, "{q=(q_1,q_2)}"] \arrow[dr, "r"'] & & \mathcal{C}^{\times} \times \mathcal{D}^{\mathrm{op}} \arrow[dl] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

where $\mathcal{C}^{\times} \to \mathrm{Fin}_*$ is the Cartesian operad associated to the ∞-category with products $\mathcal{C}$ (which is the fiberwise opposite of $(\mathcal{C}^{\mathrm{op}})^{\amalg}$; see [26, Variant 2.4.3.12]). Here the map q has the following properties (which correspond to the properties of p under taking fiberwise opposites over Fin_*):

(1) q is a map of co-Cartesian fibrations over Fin_* preserving co-Cartesian arrows.

(2) For each $\langle n\rangle \in \mathrm{Fin}_*$, the n inert maps $\sigma_i\colon \langle n\rangle \to \langle 1\rangle$ induce an equivalence

$$\mathcal{X}^{\otimes}_{\Psi,\langle n\rangle} \to \mathcal{X}^{\otimes}_{\Psi,\langle 1\rangle} \times_{\mathcal{D}} \cdots \times_{\mathcal{D}} \mathcal{X}^{\circ,\otimes}_{\Psi,\langle 1\rangle}.$$

(3) For each $\langle n\rangle$ in Fin_*, the map on fibers $\mathcal{X}^{\otimes}_{\Psi,\langle n\rangle} \to \mathcal{C}^{\times}_{\langle n\rangle} \times \mathcal{D}^{\mathrm{op}} \simeq \mathcal{C}^{\times n} \times \mathcal{D}^{\mathrm{op}}$ is a Cartesian fibration classified by the functor sending $(c_1,\ldots,c_n,d) \mapsto \Psi(c_1,d)\times\cdots\times\Psi(c_n,d)$.

In particular, the map $(r,q_2)\colon \mathcal{X}^{\otimes}_{\Psi} \to \mathrm{Fin}_* \times \mathcal{D}^{\mathrm{op}}$ has the property that (1) r is a co-Cartesian fibration and that q_2 sends r-co-Cartesian arrows to equivalences in $\mathcal{D}^{\mathrm{op}}$ and (2) for each $\langle n\rangle \in \mathrm{Fin}_*$, the map $q_2\colon \mathcal{X}^{\otimes}_{\Psi,\langle n\rangle} \to \mathcal{D}^{\mathrm{op}}$ is a Cartesian fibration. It follows from [19, Proposition 2.3.3] that $q_2\colon \mathcal{X}^{\otimes}_{\Psi} \to \mathcal{D}^{\mathrm{op}}$ is a Cartesian fibration and that q is a map of Cartesian fibrations over $\mathcal{D}^{\mathrm{op}}$ (preserving Cartesian arrows). We can therefore apply straightening over $\mathcal{D}$, and the above three conditions then imply that the straightening determines the desired functor $F_{\Psi}\colon \mathcal{D} \to \mathrm{SMCat}^{\mathrm{lax}}_{/\mathcal{C}^{\times}}$.

Proof of Proposition 5.3 Recall that there is a functor $\mathrm{Op}_{\infty}^{\mathrm{op}} \times \mathrm{Op}_{\infty} \to \mathrm{Op}_{\infty}$ sending $(\mathcal{O},\mathcal{P})$ to the ∞-operad of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\otimes}$. This takes values in $\mathrm{SMCat}^{\mathrm{lax}}_{\infty}$ if $\mathcal{P}$ is an SM ∞-category [26, Example 3.2.4.4], in which case restriction along $\mathcal{O} \to \mathcal{O}'$ determines an SM functor $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{P}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$. We can thus apply Construction A.3 to the functor $\mathcal{S}^{\mathrm{op}} \times \mathrm{SMCat}^{\mathrm{lax}}_{\infty} \to \mathrm{SMCat}^{\mathrm{lax}}_{\infty}$ sending $X \mapsto \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V})$. $\square$

Proof of Lemma 5.4 Unraveling Construction A.3, the structure of the SM functor $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \to \mathcal{S}$ arises as the straightening over Fin_* of the map $q_1\colon \mathrm{Alg}^{\otimes}_{\mathrm{Cat}}(\mathcal{V}) = q_2^{-1}(\mathcal{V}) \to \mathcal{S}^{\times}$. By construction, the straightening of q_1 is the pointwise opposite of the straightening of $p_1\colon p_2^{-1}(\mathcal{V}) \to (\mathcal{S}^{\mathrm{op}})^{\amalg}$ over Fin_* by taking fiberwise opposites over Fin_*. Consequently, the tensor product on $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is arises as the unstraightening of the opposite of the natural transformation (A-3), as desired. $\square$

Proof of Proposition 5.6 Construction A.3 has the following general property: for any $f\colon \mathcal{D}' \to \mathcal{D}$ and $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{SMCat}^{\mathrm{lax}}$, the functor $F_{\Psi\circ(\mathrm{id}\times f)}\colon \mathcal{D}' \to \mathrm{SMCat}^{\mathrm{lax}}_{/\mathcal{C}^{\times}}$ is naturally equivalent to the functor $F_{\psi} \circ f$. Consequently, the left-bottom composite in Proposition 5.6 arises by applying Construction A.3 to the functor $\Psi_{\mathcal{J}}\colon \mathcal{S}^{\mathrm{op}} \times \mathrm{SMCat}^{\mathrm{lax}} \to \mathrm{SMCat}^{\mathrm{lax}}$ sending $(X,\mathcal{V})$ to $\mathrm{Alg}_{\mathcal{O}_X}(\mathrm{Fun}(\mathcal{J},\mathcal{V}))^{\otimes}$.

Now notice that $\Psi_{\mathcal{J}}$ is equivalent to the functor sending $(X,\mathcal{V})$ to $\mathrm{Fun}(\mathcal{J},\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V}))$ with the levelwise tensor product (by adjunction to the Boardman–Vogt tensor product, see Remark 2.7). The result will therefore follow from the following general claim about Construction A.3: for any $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{SMCat}^{\mathrm{lax}}$ and any ∞-category $\mathcal{J}$, applying Construction A.3 to the functor $\Psi_{\mathcal{J}}(c,d) = \mathrm{Fun}(\mathcal{J},\Psi(c,d))$ results in the composite functor

$$F_{\Psi_{\mathcal{J}}}\colon \mathcal{D} \xrightarrow{F_{\Psi}} \mathrm{SMCat}^{\mathrm{lax}}_{\infty/\mathcal{C}^{\times}} \xrightarrow{\mathrm{Fun}(\mathcal{J},-)\times_{\mathrm{Fun}(\mathcal{J},\mathcal{C}^{\times})}\mathcal{C}^{\times}} \mathrm{SMCat}^{\mathrm{lax}}_{\infty/\mathcal{C}^{\times}}. \tag{A-4}$$

To see this, recall the inclusion $\mathrm{Fun}(\mathcal{D},\mathrm{SMCat}^{\mathrm{lax}}) \hookrightarrow \mathrm{CAlg}(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}})$ from Lemma A.2, which was given by unstraightening over $\mathcal{D}$. Under this inclusion, applying $\mathrm{Fun}(\mathcal{J},-)$ pointwise corresponds to sending a Cartesian fibration $\mathcal{E}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ to the Cartesian fibration $\mathrm{Fun}(\mathcal{J},\mathcal{E}^{\mathrm{op}}) \times_{\mathrm{Fun}(\mathcal{J},\mathcal{D}^{\mathrm{op}})} \mathcal{D}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. This implies that the monoid object determined by $\Psi_{\mathcal{J}}$ is given by the composite

$$(\mathcal{C}^{\mathrm{op}})^{\amalg} \to \text{co-Cart}(\mathcal{D})^{\mathrm{lax}} \to \text{co-Cart}(\mathcal{D})^{\mathrm{lax}},$$

where the first functor is the monoid object associated to Ψ and the second functor sends a co-Cartesian fibration $\mathcal{E} \to \mathcal{D}$ to $\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{E}) \times_{\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{D})} \mathcal{D} \to \mathcal{D}$ (note that we took opposite categories to pass from Cartesian to co-Cartesian fibrations). Next, applying the same reasoning to the unstraightening over $(\mathcal{C}^{\mathrm{op}})^{\amalg}$, we obtain that

$$\mathcal{X}^{\circ,\otimes}_{\Psi_{\mathcal{J}}} \simeq \mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{X}^{\circ,\otimes}_{\Psi}) \times_{\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, (\mathcal{C}^{\mathrm{op}})^{\amalg} \times \mathcal{D})} (\mathcal{C}^{\mathrm{op}})^{\amalg} \times \mathcal{D}.$$

Taking fiberwise opposite co-Cartesian fibrations over Fin_*, one then obtains an equivalence of Cartesian fibrations over $\mathcal{D}^{\mathrm{op}}$

$$\mathcal{X}^{\otimes}_{\Psi_{\mathcal{J}}} \simeq \mathrm{Fun}(\mathcal{J}, \mathcal{X}^{\otimes}_{\Psi}) \times_{\mathrm{Fun}(\mathcal{J}, \mathcal{C}^{\times} \times \mathcal{D}^{\mathrm{op}})} \mathcal{C}^{\times} \times \mathcal{D}^{\mathrm{op}}.$$

Under straightening over $\mathcal{D}^{\mathrm{op}}$, this equivalence provides the desired identification of $F_{\Psi_{\mathcal{J}}}$ as in (A-4). □

References

[1] **M Ando**, **A J Blumberg**, **D Gepner**, **M J Hopkins**, **C Rezk**, *An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology*, J. Topol. 7 (2014) 869–893 MR

[2] **C Barwick**, *From operator categories to higher operads*, Geom. Topol. 22 (2018) 1893–1959 MR Zbl

[3] **C Barwick**, **S Glasman**, **D Nardin**, *Dualizing cartesian and cocartesian fibrations*, Theory Appl. Categ. 33 (2018) 67–94 MR Zbl

[4] **M Basterra**, **M A Mandell**, *Homology and cohomology of E_∞ ring spectra*, Math. Z. 249 (2005) 903–944 MR

[5] **B Calmès**, **E Dotto**, **Y Harpaz**, **F Hebestreit**, **M Land**, **K Moi**, **D Nardin**, **T Nikolaus**, **W Steimle**, *Hermitian K-theory for stable ∞-categories, II: Cobordism categories and additivity*, preprint (2020) arXiv 2009.07224

[6] **W G Dwyer**, **D M Kan**, *An obstruction theory for diagrams of simplicial sets*, Nederl. Akad. Wetensch. Indag. Math. 46 (1984) 139–146 MR Zbl

[7] **W G Dwyer**, **D M Kan**, **J H Smith**, *An obstruction theory for simplicial categories*, Nederl. Akad. Wetensch. Indag. Math. 48 (1986) 153–161 MR Zbl

[8] **D Gepner**, **M Groth**, **T Nikolaus**, *Universality of multiplicative infinite loop space machines*, Algebr. Geom. Topol. 15 (2015) 3107–3153 MR Zbl

[9] **D Gepner**, **R Haugseng**, *Enriched ∞-categories via non-symmetric ∞-operads*, Adv. Math. 279 (2015) 575–716 MR Zbl

[10] **P G Goerss**, **M J Hopkins**, *André–Quillen (co-)homology for simplicial algebras over simplicial operads*, from "Une dégustation topologique", Contemp. Math. 265, Amer. Math. Soc., Providence, RI (2000) 41–85 MR Zbl

[11] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Progr. Math. 174, Birkhäuser, Basel (1999) MR Zbl

[12] **M Groth**, **K Ponto**, **M Shulman**, *The additivity of traces in monoidal derivators*, J. K-Theory 14 (2014) 422–494 MR Zbl

[13] **Y Harpaz**, **J Nuiten**, **M Prasma**, *The abstract cotangent complex and Quillen cohomology of enriched categories*, J. Topol. 11 (2018) 752–798 MR Zbl

[14] **Y Harpaz**, **J Nuiten**, **M Prasma**, *Quillen cohomology of $(\infty, 2)$-categories*, High. Struct. 3 (2019) 17–66 MR Zbl

[15] **Y Harpaz**, **J Nuiten**, **M Prasma**, *The tangent bundle of a model category*, Theory Appl. Categ. 34 (2019) 1039–1072 MR Zbl

[16] **Y Harpaz**, **J Nuiten**, **M Prasma**, *Tangent categories of algebras over operads*, Israel J. Math. 234 (2019) 691–742 MR Zbl

[17] **R Haugseng**, *Rectification of enriched ∞-categories*, Algebr. Geom. Topol. 15 (2015) 1931–1982 MR Zbl

[18] **R Haugseng**, *On lax transformations, adjunctions, and monads in $(\infty, 2)$-categories*, High. Struct. 5 (2021) 244–281 MR Zbl

[19] **R Haugseng**, **F Hebestreit**, **S Linskens**, **J Nuiten**, *Lax monoidal adjunctions, two-variable fibrations and the calculus of mates*, Proc. Lond. Math. Soc. 127 (2023) 889–957 MR Zbl

[20] **V Hinich**, *Rectification of algebras and modules*, Doc. Math. 20 (2015) 879–926 MR Zbl

[21] **V Hinich**, *Dwyer–Kan localization revisited*, Homology Homotopy Appl. 18 (2016) 27–48 MR Zbl

[22] **S Linskens**, **D Nardin**, **L Pol**, *Global homotopy theory via partially lax limits*, preprint (2022) arXiv 2206.01556

[23] **J Lurie**, *Higher topos theory*, Ann. of Math. Stud. 170, Princeton Univ. Press (2009) MR Zbl

[24] **J Lurie**, *On the classification of topological field theories*, from "Current developments in mathematics, 2008", International, Somerville, MA (2009) 129–280 MR Zbl

[25] **J Lurie**, *Spectral algebraic geometry*, preprint (2016) Available at `https://url.msp.org/Lurie-SAG`

[26] **J Lurie**, *Higher algebra*, preprint (2017) Available at `https://url.msp.org/Lurie-HA`

[27] **J P May**, **J Sigurdsson**, *Parametrized homotopy theory*, Math. Surv. Monogr. 132, Amer. Math. Soc., Providence, RI (2006) MR Zbl

[28] **D Nardin**, **J Shah**, *Parametrized and equivariant higher algebra*, preprint (2022) arXiv 2203.00072

[29] **E Riehl**, **D Verity**, *Homotopy coherent adjunctions and the formal theory of monads*, Adv. Math. 286 (2016) 802–888 MR Zbl

[30] **S Schwede**, *Spectra in model categories and applications to the algebraic cotangent complex*, J. Pure Appl. Algebra 120 (1997) 77–104 MR Zbl

[31] **G Segal**, *Categories and cohomology theories*, Topology 13 (1974) 293–312 MR Zbl

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Circular-orderability of 3-manifold groups

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We initiate the study of circular-orderability of 3-manifold groups, motivated by the L-space conjecture. We show that a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold has a circularly orderable fundamental group if and only if there exists a finite cyclic cover with left-orderable fundamental group, which naturally leads to a "circular-orderability version" of the L-space conjecture. We also show that the fundamental groups of almost all graph manifolds are circularly orderable, and contrast the behaviour of circular-orderability and left-orderability with respect to the operations of Dehn surgery and taking cyclic branched covers.

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1 Introduction

For an irreducible, rational homology 3-sphere M, the L-space conjecture posits a relationship between the properties of M admitting a coorientable taut foliation, M being *not* an L-space, and M having a left-orderable fundamental group (see Conjecture 3.3). While this conjecture is known to hold for some classes of manifolds, for example graph manifolds, new techniques are needed to tackle more general classes of manifolds, or, indeed, to tackle the conjecture in full generality.

With this conjecture in mind, several of the most successful techniques developed in recent years to tackle left-orderability of $\pi_1(M)$ have shared a common theme: they all begin with an action on the circle, and use cohomological techniques to pass to an action on the real line. For instance, in studying manifolds arising from Dehn surgery on a knot K in $\mathbb{S}^3$, a common technique is to study one-parameter families of representations $\rho_t\colon \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{PSL}(2,\mathbb{R})$ that are built so as to provide representations that factor through the quotient groups $\pi_1(\mathbb{S}^3_{p/q}(K))$ for certain values of $p/q \in \mathbb{Q} \cup \{\infty\}$. Controlling the Euler classes of these representations allows one to construct lifts $\tilde{\rho}_t\colon \pi_1(\mathbb{S}^3_{p/q}(K)) \to \widetilde{\mathrm{PSL}}(2,\mathbb{R})$, and these lifts show that $\pi_1(\mathbb{S}^3_{p/q}(K))$ is left-orderable since they have left-orderable image; see Boyer, Gordon and Watson [12], Hakamata and Teragaito [29; 30], Motegi and Teragaito [42], Culler and Dunfield [23] and Gao [26]. This technique has also been used to study left-orderability of cyclic branched covers of knots by Hu [34], Tran [51], Turner [52] and Gordon and Lidman [28].

In a similar vein, if one starts with an irreducible rational homology 3-sphere M admitting a coorientable taut foliation $\mathscr{F}$, Thurston's universal circle construction yields a representation $\rho_{\mathrm{univ}}\colon \pi_1(M) \to \mathrm{Homeo}_+(\mathbb{S}^1)$. With appropriate restrictions on $\mathscr{F}$, one can control the Euler class of ρ_{univ} and guarantee the existence of a lift $\widetilde{\rho}_{\text{univ}}\colon \pi_1(M) \to \widetilde{\text{Homeo}}_+(\mathbb{S}^1)$, again yielding left-orderability of $\pi_1(M)$ for similar reasons; see Calegari and Dunfield [16] and Boyer and Hu [13].

Motivated by the utility of actions on the circle by homeomorphisms in addressing the L-space conjecture, this work is a first step toward directly addressing the question of when the fundamental group of a 3-manifold acts on the circle by homeomorphisms — though we take an algebraic approach to the problem. Just as the existence of left-ordering of $\pi_1(M)$ captures whether or not there is an embedding $\rho\colon \pi_1(M) \to \text{Homeo}_+(\mathbb{R})$, we approach the problem by studying the existence of an orientation cocycle $c\colon \pi_1(M)^3 \to \{\pm 1, 0\}$ that is compatible with the group operation, called a circular ordering of $\pi_1(M)$. The existence of such a map determines whether or not there exists an embedding $\rho\colon \pi_1(M) \to \text{Homeo}_+(\mathbb{S}^1)$, analogous to the case of left-orderings. It should be noted that, throughout, we adopt the convention that the trivial group is left-orderable. We show:

Theorem 1.1 *Suppose that M is a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold. Then $\pi_1(M)$ is circularly orderable if and only if M admits a finite cyclic cover with left-orderable fundamental group.*

Our contribution here is not the existence of a finite-index left-orderable subgroup, as this fact already appears implicitly in [16], but that there is a normal, left-orderable subgroup that yields a finite cyclic group upon passing to the quotient. This motivates an obvious "circular-orderability" version of the L-space conjecture (see Conjecture 3.4), which mirrors the usual L-space conjecture up to finite cyclic covers.

This theorem is in fact a special case of a new algebraic result. Associated to every circular ordering c of G is a cohomology class $[f_c] \in H^2(G;\mathbb{Z})$, called the Euler class of the circular ordering. We show that, when a group G admits a circular ordering whose Euler class has order k in $H^2(G;\mathbb{Z})$, it also admits a left-orderable normal subgroup N such that $G/N \cong \mathbb{Z}/k\mathbb{Z}$; see Theorem 2.6.

From here we begin an exploration of exactly which fundamental groups admit circular orderings. We first tackle the case of Seifert fibred manifolds, providing the details of a claim of Calegari [15]. Note that circular-orderability of finite groups is well understood (a finite group is circularly orderable if and only if it is cyclic; see Proposition 2.5), and so we focus on infinite fundamental groups. If G is a group with circular ordering c, we use $\text{rot}_c(g)$ to denote the rotation number of $g \in G$; see Section 2.

Theorem 1.2 *Let M be a compact, connected Seifert fibred space and let h denote the class of a regular fibre in $\pi_1(M)$.*

(1) *If $\pi_1(M)$ is infinite, then there exists a circular ordering c of $\pi_1(M)$ such that $\text{rot}_c(h) = 0$; in particular, $\pi_1(M)$ is circularly orderable whenever it is infinite.*

(2) *If $\pi_1(M)$ is infinite and M is orientable and has nonorientable base orbifold, then every circular ordering c of $\pi_1(M)$ satisfies $\text{rot}_c(h) \in \{0, \frac{1}{2}\}$.*

(3) *If $\pi_1(M)$ is left-orderable and M is orientable and has base orbifold $\mathbb{S}^2(\alpha_1, \dots, \alpha_n)$ with $n \geq 3$, then, for every $p \in \mathbb{N}_{>0}$, there exists a circular ordering c of $\pi_1(M)$ such that $\text{rot}_c(h) = 1/p$.*

(4) *If M is orientable and has no exceptional fibres, then, for every $r \in \mathbb{R}/\mathbb{Z}$, there exists a circular ordering c of $\pi_1(M)$ such that $\mathrm{rot}_c(h) = r$.*

This leads naturally to the study of graph manifolds, where we show that an analogous fact holds.

Theorem 1.3 *Suppose that W is a graph manifold whose JSJ decomposition has Seifert fibred pieces $M_1, \dots, M_n$. Further suppose that, for each $1 \le i \le n$, if ∂M_i is a single torus boundary component, then there is no slope $\alpha \in H_1(\partial M_i; \mathbb{Z})/\{\pm 1\}$ such that $\pi_1(M_i(\alpha))$ is finite. Then, if $\pi_1(W)$ is infinite, it is circularly orderable.*

If W is not a rational homology sphere, then the first Betti number $b_1(W)$ is positive, so $\pi_1(W)$ is left-orderable by Boyer, Rolfsen and Wiest [14, Theorem 3.2]. On the other hand, if W is a rational homology sphere, then Theorem 1.3 is in fact a special case of a stronger, more technical result; see Theorem 6.6 and Corollary 6.7. We conjecture that, with appropriate generalizations of the techniques developed here, one can prove that the fundamental group of a graph manifold is circularly orderable whenever it is infinite. See Conjecture 6.11 and the preceding discussion for details.

Our approach to this proof is to mirror the technique of "slope detection" developed by Boyer and Clay [9] for the case of left-orderings of fundamental groups of graph manifolds; in Theorem 4.3 we develop a result in the case of circular orderings that is analogous to the main tool of Clay, Lidman and Watson [20]. This tool provides sufficient conditions that a manifold $W = M_1 \cup_\phi M_2$ have circularly orderable fundamental group, by requiring that the gluing map ϕ identify slopes on M_1 and M_2 whose fillings yield fundamental groups admitting compatible circular orderings. Using this technique, it turns out that, in many cases, it is sufficient to study fillings along rational longitudes to conclude that $W = M_1 \cup_\phi M_2$ has circularly orderable fundamental group; see Proposition 5.6.

We also deal with several notable cases not covered by Theorem 6.6 or Theorem 1.3; for instance, we also show:

Theorem 1.4 *The fundamental group of a compact, connected Sol manifold is circularly orderable.*

We close with a discussion of circular-orderability of fundamental groups of hyperbolic 3-manifolds. There is a well-known example of a hyperbolic 3-manifold whose fundamental group is not circularly orderable, which is the Weeks manifold; see Calegari and Dunfield [16, Theorem 9]. Therefore, we cannot expect the fundamental groups of hyperbolic 3-manifolds to be circularly orderable whenever they are infinite, as in the case of Seifert fibred manifolds.

Two approaches to the question of left-orderability of fundamental groups of hyperbolic 3-manifolds that have enjoyed success are via cyclic branched covers and via Dehn surgery. In both of these cases, advancements in Heegaard Floer techniques have provided guidance as to the expected behaviour of left-orderability with respect to these constructions. Over the course of several examples, including several infinite families of hyperbolic 3-manifolds having circularly orderable but non-left-orderable fundamental groups, we find that none of the behaviour exhibited by left-orderability with respect to these familiar topological constructions is shared with circular-orderability.

For example, it is suspected that, if the n-fold cyclic branched cover of a knot in $\mathbb{S}^3$ has left-orderable fundamental group, then so does the m-fold cyclic branched cover for all $m \geq n$. This does not hold for circular-orderability; see Propositions 7.1 and 7.2. Similarly, it is conjectured that the double branched cover of a quasialternating knot always has non-left-orderable fundamental group, but our examples show that there is no apparent relationship when left-orderability is replaced with circular-orderability: there exist alternating links (or, more generally, quasialternating links) whose double branched covers have non-circularly orderable fundamental groups (see Section 7.2), while other alternating (or quasialternating) links yield double branched covers with circularly orderable fundamental groups. Similar observations hold for the behaviour of circular-orderability with respect to Dehn surgery on a knot in $\mathbb{S}^3$.

We organize this paper as follows: Section 2 contains background and results relating to circular-orderability and left-orderability of groups in general. In Section 3 we relate these facts to 3-manifold fundamental groups, discuss the L-space conjecture and prove Theorem 1.1. In Section 4 we introduce our tools that are analogous to slope detection by left-orderings, and in Section 5 we show how these results can be applied to fillings along rational longitudes. In Section 6 we study circular-orderability of the fundamental groups of Seifert fibred manifolds and graph manifolds. Finally, in Section 7 we discuss circular-orderability of the fundamental groups of manifolds arising as the cyclic branched covers of links, and manifolds arising from Dehn surgery.

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2 Left- and circular-orderability

A strict total order $<$ on a group G is said to be a *left-ordering* if, for every $f, g, h \in G$, if $g < h$ then $fg < fh$. A group G is called *left-orderable* if it admits a left-ordering. Every left-ordering of G determines a subset $P = \{g \in G \mid g > \mathrm{id}\}$ called the *positive cone* of the ordering, it satisfies

(i) $P \cdot P \subset P$, and

(ii) $P \sqcup P^{-1} = G \setminus \{\mathrm{id}\}$.

Conversely any subset $P \subset G$ satisfying (i) and (ii) determines a left-ordering of G via the prescription

$$g < h \iff g^{-1}h \in P.$$

A *left-circular ordering* of a group G is a map $c \colon G^3 \to \{\pm 1, 0\}$ satisfying:

(1) If $(g_1, g_2, g_3) \in G^3$, then $c(g_1, g_2, g_3) = 0$ if and only if $\{g_1, g_2, g_3\}$ are not all distinct.

(2) For all $g_1, g_2, g_3, g_4 \in G$, we have

$$c(g_1, g_2, g_3) - c(g_1, g_2, g_4) + c(g_1, g_3, g_4) - c(g_2, g_3, g_4) = 0.$$

(3) For all $g, g_1, g_2, g_3 \in G$, we have
$$c(g_1, g_2, g_3) = c(gg_1, gg_2, gg_3).$$

If G admits such a map, then G is called *left-circularly orderable*. When no confusion will arise from doing so, we will write *circular ordering* for short and *circularly orderable*.

Every left-orderable group is circularly orderable, for if $<$ is a left-ordering of G then we may define $c\colon G^3 \to \{\pm 1, 0\}$ by $c(g_1, g_2, g_3) = \operatorname{sign}(\sigma)$ when $\{g_1, g_2, g_3\}$ are distinct and $c(g_1, g_2, g_3) = 0$ otherwise; here σ is the unique permutation such that $g_{\sigma(1)} < g_{\sigma(2)} < g_{\sigma(3)}$. When a circular ordering c of a left-orderable group G arises in this way, we will say that c is a *secret left-ordering*.

Every circular ordering c of G is evidently a homogeneous cocycle. However, from each circular ordering c, we can define an associated inhomogeneous cocycle $f_c\colon G^2 \to \{0, 1\}$ by
$$f_c(g, h) = \begin{cases} 0 & \text{if } g = \mathrm{id} \text{ or } h = \mathrm{id}, \\ 1 & \text{if } gh = \mathrm{id} \text{ and } g \neq \mathrm{id}, \\ \frac{1}{2}(1 - c(\mathrm{id}, g, gh)) & \text{otherwise;} \end{cases}$$
we call $[f_c] \in H^2(G; \mathbb{Z})$ the *Euler class* of the circular ordering c.

Construction 2.1 [54] Associated to $[f_c]$ is a central extension $\widetilde{G}_c$ of G, which is constructed by equipping the set $\mathbb{Z} \times G$ with the operation $(a, g)(b, h) = (a + b + f_c(g, h), gh)$.[1] The central extension $\widetilde{G}_c$ is easily seen to be left-orderable, as one can check that the set $P = \{(a, g) \mid a \geq 0\} \setminus \{(0, \mathrm{id})\}$ defines the positive cone of a left-ordering, which we denote by $<_c$. We call $\widetilde{G}_c$ the *left-ordered central extension associated to the circularly ordered group G with ordering c.*

Recall that a subset S of a left-ordered group $(G, <)$ is *$<$-cofinal* if, for every $g \in G$, there exist elements $s, t \in S$ such that $s < g < t$. An element $g \in G$ is called $<$-cofinal (or simply cofinal if the ordering is understood) whenever the cyclic subgroup $\langle g \rangle$ is $<$-cofinal as a set. The central element $(1, \mathrm{id}) \in \widetilde{G}_c$ is positive and cofinal in the left-ordering $<_c$ of $\widetilde{G}_c$ and is called the *canonical positive*, *cofinal*, *central element of* $\widetilde{G}_c$.

Construction 2.2 [54] The above construction is reversible, in a categorical sense made precise in [17]; the basic construction is as follows. Suppose that G is a left-ordered group with ordering $<$, and there is a central element $z \in G$ which is positive and $<$-cofinal. Then the quotient $G/\langle z \rangle$ inherits a circular ordering defined as follows. For each $g\langle z \rangle \in G/\langle z \rangle$, define the *minimal representative* $\bar{g}$ to be the unique coset representative of $g\langle z \rangle$ satisfying $\mathrm{id} \leq \bar{g} < z$. Then define a circular ordering $c_<$ on $G/\langle z \rangle$ by
$$c_<(g_1\langle z \rangle, g_2\langle z \rangle, g_3\langle z \rangle) = \operatorname{sign}(\sigma),$$
where σ is the unique permutation satisfying $\bar{g}_{\sigma(1)} < \bar{g}_{\sigma(2)} < \bar{g}_{\sigma(3)}$.

[1] This is just an application of the standard construction associating elements of $H^2(G; \mathbb{Z})$, represented by inhomogeneous 2-cocycles, to equivalence classes of central extensions $1 \to \mathbb{Z} \to \widetilde{G} \to G \to 1$.

When G admits a circular ordering c with $[f_c] = \mathrm{id} \in H^2(G;\mathbb{Z})$, then G is left-orderable, because the left-orderable central extension $\widetilde{G}_c$ is isomorphic to $G \times \mathbb{Z}$ (though the ordering c need not be a secret left-ordering for this to happen). It happens that c is a secret left-ordering if and only if $[f_c] = \mathrm{id} \in H_b^2(G;\mathbb{Z})$, where $H_b^2(G;\mathbb{Z})$ is the second bounded cohomology group (see [5] for details).

We also recall the notion of *rotation number* of an orientation-preserving homeomorphism $f\colon \mathbb{S}^1 \to \mathbb{S}^1$, which is connected to a circular ordering and the lift $\widetilde{G}_c$ as follows. For an orientation-preserving homeomorphism $f\colon \mathbb{S}^1 \to \mathbb{S}^1$, one may choose a preimage $\tilde{f} \in \widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$ of $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$ and define the rotation number of f to be

$$\lim_{n\to\infty} \frac{\tilde{f}^n(0)}{n} \mod \mathbb{Z}.$$

We can define the rotation number for an element g of a circularly ordered group (G,c), following [2], in a similar way. Let $z = (1,\mathrm{id}) \in \widetilde{G}_c$ denote the cofinal, central element of $\widetilde{G}_c$ relative to the ordering $<_c$. Choose an element $\tilde{g} \in \widetilde{G}_c$ such that $q(\tilde{g}) = g$, where $q\colon \widetilde{G}_c \to G$ is the quotient map. For each $n \in \mathbb{Z}$, let a_n denote the unique integer such that

$$z^{a_n} \le \tilde{g}^n < z^{a_n+1},$$

and define

$$\mathrm{rot}_c(g) = \lim_{n\to\infty} \frac{a_n}{n} \mod \mathbb{Z}.$$

Note that this limit always exists by Fekete's lemma, as one can check that the sequence $\{a_n\}_{n=1}^{\infty}$ is superadditive. Is it not difficult, though rather tedious, to show that this notion of rotation number agrees with the "traditional definition" if one uses the circular ordering c of G to create a dynamical realization $\rho_c\colon G \to \mathrm{Homeo}_+(\mathbb{S}^1)$ such that the circular ordering c of G agrees with the circular ordering of the orbit of 0 inherited from the natural circular ordering of $\mathbb{S}^1$ (see [19, Sections 2.2 and 2.4]). In particular, this implies that rotation number is invariant under conjugation, and is a homomorphism from $A \to S^1$ when restricted to any abelian subgroup $A \subset G$. Moreover, the induced homomorphism $A/\ker(\mathrm{rot}_c) \to \mathbb{S}^1$ is order-preserving with left-ordered kernel; see [27, Propositions 5.3 and 6.17; 21, Section 2].

A fundamental tool in constructing circular orderings on a given group G is the lexicographic construction, which we use often throughout this work.

Proposition 2.3 *Let*

$$1 \to K \to G \xrightarrow{q} H \to 1$$

be a short exact sequence of groups.

(1) *If K and H are left-orderable, then G is left-orderable.*

(2) *If K is left-orderable and H admits a circular ordering d, then G admits a circular ordering c satisfying $\mathrm{rot}_d(q(g)) = \mathrm{rot}_c(g)$ for all $g \in G$, and whose restriction to K is secretly a left-ordering.*

Proof Claim (1) is a straightforward exercise and is common in the literature. Claim (2) is less common, so we outline a lexicographic construction following [15, Lemma 2.2.12] and verify the claimed properties of the resulting circular ordering. Suppose K is equipped with the left-ordering $<$. Define a circular ordering $c\colon G \to \{0, \pm 1\}$ as follows. Given three distinct elements $g_1, g_2, g_3 \in G$:

Case 1 If $q(g_i)$ are all distinct, set $c(g_1, g_2, g_3) = d(q(g_1), q(g_2), q(g_3))$.

Case 2 If exactly two of $\{q(g_1), q(g_2), q(g_3)\}$ are equal, we may (by cyclically permuting the arguments and relabelling if necessary) assume that $q(g_1) = q(g_2)$, in which case we declare $c(g_1, g_2, g_3) = 1$ if $g_1^{-1}g_2 > \mathrm{id}$ and $c(g_1, g_2, g_3) = -1$ otherwise.

Case 3 If all of $\{q(g_1), q(g_2), q(g_3)\}$ are equal, then declare $c(g_1, g_2, g_3) = 1$ if and only if $\mathrm{id} < g_1^{-1}g_2 < g_1^{-1}g_3$, up to cyclic permutation.

Note that, if $g_1, g_2, g_3 \in K$, then we define $c(g_1, g_2, g_3)$ by appealing to Case 3, so $c(g_1, g_2, g_3) = 1$ if and only if $g_1 < g_2 < g_3$ up to cyclic permutation. Thus the circular ordering c is a secret left-ordering upon restriction to K.

That $\mathrm{rot}_d(q(g)) = \mathrm{rot}_c(g)$ for all $g \in G$ is proved in [2, Proof of Proposition 4.10]. □

Left-orderability and circular-orderability are also well behaved with respect to free products:

Proposition 2.4 *Let $\{G_i\}_{i \in I}$ be a family of groups. Then*:

1. [53] *The free product $\ast_{i \in I} G_i$ is left-orderable if and only if each group G_i is left-orderable. Moreover, if $<_i$ is a left-ordering of G_i for each $i \in I$, then there exists a left-ordering of $\ast_{i \in I} G_i$ extending the orderings $<_i$.*
2. [3, Theorem 4.2] *The free product $\ast_{i \in I} G_i$ is circularly orderable if and only if each group G_i is circularly orderable. Moreover, if c_i is a circular ordering of G_i for each $i \in I$, then there exists a circular ordering of $\ast_{i \in I} G_i$ extending the orderings c_i.*

Free products with amalgamation are much more finicky, with necessary and sufficient conditions that a free product with amalgamation be left-orderable (resp. circularly orderable) appearing in [6] (resp. [17]).

Tools to obstruct circular-orderability are somewhat rarer than the tools commonly used to obstruct left-orderability. One of the basic tools in this regard is the following fact:

Proposition 2.5 *A finite group is circularly orderable if and only if it is cyclic.*

For a proof from an algebraic point of view, see [18, Proposition 2.8]. A circular ordering c of G may also give rise to a left-orderable subgroup of G, depending on whether or not the corresponding Euler class $[f_c] \in H^2(G; \mathbb{Z})$ has finite order. This allows one to obstruct circular-orderability of a group G by reducing the problem to obstructing left-orderability of certain finite-index subgroups. Calegari and Dunfield [16] use a variant of this theorem, for instance, to show that the fundamental group of the Weeks manifold is not circularly orderable.

Theorem 2.6 *Suppose that c is a circular ordering of G whose Euler class $[f_c] \in H^2(G;\mathbb{Z})$ has order k. Then G contains a left-orderable normal subgroup H such that $G/H \cong \mathbb{Z}/k\mathbb{Z}$.*

Proof Consider the cocycle $kf_c\colon G^2 \to \{0,k\}$ defined by taking k times the inhomogeneous cocycle f_c, and the corresponding central extension $\widetilde{G}_{kf_c}$ constructed as $\mathbb{Z}\times G$ with multiplication $(a,g)(b,h) = (a+b+kf_c(g,h), gh)$. Define a map $\phi\colon \widetilde{G}_c \to \widetilde{G}_{kf_c}$ by $\phi(a,g) = (ka,g)$; one can check that this is an injective homomorphism. Since we assume $[f_c]$ has order k, we know that $[kf_c] = \mathrm{id} \in H^2(G;\mathbb{Z})$, and thus there exists a map $\eta\colon G\to\mathbb{Z}$ satisfying $\eta(\mathrm{id}) = 0$ and $kf_c(g,h) = \eta(g) - \eta(gh) + \eta(h)$ for all $g,h \in G$. This map η allows us to define an isomorphism $\psi\colon \widetilde{G}_{kf_c} \to \mathbb{Z}\times G$ by $\psi(a,g) = (a+\eta(g), g)$.

Let H denote the subgroup $(\psi\circ\phi)(\widetilde{G}_c) \cap (\{0\}\times G)$ of $\{0\}\times G$, meaning that

$$H = \{(0,g) \mid \exists a \in \mathbb{Z} \text{ such that } ka + \eta(g) = 0\}.$$

Then H is clearly left-orderable since it is the image under an injective map of a left-orderable group.

Let $q_k\colon \mathbb{Z}\to\mathbb{Z}/k\mathbb{Z}$ denote the quotient map. To the equation $\eta(gh) = \eta(g) + \eta(h) - kf_c(g,h)$ we apply the homomorphism q_k to arrive at $(q_k\circ\eta)(gh) = (q_k\circ\eta)(g) + (q_k\circ\eta)(h)$ and conclude that $q_k\circ\eta\colon G\to\mathbb{Z}/k\mathbb{Z}$ is a homomorphism. Moreover, $g \in \ker(q_k\circ\eta)$ if and only if $\eta(g) \equiv 0 \bmod k$, meaning that $(0,g) \in H$. Thus the obvious isomorphism $\{0\}\times G \cong G$ carries H to $\ker(q_k\circ\eta)$.

As $q_k\circ\eta$ has image in $\mathbb{Z}/k\mathbb{Z}$, it remains to argue that $q_k\circ\eta$ is surjective. Suppose the order of $\mathrm{im}(q_k\circ\eta)$ is m; then m divides k and $\frac{k}{m}$ is the largest divisor of k such that $\eta(g) \in \frac{k}{m}\mathbb{Z}$ for all $g\in G$. This means that the function $\zeta(g) = \frac{m}{k}\eta(g)$ satisfies $\zeta(g) - \zeta(gh) + \zeta(h) = mf_c(g,h)$, meaning that $[f_c] \in H^2(G;\mathbb{Z})$ has order dividing m. But $m \le k$ and the order of $[f_c]$ is k, so this forces $m = k$, implying that $q_k\circ\eta$ is surjective. $\square$

3 Fundamental groups of 3-manifolds and orderability

First, we note that every compact 3-manifold other than $\mathbb{S}^3$ admits a decomposition

$$M \cong M_1 \# M_2 \# \cdots \# M_n$$

into prime 3-manifolds, and, as such, the fundamental group of a nonprime 3-manifold M can be expressed as a free product

$$\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_n).$$

In light of Proposition 2.4(2), the question of circular-orderability of fundamental groups of 3-manifolds reduces to considering the fundamental groups of prime 3-manifolds. In fact, since the only reducible orientable prime 3-manifold is $\mathbb{S}^1\times\mathbb{S}^2$, whose fundamental group is clearly circularly orderable, in the case of orientable 3-manifolds it suffices to consider only the fundamental groups of irreducible orientable 3-manifolds. In the case of a nonorientable 3-manifold M, the first Betti number is positive whenever M is $\mathbb{P}^2$-irreducible, so M has circularly orderable fundamental group [14, Theorem 1.1].

Remark 3.1 On the other hand, if M is $\mathbb{P}^2$-reducible, then we do not know whether or not $\pi_1(M)$ is circularly orderable in general. Our techniques for producing circular orderings of 3-manifold groups make frequent use of algebraic properties that depend heavily on the situation at hand. For example, our main tools use left-orderability of infinite-index subgroups of $\pi_1(M)$ as in Proposition 3.2, or a decomposition of $\pi_1(M)$ as a free product with amalgamation of groups whose circular orderings are well understood as in Theorem 4.3 (whose proof also uses Proposition 3.2). Neither of these properties hold for the fundamental groups of $\mathbb{P}^2$-reducible manifolds; in particular, their fundamental groups always contain a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [25, Theorem 8.2], and so any argument that relies on left-orderability of infinite-index subgroups will fail.

It is well known that left-orderability behaves in a very special way with respect to fundamental groups of irreducible 3-manifolds. One of the key results in the area is the following, which we present alongside a generalization to the case of circular-orderability:

Proposition 3.2 *Let M be a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold, let G be a nontrivial group and suppose there exists an epimorphism $\rho\colon \pi_1(M) \to G$.*

(1) *If G is left-orderable, then $\pi_1(M)$ is left-orderable.*

(2) *If G is infinite and circularly orderable, then $\pi_1(M)$ is circularly orderable.*

Proof The first claim is [14, Theorem 3.2]. For the second, since $G = \rho(\pi_1(M))$ is infinite, $K = \ker(\rho)$ is an infinite-index normal subgroup of $\pi_1(M)$. From [14, Proof of Theorem 3.2], the subgroup K is therefore locally indicable and thus left-orderable, so we can use the short exact sequence $1 \to K \to \pi_1(M) \to G \to 1$ to construct a lexicographic circular ordering of $\pi_1(M)$ as in Proposition 2.3(2). □

This implies, for instance, that $\pi_1(M)$ is left-orderable whenever M satisfies the hypotheses of Proposition 3.2 and $H_1(M;\mathbb{Z})$ is infinite. The case of interest is therefore when $H_1(M;\mathbb{Z})$ is finite, where the L-space conjecture posits a connection between left-orderability of $\pi_1(M)$, the existence of coorientable taut foliations in M, and whether or not M is a Heegaard Floer homology L-space (that is, a manifold whose Heegaard Floer homology is of minimal rank).

Conjecture 3.3 (the L-space conjecture [12; 37]) If M is an irreducible, rational homology 3-sphere other than $\mathbb{S}^3$, then the following are equivalent:[2]

(1) The fundamental group of M is left-orderable.

(2) The manifold M supports a coorientable taut foliation.

(3) The manifold M is not an L-space.

Surrounding this conjecture, there are many tools and techniques to obstruct left-orderability of fundamental groups — see [16; 24; 22; 1], to name a few — and many more to prove that fundamental groups are left-orderable. Our main contribution in this section is to connect circular-orderability to this conjecture

[2]We require the caveat that $M \neq \mathbb{S}^3$ because, by our definition, the trivial group is left-orderable.

as in Theorem 1.1 of the introduction, by applying Theorem 2.6 and Proposition 3.2. Recall that a *finite cyclic cover* is, by definition, a regular covering space for which the group of deck transformations is a finite cyclic group.

Proof of Theorem 1.1 If $H_1(M;\mathbb{Z})$ is infinite, then there exists a surjection $\pi_1(M) \to \mathbb{Z}$. Therefore $\pi_1(M)$ is left-orderable by [14, Theorem 1.1]. It follows that $\pi_1(M)$ is circularly orderable, and all finite cyclic covers (including the trivial cover) have left-orderable fundamental group, so there is nothing to prove in this case.

On the other hand, suppose $H_1(M;\mathbb{Z})$ is finite and $\pi_1(M)$ is circularly orderable with circular ordering c. Note that, by a standard Euler characteristic argument (see for example [14, Lemma 3.3]), either M is closed and orientable, or M has nonempty boundary containing only $\mathbb{S}^2$ and $\mathbb{P}^2$ components. Since M is $\mathbb{P}^2$-irreducible, the latter case does not occur.

We conclude $H_1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$ by Poincaré duality, and then $H^2(M;\mathbb{Z}) \cong H^2(\pi_1(M);\mathbb{Z})$ since M is irreducible. Thus $[f_c]$ has finite order; say it has order k. Then, by Theorem 2.6, $\pi_1(M)$ admits a normal, left-orderable subgroup H such that $\pi_1(M)/H$ is cyclic. In this case, the cover $\widetilde{M}$ of M with $\pi_1(\widetilde{M}) = H$ has the desired properties.

To show the converse, suppose that $p\colon \widetilde{M} \to M$ is a finite cyclic cover with left-orderable fundamental group. Then there is a short exact sequence

$$1 \to p_*(\pi_1(\widetilde{M})) \to \pi_1(M) \to \mathbb{Z}/k\mathbb{Z} \to 1$$

for some $k \geq 1$, where the kernel is left-orderable and the quotient (if nontrivial) is circularly orderable. If the quotient is trivial, this means that $\pi_1(M)$ itself is left-orderable, and the conclusion follows. If the quotient is nontrivial, then Proposition 2.3(2) finishes the proof. $\square$

Circular-orderability is therefore one possible approach to the L-space conjecture, by first tackling the conjecture up to finite cyclic covers. That is, we have the following "circular-orderability" version of Conjecture 3.3:

Conjecture 3.4 (the L-space conjecture, circular-orderability version) If M is an irreducible, rational homology 3-sphere that is not a lens space, then the following are equivalent:[3]

(1) The fundamental group of M is circularly orderable.

(2) There exists a finite cyclic cover $\widetilde{M}$ of M that supports a coorientable taut foliation.

(3) There exists a finite cyclic cover $\widetilde{M}$ of M that is not an L-space.

Note that this has conjectural implications beyond what would follow from the "left-orderability" version of the conjecture. For instance, the following theorem connects certain topological properties directly to circular-orderability (without passing via left-orderability):

[3] If we allow M to be a lens space, then the conjecture as stated would not be true, again because the trivial group is left-orderable by our conventions.

Theorem 3.5 [16, Theorem 3.1, Corollary 3.9] *If M is an orientable, atoroidal 3-manifold containing a very full tight essential lamination or supporting a pseudo-Anosov flow, then $\pi_1(M)$ is circularly orderable.*

Conjecture 3.4 therefore predicts that one can obstruct the existence of a pseudo-Anosov flow on a rational homology 3-sphere M, or the existence of a very full tight essential lamination in M, by showing, for example, that all finite cyclic covers are L-spaces or that no finite cyclic cover supports a coorientable taut foliation.

4 Slope detection and 3-manifolds having nontrivial JSJ decomposition

One of the key techniques in the analysis of manifolds admitting a nontrivial JSJ decomposition is that of *slope detection* by left-orderings. This idea first appeared in a basic form in [20, Definition 2.6], was further developed in [9] as a key technique in proving Conjecture 3.3 for graph manifolds [9; 31], and appears also in [10; 11]. We recall the central technique from [20] in the following theorem:

Theorem 4.1 [20, Theorem 2.7] *Suppose that M_1 and M_2 are 3-manifolds with incompressible torus boundaries ∂M_i, and that $\phi\colon \partial M_1 \to \partial M_2$ is a homeomorphism such that $W = M_1 \cup_\phi M_2$ is irreducible. If there exists a slope α such that both $\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$ are left-orderable, then $\pi_1(W)$ is left-orderable.*

In what follows, we develop a generalization of this technique that applies to circular-orderability and present applications to various classes of manifolds. To prepare, we recall the following construction.

If (G, c) and (H, d) are circularly ordered groups and $\phi\colon G \to H$ is a homomorphism satisfying

$$c(g_1, g_2, g_3) = d(\phi(g_1), \phi(g_2), \phi(g_3)) \quad \text{for all } g_1, g_2, g_3 \in G,$$

then we say that ϕ is *order-preserving* or *compatible with the pair* (c, d). Note that in this case ϕ is necessarily injective. Then we can define $\tilde{\phi}\colon \widetilde{G}_c \to \widetilde{H}_d$ by $\tilde{\phi}(n, g) = (n, \phi(g))$, so that $\tilde{\phi}$ is order-preserving with respect to the left-orderings $<_c$ and $<_d$ of $\widetilde{G}_c$ and $\widetilde{H}_d$, respectively (or *compatible with the pair* $(<_c, <_d)$).

Proposition 4.2 *Suppose that $\{(G_i, c_i)\}_{i \in I}$ are circularly ordered groups each containing a subgroup H_i. If (D, d) is a cyclic circularly ordered group and $\phi_i : D \to H_i$ is an isomorphism compatible with the pair (d, c_i) for every i, then the free product with amalgamation $\mathop{\ast}_{i \in I} G_i(D \cong_{\phi_i} H_i)$ is circularly orderable.*

Proof The homomorphism $\phi_i \phi_j^{-1}$ is compatible with the pair (c_j, c_i) for all $i, j \in I$, and therefore the map $\widetilde{\phi_i \phi_j^{-1}}$ is also compatible with the pair $(<_{c_j}, <_{c_i})$ for all $i, j \in I$.

Now suppose that $D \cong \mathbb{Z}$, in which case $\widetilde{D}_d \cong \mathbb{Z} \times \mathbb{Z}$ and the images $\tilde{\phi}_i(\widetilde{D}_d)$ and $\tilde{\phi}_j(\widetilde{D}_d)$ are bounded neither from above nor below in $(\widetilde{G_i})_{c_i}$ and $(\widetilde{G_j})_{c_j}$, respectively, since each image contains the cofinal central element of the respective extension. We may therefore apply [17, Proposition 5.6] to conclude that the group $\mathop{\ast}_{i\in I} (\widetilde{G_i})_{c_i}(\widetilde{D}_d \cong_{\widetilde{\phi}_i} (\widetilde{H_i})_{c_i})$ is left-orderable, and then apply [17, Theorem 1] to conclude that $\mathop{\ast}_{i\in I} G_i(D \cong_{\phi_i} H_i)$ is circularly orderable in this case.

Next suppose that D is finite, in which case $\widetilde{D}_d$ is infinite cyclic. Then $\mathop{\ast}_{i\in I} (\widetilde{G_i})_{c_i}(\widetilde{D}_d \cong_{\widetilde{\phi}_i} (\widetilde{H_i})_{c_i})$ is an amalgamation of left-orderable groups along a cyclic subgroup, which is always left-orderable [6]. As above, that $\mathop{\ast}_{i\in I} G_i(D \cong_{\phi_i} H_i)$ is circularly orderable then follows from [17, Theorem 1]. $\square$

In preparation for the next theorem, suppose that G is a group and let $g \in G$. We denote by $\langle\langle g\rangle\rangle$ the normal closure of $g \in G$, that is, the smallest (with respect to inclusion) normal subgroup of G containing g.

Theorem 4.3 *Let M_1 and M_2 be two 3-manifolds with incompressible torus boundaries, and let $\phi\colon \partial M_1 \to \partial M_2$ be a homeomorphism such that $M = M_1 \cup_\phi M_2$ is $\mathbb{P}^2$-irreducible. If there exists a slope $\alpha \in H_1(\partial M_1;\mathbb{Z})/\{\pm 1\}$ such that $\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$ are infinite circularly orderable groups, and either*

1. *at least one of $\pi_1(\partial M_1) \subset \langle\langle\alpha\rangle\rangle$ or $\pi_1(\partial M_2) \subset \langle\langle\phi_*(\alpha)\rangle\rangle$ holds, or*
2. *$\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$ admit circular orderings c_1 and c_2, respectively, such that the induced map*

$$\bar{\phi}_*\colon \pi_1(\partial M_1)/(\langle\langle\alpha\rangle\rangle \cap \pi_1(\partial M_1)) \to \pi_1(\partial M_2)/(\langle\langle\phi_*(\alpha)\rangle\rangle \cap \pi_1(\partial M_2))$$

is an isomorphism between nontrivial groups which is compatible with the pair (c_1, c_2),

then $\pi_1(M)$ is circularly orderable.

Proof Suppose first that there exists a slope α satisfying condition (1) of the theorem; without loss of generality we assume that $\pi_1(\partial M_1) \subset \langle\langle\alpha\rangle\rangle$. To simplify notation, let $G_i = \pi_1(M_i)$ for $i = 1, 2$, each equipped with an inclusion homomorphism $f_i\colon \mathbb{Z}\oplus\mathbb{Z} \to G_i$ that identifies the peripheral subgroup $\pi_1(\partial M_i)$ with $\mathbb{Z}\oplus\mathbb{Z}$, satisfying $\phi_* \circ f_1 = f_2$, and let $q_1\colon G_1 \to G_1/\langle\langle\alpha\rangle\rangle$ and $q_2\colon G_2 \to G_2/\langle\langle\phi_*(\alpha)\rangle\rangle$ be the quotient maps.

In this case, there exists a unique map f such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} \arrow[r, "f_2"] \arrow[d, "f_1"'] & G_2 \arrow[d] \arrow[rdd, bend left, "1"] & \\
G_1 \arrow[r] \arrow[rrd, bend right, "q_1"'] & G_1 *_{\phi_*} G_2 \arrow[rd, dashed, "f"] & \\
& & G_1/\langle\langle\alpha\rangle\rangle
\end{tikzcd}$$

As the image of the map f is an infinite circularly orderable group, that $\pi_1(M) \cong G_1 *_{\phi_*} G_2$ is circularly orderable follows from Proposition 3.2(2).

Suppose there exists a slope α satisfying condition (2) of the theorem. Let $\bar{\phi}_*$ denote the map induced by ϕ, as in the statement of the theorem. There are two possibilities for the subgroup $\langle\langle\alpha\rangle\rangle \cap \pi_1(\partial M_1)$ of

$\pi_1(\partial M_1) \cong \mathbb{Z} \oplus \mathbb{Z}$: it is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \oplus n\mathbb{Z}$ for some $n > 1$. In both cases, $\bar{\phi}_*$ is an isomorphism between cyclic subgroups of G_1 and G_2 that is compatible with (c_1, c_2), and so $G_1/\langle\langle\alpha\rangle\rangle *_{\bar{\phi}_*} G_2/\langle\langle\phi_*(\alpha)\rangle\rangle$ is circularly orderable by Proposition 4.2.

Then, as before, there is a unique map f such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z} \arrow[r, "f_2"] \arrow[d, "f_1"'] & G_2 \arrow[d] \arrow[ddr, bend left, "q_2"] & \\
G_1 \arrow[r] \arrow[drr, bend right, "q_1"'] & G_1 *_{\phi_*} G_2 \arrow[dr, dashed, "f"] & \\
& & G_1/\langle\langle\alpha\rangle\rangle *_{\bar{\phi}_*} G_2/\langle\langle\phi_*(\alpha)\rangle\rangle
\end{tikzcd}$$

Now, since the groups $\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$ are infinite, so is $G_1/\langle\langle\alpha\rangle\rangle *_{\bar{\phi}_*} G_2/\langle\langle\phi_*(\alpha)\rangle\rangle$. That $\pi_1(M) \cong G_1 *_{\phi_*} G_2$ is circularly orderable follows from Proposition 3.2. $\square$

Note that this recovers Theorem 4.1 in the case that the quotients are both left-orderable.

Recall that $\mathrm{rot}_c : G \to \mathbb{S}^1$ is an order-preserving homomorphism upon restriction to any abelian subgroup, so long as it is injective. From this observation and Theorem 4.3, we arrive at the following:

Corollary 4.4 *Let M_1, M_2, $\phi\colon \partial M_1 \to \partial M_2$ and $M = M_1 \cup_\phi M_2$ be as in Theorem 4.3, and $\alpha \in H_1(\partial M_1; \mathbb{Z})/\{\pm 1\}$ be such that $\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$ are infinite circularly orderable groups. Let $\beta \in \pi_1(\partial M_1)$ denote a dual class to α, and let $q_1\colon \pi_1(M_1) \to \pi_1(M_1(\alpha))$ and $q_2\colon \pi_1(M_2) \to \pi_1\big(M_2(\phi_*(\alpha))\big)$ denote the quotient maps. If there exist circular orderings c_1 and c_2 of $\pi_1(M_1(\alpha))$ and $\pi_1\big(M_2(\phi_*(\alpha))\big)$, respectively, such that*

$$c_1(q_1(\beta^j), q_1(\beta^k), q_1(\beta^l)) = c_2\big(q_2(\phi_*(\beta^j)), q_2(\phi_*(\beta^k)), q_2(\phi_*(\beta^l))\big)$$

for all $j, k, l \in \mathbb{Z}$, then $\pi_1(M)$ is circularly orderable. In particular, if

$$\mathrm{rot}_{c_1}(q_1(\beta)) = \mathrm{rot}_{c_2}\big(q_2(\phi_*(\beta))\big)$$

and rot_{c_i} are injective, then $\pi_1(M)$ is circularly orderable.

Applications of Theorem 4.3 or Corollary 4.4 therefore hinge upon being able to construct circular orderings of fundamental groups of 3-manifolds where certain elements have prescribed rotation number.

5 Rational longitudes and knot manifolds

Recall that a knot manifold is a compact, connected, irreducible and orientable 3-manifold with boundary an incompressible torus. In this section we demonstrate a technique for creating circular orderings of fundamental groups of knot manifolds, where the cyclic subgroup generated by a class dual to the rational longitude has a prescribed circular ordering.

Lemma 5.1 *Suppose that C is a cyclic group, D is a nontrivial subgroup of C, and c is a circular ordering of D. Then there exists a circular ordering c' of C such that*

$$c'(r,s,t) = c(r,s,t) \quad \text{for all } r,s,t \in D.$$

Proof We first consider the case where $C = \mathbb{Z}$ is infinite cyclic and $D = k\mathbb{Z}$. Let c be an arbitrary circular ordering of $k\mathbb{Z}$ and let d denote the standard circular ordering of $\mathbb{S}^1$. Consider the rotation number homomorphism $\mathrm{rot}_c\colon k\mathbb{Z} \to \mathbb{S}^1$ corresponding to the circular ordering c. Suppose that $\mathrm{rot}_c(k) = \exp(2\pi i\alpha)$ with $\alpha \in [0,1]$, and define $\phi\colon \mathbb{Z} \to \mathbb{S}^1$ by $\phi(1) = \exp(2\pi i\alpha/k)$. There are two cases to consider.

First, if rot_c is injective, then $c(r,s,t) = d(\mathrm{rot}_c(r), \mathrm{rot}_c(s), \mathrm{rot}_c(t))$ for all $r,s,t \in k\mathbb{Z}$. Note that ϕ is injective because rot_c is injective, so we can define a circular ordering c' on $\mathbb{Z}$ by $c'(r,s,t) = d(\phi(r), \phi(s), \phi(t))$, which clearly extends c as required.

On the other hand, suppose that rot_c is not injective; say $\alpha = p/q$ with $\gcd(p,q) = 1$. Then c arises lexicographically from a short exact sequence

$$1 \to H \to k\mathbb{Z} \xrightarrow{\mathrm{rot}_c} \mathbb{Z}/q\mathbb{Z} \to 1,$$

where $\mathbb{Z}/q\mathbb{Z}$ is identified naturally with the q^{th} roots of unity and equipped with the restriction of the natural circular ordering d of $\mathbb{S}^1$. In this case ϕ yields a short exact sequence

$$1 \to H \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/qk\mathbb{Z} \to 1,$$

where $\mathbb{Z}/qk\mathbb{Z}$ is again equipped with the natural circular ordering arising from the natural embedding into $\mathbb{S}^1$. Thus, if we use the latter short exact sequence to lexicographically define a circular ordering c' of $\mathbb{Z}$, using the same left-ordering of H as in the former short exact sequence, then c' will be an extension of the given circular ordering c of $k\mathbb{Z}$.

When C is finite, rot_c is injective, so we can use the same construction as in the first case above. □

The next lemma is a standard result, but it is essential to our arguments and so we include a proof.

Lemma 5.2 *Let M be a compact, connected, orientable 3-manifold with a torus boundary, and $\mathbb{F}$ be a field. If $i\colon \partial M \to M$ denotes the inclusion map, then the image of the map*

$$i_*^1\colon H_1(\partial M;\mathbb{F}) \to H_1(M;\mathbb{F})$$

is of rank one.

Proof Considering the pair $(M, \partial M)$, we have the long exact sequence

$$\cdots \to H_1(\partial M;\mathbb{F}) \xrightarrow{i_*^1} H_1(M;\mathbb{F}) \xrightarrow{p_*^1} H_1(M,\partial M;\mathbb{F}) \xrightarrow{\partial_1} H_0(\partial M;\mathbb{F}) \xrightarrow{i_*^0} H_0(M;\mathbb{F}) \xrightarrow{\partial_0} 0.$$

Since M is connected, it is also path-connected, and hence $H_0(M,\partial M;\mathbb{F}) = 0$. By exactness, we have $\mathrm{im}(i_*^1) = \ker(p_*^1)$, $\mathrm{im}(p_*^1) = \ker(\partial_1)$, $\mathrm{im}(\partial_1) = \ker(i_*^0)$ and $\mathrm{im}(i_*^0) = H_0(M;\mathbb{F})$. By the rank–nullity theorem,

$$\dim_{\mathbb{F}}(\mathrm{im}(i_*^1)) = \dim_{\mathbb{F}}(H_1(M;\mathbb{F})) - \dim_{\mathbb{F}}(H_1(M,\partial M;\mathbb{F})) + \dim_{\mathbb{F}}(H_0(\partial M;\mathbb{F})) - \dim_{\mathbb{F}}(H_0(M;\mathbb{F})).$$

Since M is compact, by universal coefficient theorems and duality, we have

$$\dim_{\mathbb{F}}(H_1(M,\partial M;\mathbb{F})) = \dim_{\mathbb{F}}(H_2(M;\mathbb{F})).$$

Hence,

$$\begin{aligned}\dim_{\mathbb{F}}(\mathrm{im}(i_*^1)) &= \dim_{\mathbb{F}}(H_1(M;\mathbb{F})) - \dim_{\mathbb{F}}(H_1(M,\partial M;\mathbb{F})) + \dim_{\mathbb{F}}(H_0(\partial M;\mathbb{F})) - \dim_{\mathbb{F}}(H_0(M;\mathbb{F})) \\ &= -\chi(M) + \dim_{\mathbb{F}}(H_3(M;\mathbb{F})) + \dim_{\mathbb{F}}(H_0(\partial M;\mathbb{F})).\end{aligned}$$

Since the boundary of M is not empty, $\dim_{\mathbb{F}}(H_3(M;\mathbb{F})) = 0$. Thus,

$$\dim_{\mathbb{F}}(\mathrm{im}(i_*^1)) = -\chi(M) + \dim_{\mathbb{F}}(H_0(\partial M;\mathbb{F})).$$

Since $\chi(M) = \frac{1}{2}\chi(\partial M) = 0$, we conclude that $\dim_{\mathbb{F}}(\mathrm{im}(i_*^1)) = \dim_{\mathbb{F}}(H_0(\partial M;\mathbb{F})) = 1$. □

It follows that the image of $i_*^1\colon H_1(\partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is also of rank one when M is orientable. The unique primitive element in $H_1(\partial M;\mathbb{Z})$ whose image is of finite order in $H_1(M;\mathbb{Z})$ is referred to as the *rational longitude* of M, and is denoted λ_M.

Remark 5.3 Consider the long exact sequence, with M as in the previous lemma,

$$\cdots \to H_1(\partial M;\mathbb{Z}) \xrightarrow{i_*^1} H_1(M;\mathbb{Z}) \xrightarrow{p_*^1} H_1(M,\partial M;\mathbb{Z}) \xrightarrow{\partial_1} H_0(\partial M;\mathbb{Z}) \xrightarrow{i_*^0} H_0(M;\mathbb{Z}) \xrightarrow{\partial_0} H_0(M,\partial M;\mathbb{Z}) \to 0.$$

Since M and ∂M are both connected, they are also path-connected, and hence $H_0(M,\partial M;\mathbb{Z}) = 0$, $H_0(\partial M;\mathbb{Z}) \cong \mathbb{Z}$ and $H_0(M;\mathbb{Z}) = \mathbb{Z}$. Hence i_*^0 is an isomorphism, and $\mathrm{im}(\partial_1) = \ker(i_*^0) = 0$. This implies that $\mathrm{im}(p_*^1) = \ker(\partial_1) = H_1(M,\partial M;\mathbb{Z})$, and that i_*^1 is surjective if and only if $H_1(M,\partial M;\mathbb{Z}) = 0$. Thus we do not assume surjectivity of i_*^1, which necessitates the use of Lemma 5.1 in the proofs below.

Remark 5.4 If M is a compact, nonorientable, $\mathbb{P}^2$-irreducible 3-manifold, then $\pi_1(M)$ is left-orderable by [14, Lemma 3.3 and Theorem 3.1]; on the other hand, if M is $\mathbb{P}^2$-reducible, then our techniques do not apply in general; see Remark 3.1. Hence, from now on, we assume that all 3-manifolds are orientable.

Proposition 5.5 *Suppose that M is a knot manifold, let μ be the class of any closed curve in ∂M that is dual to λ_M and let q denote the quotient map $q\colon \pi_1(M) \to \pi_1(M(\lambda_M))$. If $M(\lambda_M)$ is irreducible, then, for every circular ordering c' of the cyclic subgroup $\langle q(\mu)\rangle$, there exists a circular ordering c of $\pi_1(M(\lambda_M))$ such that*

$$c'(q(\mu^j), q(\mu^k), q(\mu^l)) = c(q(\mu^j), q(\mu^k), q(\mu^l))$$

for all $j,k,l \in \mathbb{Z}$.

Proof For such a manifold M, it follows that $|H_1(M(\lambda_M);\mathbb{Z})|$ is infinite, with the class of μ being of infinite order by Lemma 5.2. Therefore there exists a map $\psi\colon \pi_1(M(\lambda_M)) \to \mathbb{Z}$ with the image of μ being nontrivial; say $\psi(\mu) = k$. Let c' be a given circular ordering of $\langle q(\mu)\rangle$.

Denote the standard circular ordering of $\mathbb{S}^1$ by d and suppose that $\mathrm{rot}_{c'}(q(\mu)) = r$. Define a map $\phi\colon k\mathbb{Z} \to \mathbb{S}^1$ by $\phi(k) = \exp(2\pi i r)$, and first suppose that $\mathrm{rot}_{c'}$ is injective. Then ϕ is injective, so we may define a circular ordering c'' of $k\mathbb{Z}$ by $c''(r,s,t) = d(\phi(r),\phi(s),\phi(t))$. Next suppose that $\mathrm{rot}_{c'}$ is not injective, so that $r = s/t \in \mathbb{Q}$ (with s/t in lowest terms) and the circular ordering c' of $\langle q(\mu)\rangle$ is lexicographic with respect to the short exact sequence

$$0 \to \ker(\mathrm{rot}_{c'}) \to \langle q(\mu)\rangle \to \mathbb{Z}/t\mathbb{Z} \to 0$$

for some choice of left-ordering of $\ker(\mathrm{rot}_{c'})$. Then ϕ is not injective, and we may use Proposition 2.3(2) to create a circular ordering c'' of $k\mathbb{Z}$ from the sequence $1 \to K \to k\mathbb{Z} \xrightarrow{\phi} \mathbb{S}^1$, where K is the kernel of ϕ, such that our choice of left-ordering of K agrees with the left-ordering of $\ker(\mathrm{rot}_{c'})$ under the isomorphism $k\mathbb{Z} \cong \langle q(\mu)\rangle$ given by $k \mapsto q(\mu)$.

In either case, by Lemma 5.1, there is a circular ordering $\hat{c}$ of $\psi\big(\pi_1(M(\lambda_M))\big) \cong \mathbb{Z}$ that extends c''. That $\pi_1(M(\lambda_M))$ admits a circular ordering c with the required property now follows from Proposition 2.3(2) and the proof of Proposition 3.2(2). □

Following [50, pages 428–431], recall that a Seifert fibred space is a 3-manifold which is foliated by circles, called fibres, such that each circle S has a closed tubular neighbourhood which is a union of fibres and is isomorphic to a fibred solid torus if S preserves the orientation, or a fibred solid Klein bottle if S reverses the orientation. We use h throughout to denote the class of the regular fibre.

Proposition 5.6 *Suppose that M_1 and M_2 are compact, connected and orientable 3-manifolds with torus boundary, $\phi\colon \partial M_1 \to \partial M_2$ is a homeomorphism, and $M = M_1 \cup_\phi M_2$ is irreducible.*

(1) *If $\mathrm{rk}(H_1(M_1;\mathbb{Q})) \geq 2$, then $\pi_1(M)$ is left-orderable.*

(2) *Suppose $\mathrm{rk}(H_1(M_1;\mathbb{Q})) = 1$ and let λ_1 denote the rational longitude of M_1. If $\pi_1\big(M_2(\phi_*(\lambda_1))\big)$ is infinite and circularly orderable and either $M_1(\lambda_1)$ is irreducible or M_1 is Seifert fibred with incompressible boundary, then $\pi_1(M)$ is circularly orderable.*

Proof In what follows, let $i : \partial M_1 \to M_1$ denote the inclusion map.

To prove (1), assume $\mathrm{rk}(H_1(M_1;\mathbb{Q})) \geq 2$. In this case, the image of $H_1(\partial M_1;\mathbb{Z}) \xrightarrow{i_*} H_1(M_1;\mathbb{Z})$ has rank one by Lemma 5.2. Therefore we may compose the Hurewicz map $\pi_1(M_1) \to H_1(M;\mathbb{Z})$ with the quotient $H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{Z})/\mathrm{im}(i_*)$ and obtain a map $\psi : \pi_1(M_1) \to A$, where A is an abelian group of positive rank, satisfying $\psi\big(i_*(\pi_1(\partial M_1))\big) = 0$. In this case, since M is a free product with amalgamation of $\pi_1(M_1)$ and $\pi_1(M_2)$, there exists a surjective homomorphism $\pi_1(M) \to A$ induced by $\psi : \pi_1(M_1) \to A$ and the zero homomorphism $\pi_1(M_2) \to A$. Thus $\pi_1(M)$ admits a torsion-free abelian quotient, so $\pi_1(M)$ is left-orderable by [14, Theorem 3.2].

For (2), suppose that $\mathrm{rk}(H_1(M_1;\mathbb{Q})) = 1$ and let $q_1\colon \pi_1(M_1) \to \pi_1(M_1(\lambda_1))$ and $q_2\colon \pi_1(M_2) \to \pi_1\big(M_2(\phi_*(\lambda_1))\big)$ denote the quotient maps. Equip $\pi_1\big(M_2(\phi_*(\lambda_1))\big)$ with a circular ordering c_2, and let μ_2 denote the class of a curve dual to $\phi(\lambda_1)$. Let μ_1 denote the class of a curve dual to λ_1. In this case, if $M_1(\lambda_1)$ is irreducible, then, by Proposition 5.5, we can equip $\pi_1(M_1(\lambda_1))$ with a circular ordering c_1 that agrees with c_2 under the identification of $\langle q_1(\mu_1)\rangle$ and $\langle q_2(\mu_2)\rangle$ induced by ϕ_*. The result then follows from Corollary 4.4.

On the other hand, if M_1 is Seifert fibred, then $M_1(\lambda_1)$ is irreducible by [33] unless λ_1 is the class of a regular fibre, so we need only consider the case that λ_1 is the class of a regular fibre. For λ_1 to be the class of a regular fibre, M_1 must have nonorientable base orbifold. In this case, since $\mathrm{rk}(H_1(M_1;\mathbb{Q}))=1$ we know the base orbifold must be a once-punctured projective plane and we compute

$$\pi_1(M_1)=\langle a,\gamma_1,\dots,\gamma_n,\mu,h \mid aha^{-1}=h^{-1},\, [\gamma_i,h]=1,\, [\mu,h]=1,\, \gamma_i^{\alpha_i}=h^{\beta_i},\, a^2\mu\gamma_1\dots\gamma_n=1\rangle,$$

where μ is a class dual to $\lambda_1=h$ on ∂M_1. Therefore

$$\pi_1(M_1(\lambda_1))=\langle a,\gamma_1,\dots,\gamma_n,\mu \mid \gamma_i^{\alpha_i}=1,\, a^2\mu\gamma_1\dots\gamma_n=1\rangle,$$

which is isomorphic to the free product with amalgamation

$$\langle\gamma_1 \mid \gamma_1^{\alpha_1}=1\rangle * \cdots * \langle\gamma_n \mid \gamma_n^{\alpha_n}=1\rangle * \langle\mu\rangle *_{\mu\gamma_1\dots\gamma_n=a^{-2}} \langle a\rangle.$$

In this case, we can first equip the infinite cyclic group $\langle\mu\rangle$ with a circular ordering c that agrees with c_2 under the identification of $\langle\mu\rangle$ and $\langle q_2(\mu_2)\rangle$ induced by ϕ_*. By Proposition 2.4, this circular ordering extends to the free product $\langle\gamma_1 \mid \gamma_1^{\alpha_1}=1\rangle * \cdots * \langle\gamma_n \mid \gamma_n^{\alpha_n}=1\rangle * \langle\mu\rangle$, which in turn extends to a circular ordering of $\pi_1(M_1(\lambda_1))$ by combining Lemma 5.1 and Proposition 4.2. Thus the claim follows from Corollary 4.4. □

6 Seifert fibred manifolds and graph manifolds

Aside from this last tool, there are other situations where we can control the rotation numbers of certain elements in circular orderings of fundamental groups. We next investigate Seifert fibred manifolds, where our goal is to show that the fundamental group of a Seifert fibred space is always circularly orderable whenever it is infinite, and to describe the possible circular orderings in terms of the rotation numbers of the class of a regular fibre.

6.1 Seifert fibred manifolds

It was first claimed by Calegari [15, Remark 4.3.2] that, if M is Seifert fibred and $\pi_1(M)$ is infinite, then it is circularly orderable. We provide the details of this claim below.

Recall that, when M is an orientable Seifert fibred manifold over an orientable closed surface of genus $g\geq 0$, the fundamental group has presentation

$$\pi_1(M)=\langle a_1,b_1,\dots,a_g,b_g,\gamma_1,\dots,\gamma_n,h \mid h \text{ central},\, \gamma_i^{\alpha_i}=h^{\beta_i},\, [a_1,b_1]\dots[a_g,b_g]\gamma_1\dots\gamma_n=h^b\rangle$$

and, if the surface is nonorientable, then

$$\pi_1(M)=\langle a_1,\dots,a_g,\gamma_1,\dots,\gamma_n,h \mid a_jha_j^{-1}=h^{-1},\, [\gamma_i,h]=1,\, \gamma_i^{\alpha_i}=h^{\beta_i},\, a_1^2\dots a_g^2\gamma_1\dots\gamma_n=h^b\rangle.$$

In either case, quotienting by the normal subgroup $\langle h \rangle$ (the cyclic subgroup generated by the class of the regular fibre) yields the orbifold fundamental group of the underlying orbifold. Consequently we observe the following lemma:

Lemma 6.1 *Suppose that $\alpha_1, \dots, \alpha_n \geq 2$ are integers and that $\mathcal{B}$ is an orbifold of type $\mathbb{S}^2(\alpha_1, \dots, \alpha_n)$ or $\mathbb{P}^2(\alpha_1, \dots, \alpha_n)$. Assume that*

(1) *$n \geq 3$ and $\sum_{i=1}^n 1/\alpha_i < n-2$ if $\mathcal{B} = \mathbb{S}^2(\alpha_1, \dots, \alpha_n)$, and*

(2) *$n \geq 2$ if $\mathcal{B} = \mathbb{P}^2(\alpha_1, \dots, \alpha_n)$.*

Then $\pi_1^{\mathrm{orb}}(\mathcal{B})$ is circularly orderable.

Proof (1) With the assumptions in (1), as $\mathcal{B}$ is hyperbolic, the group $\pi_1^{\mathrm{orb}}(\mathcal{B})$ embeds in $\mathrm{PSL}_2(\mathbb{R})$. As $\mathrm{PSL}_2(\mathbb{R})$ embeds in $\mathrm{Homeo}_+(\mathbb{S}^1)$, it is circularly orderable, so the result follows.

(2) In this case, for the group $\mathbb{Z}/\alpha_1\mathbb{Z} * \cdots * \mathbb{Z}/\alpha_n\mathbb{Z}$, suppose that the generator of $\mathbb{Z}/\alpha_i\mathbb{Z}$ is x_i, and use x to denote the product $x_1 \dots x_n$. Then

$$\pi_1^{\mathrm{orb}}(\mathcal{B}) = (\mathbb{Z}/\alpha_1\mathbb{Z} * \cdots * \mathbb{Z}/\alpha_n\mathbb{Z}) *_{\langle x \rangle = 2\mathbb{Z}} \mathbb{Z},$$

and the group $\mathbb{Z}/\alpha_1\mathbb{Z} * \cdots * \mathbb{Z}/\alpha_n\mathbb{Z}$ is circularly orderable by Proposition 2.4. That $\pi_1^{\mathrm{orb}}(\mathcal{B})$ is circularly orderable then follows from Proposition 4.2. □

Last, we note the following holds for all left-ordered groups admitting a cofinal, central element:

Proposition 6.2 *Suppose that $(G, <)$ is a left-ordered group, and that $z \in G$ is a cofinal, central element. Then, for every $p \in \mathbb{N}_{>0}$, the group G admits a circular ordering c such that $\mathrm{rot}_c(z) = 1/p$.*

Proof Let $p \in \mathbb{N}_{>0}$ be given, and first construct a circular ordering d of $G/\langle z^p \rangle$ by mimicking Construction 2.2 as follows. Define the minimal representative $\bar{g}$ of each $g\langle z^p \rangle \in G/\langle z^p \rangle$ to be the unique coset representative satisfying $\mathrm{id} \leq \bar{g} < z^p$, and set

$$d(g_1\langle z^p \rangle, g_2\langle z^p \rangle, g_3\langle z^p \rangle) = \mathrm{sign}(\sigma),$$

where σ is the unique permutation such that the minimal representatives satisfy $\bar{g}_{\sigma(1)} < \bar{g}_{\sigma(2)} < \bar{g}_{\sigma(3)}$. Then construct a circular ordering c of G by using the short exact sequence

$$1 \to \langle z^p \rangle \to G \to G/\langle z^p \rangle \to 1,$$

the orderings $<$ and d of the kernel and quotient, respectively, and Proposition 2.3. The circular ordering c satisfies $\mathrm{rot}_c(z) = 1/p$ (see [2, Proof of Theorem 6.2]). □

Proposition 6.3 *Suppose that M is a compact, connected, orientable Seifert fibred space with base orbifold $\mathbb{S}^2(\alpha_1, \dots, \alpha_n)$ with $n \geq 3$, and let h denote the class of a regular fibre in $\pi_1(M)$. If $\pi_1(M)$ is left-orderable, then $\pi_1(M)$ admits a left-ordering relative to which h is cofinal.*

Proof This result is essentially a restatement of [9, Proposition 4.7]. □

Proof of Theorem 1.2 We first prove (1). By [14, Theorem 1.3], $\pi_1(M)$ is left-orderable whenever the first Betti number $b_1(M)$ is positive and $M \not\cong \mathbb{P}^2 \times \mathbb{S}^1$. When $M = \mathbb{P}^2 \times \mathbb{S}^1$, the fundamental group $\pi_1(M)$ clearly admits a circular ordering of the required kind, via a lexicographic construction. This proves the claim for Seifert fibred manifolds M with $b_1(M) > 0$.

Thus we can assume that $b_1(M) = 0$, which implies that M is closed and orientable (see eg [14, Lemma 3.3]), and the base orbifold of M is either $\mathbb{S}^2(\alpha_1, \alpha_2, \dots, \alpha_n)$ or $\mathbb{P}^2(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Assume that either condition (1) or (2) of Lemma 6.1 holds. Since we also assume that $\pi_1(M)$ is infinite, the class of the fibre h is of infinite order (see eg [14, Proposition 4.1(1)]). Therefore, from the short exact sequence

$$1 \to \langle h \rangle \to \pi_1(M) \to \pi_1^{\text{orb}}(\mathcal{B}) \to 1,$$

we can lexicographically construct the required circular ordering of $\pi_1(M)$ using Lemma 6.1 and Proposition 2.3, completing the proof in these cases.

For the remaining cases, first suppose that $\mathcal{B} = \mathbb{S}^2(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\sum_{i=1}^{n} 1/\alpha_i \geq n-2$. Note that necessarily $n \leq 4$, and our assumption that $\pi_1(M)$ is infinite and $b_1(M) = 0$ rules out $n = 0$, $n = 1$ and $n = 2$. When $n = 3$ and $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 > 1$, the group $\pi_1(M)$ is finite, so we need not consider this case. On the other hand, if $\sum_{i=1}^{n} 1/\alpha_i = n-2$, then $\pi_1(M)$ has infinite abelianization [32, Proposition 2; 36, VI.13 Example], and so is left-orderable. Last, if M has base orbifold $\mathbb{P}^2$ or $\mathbb{P}^2(\alpha_1)$, then $\pi_1(M)$ is finite, $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$ [36, VI.11(c)]. Since we have assumed that $\pi_1(M)$ is infinite, $\pi_1(M)$ is either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$, which is circularly orderable in both cases. We construct a circular ordering of $\pi_1(M)$ for which $\text{rot}(h)$ is zero in either case as follows: when $\pi_1(M)$ is $\mathbb{Z}$, we equip $\mathbb{Z}$ with a secret left-ordering, and, when $\pi_1(M) \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle a, h \mid a^2 = \text{id}, aha^{-1} = h^{-1} \rangle$, we circularly order $\pi_1(M)$ lexicographically using

$$1 \to \langle h \rangle \to \pi_1(M) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

We observe that (2) follows from the defining relations of the fundamental group $\pi_1(M)$. When M has nonorientable base orbifold and admits at least one exceptional fibre, the relation $\gamma h \gamma^{-1} = h^{-1}$ holds in $\pi_1(M)$. As rotation number is invariant under conjugation, we get that $\text{rot}_c(h) = \text{rot}_c(h^{-1})$ for every circular ordering c of $\pi_1(M)$, from which it follows that $\text{rot}_c(h) \in \{0, \frac{1}{2}\}$.

To prove (3), suppose that $\pi_1(M)$ is left-orderable. Using Proposition 6.3, choose a left-ordering of $\pi_1(M)$ relative to which the class of the fibre is cofinal. The result now follows from Proposition 6.2.

Finally, (4) follows from the short exact sequence of the fibration

$$1 \to \pi_1(F) \to \pi_1(M) \to \pi_1(\mathbb{S}^1) \to 1$$

and the observation that we may use any circular ordering of $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ we please in applying Proposition 2.3. □

6.2 Graph manifolds

We begin with a few preliminaries to establish notation and some well-known facts. Recall that every compact, orientable, irreducible 3-manifold M admits a unique minimal family of disjoint incompressible tori $\mathcal{T}$ such that $M \setminus \mathcal{T}$ consists of Seifert fibred 3-manifolds and atoroidal 3-manifolds, called the JSJ decomposition. By a graph manifold, we mean a compact, connected, orientable, irreducible 3-manifold admitting a JSJ decomposition into Seifert fibred pieces. By using the tori of the JSJ decomposition to cut our graph manifolds into Seifert fibred pieces, we know that the collection of such tori is minimal for each graph manifold, and each torus is incompressible and thus π_1-injective.

Note that it follows from Proposition 3.2 that graph manifolds with infinite first homology always have left-orderable fundamental group; in particular, their fundamental groups are always circularly orderable. Thus, when it comes to circular-orderability, we will only consider the case of rational homology sphere graph manifolds.

Lemma 6.4 *Let W be a graph manifold with a torus boundary that is not homeomorphic to a Seifert fibred manifold. If $\alpha \in H_1(\partial W;\mathbb{Z})/\{\pm 1\}$ is not the slope of a regular fibre, then $W(\alpha)$ is a graph manifold.*

Proof Let $M_1, M_2, \dots, M_p$ be the Seifert pieces of the JSJ decomposition of W. Assume that the JSJ decomposition of W has only two pieces, M_1 and M_2. Without loss of generality, assume that the torus boundary of W is contained in M_2. Since α is not the slope of a regular fibre, $M_2(\alpha)$ is a Seifert fibred space with boundary $\partial M_2 \setminus \partial W$, by [33; 8, Theorem 5.1]. Therefore, $M_2(\alpha)$ is irreducible (because it is not $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$ [14, Proposition 4.1(3)]). Hence, $W(\alpha)$ is a graph manifold with Seifert pieces M_1 and $M_2(\alpha)$. Now assume that $p > 2$. Let M_k be the Seifert piece of W containing the torus boundary of W. Since α is not the slope of a regular fibre, $M_k(\alpha)$ is a Seifert fibred space with boundary $\partial M_k \setminus \partial W$, by [33; 8, Theorem 5.1]. Therefore, $M_k(\alpha)$ is irreducible (because it is not $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$ [14, Proposition 4.1(3)]). Hence $W(\alpha)$ is a graph manifold with Seifert pieces $M_1, M_2, \dots, M_k(\alpha), \dots, M_p$. □

Lemma 6.5 *Suppose that W is a graph manifold and that ∂W is a torus. Suppose W has Seifert fibred pieces $M_1, \dots, M_l$ and that $\partial W \subset M_1$. Then, if $W(\alpha)$ is reducible, α must be the slope of the regular fibre in a Seifert fibration of M_1 unless W is a Seifert fibred space with base orbifold $\mathcal{B}_1(\alpha_1, \dots, \alpha_{s_1})$ such that $\alpha_1 = \dots = \alpha_{s_1} = 1$ or $s_1 = 0$, and the geometric intersection number satisfies $\Delta(\alpha, h) = 1$, where h is the slope of a regular fibre.*

Proof Let $\alpha \in H_1(\partial W;\mathbb{Z})/\{\pm 1\}$ be a slope. We have two cases:

Case 1 Assume that α is not the slope of a regular fibre. Then the geometric intersection number satisfies $\Delta(\alpha, h) = d > 0$, where h is the slope of a regular fibre in M_1. Hence, if the base orbifold of M_1 is $\mathcal{B}_1(\alpha_1, \dots, \alpha_{s_1})$, then the base orbifold of $M_1(\alpha)$ is $\mathcal{B}'_1(\alpha_1, \dots, \alpha_{s_1}, d)$, where $\mathcal{B}'$ is obtained from $\mathcal{B}$ by filling a disk. We have two subcases:

Subcase 1 Assume that $M_1 = W$; that is, W is a Seifert fibred space. Then $W(\alpha)$ is a Seifert fibred space by [33; 8, Theorem 5.1]. Hence $W(\alpha)$ is irreducible unless it is $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$ [14, Proposition 4.1(3)]. (This is possible only in the case where $\alpha_1 = \cdots = \alpha_{s_1} = 1$ or $s_1 = 0$, and $d = 1$.)

Subcase 2 If W is not a Seifert fibred space, then $W(\alpha)$ is a graph manifold by Lemma 6.4. Hence, $W(\alpha)$ is irreducible by definition of a graph manifold.

Case 2 If α is the slope of a regular fibre, then $W(\alpha)$ may be reducible [33; 8, Theorem 5.1]. $\square$

Given a rational homology sphere graph manifold W, note that the underlying graph must be a tree. In this situation, for a fixed JSJ torus T, note that $W \setminus T$ has two components. We denote the closure of these components by W_1 and W_2, and by $\phi\colon \partial W_1 \to \partial W_2$ the homeomorphism such that $W = W_1 \cup_\phi W_2$. We use this fixed notation for the discussion and proof below.

Define a class $\mathscr{C}$ of rational homology sphere graph manifolds to be the smallest collection of 3-manifolds satisfying:

(1) All connected, irreducible rational homology sphere Seifert fibred manifolds belong to $\mathscr{C}$.

(2) A rational homology sphere graph manifold W is in $\mathscr{C}$ if and only if, for every JSJ torus $T \subset W$,
   (a) at least one of $W_1(\phi_*^{-1}(\lambda_{W_2}))$ and $W_2(\phi_*(\lambda_{W_1}))$ has infinite fundamental group, and
   (b) every irreducible manifold of the form $W_i(\alpha)$ satisfying $|H_1(W_i(\alpha);\mathbb{Z})| < \infty$ lies in $\mathscr{C}$.

Theorem 6.6 *For $W \in \mathscr{C}$, if $\pi_1(W)$ is infinite, then it is circularly orderable.*

Proof For every graph manifold W, let n_W denote the minimal number of tori required to cut W into Seifert fibred pieces (ie the number of tori in its JSJ decomposition). When $n_W = 0$, if $\pi_1(W)$ is infinite, then it is circularly orderable by Theorem 1.2. For induction, assume that $k \geq 0$ and that $\pi_1(W)$ is circularly orderable for all $W \in \mathscr{C}$ with $n_W \leq k$, and consider the case of $n_W = k+1$.

For a manifold $W \in \mathscr{C}$ with $n_W = k+1$ and infinite fundamental group, choose a JSJ torus T such that $W \setminus T$ results in two pieces W_1 and W_2 such that W_1 is Seifert fibred.

First suppose that $W_2(\phi_*(\lambda_{W_1}))$ has infinite fundamental group and let $M_1, \ldots, M_l$ denote the Seifert fibred pieces of the JSJ decomposition of W_2; assume that $\partial W_2 \subset M_1$.

Next suppose that $W_2(\phi_*(\lambda_{W_1}))$ is irreducible. If $H_1\big(W_2(\phi_*(\lambda_{W_1}));\mathbb{Z}\big)$ is finite, then we conclude $W_2(\phi_*(\lambda_{W_1})) \in \mathscr{C}$ by property (2)(b). Then $\phi_*(\lambda_{W_1})$ is not the slope of a regular fibre in the outermost piece of M_1 and the JSJ components of $W_2(\phi_*(\lambda_{W_1}))$ are precisely the Seifert fibred manifolds $M_1(\phi_*(\lambda_{W_1})), M_2, \ldots, M_l$ by Lemma 6.4. Thus $W_2(\phi_*(\lambda_{W_1}))$ is a graph manifold in $\mathscr{C}$ with $n_{W_2(\phi_*(\lambda_{W_1}))} < n_W$, so $\pi_1\big(W_2(\phi_*(\lambda_{W_1}))\big)$ is circularly orderable by induction. On the other hand, if $H_1\big(W_2(\phi_*(\lambda_{W_1}));\mathbb{Z}\big)$ is infinite, then $\pi_1\big(W_2(\phi_*(\lambda_{W_1}))\big)$ is left-orderable, and hence circularly orderable, by [14, Theorem 3.2]. In either case it follows that $\pi_1(W)$ is circularly orderable by Proposition 5.6(2).

Now suppose that $W_2(\phi_*(\lambda_{W_1}))$ is reducible, in which case $\phi_*(\lambda_{W_1})$ must be the slope of a regular fibre in ∂M_1, or $W_2(\phi_*(\lambda_{W_1}))$ is $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$, by Lemma 6.5. If $W_2(\phi_*(\lambda_{W_1}))$ is $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$, then, as each has circularly orderable fundamental group, $\pi_1(W)$ is circularly orderable by Proposition 5.6(2).

Next, in the case that $\phi_*(\lambda_{W_1})$ is the slope of a regular fibre in ∂M_1, recall that $H_1(W;\mathbb{Z})$ is finite and so the surface underlying the base orbifold of M_1 has genus zero. Further suppose that M_1 has r exceptional fibres and boundary tori $T, T_1, \dots, T_m$, and that $W_2 \setminus M_1$ has components $Y_1, \dots, Y_m$, where each Y_j is a graph manifold with torus boundary. Let $\phi_j : \partial Y_j \to T_j \subset M_1$ for $j = 1, \dots, m$ denote the gluing maps that recover W_2 from the pieces $M_1, Y_1, \dots, Y_m$. In this case, by [8, Theorem 5.1; 33], filling M_1 along $\phi_*(\lambda_{W_1})$ yields

$$M_1(\phi_*(\lambda_{W_1})) \cong L_1 \# \cdots \# L_r \# (\mathbb{S}^1 \times \mathbb{D}^2) \# \cdots \# (\mathbb{S}^1 \times \mathbb{D}^2)$$

or

$$M_1(\phi_*(\lambda_{W_1})) \cong L_1 \# \cdots \# L_r \# (\mathbb{S}^1 \times \mathbb{D}^2) \# \cdots \# (\mathbb{S}^1 \times \mathbb{D}^2) \# (\mathbb{S}^1 \times \mathbb{S}^2),$$

depending on whether or not the underlying manifold of the base orbifold is a punctured $\mathbb{S}^2$ or $\mathbb{P}^2$. Here, $L_1, \dots, L_r$ are lens spaces, there are m copies of $\mathbb{S}^1 \times \mathbb{D}^2$, each arising from a torus component of $\partial M_1 \setminus T$, and each $\partial \mathbb{D}^2$ is path-homotopic to (the image of) a regular fibre in the Dehn filled manifold $M_1(\phi_*(\lambda_{W_1}))$.

Therefore, if we denote the slope of a regular fibre on ∂M_1 by h, it follows that

$$W_2(\phi_*(\lambda_{W_1})) \cong L_1 \# \cdots \# L_r \# Y_1((\phi_1^{-1})_*(h)) \# \cdots \# Y_m((\phi_m^{-1})_*(h))$$

or

$$W_2(\phi_*(\lambda_{W_1})) \cong L_1 \# \cdots \# L_r \# Y_1((\phi_1^{-1})_*(h)) \# \cdots \# Y_m((\phi_m^{-1})_*(h)) \# (\mathbb{S}^1 \times \mathbb{S}^2),$$

again depending on whether or not the base orbifold is orientable.

In either case, we may proceed as follows. Suppose that every manifold $Y_j((\phi_j^{-1})_*(h))$ for $j = 1, \dots, m$ has infinite fundamental group. If Y_j is homeomorphic to a Seifert fibred manifold, it follows that its fundamental group of $Y_j((\phi_j^{-1})_*(h))$ is circularly orderable by Theorem 1.2. On the other hand, if Y_j is not homeomorphic to a Seifert fibred manifold, then, as $(\phi_j^{-1})_*(h)$ is not the slope of a regular fibre in the outermost Seifert fibred piece of Y_j, we know that $Y_j((\phi_j^{-1})_*(h))$ is irreducible by Lemma 6.4. Thus either $\left|H_1\big(Y_j((\phi_j^{-1})_*(h));\mathbb{Z}\big)\right| = \infty$ or $\left|H_1\big(Y_j((\phi_j^{-1})_*(h));\mathbb{Z}\big)\right| < \infty$ and $Y_j((\phi_j^{-1})_*(h)) \in \mathscr{C}$ by property (2)(b). In the former case, the fundamental group of $Y_j((\phi_j^{-1})_*(h))$ is circularly orderable since it is in fact left-orderable by [14, Theorem 3.2]. In the latter case, we may argue that $n_{Y_j((\phi_j^{-1})_*(h))} < n_W$, so the fundamental group of $Y_j((\phi_j^{-1})_*(h))$ is circularly orderable by induction.

It follows that $\pi_1\big(W_2(\phi_*(\lambda_{W_1}))\big)$ is a free product of circularly orderable groups, and thus is circularly orderable. Note it is also infinite by assumption. Last, note that λ_{W_1} is not the slope of a regular fibre in ∂W_1, since it is glued via ϕ_* to the slope of a regular fibre in ∂M_1, and thus $W_1(\lambda_{W_1})$ is irreducible by [33]. Therefore $\pi_1(W)$ is circularly orderable by Proposition 5.6(2).

Now suppose that there exists j_0 such that the fundamental group of $Y_{j_0}((\phi_{j_0}^{-1})_*(h))$ is finite, in which case Y_{j_0} is Seifert fibred and $(\phi_{j_0}^{-1})_*(h)$ is not equal to the rational longitude $\lambda_{Y_{j_0}}$ of Y_{j_0}. One of the components of $W \setminus T_{j_0}$ is Y_{j_0}; we will call the other component W'', so that $W = Y_{j_0} \cup_{\phi_{j_0}} W''$. Note that $W''((\phi_{j_0})_*(\lambda_{Y_{j_0}}))$ is irreducible by Lemma 6.4, since $(\phi_{j_0})_*(\lambda_{Y_{j_0}})$ is not the slope a regular fibre in the outermost piece of W''. Thus, either the homology of $W''((\phi_{j_0})_*(\lambda_{Y_{j_0}}))$ is infinite and so it has left-orderable fundamental group, or the homology is finite and so $W''((\phi_{j_0})_*(\lambda_{Y_{j_0}})) \in \mathscr{C}$. In the latter case, $W''((\phi_{j_0})_*(\lambda_{Y_{j_0}}))$ has infinite fundamental group since it is an irreducible graph manifold whose JSJ decomposition consists of two or more Seifert fibred pieces, and its fundamental group is circularly orderable by induction. Thus we may apply Proposition 5.6(2) with Y_{j_0} in place of M_1 and W'' in place of M_2 in order to conclude that $\pi_1(W)$ is circularly orderable.

Last, suppose $W_2(\phi_*(\lambda_{W_1}))$ has finite fundamental group (and thus W_2 is Seifert fibred), so that $W_1(\phi_*^{-1}(\lambda_{W_2}))$ is infinite by property (2)(a). Then $\pi_1\big(W_1(\phi_*^{-1}(\lambda_{W_2}))\big)$ is circularly orderable by Theorem 1.2. Thus the result follows from Proposition 5.6(2). □

We can more precisely codify the manifolds covered by Theorem 6.6 as follows. The possible base orbifolds of a Seifert fibred manifold admitting a single incompressible torus boundary component and a finite filling are

$$\mathscr{A} := \{\mathbb{D}^2(p,q), \mathbb{D}^2(2,2,r), \mathbb{D}^2(2,3,3), \mathbb{D}^2(2,3,4), \mathbb{D}^2(2,3,5) \mid r \geq 1,\ p \geq 2,\ q \geq 2\}.$$

Proof of Theorem 1.3 First, if W is not a rational homology sphere, then the first Betti number $b_1(W)$ is positive, so $\pi_1(W)$ is left-orderable by [14, Theorem 3.2]. Second, if W is a graph manifold satisfying the assumptions of Theorem 1.3 and W is Seifert fibred, then Theorem 1.2 finishes the proof.

From here, we complete the proof by showing that, if W is a rational homology sphere graph manifold satisfying the hypotheses of Theorem 1.3, then $W \in \mathscr{C}$, so Theorem 6.6 applies.

To do this, we induct on the number n_W of tori in the JSJ decomposition of W, noting that, if W satisfies the hypotheses of Theorem 1.3 and $n_W = 0$, then $W \in \mathscr{C}$ by definition. Next suppose that every rational homology sphere graph manifold W satisfying the hypotheses of Theorem 1.3 with $n_W < k$ lies in $\mathscr{C}$, and consider W with $n_W = k$.

Suppose W has Seifert fibred pieces $M_1, \dots, M_l$. Choose an arbitrary JSJ torus $T \subset W$ and cut along T to arrive at $W = W_1 \cup_\phi W_2$.

Considering $W_1(\phi_*^{-1}(\lambda_{W_2}))$ and $W_2(\phi_*(\lambda_{W_1}))$, if either admits a JSJ decomposition with one or more JSJ tori, then its fundamental group is infinite. On the other hand, if either is Seifert fibred, then again the fundamental group is infinite since the hypotheses of Theorem 1.3 imply the base orbifold of M_i is not in $\mathscr{A}$ for $i = 1, \dots, l$. In any event, W satisfies property (2)(a) in the definition of $\mathscr{C}$.

Next suppose that $W_1(\alpha)$ is irreducible for some $\alpha \in H_1(W_1;\mathbb{Z})/\{\pm 1\}$ and that $H_1(W(\alpha);\mathbb{Z})$ is finite. If W_1 is Seifert fibred, then $W_1(\alpha)$ is Seifert fibred and so lies in $\mathscr{C}$ by definition. On the other hand, if W_1 is not Seifert fibred, then, by Lemma 6.5, α must not be the slope of the regular fibre in the outermost piece of W_1 and so $W_1(\alpha)$ is a graph manifold by Lemma 6.4. Moreover, $n_{W_1(\alpha)} < k$ and so $W_1(\alpha) \in \mathscr{C}$ by our induction assumption. As the same arguments hold for W_2, we conclude that W satisfies property (2)(b) in the definition of $\mathscr{C}$, and thus $W \in \mathscr{C}$. $\square$

In fact, suppose W is an arbitrary rational homology sphere graph manifold admitting a JSJ decomposition $W = M_1 \cup_\phi M_2$ such that the base orbifold of M_i lies in $\mathscr{A}$. If $\pi_1(W)$ is circularly orderable for every such W, then Theorem 6.6 and its proof can be used, mutatis mutandis, to show that the fundamental group of *every* rational homology sphere graph manifold is circularly orderable whenever it is infinite. We thus make explicit exactly which graph manifolds having two pieces in their JSJ decomposition are covered by Theorem 6.6.

Let $\mathscr{E}$ be the set of graph manifolds whose JSJ decomposition has at least one Seifert piece with base orbifold in $\mathscr{A}$. Further, set

$$\mathscr{F} = \{\mathbb{D}^2(2,2), \mathbb{D}^2(2,3), \mathbb{D}^2(3,3), \mathbb{D}^2(3,4), \mathbb{D}^2(3,5)\} \subset \mathscr{A}.$$

Corollary 6.7 *Let W be a rational homology sphere graph manifold whose JSJ decomposition has only two Seifert pieces M_1 and M_2 with base orbifolds $\mathscr{B}_1(p_1^1, \dots, p_{s_1}^1)$ and $\mathscr{B}_2(p_1^2, \dots, p_{s_2}^2)$, respectively. Let $\phi\colon \partial M_1 \to \partial M_2$ denote the gluing map that recovers W from the pieces M_1 and M_2, and let λ_i and h_i denote the rational longitude and the slope of a regular fibre on ∂M_i for each of $i = 1, 2$. Suppose that W satisfies*

(1) *$W \notin \mathscr{E}$; or*

(2) *the base orbifold of M_1 lies in $\mathscr{A}$ and the base orbifold of M_2 is not in $\mathscr{A}$; or*

(3) *the base orbifolds of both M_1 and M_2 lie in $\mathscr{A}$ with $\mathscr{B}_1(p_1^1, \dots, p_{s_1}^1) \notin \mathscr{F}$, and*

  (a) *if $\mathscr{B}_2(p_1^2, \dots, p_{s_2}^2) \notin \mathscr{F}$, then either $\Delta(\phi_*(\lambda_1), h_2) > 1$ or $\Delta(\phi_*^{-1}(\lambda_2), h_1) > 1$;*

  (b) *if $\mathscr{B}_2(p_1^2, \dots, p_{s_2}^2) \in \mathscr{F}$, then either*
  - *$\Delta(\phi_*^{-1}(\lambda_2), h_1) > 1$, or*
  - *$\mathscr{B}_2(p_1^2, \dots, p_{s_2}^2) = \mathbb{D}^2(2,3)$ and $\Delta(\phi_*(\lambda_1), h_2) > 5$, or*
  - *$\mathscr{B}_2(p_1^2, \dots, p_{s_2}^2) = \mathbb{D}^2(3,3)$ and $\Delta(\phi_*(\lambda_1), h_2) > 2$, or*
  - *$\mathscr{B}_2(p_1^2, \dots, p_{s_2}^2) = \mathbb{D}^2(3,4)$ and $\Delta(\phi_*(\lambda_1), h_2) > 2$, or*
  - *$\mathscr{B}_2(p_1^2, \dots, p_{s_2}^2) = \mathbb{D}^2(3,5)$ and $\Delta(\phi_*(\lambda_1), h_2) > 2$;*

then $\pi_1(W)$ is circularly orderable.

Proof We prove only case (3)(b), as the other statements claimed are all consequences of Theorem 6.6, since any such manifold lies in $\mathscr{C}$.

Assume that $\mathscr{B}_1(p_1^1, \dots, p_{s_1}^1) \notin \mathscr{F}$, $\mathscr{B}_2(p_1^2, \dots, p_{s_2}^2) \in \mathscr{F}$ and $\Delta(\phi_*^{-1}(\lambda_2), h_1) > 1$. Let $\varphi = \phi^{-1}$. Since $\mathscr{B}_1(p_1^1, \dots, p_{s_1}^1) \notin \mathscr{F}$ and $\Delta(\phi_*^{-1}(\lambda_2), h_1) > 1$, $\pi_1\big(M_1(\varphi(\lambda_2))\big)$ is infinite and circularly orderable. Since $\mathscr{B}_2(p_1^2, \dots, p_{s_2}^2) \in \mathscr{F}$, M_2 is Seifert fibred with incompressible boundary. Therefore, $\pi_1(M)$ is circularly orderable by Proposition 5.6.

Next assume that $\mathcal{B}_2(p_1^2,\dots,p_{s_2}^2)=\mathbb{D}^2(2,3)$ (resp. $\mathbb{D}^2(3,3)$, $\mathbb{D}^2(3,4)$, $\mathbb{D}^2(3,5)$) and $\Delta(\phi_*(\lambda_1),h_2)>5$ (resp. $\Delta(\phi_*(\lambda_1),h_2)>2$). Then the base orbifold of $M_2(\phi(\lambda_1))$ is $\mathbb{S}^2(2,3,a)$ (resp. $\mathbb{S}^2(3,3,a)$, $\mathbb{S}^2(3,4,a)$, $\mathbb{S}^2(3,5,a)$) with $a\geq 6$ (resp. $a\geq 3$). Hence, $\pi_1\big(M_2(\phi(\lambda_1))\big)$ is infinite and circularly orderable. Since $\mathcal{B}_1(p_1^1,\dots,p_{s_1}^1)\in\mathcal{A}$, M_1 is Seifert fibred with incompressible boundary. Therefore, $\pi_1(M)$ is circularly orderable by Proposition 5.6. $\square$

6.3 Graph manifolds with two pieces

Our goal in this section is to show that we can circularly order many of the fundamental groups of graph manifolds having two pieces in their JSJ decomposition that are not covered by Corollary 6.7.

Lemma 6.8 *Let W be a 3-manifold obtained by gluing two knot exteriors in some integer homology 3-spheres on their torus boundary by some orientation-reversing homeomorphism. Then $H_1(W,\mathbb{Z})$ is cyclic.*

Proof Let K_1 and K_2 be two knots in some integer homology 3-spheres W_1 and W_2, respectively. Let M_1 and M_2 be the knot exteriors of K_1 and K_2, respectively. Let W be the 3-manifold obtained by identifying ∂M_1 and ∂M_2 by some orientation-reversing homeomorphism φ. Let (μ_1,λ_1) and (μ_2,λ_2) be the meridian–longitude slope pairs of ∂M_1 and ∂M_2, respectively. Let $\varphi_*(\mu_1)=a\mu_2+b\lambda_2$ and $\varphi_*(\lambda_1)=c\mu_2+d\lambda_2$.

Let $T=\partial M_1\cong\partial M_2$. We have the Mayer–Vietoris sequence

$$\cdots\to H_1(T;\mathbb{Z})\xrightarrow{\phi_1}H_1(M_1;\mathbb{Z})\oplus H_1(M_2;\mathbb{Z})\xrightarrow{\psi_1}H_1(W;\mathbb{Z})\xrightarrow{\partial_1}H_0(T;\mathbb{Z})$$
$$\xrightarrow{\phi_0}H_0(M_1;\mathbb{Z})\oplus H_0(M_2;\mathbb{Z})\xrightarrow{\psi_0}H_0(W;\mathbb{Z})\to 0.$$

We have that $\lambda_1=0$ in $H_1(M_1;\mathbb{Z})$, and $H_1(M_1;\mathbb{Z})$ is generated by μ_1. We have also that $\lambda_2=0$ in $H_1(M_2;\mathbb{Z})$, and $H_1(M_2;\mathbb{Z})$ is generated by μ_2. We consider $\{\mu_1,\lambda_1\}$ to be the generators of $H_1(T;\mathbb{Z})$. Hence, $\phi_1(\mu_1)=i_T(\mu_1)\oplus -i_T(\mu_1)=(\mu_1,-a\mu_2)$ and $\phi_1(\lambda_1)=i_T(\lambda_1)\oplus -i_T(\lambda_1)=(0,-c\mu_2)$. By exactness, $\mathrm{im}(\phi_1)=\ker(\psi_1)$, $\mathrm{im}(\psi_1)=\ker(\partial_1)$ and $\mathrm{im}(\partial_1)=\ker(\phi_0)$. Since T, M_1 and M_2 are connected and hence path-connected, ϕ_0 is injective. Therefore, $\mathrm{im}(\partial_1)=0$ and $\mathrm{im}(\psi_1)=\ker(\partial_1)=H_1(W;\mathbb{Z})$. So $H_1(W;\mathbb{Z})=\mathrm{im}(\psi_1)\cong H_1(M_1;\mathbb{Z})\oplus H_1(M_2;\mathbb{Z})/\ker(\psi_1)=H_1(M_1;\mathbb{Z})\oplus H_1(M_2;\mathbb{Z})/\mathrm{im}(\phi_1)=\mathbb{Z}\oplus\mathbb{Z}/\mathrm{im}(\phi_1)$. We have that $\mathrm{im}(\phi_1)=\langle(\mu_1,-a\mu_2),(0,-c\mu_2)\rangle\cong\langle(1,-a),(0,-c)\rangle$. Consider the matrix $\left(\begin{smallmatrix}1&0\\-a&-c\end{smallmatrix}\right)$. By adding a times the first row to the second row, we obtain $\left(\begin{smallmatrix}1&0\\0&-c\end{smallmatrix}\right)$. Hence $\mathrm{im}(\phi_1)=\langle(\mu_1,-a\mu_2),(0,-c\mu_2)\rangle\cong\langle(1,-a),(0,-c)\rangle\cong\langle(1,0),(0,-c)\rangle\cong\mathbb{Z}\oplus|c|\mathbb{Z}$, and so

$$H_1(W;\mathbb{Z})=\mathrm{im}(\psi_1)\cong H_1(M_1;\mathbb{Z})\oplus H_1(M_2;\mathbb{Z})/\ker(\psi_1)=H_1(M_1;\mathbb{Z})\oplus H_1(M_2;\mathbb{Z})/\mathrm{im}(\phi_1),$$

with this final group being isomorphic to $\mathbb{Z}\oplus\mathbb{Z}/(\mathbb{Z}\oplus|c|\mathbb{Z})\cong\mathbb{Z}_{|c|}$. $\square$

If W is a rational homology sphere graph manifold, we can construct a graph called the *splice diagram* $\Gamma(W)$ as follows: nodes are in one-to-one correspondence with the Seifert pieces of W. Two nodes are

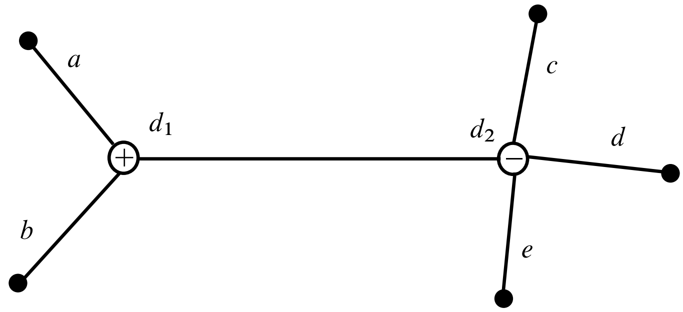

Figure 1: Example of a splice diagram.

connected by an edge if the corresponding Seifert fibred pieces are glued together along a common torus boundary component. To each node, one attaches a leaf for each singular fibre of the corresponding Seifert fibred piece.

We then decorate the graph so constructed as follows: Each edge corresponding to a leaf is labelled with the multiplicity of the corresponding singular fibre. Let v be a node of $\Gamma(W)$ and e be an edge of $\Gamma(W)$ connecting v. Let N and K be the two pieces of W obtained by cutting W along the torus corresponding to e such that K does not contain M_v, where M_v is the Seifert piece of W corresponding to v. Let D be the manifold obtained by Dehn filling K with a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ by identifying a regular fibre of ∂K with a meridian of $\mathbb{D}^2 \times \mathbb{S}^1$. Let $d_v = |H_1(D)|$, and take d_v to be the label of the edge e at the node v. We also decorate the nodes of $\Gamma(W)$ with signs $+$ or $-$ corresponding to the sign of the linking number of two nonsingular fibres in the Seifert fibration (see [46, Section 2] for more details).

Proposition 6.9 *Suppose that M_1 and M_2 are Seifert fibred spaces with incompressible boundaries and base orbifolds $\mathbb{D}^2(a_1, \dots, a_s)$ and $\mathbb{D}^2(b_1, \dots, b_t)$, respectively, and that each is the exterior of a knot in an integer homology sphere. Let W be a graph manifold obtained by gluing M_1 and M_2 along their torus boundaries by some orientation-reversing homeomorphism. If $\gcd(a_i, a_l) = 1$ and $\gcd(b_j, b_k) = 1$ for $1 \le i \ne l \le s$ and $1 \le j \ne k \le t$, and $\gcd(a_i, b_k) = 1$ for all $i = 1, \dots, s$ and $k = 1, \dots t$, then $\pi_1(W)$ is circularly orderable.*

Proof With the restrictions on a_i and b_k as in the statement of the theorem, the manifold W has a corresponding splice diagram $\Gamma(W)$ with two nodes. The edge labels around any node in the diagram are all pairwise coprime, so, by [46, Corollary 6.3], the universal abelian cover of W is an integer homology sphere graph manifold. Hence, the commutator group $[\pi_1(W), \pi_1(W)]$ is left-orderable by [20]. Since $H_1(W)$ is cyclic by Lemma 6.8, $\pi_1(W)$ is circularly orderable by Proposition 2.3. □

In particular, Proposition 6.9 applies to manifolds $W = M_1 \cup_\phi M_2$ where M_i are torus knot exteriors whose cone points have relatively coprime orders. Manifolds of this form are not completely covered by Corollary 6.7 or Theorem 6.6. We can further deal with other special cases of interest not covered by these theorems.

Proposition 6.10 *Suppose that $W = M_1 \cup_\phi M_2$ where each M_i is a twisted I-bundle over the Klein bottle. Then $\pi_1(W)$ is circularly orderable.*

Proof The fundamental group of W is an amalgamated free product of two Klein bottle groups $K_1 = \langle a, b \mid a^2 = b^2 \rangle$ and $K_2 = \langle c, d \mid c^2 = d^2 \rangle$, whose peripheral subgroups are $\langle a^2, ab \rangle$ and $\langle c^2, cd \rangle$, respectively. The amalgamation is with respect to an isomorphism $\phi\colon \langle a^2, ab \rangle \to \langle c^2, cd \rangle$. Observe that K_1 admits a lexicographic circular ordering arising from the short exact sequence

$$1 \to \langle a^2, ab \rangle \to K_1 \to \mathbb{Z}/2\mathbb{Z} \to 1,$$

and that K_2 admits a similar lexicographic circular ordering, with the choice of left-ordering on the subgroup $\langle a^2, ab \rangle$ (resp. $\langle c^2, cd \rangle$) being arbitrary. As such, we can construct circular orderings of K_1 and K_2 so that the homomorphism ϕ is order-preserving, and the subgroups $\langle a^2, ab \rangle$ and $\langle c^2, cd \rangle$ are convex.[4] Then, by [17, Proposition 1.1], $\pi_1(W)$ is circularly orderable. □

Proof of Theorem 1.4 Suppose M is such a manifold. By [14, Theorem 1.7(1)], if the boundary of M is not empty, or M is nonorientable, or M is a torus bundle over the circle, then $\pi_1(M)$ is left-orderable. Thus $\pi_1(M)$ is circularly orderable. By [14, Section 9], the only case which is left to check is when M is orientable and the union of two twisted I-bundles over the Klein bottle K, which are glued together along their torus boundaries. Therefore Proposition 6.10 finishes the proof. □

It seems, however, that the special case of a graph manifold consisting of two Seifert fibred pieces, each admitting finite fillings, is out of reach of our current technology. A general notion of "slope detection by a circular ordering" is likely needed to deal with these last few cases, though our results thusfar contribute ample evidence for the truth of the following conjecture:

Conjecture 6.11 Suppose W is a rational homology sphere graph manifold. If $\pi_1(W)$ is infinite, then it is circularly orderable.

7 Cyclic branched covers and Dehn surgery

With respect to certain well-known geometric constructions, left-orderability is conjectured to exhibit certain predictable behaviours. In this section we contrast the conjectured "predictable behaviours" of left-orderability with the behaviour of circular-orderability, which is strikingly different and whose expected behaviour at this time is completely unknown.

7.1 Cyclic branched covers

We recall the standard construction of the cyclic covers and cyclic branched covers of a knot in $\mathbb{S}^3$ in order to establish notation.

[4] Convexity here means that the quotient group inherits a circular ordering. For a more general definition and discussion of convexity in the context of circular orderings, see [17].

Let K be an oriented knot in $\mathbb{S}^3$. Let M_K be the exterior of K and S be a Seifert surface for K. Isotope S so that $S \cap \partial M_K$ is a longitude of K and let $F = S \cap M_K$. Let C be a tubular neighbourhood of F in M_K, so that C is homeomorphic to $F \times [-1, 1]$.

Set $Y = M_K \setminus F \times (-1, 1)$. The boundary of Y contains two copies of F, which we denote by $F^- \cong F \times \{-1\}$ and $F^+ \cong F \times \{1\}$; use these to create a triple (Y, F^+, F^-). Consider n copies of this triple, denoted by (Y_i, F_i^+, F_i^-) for $i = 0, \dots, n-1$, and glue them together by identifying $F_0^+ \subset Y_0$ with $F_1^- \subset Y_1$, $F_1^+ \subset Y_1$ with $F_2^- \subset Y_2, \dots, F_{n-2}^+ \subset Y_{n-2}$ with $F_{n-1}^- \subset Y_{n-1}$ and $F_{n-1}^+ \subset Y_{n-1}$ with $F_0^- \subset Y_0$. Call the resulting space X_n.

There is a regular covering map $g\colon X_n \to M_K$ and its group of deck transformations is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. The manifold X_n is called the n-fold cyclic cover of M_K and its fundamental group is isomorphic to $\ker(\pi_1(M_K) \to \mathbb{Z}/n\mathbb{Z})$. To construct the n-fold cyclic branched cover $\Sigma_n(K)$, we glue a solid torus $V \cong D^2 \times \mathbb{S}^1$ to Y_n by identifying the meridian $\partial D^2 \times \{1\}$ of V with the preimage of the meridian μ of ∂M_K under the map $g\colon X_n \to M_K$. The manifold $\Sigma_n(K)$ that results is a closed, oriented 3-manifold. For any $n \in \mathbb{N}$, let $q_n\colon X_n \to \Sigma_n(K)$ be the inclusion map. The map q_n induces a homomorphism $(q_n)_*\colon \pi_1(X_n) \to \pi_1(\Sigma_n(K))$ and $\ker(q_n)_* = \langle\langle \mu^n \rangle\rangle$. Therefore we have a short exact sequence $1 \to \langle\langle \mu^n \rangle\rangle \to \pi_1(X_n) \to \pi_1(\Sigma_n(K)) \to 1$, which identifies the fundamental group of $\Sigma_n(K)$ as the quotient of a certain subgroup of the knot group $\pi_1(M_K)$.

If $L \subset \mathbb{S}^3$ is an oriented link, then the n-fold cyclic branched cover of L, $\Sigma_n(L)$, can be also constructed [7].

Set

$$\mathcal{LO}_{\mathrm{br}}(K) = \{n \geq 2 \mid \pi_1(\Sigma_n(K)) \text{ is left-orderable}\}$$

and

$$\mathcal{CO}_{\mathrm{br}}(K) = \{n \geq 2 \mid \pi_1(\Sigma_n(K)) \text{ is circularly orderable}\}.$$

Note that $\mathcal{LO}_{\mathrm{br}}(K) \subset \mathcal{CO}_{\mathrm{br}}(K)$.

Motivated by the L-space conjecture, in [52, Question 1.8; 7], the authors ask whether or not the set $\mathcal{LO}_{\mathrm{br}}(K)$ is always of the form $\{n \mid n \geq N\}$ for some $N \geq 2$, or empty. In contrast, circular-orderability does not behave this way, with this behaviour being evident upon examining the torus knots. For example, considering the trefoil, we have the following:

Proposition 7.1 *With notation as above,* $\mathcal{CO}_{\mathrm{br}}(3_1) = \{2\} \cup \{n \mid n \geq 6\}$.

Proof The double branched cover of 3_1 is the lens space $L(3, 1)$, so its fundamental group is circularly orderable. On the other hand, the 3-, 4- and 5-fold branched cyclic covers of 3_1 have fundamental group the quaternion group, the binary tetrahedral group, and the binary icosahedral group, respectively, all of which are finite and noncyclic. It follows from Proposition 2.5 that none of these groups are circularly orderable. For $n \geq 6$ we know that $n \in \mathcal{LO}_{\mathrm{br}}(3_1) \subset \mathcal{CO}_{\mathrm{br}}(3_1)$ by [28, Theorem 1.2(i)]. □

This behaviour is not confined to torus knots.

Proposition 7.2 *With notation as above,*

$$2 \in \mathscr{C}\mathbb{O}_{\mathrm{br}}(5_2) \quad \text{and} \quad 3 \notin \mathscr{C}\mathbb{O}_{\mathrm{br}}(5_2), \quad \text{and} \quad \{n \mid n \geq 9\} \subset \mathscr{C}\mathbb{O}_{\mathrm{br}}(5_2).$$

Proof The knot 5_2 is a two-bridge knot, corresponding to the fraction $\frac{7}{4}$. As such, its double branched cover is the lens space $L(7,4)$, which has fundamental group $\mathbb{Z}/7\mathbb{Z}$, and is therefore circularly orderable.

On the other hand, the fundamental group of the Weeks manifold W is not circularly orderable by [16, Theorem 9.2], yet W is homeomorphic to $\Sigma_3(5_2)$ by [39, Main result].

Last, $n \in \mathscr{L}\mathbb{O}_{\mathrm{br}}(5_2) \subset \mathscr{C}\mathbb{O}_{\mathrm{br}}(5_2)$ for all $n \geq 9$ by [34]. □

There are also examples of knots for which $\mathscr{L}\mathbb{O}_{\mathrm{br}}(K)$ is empty. Notable examples are the two-bridge knots $p/q = 2m + 1/(2k)$, or $L_{[2k,2m]}$ with $k, m > 0$ in Conway's notation [24]. However, for these knots, $\mathscr{C}\mathbb{O}_{\mathrm{br}}(K)$ is never empty, because the double branched cover is the lens space $L(p,q)$, for which $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$, and is thus circularly orderable. Indeed, for the figure eight knot 4_1, which is $L_{[2,2]}$, we can show that infinitely many of the cyclic branched covers have circularly orderable fundamental group. We first require a proposition.

Proposition 7.3 *Let K be a prime knot in $\mathbb{S}^3$. If $n \geq 2$ and $\pi_1(\Sigma_n(K))$ is circularly orderable and infinite, then $\pi_1(\Sigma_m(K))$ is circularly orderable for all m divisible by n.*

Proof By [28, Lemma 2.11], there exists a surjective group homomorphism

$$q_{m,n} \colon \pi_1(\Sigma_m(K)) \to \pi_1(\Sigma_n(K))$$

for any positive integer m divisible by n. By [47], the manifolds $\Sigma_m(K)$ are irreducible and so Proposition 3.2(2) applies; we conclude that $\pi_1(\Sigma_m(K))$ is circularly orderable. □

Proposition 7.4 *If n is divisible by 3, then $\pi_1(\Sigma_n(4_1))$ is circularly orderable. In particular, $3\mathbb{N} \subset \mathscr{C}\mathbb{O}_{\mathrm{br}}(4_1)$.*

Proof The manifold $\Sigma_3(4_1)$ is homeomorphic to the Hantzsche–Wendt manifold, which is a Seifert fibred manifold with infinite fundamental group. Therefore $\pi_1(\Sigma_3(4_1))$ is circularly orderable by Theorem 1.2. The result now follows from Proposition 7.3. □

These observations naturally lead to the following questions:

Question 7.5 What subsets of $\mathbb{N}$ can occur as $\mathscr{C}\mathbb{O}_{\mathrm{br}}(K)$ for a knot K in $\mathbb{S}^3$?

Question 7.6 Is there a knot K in $\mathbb{S}^3$ for which $\mathscr{C}\mathbb{O}_{\mathrm{br}}(K) = \varnothing$?

7.2 Double branched covers

In this section we study the circular-orderability of the double branched cover of links, in particular the case of double branched covers of alternating links. We start with the observation that, if the L-space conjecture is true, then the fundamental group of the double branched cover of a quasialternating knot is never left-orderable (the double branched cover of a quasialternating link is an L-space [45]).

In contrast to this, there are alternating Montesinos links whose double branched cover are prism manifolds by [49; 4], so their fundamental groups are not circularly orderable by Proposition 2.5. Similarly, it turns out that the Weeks manifold is homeomorphic to $\Sigma_2(9_{49})$ [39, Main result], and so the double branched cover of 9_{49} has non-circularly orderable fundamental group. Yet 9_{49} is a quasialternating knot [35].

On the other hand, there are many examples of alternating and quasialternating knots whose double branched covers do have circularly orderable fundamental groups, as the next two results show.

7.2.1 Generalized Fibonacci groups Let $M(k,m)$ denote the double branched cover of the alternating link which is the closure of the 3-strand braid $(\sigma_1^k\sigma_2^{-k})^m$, where m and k are positive integers. By [38, page 169], the fundamental group of $M(k,m)$ is isomorphic to the generalized Fibonacci group

$$F^k_{2m} = \langle x_1,\dots,x_{2m} \mid x_i x^k_{i+1} = x_{i+2} \text{ for any } i = 1,\dots,2m\rangle,$$

where the indices are taken mod $2m$.

Proposition 7.7 *For any $k \geq 2$, the fundamental group of $M(k,m)$ is circularly orderable.*

Proof For any $k \geq 2$, $M(k,m)$ is irreducible by [38, Lemma 6 and page 171]. Moreover we can define an epimorphism $\rho\colon F^k_{2m} \to \mathbb{Z}_k * \mathbb{Z}_k$ by $\rho(x_{2i}) \mapsto x$ and $\rho(x_{2i+1}) \mapsto y$, where x and y are the generators of $\mathbb{Z}_k * \mathbb{Z}_k$. Since $\mathbb{Z}_k * \mathbb{Z}_k$ is circularly orderable by Proposition 2.4(2), $\pi_1(M(k,m))$ is circularly orderable by Proposition 3.2(2). □

Remark 7.8 For $k \geq 2$, the manifolds $M(k,2)$ are obtained by identifying two Seifert fibred spaces along a common torus boundary [38, page 171]. Thus $M(k,2)$ is a graph manifold. For $k \geq 2$ and $m \geq 3$, the manifolds $M(k,m)$ are irreducible, Haken and atoroidal hyperbolic 3-manifolds by [38, Lemma 6].

7.2.2 Generalized Takahashi manifolds Fix two positive integers n and m and a collection of integers $\{p_{k,j}, q_{k,j}, r_{k,j}, s_{k,j}\}$ satisfying $\gcd(p_{k,j}, q_{k,j}) = 1$, $\gcd(r_{k,j}, s_{k,j}) = 1$ and $p_{k,j}, r_{k,j} \geq 0$ for all $1 \leq k \leq n$ and $1 \leq j \leq m$. The generalized Takahashi manifold $T_{n,m}(p_{k,j}/q_{k,j}; r_{k,j}/s_{k,j})$ is the double branched cover of $\mathbb{S}^3$, branched over the closure of the braid appearing in Figure 2 [43, Theorem 3], first defined in [43, Section 2]. We will denote the closure of this braid by $L_{n,m}(p_{k,j}/q_{k,j}; r_{k,j}/s_{k,j})$.

This family of manifolds contains many well-known 3-manifolds, such as all n-fold cyclic branched covers of 2-bridge knots, the Weeks manifold, and some graph manifolds. When $p_{k,j} = p_j$, $q_{k,j} = q_j$, $r_{k,j} = r_j$ and $s_{k,j} = s_j$ for all $1 \leq k \leq n$ and $1 \leq j \leq m$, the manifold $T_{n,m}(p_j/q_j; r_j/s_j)$ is called a generalized periodic Takahashi manifold; correspondingly, it is the double branched cover of the link $L_{n,m}(p_j/q_j; r_j/s_j)$.

Each generalized periodic Takahashi manifold can also be viewed as a cyclic branched cover over a knot in a connected sum of lens spaces.

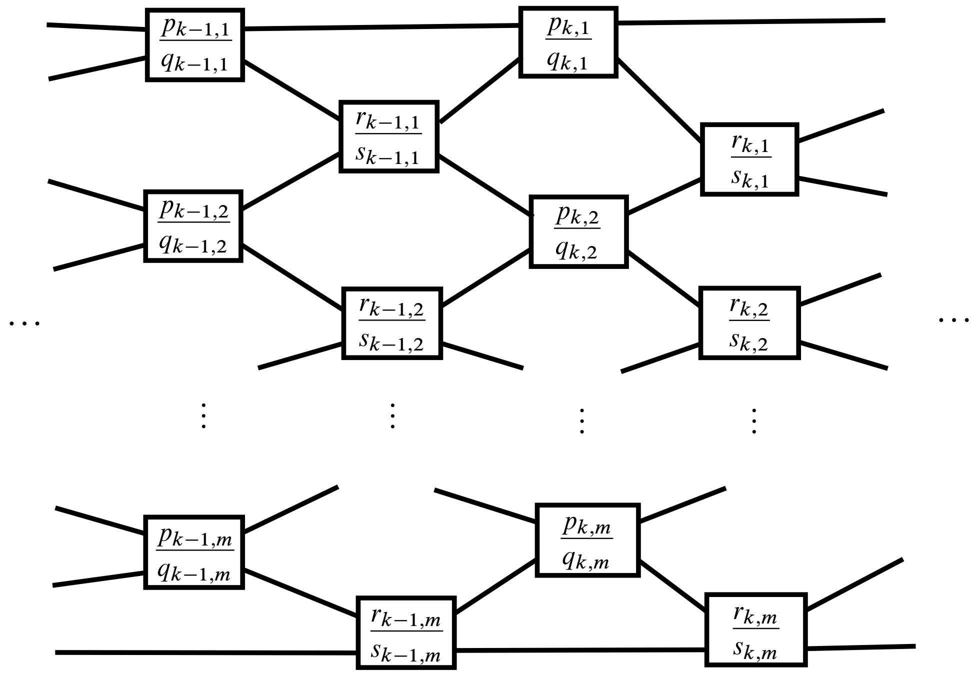

Figure 2: The braid that defines the generalized Takahashi manifold $T_{n,m}(p_{k,j}/q_{k,j};r_{k,j}/s_{k,j})$. The fraction used to label each box determines the rational tangle used in that box to create $L_{n,m}(p_{k,j}/q_{k,j};r_{k,j}/s_{k,j})$.

Theorem 7.9 [43, Theorem 6] *The generalized periodic Takahashi manifold $T_{n,m}(p_j/q_j;r_j/s_j)$ is the n-fold cyclic branched cover of the connected sum of $2m$ lens spaces*

$$L(p_1,q_1)\,\#\,L(r_1,s_1)\,\#\cdots\#\,L(p_m,q_m)\,\#\,L(r_m,s_m)$$

branched over a knot which does not depend on n.

We use this as follows:

Theorem 7.10 *The fundamental group of a generalized periodic Takahashi manifold $T_{n,m}(p_j/q_j;r_j/s_j)$ is circularly orderable if the set $\{p_j,r_j \mid 1\le j\le m\}$ contains at least two elements different from 1 and the link $L_{n,m}(p_j/q_j;r_j/s_j)$ is prime.*

Proof Since the link $L_{n,m}(p_j/q_j;r_j/s_j)$ is prime, $T_{n,m}(p_j/q_j;r_j/s_j)$ is irreducible by the equivariant sphere theorem [40] and the positive answer of the Smith conjecture [41]. By Theorem 7.9 and [28, Lemma 2.11], there exists a surjective homomorphism from the fundamental group of the generalized periodic Takahashi manifold $\pi_1(T_{n,m}(p_j/q_j;r_j/s_j))$ to the free product $\mathbb{Z}_{p_1}*\mathbb{Z}_{r_1}*\cdots*\mathbb{Z}_{p_m}*\mathbb{Z}_{r_m}$. Therefore, if the set $\{p_j,r_j\mid 1\le j\le m\}$ contains at least two elements different from 1, this free product is infinite, and hence, by Proposition 3.2, $\pi_1(T_{n,m}(p_j/q_j;r_j/s_j))$ is circularly orderable. ☐

Remark 7.11 The family of generalized Fibonacci manifolds is a subfamily of the family of generalized periodic Takahashi manifolds.

Question 7.12 Is it possible to characterize the knots $K \subset \mathbb{S}^3$ for which $\pi_1(\Sigma_2(K))$ is circularly orderable?

7.3 Dehn surgery

In this brief section, we point out that circular-orderability of manifolds arising from Dehn surgery on a knot in an integer homology 3-sphere has already appeared in the literature under a different guise, from which we already observe different behaviour than left-orderability with respect to Dehn surgery.

Recall that, for a knot K in an irreducible integer homology 3-sphere M, the result of p/q Dehn surgery on M is denoted by $M_{p/q}(K)$. The L-space conjecture predicts that, for a knot K in $\mathbb{S}^3$, if $\pi_1(\mathbb{S}^3_{p/q}(K))$ is non-left-orderable for some $p/q > 0$, then in fact $\pi_1(\mathbb{S}^3_{p/q}(K))$ is non-left-orderable precisely when $p/q \geq 2g(K)-1$, where $g(K)$ is the genus of K [44, Proposition 2.1; 48, Theorem 1].

To contrast this with circular-orderability, we first observe that, in light of Theorem 1.1:

Proposition 7.13 *Suppose that M is a compact, connected, $\mathbb{P}^2$-irreducible 3-manifold and $H_1(M;\mathbb{Z})$ is cyclic. Then $\pi_1(M)$ is circularly orderable if and only if the universal abelian cover of M has left-orderable fundamental group.*

Consequently, for an irreducible integer homology 3-sphere M, since we have $H_1(M_{p/q}(K);\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, we precisely need to investigate the universal abelian covers of these manifolds in order to know whether or not their fundamental groups are circularly orderable. This is precisely what is done in [13].

When K is fibred, we use h to denote its monodromy and $c(h)$ the fractional Dehn twist coefficient of h; see [13, Section 4] for details.

Theorem 7.14 [13] *Suppose that K is a fibred hyperbolic knot in an irreducible integer homology 3-sphere M. Given coprime p and q with $p \geq 1$, the universal abelian cover of $M_{p/q}(K)$ has left-orderable fundamental group whenever*

(1) *$pc(h) \in \mathbb{Z}$ and $q \neq pc(h)$, or*

(2) *$pc(h) \notin \mathbb{Z}$ and $q \notin \{\lfloor pc(h) \rfloor, \lfloor pc(h) \rfloor + 1\}$.*

Consequently, for any fibred knot K in an irreducible integer homology 3-sphere M, the result of p/q surgery is a manifold with circularly orderable fundamental group whenever the surgery coefficient p/q satisfies either condition (1) or (2) of Theorem 7.14.

Question 7.15 Fix a knot K in an irreducible integer homology 3-sphere M. Is it true that the set $\{p/q \mid M_{p/q}(K) \text{ is not circularly orderable}\}$ is always finite?

References

[1] **I Ba**, *L-spaces, left-orderability and two-bridge knots*, J. Knot Theory Ramifications 28 (2019) art. id. 1950019 MR Zbl

[2] **I Ba**, **A Clay**, *The space of circular orderings and semiconjugacy*, J. Algebra 586 (2021) 582–606 MR Zbl

[3] **H Baik**, **E Samperton**, *Spaces of invariant circular orders of groups*, Groups Geom. Dyn. 12 (2018) 721–763 MR Zbl

[4] **W Ballinger**, **C C-Y Hsu**, **W Mackey**, **Y Ni**, **T Ochse**, **F Vafaee**, *The prism manifold realization problem*, Algebr. Geom. Topol. 20 (2020) 757–815 MR Zbl

[5] **J Bell**, **A Clay**, **T Ghaswala**, *Promoting circular-orderability to left-orderability*, Ann. Inst. Fourier (Grenoble) 71 (2021) 175–201 MR Zbl

[6] **V V Bludov**, **A M W Glass**, *Word problems, embeddings, and free products of right-ordered groups with amalgamated subgroup*, Proc. Lond. Math. Soc. 99 (2009) 585–608 MR Zbl

[7] **M Boileau**, **S Boyer**, **C M Gordon**, *Branched covers of quasi-positive links and L-spaces*, J. Topol. 12 (2019) 536–576 MR Zbl

[8] **S Boyer**, *Dehn surgery on knots*, from "Handbook of geometric topology" (R J Daverman, R B Sher, editors), North-Holland, Amsterdam (2002) 165–218 MR Zbl

[9] **S Boyer**, **A Clay**, *Foliations, orders, representations, L-spaces and graph manifolds*, Adv. Math. 310 (2017) 159–234 MR Zbl

[10] **S Boyer**, **A Clay**, *Order-detection of slopes on the boundaries of knot manifolds*, Groups Geom. Dyn. 18 (2024) 1317–1348 MR Zbl

[11] **S Boyer**, **C M Gordon**, **Y Hu**, *Slope detection and toroidal 3-manifolds*, preprint (2021) arXiv 2106.14378

[12] **S Boyer**, **C M Gordon**, **L Watson**, *On L-spaces and left-orderable fundamental groups*, Math. Ann. 356 (2013) 1213–1245 MR Zbl

[13] **S Boyer**, **Y Hu**, *Taut foliations in branched cyclic covers and left-orderable groups*, Trans. Amer. Math. Soc. 372 (2019) 7921–7957 MR Zbl

[14] **S Boyer**, **D Rolfsen**, **B Wiest**, *Orderable 3-manifold groups*, Ann. Inst. Fourier (Grenoble) 55 (2005) 243–288 MR Zbl

[15] **D Calegari**, *Circular groups, planar groups, and the Euler class*, from "Proceedings of the Casson Fest", Geom. Topol. Monogr. 7, Geom. Topol. Publ., Coventry (2004) 431–491 MR Zbl

[16] **D Calegari**, **N M Dunfield**, *Laminations and groups of homeomorphisms of the circle*, Invent. Math. 152 (2003) 149–204 MR Zbl

[17] **A Clay**, **T Ghaswala**, *Free products of circularly ordered groups with amalgamated subgroup*, J. Lond. Math. Soc. 100 (2019) 775–803 MR Zbl

[18] **A Clay**, **T Ghaswala**, *Circularly ordering direct products and the obstruction to left-orderability*, Pacific J. Math. 312 (2021) 401–419 MR Zbl

[19] **A Clay**, **T Ghaswala**, *Cofinal elements and fractional Dehn twist coefficients*, Int. Math. Res. Not. 2024 (2024) 7401–7420 MR Zbl

[20] **A Clay**, **T Lidman**, **L Watson**, *Graph manifolds, left-orderability and amalgamation*, Algebr. Geom. Topol. 13 (2013) 2347–2368 MR Zbl

[21] **A Clay**, **K Mann**, **C Rivas**, *On the number of circular orders on a group*, J. Algebra 504 (2018) 336–363 MR Zbl

[22] **A Clay**, **L Watson**, *Left-orderable fundamental groups and Dehn surgery*, Int. Math. Res. Not. 2013 (2013) 2862–2890 MR Zbl

[23] **M Culler**, **N M Dunfield**, *Orderability and Dehn filling*, Geom. Topol. 22 (2018) 1405–1457 MR Zbl

[24] **M K Dąbkowski**, **J H Przytycki**, **A A Togha**, *Non-left-orderable 3-manifold groups*, Canad. Math. Bull. 48 (2005) 32–40 MR Zbl

[25] **D B A Epstein**, *Projective planes in 3-manifolds*, Proc. Lond. Math. Soc. 11 (1961) 469–484 MR Zbl

[26] **X Gao**, *Orderability of homology spheres obtained by Dehn filling*, Math. Res. Lett. 29 (2022) 1387–1427 MR Zbl

[27] **É Ghys**, *Groups acting on the circle*, Enseign. Math. 47 (2001) 329–407 MR Zbl

[28] **C Gordon**, **T Lidman**, *Taut foliations, left-orderability, and cyclic branched covers*, Acta Math. Vietnam. 39 (2014) 599–635 MR Zbl

[29] **R Hakamata**, **M Teragaito**, *Left-orderable fundamental group and Dehn surgery on the knot 5_2*, Canad. Math. Bull. 57 (2014) 310–317 MR Zbl

[30] **R Hakamata**, **M Teragaito**, *Left-orderable fundamental groups and Dehn surgery on genus one 2-bridge knots*, Algebr. Geom. Topol. 14 (2014) 2125–2148 MR Zbl

[31] **J Hanselman**, **J Rasmussen**, **S D Rasmussen**, **L Watson**, *L-spaces, taut foliations, and graph manifolds*, Compos. Math. 156 (2020) 604–612 MR Zbl

[32] **W Heil**, *Almost sufficiently large Seifert fiber spaces*, Michigan Math. J. 20 (1973) 217–223 MR Zbl

[33] **W Heil**, *Elementary surgery on Seifert fiber spaces*, Yokohama Math. J. 22 (1974) 135–139 MR Zbl

[34] **Y Hu**, *Left-orderability and cyclic branched coverings*, Algebr. Geom. Topol. 15 (2015) 399–413 MR Zbl

[35] **S Jablan**, *Tables of quasi-alternating knots with at most 12 crossings*, preprint (2014) arXiv 1404.4965

[36] **W Jaco**, *Lectures on three-manifold topology*, CBMS Region. Conf. Ser. Math. 43, Amer. Math. Soc., Providence, RI (1980) MR Zbl

[37] **A Juhász**, *A survey of Heegaard Floer homology*, from "New ideas in low dimensional topology", Ser. Knots Everything 56, World Sci., Hackensack, NJ (2015) 237–296 MR Zbl

[38] **C Maclachlan**, **A W Reid**, *Generalised Fibonacci manifolds*, Transform. Groups 2 (1997) 165–182 MR Zbl

[39] **A Mednykh**, **A Vesnin**, *Visualization of the isometry group action on the Fomenko–Matveev–Weeks manifold*, J. Lie Theory 8 (1998) 51–66 MR Zbl

[40] **W Meeks, III**, **L Simon**, **S T Yau**, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. 116 (1982) 621–659 MR Zbl

[41] **J W Morgan**, **H Bass** (editors), *The Smith conjecture*, Pure Appl. Math. 112, Academic, Orlando, FL (1984) MR Zbl

[42] **K Motegi**, **M Teragaito**, *Generalized torsion elements and bi-orderability of 3-manifold groups*, Canad. Math. Bull. 60 (2017) 830–844 MR Zbl

[43] **M Mulazzani**, **A Vesnin**, *Generalized Takahashi manifolds*, Osaka J. Math. 39 (2002) 705–721 MR Zbl

[44] **P Ozsváth**, **Z Szabó**, *On knot Floer homology and lens space surgeries*, Topology 44 (2005) 1281–1300 MR Zbl

[45] **P Ozsváth**, **Z Szabó**, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. 194 (2005) 1–33 MR Zbl

[46] **H M Pedersen**, *Splice diagram determining singularity links and universal abelian covers*, Geom. Dedicata 150 (2011) 75–104 MR Zbl

[47] **S P Plotnick**, *Finite group actions and nonseparating 2-spheres*, Proc. Amer. Math. Soc. 90 (1984) 430–432 MR Zbl

[48] **J Rasmussen**, *Lens space surgeries and L-space homology spheres*, preprint (2007) arXiv 0710.2531

[49] **J Remigio-Juárez**, **Y Rieck**, *The link volumes of some prism manifolds*, Algebr. Geom. Topol. 12 (2012) 1649–1665 MR Zbl

[50] **P Scott**, *The geometries of 3-manifolds*, Bull. Lond. Math. Soc. 15 (1983) 401–487 MR Zbl

[51] **A T Tran**, *On left-orderability and cyclic branched coverings*, J. Math. Soc. Japan 67 (2015) 1169–1178 MR Zbl

[52] **H Turner**, *Left-orderability, branched covers and double twist knots*, Proc. Amer. Math. Soc. 149 (2021) 1343–1358 MR Zbl

[53] **A A Vinogradov**, *On the free product of ordered groups*, Mat. Sb. 25/67 (1949) 163–168 MR Zbl In Russian

[54] **S D Zheleva**, *Cyclically ordered groups*, Sibirsk. Mat. Zh. 17 (1976) 1046–1051 MR Zbl In Russian; translated in Siberian Math. J. 17 (1976) 773–777

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On the involutive Heegaard Floer homology of negative semidefinite plumbed 3-manifolds with $b_1 = 1$

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Ozsváth and Szabó (2003) used Heegaard Floer homology to define numerical invariants $d_{1/2}$ and $d_{-1/2}$ for 3-manifolds Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. We define involutive Heegaard Floer theoretic versions of these invariants analogous to the involutive d invariants $\bar{d}$ and $\underline{d}$ defined for rational homology spheres by Hendricks and Manolescu (2017). We prove their invariance under spin integer homology cobordism and use them to establish spin filling constraints and 0-surgery obstructions analogous to results by Ozsváth and Szabó for their Heegaard Floer counterparts $d_{1/2}$ and $d_{-1/2}$. We then apply calculation techniques of Dai and Manolescu (2019) and Rustamov (2004) to compute the involutive Heegaard Floer homology of some negative semidefinite plumbed 3-manifolds with $b_1 = 1$. By combining these calculations with the 0-surgery obstructions, we are able to produce an infinite family of small Seifert fibered spaces with weight 1 fundamental group and first homology $\mathbb{Z}$ which cannot be obtained by 0-surgery on a knot in S^3, extending a result of Hedden, Kim, Mark and Park (2019).

57K18, 57K31, 57K41

1 Introduction

Involutive Heegaard Floer homology is an extension of Heegaard Floer homology due to Hendricks and Manolescu [6]. It is constructed by considering the mapping cone of a naturally arising involution on the Heegaard Floer chain complex associated to a given Heegaard diagram. For certain 3-manifolds, involutive Heegaard Floer homology contains more information than Heegaard Floer homology. In particular, it has had success illuminating the structure of the integer homology cobordism group.

Over the past several years there has been significant progress in understanding how to calculate involutive Heegaard Floer homology. Some of the methods developed include the large surgery formula of Hendricks and Manolescu [6], the results on almost rational negative definite plumbings by Dai and Manolescu [2], the connected sum formula of Hendricks, Manolescu and Zemke [7], and most recently the involutive surgery exact triangle established by Hendricks, Hom, Stoffregen and Zemke [5].

To date, much of the focus of these calculation techniques and applications has been on rational homology 3-spheres. The goals of this paper are to

(1) establish topological applications of involutive Heegaard Floer homology for 3-manifolds with $b_1 = 1$, and

(2) find an efficient way to compute the involutive Heegaard Floer homology of a certain class of such manifolds.

For rational homology spheres, important topological information is encoded by the involutive d invariants $\bar{d}$ and $\underline{d}$ defined by Hendricks and Manolescu [6]. These are numerical invariants extracted from the plus (or equivalently minus) version of involutive Heegaard Floer homology with respect to a self-conjugate spin^c structure.

In this paper, we define analogous involutive d invariants $\bar{d}_{-1/2}$, $\bar{d}_{1/2}$, $\underline{d}_{-1/2}$, and $\underline{d}_{1/2}$ for 3-manifolds Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. These invariants are generalizations of the invariants $d_{-1/2}$ and $d_{1/2}$ defined by Ozsváth and Szabó [16] and also encode important topological information. In particular, they are spin integer homology cobordism invariants. Moreover, in Section 2, we prove the following theorems, which generalize [16, Theorem 9.11 and Proposition 4.11].

Theorem A *Suppose X is a smooth oriented negative semidefinite spin 4-manifold with boundary a 3-manifold Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$.*

(1) *If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial, then*

$$b_2(X) - 3 \leq 4\underline{d}_{-1/2}(Y).$$

(2) *If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is nontrivial, then*

$$b_2(X) + 2 \leq 4\underline{d}_{1/2}(Y).$$

Remark 1.1 Hypothesis (1) implies $b_2(X) \geq 1$.

Theorem B *Let M be an oriented integer homology 3-sphere and let Y and M' be the 3-manifolds obtained via 0 and $+1$ surgery, respectively, on a knot K in M. Then*

(1) $\underline{d}(M) - \frac{1}{2} \leq \underline{d}_{-1/2}(Y)$ *and* $\bar{d}(M) - \frac{1}{2} \leq \bar{d}_{-1/2}(Y)$;
(2) $\underline{d}_{1/2}(Y) - \frac{1}{2} \leq \underline{d}(M')$ *and* $\bar{d}_{1/2}(Y) - \frac{1}{2} \leq \bar{d}(M')$.

As a consequence of these theorems, we obtain the following two corollaries:

Corollary C *Suppose K is a knot in S^3 and Y is the result of 0-surgery on K. Then*

(1) $-\frac{1}{2} \leq \underline{d}_{-1/2}(Y)$;
(2) $\bar{d}_{1/2}(Y) \leq \frac{1}{2}$.

Corollary D *Suppose Y is a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. If*

$$\underline{d}_{-1/2}(Y) < -\tfrac{1}{2} \quad \text{and} \quad \underline{d}_{1/2}(Y) < \tfrac{1}{2}$$

then Y is not the boundary of any negative semidefinite spin manifold.

To put the above results to use, we need a practical way to calculate $\bar{d}_{\pm 1/2}$ and $\underline{d}_{\pm 1/2}$. The approach we take to achieve this is to adapt existing methods for computing $\bar{d}$ and $\underline{d}$ for rational homology spheres

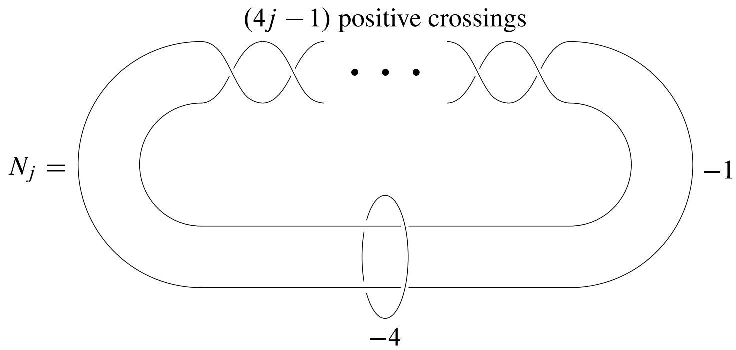

Figure 1

to the setting of 3-manifolds with $b_1 = 1$. Dai and Manolescu [2] provided a combinatorial method to compute the involutive Heegaard Floer homology of a certain class of negative definite plumbed 3-manifolds called almost rational (or AR) plumbed manifolds. In particular, their methods provide a way to compute the invariants $\bar{d}$ and $\underline{d}$ for rational homology spheres which admit such a plumbing.

Their approach utilizes the framework of lattice cohomology and graded roots introduced by Némethi [12; 13]. Lattice cohomology itself builds upon earlier work by Ozsváth and Szabó [17] in which they show how to combinatorially compute the Heegaard Floer homology of a subclass of almost rational plumbings, namely negative definite plumbings with at most one *bad* vertex. Rustamov [23] later generalized this work of Ozsváth and Szabó to the case of negative semidefinite plumbings with $b_1 = 1$ and at most one bad vertex. Subsequent works have established isomorphisms between Heegaard Floer homology and lattice (co)homology (using completed coefficients) for more general classes of plumbings, for example Ozsváth, Stipsicz and Szabó [15] and Zemke [25], the latter of which shows that the completed versions of Heegaard Floer homology and lattice homology are isomorphic for all plumbing trees.

The setting we will work in for computations of $\bar{d}_{\pm 1/2}$ and $\underline{d}_{\pm 1/2}$ is that of negative semidefinite plumbed 3-manifolds with $b_1 = 1$ and at most one bad vertex. To cohesively adapt the work of Dai and Manolescu to this setting, we first recast Rustamov's results into the language of lattice cohomology and graded roots. This requires us to slightly modify Némethi's original definition of lattice cohomology.

After establishing the above computational approach, we carry out a specific calculation of the plus version of the involutive Heegaard Floer homology of an infinite family $\{N_j\}_{j \in \mathbb{N}}$ of small Seifert fiber spaces. For $j \in \mathbb{N}$, we let

$$N_j = S^2\left(-\frac{2}{1}, \frac{-8j+1}{1}, \frac{16j-2}{8j+1}\right).$$

N_j can also be realized as surgery on a 2-component link as in Figure 1.

The family $\{N_j\}_{j \in \mathbb{N}}$ was previously studied by Hedden, Kim, Mark and Park [4]. The manifolds in this family all have first homology equal to $\mathbb{Z}$ and weight 1 fundamental groups, which are necessary conditions if said manifolds could be obtained by 0-surgery on a knot in S^3. However, by using an obstruction in terms the Rokhlin invariant, Hedden, Kim, Mark and Park [4, Theorem 7.3] proved that for all odd positive integers j, N_j cannot be obtained by 0-surgery on a knot in S^3. In the same paper, they also show that if Y is a 3-manifold that is homology cobordant to a Seifert fibered homology $S^1 \times S^2$, then Y automatically satisfies the same $d_{1/2}$ and $d_{-1/2}$ bounds as a manifold obtained by 0-surgery on a knot in S^3 (see [4, Theorem 5.2]). In other words, the noninvolutive version of Corollary C (see [16, Proposition 4.11]) cannot obstruct a Seifert fibered homology $S^1 \times S^2$ from being 0-surgery on a knot in S^3. However, it turns out that the extra information contained in involutive Heegaard Floer homology can detect Seifert fibered 0-surgery. In particular, as an application of Corollaries C and D, we are able to prove the following extension of [4, Theorem 7.3]:

Theorem E *For all positive integers j, N_j cannot be obtained by 0-surgery on a knot in S^3. In fact, N_j is not the oriented boundary of any smooth negative semidefinite spin 4-manifold.*

To provide further context for the above theorem, it is worth noting that there do exist small Seifert fiber spaces which are obtained by 0-surgery on a knot in S^3. For example, by work of Moser [10], 0-surgery on torus knots are small Seifert fibered spaces. More recently, Ichihara, Motegi and Song [8] discovered an infinite family of hyperbolic knots $\{K_n\}_{n\in\mathbb{Z}-\{0,-1,-2\}}$ with small Seifert fibered 0-surgery. These small Seifert manifolds are different from those obtained by 0-surgery on torus knots.

Interestingly, as we describe in Section 5.4,

$$HF^+(-N_1,\mathfrak{s}_0) \cong HF^+(-S^3_0(K_1),\mathfrak{s}_0)$$

where $S^3_0(K_1)$ denotes 0-surgery on K_1 and, on each side of the equation, $\mathfrak{s}_0$ is the unique self-conjugate spin^c structure. However,

$$HFI^+(-N_1,\mathfrak{s}_0) \ncong HFI^+(-S^3_0(K_1),\mathfrak{s}_0).$$

This gives a very concrete example of how involutive Heegaard Floer homology detects Seifert fibered 0-surgery whereas regular Heegaard Floer homology does not.

Organization of the paper

In Section 2, we review involutive Heegaard Floer homology and prove Theorems A and B. In Section 3, we review some basic facts about plumbed manifolds. In Section 4, we define a slightly modified version of lattice cohomology for negative semidefinite plumbings and describe how it fits with prior work of Ozsváth and Szabó, Némethi, and Rustamov. At the end of that section, we adapt [2, Theorem 3.1] to the setting of negative semidefinite plumbings with $b_1 = 1$ and at most one bad vertex. In Section 5, we compute the involutive Heegaard Floer homology of the manifolds $\{N_j\}_{j\in\mathbb{N}}$ as well as $S^3_0(K_1)$. In particular, these calculations together with the results of Section 2, enable us to prove Theorem E.

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2 Involutive Heegaard Floer homology

In this section, we briefly review the construction of involutive Heegaard Floer homology. We then recall the involutive d invariants, $\underline{d}$ and $\bar{d}$, defined by Manolescu and Hendricks for rational homology spheres and define analogous invariants, $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$, for closed oriented 3-manifolds with first homology $\mathbb{Z}$. We show that $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$ are spin integer homology cobordism invariants and use them to establish constraints on the intersection forms of negative semidefinite spin 4-manifolds whose boundary is a 3-manifold with first homology $\mathbb{Z}$. Furthermore, we establish new obstructions to a 3-manifold being realized as 0-surgery on a knot in an integer homology sphere.

We assume the reader is familiar with Heegaard Floer homology (see for example [19; 18; 21; 22]).

2.1 Notation and conventions

- We use $\mathbb{F} = \mathbb{Z}_2$ coefficients for all Heegaard Floer and involutive Heegaard Floer homology groups.
- Given a graded $\mathbb{F}[U]$-module $\mathcal{A}$, we let $\mathcal{A}[r]$ be the graded $\mathbb{F}[U]$-module defined by $\mathcal{A}[r]_k = \mathcal{A}_{k+r}$. The subscripts denote the homogeneous elements of the corresponding grading.
- We let $\mathcal{T}^+ = \mathbb{F}[U, U^{-1}]/(U \cdot \mathbb{F}[U])$ be the graded $\mathbb{F}[U]$-module where $\mathrm{gr}(U^n) = -2n$.
- We let $\mathcal{T}^+_d := \mathcal{T}^+[-d]$. In other words, $\mathcal{T}^+_d$ is the $\mathbb{F}[U]$-module $\mathcal{T}^+$ with grading shifted so that the minimal nonzero grading level is d.

2.2 Review of involutive Heegaard Floer homology

For complete details of the construction of involutive Heegaard Floer homology see [6].

Let Y be any closed, connected, oriented 3-manifold. Fix a spin^c structure $\mathfrak{s}$ on Y and let $\overline{\omega} = \{\mathfrak{s}, \bar{\mathfrak{s}}\}$ be the orbit of $\mathfrak{s}$ under the conjugation action. Let $\mathcal{H} = (H, J)$ be a Heegaard pair, ie $H = (\Sigma, \alpha, \beta, z)$ is a pointed Heegaard diagram for Y admissible with respect to $\mathfrak{s}$ and J is a generic family of almost complex structures on $\mathrm{Sym}^g(\Sigma)$. Given this setup, define

$$CF^\circ(\mathcal{H}, \overline{\omega}) = \bigoplus_{\mathfrak{t} \in \overline{\omega}} CF^\circ(\mathcal{H}, \mathfrak{t})$$

where $CF^\circ(\mathcal{H}, \mathfrak{t})$ is the usual Heegaard Floer chain complex associated to $(\mathcal{H}, \mathfrak{t})$.

We call $\overline{\mathcal{H}} = (\overline{H}, \bar{J})$ the conjugate Heegaard pair where $\overline{H} = (-\Sigma, \beta, \alpha, z)$ and where $\bar{J}$ is the corresponding conjugate family of almost complex structures. As shown by Ozsváth and Szabó [18, Theorem 2.4], there is a canonical isomorphism of chain complexes

$$\eta\colon CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\overline{\mathcal{H}}, \bar{\mathfrak{s}}).$$

Moreover, H and $\overline{H}$ both represent the same 3-manifold Y; swapping the order of the α and β curves and reversing the orientation of Σ both have the effect of reversing the orientation on Y and thus cancel each other out. One may think of $\overline{H}$ as being obtained from H by flipping the handle decomposition corresponding to H upside down.

Using naturality results of Juhász, Thurston and Zemke [9], it was observed by Hendricks and Manolescu [6, Proposition 2.3] that given two Heegaard pairs representing the same 3-manifold there is a chain homotopy equivalence between their respective Heegaard Floer chain complexes. Furthermore, these chain homotopy equivalences form a transitive system. In particular, since $\mathcal{H}$ and $\overline{\mathcal{H}}$ both represent Y, we get a chain homotopy equivalence

$$\Phi(\overline{\mathcal{H}}, \mathcal{H})\colon CF^\circ(\overline{\mathcal{H}}, \bar{\mathfrak{s}}) \to CF^\circ(\mathcal{H}, \bar{\mathfrak{s}}).$$

Taking the composition of η and Φ, we obtain a map

$$\iota = \Phi(\overline{\mathcal{H}}, \mathcal{H}) \circ \eta\colon CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\mathcal{H}, \bar{\mathfrak{s}})$$

which is uniquely determined up to chain homotopy. By swapping the roles of $\mathfrak{s}$ and $\bar{\mathfrak{s}}$ in the above discussion, we get a second map going in the opposite direction which, by an abuse of notation, we again call ι,

$$\iota\colon CF^\circ(\mathcal{H}, \bar{\mathfrak{s}}) \to CF^\circ(\mathcal{H}, \mathfrak{s}).$$

It is shown in [6] that $\iota^2\colon CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\mathcal{H}, \mathfrak{s})$ is chain homotopic to the identity.

By a further abuse of notation, we let ι also denote the direct sum of the two ι maps above, ie

$$\iota\colon CF^\circ(\mathcal{H}, \varpi) \to CF^\circ(\mathcal{H}, \varpi).$$

We then define the involutive Heegaard Floer complex, $CFI^\circ(\mathcal{H}, \varpi)$, to be the mapping cone complex

$$CF^\circ(\mathcal{H}, \varpi) \xrightarrow{Q(1+\iota)} Q \cdot CF^\circ(\mathcal{H}, \varpi)[-1].$$

Here, Q is a formal variable that shifts the grading down by 1. Therefore, as graded $\mathbb{F}[U]$-modules, $Q \cdot CF^\circ(\mathcal{H}, \varpi)[-1] \cong CF^\circ(\mathcal{H}, \varpi)$ (strictly, these are $\mathbb{Z}_2$-graded modules; there is only an absolute $\mathbb{Q}$-grading lifting the $\mathbb{Z}_2$-grading when $\mathfrak{s}$ is torsion, for example when $\mathfrak{s}$ is self-conjugate). Introducing the formal variable Q gives $CFI^\circ(\mathcal{H}, \varpi)$ the extra structure of a $\mathbb{F}[Q, U]/(Q^2)$-module rather than just an $\mathbb{F}[U]$-module. The involutive Heegaard Floer homology, $HFI^\circ(\mathcal{H}, \varpi)$, is then defined to be the homology of $CFI^\circ(\mathcal{H}, \varpi)$. It turns out that the isomorphism class of $HFI^\circ(\mathcal{H}, \varpi)$ as a graded $\mathbb{F}[Q, U]/(Q^2)$-module is independent of the choice of auxiliary data $\mathcal{H}$. Therefore, we will write $HFI^\circ(Y, \varpi)$ rather than $HFI^\circ(\mathcal{H}, \varpi)$. If $\mathfrak{s}$ is self-conjugate ($\mathfrak{s} = \bar{\mathfrak{s}}$), we write $HFI^\circ(Y, \mathfrak{s})$.

Remark 2.1 Since $HFI^\circ(Y, \overline{\omega})$ is currently only defined up to isomorphism, it is important to highlight that when one considers elements of (or maps on) HFI°, one needs to make a choice of auxiliary data. It is not known whether canonical $\mathbb{F}[Q,U]/(Q^2)$-modules can be associated to each pair $(Y, \overline{\omega})$. For that, one would need higher order naturality results. See [6, Section 2.4] for more details about this issue.

2.3 Involutive d invariants

Hendricks and Manolescu [6, Section 5] defined involutive d invariants, denoted by $\bar{d}$ and $\underline{d}$, for self-conjugate spin^c structures of rational homology spheres. Before recalling their definitions and generalizing them to 3-manifolds with $H_1 = \mathbb{Z}$, we need to review a few basic properties.

Proposition 2.2 [6, Proposition 4.6] *Suppose Y is a closed, connected, oriented 3-manifold and $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$. Then there exists an exact triangle of $\mathbb{F}[U]$-modules*

$$\begin{tikzcd}
HF^\circ(Y,\mathfrak{s}) \arrow[rr, "Q(1+\iota_*)"] & & Q \cdot HF^\circ(Y,\mathfrak{s})[-1] \arrow[dl, "g"] \\
& HFI^\circ(Y,\mathfrak{s}) \arrow[ul, "h"] &
\end{tikzcd}$$

where h decreases grading by 1 and the maps $Q(1+\iota_*)$ and g preserve grading.

Corollary 2.3 *With $(Y, \mathfrak{s})$ as in the previous proposition, if $HF_r^\circ(Y, \mathfrak{s}) \cong 0$ or $\mathbb{F}$, then the map*

$$Q(1+\iota_*)\colon HF_r^\circ(Y,\mathfrak{s}) \to Q \cdot HF_r^\circ(Y,\mathfrak{s})[-1]$$

is trivial.

Proof Since ι^2 is chain homotopic to the identity, the induced map $\iota_*^2 = 1$. In particular, ι_* is an automorphism. Since the only automorphisms of $\mathbb{F}$ or 0 are the identity, if r is a grading for which $HF_r^\circ(Y,\mathfrak{s}) \cong 0$ or $\mathbb{F}$, then ι_* is the identity. Thus, $Q(1+\iota_*) = Q(1+1) = 0$. □

Next we recall a structure result for the ∞-flavor of Heegaard Floer homology. To be consistent with [18], we phrase the next theorem in terms of $\mathbb{Z}$-coefficients. However, we will only be concerned with the mod 2 reduction of this result.

Theorem 2.4 [18, Section 10] *Let Y be a closed, connected, oriented 3-manifold. If $b_1(Y) \le 2$, then there exists an equivalence class of orientation system over Y such that for any torsion spin^c structure $\mathfrak{s}$, we have*

$$HF^\infty(Y,\mathfrak{s}) \cong \mathbb{Z}[U,U^{-1}] \otimes_{\mathbb{Z}} \Lambda^* H^1(Y;\mathbb{Z})$$

as $\mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^*(H_1(Y;\mathbb{Z})/\mathrm{Tors})$-modules.

In Heegaard Floer terminology, HF^∞ is said to be *standard* if it satisfies the conclusion of the above theorem. In other words, Theorem 2.4 says that if $b_1(Y) \in \{0,1,2\}$, then $HF^\infty(Y,\mathfrak{s})$ is automatically standard. In particular, if $b_1(Y)=0$, ie if Y is a rational homology sphere, then $HF^\infty(Y,\mathfrak{s})\cong\mathbb{F}[U,U^{-1}]$. In this case, as graded $\mathbb{F}[U]$-modules, we have the (noncanonical) splitting

$$HF^+(Y,\mathfrak{s})\cong\mathcal{T}_d^+\oplus HF_{\mathrm{red}}^+(Y,\mathfrak{s})$$

where $d=d(Y,\mathfrak{s})$ is the usual d invariant of $(Y,\mathfrak{s})$ and $\mathcal{T}_d^+$ corresponds to the image of

$$\pi_*\colon HF^\infty(Y,\mathfrak{s})\to HF^+(Y,\mathfrak{s}).$$

Similarly, if $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$ and $\mathfrak{s}_0$ is the unique torsion spin^c structure on Y, then we have that $HF^\infty(Y,\mathfrak{s}_0)\cong\mathbb{F}[U,U^{-1}]\oplus\mathbb{F}[U,U^{-1}]$ and we get the (noncanonical) splitting

$$HF^+(Y,\mathfrak{s}_0)\cong\mathcal{T}_{d_{-1/2}}^+\oplus\mathcal{T}_{d_{1/2}}^+\oplus HF_{\mathrm{red}}^+(Y,\mathfrak{s}_0)$$

where $d_{-1/2}=d_{-1/2}(Y,\mathfrak{s}_0)$ and $d_{1/2}=d_{1/2}(Y,\mathfrak{s}_0)$ are the two d invariants for $(Y,\mathfrak{s}_0)$ and $\mathcal{T}_{d_{-1/2}}^+\oplus\mathcal{T}_{d_{1/2}}^+$ corresponds to the $\mathrm{Im}(\pi_*)$. Recall, $d_{\pm1/2}\equiv\pm1/2 \bmod 2$.

Remark 2.5 The previous paragraph applies more generally to Y with $b_1(Y)=1$, not just $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$, but to simplify the exposition we will restrict to the case $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$. Ultimately, we are concerned with 0-surgery applications, so this restriction suffices for our purposes.

Proposition 2.6 *Let Y be a closed, connected oriented 3-manifold with $b_1(Y)=0$ or $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$. If $\mathfrak{s}\in\mathrm{Spin}^c(Y)$ with $\mathfrak{s}=\bar{\mathfrak{s}}$, then we get an exact triangle of $\mathbb{F}[U]$-modules:*

$$\begin{tikzcd}
HF^\infty(Y,\mathfrak{s}) \arrow[rr, "0"] & & Q\cdot HF^\infty(Y,\mathfrak{s})[-1] \arrow[dl, "g^\infty"] \\
& HFI^\infty(Y,\mathfrak{s}) \arrow[ul, "h^\infty"] &
\end{tikzcd}$$

Proof By the above discussion, if r is a grading for which $HF_r^\infty(Y,\mathfrak{s})\neq0$, then $HF_r^\infty(Y,\mathfrak{s})\cong\mathbb{F}$. The proposition then follows immediately from Corollary 2.3 and Proposition 2.2. □

We now analyze the conclusion of Proposition 2.6 in the case $b_1=0$ and recall the definition of the involutive d invariants $\bar{d}$ and $\underline{d}$. After this, we consider the case $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$. To minimize confusion, for the rest of this section we use the letter M to denote rational homology spheres and the letter Y to denote 3-manifolds with $b_1=1$.

Consider a rational homology sphere M equipped with a self-conjugate spin^c structure $\mathfrak{s}$. Then the exact triangle of Proposition 2.6 decomposes into exact sequences

$$0\to Q\cdot HF_r^\infty(M,\mathfrak{s})[-1]\xrightarrow{\sim}HFI_r^\infty(M,\mathfrak{s})\to0\quad(\text{if } r\equiv d(M,\mathfrak{s}) \bmod 2),$$
$$0\to HFI_r^\infty(M,\mathfrak{s})\xrightarrow{\sim}HF_{r-1}^\infty(M,\mathfrak{s})\to0\quad(\text{if } r\equiv d(M,\mathfrak{s})+1 \bmod 2).$$

Since the maps in the exact triangle are U-equivariant, we further get that HFI^∞ splits as a graded $\mathbb{F}[U]$-module,

$$HFI^\infty(M,\mathfrak{s})\cong Q\cdot HF^\infty(M,\mathfrak{s})[-1]\oplus HF^\infty(M,\mathfrak{s})[-1].$$

This splitting is canonical since HF_r^∞ is supported in alternating degrees. Moreover, as graded $\mathbb{F}[Q,U]/(Q^2)$-modules (up to possibly an overall grading shift) one can check that

$$HFI^\infty(M,\mathfrak{s}) \cong \mathbb{F}[Q,U,U^{-1}]/(Q^2).$$

Therefore, we may think of $HFI^\infty(M,\mathfrak{s})$ as the direct sum of two doubly infinite towers: one which is not in the image of Q, and the other which is the image of the first under multiplication by Q. Both towers have involutive grading congruent to $d(M,\mathfrak{s})$ mod $2\mathbb{Z}$.

We now recall the definition of the involutive d invariants introduced by Hendricks and Manolescu. To make sense of the definition, it is useful to recall that

$$\operatorname{Im}(\pi_*\colon HFI^\infty(M,\mathfrak{s}) \to HFI^+(M,\mathfrak{s})) = \operatorname{Im}(U^n)$$

for $n \gg 0$ (see [19, Lemma 4.6]).

Definition 2.7 [6, Definition 5.1] Let M be an oriented rational homology 3-sphere and $\mathfrak{s} \in \operatorname{Spin}^c(M)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$. Define the lower and upper involutive correction terms of $(M,\mathfrak{s})$ to be $\underline{d}(M,\mathfrak{s})$ and $\bar{d}(M,\mathfrak{s})$, respectively, where

$$\begin{aligned}
\underline{d}(M,\mathfrak{s}) &= \min\{r \mid \exists\, x \in HFI_r^+(M,\mathfrak{s}), x \in \operatorname{Im}(U^n), x \notin \operatorname{Im}(U^nQ) \text{ for } n \gg 0\} - 1,\\
\bar{d}(M,\mathfrak{s}) &= \min\{r \mid \exists\, x \in HFI_r^+(M,\mathfrak{s}), x \neq 0, x \in \operatorname{Im}(U^nQ) \text{ for } n \gg 0\}.
\end{aligned}$$

It is conceptually useful to think of $\bar{d}$ and $\underline{d}$ in terms of a splitting of HFI^+ into towers and reducible elements as follows:

Corollary 2.8 *Suppose M is an oriented rational homology 3-sphere and $\mathfrak{s} \in \operatorname{Spin}^c(M)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$. Then we get a (noncanonical) splitting as graded $\mathbb{F}[U]$-modules,*

$$HFI^+(M,\mathfrak{s}) \cong \mathcal{T}^+_{\bar{d}} \oplus \mathcal{T}^+_{\underline{d}+1} \oplus HFI^+_{\mathrm{red}}(M,\mathfrak{s}).$$

Here, $\mathcal{T}^+_{\bar{d}} \oplus \mathcal{T}^+_{\underline{d}+1}$ corresponds to $\operatorname{Im}(\pi_*)$, with $\mathcal{T}^+_{\bar{d}}$ in the image of Q.

The invariants $\underline{d}$ and $\bar{d}$ satisfy the following basic properties:

Proposition 2.9 [6, Propositions 5.1 and 5.2] *With M and $\mathfrak{s}$ as in Definition 2.7,*

1. $\underline{d}(M,\mathfrak{s}) \le d(M,\mathfrak{s}) \le \bar{d}(M,\mathfrak{s})$;
2. $\underline{d}(M,\mathfrak{s}) = -\bar{d}(-M,\mathfrak{s})$.

Additionally, Hendricks and Manolescu generalize [16, Theorem 9.6] to the involutive setting to obtain:

Theorem 2.10 [6, Theorem 1.2] *With M and $\mathfrak{s}$ as in Definition 2.7, if X is a smooth negative definite 4-manifold with boundary M and $\mathfrak{t}$ is a spin structure on X such that $\mathfrak{t}|_M = \mathfrak{s}$, then*

$$\operatorname{rank}(H^2(X;\mathbb{Z})) \le 4\underline{d}(M,\mathfrak{s}).$$

The method of proof of Theorem 2.10 is used to further show that $\underline{d}$ and $\bar{d}$ are spin rational homology cobordism invariants.

Now suppose Y is a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and let $\mathfrak{s}_0$ be the unique torsion spin^c-structure on Y. Then the exact triangle of Proposition 2.6 decomposes into short exact sequences

$$0 \to Q \cdot HF_r^\infty(Y,\mathfrak{s}_0)[-1] \xrightarrow{g^\infty} HFI_r^\infty(Y,\mathfrak{s}_0) \xrightarrow{h^\infty} HF_{r-1}^\infty(Y,\mathfrak{s}_0) \to 0.$$

These short exact sequences are of the form

$$0 \to \mathbb{F} \to \mathbb{F} \oplus \mathbb{F} \to \mathbb{F} \to 0.$$

Therefore, as vector spaces, we get a splitting

$$HFI_r^\infty(Y,\mathfrak{s}_0) \cong Q \cdot HF_r^\infty(Y,\mathfrak{s}_0)[-1] \oplus HF_{r-1}^\infty(Y,\mathfrak{s}_0)$$

where each summand is one-dimensional. Unlike in the $b_1 = 0$ case, this splitting is not canonical. However, we are still able to get the following structure result:

Proposition 2.11 *Suppose Y is a closed connected oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and $\mathfrak{s}_0 \in \mathrm{Spin}^c(Y)$ is the unique spin^c structure with $\mathfrak{s}_0 = \bar{\mathfrak{s}}_0$. Then, as graded $\mathbb{F}[Q,U]/(Q^2)$-modules,*

$$HFI^\infty(Y,\mathfrak{s}_0) \cong \mathbb{F}[Q,U,U^{-1}]/(Q^2) \oplus \mathbb{F}[Q,U,U^{-1}]/(Q^2)$$

where, on the right side of the equation, the first factor has gradings congruent to $1/2 \bmod 2$ and the second factor has gradings congruent to $-1/2 \bmod 2$.

Proof Fix a Heegaard pair $\mathcal{H} = (H,J)$ representing Y and admissible with respect to $\mathfrak{s}_0$. Let ∂^I be the boundary map on the involutive chain complex. We can compactly write ∂^I as $\partial^I = \partial + Q(1+\iota)$ where ∂ is the usual boundary map on the Heegaard Floer chain complex extended by Q-linearity.

By Theorem 2.4, $HF_{1/2}^\infty(\mathcal{H},\mathfrak{s}_0) \cong HF_{-1/2}^\infty(\mathcal{H},\mathfrak{s}_0) \cong \mathbb{F}$. Let $\alpha \in HF_{1/2}^\infty(\mathcal{H},\mathfrak{s}_0)$ and $\beta \in HF_{-1/2}^\infty(\mathcal{H},\mathfrak{s}_0)$ be the unique nonzero generators. Let $a,b \in CF^\infty(\mathcal{H},\mathfrak{s}_0)$ be representatives of α and β respectively. Then, the unique nonzero element in the image of

$$g^\infty \colon Q \cdot HF_{1/2}^\infty(\mathcal{H},\mathfrak{s}_0)[-1] \to HFI_{1/2}^\infty(\mathcal{H},\mathfrak{s}_0)$$

is $[Qa]$. Similarly, $[Qb]$ is the unique nonzero element in the image of

$$g^\infty \colon Q \cdot HF_{-1/2}^\infty(\mathcal{H},\mathfrak{s}_0)[-1] \to HFI_{-1/2}^\infty(\mathcal{H},\mathfrak{s}_0).$$

As we have observed above, $1+\iota_*$ is the zero map on homology. Therefore, there exists some $x,y \in CF^\infty(\mathcal{H},\mathfrak{s}_0)$ such that $(1+\iota)a = \partial x$ and $(1+\iota)b = \partial y$. Thus, $\partial^I(a+Qx) = 0$ and $\partial^I(b+Qy) = 0$. Furthermore, we have that $Q[a+Qx] = [Qa]$ and $Q[b+Qy] = [Qb]$. Therefore, the first summand in the decomposition can be taken to be $(\mathbb{F}[Q,U,U^{-1}]/(Q^2))[b+Qy]$ and the second to be $(\mathbb{F}[Q,U,U^{-1}]/(Q^2))[a+Qx]$. □

The isomorphism in Proposition 2.11 is not canonical with respect to a given Heegaard pair $\mathcal{H} = (H,J)$ because the elements $[a+Qx]$ and $[b+Qy]$ depend on our choice of representatives a,b,x,y. Despite this, we can still define involutive d invariants in this situation. We only need to know the $\mathbb{F}[Q,U]/(Q^2)$-module structure of HFI^∞, regardless of a canonical isomorphism.

Definition 2.12 Let Y be a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. Let $\mathfrak{s}_0$ be the unique spin^c structure on Y with $\mathfrak{s}_0 = \bar{\mathfrak{s}}_0$. Define

$$\underline{d}_{1/2}(Y,\mathfrak{s}_0) = \min\{r \mid r \equiv -\tfrac{1}{2} \bmod 2, \exists x \in HFI_r^+(Y,\mathfrak{s}_0), x \in \mathrm{Im}(U^n), x \notin \mathrm{Im}(U^n Q) \text{ for } n \gg 0\} - 1,$$
$$\underline{d}_{-1/2}(Y,\mathfrak{s}_0) = \min\{r \mid r \equiv \tfrac{1}{2} \bmod 2, \exists x \in HFI_r^+(Y,\mathfrak{s}_0), x \in \mathrm{Im}(U^n), x \notin \mathrm{Im}(U^n Q) \text{ for } n \gg 0\} - 1,$$
$$\bar{d}_{1/2}(Y,\mathfrak{s}_0) = \min\{r \mid r \equiv \tfrac{1}{2} \bmod 2, \exists x \in HFI_r^+(Y,\mathfrak{s}_0), x \neq 0, x \in \mathrm{Im}(U^n Q) \text{ for } n \gg 0\},$$
$$\bar{d}_{-1/2}(Y,\mathfrak{s}_0) = \min\{r \mid r \equiv -\tfrac{1}{2} \bmod 2, \exists x \in HFI_r^+(Y,\mathfrak{s}_0), x \neq 0, x \in \mathrm{Im}(U^n Q) \text{ for } n \gg 0\}.$$

Remark 2.13 Since $\mathfrak{s}_0$ is unique, we will often just write $\underline{d}_{\pm 1/2}(Y)$ and $\bar{d}_{\pm 1/2}(Y)$, or $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$ if Y is clear from context.

As in the $b_1 = 0$ case, it is again useful to think of these invariants in terms of a splitting of HFI^+.

Corollary 2.14 *Suppose Y is a closed, connected, oriented 3-manifold with $H_1(Y,\mathbb{Z}) \cong \mathbb{Z}$ and $\mathfrak{s}_0 \in \mathrm{Spin}^c(Y)$ is the unique Spin^c structure with $\mathfrak{s}_0 = \bar{\mathfrak{s}}_0$. Then there exists a (noncanonical) splitting*

$$HFI^+(Y,\mathfrak{s}_0) \cong \mathcal{T}^+_{\bar{d}_{1/2}} \oplus \mathcal{T}^+_{\bar{d}_{-1/2}} \oplus \mathcal{T}^+_{\underline{d}_{1/2}+1} \oplus \mathcal{T}^+_{\underline{d}_{-1/2}+1} \oplus HFI^+_{\mathrm{red}}(Y,\mathfrak{s}_0)$$

where

$$\mathcal{T}^+_{\bar{d}_{1/2}} \oplus \mathcal{T}^+_{\bar{d}_{-1/2}} \oplus \mathcal{T}^+_{\underline{d}_{1/2}+1} \oplus \mathcal{T}^+_{\underline{d}_{-1/2}+1}$$

corresponds to $\mathrm{Im}(\pi_*)$ and $\mathcal{T}^+_{\bar{d}_{1/2}} \oplus \mathcal{T}^+_{\bar{d}_{-1/2}}$ is contained in the image of multiplication by Q.

Proposition 2.15 *The involutive correction terms $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$ satisfy the basic properties*

(1) $\underline{d}_{\pm 1/2}(Y) \leq d_{\pm 1/2}(Y) \leq \bar{d}_{\pm 1/2}(Y)$;

(2) $\underline{d}_{\pm 1/2}(Y) = -\bar{d}_{\mp 1/2}(-Y)$.

Proof The proof of (1) follows from the same arguments as the proof of [6, Proposition 5.1]. The proof of (2) follows from [6, Proposition 4.4] and the same arguments as in the proof of [6, Proposition 5.2]. □

2.4 Spin filling constraints, homology cobordism invariance, and 0-surgery obstruction

Ozsváth and Szabó [16, Theorem 9.11] established constraints in terms of $d_{\pm 1/2}$ on the intersection form of a negative semidefinite 4-manifold with boundary a given 3-manifold Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. Furthermore, Ozsváth and Szabó [16, Corollary 9.14, Proposition 4.11] established 0-surgery obstructions in terms of $d_{\pm 1/2}$. In this section, we establish the analogous results in the involutive setting.

Theorem 2.16 *Suppose X is a smooth oriented negative semidefinite spin 4-manifold with boundary a 3-manifold Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$.*

(1) *If the restriction $H^1(X;\mathbb{Z})\to H^1(Y;\mathbb{Z})$ is trivial, then*
$$b_2(X)-3\le 4\underline{d}_{-1/2}(Y).$$
(2) *If the restriction $H^1(X;\mathbb{Z})\to H^1(Y;\mathbb{Z})$ is nontrivial, then*
$$b_2(X)+2\le 4\underline{d}_{1/2}(Y).$$

Proof Let $\mathfrak{s}$ be a spin structure on X. In particular, $c_1^2(\mathfrak{s})=0$. We follow the proof strategy of [16, Theorem 9.11].

(1) Suppose the restriction $H^1(X;\mathbb{Z})\to H^1(Y;\mathbb{Z})$ is trivial. First, surger out all of $b_1(X)$ without changing the nondegenerate part of the intersection form of X. Then, remove a ball from X to obtain W which we regard as a cobordism $W\colon S^3\to Y$. As observed in the proof of [16, Theorem 9.11], the map induced from the cobordism W,
$$F^{\infty}_{W,\mathfrak{s}|_W}\colon HF^{\infty}(S^3)\to HF^{\infty}(Y,\mathfrak{s}|_Y),$$
is injective with image equal to the doubly infinite tower with degrees congruent to $-1/2 \bmod 2$ and shifts degree by $\ell=\frac14(b_2(X)-3)$. Also, by [6, Section 4.5] there exists an induced map
$$F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha}\colon HFI^{\infty}(S^3)\to HFI^{\infty}(Y,\mathfrak{s}_0)$$
which also shifts degree by $\ell=\frac14(b_2(X)-3)$. Note that the involutive cobordism map $F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha}$ depends on an additional choice of auxiliary data α.

Combining the results in [6, Section 4.5] with Proposition 2.6, we see that for every even integer r, we have the following commutative diagram with exact horizontal rows:

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr, "Q(1+\iota_*)"] & & QHF^{\infty}_{r+1+\ell}(Y,\mathfrak{s}_0)[-1] \arrow[rr, "g^{\infty}_Y"] \arrow[dd, "\pi_Y"] & & HFI^{\infty}_{r+1+\ell}(Y,\mathfrak{s}_0) \arrow[rr, "h^{\infty}_Y"] \arrow[dd, "\pi^I_Y"] & & HF^{\infty}_{r+\ell}(Y,\mathfrak{s}_0) \arrow[rr, "Q(1+\iota_*)"] \arrow[dd, "\pi_Y"] & & 0 \\
& 0 \arrow[ur] \arrow[rr] \arrow[dd] & & HFI^{\infty}_{r+1}(S^3) \arrow[ur, "F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha}"] \arrow[rr, "h^{\infty}_{S^3}"] \arrow[dd, "\pi^I_{S^3}"] & & HF^{\infty}_r(S^3) \arrow[ur, "F^{\infty}_{W,\mathfrak{s}|_W}"] \arrow[rr] \arrow[dd, "\pi_{S^3}"] & & 0 \arrow[ur] & \\
& & QHF^{+}_{r+1+\ell}(Y,\mathfrak{s}_0)[-1] \arrow[rr, "g^{+}_Y"] & & HFI^{+}_{r+1+\ell}(Y,\mathfrak{s}_0) \arrow[rr, "h^{+}_Y"] & & HF^{+}_{r+\ell}(Y,\mathfrak{s}_0) & & \\
& QHF^{+}_{r+1}(S^3)[-1] \arrow[ur] \arrow[rr] & & HFI^{+}_{r+1}(S^3) \arrow[ur, "F^{I,+}_{W,\mathfrak{s}|_W,\alpha}"] \arrow[rr, "h^{+}_{S^3}"] & & HF^{+}_r(S^3) \arrow[ur, "F^{+}_{W,\mathfrak{s}|_W}"] & & &
\end{tikzcd}$$

By definition of $\underline{d}_{-1/2}(Y)$, there exists some y^+ in $HFI^{+}_{\underline{d}_{-1/2}+1}(Y,\mathfrak{s}_0)$ such that $y^+\in\mathrm{Im}(U^n)$ for $n\gg0$ and $y^+\notin\mathrm{Im}(U^nQ)$ for $n\gg0$. The condition $[y^+\in\mathrm{Im}(U^n)$ for $n\gg0]$ is equivalent to the condition $[y^+\in\mathrm{Im}(\pi^I_Y)]$. Therefore, there exists some $y^{\infty}\in HFI^{\infty}_{\underline{d}_{-1/2}+1}(Y,\mathfrak{s}_0)$ such that $\pi^I_Y(y^{\infty})=y^+$. The condition $[y^+\notin\mathrm{Im}(U^nQ)$ for $n\gg0]$ implies that $y^{\infty}\notin\mathrm{Im}(g^{\infty}_Y)$. Therefore, by exactness, $h^{\infty}_Y(y^{\infty})\neq0\in HF^{\infty}_{\underline{d}_{-1/2}}(Y,\mathfrak{s}_0)$. By assumption, the map $F^{\infty}_{W,\mathfrak{s}|_W}\colon HF^{\infty}_{\underline{d}_{-1/2}-\ell}(S^3)\to HF^{\infty}_{\underline{d}_{-1/2}}(Y,\mathfrak{s}|_Y)$

is an isomorphism. Moreover, by exactness, the map $h^{\infty}_{S^3}\colon HFI^{\infty}_{\underline{d}_{-1/2}+1-\ell}(S^3) \to HF^{\infty}_{\underline{d}_{-1/2}-\ell}(S^3)$ is also an isomorphism. Therefore, there exists some $x^{\infty} \in HFI^{\infty}_{\underline{d}_{-1/2}+1-\ell}(S^3)$ such that

$$(F^{\infty}_{W,\mathfrak{s}|_W} \circ h^{\infty}_{S^3})(x^{\infty}) = h^{\infty}_Y(y^{\infty}).$$

Let $z^{\infty} = F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha}(x^{\infty}) \in HFI^{\infty}_{\underline{d}_{-1/2}+1}(Y,\mathfrak{s}_0)$. By commutativity, $h^{\infty}_Y(z^{\infty}) = h^{\infty}_Y(y^{\infty})$. Therefore, $z^{\infty} + y^{\infty} \in \ker(h^{\infty}_Y)$. So, by exactness, there exists some $w^{\infty} \in Q \cdot HF^{\infty}_{\underline{d}_{-1/2}+1}(Y,\mathfrak{s}_0)[-1]$ such that $g^{\infty}_Y(w^{\infty}) = z^{\infty} + y^{\infty}$. If $\pi^I_Y(z^{\infty}) = 0$, then that would imply $\pi^I_Y(g^{\infty}(w^{\infty})) = y^+$. But this would be a contradiction because that would imply $y^+ \in \operatorname{Im}(U^n Q)$ for $n \gg 0$. Therefore, $\pi^I_Y(z^{\infty}) \neq 0$. Thus, $(\pi^I_Y \circ F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha})(x^{\infty}) \neq 0$. So, by commutativity, $(F^{I,+}_{W,\mathfrak{s}|_W,\alpha} \circ \pi^I_{S^3})(x^{\infty}) \neq 0$. In particular, $\pi^I_{S^3}(x^{\infty}) \neq 0$. Therefore, the element $x^+ = \pi^I_{S^3}(x^{\infty}) \in HFI^+_{\underline{d}_{-1/2}+1-\ell}(S^3)$ has the property that $x^+ \in \operatorname{Im}(U^n)$ for $n \gg 0$ and $x^+ \notin \operatorname{Im}(U^n Q)$ for $n \gg 0$. It follows that

$$\underline{d}(S^3) + 1 \leq \underline{d}_{-1/2}(Y) + 1 - \ell. \tag{2.17}$$

Observing that $\underline{d}(S^3) = 0$ and rearranging/canceling the terms, we get

$$b_2(X) - 3 \leq 4\underline{d}_{-1/2}(Y).$$

(2) Now suppose the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is nontrivial. Surger out the 1-dimensional homology of X until $b_1(X) = 1$ and so that the map $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is still nontrivial. Again, remove a ball from X to obtain a cobordism $W\colon S^3 \to Y$. In this case, the induced map

$$F^{\infty}_{W,\mathfrak{s}|_W}\colon HF^{\infty}(S^3) \to HF^{\infty}(Y,\mathfrak{s}|_Y)$$

is injective with image equal to the doubly infinite tower with degrees congruent to $+1/2 \bmod 2$. The degree shift of this map is now $\frac{1}{4}(b_2(X)+2)$. We then repeat the analogous diagram chase to establish the inequality. We leave the details to the reader. □

Corollary 2.18 *Suppose Y is a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. If*

$$\underline{d}_{-1/2}(Y) < -\tfrac{1}{2} \quad \text{and} \quad \underline{d}_{1/2}(Y) < \tfrac{1}{2}$$

then Y is not the boundary of any negative semidefinite spin manifold.

Proof Suppose X is a smooth negative semidefinite spin 4-manifold with boundary Y. If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial, then the map $H^1(Y;\mathbb{Z}) \to H^2(X,Y;\mathbb{Z})$ is injective. Since $H^1(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and $H^2(X,Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z})$, it follows that $b_2(X) \geq 1$. Hence, by Theorem 2.16, $-1/2 \leq \underline{d}_{-1/2}(Y)$. If instead $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is nontrivial, then all we can say about $b_2(X)$ is that $b_2(X) \geq 0$. Theorem 2.16 therefore implies $1/2 \leq \underline{d}_{1/2}(Y)$. The conclusion now follows. □

Proposition 2.19 *Suppose Y_1 and Y_2 are closed oriented 3-manifolds with $H_1(Y_i;\mathbb{Z}) \cong \mathbb{Z}$ for $i \in \{1,2\}$. If there exists a spin integer homology cobordism $(W,\mathfrak{s})\colon Y_1 \to Y_2$, then $\underline{d}_{\pm 1/2}(Y_1) = \underline{d}_{\pm 1/2}(Y_2)$ and $\bar{d}_{\pm 1/2}(Y_1) = \bar{d}_{\pm 1/2}(Y_2)$.*

Proof The argument is the same as in the proof of [6, Proposition 5.4], using the fact that W induces an isomorphism

$$F^{\infty}_{W,\mathfrak{s},\alpha}\colon HFI^{\infty}(Y_1,\mathfrak{s}|_{Y_1}) \to HFI^{\infty}(Y_2,\mathfrak{s}|_{Y_2}). \qquad \square$$

Theorem 2.20 *Let M be an oriented integer homology 3-sphere and let Y and M' be the 3-manifolds obtained via 0 and $+1$ surgery, respectively, on a knot K in M. Then*

(1) $\underline{d}(M) - \frac{1}{2} \le \underline{d}_{-1/2}(Y)$ *and* $\bar{d}(M) - \frac{1}{2} \le \bar{d}_{-1/2}(Y)$;

(2) $\underline{d}_{1/2}(Y) - \frac{1}{2} \le \underline{d}(M')$ *and* $\bar{d}_{1/2}(Y) - \frac{1}{2} \le \bar{d}(M')$.

Proof First, we prove the inequalities in (1).

Let $(W,\mathfrak{s})$ be the spin cobordism from M to Y obtained by attaching a 0-framed 2-handle along K and let $\mathfrak{s}_0$ be the trivial spin^c structure on Y. Then, by [16, Proposition 9.3], the induced map

$$F^{\infty}_{W,\mathfrak{s}}\colon HF^{\infty}(M) \to HF^{\infty}(Y,\mathfrak{s}_0)$$

shifts grading by $-1/2$ and is injective with image equal to the doubly infinite tower with gradings congruent to $-1/2$ mod 2. The first inequality of (1) now follows by repeating exactly the same argument as in the proof of Theorem 2.16 where now M assumes the role of S^3 and $\ell = -1/2$ (see inequality (2.17)).

To establish the second inequality in (1), we consider the rightward continuation of the commutative diagram used in the proof of Theorem 2.16 again replacing S^3 with M. Specifically, for r even, we have the following commutative diagram with exact horizontal rows:

$$\begin{tikzcd}
 & 0 \arrow[r, "Q(1+\iota_*)"] & QHF^{\infty}_{r-1/2}(Y,\mathfrak{s}_0)[-1] \arrow[r, "g^{\infty}_Y"] \arrow[dd, "\pi_Y"] & HFI^{\infty}_{r-1/2}(Y,\mathfrak{s}_0) \arrow[r, "h^{\infty}_Y"] \arrow[dd, "\pi^I_Y"] & HF^{\infty}_{r-1.5}(Y,\mathfrak{s}_0) \arrow[r, "Q(1+\iota_*)"] \arrow[dd, "\pi_Y"] & 0 \\
0 \arrow[ur] \arrow[r] & QHF^{\infty}_{r}(M)[-1] \arrow[ur, "F^{\infty}_{W,\mathfrak{s}}"] \arrow[r, "g^{\infty}_M"] \arrow[dd, "\pi_M"] & HFI^{\infty}_{r}(M) \arrow[ur, "F^{I,\infty}_{W,\mathfrak{s},\alpha}"] \arrow[r] \arrow[dd, "\pi^I_M"] & 0 \arrow[ur] \arrow[dd] & & \\
 & & QHF^{+}_{r-1/2}(Y,\mathfrak{s}_0)[-1] \arrow[r, "g^{+}_Y"] & HFI^{+}_{r-1/2}(Y,\mathfrak{s}_0) \arrow[r, "h^{+}_Y"] & HF^{+}_{r-1.5}(Y,\mathfrak{s}_0) & \\
 & QHF^{+}_{r}(M)[-1] \arrow[ur, "F^{+}_{W,\mathfrak{s}}"] \arrow[r, "g^{+}_M"] & HFI^{+}_{r}(M) \arrow[ur, "F^{I,+}_{W,\mathfrak{s},\alpha}"] \arrow[r] & HF^{+}_{r-1}(M) \arrow[ur] & &
\end{tikzcd}$$

Now we get that g^{∞}_M is an isomorphism, and we again know that $F^{\infty}_{W,\mathfrak{s}}$ is an isomorphism. Furthermore, g^{∞}_Y is injective with $\mathrm{Im}(g^{\infty}_Y) = \ker(h^{\infty}_Y)$. Thus, $F^{I,\infty}_{W,\mathfrak{s},\alpha}$ maps $HFI^{\infty}_r(M)$ isomorphically onto $\mathrm{Im}(g^{\infty}_Y)$.

By definition of $\bar{d}_{-1/2}$, there exists some nonzero $y^+ \in HFI^+(Y,\mathfrak{s}_0)$ such that $\mathrm{gr}(y^+) = \bar{d}_{-1/2}$ and $y^+ \in \mathrm{Im}(U^n Q)$ for $n \gg 0$. This implies that there exists some element

$$y^{\infty} \in \mathrm{Im}(g^{\infty}_Y) \subset HFI^{\infty}_{\bar{d}_{-1/2}}(Y,\mathfrak{s}_0)$$

such that $\pi^I_Y(y^\infty) = y^+$. Therefore, the unique nonzero element of $HFI^\infty_{\bar{d}_{-1/2}+1/2}(M)$, which we will call x^∞, maps to y^∞ under $F^{I,\infty}_{W,\mathfrak{s},\alpha}$. Since $(\pi^I_Y \circ F^{I,\infty}_{W,\mathfrak{s},\alpha})(x^\infty) = y^+ \neq 0$, the commutativity of the diagram implies $\pi^I_M(x^\infty) \neq 0$. Additionally, $\pi^I_M(x^\infty) \in \mathrm{Im}(U^n Q)$ for $n \gg 0$. Therefore,

$$\bar{d}(M) \leq \bar{d}_{-1/2}(Y) + \tfrac{1}{2}.$$

The proofs of the inequalities in (2) follow the same arguments as the proofs of (1), except that now we consider the maps

$$F^\circ_{W',\mathfrak{s}'} \colon HF^\circ(Y, \mathfrak{s}_0) \to HF^\circ(M')$$

and

$$F^{I,\circ}_{W',\mathfrak{s}',\alpha'} \colon HFI^\circ(Y, \mathfrak{s}_0) \to HFI^\circ(M')$$

induced by the spin cobordism $(W', \mathfrak{s}') \colon Y \to M'$ obtained by attaching a 2-handle to the dual of K in Y with framing so that the resulting space is M'. Analyzing the corresponding commutative diagrams and using the fact that for all r even,

$$F^\infty_{W',\mathfrak{s}'} \colon HF^\infty_{r+1/2}(Y, \mathfrak{s}_0) \to HF^\infty_r(M')$$

is an isomorphism, we get statement (2). We leave the details to the reader. □

Corollary 2.21 *Suppose K is a knot in S^3 and Y is the result of 0-surgery on K. Then*

(1) $-\frac{1}{2} \leq \underline{d}_{-1/2}(Y)$;

(2) $\bar{d}_{1/2}(Y) \leq \frac{1}{2}$.

Proof Note that $0 = d(S^3) = \underline{d}(S^3) = \bar{d}(S^3)$. Therefore, (1) follows immediately from Theorem 2.20. For (2), let $\overline{K}$ be the mirror of K. Then 0-surgery on $\overline{K}$ is $-Y$. Thus, we have $-\frac{1}{2} \leq \underline{d}_{-1/2}(-Y, \mathfrak{s}_0)$. Now by Proposition 2.15, $\underline{d}_{-1/2}(-Y, \mathfrak{s}_0) = -\bar{d}_{1/2}(Y, \mathfrak{s}_0)$. Therefore, $\bar{d}_{1/2}(Y, \mathfrak{s}_0) \leq \frac{1}{2}$. □

3 Plumbings

We now make a digression from our discussion of involutive Heegaard Floer homology to review basic properties of plumbed 3- and 4-manifolds.

Notation 3.1 Given a graph Γ, we denote the set of vertices of Γ by $\mathcal{V}(\Gamma)$ and the set of edges by $\mathcal{E}(\Gamma)$.

Definition 3.2 A *weighted graph* is a graph Γ together with a function $m \colon \mathcal{V}(\Gamma) \to \mathbb{Z}$, called a *weight function*. Given a vertex $v \in \mathcal{V}(\Gamma)$, we call $m(v)$ the weight of v. Usually we will refer to a weighted graph as Γ and not explicitly write the weight function associated to it.

For the purposes of this paper, we will use the term *plumbing graph* to mean a weighted graph Γ such that $|\mathcal{V}(\Gamma)| < \infty$ and Γ is a forest (ie a disjoint union of trees). Plumbing graphs in general can be more complicated, however for simplicity we only consider plumbing graphs of the type just described.

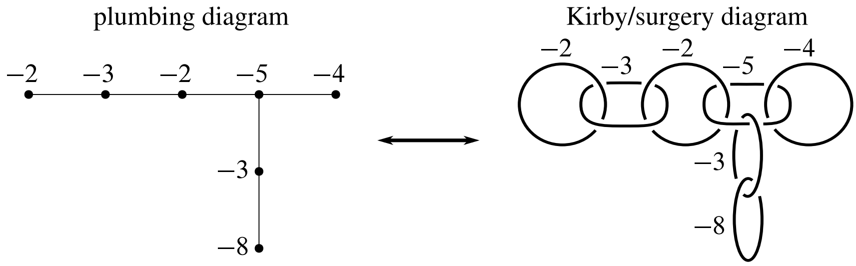


Figure 2

Given a connected plumbing graph Γ, we let $X(\Gamma)$ denote the 4-manifold obtained by plumbing disk bundles over 2-spheres according to Γ (see [3, Example 4.6.2] for details of this construction). If Γ is not connected, then we let $X(\Gamma)$ be the boundary connected sum of the plumbed 4-manifolds corresponding to the connected components of Γ. Regardless of whether Γ is connected or not, we let $Y(\Gamma)$ be the boundary of $X(\Gamma)$ and call it the *plumbed* 3-*manifold* associated to Γ.

Remark 3.3 In general, a given plumbed 3-manifold Y may bound many different plumbed 4-manifolds. Neumann [14] described a calculus for passing between different plumbing graphs that describe the same 3-manifold.

Given a plumbing graph Γ, a Kirby diagram for $X(\Gamma)$ (which is also a surgery diagram for $Y(\Gamma)$) is given by an $m(v)$-framed unknot for each $v \in \mathcal{V}(\Gamma)$ such that any pair of these unknots is either Hopf linked or unlinked depending on whether or not there is an edge between the vertices with which the unknots correspond. See Figure 2.

3.1 Algebraic topological properties of plumbings

Fix a plumbing graph Γ and let $X = X(\Gamma)$ and $Y = Y(\Gamma)$ be the associated plumbed 4- and 3-manifolds. Label the vertices of Γ by $\mathcal{V}(\Gamma) = \{v_1, \dots, v_s\}$ where $s = |\mathcal{V}(\Gamma)|$. For each $v_j \in \mathcal{V}(\Gamma)$, let $[v_j] \in H_2(X;\mathbb{Z})$ be the homology class of the 2-sphere corresponding to the 0-section of the D^2-bundle associated to v_j. Equivalently, $[v_j]$ is represented by the capped-off core of the corresponding 2-handle. In particular, it is easy to see that $H_2(X;\mathbb{Z}) \cong \bigoplus_{j=1}^{s} \mathbb{Z}[v_j]$. Given $x = \sum a_j[v_j] \in H_2(X;\mathbb{Z})$, we write $x \geq 0$ if $a_j \geq 0$ for all j. If in addition, $x \neq 0$, we write $x > 0$. Given two elements $x, y \in H_2(X;\mathbb{Z})$, we write $x \geq y$ (resp. $x > y$) if $x - y \geq 0$ (resp. $x - y > 0$).

Denote the intersection form of X by

$$(\cdot\,,\cdot)\colon H_2(X,\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}.$$

By construction,

$$([v_i],[v_j]) = \begin{cases} m(v_i) & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and there is an edge } [v_i, v_j] \text{ connecting } v_i \text{ and } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let B be the matrix of the intersection form with respect to the ordered basis $([v_1], \dots, [v_s])$. Notice, B is the incidence matrix of the graph Γ with the i^{th} diagonal entry equal to $m(v_i)$.

Definition 3.4 We define the *definiteness type* of a plumbing graph Γ to be the definiteness type of its associated intersection form $(\cdot,\cdot)$, or equivalently the definiteness type of B. For example, we say Γ is negative semidefinite if $(\cdot,\cdot)$ is negative semidefinite.

By an abuse of notation, we will also refer to the corresponding intersection pairing on cohomology as $(\cdot,\cdot)\colon H^2(X,Y;\mathbb{Z}) \times H^2(X,Y;\mathbb{Z}) \to \mathbb{Z}$. It will be useful to, in addition, consider the slightly modified intersection pairing $(\cdot,\cdot)'\colon H^2(X;\mathbb{Z}) \times H^2(X,Y;\mathbb{Z}) \to \mathbb{Z}$ with a different domain, but still defined by the usual formula: $(\alpha,\beta)' = (\alpha \cup \beta)[X]$.

Recall the set of characteristic vectors of X, denoted by $\mathrm{Char}(X)$, is defined by

$$\begin{aligned}\mathrm{Char}(X) &= \{\alpha \in H^2(X;\mathbb{Z}) \mid (\alpha,\beta)' \equiv (\beta,\beta) \bmod 2 \text{ for all } \beta \in H^2(X,Y;\mathbb{Z})\} \\ &= \{\alpha \in H^2(X;\mathbb{Z}) \mid \alpha(x) \equiv (x,x) \bmod 2 \text{ for all } x \in H_2(X;\mathbb{Z})\}.\end{aligned}$$

We now recall the relationship between the spin^c structures on X and Y and the characteristic vectors of X. The first observation is that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Spin}^c(X) \arrow[r, "|_Y"] \arrow[d, "c_1"] & \mathrm{Spin}^c(Y) \arrow[d, "c_1"] \\ \mathrm{Char}(X) \arrow[r, "\partial^*"] & H^2(Y;\mathbb{Z}) \end{tikzcd}$$

where c_1 denotes the first Chern class of the determinant line bundle of the spin^c structure, the top horizontal map is restriction to Y and the bottom horizontal map is the restriction of the map

$$\partial^*\colon H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$$

in the long exact sequence in cohomology of the pair (X,Y). The left vertical map is a bijection since $H_1(X,\mathbb{Z})$ has no 2-torsion (see [3, page 56] for details). Therefore, c_1 provides a canonical identification of $\mathrm{Spin}^c(X)$ with $\mathrm{Char}(X)$. Furthermore, since X is simply connected, we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1(Y;\mathbb{Z}) \arrow[r, "i^*"] \arrow[d, phantom, "\wr\|"] & H^2(X,Y;\mathbb{Z}) \arrow[r, "j^*"] \arrow[d, phantom, "\wr\|"] & H^2(X;\mathbb{Z}) \arrow[r, "\partial^*"] \arrow[d, phantom, "\wr\|"] & H^2(Y;\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\ 0 \arrow[r] & H_2(Y;\mathbb{Z}) \arrow[r, "i_*"] & H_2(X;\mathbb{Z}) \arrow[r, "j_*"] & H_2(X,Y;\mathbb{Z}) \arrow[r, "\partial_*"] & H_1(Y;\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

with exact rows coming from the long exact sequences in homology and cohomology of the pair (X,Y) and with vertical isomorphisms given by Poincaré/Lefschetz duality.

We have yet another commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) \arrow[r, phantom, "\cong"] & H^2(X;\mathbb{Z}) \arrow[d, phantom, "\|\wr"] \\ H_2(X;\mathbb{Z}) \arrow[u, "\phi"] \arrow[r, "j_*"] & H_2(X,Y;\mathbb{Z}) \end{tikzcd}$$

where the top row is the isomorphism coming from the universal coefficient theorem, the right vertical map is the Lefschetz duality isomorphism, and the map ϕ is defined by $\phi(x) = (x, \cdot)$.

Combining the three previous diagrams we get the following commutative diagram:

$$\begin{tikzcd}
 & & & & \mathrm{Spin}^{\mathrm{c}}(X) \arrow[r, "|_Y"] \arrow[d, "c_1"', "\wr"] & \mathrm{Spin}^{\mathrm{c}}(Y) \arrow[dd, "c_1"] & \\
 & & & & \mathrm{Char}(X) \arrow[d, hook] & & \\
0 \arrow[r] & H^1(Y;\mathbb{Z}) \arrow[r, "i^*"] \arrow[dd, leftrightarrow, "\wr"] & H^2(X,Y;\mathbb{Z}) \arrow[rr, "j^*"] \arrow[dd, leftrightarrow, "\wr"] & & H^2(X;\mathbb{Z}) \arrow[r, "\partial^*"] \arrow[dl, leftrightarrow, "\sim"'] \arrow[dd, leftrightarrow, "\wr"] & H^2(Y;\mathbb{Z}) \arrow[r] \arrow[dd, leftrightarrow, "\wr"] & 0 \\
 & & & \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) & & & \\
0 \arrow[r] & H_2(Y;\mathbb{Z}) \arrow[r, "i_*"] & H_2(X;\mathbb{Z}) \arrow[rr, "j_*"] \arrow[ur, "\phi"] & & H_2(X,Y;\mathbb{Z}) \arrow[r, "\partial_*"] & H_1(Y;\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

In addition, there is a free and transitive action of $H^2(X;\mathbb{Z})$ on $\mathrm{Char}(X)$, defined by $(\alpha, k) \mapsto k + 2\alpha$ for all $\alpha \in H^2(X;\mathbb{Z})$ and $k \in \mathrm{Char}(X)$. Restricting this action to $j^*(H^2(X,Y;\mathbb{Z}))$, we get an action of $j^*(H^2(X,Y;\mathbb{Z}))$ on $\mathrm{Char}(X)$. Let $\mathrm{Char}(X)/2j^*(H^2(X,Y;Z))$ denote the set of orbits of this action and denote the orbit $k + 2j^*(H^2(X,Y;\mathbb{Z}))$ of an element k by $[k]$.

Proposition 3.5 *The map* $\Psi\colon \mathrm{Char}(X)/2j^*(H^2(X,Y;\mathbb{Z})) \to \mathrm{Spin}^{\mathrm{c}}(Y)$ *given by*

$$\Psi([k]) = c_1^{-1}(k)|_Y$$

is well defined and is a bijection.

Notation 3.6 Justified by the above proposition, we will use $[k]$ to denote both the orbit

$$k + 2j^*(H^2(X,Y;\mathbb{Z}))$$

as well as the corresponding spin$^{\mathrm{c}}$ structure $\Psi([k])$.

Remark 3.7 From the above diagram, one can see that if k is a characteristic vector, then $[k]$ is a torsion spin$^{\mathrm{c}}$ structure on Y if and only if some integer multiple of k is in the image of j^*. Equivalently, $[k]$ is torsion if and only if there exists some $z_k \in H_2(X;\mathbb{Z}) \otimes \mathbb{Q}$ such that $k(x) = (z_k, x)$ for all $x \in H_2(X;\mathbb{Z})$.

3.2 Rationality and weight conditions

We now recall some terminology that will be useful later when we discuss lattice cohomology and Heegaard Floer homology of plumbings.

If Γ is a negative definite plumbing tree, then there is a special characteristic vector K_{can} which is called the *canonical characteristic vector*. It is defined by the equation $K_{\mathrm{can}}(v) = -m(v) - 2$ for all $v \in \mathcal{V}(\Gamma)$.

Definition 3.8 A plumbing graph Γ is called *rational* if it is a negative definite tree which satisfies the following condition: if $x \in H_2(X(\Gamma); \mathbb{Z})$ and $x > 0$, then

$$-\frac{K_{\mathrm{can}}(x) - (x, x)}{2} \geq 1.$$

Némethi [12] introduced the following generalization of rational plumbings:

Definition 3.9 [12, Definition 8.1] A negative definite plumbing tree Γ is *almost rational* if there exists a vertex $v \in \mathcal{V}(\Gamma)$ and some integer $r \leq m(v)$ such that if you replace the weight of v with r, Γ becomes rational.

A further generalization of this notion is the following:

Definition 3.10 [15, Definition 2.1] A plumbing tree Γ is *type n* if there exist n vertices of Γ such that if we reduce their weights sufficiently, the plumbing becomes rational.

Remark 3.11 A type n plumbing is not required to be negative definite.

Recall the *degree*, denoted by $\delta(v)$, of a vertex $v \in \mathcal{V}(\Gamma)$ is the number of edges adjacent to v. Following the terminology introduced in [17], we say a vertex is *bad* if $m(v) > -\delta(v)$. In particular, it can be shown that a negative definite plumbing with at most one bad vertex is almost rational.

4 Heegaard Floer homology and lattice cohomology of plumbings

In this section, we review some of the key developments in the Heegaard Floer homology and lattice cohomology of plumbed 3-manifolds. We then present a modified version of lattice cohomology that involves passing to a quotient lattice. This presentation enables us to readily adapt and combine the work of Rustamov [23] and the work of Dai and Manolescu [2] to compute HFI^+ of certain negative semidefinite plumbed 3-manifolds with $b_1 = 1$ and at most one bad vertex.

4.1 Ozsváth–Szabó description of HF^+ of negative definite plumbed 3-manifolds with at most one bad vertex

In an early paper on Heegaard Floer homology, Ozsváth and Szabó [17] provided a combinatorial description of the Heegaard Floer homology of 3-manifolds plumbed along negative definite forests with at most one bad vertex. We briefly review their description.

Given a plumbing presentation Γ of a 3-manifold Y, there is a naturally associated cobordism from S^3 to Y via attaching two handles to $S^3 \times [0, 1]$ according to the plumbing graph Γ. One can turn this cobordism around and use the fact that there is an orientation-preserving diffeomorphism from $-S^3$ to S^3 to yield a cobordism $W_\Gamma \colon -Y \to S^3$. For each spin^c structure $\mathfrak{s}$ on W_Γ, we get a U-equivariant map

$$F^+_{W_\Gamma, \mathfrak{s}} \colon HF^+(-Y, \mathfrak{s}|_Y) \to HF^+(S^3).$$

It is easy to see that the spin^c structures on W_Γ correspond in a direct way to spin^c structures on the plumbed 4-manifold $X(\Gamma)$ since W_Γ is diffeomorphic to $X(\Gamma) - D^4$. Because of this we will work with spin^c structures on $X(\Gamma)$ rather than on W_Γ.

Now by the basic facts about spin^c structures and characteristic vectors described in the previous section and the fact that $HF^+(S^3) \cong \mathcal{T}^+$ as a graded $\mathbb{F}[U]$-module, we can define a map

$$T^+ \colon HF^+(-Y) \to \mathrm{Map}(\mathrm{Char}(X(\Gamma), \mathcal{T}^+)$$

via the formula

$$T^+(\xi)(c_1(\mathfrak{s})) = F^+_{W_\Gamma,\mathfrak{s}}(\xi).$$

Here $\mathrm{Map}(\mathrm{Char}(X(\Gamma), \mathcal{T}^+)$ simply denotes the set of functions from $\mathrm{Char}(X(\Gamma))$ to $\mathcal{T}^+$.

Let $\mathrm{H}^+(\Gamma) \subset \mathrm{Map}(\mathrm{Char}(X(\Gamma), \mathcal{T}^+)$ be the functions ϕ of finite support which satisfy the following adjunction relations: For each $k \in \mathrm{Char}(X(\Gamma))$ and $v_i \in \mathcal{V}(\Gamma)$, let $2n_i = k([v_i]) + ([v_i],[v_i])$. Then,

(1) if $n_i \geq 0$, we require $U^{n_i}\phi(k + 2PDj_*[v_i]) = \phi(k)$;

(2) if $n_i < 0$, we require $U^{-n_i}\phi(k) = \phi(k + 2PDj_*[v_i])$.

The set $\mathrm{H}^+(\Gamma)$ naturally inherits an $\mathbb{F}[U]$-module structure from $\mathcal{T}^+$. One can also introduce a grading on $\mathrm{H}^+(\Gamma)$ by defining $\phi \in \mathrm{H}^+(\Gamma)$ to be a homogeneous element of degree d if $\phi(k) \in \mathcal{T}^+$ is a homogeneous element of degree $d + \frac{1}{4}(k^2 + |\mathcal{V}(\Gamma)|)$ for all $k \in \mathrm{Char}(X(\Gamma))$. Furthermore, we can decompose $\mathrm{H}^+(\Gamma)$ into a direct sum over spin^c structures of Y by defining $\mathrm{H}^+(\Gamma,[k])$ to be the elements of $\mathrm{H}^+(\Gamma)$ which are supported on the *set* $[k]$. Recall $[k]$ denotes both a spin^c structure on Y as well as a subset of $\mathrm{Char}(X(\Gamma))$ (see Notation 3.6).

Remark 4.1 In [17], $\mathrm{H}^+(\Gamma)$ is instead denoted by $\mathbb{H}^+(\Gamma)$. We have changed the notation in this paper to $\mathrm{H}^+(\Gamma)$ to avoid confusion with lattice cohomology which is denoted by $\mathbb{H}^*(\Gamma)$.

The main result (Theorem 1.2) in [17] states that if Γ is a negative definite plumbing with at most one bad vertex, then $T^+ \colon HF^+(-Y(\Gamma),[k]) \to \mathrm{H}^+(\Gamma,[k])$ is an isomorphism of graded $\mathbb{F}[U]$-modules for all spin^c structures $[k]$ on $Y(\Gamma)$. Moreover, $\mathrm{H}^+(\Gamma,[k])$ can be computed combinatorially from the data encoded by the plumbing graph. Therefore, this result enables one to compute $HF^+(-Y(\Gamma),[k])$ without having to count holomorphic disks. In particular, Ozsváth and Szabó provide a relatively simple algorithm to compute $\ker(U) \subset \mathrm{H}^+(\Gamma,[k])$.

4.2 Némethi's graded roots and lattice cohomology

Building upon the work of Ozsváth and Szabó, Némethi [12] provides an algorithm to compute the entire $\mathbb{F}[U]$-module H^+ for almost rational plumbings by adapting methods of computation sequences used in the study of normal surface singularities. On the way to computing H^+, Némethi's algorithm first computes an intermediate object called a graded root whose definition we review below (see Definition 4.19). For now, we will just mention that a graded root is weighted graph associated to $Y(\Gamma)$ from which one can easily calculate H^+ and therefore HF^+. Furthermore, by using the language of graded roots, Némethi shows that [17, Theorem 1.2] holds for almost rational plumbed manifolds, a strictly larger class of plumbed 3-manifolds than the class of negative definite trees with at most one bad vertex.

Remark 4.2 We say trees in the previous sentence because strictly speaking almost rational plumbings are typically assumed to be connected. This assumption, however, is not important. The same methods apply to yield the isomorphism if you drop the connectedness assumption in the definition of almost rational.

Motivated by questions involving complex analytic normal surface singularities and the Seiberg–Witten invariant, Némethi [13] further generalized his work on negative definite plumbed 3-manifolds by introducing the broader framework of lattice cohomology. Lattice cohomology assigns to any negative definite plumbed 3-manifold and spin^c structure a graded $\mathbb{F}[U]$-module, which we denote by $\mathbb{H}^*$.

Némethi's original definition provides two different, but equivalent, realizations of lattice cohomology. One realization is constructed by first decomposing Euclidean space $\mathbb{R}^s = \mathbb{R} \otimes H_2(X(\Gamma);\mathbb{Z})$ into cubes using the $\mathbb{Z}$-lattice $H_2(X(\Gamma);\mathbb{Z})$ with basis $[v_1],\dots,[v_s]$. Then, one considers the usual cellular cohomology of $\mathbb{R}^s$, except with the differential modified by a set of weight functions which encode information about the intersection form of $X(\Gamma)$. The other realization is built by taking the cellular cohomology of certain sublevel sets of these weight functions on cubes.

Lattice cohomology also comes equipped with an extra $\mathbb{Z}$-grading. Namely $\mathbb{H}^*$ decomposes as

$$\mathbb{H}^* = \bigoplus_{q=0}^{\infty} \mathbb{H}^q$$

such that each $\mathbb{H}^q$ is itself a $\mathbb{Z}$-graded $\mathbb{F}[U]$-module. In particular, together with his work in [12], Némethi showed that for a negative definite almost rational plumbed 3-manifold, $Y(\Gamma)$, and $\mathfrak{s} \in \mathrm{spin}^c(Y(\Gamma))$, $\mathbb{H}^0(Y(\Gamma),\mathfrak{s})$ is isomorphic to $HF^+(-Y(\Gamma),\mathfrak{s})$ as graded $\mathbb{F}[U]$-modules (up to an overall grading shift), and, moreover, $\mathbb{H}^q(Y,\mathfrak{s}) \cong 0$ for $q \geq 1$. In general, however, it is not the case that for arbitrary negative definite plumbed 3-manifolds $\mathbb{H}^q \cong 0$ for all $q \geq 1$. For example, Némethi [13, Example 4.4.1] showed the existence of a negative definite plumbed rational homology sphere with nontrivial $\mathbb{H}^1$. Of course though, this plumbing is not almost rational.

4.3 Modified formulation of lattice cohomology

In this section, we construct a modified version of lattice cohomology in order to deal with negative semidefinite plumbings. Before defining this modified version, it is important to point out that subsequent to Némethi's original definition of lattice cohomology, other variants have been defined which apply to broader classes of plumbings than those which are negative definite. In particular, Ozsváth, Stipsicz and Szabó [15] consider lattice (co)homology with completed coefficients which apply to arbitrary plumbing trees/forests including those with negative semidefinite intersection forms. The modified construction we provide is very similar to the formulation in [15]; the main difference is that we handle degenerate plumbings by passing to a certain quotient lattice rather than using completed coefficients. As in [13], we begin by giving the constructions in general terms, without reference to plumbings.

4.3.1 Construction 1 Let A be a free finitely generated $\mathbb{Z}$-module with a specified ordered basis $(e_1, \dots, e_n)$. Let $\bar{A}$ be a quotient of A with the property that $\bar{A}$ is itself a free finitely generated $\mathbb{Z}$-module. Given $a \in A$, we write $\bar{a}$ for the corresponding element of $\bar{A}$.

We define a chain complex as follows. For each $0 \le q \le n$, let C_q be the free $\mathbb{F}$-module generated by the set $\mathfrak{Q}_q = \bar{A} \times \{I \subseteq \{1, \dots, n\} \mid |I| = q\}$. Because later we will want to think of these generators as cubes in a cube complex (see Construction 2), we denote the generator of C_q and the element of $\mathfrak{Q}_q$ corresponding to $(\bar{a}, I)$ by $\square(\bar{a}, I)$. We define a differential $\partial \colon C_q \to C_{q-1}$ by the formula

$$\partial\square(\bar{a}, I) = \sum_{i \in I} \big[\square(\bar{a}, I - \{i\}) + \square(\bar{a} + \bar{e}_i, I - \{i\})\big].$$

Remark 4.3 Intuitively, it may be helpful to think of this differential as a cellular boundary map on cubes. We make this point of view precise in Construction 2.

Proposition 4.4
$$\partial^2 = 0.$$

Proof We have

$$\partial^2\square(\bar{a}, I) = \sum_{i \in I} \sum_{j \in I - \{i\}} \big[\square(\bar{a}, I - \{i, j\}) + \square(\bar{a} + \bar{e}_j, I - \{i, j\})\big] \\ + \sum_{i \in I} \sum_{j \in I - \{i\}} \big[\square(\bar{a} + \bar{e}_i, I - \{i, j\}) + \square(\bar{a} + \bar{e}_i + \bar{e}_j, I - \{i, j\})\big].$$

Now observe that the terms of the form $\square(\bar{a}, I - \{i, j\})$ cancel in pairs as i and j vary, as do the terms of the form $\square(\bar{a} + \bar{e}_i + \bar{e}_j, I - \{i, j\})$. Finally, the cross terms also cancel. Therefore, $\partial^2 = 0$. $\square$

Remark 4.5 If one wanted to work over the coefficient ring $\mathbb{Z}$ instead of $\mathbb{F}$, then signs could be introduced as follows: Given a nonempty subset I of $\{1, \dots, n\}$ with $|I| = q$, let $g_I : I \to \{1, \dots, q\}$ be the unique order-preserving bijection. Define the differential via the formula

$$\partial\square(\bar{a}, I) = \sum_{i \in I} (-1)^{g_I(i)} \big[\square(\bar{a}, I - \{i\}) - \square(\bar{a} + \bar{e}_i, I - \{i\})\big].$$

One can check that we still have $\partial^2 = 0$. For the purposes of this paper, we will stick with the coefficient ring $\mathbb{F}$.

For each $0 \le q \le s$, define $\mathscr{F}^q = \mathrm{Hom}_{\mathbb{F}}(C_q, \mathcal{T}^+)$. We endow $\mathscr{F}^q$ with an $\mathbb{F}[U]$-module structure by the formula $(U^n \cdot \phi)(\square_q) = U^n \phi(\square_q)$ for all $\square_q \in \mathfrak{Q}_q$. Our goal now is to define a differential, δ_w, on our cochain modules $\mathscr{F}^q$ by modifying the usual coboundary map by a set of weight functions w.

Definition 4.6 [13, Definition 3.1.4] A set of functions $w_q : \mathcal{Q}_q \to \mathbb{Z}$, for $0 \le q \le n$, is called a set of compatible weight functions if the following hold:

(1) For any integer $k \in \mathbb{Z}$, the set $w_0^{-1}((-\infty, k])$ is finite.

(2) For any $\square(\bar{a}, I) \in \mathcal{Q}_q$ and any $i \in I$,

$$w_q(\square(\bar{a}, I)) \ge w_{q-1}(\square(\bar{a}, I - \{i\})) \quad \text{and} \quad w_q(\square(\bar{a}, I)) \ge w_{q-1}(\square(\bar{a} + \bar{e}_i, I - \{i\})).$$

Fix a set of compatible weight functions w (we drop the subscript for simplicity). By using w, we are able to define a $\mathbb{Z}$-grading on our cochain modules $\mathcal{F}^q$. Specifically, we say that $\phi \in \mathcal{F}^q$ is homogeneous of degree $d \in \mathbb{Z}$ if $\phi(\square_q)$ is a homogeneous element of $\mathcal{T}^+$ of degree $d - 2w(\square_q)$ whenever $\phi(\square_q) \ne 0$.

4.3.2 The differential Mimicking the formula for the differential given in [13, Definition 3.1.4], we define $\delta_w : \mathcal{F}^q \to \mathcal{F}^{q+1}$ as follows:

- Let $\square_{q+1} \in \mathcal{Q}_{q+1}$ and write $\partial \square_{q+1} = \sum_k \square_q^k$.
- Given $\phi \in \mathcal{F}^q$, let

$$(\delta_w \phi)(\square_{q+1}) = \sum_k U^{w(\square_{q+1}) - w(\square_q^k)} \phi(\square_q^k).$$

Proposition 4.7
$$\delta_w^2 = 0.$$

Proof This follows directly from the definition and the fact that $\partial^2 = 0$. $\square$

Definition 4.8 The homology of the cochain complex $(\mathcal{F}^*, \delta_w)$ is called the *lattice cohomology* of the triple $(\bar{A}, (e_1, \dots, e_n), w)$ and is denoted by $\mathbb{H}^*(\bar{A}, (e_1, \dots, e_n), w)$.

Remark 4.9 (1) For each q, the $\mathbb{Z}$-grading on $\mathcal{F}^q$ induces a $\mathbb{Z}$-grading on $\mathbb{H}^q$. Therefore, $\mathbb{H}^q$ is a $\mathbb{Z}$-graded $\mathbb{F}[U]$-module.

(2) If $\bar{A} = A$, then we recover the usual lattice cohomology defined by Némethi [13].

4.3.3 Construction 2 We now give a more geometric, but equivalent formulation of the lattice cohomology theory we defined in Construction 1. This is analogous to [13, Definition 3.1.11].

First, we give a geometric realization of the chain complex C_q. For each $1 \le q \le s$, let $\boldsymbol{c}_q$ be denote the q-dimensional cube $[0, 1]^q$ oriented in the standard way. Additionally, let $\boldsymbol{c}_0$ be a fixed 0-dimensional cube (ie point) oriented positively. To each $\square(\bar{a}, I) \in \mathcal{Q}_q$ we associate a distinct copy of $\boldsymbol{c}_q$. By an abuse of notation, from now on we will regard each $\square(\bar{a}, I) \in \mathcal{Q}_q$ as both a distinct copy of $\boldsymbol{c}_q$ and a generator of C_q depending on which point of view is more convenient in a given context.

We now construct a cube complex $\mathcal{C}$ whose q-dimensional cubes are precisely the elements of $\mathcal{Q}_q$ with attaching maps defined as follows:

- First, we prescribe a method for identifying each $(q-1)$-dimensional face of $\boldsymbol{c}_q$ with $\boldsymbol{c}_{q-1}$. Let $\{x_j\}_{j=1}^{q}$ be the standard coordinate functions on $\boldsymbol{c}_q = [0,1]^q$. Each $(q-1)$-dimensional face of $\boldsymbol{c}_q$ is defined by an equation $x_i = \epsilon$ for some $\epsilon \in \{0,1\}$. Denote this face by $f_{i,\epsilon}$. For $q \geq 2$, we identify $f_{i,\epsilon}$ with $\boldsymbol{c}_{q-1}$ via the map $(x_1,\dots,x_q) \mapsto (x_1,\dots,\hat{x}_i,\dots,x_q)$. For $q=1$, we send the point $f_{i,\epsilon}$ to the point $\boldsymbol{c}_0$.
- Given $\square(\bar{a}, I) \in \mathfrak{Q}_q$, the face $f_{i,\epsilon}$ of $\square(\bar{a}, I)$ gets glued to the cube $\square(\bar{a}+\epsilon\bar{e}_i, I-\{i\})$ via the map defined in the first bullet point.

By construction the q-dimensional cellular chain group of the cube complex $\mathscr{C}$ is equal to C_q and the cellular boundary map is equal to the differential $\partial\colon C_q \to C_{q-1}$ defined in Construction 1.

Again, fix a set of compatible weight functions w. For every integer $n \geq 1$, let S_n be the subcomplex of $\mathscr{C}$ consisting of all cubes $\square$ such that $w(\square_q) \leq n$ where q ranges over all dimensions. Let

$$m_w = \min\{w(\square_q) \mid \square_q \in \mathfrak{Q}_q,\ 0 \leq q \leq n\}.$$

Define

$$\mathbb{S}^q(\bar{A}, (e_1,\dots,e_n), w) = \bigoplus_{n \geq m_w} H^q(S_n; \mathbb{F})$$

where H^q denotes the qth-cellular cohomology. For each fixed q, we give $\mathbb{S}^q(\bar{A}, (e_1,\dots,e_n), w)$ the structure of an $\mathbb{F}[U]$-module by defining the U action to be the restriction map

$$U\colon H^q(S_{n+1};\mathbb{Z}) \to H^q(S_n;\mathbb{Z}).$$

We additionally put a $\mathbb{Z}$-grading on $\mathbb{S}^q(\bar{A}, (e_1,\dots,e_n), w)$ by declaring the elements of $H^q(S_n, \mathbb{Z})$ to be homogeneous of degree $2n$.

Proposition 4.10 *As graded $\mathbb{F}[U]$-modules, $\mathbb{H}^*(\bar{A}, (e_1,\dots,e_n), w) \cong \mathbb{S}^*(\bar{A}, (e_1,\dots,e_n), w)$.*

Proof This is proved in exactly the same way as [13, Theorem 3.1.12(a)]. □

Notation 4.11 From now on we will denote lattice cohomology by $\mathbb{H}^*$ regardless of which construction we are using.

4.3.4 Lattice cohomology associated to negative semidefinite plumbings

Fix a negative semidefinite plumbing graph Γ and let k be a characteristic vector of $X(\Gamma)$ such that $[k]$ is a torsion spin^c structure on $Y(\Gamma)$.

We now show how to associate a lattice cohomology module to the pair (Γ, k). Let $L = H_2(X(\Gamma);\mathbb{Z})$ and $\bar{L} = H_2(X(\Gamma);\mathbb{Z})/\ker(j_*)$. By the long exact sequence in homology, $\bar{L}$ is isomorphic to a submodule of the free finitely generated $\mathbb{Z}$-module $H_2(X,Y;\mathbb{Z})$ and therefore is itself free and finitely generated. As in Section 3, let $s = \operatorname{rank}(H_2(X;\mathbb{Z}))$. Also, let $\sigma = s - b_1(Y)$. With this notation, we have that $\bar{L} \cong \mathbb{Z}^\sigma$. Furthermore, after choosing an ordering on the vertices, the plumbing gives us an ordered basis $([v_1],\dots,[v_s])$ of L.

We now have almost all the data we need in order to get lattice cohomology. It remains to define a set of weight functions. To do this, we rely on our choice of characteristic vector k.

4.3.5 Weight functions Let $\chi_k : L \to \mathbb{Z}$ be the function defined by $\chi_k(x) = -\frac{1}{2}(k(x) + (x, x))$.

Proposition 4.12 *$\chi_k : L \to \mathbb{Z}$ descends to a well-defined function $\bar{\chi}_k : \bar{L} \to \mathbb{Z}$.*

Proof Since $[k]$ is assumed to be a torsion spin^c structure on Y there exists, by Remark 3.7, some $z_k \in L \otimes \mathbb{Q}$ such that $k(x) = (z_k, x)$ for all $x \in L$. Now suppose $x \in L$ and $x' \in \ker(j_*)$. Then

$$\begin{aligned}
\chi_k(x + x') &= -\frac{k(x + x') + (x + x', x + x')}{2} \\
&= \chi_k(x) - \frac{k(x') + 2(x, x') + (x', x')}{2} \\
&= \chi_k(x) - \tfrac{1}{2}(z_k + 2x + x', x') \\
&= \chi_k(x) - \tfrac{1}{2}PD[j_*(x')](z_k + 2x + x') \\
&= \chi_k(x).
\end{aligned}$$

□

To make it easier to state some qualitative properties of $\bar{\chi}_k$, we now consider the extension of $\bar{\chi}_k$ by scalars to the function $\bar{\chi}_k^{\mathbb{R}} : \bar{L} \otimes \mathbb{R} \to \mathbb{R}$. Notice that the negative semidefinite intersection form $(\cdot, \cdot) : L \times L \to \mathbb{Z}$ descends to a negative definite symmetric bilinear pairing on $\bar{L}$ which we denote by $(\cdot, \cdot)_{\bar{L}}$. Extending by scalars, we get a negative definite intersection form $(\cdot, \cdot)_{\bar{L} \otimes \mathbb{R}} : (\bar{L} \otimes \mathbb{R}) \times (\bar{L} \otimes \mathbb{R}) \to \mathbb{R}$. Therefore,

$$\bar{\chi}_k^{\mathbb{R}}(\bar{x}) = -\tfrac{1}{2}\big(k(x) + (\bar{x}, \bar{x})_{\bar{L} \otimes \mathbb{R}}\big) = -\tfrac{1}{2}(\bar{z_k} + \bar{x}, \bar{x})_{\bar{L} \otimes \mathbb{R}}.$$

In particular, we see that $\bar{\chi}_k^{\mathbb{R}}$ is a positive definite quadratic form plus a linear shift. Putting these observations together yields the following proposition.

Proposition 4.13 (1) *$\bar{\chi}_k^{\mathbb{R}}$ is bounded below.*

(2) *Let $\{\bar{x}_1, \dots, \bar{x}_\sigma\}$ be any $\mathbb{R}$-basis of $\bar{L} \otimes \mathbb{R}$. Identify $\bar{L} \otimes \mathbb{R}$ with $\mathbb{R}^\sigma$ via $\bar{L} \otimes \mathbb{R} = \bigoplus_{j=1}^{\sigma} \mathbb{R}\bar{x}_j$. Then the level sets of $\bar{\chi}_k^{\mathbb{R}} : \mathbb{R}^\sigma \to \mathbb{R}$ are $(\sigma-1)$-dimensional ellipsoids and the sublevel sets are σ-dimensional balls bounded by these ellipsoids.*

Corollary 4.14 *$\bar{\chi}_k : \bar{L} \to \mathbb{Z}$ is bounded below and its sublevel sets are finite.*

Definition 4.15 Define $w_q : \mathcal{Q}_q \to \mathbb{Z}$ by

$$w(\square(\bar{l}, I)) = \max\left\{ \bar{\chi}_k(\bar{x}) \,\middle|\, \bar{x} = \bar{l} + \sum_{j \in J} \overline{[v_j]},\, J \subseteq I \right\}.$$

Note, $w : \mathcal{Q}_0 \to \mathbb{Z}$ is simply $\bar{\chi}_k$.

By Corollary 4.14, w is a valid set of weight functions.

Definition 4.16 Define $\mathbb{H}^*(\Gamma, k) = \mathbb{H}(\bar{L}, ([v_1], \dots, [v_s]), w)$.

As in the case with negative definite plumbings, different choices of representatives for $[k]$ yield isomorphic lattice cohomology up to an overall grading shift. More specifically,

Lemma 4.17 [12, Lemma 3.3.2] *If $k' = k + 2PD[j_*(l)]$ for some $l \in L$, then*

$$\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, k')[2\bar{\chi}_k(\bar{l})].$$

Remark 4.18 Némethi uses the opposite convention for grading shifts. Hence, [12, Lemma 3.3.2] is stated as $\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, k')[-2\bar{\chi}_k(\bar{l})]$.

4.4 Graded roots associated to negative semidefinite plumbings

Definition 4.19 [12, Definition 3.2] (1) Let R be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We denote by $[u, v]$ the edge with endpoints u and v. We say that R is a graded root with grading $\chi: \mathcal{V} \to \mathbb{Z}$ if

(a) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in \mathcal{E}$,

(b) $\chi(u) > \min\{\chi(u), \chi(w)\}$ for any $[u, v], [u, w] \in \mathcal{E}$ with $v \neq w$,

(c) χ is bounded below, $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$, and $\#\chi^{-1}(k) = 1$ if k is sufficiently large.

(2) We say that $v \in \mathcal{V}$ is a local minimum point of the graded root (R, χ) if $\chi(v) < \chi(w)$ for any edge $[v, w]$.

(3) If (R, χ) is a graded root, and $r \in \mathbb{Z}$, then we denote by $(R, \chi)[r]$ the same R with the new grading $\chi[r](v) := \chi(v) + r$. (This can be generalized for any $r \in \mathbb{Q}$ as well.)

Example 4.20 Figure 3 shows an example of a graded root.

We now show how to associate a graded root to a pair (Γ, k) where γ is a negative semidefinite plumbing and k is a characteristic vector of $X(\Gamma)$ such that $[k]$ is a torsion spin^c structure on $Y(\Gamma)$. For each $n \in \mathbb{Z}$, let $\bar{L}_{k,\leq n}$ be the graph whose vertex set is $\mathcal{V}(\bar{L}_{k,\leq n}) = \{\bar{x} \in \bar{L} \mid \bar{\chi}_k(\bar{x}) \leq n\}$ and such that there is an edge between two vertices $\bar{x}_1, \bar{x}_2$ if and only if $\bar{x}_1 - \bar{x}_2 = \pm\overline{[v_j]}$ where the v_j are as in Section 3.1. Now let $\pi_0(\bar{L}_{k,\leq n})$ denote the set of connected components of the graph $\bar{L}_{k,\leq n}$.

The graded root $(\bar{R}_k, \bar{\chi}_k)$ associated to Γ and k is constructed as follows:

- The vertex set is $\mathcal{V}(\bar{R}_k) = \bigsqcup_{n\in\mathbb{Z}} \pi_0(\bar{L}_{k,\leq n})$. By an abuse of notation, we denote the grading $\mathcal{V}(\bar{R}_k) \to \mathbb{Z}$ by $\bar{\chi}_k$ where now $\bar{\chi}_k|_{\pi_0(\bar{L}_{k,\leq n})} = n$.
- There is an edge between two vertices $v, v' \in \mathcal{V}(\bar{R}_k)$, which correspond to connected components C_v and $C_{v'}$, if and only if after possibly reordering v and v', we have $\bar{\chi}_k(v') = \bar{\chi}_k(v) + 1$ and $C_v \subset C_{v'}$.

Remark 4.21 When Γ is negative definite, $(\bar{R}_k, \bar{\chi}_k)$ is precisely the graded root, (R_k, χ_k), defined by Némethi [12, Section 4].

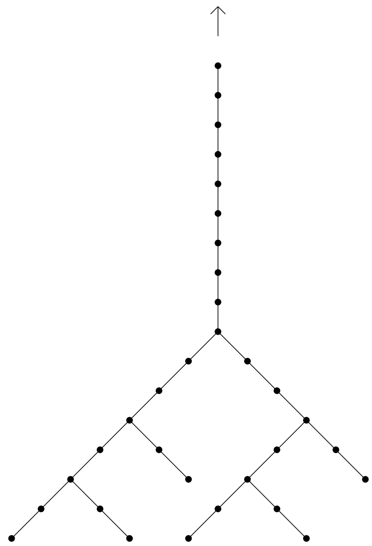


Figure 3

Remark 4.22 The graph $\bar{L}_{k,\leq n}$ is the 1-skeleton of the space S_n considered above in Construction 2 of lattice cohomology. In particular, we can think of $\pi_0(\bar{L}_{k,\leq n})$ equivalently as $\pi_0(S_n)$.

Proposition 4.23 [12, Proposition 4.3] *$(\bar{R}_k, \bar{\chi}_k)$ is a graded root.*

Proof This proof is essentially identical to the proof of [12, Proposition 4.3]. Condition (a) of Definition 4.19(1) follows immediately from the construction of $(R_k, \bar{\chi}_k)$. The proof of condition (b) is the same as in [12, Proposition 4.3]. The first two conditions of (c) follow from Corollary 4.14. The last condition of (c) follows the same argument as Némethi's proof, with mild modification. Essentially just replace the function χ_k in Némethi's proof with $\bar{\chi}_k$ and use that $\bar{\chi}_k$ has a (not necessarily unique) global minimum and that $(\cdot,\cdot)_{\bar{L}}$ is negative definite. □

Again, as in the case with negative definite plumbings, the graded roots, $(\bar{R}_k, \bar{\chi}_k)$ and $(\bar{R}_{k'}, \bar{\chi}_{k'})$ corresponding to two characteristic vectors k and k', which restrict to the same torsion spin^c structure on Y, are equal up to an overall grading shift. More specifically:

Proposition 4.24 [12, Proposition 4.4] *If $k' = k + 2PD[j_*(l)]$ for some $l \in L$ and $k \in \mathrm{Char}(X(\Gamma))$ with $[k]$ torsion, then*

$$(\bar{R}_{k'}, \bar{\chi}_{k'}) = (\bar{R}_k, \bar{\chi}_k)[\bar{\chi}_k(\bar{l})].$$

4.5 The relationship between lattice cohomology, H^+, and graded roots

In Section 4.1, we recalled the definition of the $\mathbb{F}[U]$-module $\mathrm{H}^+(\Gamma, [k])$ introduced by Ozsváth and Szabó where Γ is a negative definite plumbing and $[k]$ is a spin^c structure on $Y(\Gamma)$. The same definition makes sense for negative semidefinite plumbings and $[k]$ torsion except that we adjust the grading as follows: we say $\phi \in \mathrm{H}^+(\Gamma,[k])$ is a homogeneous element of degree d if for each $k' \in [k]$ with $\phi(k') \neq 0$, we have that $\phi(k') \in \mathcal{T}^+$ is a homogeneous element of degree

$$d + \frac{(k')^2 + |\mathcal{V}(\Gamma)| - 3b_1(Y(\Gamma))}{4}.$$

Proposition 4.25 *As graded* $\mathbb{F}[U]$*-modules,*

$$\mathrm{H}^+(\Gamma,[k]) \cong \mathbb{H}^0(\Gamma,k)\left[\frac{k^2 + |V(\Gamma)| - 3b_1(Y)}{4}\right].$$

Proof The isomorphism is induced by the map $Z\colon \mathrm{H}^+(\Gamma,[k]) \to \mathcal{F}^0$ defined by

$$Z(\phi)(\square(\bar{l},\varnothing)) = \phi(k + 2PDj_*(l)).$$

We leave the details to the reader. □

As described in [17; 23], for calculation purposes it is convenient to consider the "dual space" of $\mathrm{H}^+(\Gamma,[k])$, which we denote by $\mathrm{K}^+(\Gamma,[k])$. To recall their definition of $\mathrm{K}^+(\Gamma,[k])$, first consider the set $\mathbb{Z}_{\geq 0} \times [k]$. Write elements $(m,k') \in \mathbb{Z}_{\geq 0} \times [k]$ as $U^m \otimes k'$. Define an equivalence relation $\sim$ on $\mathbb{Z}_{\geq 0} \times [k]$ in the following way: for each $k' \in [k]$ and $v_i \in \mathcal{V}(\Gamma)$, let $2n_i = k'([v_i]) + ([v_i],[v_i])$. Then

1. if $n_i \geq 0$, we require $U^{n_i+m} \otimes (k' + 2PDj_*[v_i]) \sim U^m \otimes k'$;
2. if $n_i < 0$, we require $U^m \otimes (k' + 2PDj_*[v_i]) \sim U^{m-n_i} \otimes k'$.

In other words, two elements $U^m \otimes k'$ and $U^n \otimes k''$ are equivalent if and only if there exists a finite sequence of elements $U^{m_0} \otimes k_1, \dots, U^{m_\ell} \otimes k_\ell$ such that $U^{m_0} \otimes k_1 = U^m \otimes k'$, $U^{m_\ell} \otimes k_\ell = U^n \otimes k''$ and each adjacent pair in the sequence is related by a relation of type (1) or (2) as given above. We call such a sequence a *path* connecting $U^m \otimes k'$ and $U^n \otimes k''$.

Remark 4.26 In general, there are many different paths connecting a given pair of elements $U^m \otimes k'$ and $U^n \otimes k''$.

Write the equivalence class containing $U^m \otimes k'$ as $\underline{U^m \otimes k'}$ and define $\mathrm{K}^+(\Gamma,[k])$ to be the set of these equivalence classes. $\mathrm{K}^+(\Gamma,[k])$ is the dual of $\mathrm{H}^+(\Gamma,[k])$ (or maybe more naturally $\mathrm{H}^+(\Gamma,[k])$ is the dual of $\mathrm{K}^+(\Gamma,[k])$) in the following sense:

- Define $\mathrm{K}^+(\Gamma,[k])^*$ to be the set of finitely supported functions $\phi\colon \mathrm{K}^+(\Gamma,[k]) \to \mathcal{T}^+$ such that $\phi(\underline{U^{n+m} \otimes k'}) = U^n \phi(\underline{U^m \otimes k'})$ for all $n, m \geq 0$ and $k' \in [k]$. Endow $(\mathrm{K}^+)^*$ with an $\mathbb{F}[U]$-module structure by inheriting that of $\mathcal{T}^+$.
- Define a map $F\colon \mathrm{H}^+(\Gamma,[k]) \to \mathrm{K}^+(\Gamma,[k])^*$ by

$$F(\phi)(\underline{U^m \otimes k'}) = U^m \phi(k').$$

It is straightforward to check that F is a well-defined $\mathbb{F}[U]$-module isomorphism.

We can put more structure on $\mathrm{K}^+(\Gamma, [k])$ by thinking of it as a graph. Specifically, define $g\mathrm{K}^+(\Gamma, [k])$ to be the graph whose vertices are the elements of $\mathrm{K}^+(\Gamma, [k])$ and such that there is an edge between to vertices $\underline{U^m \otimes k'}$ and $\underline{U^n \otimes k''}$ if and only if either $\underline{U^{m+1} \otimes k'} = \underline{U^n \otimes k''}$ or $\underline{U^m \otimes k'} = \underline{U^{n+1} \otimes k''}$.

Proposition 4.27 *As graphs, $g\mathrm{K}^+(\Gamma, [k])$ is isomorphic to the graded root $(\bar{R}_k, \bar{\chi}_k)$.*

Proof This proof is essentially the same as Némethi [12, Proof of Proposition 4.7]. For completeness, we provide the details here.

By definition each element $k' \in [k]$ can be written as $k' = k + 2PD[j_*(l)]$ for some $l \in L$. Let $\bar{l}_{k'} := \bar{l} \in \bar{L}$. Define a map $p\colon \mathrm{K}^+(\Gamma, [k]) \to \mathscr{V}(\bar{R}_k)$ as follows:

$$p(\underline{U^m \otimes k'}) = \text{the connected component of } \bar{L}_{k, \leq \bar{\chi}_k(\bar{l}_{k'})+m} \text{ containing } \bar{l}_{k'}.$$

To show that p is well defined, let $2n_i = k'([v_i]) + ([v_i], [v_i])$. Suppose first that $n_i \geq 0$ so that we have $U^{n_i+m} \otimes (k' + 2PDj_*[v_i]) \sim U^m \otimes k'$. Let $k'' = k' + 2PDj_*[v_i]$. Then, $\bar{l}_{k''} = \bar{l}_{k'} + \overline{[v_i]}$. Thus,

$$\begin{aligned}
\bar{\chi}_k(\bar{l}_{k''}) + n_i + m &= \bar{\chi}_k(\bar{l}_{k'}) + \bar{\chi}_{k'}(\overline{[v_i]}) + n_i + m \\
&= \bar{\chi}_k(\bar{l}_{k'}) - n_i + n_i + m \\
&= \bar{\chi}_k(\bar{l}_{k'}) + m.
\end{aligned}$$

Therefore, $\bar{L}_{k, \leq \bar{\chi}_k(\bar{l}_{k'})+m} = \bar{L}_{k, \leq \bar{\chi}_k(\bar{l}_{k''})+n_i+m}$ and $\bar{l}_{k'}$ and $\bar{l}_{k''}$ are in the same connected component since they differ by $\overline{[v_i]}$. The case when $n_i < 0$ is similar. This establishes that p is well defined.

Next we define a map $q\colon \mathscr{V}(\bar{R}_k) \to \mathrm{K}^+(\Gamma, [k])$ which we will show is the inverse of p. Suppose $v \in \mathscr{V}(\bar{R}_k)$. Let C_v be the corresponding connected component in $\bar{L}_{k, \leq \bar{\chi}_k(v)}$ and let $\bar{l}_v$ be some element in $\bar{L} \cap C_v$. Define

$$q(v) = \underline{U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}_v)} \otimes (k + 2PDj_*(l_v))}.$$

To show q is well defined, suppose $\bar{l}'$ is some other element in $\bar{L} \cap C_v$. It suffices to consider the case that $\bar{l}' = \bar{l}_v + \overline{[v_i]}$ for some i. First note,

$$\bar{\chi}_k(\bar{l}') = \bar{\chi}_k(\bar{l}_v + \overline{[v_i]}) = \bar{\chi}_k(\bar{l}_v) + \bar{\chi}_k(\overline{[v_i]}) - ([v_i], l_v).$$

Also,

$$(k + 2PDj_*(l_v))(v_i) + (v_i, v_i) = k(v_i) + (v_i, v_i) + 2(v_i, l_v) = -2[\bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)].$$

Hence, if $-[\bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)] \geq 0$, then

$$\begin{aligned}
U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}_v)} \otimes (k + 2PDj_*(l_v)) &\sim U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}_v) - \bar{\chi}_k(\overline{[v_i]}) + (v_i, l_v)} \otimes (k + 2PDj_*(l_v + [v_i])) \\
&= U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}')} \otimes (k + 2PDj_*(l')).
\end{aligned}$$

Similarly, if $-[\bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)] < 0$, then

$$\begin{aligned}
U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}')} \otimes (k + 2PDj_*(l')) &\sim U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}') + \bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)} \otimes (k + 2PDj_*(l_v)) \\
&= U^{\bar{\chi}_k(v) - \bar{\chi}_k(\bar{l}_v)} \otimes (k + 2PDj_*(l_v)).
\end{aligned}$$

Therefore, q is well defined.

Now consider $qp(\underline{U^m \otimes k'})$ where $k' = k + 2PDj_*(\bar{l}_k)$. Let $v = p(\underline{U^m \otimes k'})$ and C_v be the connected component of $\bar{L}_{k,\leq \bar{\chi}_k(\bar{l}_{k'})+m}$ containing $\bar{l}_{k'}$. Then, by definition,

$$q(v) = \underline{U^{\bar{\chi}_k(\bar{l}_{k'})+m-\bar{\chi}_k(\bar{l}_{k'})} \otimes k + 2PDj_*(\bar{l}_{k'})} = \underline{U^m \otimes k'}.$$

Hence, $qp = \mathrm{Id}$. The other direction, ie that $pq = \mathrm{Id}$, is tautological. Therefore, p is a bijection. To see that p takes edges to edges bijectively, let $v_1 = p(\underline{U^m \otimes k'})$ and $v_2 = p(\underline{U^{m+1} \otimes k'})$. It follows directly from the definition that $C_{v_1} \subset C_{v_2}$ and $\bar{\chi}_k(v_2) - \bar{\chi}_k(v_1) = 1$. □

Remark 4.28 It is useful to point out that under the isomorphism p constructed in the above proof, we have that

$$\mathrm{gr}(p(\underline{U^m \otimes k'})) = m - \tfrac{1}{8}((k')^2 - k^2).$$

4.6 A quick review of Rustamov's results on negative semidefinite plumbings with $b_1 = 1$

Rustamov [23] generalizes the setting in which the isomorphism T^+, described in Section 4.1, holds. In particular, Rustamov proves the following theorem:

Theorem 4.29 [23, Theorem 1.2] *Let Γ be a negative semidefinite plumbing with at most one bad vertex and with $b_1(Y(\Gamma)) = 1$. Further, let $[k]$ be a torsion* spin^c *structure. Then*

(1) $T^+ \colon HF^+_{\mathrm{odd}}(-Y(\Gamma), [k]) \to \mathbb{H}^+(\Gamma, [k])$ *is an isomorphism of graded $\mathbb{F}[U]$-modules;*
(2) $HF^+_{\mathrm{even}}(-Y(\Gamma), [k]) \cong \mathcal{T}^+_d$ *where $d = d_{-1/2}(-Y(\Gamma), [k])$.*

Here $HF^+_{\mathrm{odd}}(-Y(\Gamma), [k])$ and $HF^+_{\mathrm{even}}(-Y(\Gamma), [k])$ refer to the submodules generated by elements of $HF^+(-Y(\Gamma), [k])$ of degrees congruent to $1/2 \bmod 2$ and $-1/2 \bmod 2$ respectively.

Combining Rustamov's result with the observations of the previous section, we get:

Corollary 4.30 *With Γ as above,*

$$HF^+_{\mathrm{odd}}(-Y(\Gamma), [k]) \cong \mathbb{H}^0(\Gamma, k)\left[\frac{k^2 + |V(\Gamma)| - 3}{4}\right]$$

as graded $\mathbb{F}[U]$-modules. In particular, up to an overall grading shift, $\mathbb{H}^0(\Gamma, k)$ is a topological invariant of $Y(\Gamma)$.

Remark 4.31 It is likely possible that one can prove

$$\mathbb{H}^0(\Gamma, k)\left[\frac{k^2 + |V(\Gamma)| - 3}{4}\right]$$

is a topological invariant without appealing to Heegaard Floer homology, by showing invariance under Neumann moves as in the proof of [12, Proposition 4.6].

4.7 Involutions on lattice cohomology and Heegaard Floer homology

Let Γ be a negative semidefinite plumbing with at most one bad vertex and $b_1(Y(\Gamma)) = 1$. Let $[k]$ be a self-conjugate spin^c structure on $Y(\Gamma)$. In other words, $[k] = [-k]$ or, equivalently, $k = PD[j_*(l)]$ for some $l \in L$. Note that by identifying $\bar{l}$ with k, we can think of k as an element of $\bar{L}$.

As in [2, Section 2], define $J_0 \colon \bar{L} \to \bar{L}$ by $J_0(\bar{x}) = -\bar{x} - \bar{l}$. Clearly, $J_0^2 = \mathrm{Id}$. We can extend J_0 to a cubical involution on the cube complex $\mathscr{C}$ considered in Construction 2 of lattice cohomology via the formula

$$J_0\square(\bar{a}, I) = \square\Bigl(J_0\Bigl(\bar{a} + \sum_{i \in I} \overline{[v_i]}\Bigr), I\Bigr).$$

It is straightforward to check that J_0 is compatible with the gluing of the cells. Moreover, since $\bar{\chi}_k(J_0(\bar{x})) = \bar{\chi}_k(\bar{x})$ for all $\bar{x} \in \bar{L}$, J_0 maps the subcomplex S_n of $\mathscr{C}$ to itself. Therefore, J_0 induces an involution on $H^q(S_n; \mathbb{Z})$ for each n and q, and hence on lattice cohomology. By an abuse of notation, we denote the involution on lattice cohomology again by J_0. In a similar manner, one could alternatively define J_0 by using Construction 1, but we leave the details to the reader.

Focusing our attention on the 0^{th} level of lattice cohomology, we can think of the action of J_0 on $\mathbb{H}^0$ from the dual perspective by realizing an involution on the associated graded root. More specifically, since J_0 acts continuously on S_n, J_0 also induces an involution on the connected components of S_n. Hence, J_0 induces an involution on the graded root $(\bar{R}_k, \bar{\chi}_k)$. From another perspective, under the identification of $(\bar{R}_k, \bar{\chi}_k)$ with $g\mathrm{K}^+(\Gamma, [k])$ given in Proposition 4.27, the involution J_0 sends $\underline{U^m \otimes k'}$ to $\underline{U^m \otimes -k'}$.

Dai and Manolescu [2, Theorem 3.1] showed that for negative definite almost rational plumbings, the involution J_0 on lattice cohomology is identified with the involution ι_* on Heegaard Floer homology under the isomorphism T^+ described in Section 4.1. We now show that their theorem also holds in the setting of negative semidefinite plumbed manifolds with at most one bad vertex and $b_1 = 1$.

Theorem 4.32 [2, Theorem 3.1] *Let Γ be a negative semidefinite plumbing with at most one bad vertex and $b_1(Y(\Gamma)) = 1$. If $[k]$ is a self-conjugate* spin^c *structure, then under the isomorphism T^+ given in Theorem 4.29*(1), *the maps J_0 and the restriction of ι_* to $HF^+_{\text{odd}}(-Y(\Gamma), [k])$ are identified.*

Proof First note that the isomorphism $T^+ \colon HF^+_{\text{odd}}(-Y(\Gamma), [k]) \to \mathrm{H}^+(\Gamma, [k])$ for negative semidefinite plumbed manifolds with at most one bad vertex and $b_1 = 1$ is defined in precisely the same way as the isomorphism $T^+ \colon HF^+(-Y(\Gamma), [k]) \to \mathrm{H}^+(\Gamma, [k])$ for negative definite almost rational plumbed manifolds. Therefore, as in the proof of [2, Theorem 3.1], to show that J_0 and ι_* are identified under T^+, one must show that $F^+_{W,k} = F^+_{W,-k} \circ \iota_*$. As noted in the proof of [2, Theorem 3.1], this equation follows from [20, Theorem 3.6]. $\square$

For negative semidefinite plumbed manifolds with at most one bad vertex and $b_1 = 1$, the action of ι_* on the even part of HF^+ is less interesting. Since ι_* is U-equivariant and $HF^+_{\text{even}}(-Y(\Gamma), [k]) \cong \mathscr{T}_d$ for $[k]$

self-conjugate, the restriction of ι_* to the even part must be the identity. Moreover, if one knows HF^+ and ι_*, then by using the mapping cone exact triangle in Proposition 2.2, one can completely determine HFI^+ as a graded $\mathbb{F}$-vector space.

In the context of negative definite almost rational plumbings, Dai and Manolescu [2, Sections 4 and 5]. showed that one can actually determine the entire $\mathbb{F}[U,Q]/(Q^2)$-module structure of HFI^+ just from knowing J_0. However, one encounters issues when trying to extrapolate their methods to the case of negative semidefinite plumbings with at most one bad vertex. The main difficulty is that in the negative definite almost rational case, HF^+ is supported in even gradings, whereas in the negative semidefinite case, HF^+ has both even and odd gradings which allows for the possibility of a more complicated action of ι at the chain level. Despite this issue, for negative semidefinite plumbings with at most one bad vertex whose HF^+ and ι_* are sufficiently simple, it is still possible to compute much, if not all, of the $\mathbb{F}[U,Q]/(Q^2)$-module structure of HFI^+ as well as the involutive d invariants just from the mapping cone exact triangle. We illustrate this via the examples in Section 5.

5 Small Seifert fibered space examples

In this section, we compute $HFI^+(-N_j,\mathfrak{s}_0)$ for the infinite family of small Seifert fiber spaces $\{N_j\}_{j\in\mathbb{N}}$ described in the introduction. As an application, we prove Theorem E. We also compute $HFI^+(-S^3_0(K_1),\mathfrak{s}_0)$ where $S^3_0(K_1)$ is the manifold obtained by 0-surgery on the Ichihara–Motegi–Song knot K_1 from [8]. We then compare $HFI^+(-S^3_0(K_1),\mathfrak{s}_0)$ and $HFI^+(-N_1,\mathfrak{s}_0)$.

Before computing HFI^+ of these specific manifolds, we give a brief outline in Section 5.1 of a general strategy for computing the graded root $(\bar{R}_k,\bar{\chi}_k)$, which forms a key part of our computation of HFI^+. We also, in Section 5.2, describe some combinatorial moves that will aid in the computations.

5.1 Computing the graded root

Let Γ be a negative semidefinite plumbing with at most one bad vertex and $b_1(Y(\Gamma))=1$. Let $[k]$ be a self-conjugate spin^c structure on $Y(\Gamma)$. To compute $(\bar{R}_k,\bar{\chi}_k)$, the first and main step is to determine the set

$$\mathscr{L}(\Gamma,[k]) := \{x\in \mathrm{K}^+(\Gamma,[k]) \mid x \text{ has no representative of the form } U^n\otimes k' \text{ for } n>0\}.$$

It is easy to see that the elements of $\mathscr{L}(\Gamma,[k])$ correspond to the leaves of the graded root $(\bar{R}_k,\bar{\chi}_k)$ under the isomorphism in Proposition 4.27. Moreover, from the results in Section 4.6, it follows that the leaves of $(\bar{R}_k,\bar{\chi}_k)$ correspond to a basis of the $\mathbb{F}$-vector space

$$\ker(U)\cap HF^+_{\mathrm{odd}}(-Y(\Gamma),[k]).$$

Rustamov [23, Section 3] provides an algorithm to compute $\mathscr{L}(\Gamma,[k])$ which builds on the Ozsváth and Szabó [17, Section 3] algorithm for negative definite plumbings. For our computations, rather than use

Rustamov's algorithm directly, we instead will use a simple criterion (see Proposition 5.1 below) which characterizes the elements of $\mathscr{L}(\Gamma, [k])$.

To explain this criterion, first recall from Section 4.5 that two elements $U^m \otimes k'$ and $U^n \otimes k''$ are equivalent (ie represent the same element of $\mathrm{K}^+(\Gamma, [k])$) if and only if there is a path between them. In particular, every element of $\mathscr{L}(\Gamma, [k])$ is represented by an element of the form $U^0 \otimes k'$ and every element of a path connecting $U^0 \otimes k'$ to another representative must also have 0 as the exponent on the U term. Therefore, when discussing representatives or paths for elements in $\mathscr{L}(\Gamma, [k])$, we can drop the U^0 term and instead think of a representative as an element $k' \in [k]$ and a path as a sequence of vectors $k_1, \dots, k_j \in [k]$. Furthermore, the relations defining such a path imply that for adjacent elements k_i and k_{i+1} we have that $k_{i+1} = k_i \pm 2PD[v]$ for some $v \in \mathscr{V}(\Gamma)$ with $k_i(v) = \mp m(v)$. Additionally, it follows from the definition that a representative k' of an element in $\mathscr{L}(\Gamma, [k])$ must satisfy

$$m(v) \le k'(v) \le -m(v)$$

for all $v \in \mathscr{V}(\Gamma)$. We refer to this property as $\star$ and we let $\star[k] = \{k' \in [k] \mid k' \text{ satisfies } \star\}$.

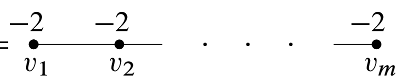
Combining these observations, we get the following proposition:

Proposition 5.1 *An element $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma, [k])$ if and only if k' satisfies $\star$ and every element on every path containing k' also satisfies $\star$.*

After using Proposition 5.1 to find elements $k_1, \dots, k_n \in [k]$ which represent the distinct elements of $\mathscr{L}(\Gamma, [k])$, it then follows that every other vertex of $(\bar{R}_k, \bar{\chi}_k)$ corresponds to an element of the form $\underline{U^m \otimes k_i}$ for some m and i. Of course, there could be relations of the form $\underline{U^{m_1} \otimes k_i} = \underline{U^{m_2} \otimes k_j}$. To determine these relations, in principle, one can write down the elements of the equivalence classes $\underline{U^{m_1} \otimes k_i}$ and $\underline{U^{m_2} \otimes k_j}$ and see whether they are equal. However, this can be quite tedious to do by hand and, in simple enough situations, there are shortcuts one can take by leveraging properties of HF^+. For example, we will use the relationship between Turaev torsion and HF^+ established in [18, Theorem 10.17] to complete the computation of $(\bar{R}_k, \bar{\chi}_k)$ for the manifolds N_j.

5.2 Moves between equivalent vectors

Let Γ be a negative semidefinite plumbing with at most one bad vertex and with $b_1 = 1$. Suppose Γ contains a linear subgraph Λ with framing -2 at each vertex, as shown below:

$\Lambda =$ 

Let $[k]$ be a self-conjugate spin^c structure on $Y(\Gamma)$. Given a characteristic vector $k' \in [k]$, let

$$k'_\Lambda = (a_1, \dots, a_m)$$

be the subvector corresponding to the vertices $v_1, \dots, v_m$. We call k'_Λ the Λ-subvector of k'.

Note, if $k' \in [k]$ and satisfies $\star$, then we must have $a_i \in \{-2, 0, 2\}$ for each $1 \leq i \leq m$. If there exists some i such that $a_i = \pm 2$, then $k'' = k' \pm 2PD[v_i]$ is an equivalent vector. In particular,

$$k''_\Lambda = (a_1, \ldots, a_{i-1} \pm 2, \mp 2, a_{i+1} \pm 2, \ldots, a_m).$$

Of course, other entries of k'' not contained in k''_Λ may also differ from those of k'. Specifically, any entry a of k' corresponding to a vertex adjacent to v_i will change from a to $a \pm 2$. We call the replacement of k' with $k'' = k' \pm 2PD[v_i]$ where $k'(v_i) = \pm 2$ a move of *type* ± 2.

Next suppose $k'_\Lambda = (a_1, \ldots, a_i, 0, \ldots, 0, 2, -2, a_j, \ldots, a_m)$. Then, by iteratively applying type $+2$ moves to the $+2$-entry, we can convert k' into an equivalent vector k'' with

$$k''_\Lambda = (a_1, \ldots, a_i, 2, -2, 0, \ldots, 0, a_j, \ldots, a_m).$$

We call the replacement of k' with k'' or k'' with k' a $(2, -2)$-*slide*. We define a $(-2, 2)$-*slide* analogously.

Lemma 5.2 *Let $k' \in [k]$ be a vector with $k'_\Lambda = (a_1, \ldots, a_i, 0, \pm 2, 0, \ldots, 0, \mp 2, a_j, \ldots, a_m)$. Then k' is equivalent to a vector k'' with $k''_\Lambda = (a_1, \ldots, a_i, \pm 2, 0, \ldots, 0, \mp 2, 0, a_j, \ldots, a_m)$.*

Proof Apply a type ± 2 move to the ± 2-entry to get an equivalent vector h' with

$$h'_\Lambda = (a_1, \ldots, a_i, \pm 2, \mp 2, \pm 2, 0, \ldots, 0, \mp 2, a_j, \ldots, a_m).$$

Now do a rightward $(\mp 2, \pm 2)$-slide to h' to convert h' into an equivalent vector h'' with

$$h''_\Lambda = (a_1, \ldots, a_i, \pm 2, 0, \ldots, 0, \mp 2, \pm 2, \mp 2, a_j, \ldots, a_m).$$

Finally apply a type ± 2 move to the rightmost ± 2-entry to get an equivalent vector k'' with

$$k''_\Lambda = (a_1, \ldots, a_i, \pm 2, 0, \ldots, 0, \mp 2, 0, a_j, \ldots, a_m). \qquad \square$$

By iterating the sequence of moves described in the above proof, we can now convert any vector $k' \in [k]$ with

$$k'_\Lambda = (a_1, \ldots, a_i, 0, \ldots, 0, \pm 2, 0, \ldots, 0, \mp 2, a_j, \ldots, a_m)$$

into an equivalent vector k'' with

$$k''_\Lambda = (a_1, \ldots, a_i, \pm 2, 0, \ldots, 0, \mp 2, 0, \ldots, 0, a_j, \ldots, a_m).$$

By an abuse of notation, we also call the replacement of k' with k'' or k'' with k' via the above sequence of moves a $(\pm 2, \mp 2)$-*slide*.

Lemma 5.3 *Suppose $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma, [k])$. Then either k'_Λ is the zero vector or it has entries which alternate between 2 and -2 with possibly 0's in between.*

Proof Suppose k' represents an element of $\mathscr{L}(\Gamma, [k])$ and k'_Λ contains a subvector of the form

$$(2, \underbrace{0, \ldots, 0}_{j}, 2)$$

where $j \geq 0$. Then, by doing a type $+2$ move on the leftmost $+2$-entry, k' is equivalent to a vector whose corresponding subvector is

$$(-2, 2, \underbrace{0, \dots, 0}_{j-1}, 2)$$

if $j \geq 1$ or $(-2, 4)$ if $j = 0$. In the latter case, the vector fails to satisfy $\star$ and thus we get a contradiction by Proposition 5.1. So we can assume the subvector is

$$(-2, 2, \underbrace{0, \dots, 0}_{j-1}, 2)$$

with $j \geq 1$. Now do a rightward $(-2, 2)$-slide to produce an equivalent vector whose corresponding subvector is

$$(\underbrace{0, \dots, 0}_{j-1}, -2, 2, 2).$$

Next apply a type $+2$ move to get an equivalent vector whose corresponding subvector is

$$(\underbrace{0, \dots, 0}_{j}, -2, 4).$$

We again get a contradiction for the same reason as before. Therefore, k'_Λ cannot contain a subvector of the form

$$(2, \underbrace{0, \dots, 0}_{j}, 2), \qquad j \geq 0.$$

By an analogous argument, k'_Λ also cannot contain a subvector of the form

$$(-2, \underbrace{0, \dots, 0}_{j}, -2), \qquad j \geq 0. \qquad \square$$

Lemma 5.4 *Suppose $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma, [k])$. Then k' is equivalent to a vector k'' such that k''_Λ is the zero vector except for possibly one nonzero entry equal to ± 2.*

Proof We induct on the number of nonzero entries of k'_Λ. Obviously the statement is true if k'_Λ is the zero vector or has only one nonzero entry. So suppose k'_Λ has $n \geq 2$ nonzero entries. Let a_i and a_{i+j} be the leftmost nonzero entries. Then by the Lemma 5.3, $a_i = \pm 2$ and $a_{i+j} = \mp 2$. For simplicity, assume $a_i = 2$. (The argument when $a_i = -2$ is identical up to sign changes.) We can write k'_Λ as

$$k'_\Lambda = (0, \dots, 0, 2, 0, \dots, 0, -2, a_{i+j+1}, \dots, a_m)$$

where there are possibly no initial 0 entries and no 0 entries between a_i and a_{i+j}. If there are initial 0 entries, then by doing a leftward $(2, -2)$-slide, k' is equivalent to a vector whose Λ-subvector is

$$(2, 0, \dots, 0, -2, 0, \dots, 0, a_{i+j+1}, \dots, a_m).$$

Now apply a type $+2$ move to the leftmost $+2$-entry to get an equivalent vector whose Λ-subvector is

$$(-2, 2, 0, \dots, 0, -2, 0, \dots, 0, a_{i+j+1}, \dots, a_m)$$

if $j > 1$, or

$$(-2, 0, \dots, 0, a_{i+2}, \dots, a_m)$$

if $j = 1$. In the latter case, we have reduced the number of nonzero entries in the Λ-subvector by 1. Hence, we can assume $j > 1$. In this case, if we do a rightward $(-2, 2)$-slide on leftmost $(-2, 2)$-pair, we get an equivalent vector whose Λ-subvector is

$$(0, \dots, 0, -2, 2, -2, 0, \dots, 0, a_{i+j+1}, \dots, a_m).$$

Finally apply a type $+2$ move to produce an equivalent vector whose Λ-subvector is

$$(0, \dots, 0, 0, -2, 0, 0, \dots, 0, a_{i+j+1}, \dots, a_m).$$

We have reduced the number of nonzero entries by 1. Therefore, by induction the result follows. □

Lemma 5.5 *Suppose $k' \in [k]$ with*

$$k'_\Lambda = (\underbrace{0, \dots, 0}_{j}, 2, \underbrace{0, \dots, 0}_{m-j-1}).$$

Then k' is equivalent to a vector k'' with

$$k''_\Lambda = (\underbrace{0, \dots, 0}_{m-j-1}, -2, \underbrace{0, \dots, 0}_{j}).$$

Proof We list the sequence of moves needed to obtain the relevant vector. In each move, we only write the resulting Λ-subvector.

(1) Type $+2$ move:

$$(\underbrace{0, \dots, 0}_{j-1}, 2, -2, 2\, \underbrace{0, \dots, 0}_{m-j-2}).$$

(2) Leftward $(2, -2)$-slide:

$$(2, -2, \underbrace{0, \dots, 0}_{j-1}, 2, \underbrace{0, \dots, 0}_{m-j-2}).$$

(3) Type $+2$ move:

$$(-2, \underbrace{0, \dots, 0}_{j}, 2, \underbrace{0, \dots, 0}_{m-j-2}).$$

(4) Rightward $(-2, 2)$-slide:

$$(\underbrace{0, \dots, 0}_{m-j-2}, -2, \underbrace{0, \dots, 0}_{j}, 2).$$

(5) Type $+2$ move:

$$(\underbrace{0, \dots, 0}_{m-j-2}, -2, \underbrace{0, \dots, 0}_{j-1}, 2, -2).$$

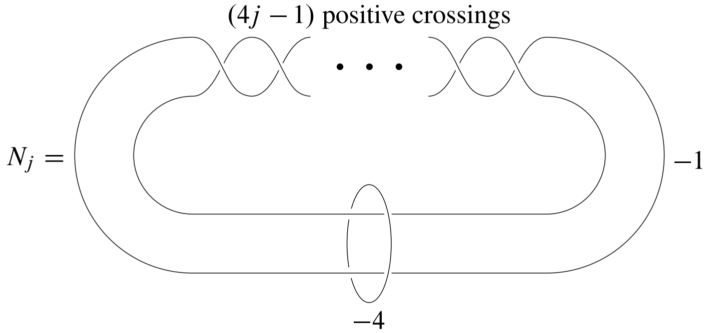


Figure 4

(6) Leftward $(2,-2)$-slide:

$$(\underbrace{0,\dots,0}_{m-j-2},-2,2,-2,\underbrace{0,\dots,0}_{j-1}).$$

(7) Type $+2$ move:

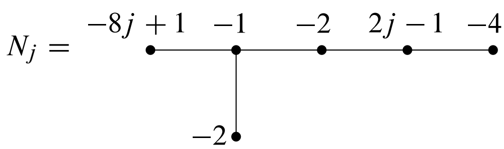
$$(\underbrace{0,\dots,0}_{m-j-1},-2,\underbrace{0,\dots,0}_{j}).$$ □

Remark 5.6 If one traces through the above sequence of moves, it is easy to see that if v is a vertex not in Λ, but is adjacent to the initial vertex v_1 or terminal vertex v_m of Λ, then $k''(v) = k'(v) + 2$.

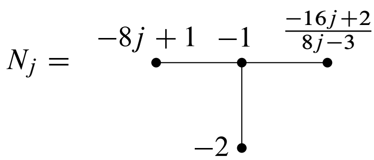
5.3 Computation of $HFI^+(-N_j, \mathfrak{s}_0)$

Recall, the 3-manifold N_j for $j \geq 1$ is given by the surgery diagram in Figure 4.

In [4, Section 7], it is shown via Kirby calculus that N_j can be represented as a plumbing as follows:



By performing two slam dunks on the rightward stem, we get:



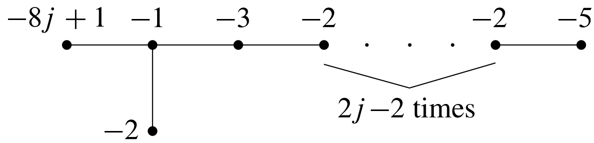
One can further check that

$$\frac{-16j+2}{8j-3} = -3 - \cfrac{1}{-2 - \cfrac{1}{\ddots -2 - \cfrac{1}{\frac{-8j+3+4r}{8j-7-4r}}}}$$

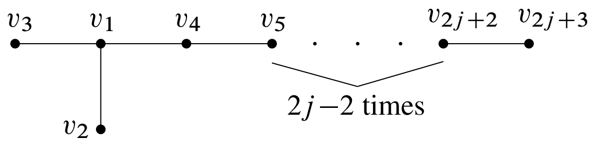
where there are r copies of -2 along the diagonal. In particular, setting $r = 2j-2$, the last term becomes

$$\frac{-8j+3+4(2j-2)}{8j-7-4(2j-2)} = -5.$$

Hence, by performing the corresponding slam dunks, we get:

$N_j =$ 

Let Γ_j be the above plumbing graph with vertices labeled as follows:



With respect to the ordered basis $([v_1], \dots, [v_{2j+3}])$, the matrix for the intersection form of $X(\Gamma_j)$ is

$$B_j = \begin{pmatrix} -1 & 1 & 1 & 1 & & & & & \\ 1 & -2 & & & & & & & \\ 1 & & -8j+1 & & & & & & \\ 1 & & & -3 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -5 \end{pmatrix}.$$

It is straightforward to check that B_j is negative semidefinite and $H_1(N_j;\mathbb{Z}) \cong \mathbb{Z}$, we leave this to the reader.

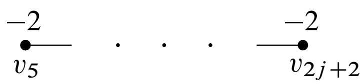
Note, the $\mathbb{Z}$-kernel of B_j is generated by the vector

$$x = (16j-2, 8j-1, 2, 8j-3, 8j-7, 8j-11, \dots, 1).$$

Therefore, the unique self-conjugate spin^c-structure $\mathfrak{s}_0$ on N_j can be thought of as

$$[k] = \{k' \in \operatorname{Char}(X(\Gamma_j)) \mid k' \cdot x = 0\}.$$

Let Λ_j be the linear subgraph of Γ_j given by:

$\Lambda_j =$ 

We write vectors $k' \in [k]$ as

$$k' = (a_1, a_2, a_3, a_4, b_5, \dots, b_{2j+2}, c_{2j+3})$$

where $k'_{\Lambda_j} = (b_5, \dots, b_{2j+2})$.

Lemma 5.7 *If $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma_j, [k])$, then k' is equivalent to a vector whose Λ_j-subvector is not equal to the zero vector.*

Proof Suppose $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma_j, [k])$. For the purpose of contradiction, suppose the Λ_j-subvector of every representative of every element of $\mathscr{L}(\Gamma_j, [k])$ is zero. Then, in particular, $k'_{\Lambda_j} = 0$. Also, since k' represents an element of $\mathscr{L}(\Gamma_j, [k])$, it must satisfy $\star$. So we must have $a_4 \in \{-3, -1, 1, 3\}$. If $a_4 = \pm 3$, then by adding $\pm 2PD[v_4]$ to k' we would obtain an equivalent vector with a nonzero Λ_j-subvector. Thus, $a_4 \in \{-1, 1\}$.

Since k' must satisfy $\star$, we also have $a_1 = \pm 1$. If $a_1 = 1$ and $a_4 = 1$, then by adding $2PD[v_1]$ to k', a_4 becomes 3. But we just showed that a_4 cannot be equal to 3. Similarly, if $a_1 = -1$ and $a_4 = -1$, then by adding $-2PD[v_1]$ to k', a_4 becomes -3, which is again a contradiction. Hence, $a_1 = \pm 1$ and $a_4 = \mp 1$. By adding $-2PD[v_1]$ if necessary, we may assume $a_1 = 1$ and $a_4 = -1$. Again, by $\star$, we must have $a_2 \in \{-2, 0, 2\}$. If $a_2 = 2$, then by adding $2PD[v_1]$ to k', we get an equivalent vector with $a_2 = 4$, which contradicts Proposition 5.1. Therefore, $a_2 \in \{0, -2\}$. If $a_2 = -2$, then by adding $-2PD[v_2]$ to k' we obtain an equivalent vector with $a_1 = -1$ and $a_4 = -1$, which we already determined cannot happen. Therefore, $a_2 = 0$. Now add $2PD[v_1]$ to k'. The result is an equivalent vector with $a_1 = -1$, $a_2 = 2$, and $a_4 = 1$. Since $a_2 = 2$, we can add $2PD[v_2]$ to get an equivalent vector with $a_1 = 1, a_2 = -2$, and $a_4 = 1$, but we have already shown that we cannot have both $a_1 = 1$ and $a_4 = 1$. Therefore, we get a contradiction and hence k' must be equivalent to some vector whose Λ_j-subvector is not equal to the zero vector. □

Somewhat counterintuitively, we are now going to use the previous lemma to find a small finite set of possible representatives of $\mathscr{L}(\Gamma_j, [k])$, all of whose Λ_j-subvectors are all equal to the zero vector.

Lemma 5.8 *If $k' \in [k]$ represents an element of $\mathscr{L}(\Gamma_j, [k])$, then k' is equivalent to a vector of the form*

$$k'' = (-1, 0, a_3, 3, 0, \dots, 0, c_{2k+3})$$

where $a_3 \in \{-8k+1, -8k+3, \dots, 8k+1\}$ and $c_{2j+3} \in \{-5, -3, -1, 1, 3\}$.

Proof Suppose k' represents an element of $\mathscr{L}(\Gamma_j, [k])$. Then, by combining Lemmas 5.4, 5.5, and 5.7, we may assume

$$k'_{\Lambda_j} = (\underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell})$$

for some $0 \le \ell \le 2j-3$. By $\star$, $a_1 = \pm 1$. If $a_1 = -1$, then we can add $-2PD[v_1]$ from k' to get an equivalent vector with $a_1 = 1$. This addition does not effect any of the entries in k'_{Λ_j}. Thus, we may assume $a_1 = 1$.

Next, by $\star$, $a_2 \in \{-2, 0, 2\}$. If $a_2 = 2$, then adding $2PD[v_1]$ to k' yields an equivalent vector with $a_2 = 4$, which violates $\star$. Therefore, $a_2 \in \{-2, 0\}$. Suppose $a_2 = -2$. Then, by adding $-2PD[v_2]$, we get an equivalent vector with $a_2 = 2$ and $a_1 = -1$. k'_{Λ_j} is unaffected by this move. If we then add $-2PD[v_1]$, we get an equivalent vector with $a_1 = 1$ and $a_2 = 0$. Again k'_{Λ_j} is unaffected. Therefore, we may assume $a_2 = 0$.

Next, with

$$k' = (1, 0, a_3, a_4, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3})$$

add $2PD[v_1]$ to k' to get the equivalent vector

$$(-1, 2, a_3 + 2, a_4 + 2, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Next, add $2PD[v_2]$ to get

$$(1, -2, a_3 + 2, a_4 + 2, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Then add another $2PD[v_1]$, to get

$$(-1, 0, a_3 + 4, a_4 + 4, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Now, if we apply the move in Lemma 5.5 and take into account Remark 5.6, one can check that we get an equivalent vector whose 4^{th} entry is $a_4 + 6$. Since we assumed k' represents an element of $\mathscr{L}(\Gamma_j, [k])$, we must therefore have that $a_4 \in \{-3, -1, 1, 3\}$ and $a_4 + 6 \in \{-3, -1, 1, 3\}$. Hence, we must have had $a_4 = -3$. To summarize, we have now shown that we can assume

$$k' = (1, 0, a_3, -3, \underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Next, by $\star$, $c_{2j+3} \in \{-5, -3, -1, 1, 3, 5\}$. If $c_{2j+3} = 5$, then again by applying the move from Lemma 5.5, one can check that we transform c_{2j+3} into 7, which violates $\star$. Therefore, we must have had $c_{2j+3} \in \{-5, -3, -1, 1, 3\}$.

Now add $-2PD[v_4]$ to get an equivalent vector (which we again call k') with $a_1 = -1$, $a_2 = 0$, $a_4 = 3$ and k'_{Λ_j} unchanged except for the first entry which decreases by 2. Also, c_{2k+3} remains unchanged. If $\ell = 0$, then k'_{Λ_j} is now the zero vector, so we are done. Thus, suppose $\ell > 0$. Then

$$k' = (-1, 0, a_3, 3, -2, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-3-\ell}, c_{2j+3}).$$

Now consider the following sequence of moves:

(1) Rightward $(-2, 2)$-slide:

$$(-1, 0, a_3, 3, \underbrace{0, \dots, 0}_{2j-3-\ell}, -2, \underbrace{0, \dots, 0}_{\ell-1}, 2, c_{2j+3}).$$

(2) Type $+2$ move:

$$(-1, 0, a_3, 3, \underbrace{0, \dots, 0}_{2j-3-\ell}, -2, \underbrace{0, \dots, 0}_{\ell-2}, 2, -2, c_{2j+3} + 2).$$

(3) Leftward $(2, -2)$-slide:

$$(-1, 0, a_3, 3, \underbrace{0, \dots, 0}_{2j-3-\ell}, -2, 2, -2 \underbrace{0, \dots, 0}_{\ell-2}, c_{2j+3} + 2).$$

(4) Type $+2$ move:

$$(-1, 0, a_3, 3, \underbrace{0, \dots, 0}_{2j-2-\ell}, -2, \underbrace{0, \dots, 0}_{\ell-1}, c_{2j+3} + 2).$$

(5) Apply Lemma 5.5 and Remark 5.6:

$$(-1, 0, a_3, 1, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-2-\ell}, c_{2j+3}).$$

(6) Add $-2PD[v_1]$:

$$(1, -2, a_3 - 2, -1, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-2-\ell}, c_{2j+3}).$$

(7) Add $-2PD[v_2]$:

$$(-1, 2, a_3 - 2, -1, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-2-\ell}, c_{2j+3}).$$

(8) Add $-2PD[v_1]$:

$$(1, 0, a_3 - 4, -3, \underbrace{0, \dots, 0}_{\ell-1}, 2, \underbrace{0, \dots, 0}_{2j-2-\ell}, c_{2j+3}).$$

The net effect of this sequence of moves is that the $+2$-entry in k'_{Λ_j} shifts one space to the left while every other entry, excluding a_3, remains the same. So now we can repeat the above process until $+2$-entry is in the first position of k'_{Λ_j}. Then add $-2PD[v_4]$ to get

$$(-1, 0, a'_3, 3, 0, \dots, 0, c_{2j+3})$$

with $c_{jk+3} \in \{-5, -3, -1, 1, 3\}$ and, by $\star$, $a'_3 \in \{-8j + 1, -8j + 3, \dots, 8j + 1\}$. □

Proposition 5.9 *If k' represents an element of $\mathscr{L}(\Gamma_j, [k])$, then k' is equivalent to*

$$k_1 = (-1, 0, 5 - 4j, 3, 0, \dots, 0, -3) \quad \textit{or} \quad k_2 = (-1, 0, 3 - 4j, 3, 0, \dots, 0, 1).$$

In particular, $|\mathscr{L}(\Gamma_j, [k])| \le 2$.

Proof Up to this point, we have not used the fact that $k' \cdot x = 0$ where x is a generator of $\ker_{\mathbb{Z}}(B_j)$ as above. So assume k' is of the form in the previous lemma. Then

$$0 = k' \cdot x = 8j - 7 + 2a_3 + c_{2j+3}$$

where $a_3 \in \{-8j+1, -8k+3, \ldots, 8j+1\}$ and $c_{2j+3} \in \{-5, -3, -1, 1, 3\}$. The only solutions to this equation with the given constraints are $(a_3, c_{2j+3}) = (5-4j, -3)$ and $(3-4j, 1)$, corresponding to k_1 and k_2, respectively. □

We have not yet proved that k_1 and k_2 represent different elements of $\mathscr{L}(\Gamma_j, [k])$. To do this we will do a similar analysis for $-N_j$ and then use Turaev torsion. However, before we undertake this task, we first compute the HF^+ grading associated to the vectors k_1 and k_2.

Corollary 5.10 $d_{1/2}(-N_j; \mathfrak{s}_0) = \frac{1}{2}$.

Proof Let

$$\alpha_1 = (-12j+6, -6j+3, -1, -6j+3, -6j, -6j-3, \ldots, 2, 1, 0),$$
$$\alpha_2 = (4j+2, 2j+1, 1, 2j-1, 2j-2, 2j-3, \ldots, 2, 1, 0).$$

Then $\alpha_1 B_j = k_1$ and $\alpha_2 B_j = k_2$. Thus,

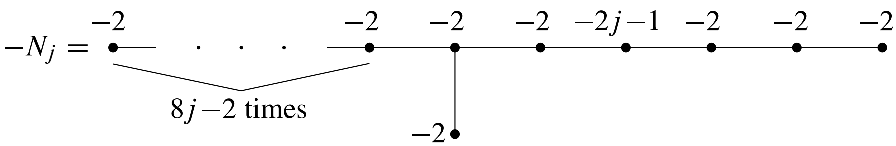
$$k_1^2 = k_1 \cdot \alpha_1 = -2j - 2,$$
$$k_2^2 = k_2 \cdot \alpha_2 = -2j - 2.$$

Hence, under the isomorphism from Corollary 4.30, the elements of $HF^+(-N_j, \mathfrak{s}_0)$ corresponding to k_1 and k_2 have gradings

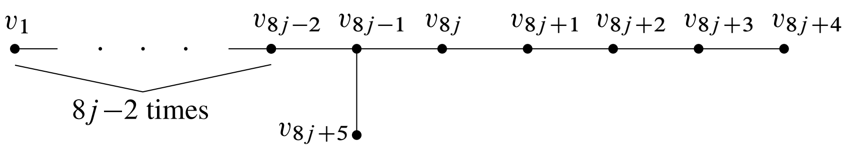
$$\mathrm{gr}(k_1) = \mathrm{gr}(k_2) = -\frac{k_2^2 + |\mathscr{V}(\Gamma_j)| - 3}{4} = -\frac{-2j - 2 + 2j + 3 - 3}{4} = \frac{1}{2}.$$ □

We now find a plumbing representation of $-N_j$ and then do Kirby calculus to make it negative semidefinite, as shown in Figure 5.

Now do slam dunks on the left and right vertices to get:



Let Γ_j' be the above plumbing graph with vertices labeled as follows:



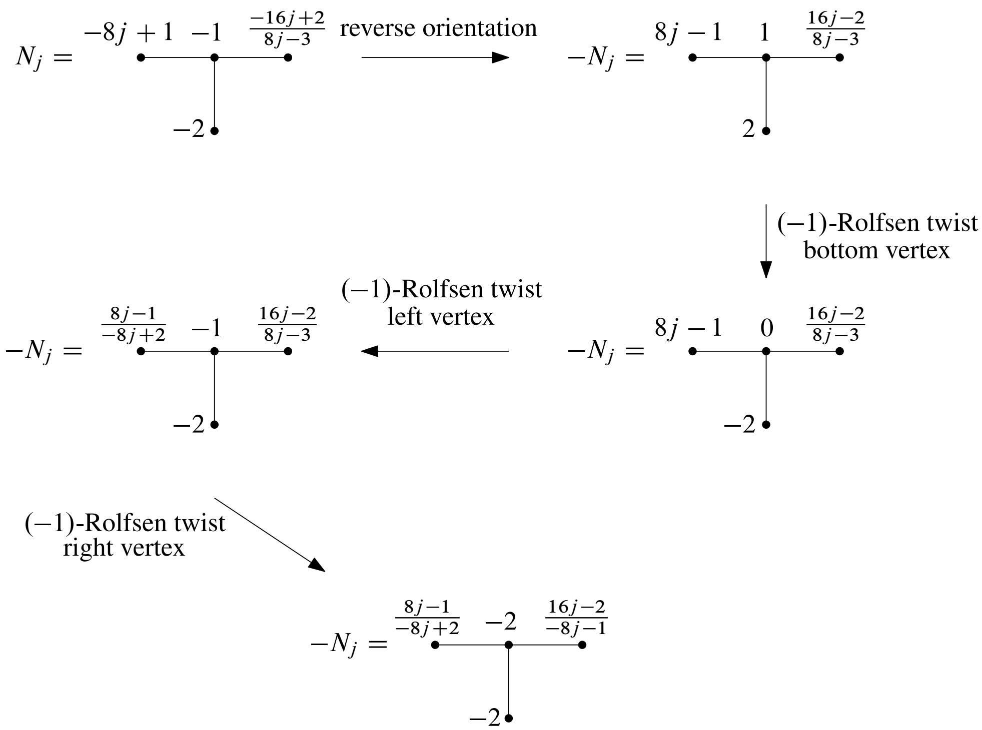


Figure 5

With respect to the ordered basis $([v_1], \dots, [v_{8j+5}])$, the matrix for the intersection form of $X(\Gamma'_j)$ is

$$B'_j = \begin{pmatrix}
-2 & 1 & & & & & & & & & & \\
1 & -2 & 1 & & & & & & & & & \\
 & & & \ddots & & & & & & & & \\
 & & & 1 & -2 & 1 & & & & & & \\
 & & & & 1 & -2 & 1 & & & & & 1 \\
 & & & & & 1 & -2 & 1 & & & & \\
 & & & & & & 1 & -2j-1 & 1 & & & \\
 & & & & & & & 1 & -2 & 1 & & \\
 & & & & & & & & 1 & -2 & 1 & \\
 & & & & & & & & & 1 & -2 & \\
 & & & & & 1 & & & & & & -2
\end{pmatrix}.$$

Again, it is straightforward to check that B'_j is negative semidefinite. Also, the $\mathbb{Z}$-kernel of B'_j is generated by the vector

$$x' = (2, 4, 6, \dots, 16j-2, 8j+1, 4, 3, 2, 1, 8j-1).$$

Let t denote a characteristic vector representing the trivial spin^c structure $\mathfrak{s}_0$. Then again, we can think of $\mathfrak{s}_0$ as

$$[t] = \{t' \in \mathrm{Char}(X(\Gamma'_j)) \mid t' \cdot x' = 0\}.$$

Let Λ'_j be the linear subgraph of Γ'_j given by:

$$\Lambda'_j = \overset{-2}{\underset{v_1}{\bullet}}\!\!-\!\!\!- \quad \cdot \quad \cdot \quad \cdot \quad -\!\!\!-\!\!\overset{-2}{\underset{v_{8j}}{\bullet}}$$

We write vectors $t' \in [t]$ as

$$t' = (a_1, a_2, \dots, a_{8j}, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, d_{8j+5})$$

where $t'_{\Lambda'_j} = (a_1, a_2, \dots, a_{8j})$.

Lemma 5.11 *If $t' \in [t]$ represents an element of $\mathscr{L}(\Gamma'_j, [t])$, then t' is equivalent to a vector whose Λ'_j-subvector is of the form*

$$(0, \dots, 0, a_{8j})$$

where $a_{8j} \in \{0, 2\}$.

Proof Suppose t' represents and element of $\mathscr{L}(\Gamma'_j, [t])$. By Lemmas 5.4 and 5.5, it suffices to consider the case when

$$t'_{\Lambda'_j} = (\underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{8j-1-\ell})$$

for some $0 \le \ell \le 8j-2$. Furthermore, by considering the linear subgraph of Γ'_j whose endpoints are $v_{\ell+1}$ and v_{8j+5}, it follows from Lemma 5.3 that $d_{8j+5} \in \{0, -2\}$.

Case 1 Suppose $d_{8j+5} = -2$ and $\ell = 8j-2$. If we add $-2PD[v_{8j+5}]$, then the Λ'_j-subvector of the resulting vector is zero, so we are done.

Case 2 Suppose $d_{8j+5} = -2$ and $\ell \le 8j-3$. Consider the following sequence of moves:

(1) Add $-2PD[v_{8j+5}]$:

$$(\underbrace{0, \dots, 0}_{\ell}, 2, \underbrace{0, \dots, 0}_{8j-3-\ell}, -2, 0, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 2).$$

(2) Rightward $(2,-2)$-slide:

$$(\underbrace{0, \dots, 0}_{\ell+1}, 2, \underbrace{0, \dots, 0}_{8j-3-\ell}, -2, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0).$$

Note, the rightmost entry of the vector changes from 2 to 0.

(3) Type -2 move on the leftmost -2:

$$\begin{aligned} &(\underbrace{0, \dots, 0}_{8j-1}, 2, b_{8j+1}-2, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0) && \text{if } \ell = 8j-3, \\ &(\underbrace{0, \dots, 0}_{\ell+1}, 2, \underbrace{0, \dots, 0}_{8j-4-\ell}, -2, 2, b_{8j+1}-2, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0) && \text{if } \ell \le 8j-4. \end{aligned}$$

If $\ell = 8j-3$ we are done. If $\ell = 8j-4$, then by applying a type -2 move on the leftmost -2, we get

$$(\underbrace{0,\dots,0}_{8j-2},2,0,b_{8j+1}-2,c_{8j+2},c_{8j+3},c_{8j+4},-2).$$

Hence, we are back to case 1. Therefore, we may assume $\ell \le 8j-5$. We now continue as follows:

(4) Leftward $(-2,2)$-slide:

$$(\underbrace{0,\dots,0}_{\ell+1},2,-2,2\underbrace{0,\dots,0}_{8j-4-\ell},b_{8j+1}-2,c_{8j+2},c_{8j+3},c_{8j+4},-2).$$

Note, the rightmost entry of the vector now changes back to -2.

(5) Type -2 move on the leftmost -2:

$$(\underbrace{0,\dots,0}_{\ell+2},2,\underbrace{0,\dots,0}_{8j-3-\ell},b_{8j+1}-2,c_{8j+2},c_{8j+3},c_{8j+4},-2).$$

We are now back to the vector we started with at the beginning of case 2, except that the $+2$ entry of the Λ'_j-subvector has shifted two positions to the right. Therefore, we can iterate this process until $\ell = 8j-3$ or $8j-4$, and we have already dealt with both of those cases.

Case 3 Suppose $d_{8j+5} = 0$ and $\ell = 8j-2$. Add $2PD[v_{8j-1}]$ to get the equivalent vector

$$(\underbrace{0,\dots,0}_{8j-3},2,-2,2,b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},2).$$

Now add $2PD[v_{8j+5}]$ to get

$$(\underbrace{0,\dots,0}_{8j-3},2,0,2,b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},-2).$$

This vector violates Lemma 5.3 and hence cannot be a representative of $\mathscr{L}(\Gamma'_j,[t])$.

Case 4 Suppose $d_{8j+5} = 0$ and $\ell \le 8j-3$, so that we start with a vector of the form

$$(\underbrace{0,\dots,0}_{\ell},2,\underbrace{0,\dots,0}_{8j-1-\ell},b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},0).$$

Now consider the following sequence of moves:

(1) Type $+2$ move:

$$(\underbrace{0,\dots,0}_{\ell-1},2,-2,2,\underbrace{0,\dots,0}_{8j-2-\ell},b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},0).$$

(2) Rightward $(-2,2)$-slide:

$$(\underbrace{0,\dots,0}_{\ell-1},2,\underbrace{0,\dots,0}_{8j-3-\ell},-2,2,0,b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},0).$$

(3) Add $2PD[v_{8j-1}]$:

$$(\underbrace{0,\dots,0}_{\ell-1},2,\underbrace{0,\dots,0}_{8j-2-\ell},-2,2,c_{8j+2},c_{8j+3},c_{8j+4},2).$$

(4) Add $2PD[v_{8j+5}]$:

$$(\underbrace{0,\dots,0}_{\ell-1},2,\underbrace{0,\dots,0}_{8j-1-\ell},2,c_{8j+2},c_{8j+3},c_{8j+4},-2).$$

Again, this vector violates Lemma 5.3 and hence cannot be a representative of $\mathscr{L}(\Gamma_j',[t])$. □

Proposition 5.12 *If $t' \in [t]$ represents an element of $\mathscr{L}(\Gamma_j',[t])$, then t' is equivalent to*

$$t_1 = (0,\dots,0,-1,0,2,0,0) \quad \textit{or} \quad t_2 = (0,\dots,0,2,-1,0,0,0,-2).$$

Proof Suppose $t' \in [k]$ represents an element of $\mathscr{L}(\Gamma_j',[t])$. By the previous lemma, we can assume

$$t' = (0,\dots,0,a_{8j},b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},d_{8j+5})$$

where $a_{8j} \in \{0,2\}$, $b_{8j+1} \in \{-2j-1,-2j+1,\dots,2j-1,2j+1\}$, $c_{8j+2},c_{8j+3},c_{8j+4} \in \{-2,0,2\}$, and $d_{8j+5} \in \{-2,0,2\}$.

Since we are assuming t' represents an element of $\mathscr{L}(\Gamma_j',[t])$, we must have

(5.13) $$0 = t' \cdot x' = (8j+1)a_{8j} + (8j-1)d_{8j+5} + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4}.$$

By Lemmas 5.4 and 5.5, we can assume $(c_{8j+2},c_{8j+3},c_{8j+4})$ is the zero vector or has exactly one nonzero entry equal to $+2$. In particular, we can assume

$$3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \in \{0,2,4,6\}.$$

Note the moves required to put the subvector $(c_{8j+2},c_{8j+3},c_{8j+4})$ into this form only effect the entry b_{8j+1} and leave all of the others unchanged.

Now suppose $a_{8j} = 2$ and $b_{8j+1} = 2j+1$. Then by adding $2PD[v_{8j+1}]$ we would obtain an equivalent vector with $a_{8j} = 4$. But this violates $\star$. Hence, if $a_{8j} = 2$, we can assume $b_{8j+1} \le 2j-1$.

Now suppose $a_{8j} = 0$ and $b_{8j+1} = 2j+1$. If $(c_{8j+2},c_{8j+3},c_{8j+4})$ is not the zero vector, but rather a vector with precisely one nonzero entry equal to $+2$, then by applying the move in Lemma 5.5 and taking into account Remark 5.6, we would obtain an equivalent vector with $b_{8j+1} = 2j+3$, which violates $\star$. Therefore, if $a_{8j} = 0$ and $b_{8j+1} = 2j+1$, we must have that $(c_{8j+2},c_{8j+3},c_{8j+4})$ is the zero vector. Plugging this into (5.13) yields

$$(8j-1)d_{8j+5} = -8j-4.$$

This clearly has no solutions with the given constraints. Therefore, we can assume $b_{8j+1} \le 2j-1$, regardless of whether $a_{8j} = 0$ or 2. In particular,

$$-8j-4 \le 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \le 8j+2.$$

Case 1 Suppose $a_{8j} = 0$ and $d_{8j+5} = -2$. Then

$$0 = t' \cdot x' = -16j + 2 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \leq -8j + 4 < 0$$

which is a contradiction.

Case 2 Suppose $a_{8j} = 0$ and $d_{8j+5} = 0$. Then

$$0 = t' \cdot x' = 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4}.$$

The only solution to this equation given the constraints we have established is

$$(b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}) = (-1, 0, 2, 0)$$

which corresponds to t_1.

Case 3 Suppose $a_{8j} = 0$ and $d_{8j+5} = 2$. Then

$$0 = t' \cdot x' = 16j - 2 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \geq 8j - 6 > 0$$

which again is a contradiction.

Case 4 Suppose $a_{8j} = 2$ and $d_{8j+5} = -2$. Then

$$0 = t' \cdot x' = 4 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4}.$$

The only solution to this equation given the constraints we have established is

$$(b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}) = (-1, 0, 0, 0)$$

which corresponds to t_2.

Case 5 Suppose $a_{8j} = 2$ and $d_{8j+5} = 0$. Then

$$0 = t' \cdot x' = 16j + 2 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \geq 8j - 2 > 0$$

which again is a contradiction. Finally:

Case 6 Suppose $a_{8j} = 2$ and $d_{8j+5} = 2$. This case is ruled out by Lemma 5.3. □

Again, we have not yet proved that t_1 and t_2 represent different elements of $\mathscr{L}(\Gamma'_j, [t])$; however, we do have:

Corollary 5.14 $$d_{1/2}(N_j, \mathfrak{s}_0) = -2j + \tfrac{1}{2}.$$

Proof Let

$$\beta_1 = (2, 4, 6, \ldots, 16j - 2, 8j + 1, 4, 2, 0, 0, 8j - 1) \quad \text{and} \quad \beta_2 = (0, \ldots, 0, -1, 0, 0, 0, 0, 1).$$

Then $\beta_1 B'_j = t_1$ and $\beta_2 B'_j = t_2$. Thus,

$$t_1^2 = t_1 \cdot \beta_1 = -4 \quad \text{and} \quad t_2^2 = t_2 \cdot \beta_2 = -4.$$

Hence, under the isomorphism in Corollary 4.30, the elements of $HF^+(N_j, \mathfrak{s}_0)$ corresponding to t_1 and t_2 have gradings

$$\operatorname{gr}(t_1) = \operatorname{gr}(t_2) = -\frac{t_2^2 + |\mathcal{V}(\Gamma'_j)| - 3}{4} = -\frac{-4 + 8j + 5 - 3}{4} = -2j + \frac{1}{2}. \qquad \square$$

Now combining Corollaries 5.10 and 5.14, and the basic fact that $d_{\pm 1/2}(-Y) = -d_{\mp 1/2}(Y)$ (see [16, Proposition 4.10]), we have

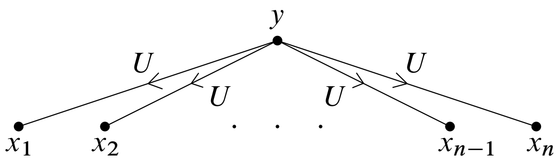
$$d_{1/2}(-N_j) = \tfrac{1}{2} \quad \text{and} \quad d_{-1/2}(-N_j) = 2j - \tfrac{1}{2}.$$

In particular, by Theorem 4.29, $HF^+_{\text{even}}(-N_j, \mathfrak{s}_0) = \mathcal{T}^+_{2j-1/2}$.

We have yet to completely determine $HF^+_{\text{odd}}(-N_j, \mathfrak{s}_0)$. So far, from Proposition 5.9, we know that $\dim_{\mathbb{F}}[\ker(U) \cap HF^+_{\text{odd}}(-N_j, \mathfrak{s}_0)] = 1$ or 2 depending on whether k_1 and k_2 represent the same element or not in $\mathcal{L}(\Gamma_j, [k])$. Therefore, as graded $\mathbb{F}[U]$-modules, we have one of the equivalences in Figure 6. Here, h_j is some positive integer depending on j which we have not yet determined.

A word of explanation is in order since on the left side of the above isomorphism we have an $\mathbb{F}[U]$-module and on the right we have one of two possible graphs. The right side is to be interpreted as follows:

- Each vertex at grading r corresponds to a basis element of the $\mathbb{F}$-vector space $HF^+_r(-N_j, \mathfrak{s}_0)$.
- If the edges emanating from a vertex y are of the form



then $Uy = x_1 + x_2 + \cdots + x_{n-1} + x_n$. In particular, if there are no edges emanating from y, then $Uy = 0$.

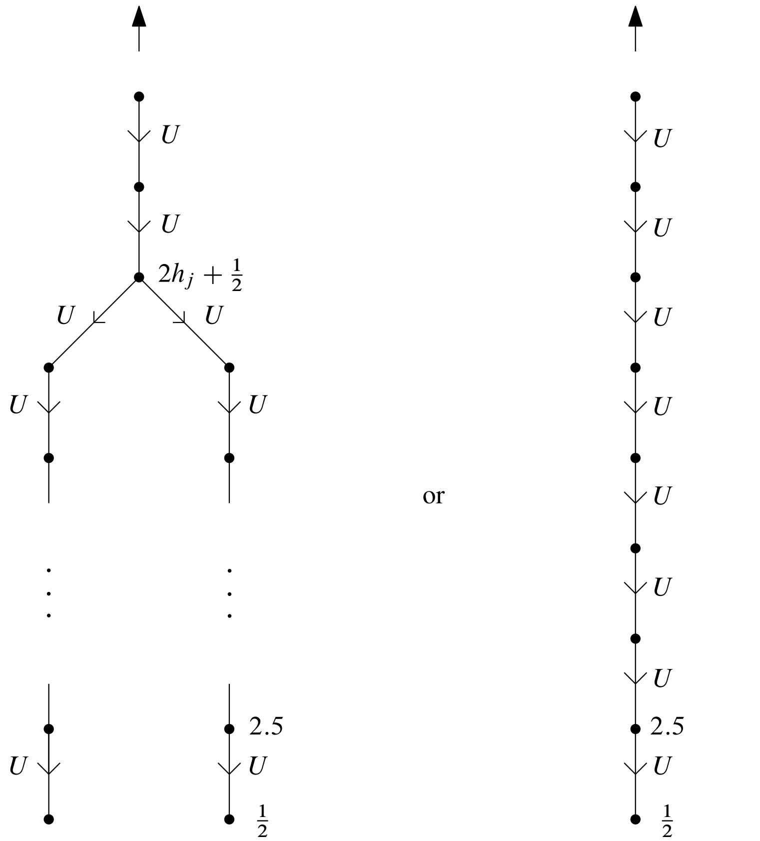
We now utilize Turaev torsion. Combining our computations thus far with [18, Theorem 10.17], we see that

$$T_{N_j}(\mathfrak{s}_0) = \begin{cases} h_j + j & \text{if } k_1 \neq k_2 \in \mathcal{L}(\Gamma_j, [k]), \\ j & \text{otherwise,} \end{cases}$$

where T_{N_j} is the Turaev torsion function associated to N_j (see [24, page 119]). Therefore, to precisely determine $HF^+_{\text{odd}}(-N_j, \mathfrak{s}_0)$, it suffices to compute $T_{N_j}(\mathfrak{s}_0)$.

There are many standard ways to compute $T_{N_j}(\mathfrak{s}_0)$. For example, Turaev [24] provided a formula in terms of a surgery description. We will now give a brief outline of how to carry out the calculation using this method, but we leave the details to the reader.

$HF^+_{\mathrm{odd}}(-N_j, \mathfrak{s}_0) \cong$



if $k_1 \neq k_2 \in \mathscr{L}(\Gamma_j, [k])$ if $k_1 = k_2 \in \mathscr{L}(\Gamma_j, [k])$

Figure 6

(1) Let $H = H_1(N_j; \mathbb{Z})$. Consider the group ring $\mathbb{Z}[H]$. Since $H \cong \mathbb{Z}$, we can think of $\mathbb{Z}[H]$ as $\mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials in the indeterminate t. Let $Q(H)$ denote the field of fractions of $\mathbb{Z}[H]$. The first step is to compute the Turaev torsion $\tau(N_j, \mathfrak{s}_0) \in Q(H)$. For this, we use the formula given in [24, VII.2, Theorem 2.2]. To apply this formula, we need to choose a surgery diagram for N_j and orient the underlying link. We use the surgery diagram in Figure 7 with underlying link L_j oriented as indicated by the arrows.
The bulk of the work in computing $\tau(N_j, \mathfrak{s}_0)$ using [24, VII.2, Theorem 2.2] is calculating the multivariable Alexander–Conway function $\nabla(L_j)$. Again, there are various approaches to computing $\nabla(L_j)$. For example, Murakami [11] provided a skein formula for ∇. Using this formula, we find that $\nabla(L_j) = yx^{4j-1} + y^{-1}x^{-4j+1}$ where the variable x corresponds to the torus knot component and the variable y corresponds to the unknot component. Plugging this into the formula for $\tau(N_j, \mathfrak{s}_0)$, we get

$$\tau(N_j, \mathfrak{s}_0) = \frac{t^{8j-1} + 1}{t^{4j-2}(t-1)^2(t+1)}.$$

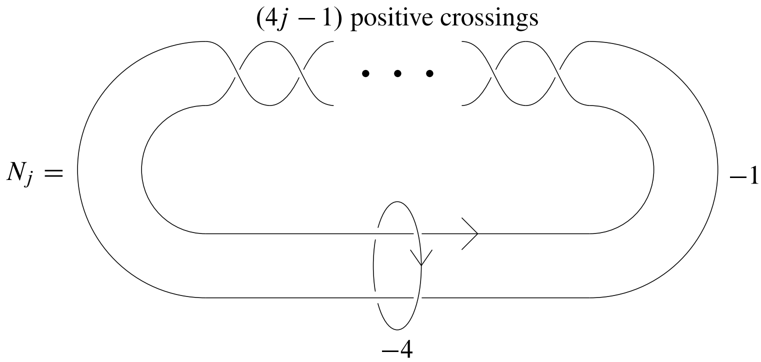


Figure 7

(2) Next, we compute $[\tau(N_j,\mathfrak{s}_0)]$ which is a Laurent polynomial obtained by truncating $\tau(N_j,\mathfrak{s}_0)$ in a certain way (see [24, page 22]). We find that

$$\begin{aligned}[\tau(N_j,\mathfrak{s}_0)] &= \frac{t^{8j-1}+1}{t^{4j-2}(t-1)^2(t+1)} - \frac{t}{(t-1)^2}\\ &= \Bigl(\sum_{i=0}^{4j-4} t^{i-4j+2}\Bigr)\Bigl(\sum_{i=1}^{2j-1} t^{2i}\Bigr) + \sum_{i=0}^{8j-4} t^{i-4j+2}\\ &= 2j + \text{nonconstant terms.}\end{aligned}$$

(3) By definition, $T_{N_j}(\mathfrak{s}_0)$ is the constant term of $[\tau(N_j,\mathfrak{s}_0)]$. Hence, $T_{N_j}(\mathfrak{s}_0) = 2j$.

Thus, we have the isomorphism of graded $\mathbb{F}[U]$-modules in Figure 8.

We now compute the involution ι_* on homology. This amounts to determining whether $-k_1$ is equivalent to k_1 or k_2. If $-k_1$ is equivalent to k_2, then the involution swaps the two legs of the left-hand graph of the above figure and leaves the right-hand graph fixed. If $-k_1$ is equivalent to k_1, then ι_* is the identity. We know show that, in fact, $-k_1$ is equivalent to k_2.

Recall, $-k_1 = (1,0,-5+4j,-3,0,\dots,0,3)$ and $k_2 = (-1,0,3-4j,3,0,\dots,0,1)$. Consider the following sequence of moves from $-k_1$ to k_2:

(1) Add $2PD[v_4]$:

$$\begin{array}{ll} (-1,0,-1,3,1) = k_2 & \text{if } j=1,\\ (-1,0,-5+4j,3,-2,\underbrace{0,\dots,0}_{2j-3},3) & \text{if } j\geq 2.\end{array}$$

So we can assume for the subsequent moves that $j \geq 2$.

(2) Apply Lemma 5.5 and Remark 5.6:

$$(-1,0,-5+4j,1,\underbrace{0,\dots,0}_{2j-3},2,1).$$

$HF^+(-N_j, \mathfrak{s}_0) \cong$

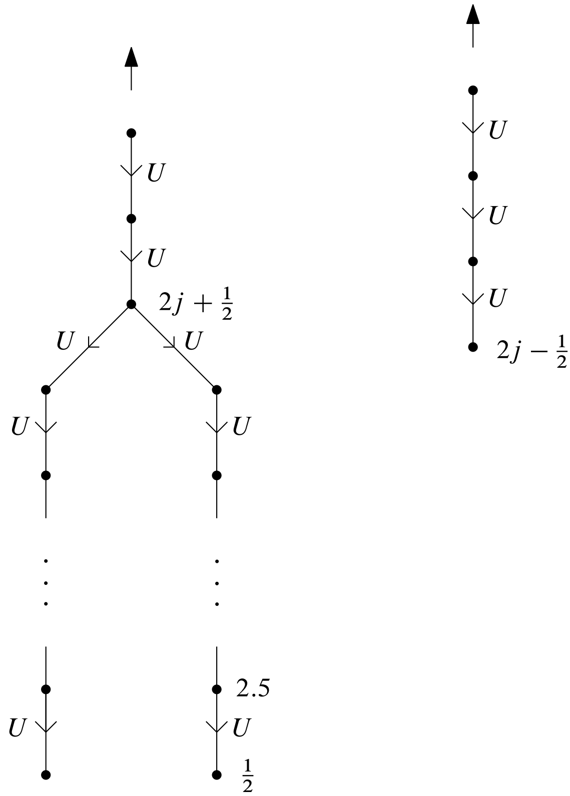


Figure 8

(3) Add $-2PD[v_1]$:

$$(1, -2, -5+4j-2, -1, \underbrace{0, \dots, 0}_{2j-3}, 2, 1).$$

(4) Add $-2PD[v_2]$:

$$(-1, 2, -5+4j-2, -1, \underbrace{0, \dots, 0}_{2j-3}, 2, 1).$$

(5) Add $2PD[v_1]$:

$$(1, 0, -5+4j-4, -3, \underbrace{0, \dots, 0}_{2j-3}, 2, 1).$$

(6) Add $-2PD[v_4]$:

$$(-1, 0, -5+4j-4, 3, -2, \underbrace{0, \dots, 0}_{2j-4}, 2, 1).$$

(7) Add $-2PD[v_5]$:

$$(-1, 0, -5+4j-4, 1, 2, -2\,\underbrace{0, \dots, 0}_{2j-5}, 2, 1).$$

(8) Rightward $(2,-2)$-slide:

$$(-1,0,-5+4j-4,1,\underbrace{0,\dots,0}_{2j-5},2,-22,1).$$

(9) Type -2 move:

$$(-1,0,-5+4j-4,1,\underbrace{0,\dots,0}_{2j-4},2,0,1).$$

Now notice that we are back to the same vector as in (2), except we have decreased the 3rd entry by 4 and shifted the $+2$ entry one slot to the left. Therefore, if we iterate this sequence of moves $(2j-4)$ more times, we get the vector

$$(-1,0,7-4j,1,2,\underbrace{0,\dots,0}_{2j-3},1).$$

Now consider the sequence of moves:

(1) Add $-2PD[v_1]$:

$$(1,-2,5-4j,-1,2,\underbrace{0,\dots,0}_{2j-3},1).$$

(2) Add $-2PD[v_2]$:

$$(-1,2,5-4j,-1,2,\underbrace{0,\dots,0}_{2j-3},1).$$

(3) Add $-2PD[v_1]$:

$$(1,0,3-4j,-3,2,\underbrace{0,\dots,0}_{2j-3},1).$$

(4) Add $-2PD[v_4]$:

$$(-1,0,3-4j,3,0,\dots,0,1)=k_2.$$

Theorem 5.15 *We have the isomorphism of graded $\mathbb{F}[U,Q]/(Q^2)$-modules given in Figure 9.*

Remark 5.16 The graph on the right-hand side of the isomorphism in Figure 9 should be interpreted as a graded $\mathbb{F}[U,Q]/(Q^2)$-module in a manner similar to what was described earlier in the context of $\mathbb{F}[U]$-modules, except now there are additional arrows labeled with Q to indicate the action of Q.

Proof For simplicity of exposition, we prove the statement for $j=1$. The proof for $j\geq 2$ is completely analogous and is left to the reader.

Fix an admissible Heegaard pair $\mathcal{H}=(H,J)$ for $(-N_1,\mathfrak{s}_0)$. We can choose representative cycles $a,b,c\in CF^+(\mathcal{H},\mathfrak{s}_0)$ such that

$$[a+b],[c]\in \mathrm{Im}[\pi_*\colon HF^\infty(\mathcal{H},\mathfrak{s}_0)\to HF^+(\mathcal{H},\mathfrak{s}_0)]$$

and the corresponding HF^+ homology generators are given in Figure 10.

$HFI^+(-N_j, \mathfrak{s}_0) \cong$

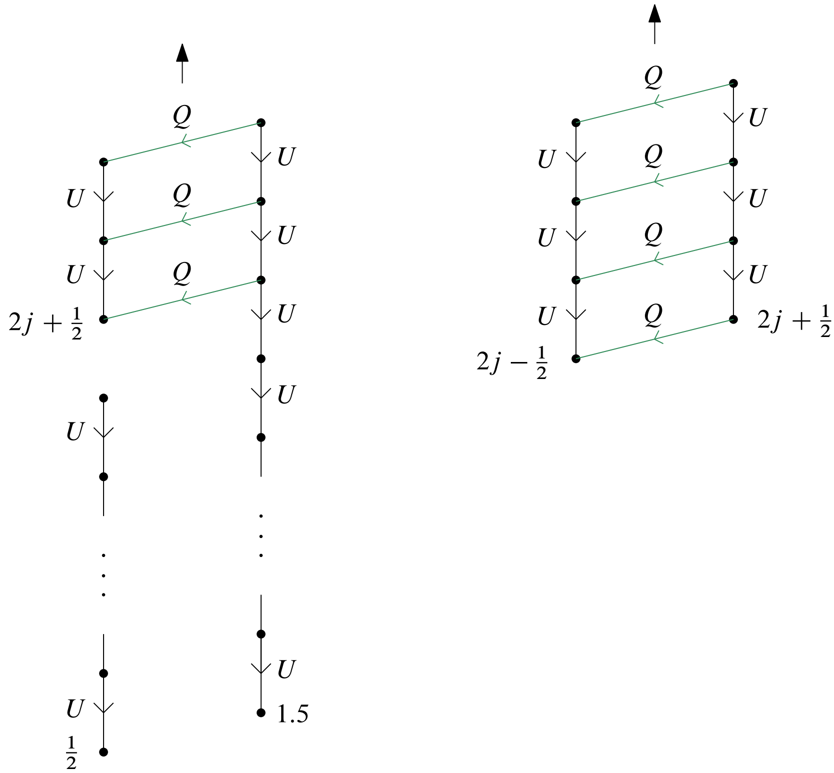


Figure 9

Since $\iota_*([a]) = \iota_*([b])$, we have that $(1+\iota_*)([a+b]) = 0$. Therefore, there exists some $d \in CF^+(\mathcal{H}, \mathfrak{s}_0)$ such that $\partial d = a+b+\iota(a+b)$. Similarly, since $(1+\iota_*)([c]) = 0$, there exists some $e \in CF^+(\mathcal{H}, \mathfrak{s}_0)$ such that $\partial e = c+\iota(c)$. It then follows from Proposition 2.2, that as graded $\mathbb{F}$-vector spaces, $HFI^+(-N_1; \mathfrak{s}_0)$ is isomorphic to that given in Figure 11.

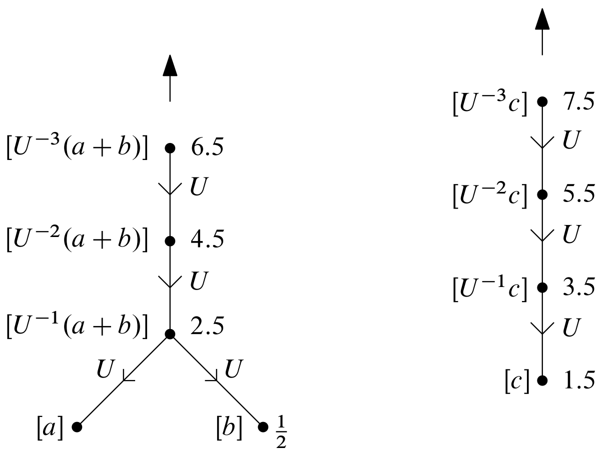


Figure 10

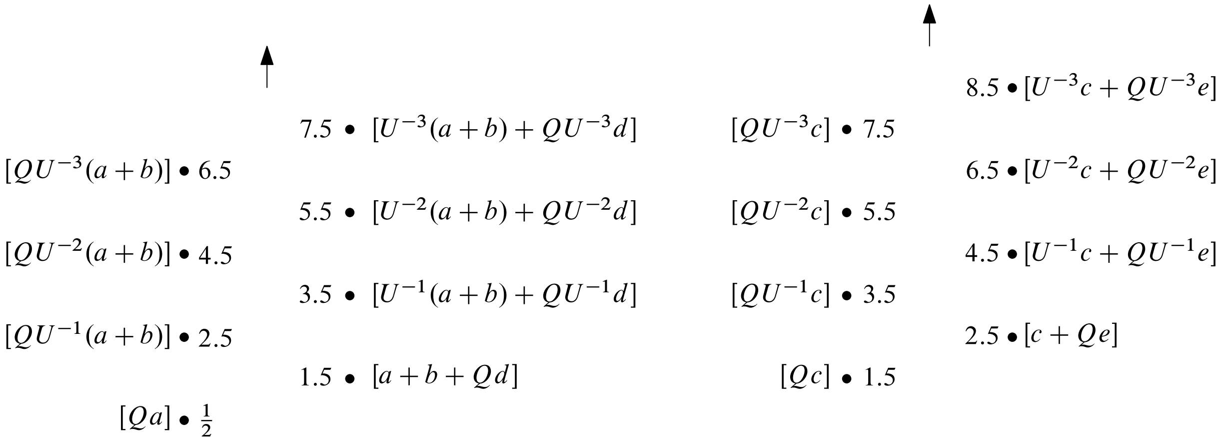


Figure 11

From this explicit description of generators, we see that for $n \geq 2$,

$$U \cdot [QU^{-n}(a+b)] = [QU^{-n+1}(a+b)],$$

and for $n \geq 1$,

$$U \cdot [U^{-n}(a+b)+QU^{-n}d] = [U^{-n+1}(a+b)+QU^{-n+1}d],$$
$$U \cdot [QU^{-n}c] = [QU^{-n+1}c],$$
$$U \cdot [U^{-n}c+QU^{-n}e] = [U^{-n+1}c+QU^{-n+1}e].$$

Next, we have

$$U \cdot [QU^{-1}(a+b)] = [Q(a+b)] = [\partial^I a] = 0.$$

Moreover, by grading considerations, we must have

$$U \cdot [Qa] = 0,$$
$$U \cdot [a+b+Qd] = 0,$$
$$U \cdot [Qc] = 0.$$

Also, either $U \cdot [c+Qe] = 0$ or $U \cdot [c+Qe] = [Qa]$. In the former case, we would have

$$\dim_{\mathbb{F}}[\ker(U : HFI^+(-N_j,\mathfrak{s}_0) \to HFI^+(-N_j,\mathfrak{s}_0))] = 5,$$
$$\dim_{\mathbb{F}}[\operatorname{coker}(U : HFI^+(-N_j,\mathfrak{s}_0) \to HFI^+(-N_j,\mathfrak{s}_0))] = 1,$$

whereas in latter we would have

$$\dim_{\mathbb{F}}[\ker(U : HFI^+(-N_j,\mathfrak{s}_0) \to HFI^+(-N_j,\mathfrak{s}_0)] = 4,$$
$$\dim_{\mathbb{F}}[\operatorname{coker}(U : HFI^+(-N_j,\mathfrak{s}_0) \to HFI^+(-N_j,\mathfrak{s}_0))] = 0.$$

Thus, by [6, Proposition 4.1], we would have either

$$\dim_{\mathbb{F}}(\widehat{HFI}(-N_j,\mathfrak{s}_0)) = 6 \text{ or } 4.$$

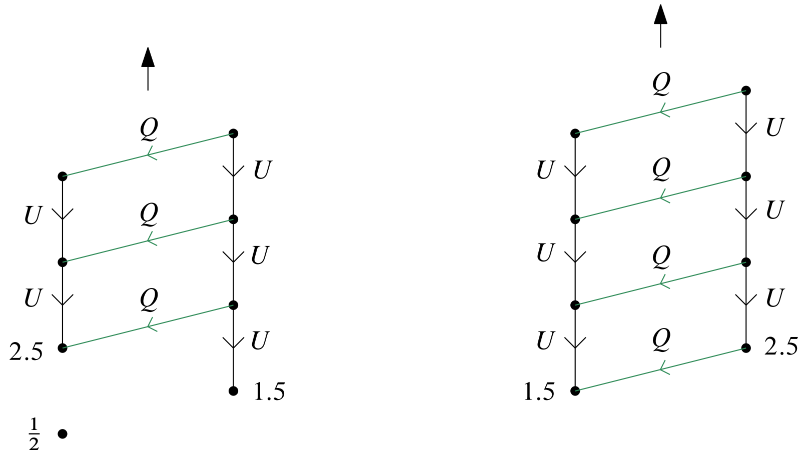
$HFI^+(-N_1, \mathfrak{s}_0) \cong$ 

Figure 12

But by [6, Corollary 4.7] we see that

$$\begin{aligned}\dim_{\mathbb{F}}(\widehat{HFI}(-N_j, \mathfrak{s}_0)) &= \dim_{\mathbb{F}}[\ker(Q(1+\iota_*)\colon \widehat{HF}(-N_j, \mathfrak{s}_0) \to Q \cdot \widehat{HF}(-N_j, \mathfrak{s}_0))] \\ &\qquad + \dim_{\mathbb{F}}[\operatorname{coker}(Q(1+\iota_*)\colon \widehat{HF}(-N_j, \mathfrak{s}_0) \to Q \cdot \widehat{HF}(-N_j, \mathfrak{s}_0))] \\ &= 3+3 \\ &= 6.\end{aligned}$$

Hence, we must have had $U \cdot [c + Qe] = 0$. We have now completely determined the U-action on $HFI^+(-N_j, \mathfrak{s}_0)$.

Next, for the Q-action, it follows from the explicit description of the generators that for $n \geq 1$,

$$Q \cdot [U^{-n}(a+b) + QU^{-1}d] = [QU^{-n}(a+b)],$$

and for $n \geq 0$,

$$Q \cdot [U^{-n}c + QU^{-n}e] = [QU^{-n}c].$$

Also,

$$Q \cdot [a+b+Qd] = [Q(a+b)] = [\partial^I a] = 0.$$

It is clear that the action of Q on all of the other generators is zero. Thus, we have the isomorphism given in Figure 12. □

Theorem 5.17 *For all positive integers j, N_j cannot be obtained by 0-surgery on a knot in S^3. In fact, N_j is not the oriented boundary of any smooth negative semidefinite spin 4-manifold.*

Proof From previous theorem, we have

$$\bar{d}_{1/2}(-N_j) = 2j + \tfrac{1}{2}, \quad \bar{d}_{-1/2}(-N_j) = 2j - \tfrac{1}{2},$$
$$\underline{d}_{1/2}(-N_j) = \tfrac{1}{2}, \qquad \underline{d}_{-1/2}(-N_j) = 2j - \tfrac{1}{2}.$$

Equivalently,

$$\underline{d}_{-1/2}(N_j) = -2j - \tfrac{1}{2}, \quad \underline{d}_{1/2}(N_j) = -2j + \tfrac{1}{2},$$
$$\bar{d}_{-1/2}(N_j) = -\tfrac{1}{2}, \qquad \bar{d}_{1/2}(N_j) = -2j + \tfrac{1}{2}.$$

The conclusion now follows immediately from Corollaries 2.18 and 2.21. □

5.4 $HFI^+(-S_0^3(K_1), \mathfrak{s}_0)$

As mentioned in the introduction, Ichihara, Motegi and Song [8] discovered an infinite family of hyperbolic knots which admit small Seifert fibered 0-surgery. In particular, for the knot K_1 in their family, they show that

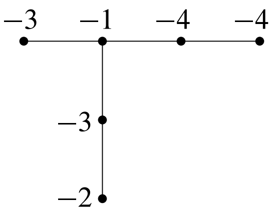
$$S_0^3(K_1) = S^2\left(\tfrac{3}{2}, -\tfrac{5}{2}, -\tfrac{15}{4}\right).$$

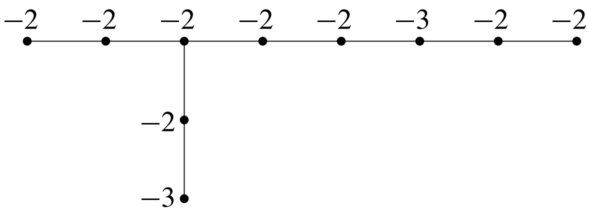
Since, by definition, $S_0^3(K_1)$ is 0-surgery on a knot in S^3, we know from Corollary 2.21 that

$$-\tfrac{1}{2} \le \underline{d}_{-1/2}(S_0^3(K_1)) \quad \text{and} \quad \bar{d}_{1/2}(S_0^3(K_1)) \le \tfrac{1}{2}.$$

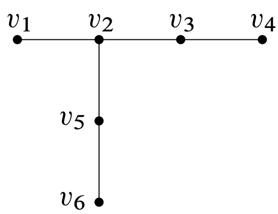
We now verify these bounds directly by computing $HFI^+(-S_0^3(K_1), \mathfrak{s}_0)$ and then we compare this to $HFI^+(-N_1, \mathfrak{s}_0)$. In the interest of brevity, we are only going to give an outline of the calculation and leave the details to the reader.

5.4.1 Step 1 We use Kirby calculus and the fact that $S_0^3(K_1)$ is a small Seifert fibered space to find negative semidefinite plumbing representations of $S_0^3(K_1)$ and $-S_0^3(K_1)$:

$S_0^3(K_1) =$ 

$-S_0^3(K_1) =$ 

Label the vertices of the above left plumbing graph as:



Using the methods of the previous section, one can show that as graded $\mathbb{F}[U]$-modules, we have the isomorphism in Figure 13, where the two leaves on the left graph correspond to the representative vectors

$$z_1 = (-1, -1, 4, -2, 1, 0) \quad \text{and} \quad z_2 = (1, -1, 0, 4, 1, 0).$$

Note that $HF^+(-S_0(K_1), \mathfrak{s}_0) \cong HF^+(-N_1, \mathfrak{s}_0)$.

$HF^+(-S^3_0(K_1), \mathfrak{s}_0) \cong$

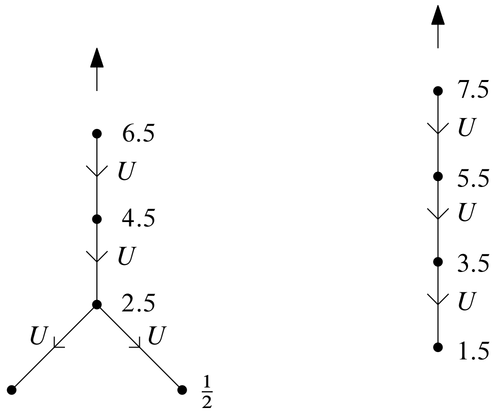


Figure 13

5.4.2 Step 2 To determine ι_*, consider the following sequence of moves starting with the vector $-z_1 = (1, 1, -4, 2, -1, 0)$:

(1) Add $-2PD[v_3]$: $(1, -1, 4, 0, -1, 0)$.

(2) Add $-2PD[v_2]$: $(-1, 1, 2, 0, -3, 0)$.

(3) Add $-2PD[v_5]$: $(-1, -1, 2, 0, 3, -2)$.

(4) Add $-2PD[v_6]$: $(-1, -1, 2, 0, 1, 2)$.

(5) Add $-2PD[v_2]$: $(-3, 1, 0, 0, -1, 2)$.

(6) Add $-2PD[v_1]$: $(3, -1, 0, 0, -1, 2)$.

(7) Add $-2PD[v_2]$: $(1, 1, -2, 0, -3, 2)$.

(8) Add $-2PD[v_5]$: $(1, -1, -2, 0, 3, 0)$.

(9) Add $-2PD[v_2]$: $(-1, 1, -4, 0, 1, 0)$.

(10) Add $-2PD[v_3]$: $(-1, -1, 4, -2, 1, 0) = z_1$.

Therefore, ι_* is the identity. In particular, unlike for $HF^+(-N_1, \mathfrak{s}_0)$, J_0 does not swap the two legs of the graded root corresponding to the odd part of HF^+. It is worth noting that this behavior of J_0 is not seen for negative definite almost rational plumbings. Specifically, in [1, Lemma 2.1] (see also [2, Section 2]) it is shown that for negative definite almost rational plumbings, the involution J_0 on the graded root of a self-conjugate spin^c structure cannot fix more than one vertex of the graded root at a given grading level. The proof of this fact relies on the result that the lattice cohomology of almost rational plumbings is concentrated in homological degree 0. However, for negative semidefinite plumbings the lattice cohomology in general will not be concentrated in homological degree 0 and hence the action of J_0 need not behave as in the almost rational case, as illustrated by this example.

5.4.3 Step 3 Applying the same methods as in the proof of Theorem 5.15, we get:

$HFI^+(-S^3_0(K_1), \mathfrak{s}_0) \cong$

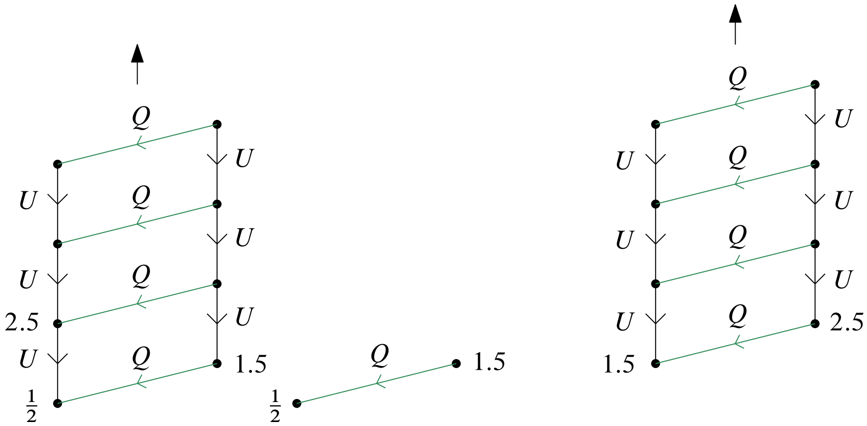


Figure 14

Theorem 5.18 *As graded $\mathbb{F}[U, Q]/(Q^2)$-modules, we have the isomorphism in Figure 14. In particular,*

$$\bar{d}_{1/2}(-S^3_0(K_1)) = \tfrac{1}{2}, \quad \bar{d}_{-1/2}(-S^3_0(K_1)) = 1.5,$$
$$\underline{d}_{1/2}(-S^3_0(K_1)) = \tfrac{1}{2}, \quad \underline{d}_{-1/2}(-S^3_0(K_1)) = 1.5.$$

In summary, even though

$$HF^+(-N_1, \mathfrak{s}_0) \cong HF^+(-S^3_0(K_1), \mathfrak{s}_0),$$

we see that

$$HFI^+(-N_1, \mathfrak{s}_0) \ncong HFI^+(-S^3_0(K_1), \mathfrak{s}_0).$$

References

[1] **I Dai**, *On the* Pin(2)*-equivariant monopole Floer homology of plumbed* 3*-manifolds*, Michigan Math. J. 67 (2018) 423–447 MR Zbl

[2] **I Dai**, **C Manolescu**, *Involutive Heegaard Floer homology and plumbed three-manifolds*, J. Inst. Math. Jussieu 18 (2019) 1115–1155 MR Zbl

[3] **R E Gompf**, **A I Stipsicz**, 4*-manifolds and Kirby calculus*, Graduate Studies in Math. 20, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[4] **M Hedden**, **M H Kim**, **T E Mark**, **K Park**, *Irreducible* 3*-manifolds that cannot be obtained by* 0*-surgery on a knot*, Trans. Amer. Math. Soc. 372 (2019) 7619–7638 MR Zbl

[5] **K Hendricks**, **J Hom**, **M Stoffregen**, **I Zemke**, *Surgery exact triangles in involutive Heegaard Floer homology*, preprint (2020) arXiv 2011.00113

[6] **K Hendricks**, **C Manolescu**, *Involutive Heegaard Floer homology*, Duke Math. J. 166 (2017) 1211–1299 MR Zbl

[7] **K Hendricks**, **C Manolescu**, **I Zemke**, *A connected sum formula for involutive Heegaard Floer homology*, Selecta Math. 24 (2018) 1183–1245 MR Zbl

[8] **K Ichihara**, **K Motegi**, **H-J Song**, *Seifert fibered slopes and boundary slopes on small hyperbolic knots*, Bull. Nara Univ. Ed. Natur. Sci. 57 (2008) 21–25 MR Zbl

[9] **A Juhász**, **D Thurston**, **I Zemke**, *Naturality and mapping class groups in Heegard Floer homology*, Mem. Amer. Math. Soc. 1338, Amer. Math. Soc., Providence, RI (2021) MR Zbl

[10] **L Moser**, *Elementary surgery along a torus knot*, Pacific J. Math. 38 (1971) 737–745 MR Zbl

[11] **J Murakami**, *A state model for the multivariable Alexander polynomial*, Pacific J. Math. 157 (1993) 109–135 MR Zbl

[12] **A Némethi**, *On the Ozsváth–Szabó invariant of negative definite plumbed 3-manifolds*, Geom. Topol. 9 (2005) 991–1042 MR Zbl

[13] **A Némethi**, *Lattice cohomology of normal surface singularities*, Publ. Res. Inst. Math. Sci. 44 (2008) 507–543 MR Zbl

[14] **W D Neumann**, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. 268 (1981) 299–344 MR Zbl

[15] **P Ozsváth**, **A I Stipsicz**, **Z Szabó**, *A spectral sequence on lattice homology*, Quantum Topol. 5 (2014) 487–521 MR Zbl

[16] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003) 179–261 MR Zbl

[17] **P Ozsváth**, **Z Szabó**, *On the Floer homology of plumbed three-manifolds*, Geom. Topol. 7 (2003) 185–224 MR Zbl

[18] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. 159 (2004) 1159–1245 MR Zbl

[19] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. 159 (2004) 1027–1158 MR Zbl

[20] **P Ozsváth**, **Z Szabó**, *Holomorphic triangles and invariants for smooth four-manifolds*, Adv. Math. 202 (2006) 326–400 MR Zbl

[21] **P Ozsváth**, **Z Szabó**, *An introduction to Heegaard Floer homology*, from “Floer homology, gauge theory, and low-dimensional topology”, Clay Math. Proc. 5, Amer. Math. Soc., Providence, RI (2006) 3–27 MR Zbl

[22] **P Ozsváth**, **Z Szabó**, *Lectures on Heegaard Floer homology*, from “Floer homology, gauge theory, and low-dimensional topology”, Clay Math. Proc. 5, Amer. Math. Soc., Providence, RI (2006) 29–70 MR Zbl

[23] **R Rustamov**, *On Heegaard Floer homology of plumbed three-manifolds with $b_1 = 1$*, preprint (2004) arXiv math/0405118

[24] **V Turaev**, *Torsions of 3-dimensional manifolds*, Progr. Math. 208, Birkhäuser, Basel (2002) MR Zbl

[25] **I Zemke**, *The equivalence of lattice and Heegaard Floer homology*, preprint (2021) arXiv 2111.14962

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Localization of a $\mathrm{KO}^*(\mathrm{pt})$-valued index and the orientability of the $\mathrm{Pin}^-(2)$ monopole moduli space

JIN MIYAZAWA

It is known that the Dirac index of a Spin^c structure is localized to the characteristic submanifold. We introduce the notion of $G^{\pm}(n, s^+, s^-)$ structure on a manifold as a common generalization of the Spin^c structure and the $H_n(s)$ structure defined by D Freed and M Hopkins, and formulate a version of characteristic submanifold for the $G^{\pm}(n, s^+, s^-)$ structure. We show that the $\mathrm{KO}^*(\mathrm{pt})$-valued index associated with the $G^{\pm}(n, s^+, s^-)$ structure is localized to the characteristic submanifold. As an application, we give a topological sufficient condition for the moduli space of $\mathrm{Pin}^-(2)$ monopoles to be orientable.

57K41, 58J20

1 Introduction

In this paper, we introduce the notion of $G^{\pm}(n, s^+, s^-)$ structure, which is a generalization of the Spin structure, the Spin^c structure, the $\mathrm{Pin}^{\pm}$ structure, and the $H_s(n)$ structure due to Freed and Hopkins [5]. We construct an elliptic differential operator associated with the $G^{\pm}(n, s^+, s^-)$ structure $\mathfrak{s}$, and we have its index $\mathrm{ind}(\mathfrak{s})$ with values in $\mathrm{KO}^{s^- - s^+ \pm n}(\mathrm{pt})$. The index $\mathrm{ind}(\mathfrak{s})$ is a generalization of the Atiyah–Milnor–Singer invariant which is defined by the Dirac type operator with the Clifford action for a Spin manifold. The index $\mathrm{ind}(\mathfrak{s})$ is also a generalization of the index of the $H_s(n)$ structure with values in $\mathrm{KO}^{-n-s}(\mathrm{pt})$ defined by Freed and Hopkins.

Our main theorem is that the index above is localized to a certain submanifold which is a generalization of a characteristic submanifold of the Spin^c structure.

Main Theorem *There exists a homomorphism f such that the diagram*

$$\begin{tikzcd} \Omega_n^{G^+(n,s^+,s^-)}(\mathrm{pt}) \arrow[r, "f"] \arrow[dr] & \Omega_{n-s^-}^{G^+(n-s^-,s^+,0)}(\mathrm{pt}) \arrow[d] \\ & \mathrm{KO}^{-n-s^++s^-}(\mathrm{pt}) \end{tikzcd}$$

commutes, where the morphisms to the KO *group are defined by the indices.*

The map f in the statement of Main Theorem is given by the localization.

Many people give generalizations of the localization of the index of the Spin^c structure to the characteristic submanifolds, for example W Zhang [18], J L Fast and S Ochanine [4], M Furuta and Y Kametani [8], and S Hayashi [9]. The methods of the localization of the index in the works of [4; 9] are to localize the topological index by using the excision theorem of K or KO theory, respectively. Our method of the localization is based on a version of the Witten deformation, which is introduced by E Witten [17] in 1982. The Witten deformation is an analytical counterpart of the excision.

The main application of the main theorem is to give a sufficient condition for the $\mathrm{Pin}^-(2)$ monopole moduli space to be orientable, which enables us to refine the $\mathrm{Pin}^-(2)$ monopole invariant. The $\mathrm{Pin}^-(2)$ monopole invariant is a variant of the Seiberg–Witten invariant introduced by N Nakamura [14; 15]. The orientability of the moduli spaces in gauge theory was originally studied for instanton by Donaldson [2; 3]. Donaldson's argument can be applied in the case of the singular instantons introduced by P B Kronheimer and T S Mrowka [12] and in the case of the $U(n)$ instantons introduced by Kronheimer [11]. In these cases, the moduli spaces are orientable. On the other hand, we show that the $\mathrm{Pin}^-(2)$ monopole moduli space may be nonorientable. Strictly speaking, we have an explicit example of a 4-manifold for which the determinant bundle on the ambient space of the moduli space is nontrivial (Corollary 5.20). We expect that our new method using the Witten deformation could be applied to other moduli spaces in gauge theory. Recently, Joyce, Tanaka and Upmeier [10] gave a new framework to deal with the orientation of moduli spaces. It is an interesting problem to understand the relation between their argument and ours.

This paper is organized as follows. In Section 2, we establish our conventions and define the index of the $G^{\pm}(n,s^+,s^-)$ structure. In Section 3, we first formulate the main theorem and give a proof in the rest of the section. The proof of the analytical details of the key localization is postponed to the appendix. In Section 4, we describe two examples in detail: the Freed–Hopkins $H_s(n)$ structure, and the $G^+(5,0,4)$ structure which we use in the next section. In Section 5, as an application, we give a topological sufficient condition for the $\mathrm{Pin}^-(2)$ monopole moduli space to be orientable and we give an example of a 4-manifold for which the determinant bundle on the ambient space which contains the $\mathrm{Pin}^-(2)$ monopole moduli space is nontrivial.

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2 Definition of the index

2.1 The $G^{\pm}(n, s^+, s^-)$ structures

Our purpose here is to establish our notation and conventions. We follow the notation of [13] for Clifford algebras and follow the definition of [1] for KO groups.

- Let Q_r denote the standard metric on $\mathbb{R}^r$. Let Q denote the quadratic form on $\mathbb{R}^l \oplus \mathbb{R}^m$ defined by

$$Q = Q_l - Q_m.$$

We will denote by $\mathrm{Cl}_{(l,m)}$ the Clifford algebra generated by $\{v \in \mathbb{R}^l \oplus \mathbb{R}^m\}$ subject to $v^2 = Q(v)$. We abbreviate $\mathrm{Cl}_{(n,0)}$ and $\mathrm{Cl}_{(0,n)}$ to Cl_n and Cl_{-n}, respectively. We write $\epsilon_1, \dots, \epsilon_l, e_1, \dots, e_m$ for the standard generators of $\mathrm{Cl}_{(l,m)}$.

- Note that $\mathrm{Cl}_{(l,m)}$ is naturally a $\mathbb{Z}/2$-graded algebra. Let

$$(\mathrm{Cl}_{(l,m)})^0 = \{a_1 a_2 \cdots a_{2k} \mid \forall i = 1, \dots, 2k,\ 0 \neq a_i \in \mathbb{R}^l \oplus \mathbb{R}^m\}$$

be the even part of $\mathrm{Cl}_{(l,m)}$ and

$$(\mathrm{Cl}_{(l,m)})^1 = \{a_1 a_2 \cdots a_{2k-1} \mid \forall i = 1, \dots, 2k-1, , 0 \neq a_i \in \mathbb{R}^l \oplus \mathbb{R}^m\}$$

be the odd part of $\mathrm{Cl}_{(l,m)}$. Here $\hat{\otimes}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product. Then there is an isomorphism $\mathrm{Cl}_{(l_1,m_1)} \hat{\otimes} \mathrm{Cl}_{(l_2,m_2)} \cong \mathrm{Cl}_{(l_1+l_2, m_1+m_2)}$.

- Let X be a compact Hausdorff space. A representative element of $\mathrm{KO}^{(b,a)}(X)$ is given by a four-tuple:

(1) The Clifford algebra $\mathrm{Cl}_{(b,a)}$.

(2) A separable $\mathbb{Z}/2$-graded Hilbert space H with a left $\mathrm{Cl}_{(b,a)}$ $\mathbb{Z}/2$-graded action. We denote by $\rho\colon \mathrm{Cl}_{(b,a)} \to \hom(H, H)$ the representation of the $\mathbb{Z}/2$-graded algebra. We assume this is continuous with respect to the norm topology of $\hom(H, H)$ which is a space of bounded operators on the Hilbert space H.

(3) Let us denote grading map by $\epsilon\colon H \to H$. Note that $\epsilon^2 = 1$.

(4) A continuous map $s\colon X \to \hom(H, H)$ (the topology of $\hom(H, H)$ is given by the operator norm) such that $s(x)$ is a bounded skew adjoint, odd and Fredholm operator which graded commutes with the $\mathrm{Cl}_{(b,a)}$ action for all $x \in X$.

We will write $(s, \epsilon, \mathrm{Cl}_{(b,a)}, (\rho, H))$ for this four-tuple. In cases where clarity permits, we use H as an abbreviation for (ρ, H).

- We define $(s, \epsilon, \mathrm{Cl}_{(b,a)}, H)$ and $(s', \epsilon', \mathrm{Cl}_{(b,a)}, H')$ as *equivalent* if they satisfy the following properties:
  - There exist four-tuples $(s_0, \epsilon_0, \mathrm{Cl}_{(b,a)}, H_0)$ and $(s_1, \epsilon_1, \mathrm{Cl}_{(b,a)}, H_1)$ such that both $\ker(s_0(x))$ and $\ker(s_1(x))$ are trivial for all $x \in X$. Note that this implies $s_0(x)$ and $s_1(x)$ are isomorphisms for all $x \in X$.
  - There exists an isometric linear map $f\colon H \oplus H_0 \to H' \oplus H_1$ such that $f \circ (\epsilon \oplus \epsilon_0) f^{-1} = (\epsilon' \oplus \epsilon_1)$ and $f \circ (s \oplus s_0) \circ f^{-1}$ and $s' \oplus s_1$ are homotopic through continuous maps from X to $\hom(H' \oplus H_1, H' \oplus H_1)$ which anticommute with $\epsilon' \oplus \epsilon_1$ and the $\mathrm{Cl}_{(b,a)}$ action.
  - The homotopy above anticommutes with $\epsilon' \oplus \epsilon_1$ and Clifford action.
- We define $\mathrm{KO}^{(b,a)}(X)$ by the set of the equivalence classes of four tuples $(s, \epsilon, \mathrm{Cl}_{(b,a)}, (\rho, H))$. We define

$$[(s, \epsilon, \mathrm{Cl}_{(b,a)}, (\rho, H))] + [(s', \epsilon', \mathrm{Cl}_{(b,a)}, (\rho, H'))] := \left[(s \oplus s', \epsilon \oplus \epsilon', \mathrm{Cl}_{(b,a)}, (\rho \oplus \rho', H \oplus H'))\right]$$

and it is easy to check that this operation is well defined. We can check that $\mathrm{KO}^{(b,a)}(X)$ is an abelian group under this operation $+$. Note that $[(s, \epsilon, \mathrm{Cl}_{(b,a)}, (\rho, H))] + [(-s, -\epsilon, \mathrm{Cl}_{(b,a)}, (-\rho, H))] = 0$, since we can deform $s \oplus -s$ to an odd, skew-adjoint, and isomorphic operator on $H \oplus H$ that graded commutes with $\mathrm{Cl}_{(b,a)}$.

- If s is an unbounded skew-adjoint Fredholm operator and

$$\tilde{s} = \frac{s}{\sqrt{1 + ss^*}}$$

satisfies the above properties, we write $(s, \epsilon, \mathrm{Cl}_{(b,a)}, H)$ instead of $(\tilde{s}, \epsilon, \mathrm{Cl}_{(b,a)}, H)$.

- We will denote by (ρ_1, V_1) the $\mathbb{Z}/2\mathbb{Z}$ graded representation of $\mathrm{Cl}_{(1,1)}$ which is given as follows. Let $V_1^0 = V_1^1 = \mathbb{R},\ V_1 = V_1^0 \oplus V_1^1$ and

$$\rho(\epsilon) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \rho(e) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$

where ϵ, e ($\epsilon^2 = 1,\ e^2 = -1$) are the generators of $\mathrm{Cl}_{(1,1)}$.

- We will let $V_n := V_1^{n\hat{\otimes}}$ and define (ρ_n, V_n) to be the representation introduced by the natural isomorphism $\mathrm{Cl}_{(n,n)} \cong \mathrm{Cl}_{(1,1)}^{n\hat{\otimes}}$. Let V_n^* denote the dual space of V_n which has a natural right $\mathrm{Cl}_{(n,n)}$ module. We will denote by ϵ^n the grading operator.

- Let $v_1^0 := 1 \in V_1^0$ and we set $v_n^0 := {v_1^0}^{n\otimes} \in V_n$. Then we see that

$$\{\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}v_n^0 \in V_n \mid i_1 < i_2 < \cdots < i_k,\ 0 \le k \le n\}$$

is basis of V_n. Also $\{{v_n^0}^* \circ \epsilon_{i_k} \circ \epsilon_{i_{k-1}} \circ \cdots \circ \epsilon_{i_1} \in V_n^* \mid i_1 < i_2 < \cdots < i_k,\ 0 \le k \le n\}$ is a basis of V_n^*.

Definition 2.1 If H is a left $\mathrm{Cl}_{(b,a)}$ module, we define the right $\mathrm{Cl}_{(a,b)}$ module structure as follows: Let ϵ be the grading operator on H and $\rho\colon \mathrm{Cl}_{(b,a)} \to \hom(H,H)$ be the representation that is given by the left $\mathrm{Cl}_{(b,a)}$ module structure. Let $\epsilon_1,\dots,\epsilon_b$ and $e_1,\dots,e_a$ be orthonormal bases of $\mathbb{R}^b$ and $\mathbb{R}^a$, respectively. Then we define the right action $\rho'\colon \mathrm{Cl}^{op}_{(a,b)} \to \hom(H,H)$ by

$$\phi\cdot\rho'(\epsilon_{i_1}\cdots\epsilon_{i_k}) := \rho(e_{i_k}\cdots e_{i_1})(\epsilon^k\phi), \quad \phi\cdot\rho'(e_{j_1}\cdots e_{j_l}) := \rho(\epsilon_{j_l}\cdots\epsilon_{j_1})(\epsilon^l\phi).$$

This is well-defined and independent of the choice of the orthonormal bases of $\mathbb{R}^b$ and $\mathbb{R}^a$. If H^* is a right $\mathrm{Cl}_{(a,b)}$ module, we can define a right $\mathrm{Cl}_{(b,a)}$ structure on H^* in the same way.

Lemma 2.2 *Assume $a \ge b$. Let $(s,\epsilon,\mathrm{Cl}_{(b,a)},H)$ be a representative element of $\mathrm{KO}^{(b,a)}(X)$. Then there exists an element $(s',\epsilon',\mathrm{Cl}_{(0,a-b)},H')$ of $\mathrm{KO}^{(0,a-b)}(X)$ which satisfies the following properties: There is an isomorphism between the Hilbert spaces*

$$f\colon H \to H' \mathbin{\hat\otimes} V_b^*$$

such that

$$\begin{aligned} s &= f^{-1}\circ s'\otimes\epsilon^b\circ f,\\ \epsilon &= f^{-1}\circ \epsilon'\otimes\epsilon^b\circ f,\\ \epsilon_i &= f^{-1}\circ 1\otimes\epsilon_i\circ f \qquad (\text{for } i=1,\dots,b),\\ e_i &= f^{-1}\circ 1\otimes e_i\circ f \qquad (\text{for } i=1,\dots,b),\\ e_{b+i} &= f^{-1}\circ e'_i\otimes\epsilon^b\circ f \qquad (\text{for } i=1,\dots,a-b), \end{aligned}$$

where $e'_1,\dots,e'_{a-b}$ are the generators of $\mathrm{Cl}_{(0,a-b)}$ and $\epsilon_1,\dots,\epsilon_b,e_1,\dots,e_b,e_{b+1},\dots,e_a$ are the generators of $\mathrm{Cl}_{(b,a)}$. Note that $\mathrm{Cl}_{(b,b)}$ action on V_n^* is the left action that is defined in Definition 2.1 by using the right $\mathrm{Cl}_{(b,b)}$ module.

Proof The subspace $H' \subset H$ is given by the intersection of $+1$-eigenspaces of $\epsilon_1e_1,\dots,\epsilon_be_b$. The actions of s,ϵ and $\mathrm{Cl}_{(0,a-b)}$ preserve H'. We write s',ϵ',e'_i $(i=1,\dots,a-b)$ for restriction of the actions of them to H'. Let us define a map g by

$$g\colon H'\mathbin{\hat\otimes}V_b^* \to H, \quad \phi'\otimes\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}{v_n^0}^* \mapsto \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}\phi'.$$

We can check that this is a map of the $\mathrm{Cl}_{(b,a)} \cong \mathrm{Cl}_{(0,a-b)}\mathbin{\hat\otimes}\mathrm{Cl}_{(b,b)}$ module. For example,

$$\begin{aligned} s\otimes\epsilon^b\cdot\phi'\otimes\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}{v_n^0}^* &= (-1)^k s\phi'\otimes\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}{v_n^0}^* \mapsto (-1)^k\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}s\phi' = s\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}\phi',\\ 1\otimes\epsilon_j\cdot\phi'\otimes\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}{v_n^0}^* &= \phi'\otimes\epsilon_j\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}{v_n^0}^* \mapsto \epsilon_j\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k}s\phi'. \end{aligned}$$

We set $f = g^{-1}$ and this proves the lemma. □

Remark 2.3 We have the option to substitute V_n for V_n^* in the assertion of Lemma 2.2. Nevertheless, in Proposition 3.11, the natural appearance is that of V_n^*. Therefore, we formulate Lemma 2.2 in the context of V_n^*.

Definition 2.4 We define a group $G^\pm(n,s^+,s^-)$ as

$$G^\pm(n,s^+,s^-)=\{g\in \mathrm{Cl}_{\pm n}\mathbin{\widehat{\otimes}}\mathrm{Cl}_{(s^+,s^-)}\mid g=v_{i_1}\cdots v_{i_{2k}},\ v_{i_j}\in\mathbb{R}^n \text{ or } \mathbb{R}^{s^+} \text{ or } \mathbb{R}^{s^-},\ |v_{i_j}|=1\}.$$

Definition 2.5 Let

$$S(O(n)\times O(s^+)\times O(s^-))=\{(A,B^+,B^-)\in O(n)\times O(s^+)\times O(s^-)\mid \det A\det B^+\det B^-=1\}.$$

The two-to-one homomorphism

$$p\colon G^\pm(n,s^+,s^-)\to S(O(n)\times O(s^+)\times O(s^-))$$

is defined by

$$p(g)v:=gvg^{-1}$$

for $g\in G^\pm(n,s^+,s^-)$, $v\in\mathbb{R}^n\oplus\mathbb{R}^{s^+}\oplus\mathbb{R}^{s^-}$.

We denote by p_n, p_{s^+} and p_{s^-} the compositions of p with the projections from $S(O(n)\times O(s^+)\times O(s^-))$ to each component $O(n)$, $O(s^+)$ and $O(s^-)$, respectively.

Definition 2.6 Let Y be an n dimensional Riemannian manifold. Let us denote by P_Y the orthogonal frame bundle of TY. A $G^\pm(n,s^+,s^-)$ structure is a tuple $(\widetilde{P},P,\pi,o,E_+,E_-)$ such that:

- $E_\pm$ is an $s^\pm$ dimensional real vector bundle such that their structure group is $O(s^\pm)$, respectively. We will denote by $P_{E_\pm}$ its frame bundle.
- o is an orientation of $TY\oplus E_+\oplus E_-$.
- P is a principal $S(O(n)\times O(s^+)\times O(s^-))$ bundle defined as the subbundle of $P_Y\times_Y P_{E_+}\times_Y P_{E_-}$:

  $$P=\{(f_n,f_+,f_-)\in P_Y\times_Y P_{E_+}\times_Y P_{E_-}\mid f_n,f_+,f_-\text{ are compatible with the orientation } o \text{ in this order}\}.$$

- We denote by $\widetilde{P}$ a principal $G^\pm(n,s^+,s^-)$ bundle and $\pi\colon\widetilde{P}\to P$ is a smooth map which satisfies the following commutative diagram:

  $$\begin{array}{ccc}\widetilde{P} & \xrightarrow{\cdot g} & \widetilde{P}\\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi}\\ P & \xrightarrow{\cdot p(g)} & P\end{array}$$

  for all $g\in G^\pm(n,s^+,s^-)$.

Definition 2.7 Let $(\widetilde{P}, P, \pi, o, E_+, E_-)$ and $(\widetilde{P}', P', \pi', o', E'_+, E'_-)$ be $G^{\pm}(n, s^+, s^-)$ structures on Y. We say that they are isomorphic if there exist isomorphisms of principal bundles $f: P \to P'$ and $\tilde{f}: \widetilde{P} \to \widetilde{P}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\tilde{f}} & \widetilde{P}' \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi'} \\ P & \xrightarrow{f} & P' \\ \big\downarrow{\scriptstyle p_n} & & \big\downarrow{\scriptstyle p_n} \\ P_Y & \xrightarrow{\mathrm{id}} & P_Y \end{array}$$

2.2 Definition of spinor bundles

Definition 2.8 A generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation of $G^+(n, s^+, s^-)$ is a pair (α, S) such that:

- S is a real vector space with a metric and a $\mathbb{Z}/2\mathbb{Z}$ grading $S = S_0 \oplus S_1$.
- α is a representation $\alpha: G^{\pm}(n, s^+, s^-) \to O(S)$ such that for all $g \in G^{\pm}(n, s^+, s^-)$, $\alpha(g)$ preserves the $\mathbb{Z}/2\mathbb{Z}$ grading of S.
- The representation space S has a $G^{\pm}(n, s^+, s^-)$ equivariant products
$$c'_n: \mathbb{R}^n \times S \to S, \quad c'_{s^+}: \mathbb{R}^{s^+} \times S \to S, \quad c'_{s^-}: \mathbb{R}^{s^-} \times S \to S,$$
such that they are anticommutative each other and odd. Here, the action of $G^{\pm}(n, s^+, s^-)$ on $\mathbb{R}^n, \mathbb{R}^{s^+}, \mathbb{R}^{s^-}$ is the left adjoint representations p_n, p_+, p_- as defined in Definition 2.4. Note that $p_n(g)v := gvg^{-1}$ for $g \in G^+(n, s^+, s^-)$ and $v \in \mathbb{R}^n$. We have similar formula for p_+ and p_-.
- The multiplications above give a $\mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)}$ module structure on S.

Moreover, if S has an additional left Clifford action of $\mathrm{Cl}_{(b,a)}$ and its generators anticommutes with c'_n, c'_{s^+}, c'_{s^-}, we call (α, S) a generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation with $\mathrm{Cl}_{(b,a)}$ action.

Definition 2.9 We define $(\rho, \mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)})$ to be the generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation with left $\mathrm{Cl}_{\mp n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^-, s^+)}$ action as follows: Take $g \in G^{\pm}(n, s^+, s^-)$ and $\phi \in \mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)}$. We define $\rho(g)\phi := g\phi$, where the right hand side is a multiplication of $\mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)}$. The representation ρ preserves the $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)}$. We define
$$c'_n: \mathbb{R}^n \times \mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)} \to \mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)}$$
by $c'_n(v)\phi = v\phi$. Replacing $\mathbb{R}^n$ with $\mathbb{R}^{s\pm}$, we define c'_{s_+}, c'_{s_-} similarly. We define the additional left $\mathrm{Cl}_{\mp n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^-, s^+)}$ action in the following way. Let ϵ be the $\mathbb{Z}/2\mathbb{Z}$ grading operator of $\mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)}$. We define $v \cdot \phi := (\epsilon\phi)v$ for $v \in \mathbb{R}^n \oplus \mathbb{R}^{s^+} \oplus \mathbb{R}^{s^-}$, where the right hand side is a multiplication of $\mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)}$. This defines a left $\mathrm{Cl}_{\mp n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^-, s^+)}$ action because the right $\mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)}$ action is odd and v anticommutes with ϵ.

Definition 2.10 Let $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^{\pm}(n, s^+, s^-)$ structure on Y and let (α, S) be a generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation of $G^+(n, s^+, s^-)$. Then they define the vector bundle
$$\not{S} = \widetilde{P} \times_\alpha S.$$
Let $\not{S}_0$ and $\not{S}_1$ be subbundles of $\not{S}$ defined by S_0 and S_1, respectively. We define the Clifford multiplication on $\not{S}$ using c'_n, c'_{s^+}, c'_{s^-}:
$$c_{TY}\colon TY \times \not{S} \to \not{S}, \quad c_{E_\pm}\colon E_\pm \times \not{S} \to \not{S}.$$
We call $\not{S}$ a generalized $\mathbb{Z}/2$ graded spinor bundle. If (α, S) is a generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation with $\mathrm{Cl}_{(b,a)}$ action, we call $\not{S}$ a generalized $\mathbb{Z}/2$ graded spinor bundle with $\mathrm{Cl}_{(b,a)}$ action.

Remark 2.11 From Definition 2.1, if $\not{S}$ has a left $\mathrm{Cl}_{(b,a)}$ action, we have a right $\mathrm{Cl}_{(a,b)}$ action on $\not{S}$ as we defined in Definition 2.1. Note that this right action commutes with $c_{TY}, c_{E_\pm}$ even if odd elements in $\mathrm{Cl}_{(a,b)}$.

Definition 2.12 Let $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^{\pm}(n, s^+, s^-)$ structure on Y. We will denote by $\not{\mathfrak{S}} = \not{\mathfrak{S}}_0 \oplus \not{\mathfrak{S}}_1$ the generalized $\mathbb{Z}/2$ graded spinor bundle with $\mathrm{Cl}_{\mp n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^-, s^+)}$ action defined by the generalized $\mathbb{Z}/2\mathbb{Z}$ graded spinor representation with $\mathrm{Cl}_{\mp n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^-, s^+)}$ action $(\rho, \mathrm{Cl}_{\pm n} \mathbin{\hat{\otimes}} \mathrm{Cl}_{(s^+, s^-)})$. We call $\not{\mathfrak{S}}$ the standard spinor bundle of $\mathfrak{s}$.

Definition 2.13 Let $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^{\pm}(n, s^+, s^-)$ structure on Y and $\not{\mathfrak{S}}$ the standard spinor bundle of $\mathfrak{s}$. We define a Dirac type operator $\not{D}$ on $\Gamma(\not{\mathfrak{S}})$ by using the Clifford multiplication c_{TY}. We will denote by ϵ the $\mathbb{Z}/2\mathbb{Z}$ grading operator on $\not{\mathfrak{S}}$.
Let us denote by $\mathrm{ind}(\mathfrak{s})$ the element in $\mathrm{KO}^{-n \pm (s^- - s^+)}(\mathrm{pt})$ defined by the representative element
$$(\not{D}, \epsilon, \mathrm{Cl}_{\mp n} \otimes \mathrm{Cl}_{(s^-, s^+)}, L^2(Y, \not{\mathfrak{S}})).$$
We call $\mathrm{ind}(\mathfrak{s})$ the index of the $G^{\pm}(n, s^+, s^-)$ structure $\mathfrak{s}$.

Remark 2.14 The index defined above coincides with the so called Atiyah–Milnor–Singer invariant index of Spin structures when $s^+ = s^- = 0$. This index is a generalization of the mod 2 index of Spin structures on a riemannian surface and the $\hat{A}$ genus. This is defined explicitly by Lawson and Michelsohn [13, Chapter II, Section 7]. In some cases, the index coincides with the index of $H_s(n)$ structure introduced in [5].

3 Proof of Main Theorem

3.1 Statement of Main Theorem

Main Theorem *There exists a homomorphism f such that the diagram*
$$\begin{tikzcd}
\Omega_n^{G^+(n,s^+,s^-)}(\mathrm{pt}) \arrow[r, "f"] \arrow[dr] & \Omega_{n-s^-}^{G^+(n-s^-,s^+,0)}(\mathrm{pt}) \arrow[d] \\
& \mathrm{KO}^{-n-s^+ + s^-}(\mathrm{pt})
\end{tikzcd}$$
commutes, where the morphisms to the KO *group are defined by the indices.*

Remark 3.1 We can find a similar commutative diagram of G^- bordism groups. In this case, the codomain target of f is $\Omega_{n-s^+}^{G^-(n-s^+,0,s^-)}(\mathrm{pt})$ instead of $\Omega_{n-s^-}^{G^+(n-s^-,s^+,0)}(\mathrm{pt})$. The proof is parallel to that of the G^+ cases. Moreover, we can prove $G^+(n,s^+,s^-) \cong G^-(n,s^-,s^+)$ and hence we will study only the G^+ cases.

First, we will construct the morphism f in the main theorem in this subsection. Second, we will prove a localization theorem of the indices in Section 3.2. Some analytic lemmas are proved in the appendix. Finally, we prove the commutativity of the diagram using the localization theorem in the final part of this section and we will complete the proof of the main theorem.

Lemma 3.2 *Let $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^\pm(n,s^+,s^-)$ structure on Y. Then there exists a canonical* Spin *structure of the vector bundle*

$$(\det TY \otimes \det E_+) \oplus TY \oplus E_+ \oplus \det E_- \oplus E_-$$

induced by $\mathfrak{s}$. Conversely, if the vector bundle above has an Spin *structure, it gives a canonical $G^+(n,s^+,s^-)$ structure on Y.*

Proof Let $\epsilon_1,\dots,\epsilon_{n+s^+}$ is a standard basis of $\mathbb{R}^n \oplus \mathbb{R}^{s^+}$ and $e_0', e_0, e_1, \dots, e_{n+s^++s^-}$ are the generators of $\mathrm{Cl}(0,(n+1)+(s^++1)+s^-)$. There is an embedding $\mathrm{Cl}(n+s^+,s^-) \to \mathrm{Cl}(0,(n+1)+(s^++1)+s^-)$ given by mapping $\epsilon_1,\dots,\epsilon_{n+s^+}$ to $e_0'e_0e_1,\dots,e_0'e_0e_{n+s^+}$. Restricting this map, we have an embedding $G^+(n,s^+,s^-) \to \mathrm{Spin}(1+n+s^++1+s^-)$. Let π be the projection $\mathrm{Spin}(1+n+s^++1+s^-) \to \mathrm{SO}(1+n+s^++1+s^-)$. The image $\pi(G^+(n,s^+,s^-))$ is isomorphic to $S(O(n)\times O(s^+)\times O(s^-))$. This image gives the representation of $S(O(n)\times O(s^+)\times O(s^-))$ whose associated vector bundle is $\det TY \otimes \det E_+ \oplus TY \oplus E_+ \oplus \det E_- \oplus E_-$. Here we define that

$$\langle e_0\rangle, \quad \langle e_1,\dots,e_n\rangle, \quad \langle e_{n+1},\dots,e_{n+s^+}\rangle, \quad \langle e_0'\rangle \quad \text{and} \quad \langle e_{n+s^++1},\dots,e_{n+s^++s^-}\rangle$$

are the representations such that the associated vector bundle of the representations are $\det TY \otimes \det E_+$, TY, E_+, $\det E_-$ and E_- respectively. Then we construct a Spin structure as desired. We have the second half of the statement by reversing the proof above. □

Remark 3.3 The isomorphism class of the $G^+(n,s^+,s^-)$ structure given by the second half of the Lemma 3.2 uniquely determined by the isomorphism class of the Spin structure of the vector bundle $(\det TY \otimes \det E_+) \oplus TY \oplus E_+ \oplus \det E_- \oplus E_-$. On the other hand, if we change the $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$ by an isomorphism of the $G^+(n,s^+,s^-)$ structure, the isomorphism class of the Spin structure given by the first half of Lemma 3.2 may change because there may exist an automorphism of the $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$ which covers a nontrivial isomorphism of $E_\pm$. We use this lemma to define the cobordism class of the $G^+(n-s^-,s^+,0)$ structure $\mathfrak{s}_C$. If two $G^+(n,s^+,s^-)$ structures $\mathfrak{s}$ and $\mathfrak{s}'$ are isomorphic they are cobordant. From the Lemma 3.9, $\mathfrak{s}_C$ and $\mathfrak{s}_C'$ are cobordant therefore this ambiguity does not matter to define the map f in the statement of the main theorem.

We will prove some lemmas in a general setting. In the following three lemmas, let M be a manifold and F, E_0, E_1 be oriented real vector bundles on M with fiber metrics.

Lemma 3.4 *An orientation preserving isometry $\alpha\colon E_0 \to E_1$ determines a canonical* Spin *structure on $E_0 \oplus E_1$.*

Proof Let n be the rank of E_0. The structure group of the subbundle

$$\{(f, \alpha(f)) \mid f \text{ is an oriented orthonormal frame of } E_0\}$$

of the frame bundle of $E_0 \oplus E_1$ is a subgroup of $\mathrm{SO}(2n)$ which is the image of the diagonal embedding $j\colon \mathrm{SO}(n) \to \mathrm{SO}(n) \times \mathrm{SO}(n) \hookrightarrow \mathrm{SO}(2n)$. The map $j_*\colon \pi_1(\mathrm{SO}(n)) \to \pi_1(\mathrm{SO}(2n))$ is trivial, and therefore there exists a homomorphism $\tilde{j}\colon \mathrm{SO}(n) \to \mathrm{Spin}(2n)$ which covers j. Let$\{g_{\alpha\beta}\}$ be the transition functions of $E_0 \oplus E_1$. We define the Spin structure on $E_0 \oplus E_1$ by using $\{\tilde{j}(g_{\alpha\beta})\}$ as transition functions. It is easy to check that this Spin structure does not depend on how to take transition functions of $E_0 \oplus E_1$. □

Lemma 3.5 *Let F, F' denote oriented real vector bundles on M with metrics. If there are* Spin *structures on $F \oplus F'$ and F', these* Spin *structures determine a canonical* Spin *structure on F.*

Proof We will denote by $\{g_{\alpha\beta}\}, \{g'_{\alpha\beta}\}$ transition functions of F and F', respectively. Let $\{\tilde{g}'_{\alpha\beta}\}$ be the lifts of $\{g'_{\alpha\beta}\}$ to the Spin group defined by the Spin structure on F'. The lift of $(g_{\alpha\beta}, g'_{\alpha\beta})$ defined by the Spin structure of $F \oplus F'$ is expressed by $[(\tilde{g}_{\alpha\beta}, \tilde{g}'_{\alpha\beta})] \in \mathrm{Spin}(n) \times \mathrm{Spin}(m)/(\mathbb{Z}/2\mathbb{Z})$, where $\tilde{g}_{\alpha\beta}$ is a lift of $g_{\alpha\beta}$. The lift $\tilde{g}_{\alpha\beta}$ is unique because the second component is fixed to be $\tilde{g}'_{\alpha\beta}$.

The lift of the transition functions $\{\tilde{g}_{\alpha\beta}\}$ satisfy the cocycle condition because $\{[(\tilde{g}_{\alpha\beta}, \tilde{g}'_{\alpha\beta})]\}$ and $\{\tilde{g}'_{\alpha\beta}\}$ do. We define the Spin structure on F by using $\{\tilde{j}(g_{\alpha\beta})\}$ as transition functions. Again it is easy to check that this Spin structure does not depend on how to take transition functions of F. □

The lemma below is clear from Lemmas 3.4 and 3.5.

Lemma 3.6 *Let $F \oplus E_0 \oplus E_1$ be a vector bundle with orientation o and* Spin *structure $\widetilde{P}$. We assume that E_1 is oriented and there is an isometry $\phi\colon E_0 \to E_1$. We will denote by o_E the orientation of E_0. Then F has a natural orientation and a* Spin *structure defined by $o, o_E, \widetilde{P}$ and ϕ.*

We use the lemmas above in our setting.

Definition 3.7 Let Y be an n dimensional Riemannian manifold and $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^+(n, s^+, s^-)$ structure on Y. We will denote by $h \in \Gamma(E_-)$ an transverse section. Let $C = h^{-1}(0)$. We call C a characteristic submanifold of $\mathfrak{s}$.

The theorem below is a generalization of the theorem that is an existence of a canonical Spin structure on a characteristic submanifold of Spin^c structure.

Theorem 3.8 *Let Y be an n dimensional Riemannian manifold and $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$ be a $G^+(n, s^+, s^-)$ structure on Y and C be a characteristic submanifold of $\mathfrak{s}$. Then we have a natural $G^+(n - s^-, s^+, 0)$ structure on C.*

Proof We restrict the vector bundle

$$\det TY \otimes \det E_+ \oplus TY \oplus E_+ \oplus \det E_- \oplus E_-$$

to C. This is naturally isomorphic to the vector bundle

$$\det TY|_C \otimes \det E_+|_C \oplus TC \oplus N \oplus E_+|_C \oplus \det E_-|_C \oplus E_-|_C,$$

where N is the normal bundle of C. We define the isomorphism $\psi\colon N \to E_-$ by using dh. By the definition of $G^+(n, s^+, s^-)$ structure, there exists an isomorphism $i\colon \det TY|_C \otimes \det E_+|_C \cong \det E_-|_C$. Let $F = TC \oplus E_+|_C$, $E_0 = \det TY|_C \otimes \det E_+|_C \oplus N$, $E_1 = \det E_-|_C \oplus E_-|_C$, and $\phi = i \oplus \psi$. They satisfy the assumption of Lemma 3.6, and hence we have the Spin structure of $TC \oplus E_+|_C$ as desired. From the latter statement of Lemma 3.2, this Spin structure defines the $G^+(n - s^-, s^+, 0)$ structure on C as desired. □

Lemma 3.9 *As an element of bordism group $\Omega^{G^+(n-s^-,s^+,0)}_{n-s^-}(\mathrm{pt})$, $[(C, \mathfrak{s}_C(h))]$ is independent of h. Moreover, if the $G^+(n, s^+, s^-)$ structure $\mathfrak{s}_0$ on a closed manifold Y_0 and $\mathfrak{s}_1$ on a closed manifold Y_1 are $G^+(n, s^+, s^-)$ cobordant, $G^+(n - s^-, s^+, 0)$ bordism class of $[(C, \mathfrak{s}_C(h))]$ given by $\mathfrak{s}_0$ and $\mathfrak{s}_1$ are same.*

Proof We first prove the first half of the statement. Let h, h' be transverse sections of E_- and let $C = h^{-1}(0)$, $C' = h^{-1}(0)$. We define a natural $G^+(n + 1, s^+, s^-)$ structure on $Y \times [0, 1]$ by using $G^+(n, s^+, s^-)$ structure on Y. Let $\tilde{h}$ be a transverse section on $Y \times [0, 1]$ such that $\tilde{h}|_{Y \times \{0\}} = h$, $\tilde{h}|_{Y \times \{1\}} = h'$. Then $\tilde{h}^{-1}(0)$ has a $G^+(n + 1 - s^-, s^+, 0)$ structure by Theorem 3.8. This manifold with $G^+(n + 1 - s^-, s^+, 0)$ structure gives the cobordism from $[(C, \mathfrak{s}_C(h))]$ to $[(C', \mathfrak{s}'_C(h'))]$. A slight change to the above proof actually shows the second half of the statement. □

We now define the map f of Main Theorem.

Definition 3.10 Let Y be an n dimensional closed manifold with $G^+(n, s^+, s^-)$ structure given by $\mathfrak{s} = (\widetilde{P}, P, \pi, o, E_+, E_-)$. Set

$$f([Y, \mathfrak{s}]) = [C, \mathfrak{s}_C].$$

3.2 Localization by Witten deformation

In this subsection, we will prove the commutativity of the diagram in Main Theorem. We prove it by using Witten deformation. We prove some lemmas in a more general setting than the setting in Main Theorem. The proofs in this section and in the appendix are based on the localization argument of Furuta [7] and Fukaya, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita [6]. We simplify and modify these arguments to suit the situation in this paper.

In the previous subsection, we define the $G^+(n-s^-,s^+,0)$ structure $\mathfrak{s}_C$ on a characteristic submanifold $C=h^{-1}(0)$ from a $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$ on Y in Theorem 3.8. Now we consider the relation between the standard spinor bundle of $\mathfrak{s}_C$ and the restriction of the standard spinor bundle $\mathfrak{S}$ of $\mathfrak{s}$ to C. First, we consider $\mathfrak{S}|_C$, the restriction of the standard $\mathbb{Z}/2\mathbb{Z}$ graded spinor bundle of $\mathfrak{s}$ to C.

Proposition 3.11 *Let Y be an n dimensional manifold with $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$. We will denote by $\mathfrak{S}$ the standard $\mathbb{Z}/2\mathbb{Z}$ graded spinor bundle of $\mathfrak{s}$. Let $C=h^{-1}(0)$ and let $\mathfrak{s}_C$ be the $G^+(n-s^-,s^+,0)$ structure on C defined in Theorem 3.8. We define a Clifford action on $\mathfrak{S}|_C$ by restricting the Clifford action $(c_{TY},c_{E_+})\colon (TY\oplus E_+)\otimes\mathfrak{S}\to\mathfrak{S}$ to the vector bundle $(TC\oplus E_+|_C)\otimes\mathfrak{S}|_C$ on C. We will denote by $(\tilde{c}_{TC},\tilde{c}_{E_+})$ this restricted Clifford action. Note that the image of $(\tilde{c}_{TC},\tilde{c}_{E_+})$ is $\mathfrak{S}|_C$. Then there exists an isomorphism Φ which has following properties: Let $\mathfrak{S}_C$ be the standard $\mathbb{Z}/2\mathbb{Z}$ graded spinor bundle of $\mathfrak{s}_C$. The isomorphism*

$$\Phi\colon \mathfrak{S}|_C\to \mathrm{Cl}_+(N(C))\mathbin{\hat{\otimes}}\mathfrak{S}_C\mathbin{\hat{\otimes}}V^*_{s^-}$$

preserves the Clifford action of $TC\oplus E_+|_C$ when we define a Clifford action to right hand side by

$$c_{TC}(x)\cdot\omega\otimes\phi\otimes v:=(-1)^{|\omega|}\omega\otimes\tilde{c}_{TC}(x)\phi\otimes v,$$
$$c_{E_+}(w)\cdot\omega\otimes\phi\otimes v:=(-1)^{|\omega|}\omega\otimes\tilde{c}_{E_+}(w)\phi\otimes v$$

for $\omega\otimes\phi\otimes v\in\mathrm{Cl}_+(N(C))\mathbin{\hat{\otimes}}\mathfrak{S}_C\mathbin{\hat{\otimes}}V^*_{s^-}$, $x\in TC$ and $w\in E_+|_C$. Moreover, Φ preserves the right $\mathrm{Cl}_{(n+s^+,s^-)}$ action if we define the right $\mathrm{Cl}_{(n+s^+,s^-)}$ action to $\mathrm{Cl}_+(N(C))\mathbin{\hat{\otimes}}\mathfrak{S}_C\mathbin{\hat{\otimes}}V^*_{s^-}$ by

$$\omega\otimes\phi\otimes v\cdot a\otimes b:=(-1)^{|a||v|}\omega\otimes\phi\cdot a\otimes vb$$

for $\omega\otimes\phi\otimes v\in\mathrm{Cl}_+(N(C))\mathbin{\hat{\otimes}}\mathfrak{S}_C\mathbin{\hat{\otimes}}V^*_{s^-}$ and $a\otimes b\in\mathrm{Cl}_{(n+s^+-s^-,0)}\mathbin{\hat{\otimes}}\mathrm{Cl}_{(s^-,s^-)}$.

Proof From the proof of Theorem 3.8, we see the structure group of the Spin structure of

$$\det N(C)\oplus\det E_-\oplus N(C)\oplus E_-$$

reduces the subgroup of $\mathrm{Spin}(2s^-+2)$ which is isomorphic to $O(s^-)$. If we write $O(s^-)$ for the subgroup, then the structure group of $\mathfrak{S}|_C$ is $G^+(n-s^-,s^+,0)\times O(s^-)$. Write $G=G^+(n-s^-,s^+,0)\times O(s^-)$.

Let us denote by $S=\mathrm{Cl}_{(n+s^+,s^-)}$ a representation space whose associated vector bundle is $\mathfrak{S}$. We describe the action of G on S. We define $\tilde{g}_v\in O(s^-)$ to be the natural lift of g_v to $\mathrm{Spin}(2s^-+2)$ given in Lemma 3.4 where g_v is the composition of the reflections of vectors$(e,0,e,0),(0,v,0,v)\in\det\mathbb{R}^{s^-}\oplus\mathbb{R}^{s^-}\oplus\det\mathbb{R}^{s^-}\oplus\mathbb{R}^{s^-}$ ($|e|=|v|=1$). The element g_v is independent of the choice of e. Elements in the group $O(s^-)\subset\mathrm{Spin}(2s^-+2)$ are the products of finitely many elements of the form $\tilde{g}_v$ for some v and ±1. The Clifford multiplications c'_n,c'_- are G equivariant and hence the action of $\tilde{g}_v$ is given by $\pm c'_n(v)c'_-(v)$.

The sign of $\tilde{g}_v=\pm c'_n(v)c'_-(v)$ is determined by how to embed the group $G^+(n,s^+,s^-)$ to a Spin group. In our convention of Lemma 3.2, the image of $c'_n(e_i)c'_-(e_i)$ of the embedding is $e_0e_ie'_0e'_i$ and this coincides with $\tilde{g}_{e_i}$ which is a lift of g_v given in Lemma 3.4. Thus we have $\tilde{g}_v=c'_n(v)c'_-(v)$.

We will denote by S_C a subspace of S which is the intersection of $+1$-eigenspaces of $\tilde{g}_v = c'_n(v)c'_-(v)$ for all $v \in S(\mathbb{R}^{s^-})$. The subspace S_C coincides with the intersections of $+1$-eigenspaces of $c'_n(e_i)c'_-(e_i)$ ($i = 1, \dots, s^-$), where $\{e_1, \dots, e_{s^-}\}$ is an orthonormal basis of $\mathbb{R}^{s^-}$. The action of $G^+(n - s^-, s^+, 0)$ preserves S_C because this action commutes with $c'_n(v)c'_-(v)$ for all v. The definition of S_C immediately implies that the action of $O(s^-)$ on S_C is trivial.

Let us show that S_C coincides with $\mathrm{Cl}_{(n+s^+-s^-,0)} \subset S$. Recall that the standard spinor bundle $\mathfrak{S}$ is the associated bundle with the representation $\mathrm{Cl}_{(n+s^+,s^-)} = \mathrm{Cl}_{(n-s^-+s^+,0)} \hat{\otimes} \mathrm{Cl}_{(s^-,s^-)}$. An element $\tilde{g}_v = c'_n(v)c'_-(v)$ acts on $\mathrm{Cl}_{(n-s^-+s^+,0)}$ trivially and on $\mathrm{Cl}_{(s^-,s^-)}$ by left multiplication of $\mathrm{Cl}_{(s^-,s^-)}$. We will denote by $\widetilde{V}$ the intersection of the $+1$-eigenspaces of $c'_n(v)c'_-(v)$ in $\mathrm{Cl}_{(s^-,s^-)}$ for all $v \in S(\mathbb{R}^{s^-})$. The dimension of $\widetilde{V}$ is 2^{s^-}. The action of $c'_n(v)c'_-(v)$ commutes with the right $\mathrm{Cl}_{(s^-,s^-)}$ action. The only irreducible representation of $\mathrm{Cl}_{(s^-,s^-)}$ is V_{s^-} and its dimension is 2^{s^-}. Hence there is an isomorphism $\psi\colon V_{s^-}^* \to \widetilde{V}$. Using ψ, we have the isomorphism $\mathrm{Cl}_{(n+s^+-s^-,0)} \hat{\otimes} V_{s^-}^* \to S_C$ by $\xi \otimes v \mapsto \xi\psi(v)$. If we give the G action on $\mathrm{Cl}_{(s^-,0)}$ by $g \cdot x \mapsto gxg^{-1}$, we have the G equivariant isomorphism

$$\Psi_0\colon \mathrm{Cl}_{(s^-,0)} \hat{\otimes} \mathrm{Cl}_{(n+s^+-s^-,0)} \hat{\otimes} V_{s^-}^* \to \mathrm{Cl}_{(n+s^+,s^-)}, \quad x \otimes \xi \otimes v \mapsto x \cdot \xi \cdot \psi(v).$$

The associated vector bundles of $\mathrm{Cl}_{(s^-,0)}, \mathrm{Cl}_{(n+s^+-s^-,0)}$ and $V_{s^-}^*$ is $\mathrm{Cl}_+(N(C))$, $\mathfrak{S}_C$ and the trivial bundle $V_{s^-}^*$, respectively. Thus we have the map

$$\Psi\colon \mathrm{Cl}_+(N(C)) \hat{\otimes} \mathfrak{S}_C \hat{\otimes} V_{s^-}^* \to \mathfrak{S}|_C.$$

We define the map Φ to be the inverse of the map Ψ. It is easy to check to see that Φ preserves the Clifford action of $TC \otimes E_+|_C$ and right $\mathrm{Cl}_{(n+s^+,s^-)}$ action. □

In this section we identify $\mathfrak{S}|_C$ with $\mathrm{Cl}_+(N(C)) \hat{\otimes} \mathfrak{S}_C \hat{\otimes} V_{s^-}^*$ by the map Φ.

Next we prove the localization of analytic index in Proposition 3.22. We identify a tubular neighborhood of C, denoted by $U(C)$, with the open disk bundle of the normal bundle of C,

$$B(N(C)) = \{v \in N(C) \mid |v| < 1\}.$$

We will denote by $\pi\colon N(C) \to C$ the projection of the normal bundle. Let us outline the proof of the localization.

Step 1 We first formulate the index of a Dirac type operator acting on the sections of the vector bundle $\pi^*(\mathfrak{S}|_C)$ over $N(C)$. Since $N(C)$ is not closed, we need behavior of its end. We will see the index of $N(C)$ coincides with the index of the $G^+(n, s^-, s^+)$ structure $\mathfrak{s}_C$ on C.

Step 2 The analytic index of the $G^+(n, s^+, s^-)$ structure $\mathfrak{s}$ on Y coincides with the index of **Step 1**. In this argument we deform the Dirac type operator in Definition 2.13. We prove some analytic lemmas and propositions of this step in the appendix.

3.2.1 We begin with Step 1 First, we consider the trivial $G^+(n,0,n)$ structure on $\mathbb{R}^n$. This structure appears in a fiberwise way when we consider the operator on $N(C)$. Let $\mathfrak{s}_0$ be a trivial $G^+(n,0,n)$ structure on $\mathbb{R}^n$. We will denote by $h(x)=x$ the section of $E_-=\mathbb{R}^n\times\mathbb{R}^n$ and we have the isomorphism from $T\mathbb{R}^n$ to E_- by using dh. We have the reduction of the structure group of $\mathfrak{s}_0$ to the subgroup $G\subset G^-(n,0,n)$ which is naturally isomorphic to $O(n)$ by using Lemma 3.4. Let $\mathfrak{S}_f$ denote the standard $\mathbb{Z}/2$ graded spinor bundle of $\mathfrak{s}_0$. Let $C=h^{-1}(0)=\{0\}$, be a characteristic submanifold. From Proposition 3.11 we identify $\mathfrak{S}_f|_C$ with $\mathrm{Cl}_+(\mathbb{R}^n)\mathbin{\hat{\otimes}}\underline{\mathbb{R}}\mathbin{\hat{\otimes}}V_n^*$. Note that $\mathfrak{S}_C\cong C\times\mathbb{R}=\underline{\mathbb{R}}$ is a vector bundle on a point. Let L be a trivial vector bundle on $\mathbb{R}^n$.

Lemma 3.12 *We will denote by $\mathcal{S}$ a set of rapidly decreasing sections of $\mathfrak{S}_f\otimes L$. We will denote by D_m' a differential operator acting on sections of $\Gamma(\mathfrak{S}_f\otimes L)$ given by*

$$D_m'=\sum_{i=1}^n\Big(c_{T\mathbb{R}^n}(dx^i)\frac{\partial}{\partial x^i}+mc_{E_-}(e_i)x^i\Big).$$

Note that D_m' preserves the subspace $\mathcal{S}\subset\Gamma(\mathfrak{S}_f\otimes L)$. The operator D_m' is independent of the choice of the orthonormal basis of $\mathbb{R}^n$. This operator commutes with the right $\mathrm{Cl}_{(n,n)}$ action, and there is an isomorphism between $\ker D_m'\cap\mathcal{S}$ and $\underline{\mathbb{R}}\mathbin{\hat{\otimes}}V_n^*\otimes L$ as right $\mathrm{Cl}_{(n,n)}$ modules.

Proof We see at once that D_m' is independent of choice of the orthonormal basis of $\mathbb{R}^n$ by direct calculation.

Let ϕ be a rapidly decreasing section. We have $D_m'\phi=0$ if and only if $(D_m')^2\phi=0$ because D_m' is a skew-symmetric operator. We have

$$(D_m')^2=\sum_{i=1}^n\Big(\Big(\frac{\partial}{\partial x^i}\Big)^2-m^2(x^i)^2+mc_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)\Big)=-H+\sum_{i=1}^n(mc_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)),$$

where H is a harmonic oscillator acting on smooth functions $\mathbb{R}^n\to\mathbb{R}$. It is well known that H has only discrete spectrum and each eigenspace is 1-dimensional. The eigenvalues are $nm,2nm,3nm,\dots$ and each eigenfunction is rapidly decreasing. In particular, the eigenfunction of nm is $e^{-m|x|^2/2}$ and this does not depend on the choice of orthonormal basis of $\mathbb{R}^n$. The eigenvalues of $mc_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)$ are $\pm m$ for all i. Hence the kernel of $(D_m')^2$ is in the intersection of the $+1$ eigenspace of $c_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)$ and nm eigenspace of H. The intersection of the $+1$ eigenspace of $c_{T\mathbb{R}^n}(dx^i)c_{E_-}(e_i)$ for all i coincides with $\underline{\mathbb{R}}\mathbin{\hat{\otimes}}V_n^*\otimes L$ from the construction of the map Φ on Proposition 3.11. Thus we have

$$\ker D_m'\cap\mathcal{S}=\underline{\mathbb{R}}\mathbin{\hat{\otimes}}V_n^*\otimes L\cdot e^{-m|x|^2/2}.$$ □

Definition 3.13 We will denote by $\tilde{u}_0$ a function on $\mathbb{R}^n$ given by

$$\frac{e^{-m|x|^2/2}}{\|e^{-m|x|^2/2}\|_{L^2(\mathbb{R}^n)}}.$$

Definition 3.14 We will follow the notation of the statement and the proof of Proposition 3.11. We deform a metric of Y if necessary and we identify a tubular neighborhood $U(C)$ of C with $B(N(C)) = \{v \in N(C) \mid |v| < 1\}$ as a Riemannian manifold. Let us denote by $\pi\colon N(C) \to C$ the projection. We define a $G^+(n,s^+,s^-)$ structure on $B(N(C))$ by restricting the $G^+(n,s^+,s^-)$ structure $\mathfrak{s}$ on Y. By abuse of notation, we use the same letter $\mathfrak{s}$ for this $G^+(n,s^+,s^-)$ structure on $B(N(C))$ and we write $\mathfrak{S}$ instead of $\mathfrak{S}|_{U(C)}$. The structure group of $\mathfrak{s}$ reduces to $G = G^+(n-s^-, s^+, 0) \times O(s^-)$ by using the isomorphism $dh\colon N(C) \cong E_-$. Let $\not{D}_m = \not{D} + mc_-(h)$ where $\not{D}$ is a Dirac operator of $\mathfrak{S}$. We abbreviate $c_-(h)$ to h. Note that $\not{D}_m$ is an antisymmetric operator. Let A be a connection such that $c_{TY} \circ A = \not{D}$. Perturbing A, we may assume that $\pi^*(A|_C) = A$ on $B(N(C))$.

Lemma 3.15 *Let* $\not{D}'_C$ *be the Dirac operator defined by the Clifford action* $c_{TC}\colon TC \otimes \mathfrak{S}|_C \to \mathfrak{S}|_C$*. Then we have a decomposition*

$$\not{D} = D_C + D_f$$

with the following property:

- *On* $U \times \mathbb{R}^{s^-}$*, which is a trivialization of* $N(C)$*,*
$$D_C = \not{D}'_C|_U, \quad D_f = \sum_{k=1}^{s^-} c_{TY}(d\xi_i)\frac{\partial}{\partial \xi_i}.$$
Here c_{TY} *is the restriction of the Clifford multiplication of* $\mathfrak{s}$ *to* $\pi^* N(C) \subset TN(C)$ *and* $(\xi_1, \dots, \xi_{s^-})$ *are coordinates of* $\mathbb{R}^{s^-}$*, which is a fiber of* $N(C)$*.*
- D_C *and* D_f *anticommute.*
- *Both* D_C *and* D_f *are antisymmetric with respect to the* L^2 *inner product on* $N(C)$*.*

Proof It is easy to see that we have the decomposition of $\not{D}$ on each trivialization of $N(C)$. The operator

$$\sum_{k=1}^{s^-} c_{TY}(d\xi_i)\frac{\partial}{\partial \xi_i}$$

is $O(n)$ invariant. Hence the operator above on each trivialization coincides with the corresponding one on another trivialization. Thus we have the operator D_f on the whole of $N(C)$. Let $D_C := \not{D} - D_f$. On each trivialization, the operator D_f coincides with $\sum_{k=1}^{s^-} c_{TY}(d\xi_i)\frac{\partial}{\partial \xi_i}$ and D_C coincides with $\not{D}'_C|_U$. We will see at once anticommutativity of these operators on each trivialization on $N(C)$. We see these operators are antisymmetric with respect to inner product L^2. It is sufficient to prove that D_f is antisymmetric because the operator $\not{D}$ is antisymmetric. To see this, we decompose an integral on $N(C)$ by $\int_{N(C)} = \int_C \int_{\text{fiber}}$, where $\int_{\text{fiber}}$ is the integration on each fiber. The operator D_f is antisymmetric with respect to the L^2 inner product on each fiber and we have D_f is antisymmetric with respect to the L^2 norm on $N(C)$. □

Definition 3.16 We will denote by u_0 the function on $N(C)$ whose restriction to each fiber of $N(C)$ coincides with the function $\tilde{u}_0$.

Lemma 3.17 *We will denote by $h \in \Gamma(\pi^* E_-)$ the tautological section of $\pi^* E_- \cong \pi^* N(C)$. Let $\not{D}_C$ be a Dirac operator on $\not{S}_C \mathbin{\hat{\otimes}} V_{s^-}^*$. Let m be a positive real number and let $a \in \Gamma(\not{S}_C \mathbin{\hat{\otimes}} V_{s^-}^*)$. We have*

$$(\not{D} + mh)\pi^* a \cdot u_0 = \pi^*(\not{D}_C a) \cdot u_0.$$

Proof Recall the decomposition $\not{D} = D_C + D_f$. The restriction of $D_f + mh$ to each trivialization of $N(C)$ coincides with the operator D'_m in Lemma 3.12. Thus we have $(D_f + mh)(\pi^* a \cdot u_0) = 0$. It is sufficient to consider the operator D_C. The operator D_C coincides with the operator $\not{D}'_C$ on each trivialization and this operator is a pullback of an operator on C. Hence we have $D_C(\pi^* a \cdot u_0) = \pi^*(\not{D}_C a) \cdot u_0$. □

Lemma 3.18 *Let $H = \{\pi^* a \cdot u_0 \in \Gamma(\not{S}) \mid a \in \Gamma(\not{S}_C \mathbin{\hat{\otimes}} V_{s^-}^*),\ \not{D}_C a = 0\}$. We will denote by λ_C the minimum of the absolute value of nonzero eigenvalues of $\not{D}_C$. Let m be a positive number such that $m \geq |\lambda_C|$. Let $\phi \in \Gamma(N(C), \not{S})$ be a section whose restriction to each fiber of $N(C)$ is rapidly decreasing function. If ϕ is orthogonal to H in the L^2 inner product, We have $\|\not{D}_m \phi\| \geq \lambda_C \|\phi\|$, where $\|\cdot\|$ is the L^2 norm on $N(C)$.*

Proof We decompose ϕ into $\phi = \phi_0 + \pi^* b \cdot u_0$, where $b \in \Gamma(\not{S}_C) \mathbin{\hat{\otimes}} V_{s^-}^*$ and ϕ_0 is a fiberwise rapidly decreasing section such that $\int_{N(C)_x} \langle \phi_0, \pi^* a \cdot u_0 \rangle = 0$ for all $x \in C$, $a \in (\not{S}_C \mathbin{\hat{\otimes}} V_{s^-}^*)_x$. From Lemma 3.12 and for L to be $(\not{S}_C)_x$, ϕ_0 is orthogonal to the kernel of $D_f + mh$ on each fiber. The section b is orthogonal to $\ker \not{D}_C$ in the L^2 inner product on C because ϕ is orthogonal to H. From the above decomposition, we have

$$(\not{D} + mh)\phi = (\not{D} + mh)\phi_0 + \pi^*(\not{D}_C b) \cdot u_0.$$

It is easy to see that

$$\int_{N(C)} \langle (\not{D} + mh)\phi_0, \pi^*(\not{D}_C b) \cdot u_0 \rangle = 0 \quad \text{and} \quad \|(\not{D} + mh)\phi\|^2 = \|(\not{D} + mh)\phi_0\|^2 + \|\pi^*(\not{D}_C b) \cdot u_0\|^2.$$

It is straightforward to see D_C and $D_f + mh$ are anticommutative.

Thus we have

$$\begin{aligned} \int_{N(C)} |(\not{D} + mh)\phi_0|^2 &= \int_{N(C)} |D_C \phi_0|^2 + \int_{N(C)} |(D_f + mh)\phi_0|^2 \\ &\geq \int_C \int_{\text{fiber}} |(D_f + mh)\phi_0|^2 \geq \int_C m^2 \int_{\text{fiber}} |\phi_0|^2 \geq \int_C \lambda_C^2 \int_{\text{fiber}} |\phi_0|^2 \end{aligned}$$

and

$$\int_{N(C)} |\pi^*(\not{D}_C b) \cdot u_0|^2 = \int_{N(C)} |\pi^*(\not{D}_C b) \cdot u_0|^2 \geq \lambda_C^2 \int_{N(C)} |\pi^* b \cdot u_0|^2.$$

Hence we have the inequality $\|\not{D}_m \phi\| \geq \lambda_C \|\phi\|$ as desired. □

Definition 3.19 The finite-dimensional vector space H has a natural $\mathbb{Z}/2\mathbb{Z}$ grading and a left $\mathrm{Cl}_{(n+s^+, s^-)}$ action induced by $\not{S}$. Let ϵ be the $\mathbb{Z}/2\mathbb{Z}$ grading operator. The four-tuple $(0, \epsilon, \mathrm{Cl}_{(n+s^+, s^-)}, H)$ defines an element of $\mathrm{KO}^{s^- - n - s^+}(\mathrm{pt})$. We write $\mathrm{ind}(N(C), \not{S})$ for this element in $\mathrm{KO}^{s^- - n - s^+}(\mathrm{pt})$.

Remark 3.20 By the definition of H and from Lemma 2.2, we have

$$\operatorname{ind}(\mathfrak{s}_C) = \big[(\not{D}_C, \epsilon, \mathrm{Cl}_{(n+s^+,s^-)}, L^2(C, \not{S}_C \mathbin{\hat{\otimes}} V^*_{s^-}))\big] = \operatorname{ind}(N(C), \not{S}) \in \mathrm{KO}^{s^- - n - s^+}(\mathrm{pt}).$$

Remark 3.21 The notion of the index of open manifold is introduced by Mikio Furuta in [7]. When the index $\operatorname{ind}(N(C), \not{S})$ values in $\mathrm{KO}^0(\mathrm{pt})$, it coincides with the index of the pair $(N(C), [h])$.

3.2.2 Now Step 2

We prove $\operatorname{ind}(\mathfrak{s})$, an analytic index of $G^+(n, s^+, s^-)$ structure on Y, is equal to $\operatorname{ind}(N(C), \not{S})$.

We introduce some notation.

- If necessary we perturb h so that h satisfies $|h| < 1$ on $U(C)$ and $|h| = 1$ on the complement set of $U(C)$.
- We will denote by $\mathcal{H}^\lambda_m$ the direct sum of eigenspaces of $\not{D}_m \colon \Gamma(Y, \not{S}) \to \Gamma(Y, \not{S})$ such that the absolute value of each eigenvalue is less than λ^2. This is a finite-dimensional subspace of $L^2(Y, \not{S})$. $\mathcal{H}^\lambda_m$ has a natural $\mathbb{Z}/2\mathbb{Z}$ grading and a left $\mathrm{Cl}_{(n+s^+,s^-)}$ action.
- Let ρ be a smooth function on Y supported in $U(C)$ such that
$$\rho(z) = \begin{cases} 1 & \text{if } |z| \le \frac{1}{2}, \\ 0 & \text{if } |z| \ge \frac{2}{3} \end{cases}$$
for $z \in U(C) \cong B(N(C)) \subset N(C)$ and monotone decreasing on $\frac{1}{2} < |z| < \frac{2}{3}$ in $|z|$, where $|\cdot|$ is a norm of $N(C)$. (We identify $U(C)$ and $B(N(C))$.)

The following proposition is proved by the general theory of Witten deformation. We prove this proposition in the appendix.

Proposition 3.22 *Assume that a positive constant λ is smaller than a constant determined by the principal symbol of D and the differential of ρ and suppose that $m > \lambda$. Let Π' be the orthogonal projection from $L^2(N(C), \not{S})$ to H. Then the map*

$$\mathcal{H}^\lambda_m \to H, \quad \phi \mapsto \Pi'(\rho\phi)$$

is an isomorphism and preserves the $\mathbb{Z}/2$ grading and the $\mathrm{Cl}_{(n+s^+,s^-)}$ action.

Proof of Main Theorem From the definition of the index of $\mathfrak{s}$,

$$\operatorname{ind}(\mathfrak{s}) = [(\not{D}_m, \epsilon, \mathrm{Cl}_{(s^-,s^++n)}, \mathcal{H}^\lambda_m)]$$

and we see

$$[(0, \epsilon, \mathrm{Cl}_{(s^-,s^++n)}, H)] = [(\not{D}_m, \epsilon, \mathrm{Cl}_{(s^-,s^++n)}, \mathcal{H}^\lambda_m)] \in \mathrm{KO}^{s^- - n - s^+}(\mathrm{pt})$$

from Proposition 3.22. We have $[(0, \epsilon, \mathrm{Cl}_{(s^-,s^++n)}, H)] = \operatorname{ind}(\mathfrak{s}_C)$ from Remark 3.20. Thus we have proved Main Theorem. $\square$

4 Examples

4.1 Freed–Hopkins $H_n(s)$

In this subsection, we consider a family of groups $H_n(s)$ defined by Freed and Hopkins [5]. The group $H_n(s)$ is given in Table 1.

Lemma 4.1 *In the case $s = 0, 1, 2, 3$, we have $H_n(s) \cong G^+(n, s, 0)$ and in the case $s = -1, -2, -3$, we have $H_n(s) \cong G^+(n, 0, -s)$.*

Proof It is obvious for the case $s = 0$ therefore it is sufficient to prove the case of $|s| = 1, 2, 3$. First we consider the case $|s| = 3$. We use the identification $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ and $(\mathrm{Cl}_{+n} \mathbin{\hat{\otimes}} (\mathbb{R} \oplus \mathbb{R}\Gamma))_0 \cong \mathrm{Cl}_{\pm n}$, where $\Gamma = e_1 e_2 e_3 \in \mathrm{Cl}_s$ and $\pm$ is the sign of s. We have an isomorphism $G^+(n, s, 0) \to H_n(s)$ (or $G^+(n, 0, -s) \to H_n(s)$) given by

$$gu \mapsto \begin{cases} [g, u] & \text{if } g \in \mathrm{Spin}(n),\ u \in \mathrm{Spin}(3), \\ [g \otimes \Gamma, \Gamma^{-1} u] & \text{otherwise} \end{cases}$$

for $g \in \mathrm{Pin}^+(n),\ u \in \mathrm{Pin}^{\pm}(3)$.

We consider the case $|s| = 1, 2$. We will denote by e a generator of Cl_s with $|e| = 1$. We use the identification $\mathrm{Cl}_{+n} \mathbin{\hat{\otimes}} (\mathbb{R} \oplus \mathbb{R}e) \cong \mathrm{Cl}_{\pm n}$ where $\pm$ is the sign of s. We have an isomorphism $G^+(n, s, 0) \to H_n(s)$ (or $G^+(n, 0, -s) \to H_n(s)$) given by

$$gu \mapsto \begin{cases} [g, u] & \text{if } g \in \mathrm{Spin}(n),\ u \in \mathrm{Spin}(|s|), \\ [g \otimes e, e^{-1} u] & \text{otherwise} \end{cases}$$

for $g \in \mathrm{Pin}^+(n),\ u \in \mathrm{Pin}^{\pm}(|s|)$. □

Remark 4.2 An index of $H_n(s)$ structure is defined in [5]. From the lemma above, we see the index of $H_n(s)$ coincides with our index of $G^+(n, s, 0)$ (or $G^+(n, 0, -s)$) structure.

s	$H_n(s)$
0	$\mathrm{Spin}(n)$
-1	$\mathrm{Pin}^+(n)$
-2	$\mathrm{Pin}^+(n) \ltimes_{\mathbb{Z}/2\mathbb{Z}} U(1)$
-3	$\mathrm{Pin}^-(n) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{SU}(2)$
4	$\mathrm{Spin}(n) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{SU}(2)$
3	$\mathrm{Pin}^+(n) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{SU}(2)$
2	$\mathrm{Pin}^-(n) \ltimes_{\mathbb{Z}/2\mathbb{Z}} U(1)$
1	$\mathrm{Pin}^-(n)$

Table 1: Groups $H_n(s)$ defined by Freed and Hopkins [5].

Remark 4.3 In the case $s=4$, $H_n(4)$ is isomorphic to a subgroup of $G^+(n,0,4)$ and we can see the index of $H_n(4)$ structure defined in [5] coincides with our index if the structure group of a $G^+(n,0,4)$ structure reduces to $H_n(4)$.

We identify Spin(4) with SU(2) × SU(2). We give an embedding of $H_n(4)$ to $G^+(n,0,4)$ by

$$\mathrm{Spin}(n)\times_{\mathbb{Z}/2\mathbb{Z}}\mathrm{SU}(2)\ni[g,u]\mapsto[g,\mathrm{diag}(u,u)]\in\mathrm{Spin}(n)\times_{\mathbb{Z}/2\mathbb{Z}}\mathrm{Spin}(4)\subset G^+(n,0,4).$$

The following proposition is a consequence of Main Theorem, Lemma 4.1 and the above remarks.

Proposition 4.4 *We assume* $s=-1,-2,-3,$ *or* 4. *Then there exists an isomorphism* f *such that the following diagram commutes*:

$$\begin{tikzcd} \Omega_n^{H_n(s)}(\mathrm{pt}) \arrow[r, "f"] \arrow[dr] & \Omega^{\mathrm{Spin}}_{n-|s|}(\mathrm{pt}) \arrow[d] \\ & \mathrm{KO}^{-n-s}(\mathrm{pt}) \end{tikzcd}$$

4.2 The case of $G^+(5,0,4)$ structure

We consider the index of the $G^+(5,0,4)$ structure especially because it is important to see the orientability of Pin$^-$(2) monopole moduli space in the next section. The following arguments can be easily generalized to the case of the index of the $G^+(8k+5,0,4)$ structure. We use only the $k=0$ case in the next section; therefore we only consider this case to avoid complications.

Definition 4.5 Let Z be a 5 dimensional closed manifold and $\mathfrak{s}$ be a $G^+(5,0,4)$ structure on Z. Let $\mathfrak{S}$ be the standard spinor bundle of $\mathfrak{s}$. Note that $\mathfrak{S}$ has a natural right $\mathrm{Cl}_{(5,4)}$ action. Let $\epsilon_0,\epsilon_1,\dots,\epsilon_4$ be generators of Cl_{+5}. and let $e_1,\dots,e_4$ be generators of Cl_{-4}. We will denote by $\widetilde{S}$ a subbundle of $\mathfrak{S}$ such that the intersection of $+1$-eigenspaces of $\epsilon_1e_1,\dots,\epsilon_4e_4$. We call $\widetilde{S}$ the spinor bundle of $\mathfrak{s}$. The Clifford multiplication of TZ and E_- preserves $\widetilde{S}$ because they commute with the right $\mathrm{Cl}_{(5,4)}$ action. We define a skew-adjoint Dirac operator $\widetilde{D}$ on $\widetilde{S}$ by using Clifford action of TZ.

We have the following lemma from Lemma 2.2:

Lemma 4.6 *The spinor bundle* $\widetilde{S}$ *is a generalized* $\mathbb{Z}/2$ *graded spinor bundle with left* Cl_{-1} *action. The index* $\mathrm{ind}(\mathfrak{s})$ *of* $\mathfrak{s}$ *coincides with* $(0,\epsilon,\mathrm{Cl}_{-1},\ker\widetilde{D})\in\mathrm{KO}^{-1}(\mathrm{pt})$. *Moreover, under the isomorphism* $\mathrm{KO}^{-1}(\mathrm{pt})\cong\mathbb{Z}/2\mathbb{Z}$, *we have* $\mathrm{ind}(\mathfrak{s})=\dim\ker(\widetilde{D}|_{\widetilde{S}^+})\bmod 2$ *where* $\widetilde{S}^+$ *is the even part of* $\widetilde{S}$.

We provide a specific construction of $\widetilde{S}$ when the structure group of $\mathfrak{s}$ is reduced from $G^+(5,0,4)$ to $\mathrm{Spin}(5)\times\mathrm{Spin}(4)/\mathbb{Z}/2\mathbb{Z}$. This construction is useful to consider the orientability of Pin$^-$(2) monopole moduli space.

Definition 4.7 We will denote by $\epsilon_0, \dots, \epsilon_4$ the generators of Cl_{+5} and by $e_1, \dots, e_4$ the generators of Cl_{-4}.

- Let $\mathbb{H}$ be the quaternion ring. We use the convention that $ijk = 1$.
- Let $\mathbb{H}(2)$ be the matrix ring of 2×2 matrices with entries in the ring $\mathbb{H}$.
- We define an isomorphism $\alpha \colon \mathrm{Cl}_{-4} \to \mathbb{H}(2)$ by
$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 \mapsto \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad e_4 \mapsto \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$
- Let $\Gamma = e_1 e_2 e_3 e_4 \in \mathrm{Cl}_{-4}$. Note that $\Gamma^2 = 1$ and
$$\alpha(\Gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
- We define an isomorphism $\beta \colon \mathrm{Cl}_{+5} \to \mathbb{H}(2) \oplus \mathbb{H}(2)$ by
$$\epsilon_0 \mapsto (\alpha(\Gamma), -\alpha(\Gamma)), \qquad \epsilon_i \mapsto (\alpha(\Gamma)\alpha(e_i), -\alpha(\Gamma)\alpha(e_i)), \quad i = 1, 2, 3, 4.$$
- We define an isomorphism $f \colon \mathrm{Cl}_{(5,4)} \to (\mathbb{H}(2) \oplus \mathbb{H}(2)) \otimes \mathbb{H}(2)$ by
$$\epsilon_p \mapsto \beta(\epsilon_p) \otimes \alpha(\Gamma), \quad e_q \mapsto 1 \otimes \alpha(e_q),$$
where $p = 0, \dots, 4,\ q = 1, \dots, 4$.
- Using isomorphisms α, β, we define the spinor representations of $\mathrm{Cl}_{+5}, \mathrm{Cl}_{-4}$ whose representation spaces are $\mathbb{H}^2 \oplus \mathbb{H}^2$, $\mathbb{H}^2$, respectively. By abuse of notation, we let α, β stand for these spinor representations, respectively. Note that
$$\beta(\epsilon_i) \cdot (\phi_0, \phi_1) = (\alpha(\Gamma)\alpha(e_i)\phi_0, -\alpha(\Gamma)\alpha(e_i)\phi_1) \quad \text{for } i = 0, \dots, 4.$$

Remark 4.8 In our notations, the $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathbb{H}(2) \oplus \mathbb{H}(2)$ induced by the isomorphism β is given as follows: The subspace $\{(A, A) \mid A \in \mathbb{H}(2)\}$ is the even part and $\{(A, -A) \mid A \in \mathbb{H}(2)\}$ is the odd part. The $\mathbb{Z}/2\mathbb{Z}$ grading operator is a map $(A, B) \mapsto (B, A)$.

The $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathbb{H}(2)$ induced by the $\mathbb{Z}/2\mathbb{Z}$ grading of Cl_{-4} using the map α is given by declaring the even part consists of diagonal matrices and the odd part of off-diagonal one. The $\mathbb{Z}/2\mathbb{Z}$ grading operator is given by $A \mapsto \alpha(\Gamma) A \alpha(\Gamma)^{-1} = \mathrm{Ad}(\alpha(\Gamma))(A)$.

Definition 4.9 We assume that the structure group of $\mathfrak{s}$ reduces to the group $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2)$.

Let $S = S_0 \oplus S_1,\ S_0 = S_1 = \mathbb{H}^2$ be a representation space of the spinor representation β of Cl_{+5}. We define the representation ρ by
$$\rho([q, (u_0, u_1)])\phi = \beta(q) \cdot (\phi_0 u_0^{-1}, \phi_1 u_1^{-1})$$

for

$$\phi = (\phi_0, \phi_1) \in S \cong \mathbb{H}^2 \oplus \mathbb{H}^2,$$
$$[q, (u_0, u_1)] \in \mathrm{Spin}(5) \times (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/(\mathbb{Z}/2) \cong \mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2).$$

We denote by S' the associated vector bundle of the representation ρ. We give a $\mathbb{Z}/2\mathbb{Z}$ grading on $S' \oplus S'$ by declaring the subspace $\{(\varphi, \varphi) \in S' \oplus S'\}$ is the even part and the subspace $\{(\varphi, -\varphi) \in S' \oplus S'\}$ is the odd part. We give the following right Cl_{+1} action: let ϵ_0 be the generator of Cl_{+1} and we define the right action of ϵ_0 by $(\varphi_0, \varphi_1) \cdot \epsilon_0 = (\varphi_0, -\varphi_1)$ for $(\varphi_0, \varphi_1) \in S' \oplus S'$. The left Cl_{-1} action is given by $e_0 \cdot \phi := (\epsilon \phi) \cdot \epsilon_0$, where ϵ is the $\mathbb{Z}/2\mathbb{Z}$ grading operator. We define the Clifford multiplication c_n' by the left Clifford multiplication of $\mathrm{Cl}_{(5,4)}$ given by $\epsilon_j \cdot (\varphi_0, \varphi_1) = (\beta(\epsilon_j)\varphi_0, -\beta(\epsilon_j)\varphi_1)$ for $j = 0, \dots, 4$. Then $S' \oplus S'$ is $\mathbb{Z}/2\mathbb{Z}$ graded generalized spinor bundle with left Cl_{-1} action of $\mathfrak{s}$.

Remark 4.10 We can define the c_{E_-} Clifford action on $S' \oplus S'$ such that that action is preserved by the isomorphism in the Lemma below. But we will not use c_- in Section 5.

Lemma 4.11 *As a $\mathbb{Z}/2\mathbb{Z}$ graded generalized spinor bundle with left* Cl_{-1} *action of $\mathfrak{s}$, $S' \oplus S' \cong \widetilde{S}$. In particular, $\widetilde{S}^+ \cong S'$.*

Proof We identify $\mathrm{Cl}_{(5,4)}$ with $(\mathbb{H}(2) \oplus \mathbb{H}(2)) \otimes_{\mathbb{R}} \mathbb{H}(2)$ by using the isomorphism f in Definition 4.7. Let Φ be a map

$$(\mathbb{H}(2) \oplus \mathbb{H}(2)) \otimes_{\mathbb{R}} \mathbb{H}(2) \ni (A_0, A_1) \otimes B \mapsto (A_0 B^*, A_1 \operatorname{Ad}(\alpha(\Gamma))(B^*)) \in \mathbb{H}(2) \oplus \mathbb{H}(2),$$

where B^* is transpose matrix of $\bar{B}$. (We will denote by $\bar{\cdot}$ the quaternion conjugate.) We define the $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z})$ action on the space left hand side by

$$[q, (u_0, u_1)](A_0, A_1) = \left(q A_0 \begin{pmatrix} u_0^{-1} & 0 \\ 0 & u_1^{-1} \end{pmatrix}, q A_1 \begin{pmatrix} u_0^{-1} & 0 \\ 0 & u_1^{-1} \end{pmatrix} \right)$$

for $[q, (u_0, u_1)] \in \mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Spin}(5) \times (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/(\mathbb{Z}/2\mathbb{Z})$. For this action, we see Φ is a $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z})$ equivariant map because $\mathrm{diag}(u_0^{-1}, u_1^{-1})$ commutes with $\alpha(\Gamma)$.

The map Φ is invariant under the right multiplication of $f(\epsilon_i e_i)$ for $i = 1, 2, 3, 4$. Therefore, the -1-eigenspaces of $f(\epsilon_i e_i)$ for some $i = 1, 2, 3, 4$ are the subspace of $\ker \Phi$. We see at once Φ is surjective and the dimension of $\mathbb{H}(2) \oplus \mathbb{H}(2)$ is equal to the dimension of $\widetilde{S}$. Thus we have that the restriction of Φ to the representation space $\widetilde{S}$ is a $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z})$ equivariant isomorphism.

We see the $\mathrm{Spin}(5) \times \mathrm{Spin}(4)/(\mathbb{Z}/2\mathbb{Z})$ representation $\mathbb{H}(2) \oplus \mathbb{H}(2)$ is equivalent to the representation $S' \oplus S'$ through an isomorphism given by identifying each column of the matrix with an element of S_0', S_1'. The $\mathbb{Z}/2\mathbb{Z}$ grading of $\widetilde{S}$ is defined by the $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathrm{Cl}_{(5,4)}$. It is easy to see that this $\mathbb{Z}/2\mathbb{Z}$ grading of $\widetilde{S}$ coincides with the $\mathbb{Z}/2\mathbb{Z}$ grading of $S' \oplus S'$. Moreover, we see at once Φ preserves the Clifford multiplication c_{TZ} and $f(\epsilon_0)$. This completes the proof. □

Remark 4.12 If we change the basis of $S' \oplus S'$ by the isomorphism $(\phi_0, \phi_1) \mapsto \left(\frac{1}{2}(\phi_0+\phi_1), \frac{1}{2}(\phi_0-\phi_1)\right)$, we change the $\mathbb{Z}/2\mathbb{Z}$ grading of $S' \oplus S'$ and the Clifford multiplication c_{TZ}. The $\mathbb{Z}/2\mathbb{Z}$ grading is given by $S' \oplus 0$ is even and $0 \oplus S'$ is odd. The Clifford multiplication c_{TZ} and the right ϵ_0 action are given by the matrix

$$c_{TZ}(\epsilon_j) = \begin{pmatrix} 0 & \beta(\epsilon_j) \\ \beta(\epsilon_j) & 0 \end{pmatrix}, \quad \epsilon_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that c_{TZ} and right ϵ_0 action commute.

5 The orientability of $\mathrm{Pin}^-(2)$ monopole moduli space

5.1 The orientability and mod 2 indices

In Section 4.1, we consider the spinor bundle and index of the $G^+(5,0,4)$ structure. In this section, we translate the orientability of the $\mathrm{Pin}^-(2)$ monopole moduli space into a mod 2 index on a five dimensional manifold.[1] From Main Theorem, we give a topological criterion for the determinant line bundle on the ambient space of the $\mathrm{Pin}^-(2)$ monopole moduli space to be trivial.

Definition 5.1 We define a group $\mathrm{Spin}^{c-}(4)$ by

$$\mathrm{Spin}^{c-}(4) := \mathrm{Spin}(4) \times \mathrm{Pin}^-(2)/(\mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{Z}/2\mathbb{Z} \cong \{(1,1),(-1,-1)\} \in \mathrm{Spin}(4) \times \mathrm{Pin}^-(2)$.

Remark 5.2 By the definition above, we see that $\mathrm{Spin}^{c-}(4)$ is a subgroup of $G^+(4,0,3)$ by using the natural embedding $\mathrm{Spin}(4) \subset \mathrm{Cl}_{(+4)}$.

Definition 5.3 We define a Spin^{c-} structure on Riemannian 4-manifold X as a $G^+(4,0,3)$ structure whose structure group reduces to Spin^{c-}.

Definition 5.4 Let X be a 4-manifold and $\mathfrak{s}_X$ be a Spin^{c-} structure on X. We will denote by $\widetilde{P}$ a principal $\mathrm{Spin}^{c-}(4)$ bundle given by the Spin^{c-} structure $\mathfrak{s}_X$.

- We identify $\mathbb{R}^4$ with $\mathbb{H}$. We define the representation of $\mathrm{Spin}^{c-}(4)$

$$\Delta^{\pm}\colon \mathrm{Spin}^{c-}(4) \to \mathrm{GL}(4,\mathbb{R}) \quad \text{and} \quad \rho\colon \mathrm{Spin}^{c-}(4) \to \mathrm{GL}(4,\mathbb{R})$$

by

$$\Delta^{\pm}([q^+,q^-,u])\phi = q^{\pm}\phi u^{-1}, \quad \rho([q^+,q^-,u])v = q^+ v (q^-)^{-1}.$$

We will denote by $S^{\pm}$ the associated vector bundle given by the representation $\Delta^{\pm}$ and the principal Spin^{c-} bundle $\widetilde{P}$. The associated vector bundle given by the representation ρ is TX.

[1] Originally, Mikio Furuta translated the orientability of $\mathrm{Pin}^-(2)$ monopole moduli space into a mod 2 index associated with a 3 dimensional submanifold with $G^+(3,0,2)$ structure. Nobuhiro Nakamura wrote Furuta's idea in the note [16]. The argument in that note is the neck stretching argument. In this paper, we use Witten deformation to determine the orientability of $\mathrm{Pin}^-(2)$ monopole moduli space.

- Let $\rho^*([q^+, q^-, u])v' := q^- v (q^+)^{-1}$. We observe that there is a natural isomorphism $\widetilde{P} \times_{\rho^*} \mathbb{H} \cong T^*X$ and we have an isomorphism $TX \to T^*X$ that is given by

$$TX \cong \widetilde{P} \times_\rho \mathbb{H} \ni [\tilde{p}, v] \mapsto [\tilde{p}, \bar{v}] \in \widetilde{P} \times_{\rho^*} \mathbb{H} \cong T^*X.$$

We denote the image of $v \in T_x X$ under this mapping as $\bar{v}$.

- We define a Clifford multiplication

$$c \colon TX \otimes S^+ \to S^-$$

by $c(v \otimes \phi) = \bar{v}\phi$. We will denote by $c(v)\phi$ this Clifford action. We define

$$c^* \colon TX \otimes S^- \to S^+$$

by $c^*(v \otimes \phi) = v\phi$. By abuse of notation, we use the same letter $c(v)\phi$. We have $c(v)^2 = |v|^2$ for $v \in T_x X$.

- A connection A on $\widetilde{P}$ is a Spin^{c-} connection if it satisfies

$$\nabla^A(c(X)\phi) = c(\nabla^{LC} X)\phi + c(X)\nabla^A \phi$$

for $X \in \Gamma(TX)$, $\phi \in \Gamma(S^+)$, where ∇^{LC} is the Levi-Civita connection of X.

- Let A be a Spin^{c-} connection. We define a Dirac operator $D_A^{\pm}$ associated to A by the composition of following maps:

$$\Gamma(S^{\pm}) \xrightarrow{\nabla^A} \Gamma(TX^* \otimes S^{\pm}) \cong \Gamma(TX \otimes S^{\pm}) \xrightarrow{c} \Gamma(S^{\mp}).$$

The notion of the Spin^{c-} structure is introduced by Nobuhiro Nakamura [14; 15] to define the Pin$^-$(2) monopole equations and the Pin$^-$(2) monopole invariant. The Pin$^-$(2) monopole invariant is a $\mathbb{Z}/2\mathbb{Z}$-valued invariant. The matter of when we determine the orientability of the Pin$^-$(2) monopole moduli space is whether a gauge transformation preserves an orientation of $\operatorname{ind} D_A$.

To define a gauge transformation, we introduce some associated vector bundles and their Clifford actions.

Definition 5.5 Let X be a 4-manifold and $\mathfrak{s}_X$ be a Spin^{c-} structure on X. Let $\widetilde{P}$ be the principal Spin^{c-}(4) bundle given by the Spin^{c-} structure $\mathfrak{s}_X$.

- We identify $\mathbb{R}^2$ with the subspace $\{z \in \mathbb{H} \mid z = x + iy,\ x, y \in \mathbb{R}\} \cong \mathbb{C}$. We define the real Spin^{c-}(4) representation

  $$\rho_0' \colon \mathrm{Spin}^{c-}(4) \to \mathrm{GL}(2, \mathbb{R})$$

  by $\rho_0'([q^+, q^-, u])z = uzu^{-1}$, where the multiplication is that of $\mathbb{H}$.

- We identify $\mathbb{R}^2$ with the subspace $\{jw \in \mathbb{H} \mid w \in \mathbb{C}\}$. We define the real Spin^{c-}(4) representation

  $$\rho_1' \colon \mathrm{Spin}^{c-}(4) \to \mathrm{GL}(2, \mathbb{R})$$

  by $\rho_1'([q^+, q^-, u])jw = ujwu^{-1}$, where the multiplication is that of $\mathbb{H}$.

It is easy to check the well-definedness of the above definition. We will denote by $\widetilde{\mathbb{C}}$, E the associated vector bundles given by $\widetilde{P}$ and the representations ρ_0', ρ_1', respectively.

Definition 5.6 We define the multiplication $S \otimes (\widetilde{\mathbb{C}} \oplus E) \to S$ as follows. Note that the representation space of $\rho_0 \oplus \rho_1$ and $\Delta^+ \oplus \Delta^-$ is $\mathbb{H}$ and $\mathbb{H}^2 = \mathbb{H}_+ \oplus \mathbb{H}_-$, respectively. We define a multiplication of elements in these representation space by

$$\phi \cdot v := \phi v$$

for $v \in \mathbb{H}$ and $\phi \in \mathbb{H}^2$. This is a $\mathrm{Spin}^{c-}(4)$ equivariant multiplication. This defines a multiplication $S = S^+ \oplus S^-$ and $\widetilde{\mathbb{C}} \oplus E$. We define the multiplication $(\widetilde{\mathbb{C}} \oplus E) \otimes (\widetilde{\mathbb{C}} \oplus E) \to \widetilde{\mathbb{C}} \oplus E$ in the same way.

Remark 5.7 Regarding the Spin^{c-} structure $\mathfrak{s}_X$ as a $G^+(4,0,3)$ structure, the vector bundle $\det E \oplus E$ is the vector bundle E_- associated with $\mathfrak{s}_X$ (see Definition 2.6). From the definition of $\widetilde{\mathbb{C}}$ and E, we have $\mathrm{Im}(\widetilde{\mathbb{C}}) \cong \det E$. From now on, we fix an isomorphism $\mathrm{Im}(\widetilde{\mathbb{C}}) \cong \det E$ and identify $\mathrm{Im}(\widetilde{\mathbb{C}})$ with $\det E$. If we take another isomorphism, arguments do not change.

Definition 5.8 We call an element of $\{u \in \Gamma(\widetilde{\mathbb{C}}) \mid |u| = 1\}$ a gauge transformation.

Now we state the topological method of evaluating the orientability of $\mathrm{Pin}^-(2)$ monopole space.

Theorem 5.9 *Let u be a gauge transformation. We perturb u if necessary and we may assume that -1 is a regular value of u. Let h be a section of E such that h is transverse to the zero section and its submanifold $u^{-1}(-1) \subset X$. Then there is a natural* Spin *structure on $h^{-1}(0) \cap u^{-1}(-1)$. This defines an element of* 1 *dimensional* Spin *bordism group $\Omega_1^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}$ and we denote by* $\text{t-ind}(\mathfrak{s}_X, u)$ *this element. Then the following statements are equivalent*:

1. *The gauge transformation u preserves the orientation of the index bundle $\{\mathrm{ind}(D_A)\}_A$ on the configuration space.*
2. *The element* $\text{t-ind}(\mathfrak{s}_X, u)$ *of* $\Omega_1^{\mathrm{Spin}}(\mathrm{pt})$ *is trivial.*

Corollary 5.10 If we have $\text{t-ind}(\mathfrak{s}_X, u) = 0$ for all gauge transformation u, the $\mathrm{Pin}^-(2)$ monopole moduli space is orientable.

We begin the preparation of the proof of Theorem 5.9.

Definition 5.11 Let X be a 4-manifold and $\mathfrak{s}_X$ be a Spin^{c-} structure on X. Let $S = S^+ \oplus S^-$ be the spinor bundle of $\mathfrak{s}_X$ and u be a gauge transformation.

- We define the vector bundle L on $X \times S^1$ as follows. Let $\pi : [0,1] \times X \to X$ be the projection. We introduce an equivalence relation on $\pi^*\widetilde{\mathbb{C}}$ by $(0,z) \sim (1, zu)$ for $z \in \widetilde{\mathbb{C}}$ and L is the quotient of this equivalence relation $L = \pi^*\widetilde{\mathbb{C}}/\sim$. Note that L has the canonical left $\pi^*\widetilde{\mathbb{C}}$ action.

- Let $\widetilde{\pi}: S^1 \times X \to X$ be the projection. By abuse of notation, we use the same letter $\widetilde{\mathbb{C}}$ for $\widetilde{\pi}^*\widetilde{\mathbb{C}}$.
- $V = V^+ \oplus V^-$ is the $\mathbb{Z}/2\mathbb{Z}$ graded vector bundle given by $V = \widetilde{\pi}^* S \otimes_{\widetilde{\mathbb{C}}} (L \oplus \widetilde{\mathbb{C}})$, $V^+ = \widetilde{\pi}^* S$ and $V^- = \widetilde{\pi}^* S \otimes_{\widetilde{\mathbb{C}}} L$, where $\otimes_{\widetilde{\mathbb{C}}}$ is a tensor product of $\widetilde{\mathbb{C}}$ modules.

Lemma 5.12 *We define a skew-adjoint Dirac type operator D on V as follows*:

- *Let A be a* Spin^{c-} *connection on X and D_A is the Dirac operator on X given by A.*
- *Let A_t be a one-parameter family of* Spin^{c-} *connection on X such that $A_t = A$ for $t < \frac{1}{3}$ and $A_t = u^* A$ for $t > \frac{2}{3}$.*
- *Let ϵ be the $\mathbb{Z}/2\mathbb{Z}$ grading operator of S. Let ϵ' be the operator on $L \oplus \widetilde{\mathbb{C}}$ given by $1 \oplus (-1)$.*
- *We define the Dirac type operator on $X \times S^1$ by*
$$D = D_t \otimes \mathrm{pr}_L + D_A \otimes \mathrm{pr}_{\widetilde{\mathbb{C}}} + \epsilon \partial_t \otimes \epsilon',$$
where t is the coordinate of S^1 and pr_L, $\mathrm{pr}_{\widetilde{\mathbb{C}}}$ are the projections to $L, \widetilde{\mathbb{C}}$, respectively.

Then the operator D is well defined on $S^1 \times X$. It is equivalent that u preserves an orientation of the index bundle $\{\mathrm{ind}(D_A)\}$ *on the configuration space and* $\dim\ker D \bmod 2 = 0$.

Proof The four-tuple $(D_t \otimes \mathrm{pr}_L + D_A \otimes \mathrm{pr}_{\widetilde{\mathbb{C}}}, \epsilon \otimes \epsilon', 0, L^2(X \times S^1, V))$ defines a family index in $\mathrm{KO}^0(S^1, \mathrm{pt})$. This family index coincides with $\mathrm{ind}(D_t) - \mathrm{ind}(D_A)$ when we use the definition of $\mathrm{KO}^0(S^1, \mathrm{pt})$ as the subgroup of the Grothendieck group of real vector bundles on S^1. We see at once the family index $\mathrm{ind}(D_t) - \mathrm{ind}(D_A)$ is trivial if and only if u preserves an orientation of $\mathrm{ind}(D_A)$ by the definition of V and D_t.

We see that this family index coincides with $\dim\ker D \bmod 2 \in \mathbb{Z}/2\mathbb{Z} \cong \mathrm{KO}^0(S^1, \mathrm{pt})$ from index theory. ☐

Definition 5.13 We define the $G^+(5,0,4)$ structure $\mathfrak{s}_Z$ on $Z = X \times S^1$ as follows: Let $\widetilde{P}$ be the principal Spin^{c-}(4) bundle on X given by Spin^{c-} structure $\mathfrak{s}_X$ and $\pi: [0,1] \times X \to X$ be the projection. We will denote by ι an embedding
$$\mathrm{Spin}^{c-}(4) \cong \mathrm{Spin}(4) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Pin}^-(2) \to \mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(4) \cong \mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} (\mathrm{Sp}(1) \times \mathrm{Sp}(1))$$
given by $[q, u'] \mapsto [i(q), (u', u')]$, where i is a embedding $\mathrm{Spin}(4) \to \mathrm{Spin}(5)$ which is a lift of a map $A \mapsto \mathrm{diag}(1, A)$. Let u be a gauge transformation of $\mathfrak{s}_X$. We define a principal $\mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(4)$ bundle $\widetilde{P}_Z$ on Z by $\pi^* \widetilde{P} \times_\iota (\mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(4))/\sim$, where $\sim$ is an equivalent relation given by
$$(0, p) \sim (1, pu), \ \ p \in \widetilde{P} \times_\iota (\mathrm{Spin}(5) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(4)).$$
It is easy to see that $\widetilde{P}_Z$ defines a Spin structure of $TZ \oplus L \oplus \pi^* E$. We will denote by $\mathfrak{s}_Z$ a $G^+(5,0,4)$ structure on Z given by $\widetilde{P}_Z$.

Lemma 5.14 *The* mod 2 *index* $\dim\ker D \bmod 2$ *of the skew-adjoint operator D coincides with* $\mathrm{ind}(\mathfrak{s}_Z) \in \mathrm{KO}^{-1}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}$ *where* $\mathrm{ind}(\mathfrak{s}_Z)$ *is the index of* $\mathfrak{s}_Z$.

Proof By the definition of V, we see that V is a spinor bundle S' of $G^+(5,0,4)$ structure $\mathfrak{s}_Z$ given in Section 4.2. This lemma follows from Lemma 4.11 and Remark 4.12. $\square$

We have the following proposition from Main Theorem.

Proposition 5.15 *Let X be a closed Riemannian 4-manifold and $\mathfrak{s}$ be a Spin^{c-} structure on X. Take a gauge transformation $u \in \Gamma(\widetilde{\mathbb{C}})$. Then the following statements are equivalent:*

1. *The gauge transformation u preserves the orientation of the index bundle $\{\mathrm{ind}(D_A)\}$.*
2. *The Spin structure induced on the zero locus of a transverse section of the vector bundle $L \oplus \pi^* E$ on $X \times S^1$ from Theorem 3.8 is trivial.*

We consider the Spin structure on the zero locus of the transverse section of $L \oplus \pi^* E$.

Lemma 5.16 *If it is necessary, we perturb u by homotopy and we may assume that -1 is a regular value of u. Let h be a section of E such that h is transverse to the zero section and its submanifold $u^{-1}(-1) \subset X$. Let $C = h^{-1}(0) \cap u^{-1}(-1)$ and $U(C)$ is its tubular neighborhood. From Theorem 3.8, we have a spin structure $\mathfrak{s}_C$ on $C \subset U(C)$ introduced by the section $\mathrm{Im}(u) \oplus h \in \Gamma((\det E \oplus E)|_{U(C)})$. Then there exists a transverse section of the vector bundle $L \oplus \pi^* E$ whose zero locus is $(h^{-1}(0) \cap u^{-1}(-1)) \times \{\frac{1}{2}\} \subset X \times \{\frac{1}{2}\}$ and the Spin structure on $C \times \{\frac{1}{2}\}$ given in Theorem 3.8 coincides with $\mathfrak{s}_C$.*

Proof The transversality of $\mathrm{Im}(u) \oplus h \in \Gamma((\det E \oplus E)|_{U(C)})$ follows from the assumptions on u and h. Then $\mathfrak{s}_C$ is well defined.

To define a section of $L \times \pi^* E$, it is sufficient to take a section $s = (s_0, s_1) \in \Gamma((\widetilde{\mathbb{C}} \oplus E) \times [0,1])$ such that $s_0(0) \cdot u = s_0(1)$. We define a section $s = (s_0, s_1)$ as

$$s_0(t) = (1-t) + tu, \quad s_1(t) = h.$$

From the definition of s, we see that $s^{-1}(0) = (h^{-1}(0) \cap u^{-1}(-1)) \times \{\frac{1}{2}\} \subset X \times \{\frac{1}{2}\}$ and s is transverse to the zero section. The normal bundle of $s^{-1}(0)$ splits into the $[0,1]$ direction and the X direction. The $[0,1]$ direction is trivial rank one vector bundle, and the X direction is isomorphic to the vector bundle $\det E \oplus E$. On $s^{-1}(0)$, the real part of s_0 is a section of a summand of the normal bundle which is the $[0,1]$ direction, and $\mathrm{Im}(s_0) \oplus s_1 = \mathrm{Im}(u)/2 \oplus h$ is a section of $\det E \oplus E$. Then we have a Spin structure on $s^{-1}(0)$ given by s and $\mathfrak{s}_Z$ coincides with $\mathfrak{s}_C$. $\square$

Proof of Theorem 5.9 From Lemma 5.16, a gauge transformation u preserves the orientation of the index bundle $\{\mathrm{ind}(D_A)\}_A$ on the configuration space if and only if the index of the Spin structure $\mathfrak{s}_C$ on $h^{-1}(0) \cap u^{-1}(-1)$ is trivial. The index gives the isomorphism $\Omega_1^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}$. $\square$

Remark 5.17 A slight change in the proof of Lemma 5.16 shows that there is a $G^+(3,0,2)$ structure ($\mathrm{Pin}_+^{\tilde{c}}$ structure) on $Y := u^{-1}(-1) \subset X$ induced by $\mathfrak{s}_X$. The vector bundle $E|_Y$ coincides with E_- given by this $G^+(3,0,2)$ structure. Thus we have that the index of the Spin structure on $h^{-1}(0) \cap Y$ coincides with the index of this $G^+(3,0,2)$ structure on Y from Main Theorem. From Theorem 5.9, u preserves an orientation of moduli space if and only if the index of this $G^+(3,0,2)$ structure on $Y = u^{-1}(-1)$ is trivial. This is the statement proved by Mikio Furuta.

5.2 Examples

In this section, we give an example of four manifold with Spin^{c-} structure such that there exists a gauge transformation which reverses an orientation of $\{\mathrm{ind}\, D_A\}$.

Proposition 5.18 *Let Y be a 3 dimensional closed Riemannian manifold and $\mathfrak{s}$ be a $G^+(3,0,2)$ structure ($\mathrm{Pin}_+^{\tilde{c}}$ structure) on Y such that the index of $\mathfrak{s}$ is nontrivial. We denote by X a four manifold given by gluing two copies of the disk bundle of $\det TY$ along boundaries. There is a Spin^{c-} structure on X and a gauge transformation which reverse the orientation of $\{\mathrm{ind}\, D_A\}$.*

Proof First, we construct a Spin^{c-} structure on X.

- Let $l = \det TY$ and $\pi : l \to Y$ be the projection. We will denote by E a vector bundle E_- associated with the $G^+(3,0,2)$ structure $\mathfrak{s}$. Note that $\det E \cong l$ and $\mathfrak{s}$ is given by a Spin structure of

$$\det TY \oplus TY \oplus \det E \oplus E.$$

- We will denote by $\mathfrak{s}'$ a Spin^{c-} structure on the total space l given by the Spin structure of

$$\pi^* l \oplus TY \oplus \pi^* l \oplus \pi^* E$$

induced by $\mathfrak{s}$.

- Let $D(l)$ be the disk bundle of l and $S(l)$ be the sphere bundle of l. We choose a canonical trivialization of $\pi^* l$ on $S(l)$. Hence we have that E and $TS(l)$ are orientable on $S(l)$. Thus the restriction of $\mathfrak{s}'$ to $S(l)$ induces a Spin^c structure.
- $S(l)$ is a double cover of Y and the covering transformation τ reverses the orientation of $S(l)$. We glue two copies of $D(l)$ along $S(l)$ by the map τ. Let $X = D(l) \cup_\tau D(l)$. X is an oriented closed manifold.
- We glue Spin^c structure on each $S(l)$ to give a Spin^{c-} structure $\mathfrak{s}_X$ on X such that the restriction of $\mathfrak{s}_X$ to each $D(l)$ coincides with $\mathfrak{s}'$.

Second, we give a gauge transformation u which reverses the orientation of $\mathrm{ind}\, D_A$.

- We will denote by f the tautological section of $\pi^* l$ on l.
- On the open subset $l \setminus Y$, $\pi^* l$ has the canonical trivialization. In this trivialization, we have $f(v) = |v|$ for $v \in l \setminus Y$.

- Deform f in the area $|v| \geq \frac{1}{2}$ and we assume that $f(v) = 1$ for $|v| \geq \frac{2}{3}$.
- We define $s(v) = -\exp(i\pi f(v))$ as follows: There is the natural isomorphism $l \otimes l \cong \mathbb{R}$ and using this isomorphism we have $f^{2n} \in \mathbb{R}$, $f^{2n+1} \in l$. We define $\exp(i\pi f(v)) \in S(\mathbb{R} \otimes \sqrt{-1}l)$ using Taylor expansion of exponential function.
- Note that $s = 1$ around $S(l)$ and we extend on X by 1 on another $D(l)$. We have $s^{-1}(-1) = Y$.
- The index of $\mathfrak{s}$ on Y is nontrivial and from Theorem 5.9 and Remark 5.17, u reverses the orientation of $\operatorname{ind} D_A$. □

We give an explicit example of Y in Proposition 5.18.

Lemma 5.19 *There is a $G^+(3,0,2)$ structure $\mathfrak{s}_0$ on $\mathbb{R}P^2 \times S^1$ whose index is nontrivial.*

Proof For simplicity of notation, we omit the notation of the pull-back of projections. From Lemma 3.4, we give a Spin structure on the vector bundle

$$\det T\mathbb{R}P^2 \oplus TS^1 \oplus T\mathbb{R}P^2 \oplus \det T\mathbb{R}P^2 \oplus T\mathbb{R}P^2$$

and we give the $G^+(3,0,2)$ structure $\mathfrak{s}_0$ by Lemma 3.2. Note that $E_- = T\mathbb{R}P^2$. We take a transverse section of $T\mathbb{R}P^2$ on $\mathbb{R}P^2$ whose zero locus is a single point on $\mathbb{R}P^2$. By pulling back this section, we have a transverse section h of E_- whose zero locus is $S^1 \times \{\mathrm{pt}\}$. We immediately see that the Spin structure on $h^{-1}(0) \cong S^1$ induced by h from Theorem 3.8 is given by the product $S^1 \times \mathrm{Spin}(1)$. This is the nontrivial element in the 1-dimensional Spin bordism group. From Main Theorem, we have that the index of $\mathfrak{s}_0$ is nontrivial. □

From Proposition 5.18 and Lemma 5.19, we deduce following corollary.

Corollary 5.20 We set $X = (\mathbb{R}P^3 \sharp \mathbb{R}P^3) \times S^1$ and $\mathfrak{s} = \mathfrak{s}_0$. The determinant bundle on the ambient space of the moduli space is not orientable.

This manifold is diffeomorphic to $P\gamma \times S^1$, where γ is the tautological bundle of $\mathbb{R}P^2$ and $P\gamma$ is its projectivization.

Remark 5.21 The gluing construction in the proof of Proposition 5.18 can be generalized in the case of gluing two 4 dimensional Spin^{c-} manifolds with boundary. If the restrictions of the Spin^{c-} structures to their boundaries induce Spin^c structures and their boundaries are diffeomorphic by a map which preserves the orientation and the Spin^c structure, our construction works.

Remark 5.22 In the case $Y = \mathbb{R}P^2 \times S^1$ and $\mathfrak{s} = \mathfrak{s}_0$, we have $S(l) = S^2 \times S^1$. We cannot glue $D(l)$ by $D^3 \times S^1$ along $S(l)$ because the first Chern class of Spin^c structure on $S(l)$ is the Euler class of TS^2 and this cannot be extended to $D^3 \times S^1$.

Appendix

Here we prove Proposition 3.22. We follow the notation of Section 3.2.

Lemma A.1 *We assume that $\phi \in \Gamma(Y, \mathfrak{S})$ satisfies $\|\not{D}_m\phi\| \le \lambda\|\phi\|$. There are functions $A_h(m,\lambda)$, $B_h(m,\lambda)$ of positive real numbers m,λ depending on h such that if we fix a value λ, they satisfy*

$$A_h(m,\lambda) \to 0, \quad B_h(m,\lambda) \to 1$$

when $m \to \infty$. The functions $A_h(m,\lambda)$, $B_h(m,\lambda)$ satisfy the inequalities

$$\int_V |\phi|^2 \le A_h(m,\lambda)\int_Y |\phi|^2, \quad B_h(m,\lambda)\int_Y |\phi|^2 \le \int_Y |\rho\phi|^2,$$

where $V = \{z \in U(C) \cong B(N(C)) \mid |z| > \frac{1}{2}\} \cup U(C)^c$.

Proof We have the following estimate:

$$\begin{aligned}
\lambda^2\int_Y |\phi|^2 &\ge \int_Y |\not{D}_m\phi|^2 \\
&= \int_Y \langle -\not{D}_m^2\phi, \phi\rangle \\
&= \int_Y \langle (-(\not{D})^2 - m\{\not{D}, h\} - m^2h^2)\phi, \phi\rangle \\
&= \int_Y |\not{D}\phi|^2 + \int_Y m^2|h\phi|^2 - \int_Y m\langle\{\not{D}, h\}, \phi\rangle \\
&\ge \int_V m^2|h\phi|^2 - \int_Y mC_0\|dh\|_\infty|\phi|^2 \\
&\ge \frac{m^2}{4}\int_V |\phi|^2 - mC_0\|dh\|_\infty\int_Y |\phi|^2.
\end{aligned}$$

We define $A_h(m,\lambda) = 2(mC_0\|dh\|_\infty + \lambda^2)/m^2$ and we have the first inequality, where C_0 is a constant only depending on the principal symbol of D (Clifford action). We have the second inequality by the following estimate:

$$\begin{aligned}
\left(\int_Y |\rho\phi|^2\right)^{\frac{1}{2}} &\ge \left(\int_Y |\phi|^2\right)^{\frac{1}{2}} - \left(\int_Y |(1-\rho)\phi|^2\right)^{\frac{1}{2}} \\
&\ge \left(\int_Y |\phi|^2\right)^{\frac{1}{2}} - \left(\int_V |\phi|^2\right)^{\frac{1}{2}} \\
&\ge \left(\int_Y |\phi|^2\right)^{\frac{1}{2}} - \left(A_h(m,\lambda)\int_Y |\phi|^2\right)^{\frac{1}{2}} \\
&= \left(1 - \sqrt{A_h(m,\lambda)}\right)\left(\int_Y |\phi|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Setting $B_h(m,\lambda) = \left(1 - \sqrt{A_h(m,\lambda)}\right)^2$, the proof is completed. □

A slight change of the proof of the above lemma actually shows the following lemma.

Lemma A.2 *We will denote by $\mathcal{S}(N(C), \mathfrak{S})$ a set of rapidly decreasing sections of $\mathfrak{S}$ on the total space of the vector bundle $N(C)$. We assume a section $\psi \in \mathcal{S}(N(C), \mathfrak{S})$ satisfies $\| \not{D}_m \psi \| \le \lambda \| \psi \|$. There are functions $A'(m,\lambda)$, $B'(m,\lambda)$ of positive real numbers m, λ such that if we fix a value λ, they satisfy*

$$A'(m,\lambda) \to 0, \quad B'(m,\lambda) \to 1$$

when $m \to \infty$. The functions $A'(m,\lambda)$, $B'(m,\lambda)$ satisfy the inequalities

$$\int_{V'} |\psi|^2 \le A'(m,\lambda) \int_{N(C)} |\psi|^2, \quad B'(m,\lambda) \int_{N(C)} |\psi|^2 \le \int_{N(C)} |\rho\psi|^2,$$

where $V' = \{z \in N(C) \mid |z| > \frac{1}{2}\}$.

Note that the perturbation term h of $\not{D}_m$ on $N(C)$ satisfies $|\{\not{D}, h\}| = 2$.

Lemma A.3 *We assume that λ is smaller than a constant given by the principal symbol of D and the differentiation of ρ. We assume that m is large enough. Let Π' be the orthogonal projection from $L^2(N(C), \mathfrak{S})$ to H. Then the map*

$$\mathcal{H}_m^\lambda \to H, \quad \phi \mapsto \Pi'(\rho\phi)$$

is injective.

Proof If a section $\phi \in \mathcal{H}_m^\lambda$ satisfies $\|\phi\|_{L^2(Y,\mathfrak{S})} = 1$ and $\Pi'(\rho\phi) = 0$, we have

$$\|\not{D}_m \rho\phi\|_{L^2(N(C),\mathfrak{S})} \ge \lambda_C \|\rho\phi\|_{L^2(N(C),\mathfrak{S})}$$

from Lemma 3.18 when m is large enough. The support of the function ρ is contained in

$$U = \{z \in N(C) \mid |z| < 1\}$$

and we identify U with $U(C) \subset Y$. We regard $\rho\phi$ as a section on Y and we have

$$\|\not{D}_m \rho\phi\|_{L^2(Y,\mathfrak{S})} \ge \lambda_C \|\rho\phi\|_{L^2(Y,\mathfrak{S})}.$$

But we have the following estimate from Lemma A.1:

$$\begin{aligned}
\int_Y |D_m \rho\phi|^2 &\le \int_Y |[\not{D}, \rho]\phi|^2 + \int_Y |\rho D_m \phi|^2 \\
&\le C_0 \|d\rho\|_\infty^2 \int_{U \cap V} |\phi|^2 + \int_Y |D_m \phi|^2 \\
&\le C_0 \|d\rho\|_\infty^2 A_h(m,\lambda) \int_Y |\phi|^2 + \lambda^2 \int_Y |\phi|^2 \\
&\le C_0 \|d\rho\|_\infty^2 (A_h(m,\lambda) + \lambda^2) B_h(m,\lambda)^{-1} \int_Y |\rho\phi|^2.
\end{aligned}$$

Provided m is large enough, the coefficient of $\int_Y |\rho\phi|^2$ tends to $C_0 \|d\rho\|_\infty^2 \lambda^2$. If this constant is smaller than λ_C, the above estimate contradicts the inequality $\|D_m \rho\phi\| \ge \lambda_C \|\rho\phi\|$. □

We fix the value λ so that $2C\|d\rho\|_\infty^2\lambda^2 < \lambda_C$.

Lemma A.4 *We assume that m is large enough. Let Π_m^λ be the orthogonal projection from $L^2(Y, \not{S})$ to $\mathcal{H}_m^\lambda$. The map*

$$H \to \mathcal{H}_m^\lambda, \quad \psi \mapsto \Pi_m^\lambda(\rho\psi)$$

is injective.

Proof If the map above is not injective, we have

$$\|D_m\rho\psi\|_{L^2(Y,\not{S})} \geq \lambda\|\rho\psi\|_{L^2(Y,\not{S})}$$

for some $\psi \in H$. On the other hand, elements in H are rapidly decreasing sections; hence we use Lemma A.2. Thus we have the following estimate by a similar argument to the proof of the Lemma A.3:

$$\int_{N(C)} |D_m\rho\psi|^2 \leq C\|d\rho\|_\infty^2 A'(m,0)B'(m,0)^{-1}\int_{N(C)} |\rho\psi|^2.$$

If m is large enough, the above estimate contradicts the inequality $\|D_m\rho\psi\| \geq \lambda\|\rho\psi\|$. □

Proof of Proposition 3.22 We have $\dim H \geq \dim \mathcal{H}_m^\lambda$ from Lemma A.3 and we have $\dim H \leq \dim \mathcal{H}_m^\lambda$ from Lemma A.4. Thus the maps of Lemmas A.3 and A.4 are isomorphisms. In particular, the map in Proposition 3.22 is the same as the map in Lemma A.4 and it is an isomorphism. Moreover, it is easy to see that this map preserves the $\mathbb{Z}/2$ gradings and the left $\mathrm{Cl}_{(s^-, n+s^+)}$ actions. □

References

[1] **M F Atiyah**, **I M Singer**, *Index theory for skew-adjoint Fredholm operators*, Inst. Hautes Études Sci. Publ. Math. 37 (1969) 5–26 MR Zbl

[2] **S K Donaldson**, *The orientation of Yang–Mills moduli spaces and 4-manifold topology*, J. Differential Geom. 26 (1987) 397–428 MR Zbl

[3] **S K Donaldson**, **P B Kronheimer**, *The geometry of four-manifolds*, Oxford Univ. Press (1990) MR Zbl

[4] **J L Fast**, **S Ochanine**, *On the KO characteristic cycle of a* Spin^c *manifold*, Manuscripta Math. 115 (2004) 73–83 MR Zbl

[5] **D S Freed**, **M J Hopkins**, *Reflection positivity and invertible topological phases*, Geom. Topol. 25 (2021) 1165–1330 MR Zbl

[6] **H Fukaya**, **M Furuta**, **S Matsuo**, **T Onogi**, **S Yamaguchi**, **M Yamashita**, *The Atiyah–Patodi–Singer index and domain-wall fermion Dirac operators*, Comm. Math. Phys. 380 (2020) 1295–1311 MR Zbl

[7] **M Furuta**, *Index theorem, I*, Transl. Math. Monogr. 235, Amer. Math. Soc., Providence, RI (2007) MR Zbl

[8] **M Furuta**, **Y Kametani**, *Equivariant version of Rochlin-type congruences*, J. Math. Soc. Japan 66 (2014) 205–221 MR Zbl

[9] **S Hayashi**, *Localization of Dirac operators on* $4n+2$ *dimensional open* Spin^c *manifolds*, preprint (2013) arXiv 1306.0389

[10] **D Joyce**, **Y Tanaka**, **M Upmeier**, *On orientations for gauge-theoretic moduli spaces*, Adv. Math. 362 (2020) art. id. 106957 MR Zbl

[11] **P B Kronheimer**, *Four-manifold invariants from higher-rank bundles*, J. Differential Geom. 70 (2005) 59–112 MR Zbl

[12] **P B Kronheimer**, **T S Mrowka**, *Knot homology groups from instantons*, J. Topol. 4 (2011) 835–918 MR Zbl

[13] **H B Lawson, Jr**, **M-L Michelsohn**, *Spin geometry*, Princeton Math. Ser. 38, Princeton Univ. Press (1989) MR Zbl

[14] **N Nakamura**, $\mathrm{Pin}^-(2)$*-monopole equations and intersection forms with local coefficients of four-manifolds*, Math. Ann. 357 (2013) 915–939 MR Zbl

[15] **N Nakamura**, $\mathrm{Pin}^-(2)$*-monopole invariants*, J. Differential Geom. 101 (2015) 507–549 MR Zbl

[16] **N Nakamura**, *A note on the orientability of the moduli space of the* $\mathrm{Pin}^-(2)$ *monopole*, preprint (2018) In Japanese. Available at `http://kansai-gauge.squares.net/misc/pin2-ori.pdf`

[17] **E Witten**, *Supersymmetry and Morse theory*, J. Differential Geom. 17 (1982) 661–692 MR Zbl

[18] **W Zhang**, Spin^c*-manifolds and Rokhlin congruences*, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993) 689–692 MR Zbl

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Verdier duality on conically smooth stratified spaces

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We prove a duality for constructible sheaves on conically smooth stratified spaces. We consider sheaves with values in a stable and bicomplete ∞-category equipped with a closed symmetric monoidal structure, and in this setting constructible means locally constant along strata and with dualizable stalks. The crucial point where we need to employ the geometry of conically smooth structures is in showing that Lurie's version of Verdier duality restricts to an equivalence between constructible sheaves and cosheaves: this requires a computation of the exit paths ∞-category of a compact stratified space, which we obtain via resolution of singularities.

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1 Introduction

Constructible sheaves are of great interest in both algebraic and differential geometry, as they provide tools to study invariants for singular spaces (such as *intersection cohomology*; see Beĭlinson, Bernstein and Deligne [8]) and have relations with D-modules (see Kashiwara [12]). Roughly speaking, constructible sheaves are sheaves on stratified spaces that behave nicely on strata (see Definition 3.2 for more details). A fundamental feature of constructible sheaves is that, assuming a finiteness condition on the stalks, they carry a duality sometimes referred to as *Verdier duality*. This is an antiequivalence from the category of constructible sheaves to itself, which is defined by taking an internal hom into the *dualizing complex*. Verdier duality has many applications; for example, using abstract trace methods it allows one to associate to any constructible sheaf a class in Borel–Moore homology. One of the interests of these classes is that they can be related to Euler characteristics via computations with the six-functor formalism (see Kashiwara and Schapira [13, Chapter 9] for a discussion on classical index formulas and their microlocal enhancements).

As far the author knows, it was an idea of MacPherson that, when the strata are manifolds, the duality should be thought of as a combination of two different equivalences of categories. The first, induced by the construction of *sections with compact support*, was expected to identify constructible sheaves with constructible *cosheaves*. Without constructibility assumptions, this was proven by Lurie [15, Theorem 5.5.5.1]. The second maps contravariantly constructible cosheaves to sheaves, and is obtained using a foreseen combinatorial description of constructible (co)sheaves, similar in spirit to monodromy for local systems. Following Barwick, Glasman and Haine [7], we refer to this combinatorial description as *exodromy* (see [15, Theorem A.9.3] and Ayala, Francis and Tanaka [5, Theorem 1.2.5]). The topological exodromy equivalence uses a generalization of the homotopy type of a stratified topological space, which keeps track of the stratification. This is known as the category of *exit paths* of a stratified topological space (see [15, Definition A.6.2] and Treumann [19]). We use here the language of ∞-categories to realize the vision of MacPherson and prove the expected duality result in a very general setting.

The first appearance of a proof of Verdier duality following the approach proposed by MacPherson is due to Curry [10, Theorem 7.7]. That work deals with derived categories of constructible (co)sheaves of vector spaces on locally finite regular CW complexes. This approach was later generalized by Aoki [1], working with spectra-valued functors on posets. In our setting, a stratum can be any smooth manifold. Hence, via exodromy, we obtain a duality for spectra-valued functors on ∞-categories that are not necessarily posets. Another mention of Verdier duality is due to Ayala, Mazel-Gee and Rozenblyum [6, Example 1.10.8], who outline a strategy to prove Verdier duality on stratified topological spaces that is essentially the same as the one we employ in this paper. However, some of the main steps in their outline lack a rigorous proof (see Remark 4.10 for a more detailed comment).

Let us spend a few words to specify more precisely the framework in which we are working. Relying on our previous paper [20], we will be able to deal with sheaves valued in any stable bicomplete ∞-category $\mathcal{C}$, equipped with a closed symmetric monoidal structure. The machinery of six functors developed in [20] supplies us with a *dualizing sheaf* $\omega_X^{\mathcal{C}}$ for any $\mathcal{C}$ as above and X a locally compact Hausdorff stratified space. More precisely, if $a\colon X \to *$ is the unique map, $\omega_X^{\mathcal{C}}$ is defined by applying the functor

$$a_{\mathcal{C}}^{!}\colon \mathcal{C} \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

to the monoidal unit of $\mathcal{C}$. Our duality functor will be given by taking an internal hom into $\omega_X^{\mathcal{C}}$, and denoted by $D_X^{\mathcal{C}}$.

Following the nomenclature of [7], we will define a sheaf with values in $\mathcal{C}$ to be *formally constructible*[1] if its restriction to each stratum is locally constant, and *constructible* if furthermore all of its stalks are dualizable. Similar definitions can be given for $\mathcal{C}$-valued cosheaves by observing that, up to passing to an opposite category, these are $\mathcal{C}^{\mathrm{op}}$-valued sheaves.

[1] In this paper we will only deal with sheaves which are constructible with respect to a fixed stratification, as opposed to [13, Chapter 8], for example.

Remark 1.1 The requirement of dualizability for stalks is unavoidable because, when X is the point, $\omega_X^{\mathcal{C}}$ is the monoidal unit of $\mathcal{C}$ and the duality functor coincides with the one coming from the monoidal structure on $\mathcal{C}$. Furthermore, this assumption is highly reasonable. For example, if $\mathcal{C} = D(R)$ and R is a commutative ring (or, more generally, a module over any E_∞-ring spectrum), it is a well-known result that a complex is dualizable if and only if it is perfect (see for example [15, Proposition 7.2.4.4] for a proof of a more general statement about E_1-ring spectra). Consequently, under our assumptions, we are able to recover the classical setting as a special case.

For the geometric side of the story, we will consider *conically smooth stratified spaces*.[2] These were introduced by Ayala, Francis and Tanaka [5], and provide a natural extension of C^∞-structures in the stratified setting. Notable examples of stratified spaces admitting a conically smooth atlas are *Whitney stratified spaces*, as proven by Nocera and Volpe [17]. The definition of conically smooth atlases is rather involved, as it relies on an elaborate inductive construction based on the *depth* of a stratification. For convenience, we recall the definition of depth.

Definition 1.2 Let $s\colon X \to P$ be a stratified space. Then the *depth* is defined as

$$\operatorname{depth}(X) = \sup_{x \in X} (\dim_x(X) - \dim_x(X_{s(x)})),$$

where dim denotes the covering dimension and $X_{s(x)}$ is the stratum of X corresponding to $s(x) \in P$.

We suggest the reader has a look at the introductions of [5; 17] to get an idea of how this works.

The main feature of conically smooth structures we will use in this paper is the *unzip* construction (see [5, Definition 7.3.11]), which allows one to functorially resolve any conically smooth stratified space into a manifold with corners. We will give a brief explanation of how this works in Example 2.15, but for now let us only mention that, if $X_k \hookrightarrow X$ is the inclusion of a stratum of maximal depth, it consists of a square

(1.3)
$$\begin{tikzcd} \operatorname{Link}_k(X) \arrow[r, hook] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Unzip}_k(X) \arrow[d] \\ X_k \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & X \end{tikzcd}$$

which is both pushout and pullback, and $\operatorname{Unzip}_k(X)$ is a conically smooth manifold with *boundary* given by $\operatorname{Link}_k(X)$ such that both its interior and $\operatorname{Link}_k(X)$ have depth strictly smaller than that of X. An interesting consequence of the existence of the pushout/pullback square (1.3) is that the notion of conically smooth map is completely determined by that of smooth maps between manifolds with corners.

We are now ready to state our main result.

[2]Most of our strategy to prove Verdier duality works more generally for C^0-stratified spaces (see [5, Definition 2.1.15]). However, the proof of Proposition 2.19 relies on the existence of blow-ups, which are not available without the presence a conically smooth structure. In a future paper [21], we will use some general facts about stratified homotopy types to show that the exit path ∞-category of a compact C^0-stratified space is a compact object in $\mathcal{C}\mathrm{at}_\infty$.

Theorem 1.4 (Theorem 4.8) *Let X be a conically smooth stratified space and let $\mathcal{S}\mathrm{hv}^c(X;\mathcal{C})$ be the full subcategory of $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ spanned by constructible sheaves. Then the functor*

$$D_X^{\mathcal{C}}\colon \mathcal{S}\mathrm{hv}(X;\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

restricts to an equivalence of ∞-categories

$$D_X^{\mathcal{C}}\colon \mathcal{S}\mathrm{hv}^c(X;\mathcal{C})^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{S}\mathrm{hv}^c(X;\mathcal{C}).$$

To conclude this introduction, let us make a short comment on how our proof strategy goes. As mentioned earlier, our first observation is that the functor $D_X^{\mathcal{C}}$ factors through the equivalence

$$\mathbb{D}_{\mathcal{C}}\colon \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}),$$

proven by Lurie [15, Theorem 5.5.5.1]. Most of the work then lies in proving that the restriction of $\mathbb{D}_{\mathcal{C}}$ to constructible sheaves factors through constructible cosheaves. We first show in Proposition 4.3 that $\omega_X^{\mathcal{C}}$ is constructible when $\mathcal{C} = \mathcal{S}\mathrm{p}$ (the ∞-category of spectra), and from the techniques developed in [20] we deduce immediately that

$$a_{\mathcal{C}}^!\colon \mathcal{C} \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

factors through formally constructible sheaves. As a consequence of this and some properties of constructible sheaves that follow from *homotopy invariance* (see Theorem 3.4), one deduces that $\mathbb{D}_{\mathcal{C}}$ maps formally constructible sheaves into formally constructible cosheaves. We stress that being able to work with such a general class of coefficients, which is closed under passing to opposite categories, makes this step extremely formal.

The missing piece is then showing that $\mathbb{D}_{\mathcal{C}}$ preserves the property of having dualizable stalks. This is the point where we have to employ the geometry of conically smooth structures. More specifically, we use the unzip constuction and an inductive argument on the depth to prove that any compact stratified space equipped with a conically smooth structure has a finite exit paths ∞-category (Proposition 2.19). For simplicity, let us explain how to use Proposition 2.19 in the special case $X = C(Z)$ with Z compact, where $C(Z)$ denotes the cone on Z. If $x \in X$ is the cone point and F is any constructible sheaf on X, there is a fiber sequence

$$\Gamma_x(X;F) \to F_x \to \Gamma(Z;F). \tag{1.5}$$

Here $\Gamma_x(X;F)$ denotes the sections of F supported at x (ie the stalk of the associated cosheaf of compactly supported sections of F) and F_x is the stalk of F at x. By Proposition 2.19 and the exodromy equivalence (which we show holds also for our general class of coefficients in Theorem 3.19), one deduces that $\Gamma(Z;F)$ is dualizable. Thus, using the fiber sequence above, F_x is dualizable if and only if $\Gamma_x(X;F)$ is, which proves our claim.

1.1 Linear overview

We now give a linear overview of the results in our paper.

Section 2 is mainly devoted to the proof Proposition 2.19. In the first part we recall the definition of a finite ∞-category, and show how these can be described in the model of quasicategories. None of these results or definitions are new, but we decided to include a few words on the subject since we could not find any reference dealing with it in our preferred fashion. In the second part we recall Lurie's definition of the simplicial set of exit paths of a stratified topological space. Given a proper stratified fiber bundle $\pi: L \to X$, we show how one can conveniently compute the exit paths of the fiberwise cone of π in terms of L and X. To prove Proposition 2.19, we cover a conically smooth stratified space with an open subset given by the locus of points of depth zero and a tubular neighborhood of its complement. By induction and Lemma 2.13, one is then left to show that the exit paths ∞-category of the former is finite. This is proven using the unzip construction. Namely, by unzipping the complement, the open subset of points of depth 0 can be identified with the interior of a compact manifold with corners.

In Section 3 we extend the results of Haine, Porta and Teyssier [11] to sheaves valued in stable bicomplete ∞-categories. This is very formal, after [20]. As a consequence, we show that the stalk at a point x of a constructible sheaf is the same as sections at any conical chart around x (Corollary 3.7). We also provide a convenient description of the restriction of a constructible sheaf along a stratum (Corollary 3.10). These two results are essential and are used very often in what follows. For example, the first immediately implies the existence of the fiber sequence (1.5). In the second part we then characterize constructible sheaves by the property of being homotopy invariant (see Proposition 3.13), and use this to deduce exodromy for conically smooth spaces with general stable bicomplete coefficients (Theorem 3.19).

In Section 4 we prove our main result. Given a C^0-stratified space X, through an inductive argument on the depth we show that $\omega_X^{\mathcal{C}}$ is constructible (Proposition 4.3). We first reduce to proving the statement in the case $\mathcal{C} = \mathcal{S}\mathrm{p}$ by employing the techniques developed in [20]. Then the only nontrivial part consists in showing that, when X is a cone, the stalk of the dualizing sheaf at the cone point is a finite spectrum. We then conclude by proving Theorem 4.8. As explained at the beginning of the introduction, our argument starts by observing that the duality functor factors through Lurie's Verdier duality. The hard part then consists in showing that the latter restricts to an equivalence between constructible (co)sheaves, for which we use all the results obtained previously in the paper.

Finally, in the appendix, we show that the shape of any proper and locally contractible ∞-topos is a compact object in the ∞-category of ∞-groupoids.

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Of course, the second person I would like to thank is Guglielmo himself. The hours of discussions we've had through Zoom during early pandemic times, and later while he visited in Regensburg, have helped me to sharpen my intuition on stratified spaces. The importance of our friendship too cannot be overestimated.

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2 Finite exit paths

This first section contains the main geometric input needed to achieve our goal. Namely, we show the exit paths ∞-category of a compact conically smooth stratified space is finite (Proposition 2.19).

2.1 Finite ∞-categories

This short section is devoted to recalling the definition of a finite ∞-category. Before going into that, we say a few words about what an ∞-category is for us. In this paper, we work in the model of quasicategories. Let $\mathrm{s}\mathcal{S}\mathrm{et}$ be the category of simplicial sets. Following [9, Example 7.10.14], we define $\mathcal{C}\mathrm{at}_\infty$ as the localization of $\mathrm{s}\mathcal{S}\mathrm{et}$ at the class of Joyal equivalences. The class of Joyal equivalences and fibrations equips $\mathrm{s}\mathcal{S}\mathrm{et}$ with the structure of a category with weak equivalences and fibrations in the sense of [9, Definition 7.4.12]. Hence, by [9, Theorem 7.5.18], any object in $\mathcal{C}\mathrm{at}_\infty$ is equivalent to the image through the localization functor $\mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}_\infty$ of a fibrant object in the Joyal model structure. For this reason, objects of $\mathcal{C}\mathrm{at}_\infty$ will be called ∞-categories.

The first definition we propose is expressed internally to the ∞-category $\mathcal{C}\mathrm{at}_\infty$ in terms of pushouts, and so in a kind of model-independent fashion. Later we prove that this is actually equivalent to a notion of finiteness that one might expect in the simplicial model. All the results appearing here are not at all original, but we still felt the need to write this section as, in the process of completing the paper, we could not locate a reference dealing with the subject. In what follows, we will denote by $\mathcal{S}$ the full subcategory of $\mathcal{C}\mathrm{at}_\infty$ spanned by ∞-groupoids.

Definition 2.1 An ∞-category is said to be *finite* if it belongs to the smallest full subcategory of $\mathcal{C}\mathrm{at}_\infty$ which contains $\varnothing$, Δ^0 and Δ^1 and is closed under pushouts. An ∞-groupoid is said to be finite if it is so as an ∞-category. We will denote by $\mathcal{C}\mathrm{at}_\infty^f$ and $\mathcal{S}^f$, respectively, the full subcategories of $\mathcal{C}\mathrm{at}_\infty$ and $\mathcal{S}$ spanned by finite objects.

Remark 2.2 Recall that the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ admits both a left and a right adjoint, and one may describe the left adjoint on objects by sending an ∞-category $\mathcal{C}$ to the localization $\mathcal{C}[\mathcal{C}^{-1}]$. Thus, since $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ preserves colimits and $\Delta^1[(\Delta^1)^{-1}] \simeq \Delta^0$, one may identify the class of finite ∞-groupoids with the objects of the smallest full subcategory of $\mathcal{S}$ which contains $\varnothing$ and Δ^0 and is closed under pushouts. This implies in particular that, for any finite ∞-category $\mathcal{C}$, the localization $\mathcal{C}[\mathcal{C}^{-1}]$ is again finite.

Lemma 2.3 *Let $\mathcal{C}$ be a finite ∞-category and let W be a finite subcategory of $\mathcal{C}$. Then the localization $\mathcal{C}[W^{-1}]$ is again finite.*

Proof We have a pushout square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ W[W^{-1}] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{C}[W^{-1}] \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$; thus, it suffices to show that $W[W^{-1}]$ is finite. This follows immediately by Remark 2.2. □

Recall that a simplicial set is said to be *finite* if it has a finite number of nondegenerate simplices. In the next proposition we reconcile this notion of finiteness with the one in Definition 2.1. We will need the following lemma, whose proof was explained to us by Sebastian Wolf.

Lemma 2.4 *Let $\mathcal{C}$ be an ∞-category and let $f: K \to \mathcal{C}$ be any map of simplicial sets, where K is finite. Moreover, suppose that there exists a finite simplicial set K' and a Joyal equivalence $g: K' \to \mathcal{C}$. Then there exists a finite simplicial set L, a Joyal equivalence $j: L \to \mathcal{C}$ and a commutative diagram in* sSet

$$\begin{tikzcd} & \mathcal{C} \\ K \arrow[ur, "f"] \arrow[r, hook, "k"] & L \arrow[u, "j"'] \end{tikzcd}$$

where k is a monomorphism.

Proof We define inductively a sequence of finite simplicial sets $\{K'_n\}_{n \in \mathbb{N}}$. We set $K'_0 = K'$, and we define K'_n via the pushout

$$\begin{tikzcd} \coprod_{\Lambda^n_j \to K'_{n-1}} \Lambda^n_j \arrow[r] \arrow[d, hook] & K'_{n-1} \arrow[d, hook] \\ \coprod_{\Lambda^n_j \to K'_{n-1}} \Delta^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & K'_n \end{tikzcd}$$

Now set

$$K'_\infty := \operatorname{colim}(K'_0 \hookrightarrow K'_1 \hookrightarrow \cdots \hookrightarrow K'_n \hookrightarrow \cdots).$$

Since all horns are finite simplicial sets, any map $\Lambda^n_j \to K'_\infty$ factors through some K'_m. By construction of the sequence, we get a commutative diagram

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d] & K'_m \arrow[r, hook] \arrow[d, hook] & K'_\infty \\ \Delta^n \arrow[r, dotted] & K'_{m+1} \arrow[ur, hook] & \end{tikzcd}$$

which implies that K'_∞ is an ∞-category. Since the class of categorical anodyne extensions is saturated (see [9, Definition 3.3.3]), $K' \hookrightarrow K'_\infty$ is a categorical anodyne extension, and in particular a Joyal equivalence. Hence, by the assumption that $\mathcal{C}$ is an ∞-category, we get a commutative triangle

$$\begin{tikzcd} K' \arrow[r, "g"] \arrow[d, hook] & \mathcal{C} \\ K'_\infty \arrow[ur, dotted, "\phi"'] & \end{tikzcd}$$

where ϕ is a Joyal equivalence by the 2-out-of-3 property. Since K'_∞ is also an ∞-category, ϕ admits a quasi-inverse

$$\psi\colon \mathcal{C} \to K'_\infty.$$

By the finiteness of K, the composition

$$\psi f\colon K \to \mathcal{C} \to K'_\infty$$

factors through some $\delta\colon K \to K'_n$. Thus, we get a triangle

$$\begin{tikzcd} & \mathcal{C} & \\ K \arrow[ur, "f"] \arrow[r, "\delta"] & K'_n \arrow[r, hook] & K'_\infty \arrow[ul, "\phi"'] \end{tikzcd}$$

which commutes up to J-homotopy, where J is the interval object for the Joyal model structure as defined in [9, Definition 3.3.3].

Now let L be the mapping cylinder of δ. Since J is a finite simplicial set, L must be finite as well. By the usual factorizations obtained via mapping cylinders, we get a triangle

$$\begin{tikzcd} & \mathcal{C} \\ K \arrow[ur, "f"] \arrow[r, hook, "i"] & L \arrow[u, "p"'] \end{tikzcd}$$

commuting up to J-homotopy, where i is a monomorphism and p is a Joyal equivalence. If H is a J-homotopy between f and pi, we may find a map $\widetilde{H}$ fitting in the diagram

$$\begin{tikzcd} K \times J \cup L \times \{1\} \arrow[r, "H \cup p"] \arrow[d, hook] & \mathcal{C} \\ L \times J \arrow[ur, dotted, "\widetilde{H}"'] & \end{tikzcd}$$

since $K \times J \cup L \times \{1\} \hookrightarrow L \times J$ is a categorical anodyne extension. Denote by j the restriction of $\widetilde{H}$ to $L \times \{0\}$. By construction, we get a commutative triangle

$$\begin{tikzcd} & \mathcal{C} \\ K \arrow[ur, "f"] \arrow[r, hook, "i"] & L \arrow[u, "j"'] \end{tikzcd}$$

Since the map j is J-homotopic to p, it is a Joyal equivalence. □

Proposition 2.5 *Let $\gamma\colon \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}_\infty$ be the localization functor. Then an ∞-category $\mathcal{C}$ is finite if and only if there exists a finite simplicial set K and an equivalence $\mathcal{C} \simeq \gamma(K)$.*

Proof Let $\mathrm{s}\mathcal{S}\mathrm{et}^f$ be the full subcategory of $\mathrm{s}\mathcal{S}\mathrm{et}$ spanned by the finite simplicial sets, and denote by $\mathcal{F}$ the essential image of the restriction of γ to $\mathrm{s}\mathcal{S}\mathrm{et}^f$. We need to show that $\mathcal{C}\mathrm{at}_\infty^f$ coincides with $\mathcal{F}$.

Let K be any finite simplicial set, so that in particular there exists some finite n such that $K = \mathrm{sk}_n(K)$. By induction on n and using the cellular decomposition

$$\begin{tikzcd} \coprod_{\partial\Delta^n \to K} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sk}_{n-1}(K) \arrow[d] \\ \coprod_{\Delta^n \to K} \Delta^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{sk}_n(K) \end{tikzcd}$$

we see that, to prove that $L(K)$ belongs to $\mathcal{C}\mathrm{at}_\infty^f$, it suffices to show that each Δ^n does. But this is clear because the n-simplex is Joyal equivalent to the n-spine. Thus, $\mathcal{F} \subseteq \mathcal{C}\mathrm{at}_\infty^f$.

Since $\mathcal{F}$ contains $\varnothing$, Δ^0 and Δ^1, we are now only left to show that $\mathcal{F}$ is closed under pushouts. Let

$$\mathcal{D} \leftarrow \mathcal{C} \rightarrow \mathcal{E}$$

be any cospan of ∞-categories in $\mathcal{F}$, and let $K \to \mathcal{C}$ be any Joyal equivalence, where K is a finite simplicial set. By applying Lemma 2.4 twice, we get a map of cospans

$$\begin{tikzcd} L \arrow[d] & K \arrow[l, hook'] \arrow[r, hook] \arrow[d] & M \arrow[d] \\ \mathcal{D} & \mathcal{C} \arrow[l] \arrow[r] & \mathcal{E} \end{tikzcd}$$

where the vertical arrows are Joyal equivalences and the upper horizontal arrows are monomorphisms. We then get a Joyal equivalence between the respective homotopy pushouts, and thus the desired conclusion. □

2.2 Finiteness properties of compact conically smooth spaces

The main goal of this section is to show that the exit paths ∞-category of a compact conically smooth stratified space is finite (Proposition 2.19). For this purpose, we make use of Lurie's model of the exit paths ∞-category, whose definition we now recall.

By a slight abuse of notation, for a poset P we still denote by P the topological space obtained by equipping the poset with the Alexandroff topology. If $X \to P$ is a stratified topological space, then we define $\mathrm{Exit}_P(X)$ by forming the pullback

$$\begin{tikzcd} \mathrm{Exit}_P(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sing}(X) \arrow[d] \\ N(P) \arrow[r] & \mathrm{Sing}(P) \end{tikzcd}$$

in the category of simplicial sets. Lurie showed that if the stratification $X \to P$ is conical, then $\mathrm{Exit}_P(X)$ is an ∞-category [15, Theorem A.6.4].

Example 2.6 Consider the stratified space $\mathbb{R}^2 \to \{0 < 1\}$, where the closed stratum is given by the origin. This can be identified, up to stratified homeomorphism, with the cone on S^1, with its natural stratification. Using Lemma 2.13, one can show that $\mathrm{Exit}_{\{0<1\}}(\mathbb{R}^2)$ is equivalent to the ∞-category $(B\mathbb{Z})^{\triangleleft}$. This is given by formally adding an initial object to the classifying space $B\mathbb{Z}$. More explicitly, let us denote by x the origin, and y any point different from x. Then one can describe $\mathrm{Exit}_{\{0<1\}}(\mathbb{R}^2)$ as a 1-category with two distinct objects x and y, where x is initial and the monoid of endomorphisms of y is given by $\mathbb{Z}$.

Remark 2.7 In what follows, we often consider a stratified space X without specifying any particular notation for its stratifying poset. In that case, by a slight abuse of notation, we write $\mathrm{Exit}(X)$ for the ∞-category of exit paths of X.

In [5, Definition 1.1.5], the authors propose an alternative model of the opposite of the ∞-category of exit paths of a conically smooth stratified space, called the *enter paths ∞-category*. Let us briefly recall this definition, as it also allows us to introduce some notation that is used throughout our paper.

Let Snglr be the 1-category whose objects are conically smooth stratified spaces and morphisms are conically smooth open immersions, and Bsc the full subcategory of Snglr spanned by *basic* conically smooth stratified spaces, ie those which are isomorphic to one of the type $\mathbb{R}^n \times C(Z)$, where Z is compact and conically smooth.

Lemma 4.1.4 of [5] shows that Snglr (and therefore Bsc) admits an enrichment in Kan complexes. By passing to homotopy coherent nerves, one gets ∞-categories that we denote by

(2.8) $$\mathcal{B}\mathrm{sc} \to \mathcal{S}\mathrm{nglr}.$$

For any conically smooth stratified space, the authors of [5] then define $\mathrm{Entr}(X)$ as the slice $\mathcal{B}\mathrm{sc}_{/X}$. More precisely, this is defined to be the pullback

$$\begin{tikzcd} \mathrm{Entr}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{S}\mathrm{nglr}_{/X} \arrow[d] \\ \mathcal{B}\mathrm{sc} \arrow[r] & \mathcal{S}\mathrm{nglr} \end{tikzcd}$$

Their proof of the exodromy equivalence for constructible sheaves of spaces, combined with the one in [15], implies that Lurie's exit paths ∞-category has to be equivalent to $\mathrm{Entr}(X)^{\mathrm{op}}$ (see [5, Corollary 1.2.10]).

In this section, we prefer to use Lurie's model, because it has an evident much richer functoriality. Indeed, one sees by the functoriality of Sing that Exit is functorial with respect to general stratified maps. On the other hand, the one in [5] is only functorial with respect to conically smooth open embeddings. We also see immediately that, if we stratify P over itself through the identity, then $\mathrm{Exit}_P(P) = N(P)$.

Definition 2.9 Let $f\colon (X \to P) \to (Y \to Q)$ be a map of stratified spaces. We say that f is a *full inclusion of strata* if the underlying map of posets $P \to Q$ is injective and full, and the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ P \arrow[r] & Q \end{tikzcd}$$

is a pullback of topological spaces.

We will also need the following lemma.

Lemma 2.10 *Let $X \to P$ and $Y \to Q$ be stratified spaces, and assume that the stratification on X is conical. Assume that we have a stratified embedding $Y \hookrightarrow X$ which is a full inclusion of strata. Then* $\mathrm{Exit}_Q(Y)$ *is an ∞-category and the induced functor* $\mathrm{Exit}_Q(Y) \to \mathrm{Exit}_P(X)$ *is fully faithful.*

Proof Since the functor Sing from topological spaces to simplicial sets preserves limits and since $Y \hookrightarrow X$ is a full inclusion of strata, we get a pullback square

(2.11)
$$\begin{tikzcd} \mathrm{Exit}_Q(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Exit}_P(X) \arrow[d] \\ N(Q) \arrow[r] & N(P) \end{tikzcd}$$

of simplicial sets. By [15, Theorem A.6.4(1)], the functor $\mathrm{Exit}_P(X) \to N(P)$ is an inner fibration. Using the pullback square (2.11), one sees that the map $\mathrm{Exit}_Q(Y) \to N(Q)$ is also an inner fibration, which implies that $\mathrm{Exit}_Y(Y)$ is an ∞-category. Moreover, since the inclusion is full, the functor $N(Q) \to N(P)$ is fully faithful, and thus we may conclude again by (2.11). □

Recall that, for a proper conically smooth fiber bundle $\pi\colon L \to X$, we define the *fiberwise cone* of π as the pushout

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L \times \mathbb{R}_{\geq 0} \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C(\pi) \end{tikzcd} \tag{2.12}$$

taken in the category of conically smooth stratified spaces (see [5, Example 3.6.3]). By definition, we get a new fiber bundle $C(\pi) \to X$ whose fibers are isomorphic to basics. We now show how to compute the exit paths of $C(\pi)$ in terms of L and X.

Lemma 2.13 *Let $\pi\colon L \to X$ be a proper conically smooth fiber bundle. Then the commutative square*

$$\begin{tikzcd} \mathrm{Exit}(L) \arrow[r] \arrow[d] & \mathrm{Exit}(L \times \mathbb{R}_{\geq 0}) \arrow[d] \\ \mathrm{Exit}(X) \arrow[r] & \mathrm{Exit}(C(\pi)) \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$ induced by (2.12) is a pushout.

Proof By the Van Kampen theorem for exit paths [15, Theorem A.7.1], we may assume that X is a basic. Thus, by [5, Corollary 7.1.4], we may also assume that π is a trivial bundle. Since Exit commutes with finite products, we may assume that $X = *$, and hence we are only left to prove that, for any compact conically smooth space L, the square

$$\begin{tikzcd} \mathrm{Exit}(L) \arrow[r] \arrow[d] & \mathrm{Exit}(L \times \mathbb{R}_{\geq 0}) \arrow[d] \\ \Delta^0 \arrow[r] & \mathrm{Exit}(C(L)) \end{tikzcd}$$

is a pushout in $\mathcal{C}\mathrm{at}_\infty$. This follows from [5, Lemma 6.1.4]. $\square$

We will also need to use the *unzip* and *link* construction, as defined in [5, Definition 7.3.11]. By [5, Proposition 7.3.10], for any proper constructible embedding $X \hookrightarrow Y$ we have a pullback square

$$\begin{tikzcd} \mathrm{Link}_X(Y) \arrow[r, hook] \arrow[d, "\pi_X"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Unzip}_X(Y) \arrow[d] \\ X \arrow[r, hook] & Y \end{tikzcd} \tag{2.14}$$

Here $\mathrm{Unzip}_X(Y)$ is a conically smooth manifold with corners whose interior is identified with $Y \setminus X$, and $\mathrm{Unzip}_X(Y) \to Y$ and $\pi_X\colon \mathrm{Link}_X(Y) \to X$ are proper constructible bundles.

Example 2.15 To get a feeling for how $\mathrm{Unzip}_X(Y)$ works, one may think of it as a generalization of the spherical blow-up (see [3]). More precisely, when Y is a smooth manifold stratified with a closed submanifold X and its open complement, the unzip of $X \hookrightarrow Y$ coincides with the spherical blow-up of X in Y, and the link is diffeomorphic to the boundary of any normalized tubular neighborhood of X in Y.

For a visual representation of the unzip construction, we refer to [5, Figure 7.3.1].

The link of a proper constructible embedding is used to provide tubular neighborhoods in the stratified setting. Proposition 8.2.5 of [5] shows that there is a conically smooth map

$$C(\pi_X) \hookrightarrow Y \tag{2.16}$$

under X which is a refinement onto its image and whose image is open in Y. Here we are using the same notation as in (2.14). Denote by $\widetilde{\mathrm{Link}_X(Y)} \times \mathbb{R}_{>0}$ and $\widetilde{C(\pi_X)}$ the respective refinements of $\mathrm{Link}_X(Y) \times \mathbb{R}_{>0}$ and $C(\pi_X)$ through the embedding (2.16).

Corollary 2.17 *Let $X \hookrightarrow Y$ be a proper conically smooth constructible embedding. Then the square*

$$\begin{tikzcd}
\mathrm{Exit}(\widetilde{\mathrm{Link}_X(Y)} \times \mathbb{R}_{>0}) \arrow[r] \arrow[d] & \mathrm{Exit}(Y \setminus X) \arrow[d] \\
\mathrm{Exit}(\widetilde{C(\pi_X)}) \arrow[r] & \mathrm{Exit}(Y)
\end{tikzcd}$$

is a pushout in $\mathcal{C}\mathrm{at}_\infty$.

Proof This follows immediately by the Van Kampen theorem for exit paths ∞-categories in [15, Theorem A.7.1]. $\square$

Remark 2.18 For the existence of tubular neighborhoods, one may relax the assumption of properness for a constructible embedding $i \colon X \hookrightarrow Y$ to just requiring that there is a factorization

$$\begin{tikzcd}
& Y' \arrow[dr, hook, "j"] & \\
X \arrow[ur, hook, "i'"] \arrow[rr, "i"] & & Y
\end{tikzcd}$$

where i' is a proper constructible embedding and j is a conically smooth open embedding. For example, if P is the stratifying poset of Y and $X = Y_\alpha$ for some $\alpha \in P$, one may pick $Y' = Y_{\geq\alpha}$ and thus get a tubular neighborhood of Y_p.

We are now ready to prove the main result of the section.

Proposition 2.19 *Let X be any compact conically smooth stratified space. Then $\mathrm{Exit}(X)$ is a finite ∞-category.*

Proof Since X is compact, X is finite-dimensional and hence also has finite depth. We then argue by induction on $\mathrm{depth}(X) = k$.

If $k = 0$, it is well known that $\mathrm{Exit}(X) = \mathrm{Sing}(X)$ is a finite ∞-groupoid. For example, this follows by the Van Kampen theorem [15, Theorem A.3.1] and the existence of finite good covers for X.

Assume now that k is positive. Denote by X_0 the union of strata of minimal depth, and by $X_{>0}$ its complement in X. One sees that $X_{>0} \hookrightarrow X$ is a proper constructible embedding, and hence, by Corollary 2.17, we get a pushout

$$\begin{tikzcd}
\mathrm{Exit}(\widetilde{\mathrm{Link}_{>0}(X)} \times \mathbb{R}_{>0}) \arrow[r] \arrow[d] & \mathrm{Exit}(X_0) \arrow[d] \\
\mathrm{Exit}(\widetilde{C(\pi_{>0})}) \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{Exit}(X).
\end{tikzcd}$$

By Lemma 2.13, to conclude the proof it suffices to show that $\mathrm{Exit}(\widetilde{\mathrm{Link}_{>0}(X)})$, $\mathrm{Exit}(\widetilde{C(\pi_{>0})})$ and $\mathrm{Exit}(X_0)$ are finite.

Being a closed subset of X, the space $X_{>0}$ is compact. By the pullback square (2.14), $\mathrm{Link}_{>0}(X)$ is compact too. Since the depths of both are strictly less than $\mathrm{depth}(X)$, by the inductive hypothesis both $X_{>0}$ and $\mathrm{Link}_{>0}(X)$ have finite exit paths ∞-categories. Notice that we have a stratified embedding

$$\widetilde{\mathrm{Link}_{>0}(X)} \times \mathbb{R}_{>0} \hookrightarrow X_0$$

and X_0 is a smooth manifold, so the stratification on $\widetilde{\mathrm{Link}_{>0}(X)}$ is trivial. Thus, $\mathrm{Exit}(\widetilde{\mathrm{Link}_{>0}(X)})$ is a localization of $\mathrm{Exit}(\mathrm{Link}_{>0}(X))$ at all maps and, by Lemma 2.3, $\mathrm{Exit}(\widetilde{\mathrm{Link}_{>0}(X)})$ is finite.

By Lemma 2.13 and the inductive hypothesis, we also know that $\mathrm{Exit}(C(\pi_{>0}))$ is finite. Using [5, Proposition 1.2.13], the canonical functor

$$\phi\colon \mathrm{Exit}(C(\pi_{>0})) \to \mathrm{Exit}(\widetilde{C(\pi_{>0})})$$

is a localization at the class of exit paths that are inverted by ϕ. Since the inclusion $C(\pi_{>0}) \to X$ lies under $X > 0$, the same argument as above shows that a noninvertible exit path is inverted by ϕ if and only if it lies inside $\mathrm{Link}_{>0}(X) \times \mathbb{R}_{>0} \hookrightarrow C(\pi_{>0})$. Notice that $\mathrm{Link}_{>0}(X) \times \mathbb{R}_{>0} \hookrightarrow C(\pi_{>0})$ is a full inclusion of strata. Therefore, by Lemma 2.10, the induced functor on exit paths is the inclusion of a full subcategory. This implies that one can identify $\mathrm{Exit}(\widetilde{C(\pi_{>0})})$ with the localization of $\mathrm{Exit}(C(\pi_{>0}))$ at $\mathrm{Exit}(\mathrm{Link}_{>o}(X)) \times \mathbb{R}_{>0}$. Hence, by Lemma 2.3, $\mathrm{Exit}(\widetilde{C(\pi_{>0})})$ is finite as well.

We know that X_0 is the interior of the compact manifold with corners $\mathrm{Unzip}_{>0}(X)$. One can show that the existence of collaring for corners [5, Lemma 8.2.1] implies that the inclusion $X_0 \hookrightarrow \mathrm{Unzip}_{>0}(X)$ is a homotopy equivalence. In the proof of [6, Lemma 2.1.3] one may find a construction of a homotopy inverse of the inclusion $X_0 \hookrightarrow \mathrm{Unzip}_{>0}(X)$ in the special case where the corner structure is a boundary. However, the construction of such an inverse in the more general case is completely analogous. Hence, to conclude the proof it suffices to show that $\mathrm{Sing}(\mathrm{Unzip}_{>0}(X))$ is finite. This follows by the existence of good covers for manifolds with corners. $\square$

Corollary 2.20 *Let X be a finitary conically smooth stratified space* (see [5, Definition 8.3.6]). *Then* $\mathrm{Exit}(X)$ *is a finite ∞-category. In particular, if X is the interior of a compact conically smooth manifold with corners, then* $\mathrm{Exit}(X)$ *is a finite ∞-category.*

Proof By Proposition 2.19 and Lemma 2.13, the class of conically smooth spaces with finite exit paths ∞-category contains all basics. Thus it suffices to show that it is closed under taking collar gluings. Suppose there is a collar gluing $f\colon Y \to [-1, 1]$ such that $f^{-1}\big([-1, 1)\big)$, $f^{-1}\big((-1, 1]\big)$ and $f^{-1}(0)$ are finitary. Then we get an open covering of Y given by $f^{-1}\big([-1, 1)\big)$, $f^{-1}\big((-1, 1]\big)$ and $\mathbb{R} \times f^{-1}(0)$, which implies that $\mathrm{Exit}(Y)$ is finite by Van Kampen.

The last part of the statement follows by [5, Theorem 8.3.10(1)]. □

3 Homotopy invariance and exodromy with general coefficients

In this section we explain how to use [20] to prove *homotopy invariance* and the *exodromy equivalence* for (formally) constructible sheaves (see Definition 3.2) valued in stable and bicomplete ∞-categories (Theorem 3.19).

A proof of homotopy invariance for constructible sheaves with presentable coefficients can be found in [11]. Our argument in Theorem 3.4 follows precisely the one in [11]. Nevertheless, we will try to quickly outline the main steps to convince the reader that all the results in [11], after [20], generalize to the setting of stable bicomplete coefficients, at least if we restrict ourselves to locally compact Hausdorff spaces.

The exodromy equivalence was first proven by Lurie [15, Theorem A.9.3] and later generalized in [18]. For constructible sheaves of ∞-groupoids on conically smooth stratified spaces, this was proven in [5, Theorem 1.2.5]. Here we stick with conically smooth stratified spaces, and we provide a short argument that works for constructible sheaves in stable and bicomplete ∞-categories. This is essentially a combination of the homotopy invariance and [5, Lemma 4.5.1].

With these at hand, we show that global sections of constructible sheaves on compact conically smooth stratified spaces are dualizable (Corollary 3.22).

3.1 Homotopy invariance of constructible sheaves

From now on, all ∞-categories appearing as coefficients for sheaves will be assumed to be stable and bicomplete, all topological spaces locally compact Hausdorff and all posets Noetherian. The next lemma shows in particular that the stratifying poset of any C^0-stratified space (see [5, Definition 2.1.15]) is Noetherian. Since any conically smooth stratified space is by definition C^0-stratified, all our stratified spaces of interest will have Noetherian stratifying posets.

Lemma 3.1 *Let $X \to P$ be a C^0-stratified space. Then P is locally finite and therefore Noetherian.*

Proof Recall that, by [5, Lemma 2.2.2], any C^0-stratified space admits a basis given by its open subsets isomorphic as stratified spaces to ones of type $\mathbb{R}^n \times C(Z)$, where Z is a compact C^0-stratified space. Therefore, it will suffice to show P is finite when X is compact. We prove this by induction on the depth of X.

When the depth of X is 0, it follows by [17, Lemma 2.22] that P is discrete. Since X is compact, it can have only a finite number of connected components, and therefore P has to be finite.

Assume that the depth of X is n. Since X is compact and C^0-stratified, one may find a finite cover of X by open subsets isomorphic as stratified spaces to $\mathbb{R}^n \times C(Z)$, where Z is a compact C^0-stratified space. But the depth of Z is smaller than n, and thus by the inductive assumption its stratifying poset has to be finite. Therefore, P has to be finite. □

Definition 3.2 Let X be any locally compact Hausdorff topological space, and let $a\colon X \to *$ be the unique map. We say that a sheaf $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is *constant* if there exists an object $M \in \mathcal{C}$ and an equivalence $F \simeq a^*_{\mathcal{C}} M$. We say that F is *locally constant* if there exists an open covering $\{U_i\}_{i\in I}$ of X such that $F|_{U_i}$ is constant.

Let $X \to P$ be a stratified space. We say that a sheaf $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is *formally constructible* if, for any $\alpha \in P$, the restriction of F to the stratum X_α is locally constant.

Assume now that $\mathcal{C}$ admits a closed symmetric monoidal structure, and denote by $\mathcal{C}^{\mathrm{dual}}$ the full subcategory of $\mathcal{C}$ spanned by dualizable objects. We say that F is *constructible* if F is formally constructible and each stalk of F belongs to $\mathcal{C}^{\mathrm{dual}}$.

We denote by $\mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;C)$ and $\mathcal{S}\mathrm{hv}^{c}(X;\mathcal{C})$ the full subcategories of $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ spanned respectively by formally constructible and constructible sheaves. Dually, we define formally constructible and constructible cosheaves on X as $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C}) := \mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ and $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}^{c}(X;\mathcal{C}) := \mathcal{S}\mathrm{hv}^{c}(X;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

In this paper we only deal with constructible sheaves with respect to a specified stratification. Therefore, we will take the liberty of omitting the stratifying poset from our notation for constructible sheaves.

We first recall the proof of the homotopy invariance of constructible sheaves and, before that, the definition of stratified homotopy equivalence.

Definition 3.3 Let $X \to P$ and $Y \to Q$ be stratified spaces, and let $[0,1] \subseteq \mathbb{R}$ be the closed interval, considered as a stratified space with a single stratum. A *stratified homotopy* is a map of stratified spaces $H\colon X \times [0,1] \to Y$.

We say that a stratified map $f\colon X \to Y$ is a *stratified homotopy equivalence* if there exists a stratified map $g\colon Y \to X$ and stratified homotopies $H\colon X \times [0,1] \to X$ and $K\colon Y \times [0,1] \to Y$ such that $H|_{X\times\{0\}} = \mathrm{id}_X$, $H|_{X\times\{1\}} = gf$, $K|_{Y\times\{0\}} = \mathrm{id}_Y$ and $K|_{Y\times\{1\}} = fg$.

Theorem 3.4 (homotopy invariance) *Let $X \to P$ be a stratified space. Let $p\colon X \times [0,1] \to X$ be the canonical projection. Then $p^*\colon \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{S}\mathrm{hv}(X \times [0,1];\mathcal{C})$ restricts to an equivalence*

$$\mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C}) \simeq \mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X \times [0,1];\mathcal{C}).$$

As a consequence, if $Y \to P$ is another stratified space and $f\colon X \to Y$ is a stratified homotopy equivalence, then the functor

$$f^*\colon \mathcal{S}\mathrm{hv}^{\mathrm{fc}}(Y;\mathcal{C}) \to \mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C}).$$

is an equivalence.

Proof We first treat the case of locally constant sheaves, ie when $P = *$. By [15, Lemma A.2.9; 20, Corollary 5.2], p^* is fully faithful. We start by showing that p_* preserves constant sheaves. If $a\colon X \to *$ and $b\colon X \times [0,1] \to *$ are the unique maps, then, for any object $M \in \mathcal{C}$, the fully faithfulness of p^* implies that we have equivalences

$$p_* b^* M \simeq p_* p^* a^* M \simeq a^* M.$$

Now let F be any locally constant sheaf on $X \times [0,1]$. By [11, Lemma 4.9], there exists an open cover $\{U_i\}_{i \in I}$ of X such that $F|_{U_i \times [0,1]}$ is constant. Therefore, since $[0,1]$ is compact, by proper base change (see [20, Proposition 6.1]) we see that $(p_* F)|_{U_i}$ is constant. Hence, p_* preserves locally constant sheaves. Thus, to conclude we only need to show that, for any locally constant sheaf F on $X \times [0,1]$, the counit map $p^* p_* F \to F$ is an equivalence.

Again by base change and [11, Lemma 4.9], we may reduce to the case when $F \simeq b^* M$ is constant. In this case we have a commutative diagram

$$\begin{tikzcd}
p^* p_* b^* M \arrow[r, "\mathrm{counit}_{b^* M}"] \arrow[d, "\simeq"] & b^* M \arrow[d, "\simeq"] \\
p^* p_* p^* a^* M \arrow[r, "\mathrm{counit}_{p^* a^* M}"] & p^* a^* M \\
p^* a^* M \arrow[u, "p^*(\mathrm{unit}_{a^* M})", "\simeq"'] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

which implies the desired result.

Now assume that P is any Noetherian poset. Using base change in a similar way as before, one sees that p_* preserves formally constructible sheaves, and thus we are left to show that, for any F constructible, $p^* p_* F \to F$ is an equivalence. By [20, Corollary 4.2], any stable and bicomplete ∞-category respects gluing in the sense of [11, Definition 5.17]. Hence, Lemma 5.19 of [11] implies that the functors given by restricting to the strata of X are jointly conservative. By base change we may thus assume that F is locally constant, whence the counit is known to be an equivalence by the previous step.

The last part of the statement then follows by a standard argument analogous to the proof of [20, Corollary 3.1]. □

We now present a couple of useful corollaries of homotopy invariance.

Corollary 3.5 *Let $f\colon X \to Y$ be a stratified homotopy equivalence and $F \in \mathcal{S}\mathrm{hv}^{\mathrm{fc}}(Y;\mathcal{C})$. The natural map*

$$\Gamma(Y;F) \to \Gamma(X;f^*F)$$

is an equivalence.

Proof The commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "a"'] & & Y \arrow[dl, "b"] \\ & * & \end{tikzcd}$$

induces an invertible natural transformation $f^*b^* \simeq a^*$. Since both a^* and b^* factor through formally constructible sheaves, we get $b^* \simeq \eta a^*$, where η is any adjoint inverse of the restriction of f^* to $\mathcal{S}\mathrm{hv}^{\mathrm{fc}}(Y;\mathcal{C})$. Thus, by passing to right adjoints, we get the desired equivalence. □

We will need the following lemma.

Lemma 3.6 *Let Z be a compact topological space. Let $\mathbb{R}_{\geq 0} \times Z \to C(Z)$ be the quotient map, and for each $\epsilon > 0$ denote by $C_\epsilon(Z)$ the image of the open subset $[0,\epsilon) \times Z$. Then the family of open subsets*

$$\{C_\epsilon(Z) \mid \epsilon \in \mathbb{R}_{>0}\}$$

forms a basis at the cone point.

Proof We will prove that, for every open subset W of $\mathbb{R}_{\geq 0} \times Z$ containing $\{0\} \times Z$, there exists some $\epsilon > 0$ such that $[0,\epsilon) \times Z \subseteq W$. Since Z is compact, one can obtain a finite covering of $\{0\} \times Z$ with opens of type $[0,\epsilon_i) \times V_i \subseteq W$, and thus, by taking ϵ to be the minimum of the ϵ_i, we get the claim. □

Corollary 3.7 *Let X be a C^0-stratified space and $F \in \mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C})$. For any point $x \in X$ and any conical chart $\mathbb{R}^n \times C(Z)$ centered at x, the natural map*

$$\Gamma(\mathbb{R}^n \times C(Z);F) \to F_x$$

is an equivalence.

Proof First of all, notice that there is a homeomorphism $\mathbb{R}^n \cong C(S^{n-1})$ sending 0 to the cone point, under which the subsets $C_\epsilon(S^{n-1})$ on the right-hand side are identified with open balls centered at zero with radius ϵ. Let us denote these subsets by $B_\epsilon(0)$. By Lemma 3.6, the family of open subsets $\{B_\epsilon(0) \times C_\epsilon(Z)\}_{\epsilon>0}$ is cofinal in the family of all open subsets of $\mathbb{R}^n \times C(Z)$ containing $(0,\text{cone point})$. Hence,

$$\varinjlim_{\epsilon>0} \Gamma(B_\epsilon(0) \times C_\epsilon(Z);F) \simeq F_x.$$

By Corollary 3.5, we have

$$\Gamma(\mathbb{R}^n \times C(Z);F) \simeq \varinjlim_{\epsilon>0} \Gamma(\mathbb{R}^n \times C_\epsilon(Z);F)$$ □

Now let $X \to P$ be a conically smooth stratified space, and let $\alpha \in P$. By Remark 2.18, we get a commutative triangle

$$\begin{tikzcd} & X_\alpha \arrow[dl, hook', "i"'] \arrow[dr, hook, "i_\alpha"] & \\ C(\pi_\alpha) \arrow[rr, hook, "j"'] & & X \end{tikzcd}$$

where π_α is the fiber bundle $\mathrm{Link}_{X_\alpha}(X_{\geq\alpha}) \to X_\alpha$, i is the cone-point section of the fiber bundle $p\colon C(\pi_\alpha) \to X_\alpha$ and j is a conically smooth open immersion. For any $F \in \mathcal{S}\mathrm{hv}(X; \mathcal{C})$, the unit of the adjunction $i^* \dashv i_*$ gives a natural map

(3.8) $$p_* j^* F \to p_* i_* i^* j^* F \simeq i_\alpha^* F.$$

Furthermore, the map (3.8) can be obtained by applying the sheafification functor to

(3.9) $$p_* j^* F \to p_* i_* (i^*)^{\mathrm{pre}} j^* F \simeq (i_\alpha^*)^{\mathrm{pre}} F,$$

where $(i^*)^{\mathrm{pre}}$ and $(i_\alpha^*)^{\mathrm{pre}}$ denote the corresponding presheaf pullback functors.

Corollary 3.10 *Let $X_\alpha \hookrightarrow X$ be the inclusion of a stratum in a conically smooth stratified space, and let F be any formally constructible sheaf on X. Then the map (3.8) is an equivalence.*

Proof Since F is a sheaf, it suffices to show that (3.9) is an equivalence. As usual, it suffices to prove that it is an equivalence after taking sections on any euclidean chart U of X_α. For any such U, by [5, Corollary 7.1.4],

$$\Gamma(U; p_* j^* F) = \Gamma(U \times C(Z); F)$$

for some compact conically smooth stratified space Z. Thus we are left to show that the natural map

(3.11) $$\Gamma(U \times C(Z); F) \to \varinjlim_{U \subseteq V} \Gamma(V; F)$$

is an equivalence.

By a cofinality argument, the map (3.11) factors through an equivalence

$$\varinjlim_{\epsilon > 0} \Gamma(U \times C_\epsilon(Z); F) \simeq \varinjlim_{U \subseteq V} \Gamma(V; F),$$

and thus we are left to show that

$$\Gamma(U \times C(Z); F) \to \varinjlim_{\epsilon > 0} \Gamma(U \times C_\epsilon(Z); F)$$

is an equivalence. This last assertion then follows by Corollary 3.5. □

3.2 Exodromy

This subsection is devoted to giving a proof of the exodromy equivalence on conically smooth stratified spaces for constructible sheaves valued in stable and bicomplete ∞-categories. To do this we use the model of the exit paths ∞-category of a conically smooth stratified space given in [5, Definition 1.1.5]. As an intermediate step, we provide a useful characterization of the property of being formally constructible for a sheaf on a conically smooth stratified space. In short, this says that a sheaf is formally constructible if and only if it is homotopy invariant (see Proposition 3.13 for a precise statement). We also collect a couple of useful corollaries of this fact, which are used later in the following section to prove a crucial step of our main result.

One has functors

$$\begin{tikzcd} \mathrm{Bsc}_{/X} \arrow[r, "\gamma"] \arrow[d, "\mathrm{im}"] & \mathcal{B}\mathrm{sc}_{/X} \\ \mathcal{U}(X) \end{tikzcd}$$

where $\mathcal{U}(X)$ denotes the poset of open subsets of X, and im sends an open immersion to its image in X. Lemma 4.5.1 of [5] shows that γ is a localization at the class $\mathcal{W}$ of open immersions of basics $U \hookrightarrow V$ such that U and V are abstractly isomorphic in Strat. That is, precomposing with γ gives an equivalence

$$\gamma^*\colon \mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C}) \to \mathrm{Fun}_{\mathcal{W}}(\mathrm{Bsc}^{\mathrm{op}}_{/X}, \mathcal{C}), \tag{3.12}$$

where the right-hand side denotes the full subcategory of $\mathrm{Fun}(\mathrm{Bsc}^{\mathrm{op}}_{/X}, \mathcal{C})$ spanned by functors which send all morphisms in $\mathcal{W}$ to equivalences. In the next proposition we show that $\mathcal{W}$ coincides with the class of open immersions which are stratified homotopy equivalences, and then characterize the property of being formally constructible through these maps.

Proposition 3.13 *Let $X \to P$ be a conically smooth stratified space and let $F \in \mathcal{S}\mathrm{hv}(X; \mathcal{C})$. Then the following assertions are equivalent*:

(i) *F is formally constructible.*

(ii) *For any inclusion $V \hookrightarrow U$ of basic open subsets of X which is a stratified homotopy equivalence*,

$$\Gamma(U; F) \to \Gamma(V; F)$$

is an equivalence.

(iii) *For any inclusion $V \hookrightarrow U$ of basic open subsets of X which are abstractly isomorphic*,

$$\Gamma(U; F) \to \Gamma(V; F)$$

is an equivalence.

Proof We first prove that (ii) is equivalent to (iii) by showing that an open immersion $j: V \hookrightarrow U$ of basic open subsets of X is a stratified homotopy equivalence if and only if U and V are abstractly isomorphic.

First of all, observe that j is conically smooth, because the conically smooth structures of U and V are restricted from that of X. If j is a stratified homotopy equivalence, it follows that U and V are stratified over the same subposet of P, and so in particular V intersects the stratum of maximal depth of U. Therefore, by the equivalence of conditions (2) and (4) in [5, Lemma 4.3.7], U and V are isomorphic.

Conversely, assume that U is abstractly isomorphic to V. Therefore, up to composing with isomorphisms on both sides, we may assume j is of the form $j\colon \mathbb{R}^n \times C(Z) \hookrightarrow \mathbb{R}^n \times C(Z)$. Then, by [5, Lemma 4.3.6], j is homotopy equivalent to $D_0 j$, where $D_0 j$ denotes the differential of j at the point $(0, \text{cone point})$ (see [5, Definition 3.1.4]). Since $D_0 j$ is a stratified homotopy equivalence, the same is true for j.

By Theorem 3.4, (i) implies (ii), so we are left to show that (iii) implies (i). Let $i\colon Y \to X$ be the inclusion of a stratum, and let $V \hookrightarrow U$ be an inclusion of euclidean charts of Y. By Corollary 3.10, the horizontal arrows in the commutative square

$$\begin{array}{ccc} \Gamma(U \times C(Z); F) & \longrightarrow & \Gamma(U; i^* F) \\ \downarrow & & \downarrow \\ \Gamma(V \times C(Z); F) & \longrightarrow & \Gamma(V; i^* F) \end{array}$$

are invertible, and thus $\Gamma(U; i^* F) \to \Gamma(V; i^* F)$ is invertible too. Therefore, to deduce that $i^* F$ is locally constant, we just need to show that (iii) implies (i) in the special case when X has a trivial stratification, ie when X is a smooth manifold. The result is now a very special case of [11, Proposition 3.1]. For the reader's convenience, we review and adapt the proof of [loc. cit.] to our setting in the following proposition. □

Proposition 3.14 *Let X be a smooth manifold and let $F \in \mathcal{S}\mathrm{hv}(X; \mathcal{C})$. Then the following assertions are equivalent*:

(i) *F is locally constant.*

(ii) *For any inclusion $V \hookrightarrow U$ of euclidean charts of X, the restriction*

$$\Gamma(U; F) \to \Gamma(V; F)$$

is an equivalence.

Proof Since the question is local, we may assume that $X = \mathbb{R}^n$, where we will prove that condition (ii) implies that F is constant. More precisely, we will show that, if $a\colon \mathbb{R}^n \to *$ is the unique map, the counit morphism

(3.15) $$a^* a_* F \to F$$

is an equivalence. Since $\mathbb{R}^n$ is hypercomplete and admits a basis given by those open subsets diffeomorphic to itself, it then suffices to check that, for any such open $j\colon U \hookrightarrow \mathbb{R}^n$, the map $a_* j_* j^* a^* a_* F \to a_* j_* j^* F$

obtained by applying to (3.15) the functor of sections at U is invertible. Notice that we have a commutative triangle

$$\begin{tikzcd} a_* j_* j^* a^* a_* F \arrow[r, "a_* j_* j^* \mathrm{counit}_F"] \arrow[d, "\mathrm{counit}_{a_* F}"'] & a_* j_* j^* F \\ a_* F \arrow[ur, "\simeq", "a_*(\mathrm{unit}_F)"'] & \end{tikzcd}$$

where the diagonal map is invertible by the assumption in (iii). Thus, to conclude the proof it suffices to show that a^* is fully faithful, which follows by the homotopy invariance of the shape (see [20, Corollary 3.1]). □

Corollary 3.16 *For $f: L \to X$ a conically smooth fiber bundle, the pushforward $f_*^{\mathcal{C}}: \mathcal{S}\mathrm{hv}(L;\mathcal{C}) \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ preserves formally constructible sheaves.*

Proof By definition of a conically smooth fiber bundle, for any point $x \in X$ there exists an open neighborhood U of x, a conically smooth stratified space Y and a pullback square

$$\begin{tikzcd} U \times Y \arrow[r, hook] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "f"] \\ U \arrow[r, hook] & X \end{tikzcd}$$

where p is the canonical projection. Let F be a formally constructible sheaf on L. To prove that $f_*^{\mathcal{C}} F$ is formally constructible, it suffices to show that its restriction to any U as above is formally constructible. Therefore, since any open immersion gives a locally contractible geometric morphism, by smooth base change (see [20, Lemma 3.25]) it suffices to show that $p_*^{\mathcal{C}}$ preserves formally constructible sheaves.

Let G be any formally constructible sheaf on $U \times Y$, and let $j: V \hookrightarrow W$ be any open immersion of basics in U which is a stratified homotopy equivalence. By Proposition 3.13, we need to show that the restriction of $p_*^{\mathcal{C}} G$ corresponding to j is an equivalence. We have a commutative square

$$\begin{tikzcd} \Gamma(W; p_*^{\mathcal{C}} G) \arrow[r] \arrow[d, "\simeq"] & \Gamma(V; p_*^{\mathcal{C}} G) \arrow[d, "\simeq"] \\ \Gamma(W \times Y; G) \arrow[r] & \Gamma(V \times Y; G) \end{tikzcd}$$

Since G is constructible and $j \times \mathrm{id}_Y: V \times Y \hookrightarrow W \times Y$ is again a stratified homotopy equivalence, by Corollary 3.5 we see that the lower horizontal arrow is an equivalence. The proof is then concluded by observing that both vertical arrows are equivalences. □

Corollary 3.17 *Let X be a conically smooth stratified space and let $i: X_\alpha \hookrightarrow X$ be the inclusion of a stratum. Let $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ be a formally constructible sheaf. Then $i_{\mathcal{C}}^! F$ is locally constant.*

Proof We have a fiber sequence

(3.18) $$i_{\mathcal{C}}^! F \to i_{\mathcal{C}}^* F \to i_{\mathcal{C}}^* j_*^{\mathcal{C}} j_{\mathcal{C}}^* F,$$

where j is the open immersion $X_{>\alpha} \hookrightarrow X$. Hence, to conclude it suffices to show that $i^*_{\mathcal{C}} j^{\mathcal{C}}_* j^*_{\mathcal{C}} F$ is locally constant.

Let $\pi_\alpha\colon \mathrm{Link}_{X_\alpha}(X_{\geq\alpha}) \to X_\alpha$ be the projection from the link of X_α in $X_{\geq\alpha}$. Denote by k the open immersion

$$k\colon \mathrm{Link}_{X_\alpha}(X_{\geq\alpha}) \times \mathbb{R}_{>0} \hookrightarrow C(\pi_\alpha) \hookrightarrow X,$$

and by p the conically smooth fiber bundle

$$p\colon \mathrm{Link}_{X_\alpha}(X_{\geq\alpha}) \times \mathbb{R}_{>0} \to \mathrm{Link}_{X_\alpha}(X_{\geq\alpha}) \xrightarrow{\pi_\alpha} X_\alpha,$$

where the first arrow is the canonical projection. Then, by Corollary 3.10, we have an equivalence $i^*_{\mathcal{C}} j^{\mathcal{C}}_* j^*_{\mathcal{C}} F \simeq p^{\mathcal{C}}_* k^*_{\mathcal{C}} F$. But, since $k^*_{\mathcal{C}} F$ is formally constructible and p is a conically smooth fiber bundle, we can apply Corollary 3.16 to deduce that $i^*_{\mathcal{C}} j^{\mathcal{C}}_* j^*_{\mathcal{C}} F$ is locally constant. □

Theorem 3.19 (exodromy) *The composition*

$$\mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C}) \xrightarrow{\gamma^*} \mathrm{Fun}_{\mathcal{W}}((\mathrm{Bsc}_{/X})^{\mathrm{op}}, \mathcal{C}) \xrightarrow{\mathrm{im}_*} \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathcal{C})$$

is fully faithful with essential image $\mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C})$. *Moreover, if we assume that* $\mathcal{C}$ *has a closed symmetric monoidal structure, the statement remains true if we replace* $\mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C})$ *by* $\mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C}^{\mathrm{dual}})$ *and* $\mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C})$ *by* $\mathcal{S}\mathrm{hv}^{c}(X;\mathcal{C})$.

Proof By Proposition 3.13, it suffices to show that the restriction of im_* to $\mathrm{Fun}_{\mathcal{W}}(\mathrm{Bsc}_{/X}, \mathcal{C})$ factors through $\mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C})$.

Let U be a basic open subset of X and let $\kappa\colon T \hookrightarrow \mathrm{Bsc}_{/U}$ be a covering sieve. There is at least one $V \in T$ whose image in U intersects the deepest stratum. By the equivalence of conditions (2) and (4) in [5, Lemma 4.3.7], U and V are abstractly isomorphic. Then, for any $F \in \mathrm{Fun}_{\mathcal{W}}(\mathrm{Bsc}_{/X}, \mathcal{C})$, we have a commutative triangle

$$\begin{tikzcd} \Gamma(U;F) \arrow[r] \arrow[dr, "\simeq"'] & \varprojlim_{O\in T} \Gamma(O;F) \arrow[d] \\ & \Gamma(V;F) \end{tikzcd}$$

where the diagonal map is invertible by assumption. By [5, Proposition 3.2.23], open subsets isomorphic to basics form a basis for the topology of X. Since X is hypercomplete, to show that $\mathrm{im}_* F$ is a sheaf it suffices, by [2, Theorem A.6], to check the sheaf condition on open covers formed by basics of basic open subsets. More succinctly, we have to show that the horizontal map in the triangle above is invertible. Moreover, by the 2-out-of-3 property, it suffices to show that the vertical map is invertible.

Let $\delta\colon T \to \mathcal{T}$ be the localization of T at $\mathcal{W}$. Since δ is final and κ^* sends maps in X to equivalences, the result then follows by observing that V is a terminal object in $\mathcal{T}$.

For the second part of the statement, we just need to show that, for any functor $F\colon \mathrm{Exit}(X) \to \mathcal{C}^{\mathrm{dual}}$, the stalks of $\mathrm{im}_* \gamma^* F$ are dualizable. By Corollary 3.7, it suffices to prove that, for any open U which is isomorphic to a basic, $\Gamma(U; \mathrm{im}_* \gamma^* F)$ is dualizable. But $\Gamma(U; \mathrm{im}_* \gamma^* F)$ is equivalent to the value of F at the object

$$(U \subseteq X) \in \mathrm{Exit}(X) = (\mathcal{B}\mathrm{sc}_{/X})^{\mathrm{op}},$$

and therefore is dualizable by assumption. □

Example 3.20 Consider again the stratified space $(X \to P) := (\mathbb{R}^2 \to \{0 < 1\})$ as in Example 2.6, and let $\mathcal{C}$ be a stable and bicomplete ∞-category. It follows from Theorem 3.19 that giving a formally constructible sheaf on X is essentially the same as providing two objects M and N in $\mathcal{C}$, a $\mathbb{Z}$-action on N and a $\mathbb{Z}$-equivariant map $\alpha\colon M \to N$, where M is equipped with the trivial action. One may equivalently supply a $\mathbb{Z}$-equivariant object N and a map $\tilde{\alpha}\colon M \to N^{h\mathbb{Z}}$, where the target of $\tilde{\alpha}$ denotes the homotopy fixed points.

Remark 3.21 Even though we assumed from the beginning that the coefficients are stable and bicomplete, all the arguments we have discussed work whenever $\mathcal{S}\mathrm{hv}(X; \mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathcal{C})$ admits a left adjoint and $\mathcal{C}$ respects gluings in the sense of [11, Definition 5.17]. In particular, our proof also recovers the case $\mathcal{C} = \mathcal{S}$. A proof of the exodromy equivalence with presentable coefficients but on a much bigger class of stratified spaces can be found in [18].

Corollary 3.22 *Let Z be any compact conically smooth stratified space and let $F \in \mathcal{S}\mathrm{hv}^c(Z; \mathcal{C})$. Then $\Gamma(Z; F)$ is dualizable.*

Proof By Theorem 3.19, we know that there exists an essentially unique functor $G\colon \mathrm{Exit}(Z) \to \mathcal{C}^{\mathrm{dual}}$ such that $\mathrm{im}_* \gamma^* G \simeq F$. Therefore, since γ is final, global sections of F are equivalent to the limit of G. The proof is then concluded by applying Proposition 2.19 and observing that, since $\mathcal{C}$ is stable and its monoidal structure is closed, $\mathcal{C}^{\mathrm{dual}}$ is itself stable. □

Remark 3.23 Let Z be a stratified space such that $\mathrm{Exit}(Z)$ is a retract in $\mathcal{C}\mathrm{at}_\infty$ of a finite ∞-category, and assume that the exodromy equivalence holds for constructible sheaves on Z. Since $\mathcal{C}^{\mathrm{dual}}$ is idempotent complete, the same argument as in Corollary 3.22 shows that, for any $F \in \mathcal{S}\mathrm{hv}^c(X; \mathcal{C})$, the object $\Gamma(Z; F)$ is dualizable.

4 Verdier duality

This final section is devoted to proving Verdier duality for conically smooth spaces (Theorem 4.8). For this reason, from now on our ∞-categories of coefficients are assumed to be equipped with a closed symmetric monoidal structure. We first introduce the Verdier duality functor, and then recall the definition of Lurie's *covariant Verdier duality*. A crucial observation for the proof stratergy that we adopt is that these two functors are closely related.

For any locally compact Hausdorff topological space X, we will denote by $\omega_X^{\mathcal{C}}$ the sheaf $a^!(\mathbb{1}_{\mathcal{C}})$, where $a\colon X \to *$ is the unique map and $\mathbb{1}_{\mathcal{C}}$ is the monoidal unit in $\mathcal{C}$. The sheaf $\omega_X^{\mathcal{C}}$ will be called the $\mathcal{C}$-valued *dualizing sheaf* of X. We denote the functor

$$\underline{\mathrm{Hom}}_X(-,\omega_X^{\mathcal{C}})\colon \mathcal{S}\mathrm{hv}(X;\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

simply by $D_X^{\mathcal{C}}$ and, when $X = *$, we will only write $D^{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. In this case, $D^{\mathcal{C}}$ sends an object $M \in \mathcal{C}$ to its dual $\underline{\mathrm{Hom}}_{\mathcal{C}}(M,\mathbb{1}_{\mathcal{C}})$. Therefore, $D^{\mathcal{C}}$ gives an equivalence between $\mathcal{C}^{\mathrm{dual}}$ and its opposite.

Recall that, for any $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ and $V \in \mathcal{U}(X)$, one defines the *compactly supported sections* of F at V by

$$\Gamma_c(V;F) := \varinjlim_{K\subseteq V} \Gamma_K(V;F).$$

In the colimit above, K ranges through the compact subsets of V, and $\Gamma_K(V;F)$ denotes the fiber of the restriction $\Gamma(V;F) \to \Gamma(V\setminus K;F)$ (see for example [20, Definition 5.6] and the whole section there for a more detailed discussion). The association $F \mapsto \Gamma_c(X;F)$ gives a left adjoint to the functor $a^!$. The above construction can be upgraded to a functor

$$\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \xrightarrow{\mathbb{D}_X^{\mathcal{C}}} \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}), \qquad F \mapsto (U \mapsto \Gamma_c(U;F)).$$

Lurie [15, Theorem 5.5.5.1] shows that $\mathbb{D}$ is an equivalence of ∞-categories. This equivalence is referred to as *covariant Verdier duality*. Our next lemma explains the relation between covariant Verdier duality and the contravariant functor $D_X^{\mathcal{C}}$.

Lemma 4.1 *Let X be any locally compact Hausdorff topological space and let $\mathcal{C}$ be any stable bicomplete ∞-category equipped with a closed symmetric monoidal structure. Then there is a factorization*

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X;\mathcal{C})^{\mathrm{op}} \arrow[rr, "\underline{\mathrm{Hom}}_X(-{,}\omega_X)"] \arrow[dr, "\mathbb{D}"', "\simeq"] & & \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \\
& \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C})^{\mathrm{op}} \arrow[ur, "D_\bullet^{\mathcal{C}}"'] &
\end{tikzcd}$$

where $D_\bullet^{\mathcal{C}}$ denotes the functor obtained by postcomposing with $D^{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$.

Proof Let $j\colon U \hookrightarrow X$ be any open subset of X. Then, for any $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$, by applying [20, Corollary 3.26, Lemma 6.5 and Proposition 6.12], we get functorial equivalences

$$\Gamma(U;\underline{\mathrm{Hom}}_X(F,\omega_X)) \simeq \Gamma(U;\underline{\mathrm{Hom}}_U(j^*F,j^*\omega_X^{\mathcal{C}})) \simeq \Gamma(U;\underline{\mathrm{Hom}}_U(j^*F,\omega_U^{\mathcal{C}})) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(\Gamma_c(U;F),\mathbb{1}_{\mathcal{C}})$$

and thus we have the desired factorization. □

Our next goal toward proving Theorem 4.8 is to show that the covariant Verdier duality functor $\mathbb{D}$ sends constructible sheaves to constructible cosheaves. This is a consequence of constructibility of the dualizing sheaf, which we prove in Proposition 4.3. We first compute the stalk at the cone point of the dualizing sheaf of a cone on a compact C^0-stratified space.

Lemma 4.2 *Let Z be a compact C^0-stratified space. Denote by X the cone $C(Z)$ and let $x \in C(Z)$ be the cone point. Let $\mathbb{1} \in \mathcal{C}$ be the monoidal unit and let $\mathbb{1}_X \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ be the constant sheaf at $\mathbb{1}$. Then we have an equivalence*

$$(\omega_X^{\mathcal{C}})_x \simeq D^{\mathcal{C}}(\Gamma_{\{x\}}(X;\mathbb{1}_X)).$$

Proof We have equivalences

$$(\omega_X^{\mathcal{C}})_x = \varinjlim_{x \in U} \Gamma(U;\omega_X^{\mathcal{C}}) \simeq \varinjlim_{0<\epsilon\leq\infty} \Gamma(C_\epsilon(Z);\omega_X^{\mathcal{C}}) \simeq \varinjlim_{0<\epsilon\leq\infty} D^{\mathcal{C}}\big(\Gamma_c(C_\epsilon(Z);\mathbb{1}_X)\big),$$

where the second follows from Lemma 3.6 and the third by [20, Proposition 6.12]. Here $C_\infty(Z)$ refers to $C(Z)$. Notice that the colimit above is indexed by a weakly contractible ∞-category. Therefore, it will suffice to show that, for any ϵ, the map

$$\Gamma_{\{x\}}(X;\mathbb{1}_X) \simeq \Gamma_{\{x\}}(C_\epsilon(Z);\mathbb{1}_X) \to \Gamma_c(C_\epsilon(Z);\mathbb{1}_X)$$

is invertible (see [20, Remark 5.7] for a proof of why the first equivalence holds).

First of all, notice that, for any $K \subseteq C_\epsilon(Z)$ compact containing the cone point, there exists a $T \geq 0$ such that $K \subseteq \overline{C_T(Z)}$ (namely, take T to be the maximum in the image of K through the projection $C_\epsilon(Z) \to \mathbb{R}_{\geq 0}$). Hence, by a cofinality argument, we have a commutative triangle

$$\begin{tikzcd}
\Gamma_{\{x\}}(X;\mathbb{1}_X) \arrow[d] \arrow[dr] & \\
\varinjlim_{0\leq T<\epsilon} \Gamma_{\overline{C_T(Z)}}(X;\mathbb{1}_X) \arrow[r, "\simeq"] & \Gamma_c(C_\epsilon(Z);\mathbb{1}_X)
\end{tikzcd}$$

where we fix $\{x\} := \overline{C_T(Z)}$. Notice that 0 is the initial object in the indexing poset of the colimit appearing above. Hence, to conclude our proof it suffices to prove that, for any T, the map

$$\Gamma_{\{x\}}(X;\mathbb{1}_X) \to \Gamma_{\overline{C_T(Z)}}(X;\mathbb{1}_X)$$

is invertible. By definition, this holds if and only if the restriction

$$\Gamma(X \setminus \{x\};\mathbb{1}_X) \to \Gamma(X \setminus \overline{C_T(Z)};\mathbb{1}_X)$$

is invertible. But the inclusion $X \setminus \overline{C_T(Z)} \hookrightarrow X \setminus \{x\}$ is a homotopy equivalence, and so we may conclude by the homotopy invariance of the shape. □

Proposition 4.3 *Let X be any C^0-stratified topological space. Then the dualizing sheaf $\omega_X^{\mathcal{C}}$ is constructible.*

Proof We will proceed by induction on the depth of X. If X has depth 0, then X is a topological manifold (see [17, Lemma 2.22]), and hence, by [20, Proposition 6.18], $\omega_X^{\mathcal{C}}$ is locally equivalent to $\Sigma^{\dim(X)}\mathbb{1}_X$. Now assume that X has finite nonzero depth. Since the question is local on X, by [5, Lemma 2.2.2] we may assume that $X = \mathbb{R}^n \times C(Z)$, where Z is a compact C^0-stratified space with $\mathrm{depth}(Z) < \mathrm{depth}(X)$.

Let $p\colon \mathbb{R}^n \times C(Z) \to C(Z)$ be the projection and $b\colon C(Z) \to *$ the unique map. By [20, Proposition 6.18], for any sheaf F on $C(Z)$ we have a functorial equivalence $p^! F \simeq \Sigma^n p^* F$, so it suffices to show that $b^!_{\mathcal{C}} \mathbb{1} = \omega^{\mathcal{C}}_{C(Z)}$ is constructible. Hence we may assume that $X = C(Z)$.

Let x be the cone point and $j : U \hookrightarrow X$ its open complement. Since x is the point at which the depth is maximal, we have $\operatorname{depth}(U) < \operatorname{depth}(X)$. Moreover, we have an equivalence $j^* \omega^{\mathcal{C}}_X \simeq \omega^{\mathcal{C}}_U$, and so, by the inductive hypothesis, $j^* \omega^{\mathcal{C}}_X$ is constructible. Thus, for every stratum $T \subseteq X$ which does not contain the cone point, the restriction of ω_X is locally constant with dualizable stalks. Hence it remains to prove that the stalk of ω_X at the cone point is dualizable.

Since the dual of a dualizable object is again dualizable, by Lemma 4.2 it suffices to show that $\Gamma_{\{x\}}(X; \mathbb{1}_X)$ is dualizable. By definition, $\Gamma_{\{x\}}(X; \mathbb{1}_X)$ is the fiber of the restriction $\Gamma(X; \mathbb{1}_X) \to \Gamma(X \setminus \{x\}; \mathbb{1}_X)$. Using the homotopy invariance of the shape, we get a commutative square

$$\begin{tikzcd} \Gamma(X;\mathbb{1}_X) \arrow[r] \arrow[d, "\simeq"] & \Gamma(X\setminus\{x\};\mathbb{1}_X) \arrow[d, "\simeq"] \\ \mathbb{1} \arrow[r] & \Gamma(Z;\mathbb{1}_Z) \end{tikzcd}$$

where the right vertical arrow is induced by any inclusion $Z \hookrightarrow X$ inducing an homotopy equivalence between Z and $X \setminus \{x\}$, and the left vertical arrow by the inclusion of x in X. Since $\mathcal{C}^{\mathrm{dual}}$ is stable, it suffices to show that $\Gamma(Z; \mathbb{1}_Z)$ is dualizable. But this follows from Corollary A.7. □

Corollary 4.4 *Let X be a conically smooth stratified space. Then the covariant Verdier duality functor $\mathbb{D}^{\mathcal{C}}_X$ restricts to an equivalence*

$$\mathbb{D} : \mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C}) \simeq \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C}). \tag{4.5}$$

Proof The inverse of the covariant Verdier functor $\mathbb{D}^{\mathcal{C}}_X$ is given by $(\mathbb{D}^{\mathcal{C}^{\mathrm{op}}}_X)^{\mathrm{op}}$ (see for example the proof of [20, Theorem 5.10]). Therefore, it suffices to show that $\mathbb{D}^{\mathcal{C}}_X$ preserves formally constructible objects for any $\mathcal{C}$ stable and bicomplete.

First of all, we prove that, if $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is locally constant, then $\mathbb{D}F$ is a formally constructible cosheaf. Since restricting along an open immersion commutes with $\mathbb{D}$ (see [20, Lemma 6.5]) and the property of being formally constructible can be checked on an open cover, it suffices to show that $\mathbb{D}$ sends constant sheaves to formally constructible sheaves.

Assume $F \simeq a^* M$, where $a\colon X \to *$ is the unique map. In this case, by [20, Definition 6.1], $\mathbb{D}F \simeq a^!_{\mathcal{C}^{\mathrm{op}}} M$. Moreover, by [20, Proposition 6.16], $a^!_{\mathcal{C}^{\mathrm{op}}} M \simeq \omega^{\mathcal{C}^{\mathrm{op}}}_X \otimes a^* M$. Therefore, by Proposition 4.3, $\mathbb{D}F$ is constructible.

Assume now that F is any formally constructible sheaf, and let $i : X_\alpha \hookrightarrow X$ be the inclusion of a stratum of X, with complement $j : U \hookrightarrow X$. We need to show that $i^*_{\mathcal{C}^{\mathrm{op}}} \mathbb{D}F \simeq \mathbb{D} i^!_{\mathcal{C}} F$ is locally constant. Notice that it suffices to show that $i^!_{\mathcal{C}}F$ is locally constant. Indeed, since X_α is a smooth manifold, it is in particular conically smooth. Therefore, by what we have proven before, if G is any locally constant sheaf on X_α, $\mathbb{D}G$ must be formally constructible. But X_α is unstratified, and hence being formally constructible on X_α is equivalent to being locally constant. Thus, by Corollary 3.17, $i^!_{\mathcal{C}}F$ is locally constant. □

Proposition 4.6 *Let X be a conically smooth stratified space. Then the covariant Verdier duality functor $\mathbb{D}^{\mathcal{C}}_X$ restricts to an equivalence*

$$\mathbb{D}\colon \mathcal{S}\mathrm{hv}^c(X;\mathcal{C}) \simeq \mathcal{C}o\mathcal{S}\mathrm{hv}^c(X;\mathcal{C}). \tag{4.7}$$

Proof By Corollary 4.4, it suffices to show that, for any $x \in X$ and $F \in \mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C})$, $F_x \in \mathcal{C}$ is dualizable if and only if $(\mathbb{D}F)_x$ is dualizable, where the latter denotes the costalk of $\mathbb{D}F$ at x.

Let $x\colon * \hookrightarrow X$ be the inclusion of a point $x \in X$. By definition, there are equivalences $(\mathbb{D}F)_x \simeq x^*_{\mathcal{C}^{\mathrm{op}}}(\mathbb{D}F) \simeq x^!_{\mathcal{C}}F$. Thus, by applying global sections to the localization sequence associated to i and j, we obtain a fiber sequence

$$x^!F \simeq \Gamma(X; i_*i^!F) \to \Gamma(X;F) \to \Gamma(U;F)$$

and hence an equivalence

$$\Gamma_{\{x\}}(X;F) \simeq i^!F.$$

Thus, by choosing a conical chart $\mathbb{R}^n \times C(Z)$ around x and applying Corollary 3.7, we get a fiber sequence

$$(\mathbb{D}F)_x \to F_x \to \Gamma((\mathbb{R}^n \times C(Z)) \setminus (0,*); F),$$

where $* \in C(Z)$ denotes the cone point. Therefore, arguing as in the proof of Corollary 3.22, it suffices to show that $\mathrm{Exit}\big((\mathbb{R}^n \times C(Z)) \setminus (0,*)\big)$ is finite. But, by Van Kampen for exit paths, one has a pushout

$$\begin{tikzcd}
\mathrm{Exit}(\mathbb{R}^n \setminus \{0\} \times \mathbb{R}_{>0} \times Z) \arrow[r] \arrow[d] & \mathrm{Exit}(\mathbb{R}^n \times \mathbb{R}_{>0} \times Z) \arrow[d] \\
\mathrm{Exit}(\mathbb{R}^n \setminus \{0\} \times C(Z)) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{Exit}\big((\mathbb{R}^n \times C(Z)) \setminus (0,*)\big)
\end{tikzcd}$$

The result then follows by observing that Exit commutes with products, $\mathrm{Exit}(Z)$ and $\mathrm{Exit}(C(Z))$ are both finite by Proposition 2.19 and Lemma 2.13, and $\mathrm{Exit}(\mathbb{R}^n \setminus \{0\}) \simeq \mathrm{Sing}(S^{n-1})$ is finite. □

Theorem 4.8 *Let X be a conically smooth stratified space. Then the restriction to $\mathcal{S}\mathrm{hv}^c(X;\mathcal{C})^{\mathrm{op}}$ of the functor $D^{\mathcal{C}}_X$ factors through an equivalence*

$$D^{\mathcal{C}}_X\colon \mathcal{S}\mathrm{hv}^c(X;\mathcal{C})^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{S}\mathrm{hv}^c(X;\mathcal{C}).$$

Proof By Lemma 4.1 and Proposition 4.6, we only need to show that

$$D^{\mathcal{C}}_\bullet\colon \mathcal{C}o\mathcal{S}\mathrm{hv}^c(X;\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{hv}^c(X;\mathcal{C})$$

is an equivalence. Denote again by $\mathrm{im}\colon \mathrm{Bsc}_{/X} \to \mathcal{U}(X)$ the functor taking a conically smooth open immersion into X to its image. The diagram

$$\begin{array}{ccc}
\mathrm{Fun}(\mathrm{Exit}(X), (\mathcal{C}^{\mathrm{dual}})^{\mathrm{op}}) & \xrightarrow{D^{\mathcal{C}}_{\bullet}} & \mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C}^{\mathrm{dual}}) \\
\simeq\big\downarrow \gamma^* & & \simeq\big\downarrow \gamma^* \\
\mathrm{Fun}_{\mathcal{W}}((\mathrm{Bsc}_{/X})^{\mathrm{op}}, (\mathcal{C}^{\mathrm{dual}})^{\mathrm{op}}) & \xrightarrow{D^{\mathcal{C}}_{\bullet}} & \mathrm{Fun}_{\mathcal{W}}((\mathrm{Bsc}_{/X})^{\mathrm{op}}, \mathcal{C}^{\mathrm{dual}}) \\
\simeq\big\uparrow \mathrm{im}^* & & \simeq\big\uparrow \mathrm{im}^* \\
\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}^c(X;\mathcal{C})^{\mathrm{op}} & \xrightarrow{D^{\mathcal{C}}_{\bullet}} & \mathcal{S}\mathrm{hv}^c(X;\mathcal{C})
\end{array}$$

commutes, since the horizontal arrows are given by postcompositions and the vertical arrows by pre-compositions. Moreover, since the restriction of $D^{\mathcal{C}}$ induces a duality on $\mathcal{C}^{\mathrm{dual}}$, the upper horizontal arrows are equivalences, and thus we get the desired conclusion. □

Example 4.9 Let us give an explicit description of what Verdier duality looks like for the stratified space X appearing in Example 2.6. Let $\mathcal{C}$ be any stable and bicomplete ∞-category. For any map $\alpha\colon M \to N^{h\mathbb{Z}}$ in $\mathcal{C}$ as in Example 3.20, we get a map $\Omega N^{h\mathbb{Z}} \to \mathrm{fib}(\alpha)$. It follows by Poincaré duality for manifolds (see [20, Proposition 6.18]) that there is an equivalence $N^{h\mathbb{Z}} \simeq \Omega N_{h\mathbb{Z}}$. Therefore, we have a map $\Omega^2 N_{h\mathbb{Z}} \to \mathrm{fib}(\alpha)$. This can be upgraded to a functor

$$\mathrm{Fun}(B\mathbb{Z}^{\triangleleft}, \mathcal{C}) \to \mathrm{Fun}(B\mathbb{Z}^{\triangleright}, \mathcal{C}), \quad (\alpha\colon M \to N^{h\mathbb{Z}}) \mapsto (\tilde{\alpha}\colon \Omega^2 N_{h\mathbb{Z}} \to \mathrm{fib}(\alpha)),$$

which is easily seen to be an equivalence. We invite the interested reader to work out the details to show that the functor given above coincides with (4.5), after applying the exodromy equivalence.

Remark 4.10 In [6, Example 1.10.8], the authors propose a strategy to prove Verdier duality. However, they do not provide proofs for some of the major steps in their outline. We specify here the main missing points in [loc. cit.]. Let $X \to P$ be any stratified topological space. First of all, in [loc. cit.] there is no explanation of why the stratification on $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ restricts to a stratification on $\mathcal{S}\mathrm{hv}^{\mathrm{fc}}(X;\mathcal{C})$. We verify this for conically smooth stratified spaces in the proof of Corollary 3.17. Secondly, in [loc. cit.] the authors claim without proof that, if ω_X is formally constructible, then the covariant Verdier duality functor preserves formally constructible objects. We prove this claim in Corollary 4.4. The authors also do not explain for which kind of stratified topological spaces one should expect the dualizing sheaf to be formally constructible. We show that this is the case for C^0-stratified spaces in Proposition 4.3.

Remark 4.11 The equivalences (4.5) and (4.7) are already interesting on their own, because they imply that, for any stratified map $f\colon X \to Y$, $f_*^{\mathcal{C}}$ or $f^*_{\mathcal{C}}$ preserves (formal) constructibility if and only if $f_!^{\mathcal{C}}$ or $f^!_{\mathcal{C}}$ does. In particular, $f^!_{\mathcal{C}}$ always preserves (formally) constructible sheaves.

Remark 4.12 Any μ-stratification of an analytic manifold in the sense of [13] satisfies the Whitney conditions, and hence by [17] defines a conically smooth structure. Thus, Theorem 4.8 recovers and generalizes the duality on constructible sheaves on analytic manifolds as defined in [13] (ie sheaves which are constructible in our sense with respect to *some* μ-stratification).

Appendix The shape of a proper locally contractible ∞-topos

In this appendix, we present a couple of topos-theoretic results that provide an elegant argument to conclude the last step in the proof of Proposition 4.3. While the content of this appendix is not fundamental for our primary purposes, we have decided to include it because we believe it is interesting on its own. More precisely, in this appendix we prove that the *shape* of any *proper* and *locally contractible* ∞-topos is a compact ∞-groupoid.

We start by recalling the definition of the shape of a locally contractible ∞-topos. For a more general and detailed discussion about shape and locally contractible geometric morphisms, see [20, Section 3].

Definition A.1 Let $\mathcal{X}$ be an ∞-topos and let $a\colon \mathcal{X} \to \mathcal{S}$ be the unique geometric morphism. We say that $\mathcal{X}$ is *locally contractible* if $a^*\colon \mathcal{S} \to \mathcal{X}$ admits a left adjoint, denoted by $a_\sharp\colon \mathcal{X} \to \mathcal{S}$.

If $\mathcal{X}$ is any locally contractible ∞-topos, we define the *shape* of $\mathcal{X}$, denoted by $\Pi_\infty(\mathcal{X})$, as the ∞-groupoid $a_\sharp(1_{\mathcal{X}})$, where $1_{\mathcal{X}}$ denotes the terminal object of $\mathcal{X}$.

We now show that sheaf topoi associated to C^0-stratified spaces are locally contractible. We need the following preliminary lemma.

Lemma A.2 *Let X be a C^0-stratified space. Then the ∞-topos $\mathcal{S}\mathrm{hv}(X;\mathcal{S})$ is hypercomplete.*

Proof By [5, Lemma 2.2.2], X admits an open cover given by its open subsets isomorphic as stratified spaces to ones of type $\mathbb{R}^n \times C(Z)$, where Z is a compact C^0-stratified space. Therefore, X is locally paracompact and of finite covering dimension. By [14, Theorem 7.2.3.6], the covering dimension of a paracompact space agrees with its *homotopy dimension* (see [14, Definition 7.2.1.1]). Moreover, by [14, Corollary 7.2.1.12], any ∞-topos which is locally of finite homotopy dimension is hypercomplete. Therefore, we conclude that $\mathcal{S}\mathrm{hv}(X;\mathcal{S})$ is hypercomplete. □

Corollary A.3 *Let X be a C^0-stratified space. Then the ∞-topos $\mathcal{S}\mathrm{hv}(X;\mathcal{S})$ is locally contractible. Moreover, we have an equivalence of ∞-groupoids $\Pi_\infty(\mathcal{S}\mathrm{hv}(X;\mathcal{S})) \simeq \mathrm{Sing}(X)$.*

Proof Since, by [5, Lemma 2.2.2], the topological space X is locally contractible, the result follows from Lemma A.2 and [20, Corollary 3.19]. □

Definition A.4 Let $\mathcal{X}$ be an ∞-topos and let $a\colon \mathcal{X} \to \mathcal{S}$ be the unique geometric morphism. We say that $\mathcal{X}$ is *proper* if $a_*\colon \mathcal{X} \to \mathcal{S}$ preserves filtered colimits.

Remark A.5 It would be very natural to define an ∞-topos to be proper by requiring the unique geometric morphism $a\colon \mathcal{X} \to \mathcal{S}$ to be proper in the sense of [14, Definition 7.3.1.4]. This alternative definition is proven to be equivalent to Definition A.4 in [16].

Proposition A.6 *Let $\mathcal{X}$ be a proper and locally contractible ∞-topos. Then $\Pi_\infty(\mathcal{X})$ is a compact object in $\mathcal{S}$.*

Proof Let $a\colon \mathcal{X} \to \mathcal{S}$ be the unique geometric morphism. Observe that $a_*\colon \mathcal{X} \to \mathcal{S}$ is corepresented by the terminal object $1_\mathcal{X}$. Since $\mathcal{X}$ is assumed to be proper, $1_\mathcal{X}$ must be a compact object in $\mathcal{X}$. Therefore, to conclude the proof it suffices to show that $a_\sharp\colon \mathcal{X} \to \mathcal{S}$ preserves compact objects. But this is clear because its right adjoint a^* preserves (filtered) colimits. □

Corollary A.7 *Let X be any compact Hausdorff topological space and assume that $\mathcal{S}\mathrm{hv}(X;\mathcal{S})$ is locally contractible. Let $\mathcal{C}$ be any stable bicomplete ∞-category equipped with a closed symmetric monoidal structure. Let $M \in \mathcal{C}$ be any dualizable object and denote by M_X the constant sheaf at M. Then $\Gamma(X; M_X)$ is dualizable.*

Proof Let $a\colon X \to *$ be the unique map. Recall that, by [20, Corollary 5.16], we have an equivalence $\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \simeq \mathcal{S}\mathrm{hv}(X;\mathcal{S}) \otimes \mathcal{C}$, where the $\otimes$ denotes Lurie's tensor product of cocomplete ∞-categories. Since X is locally contractible, by combining [20, Corollaries 5.16 and 5.20] we see that $a^*_\mathcal{C}\colon \mathcal{C} \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ admits a left adjoint $a^\mathcal{C}_\sharp$ obtained by tensoring with $\mathcal{C}$ the cocontinuous functor $a_\sharp\colon \mathcal{S}\mathrm{hv}(X;\mathcal{S}) \to \mathcal{S}$. In particular, if we denote by $\mathbb{1}_X$ the constant sheaf at the monoidal unit $\mathbb{1} \in \mathcal{C}$,

$$a^\mathcal{C}_\sharp(\mathbb{1}_X) \simeq \varinjlim_{\Pi_\infty(X)} \mathbb{1}.$$

Here $\Pi_\infty(X)$ denotes the shape of the locally contractible ∞-topos $\mathcal{S}\mathrm{hv}(X;\mathcal{S})$. Moreover, it follows from the dual version of the smooth projection formula (see [20, Corollary 3.26]) that there is an equivalence $\Gamma(X; M_X) \simeq \underline{\mathrm{Hom}}_\mathcal{C}(a^\mathcal{C}_\sharp(\mathbb{1}_X), M)$. Hence,

$$\Gamma(X; M_X) \simeq \varprojlim_{\Pi_\infty(X)} M.$$

Since $\mathcal{C}^{\mathrm{dual}}$ is an idempotent complete stable ∞-category, we can conclude by Proposition A.6. □

References

[1] **K Aoki**, *Posets for which Verdier duality holds*, Selecta Math. 29 (2023) art. id. 78 MR Zbl

[2] **K Aoki**, *Tensor triangular geometry of filtered objects and sheaves*, Math. Z. 303 (2023) art. id. 62 MR Zbl

[3] **G Arone**, **M Kankaanrinta**, *On the functoriality of the blow-up construction*, Bull. Belg. Math. Soc. Simon Stevin 17 (2010) 821–832 MR Zbl

[4] **D Ayala**, **J Francis**, **N Rozenblyum**, *A stratified homotopy hypothesis*, J. Eur. Math. Soc. 21 (2019) 1071–1178 MR Zbl

[5] **D Ayala**, **J Francis**, **H L Tanaka**, *Local structures on stratified spaces*, Adv. Math. 307 (2017) 903–1028 MR Zbl

[6] **D Ayala**, **A Mazel-Gee**, **N Rozenblyum**, *Stratified noncommutative geometry*, Mem. Amer. Math. Soc. 1485, Amer. Math. Soc., Providence, RI (2024) MR Zbl

[7] **C Barwick**, **S Glasman**, **P Haine**, *Exodromy*, preprint (2018) arXiv 1807.03281

[8] **A A Beĭlinson**, **J Bernstein**, **P Deligne**, *Faisceaux pervers*, from "Analysis and topology on singular spaces, I", Astérisque 100, Soc. Math. France, Paris (1982) 5–171 MR Zbl

[9] **D-C Cisinski**, *Higher categories and homotopical algebra*, Cambridge Stud. Adv. Math. 180, Cambridge Univ. Press (2019) MR Zbl

[10] **J M Curry**, *Dualities between cellular sheaves and cosheaves*, J. Pure Appl. Algebra 222 (2018) 966–993 MR Zbl

[11] **P J Haine**, **M Porta**, **J-B Teyssier**, *The homotopy-invariance of constructible sheaves*, Homology Homotopy Appl. 25 (2023) 97–128 MR Zbl

[12] **M Kashiwara**, *The Riemann–Hilbert problem for holonomic systems*, Publ. Res. Inst. Math. Sci. 20 (1984) 319–365 MR Zbl

[13] **M Kashiwara**, **P Schapira**, *Sheaves on manifolds*, Grundl. Math. Wissen. 292, Springer (1990) MR Zbl

[14] **J Lurie**, *Higher topos theory*, Ann. of Math. Stud. 170, Princeton Univ. Press (2009) MR Zbl

[15] **J Lurie**, *Higher algebra*, preprint (2017) Available at `https://url.msp.org/Lurie-HA`

[16] **L Martini**, **S Wolf**, *Proper morphisms of ∞-topoi*, preprint (2023) arXiv 2311.08051

[17] **G Nocera**, **M Volpe**, *Whitney stratifications are conically smooth*, Selecta Math. 29 (2023) art. id. 68 MR Zbl

[18] **M Porta**, **J-B Teyssier**, *Topological exodromy with coefficients*, preprint (2022) arXiv 2211.05004

[19] **D Treumann**, *Exit paths and constructible stacks*, Compos. Math. 145 (2009) 1504–1532 MR Zbl

[20] **M Volpe**, *The six operations in topology*, preprint (2021) arXiv 2110.10212

[21] **M Volpe**, *Finiteness and finite domination in stratified homotopy theory*, preprint (2024) arXiv 2412.04745

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Toward a topological description of Legendrian contact homology of unit conormal bundles

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For a smooth compact submanifold K of a Riemannian manifold Q, its unit conormal bundle Λ_K is a Legendrian submanifold of the unit cotangent bundle of Q with a canonical contact structure. Using pseudoholomorphic curve techniques, the Legendrian contact homology of Λ_K is defined when, for instance, $Q = \mathbb{R}^n$. Aiming at giving another description of this homology, we define a graded $\mathbb{R}$-algebra for any pair (Q, K) with orientations from a perspective of string topology and prove its invariance under smooth isotopies of K. We conjecture that it is isomorphic to the Legendrian contact homology of Λ_K with coefficients in $\mathbb{R}$ in all degrees. This is a reformulation of a homology group, called string homology, introduced by Cieliebak, Ekholm, Latschev and Ng when the codimension of K is 2, though the coefficient is reduced from the original $\mathbb{Z}[\pi_1(\Lambda_K)]$ to $\mathbb{R}$. We compute our invariant (i) in all degrees for specific examples, and (ii) in the 0^{th} degree when the normal bundle of K is a trivial 2-plane bundle.

53D42; 55P50, 57R17

1 Introduction

Convention Throughout this paper, all manifolds are of class C^∞ without boundary and second countable, and all submanifolds are of class C^∞ without boundary, unless otherwise specified.

Background Let Q be a manifold with a Riemannian metric, and K be a compact submanifold of Q. To any pair (Q, K), one can associate the unit cotangent bundle UT^*Q of Q and the unit conormal bundle Λ_K of K. It is well known that UT^*Q has a canonical contact structure and Λ_K is a Legendrian submanifold of UT^*Q.

As an invariant of Legendrian submanifolds, the *Legendrian contact homology* has been studied for pairs (M,Λ) of a contact manifold M and its compact Legendrian submanifold Λ. It is the homology of a differential graded algebra generated by Reeb chords of Λ, and was introduced by Chekanov [4] and Eliashberg [12]. The differential is defined by using pseudoholomorphic curves in the symplectization of M. A rigorous definition was given by Ekholm, Etnyre and Sullivan [9; 11] when there is a diffeomorphism from M to the contactization of a Liouville manifold which preserves contact forms. As is mentioned in [11, Section 5.1], this included the case of $M = UT^*\mathbb{R}^n$. (The definition of [11] is given by pseudoholomorphic curves in the Liouville manifold. These curves can be lifted to pseudoholomorphic curves in the symplectization of M. See Dimitroglou Rizell [7].)

Suppose conceptually that we have an algebraic invariant in symplectic or contact topology defined by using pseudoholomorphic curves, and apply it to an object related to the cotangent bundle T^*Q. For instance, we consider the symplectic homology of T^*Q, or the wrapped Floer homology of the conormal bundle L_K of K in T^*Q. In this case, it is known by the following results that these invariants have another view from the topology of the loop or path space of Q, without using pseudoholomorphic curves (here we assume that Q is a closed spin manifold and all homology groups have $\mathbb{Z}$-coefficients):

- The symplectic homology $SH_*(T^*Q)$ of T^*Q is isomorphic to the singular homology of the free loop space of Q; see Abbondandolo and Schwarz [2], Abouzaid [3] and Viterbo [20].
- The wrapped Floer homology $WF_*(L_K, L_K)$ of L_K is isomorphic to the singular homology of the space of paths in Q with endpoints in K; see Abbondandolo, Portaluri and Schwarz [1].

These results lead us to an expectation that if the Legendrian contact homology of a pair (UT^*Q,Λ_K) is defined, it has another description in terms of the topology of the path space of Q. This expectation has already been confirmed in particular cases. When the codimension of K is 2, Cieliebak, Ekholm, Latschev and Ng [6] defined a graded $\mathbb{Z}[\pi_1(\Lambda_K)]$-algebra, called *string homology*, which is inspired by string topology of the path space of Q. They showed that when Q is equal to $\mathbb{R}^3$ with the standard metric and K is a knot, the 0^{th} degree part of this algebra is isomorphic to the 0^{th} degree part of the *fully noncommutative* Legendrian contact homology of $(UT^*\mathbb{R}^3,\Lambda_K)$ with coefficients in $\mathbb{Z}[\pi_1(\Lambda_K)]$. However, such topological descriptions have not yet been defined in higher degrees or for K with codim $K \neq 2$.

Main results Let Q be an oriented manifold and K be its compact oriented submanifold of codimension $d \geq 1$. The main purpose of this paper is to define a graded $\mathbb{R}$-algebra $H_*^{\text{string}}(Q,K)$ and observe its basic properties. This graded $\mathbb{R}$-algebra can be regarded as a reformulation of the string homology of [6], whose coefficient is reduced from $\mathbb{Z}[\pi_1(\Lambda_K)]$ to $\mathbb{R}$. The feature of our formulation is that $H_*^{\text{string}}(Q,K)$ is defined for K of an arbitrary codimension and in all degrees, compared to the string homology defined for K of codimension 2 and generated by singular chains of degree less than or equal to 2. The two main differences from the string homology in its construction are the reduction of the coefficient and the substitution of singular chains by *de Rham chains* explained below.

The construction of $H_*^{\mathrm{string}}(Q,K)$ can be briefly summarized as follows: We first choose auxiliary data including a complete Riemannian metric on Q. As a graded $\mathbb{R}$-vector space, it is defined to be

$$H_*^{\mathrm{string}}(Q,K) := \varinjlim_{a\to\infty} \varprojlim_{\substack{(\varepsilon,\delta)\in\mathcal{T}_a\\ \varepsilon\to 0}} H_*^{<a}(\varepsilon,\delta), \tag{1}$$

where $H_*^{<a}(\varepsilon,\delta)$ for $a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K)$ and $(\varepsilon,\delta)\in\mathcal{T}_a$ is the homology of a chain complex

$$\left(C_*^{<a}(\varepsilon) := \bigoplus_{m=0}^{\infty} C^{\mathrm{dR}}_{*-m(d-2)}(\Sigma_m^{a+m\varepsilon},\Sigma_m^0),\, D_\delta\right). \tag{2}$$

An explanation for each piece of the above definition is the following:

(A) $C_*^{\mathrm{dR}}(X,A)$ is the $\mathbb{R}$-vector space of de Rham chains defined for a pair of differentiable spaces (X,A). Together with the boundary operator

$$\partial\colon C_*^{\mathrm{dR}}(X,A)\to C_{*-1}^{\mathrm{dR}}(X,A),$$

$(C_*^{\mathrm{dR}}(X,A),\partial)$ becomes a chain complex. De Rham chains can be used as substitutions of singular chains over $\mathbb{R}$. Their basic properties are summarized in Section 2. The advantage is that the fiber product of de Rham chains can be defined in a natural way. The main references are Irie [15; 16].

(B) For $a\in\mathbb{R}_{>0}$ and $m\in\mathbb{Z}_{\geq 1}$, Σ_m^a is a differentiable space of sequences $(\gamma_1,\dots,\gamma_m)$ of paths $\gamma_k\colon[0,T_k]\to Q$ for $k=1,\dots,m$ with endpoints in K. It includes all $(\gamma_k)_{k=1,\dots,m}$ whose total length is less than a. For the precise definition, see Section 3.1. Exceptionally, Σ_0^a is the one-point set for $a>0$ and Σ_0^0 is the empty set.

(C) D_δ is defined by $D_\delta(x) := \partial x + \sum_{k=1}^m (-1)^{p+kd+1} f_{k,\delta}(x)$ for $x\in C^{\mathrm{dR}}_{p-m(d-2)}(\Sigma_m^{a+m\varepsilon},\Sigma_m^0)$. Here

$$f_{k,\delta}\colon C_*^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon},\Sigma_m^0)\to C^{\mathrm{dR}}_{*+1-d}(\Sigma_{m+1}^{a+(m+1)\varepsilon},\Sigma_{m+1}^0)\quad\text{for } k=1,\dots,m$$

are operators which play the key role in our construction. The idea comes from an operation of string topology explained by three steps:

(i) Fix a pair of short paths $(\sigma_i\colon[0,\varepsilon_i]\to N_\varepsilon)_{i=1,2}$ in a tubular neighborhood N_ε of K such that $\sigma_1(\varepsilon_1),\sigma_2(0)\in K$ and $\sigma_1(0)=\sigma_2(\varepsilon_2)$.

(ii) For any sequence of m paths $(\gamma_k)_{k=1,\dots,m}$, we split the k^{th} path $\gamma_k\colon[0,T_k]\to Q$ at a time, say τ, if the image $\gamma_k(\tau)$ coincides with $\sigma_1(0)$. We then concatenate $\gamma_k|_{[0,\tau]}$ (resp. $\gamma_k|_{[\tau,T_k]}$) with σ_1 (resp. σ_2) to get a new sequence of $m+1$ paths

$$(\gamma_1,\dots,\gamma_{k-1},(\gamma_k|_{[0,\tau]}\cdot\sigma_1),(\sigma_2\cdot\gamma_k|_{[\tau,T_k]}),\gamma_{k+1},\dots,\gamma_m).$$

(iii) We extend the procedures (i)–(ii) for families (or chains) of paths parametrized over manifolds.

For the precise definition, we need to take fiber products of chains. See Setions 3.3 and 3.4. The operator $f_{k,\delta}$ depends on a chain $\delta\in C^{\mathrm{dR}}_{n-d}(S_\varepsilon)$, where S_ε for $\varepsilon>0$ is a differentiable space of pairs of short paths in N_ε introduced in Section 3.1. For $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$, where $\mathcal{L}(K)$ is a closed subset of Lebesgue measure 0, a set $\mathcal{T}_a$ consisting of pairs (ε, δ) is defined in Definition 4.4. It is necessary to prove $D_\delta \circ D_\delta = 0$ for any $(\varepsilon, \delta) \in \mathcal{T}_a$, which we do in Proposition 4.1, to define the chain complex (2).

(D) The inverse limit in (1) is defined from an inverse system

$$\bigl(\{H_*^{<a}(\varepsilon, \delta)\}_{(\varepsilon,\delta)\in\mathcal{T}_a}, \{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon'\leq\varepsilon}\bigr).$$

Its construction is given in Section 4.3. To define the linear map $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\colon H_*^{<a}(\varepsilon', \delta') \to H_*^{<a}(\varepsilon, \delta)$, we need to factor through another homology group constructed from "$[-1, 1]$-modeled de Rham chains". Furthermore, to check its well-definedness and a claim about composition, we need one more homology group constructed from "$[-1, 1]^2$-modeled de Rham chains". These variants of de Rham chains are introduced in Section 3.5.

(E) The inverse limit is denoted by $H_*^{<a}(Q, K) := \varprojlim_{\varepsilon\to 0} H_*^{<a}(\varepsilon, \delta)$. The direct limit in (1) is defined from $\bigl(\{H_*^{<a}(Q, K)\}_{a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K)}, \{I^{a,b}\}_{a\leq b}\bigr)$, where $I^{a,b}\colon H_*^{<a}(Q, K) \to H_*^{<b}(Q, K)$ is induced by the inclusion maps $\Sigma_m^{a+m\varepsilon} \to \Sigma_m^{b+m\varepsilon}$ for all $m \in \mathbb{Z}_{\geq 0}$. See Section 4.4.

(F) A graded associative product structure on $H_*^{\mathrm{string}}(Q, K)$ is induced by natural maps $\Sigma_m^a \times \Sigma_{m'}^{a'} \to \Sigma_{m+m'}^{a+a'}$ for all $m, m' \in \mathbb{Z}_{\geq 0}$. The unit comes from $1 \in \mathbb{R} = C_0^{\mathrm{dR}}(\Sigma_0^a, \Sigma_0^0)$ for $a > 0$. See Section 4.4.2.

A fundamental property of $H_*^{\mathrm{string}}(Q, K)$ is the invariance under C^∞ isotopies of K.

Theorem 1.1 *The unital graded $\mathbb{R}$-algebra $H_*^{\mathrm{string}}(Q, K)$ is independent up to isomorphism of the auxiliary data and invariant under changing the orientation of K. Moreover, it is invariant under C^∞ isotopies of K.* (See Proposition 4.20.)

We also give nontrivial computations when $Q = \mathbb{R}^{2d-1}$ for $d \geq 2$. For two specific submanifolds in $\mathbb{R}^{2d-1}$ both of which are diffeomorphic to $S^{d-1} \sqcup S^{d-1}$, we prove that our invariant is isomorphic to the homology of a finitely generated differential graded algebra. Using this computation, we obtain the next result.

Theorem 1.2 *For every $d \geq 2$, there are two nonisotopic oriented submanifolds K and K' in $\mathbb{R}^{2d-1}$ of codimension d such that Λ_K is isotopic to $\Lambda_{K'}$ as a C^∞ submanifold with a spin structure in $UT^*\mathbb{R}^{2d-1}$, while $H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K) \not\cong H_*^{\mathrm{string}}(\mathbb{R}^{2d-1}, K')$.* (See Corollary 5.10.)

The spin structure on Λ_K for any submanifold K in a spin manifold Q is explained in Proposition 5.11.

Another purpose of this paper is to shed light on the relation to Legendrian contact homology. The following result is nontrivial from the construction.

Theorem 1.3 *When the codimension of K is 2 and the normal bundle of K is trivial, $H_0^{\mathrm{string}}(Q, K)$ is isomorphic to the cord algebra of (Q, K) over $\mathbb{R}$.* (See Theorem 6.11.)

If K is connected, the cord algebra over $\mathbb{R}$ we consider in this paper is a reduction of the cord algebra over $\mathbb{Z}[H_1(\Lambda_K)]$ defined by Ng [19]. Combined with a result of Ekholm, Etnyre, Ng and Sullivan [8], the cord algebra for a knot K in $\mathbb{R}^3$ was proved to be isomorphic to the 0^{th} degree part of the Legendrian contact homology of $(UT^*\mathbb{R}^3, \Lambda_K)$. Later, another direct proof was given in [6].

The author makes the following more radical conjecture when $Q = \mathbb{R}^n$.

Conjecture 1.4 *For any compact oriented submanifold K in $\mathbb{R}^n$, $H_*^{\text{string}}(\mathbb{R}^n, K)$ is isomorphic to the Legendrian contact homology of $(UT^*\mathbb{R}^n, \Lambda_K)$ with coefficients in $\mathbb{R}$.*

The Legendrian contact homology with coefficients in $\mathbb{R}$ is an invariant of Legendrian submanifolds with a spin structure; see Ekholm, Etnyre and Sullivan [10; 11]. If Conjecture 1.4 is true, then our invariant can be applied to study the contact topology of $UT^*\mathbb{R}^n$. For instance, assuming this conjecture, Theorem 1.2 would imply that Λ_K is not isotopic to $\Lambda_{K'}$ as a Legendrian submanifold with a spin structure.

Organization of paper In Section 2, general notions of a differentiable space and its de Rham chain complex are introduced. In Section 3.1, the differentiable spaces Σ_m^a and S_ε are defined. Their de Rham chain complexes are observed in Section 3.2. Through Sections 3.3 and 3.4, the operator $f_{k,\delta}$ is defined. In Section 3.5, $[-1,1]$-modeled and $[-1,1]^2$-modeled de Rham chains for path spaces are introduced. In Section 4.1, we define the chain complexes (2) and give a couple of computations. In Section 4.2, we consider their variants using those chains in Section 3.5. They are necessary to define the map $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}$ in Section 4.3. The definition of $H_*^{\text{string}}(Q, K)$ is given in Section 4.4. The independence on auxiliary data is checked in Section 4.5, from which the isotopy invariance follows immediately. In Section 5, we examine the algebraic structure of $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K)$ when K is a higher-dimensional generalization of the Hopf link or the unlink in $\mathbb{R}^3$. In Section 6.1, referring to [6], we define the cord algebra and give another description as the 0^{th} degree part of the string homology. In Section 6.2, we construct a graded map from the string homology to $H_*^{\text{string}}(Q, K)$. In Section 6.3, this map is proved to be an isomorphism on the 0^{th} degree part.

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2 Differentiable space and de Rham chains

In this section, the notions of differentiable spaces and de Rham chains are introduced. We also summarize results applied in the latter sections.

Remark 2.1 The notion of differentiable space goes back to [5] by K-T Chen. The notion of de Rham chains was proposed by Fukaya [14], and later Irie gave the definition in [15; 16]. We mainly refer, especially about sign conventions, to [16]. As is mentioned in [15, Remark 4.1], the definition of plots (elements of a differentiable structure) in this paper is different from that of [5].

2.1 Notation and conventions

For $m, N \in \mathbb{Z}_{\geq 0}$, let $\mathcal{U}_{m,N}$ be the set of oriented m-dimensional submanifolds of $\mathbb{R}^N$. We then define $\mathcal{U} := \bigcup_{m,N \in \mathbb{Z}_{\geq 0}} \mathcal{U}_{m,N}$. Let us fix a few conventions about orientations. If we write $\mathbb{R}^n$ for $n \in \mathbb{Z}_{\geq 1}$, this means the manifold $\mathbb{R}^n \in \mathcal{U}_{n,n}$ whose orientation is given so that $dx_1 \wedge \cdots \wedge dx_n$ is a positive volume form when $(x_1, \ldots, x_n)$ is the standard coordinate of $\mathbb{R}^n$. If we write $\{0\}$, this means $\{0\} \in \mathcal{U}_{0,0}$ with a positive sign assigned.

Let us think about the orientation of fiber products of oriented manifolds. For $U, V, M \in \mathcal{U}$, suppose that there are two C^∞ maps $f : U \to M$ and $g : V \to M$. We also assume that g is a submersion. (Hereafter, all submersions are of class C^∞.) Then the fiber product

$$U \ {}_f\!\times_g \ V := \{(u, v) \in U \times V \mid f(u) = g(v)\}$$

is a C^∞ submanifold of $U \times V$. In order to determine the orientation at $(u, v) \in U \ {}_f\!\times_g \ V$, we take a right inverse $s : T_{g(v)}M \to T_v V$ of $(dg)_v$ (ie $(dg)_v \circ s = \mathrm{id}_{T_{g(v)}M}$). Then there are two isomorphisms

$$\begin{aligned} T_{g(v)}M \times \mathrm{Ker}(dg)_v &\to T_v V, & (z, y) &\mapsto s(z) + y, \\ T_u U \times \mathrm{Ker}(dg)_v &\to T_{(u,v)}(U \ {}_f\!\times_g \ V), & (x, y) &\mapsto (x, y + s \circ (df)_u(x)). \end{aligned}$$

The orientations of $\mathrm{Ker}(dg)_v$ and $T_{(u,v)}(U \ {}_f\!\times_g \ V)$ are determined so that the above isomorphisms preserve orientations. Of course, when X and Y are oriented $\mathbb{R}$-vector spaces, we assign the product orientation on $X \times Y$. In particular, when $M = \{0\}$, this gives the orientation of the product manifold $U \times V$.

For $U \in \mathcal{U}$, $\Omega_c^p(U)$ is the vector space of compactly supported C^∞ differential p-forms on U. When $p < 0$ or $p > \dim U$, we define $\Omega_c^p(U) := 0$. For $U, U' \in \mathcal{U}$ and a submersion $\pi : U' \to U$, we have an $\mathbb{R}$-linear map

$$\pi_! : \Omega_c^p(U') \to \Omega_c^{p-(\dim U' - \dim U)}(U),$$

called the *integration along fibers*. When $U' = \mathbb{R}^d \times \mathbb{R}^k$, $U = \mathbb{R}^k$ and $\pi(t, x) = x$ for $(t, x) \in U'$, this map is characterized by the following: for $f \in \Omega_c^0(U')$, $1 \leq i_1 < \cdots < i_a \leq d$ and $1 \leq j_1 < \cdots < j_b \leq k$, if we take $\omega := f(dt_{i_1} \wedge \cdots \wedge dt_{i_a} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_b})$, then, for every $x \in U$,

$$(\pi_!(\omega))_x = \begin{cases} 0 & \text{if } a < d, \\ \left(\int_{\mathbb{R}^d} f(\cdot, x)\, dt_1 \wedge \cdots \wedge dt_d\right)(dx_{i_1} \wedge \cdots \wedge dx_{i_a})_x & \text{if } a = d. \end{cases}$$

For an arbitrary submersion $\pi : U' \to U$, $\pi_!$ is defined by taking local charts and a partition of unity on U.

2.2 De Rham chain complex

2.2.1 Differentiable space We proceed to the definition of differentiable spaces.

Definition 2.2 Let X be a set and P_X be a set of pairs (U,φ) of $U \in \mathcal{U}$ and a map $\varphi\colon U \to X$. We say P_X is a *differentiable structure* on X if it satisfies the following condition:

- For any $(U,\varphi) \in P_X$, $U' \in \mathcal{U}$ and a submersion $\pi : U' \to U$, the pair $(U', \varphi \circ \pi)$ is also an element of P_X.

We call such a pair (X, P_X) a *differentiable space*. An element of P_X is called a *plot* of (X, P_X).

Example 2.3 Let M be a manifold. There are two types of canonical differentiable structures on M:

$$P_M := \{(U,\varphi) \mid \varphi\colon U \to M \text{ is a } C^\infty \text{ map}\}, \quad P_M^{\mathrm{reg}} := \{(U,\varphi) \mid \varphi\colon U \to M \text{ is a submersion}\}.$$

Clearly, (M, P_M) and (M, P_M^{reg}) are differentiable spaces. The latter is denoted by M^{reg}. We consider the differentiable structure P_M for any manifold M, unless we declare we use M^{reg}.

Definition 2.4 Let (X, P_X) and (Y, P_Y) be differentiable spaces and Z be a subset of X. Denote the projection map from $X \times Y$ to X (resp. Y) by pr_X (resp. pr_Y), and the inclusion map from Z to X by ι_Z.

(a) We define differentiable structures on $X \times Y$ and Z by

$$P_{X\times Y} := \{(U,\varphi) \mid (U, \mathrm{pr}_X \circ \varphi) \in P_X \text{ and } (U, \mathrm{pr}_Y \circ \varphi) \in P_Y\}, \quad P_Z := \{(U,\varphi) \mid (U, \iota_Z \circ \varphi) \in P_X\}.$$

(b) Let $f : X \to Y$ be a map. We say f is a *smooth map* if $(U, f \circ \varphi) \in P_Y$ for any $(U,\varphi) \in P_X$.

In the case of the above definition, we simply call (Z, P_Z) a *subspace* of (X, P_X). Note that, given a set W and two maps $f : X \to W$ and $g\colon Y \to W$, the fiber product $X \,{}_f\!\times_g Y$ becomes a differentiable space as a subspace of $(X \times Y, P_{X\times Y})$.

2.2.2 De Rham chains Next we introduce the notion of de Rham chain complex. Hereafter, if we say that X is a differentiable space, this means that X is equipped with a differentiable structure, denoted by P_X.

Let X be a differentiable space. We consider a graded $\mathbb{R}$-vector space

$$A_*(X) := \bigoplus_{(U,\varphi)\in P_X} \Omega_c^{\dim U - *}(U).$$

For $(U,\varphi) \in P_X$ and $\omega \in \Omega_c^{\dim U - *}(U)$, let (U,φ,ω) denote the element of $A_*(X)$ whose component for $(V,\psi) \in P_X$ is

$$(U,\varphi,\omega)_{(V,\psi)} = \begin{cases} \omega & \text{if } (V,\psi) = (U,\varphi), \\ 0 & \text{if } (V,\psi) \neq (U,\varphi). \end{cases}$$

We take a linear subspace $Z_*(X)$ of $A_*(X)$ generated by

$$\{(U', \varphi \circ \pi, \omega) - (U, \varphi, \pi_! \omega) \mid (U,\varphi) \in P_X \text{ and } \pi : U' \to U \text{ is a submersion}\}.$$

Then we define a quotient vector space

$$C_*^{\mathrm{dR}}(X) := A_*(X)/Z_*(X).$$

The equivalence class of $(U,\varphi,\omega)\in A_*(X)$ in $C_*^{\mathrm{dR}}(X)$ is denoted by $[U,\varphi,\omega]$. We also define an $\mathbb{R}$-linear map $\partial\colon C_*^{\mathrm{dR}}(X)\to C_{*-1}^{\mathrm{dR}}(X)$ of degree -1 by

$$\partial[U,\varphi,\omega]:=(-1)^{|\omega|+1}[U,\varphi,d\omega].$$

This map is well defined and $\partial\circ\partial=0$. $(C_*^{\mathrm{dR}}(X),\partial)$ is called the *de Rham chain complex* of a differentiable space X, and its elements are called *de Rham chains* of X. By taking its homology, we obtain

$$H_*^{\mathrm{dR}}(X):=H_*(C_*^{\mathrm{dR}}(X),\partial).$$

In addition, a functoriality holds. Namely, any smooth map $f:X\to Y$ induces a chain map

$$f_*\colon C_*^{\mathrm{dR}}(X)\to C_*^{\mathrm{dR}}(Y),\quad [U,\varphi,\omega]\mapsto[U,f\circ\varphi,\omega].$$

Remark 2.5 The following are fundamental techniques to compute de Rham chains:

(a) For $[U,\varphi,\omega]\in C_*^{\mathrm{dR}}(X)$, suppose that $V\subset U$ is an open subset containing $\operatorname{supp}\omega$. Then $[U,\varphi,\omega]=[V,\varphi|_V,\omega|_V]\in C_*^{\mathrm{dR}}(X)$.

(b) If $(\mathbb{R}\times U,\varphi)\in P_X$ satisfies $\varphi(s,\cdot)=\varphi_0$ when $s\le 0$ and $\varphi(s,\cdot)=\varphi_1$ when $s\ge 1$ for some $(U,\varphi_0),(U,\varphi_1)\in P_X$, then

$$\partial[\mathbb{R}\times U,\varphi,(-1)^{|\omega|}\chi\times\omega]=[U,\varphi_1,\omega]-[U,\varphi_0,\omega]\in C_*^{\mathrm{dR}}(X)$$

for a closed form $\omega\in\Omega_c^{\dim U-*}(U)$ and $\chi\colon\mathbb{R}\to[0,1]$ such that $\operatorname{supp}\chi$ is compact and $\chi(s)=1$ for every $s\in[0,1]$.

Example 2.6 Let M be an oriented smooth manifold. The de Rham chain complex of M^{reg} is naturally isomorphic to $(\Omega_c^{\dim M-*}(M),d)$ through the map

$$C_p^{\mathrm{dR}}(M^{\mathrm{reg}})\to\Omega_c^{\dim M-p}(M),\quad [U,\varphi,\omega]\mapsto(-1)^{s(p)}\varphi_!\omega.$$

Here $s(p):=\frac{1}{2}(p-\dim M)(p-\dim M-1)$. Hence $H_*^{\mathrm{dR}}(M^{\mathrm{reg}})$ is isomorphic to the compactly supported de Rham cohomology $H_{c,\mathrm{dR}}^{\dim M-*}(M)$.

Let us define the de Rham chain complex for a pair of differentiable spaces. A smooth map $f:X\to\mathbb{R}$ is said to be *approximately smooth* if there exists a decreasing sequence $(f_j)_{j\in\mathbb{Z}_{\ge 1}}$ of smooth maps from X to $\mathbb{R}$ such that $\lim_{j\to\infty}f_j(x)=f(x)$ for every $x\in X$. The following lemma is proved in [15, Lemma 4.11].

Lemma 2.7 *For an approximately smooth function $f:X\to\mathbb{R}$, let $X^a:=f^{-1}((-\infty,a))$ for every $a\in\mathbb{R}\cup\{\infty\}$. Then, for $a,b\in\mathbb{R}\cup\{\infty\}$ with $a\le b$, the linear map $i_*\colon C_*^{\mathrm{dR}}(X^a)\to C_*^{\mathrm{dR}}(X^b)$, which is induced by the inclusion map $i:X^a\to X^b$, is injective.*

In the setting of the above lemma, we define a quotient complex

$$C_*^{\mathrm{dR}}(X^b,X^a):=C_*^{\mathrm{dR}}(X^b)/i_*(C_*^{\mathrm{dR}}(X^a)).$$

Its homology is denoted by $H_*^{\mathrm{dR}}(X^b,X^a)$.

Next, we define a fiber product of de Rham chains.

Definition 2.8 Let (X, P_X) and (Y, P_Y) be differentiable spaces. Suppose that we have an oriented manifold M of dimension n and two smooth maps

$$f\colon (X, P_X) \to (M, P_M), \quad g\colon (Y, P_Y) \to (M, P_M^{\mathrm{reg}}) = M^{\mathrm{reg}}.$$

Then we define a linear map

$$C^{\mathrm{dR}}_{p+n}(X) \otimes C^{\mathrm{dR}}_{q+n}(Y) \to C^{\mathrm{dR}}_{p+q+n}(X \,{}_f\times_g\, Y), \quad x \otimes y \mapsto x \,{}_f\times_g\, y$$

by

$$x \,{}_f\times_g\, y := (-1)^{p|\eta|}[W, (\varphi \times \psi)|_W, (\omega \times \eta)|_W]$$

for $x = [U, \varphi, \omega] \in C^{\mathrm{dR}}_{p+n}(X)$ and $y = [V, \psi, \eta] \in C^{\mathrm{dR}}_{q+n}(Y)$. Here, $W := U \,{}_{f\circ\varphi}\times_{g\circ\psi}\, V$ is a fiber product over M.

It is straightforward to check the well-definedness of $x \,{}_f\times_g\, y$. It can also be checked that

$$\partial(x \,{}_f\times_g\, y) = (\partial x) \,{}_f\times_g\, y + (-1)^p x \,{}_f\times_g\, (\partial y)$$

for any $x \in C^{\mathrm{dR}}_{p+n}(X)$ and $y \in C^{\mathrm{dR}}_{q+n}(Y)$. When $M = \{0\}$, we simply write $x \,{}_f\times_g\, y$ as $x \times y$.

2.2.3 Collection of results about the de Rham chain complex In the rest of this section, let us summarize some basic results about the de Rham chain complex. The first result can be compared with the computation for M^{reg} in Example 2.6. Hereafter, $H^{\mathrm{sing}}(\cdot)$ denotes the singular homology with coefficients in $\mathbb{R}$.

Proposition 2.9 *For every oriented manifold M, there exists a canonical isomorphism*

$$\Psi_M\colon H^{\mathrm{sing}}_*(M) \to H^{\mathrm{dR}}_*(M)$$

such that, for any C^∞ map $f: M \to N$ between oriented manifolds, $\Psi_N \circ f_* = f_* \circ \Psi_M$.

For the details of the construction of Ψ_M, see [15, Section 4.7]. It is the composition of a natural isomorphism between $H^{\mathrm{sing}}_*(M)$ and $H^{\mathrm{sm}}_*(M)$ (the homology of smooth singular chains in M) and a canonical map from $H^{\mathrm{sm}}_*(M)$ to $H^{\mathrm{dR}}_*(M)$. For the proof that Ψ_M is an isomorphism, see [15, Section 5]. This result can be extended to relative homology groups for (M, N), where N is an open submanifold of M such that $N = f^{-1}((-\infty, a))$ for some approximately smooth map $f: M \to \mathbb{R}$.

Next, let $f, g\colon X \to Y$ be smooth maps between differentiable spaces. We say f is *homotopic* to g if there exists a smooth map $H\colon \mathbb{R} \times X \to Y$ such that $H(t,x) = f(x)$ for $t \le 0$ and $H(t,x) = g(x)$ for $t \ge 1$. Then we have the following result. For the proof, see [15, Proposition 4.7].

Proposition 2.10 *For two smooth map $f, g\colon X \to Y$, if f is homotopic to g, then there exists a chain homotopy $K\colon C^{\mathrm{dR}}_*(X) \to C^{\mathrm{dR}}_{*+1}(Y)$ such that $\partial K + K\partial = f_* - g_*$. In particular, $f_* = g_*\colon H^{\mathrm{dR}}_*(X) \to H^{\mathrm{dR}}_*(Y)$.*

Remark 2.11 For three smooth maps $f, g, h\colon X \to Y$ such that f is homotopic to g and g is homotopic to h, we can ask whether f is homotopic to h. In fact, if the differentiable structure P_Y of Y satisfies the following condition, such transitivity holds (the proof is straightforward):

- For any $U \in \mathcal{U}$ and $(U_1, \varphi_1), (U_2, \varphi_2) \in P_Y$ such that $(U_i)_{i=1,2}$ is an open cover of U and $\varphi_1|_{U_1 \cap U_2} = \varphi_2|_{U_1 \cap U_2}$, $(U, \varphi) \in P_Y$ holds for $\varphi\colon U \to Y$ which maps $u \in U_i$ to $\varphi_i(u)$ for $i = 1, 2$.

All differentiable spaces appearing after Section 3 satisfy this condition. However, as mentioned in [15, Remark 4.4], it seems difficult in the general case to prove such transitivity.

The last one is a result about excisions.

Proposition 2.12 *Let X be a differentiable space and $Y = f^{-1}((-\infty, a)) \subset X$ for some approximately smooth function $f\colon X \to \mathbb{R}$ and $a \in \mathbb{R}$. Suppose there is another approximately smooth function $g\colon X \to \mathbb{R}$ and $b_0 \in \mathbb{R}$ such that $g^{-1}((b_0, \infty)) \subset Y$. For every $b > b_0$, let $X^b := g^{-1}((-\infty, b))$ and $Y^b := (g|_Y)^{-1}((-\infty, b))$. Then the inclusion map of pairs $i\colon (X^b, Y^b) \to (X, Y)$ induces an isomorphism $i_*\colon C_*^{\mathrm{dR}}(X^b, Y^b) \to C_*^{\mathrm{dR}}(X, Y)$.*

Proof We first prove the assertion when $g\colon X \to \mathbb{R}$ is a smooth map. For $b > b_0$, choose $\delta > 0$ and a smooth function $\kappa\colon \mathbb{R} \to [0, 1]$ such that $2\delta < b - b_0$ and $\kappa(b') = 1$ if $b' \le b_0 + \delta$ while $\kappa(b') = 0$ if $b' \ge b - \delta$. Then we define a linear map

$$r\colon C_*^{\mathrm{dR}}(X) \to C_*^{\mathrm{dR}}(X^b), \quad [U, \varphi, \omega] \mapsto [U^b, \varphi|_{U^b}, (\kappa \circ g \circ \varphi) \cdot \omega|_{U^b}],$$

where $U^b := (g \circ \varphi)^{-1}((-\infty, b))$. This reduces to a map $\bar{r}\colon C_*^{\mathrm{dR}}(X, Y) \to C_*^{\mathrm{dR}}(X^b, Y^b)$. We claim that $\bar{r}$ is the inverse map of i_*. Indeed, for any $x = [U, \varphi, \omega] \in C_*^{\mathrm{dR}}(X)$, we have

$$\begin{aligned} x - i_* \circ \bar{r}(x) &= [U, \varphi, \omega] - [U, \varphi, (\kappa \circ g \circ \varphi) \cdot \omega] \\ &= [U_0, \varphi|_{U_0}, ((1 - \kappa) \circ g \circ \varphi) \cdot \omega|_{U_0}] \in C_*^{\mathrm{dR}}(Y) \quad \text{for } U_0 := (g \circ \varphi)^{-1}((b_0, \infty)). \end{aligned}$$

Similarly, we can show that $x - \bar{r} \circ i_*(x) \in C_*^{\mathrm{dR}}(Y^b)$ for $x \in C_*^{\mathrm{dR}}(X^b)$.

In the general case, there exists a decreasing sequence $(g_j)_{j \ge 1}$ of smooth maps $g_j\colon X \to \mathbb{R}$ such that $g_j(x) \to g(x)$ as $j \to \infty$ for every $x \in X$. For $b > b_0$, let $X_j^b := g_j^{-1}((-\infty, b))$ and $Y_j^b := (g_j|_Y)^{-1}((-\infty, b))$. From [15, Corollary 4.12(i)], $\varinjlim_{j \to \infty} C_*^{\mathrm{dR}}(X_j^b, Y_j^b) \to C_*^{\mathrm{dR}}(X^b, Y^b)$, induced by inclusion maps, is an isomorphism. We have shown that $(i|_{(X_j^b, Y_j^b)})_*\colon C_*^{\mathrm{dR}}(X_j^b, Y_j^b) \to C_*^{\mathrm{dR}}(X, Y)$ is an isomorphism for every $j \ge 1$, so i_* is also an isomorphism. □

3 Differentiable space of paths and operations from string topology

Throughout this paper, Q is a manifold of dimension n, and K is a compact submanifold of Q of codimension $d \ge 1$. In addition, both Q and K are required to have fixed orientations. The construction of $H_*^{\mathrm{string}}(Q, K)$ depends on the following auxiliary data:

(a) a complete Riemannian metric g on Q (we write $g(v,w) = \langle v,w\rangle_g$ and $\sqrt{g(v,v)} = |v|_g$);

(b) a constant $C_0 \geq 1$;

(c) a positive real number ε_0 for which the map

$$\{(x,v) \in (TK)^\perp \mid |v|_g < \varepsilon_0\} \to Q, \quad (x,v) \mapsto \exp_x(v), \tag{3}$$

is an open embedding;

(d) a C^∞ function $\mu\colon \left[0,\frac{3}{2}\right] \to [0,1]$ such that $\mu(t) = t$ near $t = 0$, $\mu(t) = 1$ near $t = \frac{3}{2}$, and $0 \leq \mu'(t) \leq 1$ for every $t \in \left[0,\frac{3}{2}\right]$.

The independence of $H_*^{\mathrm{string}}(Q,K)$ on these data up to isomorphism is proved in Section 4.5. Until then, these data are fixed, so $\langle v,w\rangle_g$ and $|v|_g$ are denoted by $\langle v,w\rangle$ and $|v|$, respectively.

We define $\mathscr{C}(K)$ to be the set of geodesics $\gamma\colon [0,T] \to Q$ with unit speed such that $\gamma(0), \gamma(T) \in K$ and $\gamma'(0) \in (T_{\gamma(0)}K)^\perp$ and $\gamma'(T) \in (T_{\gamma(T)}K)^\perp$. Such geodesics are called *binormal chords* of K. We also define, for $m \in \mathbb{Z}_{\geq 1}$,

$$\mathscr{L}_m(K) := \left\{ \sum_{k=1}^m \operatorname{length}\gamma_m \;\middle|\; \gamma_1,\ldots,\gamma_m \in \mathscr{C}(K) \right\}, \quad \mathscr{L}(K) := \bigcup_{m=1}^\infty \mathscr{L}_m(K).$$

These are closed subsets of $\{a \in \mathbb{R} \mid a \geq 2\varepsilon_0\}$, since K is compact. Moreover, they are null sets with respect to Lebesgue measure. For the proof, see Lemma 3.10.

3.1 Differentiable space of paths

In this section, we introduce two differentiable spaces of paths, Σ_m^a and S_ε. Let $\Omega_K(Q)$ be the set of C^∞ paths $\gamma\colon [0,T] \to Q$ with $T > 0$ such that $\gamma(0), \gamma(T) \in K$ and $|\gamma'(t)| \leq C_0$ for any $t \in [0,T]$. For any C^∞ path $\gamma\colon [0,T] \to Q$, we define

$$\operatorname{length}\gamma := \int_0^T |\gamma'(t)|\,dt.$$

For $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $m \in \mathbb{Z}_{\geq 1}$, we define Σ_m^a to be the subset of $\Omega_K(Q)^{\times m}$ which consists of $(\gamma_k\colon [0,T_k] \to Q)_{k=1,\ldots,m}$ satisfying *either* of the following two conditions:

- $\sum_{k=1}^m \operatorname{length}\gamma_k < a$.
- $\min_{1\leq k\leq m} \operatorname{length}\gamma_k < \varepsilon_0$.

The differentiable structure on Σ_m^a is defined by

$$P_{\Sigma_m^a} := \{(U,\varphi) \mid U \in \mathscr{U} \text{ and } \varphi\colon U \to \Sigma_m^a \text{ is smooth}\}.$$

Here we say φ is smooth in the following sense: if we write $\varphi(u) = (\gamma_k^u\colon [0,T_k^u] \to Q)_{k=1,\ldots,m}$ for $u \in U$, then, for each $k \in \{1,\ldots,m\}$, the function $U \to \mathbb{R}_{>0}$, $u \mapsto T_k^u$, is of class C^∞ and

$$\{(u,t) \in U \times \mathbb{R} \mid 0 \leq t \leq T_k^u\} \to Q, \quad (u,t) \mapsto \gamma_k^u(t),$$

is a C^∞ map. As an exception, let us define $\Sigma_0^a := \{*\}$ if $a > 0$ and $\Sigma_0^0 := \varnothing$, together with the differentiable structure $P_{\Sigma_0^a} := \{(U,\varphi) \mid U \in \mathscr{U}, \varphi\colon U \to \Sigma_0^a\}$.

We consider the de Rham chain complex $(C_*^{\mathrm{dR}}(\Sigma_m^a), \partial)$ for $a \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 0}$. Lemma 2.7 implies that we may think of $C_*^{\mathrm{dR}}(\Sigma_m^a)$ as a linear subspace of $C_*^{\mathrm{dR}}(\Sigma_m^b)$ when $a \leq b$, since

$$\Sigma_m^b \to \mathbb{R}, \qquad (\gamma_l)_{l=1,\dots,m} \mapsto \operatorname{length} \gamma_k \quad \text{for } k = 1, \dots, m,$$
$$\mathbb{R}^m \to \mathbb{R}, \qquad (a_k)_{k=1,\dots,m} \mapsto \min_{1 \leq k \leq m} a_k,$$

are approximately smooth functions. Thus the quotient complex $(C_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a), \partial)$ is defined.

Remark 3.1 When $a \leq m\varepsilon_0$, the condition that $\sum_{k=1}^m \operatorname{length} \gamma_k < a$ implies that one of γ_k for $k = 1, \dots, m$ has length less than ε_0. Thus, $\Sigma_m^a = \Sigma_m^0$ if $a \leq m\varepsilon_0$. When $a = \infty$, $\Sigma_m^\infty = \Omega_K(Q)^{\times m}$, which will be used only in Section 6.

Next, we define another differentiable space of paths. For every $\varepsilon \in (0, \varepsilon_0]$, the open subset in Q

$$N_\varepsilon := \left\{\exp_x(v) \mid x \in K,\, v \in (T_x K)^\perp \text{ and } |v| < \tfrac{1}{2}\varepsilon\right\}$$

is a tubular neighborhood of K in Q. Then we define a set S_ε which consists of pairs of C^∞ paths $(\sigma_i \colon [0, \varepsilon_i] \to N_\varepsilon)_{i=1,2}$ satisfying:

- $0 < \varepsilon_i \leq \frac{1}{2}\varepsilon$ for $i = 1, 2$.
- $\sigma_1(\varepsilon_1), \sigma_2(0) \in K$ and $\sigma_1(0) = \sigma_2(\varepsilon_2)$.
- $|\sigma_i'(t)| \leq 1$ for $i = 1, 2$ and any $t \in [0, \varepsilon_i]$.

On this set, the evaluation map ev_0 is defined by

$$\mathrm{ev}_0 \colon S_\varepsilon \to N_\varepsilon, \quad (\sigma_1, \sigma_2), \mapsto \sigma_1(0).$$

The differentiable structure on S_ε is defined by

$$P_{S_\varepsilon} := \{(V, \psi) \mid V \in \mathfrak{U},\, \psi \text{ is a smooth map such that } \mathrm{ev}_0 \circ \psi \colon V \to N_\varepsilon \text{ is a submersion}\}.$$

Here we say ψ is smooth in the following sense: if we write $\psi(v) = (\sigma_i^v \colon [0, \varepsilon_i^v] \to N_\varepsilon)_{i=1,2}$ for $v \in V$, then, for $i \in \{1, 2\}$, the function $V \to \mathbb{R}_{>0}$, $v \mapsto \varepsilon_i^v$, is of class C^∞ and

$$\{(v, t) \in V \times \mathbb{R} \mid 0 \leq t \leq \varepsilon_i^v\} \to N_\varepsilon, \quad (v, t) \mapsto \sigma_i^v(t),$$

is a C^∞ map. Note that ev_0 is a smooth map from $(S_\varepsilon, P_{S_\varepsilon})$ to $(N_\varepsilon, P_{N_\varepsilon}^{\mathrm{reg}}) = N_\varepsilon^{\mathrm{reg}}$, defined in Example 2.3.

3.2 Homology groups

In this section, we examine the homology groups $H_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a)$ and $H_*^{\mathrm{dR}}(S_\varepsilon)$. The main results are Propositions 3.7 and 3.9. At the end, several additional results are proved.

3.2.1 Finite-dimensional approximation of Σ_m^a Let us fix $b_0 \in \mathbb{R}_{>0}$ and prepare some notation related to the Riemannian metric g. We note that there is a compact subset of Q which contains the images of all paths $\gamma \in \Omega_K(Q)$ with $\operatorname{length} \gamma \leq b_0$, since K is compact and g is complete. For any two points

$q, q' \in Q$, let $d(q, q')$ be the distance between q and q'. Let us also fix $\varepsilon_g > 0$ such that, if q and q' in this compact set satisfy $d(q, q') < \varepsilon_g$, then there exists a unique geodesic path on $[0, 1]$ of length $d(q, q')$ from q to q'. We write this geodesic by $\overline{qq'}\colon [0, 1] \to Q$.

For every $a \in [0, b_0)$ and $m \in \mathbb{Z}_{\geq 1}$, let $\overline{\Sigma}^a_m$ be the subspace of Σ^a_m which consists of $(\gamma_k)_{k=1,\dots,m}$ satisfying $\sum_{k=1}^m \operatorname{length} \gamma_k < b_0$. From Proposition 2.12 about excision, the inclusion map $\iota\colon \overline{\Sigma}^b_m \to \Sigma^b_m$ induces an isomorphism

$$\iota_*\colon H^{\mathrm{dR}}_*(\overline{\Sigma}^b_m, \overline{\Sigma}^a_m) \to H^{\mathrm{dR}}_*(\Sigma^b_m, \Sigma^a_m) \tag{4}$$

for $a, b \in [0, b_0)$ with $a \leq b$. This means that we cut out a subset

$$\left\{ (\gamma_k)_{k=1,\dots,m} \;\middle|\; \min_{1 \leq k \leq m} \operatorname{length} \gamma_k < \varepsilon_0 \text{ and } \sum_{k=1}^m \operatorname{length} \gamma_k \geq b_0 \right\}$$

to compute the homology group.

First, we approximate $\overline{\Sigma}^a_m$ for $a \in [0, b_0)$ by finite-dimensional manifolds. For every $\nu \in \mathbb{Z}_{\geq 1}$, let us define

$$\overline{\Sigma}^a_m(\nu) := \{(\gamma_k\colon [0, T_k] \to Q)_{k=1,\dots,m} \in \overline{\Sigma}^a_m \mid \max_{1 \leq k \leq m} T_k < C_0^{-1} \varepsilon_g \nu\}, \tag{5}$$

so that $\bigcup_{\nu=1}^\infty \overline{\Sigma}^a_m(\nu) = \overline{\Sigma}^a_m$. Let us also define $B_m(\nu)$ to be a submanifold of $(Q^{\times(\nu+1)})^{\times m}$ which consists of $(q^l_k)^{l=0,\dots,\nu}_{k=1,\dots,m}$ satisfying:

- $q^0_k, q^\nu_k \in K$ for every $k = 1, \dots, m$.
- $\sum_{k=1}^m \sum_{l=0}^{\nu-1} d(q^l_k, q^{l+1}_k) < b_0$.
- $d(q^l_k, q^{l+1}_k) < \varepsilon_g$ for every $k = 1, \dots, m$ and $l = 0, \dots, \nu - 1$.

We then define $B^a_m(\nu)$ for $a < b_0$ to be the open submanifold of $B_m(\nu)$ which consists of $(q^l_k)^{l=0,\dots,\nu}_{k=1,\dots,m}$ satisfying *either* of the following two conditions:

- $\sum_{k=1}^m \sum_{l=0}^{\nu-1} d(q^l_k, q^{l+1}_k) < a$.
- $\min_{1 \leq k \leq m} \bigl(\sum_{l=0}^{\nu-1} d(q^l_k, q^{l+1}_k)\bigr) < \varepsilon_0$.

The differentiable structures on this manifold is $P_{B^a_m(\nu)}$ defined in Example 2.3. For every $\nu \in Z_{\geq 1}$, there are two maps: $\iota_{\Sigma,\nu}\colon \overline{\Sigma}^a_m(\nu) \to \overline{\Sigma}^a_m(2\nu)$ is just the inclusion map, and $\iota_{B,\nu}\colon B^a_m(\nu) \to B^a_m(2\nu)$ is an embedding of a manifold which maps $(q^l_k)^{l=0,\dots,\nu}_{k=1,\dots,m} \in B^a_m(\nu)$ to $(\bar{q}^{l'}_k)^{l'=0,\dots,2\nu}_{k=1,\dots,m} \in B^a_m(2\nu)$, where

$$\bar{q}^{l'}_k = \begin{cases} q^l_k & \text{if } l' \text{ is even and } l' = 2l, \\ \overline{q^l_k q^{l+1}_k}\bigl(\tfrac{1}{2}\bigr) & \text{if } l' \text{ is odd and } l' = 2l + 1. \end{cases}$$

In addition, we define two maps

$$f_\nu\colon \overline{\Sigma}^a_m(\nu) \to B^a_m(\nu), \quad g_\nu\colon B^a_m(\nu) \to \overline{\Sigma}^a_m(2\nu),$$

as follows: f_ν maps $(\gamma_k\colon[0,T_k]\to Q)_{k=1,\ldots,m}\in\overline{\Sigma}^a_m(\nu)$ to $(\gamma_k((l/\nu)T_k))^{l=0,\ldots,\nu}_{k=1,\ldots,m}\in B^a_m(\nu)$. Note that for $l=0,\ldots,\nu-1$ and $k=1,\ldots,m$,

$$d\Bigl(\gamma_k\Bigl(\frac{l}{\nu}T_k\Bigr),\gamma_k\Bigl(\frac{l+1}{\nu}T_k\Bigr)\Bigr)\le \operatorname{length}\gamma_k|_{[(l/\nu)T_k,((l+1)/\nu)T_k]}\le\frac{T_k}{\nu}\cdot\sup_{t\in[0,T_k]}|\gamma'(t)|<\varepsilon_g.$$

On the other hand, g_ν maps $(q^l_k)^{l=0,\ldots,\nu}_{k=1,\ldots,m}\in B^a_m(\nu)$ to

$$\bigl(\gamma_k\colon\bigl[0,\tfrac{3}{2}C_0^{-1}\varepsilon_g\nu\bigr]\to Q\bigr)_{k=1,\ldots,m}\in\overline{\Sigma}^a_m(2\nu),$$

where, for $l=0,\ldots,\nu-1$,

$$\gamma_k(t):=\overline{q^l_kq^{l+1}_k}\circ\chi\Bigl(\frac{C_0}{\varepsilon_g}\cdot t-\tfrac{3}{2}l\Bigr)\quad\text{if }\tfrac{3}{2}C_0^{-1}\varepsilon_g l\le t\le\tfrac{3}{2}C_0^{-1}\varepsilon_g(l+1).$$

Here $\chi\colon\bigl[0,\frac{3}{2}\bigr]\to[0,1]$ is a C^∞ function such that $\chi(t)=0$ near $t=0$, $\chi(t)=\frac{1}{2}$ near $t=\frac{3}{4}$, $\chi(t)=1$ near $t=\frac{3}{2}$, and $0\le\chi'(t)\le 1$ for $t\in\bigl[0,\frac{3}{2}\bigr]$. Note that $|(\gamma_k)'(t)|\le d(q^l_k,q^{l+1}_k)\cdot\sup_{t\in[0,3/2]}|\chi'(t)|\cdot C_0/\varepsilon_g\le C_0$. The next lemma shows that $B^a_m(\nu)$ approximates $\overline{\Sigma}^a_m(\nu)$ as $\nu\to\infty$. See [15, Lemma 6.3].

Lemma 3.2 *The following diagram commutes up to homotopy*:

$$\begin{tikzcd}
\overline{\Sigma}^a_m(\nu) \arrow[r, "\iota_{\Sigma,\nu}"] \arrow[d, "f_\nu"'] & \overline{\Sigma}^a_m(2\nu) \arrow[d, "f_{2\nu}"] \\
B^a_m(\nu) \arrow[r, "\iota_{B,\nu}"'] \arrow[ur, "g_\nu"] & B^a_m(2\nu)
\end{tikzcd}$$

Proof The lower right triangle commutes in the strict sense. For the upper left triangle, we need to show that $\iota_{\Sigma,\nu}$ is homotopic to $g_\nu\circ f_\nu$.

We abbreviate $C_0^{-1}\cdot\varepsilon_g$ by c_0. For $(\gamma_k\colon[0,T_k]\to Q)_{k=1\ldots,m}\in\overline{\Sigma}^a_m(\nu)$, let us define a path $\gamma^s_k\colon[0,c_0\nu]\to Q$ for $k=1,\ldots,m$ and $s\in[0,1]$ by

$$\gamma^s_k(t):=\begin{cases}\overline{\gamma_k((l/\nu)T_k)\gamma_k\bigl(((l+s)/\nu)T_k\bigr)}(t-c_0l) & \text{if }c_0l\le t\le c_0(l+s),\\ \gamma_k\bigl((T_k/(c_0\nu))(t-c_0l)+(l/\nu)T_k\bigr) & \text{if }c_0(l+s)\le t\le c_0(l+1),\end{cases}\qquad\text{for }l=0,\ldots,l-1.$$

Then γ^0_k is equal to $\gamma_k(T_k/(c_0\nu)\,\cdot)$ and γ^1_k is a broken geodesic connecting $\bigl(\gamma_k((l/\nu)T_k)\bigr)^{l=0,\ldots,\nu}$. We modify γ^s_k to a C^∞ path. For instance, we take a C^∞ function $\widetilde{\chi}\colon[0,1]\times\bigl[0,\frac{3}{2}c_0\nu\bigr]\to[0,c_0\nu]$ satisfying $0\le\partial\widetilde{\chi}(s,t)/\partial t\le 1$ and

$$\widetilde{\chi}(s,t)=\begin{cases}c_0l & \text{on a neighborhood of }\bigl\{t=\frac{3}{2}c_0l\bigr\},\\ \frac{2}{3}\bigl(t-\frac{1}{4}c_0\bigr) & \text{on a neighborhood of }\bigl\{t=s+\frac{3}{2}c_0l+\frac{1}{4}c_0\bigr\}.\end{cases}$$

Then we define $\widetilde{\gamma}^s_k:=\gamma^s_k\circ\widetilde{\chi}(s,\cdot)\colon\bigl[0,\frac{3}{2}c_0\nu\bigr]\to Q$. If we take a C^∞ function $\kappa\colon\mathbb{R}\to[0,1]$ such that $\kappa(t)=0$ if $t\le 0$ and $\kappa(t)=1$ if $t\ge 1$, then we get a smooth map

$$H\colon\mathbb{R}\times\overline{\Sigma}^a_m(\nu)\to\overline{\Sigma}^a_m(2\nu),\quad(s,(\gamma_k)_{k=1,\ldots,m})\mapsto(\widetilde{\gamma}^{\kappa(s)}_k)_{k=1,\ldots,m}.$$

This gives a homotopy from $H(0,\cdot)$ to $H(1,\cdot)$. Moreover, $\iota_{\Sigma,\nu}$ is homotopic to $H(0,\cdot)$ since the paths of $\iota_{\Sigma,\nu}((\gamma_k)_{k=1,\dots,m})$ and those of $H(0,(\gamma_k)_{k=1,\dots,m})$ differ only by parametrizations, so the homotopy can be constructed by interpolating these parametrizations. For the same reason, $g_\nu \circ f_\nu$ is homotopic to $H(1,\cdot)$. Therefore, $\iota_{\Sigma,\nu}$ is homotopic to $g_\nu \circ f_\nu$. □

From Lemma 3.2, it follows that, for any $a,b \in \mathbb{R}_{\geq 0}$ with $a \leq b < b_0$,

$$\varinjlim_{j\to\infty} (f_{2^j})_* \colon H_*^{\mathrm{dR}}(\overline{\Sigma}_m^b, \overline{\Sigma}_m^a) = \varinjlim_{j\to\infty} H_*^{\mathrm{dR}}(\overline{\Sigma}_m^b(2^j), \overline{\Sigma}_m^a(2^j)) \to \varinjlim_{j\to\infty} H_*^{\mathrm{dR}}(B_m^b(2^j), B_m^a(2^j))$$

is an isomorphism. Combining with (4), we get an isomorphism

$$\Bigl(\varinjlim_{j\to\infty} (f_{2^j})_*\Bigr) \circ (\iota_*)^{-1} \colon H_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a) \to \varinjlim_{\nu\to\infty} H_*^{\mathrm{dR}}(B_m^b(\nu), B_m^a(\nu)). \tag{6}$$

Furthermore, from Proposition 2.9, $H_*^{\mathrm{sing}}(B_m^b(\nu), B_m^a(\nu)) \cong H_*^{\mathrm{dR}}(B_m^b(\nu), B_m^a(\nu))$.

3.2.2 Computation of homology by Morse theory Next, we examine the singular homology group $H_*^{\mathrm{sing}}(B_m^b(\nu), B_m^a(\nu))$ in terms of Morse theory. Fix $m \in \mathbb{Z}_{\geq 1}$ and $\nu \in \mathbb{Z}_{\geq 1}$. For $k \in \{1,\dots,m\}$ and $l \in \{0,\dots,\nu-1\}$, we set

$$h_k^l \colon B_m(\nu) \to \mathbb{R}, \quad (q_k^l)_{k=1,\dots,m}^{l=0,\dots,\nu} \mapsto d(q_k^l, q_k^{l+1})^2.$$

For every $r > 0$, let us introduce

- a C^∞ function $\sigma_r \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, $z \mapsto \sqrt{z+r}$;
- a C^∞ function
$$L_r \colon B_m(\nu) \to \mathbb{R}, \quad \boldsymbol{q} \mapsto \sum_{k=1}^{m} \sum_{l=0}^{\nu-1} \sigma_r \circ h_k^l(\boldsymbol{q});$$
- compact subsets of $B_m(\nu)$
$$Z_r := \{\boldsymbol{q} \in L_r^{-1}([0,b_0]) \mid \sigma_r \circ h_k^l(\boldsymbol{q}) \leq \varepsilon_g \text{ for every } k = 1,\dots,m \text{ and } l = 0,\dots,\nu-1\},$$
$$Z_r^0 := \Bigl\{\boldsymbol{q} \in Z_r \Bigm| \min_{1\leq k\leq m} \sum_{l=1}^{\nu-1} \sigma_r \circ h_k^l(\boldsymbol{q}) \leq \varepsilon_0\Bigr\}.$$

The role of $\{\sigma_r\}_{r>0}$ is to approximate $\sqrt{z}$ by C^∞ functions. We define, for every $a \in [0,b_0)$ and $r > 0$,

$$Z_r^a := (L_r|_{Z_r})^{-1}\bigl([0,a)\bigr) \cup Z_r^0.$$

Then $Z_r^a \subset Z_{r'}^a$ if $0 < r' < r$. Furthermore, $\bigcup_{r>0} Z_r^a = B_m^a(\nu)$. Therefore, we have an isomorphism induced by the inclusion maps

$$\varinjlim_{r\to 0} H_*^{\mathrm{sing}}(Z_r^b, Z_r^a) \to H_*^{\mathrm{sing}}(B_m^b(\nu), B_m^a(\nu))$$

for $a,b \in \mathbb{R}_{\geq 0}$ with $a \leq b < b_0$. In order to apply Morse theory to L_r, we need to determine its critical points. The next lemma is fundamental. We omit the proof, but a similar result is proved, for instance in [17], when there is no boundary condition.

Lemma 3.3 *For $k_0 \in \{1,\dots,m\}$ and $l_0 \in \{0,\dots,\nu-1\}$, let $X_{k_0}^{l_0}$ be the gradient vector field of $h_{k_0}^{l_0}$. Define π^{l_0} to be the orthogonal projection from $TQ|_K$ to TK if $l_0 = 0$ or ν, and otherwise $\pi^{l_0} := \mathrm{id}_{TQ}$. Then each component of $X_{k_0}^{l_0} = (v_k^l)_{k=1,\dots,m}^{l=0,\dots,\nu}$ is determined by the following: If $(k,l) \neq (k_0,l_0),(k_0,l_0+1)$, then $v_k^l = 0$. Otherwise,*

$$v_{k_0}^{l_0} = \pi^{l_0}\Big(-\frac{d}{dt}\Big|_{t=0} \overline{q_{k_0}^{l_0} q_{k_0}^{l_0+1}}\Big), \qquad v_{k_0}^{l_0+1} = \pi^{l_0+1}\Big(\frac{d}{dt}\Big|_{t=1} \overline{q_{k_0}^{l_0} q_{k_0}^{l_0+1}}\Big).$$

Proposition 3.4 *Suppose that $a \in [0,b_0) \setminus \mathscr{L}_m(K)$ and $r > 0$ is sufficiently small. Then $\boldsymbol{q} \in Z_r \setminus Z_r^0$ is a critical point of L_r with its value in $[0,a]$ if and only if there exist $\gamma_1,\dots,\gamma_m \in \mathscr{C}(K)$ such that $(\gamma_k)_{k=1,\dots,m} \in \overline{\Sigma}_m^a(\nu)$ and $\boldsymbol{q} = f_\nu((\gamma_k)_{k=1,\dots,m})$.*

Proof From Lemma 3.3, $\boldsymbol{q} = (q_k^l)_{k=1,\dots,m}^{l=0,\dots,\nu} \in B_m(\nu)$ is a critical point of L_r if and only if the following conditions hold for every $k \in \{1,\dots,m\}$:

$$\begin{aligned} &(\sigma_r)'(h_k^0(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=0} \overline{q_k^0 q_k^1} \in (T_{q_k^0}K)^\perp \quad \text{and} \quad (\sigma_r)'(h_k^{\nu-1}(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=1} \overline{q_k^{\nu-1} q_k^\nu} \in (T_{q_k^\nu}K)^\perp, \\ &(\sigma_r)'(h_k^{l-1}(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=1} \overline{q_k^{l-1} q_k^l} = (\sigma_r)'(h_k^l(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=0} \overline{q_k^l q_k^{l+1}} \quad \text{for every } l \in \{1,\dots,\nu-1\}. \end{aligned} \tag{7}$$

Comparing the norms of both sides in the second line of (7), we have, for $l \in \{1,\dots,\nu-1\}$,

$$(\sigma_r)'(h_k^{l-1}(\boldsymbol{q})) \cdot \sqrt{h_k^{l-1}(\boldsymbol{q})} = (\sigma_r)'(h_k^l(\boldsymbol{q})) \cdot \sqrt{h_k^l(\boldsymbol{q})}.$$

Since $(\sigma_r)'(z)\sqrt{z}$ is a strictly increasing function of z, this equation means that $d(q_k^{l-1},q_k^l) = d(q_k^l,q_k^{l+1})$ for $l \in \{1,\dots,\nu-1\}$. Since $(\sigma_r)' > 0$, the conditions of (7) are equivalent to

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} \overline{q_k^0 q_k^1} \in (T_{q_k^0}K)^\perp \quad \text{and} \quad \frac{d}{dt}\Big|_{t=1} \overline{q_k^{\nu-1} q_k^\nu} \in (T_{q_k^\nu}K)^\perp, \\ &\frac{d}{dt}\Big|_{t=1} \overline{q_k^{l-1} q_k^l} = \frac{d}{dt}\Big|_{t=0} \overline{q_k^l q_k^{l+1}} \quad \text{for every } l \in \{1,\dots,\nu-1\}. \end{aligned} \tag{8}$$

For every $\boldsymbol{q} \in B_m(\nu)$, let us define $T_{k,\boldsymbol{q}} := \sum_{l=0}^{\nu-1} d(q_k^l,q_k^{l+1})$ and a path $\gamma_{k,\boldsymbol{q}} : [0,T_{k,\boldsymbol{q}}] \to Q$ determined by

$$\gamma_{k,\boldsymbol{q}}(T_{k,\boldsymbol{q}}t) := \overline{q_k^l q_k^{l+1}}(\nu t - l) \quad \text{if } \frac{l}{\nu} \le t \le \frac{l+1}{\nu} \text{ for } l \in \{0,\dots,\nu-1\}.$$

Note that, if $(\gamma_{k,\boldsymbol{q}})_{k=1,\dots,m} \in \overline{\Sigma}_m^a(\nu)$, then $\boldsymbol{q} = f_\nu((\gamma_{k,\boldsymbol{q}})_{k=1,\dots,m})$. We take $r > 0$ so small that $T_{k,\boldsymbol{q}} > 0$ for every $k = 1,\dots,m$ and $\boldsymbol{q} \in Z_r \setminus Z_r^0$. Then, for $\boldsymbol{q} \in Z_r \setminus Z_r^0$, condition (8) is equivalent to $\gamma_{k,\boldsymbol{q}}$ being a binormal chord of K in Q, that is, $\gamma_{k,\boldsymbol{q}} \in \mathscr{C}(K)$. In addition,

$$\lim_{r\to 0} L_r(\boldsymbol{q}) = \sum_{k=1}^m \operatorname{length} \gamma_{k,\boldsymbol{q}}$$

for every $\boldsymbol{q} \in Z_r$ uniformly. Recall that $\mathscr{L}_m(K)$ is a closed subset of $\mathbb{R}_{>0}$. Assuming that $a \notin \mathscr{L}_m(K)$ and $r > 0$ is sufficiently small, it follows that a critical point $\boldsymbol{q} \in Z_r \setminus Z_r^0$ of L_r satisfies $L_r(\boldsymbol{q}) \le a$ if and only if the binormal chords $(\gamma_{k,\boldsymbol{q}})_{k=1,\dots,m}$ satisfy $\sum_{k=1}^m \operatorname{length} \gamma_{k,\boldsymbol{q}} < a$. □

Let Y_r be the gradient vector field of L_r. To prove the next lemma, which is rather technical, let us prepare a few computations. $X_{k,r}^l := ((\sigma_r)' \circ h_k^l) \cdot X_k^l$ is the gradient vector field of $\sigma_r \circ h_k^l$. For every

$k \in \{1, \dots, m\}$, $l \in \{1, \dots, \nu-2\}$ and $\boldsymbol{q} \in B_m(\nu)$, we have

$$\begin{aligned}\langle Y_r, X_k^l\rangle(\boldsymbol{q}) = \Big\langle (\sigma_r)'(h_k^{l-1}(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=1} \overline{q_k^{l-1} q_k^l}, -\frac{d}{dt}\Big|_{t=0} \overline{q_k^l q_k^{l+1}} \Big\rangle \\ + (\sigma_r)'(h_k^l(\boldsymbol{q})) \cdot \Big|\frac{d}{dt}\Big|_{t=0} \overline{q_k^l q_k^{l+1}}\Big|^2 + (\sigma_r)'(h_k^l(\boldsymbol{q})) \cdot \Big|\frac{d}{dt}\Big|_{t=1} \overline{q_k^l q_k^{l+1}}\Big|^2 \\ + \Big\langle -(\sigma_r)'(h_k^{l+1}(\boldsymbol{q})) \cdot \frac{d}{dt}\Big|_{t=0} \overline{q_k^{l+1} q_k^{l+2}}, \frac{d}{dt}\Big|_{t=1} \overline{q_k^l q_k^{l+1}} \Big\rangle.\end{aligned}$$

We abbreviate the increasing function $(\sigma_r)'\sqrt{z}$ by τ_r. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\langle Y_r, X_{k,r}^l\rangle(\boldsymbol{q}) &\geq -\tau_r(h_k^{l-1}(\boldsymbol{q})) \cdot \tau(h_k^l(\boldsymbol{q})) + 2\big(\tau_r(h_k^l(\boldsymbol{q}))\big)^2 - \tau_r(h_k^{l+1}(\boldsymbol{q})) \cdot \tau_r(h_k^l(\boldsymbol{q})) \\ &= \big(2\tau(h_k^l(\boldsymbol{q})) - \tau_r(h_k^{l-1}(\boldsymbol{q})) - \tau_r(h_k^{l+1}(\boldsymbol{q}))\big) \cdot \tau_r(h_k^l(\boldsymbol{q})).\end{aligned}$$

Since σ_r and τ_r are increasing functions, we have, for $k_0 \in \{1, \dots, m\}$ and $l_0 \in \{1, \dots, \nu-2\}$,

$$\sigma_r \circ h_{k_0}^{l_0}(\boldsymbol{q}) = \max_{k,l} \sigma_r \circ h_k^l(\boldsymbol{q}) \implies \langle -Y_r, X_{k_0,r}^{l_0}\rangle(\boldsymbol{q}) \leq 0. \tag{9}$$

The same result holds when $l_0 = 0$ or $\nu - 1$. We also note that, for every $k_0 \in \{1, \dots, m\}$,

$$\Big\langle -Y_r, \sum_{l=1}^{\nu-1} X_{k_0,r}^l \Big\rangle(\boldsymbol{q}) \leq 0. \tag{10}$$

Lemma 3.5 *The trajectory of any point in Z_r (resp. Z_r^0) along $-Y_r$ never goes outside Z_r (resp. Z_r^0) at positive time.*

Proof Suppose that $\Gamma\colon [0,T] \to B_m(\nu)$ is a trajectory along $-Y_r$. Let us consider two continuous functions $f, g\colon [0,T] \to \mathbb{R}$ defined by

$$f(t) := \max_{k,l} \sigma_r \circ h_k^l(\Gamma(t)), \quad g(t) := \min_k \sum_{l=1}^{\nu-1} \sigma_r \circ h_k^l(\Gamma(t)).$$

To prove this lemma, it suffices to show that they are decreasing functions. Indeed, there exists a discrete subset $A \subset [0,T]$ such that f and g are differentiable at every $t \in [0,T] \setminus A$. Inequalities (9) and (10) imply that $f'(t) \leq 0$ and $g'(t) \leq 0$ for every $t \in [0,T] \setminus A$. Hence, f and g are decreasing on $[0,T]$. □

We apply a general result from Morse theory.

Lemma 3.6 *Let B be a manifold and $L\colon B \to \mathbb{R}$ be a C^∞ function. For $a, b \in \mathbb{R}$ with $a \leq b$ and two compact subsets $Z, Z^0 \subset B$, suppose that there is no critical point of L in $(L|_Z)^{-1}([a,b]) \setminus Z^0$ and that the trajectory of any point in Z (resp. Z^0) along the negative gradient vector field of L never goes outside Z (resp. Z^0) at positive times. Let us define $Z^{a'} := (L|_Z)^{-1}((-\infty, a')) \cup Z^0$ for $a' \in \{a, b\}$. Then $H_*^{\mathrm{sing}}(Z^b, Z^a) = 0$.*

Proof The conditions on a, b, Z and Z^0 show that Z^b can be deformed into Z^a along the negative gradient flow of L. Therefore, we get a map from (Z^b, Z^a) to (Z^a, Z^a) which gives the inverse map of the inclusion map up to homotopy. □

Combining the above results, we prove the first main proposition in this section.

Proposition 3.7 *If $\mathscr{L}_m(K) \cap [a,b] = \varnothing$, then $H_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a) = 0$.*

Proof From Proposition 3.4 and Lemma 3.5, we can apply Lemma 3.6 to show that, if $\mathscr{L}_m(K) \cap [a,b] = \varnothing$ and $r > 0$ is sufficiently small, then $H_*^{\mathrm{sing}}(Z_r^b, Z_r^a) = 0$, and thus

$$H_*^{\mathrm{sing}}(B_m^b(\nu), B_m^a(\nu)) \cong \varinjlim_{r \to 0} H_*^{\mathrm{sing}}(Z_r^b, Z_r^a) = 0.$$

From (6), it follows that $H_*^{\mathrm{dR}}(\Sigma_m^b, \Sigma_m^a) \cong \varinjlim_{j\to\infty} H_*^{\mathrm{sing}}(B_m^b(2^j), B_m^a(2^j)) = 0$. □

3.2.3 $H_*^{\mathrm{dR}}(S_\varepsilon)$ and the evaluation map Next, we examine $H_*^{\mathrm{dR}}(S_\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0]$. Choose a Riemannian metric g' on N_{ε_0} for which K is a totally geodesic submanifold of N_{ε_0}. Then we can take a constant $C_1 \geq 1$ such that, for any $q, q' \in N_{\varepsilon_0/C_1}$ with $d(q,q') < \varepsilon_0/C_1$, there exists a unique shortest geodesic in N_{ε_0} with respect to g' from q to q'. Let us write this geodesic by $\widetilde{qq'}\colon [0,1] \to N_{\varepsilon_0}$. In this subsection, exp denotes the exponential map with respect to g.

There is a smooth map between differentiable spaces $s_\varepsilon\colon N_\varepsilon^{\mathrm{reg}} \to S_\varepsilon$ which maps $\exp_x v \in N_\varepsilon$ for $x \in K$ and $v \in (T_x K)^\perp$ with $|v| < \frac{1}{2}\varepsilon$ to

$$s_\varepsilon(\exp_x v) := \left(\sigma_i^v \colon \left[0, \tfrac{1}{2}\varepsilon\right] \to N_\varepsilon\right)_{i=1,2},$$

where $\sigma_1^v(t) = \exp_x\bigl(((\varepsilon - 2t)/\varepsilon)v\bigr)$ and $\sigma_2^v(t) = \exp_x((2t/\varepsilon)v)$ for $t \in \left[0, \frac{1}{2}\varepsilon\right]$. This satisfies $\mathrm{ev}_0 \circ s_\varepsilon = \mathrm{id}_{N_\varepsilon}$. For $\varepsilon, \bar\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon \leq \bar\varepsilon$, let $i_{\varepsilon,\bar\varepsilon}\colon S_\varepsilon \to S_{\bar\varepsilon}$ denote the inclusion map.

Lemma 3.8 *There exists a constant $C \geq C_1$ such that, for any $\varepsilon \in (0, \varepsilon_0/C]$, the inclusion map $i_{\varepsilon, C\varepsilon}\colon S_\varepsilon \to S_{C\varepsilon}$ is homotopic to $i_{\varepsilon,C\varepsilon} \circ s_\varepsilon \circ \mathrm{ev}_0$.*

Proof We define a C^∞ map

$$G\colon \left\{(q,q') \in N_{\varepsilon_0/C_1} \times N_{\varepsilon_0/C_1} \;\middle|\; d(q,q') < \frac{\varepsilon_0}{C_1}\right\} \times [0,1] \to N_{\varepsilon_0}, \quad ((q,q'),s) \mapsto \widetilde{qq'}(s).$$

Then there is a constant $C \geq C_1$ such that

$$|d(G(\,\cdot\,,s))_{(q,q')}(v,v')|_g \leq \tfrac{1}{2}C(|v|_g + |v'|_g)$$

for every $s \in [0,1]$ and $(v,v') \in T_q N_{\varepsilon_0/C} \times T_{q'} N_{\varepsilon_0/C}$ with $d(q,q') < \varepsilon_0/C_1$. For any $\varepsilon \in (0, \varepsilon_0/C]$ and $(\sigma_i\colon [0,\varepsilon_i] \to N_\varepsilon)_{i=1,2} \in S_\varepsilon$, we set $\bar\varepsilon := C\varepsilon$ and define $(\sigma_i^{(s)}\colon [0, \varepsilon_i^s] \to N_{\bar\varepsilon})_{i=1,2} \in S_{\bar\varepsilon}$ for $s \in \mathbb{R}$ as follows: Take $x \in K$ and $v \in T_x K$ such that $\sigma_1(0) = \exp_x v$ and $|v| < \frac{1}{2}\varepsilon$. Then we define

$$\varepsilon_i^s := \begin{cases} (1-\kappa(s))\varepsilon_i + \kappa(s)\bar\varepsilon & \text{if } s \leq \frac{1}{3}, \\ \bar\varepsilon & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3}, \\ \bigl(1-\kappa\bigl(s-\frac{2}{3}\bigr)\bigr)\bar\varepsilon + \kappa\bigl(s-\frac{2}{3}\bigr)\varepsilon & \text{if } s \leq \frac{1}{3}, \end{cases}$$

$$\sigma_i^{(s)}(t) := \begin{cases} \sigma_i(\varepsilon_i t/\varepsilon_i^s) & \text{if } s \leq \frac{1}{3}, \\ G\bigl(\sigma_i(\varepsilon_i t/\bar\varepsilon), \sigma_i^v(t/C), \kappa\bigl(s-\frac{1}{3}\bigr)\bigr) & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3}, \\ \sigma_i^v(\varepsilon t/\varepsilon_i^s) & \text{if } \frac{2}{3} \leq s. \end{cases}$$

Here, $\kappa\colon \mathbb{R} \to [0,1]$ is a C^∞ function such that $\kappa(s) = 0$ if $s \le 0$ and $\kappa(s) = 1$ if $s \ge \frac{1}{3}$. Note that, when $\frac{1}{3} \le s \le \frac{2}{3}$,

$$|(\sigma_i^{(s)})'(t)|_g \le \tfrac{1}{2} C \left(\frac{\varepsilon_i}{\bar{\varepsilon}} \sup |\sigma_i'|_g + \frac{1}{C} \sup |(\sigma_i^v)'|_g \right) \le \tfrac{1}{2} C \left(\frac{\varepsilon_i}{\bar{\varepsilon}} + \frac{1}{C} \right) \le 1.$$

Now the homotopy from $i_{\varepsilon,\bar{\varepsilon}}$ to $i_{\varepsilon,\bar{\varepsilon}} \circ s_\varepsilon \circ \mathrm{ev}_0$ is given by the map

$$\mathbb{R} \times S_\varepsilon \to S_{\bar{\varepsilon}} \colon (s, (\sigma_i)_{i=1,2}) \to (\sigma_i^{(s)}, \quad [0, \varepsilon_i^s] \mapsto N_{\bar{\varepsilon}})_{i=1,2}. \qquad \square$$

Proposition 3.9 *Let C be the constant of Lemma 3.8. For any $\varepsilon \in (0, \varepsilon_0/C]$ and $x \in H_*^{\mathrm{dR}}(S_\varepsilon)$,*

$$(\mathrm{ev}_0)_*(x) = 0 \in H_*^{\mathrm{dR}}(N_\varepsilon) \implies (i_{\varepsilon, C\varepsilon})_*(x) = 0 \in H_*^{\mathrm{dR}}(S_{C\varepsilon}).$$

Proof By Lemma 3.8, $(i_{\varepsilon,C\varepsilon})_*(x) = (i_{\varepsilon,C\varepsilon} \circ s_\varepsilon)_*((\mathrm{ev}_0)_*(x)) = 0$ if $(\mathrm{ev}_0)_*(x) = 0 \in H_*^{\mathrm{dR}}(N_\varepsilon)$. $\square$

3.2.4 Additional results Using the computations obtained in the former subsections, we prove several additional results. As before, b_0 is a fixed real number, and $a \in [0, b_0)$, which may belong to $\mathscr{L}(K)$.

Lemma 3.10 *There exists a manifold Z and a C^∞ function $f\colon Z \to \mathbb{R}$ such that $\mathscr{L}_1(K) \cap [0,a)$ is contained in the set of critical values of f.*

Proof We use the notation in the proof of Proposition 3.4 for $m = 1$. For every critical point $\boldsymbol{q} \in Z_r \setminus Z_r^0$ of L_r, length $\gamma_{1,\boldsymbol{q}} = f_r(L_r(\boldsymbol{q})) \in (0, f_r(b_0))$, where f_r is determined by $f_r(\nu\sigma_r(l)) = \nu\sqrt{l}$ for every $l \in \mathbb{R}_{>0}$. We choose $r > 0$ so that $a < f_r(b_0)$. Then, from the correspondence between binormal chords and critical points of L_r, $\mathscr{L}_m(K) \cap [0,a)$ is contained in the critical value of $f := f_r \circ L_r$ on $Z := Z_r \setminus Z_r^0$. $\square$

This proves that $\mathscr{L}(K) = \bigcup_{m=1}^\infty \mathscr{L}_m(K)$ is a null set. Indeed, for any $m \in \mathbb{Z}$, $\mathscr{L}_m(K) \cap [0,a)$ is contained in the set of critical values of the C^∞ function $Z^{\times m} \to \mathbb{R}$, $(x_1, \dots, x_m) \mapsto f(x_1) + \cdots + f(x_m)$. By Sard's theorem, it is a null set in $\mathbb{R}$. Since b_0 was chosen arbitrarily, it follows that $\mathscr{L}_m(K)$ is a null set.

Next, recall that the definition of Σ_m^a depends on auxiliary data C_0 and ε_0.

Lemma 3.11 *$H_*^{\mathrm{dR}}(\Sigma_m^a)$ does not depend on the choice of C_0 and ε_0. More precisely, the following hold:*

- *If we write Σ_m^a as Σ_{m,C_0}^a to clarify the dependence on C_0, the inclusion map $\Sigma_{m,C_0}^a \to \Sigma_{m,C_0'}^a$ for $C_0 \le C_0'$ induces an isomorphism on homology.*
- *If we write Σ_m^a as $\Sigma_{m,\varepsilon_0}^a$ to clarify the dependence on ε_0, the inclusion map $\Sigma_{m,\varepsilon_0'}^a \to \Sigma_{m,\varepsilon_0}^a$ for $\varepsilon_0' < \varepsilon_0$ induces an isomorphism on homology.*

Proof We define a smooth map $\Sigma_{m,C_0'}^a \to \Sigma_{m,C_0}^a$ which maps $(\gamma_k\colon [0,T_k] \to Q)_{k=1,\dots,m}$ to

$$\big([0, C_0' T_k / C_0] \to Q, \quad t \mapsto \gamma_k(C_0 t / C_0')\big)_{k=1,\dots,m}.$$

This gives the inverse map of the inclusion map up to homotopy. This proves the assertion for C_0.

To prove the assertion for ε_0, let us write Z^a_r by Z^a_{r,ε_0}. Then there is no critical point of $\sum_{l=0}^{\nu-1}\sigma_r\circ h^l_k$ in $Z^a_{r,\varepsilon_0}\setminus Z^a_{r,\varepsilon'_0}$ for every $k\in\{1,\dots,m\}$. By deforming along the negative gradient vector field $-\sum_{l=0}^{\nu-1}X^l_{k,r}$ inductively on $k=1,2,\dots,m$, we can see that Z^a_{r,ε'_0} is a deformation retract of Z^a_{r,ε_0}. This implies that $H^{\mathrm{sing}}_*(Z^a_{r,\varepsilon_0},Z^a_{r,\varepsilon'_0})=0$ and thus $H^{\mathrm{dR}}_*(\Sigma^a_{m,\varepsilon_0},\Sigma^a_{m,\varepsilon'_0})=0$. □

For the sake of discussions in Section 6, let us fix a topology on the set Σ^a_m for $a\in\mathbb{R}_{\geq 0}\cup\{\infty\}$ as follows: $\Omega_K(Q)$ becomes a topological space such that the injection

$$\Omega_K(Q)\to C^\infty([0,1],Q)\times\mathbb{R}_{>0},\quad (\gamma\colon[0,T]\to Q)\mapsto(\gamma(T^{-1}\cdot),T),$$

is a homeomorphism onto its image when $C^\infty([0,1],Q)$ is equipped with the C^∞-topology. We give Σ^a_m the restricted topology from $\Omega_K(Q)^{\times m}$. Then we can consider singular homology groups, such as $H^{\mathrm{sing}}_*(\Sigma^b_m,\Sigma^a_m)$ for $a\leq b$.

Suppose that $a,b\in\mathbb{R}_{\geq 0}$ and $a\leq b<b_0$. By the excision theorem, $\iota_*\colon H^{\mathrm{sing}}_*(\overline{\Sigma}^b_m,\overline{\Sigma}^a_m)\to H^{\mathrm{sing}}_*(\Sigma^b_m,\Sigma^a_m)$ is an isomorphism. All maps in Lemma 3.2 are continuous and the diagram commutes up to continuous homotopy. Therefore, we have an isomorphism

$$\Big(\varinjlim_{j\to\infty}(f_{2^j})_*\Big)\circ(\iota_*)^{-1}\colon H^{\mathrm{sing}}_*(\Sigma^b_m,\Sigma^a_m)\to\varinjlim_{j\to\infty}H^{\mathrm{sing}}_*(B^b_m(2^j),B^a_m(2^j)).\tag{11}$$

3.3 Splitting and concatenating paths

For $\varepsilon\in(0,\varepsilon_0/(5C_0)]$, we define an open subset of $\mathbb{R}^2$,

$$A_\varepsilon:=\{(T,\tau)\mid T>4\varepsilon\text{ and }2\varepsilon<\tau<T-2\varepsilon\}.$$

This becomes a differentiable space as a subspace of $(\mathbb{R}^2)^{\mathrm{reg}}$. For $a\in\mathbb{R}_{\geq 0}$, $m\in\mathbb{Z}_{\geq 1}$ and $k\in\{1,\dots,m\}$, there are smooth maps

$$\mathrm{tl}_k\colon\Sigma^a_m\to\mathbb{R},\quad(\gamma_l\colon[0,T_l]\to Q)_{l=1,\dots,m}\mapsto T_k,\ \mathrm{pr}_T\colon A_\varepsilon\to\mathbb{R}^{\mathrm{reg}},\quad(T,\tau)\mapsto T.$$

(Here tl stands for the time length.) These maps define a fiber product $\Sigma^a_m\,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}A_\varepsilon$ over $\mathbb{R}$, and the k^{th} evaluation map ev_k is defined on it by

$$\mathrm{ev}_k\colon\Sigma^a_m\,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}A_\varepsilon\to Q\quad((\gamma_l)_{l=1,\dots,m},(T,\tau))\mapsto\gamma_k(\tau).$$

From ev_k and $\mathrm{ev}_0\colon S_\varepsilon\to Q^{\mathrm{reg}}$, we obtain a fiber product over Q. We define a map on this fiber product

$$\mathrm{con}_k\colon(\Sigma^a_m\,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}A_\varepsilon)\,{}_{\mathrm{ev}_k}\!\times_{\mathrm{ev}_0}S_\varepsilon\to\Sigma^{a+\varepsilon}_{m+1},$$

which maps $\big((\gamma_l)_{l=1,\dots,m},(T,\tau),(\sigma_i\colon[0,\varepsilon_i]\to N_\varepsilon)_{i=1,2}\big)$ to $(\gamma_1,\dots,\gamma_{k-1},\tilde\gamma^1_k,\tilde\gamma^2_k,\gamma_{k+1},\dots,\gamma_m)$, where $\tilde\gamma^i_k$ for $i=1,2$ are the paths

$$\begin{aligned}&\tilde\gamma^1_k\colon[0,\tau+2\varepsilon_1]\to Q,&& t\mapsto\begin{cases}\gamma_k(t)&\text{if }0\leq t\leq\tau-\varepsilon_1,\\ \gamma_k\big(\tau-\varepsilon_1+\varepsilon_1\mu((t-\tau+\varepsilon_1)/\varepsilon_1)\big)&\text{if }\tau-\varepsilon_1\leq t\leq\tau+\frac12\varepsilon_1,\\ \sigma_1\big(\varepsilon_1-\varepsilon_1\mu((\tau+2\varepsilon_1-t)/\varepsilon_1)\big)&\text{if }\tau+\frac12\varepsilon_1\leq t\leq\tau+2\varepsilon_1,\end{cases}\\ &\tilde\gamma^2_k\colon[0,T-\tau+2\varepsilon_2]\to Q,&& t\mapsto\begin{cases}\sigma_2(\varepsilon_2\mu(t/\varepsilon_2))&\text{if }0\leq t\leq\frac32\varepsilon_2,\\ \gamma_k\big(\tau+\varepsilon_2-\varepsilon_2\mu((3\varepsilon_2-t)/\varepsilon_2)\big)&\text{if }\frac32\varepsilon_2\leq t\leq 3\varepsilon_2,\\ \gamma_k(t+\tau-2\varepsilon_2)&\text{if }3\varepsilon_2\leq t\leq T-\tau+2\varepsilon_2.\end{cases}\end{aligned}\tag{12}$$

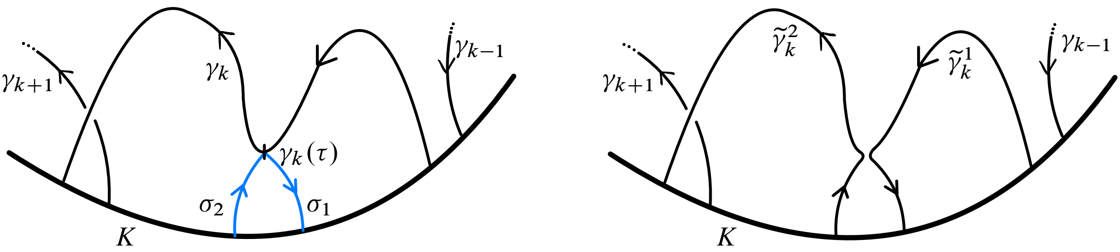

Figure 1: The process to define $\tilde{\gamma}_k^1$ and $\tilde{\gamma}_k^2$.

Here, $\mu\colon \left[0, \frac{3}{2}\right] \to [0,1]$ is one of fixed data we have chosen in the beginning of Section 3.

This definition can be explained as follows (see Figure 1): We split the k^{th} path $\gamma_k\colon [0,T] \to Q$ at $\tau \in (2\varepsilon, T-2\varepsilon)$, where $\gamma_k(\tau) = \sigma_1(0)\ (= \sigma_2(\varepsilon_2)) \in N_\varepsilon$, and then concatenate $\gamma_k|_{[0,\tau]}$ (resp. $\gamma_k|_{[\tau,T]}$) with σ_1 (resp. σ_2). The reparametrizations via μ are necessary in order to modify them to C^∞ paths. Note that

$$\operatorname{length}\tilde{\gamma}_k^1 + \operatorname{length}\tilde{\gamma}_k^2 = \operatorname{length}\gamma_k + \operatorname{length}\sigma_1 + \operatorname{length}\sigma_2 < \operatorname{length}\gamma_k + \varepsilon.$$

The following lemma shows the cases where an element in the fiber product is mapped by con_k into Σ^0_{m+1}.

Lemma 3.12 *For* $((\gamma_l)_{l=1,\dots,m}, (T,\tau), (\sigma_i)_{i=1,2}) \in (\Sigma^a_m\, {}_{\mathrm{tl}_k}{\times}_{\mathrm{pr}_T} A_\varepsilon)\, {}_{\mathrm{ev}_k}{\times}_{\mathrm{ev}_0} S_\varepsilon$, *we have*

$$\mathrm{con}_k((\gamma_l)_{l=1,\dots,m}, (T,\tau), (\sigma_i)_{i=1,2}) \in \Sigma^0_{m+1}$$

if one of the following three conditions holds:

(i) $(\gamma_l)_{l=1,\dots,m} \in \Sigma^0_m$.

(ii) $\tau < 4\varepsilon_0/(5C_0)$ *or* $T - 4\varepsilon_0/(5C_0) < \tau$.

(iii) γ_k *satisfies, for every* $\tau' \in (\gamma_k)^{-1}(N_\varepsilon)$, *that either* $\gamma_k|_{[0,\tau']}$ *or* $\gamma_k|_{[\tau',T]}$ *has length less than* $\frac{4}{5}\varepsilon_0$.

Proof As in the definition of con_k, let us write

$$\mathrm{con}_k((\gamma_l)_{l=1,\dots,m}, (T,\tau), (\sigma_i)_{i=1,2}) = (\gamma_1, \dots, \tilde{\gamma}_k^1, \tilde{\gamma}_k^2, \dots, \gamma_m).$$

Under condition (i), $\operatorname{length}\gamma_l < \varepsilon_0$ for some $l \in \{1,\dots,m\}$. If $l \neq k$, the assertion is trivial. If $l = k$, either $\gamma_k|_{[0,\tau]}$ or $\gamma_k|_{[\tau,T]}$ has length less than $\frac{1}{2}\varepsilon_0$, and thus either $\tilde{\gamma}_k^1$ or $\tilde{\gamma}_k^2$ has length less than

$$\tfrac{1}{2}\varepsilon_0 + \max\{\operatorname{length}\sigma_i \mid i = 1,2\},$$

and this value is smaller than ε_0. Under condition (ii), if $\tau < 4\varepsilon_0/(5C_0)$ (resp. $T - 4\varepsilon_0/(5C_0) < \tau$), then $\operatorname{length}\gamma_k|_{[0,\tau]} < \frac{4}{5}\varepsilon_0$ (resp. $\operatorname{length}\gamma_k|_{[\tau,T]} < \frac{4}{5}\varepsilon_0$). Therefore, either $\tilde{\gamma}_k^1$ or $\tilde{\gamma}_k^2$ has length smaller than ε_0. The same result also holds under condition (iii). □

3.4 Operation on de Rham chains

For $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, let us choose a C^∞ cutoff function $\rho_\varepsilon\colon A_\varepsilon \to [0,1]$ such that

$$\rho_\varepsilon(T,\tau) = \begin{cases} 0 & \text{if } t \le \frac{10}{3}\varepsilon \text{ or } T - \frac{10}{3}\varepsilon \le \tau, \\ 1 & \text{if } \frac{11}{3}\varepsilon \le \tau \le T - \frac{11}{3}\varepsilon. \end{cases}$$

In particular, $\rho_\varepsilon(T,\tau)=0$ if $T\le 5\varepsilon$. We also choose a C^∞ function $\chi_\nu\colon A_\varepsilon\to[0,1]$ for every $\nu\in\mathbb{Z}_{\ge1}$ such that $\chi_\nu(T,\tau)=1$ if $T\le\nu$ and $\chi_\nu(T,\tau)=0$ if $T\ge\nu+1$. The support of $\chi_\nu\rho_\varepsilon$ is compact, so we obtain a de Rham chain

$$\alpha_{\varepsilon,\nu}:=[A_\varepsilon,\mathrm{id}_{A_\varepsilon},\chi_\nu\rho_\varepsilon]\in C_2^{\mathrm{dR}}(A_\varepsilon).$$

In addition, we define $\Sigma_m^a(\nu):=\bigcap_{k=1}^m(\mathrm{tl}_k)^{-1}\big([0,\nu)\big)$, which is a subspace of Σ_m^a.

For $m\in\mathbb{Z}_{\ge1}$, $k\in\{1,\dots,m\}$ and $\xi\in C_q^{\mathrm{dR}}(S_\varepsilon)$, we define a linear map

$$f_{k,\xi}\colon C_*^{\mathrm{dR}}(\Sigma_m^a)\to C_{*+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon})$$

such that, for $x\in C_*^{\mathrm{dR}}(\Sigma_m^a(\nu))\subset C_*^{\mathrm{dR}}(\Sigma_m^a)$ with $\nu\in\mathbb{Z}_{\ge1}$,

$$f_{k,\xi}(x)=(\mathrm{con}_k)_*((x\,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}\alpha_{\varepsilon,\nu})\,{}_{\mathrm{ev}_k}\!\times_{\mathrm{ev}_0}\xi).$$

This map is well defined since $x\,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}(\alpha_{\varepsilon,\nu}-\alpha_{\varepsilon,\nu'})=0$ when $x\in C_*^{\mathrm{dR}}(\Sigma_m^a(\nu))$ and $\nu'\ge\nu$.

Returning to Definition 2.8, we can describe the de Rham chain $f_{k,\xi}(x)$ explicitly. For $x=[U,\varphi,\omega]\in C_p^{\mathrm{dR}}(\Sigma_m^a)$, we write $\varphi(u)=(\gamma_l^u\colon[0,T_l^u]\to Q)_{l=1,\dots,m}$ for $u\in U$. If we take $\nu>\sup_{u\in\operatorname{supp}\omega}T_k^u$, then $x\in C_p^{\mathrm{dR}}(\Sigma_m^a(\nu))$. First, we have $x\,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}\alpha_{\varepsilon,\nu}=[\widetilde U_k,\widetilde\varphi_k,\widetilde\omega_k]$, where

$$\begin{gathered}\widetilde U_k:=\{(u,\tau)\in U\times\mathbb{R}\mid 2\varepsilon<\tau<T_k^u-2\varepsilon\},\\ \widetilde\varphi_k\colon\widetilde U_k\to\Sigma_m^a\,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}A_\varepsilon,\quad (u,\tau)\mapsto(\varphi(u),(T_k^u,\tau)),\\ \widetilde\omega_k\in\Omega_c^*(\widetilde U),\quad (\widetilde\omega_k)_{(u,\tau)}:=\rho_\varepsilon(T_k^u,\tau)\cdot\chi_\nu(T_k^u,\tau)\cdot\omega_u=\rho_\varepsilon(T_k^u,\tau)\cdot\omega_u.\end{gathered}$$

Here, $\widetilde U_k$ is oriented as an open submanifold of $U\times\mathbb{R}$. The last equality holds since $\chi_\nu(T_k^u,\tau)=1$ for $u\in\operatorname{supp}\omega$. This shows the independence of $f_{k,\xi}(x)$ on the choice of χ_ν. For $\xi=[V,\psi,\eta]\in C_q^{\mathrm{dR}}(S_\varepsilon)$, we write $\psi(v)=(\sigma_i^v)_{i=1,2}$ for $v\in V$. Then $f_{k,\xi}(x)=(-1)^r[W_k,\Phi_k,\zeta_k]$, where

$$\begin{gathered}W_k:=\{(u,\tau,v)\in U\times\mathbb{R}\times V\mid 2\varepsilon<\tau<T_k^u-2\varepsilon,\ \gamma_k^u(\tau)=\sigma_1^v(0)\},\\ \Phi_k\colon W_k\to\Sigma_{m+1}^{a+\varepsilon},\quad (u,\tau,v)\mapsto\mathrm{con}_k(\varphi(u),(T_k^u,\tau),\psi(v)),\\ \zeta_k\in\Omega_c^*(W_k),\quad (\zeta_k)_{(u,\tau,v)}:=\rho_\varepsilon(T_k^u,\tau)\cdot(\omega_u\times\eta_v),\\ r:=(p+1-n)|\eta|.\end{gathered}\tag{13}$$

Here W_k is oriented as a fiber product over Q of $\widetilde U_k\to Q$, $(u,\tau)\mapsto\gamma_k^u(\tau)$, and $\mathrm{ev}_0\circ\psi\colon V\to Q$.

Lemma 3.13 *For $x\in C_p^{\mathrm{dR}}(\Sigma_m^a)$ and $\xi\in C_q^{\mathrm{dR}}(S_\varepsilon)$,*

$$\partial\circ f_{k,\xi}(x)-f_{k,\xi}\circ\partial(x)-(-1)^{p+1-n}f_{k,\partial\xi}(x)\in C_{p+q-n}^{\mathrm{dR}}(\Sigma_m^0).$$

Proof Using the notation of (13) for $x=[U,\varphi,\omega]\in C_p^{\mathrm{dR}}(\Sigma_m^a)$ and $\xi=[V,\psi,\eta]\in C_q^{\mathrm{dR}}(S_\varepsilon)$, we have

$$\begin{aligned}\partial\circ f_{k,\xi}(x)-f_{k,\xi}\circ\partial(x)-(-1)^{p+1-n}f_{k,\partial\xi}(x)&=(-1)^{p-1}(\mathrm{con}_k)_*\big((x\,{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}(\partial\alpha_{\varepsilon,\nu}))\,{}_{\mathrm{ev}_k}\!\times_{\mathrm{ev}_0}\xi\big)\\&=(-1)^{(p-n)|\eta|+1}[W_k,\Phi_k,\theta_k],\end{aligned}$$

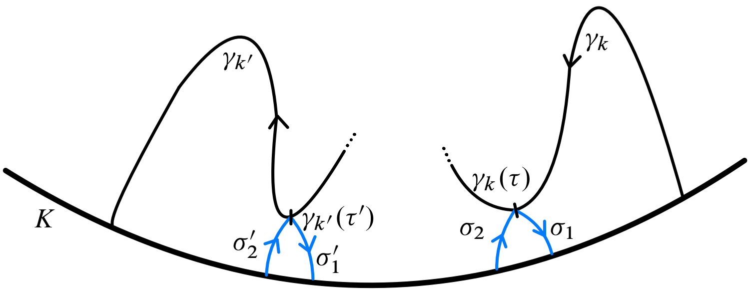

Figure 2: The case where $(\gamma_l)_{l=1,\dots,m}$ intersects both $(\sigma_i)_{i=1,2}$ and $(\sigma'_i)_{i=1,2}$.

where $\theta_k \in \Omega^*_c(W_k)$ is defined by $(\theta_k)_{(u,\tau,v)} = \partial\rho_\varepsilon(T^u_k,\tau)/\partial\tau \cdot (\omega_u \times d\tau \times \eta_v)$. From the condition on ρ_ε, the support of θ lies in an open subset

$$\overline{W}_k := \{(u,\tau,v) \in W_k \mid \tau < 4\varepsilon \text{ or } T^u_k - 4\varepsilon < \tau\}.$$

Since $(\varphi(u),(T^u_k,\tau),\psi(v)) \in (\Sigma^a_m\ {}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T} A_\varepsilon)\ {}_{\mathrm{ev}_k}\!\times_{\mathrm{ev}_0} S_\varepsilon$ for $(u,\tau,v) \in \overline{W}_k$ satisfies condition (ii) of Lemma 3.12, it follows that $\Phi_k(\overline{W}_k) \subset \Sigma^0_m$. Therefore,

$$[W_k,\Phi_k,\theta] = [\overline{W}_k,\Phi_k|_{\overline{W}_k},\theta|_{\overline{W}_k}] \in C^{\mathrm{dR}}_{p+q-n}(\Sigma^0_m). \qquad \square$$

The next lemma is crucial to define chain complexes in Section 4.1. Before stating it, let us give an observation. Suppose that we have

$$(\gamma_l)_{l=1,\dots,m} \in \Sigma^a_m, \quad (\sigma_i : [0,\varepsilon_i] \to N_\varepsilon)_{i=1,2}, \quad (\sigma'_i : [0,\varepsilon'_i] \to N_\varepsilon)_{i=1,2} \in S_\varepsilon,$$
$$k,k' \in \{1,\dots,m\} \quad \text{with } k \le k',$$

and there exist two points $\tau \in (2\varepsilon, T_k - 2\varepsilon)$ and $\tau' \in (2\varepsilon, T_{k'} - 2\varepsilon)$ such that

$$\gamma_k(\tau) = \sigma_1(0), \quad \gamma_{k'}(\tau') = \sigma'_1(0).$$

When $k = k'$, we additionally assume that $\tau + 2\varepsilon < \tau'$. (Figure 2 describes the situation we consider.) Then we can split γ_k at $t = \tau$ and $\gamma_{k'}$ at $t = \tau'$, and concatenate them with $(\sigma_i)_{i=1,2}$ and $(\sigma'_i)_{i=1,2}$, respectively. Depending on which point we use first, there are two elements

$$\Phi := \mathrm{con}_k((\gamma_l)_{l=1,\dots,m},(T_k,\tau),(\sigma_i)_{i=1,2}) \in \Sigma^{a+\varepsilon}_{m+1},$$
$$\Phi' := \mathrm{con}_{k'}((\gamma_l)_{l=1,\dots,m},(T_{k'},\tau'),(\sigma'_i)_{i=1,2}) \in \Sigma^{a+\varepsilon}_{m+1}.$$

In either case, there remains another point which we have not yet used. When $k < k'$, we can split the $(k'+1)^{\text{st}}$ path of Φ at $t = \tau'$ and the k^{th} path of Φ' at $t = \tau$, and concatenate them with $(\sigma'_i)_{i=1,2}$ and $(\sigma_i)_{i=1,2}$, respectively. When $k = k'$, we can split the $(k+1)^{\text{st}}$ path of Φ at $t = \tau' - \tau + 2\varepsilon_2$ and the k^{th} path of Φ' at $t = \tau$, and concatenate them with $(\sigma'_i)_{i=1,2}$ and $(\sigma_i)_{i=1,2}$, respectively. After these two steps, we get the equations

$$\begin{cases} \mathrm{con}_{k'+1}(\Phi,(T_{k'},\tau'),(\sigma'_i)_{i=1,2}) = \mathrm{con}_k(\Phi',(T_k,\tau),(\sigma_i)_{i=1,2}) & \text{if } k < k', \\ \mathrm{con}_{k+1}(\Phi,(\tilde{T}^2_k,\tau'-\tau+2\varepsilon_2),(\sigma'_i)_{i=1,2}) = \mathrm{con}_k(\Phi',(\tilde{T}^1_k,\tau),(\sigma_i)_{i=1,2}) & \text{if } k = k'. \end{cases} \tag{14}$$

Here, $\widetilde{T}_k^2 := T_k - \tau + 2\varepsilon_2$ and $\widetilde{T}_k^1 := \tau' + 2\varepsilon_1'$. This observation leads us to the following lemma about de Rham chains.

Lemma 3.14 *For $x \in C_p^{\mathrm{dR}}(\Sigma_m^a)$, $\xi \in C_q^{\mathrm{dR}}(S_\varepsilon)$ and $k, k' \in \{1, \dots, m\}$ with $k \le k'$, the following hold:*

$$\begin{cases} f_{k'+1,\xi} \circ f_{k,\xi}(x) + (-1)^{q-n} f_{k,\xi} \circ f_{k',\xi}(x) = 0 & \text{if } k < k', \\ f_{k'+1,\xi} \circ f_{k,\xi}(x) + (-1)^{q-n} f_{k,\xi} \circ f_{k',\xi}(x) \in C^{\mathrm{dR}}_{p+2+2q-2n}(\Sigma^0_{m+2}) & \text{if } k = k'. \end{cases}$$

Proof We use the notation of (13) for $x = [U, \varphi, \omega]$ and $\xi = [V, \psi, \eta]$. For short, let us abbreviate for $(T, \tau, T', \tau') \in A_\varepsilon \times A_\varepsilon$

$$\rho(T, \tau, T', \tau) := \rho_\varepsilon(T, \tau) \cdot \rho_\varepsilon(T', \tau').$$

Case 1 We consider the case $k < k'$. We have $f_{k'+1,\xi} \circ f_{k,\xi}(x) = (-1)^s [W_{k,k'}, \Phi_{k,k'}, \zeta_{k,k'}]$ for

$$\begin{gathered} W_{k,k'} := \{(u, \tau, v, \tau', v') \mid (u, \tau, v) \in W_k,\ 2\varepsilon < \tau' < T^u_{k'} - 2\varepsilon,\ \gamma^u_{k'}(\tau') = \sigma_1^{v'}(0)\}, \\ \Phi_{k,k'} \colon W_{k,k'} \to \Sigma^{a+2\varepsilon}_{m+2}, \quad (u, \tau, v, \tau', v') \mapsto \mathrm{con}_{k'+1}(\Phi_k(u, \tau, v), (T^u_{k'}, \tau'), \psi(v')), \\ \zeta_{k,k'} \in \Omega^*_c(W_{k,k'}), \quad (\zeta_{k,k'})_{(u,\tau,v,\tau',v')} := \rho(T^u_k, \tau, T^u_{k'}, \tau') \cdot (\omega_u \times \eta_v \times \eta_{v'}), \\ s := (q + 1 - n)|\eta|, \end{gathered}$$

by substituting k and $[U, \varphi, \omega]$ in (13) with k' and $[W_k, \Phi_k, (-1)^r \zeta_k]$.

Similarly, $f_k \circ f_{k'}(x) = (-1)^s [W'_{k,k'}, \Phi'_{k,k'}, \zeta'_{k,k'}]$ for

$$\begin{gathered} W'_{k,k'} := \{(u, \tau', v', \tau, v) \mid (u, \tau', v') \in W_{k'},\ 2\varepsilon < \tau < T^u_k - 2\varepsilon,\ \gamma^u_k(\tau) = \sigma_1^v(0)\}, \\ \Phi'_{k,k'} \colon W'_{k,k'} \to \Sigma^{a+2\varepsilon}_{m+2}, \quad (u, \tau', v', \tau, v) \mapsto \mathrm{con}_k(\Phi_{k'}(u, \tau', v'), (T^u_k, \tau), \psi(v)), \\ \zeta'_{k,k'} \in \Omega^*_c(W'_{k,k'}), \quad (\zeta'_{k,k'})_{(u,\tau',v'\tau,v)} = \rho(T^u_{k'}, \tau', T^u_k, \tau) \cdot (\omega_u \times \eta_{v'} \times \eta_v). \end{gathered}$$

We define a diffeomorphism

$$h \colon W_{k,k'} \to W'_{k,k'}, \quad (u, \tau, v, \tau', v') \mapsto (u, \tau', v', \tau, v),$$

which changes the sign of orientation by $(-1)^{(1+\dim V - n)^2}$. From (14), it follows that $\Phi'_{k,k'} \circ h = \Phi_{k,k'}$. Moreover, $h_*(\zeta_{k,k'}) = (-1)^{|\eta|^2} \zeta'_{k,k'}$. Combining these computations,

$$f_{k'+1,\xi} \circ f_{k,\xi}(x) = (-1)^{s+(1+\dim V-n+|\eta|)} [W'_{k,k'}, \Phi'_{k,k'}, \zeta'_{k,k'}] = (-1)^{q-n+1} f_{k,\xi} \circ f_{k',\xi}(x).$$

Case 2 We consider the case $k = k'$. We have $f_{k+1} \circ f_k(x) = (-1)^s [W_{k,k}, \Phi_{k,k}, \zeta_{k,k}]$ for

$$\begin{gathered} W_{k,k} := \{(u, \tau, v, \tau', v') \mid (u, \tau, v) \in W_k,\ 2\varepsilon < \tau' < \widetilde{T}^2_k(u, \tau, v) - 2\varepsilon,\ \gamma^u_k(\tau' + \tau - 2\varepsilon^v_2) = \sigma_2^{v'}(0)\}, \\ \Phi_{k,k} \colon W_{k,k} \to \Sigma^{a+2\varepsilon}_{m+2}, \quad (u, \tau, v, \tau', v') \mapsto \mathrm{con}_{k+1}\big(\Phi_k(u, \tau, v), (\widetilde{T}^2_k(u, \tau, v), \tau'), \psi(v')\big), \\ \zeta_{k,k} \in \Omega^*_c(W_{k,k}), \quad (\zeta_{k,k})_{(u,\tau,v,\tau',v')} = \rho(T^u_k, \tau, \widetilde{T}^2_k(u, \tau, v), \tau') \cdot (\omega_u \times \eta_v \times \eta_{v'}), \end{gathered}$$

where $\widetilde{T}^2_k(u, \tau, v) := T^u_k - \tau + 2\varepsilon^v_2$.

Similarly, $f_{k,\xi} \circ f_{k,\xi}(x) = (-1)^s [W'_{k,k}, \Phi'_{k,k}, \zeta'_{k,k}]$ for

$$\begin{aligned} W'_{k,k} &:= \{(u,\tau',v',\tau,v) \mid (u,\tau',v') \in W_k,\ 2\varepsilon < \tau < \widetilde{T}^1_k(\tau',v') - 2\varepsilon,\ \gamma^u_k(\tau) = \sigma^v_1(0)\}, \\ \Phi'_{k,k} &: W'_{k,k} \to \Sigma^{a+2\varepsilon}_{m+2}, \quad (u,\tau',v',\tau,v) \mapsto \mathrm{con}_k\big(\Phi_k(u,\tau',v'), (\widetilde{T}^1_k(\tau',v'),\tau), \psi(v)\big), \\ \zeta'_{k,k} &\in \Omega^*_c(W_{k,k}), \quad (\zeta'_{k,k})_{(u,\tau',v',\tau,v)} = \rho(T^u_k, \tau', \widetilde{T}^1_k(\tau',v'), \tau) \cdot (\omega_u \times \eta_{v'} \times \eta_v), \end{aligned}$$

where $\widetilde{T}^1_k(\tau',v') := \tau' + 2\varepsilon^{v'}_1$. Since $\rho(T_k, \tau, \widetilde{T}^2_k(u,\tau,v), \tau') = 0$ for $\tau' \le \frac{10}{3}\varepsilon$, we have $f_{k+1} \circ f_k(x) = (-1)^s [\overline{W}_{k,k}, \Phi_{k,k}, \zeta_{k,k}]$ for $\overline{W}_{k,k} := W_{k,k} \cap \{\tau' > 3\varepsilon\}$.

This time, we define a map

$$h\colon \overline{W}_{k,k} \to W'_{k,k}, \quad (u,\tau,v,\tau',v') \mapsto (u, \tau'+\tau-2\varepsilon^v_2, v', \tau, v),$$

which is an open embedding and changes the sign of the orientation by $(-1)^{(1+\dim V - n)^2}$. From (14), it follows that $\Phi'_{k,k} \circ h = \Phi_{k,k}$. Indeed, if we set $\tau'_* := \tau' + \tau - 2\varepsilon^v_2$, then

$$\begin{aligned} \Phi'_{k,k} \circ h(u,\tau,v,\tau',v') &= \mathrm{con}_k\big(\Phi_k(u,\tau'_*,v'), (\widetilde{T}^1_k(\tau'_*,v'),\tau), \psi(v)\big) \\ &= \mathrm{con}_{k+1}\big(\Phi_k(u,\tau,v), (\widetilde{T}^1_k(u,\tau,v), \tau'_* - \tau + 2\varepsilon^v_2), \psi(v')\big) \\ &= \Phi_{k,k}(u,\tau,v,\tau',v') \end{aligned}$$

for every $(u,\tau,v,\tau',v') \in \overline{W}_{k,k}$. Therefore,

$$(-1)^{q-n+1} f_{k+1,\xi} \circ f_{k,\xi}(x) - f_{k,\xi} \circ f_{k,\xi}(x) = (-1)^s [W'_{k,k}, \Phi'_{k,k}, (-1)^{|\eta|^2} h_*(\zeta_{k,k}) - \zeta'_{k,k}].$$

For $(u,\tau',v',\tau,v) \in W'_{k,k}$,

$$\begin{aligned} &(-1)^{|\eta|^2} (h_*(\zeta_{k,k}))_{(u,\tau',v',\tau,v)} - (\zeta'_{k,k})_{(u,\tau',v',\tau,v)} \\ &\qquad = \big(\rho(T^u_k, \tau, \widetilde{T}^2_k(u,\tau,v), \tau'-\tau+2\varepsilon^v_2) - \rho(T^u_k, \tau', \widetilde{T}^1_k(\tau',v'), \tau)\big) \cdot (\omega_u \times \eta_{v'} \times \eta_v). \end{aligned}$$

If $\frac{11}{3}\varepsilon \le \tau' \le T^u_k - \frac{11}{3}\varepsilon$ and $\frac{11}{3}\varepsilon \le \tau \le \widetilde{T}^1_k(\tau',v') - \frac{11}{3}\varepsilon$, it can be checked that $\frac{11}{3}\varepsilon \le \tau' - \tau + 2\varepsilon^v_2 \le \widetilde{T}^2_k(u,\tau,v) - \frac{11}{3}\varepsilon$ and $\frac{11}{3} \le \tau \le T^u_k - \frac{11}{3}\varepsilon$ hold, and thus

$$\rho(T^u_k, \tau, \widetilde{T}^2_k(u,\tau,v), \tau'-\tau+2\varepsilon^v_2) = 1 = \rho(T^u_k, \tau', \widetilde{T}^1_k(\tau',v'), \tau).$$

Therefore, $\mathrm{supp}((-1)^{|\eta|^2} h_*(\zeta_{k,k}) - \zeta'_{k,k})$ lies in $W^1_{k,k} \cup W^2_{k,k}$, where

$$\begin{aligned} W^1_{k,k} &:= \{(u,\tau',v',\tau,v) \in W'_{k,k} \mid \tau' < 4\varepsilon \text{ or } T^u_k - 4\varepsilon < \tau'\}, \\ W^2_{k,k} &:= \{(u,\tau',v',\tau,v) \in W'_{k,k} \mid \tau < 4\varepsilon \text{ or } \widetilde{T}^1_k(\tau',v') - 4\varepsilon < \tau\}. \end{aligned}$$

From Lemma 3.12, $\Phi_k(u,\tau',v') = \mathrm{con}_k(\varphi(u), (T^u_k, \tau'), \psi(v')) \in \Sigma^0_{m+1}$ for all $(u,\tau',v',\tau,v) \in W^1_{k,k}$. Then Lemma 3.12 is applied again to show that

$$\Phi'_{k,k}(u,\tau',v',\tau,v) = \mathrm{con}_k\big(\Phi_k(u,\tau',v'), (\widetilde{T}^1_k(\tau',v'),\tau), \psi(v)\big)$$

is an element of Σ^0_{m+2} for every $(u,\tau',v',\tau,v)\in W^1_{k,k}\cup W^2_{k,k}$. Indeed, we can apply case (i) of Lemma 3.12 for $(u,\tau',v',\tau,v)\in W^1_{k,k}$, and case (ii) for $(u,\tau',v',\tau,v)\in W^2_{k,k}$. As a consequence,

$$\begin{aligned}&(-1)^{q-n+1} f_{k+1,\xi}\circ f_{k,\xi}(x)-f_{k,\xi}\circ f_{k,\xi}(x)\\ &\quad=[W^1_{k,k}\cup W^2_{k,k},\Phi'_{k,k}|_{W^1_{k,k}\cup W^2_{k,k}},((-1)^{|\eta|^2}h_*(\zeta_{k,k})-\zeta'_{k,k})|_{W^1_{k,k}\cup W^2_{k,k}}]\in C^{\mathrm{dR}}_{p+2+2q-2n}(\Sigma^0_{m+2}).\qquad\square\end{aligned}$$

In the definition of $f_{k,\xi}$ for $\xi\in C^{\mathrm{dR}}_q(S_\varepsilon)$, there is an ambiguity about the choice of $\rho_\varepsilon\colon A_\varepsilon\to[0,1]$. Suppose that we choose another ρ'_ε, and define $\alpha'_{\varepsilon,\nu}$ and $f'_{k,\xi}$ in the same way as $\alpha_{\varepsilon,\nu}$ and $f_{k,\xi}$. Take an arbitrary chain $x\in C^{\mathrm{dR}}_p(\Sigma^a_m)$. Since $\operatorname{supp}(\rho'_\varepsilon-\rho_\varepsilon)$ lies in $\{(T,\tau)\in A_\varepsilon\mid \tau<4\varepsilon \text{ or } T-4\varepsilon<\tau\}$, $x\;{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}(\alpha_{\varepsilon,\nu}-\alpha'_{\varepsilon,\nu})$ is a chain in the subspace

$$\{((\gamma_k)_{k=1,\dots,m},(T,\tau))\in\Sigma^a_m\;{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}A_\varepsilon\mid \tau<4\varepsilon \text{ or } T-4\varepsilon<\tau\}.$$

From Lemma 3.12, we have

$$f_{k,\xi}(x)-f'_{k,\xi}(x)=(\mathrm{con}_k)_*\big((x\;{}_{\mathrm{tl}_k}\!\times_{\mathrm{pr}_T}(\alpha_{\varepsilon,\nu}-\alpha'_{\varepsilon,\nu}))\;{}_{\mathrm{ev}_k}\!\times_{\mathrm{ev}_0}\xi\big)\in C^{\mathrm{dR}}_{p+1+q-n}(\Sigma^0_{m+1}).$$

Therefore, for $a\in\mathbb{R}_{\ge 0}$, the induced map on the quotient spaces, which is denoted by the same symbol

$$f_{k,\xi}\colon C^{\mathrm{dR}}_*(\Sigma^a_m,\Sigma^0_m)\to C^{\mathrm{dR}}_{*+1+q-n}(\Sigma^{a+\varepsilon}_{m+1},\Sigma^0_{m+1}),$$

is independent of the choice of ρ_ε. For this map, the equation

$$\partial\circ f_{k,\xi}-f_{k,\xi}\circ\partial=(-1)^{p+1-n}f_{k,\partial\xi}\colon C^{\mathrm{dR}}_p(\Sigma^a_m,\Sigma^0_m)\to C^{\mathrm{dR}}_{p+1+q-n}(\Sigma^{a+\varepsilon}_{m+1},\Sigma^0_{m+1})\tag{15}$$

follows from Lemma 3.13, and the equation for $k'\ge k$,

$$f_{k'+1,\xi}\circ f_{k,\xi}+(-1)^{q-n}f_{k,\xi}\circ f_{k',\xi}=0\colon C^{\mathrm{dR}}_p(\Sigma^a_m,\Sigma^0_m)\to C^{\mathrm{dR}}_{p+2+2q-2n}(\Sigma^{a+2\varepsilon}_{m+2},\Sigma^0_{m+2}),\tag{16}$$

follows from Lemma 3.14. When $\partial\xi=0$, (15) implies that $f_{k,\xi}$ is a chain map shifting the degree by $1+q-n$.

3.5 $[-1,1]$- and $[-1,1]^2$-modeled de Rham chains

In this section, we introduce two types of variants of de Rham chains. In this paper, they are necessary for only four kinds of differentiable spaces: Σ^a_m, $\Sigma^a_m\times\Sigma^{a'}_{m'}$, S_ε and (M,P_M) for a manifold M. Throughout this section, X stands for one of these differentiable spaces.

3.5.1 $[-1,1]$-modeled de Rham chains We introduce chains in $\mathbb{R}\times X$. We define $\overline{P}_X$ as the set of tuples $(U,\varphi,(\tau_+,\tau_-))$ such that:

- $(U,\varphi)\in P_{\mathbb{R}^{\mathrm{reg}}\times X}$. If $X=S_\varepsilon$, we additionally require that $(\mathrm{id}_{\mathbb{R}}\times\mathrm{ev}_0)\circ\varphi\colon U\to\mathbb{R}\times Q$ is a submersion. Let us write $\varphi=(\varphi_{\mathbb{R}},\varphi_X)\colon U\to\mathbb{R}\times X$ and $U_I:=\varphi_{\mathbb{R}}^{-1}(I)$ for any subset $I\subset\mathbb{R}$.
- $\tau_+\colon U_{\mathbb{R}_{\ge 1}}\to\mathbb{R}_{\ge 1}\times U_{\{1\}}$ and $\tau_-\colon U_{\mathbb{R}_{\le -1}}\to\mathbb{R}_{\le -1}\times U_{\{-1\}}$ are diffeomorphisms such that
$$\varphi\circ\tau_+^{-1}=i_{\mathbb{R}_{\ge 1}}\times\varphi_X|_{U_{\{1\}}},\quad \varphi\circ\tau_-^{-1}=i_{\mathbb{R}_{\le -1}}\times\varphi_X|_{U_{\{-1\}}}.$$
Here, $i_{\mathbb{R}_{\ge 1}}$ (resp. $i_{\mathbb{R}_{\le -1}}$) is the inclusion map from $\mathbb{R}_{\ge 1}$ (resp. $\mathbb{R}_{\le -1}$) to $\mathbb{R}$.

Remark 3.15 When $X = S_\varepsilon$, the condition that $(U, \varphi) \in P_{\mathbb{R}^{\mathrm{reg}} \times S_\varepsilon}$ implies only that the composition of $(\mathrm{id}_{\mathbb{R}} \times \mathrm{ev}_0) \circ \varphi \colon U \to \mathbb{R} \times Q$ with $\mathrm{pr}_{\mathbb{R}}$ (resp. pr_Q) is a submersion to $\mathbb{R}$ (resp. Q). The condition that $(\mathrm{id}_{\mathbb{R}} \times \mathrm{ev}_0) \circ \varphi$ itself is a submersion is necessary to define a fiber product of $[-1,1]$-modeled de Rham chains in the latter subsection.

For $(U, \varphi, (\tau_+, \tau_-)) \in \overline{P}_X$, we define a linear subspace $\Omega_c^p(U, \varphi, (\tau_+, \tau_-))$ of $\Omega^p(U)$ which consists of p-forms ω on U such that $\operatorname{supp} \omega \cap U_{[-1,1]}$ is compact, $(\tau_+^{-1})^* \omega = 1 \times \omega|_{U_{\{1\}}}$ and $(\tau_-^{-1})^* \omega = 1 \times \omega|_{U_{\{-1\}}}$. We consider the graded $\mathbb{R}$-vector space

$$\overline{A}_*(X) := \bigoplus_{(U,\varphi,(\tau_+,\tau_-)) \in \overline{P}_X} \Omega_c^{\dim U - 1 - *}(U, \varphi, (\tau_+, \tau_-)).$$

For $\boldsymbol{U} = (U, \varphi, (\tau_+, \tau_-)) \in \overline{P}_X$ and $\omega \in \Omega_c^p(U, \varphi, (\tau_+, \tau_-))$, let $(U, \varphi, (\tau_+, \tau_-), \omega)$ denote the element of $\overline{A}_*(X)$ whose component for $\boldsymbol{V} \in \overline{P}_X$ is

$$(U, \varphi, (\tau_+, \tau_-), \omega)_{\boldsymbol{V}} = \begin{cases} \omega & \text{if } \boldsymbol{V} = \boldsymbol{U}, \\ 0 & \text{if } \boldsymbol{V} \neq \boldsymbol{U}. \end{cases}$$

We take the linear subspace $\overline{Z}_*(X)$ of $\overline{A}_*(X)$ generated by vectors

$$(V, \varphi \circ \pi, (\sigma_+, \sigma_-), \omega) - (U, \varphi, (\tau_+, \tau_-), \pi_! \omega)$$

for any submersion $\pi \colon V \to U$ such that

$$(\mathrm{id}_{\mathbb{R}_{\geq 1}} \times \pi|_{V_{\{1\}}}) \circ \sigma_+ = \tau_+ \circ \pi, \quad (\mathrm{id}_{\mathbb{R}_{\leq -1}} \times \pi|_{V_{\{-1\}}}) \circ \sigma_- = \tau_- \circ \pi.$$

We define the quotient vector space

$$\overline{C}_*^{\mathrm{dR}}(X) := \overline{A}_*(X) / \overline{Z}_*(X),$$

whose elements we call $[-1,1]$-*modeled de Rham chains*. $[U, \varphi, (\tau_+, \tau_-), \omega]$ denotes the equivalence class of $(U, \varphi, (\tau_+, \tau_-), \omega)$. We define a degree -1 linear map $\partial \colon \overline{C}_*^{\mathrm{dR}}(X) \to \overline{C}_{*-1}^{\mathrm{dR}}(X)$ by

$$\partial [U, \varphi, (\tau_+, \tau_-), \omega] := (-1)^{|\omega|+1} [U, \varphi, (\tau_+, \tau_-), d\omega].$$

Obviously $\partial \circ \partial = 0$, and we obtain a chain complex $(\overline{C}_*^{\mathrm{dR}}(X), \partial)$. Its homology is denoted by $\overline{H}_*^{\mathrm{dR}}(X)$.

Naturally, there are three chain maps

$$\bar{\imath} \colon C_*^{\mathrm{dR}}(X) \to \overline{C}_*^{\mathrm{dR}}(X), \quad [V, \psi, \omega] \mapsto (-1)^{\dim V} [\mathbb{R} \times V, \mathrm{id}_{\mathbb{R}} \times \psi, (\mathrm{id}_{\mathbb{R}_{\geq 1} \times V}, \mathrm{id}_{\mathbb{R}_{\leq -1} \times V}), 1 \times \omega], \tag{17}$$

and

$$\begin{aligned} e_+ &\colon \overline{C}_*^{\mathrm{dR}}(X) \to C_*^{\mathrm{dR}}(X), \quad [U, \varphi, (\tau_+, \tau_-), \omega] \mapsto (-1)^{\dim U - 1} [U_{\{1\}}, \varphi_X|_{U_{\{1\}}}, \omega|_{U_{\{1\}}}], \\ e_- &\colon \overline{C}_*^{\mathrm{dR}}(X) \to C_*^{\mathrm{dR}}(X), \quad [U, \varphi, (\tau_+, \tau_-), \omega] \mapsto (-1)^{\dim U - 1} [U_{\{-1\}}, \varphi_X|_{U_{\{-1\}}}, \omega|_{U_{\{-1\}}}]. \end{aligned} \tag{18}$$

Here, $U_{\{1\}}$ and $U_{\{-1\}}$ are oriented so that τ_+ and τ_- preserve orientations. Clearly, $e_+ \circ \bar{\imath} = e_- \circ \bar{\imath} = \mathrm{id}_{C_*^{\mathrm{dR}}(X)}$. For $\bar{\imath} \circ e_+$ and $\bar{\imath} \circ e_-$, the next result holds.

Lemma 3.16 *$\bar{\imath} \circ e_+$ and $\bar{\imath} \circ e_-$ are chain homotopic to the identity map $\mathrm{id}_{\overline{C}^{\mathrm{dR}}_*(X)}$.*

Proof This assertion is essentially proved in [16, Lemma 4.8]. We should note that the result in the reference is proved for a specific differentiable space $\mathscr{L}_{k+1}(a)$ (a differentiable space of marked Moore loops in a manifold). However, even for a differentiable space X considered in this section, we can extend the definition of a chain

$$K\big([U,\varphi,(\tau_+,\tau_-),\omega]\big) := (-1)^{|\omega|+1}[\mathbb{R}\times U,\bar{\varphi},(\bar{\tau}_+,\bar{\tau}_-),\bar{\omega}] \in \overline{C}^{\mathrm{dR}}_{*+1}(X)$$

for any $[U,\varphi,(\tau_+,\tau_-),\omega] \in \overline{C}^{\mathrm{dR}}_*(X)$, which appears in the proof of [16, Lemma 4.8]. Then $K\colon \overline{C}^{\mathrm{dR}}_*(X) \to \overline{C}^{\mathrm{dR}}_{*+1}(X)$ gives a chain homotopy from $\mathrm{id}_{\overline{C}^{\mathrm{dR}}_*(X)}$ to $\bar{\imath}\circ e_+$. The proof for $\bar{\imath}\circ e_-$ is completely parallel. □

3.5.2 $[-1,1]^2$-modeled de Rham chains Let us define the smooth map

$$\iota\colon (\mathbb{R}^2)^{\mathrm{reg}} \times X \to (\mathbb{R}^2)^{\mathrm{reg}} \times X, \quad ((r_1,r_2),x) \mapsto ((r_2,r_1),x).$$

We often use the coordinate (r_1,r_2) of $\mathbb{R}^2$ to denote its subsets, for instance $\mathbb{R}_{\geq 1}\times\mathbb{R} = \{r_1 \geq 1\}$.

We introduce chains in $\mathbb{R}^2\times X$. We define $\widehat{P}_X$ as the set of tuples $(U,\varphi,(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-))$ such that:

- $(U,\varphi) \in P_{(\mathbb{R}^2)^{\mathrm{reg}}\times X}$. If $X = S_\varepsilon$, we additionally require that $(\mathrm{id}_{\mathbb{R}^2}\times \mathrm{ev}_0)\circ\varphi\colon U \to \mathbb{R}^2\times Q$ is a submersion. Let us write $\varphi = ((\varphi^1_{\mathbb{R}},\varphi^2_{\mathbb{R}}),\varphi_X)\colon U \to \mathbb{R}^2\times X$ and $U_D := \{u \in U \mid (\varphi^1_{\mathbb{R}}(u),\varphi^2_{\mathbb{R}}(u)) \in D\}$ for any subset $D \subset \mathbb{R}^2$.
- τ^j_+ and τ^j_- for $j = 1,2$ are diffeomorphisms such that

$$\varphi\circ(\tau^1_+)^{-1} = i_{\mathbb{R}_{\geq 1}} \times (\varphi^2_{\mathbb{R}}\times\varphi_X)|_{U_{\{r_1=1\}}}, \qquad \varphi\circ(\tau^1_-)^{-1} = i_{\mathbb{R}_{\leq -1}} \times (\varphi^2_{\mathbb{R}}\times\varphi_X)|_{U_{\{r_1=-1\}}},$$
$$\iota\circ\varphi\circ(\tau^2_+)^{-1} = i_{\mathbb{R}_{\geq 1}} \times (\varphi^1_{\mathbb{R}}\times\varphi_X)|_{U_{\{r_2=1\}}}, \quad \iota\circ\varphi\circ(\tau^2_-)^{-1} = i_{\mathbb{R}_{\leq -1}} \times (\varphi^1_{\mathbb{R}}\times\varphi_X)|_{U_{\{r_2=-1\}}}.$$

For $(U,\varphi,(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-)) \in \widehat{P}_X$, we define the linear subspace $\Omega^p_c(U,\varphi,(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-))$ of $\Omega^p(U)$ which consists of p-forms ω on U such that $\operatorname{supp}\omega \cap U_{[-1,1]\times[-1,1]}$ is compact and

$$((\tau^j_+)^{-1})^*\omega = 1\times\omega|_{U_{\{r_j=1\}}}, \quad ((\tau^j_-)^{-1})^*\omega = 1\times\omega|_{U_{\{r_j=-1\}}} \quad \text{for } j=1,2.$$

We consider the graded $\mathbb{R}$-vector space

$$\widehat{A}_*(X) := \bigoplus_{(U,\varphi,(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-))\in\widehat{P}_X} \Omega^{\dim U - 2 - *}_c(U,\varphi,(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-)).$$

For $\boldsymbol{U} = (U,\varphi,(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-)) \in \widehat{P}_X$ and $\omega \in \Omega^p_c(U,\varphi,(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-))$, let

$$(U,\varphi,(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-),\omega)$$

denote the element of $\widehat{A}_*(X)$ whose component for $\boldsymbol{V} \in \widehat{P}_X$ is

$$(U,\varphi,(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-),\omega)_{\boldsymbol{V}} = \begin{cases} \omega & \text{if } \boldsymbol{V} = \boldsymbol{U}, \\ 0 & \text{if } \boldsymbol{V} \neq \boldsymbol{U}. \end{cases}$$

We take the linear subspace $\widehat{Z}_*(X)$ of $\widehat{A}_*(X)$ generated by vectors

$$(V, \varphi\circ\pi, (\sigma^1_+, \sigma^1_-), (\sigma^2_+, \sigma^2_-), \omega) - (U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \pi_!\omega)$$

for any submersion $\pi\colon V \to U$ such that, for $j = 1, 2$,

$$(\mathrm{id}_{\mathbb{R}_{\geq 1}} \times \pi|_{V_{\{r_j=1\}}}) \circ \sigma^j_+ = \tau^j_+ \circ \pi, \quad (\mathrm{id}_{\mathbb{R}_{\leq -1}} \times \pi|_{V_{\{r_j=-1\}}}) \circ \sigma^j_- = \tau^j_- \circ \pi.$$

Now we define the quotient vector space

$$\widehat{C}^{\mathrm{dR}}_*(X) := \widehat{A}_*(X)/\widehat{Z}_*(X),$$

whose elements we call $[-1,1]^2$*-modeled de Rham chains*. $[U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega]$ denotes the equivalence class of $(U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega)$. We define a degree -1 linear map $\partial\colon \widehat{C}^{\mathrm{dR}}_*(X) \to \widehat{C}^{\mathrm{dR}}_{*-1}(X)$ by

$$\partial[U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega] := (-1)^{|\omega|+1}[U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), d\omega].$$

Obviously $\partial\circ\partial = 0$, and we obtain a chain complex $(\widehat{C}^{\mathrm{dR}}_*(X), \partial)$. Its homology is denoted by $\widehat{H}^{\mathrm{dR}}_*(X)$.

Naturally, there are six chain maps

$$\hat{\imath}^1, \hat{\imath}^2\colon \overline{C}^{\mathrm{dR}}_*(X) \to \widehat{C}^{\mathrm{dR}}_*(X), \quad e^1_+, e^2_+, e^1_-, e^2_-\colon \widehat{C}^{\mathrm{dR}}_*(X) \to \overline{C}^{\mathrm{dR}}_*(X), \tag{19}$$

defined as follows: $\hat{\imath}^1$ and $\hat{\imath}^2$ map $x = [V, \psi, (\tau_+, \tau_-), \omega] \in \overline{C}^{\mathrm{dR}}_*(X)$ to

$$\hat{\imath}^1 x := (-1)^{\dim V-1}[\mathbb{R}\times V, \mathrm{id}_{\mathbb{R}}\times\psi, (\mathrm{id}_{\mathbb{R}_{\geq 1}\times V}, \mathrm{id}_{\mathbb{R}_{\leq -1}\times V}), (\hat{\tau}_+, \hat{\tau}_-), 1\times\omega],$$
$$\hat{\imath}^2 x := (-1)^{\dim V}[\mathbb{R}\times V, \iota\circ(\mathrm{id}_{\mathbb{R}}\times\psi), (\hat{\tau}_+, \hat{\tau}_-), (\mathrm{id}_{\mathbb{R}_{\geq 1}\times V}, \mathrm{id}_{\mathbb{R}_{\leq -1}\times V}), 1\times\omega],$$

where $\hat{\tau}_+\colon \mathbb{R}\times V_{\mathbb{R}_{\geq 1}} \to \mathbb{R}_{\geq 1}\times(\mathbb{R}\times V_{\{1\}})$ and $\hat{\tau}_-\colon \mathbb{R}\times V_{\mathbb{R}_{\leq -1}} \to \mathbb{R}_{\leq -1}\times(\mathbb{R}\times V_{\{-1\}})$ are determined by

$$\hat{\tau}_+(r', \tau_+^{-1}(r, u_+)) = (r, (r', u_+)) \quad \text{for } r' \in \mathbb{R} \text{ and } (r, u_+) \in \mathbb{R}_{\geq 1}\times V_{\{1\}},$$
$$\hat{\tau}_-(r', \tau_-^{-1}(r, u_-)) = (r, (r', u_-)) \quad \text{for } r' \in \mathbb{R} \text{ and } (r, u_-) \in \mathbb{R}_{\leq -1}\times V_{\{-1\}}.$$

In addition, e^j_+ and e^j_- for $j = 1, 2$ map $y = [U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega] \in \widehat{C}^{\mathrm{dR}}_*(X)$ to

$$e^1_+ y := (-1)^{\dim U}[U_{\{1\}\times\mathbb{R}}, (\varphi^2_{\mathbb{R}}, \varphi_X)|_{U_{\{1\}\times\mathbb{R}}}, (\tau^2_+|_{U_{\{1\}\times\mathbb{R}_{\geq 1}}}, \tau^2_-|_{U_{\{1\}\times\mathbb{R}_{\leq -1}}}), \omega|_{U_{\{1\}\times\mathbb{R}}}],$$
$$e^2_+ y := (-1)^{\dim U-1}[U_{\mathbb{R}\times\{1\}}, (\varphi^1_{\mathbb{R}}, \varphi_X)|_{U_{\mathbb{R}\times\{1\}}}, (\tau^1_+|_{U_{\mathbb{R}_{\geq 1}\times\{1\}}}, \tau^1_-|_{U_{\mathbb{R}_{\leq -1}\times\{1\}}}), \omega|_{U_{\mathbb{R}\times\{1\}}}],$$
$$e^1_- y := (-1)^{\dim U}[U_{\{-1\}\times\mathbb{R}}, (\varphi^2_{\mathbb{R}}, \varphi_X)|_{U_{\{-1\}\times\mathbb{R}}}, (\tau^2_+|_{U_{\{-1\}\times\mathbb{R}_{\geq 1}}}, \tau^2_-|_{U_{\{-1\}\times\mathbb{R}_{\leq -1}}}), \omega|_{U_{\{-1\}\times\mathbb{R}}}],$$
$$e^2_- y := (-1)^{\dim U-1}[U_{\mathbb{R}\times\{-1\}}, (\varphi^1_{\mathbb{R}}, \varphi_X)|_{U_{\mathbb{R}\times\{-1\}}}, (\tau^1_+|_{U_{\mathbb{R}_{\geq 1}\times\{-1\}}}, \tau^1_-|_{U_{\mathbb{R}_{\leq -1}\times\{-1\}}}), \omega|_{U_{\mathbb{R}\times\{-1\}}}].$$

Here, the orientations of U_D for $D = \{r_j = 1\}, \{r_j = -1\}$ for $j = 1, 2$ are determined so that τ^j_+ and τ^j_- preserve orientations. The signs are chosen so that

$$e_+\circ e^1_+ = e_+\circ e^2_+, \quad e_-\circ e^1_+ = e_+\circ e^2_-, \quad e_+\circ e^1_- = e_-\circ e^2_+, \quad e_-\circ e^1_- = e_-\circ e^2_-. \tag{20}$$

Clearly, $e^j_+\circ\hat{\imath}^j = e^j_-\circ\hat{\imath}^j = \mathrm{id}_{\overline{C}^{\mathrm{dR}}_*(X)}$ for $j = 1, 2$. For $\hat{\imath}^j\circ e^j_+$ and $\hat{\imath}^j\circ e^j_-$ for $j = 1, 2$, the next result holds.

Lemma 3.17 *$\hat{\imath}^j \circ e^j_+$ and $\hat{\imath}^j \circ e^j_-$ for $j = 1, 2$ are chain homotopic to $\mathrm{id}_{\widehat{C}^{\mathrm{dR}}_*(X)}$.*

Proof We omit the detailed proof and refer to Lemma 3.16. For any $x = [U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega] \in \widehat{C}^{\mathrm{dR}}_*(X)$, let us define diffeomorphisms, for $j = 1, 2$,

$$\tilde{\tau}^j_+ \colon \mathbb{R} \times U_{\{r_j \geq 1\}} \to \mathbb{R}_{\geq 1} \times (\mathbb{R} \times U_{\{r_j = 1\}}), \quad \tilde{\tau}^j_- \colon \mathbb{R} \times U_{\{r_j \leq -1\}} \to \mathbb{R}_{\leq -1} \times (\mathbb{R} \times U_{\{r_j = -1\}})$$

such that $(\tilde{\tau}^j_+)^{-1}(r_j, (r, u)) = (r, (\tau^j_+)^{-1}(r_j, u))$ and $(\tilde{\tau}^j_-)^{-1}(r_j, (r, u)) = (r, (\tau^j_-)^{-1}(r_j, u))$. Referring to the proof of [16, Lemma 4.8], we can find $\bar{\varphi}^j$, $\bar{\tau}^j_\pm$ and $\bar{\omega}^j$ for $j = 1, 2$ to define

$$K^1(x) := (-1)^{|\omega|+1}[\mathbb{R} \times U, \bar{\varphi}^1, (\bar{\tau}^1_+, \bar{\tau}^1_-), (\tilde{\tau}^2_+, \tilde{\tau}^2_-), \bar{\omega}^1],$$
$$K^2(x) := (-1)^{|\omega|+1}[\mathbb{R} \times U, \bar{\varphi}^2, (\tilde{\tau}^1_+, \tilde{\tau}^1_-), (\bar{\tau}^2_+, \bar{\tau}^2_-), \bar{\omega}^2]$$

such that $K^j \colon \widehat{C}^{\mathrm{dR}}_*(X) \to \widehat{C}^{\mathrm{dR}}_{*+1}(X)$ is a chain homotopy from $\mathrm{id}_{\widehat{C}^{\mathrm{dR}}_*(X)}$ to $\hat{\imath}^j \circ e^j_+$ for $j = 1, 2$. The proof for $\hat{\imath}^j \circ e^j_-$ is completely parallel. $\square$

3.5.3 Collection of analogies with ordinary de Rham chains As above, X and Y are chosen from the differentiable spaces Σ^a_m, $\Sigma^a_m \times \Sigma^{a'}_{m'}$, S_ε and (M, P_M) for a manifold M. Let $f \colon X \to Y$ be a smooth map. If $Y = S_\varepsilon$, we require that $X = S_{\varepsilon'}$ and $\mathrm{ev}_0 \circ f = \mathrm{ev}_0$. Then f induces chain maps

$$f_* \colon \overline{C}^{\mathrm{dR}}_*(X) \to \overline{C}^{\mathrm{dR}}_*(Y), \qquad [U, \varphi, (\tau_+, \tau_-), \omega] \mapsto [U, f \circ \varphi, (\tau_+, \tau_-), \omega],$$
$$f_* \colon \widehat{C}^{\mathrm{dR}}_*(X) \to \widehat{C}^{\mathrm{dR}}_*(Y), \quad [U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega] \mapsto [U, f \circ \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega].$$

If $X = \Sigma^a_m$ and $Y = \Sigma^b_m$ for $a \leq b$ and f is the inclusion map, then we claim that the above maps are injective. This can be proved in the same way as Lemma 2.7, so we omit the proof. As a consequence, we can define $\overline{C}^{\mathrm{dR}}_*(\Sigma^b_m, \Sigma^a_m)$ and $\widehat{C}^{\mathrm{dR}}_*(\Sigma^b_m, \Sigma^a_m)$ as quotient complexes.

Next, let $(X, Y) = (\Sigma^a_m, \Sigma^{a'}_{m'})$ or $(\{0\}, S_\varepsilon)$. We identify $\{0\} \times S_\varepsilon$ with S_ε. Then a cross product $x \times y \in \overline{C}^{\mathrm{dR}}_{p+q}(X \times Y)$ is defined for $x = [U, \varphi, (\tau_+, \tau_-), \omega] \in \overline{C}^{\mathrm{dR}}_p(X)$ and $y = [V, \psi, (\sigma_+, \sigma_-), \eta] \in \overline{C}^{\mathrm{dR}}_q(Y)$ by

$$x \times y := (-1)^{p|\eta|}[W, \tilde{\varphi}, (\tilde{\tau}_+, \tilde{\tau}_-), \omega \times \eta].$$

Here, $W := U \;{}_{\varphi_{\mathbb{R}}}\!\times_{\psi_{\mathbb{R}}} V$ is a fiber product over $\mathbb{R}$ and $\tilde{\varphi}$, $\tilde{\tau}_+$ and $\tilde{\tau}_-$ are determined by

$$\tilde{\varphi} \colon W \to \mathbb{R} \times (X \times Y), \quad (u, v) \mapsto (\varphi_{\mathbb{R}}(u), \varphi_X(u), \psi_Y(v)),$$
$$\tilde{\tau}_+(u, v) = (r, (u_+, v_+)) \quad \text{for } (u, v) = ((\tau_+)^{-1}(r, u_+), (\sigma_+)^{-1}(r, v_+)) \in W_{\mathbb{R}_{\geq 1}},$$
$$\tilde{\tau}_-(u, v) = (r, (u_-, v_-)) \quad \text{for } (u, v) = ((\tau_-)^{-1}(r, u_-), (\sigma_-)^{-1}(r, v_-)) \in W_{\mathbb{R}_{\leq -1}}.$$

Similarly, a cross product $x \times y \in \widehat{C}^{\mathrm{dR}}_{p+q}(X \times Y)$ is defined for $x = [U, \varphi, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \omega] \in \widehat{C}^{\mathrm{dR}}_p(X)$ and $y = [V, \psi, (\sigma^1_+, \sigma^1_-), (\sigma^2_+, \sigma^2_-), \eta] \in \widehat{C}^{\mathrm{dR}}_q(Y)$ by

$$x \times y := (-1)^{p|\eta|}[W, \tilde{\varphi}, (\tilde{\tau}^1_+, \tilde{\tau}^1_-), (\tilde{\tau}^2_+, \tilde{\tau}^2_-), \omega \times \eta].$$

Here $W := U \;{}_{(\varphi^1_{\mathbb{R}}, \varphi^2_{\mathbb{R}})}\!\times_{(\psi^1_{\mathbb{R}}, \psi^2_{\mathbb{R}})} V$ is a fiber product over $\mathbb{R}^2$ and $\tilde{\varphi}$, $\tilde{\tau}^j_+$ and $\tilde{\tau}^j_-$ for $j = 1, 2$ are determined by

$$\tilde{\varphi} \colon W \to \mathbb{R} \times (X \times Y), \quad (u, v) \mapsto (\varphi_{\mathbb{R}}(u), \varphi_X(u), \psi_Y(v)),$$
$$\tilde{\tau}^j_+(u, v) = (r, (u_+, v_+)) \quad \text{for } (u, v) = ((\tau^j_+)^{-1}(r, u_+), (\sigma^j_+)^{-1}(r, v_+)) \in W_{\{r_j = 1\}},$$
$$\tilde{\tau}^j_-(u, v) = (r, (u_-, v_-)) \quad \text{for } (u, v) = ((\tau^j_-)^{-1}(r, u_-), (\sigma^j_-)^{-1}(r, v_-)) \in W_{\{r_j = -1\}}.$$

The next result is analogous to Propositions 3.7 and 3.9. It follows immediately from the fact that $(e_+)_*\colon \overline{H}^{\mathrm{dR}}_*(X) \to H^{\mathrm{dR}}_*(X)$ is an isomorphism (see Lemma 3.16).

Proposition 3.18 *Let $a, b \in \mathbb{R}_{>0}$ with $a \le b$ and $\varepsilon \in (0, \varepsilon_0/C]$ for the constant C of Lemma 3.8. Then the following hold*:

- *If $\mathscr{L}_m(K) \cap [a,b] = \varnothing$, then $\overline{H}^{\mathrm{dR}}_*(\Sigma^b_m, \Sigma^a_m) = 0$.*
- *For any $x \in \overline{H}^{\mathrm{dR}}_*(S_\varepsilon)$,*

$$(\mathrm{ev}_0)_* \circ (e_+)_*(x) = 0 \in H^{\mathrm{dR}}_*(N_\varepsilon) \implies (i_{\varepsilon, C\varepsilon})_*(x) = 0 \in \overline{H}^{\mathrm{dR}}_*(S_{C\varepsilon}).$$

3.5.4 Operations on $[-1,1]$- and $[-1,1]^2$-modeled de Rham chains In the rest of this section, let us define operators corresponding to $f_{k,\xi}$. We prefer to refer to the explicit description (13) of $f_{k,\xi}(x)$ rather than its original definition using fiber products of chains. Let $\varepsilon \in (0, \varepsilon_0/(5C_0)]$ and $\rho_\varepsilon\colon A_\varepsilon \to [0,1]$ be the C^∞ function we have chosen in the beginning of Section 3.4

For $k = 1, \dots, m$ and $\bar{\xi} \in \overline{C}^{\mathrm{dR}}_q(S_\varepsilon)$, we define a linear map

$$\bar{f}_{k,\bar{\xi}}\colon \overline{C}^{\mathrm{dR}}_*(\Sigma^a_m) \to \overline{C}^{\mathrm{dR}}_{*+1+q-n}(\Sigma^{a+\varepsilon}_{m+1})$$

as follows: Let

$$x = [U, \varphi, (\tau_+, \tau_-), \omega] \in \overline{C}^{\mathrm{dR}}_p(\Sigma^a_m), \quad \bar{\xi} = [V, \psi, (\sigma_+, \sigma_-), \eta] \in \overline{C}^{\mathrm{dR}}_q(S_\varepsilon),$$

and

$$\begin{aligned} \varphi(u) &= (\varphi_{\mathbb{R}}(u), \varphi_\Sigma(u)) = (\varphi_{\mathbb{R}}(u), (\gamma^u_l\colon [0, T^u_l] \to Q)_{l=1,\dots,m}) \in \mathbb{R} \times \Sigma^a_m, \\ \psi(v) &= (\psi_{\mathbb{R}}(v), \psi_S(v)) = (\psi_{\mathbb{R}}(v), (\sigma^v_i)_{i=1,2}) \in \mathbb{R} \times S_\varepsilon \end{aligned}$$

for every $u \in U$ and $v \in V$. Then we define $\bar{f}_{k,\bar{\xi}}(x) := (-1)^s [W_k, \Phi_k, (\tilde{\tau}_+, \tilde{\tau}_-), \zeta_k]$, where

$$\begin{aligned} W_k &:= \{(u, \tau, v) \in U \times \mathbb{R} \times V \mid 2\varepsilon < \tau < T^u_k - 2\varepsilon,\ (\varphi_{\mathbb{R}}(u), \gamma^u_k(\tau)) = (\psi_{\mathbb{R}}(u), \sigma^v_1(0))\}, \\ \Phi_k &\colon W_k \to \mathbb{R} \times \Sigma^{a+\varepsilon}_{m+1}, \quad (u, \tau, v) \mapsto \big(\varphi_{\mathbb{R}}(u), \mathrm{con}_k(\varphi_\Sigma(u), (T^u_k, \tau), \psi_S(v))\big), \\ \zeta_k &\in \Omega^*_c(W_k), \quad (\zeta_k)_{(u,\tau,v)} := \rho_\varepsilon(T^u_k, \tau) \cdot (\omega_u \times \eta_v), \\ s &:= (p+1-n)|\eta| + n + 1, \end{aligned} \tag{21}$$

and

$$\tilde{\tau}_+(u, \tau, v) := (r, (u_+, \tau, v_+)) \in \mathbb{R} \times (W_k)_{\{r=1\}}$$

for $(u, \tau, v) = (\tau_+^{-1}(r, u_+), \tau, \sigma_+^{-1}(r, v_+)) \in (W_k)_{\{r \ge 1\}}$, and

$$\tilde{\tau}_-(u, \tau, v) := (r, (u_-, \tau, v_-)) \in \mathbb{R} \times (W_k)_{\{r=-1\}}$$

for $(u, \tau, v) = (\tau_-^{-1}(r, u_-), \tau, \sigma_-^{-1}(r, v_-)) \in (W_k)_{\{r \le -1\}}$. Here, W_k is oriented as a fiber product over $\mathbb{R} \times Q$ of the map

$$\{(u, \tau) \in U \times \mathbb{R} \mid 2\varepsilon < \tau < T^u_k - 2\varepsilon\} \to \mathbb{R} \times Q, \quad (u, \tau) \mapsto (\varphi_{\mathbb{R}}(u), \gamma^u_k(\tau)),$$

and the submersion $(\mathrm{id}_{\mathbb{R}}\times \mathrm{ev}_0)\circ\psi\colon V\to\mathbb{R}\times Q$. It can be checked that $\bar\imath$ and e_+, e_- intertwine this operator and $f_{k,\xi}$ for $\xi\in C_q^{\mathrm{dR}}(S_\varepsilon)$. Namely,

$$\bar\imath\circ f_{k,\xi}=f_{k,\bar\imath\xi}\circ\bar\imath,\quad e_+\circ\bar f_{k,\bar\xi}=f_{k,e_+\bar\xi}\circ e_+,\quad e_-\circ\bar f_{k,\bar\xi}=f_{k,e_-\bar\xi}\circ e_-.$$

Similar results as for $f_{k,\xi}$ are the following: $\bar f_{k,\bar\xi}$ induces a linear map

$$\bar f_{k,\bar\xi}\colon \overline{C}{}_*^{\mathrm{dR}}(\Sigma_m^a,\Sigma_m^0)\to\overline{C}{}_{*+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon},\Sigma_{m+1}^0),$$

which is independent of ρ_ε. The next equations are variants of (15) and (16), and they follow from similar computations as Lemmas 3.13 and 3.14, so we omit the proof.

Proposition 3.19 *For $k'\geq k$,*

$$\partial\circ\bar f_{k,\bar\xi}-\bar f_{k,\bar\xi}\circ\partial=(-1)^{p+1-n}\bar f_{k,\partial\bar\xi}\colon \overline{C}{}_p^{\mathrm{dR}}(\Sigma_m^a,\Sigma_m^0)\to\overline{C}{}_{p+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon},\Sigma_{m+1}^0),$$
$$\bar f_{k'+1,\bar\xi}\circ\bar f_{k,\bar\xi}+(-1)^{q-n}\bar f_{k,\bar\xi}\circ\bar f_{k',\bar\xi}=0\colon \overline{C}{}_p^{\mathrm{dR}}(\Sigma_m^a,\Sigma_m^0)\to\overline{C}{}_{p+2+2q-2n}^{\mathrm{dR}}(\Sigma_{m+2}^{a+2\varepsilon},\Sigma_{m+2}^0).$$

Next, for $k=1,\dots,m$ and $\hat\xi\in\widehat{C}{}_q^{\mathrm{dR}}(S_\varepsilon)$, we define a linear map

$$\hat f_{k,\hat\xi}\colon\overline{C}{}_*^{\mathrm{dR}}(\Sigma_m^a)\to\overline{C}{}_{*+1+q-n}^{\mathrm{dR}}(\Sigma_{m+1}^{a+\varepsilon})$$

as follows: Let

$$x=[U,\varphi,(\tau_+^1,\tau_-^1),(\tau_+^2.\tau_-^2),\omega]\in\widehat{C}{}_p^{\mathrm{dR}}(\Sigma_m^a),\quad \hat\xi=[V,\psi,(\sigma_+^1,\sigma_-^1),(\sigma_+^2,\sigma_-^2),\eta]\in\widehat{C}{}_q^{\mathrm{dR}}(S_\varepsilon),$$

and

$$\varphi(u)=(\varphi_{\mathbb{R}^2}(u),\varphi_\Sigma(u))=\bigl((\varphi_{\mathbb{R}}^1(u),\varphi_{\mathbb{R}}^2(u)),(\gamma_l^u\colon[0,T_l^u]\to Q)_{l=1,\dots,m}\bigr)\in\mathbb{R}^2\times\Sigma_m^a,$$
$$\psi(v)=(\psi_{\mathbb{R}^2}(v),\psi_S(v))=\bigl((\psi_{\mathbb{R}}^1(v),\psi_{\mathbb{R}}^2(v)),(\sigma_i^v)_{i=1,2}\bigr)\in\mathbb{R}^2\times S_\varepsilon$$

for every $u\in U$ and $v\in V$. Then we define $\hat f_{k,\hat\xi}(x):=(-1)^s[W_k,\Phi_k,(\tilde\tau_+^1,\tilde\tau_-^1),(\tilde\tau_+^2,\tilde\tau_-^2),\zeta_k]$, where

$$W_k:=\{(u,\tau,v)\in U\times\mathbb{R}\times V\mid 2\varepsilon<\tau<T_k^u-2\varepsilon,\ (\varphi_{\mathbb{R}^2}(u),\gamma_k^u(\tau))=(\psi_{\mathbb{R}^2}(u),\sigma_1^v(0))\},$$
$$\Phi_k\colon W_k\to\mathbb{R}^2\times\Sigma_{m+1}^{a+\varepsilon},\quad (u,\tau,v)\mapsto\bigl(\varphi_{\mathbb{R}^2}(u),\mathrm{con}_k(\varphi_\Sigma(u),(T_k^u,\tau),\psi_S(v))\bigr),$$
$$\zeta_k\in\Omega_c^*(W_k),\quad (\zeta_k)_{(u,\tau,v)}:=\rho_\varepsilon(T_k^u,\tau)\cdot(\omega_u\times\eta_v),$$
$$s:=(p+1-n)|\eta|,$$

and, for $j=1,2$,

$$\tilde\tau_+^j(u,\tau,v):=(r_j,(u_+^j,\tau,v_+^j))\in\mathbb{R}_{\geq1}\times(W_k)_{\{r_j=1\}}$$

for $(u,\tau,v)=((\tau_+^j)^{-1}(r_j,u_+^j),\tau,(\sigma^j)_+^{-1}(r_j,v_+^j))\in(W_k)_{\{r_j\geq1\}}$ and

$$\tilde\tau_-^j(u,\tau,v):=(r_j,(u_-^j,\tau,v_-^j))\in\mathbb{R}_{\leq-1}\times(W_k)_{\{r_j=-1\}}$$

for $(u,\tau,v)=((\tau_-^j)^{-1}(r_j,u_-^j),\tau,(\sigma^j)_-^{-1}(r_j,v_-^j))\in(W_k)_{\{r_j\leq-1\}}$. Here, W_k is oriented as a fiber product over $\mathbb{R}^2\times Q$ of the map

$$\{(u,\tau)\in U\times\mathbb{R}\mid 2\varepsilon<\tau<T_k^u-2\varepsilon\}\to\mathbb{R}^2\times Q\colon(u,\tau)\to(\varphi_{\mathbb{R}^2}(u),\gamma_k^u(\tau))$$

and the submersion $(\mathrm{id}_{\mathbb{R}^2}\times\mathrm{ev}_0)\circ\psi\colon V\to\mathbb{R}^2\times Q$. It can be checked that $\hat\imath^j$ and e_+^j and e_-^j for $j=1,2$ intertwine this operator and $\bar f_{k,\bar\xi}$ for $\bar\xi\in\overline{C}{}_q^{\mathrm{dR}}(S_\varepsilon)$.

Similar results as for $f_{k,\xi}$ are the following: $\hat{f}_{k,\hat{\xi}}$ induces a linear map

$$\hat{f}_{k,\hat{\xi}}\colon \widehat{C}^{\mathrm{dR}}_{*}(\Sigma^a_m, \Sigma^0_m) \to \widehat{C}^{\mathrm{dR}}_{*+1+q-n}(\Sigma^{a+\varepsilon}_{m+1}, \Sigma^0_{m+1}),$$

which is independent of ρ_ε. The next equations are variants of (15) and (16), and they follow from similar computations as Lemmas 3.13 and 3.14, so we omit the proof.

Proposition 3.20 *For $k' \geq k$,*

$$\partial \circ \hat{f}_{k,\hat{\xi}} - \hat{f}_{k,\hat{\xi}} \circ \partial = (-1)^{p+1-n} \hat{f}_{k,\partial\hat{\xi}}\colon \widehat{C}^{\mathrm{dR}}_{p}(\Sigma^a_m, \Sigma^0_m) \to \widehat{C}^{\mathrm{dR}}_{p+1+q-n}(\Sigma^{a+\varepsilon}_{m+1}, \Sigma^0_{m+1}),$$

$$\hat{f}_{k'+1,\hat{\xi}} \circ \hat{f}_{k,\hat{\xi}} + (-1)^{q-n} \hat{f}_{k,\hat{\xi}} \circ \hat{f}_{k',\hat{\xi}} = 0\colon \widehat{C}^{\mathrm{dR}}_{p}(\Sigma^a_m, \Sigma^0_m) \to \widehat{C}^{\mathrm{dR}}_{p+2+2q-2n}(\Sigma^{a+2\varepsilon}_{m+2}, \Sigma^0_{m+2}).$$

4 Construction of $H^{\mathrm{string}}_*(Q, K)$

4.1 Definition of chain complex

For $a \in \mathbb{R}_{>0}$ and $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, we define the graded $\mathbb{R}$-vector space

$$C^{<a}_*(\varepsilon) := \bigoplus_{m=0}^{\infty} C^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m).$$

If $m \geq 2a/\varepsilon_0$, then $a + m\varepsilon \leq m\left(\frac{1}{2}\varepsilon_0 + \varepsilon\right) \leq m\varepsilon_0$. In this case, $\Sigma^{a+m\varepsilon}_m = \Sigma^0_m$ by Remark 3.1. Therefore, the component for $m \in \mathbb{Z}_{\geq 0}$ vanishes if $m \geq 2a/\varepsilon_0$.

For each $m \in \mathbb{Z}_{\geq 0}$, we think of $C^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m)$ as a linear subspace of $C^{<a}_*(\varepsilon)$ in a natural way. For $\delta \in C^{\mathrm{dR}}_{n-d}(S_\varepsilon)$, we define a degree -1 linear map

$$D_\delta\colon C^{<a}_*(\varepsilon) \to C^{<a}_{*-1}(\varepsilon)$$

such that, for $x \in C^{\mathrm{dR}}_{p-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m)$,

$$D_\delta(x) = \partial x + \sum_{k=1}^{m} (-1)^{p+1+kd} f_{k,\delta}(x) \in C^{<a}_{p-1}(\varepsilon).$$

When $m = 0$, the right-hand side is just equal to ∂x.

Proposition 4.1 *If $\delta \in C^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ satisfies $\partial\delta = 0$, then $D_\delta \circ D_\delta = 0$.*

Proof Take an arbitrary $x \in C^{\mathrm{dR}}_{p-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m)$. Since $\partial \circ \partial = 0$,

$$D_\delta \circ D_\delta(x) = \sum_{k=1}^{m} ((-1)^{p+1+kd} \partial \circ f_{k,\delta}(x) + (-1)^{p+kd} f_{k,\delta} \circ \partial(x)) + \sum_{k'=1}^{m+1} \sum_{k=1}^{m} (-1)^{(k+k')d-1} f_{k',\delta} \circ f_{k,\delta}(x).$$

Applying (15) for $q=n-d$ and $\xi=\delta$ for which $\partial\delta=0$, we can see that the first summand is equal to 0. For the second summand, we apply (16) for $q=n-d$ and $\xi=\delta$. Then

$$\begin{aligned}\sum_{k'=1}^{m+1}\sum_{k=1}^{m}(-1)^{(k+k')d-1} f_{k',\delta}\circ f_{k,\delta}(x) &= \sum_{1\le k\le k'\le m}((-1)^{(k+k'+1)d-1} f_{k'+1,\delta}\circ f_{k,\delta}(x)+(-1)^{(k+k')d-1} f_{k,\delta}\circ f_{k',\delta}(x))\\ &=0.\end{aligned}$$

This shows that $D_\delta\circ D_\delta(x)=0$. □

In summary, for $a\in\mathbb{R}_{>0}$, $\varepsilon\in(0,\varepsilon_0/(5C_0)]$ and $\delta\in C^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ with $\partial\delta=0$, a chain complex $(C_*^{<a}(\varepsilon), D_\delta)$ is defined. Let $H_*^{<a}(\varepsilon,\delta)$ denote its homology.

The chain complex $(C_*^{<a}(\varepsilon), D_\delta)$ is filtered by subcomplexes $\{\mathscr{F}^{<a}_{\varepsilon,p}\}_{p\in\mathbb{Z}}$ defined by

$$\mathscr{F}^{<a}_{\varepsilon,p} := \bigoplus_{m\ge -p}^{\infty} C^{\mathrm{dR}}_{*-m(d-2)}(\Sigma_m^{a+m\varepsilon},\Sigma_m^0). \tag{22}$$

Let $E^{<a}_{(\varepsilon,\delta)} := \bigl(\{(E^{<a}_{(\varepsilon,\delta)})^r_{p,q}\},\{(d^{<a}_{(\varepsilon,\delta)})^r_{p,q}\}\bigr)$ be the spectral sequence determined by $\{\mathscr{F}^{<a}_{\varepsilon,p}\}_{p\in\mathbb{Z}}$. Note that $\mathscr{F}^{<a}_{\varepsilon,p}=0$ for $p\le -2a/\varepsilon_0$ and $\mathscr{F}^{<a}_{\varepsilon,p}=\mathscr{F}^{<a}_{\varepsilon,0}$ for $p\ge 0$, and thus this spectral sequence converges to $H_*^{<a}(\varepsilon,\delta)$ in the sense of [21, Bounded Convergence 5.2.5]. The first page is given by

$$(E^{<a}_{(\varepsilon,\delta)})^1_{-m,q} = \begin{cases} H^{\mathrm{dR}}_{(q-m)-m(d-2)}(\Sigma_m^{a+m\varepsilon},\Sigma_m^0) = H^{\mathrm{dR}}_{q-m(d-1)}(\Sigma_m^{a+m\varepsilon},\Sigma_m^0) & \text{if } m\ge 0,\\ 0 & \text{if } m<0.\end{cases}$$

Let us state an abstract lemma about morphisms in the category of spectral sequences. This result, which is a refinement of [21, Comparison Theorem 5.2.12], will be repeatedly used in the rest of this paper.

Lemma 4.2 *Let $E=(\{E^r_{p,q}\},\{d^r_{p,q}\})$ and $E'=(\{E'^r_{p,q}\},\{d'^r_{p,q}\})$ be bounded spectral sequences which converge to H_* and H'_*, respectively, in the sense of* [21, Bounded Convergence 5.2.5]. *Let $f=\{f^r_{p,q}\}$ be a morphism from E to E' which is compatible with $\{h_n\colon H_n\to H'_n\}_{n\in\mathbb{Z}}$. Then the following assertion holds*:

- *Suppose that, for some $r_0\ge 1$ and $n_0\in\mathbb{Z}$, $f^{r_0}_{p,q}$ is an isomorphism if $p+q<n_0$, and a surjection if $p+q=n_0$. Then h_n is an isomorphism if $n<n_0$ and a surjection if $n=n_0$. In particular, if $f^{r_0}_{p,q}$ is an isomorphism for every $p,q\in\mathbb{Z}$ for some $r_0\ge 1$, then h_n is an isomorphism for every $n\in\mathbb{Z}$.*

Proof Suppose that $\{f^r_{p,q}\}$ satisfy the condition of the assertion for $r_0\ge 1$ and $n_0\in\mathbb{Z}$. Note that, for any $r\ge 1$ and $p,q\in\mathbb{Z}$,

$$f^{r+1}_{p,q}\colon E^{r+1}_{p,q}\cong \operatorname{Ker} d^r_{p,q}/\operatorname{Im} d^r_{p+r,q-r+1}\to E'^{r+1}_{p,q}\cong \operatorname{Ker} d'^r_{p,q}/\operatorname{Im} d'^r_{p+r,q-r+1}$$

is induced by $\{f^r_{p,q}\}_{p,q}$. Therefore, by inductive arguments about $\{f^r_{p,q}\}$ on $r=r_0,r_0+1,\dots$, we can prove that $f^\infty_{p,q}\colon E^\infty_{p,q}\to E'^\infty_{p,q}$ is an isomorphism if $p+q<n_0$, and a surjection if $p+q=n_0$. We omit the concluding argument about h_n, since it is parallel to [21, Comparison Theorem 5.2.12]. □

For $\varepsilon, \bar{\varepsilon} \in (0, \varepsilon_0/(5C_0)]$ with $\varepsilon \le \bar{\varepsilon}$, let $j_{\varepsilon,\bar{\varepsilon}} \colon \Sigma_m^{a+m\varepsilon} \to \Sigma_m^{a+m\bar{\varepsilon}}$ for $m \in \mathbb{Z}_{\ge 0}$ be the inclusion maps. These maps induce a linear map

$$(j_{\varepsilon,\bar{\varepsilon}})_* \colon C_*^{<a}(\varepsilon) \to C_*^{<a}(\bar{\varepsilon}).$$

For any $\delta \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ with $\partial\delta = 0$, $(j_{\varepsilon,\bar{\varepsilon}})_*$ is a chain map from $(C_*^{<a}(\varepsilon), D_\delta)$ to $(C_*^{<a}(\bar{\varepsilon}), D_{(i_{\varepsilon,\bar{\varepsilon}})_*\delta})$ and preserves the filtrations $\{\mathscr{F}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$ and $\{\mathscr{F}_{\bar{\varepsilon},p}^{<a}\}_{p\in\mathbb{Z}}$.

Lemma 4.3 *Suppose that $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$. Then the induced map on homology*

$$(j_{\varepsilon,\bar{\varepsilon}})_* \colon H_*^{<a}(\varepsilon, \delta) \to H_*^{<a}(\bar{\varepsilon}, (i_{\varepsilon,\bar{\varepsilon}})_*\delta)$$

is an isomorphism if $\bar{\varepsilon}$ satisfies $[a, a + (2a/\varepsilon_0)\bar{\varepsilon}] \cap \mathscr{L}(K) = \varnothing$.

Proof If $m \ge 2a/\varepsilon_0$, then $\Sigma_m^{a+m\bar{\varepsilon}} = \Sigma_m^{a+m\varepsilon} = \Sigma_m^0$. If $0 \le m \le 2a/\varepsilon_0$, then $[a+m\varepsilon, a+m\bar{\varepsilon}] \cap \mathscr{L}(K) = \varnothing$ from the condition on $\bar{\varepsilon}$. Thus, Proposition 3.7 is applied to show that $H_*^{\mathrm{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}, \Sigma_m^{a+m\varepsilon}) = 0$ for all $m \in \mathbb{Z}_{\ge 0}$. Therefore, the induced map on the $(-m, q)$-term for $m \ge 0$ of the first page

$$(j_{\varepsilon,\bar{\varepsilon}})_* \colon (E_{(\varepsilon,\delta)}^{<a})_{-m,q}^1 = H_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) \to (E_{(\bar{\varepsilon},(i_{\varepsilon,\bar{\varepsilon}})_*\delta)}^{<a})_{-m,q}^1 = H_{q-m(d-1)}^{\mathrm{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}, \Sigma_m^0)$$

is an isomorphism. Now the assertion follows from Lemma 4.2. □

As we have seen in Example 2.6, $H_*^{\mathrm{dR}}(N_\varepsilon^{\mathrm{reg}}) \cong H_c^d(N_\varepsilon; \mathbb{R})$. Therefore, we can determine a unique homology class $\mathrm{Th}_\varepsilon \in H_{n-d}^{\mathrm{dR}}(N_\varepsilon^{\mathrm{reg}})$ which corresponds to the Thom class of $(TK)^\perp$ through the diffeomorphism $\{(x, v) \in (TK)^\perp \mid v < \varepsilon\} \to N_\varepsilon$, $(x, v) \mapsto \exp_x v$.

The above lemma leads us to define a set of data (ε, δ) as follows.

Definition 4.4 Let $C \ge 1$ be the constant of Lemma 3.8. We define $\mathscr{T}_a$ for every $a \in \mathbb{R}_{\ge 0} \setminus \mathscr{L}(K)$ to be the set of pairs (ε, δ) of $\varepsilon \in (0, \varepsilon_0/(5C^4)]$ and $\delta \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ such that:

(a) $[a, a + (2a/\varepsilon_0)\hat{\varepsilon}] \cap \mathscr{L}(K) = \varnothing$ for $\hat{\varepsilon} := C^3\varepsilon$.

(b) $\partial\delta = 0$ and $(\mathrm{ev}_0)_*[\delta] = \mathrm{Th}_\varepsilon \in H_{n-d}^{\mathrm{dR}}(N_\varepsilon^{\mathrm{reg}})$.

Let $a, b \in \mathbb{R}_{>0}$ with $a < b$. For $\varepsilon \in (0, \varepsilon_0/(5C_0)]$ and $\delta \in C_{n-d}^{\mathrm{dR}}(S_\varepsilon)$ with $\partial\delta = 0$, there exists a chain map $(I_\varepsilon^{a,b})_*$ from $(C_*^{<a}(\varepsilon), D_\delta)$ to $(C_*^{<b}(\varepsilon), D_\delta)$ induced by inclusion maps $I_\varepsilon^{a,b} \colon \Sigma_m^{a+m\varepsilon} \to \Sigma_m^{b+m\varepsilon}$ for all $m \in \mathbb{Z}_{\ge 0}$. We define the quotient complex

$$(C_*^{[a,b)}(\varepsilon) := C_*^{<b}(\varepsilon)/C_*^{<a}(\varepsilon), D_\delta).$$

Let $H_*^{[a,b)}(\varepsilon, \delta)$ denote its homology. Obviously, there exists a long exact sequence

$$\cdots \to H_*^{<a}(\varepsilon, \delta) \xrightarrow{(I_\varepsilon^{a,b})_*} H_*^{<b}(\varepsilon, \delta) \to H_*^{[a,b)}(\varepsilon, \delta) \to H_{*-1}^{<a}(\varepsilon, \delta) \xrightarrow{(I_\varepsilon^{a,b})_*} \cdots. \tag{23}$$

The next result is a trivial computation from the spectral sequence.

Proposition 4.5 *For $a, b \in \mathbb{R} \setminus \mathscr{L}(K)$ with $a < b$ and $(\varepsilon, \delta) \in \mathscr{T}_a \cap \mathscr{T}_b$, the following hold:*

- *If $[a,b] \cap \mathscr{L}(K) = \varnothing$, then $H_*^{[a,b)}(\varepsilon, \delta) = 0$.*
- *If there exist $c \in \mathscr{L}(K)$ and $m_0 \in \mathbb{Z}_{\geq 1}$ such that $[a,b] \cap \mathscr{L}_m(K) = \{c\}$ if $m = m_0$ and $[a,b] \cap \mathscr{L}_m(K) = \varnothing$ otherwise, then*

$$H_*^{[a,b)}(\varepsilon, \delta) \cong H^{\mathrm{dR}}_{*-m_0(d-2)}(\Sigma^b_{m_0}, \Sigma^a_{m_0}).$$

Proof Let $E^{[a,b)}_{(\varepsilon,\delta)}$ be the spectral sequence determined by a filtration $\{\mathscr{F}^{<b}_{\varepsilon,m}/\mathscr{F}^{<a}_{\varepsilon,m}\}_{m\in\mathbb{Z}}$. We apply Proposition 3.7 to the first page. For the first case, $(E^{[a,b)}_{(\varepsilon,\delta)})^1_{p,q} = 0$ for every $p, q \in \mathbb{Z}$, so the assertion is trivial. For the second case, $(E^{[a,b)}_{(\varepsilon,\delta)})^1_{p,q} = 0$ for every $p \neq -m_0$, so all differentials are the zero map. Therefore, $H_q^{[a,b)}(\varepsilon, \delta) \cong (E^{[a,b)}_{(\varepsilon,\delta)})^1_{-m_0,q+m_0}$ and the assertion follows from

$$(E^{[a,b)}_{(\varepsilon,\delta)})^1_{-m_0,q+m_0} = H^{\mathrm{dR}}_{q-m_0(d-2)}(\Sigma^{b+m_0\varepsilon}_{m_0}, \Sigma^{a+m_0\varepsilon}_{m_0}) \cong H^{\mathrm{dR}}_{q-m_0(d-2)}(\Sigma^b_{m_0}, \Sigma^a_{m_0}).$$

Here, the last isomorphism comes from Proposition 3.7. $\square$

4.2 Variants from $[-1,1]$- and $[-1,1]^2$-modeled de Rham chains

In Section 3.5, we introduced $[-1,1]$-modeled and $[-1,1]^2$-modeled de Rham chains. In this section, we define chain complexes as in Section 4.1 by using these types of chains. Their constructions and some computations are parallel to the former section, so we often omit proofs.

First, we deal with $[-1,1]$-modeled chains. For $a \in \mathbb{R}_{>0}$ and $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, we consider the graded $\mathbb{R}$-vector space

$$\overline{C}^{<a}_*(\varepsilon) := \bigoplus_{m=0}^{\infty} \overline{C}^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m).$$

For $\bar{\delta} \in \overline{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$, we define a degree -1 map $\overline{D}_{\bar{\delta}} \colon \overline{C}^{<a}_*(\varepsilon) \to \overline{C}^{<a}_{*-1}(\varepsilon)$ by

$$\overline{D}_{\bar{\delta}}(x) := \partial x + \sum_{k=1}^{m} (-1)^{p+1+kd} \bar{f}_{k,\bar{\delta}}(x)$$

for $x \in \overline{C}^{\mathrm{dR}}_{p-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m)$.

Proposition 4.6 *If $\bar{\delta} \in \overline{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ satisfies $\partial\bar{\delta} = 0$, then $\overline{D}_{\bar{\delta}} \circ \overline{D}_{\bar{\delta}} = 0$.*

This is analogous to Proposition 4.1 and can be deduced from the two equations of Proposition 3.19. From this proposition, for $\bar{\delta} \in \overline{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ with $\partial\delta = 0$, we obtain a chain complex $(\overline{C}^{<a}_*(\varepsilon), \overline{D}_{\bar{\delta}})$. Let $\overline{H}^{<a}_*(\varepsilon, \bar{\delta})$ denote its homology.

Let us consider a relation to the chain complex defined in Section 4.1. The linear maps (18) for $X = \Sigma^{a+m\varepsilon}_m$

$$e_+, e_- \colon \overline{C}^{\mathrm{dR}}_*(\Sigma^{a+m\varepsilon}_m) \to C^{\mathrm{dR}}_*(\Sigma^{a+m\varepsilon}_m) \quad \text{for } m \in \mathbb{Z}_{\geq 0}$$

naturally induce linear maps

$$e_{\varepsilon,+}, e_{\varepsilon,-} \colon \overline{C}^{<a}_*(\varepsilon) \to C^{<a}_*(\varepsilon),$$

and these are chain maps from $(\overline{C}^{<a}_*(\varepsilon), \overline{D}_{\bar{\delta}})$ to $(C^{<a}_*(\varepsilon), D_{e_+\bar{\delta}})$ and $(C^{<a}_*(\varepsilon), D_{e_-\bar{\delta}})$, respectively.

We define a filtration $\{\overline{\mathcal{F}}^{<a}_{\varepsilon,p}\}_{p\in\mathbb{Z}}$ by

$$\overline{\mathcal{F}}^{<a}_{\varepsilon,p} := \bigoplus_{m\geq -p} \overline{C}^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m).$$

Let $\overline{E}^{<a}_{(\varepsilon,\bar{\delta})}$ be the spectral sequence determined by this filtration.

Lemma 4.7 *$e_{\varepsilon,+}$ and $e_{\varepsilon,-}$ are quasi-isomorphisms.*

Proof We prove this assertion for only $e_{\varepsilon,+}$. The proof for $e_{\varepsilon,-}$ is parallel. Since $e_{\varepsilon,+}$ preserves filtrations $\{\overline{\mathcal{F}}^{<a}_{\varepsilon,p}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{F}^{<a}_{\varepsilon,p}\}_{p\in\mathbb{Z}}$, this induces a map on the first page $(e_{\varepsilon,+})_*\colon (\overline{E}^{<a}_{(\varepsilon,\bar{\delta})})^1_{p,q} \to (E^{<a}_{(\varepsilon,e_+\bar{\delta})})^1_{p,q}$. For $p=-m\leq 0$, this coincides with

$$(e_+)_*\colon \overline{H}^{\mathrm{dR}}_{q-m(d-1)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m) \to H^{\mathrm{dR}}_{q-m(d-1)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m).$$

From Lemma 3.16, this map is an isomorphism. Now the assertion follows from Lemma 4.2. □

For $\bar{\varepsilon}, \widehat{\varepsilon} \in (0, \varepsilon_0/(5C_0)]$ with $\bar{\varepsilon} \leq \widehat{\varepsilon}$, the linear map $(j_{\bar{\varepsilon},\widehat{\varepsilon}})_*\colon \overline{C}^{<a}_*(\bar{\varepsilon}) \to \overline{C}^{<a}_*(\widehat{\varepsilon})$, induced by the inclusion maps $j_{\bar{\varepsilon},\widehat{\varepsilon}}\colon \Sigma^{a+m\bar{\varepsilon}}_m \to \Sigma^{a+m\widehat{\varepsilon}}_m$ for all $m\in\mathbb{Z}_{\geq 0}$, is a chain map from $(\overline{C}^{<a}_*(\bar{\varepsilon}), \overline{D}_{\bar{\delta}})$ to $(\overline{C}^{<a}_*(\widehat{\varepsilon}), \overline{D}_{(i_{\bar{\varepsilon},\widehat{\varepsilon}})_*\bar{\delta}})$.

Lemma 4.8 *Suppose that $a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K)$. Then the induced map on homology*

$$(j_{\bar{\varepsilon},\widehat{\varepsilon}})_*\colon \overline{H}^{<a}_*(\bar{\varepsilon},\bar{\delta}) \to \overline{H}^{<a}_*(\widehat{\varepsilon}, (i_{\bar{\varepsilon},\widehat{\varepsilon}})_*\bar{\delta})$$

is an isomorphism if $\widehat{\varepsilon}$ satisfies $[a, a+(2a/\varepsilon_0)\widehat{\varepsilon}]\cap\mathcal{L}(K)=\varnothing$.

Proof The proof is parallel to that of Lemma 4.3. The chain map $(j_{\bar{\varepsilon},\widehat{\varepsilon}})_*$ preserves the filtrations $\{\overline{\mathcal{F}}^{<a}_{\bar{\varepsilon},p}\}_{p\in\mathbb{Z}}$ and $\{\overline{\mathcal{F}}^{<a}_{\widehat{\varepsilon},p}\}_{p\in\mathbb{Z}}$. This induces an isomorphism on the first page since, for every $m\in\mathbb{Z}_{\geq 0}$,

$$\overline{H}_*(\Sigma^{a+m\widehat{\varepsilon}}_m, \Sigma^{a+m\bar{\varepsilon}}_m) = 0$$

by Proposition 3.18. Now the assertion follows from Lemma 4.2. □

The above lemma leads us to the following definition.

Definition 4.9 Let $C\geq 1$ be the constant of Lemma 3.8. We define $\overline{\mathcal{T}}_a$ for $a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K)$ to be the set of pairs $(\bar{\varepsilon},\bar{\delta})$ of $\bar{\varepsilon}\in(0,\varepsilon_0/(5C^3)]$ and $\bar{\delta}\in\overline{C}^{\mathrm{dR}}_{n-d}(S_{\bar{\varepsilon}})$ such that:

(a) $[a, a+(2a/\varepsilon_0)\widehat{\varepsilon}]\cap\mathcal{L}(K)=\varnothing$ for $\widehat{\varepsilon} := C^2\bar{\varepsilon}$.

(b) $\partial\bar{\delta}=0$ and $(\mathrm{ev}_0)_*[e_+\bar{\delta}] = \mathrm{Th}_{\bar{\varepsilon}} \in H^{\mathrm{dR}}_{n-d}(N^{\mathrm{reg}}_{\bar{\varepsilon}})$.

Next, we deal with $[-1,1]^2$-modeled chains. For $a\in\mathbb{R}_{>0}$ and $\varepsilon\in(0,\varepsilon_0/(5C_0)]$, we consider the graded $\mathbb{R}$-vector space

$$\widehat{C}^{<a}_*(\varepsilon) := \bigoplus_{m=0}^{\infty} \widehat{C}^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m).$$

For $\widehat{\delta} \in \widehat{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$, we define a degree -1 map $\widehat{D}_{\widehat{\delta}} \colon \widehat{C}^{<a}_*(\varepsilon) \to \widehat{C}^{<a}_{*-1}(\varepsilon)$ by

$$\widehat{D}_{\widehat{\delta}}(x) := \partial x + \sum_{k=1}^{m} (-1)^{p+1+kd} \hat{f}_{k,\widehat{\delta}}(x)$$

for $x \in \widehat{C}^{\mathrm{dR}}_{p-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m)$.

Proposition 4.10 *If $\widehat{\delta} \in \widehat{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ satisfies $\partial\widehat{\delta} = 0$, then $\widehat{D}_{\widehat{\delta}} \circ \widehat{D}_{\widehat{\delta}} = 0$.*

This is analogous to Proposition 4.1 and can be deduced from the two equations of Proposition 3.20. From this proposition, for $\widehat{\delta} \in \widehat{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ with $\partial\widehat{\delta} = 0$, we obtain a chain complex $(\widehat{C}^{<a}_*(\varepsilon), \widehat{D}_{\widehat{\delta}})$. Let $\widehat{H}^{<a}_*(\varepsilon, \widehat{\delta})$ denote its homology.

Let us consider a relation to the chain complex defined by $[-1,1]$-modeled de Rham chains. For $j = 1, 2$, the linear maps of (19) for $X = \Sigma^{a+m\varepsilon}_m$,

$$e^j_+, e^j_- \colon \widehat{C}^{\mathrm{dR}}_*(\Sigma^{a+m\varepsilon}_m) \to \overline{C}^{\mathrm{dR}}_*(\Sigma^{a+m\varepsilon}_m) \quad \text{for } m \in \mathbb{Z}_{\geq 0},$$

naturally induce linear maps

$$e^j_{\varepsilon,+}, e^j_{\varepsilon,-} \colon \widehat{C}^{<a}_*(\varepsilon) \to \overline{C}^{<a}_*(\varepsilon),$$

and these are chain maps from $(\widehat{C}^{<a}_*(\varepsilon), \widehat{D}_{\widehat{\delta}})$ to $(\overline{C}^{<a}_*(\varepsilon), \overline{D}_{e^j_+\widehat{\delta}})$ and $(\overline{C}^{<a}_*(\varepsilon), \overline{D}_{e^j_-\widehat{\delta}})$, respectively.

Lemma 4.11 *$e^j_{\varepsilon,+}$ and $e^j_{\varepsilon,-}$ for $j = 1, 2$ are quasi-isomorphisms.*

Proof The proof is parallel to that of Lemma 4.7. This time, we use the spectral sequence determined by a filtration $\{\widehat{\mathcal{F}}^a_{\varepsilon,p}\}_{p\in\mathbb{Z}}$, which is defined by

$$\widehat{\mathcal{F}}^a_{\varepsilon,p} := \bigoplus_{m \geq -p} \widehat{C}^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m, \Sigma^0_m).$$

Then $e^j_{\varepsilon,+}$ and $e^j_{\varepsilon,-}$ for $j = 1, 2$ preserve the filtrations $\{\widehat{\mathcal{F}}^a_{\varepsilon,p}\}_{p\in\mathbb{Z}}$ and $\{\overline{\mathcal{F}}^a_{\varepsilon,p}\}_{p\in\mathbb{Z}}$. By Lemma 3.17, they induce an isomorphism on the first page. Now the assertion follows from Lemma 4.2. □

4.3 The limit of $\varepsilon \to 0$

In this section, we define a transition map

$$k_{(\varepsilon',\delta'),(\varepsilon,\delta)} \colon H^{<a}_*(\varepsilon', \delta') \to H^{<a}_*(\varepsilon, \delta)$$

for every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ and $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$, by using $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathcal{T}}_a$ satisfying

$$(24) \qquad \varepsilon \leq \bar{\varepsilon}, \quad e_+\bar{\delta} = (i_{\varepsilon,\bar{\varepsilon}})_*\delta, \quad e_-\bar{\delta} = (i_{\varepsilon',\bar{\varepsilon}})_*\delta'.$$

In fact, $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}$ is an isomorphism. We also prove that $\big(\{H^{<a}_*(\varepsilon, \delta)\}_{(\varepsilon,\delta)\in\mathcal{T}_a}, \{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon'\leq\varepsilon}\big)$ forms an inverse system.

4.3.1 Construction of transition maps Let us first prove the existence of the above $(\bar{\varepsilon}, \bar{\delta})$.

Lemma 4.12 *For $(\varepsilon, \delta), (\varepsilon', \delta) \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$, there exists $(\bar{\varepsilon}, \bar{\delta}) \in \overline{\mathcal{T}}_a$ satisfying (24).*

Proof Let us take $\bar{\varepsilon} := C\varepsilon$ for the constant C in Lemma 3.8, and rewrite $\delta_+ := (i_{\varepsilon,\bar{\varepsilon}})_*\delta$ and $\delta_- := (i_{\varepsilon',\bar{\varepsilon}})_*\delta'$ for short. Since

$$(\mathrm{ev}_0)_*[\delta - (i_{\varepsilon',\varepsilon})_*\delta'] = \mathrm{Th}_\varepsilon - (i_{\varepsilon',\varepsilon})_*\mathrm{Th}_{\varepsilon'} = 0 \in H^{\mathrm{dR}}_{n-d}(N_\varepsilon),$$

Proposition 3.9 shows that there exists $\theta \in C^{\mathrm{dR}}_{n-d+1}(S_{\bar{\varepsilon}})$ such that

$$\partial\theta = (i_{\varepsilon,\bar{\varepsilon}})_*(\delta - (i_{\varepsilon',\varepsilon})_*\delta') = \delta_+ - \delta_-.$$

Let $\kappa\colon \mathbb{R} \to [0,1]$ be a C^∞ function such that $\kappa(r) = 1$ if $1 \le r$ and $\kappa(r) = 0$ if $r \le -1$. We take chains $\beta_+, \beta_- \in \bar{C}^{\mathrm{dR}}_0(\{0\})$ defined by

$$\beta_+ := [\mathbb{R}, \mathrm{id}_{\mathbb{R}}, (\mathrm{id}_{\mathbb{R}_{\ge 1}}, \mathrm{id}_{\mathbb{R}_{\le -1}}), \kappa], \quad \beta_- := [\mathbb{R}, \mathrm{id}_{\mathbb{R}}, (\mathrm{id}_{\mathbb{R}_{\ge 1}}, \mathrm{id}_{\mathbb{R}_{\le -1}}), 1-\kappa].$$

Now we define $\bar{\delta}$ by

$$\bar{\delta} := \beta_+ \times (\bar{\imath}\delta_+) + \beta_- \times (\bar{\imath}\delta_-) + (\partial\beta_+) \times (\bar{\imath}\theta) \in \bar{C}^{\mathrm{dR}}_{n-d}(S_{\bar{\varepsilon}}).$$

This satisfies condition (24). Moreover, $\partial\bar{\delta} = 0$ and $(\mathrm{ev}_0)_*[e_+\bar{\delta}] = (\mathrm{ev}_0)_*[\delta_+] = \mathrm{Th}_{\bar{\varepsilon}}$. Now it is clear that $(\bar{\varepsilon}, \bar{\delta}) = (C\varepsilon, \bar{\delta})$ satisfies the two conditions to be an element of $\bar{\mathcal{T}}_a$. $\square$

From Lemmas 4.3 and 4.7, we can define isomorphisms

$$f_{(\bar{\varepsilon},\bar{\delta}),+} := (j_{\varepsilon,\bar{\varepsilon}})_*^{-1} \circ (e_{\bar{\varepsilon},+})_*\colon \bar{H}^{<a}_*(\bar{\varepsilon},\bar{\delta}) \to H^{<a}_*(\varepsilon,\delta),$$
$$f_{(\bar{\varepsilon},\bar{\delta}),-} := (j_{\varepsilon',\bar{\varepsilon}})_*^{-1} \circ (e_{\bar{\varepsilon},-})_*\colon \bar{H}^{<a}_*(\bar{\varepsilon},\bar{\delta}) \to H^{<a}_*(\varepsilon',\delta')$$

such that the following diagrams commute:

$$\begin{tikzcd}
\bar{H}^{<a}_*(\bar{\varepsilon},\bar{\delta}) \arrow[r, "f_{(\bar{\varepsilon},\bar{\delta}),+}"] \arrow[d, "(e_{\bar{\varepsilon},+})_*"'] & H^{<a}_*(\varepsilon,\delta) \arrow[d, "(j_{\varepsilon,\bar{\varepsilon}})_*"] \\
H^{<a}_*(\bar{\varepsilon}, e_+\bar{\delta}) \arrow[r, equal] & H^{<a}_*(\bar{\varepsilon}, (i_{\varepsilon,\bar{\varepsilon}})_*\delta)
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{H}^{<a}_*(\bar{\varepsilon},\bar{\delta}) \arrow[r, "f_{(\bar{\varepsilon},\bar{\delta}),-}"] \arrow[d, "(e_{\bar{\varepsilon},-})_*"'] & H^{<a}_*(\varepsilon',\delta') \arrow[d, "(j_{\varepsilon',\bar{\varepsilon}})_*"] \\
H^{<a}_*(\bar{\varepsilon}, e_-\bar{\delta}) \arrow[r, equal] & H^{<a}_*(\bar{\varepsilon}, (i_{\varepsilon',\bar{\varepsilon}})_*\delta')
\end{tikzcd}$$

We define an isomorphism

$$k_{(\bar{\varepsilon},\bar{\delta})} := f_{(\bar{\varepsilon},\bar{\delta}),+} \circ (f_{(\bar{\varepsilon},\bar{\delta}),-})^{-1}\colon H^{<a}_*(\varepsilon',\delta') \to H^{<a}_*(\varepsilon,\delta).$$

Later, we will prove the independence on $(\bar{\varepsilon},\bar{\delta})$ (Corollary 4.16), and this is the transition map $k_{(\varepsilon',\delta),(\varepsilon,\delta)}$ we need.

Lemma 4.13 *When $(\varepsilon',\delta') = (\varepsilon,\delta) \in \mathcal{T}_a$, we may take $(\varepsilon, \bar{\imath}\delta) \in \bar{\mathcal{T}}_a$ as an element satisfying (24). In this case, we have*

$$k_{(\varepsilon,\bar{\imath}\delta)} = \mathrm{id}_{H^{<a}_*(\varepsilon,\delta)}.$$

Proof To prove this assertion, let us introduce a chain map $\bar{\imath}_\varepsilon$ from $(C^{<a}_*(\varepsilon), D_\delta)$ to $(\bar{C}^{<a}_*(\varepsilon), \bar{D}_{\bar{\imath}\delta})$ induced by $\bar{\imath}\colon C^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m) \to \bar{C}^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^{a+m\varepsilon}_m)$ (see (17)) for all $m \in \mathbb{Z}_{\ge 0}$. This satisfies $e_{\varepsilon,+} \circ \bar{\imath}_\varepsilon = \mathrm{id}_{C^{<a}_*(\varepsilon)} = e_{\varepsilon,-} \circ \bar{\imath}_\varepsilon$, and thus

$$(e_{\varepsilon,+})_* = (\bar{\imath}_\varepsilon)_*^{-1} = (e_{\varepsilon,-})_*\colon \bar{H}^{<a}_*(\varepsilon,\bar{\imath}\delta) \to H^{<a}_*(\varepsilon,\delta). \tag{25}$$

Therefore, $k_{(\varepsilon,\bar{\imath}\delta)} = (e_{\varepsilon,+})_* \circ (e_{\varepsilon,-})_*^{-1} = \mathrm{id}_{H^{<a}_*(\varepsilon,\delta)}$. $\square$

4.3.2 Compositions Next, we think about compositions of maps in the set $\{k_{(\bar\varepsilon,\bar\delta)}\}_{(\bar\varepsilon,\bar\delta)\in\overline{\mathscr{T}}_a}$. For $(\varepsilon,\delta),(\varepsilon',\delta'),(\varepsilon'',\delta'')\in\mathscr{T}_a$ with $\varepsilon''\le\varepsilon'\le\varepsilon$, suppose that we have chosen $(\bar\varepsilon,\bar\delta),(\bar\varepsilon',\bar\delta'),(\widetilde{\varepsilon},\widetilde{\delta})\in\overline{\mathscr{T}}_a$ satisfying

$$e_+\bar\delta=(i_{\varepsilon,\bar\varepsilon})_*\delta,\qquad e_+\bar\delta'=(i_{\varepsilon',\bar\varepsilon'})_*\delta',\qquad e_+\widetilde{\delta}=(i_{\varepsilon,\widetilde{\varepsilon}})_*\delta,$$
$$e_-\bar\delta=(i_{\varepsilon',\bar\varepsilon})_*\delta',\quad e_-\bar\delta'=(i_{\varepsilon'',\bar\varepsilon'})_*\delta'',\quad e_-\widetilde{\delta}=(i_{\varepsilon'',\widetilde{\varepsilon}})_*\delta''.$$

In this situation, let us first prove the following lemma.

Lemma 4.14 *There exists $\widehat{\varepsilon}\in(0,\varepsilon_0/(5C_0)]$ and $\widehat{\delta}\in\widehat{C}^{\mathrm{dR}}_{n-d}(S_{\widehat{\varepsilon}})$ such that $\partial\widehat{\delta}=0$ and*

$$e^1_+\widehat{\delta}=(i_{\widetilde{\varepsilon},\widehat{\varepsilon}})_*\widetilde{\delta},\quad e^1_-\widehat{\delta}=(i_{\bar\varepsilon',\widehat{\varepsilon}})_*\bar\delta',\quad e^2_+\widehat{\delta}=(i_{\bar\varepsilon,\widehat{\varepsilon}})_*\bar\delta,\quad e^2_-\widehat{\delta}=(i_{\varepsilon'',\widehat{\varepsilon}})_*(\bar\imath\delta'').$$

Proof Let us take $\rho:=C\cdot\max\{\bar\varepsilon,\bar\varepsilon',\widetilde{\varepsilon}\}$, $\widehat{\varepsilon}:=C\rho$ and rewrite $\bar\delta_+:=(i_{\widetilde{\varepsilon},\rho})_*\widetilde{\delta}$ and $\bar\delta_-:=(i_{\bar\varepsilon',\rho})_*\bar\delta'$ for short. Since

$$(\mathrm{ev}_0)_*\circ(e_+)_*[(i_{\widetilde{\varepsilon},C^{-1}\rho})_*\widetilde{\delta}-(i_{\bar\varepsilon',C^{-1}\rho})_*\bar\delta']=(i_{\widetilde{\varepsilon},C^{-1}\rho})_*(\mathrm{Th}_{\widetilde{\varepsilon}})-(i_{\bar\varepsilon',C^{-1}\rho})_*(\mathrm{Th}_{\bar\varepsilon'})=0\in H^{\mathrm{dR}}_*(N_{C^{-1}\rho}),$$

Proposition 3.18 shows that there exists $\bar\theta_1\in\overline{C}^{\mathrm{dR}}_{n-d+1}(S_\rho)$ such that $\partial\bar\theta_1=\bar\delta_+-\bar\delta_-$. We define $\widehat{\kappa}^1:\mathbb{R}\times\mathbb{R}\to[0,1]$, $(r_1,r_2)\mapsto\kappa(r_1)$, where κ is the function that appeared in the proof of Lemma 4.12. Then we take chains $\widehat{\beta}_+,\widehat{\beta}_-\in\widehat{C}^{\mathrm{dR}}_0(\{0\})$ defined by

$$\widehat{\beta}^1_+:=[\mathbb{R}^2,\mathrm{id}_{\mathbb{R}^2},(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-),\widehat{\kappa}],\quad\widehat{\beta}^1_-:=[\mathbb{R}^2,\mathrm{id}_{\mathbb{R}^2},(\tau^1_+,\tau^1_-),(\tau^2_+,\tau^2_-),1-\widehat{\kappa}],$$

where $\tau^j_+=\mathrm{id}_{\{r_j\ge1\}}$ and $\tau^j_-=\mathrm{id}_{\{r_j\le-1\}}$ for $j=1,2$. We define

$$\xi:=\widehat{\beta}^1_+\times(\hat\imath^1\bar\delta_+)+\widehat{\beta}^1_-\times(\hat\imath^1\bar\delta_-)+(\partial\widehat{\beta}^1_+)\times(\hat\imath^1(\bar\theta_1-\bar\imath e_-\bar\theta_1))\in\widehat{C}^{\mathrm{dR}}_{n-d}(S_\rho).$$

This chain satisfies $\partial\xi=0$ (note that $e_-(\partial\bar\theta_1)=e_-\bar\delta_+-e_-\bar\delta_-=0$). Moreover,

$$e^1_+\xi=(i_{\widetilde{\varepsilon},\rho})_*\widetilde{\delta},\quad e^1_-\xi=(i_{\bar\varepsilon',\rho})_*\bar\delta',\quad e^2_-\xi=(i_{\varepsilon'',\rho})_*(\bar\imath\delta'')$$

hold. The former two equations are easy to check. The third equation can be checked as follows: since $e^2_-\circ\hat\imath^1=\bar\imath\circ e_-$, we have $e^2_-(\hat\imath^1(\bar\theta_1-\bar\imath e_-\bar\theta_1))=0$ and thus

$$e^2_-\xi=(e^2_-\widehat{\beta}^1_+)\times\bigl(\bar\imath((i_{\varepsilon'',\rho})_*\delta'')\bigr)+(e^2_-\widehat{\beta}^1_-)\times\bigl(\bar\imath((i_{\varepsilon'',\rho})_*\delta'')\bigr)=(i_{\varepsilon'',\rho})_*(\bar\imath\delta'').$$

Let us write $\underline{\delta}:=e^2_+\xi\in\overline{C}^{\mathrm{dR}}_{n-d}(S_\rho)$ and consider the difference of chains $\underline{\delta}-(i_{\bar\varepsilon,\rho})_*\bar\delta$. We claim that there exists $\bar\theta_2\in\overline{C}^{\mathrm{dR}}_{n-d+1}(S_{\widehat{\varepsilon}})$ such that

$$\partial\bar\theta_2=(i_{\rho,\widehat{\varepsilon}})_*\underline{\delta}-(i_{\bar\varepsilon,\widehat{\varepsilon}})_*\bar\delta,\quad e_+\bar\theta_2=e_-\bar\theta_2=0.\tag{26}$$

We prove this claim. Since $e_+\circ e^2_+=e_+\circ e^1_+$ and $e_-\circ e^2_+=e_+\circ e^1_-$, we have

$$(e_+)_*[\underline{\delta}-(i_{\bar\varepsilon,\rho})_*\bar\delta]=\ (e_+\circ e^1_+)_*[\xi]-(i_{\varepsilon,\rho})_*[\delta]\ =0\in H^{\mathrm{dR}}_{n-d}(S_\rho),$$
$$(e_-)_*[\underline{\delta}-(i_{\bar\varepsilon,\rho})_*\bar\delta]=(e_+\circ e^1_-)_*[\xi]-(i_{\varepsilon',\rho})_*[\delta']=0\in H^{\mathrm{dR}}_{n-d}(S_\rho).$$

From Lemma 3.16, there exists $\bar\theta'_2\in\overline{C}^{\mathrm{dR}}_{n-d+1}(S_\rho)$ such that $\partial\bar\theta'_2=\underline{\delta}-(i_{\bar\varepsilon,\rho})_*\bar\delta$ and $\partial(e_+\bar\theta'_2)=\partial(e_-\bar\theta'_2)=0$. Since $(\mathrm{ev}_0)_*[e_+\bar\theta'_2]$ and $(\mathrm{ev}_0)_*[e_-\bar\theta'_2]$ are contained in $H^{\mathrm{dR}}_{n-d+1}(N_\rho)=\{0\}$, Proposition 3.18 shows that there exist $\varphi_+, \varphi_- \in C^{\mathrm{dR}}_{n-d+2}(S_{\hat{\varepsilon}})$ such that $\partial\varphi_+ = (i_{\rho,\hat{\varepsilon}})_*(e_+\bar{\theta}'_2)$ and $\partial\varphi_- = (i_{\rho,\hat{\varepsilon}})_*(e_-\bar{\theta}'_2)$. Then, by using β_+ and β_- in the proof of Lemma 4.12, we define

$$\bar{\theta}_2 := (i_{\rho,\hat{\varepsilon}})_*\bar{\theta}'_2 - \partial(\beta_+ \times \bar{\imath}\varphi_+) - \partial(\beta_- \times \bar{\imath}\varphi_-) \in \overline{C}^{\mathrm{dR}}_{n-d+1}(S_{\hat{\varepsilon}}),$$

and this chain satisfies (26).

We take $\hat{\kappa}^2 \colon \mathbb{R}^2 \to [0,1]$, $(r_1, r_2) \mapsto \kappa(r_2)$, and

$$\hat{\beta}^2_+ := [\mathbb{R}^2, \mathrm{id}_{\mathbb{R}^2}, (\tau^1_+, \tau^1_-), (\tau^2_+, \tau^2_-), \hat{\kappa}^2] \in \hat{C}^{\mathrm{dR}}_0(\{0\}).$$

Finally, we define a chain

$$\hat{\delta} := (i_{\rho,\hat{\varepsilon}})_*\xi - \partial(\hat{\beta}^2_+ \times \hat{\imath}^2\bar{\theta}_2) \in \hat{C}^{\mathrm{dR}}_{n-d}(S_{\hat{\varepsilon}}).$$

This satisfies $\partial\hat{\delta} = 0$ and the required four equations. $\square$

Lemma 4.14 is applied to prove the next proposition.

Proposition 4.15 $k_{(\bar{\varepsilon},\bar{\delta})} \circ k_{(\bar{\varepsilon}',\bar{\delta}')} = k_{(\tilde{\varepsilon},\tilde{\delta})} \colon H^{<a}_*(\varepsilon'', \delta'') \to H^{<a}_*(\varepsilon, \delta).$

Proof Let $(\hat{\varepsilon}, \hat{\delta})$ be the pair of Lemma 4.14. From Lemmas 4.8 and 4.11, we can define isomorphisms

$$\begin{aligned}
f^1_{(\hat{\varepsilon},\hat{\delta}),+} &:= (j_{\tilde{\varepsilon},\hat{\varepsilon}})^{-1}_* \circ (e^1_{\hat{\varepsilon},+})_* \colon \hat{H}^{<a}_*(\hat{\varepsilon}, \hat{\delta}) \to \overline{H}^{<a}_*(\tilde{\varepsilon}, \tilde{\delta}),\\
f^1_{(\hat{\varepsilon},\hat{\delta}),-} &:= (j_{\bar{\varepsilon}',\hat{\varepsilon}})^{-1}_* \circ (e^1_{\hat{\varepsilon},-})_* \colon \hat{H}^{<a}_*(\hat{\varepsilon}, \hat{\delta}) \to \overline{H}^{<a}_*(\bar{\varepsilon}', \bar{\delta}'),\\
f^2_{(\hat{\varepsilon},\hat{\delta}),+} &:= (j_{\bar{\varepsilon},\hat{\varepsilon}})^{-1}_* \circ (e^2_{\hat{\varepsilon},+})_* \colon \hat{H}^{<a}_*(\hat{\varepsilon}, \hat{\delta}) \to \overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}).
\end{aligned}$$

From the definitions of $k_{(\bar{\varepsilon},\bar{\delta})}$, $k_{(\bar{\varepsilon}',\bar{\delta}')}$ and $k_{(\tilde{\varepsilon},\tilde{\delta})}$, it suffices to show that the following diagram commutes:

$$\begin{tikzcd}
 & & & \overline{H}^{<a}_*(\tilde{\varepsilon}, \tilde{\delta}) \arrow[dddlll, "f_{(\tilde{\varepsilon},\tilde{\delta}),-}"'] \arrow[dddrrr, "f_{(\tilde{\varepsilon},\tilde{\delta}),+}"] & & & \\
 & & & \hat{H}^{<a}_*(\hat{\varepsilon}, \hat{\delta}) \arrow[u, "f^1_{(\hat{\varepsilon},\hat{\delta}),+}"'] \arrow[ddll, "f^1_{(\hat{\varepsilon},\hat{\delta}),-}"'] \arrow[ddrr, "f^2_{(\hat{\varepsilon},\hat{\delta}),+}"] & & & \\
 & & & & & & \\
H^{<a}_*(\varepsilon'', \delta'') & & & H^{<a}_*(\varepsilon', \delta') & & & H^{<a}_*(\varepsilon, \delta) \\
 & \overline{H}^{<a}_*(\bar{\varepsilon}', \bar{\delta}') \arrow[ul, "f_{(\bar{\varepsilon}',\bar{\delta}'),-}"] \arrow[urr, "f_{(\bar{\varepsilon}',\bar{\delta}'),+}"'] & & & & \overline{H}^{<a}_*(\bar{\varepsilon}, \bar{\delta}) \arrow[ull, "f_{(\bar{\varepsilon},\bar{\delta}),-}"] \arrow[ur, "f_{(\bar{\varepsilon},\bar{\delta}),+}"'] &
\end{tikzcd}$$

Note that all maps appearing in the diagram are isomorphisms. We need to prove the three equations

$$\begin{aligned}
f_{(\tilde{\varepsilon},\tilde{\delta}),+} \circ f^1_{(\hat{\varepsilon},\hat{\delta}),+} &= f_{(\bar{\varepsilon},\bar{\delta}),+} \circ f^2_{(\hat{\varepsilon},\hat{\delta}),+},\\
f_{(\bar{\varepsilon},\bar{\delta}),-} \circ f^2_{(\hat{\varepsilon},\hat{\delta}),+} &= f_{(\bar{\varepsilon}',\bar{\delta}'),+} \circ f^1_{(\hat{\varepsilon},\hat{\delta}),-},\\
f_{(\tilde{\varepsilon},\tilde{\delta}),-} \circ f^1_{(\hat{\varepsilon},\hat{\delta}),+} &= f_{(\bar{\varepsilon}',\bar{\delta}'),-} \circ f^1_{(\hat{\varepsilon},\hat{\delta}),-}.
\end{aligned} \tag{27}$$

Let us prove the first equation. Returning to the definition of $f_{(\widetilde{\varepsilon},\widetilde{\delta}),+}$ and $f^1_{(\widehat{\varepsilon},\widehat{\delta}),+}$,

$$\begin{aligned}
f_{(\widetilde{\varepsilon},\widetilde{\delta}),+} \circ f^1_{(\widehat{\varepsilon},\widehat{\delta}),+} &= (j_{\varepsilon,\widetilde{\varepsilon}})_*^{-1} \circ (e_{\widetilde{\varepsilon},+})_* \circ (j_{\widetilde{\varepsilon},\widehat{\varepsilon}})_*^{-1} \circ (e^1_{\widehat{\varepsilon},+})_* \\
&= (j_{\varepsilon,\widehat{\varepsilon}})_*^{-1} \circ (e_{\widehat{\varepsilon},+} \circ e^1_{\widehat{\varepsilon},+})_* \\
&= (j_{\varepsilon,\widehat{\varepsilon}})_*^{-1} \circ (e_{\widehat{\varepsilon},+} \circ e^2_{\widehat{\varepsilon},+})_* \\
&= (j_{\varepsilon,\bar{\varepsilon}})_*^{-1} \circ (e_{\bar{\varepsilon},+})_* \circ (j_{\bar{\varepsilon},\widehat{\varepsilon}})_*^{-1} \circ (e^2_{\widehat{\varepsilon},+})_* \\
&= f_{(\bar{\varepsilon},\bar{\delta}),+} \circ f^2_{(\widehat{\varepsilon},\widehat{\delta}),+}.
\end{aligned}$$

Here, the second and fourth equalities follow from the obvious equations

$$(j_{\widetilde{\varepsilon},\widehat{\varepsilon}})_* \circ (e_{\widetilde{\varepsilon},+})_* = (e_{\widehat{\varepsilon},+})_* \circ (j_{\widetilde{\varepsilon},\widehat{\varepsilon}})_*, \quad (j_{\bar{\varepsilon},\widehat{\varepsilon}})_* \circ (e_{\bar{\varepsilon},+})_* = (e_{\widehat{\varepsilon},+})_* \circ (j_{\bar{\varepsilon},\widehat{\varepsilon}})_*.$$

The third equality follows from

$$e_{\widehat{\varepsilon},+} \circ e^1_{\widehat{\varepsilon},+} = e_{\widehat{\varepsilon},+} \circ e^2_{\widehat{\varepsilon},+},$$

which comes from the relation $e_+ \circ e^1_+ = e_+ \circ e^2_+$ of (20). The second equation of (27) can be proved similarly by applying

$$e_{\widehat{\varepsilon},-} \circ e^2_{\widehat{\varepsilon},+} = e_{\widehat{\varepsilon},+} \circ e^1_{\widehat{\varepsilon},-},$$

which comes from the relation $e_- \circ e^2_+ = e_+ \circ e^1_-$ of (20). To prove the third equation of (27), there is one nontrivial matter: we need to apply

$$(e_{\widehat{\varepsilon},-} \circ e^1_{\widehat{\varepsilon},+})_* = (e_{\widehat{\varepsilon},-} \circ e^1_{\widehat{\varepsilon},-})_* \colon \widehat{H}^{<a}_*(\widehat{\varepsilon},\widehat{\delta}) \to H^{<a}_*(\widehat{\varepsilon},(i_{\varepsilon'',\widehat{\varepsilon}})_*\delta''),$$

which does not follow from (20) directly. To check this equation, we consider the following diagram including $\overline{H}^{<a}_*(\widehat{\varepsilon},(i_{\varepsilon'',\widehat{\varepsilon}})_*(\bar{\imath}\delta''))$:

$$\begin{tikzcd}
\widehat{H}^{<a}_*(\widehat{\varepsilon},\widehat{\delta}) \arrow[rr, "(e^1_{\widehat{\varepsilon},+})_*"] \arrow[dr, "(e^2_{\widehat{\varepsilon},-})_*"] \arrow[dd, "(e^1_{\widehat{\varepsilon},-})_*"'] & & \overline{H}^{<a}_*(\widehat{\varepsilon},(i_{\widetilde{\varepsilon},\widehat{\varepsilon}})_*\widetilde{\delta}) \arrow[dd, "(e_{\widehat{\varepsilon},-})_*"] \\
& \overline{H}^{<a}_*(\widehat{\varepsilon},(i_{\varepsilon'',\widehat{\varepsilon}})_*(\bar{\imath}\delta'')) \arrow[dr, shift left, "(e_{\widehat{\varepsilon},+})_*"] \arrow[dr, shift right, "(e_{\widehat{\varepsilon},-})_*"'] & \\
\overline{H}^{<a}_*(\widehat{\varepsilon},(i_{\bar{\varepsilon}',\widehat{\varepsilon}})_*\bar{\delta}') \arrow[rr, "(e_{\widehat{\varepsilon},-})_*"] & & H^{<a}_*(\widehat{\varepsilon},(i_{\varepsilon'',\widehat{\varepsilon}})_*\delta'')
\end{tikzcd}$$

Then

$$e_{\widehat{\varepsilon},-} \circ e^1_{\widehat{\varepsilon},+} = e_{\widehat{\varepsilon},+} \circ e^2_{\widehat{\varepsilon},-}, \quad e_{\widehat{\varepsilon},-} \circ e^1_{\widehat{\varepsilon},-} = e_{\widehat{\varepsilon},-} \circ e^2_{\widehat{\varepsilon},-}$$

follow directly from the relations $e_- \circ e^1_+ = e_+ \circ e^2_-$ and $e_- \circ e^1_- = e_- \circ e^2_-$ of (20). If we rewrite $(i_{\varepsilon'',\widehat{\varepsilon}})_*\delta''$ by $\delta^{\#}$, equation (25) shows that

$$(e_{\widehat{\varepsilon},+})_* = (e_{\widehat{\varepsilon},-})_* \colon \overline{H}^{<a}_*(\widehat{\varepsilon},\bar{\imath}\delta^{\#}) \to H^{<a}_*(\widehat{\varepsilon},\delta^{\#}).$$

From the above diagram, we get $(e_{\widehat{\varepsilon},-})_* \circ (e^1_{\widehat{\varepsilon},+})_* = (e_{\widehat{\varepsilon},-})_* \circ (e^1_{\widehat{\varepsilon},-})_*$. $\square$

Corollary 4.16 *For $(\varepsilon,\delta),(\varepsilon',\delta')\in\mathcal{T}_a$ with $\varepsilon'\le\varepsilon$, the isomorphism*

$$k_{(\bar\varepsilon,\bar\delta)}\colon H_*^{<a}(\varepsilon',\delta')\to H_*^{<a}(\varepsilon,\delta)$$

does not depend on the choice of $(\bar\varepsilon,\bar\delta)\in\overline{\mathcal{T}}_a$ satisfying (24).

Proof Let $(\bar\varepsilon,\bar\delta)$ and $(\tilde\varepsilon,\tilde\delta)$ be two arbitrary choices from $\overline{\mathcal{T}}_a$ satisfying (24). We apply Proposition 4.15 to the case where $(\varepsilon'',\delta'')=(\varepsilon',\delta')$ and $(\bar\varepsilon',\bar\delta')=(\varepsilon',\bar\imath\delta')$. By Lemma 4.13, $k_{(\varepsilon',\bar\imath\delta')}$ is equal to the identity map on $H_*^{<a}(\varepsilon',\delta')$, so we get

$$k_{(\bar\varepsilon,\bar\delta)}\circ\mathrm{id}_{H_*^{<a}(\varepsilon',\delta')}=k_{(\tilde\varepsilon,\tilde\delta)}.\qquad\square$$

From this result, we may rewrite $k_{(\bar\varepsilon,\bar\delta)}\colon H_*^{<a}(\varepsilon',\delta')\to H_*^{<a}(\varepsilon,\delta)$ as $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}$. The equations of Lemma 4.13 and Proposition 4.15 can be rewritten as

$$k_{(\varepsilon,\delta),(\varepsilon,\delta)}=\mathrm{id}_{H_*^{<a}(\varepsilon,\delta)},\quad k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\circ k_{(\varepsilon'',\delta''),(\varepsilon',\delta')}=k_{(\varepsilon'',\delta''),(\varepsilon,\delta)}.$$

Now, $\mathcal{T}_a$ becomes a directed set by the relation $(\varepsilon',\delta')\le(\varepsilon,\delta)$ if and only if $\varepsilon'\le\varepsilon$. Then we obtain an inverse system

$$\left(\{H_*^{<a}(\varepsilon,\delta)\}_{(\varepsilon,\delta)\in\mathcal{T}_a},\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon'\le\varepsilon}\right)$$

and its inverse limit

$$H_*^{<a}(Q,K):=\varprojlim_{\varepsilon\to0}H_*^{<a}(\varepsilon,\delta)$$

is defined.

4.3.3 Spectral sequence Lastly, we extend the above work to spectral sequences. For $(\varepsilon,\delta),(\varepsilon',\delta')\in\mathcal{T}_a$ with $\varepsilon'\le\varepsilon$, we take $(\bar\varepsilon,\bar\delta)\in\overline{\mathcal{T}}_a$ satisfying (24). Chain maps $(j_{\varepsilon,\bar\varepsilon})_*$, $e_{\bar\varepsilon+}$, $e_{\bar\varepsilon,-}$ and $(j_{\varepsilon',\bar\varepsilon})_*$ induce a zig zag of morphisms of spectral sequences

$$E^{<a}_{(\varepsilon',\delta')}\xrightarrow{(j_{\varepsilon',\bar\varepsilon})_*}E^{<a}_{(\bar\varepsilon,(i_{\varepsilon',\bar\varepsilon})_*\delta')}\xleftarrow{(e_{\bar\varepsilon-})_*}\overline{E}^{<a}_{(\bar\varepsilon,\bar\delta)}\xrightarrow{(e_{\bar\varepsilon,+})_*}E^{<a}_{(\bar\varepsilon,(i_{\varepsilon,\bar\varepsilon})_*\delta)}\xleftarrow{(j_{\varepsilon,\bar\varepsilon})_*}E^{<a}_{(\varepsilon,\delta)}.\tag{28}$$

All of them are isomorphisms. Let $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\colon E^{<a}_{(\varepsilon',\delta')}\to E^{<a}_{(\varepsilon,\delta)}$ denote the composition of these maps. (The independence on (ε,δ) can be proved in the same way as Corollary 4.16.)

Proposition 4.17 *There exists a spectral sequence $E^{<a}=\left(\{(E^{<a})^r_{p,q}\},\{(d^{<a})^r_{p,q}\}\right)$ which converges to $H_*^{<a}(Q,K)$ in the sense of [21, Bounded Convergence 5.2.5] such that*

$$(E^{<a})^1_{-m,q}=\begin{cases}H^{\mathrm{dR}}_{q-m(d-1)}(\Sigma^a_m,\Sigma^0_m) & \text{if } m\ge0,\\ 0 & \text{if } m<0\end{cases}$$

Proof On the first page, the middle two maps of (28) have the form

$$H_{q+p(d-1)}(\Sigma^{a-p\bar\varepsilon}_{-p},\Sigma^0_{-p})\xleftarrow{(e_+)_*}\overline{H}_{q+p(d-1)}(\Sigma^{a-p\bar\varepsilon}_{-p},\Sigma^0_{-p})\xrightarrow{(e_-)_*}H_{q+p(d-1)}(\Sigma^{a-p\bar\varepsilon}_{-p},\Sigma^0_{-p}).$$

Since $(e_-)_*=(\bar\imath)_*^{-1}=(e_+)_*$, this composition is equal to the identity map. Therefore, $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}$ is equal to $(j_{\varepsilon',\varepsilon})_*\colon H_{q+p(d-1)}(\Sigma^{a-p\varepsilon'}_{-p},\Sigma^0_{-p})\to H_{q+p(d-1)}(\Sigma^{a-p\varepsilon}_{-p},\Sigma^0_{-p})$ on the (p,q)-term with $p\le0$ of the first page.

By $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon'\le\varepsilon}$, we define $(E^{<a})^r_{p,q} := \varprojlim_{\varepsilon\to 0}(E^{<a}_{(\varepsilon,\delta)})^r_{p,q}$. Moreover,

$$(d^{<a})^r_{p,q}\colon (E^{<a})^r_{p,q} \to (E^{<a})^r_{p-r,q+r-1}$$

is defined to be the map induced by $\{(d^{<a}_{(\varepsilon,\delta)})^r_{p,q}\}_{(\varepsilon,\delta)\in\mathscr{T}_a}$. Then the (p,q)-term of the first page is given by

$$(E^{<a})^1_{p,q} = \varprojlim_{\varepsilon\to 0} H_{q+p(d-1)}(\Sigma^{a-p\varepsilon}_{-p}, \Sigma^0_{-p}) = H_{q+p(d-1)}(\Sigma^a_{-p}, \Sigma^0_{-p})$$

for $p \le 0$ and $(E^{<a})^1_{p.q} = 0$ for $p > 0$. Since $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon'\le\varepsilon}$ consists of isomorphisms, it is clear that $E^{<a}$ is a spectral sequence which converges to $H^{<a}_*(Q,K)$. □

4.4 Definition of $H^{\mathrm{string}}_*(Q,K)$

In this section, we define $I^{a,b}\colon H^{<a}_*(Q,K) \to H^{<b}_*(Q,K)$ for $a, b \in \mathbb{R}_{>0}\setminus\mathscr{L}(K)$ with $a \le b$ to get a direct system $\big(\{H^{<a}_*(Q,K)\}_{a\in\mathbb{R}_{>0}\setminus\mathscr{L}(K)}, \{I^{a,b}\}_{a\le b}\big)$. After defining $H^{\mathrm{string}}_*(Q,K)$ as its direct limit, we will give a structure of a unital graded $\mathbb{R}$-algebra.

4.4.1 The limit of $a\to\infty$ Let a and b be the above real numbers. For any $(\varepsilon,\delta)\in\mathscr{T}_a\cap\mathscr{T}_b$, we have considered in Section 4.1 a linear map

$$(I^{a,b}_\varepsilon)_*\colon H^{<a}_*(\varepsilon,\delta) \to H^{<b}_*(\varepsilon,\delta),$$

which is induced by the inclusion maps $I^{a,b}_\varepsilon\colon \Sigma^{a+m\varepsilon}_m \to \Sigma^{b+m\varepsilon}_m$ for all $m \in Z_{\ge 0}$.

Lemma 4.18 *Suppose that $(\bar\varepsilon,\bar\delta)\in\overline{\mathscr{T}}_a\cap\overline{\mathscr{T}}_b$ satisfies (24) for $(\varepsilon,\delta), (\varepsilon',\delta)\in\mathscr{T}_a\cap\mathscr{T}_b$ with $\varepsilon'\le\varepsilon$. Then the following diagram commutes*:

$$\begin{tikzcd}
H^{<a}_*(\varepsilon,\delta) \arrow[r, "(I^{a,b}_\varepsilon)_*"] & H^{<b}_*(\varepsilon,\delta) \\
H^{<a}_*(\varepsilon',\delta') \arrow[u, "k_{(\varepsilon',\delta'),(\varepsilon,\delta)}"] \arrow[r, "(I^{a,b}_{\varepsilon'})_*"] & H^{<b}_*(\varepsilon',\delta') \arrow[u, "k_{(\varepsilon',\delta'),(\varepsilon,\delta)}"']
\end{tikzcd}$$

Proof $I^{a,b}_{\bar\varepsilon}$ induces a linear map $(I^{a,b}_{\bar\varepsilon})_*\colon \overline{H}^{<a}_*(\bar\varepsilon,\bar\delta)\to\overline{H}^{<b}_*(\bar\varepsilon,\bar\delta)$. It suffices to show that

$$(I^{a,b}_\varepsilon)_*\circ f_{(\bar\varepsilon,\bar\delta),+} = f_{(\bar\varepsilon,\bar\delta),+}\circ(I^{a,b}_{\bar\varepsilon})_*\colon \overline{H}^{<a}_*(\bar\varepsilon,\bar\delta)\to H^{<b}_*(\varepsilon,\delta),$$
$$(I^{a,b}_{\varepsilon'})_*\circ f_{(\bar\varepsilon,\bar\delta),-} = f_{(\bar\varepsilon,\bar\delta),-}\circ(I^{a,b}_{\bar\varepsilon})_*\colon \overline{H}^{<a}_*(\bar\varepsilon,\bar\delta)\to H^{<b}_*(\varepsilon',\delta').$$

Let us check the first equation. Since $j_{\varepsilon,\bar\varepsilon}\circ I^{a,b}_\varepsilon = I^{a,b}_{\bar\varepsilon}\circ j_{\varepsilon,\bar\varepsilon}\colon \Sigma^{a+m\varepsilon}_m\to\Sigma^{b+m\bar\varepsilon}_m$,

$$\begin{aligned}
(I^{a,b}_\varepsilon)_*\circ f_{(\bar\varepsilon,\bar\delta),+} &= (I^{a,b}_\varepsilon)_*\circ(j_{\varepsilon,\bar\varepsilon})^{-1}_*\circ(e_{\bar\varepsilon,+})_* \\
&= (j_{\varepsilon,\bar\varepsilon})^{-1}_*\circ(I^{a,b}_{\bar\varepsilon})_*\circ(e_{\bar\varepsilon,+})_* \\
&= (j_{\varepsilon,\bar\varepsilon})^{-1}_*\circ(e_{\bar\varepsilon,+})_*\circ(I^{a,b}_{\bar\varepsilon})_* \\
&= f_{(\bar\varepsilon,\bar\delta),+}\circ(I^{a,b}_{\bar\varepsilon})_*.
\end{aligned}$$

The second equation can be proved by replacing ε and $+$ in the above computation by ε' and $-$. □

This lemma implies that, after taking the limits of $(\varepsilon,\delta)\in\mathcal{T}_a\cap\mathcal{T}_b$ as $\varepsilon\to 0$, we get a linear map

$$I^{a,b} := \varprojlim_{\varepsilon\to 0}(I^{a,b}_\varepsilon)_*\colon H^{<a}_*(Q,K)\to H^{<b}_*(Q,K).$$

Furthermore, for $b\ge a$ and $(\varepsilon,\delta),(\varepsilon',\delta')\in\mathcal{T}_a\cap\mathcal{T}_b$ with $\varepsilon'\le\varepsilon$, $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}$ induces an isomorphism from $H^{[a,b)}_*(\varepsilon',\delta')$ to $H^{[a,b)}_*(\varepsilon,\delta)$. Thus, we can also define

$$H^{[a,b)}_*(Q,K) := \varprojlim_{\varepsilon\to 0} H^{[a,b)}_*(\varepsilon,\delta).$$

Note that a long exact sequence

$$\cdots\to H^{<a}_*(Q,K)\xrightarrow{I^{a,b}} H^{<b}_*(Q,K)\to H^{[a,b)}_*(Q,K)\to H^{<a}_{*-1}(Q,K)\xrightarrow{I^{a,b}}\cdots \tag{29}$$

is induced by (23).

From the direct system $\big(\{H^{<a}_*(Q,K)\}_{a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K)},\{I^{a,b}\}_{a\le b}\big)$, we finally define a graded $\mathbb{R}$-vector space

$$H^{\mathrm{string}}_*(Q,K) := \varinjlim_{a\to\infty} H^{<a}_*(Q,K).$$

4.4.2 Product structure Let us see that $H^{\mathrm{string}}_*(Q,K)$ has the structure of a unital associative graded $\mathbb{R}$-algebra. For any $a,a'\in\mathbb{R}_{>0}\cup\{\infty\}$, $m,m'\in\mathbb{Z}_{\ge 0}$ and $\varepsilon\in\big(0,\frac{1}{5}\varepsilon_0\big]$, there is a map

$$\Pi\colon \Sigma^{a+m\varepsilon}_m\times\Sigma^{a'+m'\varepsilon}_{m'}\to\Sigma^{(a+a')+(m+m')\varepsilon}_{m+m'},\quad ((\gamma_k)_{k=1,\dots,m},(\gamma'_l)_{l=1,\dots,m'})\mapsto(\gamma_1,\dots,\gamma_m,\gamma'_1,\dots,\gamma'_{m'}).$$

When $m=0$ or $m'=0$, Π is identified with the identity map. We define a linear map

$$C^{<a}_p(\varepsilon)\otimes C^{<a'}_q(\varepsilon)\to C^{<a+a'}_{p+q}(\varepsilon),\quad x\otimes y\mapsto x\star y, \tag{30}$$

such that, for $x\in C^{\mathrm{dR}}_{p-m(d-2)}(\Sigma^{a+m\varepsilon}_m,\Sigma^0_m)$ and $y\in C^{\mathrm{dR}}_{q-m'(d-2)}(\Sigma^{a'+m'\varepsilon}_{m'},\Sigma^0_{m'})$,

$$x\star y = (-1)^{mqd}\,\Pi_*(x\times y).$$

We note that the associative relation $(x\star y)\star z = x\star(y\star z)$ holds. Suppose that $a,a',a+a'\notin\mathcal{L}(K)$. For any $(\varepsilon,\delta)\in\mathcal{T}_a\cap\mathcal{T}_{a'}\cap\mathcal{T}_{a+a'}$, the above map is compatible with the differential D_δ. Indeed, for $x\in C^{\mathrm{dR}}_{p-m(d-2)}(\Sigma^{a+m\varepsilon}_m,\Sigma^0_m)$ and $y\in C^{\mathrm{dR}}_{q-m'(d-2)}(\Sigma^{a'+m'\varepsilon}_{m'},\Sigma^0_{m'})$,

$$\begin{aligned}(-1)^{mqd}D_\delta(x\star y) &= \partial(\Pi_*(x\times y)) + \sum_{k=1}^{m+m'}(-1)^{p+q+1+kd} f_{k,\delta}(\Pi_*(x\times y))\\ &= \Pi_*(\partial x\times y) + \sum_{k=1}^{m}(-1)^{p+q+1+kd+s_0}\Pi_*(f_{k,\delta}(x)\times y)\\ &\qquad + (-1)^{p-m(d-2)}\Pi_*(x\times\partial y) + \sum_{l=1}^{m'}(-1)^{p+q+1+(l+m)d}\Pi_*(x\times f_{l,\delta}(y))\\ &= (-1)^{mqd}(D_\delta(x))\star y + (-1)^{p+mqd}x\star(D_\delta(y)).\end{aligned}$$

Here, $s_0 := (q - m'(d-2))(d+1)$. This computation shows that

$$D_\delta(x \star y) = (D_\delta(x)) \star y + (-1)^p x \star (D_\delta(y))$$

for $x \in C_p^{<a}(\varepsilon)$ and $y \in C_q^{<a'}(\varepsilon)$. Therefore, (30) induces a linear map on homology,

$$H_p^{<a}(\varepsilon,\delta) \otimes H_q^{<a'}(\varepsilon,\delta) \to H_{p+q}^{<a+a'}(\varepsilon,\delta),$$

for every $(\varepsilon,\delta) \in \mathcal{T}_a \cap \mathcal{T}_{a'} \cap \mathcal{T}_{a+a'}$.

Likewise, let us define $x \mathbin{\bar\star} y := (-1)^{mqd}\Pi_*(x \times y) \in \overline{C}_{p+q}^{<a+a'}(\bar\varepsilon)$ for $x \in \overline{C}^{\mathrm{dR}}_{p-m(d-2)}(\Sigma_m^{a+m\bar\varepsilon}, \Sigma_m^0)$ and $y \in \overline{C}^{\mathrm{dR}}_{q-m'(d-2)}(\Sigma_{m'}^{a'+m'\bar\varepsilon}, \Sigma_{m'}^0)$. Then $e_{\bar\varepsilon,+}$ and $e_{\bar\varepsilon,-}$ intertwine the $\star$-operation and the $\bar\star$-operation. This shows that the $\star$-operation is compatible with $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon'\le\varepsilon}$. Therefore, in the limit $\varepsilon \to 0$, a linear map

$$H_p^{<a}(Q,K) \otimes H_q^{<a'}(Q,K) \to H_{p+q}^{<a+a'}(Q,K)$$

is induced. The commutativity with $\{I^{a,b}\}_{a\le b}$ naturally holds. As a result, we get an associative product structure on $H_*^{\mathrm{string}}(Q,K)$. The element $1 \in H_0^{\mathrm{string}}(Q,K)$, which comes from $1 \in \mathbb{R} = C_0^{\mathrm{dR}}(\Sigma_0^a, \Sigma_0^0) \subset C_0^{<a}(\varepsilon,\delta)$, is the unit of this graded algebra.

4.4.3 Explicit choice of (ε,δ) Lastly, let us introduce a condition on $(\varepsilon,\delta) \in \mathcal{T}_a$ so that δ can be written explicitly. The concrete computations in Sections 5 and 6 become easier by choosing (ε,δ) satisfying this condition.

Suppose that there exists a fixed trivialization $\mathbb{R}^d \times K \to (TK)^\perp$ of the normal bundle of K which preserves orientations and fiber metrics. For every $\varepsilon \le \varepsilon_0$, let us write $\mathbb{O}_\varepsilon := \{w \in \mathbb{R}^d \mid |w| < \frac{1}{2}\varepsilon\}$. Composing with the exponential map (3), we obtain a diffeomorphism

$$h \colon \mathbb{O}_\varepsilon \times K \to N_\varepsilon,$$

which preserves orientations. In this case, we say $(\varepsilon,\delta) \in \mathcal{T}_a$ is *standard with respect to* h if $\delta \in C^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ has the form

$$\delta = [N_\varepsilon, \psi_\varepsilon, h_*(\nu_\varepsilon \times 1)] \tag{31}$$

satisfying:

- $\psi_\varepsilon \colon N_\varepsilon \to S_\varepsilon$, $v \to (\sigma_j^v)_{j=1,2}$, is defined by
$$\sigma_1^v \colon [0, \tfrac{1}{2}\varepsilon] \to N_\varepsilon, \quad t \mapsto h\Big(\frac{\varepsilon - 2t}{\varepsilon} w, x\Big), \quad \sigma_2^v \colon [0, \tfrac{1}{2}\varepsilon] \to N_\varepsilon, \quad t \mapsto h\Big(\frac{2t}{\varepsilon} w, x\Big),$$
for $v = h(w,x) \in N_\varepsilon$.
- $h_*(\nu_\varepsilon \times 1) \in \Omega_c^{n-d}(N_\varepsilon)$ for some $\nu_\varepsilon \in \Omega_c^d(\mathbb{O}_\varepsilon)$ with $\int_{\mathbb{O}_\varepsilon} \nu_\varepsilon = 1$.

Suppose that $(\bar\varepsilon, \bar\delta) \in \overline{\mathcal{T}}_a$ satisfies (24) for $(\varepsilon,\delta), (\varepsilon',\delta') \in \mathcal{T}_a$ with $\varepsilon' \le \varepsilon$ which are given as above. Then we say $(\bar\varepsilon,\bar\delta)$ is *standard with respect to* h if $\bar\varepsilon = \varepsilon$ and $\bar\delta \in \overline{C}^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ has the form

$$\bar\delta = (-1)^n [\mathbb{R} \times N_\varepsilon, \bar\psi_{\varepsilon',\varepsilon}, (\mathrm{id}_{\mathbb{R}_{\ge 1}\times N_\varepsilon}, \mathrm{id}_{\mathbb{R}_{\le -1}\times N_\varepsilon}), \bar\eta_{\varepsilon',\varepsilon}] \tag{32}$$

such that, for some C^∞ function $\kappa\colon \mathbb{R} \to [0,1]$ with $\kappa(r) = 0$ if $r \le -1$ and $\kappa(r) = 1$ if $r \ge 1$, the following hold:

- $\bar{\psi}_{\varepsilon',\varepsilon}(r,v) = (r, (\sigma_j^{(r,v)})_{j=1,2}) \in \mathbb{R} \times S_\varepsilon$ is defined by
$$\sigma_1^{(r,v)}\colon \left[0, \tfrac{1}{2}\varepsilon_r\right] \to N_\varepsilon, \quad t \mapsto h\left(\frac{\varepsilon_r - 2t}{\varepsilon_r} w, x\right), \qquad \sigma_2^{(r,v)}\colon \left[0, \tfrac{1}{2}\varepsilon_r\right] \to N_\varepsilon, \quad t \mapsto h\left(\frac{2t}{\varepsilon_r} w, x\right),$$
for $r \in \mathbb{R}$, $v = h(w,x) \in N_\varepsilon$ and $\varepsilon_r := \kappa(r)\varepsilon + (1-\kappa(r))\varepsilon'$.
- $(\mathrm{id}_{\mathbb{R}} \times h)^* \bar{\eta}_{\varepsilon',\varepsilon} = \kappa \times (\nu_\varepsilon \times 1) + (1-\kappa) \times (\nu_{\varepsilon'} \times 1) + (d\kappa) \times (\theta \times 1)$ for some $\theta \in \Omega_c^{d-1}(\mathbb{O}_\varepsilon)$ satisfying $d\theta = \nu_\varepsilon - \nu_{\varepsilon'}$.

In summary, in order to compute $H_*^{\mathrm{string}}(Q,K)$ when a trivialization h is given, we only need to deal with $(\varepsilon,\delta) \in \mathcal{T}_a$ and $(\varepsilon,\bar{\delta}) \in \overline{\mathcal{T}}_a$ which are standard with respect to h.

4.5 Invariance

In this section, we prove the invariance of $H_*^{\mathrm{string}}(Q,K)$ up to isomorphism under changing auxiliary data. More precisely, we consider the dependence of the construction on the following data (see the beginning of Section 3):

(a) a complete Riemannian metric g on Q;

(b) a constant $C_0 \ge 1$ which bounds the speed of all $\gamma \in \Omega_K(Q)$;

(c) a real number $\varepsilon_0 > 0$ which is the diameter (in the direction of fibers) of a tubular neighborhood N_{ε_0} of K;

(d) a C^∞ function $\mu\colon \left[0, \tfrac{3}{2}\right] \to [0,1]$ which is used to define con_k.

Notation Let X be an arbitrary notation, which we have defined in the former sections. As a rule, in this section, if its definition depends on some auxiliary data S, we rewrite X as X_S when discussing the dependence on S.

Independence on $\boldsymbol{\mu}$ We choose a C^∞ family $\bar{\mu} := (\mu_r)_{r \in \mathbb{R}}$ such that each μ_r satisfies the same condition as μ, and $\mu_r = \mu_{-1}$ if $r \le -1$ while $\mu_r = \mu_1$ if $r \ge 1$. Then a map con_{k,μ_r} is defined for each $r \in \mathbb{R}$. For $(\bar{\varepsilon},\bar{\delta}) \in \mathcal{T}_a$, let us define $\bar{f}_{k,\bar{\delta},\bar{\mu}}\colon \overline{C}_*^{\mathrm{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}) \to \overline{C}_*^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\bar{\varepsilon}})$ by replacing con_k in the definition of Φ_k of (21) by $\mathrm{con}_{k,\mu_{r(u)}}$. We also replace $\bar{f}_{k,\bar{\delta}}$ in the definition of $\overline{D}_{\bar{\delta}}$ by $\bar{f}_{k,\bar{\delta},\bar{\mu}}$ to define a linear map
$$\overline{D}_{\bar{\delta},\bar{\mu}}\colon \overline{C}_*^{<a}(\bar{\varepsilon}) \to \overline{C}_{*-1}^{<a}(\bar{\varepsilon}).$$

This satisfies $\overline{D}_{\bar{\delta},\bar{\mu}} \circ \overline{D}_{\bar{\delta},\bar{\mu}} = 0$, so we get a chain complex $(\overline{C}_*^{<a}(\bar{\varepsilon}), \overline{D}_{\bar{\delta},\bar{\mu}})$. Let $\overline{H}_*^{<a}(\varepsilon,\bar{\delta},\bar{\mu})$ denote its homology group.

We rewrite $e_{\varepsilon,+}, e_{\varepsilon,-}\colon \overline{C}_*^{<a}(\varepsilon) \to C_*^{<a}(\varepsilon)$ by $e_{\varepsilon,\bar{\mu},+}$ and $e_{\varepsilon,\bar{\mu},-}$, respectively. They induce
$$(e_{\varepsilon,\bar{\mu},+})_*\colon \overline{H}_*^{<a}(\varepsilon,\bar{\delta},\bar{\mu}) \to H_*^{<a}(\varepsilon, e_+\delta)_{\mu_1}, \quad (e_{\varepsilon,\bar{\mu},-})_*\colon \overline{H}_*^{<a}(\varepsilon,\bar{\delta},\bar{\mu}) \to H_*^{<a}(\varepsilon, e_-\delta)_{\mu_{-1}}.$$

We can prove, just as Lemma 4.7, that they are isomorphisms. When $(\bar\varepsilon,\bar\delta)\in\overline{\mathcal{T}}_{[a,b)}$ satisfies (24), we define an isomorphism $k_{(\bar\varepsilon,\bar\delta,\bar\mu)}\colon H^{<a}_*(\varepsilon',\delta')_{\mu_{-1}}\to H^{<a}_*(\varepsilon,\delta)_{\mu_1}$ by

$$k_{(\bar\varepsilon,\bar\delta,\bar\mu)}:=((j_{\varepsilon,\bar\varepsilon})_*^{-1}\circ(e_{\bar\varepsilon,\bar\mu,+})_*)\circ((j_{\varepsilon',\bar\varepsilon})_*^{-1}\circ(e_{\bar\varepsilon,\bar\mu,-})_*)^{-1}.$$

As in Proposition 4.15, the two triangles in the following diagram commute:

$$\begin{tikzcd}
H^{<a}_*(\varepsilon,\delta)_{\mu_{-1}} \arrow[r, "k_{(\varepsilon,\bar{\imath}\delta,\bar\mu)}"] & H^{<a}_*(\varepsilon,\delta)_{\mu_1} \\
H^{<a}_*(\varepsilon',\delta')_{\mu_{-1}} \arrow[u, "k_{(\varepsilon',\delta'),(\varepsilon,\delta)}"] \arrow[r, "k_{(\varepsilon',\bar{\imath}\delta',\bar\mu)}"'] \arrow[ur, "k_{(\bar\varepsilon,\bar\delta,\bar\mu)}"] & H^{<a}_*(\varepsilon',\delta')_{\mu_1} \arrow[u, "k_{(\varepsilon',\delta'),(\varepsilon,\delta)}"']
\end{tikzcd}$$

Therefore, $\{k_{(\varepsilon,\bar{\imath}\delta,\bar\mu)}\}_{(\varepsilon,\delta)\in\mathcal{T}_a}$ induces an isomorphism on the limits of $\varepsilon\to 0$,

$$k^a_{\bar\mu}\colon H^{<a}_*(Q,K)_{\mu_{-1}}\to H^{<a}_*(Q,K)_{\mu_1}.$$

It is easy to see, as in Lemma 4.18, that $\{k^a_{\bar\mu}\}_{a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K)}$ commute with $\{I^{a,b}\}_{a\le b}$, so we get an isomorphism from $H^{\mathrm{string}}_*(Q,K)_{\mu_{-1}}$ to $H^{\mathrm{string}}_*(Q,K)_{\mu_1}$. It is also possible to prove, in the same way as Corollary 4.16, that this isomorphism does not depend on the choice of $\bar\mu$.

Independence on ε_0 For $\varepsilon_0'\le\varepsilon_0$, we consider the inclusion maps $\mathrm{j}_{\varepsilon_0',\varepsilon_0}\colon \Sigma^{a+m\varepsilon}_{m,\varepsilon_0'}\to\Sigma^{a+m\varepsilon}_{m,\varepsilon_0}$ for all $m\in\mathbb{Z}_{\ge 0}$. They induce a chain map $(\mathrm{j}_{\varepsilon_0',\varepsilon_0})_*$ from $(C^{<a}_*(\varepsilon)_{\varepsilon_0'},D_\delta)$ to $(C^{<a}_*(\varepsilon)_{\varepsilon_0},D_\delta)$ for every $a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K)$ and $(\varepsilon,\delta)\in\mathcal{T}_{a,\varepsilon_0'}(\subset\mathcal{T}_{a,\varepsilon_0})$. This preserves filtrations $\{\mathcal{F}^{<a}_{\varepsilon,p,\varepsilon_0}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{F}^{<a}_{\varepsilon,p,\varepsilon_0'}\}_{p\in\mathbb{Z}}$, so it induces an morphism of the spectral sequences. On the $(-m,q)$-term for $m\ge 0$ of the first page, it is equal to the map

$$(\mathrm{j}_{\varepsilon_0',\varepsilon_0})_*\colon H^{\mathrm{dR}}_{q-m(d-1)}(\Sigma^{a+m\varepsilon}_{m,\varepsilon_0'},\Sigma^0_{m,\varepsilon_0'})\to H^{\mathrm{dR}}_{q-m(d-1)}(\Sigma^{a+m\varepsilon}_{m,\varepsilon_0},\Sigma^0_{m,\varepsilon_0}),$$

which is an isomorphism by Lemma 3.11. Therefore, by Lemma 4.2, $(\mathrm{j}_{\varepsilon_0',\varepsilon_0})_*\colon H^{<a}_*(\varepsilon,\delta)_{\varepsilon_0'}\to H^{<a}_*(\varepsilon,\delta)_{\varepsilon_0}$ is also an isomorphism.

We can prove its commutativity with $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon'\le\varepsilon}$ as in Lemma 4.18, so we get an isomorphism on the limit of $\varepsilon\to 0$,

$$\mathfrak{J}^a_{\varepsilon_0',\varepsilon_0}\colon H^{<a}_*(Q,K)_{\varepsilon_0'}\to H^{<a}_*(Q,K)_{\varepsilon_0}.$$

It holds naturally that $\{\mathfrak{J}^a_{\varepsilon_0',\varepsilon_0}\}_{a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K)}$ commutes with $\{I^{a,b}\}_{a\le b}$. Therefore, in the limit $a\to\infty$, we get an isomorphism from $H^{\mathrm{string}}_*(Q,K)_{\varepsilon_0'}$ to $H^{\mathrm{string}}_*(Q,K)_{\varepsilon_0}$.

Independence on C_0 For $C_0'\ge C_0\ge 1$, we consider the inclusion maps $\mathrm{j}_{C_0,C_0'}\colon\Sigma^a_{m,C_0}\to\Sigma^a_{m,C_0'}$ for all $m\in\mathbb{Z}_{\ge 0}$. Parallel to the proof of the independence on ε_0, we apply Lemma 3.11 to show that an isomorphism

$$\mathfrak{J}^a_{C_0,C_0'}\colon H^{<a}_*(Q,K)_{C_0}\to H^{<a}_*(Q,K)_{C_0'}$$

is induced. It holds naturally that $\{\mathfrak{J}^a_{C_0,C_0'}\}_{a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K)}$ commutes with $\{I^{a,b}\}_{a\le b}$. Therefore, in the limit $a\to\infty$, we get an isomorphism from $H^{\mathrm{string}}_*(Q,K)_{C_0}$ to $H^{\mathrm{string}}_*(Q,K)_{C_0'}$.

Independence on g First, let us introduce an graded algebra $\check{H}^{\mathrm{string}}_*(Q,K)_g$ which is isomorphic to $H^{\mathrm{string}}_*(Q,K)_{(g,C_0,\varepsilon_0)}$, but whose definition does not depend on ε_0 and C_0. For every $a\in\mathbb{R}_{>0}\setminus\mathscr{L}(K)_g$, we define the limit of $\varepsilon_0\to 0$ and $C_0\to\infty$,

$$\check{H}^{<a}_*(Q,K)_g := \varinjlim_{C_0\to\infty}\varprojlim_{\varepsilon_0\to 0} H^{<a}_*(Q,K)_{(g,C_0,\varepsilon_0)},$$

by the $\{\mathfrak{I}^a_{\varepsilon'_0,\varepsilon_0}\}_{\varepsilon'_0\le\varepsilon_0}$ and $\{\mathfrak{I}^a_{C_0,C'_0}\}_{C_0\le C'_0}$ defined above. Then $\{I^{a,b}\}_{a\le b}$ induces a family of maps

$$\{\check{I}^{a,b}_g\colon \check{H}^{<a}_*(Q,K)_g\to\check{H}^{<b}_*(Q,K)_g\}_{a\le b},$$

and we can take the limit as $a\to\infty$ to define $\check{H}^{\mathrm{string}}_*(Q,K)_g := \varinjlim_{a\to\infty}\check{H}^{<a}_*(Q,K)_g$.

Suppose that g and g' are complete Riemannian metrics on Q. For $a>0$, there exists a compact subset Z_a which contains the images of all $\gamma\in\bigcup_{C_0\ge 1}\Omega_K(Q)_{(g,C_0)}$ with $\mathrm{length}_g\,\gamma<a$ and the images of all $\gamma\in\bigcup_{C_0\ge 1}\Omega_K(Q)_{(g',C_0)}$ with $\mathrm{length}_{g'}\,\gamma<a$. For any $a\in\mathbb{R}_{>0}\setminus(\mathscr{L}(K)_g\cup\mathscr{L}(K)_{g'})$, there exists a constant $c_a\ge 1$ such that $|\cdot|_{g'}\le c_a|\cdot|_g$ and $|\cdot|_g\le c_a|\cdot|_{g'}$ on Z_a. We may additionally assume that $ac_a\notin\mathscr{L}(K)_g\cup\mathscr{L}(K)_{g'}$.

Let $a\in\mathbb{R}_{>0}\setminus(\mathscr{L}(K)_g\cup\mathscr{L}(K)_{g'})$. For any $C_0\ge 1$ and $\varepsilon_0>0$, we have the inclusion map

$$\mathrm{j}_{(C_0,\varepsilon_0)}\colon \Sigma^a_{m,(g,C_0,\varepsilon_0)}\to\Sigma^{ac_a}_{m,(g',C_0c_a,\varepsilon_0c_a)}.$$

In addition, let $S^a_{\varepsilon,g}$ be the subspace of $S_{\varepsilon,g}$ which consists of $(\sigma_i)_{i=1,2}$ satisfying $|(\sigma_i)'(t)|_g\le c_a^{-1}$ for $i=1,2$. If ε is sufficiently small, we have the inclusion map

$$\mathrm{i}_\varepsilon\colon S^a_{\varepsilon,g}\to S_{\varepsilon c_a,g'}.$$

These maps induces a linear map on the homology groups

$$(\mathrm{j}_{(C_0,\varepsilon_0)})_*\colon H^{<a}_*(\varepsilon,\delta)_{(g,C_0,\varepsilon_0)}\to H^{<ac_a}_*(\varepsilon c_a,(\mathrm{i}_\varepsilon)_*\delta)_{(g',C_0c_a,\varepsilon_0c_a)}$$

for $(\varepsilon,\delta)\in\mathscr{T}_{a,(g,C_0,\varepsilon_0)}$ such that $\delta\in C^{\mathrm{dR}}_{n-d}(S^a_{\varepsilon,g})$. Its commutativity with $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon'\le\varepsilon}$ can be proved as in Lemma 4.18. Let us write the induced map on the limits of $\varepsilon\to 0$ by

$$\mathfrak{J}^a_{(C_0,\varepsilon_0)}\colon H^{<a}_*(Q,K)_{(g,C_0,\varepsilon_0)}\to H^{<ac_a}_*(Q,K)_{(g',C_0c_a,\varepsilon_0c_a)}.$$

Moreover, it holds naturally that the maps of $\{\mathfrak{J}^a_{(C,\varepsilon_0)}\}_{C_0\ge 1,\varepsilon_0>0}$ commute with those of $\{\mathfrak{I}^a_{\varepsilon'_0,\varepsilon_0}\}_{\varepsilon'_0\le\varepsilon_0}$ and $\{\mathfrak{I}^a_{C_0,C'_0}\}_{C_0\le C'_0}$, so we get a map on the limit of $\varepsilon_0\to 0$ and $C_0\to\infty$,

$$\mathfrak{J}^a := \varinjlim_{C_0\to\infty}\varprojlim_{\varepsilon_0\to 0}\mathfrak{J}^a_{(C_0,\varepsilon_0)}\colon \check{H}^{<a}_*(Q,K)_g\to\check{H}^{<c_aa}_*(Q,K)_{g'}.$$

Lastly, $\{\mathfrak{J}^a\}_{a\in\mathbb{R}_{>0}\setminus(\mathscr{L}(K)_g\cup\mathscr{L}(K)_{g'})}$ is compatible with $\{\check{I}^{a,b}_g\}_{a\le b}$, so it induces a map on the limit of $a\to\infty$,

$$\mathfrak{J}\colon \check{H}^{\mathrm{string}}_*(Q,K)_g\to\check{H}^{\mathrm{string}}_*(Q,K)_{g'}.$$

If we exchange g and g', we can also define the map $(\mathfrak{J}')^a \colon \check{H}_*^{<a}(Q,K)_{g'} \to \check{H}_*^{<ac_a}(Q,K)_g$ for $a \in \mathbb{R}_{>0} \setminus (\mathcal{L}(K)_g \cup \mathcal{L}(K)_{g'})$. For $b := ac_a$, we have

$$(\mathfrak{J}')^b \circ \mathfrak{J}^a = \check{I}_g^{a,bc_b}, \quad \mathfrak{J}^b \circ (\mathfrak{J}')^a = \check{I}_{g'}^{a,bc_b}.$$

Therefore, $\varinjlim_{a\to\infty} (\mathfrak{J}')^a$ is the inverse map of $\mathfrak{J}$. This proves the independence on g.

This completes the proof of the independence on auxiliary data.

Finally, let us prove the invariance under changing the orientation of K. Suppose that $K = \bigsqcup_{\alpha\in A} K_\alpha$ for connected components $\{K_\alpha\}_{\alpha\in A}$. Then $N_\varepsilon = \bigsqcup_{\alpha\in A} N_{\varepsilon,\alpha}$ and $S_\varepsilon = \bigsqcup_{\alpha\in A} S_{\varepsilon,\alpha}$, where $N_{\varepsilon,\alpha}$ is a tubular neighborhood of K_α and $S_{\varepsilon,\alpha}$ consists of pairs of paths in $N_{\varepsilon,\alpha}$. In addition, for every $\alpha_1,\dots,\alpha_{2m} \in A$, let $\Sigma^a_{m,(\alpha_1,\dots,\alpha_{2m})}$ be the subspace of Σ^a_m consisting of $(\gamma_k \colon [0,T_k] \to Q)_{k=1,\dots,m}$ such that $\gamma_k(0) \in K_{\alpha_{2k-1}}$ and $\gamma_k(T_k) \in K_{\alpha_{2k}}$ for $k = 1,\dots,m$. Then $C^{\mathrm{dR}}_{n-d}(S_\varepsilon) = \bigoplus_{\alpha\in A} C^{\mathrm{dR}}_{n-d}(S_{\varepsilon,\alpha})$ and $C^{\mathrm{dR}}_*(\Sigma^a_m) = \bigoplus_{\alpha_1,\dots,\alpha_{2m}\in A} C^{\mathrm{dR}}_*(\Sigma^a_{m,(\alpha_1,\dots,\alpha_{2m})})$.

For any subset $B \subset A$, let K_B be an oriented submanifold obtained from K by reversing the orientations of $\{K_\alpha\}_{\alpha\in B}$. For every $\delta = \sum_{\alpha\in A} \delta_\alpha \in C^{\mathrm{dR}}_{n-d}(S_\varepsilon)$ (where δ_α is a chain in $S_{\varepsilon,\alpha}$), let us write $\delta_B := \sum_{\alpha\in A\setminus B} \delta_\alpha - \sum_{\alpha\in B} \delta_\alpha$. By using the notation

$$s(\alpha_1,\dots,\alpha_{2m}) := \#\{k \in \{1,\dots,m\} \mid \alpha_{2k} \in B\},$$

we define a linear map $F^a_\varepsilon \colon C^{<a}_*(\varepsilon) \to C^{<a}_*(\varepsilon)$ such that $F^a_\varepsilon(x) = (-1)^{s(\alpha_1,\dots,\alpha_{2m})}x$ for every $x \in C^{\mathrm{dR}}_{*-m(d-2)}(\Sigma^a_{m,(\alpha_1,\dots,\alpha_{2m})}, \Sigma^0_{m,(\alpha_1,\dots,\alpha_{2m})})$ for $m \geq 1$, and $F^a_\varepsilon(x) = x$ for $x \in C^{\mathrm{dR}}_*(\Sigma^a_0, \Sigma^0_0)$. For every $(\varepsilon,\delta) \in \mathcal{T}_a$, (ε,δ_B) satisfies the conditions of Definition 4.4 for (Q, K_B), and F^a_ε is a chain map from $(C^{<a}_*(\varepsilon), D_\delta)$ to $(C^{<a}_*(\varepsilon), D_{\delta_B})$. Moreover, F^a_ε is compatible with the $\star$-operation. By taking the limit of $\varepsilon \to 0$ and $a \to \infty$, we obtain an isomorphism of graded $\mathbb{R}$-algebras

$$F \colon H^{\mathrm{string}}_*(Q,K) \to H^{\mathrm{string}}_*(Q,K_B).$$

Remark 4.19 Similarly, one can prove the invariance of $H^{\mathrm{string}}_*(Q,K)$ under changing the orientation of Q.

Proposition 4.20 *$H^{\mathrm{string}}_*(Q,K)$ is invariant under a C^∞ isotopy of K.*

Proof For two oriented compact submanifolds K_0 and K_1 of Q, suppose that there exists a C^∞ family of embeddings $\{f_t \colon K_0 \to Q\}_{t\in[0,1]}$ such that f_0 is the inclusion map of K_0 and $f_1(K_0) = K_1$. Then this isotopy can be extended to an ambient isotopy $\{F_t\}_{t\in[0,1]}$ such that $F_0 = \mathrm{id}_Q$ and $F_1(K_0) = K_1$. Since F_1 is an isometry from $(Q,(F_1)^*g)$ to (Q,g), it naturally induces an isomorphism

$$H^{\mathrm{string}}_*(Q,K_0)_{(F_1)^*g} \to H^{\mathrm{string}}_*(Q,K_1)_g.$$

The assertion follows from the independence on the Riemannian metric on Q. ◻

5 Examples

In this section we determine the algebraic structure of $H_*^{\mathrm{string}}(Q,K)$ for two examples when $Q=\mathbb{R}^{2d-1}$ with $d\geq 2$. These examples are higher-dimensional generalizations of the Hopf link and the unlink in $\mathbb{R}^3$.

The manifold $Q=\mathbb{R}^{2d-1}$ has the standard orientation. Let us use the coordinate $(z_0,z_1,z_2)\in\mathbb{R}^{d-1}\times\mathbb{R}\times\mathbb{R}^{d-1}=\mathbb{R}^{2d-1}$. The unit sphere $S^{d-1}\subset\mathbb{R}^d$ is oriented as the boundary of the unit ball. We consider three ways of embedding the unit sphere $S^{d-1}\subset\mathbb{R}^d$ into $\mathbb{R}^{2d-1}$:

$$
\begin{aligned}
S^{d-1}\subset\mathbb{R}^d&=\mathbb{R}^{d-1}\times\mathbb{R}\to\mathbb{R}^{2d-1}, &&(z_0,z_1)\mapsto(z_0,z_1,0),\\
S^{d-1}\subset\mathbb{R}^d&=\mathbb{R}\times\mathbb{R}^{d-1}\to\mathbb{R}^{2d-1}, &&(z_1,z_2)\mapsto(0,z_1+1,z_2),\\
S^{d-1}\subset\mathbb{R}^d&=\mathbb{R}^{d-1}\times\mathbb{R}\to\mathbb{R}^{2d-1}, &&(z_0,z_1)\mapsto(z_0,z_1,z_2^*),
\end{aligned}
$$

for a fixed vector $z_2^*\in\mathbb{R}^{d-1}\setminus\{0\}$. Their images are written as K_0, K_1, K_2 in order. These submanifolds are oriented so that the diffeomorphisms $S^{d-1}\to K_i$ from the above maps change the sign of the orientation by $(-1)^{d-1}$.

As notation, given a set $\mathcal{S}$ and a map $\mathcal{S}\to\mathbb{Z}$, $s\mapsto|s|$, let $\mathcal{A}_*(\mathcal{S})$ denote the unital noncommutative graded $\mathbb{R}$-algebra freely generated by $\mathcal{S}$ such that $s\in\mathcal{A}_{|s|}(\mathcal{S})$ for every $s\in\mathcal{S}$.

5.1 Computation of $H_*^{\mathrm{string}}(\mathbb{R}^{2d-1},K_0\cup K_1)$

Let us define $\mathcal{A}_*^{\mathrm{Hopf}}:=\mathcal{A}_*(\mathscr{C}\cup\mathscr{D}\cup\mathscr{E})$ by the three sets

$$
\mathscr{C}:=\{c^0_{i,j}\}_{i\neq j}\cup\{c^1_{i,i}\}_i\cup\{c^1_{i,j},\bar c^1_{i,j}\}_{i\neq j}\cup\{c^2_{i,j}\}_{i,j},\quad \mathscr{D}:=\{d^1_{i,i}\}_i\cup\{d^2_{i,j}\}_{i,j},\quad \mathscr{E}:=\{e^1_{i,i}\}_i\cup\{e^2_{i,j}\}_{i,j},
$$

where i and j run over $\{0,1\}$. The degree of each element is given by

$$
|c^0_{i,j}|=d-2,\qquad |c^1_{i,j}|=|\bar c^1_{i,j}|=2d-3\quad\text{for } i\neq j,
$$
$$
|c^1_{i,i}|=2d-3,\quad |c^2_{i,j}|=3d-4,\quad |d^1_{i,i}|=2d-3,\quad |d^2_{i,j}|=3d-4,\quad |e^1_{i,i}|=2d-4,\quad |e^2_{i,j}|=3d-5.
$$

We define a graded derivation $\partial\colon\mathcal{A}_*^{\mathrm{Hopf}}\to\mathcal{A}_{*-1}^{\mathrm{Hopf}}$ such that

$$
\partial c^0_{i,j}=0,\quad \partial c^1_{i,i}=\partial c^1_{i,j}=\partial\bar c^1_{i,j}=0,\quad \partial c^2_{i,j}=0,\quad \partial d^1_{i,i}=e^1_{i,i},\quad \partial d^2_{i,j}=e^2_{i,j},\quad \partial e^1_{i,i}=0,\quad \partial e^2_{i,j}=0,
$$

and they are extended by the Leibniz rule. We also define another graded derivation $F:\mathcal{A}_*^{\mathrm{Hopf}}\to\mathcal{A}_{*-1}^{\mathrm{Hopf}}$ such that

$$
Fc^0_{i,j}=Fc^1_{i,j}=F\bar c^1_{i,j}=0\quad\text{for } i\neq j,
$$

$$
\begin{aligned}
Fc^1_{0,0}&=(-1)^d e^1_{0,0}+(-1)^d c^0_{0,1}c^0_{1,0}, & Fc^1_{1,1}&=(-1)^d e^1_{1,1}+c^0_{1,0}c^0_{0,1},\\
Fc^2_{0,0}&=-e^2_{0,0}-(\bar c^1_{0,1}c^0_{1,0}+(-1)^d c^1_{0,1}c^0_{1,0}), & Fc^2_{1,1}&=-e^2_{1,1}-((-1)^d\bar c^1_{1,0}c^0_{0,1}+c^1_{1,0}c^0_{0,1}),\\
Fc^2_{0,1}&=-e^2_{0,1}, & Fc^2_{1,0}&=-e^2_{1,0},\\
Fd^1_{i,i}&=0, & Fd^2_{i,j}&=0,\\
Fe^1_{i,i}&=0, & Fe^2_{i,j}&=0,
\end{aligned}
$$

and they are extended by the Leibniz rule. It is easy to see that $\partial\circ\partial=0$, $\partial\circ F+F\circ\partial=0$ and $F\circ F=0$. Therefore, we obtain a differential graded $\mathbb{R}$-algebra $(\mathscr{A}_*^{\mathrm{Hopf}},\partial+F)$. Note that the differential graded $\mathbb{R}$-algebra $(\mathscr{A}_*^{\mathrm{Hopf}},\partial)$ is obtained from $(\mathscr{A}_*(\mathscr{C}),0)$ by *stabilizations* (see [13, Definition 3.9]). Thus,

$$(\mathscr{A}_*(\mathscr{C}),0)\xrightarrow{i}(\mathscr{A}_*^{\mathrm{Hopf}},\partial)\xleftarrow{\tau}(\mathscr{A}_*(\mathscr{C}),0), \tag{33}$$

where i is the inclusion map and τ is the projection map, are quasi-isomorphisms. For the proof, see [13, Corollary 3.11].

The goal of this section is to prove the next theorem.

Theorem 5.1 *There exists an isomorphism of graded* $\mathbb{R}$*-algebras*

$$H_*(\mathscr{A}_*^{\mathrm{Hopf}},\partial+F)\cong H_*^{\mathrm{string}}(\mathbb{R}^{2d-1},K_0\cup K_1).$$

To compute $H_*^{\mathrm{string}}(Q,K_0\cup K_1)$, we fix auxiliary data such that g is the standard Riemannian metric on $\mathbb{R}^{2d-1}$. The constant C_0 is required to be $C_0>3$. The other data, ε_0 and μ, are not specified. The proof is divided into three steps.

Step 1 We first observe de Rham chains in $C_*^{\mathrm{dR}}(\Sigma_m^a,\Sigma_m^0)$. We define a map

$$\varphi\colon(K_0\cup K_1)^2\to\Omega_{K_0\cup K_1}(\mathbb{R}^{2d-1})$$

such that each $(p,p')\in(K_0\cup K_1)^2$ is mapped to a path of a segment

$$\varphi(p,p')\colon[0,1]\to\mathbb{R}^{2d-1},\quad t\mapsto(1-t)p+tp'.$$

We fix two points $p_0:=(0,1,0)\in K_0$ and $p_1:=(0,0,0)\in K_1$. Then we define de Rham chains

$$\begin{aligned}
x_{i,j}^0&:=[\{(p_i,p_j)\},\varphi|_{\{(p_i,p_j)\}},1]\in C_0^{\mathrm{dR}}(\Sigma_1^a,\Sigma_1^0)&&\text{if } i\neq j,\\
x_{i,i}^1&:=[\{p_i\}\times K_i,\varphi|_{\{p_i\}\times K_i},1]\in C_{d-1}^{\mathrm{dR}}(\Sigma_1^a,\Sigma_1^0),\\
x_{i,j}^1&:=[\{p_i\}\times K_j,\varphi|_{\{p_i\}\times K_j},1]\in C_{d-1}^{\mathrm{dR}}(\Sigma_1^a,\Sigma_1^0)&&\text{if } i\neq j,\\
\bar{x}_{i,j}^1&:=[K_i\times\{p_j\},\varphi|_{K_i\times\{p_j\}},1]\in C_{d-1}^{\mathrm{dR}}(\Sigma_1^a,\Sigma_1^0)&&\text{if } i\neq j,\\
x_{i,j}^2&:=[K_i\times K_j,\varphi|_{K_i\times K_j},1]\in C_{2d-2}^{\mathrm{dR}}(\Sigma_1^a,\Sigma_1^0).
\end{aligned}\tag{34}$$

Here, $a>3$ and i and j run over $\{0,1\}$. Obviously, they are cycle chains for ∂. We write the set of these chains as

$$\mathscr{X}:=\{x_{i,j}^0\}_{i\neq j}\cup\{x_{i,i}^1\}_i\cup\{x_{i,j}^1,\bar{x}_{i,j}^1\}_{i\neq j}\cup\{x_{i,j}^2\}_{i,j}.$$

If we define a function $\mathfrak{l}\colon\mathscr{X}\to\mathbb{R}_{>0}$ by

$$\mathfrak{l}(x_{i,j}^0)=\mathfrak{l}(x_{i,j}^1)=\mathfrak{l}(\bar{x}_{i,j}^1)=1\quad\text{for } i\neq j,\qquad\mathfrak{l}(x_{i,i}^1)=\mathfrak{l}(x_{i,i}^2)=2,\qquad\mathfrak{l}(x_{i,j}^2)=3\quad\text{for } i\neq j,$$

then each $x\in\mathscr{X}$ satisfies $x\in C_*^{\mathrm{dR}}(\Sigma_1^{\mathfrak{l}(x)+\varepsilon},\Sigma_1^0)$ for any $\varepsilon>0$.

For every $a\in\mathbb{R}_{>0}$ and $m\in\mathbb{Z}_{\geq1}$, let us consider the manifold

$$B_m^a:=\left\{(q_1^0,q_1^1,\dots,q_m^0,q_m^1)\in(K_0\cup K_1)^{2m}\;\middle|\;\sum_{k=1}^m|q_k^0-q_k^1|<a\text{ or }\min_{1\leq k\leq m}|q_k^0-q_k^1|<\varepsilon_0\right\}.$$

This is homotopy equivalent to Σ^a_m by two smooth maps

$$\begin{aligned} \pi_m\colon \Sigma^a_m \to B^a_m, &\quad (\gamma_k\colon [0,T_k] \to \mathbb{R}^{2d-1})_{k=1,\dots,m} \mapsto (\gamma_1(0),\gamma_1(T_1),\dots,\gamma_m(0),\gamma_m(T_m)), \\ i_m\colon B^a_m \to \Sigma^a_m, &\quad (q^0_1,q^1_1,\dots,q^0_m,q^1_m) \mapsto (\varphi(q^0_k,q^1_k))_{k=1,\dots,m}, \end{aligned}$$

for which $\pi_m \circ i_m = \mathrm{id}_{B^a_m}$ and $i_m \circ \pi_m$ is homotopic to $\mathrm{id}_{\Sigma^a_m}$ (see Lemma 3.2).

Notation In this section, if N is a submanifold of a manifold M, then the inclusion map $N \to M$ is denoted by ι_N.

Lemma 5.2 *Let M be an oriented manifold with open submanifold N. Suppose that there exists an approximately smooth function $f\colon M \to \mathbb{R}$ such that $N = f^{-1}((-\infty,0))$. In addition, assume that $H^{\mathrm{sing}}_*(M,N)$ has a finite dimension. Then there exists an isomorphism between $H^{\mathrm{sing}}_*(M,N)$ and $H^{\mathrm{dR}}_*(M,N)$ such that, for every closed oriented k-dimensional submanifold K of M, the fundamental class $[K] \in H^{\mathrm{sing}}_k(M,N)$ corresponds to $(-1)^{s(k)}[K,\iota_K,1] \in H^{\mathrm{dR}}_k(M,N)$, where $s(k) := \frac{1}{2}(k-\dim M)(k-\dim M-1)$.*

Proof We consider the correspondence through the isomorphisms

$$H^{\mathrm{sing}}_*(M,N) \cong H^{\dim M-*}_{c,\mathrm{dR}}(M,N) \cong H^{\mathrm{dR}}_*(M^{\mathrm{reg}},N^{\mathrm{reg}}) \to H^{\mathrm{dR}}_*(M,N).$$

The first isomorphism is defined by Poincaré duality. The second isomorphism was given in Example 2.6. The last isomorphism is induced by $\mathrm{id}_M\colon M^{\mathrm{reg}} \to M$ [15, Proposition 5.2]. Let us identify the tubular neighborhood N_K of K with the normal bundle of K. Then $[K] \in H^{\mathrm{sing}}_k(M,N)$ corresponds to $[\eta] \in H^{\dim M-k}_{c,\mathrm{dR}}(M,N)$, where $\eta \in \Omega^{\dim M-k}_c(M)$ has its support in N_K and represents the Thom class of the normal bundle. Recalling Example 2.6, we can see that this cohomology class corresponds to $(-1)^{s(k)}[M,\mathrm{id}_M,\eta] = (-1)^{s(k)}[N_K,\mathrm{id}_{N_K},\eta] \in H^{\mathrm{dR}}_k(M^{\mathrm{reg}},N^{\mathrm{reg}})$. Let $\pi_{N_K}\colon N_K \to K$ be the bundle projection. Then, as a de Rham chain in $C^{\mathrm{dR}}_k(M,N)$, $[N_K,\iota_{N_K},\eta]$ is homologous to $[N_K,\iota_K\circ\pi_{N_K},\eta]$ since $\iota_{N_K}\colon N_K \to M$ is homotopic to $\iota_K\circ\pi_{N_K}\colon N_K \to K \subset M$. Now the assertion follows since

$$[N_K,\iota_K\circ\pi_{N_K},\eta] = [K,\iota_K,(\pi_{N_K})_!\eta] = [K,\iota_K,1] \in H^{\mathrm{dR}}_k(M,N). \qquad \square$$

Let us see that a basis of $H^{\mathrm{dR}}_*(\Sigma^a_m,\Sigma^0_m)$ is given by $\mathscr{X}$ and the $\star$-operation. First, suppose that $m=1$ and $a>3$. Then $B^a_1 = K\times K$. Through the isomorphism $H^{\mathrm{dR}}_*(B^a_1,B^0_1) \cong H^{\mathrm{sing}}_*(B^a_1,B^0_1)$ of Lemma 5.2, $\{(\pi_1)_*[x] \mid x \in \mathscr{X}\}$ corresponds to the set of singular homology classes

$$\bigl\{[\{(p_i,p_j)\}]\bigr\}_{i\neq j} \cup \bigl\{[\{p_i\}\times K_i]\bigr\}_i \cup \bigl\{[\{p_i\}\times K_j]\bigr\}_{i\neq j} \cup \bigl\{[K_i\times\{p_j\}]\bigr\}_{i\neq j} \cup \{[K_i\times K_j]\}_{i,j},$$

which is a basis of $H^{\mathrm{sing}}_*(K\times K,B^0_1)$. Therefore, $\{[x] \mid x \in \mathscr{X}\}$ is a basis of $H^{\mathrm{dR}}_*(\Sigma^a_1,\Sigma^0_1)$ for $a>3$.

For $a \in (0,3]$, we consider the deformation along the negative gradient vector field of a C^∞ function $E_{i,j}\colon (K_i\cup K_j)^2 \to \mathbb{R}\colon (p,p') \to |p-p'|^2$. If $i=j$, then $\max E_{i,j} = 4$, $\min E_{i,j} = 0$, and the subset $\{(p,p') \in K_i\times K_i \mid |p-p'|<a\}$ for $a \le 2$ has $E^{-1}_{i,i}(0) = \{(p,p') \in K_i\times K_i \mid p=p'\}$ (the diagonal) as a deformation retract. If $i\neq j$, then $\max E_{i,j} = 9$, $\min E_{i,j} = 1$, and the subset $\{(p,p') \in K_i\times K_j \mid |p-p'|<a\}$ for $a \le 3$ has $E^{-1}_{i,j}(0) = (K_i\times\{p_j\}) \cup (\{p_i\}\times K_j)$ (a bouquet) as

a deformation retract. From these observations, we can see that $\{[x] \mid x \in \mathscr{X}, \mathfrak{l}(x) < a\}$ is a basis of $H_*^{\mathrm{dR}}(\Sigma_1^a, \Sigma_1^0)$. In general, for any $m \in \mathbb{Z}_{\geq 1}$ and $a \in \mathbb{R}_{>0}$,

$$\{[x_1 \star \cdots \star x_m] \mid x_1, \dots, x_m \in \mathscr{X}, \mathfrak{l}(x_1) + \cdots + \mathfrak{l}(x_m) < a\} \tag{35}$$

is a basis of $H_*^{\mathrm{dR}}(\Sigma_m^a, \Sigma_m^0)$.

We fix a trivialization $h\colon \mathbb{O}_{\varepsilon_0} \times (K_0 \cup K_1) \to N_{\varepsilon_0}$ such that $h(w, p_0) = p_0 + (0, w)$ and $h(w, p_1) = p_1 + (-w, 0)$ for every $w \in \mathbb{O}_{\varepsilon_0} \subset \mathbb{R}^d$. (The orientations of K_0 and K_1 are chosen so that this map preserves orientations.)

Suppose that $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K_0 \cup K_1)$ $(= \mathbb{R}_{>0} \setminus \mathbb{Z}_{\geq 1})$. In the following series of three lemmas, we will observe the chains $f_{1,\delta}(x)$ for each $x \in \mathscr{X}$. Hereafter, we assume that $(\varepsilon, \delta) \in \mathscr{T}_a$ is standard with respect to h (see Section 4.4.3).

Lemma 5.3 *For $i \neq j$,*

$$f_{1,\delta}(x_{i,j}^0) = 0, \quad f_{1,\delta}(x_{i,j}^1) = f_{1,\delta}(\bar{x}_{i,j}^1) = 0.$$

Proof For $(p, p') \in K_i \times K_j$ with $i \neq j$ such that either $p = p_i$ or $p' = p_j$, $\varphi(p, p')$ satisfies condition (iii) of Lemma 3.12 for $m = k = 1$. The equations follow from Lemma 3.12. □

Lemma 5.4 *For $i \in \{0, 1\}$, there exist $y_{i,i}^1 \in C_1^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ and $y_{i,i}^2 \in C_d^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ such that*

$$\begin{aligned} \partial y_{0,0}^1 &= f_{1,\delta}(x_{0,0}^1) - x_{0,1}^0 \star x_{1,0}^0, \\ \partial y_{1,1}^1 &= f_{1,\delta}(x_{1,1}^1) - (-1)^d x_{1,0}^0 \star x_{0,1}^0, \\ \partial y_{0,0}^2 &= f_{1,\delta}(x_{0,0}^2) - (\bar{x}_{0,1}^1 \star x_{1,0}^0 + (-1)^d x_{0,1}^1 \star x_{1,0}^0), \\ \partial y_{1,1}^2 &= f_{1,\delta}(x_{1,1}^2) - ((-1)^d \bar{x}_{1,0}^1 \star x_{0,1}^0 + x_{1,0}^1 \star x_{0,1}^0), \end{aligned} \tag{36}$$

and

$$(f_{1,\delta} + (-1)^d f_{2,\delta})(y_{i,i}^1) = 0, \quad (f_{1,\delta} + (-1)^d f_{2,\delta})(y_{i,i}^2) = 0. \tag{37}$$

Proof We only show the existence of $y_{0,0}^2$ and $y_{0,0}^1$. Replacing (K_0, p_0) by (K_1, p_1), $y_{1,1}^2$ and $y_{1,1}^1$ are constructed in a parallel way, except for the difference of signs.

Since (ε, δ) is standard, δ has the form (31). Using the notation of (13), we can write $f_{1,\delta}(x_{0,0}^2) = [W_1, \Phi_1, \zeta_1]$, where

$$\begin{gathered} W_1 = \{((p, p'), \tau, v) \in (K_0 \times K_0) \times \mathbb{R} \times N_\varepsilon \mid 2\varepsilon < \tau < 1 - 2\varepsilon, (1-\tau)p + \tau p' = v\}, \\ \Phi_1 \colon W_1 \to \Sigma_2^{2+2\varepsilon}, \quad ((p, p'), \tau, v) \mapsto \mathrm{con}_1(\varphi(p, p'), (1, \tau), \psi_\varepsilon(v)), \\ \zeta_1 \in \Omega_c^d(W_1), \quad (\zeta_1)_{((p,p'),\tau,v)} = \rho_\varepsilon(1, \tau) \cdot (h_*(\nu_\varepsilon \times 1))_v. \end{gathered}$$

N_ε is a disjoint union of $h(\mathbb{O}_\varepsilon \times K_0)$ and $h(\mathbb{O}_\varepsilon \times K_1)$. Corresponding to this division, we define $W_i' := \{((p, p'), \tau, v) \in W_1 \mid v \in h(\mathbb{O}_\varepsilon \times K_i)\}$ for $i = 0, 1$. If $((p, p'), \tau, v) \in W_0'$, then τ satisfies condition (ii)

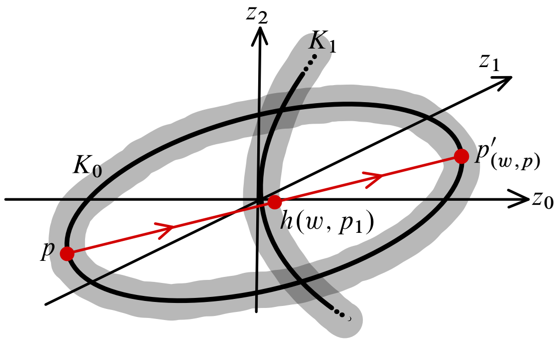

Figure 3: The path $\varphi(p, p'_{(w,p)})$. The gray region is the tubular neighborhood N_ε of $K_0 \cup K_1$.

of Lemma 3.12, so $[W'_0, \Phi_1, \zeta_1] = 0 \in C^{\mathrm{dR}}_{d-1}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$. For $((p, p'), \tau, v) \in W'_1$, we have $\rho_\varepsilon(1, \tau) = 1$. Moreover, there is an diffeomorphism

$$I : \mathbb{O}_\varepsilon \times K_0 \to W'_1, \quad (w, p) \mapsto ((p, p'_{(w,p)}), |p - h(w, p_1)|, h(w, p_1)).$$

Here $p'_{(w,p)} \in K_0 \setminus \{p\}$ is determined by $h(w, p_1) \in \mathrm{Im}(\varphi(p, p'_{(w,p)}))$, as described in Figure 3. The diffeomorphism I preserves orientations and $I^*\zeta_1 = \nu_\varepsilon \times 1$.

Likewise, we consider an explicit description of $f_{1,\delta}(x^1_{0,0})$. Then

$$f_{1,\delta}(x^2_{0,0}) = [\mathbb{O}_\varepsilon \times K_0, \Phi_1 \circ I, \nu_\varepsilon \times 1] \in C^{\mathrm{dR}}_{d-1}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0),$$
$$f_{1,\delta}(x^1_{0,0}) = [\mathbb{O}_\varepsilon \times \{p_0\}, \Phi_1 \circ I|_{\mathbb{O}_\varepsilon \times \{p_0\}}, \nu_\varepsilon \times 1] \in C^{\mathrm{dR}}_0(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0).$$

Let us define $\widetilde{\Phi}_1 : \mathbb{R} \times (\mathbb{O}_\varepsilon \times K_0) \to \Sigma_2^{2+2\varepsilon}$ as follows: Choose a C^∞ function $\kappa : \mathbb{R} \to [0, 1]$ such that $\kappa(s) = 0$ if $s \le \frac{1}{2}$ and $\kappa(s) = 1$ if $s \ge 1$. For $s \ge \frac{1}{2}$, we define $\widetilde{\Phi}_1(s, (w, p) := \Phi_1 \circ I(\kappa(s) \cdot w, p)$. Then the first path (resp. the second path) of $\widetilde{\Phi}_1\big(\frac{1}{2}, (w, p)\big)$ is equal to $\varphi(p, p_1)$ (resp. $\varphi(p_1, -p)$) up to a reparametrization, so we define $\widetilde{\Phi}(s, (w, p))$ for $s \le \frac{1}{2}$ by interpolating the parametrizations so that $\widetilde{\Phi}(s, (w, p)) = (\varphi(p, p_1), \varphi(p_1, -p))$ for $s \le 0$. Now we define

$$\tilde{y}^2_{0,0} := (-1)^d [\mathbb{R} \times (\mathbb{O}_\varepsilon \times K_0), \widetilde{\Phi}_1, \chi \times (\nu_\varepsilon \times 1)] \in C^{\mathrm{dR}}_d(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0),$$
$$\tilde{y}^2_{0,0} := (-1)^d [\mathbb{R} \times (\mathbb{O}_\varepsilon \times \{p_0\}), \widetilde{\Phi}_1|_{\mathbb{R} \times (\mathbb{O}_\varepsilon \times \{p_0\})}, \chi \times (\nu_\varepsilon \times 1)] \in C^{\mathrm{dR}}_1(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0),$$

where $\chi : \mathbb{R} \to [0, 1]$ is a C^∞ function with compact support such that $\chi \equiv 1$ on $[0, 1]$.

From the constructions of $\tilde{y}^2_{0,0}$ and $\tilde{y}^1_{0,0}$, we can compute their boundary chains as follows: Let us introduce two maps $\varphi_0 : K_0 \to K_0 \times K_0 : p \mapsto (p, -p)$ and $\tilde{\imath}_2 : K_0 \times K_0 \to \Sigma_2^{2+2\varepsilon}, (p, p') \mapsto (\varphi(p, p_1), \varphi(p_1, p'))$. Then

$$\partial \tilde{y}^2_{0,0} = f_{1,\delta}(x^2_{0,0}) - (\tilde{\imath}_2)_*[\mathbb{O}_\varepsilon \times K_0, \varphi_0 \circ \mathrm{pr}_{K_0}, \nu_\varepsilon] = f_{1,\delta}(x^2_{0,0}) - (\tilde{\imath}_2)_*[\varphi_0(K_0), \iota_{\varphi_0(K_0)}, 1],$$
$$\partial \tilde{y}^1_{0,0} = f_{1,\delta}(x^1_{0,0}) - (\tilde{\imath}_2)_*[\mathbb{O}_\varepsilon \times \{p_0\}, \varphi_0 \circ \mathrm{pr}_{\{p_0\}}, \nu_\varepsilon \times 1] = f_{1,\delta}(x^1_{0,0}) - (\tilde{\imath}_2)_*[\{\varphi_0(p_0)\}, \iota_{\{\varphi_0(p_0)\}}, 1].$$

Here we used the condition that $\int_{\mathbb{O}_\varepsilon} \nu_\varepsilon = 1$. Moreover, we can check from the definition that $\widetilde{\Phi}_1(s, (w, p))$ satisfies condition (iii) of Lemma 3.12 for $m = 2$ and $k = 1, 2$. Therefore, $f_{k,\delta}(\tilde{y}^2_{0,0}) = 0$ and $f_{k,\delta}(\tilde{y}^1_{0,0}) = 0$ hold for $k = 1, 2$.

As homology classes in $H_*^{\mathrm{sing}}(K_0 \times K_0)$,

$$[\varphi_0(K_0)] = [K_0 \times \{p_0\}] + (-1)^d [\{p_0\} \times K_0] \in H_{d-1}^{\mathrm{sing}}(K_0 \times K_0),$$
$$[\{\varphi_0(p_0)\}] = [\{(p_0, p_0)\}] \in H_0^{\mathrm{sing}}(K_0 \times K_0).$$

Therefore, by Lemma 5.2, there exist $z_{0,0}^2 \in C_d^{\mathrm{dR}}(K_0 \times K_0)$ and $z_{0,0}^1 \in C_1^{\mathrm{dR}}(K_0 \times K_0)$ such that

$$\partial z_{0,0}^2 = [\varphi_0(K_0), \iota_{\varphi_0(K_0)}, 1] - \big([K_0 \times \{p_0\}, \iota_{K_0 \times \{p_0\}}, 1] + (-1)^d [\{p_0\} \times K_0, \iota_{\{p_0\} \times K_0}, 1]\big),$$
$$\partial z_{0,0}^1 = [\{\varphi_0(p_0)\}, \iota_{\{\varphi_0(p_0)\}}, 1] - [\{(p_0, p_0)\}, \iota_{\{(p_0,p_0)\}}, 1].$$

It is clear from the definition of each $x \in \mathscr{X}$ that

$$(\tilde{\imath}_2)_*[K_0 \times \{p_0\}, \iota_{K_0 \times \{p_0\}}, 1] = \bar{x}_{0,1}^1 \star x_{1,0}^0,$$
$$(\tilde{\imath}_2)_*[\{p_0\} \times K_0, \iota_{\{p_0\} \times K_0}, 1] = x_{0,1}^1 \star x_{1,0}^0, \ (\tilde{\imath}_2)_*[\{(p_0, p_0)\}, \iota_{\{(p_0,p_0)\}}, 1] = x_{0,1}^0 \star x_{1,0}^0.$$

Therefore, $y_{0,0}^2 := \tilde{y}_{0,0}^2 + (\tilde{\imath}_2)_* z_{0,0}^2 \in C_d^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ and $y_{0,0}^1 := \tilde{y}_{0,0}^1 + (\tilde{\imath}_2)_* z_{0,0}^1 \in C_0^{\mathrm{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ satisfy the first and the second equations of (36). Moreover, any path in the image of $\tilde{\imath}_2$ satisfies condition (iii) of Lemma 3.12 for $m = 2$ and $k = 1, 2$. Therefore, $f_{k,\delta}((\tilde{\imath}_2)_* z_{0,0}^2) = 0$ and $f_{k,\delta}((\tilde{\imath}_2)_* z_{0,0}^1) = 0$ hold for $k = 1, 2$, and thus $y_{0,0}^2$ and $y_{0,0}^1$ satisfy (37). □

Lemma 5.5 *There exist $y_{0,1}^2, y_{1,0}^2 \in C_{2d-1}^{\mathrm{dR}}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0)$ such that*

$$\partial y_{0,1}^2 = f_{1,\delta}(x_{0,1}^2), \quad \partial y_{1,0}^2 = f_{1,\delta}(x_{1,0}^2) \tag{38}$$

and

$$(f_{1,\delta} + (-1)^d f_{2,\delta})(y_{0,1}^2) = 0, \quad (f_{1,\delta} + (-1)^d f_{2,\delta})(y_{1,0}^2) = 0. \tag{39}$$

Proof We only show the existence of $y_{0,1}^2$. Exchanging K_0 and K_1, $y_{1,0}^2$ is constructed in a parallel way.

Since (ε, δ) is standard, δ has the form (31). Using the notation of (13), we can write $f_{1,\delta}(x_{0,1}^2) = [W_1, \Phi_1, \zeta_1]$, where

$$W_1 = \{((p, p'), \tau, v) \in (K_0 \times K_1) \times \mathbb{R} \times N_\varepsilon \mid 2\varepsilon < \tau < 1 - 2\varepsilon, (1 - \tau)p + \tau p' = v\},$$
$$\Phi_1 \colon W_1 \to \Sigma_2^{3+2\varepsilon}, \quad ((p, p'), \tau, v) \mapsto \mathrm{con}_1(\varphi(p, p'), (1, \tau), \psi_\varepsilon(v)),$$
$$\zeta_1 \in \Omega_c^d(W_1), \quad (\zeta_1)_{((p,p'),\tau,v)} = \rho_\varepsilon(1, \tau) \cdot (h_*(\nu_\varepsilon \times 1))_v.$$

If $p \in K_0$ is sufficiently close to p_0, then $\varphi(p, p')$ satisfies condition (iii) of Lemma 3.12 for any $p' \in K_1$. Symmetrically, if $p' \in K_1$ is sufficiently close to p_1, then $\varphi(p, p')$ satisfies the same condition for any $p \in K_0$. See Figure 4. Therefore, for any bump function $b \colon K_0 \times K_1 \to \mathbb{R}$ whose support is localized near $(K_0 \times \{p_1\}) \cup (\{p_0\} \times K_1)$, we have $[W_1, \Phi_1, \zeta_1'] = 0$ for

$$\zeta_1' \in \Omega_c^d(W_1), \quad (\zeta_1')_{((p,p'),\tau,v)} = b(p, p') \cdot \rho_\varepsilon(1, \tau) \cdot (h_*(\nu_\varepsilon \times 1))_v.$$

We remark that $[W_1, \Phi_1, \zeta_1']$ is an explicit description (13) of $f_{1,\delta}([K_0 \times K_1, \varphi|_{K_0 \times K_1}, b])$.

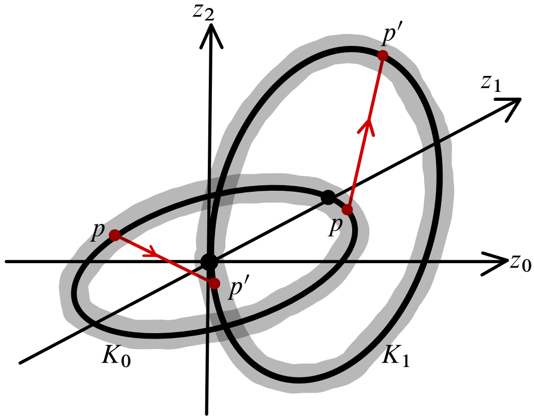

Figure 4: The path $\varphi(p, p')$ when p is close to p_0 or p' is close to p_1. The gray region is the tubular neighborhood N_ε of $K_0 \cup K_1$

Now we choose b so that it is constant to 1 on a neighborhood of $(K_0 \times \{p_1\}) \cup (\{p_0\} \times K_1)$. Then the above computation shows that

$$\begin{aligned} f_{1,\delta}(x^2_{0,1}) &= f_{1,\delta}([K_0 \times K_1, \varphi|_{K_0\times K_1}, 1-b]) + f_{1,\delta}([K_0 \times K_1, \varphi|_{K_0\times K_1}, b]) \\ &= f_{1,\delta}([K'_0 \times K'_1, \varphi|_{K'_0\times K'_1}, 1-b|_{K'_0\times K'_1}]), \end{aligned}$$

where K'_i for $i = 0, 1$ is the complement of a small closed ball containing p_i. Since $K'_0 \times K'_1$ is contractible, there exists a map $R\colon \mathbb{R} \times K'_0 \times K'_1 \to K'_0 \times K'_1$ such that $R(s, \cdot) = \mathrm{id}_{K'_0\times K'_1}$ for $s \geq 1$ and $R(s, \cdot)$ is constant to some point in $K'_0 \times K'_1$ for $s \leq 0$. Using the function χ in the proof of Lemma 5.4, let us define a chain

$$\tilde{x}^2_{0,1} := [\mathbb{R} \times (K'_0 \times K'_1), \varphi \circ R, \chi \times (1 - b|_{K'_0\times K'_1})] \in C^{\mathrm{dR}}_{2d-1}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0).$$

This chain satisfies

$$\partial \tilde{x}^2_{0,1} = [K'_0 \times K'_1, \varphi|_{K'_0\times K'_1}, 1-b|_{K'_0\times K'_1}] + [\mathbb{R} \times (K'_0 \times K'_1), \varphi \circ R, \chi \times db|_{K'_0\times K'_1})].$$

Note that $[K'_0 \times K'_1, \varphi \circ R(0, \cdot), 1-b|_{K'_0\times K'_1}] = 0$ since $\varphi \circ R(0, \cdot)$ is constant. The second chain of the right-hand side is mapped by $f_{1,\delta}$ to 0 since the support of db is localized near $(K_0 \times \{p_1\}) \cup (\{p_0\} \times K_1)$.

Now we define $y^2_{0,1} := f_{1,\delta}(\tilde{x}^2_{0,1}) \in C^{\mathrm{dR}}_{2d-1}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0)$. Then the first equation of (38) holds since $\partial y^2_{0,1} = f_{1,\delta}(\partial \tilde{x}^2_{0,1}) = f_{1,\delta}(x^2_{0,1})$. The first equation (39) follows from $(f_{1,\delta} + (-1)^d f_{2,\delta}) \circ f_{1,\delta} = 0$ (see the proof of Proposition 4.1). □

Step 2 We define a function $\mathfrak{l}\colon \mathscr{C} \cup \mathscr{D} \cup \mathscr{E} \to \mathbb{R}_{>0}$ by

$$\mathfrak{l}(c^0_{i,j}) = \mathfrak{l}(c^1_{i,j}) = \mathfrak{l}(\bar{c}^1_{i,j}) = 1 \quad \text{for } i \neq j, \qquad \mathfrak{l}(c^1_{i,i}) = 2, \qquad \mathfrak{l}(c^2_{i,j}) = \mathfrak{l}(d^2_{i,j}) = \mathfrak{l}(e^2_{i,j}) = \begin{cases} 2 & \text{if } i = j, \\ 3 & \text{if } i \neq j. \end{cases}$$

For every $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K_0 \cup K_1)$, let $\mathscr{A}^{<a}_*$ be an $\mathbb{R}$-subspace of $\mathscr{A}^{\mathrm{Hopf}}_*$ spanned by words of elements in $\mathscr{C} \cup \mathscr{D} \cup \mathscr{E}$ such that the sum of the values of $\mathfrak{l}$ is less than a. Then $(\partial + F)(\mathscr{A}^{<a}_*) \subset \mathscr{A}^{<a}_*$, so we get a subcomplex $(\mathscr{A}^{<a}_*, \partial + F)$.

We continue to use $(\varepsilon,\delta)\in\mathcal{T}_a$ which is standard with respect to h. The second step is to construct a chain map from $(\mathcal{A}_*^{<a},\partial+F)$ to $(C_*^{<a}(\varepsilon),D_\delta)$, and prove that it is a quasi-isomorphism.

Let $y^1_{i,i}$ and $y^2_{i,j}$ be the chains of Lemmas 5.4 and 5.5, which depend on (ε,δ). We define a linear map $\Phi^{<a}_{(\varepsilon,\delta)}\colon \mathcal{A}_*^{<a}\to C_*^{<a}(\varepsilon)$ such that

$$\begin{gathered}\Phi^{<a}_{(\varepsilon,\delta)}(c^0_{i,j}):=x^0_{i,j},\qquad \Phi^{<a}_{(\varepsilon,\delta)}(c^1_{i,j}):=x^0_{i,j},\quad \Phi^{<a}_{(\varepsilon,\delta)}(\bar{c}^1_{i,j}):=\bar{x}^1_{i,j}\quad\text{for } i\neq j,\\ \Phi^{<a}_{(\varepsilon,\delta)}(c^1_{i,i}):=x^1_{i,i},\quad \Phi^{<a}_{(\varepsilon,\delta)}(c^2_{i,j}):=x^2_{i,j},\quad \Phi^{<a}_{(\varepsilon,\delta)}(d^1_{i,i}):=y^1_{i,i},\\ \Phi^{<a}_{(\varepsilon,\delta)}(d^2_{i,j}):=y^2_{i,j},\quad \Phi^{<a}_{(\varepsilon,\delta)}(e^1_{i,i}):=\partial y^1_{i,i},\quad \Phi^{<a}_{(\varepsilon,\delta)}(e^2_{i,j}):=\partial y^1_{i,j},\end{gathered}\tag{40}$$

and extend them naturally by the product map on $\mathcal{A}_*^{\mathrm{Hopf}}$ and the $\star$-operation.

Proposition 5.6 *$\Phi^{<a}_{(\varepsilon,\delta)}$ is a chain map from $(\mathcal{A}_*^{<a},\partial+F)$ to $(C_*^{<a}(\varepsilon),D_\delta)$.*

Proof This follows immediately from the series of three lemmas in Step 1. For each $\xi\in\mathscr{C}\cup\mathscr{D}\cup\mathscr{E}$, $D_\delta\circ\Phi^{<a}_{(\varepsilon,\delta)}(\xi)=(\partial+F)\circ\Phi^{<a}_{(\varepsilon,\delta)}(\xi)$ is proved by

- Lemma 5.3 if $\xi=c^0_{i,j},c^1_{i,j},\bar{c}^1_{i,j}$ with $i\neq j$;
- the equations (36) if $\xi=c^1_{i,i},c^2_{i,i}$;
- the equations (38) if $\xi=c^2_{i,j}$ with $i\neq j$;
- the equations (37) if $\xi=d^1_{i,i},d^2_{i,i},e^1_{i,i},e^2_{i,i}$;
- the equations (39) if $\xi=d^2_{i,j},e^2_{i,j}$ with $i\neq j$. □

Therefore, we have a linear map on the homology groups

$$(\Phi^{<a}_{(\varepsilon,\delta)})_*\colon H_*(\mathcal{A}_*^{<a},\partial+F)\to H_*^{<a}(\varepsilon,\delta).$$

Proposition 5.7 *$(\Phi^{<a}_{(\varepsilon,\delta)})_*$ is an isomorphism.*

Proof We introduce a function $\mathfrak{m}\colon\mathscr{C}\cup\mathscr{D}\cup\mathscr{E}\to\mathbb{Z}_{\geq 1}$ such that $\mathfrak{m}(\mathscr{C})=\{1\}$ and $\mathfrak{m}(\mathscr{D})=\mathfrak{m}(\mathscr{E})=\{2\}$. For every $m\in\mathbb{Z}_{\geq 0}$, let $\mathcal{A}_*^{<a}(m)$ be an $\mathbb{R}$-subspace of $\mathcal{A}_*^{<a}$ generated by words of elements of $\mathscr{C}\cup\mathscr{D}\cup\mathscr{E}$ such that the sum of the values of $\mathfrak{m}$ is equal to m. (When $m=0$, $\mathcal{A}_*^{<a}(0):=\mathbb{R}\cdot 1$.) Then the chain complex $(\mathcal{A}_*^{<a},\partial+F)$ is filtered by subcomplexes $\{\mathcal{G}_p^{<a}\}_{p\in\mathbb{Z}}$ defined by $\mathcal{G}_p^{<a}:=\bigoplus_{m\geq -p}\mathcal{A}_*^{<a}(m)$. Let us consider the spectral sequence determined by this filtration. The $(-m,q)$-term for $m\geq 0$ of its first page is given by

$$H_{q-m}(\mathcal{A}_*^{<a}(m),\partial)\cong H_{q-m}(\mathcal{A}_*^{<a}(m,\mathscr{C}),0)=\mathcal{A}_{q-m}^{<a}(m,\mathscr{C}).$$

Here, $\mathcal{A}_*^{<a}(m,\mathscr{C}):=\mathcal{A}_*^{<a}(m)\cap\mathcal{A}_*(\mathscr{C})$ and the first isomorphism is induced by restricting the quasi-isomorphisms (33). $\Phi^{<a}_{(\varepsilon,\delta)}$ preserves the filtrations $\{\mathcal{G}_p^{<a}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{F}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$. The induced map on the $-m^{\text{th}}$ column for $m\geq 0$ of the first page has the form

$$(\Phi^{<a}_{(\varepsilon,\delta)})_*\colon\mathcal{A}_{*-m}^{<a}(m,\mathscr{C})\to H^{\mathrm{dR}}_{*-m(d-1)}(\Sigma_m^{a+m\varepsilon},\Sigma_m^0).$$

This map is an isomorphism since the basis $\{c_1\cdots c_m \mid c_1,\dots,c_m \in \mathscr{C},\ \mathfrak{l}(c_1)+\cdots+\mathfrak{l}(c_m)<a\}$ of $\mathscr{A}^{<a}_{*-m}(m,\mathscr{C})$ is mapped to the basis (35). In the $-m^{\text{th}}$ column for $m<0$, $(\Phi^{<a}_{(\varepsilon,\delta)})_*$ is a map between the zero vector spaces. The proposition now follows from Lemma 4.2. $\square$

Step 3 The last step is to show that the family of maps

$$\{(\Phi^{<a}_{(\varepsilon,\delta)})_* \mid a\in\mathbb{R}_{>0}\setminus\mathscr{L}(K),\ (\varepsilon,\delta)\in\mathscr{T}_a \text{ is standard with respect to } h\}$$

induces an isomorphism on the limit of $\varepsilon\to 0$ and $a\to\infty$.

Lemma 5.8 *For $(\varepsilon,\delta),(\varepsilon',\delta')\in\mathscr{T}_a$ with $\varepsilon'\le\varepsilon$ which are standard with respect to h,*

$$k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\circ(\Phi^{<a}_{(\varepsilon',\delta')})_* = (\Phi^{<a}_{(\varepsilon,\delta)})_*.$$

Proof We have defined $\Phi^{<a}_{(\varepsilon,\delta)}$ by the chains $\{y^1_{i,i}, y^2_{i,j}\}$, which depends on (ε,δ). As notation, let $\{(y^1_{i,i})', (y^2_{i,j})'\}$ be the corresponding chains constructed from (ε',δ'), by which $\Phi^{<a}_{(\varepsilon',\delta')}$ is defined.

We take $(\bar\varepsilon,\bar\delta)\in\overline{\mathscr{T}}_a$ satisfying (24) for (ε,δ) and (ε',δ'). We may assume that it is standard with respect to h, and thus $\bar\varepsilon=\varepsilon$. As in Lemma 5.3, $\bar f_{1,\bar\delta}(\bar\iota(x))=0$ holds for $x=x^0_{i,j}, x^1_{i,j}, \bar x^1_{i,j}$ with $i\neq j$, where $\bar\iota$ is the map (17). We claim that there exist $[-1,1]$-modeled chains $\bar y^1_{i,i}\in\overline{C}{}^{\mathrm{dR}}_1(\Sigma_2^{2+2\varepsilon},\Sigma_2^0)$, $\bar y^2_{i,i}\in\overline{C}{}^{\mathrm{dR}}_d(\Sigma_2^{2+2\varepsilon},\Sigma_2^0)$ and $\bar y^2_{i,j}\in\overline{C}{}^{\mathrm{dR}}_d(\Sigma_2^{3+2\varepsilon},\Sigma_2^0)$ with $i\neq j$ which satisfy the following equations:

- the variants of the equations (36), (37), (38) and (39) determined by replacing $\{y^1_{i,i}, y^2_{i,j}, f_{k,\delta}, \star\}$ by $\{\bar y^1_{i,i}, \bar y^2_{i,j}, \bar f_{k,\bar\delta}, \bar\star\}$ and $x\in\mathscr{X}$ by $\bar\iota(x)$;
- $e_+\bar y^1_{i,i}=y^1_{i,i}$, $e_+\bar y^2_{i,j}=y^2_{i,j}$, $e_-\bar y^1_{i,i}=(j_{\varepsilon',\varepsilon})_*(y^1_{i,i})'$, and $e_-\bar y^2_{i,j}=(j_{\varepsilon',\varepsilon})_*(y^2_{i,j})'$.

This claim is proved by rewriting the proofs of Lemmas 5.4 and 5.5 for $[-1,1]$-modeled chains. We omit the proof.

We define a linear map $\bar\Phi^{<a}_\varepsilon\colon\mathscr{A}^{<a}_*\to\overline{C}{}^{<a}_*(\varepsilon)$ as in (40) by replacing $x\in\mathscr{X}$ by $\bar\iota(x)$ and $\{y^1_{i,i}, y^2_{i,j}\}$ by $\{\bar y^1_{i,i}, \bar y^2_{i,j}\}$, and extend naturally by the product on $\mathscr{A}^{\mathrm{Hopf}}_*$ and the $\bar\star$-operation. The former equations about $\bar y^1_{i,i}$ and $\bar y^2_{i,j}$ ensure that $\bar\Phi^{<a}_\varepsilon$ is a chain map from $(\mathscr{A}^{<a}_*,\partial+F)$ to $(\overline{C}{}^{<a}_*(\varepsilon),\overline{D}_{\bar\delta})$, as in Proposition 5.6. The latter equations about $\bar y^1_{i,i}$ and $\bar y^2_{i,j}$ ensure the commutativity of the diagram

$$\begin{tikzcd}
 & & H^{<a}_*(\varepsilon,\delta) \\
H_*(\mathscr{A}^{<a}_*,\partial+F) \arrow[urr, bend left, "(\Phi^a_{(\varepsilon,\delta)})_*"] \arrow[r, "(\bar\Phi^a_\varepsilon)_*"] \arrow[drr, bend right, "(\Phi^a_{(\varepsilon',\delta')})_*"'] & \overline{H}{}^{<a}_*(\varepsilon,\bar\delta) \arrow[ur, "f_{(\varepsilon,\bar\delta),+}"] \arrow[dr, "f_{(\varepsilon,\bar\delta),-}"'] & \\
 & & H^{<a}_*(\varepsilon',\delta') \arrow[uu, "k_{(\varepsilon',\delta'),(\varepsilon,\delta)}"']
\end{tikzcd}$$

$\square$

Therefore, the family of maps $\{(\Phi^{<a}_{(\varepsilon,\delta)})_* \mid (\varepsilon,\delta) \in \mathcal{T}_a$ is standard with respect to $h\}$ induces an isomorphism on the limit of $\varepsilon \to 0$,

$$\Phi^{<a}_*\colon H_*(\mathcal{A}^{<a}_*, \partial + F) \to H^{<a}_*(\mathbb{R}^{2d-1}, K_0 \cup K_1)$$

for every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K_0 \cup K_1)$. Moreover, $\{\Phi^{<a}_*\}_{a\in\mathbb{R}_{>0}\setminus\mathcal{L}(K_0\cup K_1)}$ is naturally compatible with $\{I^{a,b}\}_{a\le b}$ and the family of linear maps

$$\{H_*(\mathcal{A}^{<a}_*, \partial + F) \to H_*(\mathcal{A}^{<b}_*, \partial + F)\}_{a\le b}$$

which is induced by the inclusion map $\mathcal{A}^{<a}_* \to \mathcal{A}^{<b}_*$. Therefore, on the limit of $a \to \infty$, we obtain an isomorphism

$$H_*(\mathcal{A}^{\mathrm{Hopf}}_*, \partial + F) = \varinjlim_{a\to\infty} H_*(\mathcal{A}^{<a}_*, \partial + F) \to H^{\mathrm{string}}_*(\mathbb{R}^{2d-1}, K_0 \cup K_1).$$

This finishes the proof of Theorem 5.1.

5.2 Computation of $H^{\mathrm{string}}_*(\mathbb{R}^{2d-1}, K_0 \cup K_2)$

We define $\mathcal{A}^{\mathrm{unlink}}_* := \mathcal{A}_*(\mathcal{C}')$ by the set

$$\mathcal{C}' := \{c^0_{i,j}\}_{i\ne j} \cup \{c^1_{i,i}\}_i \cup \{c^1_{i,j}, \bar{c}^1_{i,j}\}_{i\ne j} \cup \{c^2_{i,j}\}_{i,j},$$

where i and j run over $\{0,2\}$. The degree of each element is given by

$$|c^0_{i,j}| = d-2, \qquad |c^1_{i,j}| = |c^1_{i,j}| = |\bar{c}^1_{i,j}| = 2d-3 \quad \text{for } i \ne j, \qquad |c^1_{i,i}| = 2d-3, \qquad |c^2_{i,j}| = 3d-4.$$

(Obviously, there exists an isomorphism $\mathcal{A}_*(\mathcal{C}) \cong \mathcal{A}^{\mathrm{unlink}}_*$ as graded $\mathbb{R}$-algebras.) We define a graded derivation $\partial := 0\colon \mathcal{A}^{\mathrm{unlink}}_* \to \mathcal{A}^{\mathrm{unlink}}_{*-1}$. For a differential graded algebra $(\mathcal{A}^{\mathrm{unlink}}_*, \partial)$, we have the following result.

Theorem 5.9 *There exists an isomorphism of graded* $\mathbb{R}$*-algebras*

$$H_*(\mathcal{A}^{\mathrm{unlink}}_*, \partial) \cong H^{\mathrm{string}}_*(\mathbb{R}^{2d-1}, K_0 \cup K_2).$$

To compute $H^{\mathrm{string}}_*(\mathbb{R}^{2d-1}, K_0 \cup K_2)$, we fix auxiliary data such that g is the standard Riemannian metric on $\mathbb{R}^{2d-1}$. The constant C_0 is required to be $C_0 > \sqrt{|z^*_2|^2 + 4}$. The other data, ε_0 and μ, are not specified. The strategy of the proof is the same as for Theorem 5.1, but it is much more simple. We only give the outline of each step.

Step 1 We may assume that $|z^*_2| > 2$. Let us fix points $p_0 := (0,1,0) \in K_0$ and $p_2 := (0,1,z^*_2) \in K_2$, and define submanifolds of $(K_0 \cup K_2)^2$,

$$K_{0,2} := \{(p,p') \in K_0 \times K_2 \mid p' = p + (0,0,z^*_2)\}, \quad K_{2,0} := \{(p,p') \in K_2 \times K_0 \mid p' = p - (0,0,z^*_2)\}.$$

Let $\varphi\colon (K_0 \cup K_2)^2 \to \Omega_{K_0\cup K_2}(\mathbb{R}^{2d-1})$ be the map defined as in Section 5.1 by replacing K_1 by K_2. Then we define the set of chains

$$\mathcal{X}' := \{x^0_{i,j}\}_{i\ne j} \cup \{x^1_{i,i}\}_i \cup \{x^1_{i,j}, \bar{x}^1_{i,j}\}_{i\ne j} \cup \{x^2_{i,j}\}_{i,j},$$

where i and j run over $\{0,2\}$, as follows:

$$\begin{aligned}
x^0_{i,j} &:= [\{(p_i,p_j)\}, \varphi|_{\{(p_i,p_j)\}}, 1] \in C^{\mathrm{dR}}_0(\Sigma^a_1, \Sigma^0_1) && \text{if } (i \neq j),\\
x^1_{i,i} &:= [\{p_i\}\times K_i, \varphi|_{\{p_i\}\times K_i}, 1] \in C^{\mathrm{dR}}_{d-1}(\Sigma^a_1, \Sigma^0_1), &&\\
x^1_{i,j} &:= [K_{i,j}, \varphi|_{K_{i,j}}, 1] \in C^{\mathrm{dR}}_{d-1}(\Sigma^a_1, \Sigma^0_1) && \text{if } (i \neq j),\\
\bar{x}^1_{i,j} &:= [K_i\times\{p_j\}, \varphi|_{K_i\times\{p_j\}}, 1] \in C^{\mathrm{dR}}_{d-1}(\Sigma^a_1, \Sigma^0_1) && \text{if } (i \neq j),\\
x^2_{i,j} &:= [K_i\times K_j, \varphi|_{K_i\times K_j}, 1] \in C^{\mathrm{dR}}_{2d-2}(\Sigma^a_1, \Sigma^0_1). &&
\end{aligned}$$

Here, $a > \sqrt{|z^*_2|^2+4}$. If we define $\mathfrak{l}\colon \mathscr{X}' \to \mathbb{R}_{>0}$ by

$$\mathfrak{l}(x^0_{i,j}) = \mathfrak{l}(x^1_{i,j}) = |z^*_2| \quad \text{and} \quad \mathfrak{l}(\bar{x}^1_{i,j}) = \mathfrak{l}(x^2_{i,j}) = \sqrt{|z^*_2|^2+4} \quad \text{for } i \neq j, \qquad \mathfrak{l}(x^1_{i,i}) = \mathfrak{l}(x^2_{i,i}) = 2,$$

then each $x \in \mathscr{X}'$ satisfies $x \in C^{\mathrm{dR}}_*(\Sigma^{\mathfrak{l}(x)+\varepsilon}_1, \Sigma^0_1)$ for any $\varepsilon > 0$. Furthermore, a basis of $H^{\mathrm{dR}}_*(\Sigma^a_m, \Sigma^0_m)$ for $a \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 1}$ is given by the set of homology classes

$$\{[x_1 \star \cdots \star x_m] \mid x_1,\dots,x_m \in \mathscr{X}',\ \mathfrak{l}(x_1)+\cdots+\mathfrak{l}(x_m) < a\}.$$

The reason this case is simpler is the following: for any $x \in \mathscr{X}'$ and $(\varepsilon,\delta) \in \mathscr{T}_a$ with $a > \mathfrak{l}(x)$, the equation

$$f_{1,\delta}(x) = 0 \in C^{\mathrm{dR}}_*(\Sigma^{\mathfrak{l}(x)+2\varepsilon}_2, \Sigma^0_2)$$

holds since the path $\varphi(p,p')$ satisfies condition (iii) of Lemma 3.12 for any $(p,p') \in (K_0 \cup K_2)^2$ (see Lemma 5.3).

Step 2 There exists a bijection $\mathscr{C}' \to \mathscr{X}'$ which maps $c^k_{i,j}$ to $x^k_{i,j}$ and $\bar{x}^1_{i,j}$ to $\bar{c}^1_{i,j}$ for $k \in \{0,1,2\}$ and $i,j \in \{0,2\}$. Composing $\mathfrak{l}\colon \mathscr{X}' \to \mathbb{R}_{>0}$ with this bijection, a function $\mathfrak{l}\colon \mathscr{C}' \to \mathbb{R}_{>0}$ is defined. Similar to $\mathscr{A}^{\mathrm{Hopf}}_*$, $\mathscr{A}^{\mathrm{unlink}}_*$ is filtered by subcomplexes $(\mathscr{A}^{<a}_*, \partial)$ for all $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K_0 \cup K_2)$ which is defined by using $\mathfrak{l}\colon \mathscr{C}' \to \mathbb{R}_{>0}$.

Now $\Phi^{<a}_\varepsilon\colon \mathscr{A}^{<a}_* \to C^{<a}_*(\varepsilon)$ is defined so that $c \in \mathscr{C}'$ is mapped to $x \in \mathscr{X}'$ which corresponds to c via the above bijection, and extended naturally via the product map on $\mathscr{A}^{\mathrm{unlink}}_*$ and the $\star$-operation. It is clear in this case that $\Phi^{<a}_\varepsilon$ is a chain map from $(\mathscr{A}^{<a}_*, \partial = 0)$ to $(C^{<a}_*(\varepsilon), D_\delta)$. The fact that this map is a quasi-isomorphism is proved by a similar argument as in Proposition 5.7 about spectral sequences.

Step 3 We check that the family of maps $(\Phi^{<a}_\varepsilon)_*$ induces an isomorphism on the limit of $\varepsilon \to 0$ and $a \to \infty$. This finishes the proof of Theorem 5.9.

5.3 A corollary and its potential application

The next result is a corollary from the above computations

Corollary 5.10 *As graded $\mathbb{R}$-algebras,*

$$H^{\mathrm{string}}_*(\mathbb{R}^{2d-1}, K_0 \cup K_1) \not\cong H^{\mathrm{string}}_*(\mathbb{R}^{2d-1}, K_0 \cup K_2).$$

Proof From Theorems 5.1 and 5.9, it suffices to show that $H_*(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F)$ is not isomorphic to $\mathscr{A}_*^{\mathrm{unlink}}$ as a graded $\mathbb{R}$-algebra. Let us rewrite $(c^0_{0,1}, c^0_{1,0}, e^1_{0,0}, e^1_{1,1})$ by (a_0, a_1, b_0, b_1) and $(c^0_{0,2}, c^0_{2,0})$ by (a'_0, a'_1). In addition, we define $C_0 := b_0 + a_0 a_1$ and $C_1 := b_1 + (-1)^d a_1 a_0$.

If $d = 2$, $H_0(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F)$ is the a priori noncommutative $\mathbb{R}$-algebra generated by $\{a_0, a_1, b_0, b_1\}$ modulo the ideal generated by $\{b_0, b_1, C_0, C_1\}$. This is isomorphic to the commutative algebra $\mathbb{R}[a_0, a_1]/(a_0 a_1)$. On the other hand, $\mathscr{A}_0^{\mathrm{unlink}}$ is the noncommutative algebra freely generated by $\{a'_0, a'_1\}$. Therefore, $H_0(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F) \not\cong \mathscr{A}_0^{\mathrm{unlink}}$ as $\mathbb{R}$-algebras.

If $d \geq 3$, the lower-degree parts are isomorphic as vector spaces. Indeed, for $p \leq 2d - 5$,

$$H_p(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F) \cong \mathscr{A}_p^{\mathrm{unlink}} \cong \begin{cases} \mathbb{R} & \text{if } p = 0, \\ \mathbb{R}a_0 \oplus \mathbb{R}a_1 & \text{if } p = d - 2, \\ 0 & \text{otherwise.} \end{cases}$$

However, $H_{2d-4}(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F)$ is the $\mathbb{R}$-vector space spanned by $\{a_i a_j \mid i, j \in \{0, 1\}\} \cup \{b_0, b_1\}$ modulo the subspace generated by $\{b_0, b_1, C_0, C_1\}$, so its dimension is equal to 2. On the other hand, $\mathscr{A}_{2d-4}^{\mathrm{unlink}}$ is the $\mathbb{R}$-vector space spanned by $\{a'_i a'_j \mid i, j \in \{0, 1\}\}$, so its dimension is equal to 4. Therefore, $H_{2d-4}(\mathscr{A}_*^{\mathrm{Hopf}}, \partial + F) \not\cong \mathscr{A}_{2d-4}^{\mathrm{unlink}}$ as $\mathbb{R}$-vector spaces. □

Let us see a potential application of this result. First we determine spin structures on unit conormal bundles

Proposition 5.11 *Let Q be an n-dimensional Riemannian manifold with a fixed spin structure. Then, for every submanifold K in Q, we can assign a spin structure on its unit conormal bundle Λ_K so that, if K is isotopic to K' as a submanifold in Q, then Λ_K is isotopic to $\Lambda_{K'}$ as a Legendrian submanifold with a spin structure.*

Proof Let us identify T^*Q with TQ via the Riemannian metric. We also identify Q with the zero section of TQ. Let L_K be the conormal bundle of K. Note that the tangent space of TQ at $(q, 0) \in Q$ is equal to $T_q Q \oplus T_q Q$, where the first component is the tangent space of the base space Q, and the second component is the tangent space of the fiber $T_q Q$. For every $q \in K$, $T_{(q,0)}(L_K) = T_q K \oplus (T_q K)^{\perp}$. Thus the vector bundle $T(L_K)|_K$ has a spin structure induced by $TQ|_K$. Since K is a deformation retract of L_K, this spin structure is extended to $T(L_K)$. By using the diffeomorphism $\mathbb{R}_{>0} \times \Lambda_K \to L_K \setminus K$, $(r, (q, p)) \mapsto (q, r \cdot p)$, we can determine a spin structure on $T(\Lambda_K)$ such that the spin structure on $\underline{\mathbb{R}} \oplus T\Lambda_K$, induced by the inclusion map $\mathrm{Spin}(2n-1) \to \mathrm{Spin}(2n)$, is equal to the spin structure on $T(L_K)|_{\Lambda_K} \cong \underline{\mathbb{R}} \oplus T\Lambda_K$. This spin structure on Λ_K for every submanifold K clearly satisfies the condition of this proposition. □

Let us consider the unit conormal bundles of $K_0 \cup K_1$ and $K_0 \cup K_2$.

Proposition 5.12 *As a $(2d-2)$-dimensional submanifold of $UT^*\mathbb{R}^{2d-1}$ with the spin structure determined by Proposition 5.11, $\Lambda_{K_0 \cup K_1}$ is isotopic to $\Lambda_{K_0 \cup K_2}$.*

Proof For $s \in [0,1]$, we define $K_1^s := \{q + (0, 2s, 0) \in \mathbb{R}^{2d-1} \mid q \in K_1\}$. We also choose a C^∞ function $[0,1] \to [0,\pi]$, $s \mapsto \theta_s$, such that $\theta_0 = \theta_1 = 0$ and $\theta_{1/2} = \frac{\pi}{2}$, and define $R_s \in \mathrm{SO}(2d-1)$ for $s \in [0,1]$ by $R_s(v_0, v_1, v_2) := ((\cos\theta_s)v_0 - (\sin\theta_s)v_2, v_1, (\sin\theta_s)v_0 + (\cos\theta_s)v_2)$ for every $(v_0, v_1, v_2) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1}$. We then define an isotopy $(\Lambda_s)_{s\in[0,1]}$ from $\Lambda_{K_1^0} = \Lambda_{K_1}$ to $\Lambda_{K_1^1}$ by

$$\Lambda_s := \{(q, p) \in UT^*\mathbb{R}^{2d-1} \mid q \in K_s,\ p \circ R_s|_{T_q K_s^1} = 0\}.$$

Λ_s intersects Λ_{K_0} if and only if $s = \frac{1}{2}$, and $\Lambda_{1/2} \cap \Lambda_{K_0} = \Lambda_{K_0} \cap UT^*_{p_0}\mathbb{R}^{2d-1}$, where $p_0 = (0,1,0) \in \mathbb{R}^{2d-1}$. We can slightly perturb $(\Lambda_s)_{s\in[0,1]}$ around $s = \frac{1}{2}$ to an isotopy $(\Lambda'_s)_{s\in[0,1]}$ such that Λ'_s does not intersect Λ_{K_0} for every $s \in [0,1]$. This isotopy is homotopic to an isotopy $(\Lambda_{K_1^s})_{s\in[0,1]}$, which preserves the spin structure of Proposition 5.11. In addition, $K_0 \cup K_1^1$ is isotopic to $K_0 \cup K_2$ in $\mathbb{R}^{2d-1}$. Therefore, as a C^∞ submanifold with a spin structure, $\Lambda_{K_0} \cup \Lambda_{K_1}$ is isotopic to $\Lambda_{K_0} \cup \Lambda_{K_2}$. □

If Conjecture 1.4 in the introduction is true, then Corollary 5.10 can be applied to show that the unit conormal bundle $\Lambda_{K_0 \cup K_1}$ is not isotopic to $\Lambda_{K_0 \cup K_2}$ as a Legendrian submanifold with a spin structure in $UT^*\mathbb{R}^{2d-1}$, though they are isotopic as C^∞ submanifolds with spin structures by the above proposition.

6 Cord algebra and $H_0^{\mathrm{string}}(Q, K)$

Throughout this section, we consider the case where the codimension of K is 2 (ie $d = 2$) and the normal bundle $(TK)^\perp$ is trivial. The purpose is to show that $H_0^{\mathrm{string}}(Q, K)$ is isomorphic to an isotopy invariant of K, called *cord algebra*.

6.1 Cord algebra and string homology

In this section, we refer to [6; 19] and give a definition of cord algebra and string homology. Note that, in this paper, their coefficients are reduced from the original $\mathbb{Z}[\pi_1(\partial N_{\varepsilon_0})]$ to $\mathbb{R}$.

We fix a frame of $(TK)^\perp$ to give an isomorphism $\mathbb{R}^2 \times K \cong (TK)^\perp$ of vector bundles over K which preserves their fiber metrics and orientations. Combining with the map (3), we obtain a diffeomorphism

$$h\colon \mathbb{O}_{\varepsilon_0} \times K \to N_{\varepsilon_0},$$

which preserves orientations. Here, $\mathbb{O}_\varepsilon = \left\{w \in \mathbb{R}^2 \mid |w| < \frac{1}{2}\varepsilon\right\}$ for every $\varepsilon \le \varepsilon_0$, as defined in Section 4.4.3.

First, we define an $\mathbb{R}$-algebra $\mathrm{Cord}(Q, K; \mathbb{R})$. Its relation to the cord algebra defined in [6; 19] is discussed later in Remark 6.2. Let us prepare some notation. We fix $w_0 \in \mathbb{O}_{\varepsilon_0} \setminus \{0\}$ and define a submanifold disjoint from K,

$$K' := \{h(w_0, x) \mid x \in K\} \subset N_{\varepsilon_0}.$$

For every $x \in K$, we define $c_x\colon [0,1] \to Q \setminus K$ to be the constant path at $h(w_0, x) \in K'$. We also define $m_x\colon [0,1] \to Q \setminus K$ to be a loop in a punctured disk $h((\mathbb{O}_{\varepsilon_0} \setminus \{0\}) \times \{x\}) \subset N_{\varepsilon_0} \setminus K$ based at $h(w_0, x) \in K'$, whose winding number around $h(0, x)$ is equal to 1. In addition, let $\pi_1(Q \setminus K, K')$ be the set of homotopy classes of continuous paths $\gamma\colon [0,1] \to Q \setminus K$ such that $\gamma(\{0,1\}) \subset K'$.

Definition 6.1 Let $\mathscr{A}$ be the unital noncommutative $\mathbb{R}$-algebra freely generated by the set $\pi_1(Q \setminus K, K')$. We define the two-sided ideal $\mathscr{I}$ generated by the elements

$$[c_x], \quad [\gamma_1 \cdot \gamma_2] - [\gamma_1 \cdot m_x \cdot \gamma_2] - [\gamma_1][\gamma_2]$$

for all $x \in K$ and $\gamma_j : [0,1] \to Q \setminus K$ for $j = 1, 2$ such that $\gamma_j(\{0,1\}) \subset K'$ and $\gamma_1(1) = h(w_0, x) = \gamma_2(0)$. Then we define an $\mathbb{R}$-algebra $\mathrm{Cord}(Q, K; \mathbb{R}) := \mathscr{A}/\mathscr{I}$, and call it the *cord algebra* of (Q, K).

Remark 6.2 When K is 1-dimensional and connected (ie K is an oriented knot in a 3-manifold Q), we fix a basepoint $* \in K'$. A *cord* is a path $\gamma : [0,1] \to Q$ such that $\gamma([0,1]) \cap K = \varnothing$ and $\gamma(0), \gamma(1) \in K' \setminus \{*\}$. The notion of cord algebra (or *cord ring*) of knots was defined in [6; 18; 19], for instance. The most refined one is [6, Definition 2.6], which is defined as a noncommutative algebra over $\mathbb{Z}$ generated by the set of homotopy classes of cords and $\{\lambda^{\pm}, \mu^{\pm}\}$, modulo the relations about $\{\lambda^{\pm}, \mu^{\pm}\}$ and the "skein relations". If we replace both λ and μ by $1 \in \mathbb{Z}$ and tensor this $\mathbb{Z}$-algebra with $\mathbb{R}$, we obtain an $\mathbb{R}$-algebra isomorphic to $\mathrm{Cord}(Q, K; \mathbb{R})$. (The isomorphism is induced by a natural map from the set of homotopy classes of cords to $\pi_1(Q \setminus K, K')$.)

We should also note that, in [19, Definition 2.1], the cord algebra over $\mathbb{Z}[H_1(\partial N_{\varepsilon_0})]$ was defined when K is a connected codimension 2 submanifold of an arbitrary manifold and its normal bundle is oriented. In our setting, we have an isomorphism $(h^{-1})_* : H_1(\partial N_{\varepsilon_0}) \to H_1(S^1 \times K) \cong H_1(S^1) \oplus H_1(K)$. There exists a ring homomorphism $\varphi : \mathbb{Z}[H_1(\partial N_{\varepsilon_0})] \to \mathbb{R}$ determined by $\varphi(h_*([S^1])) = -1$ and $\varphi(h_*(c)) = 1$ for every $c \in H_1(K)$. If the base change of the cord algebra of [19, Definition 2.1] is done by φ, we obtain an $\mathbb{R}$-algebra isomorphic to $\mathrm{Cord}(Q, K; \mathbb{R})$.

Next, we refer to [6, Section 2.1] and define the string homology, which is simplified for our purpose. For $m \in \mathbb{Z}_{\geq 1}$, $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $p = 0, 1$, we define an $\mathbb{R}$-subspace $C_p^{\pitchfork}(m, a) \subset C_p^{\mathrm{sing}}(\Sigma_m^a)$ consisting of generic singular p-chains satisfying jet transversality conditions. (Recall that we have fixed a topology of Σ_m^a in Section 3.2.4.)

In the case of $p = 0$, $C_0^{\pitchfork}(m, a)$ is generated by $(\gamma_k : [0, T_k] \to Q)_{k=1,\dots,m} \in \Sigma_m^a$ satisfying the following conditions:

(0a) $(\gamma_k)'(0), (\gamma_k)'(T_k) \notin TK$ for every $k \in \{1, \dots, m\}$.

(0b) $\gamma_k(t) \notin K$ for every $k \in \{1, \dots, m\}$ and $t \neq 0, T_k$.

In the case of $p = 1$, $C_1^{\pitchfork}(m, a)$ is generated by 1-parameter families of paths

$$[0,1] \to \Sigma_m^a, \quad u \mapsto (\gamma_k^u : [0, T_k^u] \to Q)_{k=1,\dots,m},$$

such that $[0,1] \to \mathbb{R}_{>0}$, $u \mapsto T_k^u$, is a C^∞ function and

$$\Gamma_k : \{(u,t) \mid 0 \leq u \leq 1,\ 0 \leq t \leq T_k^u\} \to Q$$

is a C^∞ map for every $k \in \{1, \dots, m\}$, and satisfies the following conditions:

(1a) $(\gamma_k^0)_{k=1,\ldots,m}$ and $(\gamma_k^1)_{k=1,\ldots,m}$ satisfy (0a)–(0b).

(1b) $(\gamma_k^u)'(0), (\gamma_k^u)'(T_k^u) \notin TK$ for every $u \in [0,1]$ and $\Gamma_k^{\mathrm{int}} := \Gamma_k|_{\{(u,t) \mid t \neq 0, T_k^u\}}$ is transverse to K for every $k \in \{1,\ldots,m\}$.

(1c) If $(u_*,t_*), (u'_*,t'_*) \in \coprod_{k=1}^m (\Gamma_k^{\mathrm{int}})^{-1}(K)$ are distinct points, then $u_* \neq u'_*$.

Note that condition (1b) implies that $(\Gamma_k^{\mathrm{int}})^{-1}(K)$ is a finite set. In addition, we define $C_p^{\pitchfork}(0,a) := C_p^{\mathrm{sing}}(\Sigma_0^a)$ for $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $p = 0, 1$.

By (1a), the boundary operator of the singular chain complex $\partial^{\mathrm{sing}} \colon C_1^{\mathrm{sing}}(\Sigma_m^a) \to C_0^{\mathrm{sing}}(\Sigma_m^a)$ is restricted to the map

$$\partial^{\mathrm{sing}} \colon C_1^{\pitchfork}(m,a) \to C_0^{\pitchfork}(m,a), \quad (\gamma_k^u)_{k=1,\ldots,m}^{u\in[0,1]} \mapsto (\gamma_k^1)_{k=1,\ldots,m} - (\gamma_k^0)_{k=1,\ldots,m}.$$

We also define a linear map $f_k^{\pitchfork} \colon C_1^{\pitchfork}(m,a) \to C_0^{\pitchfork}(m+1,a)$ for $m \in \mathbb{Z}_{\geq 1}$ such that, for any 1-chain $x = (\gamma_k^u \colon [0,T_k^u] \to Q)_{k=1,\ldots,m}^{u\in[0,1]} \in C_1^{\pitchfork}(m,a)$,

$$f_k^{\pitchfork}(x) := \sum_{(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} \mathrm{sign}(u_*,t_*) \cdot (\gamma_1^{u_*} \ldots, \gamma_{k-1}^{u_*}, \widehat{\gamma_k^{u_*}1}, \widehat{\gamma_k^{u_*}2}, \gamma_{k+1}^{u_*}, \ldots, \gamma_m^{u_*}).$$

Here

$$\widehat{\gamma_k^{u_*}1} := \gamma_k^{u_*}|_{[0,t_*]} \colon [0,t_*] \to Q, \quad \widehat{\gamma_k^{u_*}2} := \gamma_k^{u_*}|_{[t_*,T_k^{u_*}]}(\cdot - t_*) \colon [0, T_k^{u_*} - t_*] \to Q,$$

and $\mathrm{sign}(u_*,t_*) \in \{\pm 1\}$ is the orientation sign of the open embedding into $\mathbb{O}_{\varepsilon_0} \subset \mathbb{R}^2$

$$\Gamma_k^{\mathrm{fiber}} := \mathrm{pr}_{\mathbb{R}^2} \circ h^{-1} \circ \Gamma_k$$

defined on a small neighborhood of $(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K) \subset (0,1) \times \mathbb{R}_{>0}$. For convenience, let us define, for $p \notin \{0,1\}$, $C_p^{\pitchfork}(m,a) := 0$, $\partial^{\mathrm{sing}} := 0 \colon C_p^{\pitchfork}(m,a) \to C_{p-1}^{\pitchfork}(m,a)$ and $f_k^{\pitchfork} := 0 \colon C_p^{\pitchfork}(m,a) \to C_{p-1}^{\pitchfork}(m+1,a)$.

For $a \in \mathbb{R}_{>0} \cup \{\infty\}$ and $m \in \mathbb{Z}_{\geq 0}$, we define the quotient vector space

$$C_*^{\pitchfork,<a}(m) := C_*^{\pitchfork}(m,a)/C_*^{\pitchfork}(m,0).$$

Then ∂^{sing} and $f_k^{\pitchfork}$ induce linear maps

$$\partial^{\mathrm{sing}} \colon C_*^{\pitchfork,<a}(m) \to C_{*-1}^{\pitchfork,<a}(m), \quad f_k^{\pitchfork} \colon C_*^{\pitchfork,<a}(m) \to C_{*-1}^{\pitchfork,<a}(m+1).$$

Now we define the graded $\mathbb{R}$-vector space

$$C_*^{\pitchfork,<a} := \bigoplus_{m=0}^{\infty} C_*^{\pitchfork,<a}(m).$$

For each $m \in \mathbb{Z}_{\geq 0}$, $C_*^{\pitchfork,<a}(m)$ is considered to be its linear subspace in a natural way. Then we define a degree -1 map $D^{\pitchfork} \colon C_*^{\pitchfork,<a} \to C_{*-1}^{\pitchfork,<a}$ by

$$D^{\pitchfork}(x) := \partial^{\mathrm{sing}} x + \sum_{k=1}^{m} f_k^{\pitchfork}(x) \tag{41}$$

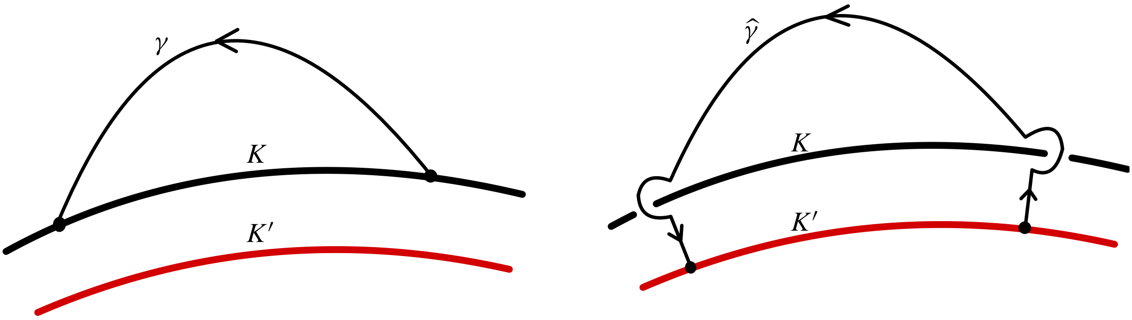

Figure 5: The picture of $\widehat{\gamma}$ defined for $(\gamma)_{k=1} \in \Sigma_1^\infty$.

for $x \in C_*^{\pitchfork,<a}(m)$. For $x \in C_*^{\pitchfork,<a}(0)$, the right-hand side is just equal to $\partial^{\mathrm{sing}} x$. Then we obtain a chain complex $(C_*^{\pitchfork,<a}, D^{\pitchfork})$. Let $H_*^{\pitchfork,<a}$ denote its homology group. In addition, for $a, b \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ with $a \leq b$, we define a linear map $J^{a,b} \colon H_*^{\pitchfork,<a} \to H_*^{\pitchfork,<b}$ induced by the inclusion maps $C_*^{\pitchfork}(m, a) \to C_*^{\pitchfork}(m, b)$ for all $m \in \mathbb{Z}_{\geq 0}$.

In this section, we call $H_*^{\pitchfork,<\infty}$ the *string homology* of (Q, K). Note that the direct limit $\varinjlim_{a\to\infty} H_*^{\pitchfork,<a}$ defined by $\{J^{a,b}\}_{a\leq b}$ is isomorphic to $H_0^{\pitchfork,<\infty}$. Furthermore, $H_0^{\pitchfork,<\infty}$ has a associative product structure induced by $\Pi \colon \Sigma_m^\infty \times \Sigma_{m'}^\infty \to \Sigma_{m+m'}^\infty$. Thus $H_0^{\pitchfork,<\infty}$ is a unital associative $\mathbb{R}$-algebra, whose unit comes from $* \in C_0^{\pitchfork,<\infty}(0)$.

The next proposition is essentially proved in [6, Proposition 2.9].

Proposition 6.3 *As an* $\mathbb{R}$*-algebra,* $\mathrm{Cord}(Q, K; \mathbb{R})$ *is isomorphic to* $H_0^{\pitchfork,<\infty}$.

Proof For every homotopy class $z \in \pi_1(Q \setminus K, K')$, we choose a C^∞ path γ which represents z. We then define a path $\bar{\gamma}$ as follows: for $x_0, x_1 \in K$ such that $\gamma(i) = h(w_0, x_i)$ for $i \in \{0, 1\}$,

$$\bar{\gamma} \colon [0,3] \to Q, \qquad t \mapsto \begin{cases} h(t \cdot w_0, x_0) & \text{if } 0 \leq t \leq 1, \\ \gamma(t-1) & \text{if } 1 \leq t \leq 2, \\ h((3-t) \cdot w_0, x_1) & \text{if } 2 \leq t \leq 3. \end{cases}$$

We modify $\bar{\gamma}$ to $\widetilde{\gamma}$ by a reparametrization so that $(\widetilde{\gamma})_{k=1} \in \Sigma_1^\infty$ and satisfies (0a) and (0b). Then a homomorphism of unital $\mathbb{R}$-algebras

$$F \colon \mathcal{A} \to H_0^{\pitchfork,<\infty}$$

is defined so that $z = [\gamma]$ is mapped to $[(\widetilde{\gamma})_{k=1}]$. From the definition of $D^{\pitchfork}$, it can be checked that F is a well-defined surjection and maps the ideal $\mathcal{I}$ into 0. Therefore, we obtain a surjective homomorphism of unital $\mathbb{R}$-algebras $\overline{F} \colon \mathcal{A}/\mathcal{I} \to H_0^{\pitchfork,<\infty}$.

We prove that $\overline{F}$ is injective by describing its inverse map. For $(\gamma \colon [0,T] \to Q)_{k=1} \in \Sigma_1^\infty$ satisfying (0a) and (0b), let us define

$$\check{\gamma} \colon [0,1] \to Q, \qquad t \mapsto \begin{cases} h((1-3t) \cdot w_0, \gamma(0)) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \gamma(3Tt - T) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ h((3t-2) \cdot w_0, \gamma(T)) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

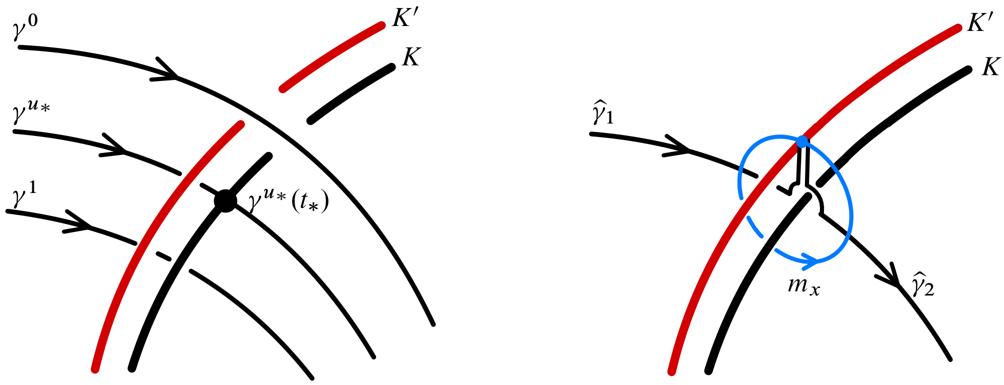

Figure 6: The left-hand side describes the 1-chain $(\gamma^u)_{k=1}^{u\in[0,1]}$ such that $\operatorname{sign}(u_*,t_*)=+1$ at $(u_*,t_*)\in(\Gamma_1^{\mathrm{int}})^{-1}(K)$. From the right-hand side, we can see that $[\hat{\gamma}^0]=[\hat{\gamma}_1\cdot\hat{\gamma}_2]$ and $[\hat{\gamma}^1]=[\hat{\gamma}_1\cdot m_x\cdot\hat{\gamma}_2]$ as elements of $\pi_1(Q\setminus K,K')$.

As described in Figure 5, we change $\check{\gamma}$ into $\hat{\gamma}$ by small perturbations inside N_{ε_0} around $t\in\{\frac{1}{3},\frac{2}{3}\}$ so that $\hat{\gamma}$ does not intersect K. We then obtain a homotopy class $[\hat{\gamma}]\in\pi_1(Q\setminus K,K')$. Note that, for any $\hat{\gamma}^j$ for $j=1,2$ from two choices of perturbations, there exist $l_0,l_1\in\{0,+1,-1\}$ such that

$$[\hat{\gamma}^2]=[(m_{\gamma(0)})^{l_0}\cdot\hat{\gamma}^1\cdot(m_{\gamma(T)})^{l_1}]\in\pi_1(Q\setminus K,K').$$

Here, $(m_x)^1:=m_x$, $(m_x)^0:=c_x$, and $(m_x)^{-1}$ denotes the inverse path of m_x. (As a natural extension, $(m_x)^l$ for $l\in\mathbb{Z}$ is defined.) Thus, $[\hat{\gamma}^1]=[\hat{\gamma}^2]$ as an element of $\mathcal{A}/\mathcal{I}$. If $\gamma\in\Sigma_1^0$ (ie length $\gamma<\varepsilon_0$), then $[\hat{\gamma}]=[(m_x)^l]\in\pi_1(Q\setminus K,K')$ for some $x\in K$ and $l\in\mathbb{Z}$. In this case, $[\hat{\gamma}]=0$ as an element of $\mathcal{A}/\mathcal{I}$. Therefore, we have a well-defined linear map

$$G\colon C_0^{\pitchfork,<\infty}\to\mathcal{A}/\mathcal{I},\qquad\begin{cases}C^{\pitchfork,<\infty}(m)\ni(\gamma_k)_{k=1,\dots,m}\mapsto[\hat{\gamma}_1]\cdots[\hat{\gamma}_m] & \text{for } m\geq 1,\\ \qquad C^{\pitchfork,<\infty}(0)\ni 1\mapsto\text{the unit.}\end{cases}$$

From the transversality condition (1b) together with (1c), it follows that $\operatorname{Im}D^{\pitchfork}$ is mapped into 0. Indeed, in a simple case, for $(\gamma^u)_{k=1}^{u\in[0,1]}\in C_1^{\pitchfork}(1,\infty)$ such that $(\Gamma_1^{\mathrm{int}})^{-1}(K)=\{(u_*,t_*)\}$, we can see that

$$G(D^{\pitchfork}(x))=\begin{cases}[\hat{\gamma}_1\cdot m_x\cdot\hat{\gamma}_2]-[\hat{\gamma}_1\cdot\hat{\gamma}_2]+\operatorname{sign}(u_*,t_*)[\hat{\gamma}_1][\hat{\gamma}_2] & \text{if } \operatorname{sign}(u_*,t_*)=+1,\\ [\hat{\gamma}_1\cdot\hat{\gamma}_2]-[\hat{\gamma}_1\cdot m_x\cdot\hat{\gamma}_2]+\operatorname{sign}(u_*,t_*)[\hat{\gamma}_1][\hat{\gamma}_2] & \text{if } \operatorname{sign}(u_*,t_*)=-1,\end{cases}$$

for $x:=\gamma^{u_*}(t_*)$, $\gamma_1:=\gamma^{u_*}|_{[0,t_*]}$ and $\gamma_2:=\gamma^{u_*}|_{[t_*,T^{u_*}]}(\cdot-t_*)$. Figure 6 describes the case where $\operatorname{sign}(u_*,t_*)=+1$. Thus $G\big(D^{\pitchfork}((\gamma^u)_{k=1}^{u\in[0,1]})\big)=0\in\mathcal{A}/\mathcal{I}$. Condition (1c) implies that the general case can be reduced to this simple case. Therefore, we obtain a well-defined linear map $\bar{G}\colon H_0^{\pitchfork,<\infty}\to\mathcal{A}/\mathcal{I}$. Finally, for any $[\gamma]\in\pi_1(Q\setminus K,K')$ such that $\gamma(i)=h(w_0,x_i)$ for $i\in\{0,1\}$, there exist $l_0,l_1\in\{0,+1,-1\}$ such that

$$\bar{G}\circ\bar{F}([\gamma])=[(m_{x_0})^{l_0}\cdot\gamma\cdot(m_{x_1})^{l_1}],$$

which is equal to $[\gamma]$ in $\mathcal{A}/\mathcal{I}$. This implies that $\bar{F}$ is injective. ◻

6.2 Connecting string homology and $H_*^{\text{string}}(Q, K)$

The purpose of this section is to construct a linear map from $H_*^{\pitchfork,<a}$ to $H_*^{<a}(Q, K)$ for all $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$. On the limit of $a \to \infty$, an $\mathbb{R}$-algebra homomorphism from $H_*^{\pitchfork,<\infty}$ to $H_*^{\text{string}}(Q, K)$ is defined. Before constructing this map, we will prepare two things.

6.2.1 Preliminaries First, we define a map Ψ which associates de Rham chains with singular chains. Let $\kappa, \chi \colon \mathbb{R} \to [0, 1]$ be C^∞ functions such that:

- $\kappa(u) = 0$ if $u \le 0$ and $\kappa(u) = 1$ if $u \ge 1$. In addition, $\kappa'(u) > 0$ if $0 < u < 1$.
- $\chi \colon \mathbb{R} \to [0, 1]$ has compact support and $\chi(s) = 1$ for every $s \in [0, 1]$.

For $p \in \mathbb{Z}$, a linear map
$$\Psi \colon C_p^{\pitchfork}(m, a) \to C_p^{\text{dR}}(\Sigma_m^a)$$
is defined by
$$\begin{cases} \Psi((\gamma_k)_{k=1,\dots,m}) := [\{0\}, c_0, 1] & \text{if } p = 0, \\ \Psi((\gamma_k^u)_{k=1,\dots,m}^{u\in[0,1]}) := [\mathbb{R}, c_1, \chi] & \text{if } p = 1, \\ \Psi = 0 & \text{otherwise}, \end{cases}$$
where c_0 is the constant map to $(\gamma_k)_{k=1,\dots,m}$ and $c_1 \colon \mathbb{R} \to \Sigma_m^a$, $u \mapsto (\gamma_k^{\kappa(u)})_{k=1,\dots,m}$. These maps commute with boundary operators, namely
$$\partial \circ \Psi = \Psi \circ \partial^{\text{sing}} \colon C_p^{\pitchfork}(m, a) \to C_{p-1}^{\text{dR}}(\Sigma_m^a).$$

Next, we define a filtration $\{C_p^{\pitchfork}(m, a, \varepsilon)\}_{\varepsilon>0}$ of $C_p^{\pitchfork}(m, a)$. When $p = 0$, $C_0^{\pitchfork}(m, a, \varepsilon)$ is an $\mathbb{R}$-subspace generated by $(\gamma_k \colon [0, T_k] \to Q)_{k=1,\dots,m} \in \Sigma_m^a$ satisfying (0a)–(0b) and the following condition:

(0c) There exists $\tau_0 \in (0, \varepsilon_0/(5C_0)]$ such that $\gamma_k([\tau_0, T_k - \tau_0]) \cap N_\varepsilon = \varnothing$ for every $k \in \{1, \dots, m\}$.

In the case of $p = 1$, $C_1^{\pitchfork}(m, a, \varepsilon)$ is an $\mathbb{R}$-subspace generated by $(\gamma_k^u \colon [0, T_k^u] \to Q)_{k=1,\dots,m}^{u\in[0,1]}$ satisfying (1a)–(1c) and the following conditions:

(1d) $(\gamma_k^0)_{k=1,\dots,m}$ and $(\gamma_k^1)_{k=1,\dots,m}$ satisfy (0c).

(1e) There exist $\tau_0 \in (0, \varepsilon_0/(5C_0)]$ and an open neighborhood $U_{(u_*,t_*)}$ for each $(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)$ for $k = 1, \dots, m$ such that
$$(u_*, t_*) \in U_{(u_*,t_*)} \subset \{(u, t) \mid 0 < u < 1,\ \tau_0 < t < T_k^u - \tau_0\},$$
and the following hold:

(1e-1) $U_{(u_*,t_*)} \subset \{(u, t) \mid |t - t_*| < \tau_0\}$.

(1e-2) For any two distinct points $(u_*, t_*), (u'_*, t'_*) \in \coprod_{k=1}^m (\Gamma_k^{\text{int}})^{-1}(K)$, the projections of the sets $U_{(u_*,t_*)}, U_{(u'_*,t'_*)} \subset [0, 1] \times \mathbb{R}_{>0}$ to $[0, 1]$,
$$\mathrm{pr}_{[0,1]}(U_{(u_*,t_*)}), \mathrm{pr}_{[0,1]}(U_{(u'_*,t'_*)}) \subset [0, 1],$$
are disjoint.

(1e-3) For every $k \in \{1, \dots, m\}$,

$$(\Gamma_k|_{\{(u,t) \mid \tau_0 \le t \le T_k^u - \tau_0\}})^{-1}(N_\varepsilon) = \bigcup_{(u_*, t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)} U_{(u_*, t_*)}.$$

Moreover, for each $(u_*, t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)$,

$$\Gamma_k^{\mathrm{fiber}} \colon U_{(u_*, t_*)} \to \mathbb{O}_\varepsilon$$

is a diffeomorphism. (Recall that $\Gamma_k^{\mathrm{fiber}} = \mathrm{pr}_{\mathbb{R}^2} \circ h^{-1} \circ \Gamma_k$.)

In addition, we define $C_p^{\pitchfork}(m, a, \varepsilon) := C_p^{\pitchfork}(m, a)$ if $p \notin \{0, 1\}$ or $m = 0$.

Roughly speaking, (0c) means that $\gamma_k(t)$ is far from K by a distance at least $\frac{1}{2}\varepsilon$, except when t is close to $\{0, T_k\}$. Condition (1e) means that $\gamma_k^u(t)$ is far from K by a distance at least $\frac{1}{2}\varepsilon$, except when t is close to $\{0, T_k^u\}$ or when (u, t) is close to some point in $(\Gamma_k^{\mathrm{int}})^{-1}(K)$. Note that, when $0 < \varepsilon' \le \varepsilon$, we have $C_*^{\pitchfork}(m, a, \varepsilon) \subset C_*^{\pitchfork}(m, a, \varepsilon')$, and

$$\bigcup_{\varepsilon > 0} C_*^{\pitchfork}(m, a, \varepsilon) = C_*^{\pitchfork}(m, a). \tag{42}$$

Moreover, ∂^{sing} and $f_k^{\pitchfork}$ for $k = 1, \dots, m$ are restricted to linear maps

$$\partial^{\mathrm{sing}} \colon C_*^{\pitchfork}(m, a, \varepsilon) \to C_{*-1}^{\pitchfork}(m, a, \varepsilon), \quad f_k^{\pitchfork} \colon C_*^{\pitchfork}(m, a, \varepsilon) \to C_{*-1}^{\pitchfork}(m+1, a, \varepsilon).$$

For every $\varepsilon > 0$, we define $C_*^{\pitchfork, <a}(m, \varepsilon) := C_*^{\pitchfork}(m, a, \varepsilon) / C_*^{\pitchfork}(m, 0, \varepsilon)$ and

$$C_*^{\pitchfork, <a}(\varepsilon) := \bigoplus_{m=0}^{\infty} C_*^{\pitchfork, <a}(m, \varepsilon).$$

A linear map $D_\varepsilon^{\pitchfork} \colon C_*^{\pitchfork, <a}(\varepsilon) \to C_{*-1}^{\pitchfork, <a}(\varepsilon)$ is defined by the same form as (41). Then we obtain a chain complex $(C_*^{\pitchfork, <a}(\varepsilon), D_\varepsilon^{\pitchfork})$. Let $H_*^{\pitchfork, <a}(\varepsilon)$ denote its homology. When $0 < \varepsilon' \le \varepsilon$, the inclusion maps $C_*^{\pitchfork}(m, a, \varepsilon) \to C_*^{\pitchfork}(m, a, \varepsilon')$ for all $m \in \mathbb{Z}_{\ge 0}$ induce a linear map

$$l_{\varepsilon, \varepsilon'} \colon H_*^{\pitchfork, <a}(\varepsilon) \to H_*^{\pitchfork, <a}(\varepsilon'),$$

and we have a direct system $\big(\{H_*^{\pitchfork, <a}(\varepsilon)\}_{\varepsilon > 0}, \{l_{\varepsilon, \varepsilon'}\}_{\varepsilon \ge \varepsilon'}\big)$. From the relation (42), we have

$$\varinjlim_{\varepsilon \to 0} H_*^{\pitchfork, <a}(\varepsilon) = H_*^{\pitchfork, <a}.$$

6.2.2 Construction of the chain map

With the above preparations, we consider the maps

$$C_p^{\pitchfork}(m, a, \varepsilon) \to C_p^{\mathrm{dR}}(\Sigma_m^{a + m\varepsilon}), \quad x \mapsto \Psi(x), \quad \text{for } m \in \mathbb{Z}_{\ge 0}$$

for $\varepsilon \in (0, \varepsilon_0/(5C_0)]$. They induce a linear map from $C_*^{\pitchfork, <a}(\varepsilon)$ to $C_*^{<a}(\varepsilon)$, but this is not a chain map. In order to fill in the gap, we need to prove the following lemma for $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$.

Lemma 6.4 *Suppose that* $(\varepsilon,\delta)\in\mathscr{T}_a$ *is standard with respect to* h. *Then, for* $m\in\mathbb{Z}_{\geq 1}$ *and* $k\in\{1,\dots,m\}$, *there exists a linear map*

$$o_{k,(\varepsilon,\delta)}\colon C_1^{\pitchfork}(m,a,\varepsilon)\to C_1^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon})$$

such that the following hold for any $x\in C_1^{\pitchfork}(m,a,\varepsilon)$:

(i) $\partial(o_{k,(\varepsilon,\delta)}(x))-(f_{k,\delta}\circ\Psi(x)-\Psi\circ f_k^{\pitchfork}(x))\in C_0^{\mathrm{dR}}(\Sigma_{m+1}^0)$.

(ii) $f_{l,\delta}(o_{k,(\varepsilon,\delta)}(x))\in C_0^{\mathrm{dR}}(\Sigma_{m+2}^0)$ *for every* $l\in\{1,\dots,m+1\}$.

Proof It suffices to define $o_{k,(\varepsilon,\delta)}(x)$ for $x=(\gamma_l^u)_{l=1,\dots,m}^{u\in[0,1]}$ satisfying (1a),…,(1e). The proof is divided into three steps: We define de Rham chains in the first two steps. In the last step, $o_{k,(\varepsilon,\delta)}(x)$ is defined as the sum of these chains and we check conditions (i) and (ii).

Step 1 From the definitions of Ψ and $f_k^{\pitchfork}$,

$$f_{k,\delta}\circ\Psi(x)-\Psi\circ f_k^{\pitchfork}(x)=f_{k,\delta}(\Psi(x))-\sum_{(u_*,t_*)\in(\Gamma_k^{\mathrm{int}})^{-1}(K)}\mathrm{sign}(u_*,t_*)\cdot[\{0\},c_0(u_*,t_*),1],$$

where $c_0(u_*,t_*)$ is a constant map to $(\gamma_1^{u_*},\dots,\widehat{\gamma_k^{u_*1}},\widehat{\gamma_k^{u_*2}},\dots,\gamma_m^{u_*})$. Since (ε,δ) is standard, it has the form (31). Using the notation of (13), we can explicitly write $f_{k,\delta}(\Psi(x))=[W_k,\Phi_k,\zeta_k]$, where

$$\begin{gathered}W_k=\{(u,\tau,v)\in\mathbb{R}\times\mathbb{R}\times N_\varepsilon\mid 2\varepsilon<\tau<T_k^{\mu(u)}-2\varepsilon,\ \gamma_k^{\mu(u)}(\tau)=\sigma_1^v(0)\},\\ \Phi_k\colon W_k\to\Sigma_{m+1}^{a+(m+1)\varepsilon},\quad (u,\tau,v)\mapsto\mathrm{con}_k((\gamma_l^{\mu(u)})_{l=1,\dots,m},(T_k^{\mu(u)},\tau),\psi_\varepsilon(v)),\\ \zeta_k\in\Omega_c^2(W_k),\quad (\zeta_k)_{(u,\tau,v)}=\rho_\varepsilon(T_k^{\mu(u)},\tau)\cdot\chi(u)\cdot(h_*(\nu_\varepsilon\times 1))_v.\end{gathered}$$

Recall condition (1e-3) and note that $\sigma_1^v(0)=v$. We define $\overline{W}_k:=W_k\cap\{\tau_0<\tau<T_k^{\mu(u)}-\tau_0\}$. Then

$$\overline{W}_k\to\bigcup_{(u_*,t_*)\in(\Gamma_k^{\mathrm{int}})^{-1}(K)}U_{(u_*,t_*)},\quad (u,\tau,v)\mapsto(\mu(u),\tau),$$

is an orientation-preserving diffeomorphism. Moreover, $\rho_\varepsilon(T_k^{\mu(u)},\tau)\cdot\chi(u)=1$ for $(u,\tau,v)\in\overline{W}_k$. On the other hand, it follows from (1e-1) and Lemma 3.12 that $\Phi_k(u,\tau,v)\in\Sigma_{m+1}^0$ for $(u,\tau,v)\in W_k\setminus\overline{W}_k$. Therefore,

$$\begin{aligned}f_{k,\delta}(\Psi(x))-\sum_{(u_*,t_*)\in(\Gamma_k^{\mathrm{int}})^{-1}(K)}[U_{(u_*,t_*)},\Phi'_{(u_*,t_*)},\zeta'_{(u_*,t_*)}]&=f_{k,\delta}(\Psi(x))-[\overline{W}_k,\Phi_k|_{\overline{W}_k},\zeta_k|_{\overline{W}_k}]\\&\in C_0^{\mathrm{dR}}(\Sigma_{m+1}^0),\end{aligned}$$

where

$$\begin{gathered}\Phi'_{(u_*,t_*)}\colon U_{(u_*,t_*)}\to\Sigma_{m+1}^{a+(m+1)\varepsilon},\quad (u,\tau)\mapsto\mathrm{con}_k\big((\gamma_l^u)_{l=1,\dots,m},(T_k^u,\tau),\psi_\varepsilon(\gamma_k^u(\tau))\big),\\ \zeta'_{(u_*,t_*)}=(\Gamma_k^{\mathrm{int}}|_{U_{(u_*,t_*)}})^*(h_*(\nu_\varepsilon\times 1))\in\Omega_c^2(U_{(u_*,t_*)}).\end{gathered}$$

As a result, $f_{k,\delta} \circ \Psi(x) - \Psi \circ f_k^{\pitchfork}(x)$ is equal to the chain

$$\sum_{(u_*,t_*)\in(\Gamma_k^{\mathrm{int}})^{-1}(K)} \big([U_{(u_*,t_*)}, \Phi'_{(u_*,t_*)}, \zeta'_{(u_*,t_*)}] - \operatorname{sign}(u_*,t_*)\cdot[\{0\}, c_0(u_*,t_*), 1]\big) \tag{43}$$

modulo $C_0^{\mathrm{dR}}(\Sigma_{m+1}^{0})$.

For each $(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)$, consider a diffeomorphism

$$\Gamma_k^{\mathrm{fiber}} = \mathrm{pr}_{\mathbb{R}^2} \circ h^{-1} \circ \Gamma_k^{\mathrm{int}}|_{U_{(u_*,t_*)}} \colon U_{(u_*,t_*)} \to \mathbb{O}_\varepsilon$$

and a scalar multiplication $m_s \colon \mathbb{O}_\varepsilon \to \mathbb{O}_\varepsilon$, $w \mapsto \kappa(s)\cdot w$, for $s \in \mathbb{R}$. We define a deformation retraction to $\{(u_*,t_*)\}$,

$$\mathbb{R} \times U_{(u_*,t_*)} \to U_{(u_*,t_*)}, \quad (s,(u,\tau)) \mapsto (u_s,\tau_s) := (\Gamma_k^{\mathrm{fiber}})^{-1} \circ m_s \circ (\Gamma_k^{\mathrm{fiber}})(u,\tau).$$

Now we define a map,

$$\widetilde{\Phi}'_{(u_*,t_*)} \colon \mathbb{R} \times U_{(u_*,t_*)} \to \Sigma_{m+1}^{a+(m+1)\varepsilon}, \quad (s,(u,\tau)) \mapsto \Phi'_{(u_*,t_*)}(u_s,\tau_s),$$

and the de Rham chain

$$o_k^1 := \sum_{(u_*,t_*)\in(\Gamma_k^{\mathrm{int}})^{-1}(K)} [\mathbb{R}\times U_{(u_*,t_*)}, \widetilde{\Phi}'_{(u_*,t_*)}, \chi \times \zeta'_{(u_*,t_*)}] \in C_1^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon}).$$

If $s \geq 1$, $\widetilde{\Phi}'_{(u_*,t_*)}(s,(u,\tau)) = \Phi'_{(u_*,t_*)}(u,\tau)$. If $s \leq 0$, $\widetilde{\Phi}'_{(u_*,t_*)}(s,(u,\tau))$ is constant to

$$(\gamma_1^{u_*}, \dots, \widetilde{\gamma_k^{u_*1}}, \widetilde{\gamma_k^{u_*2}}, \dots, \gamma_m^{u_*}) =: c_0'(u_*,t_*),$$

which is defined by (12) for $\gamma_k = \gamma_k^{u_*}$, $(T_k,\tau) = (T_k^{u_*}, t_*)$ and $\sigma_i \colon \left[0, \frac{1}{2}\varepsilon\right] \to \{\gamma_k^{u_*}(t_*)\} \subset Q$. Therefore, the boundary chain ∂o_k^1 is equal to

$$\sum_{(u_*,t_*)\in(\Gamma_k^{\mathrm{int}})^{-1}(K)} \left([U_{(u_*,t_*)}, \Phi'_{(u_*,t_*)}, \zeta'_{(u_*,t_*)}] - \left[\{0\}, c_0'(u_*,t_*), \int_{U_{(u_*,t_*)}} (\Gamma_k)^*(h_*(\nu_\varepsilon \times 1))\right]\right).$$

Since $\int_{\mathbb{O}_\varepsilon} \nu_\varepsilon = 1$, we can compute that

$$\int_{U_{(u_*,t_*)}} (\Gamma_k)^*(h_*(\nu_\varepsilon\times 1)) = \int_{U_{(u_*,t_*)}} (h^{-1}\circ\Gamma_k)^*(\nu_\varepsilon \times 1) = \int_{U_{(u_*,t_*)}} (\Gamma_k^{\mathrm{fiber}})^*\nu_\varepsilon = \operatorname{sign}(u_*,t_*) \in \{\pm 1\}.$$

Thus,

$$\partial(o_k^1) = \sum_{(u_*,t_*)\in(\Gamma_k^{\mathrm{int}})^{-1}(K)} \big([U_{(u_*,t_*)}, \Phi'_{(u_*,t_*)}, \zeta'_{(u_*,t_*)}] - \operatorname{sign}(u_*,t_*)\cdot[\{0\}, c_0'(u_*,t_*), 1]\big). \tag{44}$$

Step 2 For each $(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)$, $c_0'(u_*,t_*)$ coincides with $c_0(u_*,t_*)$ up to reparametrizations of the k^{th} and $(k+1)^{\text{st}}$ paths. We define by interpolating parametrizations

$$c_1(u_*,t_*) \colon [0,1] \to \Sigma_{m+1}^{a+(m+1)\varepsilon}, \quad s \mapsto (\gamma_1^{u_*}, \dots, \gamma_{k-1}^{u_*}, \widehat{\gamma_k^{u_*1}} \circ \mu_s^1, \widehat{\gamma_k^{u_*2}} \circ \mu_s^2, \gamma_{k+1}^{u_*}, \dots, \gamma_m^{u_*}),$$

so that $c_1(u_*, t_*)(0) = c_0(u_*, t_*)$ and $c_1(u_*, t_*)(1) = c'_0(u_*, t_*)$. Then we obtain a chain

$$o^2_{(u_*,t_*)} := [\mathbb{R}, c_1(u_*, t_*) \circ \kappa, \chi] \in C_1^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon}),$$

which satisfies $\partial(o^2_{(u_*,t_*)}) = [\{0\}, c'_0(u_*, t_*), 1] - [\{0\}, c_0(u_*, t_*), 1]$.

Step 3 We define a chain

$$o_{k,(\varepsilon,\delta)}(x) := o^1_k + \sum_{(u_*,t_*)\in(\Gamma_k^{\mathrm{int}})^{-1}(K))} \operatorname{sign}(u_*, t_*) \cdot o^2_{(u_*,t_*)}.$$

From (44), $\partial(o_{k,(\varepsilon,\delta)}(x))$ is equal to the chain of (43). Therefore, $o_{k,\delta}(x)$ satisfies condition (i). Condition (ii) can be checked as follows: From conditions (1e-2) and (1e-4), those paths in $\Phi'_{(u_*,t_*)}(s, (u, \tau))$ and $c_1(u_*, t_*)(s)$ satisfy condition (iii) of Lemma 3.12. Therefore, $f_{l,\delta}(o^1_k)$ and $f_{l,\delta}(o^2_{(u_*,t_*)})$ belongs to $C_0^{\mathrm{dR}}(\Sigma^0_{m+2})$ for $l = 1, \dots, m+1$. □

For $(\varepsilon, \delta) \in \mathcal{T}_a$ which is standard, we define a linear map $\Phi^{<a}_{(\varepsilon,\delta)} \colon C_*^{\pitchfork,<a}(\varepsilon) \to C_*^{<a}(\varepsilon)$ such that, for $x \in C_p^{\pitchfork}(m, a, \varepsilon)$, the equivalence class $[x] \in C_p^{\pitchfork,<a}(m, \varepsilon)$ is mapped to

$$\Phi^{<a}_{(\varepsilon,\delta)}([x]) = \begin{cases} [\Psi(x)] & \text{if } p = 0, \\ [\Psi(x)] - \sum_{k=1}^m [o_{k,(\varepsilon,\delta)}(x)] & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The two properties of $o_{k,(\varepsilon,\delta)}$ show that $\Phi^{<a}_{(\varepsilon,\delta)}$ is a chain map from $(C_*^{\pitchfork,<a}(\varepsilon), D_\varepsilon^{\pitchfork})$ to $(C_*^{<a}(\varepsilon), D_\delta)$. Therefore, we obtain a linear map on homology,

$$(\Phi^{<a}_{(\varepsilon,\delta)})_* \colon H_*^{\pitchfork,<a}(\varepsilon) \to H_*^{<a}(\varepsilon, \delta).$$

6.2.3 Commutativity with transition maps

We need to check the relation of $\Phi^{<a}_{(\varepsilon,\delta)}$ for $\{k_{(\varepsilon',\delta),(\varepsilon,\delta)}\}_{\varepsilon'\le\varepsilon}$ and $\{l_{\varepsilon,\varepsilon'}\}_{\varepsilon\ge\varepsilon'}$.

Proposition 6.5 *For $(\varepsilon, \delta), (\varepsilon', \delta) \in \mathcal{T}_a$ with $\varepsilon' \le \varepsilon$ which are standard with respect to h, the following diagram commutes:*

$$\begin{tikzcd} H_*^{\pitchfork,<a}(\varepsilon) \arrow[r, "(\Phi^{<a}_{(\varepsilon,\delta)})_*"] \arrow[d, "l_{\varepsilon,\varepsilon'}"'] & H_*^{<a}(\varepsilon, \delta) \\ H_*^{\pitchfork,<a}(\varepsilon') \arrow[r, "(\Phi^{<a}_{(\varepsilon',\delta')})_*"] & H_*^{<a}(\varepsilon', \delta) \arrow[u, "k_{(\varepsilon',\delta'),(\varepsilon,\delta)}"'] \end{tikzcd}$$

To prove this proposition, we return to the definition $k_{(\varepsilon',\delta'),(\varepsilon,\delta)} = k_{(\bar\varepsilon,\bar\delta)}$ by $(\bar\varepsilon, \bar\delta) \in \overline{\mathcal{T}}_a$ satisfying (24) for (ε, δ) and (ε', δ'). We require $(\bar\varepsilon, \bar\delta)$ to be standard with respect to h, and thus $\bar\varepsilon = \varepsilon$.

We set $\bar\Psi := \bar\imath \circ \Psi \colon C_*^{\pitchfork}(m, a) \to \bar C_*^{\mathrm{dR}}(\Sigma_m^a)$ for all $m \in \mathbb{Z}_{\ge 0}$. Again, the induced map from $C_*^{\pitchfork,<a}(\varepsilon)$ to $\bar C_*^{<a}(\varepsilon)$ is not a chain map. To fill in the gap, we need the following lemma.

Lemma 6.6 *For $m \in \mathbb{Z}_{\ge 1}$ and $k \in \{1, \dots, m\}$, suppose that we have taken maps $o_{k,(\varepsilon,\delta)}$ and $o_{k,(\varepsilon',\delta')}$ of Lemma 6.4 for (ε, δ) and (ε', δ'). Then there exists a linear map*

$$\bar o_{k,(\varepsilon,\bar\delta)} \colon C_1^{\pitchfork}(m, a, \varepsilon) \to \bar C_1^{\mathrm{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon})$$

such that the following hold for any $x \in C_1^{\pitchfork}(m,a,\varepsilon)$*:*

- $\partial(\bar{o}_{k,(\varepsilon,\bar\delta)}) - (\bar{f}_{k,\bar\delta} \circ \bar\Psi(x) - \bar\Psi \circ f_k^{\pitchfork}(x)) \in \bar{C}_0^{\mathrm{dR}}(\Sigma_{m+1}^0)$.
- $\bar{f}_{l,\bar\delta}(\bar{o}_{k,(\varepsilon,\bar\delta)}(x)) \in \bar{C}_0^{\mathrm{dR}}(\Sigma_{m+2}^0)$ *for every* $l \in \{1,\dots,m+1\}$.
- $e_+(\bar{o}_{k,(\varepsilon,\bar\delta)}(x)) = o_{k,(\varepsilon,\delta)}(x)$ *and* $e_-(\bar{o}_{k,(\varepsilon,\bar\delta)}(x)) = (j_{\varepsilon',\varepsilon})_*(o_{k,(\varepsilon',\delta')}(x))$.

Proof We omit the detailed proof. Note that $\bar\delta$ has the form (32). For any $x = (\gamma_k)_{k=1,\dots,m}$, we can compute explicitly that the chain $\bar{f}_{k,\bar\delta} \circ \bar\Psi(x) - \bar\Psi \circ f_k^{\pitchfork}(x)$ is equal to the sum of chains, for all $(u_*,t_*) \in (\Gamma_k^{\mathrm{int}})^{-1}(K)$,

$$[\mathbb{R} \times U_{(u_*,t_*)}, \bar\Phi'_{(u_*,t_*)}, (\mathrm{id}_{\mathbb{R}_{\ge 1}} \times U_{(u_*,t_*)}, \mathrm{id}_{\mathbb{R}_{\le -1}} \times U_{(u_*,t_*)}), \bar\zeta'_{(u_*,t_*)}] \\ - \mathrm{sign}(u_*,t_*) \cdot [\mathbb{R}, c_0(u_*,t_*), (\mathrm{id}_{\mathbb{R}_{\ge 1}}, \mathrm{id}_{\mathbb{R}_{\le -1}}), 1]$$

modulo $\bar{C}_0^{\mathrm{dR}}(\Sigma_{m+1}^0)$. Here $\bar\Phi'_{(u_*,t_*)} \colon \mathbb{R} \times U_{(u_*,t_*)} \to \mathbb{R} \times \Sigma_{m+1}^{a+(m+1)\varepsilon}$ is determined by

$$\bar\Phi'_{(u_*,t_*)}(r,(u,\tau)) := \big(r, \mathrm{con}_k\big((\gamma_l^u)_{l=1,\dots,m}, (T_k^u,\tau), \bar\psi_{\varepsilon',\varepsilon}(r,\gamma_k^u(\tau))\big)\big)$$

and $(\bar\zeta'_{(u_*,t_*)}) := (\mathrm{id}_{\mathbb{R}} \times \Gamma_k^{\mathrm{int}}|_{U_{(u_*,t_*)}})^*(1 \times \bar\eta_{\varepsilon',\varepsilon}) \in \Omega^2(\mathbb{R} \times U_{(u_*,t_*)})$. The $[-1,1]$-modeled chain $\bar{o}_{k,(\varepsilon,\bar\delta)}$ is defined in a similar way as $o_{k,(\varepsilon,\delta)}$ in Lemma 6.4, and we can check that this chain satisfies the required three conditions. □

Proof of Proposition 6.5 We define a linear map $\bar\Phi^{<a}_{(\varepsilon,\bar\delta)} \colon C_*^{\pitchfork,<a}(\varepsilon) \to \bar{C}_*^{<a}(\varepsilon)$ such that, for $x \in C_p^{\pitchfork,<a}(m,a,\varepsilon)$,

$$\bar\Phi^{<a}_{(\varepsilon,\bar\delta)}([x]) = \begin{cases} [\bar\Psi(x)] & \text{if } p=0, \\ [\bar\Psi(x)] - \sum_{k=1}^m [\bar{o}_{k,(\varepsilon,\bar\delta)}(x)] & \text{if } p=1, \\ 0 & \text{otherwise.} \end{cases}$$

The first two properties of $\bar{o}_{k,(\varepsilon,\bar\delta)}$ show that this is a chain map from $(C_*^{\pitchfork,<a}(\varepsilon), D_\varepsilon^{\pitchfork})$ to $(\bar{C}_*^{<a}(\varepsilon), \bar{D}_{\bar\delta})$. Therefore, we get a linear map on homology,

$$(\bar\Phi^{<a}_{(\varepsilon,\bar\delta)})_* \colon H_*^{\pitchfork,<a}(\varepsilon) \to \bar{H}_*^{<a}(\varepsilon,\bar\delta).$$

The third property of $\bar{o}_{k,(\varepsilon,\bar\delta)}$ implies that the following diagram commutes:

$$\begin{tikzcd}
H_*^{\pitchfork,<a}(\varepsilon) \arrow[r, "(\Phi^{<a}_{(\varepsilon,\delta)})_*"] \arrow[dr, "(\bar\Phi^{<a}_{(\varepsilon,\bar\delta)})_*"'] \arrow[dd, "l_{\varepsilon,\varepsilon'}"'] & H_*^{<a}(\varepsilon,\delta) \arrow[r, "(j_{\varepsilon,\varepsilon})_*=\mathrm{id}"] & H_*^{<a}(\varepsilon,\delta) \\
& \bar{H}_*^{<a}(\varepsilon,\bar\delta) \arrow[ur, "(e_{\varepsilon,+})_*"'] \arrow[dr, "(e_{\varepsilon,-})_*"] & \\
H_*^{\pitchfork,<a}(\varepsilon') \arrow[r, "(\Phi^{<a}_{(\varepsilon',\delta')})_*"] & H_*^{<a}(\varepsilon',\delta') \arrow[r, "(j_{\varepsilon',\varepsilon})_*"] & H_*^{<a}(\varepsilon,(i_{\varepsilon',\varepsilon})_*\delta')
\end{tikzcd}$$

The proposition is now proved since $k_{(\varepsilon,\bar\delta)} = ((j_{\varepsilon,\varepsilon})_*^{-1} \circ (e_{\varepsilon,+})_*) \circ ((j_{\varepsilon',\varepsilon})_*^{-1} \circ (e_{\varepsilon_-})_*)^{-1}$. □

Let $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$. Proposition 6.5 shows that the family of maps

$$\{(\Phi^{<a}_{(\varepsilon,\delta)})_* \mid (\varepsilon,\delta) \in \mathscr{T}_a \text{ is standard with respect to } h\}$$

induces a linear map on the limits of $\varepsilon \to 0$,

$$\Phi^{<a}\colon H_*^{\pitchfork,<a} = \varinjlim_{\varepsilon\to 0} H_*^{\pitchfork,<a}(\varepsilon) \to H_*^{<a}(Q,K) = \varprojlim_{\varepsilon\to 0} H_*^{<a}(\varepsilon,\delta).$$

Naturally, those maps of $\{\Phi^{<a}\}_{a\in\mathbb{R}_{>0}\setminus\mathscr{L}(K)}$ commutes with $\{I^{a,b}\colon H_*^{<a}(Q,K) \to H_*^{<b}(Q,K)\}_{a\le b}$ and $\{J^{a,b}\colon H_*^{\pitchfork,<a} \to H_*^{\pitchfork,<b}\}_{a\le b}$. Therefore, on the limit of $a \to \infty$, we have a linear map

$$\Phi\colon H_*^{\pitchfork,\infty} \to H_*^{\mathrm{string}}(Q,K).$$

Moreover, it is straightforward to check that Φ is a homomorphism of unital $\mathbb{R}$-algebras.

6.3 Proof of isomorphism

In this section, we prove that for every $a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)$, $\Phi^{<a}$ is an isomorphism in the 0^{th} degree. As an immediate consequence, it is shown that the cord algebra of (Q,K) is isomorphic to $H_0^{\mathrm{string}}(Q,K)$.

For each $m \in \mathbb{Z}_{\ge 0}$, let $\partial_m^{\mathrm{sing}}\colon C_1^{\pitchfork,<a}(m) \to C_0^{\pitchfork,<a}(m)$ denote the singular boundary operator. We also write

$$\begin{aligned} \Psi_{0,m}\colon \operatorname{Coker}\partial_m^{\mathrm{sing}} &\to H_0^{\mathrm{dR}}(\Sigma_m^a,\Sigma_m^0), \quad [x]\mapsto[\Psi(x)],\\ \Psi_{1,m}\colon \operatorname{Ker}\partial_m^{\mathrm{sing}} &\to H_1^{\mathrm{dR}}(\Sigma_m^a,\Sigma_m^0), \quad\ \ x\mapsto[\Psi(x)]. \end{aligned} \tag{45}$$

Lemma 6.7 *$\Psi_{0,m}$ is an isomorphism and $\Psi_{1,m}$ is a surjection.*

Proof Naturally, there are two maps

$$\operatorname{Coker}\partial_m^{\mathrm{sing}} \to H_0^{\mathrm{sing}}(\Sigma_m^a,\Sigma_m^0), \quad \operatorname{Ker}\partial_m^{\mathrm{sing}} \to H_1^{\mathrm{sing}}(\Sigma_m^a,\Sigma_m^0).$$

induced by the inclusions $C_p^{\pitchfork}(m,a) \to C_p^{\mathrm{sing}}(\Sigma_m^a)$ for $p=0,1$. The subset of Σ_m^a (resp. $C^0([0,1],\Sigma_m^a)$) consisting of elements satisfying conditions (0a)–(0b) (resp. (1a)–(1c)) is open dense. This fact implies that the first map is an isomorphism and the second map is a surjection. Then we consider the following diagram for $p=0,1$:

$$\begin{tikzcd} K_{p,m} \arrow[r, "\Psi_{p,m}"] \arrow[d] & H_p^{\mathrm{dR}}(\Sigma_m^a,\Sigma_m^0) \arrow[r, "(6)"] & \varinjlim_{j\to\infty} H_p^{\mathrm{dR}}(B_m^a(2^j),B_m^0(2^j)) \\ H_p^{\mathrm{sing}}(\Sigma_m^a,\Sigma_m^0) \arrow[rr, "(11)"] & & \varinjlim_{j\to\infty} H_p^{\mathrm{sing}}(B_m^a(2^j),B_m^0(2^j)) \arrow[u] \end{tikzcd}$$

Here, $K_{0,m} := \operatorname{Coker}\partial_m^{\mathrm{sing}}$ and $K_{1,m} := \operatorname{Ker}\partial_m^{\mathrm{sing}}$. The left vertical map is defined as above. The right vertical map is an isomorphism from Proposition 2.9. The horizontal maps are the isomorphisms of (6) and (11). The commutativity follows from the definition of the right vertical map. See [15, Section 4.7]. Now the assertion of the lemma follows from the diagram. □

For the chain complexes $(C_*^{\pitchfork,<a}, D^{\pitchfork})$ and $(C_*^{\pitchfork,<a}(\varepsilon), D_\varepsilon^{\pitchfork})$, we define their filtrations $\{\mathcal{H}_p^{<a}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{H}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$ by

$$\mathcal{H}_p^{<a} := \bigoplus_{m\geq -p} C_*^{\pitchfork,<a}(m), \quad \mathcal{H}_{\varepsilon,p}^{<a} := \bigoplus_{m\geq -p} C_*^{\pitchfork,<a}(m,\varepsilon).$$

Let $E^{\pitchfork,<a}$ and $E_\varepsilon^{\pitchfork,<a}$ be the spectral sequences determined by $\{\mathcal{H}_p^{<a}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{H}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$, respectively. The $(-m,q)$-terms of their first pages are given by

$$(E^{\pitchfork,<a})^1_{-m,q} = \begin{cases} \operatorname{Coker} \partial_m^{\mathrm{sing}} & \text{if } q-m=0 \text{ and } m\geq 0, \\ \operatorname{Ker} \partial_m^{\mathrm{sing}} & \text{if } q-m=1 \text{ and } m\geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$(E_\varepsilon^{\pitchfork,<a})^1_{-m,q} = \begin{cases} \operatorname{Coker} \partial_{\varepsilon,m}^{\mathrm{sing}} & \text{if } q-m=0 \text{ and } m\geq 0, \\ \operatorname{Ker} \partial_{\varepsilon,m}^{\mathrm{sing}} & \text{if } q-m=1 \text{ and } m\geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\partial_{\varepsilon,m}^{\mathrm{sing}}\colon C_1^{\pitchfork,<a}(m,\varepsilon) \to C_0^{\pitchfork,<a}(m,\varepsilon)$ also denotes the singular boundary operator. If $0<\varepsilon'\leq\varepsilon$, there exists a morphism $l_{\varepsilon,\varepsilon'}\colon E_\varepsilon^{\pitchfork,<a} \to E_{\varepsilon'}^{\pitchfork,<a}$ induced by the inclusion maps $C_*^{\pitchfork}(m,a,\varepsilon) \to C_*^{\pitchfork}(m,a,\varepsilon')$ for all $m\in\mathbb{Z}_{\geq 0}$. Naturally, $\varinjlim_{\varepsilon\to 0} E_\varepsilon^{\pitchfork,<a} \cong E^{\pitchfork,<a}$.

For $(\varepsilon,\delta)\in\mathcal{T}_a$ which is standard with respect to h, the chain map $\Phi_{(\varepsilon,\delta)}^{<a}$ preserves the filtrations $\{\mathcal{H}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$ and $\{\mathcal{F}_{\varepsilon,p}^{<a}\}_{p\in\mathbb{Z}}$, so it induces a morphism of spectral sequences

$$(\Phi_{(\varepsilon,\delta)}^{<a})_*\colon E_\varepsilon^{\pitchfork,<a} \to E_{(\varepsilon,\delta)}^{<a}.$$

Note that, on the $(-m,q)$-term for $m\geq 0$ of their first pages, this can be written as

$$(46)\qquad \begin{aligned} (\Phi_{(\varepsilon,\delta)}^{<a})_*\colon \operatorname{Coker} \partial_{\varepsilon,m}^{\mathrm{sing}} &\to H_0^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0), & [x] &\mapsto [\Psi(x)] & &\text{if } q=m, \\ (\Phi_{(\varepsilon,\delta)}^{<a})_*\colon \operatorname{Ker} \partial_{\varepsilon,m}^{\mathrm{sing}} &\to H_1^{\mathrm{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0), & x &\mapsto [\Psi(x)] & &\text{if } q=m+1. \end{aligned}$$

Recall that we have defined $k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\colon E_{(\varepsilon',\delta')}^{<a} \to E_{(\varepsilon,\delta)}^{<a}$ by the composition of the maps of (28). The next result is a variant of Proposition 6.5 for spectral sequences.

Proposition 6.8 *The following diagram commutes*:

$$\begin{tikzcd} E_\varepsilon^{\pitchfork,<a} \arrow[r, "(\Phi_{(\varepsilon,\delta)}^{<a})_*"] \arrow[d, "l_{\varepsilon,\varepsilon'}"'] & E_{(\varepsilon,\delta)}^{<a} \\ E_{\varepsilon'}^{\pitchfork,<a} \arrow[r, "(\Phi_{(\varepsilon',\delta')}^{<a})_*"] & E_{(\varepsilon',\delta')}^{<a} \arrow[u, "k_{(\varepsilon',\delta'),(\varepsilon,\delta)}"'] \end{tikzcd}$$

This can be proved as Proposition 6.5 by taking $(\bar{\Phi}_{(\varepsilon,\bar{\delta})}^{<a})_*\colon E_\varepsilon^{\pitchfork,<a} \to \bar{E}_{(\varepsilon,\bar{\delta})}^{<a}$ into consideration. We omit the proof.

We use the spectral sequence $E^{<a}$ of Proposition 4.17. The above proposition and (46) immediately implies the existence of the following morphism of spectral sequences.

Proposition 6.9 *There exists a morphism of spectral sequences $\Phi^{<a}\colon E^{\pitchfork,<a} \to E^{<a}$ such that, on the $(-m,q)$-term for $m \geq 0$ of the first page,*

$$\Phi^{<a} = \Psi_{0,m}\colon (E^{\pitchfork,<a})^1_{-m,q} = \operatorname{Coker} \partial^{\mathrm{sing}}_m \to (E^{<a})^1_{-m,q} = H^{\mathrm{dR}}_0(\Sigma^a_m, \Sigma^0_m) \quad \text{if } q = m,$$
$$\Phi^{<a} = \Psi_{1,m}\colon (E^{\pitchfork,<a})^1_{-m,q} = \operatorname{Ker} \partial^{\mathrm{sing}}_m \to (E^{<a})^1_{-m,q} = H^{\mathrm{dR}}_1(\Sigma^a_m, \Sigma^0_m) \quad \text{if } q = m+1.$$

This property of $\Phi^{<a}\colon E^{\pitchfork,<a} \to E^{<a}$ implies a result on the compatible map $\Phi^{<a}\colon H^{\pitchfork,<a}_p \to H^{<a}_p(Q,K)$.

Proposition 6.10 *$\Phi^{<a}\colon H^{\pitchfork,<a}_p \to H^{<a}_p(Q,K)$ is an isomorphism if $p = 0$ and a surjection if $p = 1$.*

Proof By Lemma 6.7 and Proposition 6.9, $\Phi^{<a}\colon (E^{\pitchfork,<a})^1_{p,q} \to (E^{<a})^1_{p,q}$ is an isomorphism if $p+q \leq 0$ and a surjection if $p+q = 1$. Since $E^{\pitchfork,<a}$ converges to $H^{\pitchfork,<a}_*$ and $E^{<a}$ converges to $H^{<a}_*(Q,K)$, we can apply Lemma 4.2 to prove the above assertion for $\Phi^{<a}\colon H^{\pitchfork,<a}_* \to H^{<a}_*(Q,K)$. □

On their limits of $a \to \infty$, $\{\Phi^{<a}\}_{a \in \mathbb{R}_{>0} \setminus \mathscr{L}(K)}$ induces an isomorphism

$$\Phi\colon H^{\pitchfork,<\infty}_0 \to H^{\mathrm{string}}_0(Q,K).$$

Combining with Proposition 6.3, we finally obtain the following result.

Theorem 6.11 *As a unital $\mathbb{R}$-algebra, $\operatorname{Cord}(Q,K;\mathbb{R})$ is isomorphic to $H^{\mathrm{string}}_0(Q,K)$.*

References

[1] **A Abbondandolo**, **A Portaluri**, **M Schwarz**, *The homology of path spaces and Floer homology with conormal boundary conditions*, J. Fixed Point Theory Appl. 4 (2008) 263–293 MR Zbl

[2] **A Abbondandolo**, **M Schwarz**, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. 59 (2006) 254–316 MR Zbl

[3] **M Abouzaid**, *Symplectic cohomology and Viterbo's theorem*, from "Free loop spaces in geometry and topology", IRMA Lect. Math. Theor. Phys. 24, Eur. Math. Soc., Zürich (2015) 271–485 MR Zbl

[4] **Y Chekanov**, *Differential algebra of Legendrian links*, Invent. Math. 150 (2002) 441–483 MR Zbl

[5] **K T Chen**, *On differentiable spaces*, from "Categories in continuum physics", Lecture Notes in Math. 1174, Springer (1986) 38–42 MR Zbl

[6] **K Cieliebak**, **T Ekholm**, **J Latschev**, **L Ng**, *Knot contact homology, string topology, and the cord algebra*, J. Éc. Polytech. Math. 4 (2017) 661–780 MR Zbl

[7] **G Dimitroglou Rizell**, *Lifting pseudo-holomorphic polygons to the symplectisation of $P \times \mathbb{R}$ and applications*, Quantum Topol. 7 (2016) 29–105 MR Zbl

[8] **T Ekholm**, **J B Etnyre**, **L Ng**, **M G Sullivan**, *Knot contact homology*, Geom. Topol. 17 (2013) 975–1112 MR Zbl

[9] **T Ekholm**, **J Etnyre**, **M Sullivan**, *The contact homology of Legendrian submanifolds in $\mathbb{R}^{2n+1}$*, J. Differential Geom. 71 (2005) 177–305 MR Zbl

[10] **T Ekholm**, **J Etnyre**, **M Sullivan**, *Orientations in Legendrian contact homology and exact Lagrangian immersions*, Int. J. Math. 16 (2005) 453–532 MR Zbl

[11] **T Ekholm**, **J Etnyre**, **M Sullivan**, *Legendrian contact homology in $P \times \mathbb{R}$*, Trans. Amer. Math. Soc. 359 (2007) 3301–3335 MR Zbl

[12] **Y Eliashberg**, *Invariants in contact topology*, from "Proceedings of the International Congress of Mathematicians, II" (G Fischer, U Rehmann, editors), Deutsche Mathematiker Vereinigung, Berlin (1998) 327–338 MR Zbl

[13] **J B Etnyre**, **L L Ng**, **J M Sabloff**, *Invariants of Legendrian knots and coherent orientations*, J. Symplectic Geom. 1 (2002) 321–367 MR Zbl

[14] **K Fukaya**, *Application of Floer homology of Langrangian submanifolds to symplectic topology*, from "Morse theoretic methods in nonlinear analysis and in symplectic topology", NATO Sci. Ser. II Math. Phys. Chem. 217, Springer (2006) 231–276 MR Zbl

[15] **K Irie**, *A chain level Batalin–Vilkovisky structure in string topology via de Rham chains*, Int. Math. Res. Not. 2018 (2018) 4602–4674 MR Zbl

[16] **K Irie**, *Chain level loop bracket and pseudo-holomorphic disks*, J. Topol. 13 (2020) 870–938 MR Zbl

[17] **J Milnor**, *Morse theory*, Ann. of Math. Stud. 51, Princeton Univ. Press (1963) MR Zbl

[18] **L Ng**, *Knot and braid invariants from contact homology, II*, Geom. Topol. 9 (2005) 1603–1637 MR Zbl

[19] **L Ng**, *Framed knot contact homology*, Duke Math. J. 141 (2008) 365–406 MR Zbl

[20] **C Viterbo**, *Generating functions, symplectic geometry, and applications*, from "Proceedings of the International Congress of Mathematicians, I" (S D Chatterji, editor), Birkhäuser, Basel (1995) 537–547 MR Zbl

[21] **C A Weibel**, *An introduction to homological algebra*, Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press (1994) MR Zbl

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Enriched quasicategories and the templicial homotopy coherent nerve

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We lay the foundations for a theory of quasicategories in a monoidal category $\mathcal{V}$ replacing Set, aimed at realising weak enrichment in the category $S\mathcal{V}$ of simplicial objects in $\mathcal{V}$. To accommodate noncartesian monoidal products, we make use of an ambient category $S_{\otimes}\mathcal{V}$ of templicial, or "tensor-simplicial", objects in $\mathcal{V}$, which are certain colax monoidal functors, following Leinster. Inspired by the description of the categorification functor due to Dugger and Spivak, we construct a templicial analogue of the homotopy coherent nerve functor which goes from $S\mathcal{V}$-enriched categories to $S_{\otimes}\mathcal{V}$. We show that an $S\mathcal{V}$-enriched category whose underlying simplicial category is locally Kan is turned into a quasicategory in $\mathcal{V}$ by this nerve functor.

18D20, 18N60; 18M05, 18N50

1 Introduction

1.A The main goal

The theory of $(\infty, 1)$-categories (or simply ∞-categories) is by now well established, with notable models including simplicial categories by Bergner [6], Segal categories by Hirschowitz and Simpson [20], complete Segal spaces by Rezk [37] and quasicategories by Joyal [23]. These models were all shown to be homotopically equivalent by the work of Bergner [7], Joyal [24; 25] and Lurie [28].

One may view ∞-categories as being "weakly enriched in spaces"; that is, between objects we have morphism spaces (usually formalised as simplicial sets) along with compositions that are only well defined and associative up to coherent homotopy. In analogy with ordinary enriched categories, one may thus conceive (weakly) enriched ∞-categories by replacing SSet by a suitable category $\mathcal{M}$ possessing some weak or higher structure, eg a monoidal model category or a monoidal ∞-category.

A general approach to enrichment is due to Gepner and Haugseng [16], who developed a theory of ∞-categories weakly enriched in a monoidal ∞-category. Their theory is built on Lurie's ∞-operads [29] and, as such, on the extensive framework of quasicategories.

Alternatively, one can consider enriched counterparts of each of the classical models for ∞-categories listed above. A particularly easy one is the "strict model" of simplicial categories, which one may replace by categories strictly enriched in a suitable monoidal model category $\mathcal{M}$. See the work of Berger and Moerdijk [5], Stanculescu [40] or Muro [35]. Further, in the case where $\mathcal{M}$ is cartesian (ie its monoidal structure is given by the cartesian product), Simpson [39] introduced $\mathcal{M}$-enriched Segal categories as certain simplicial objects in $\mathcal{M}$. Haugseng [19, Theorems 5.8 and 6.17] showed that both the strict model and Simpson's model are presentations of enriched ∞-categories in the sense of [16].

Building on Simpson's work and generalising Leinster's homotopy monoids [26], Bacard [3] defined $\mathcal{M}$-enriched Segal categories over a general monoidal model category $\mathcal{M}$ in order to encompass enriching categories of interest, like chain complexes over a commutative ring. Recently, an approach to complete dg-Segal spaces of a quite different flavour was put forward by Dimitriadis Bermejo [13], replacing the simplex category by a category of free dg-categories of finite type.

Finally, there are the particularly tangible quasicategories which have proven very successful, and whose theory has seen extensive development due to the work of Joyal, Lurie and many others. The main goal of the present paper is to lay the basic foundations for a concrete model of "enriched quasicategories", which stand to quasicategories as Bacard's enriched Segal categories stand to Segal categories. While the development of the homotopy theory of these objects is relegated to subsequent work, our constructions are motivated by homotopy-theoretic considerations, as we further explain.

For a suitable monoidal category $\mathcal{V}$, we define *quasicategories in* $\mathcal{V}$. Here, like the category Set of sets, $\mathcal{V}$ is a category not necessarily having any weak or higher structure. Instead, quasicategories in $\mathcal{V}$ should be viewed as being weakly enriched in the monoidal category $S\mathcal{V}$ of simplicial objects in $\mathcal{V}$ and, as such, they have a higher categorical nature. Here, we consider $S\mathcal{V}$ with the right-transferred model structure from SSet. This model structure exists for example if the monoidal unit I is a projective generator of $\mathcal{V}$, which goes back to Quillen [36, Section II.4]. The restriction to the case $\mathcal{M} = S\mathcal{V}$ allows us to keep our model tangible and elementary.

Like in the classical situation, quasicategories in $\mathcal{V}$ arise as a subclass of a larger category. We denote this category by $S_{\otimes}\mathcal{V}$ and call its objects *templicial* (short for *tensor-simplicial*) *objects* in $\mathcal{V}$. It is important to note that, while the hom-spaces are purported to be simplicial objects, templicial objects themselves are not. This change of perspective is necessary in order to make sense of basic constructions like the nerve, as we will explain in Section 1.B. Nonetheless, when $\mathcal{V} = \text{Set}$, both simplicial and templicial objects recover simplicial sets. Another case of particular interest is $\mathcal{V} = \mathrm{Mod}(k)$, the category of modules over a commutative ring k. Motivated by (noncommutative) algebraic geometry, where higher categorical structures like dg- and A_∞-categories play prominent roles as models for spaces, we focus on this case in subsequent work [27].

Classically, the Joyal model structure for quasicategories on SSet [23] and the model structure for simplicial categories by Bergner [6] are related by the homotopy coherent nerve functor, which is the right-adjoint in a Quillen equivalence. The construction of the homotopy coherent nerve goes back to Cordier [10] and in fact it was already shown by Cordier and Porter [11, Theorem 2.1] (though not with this terminology) that it preserves fibrant objects. Indeed, given a locally Kan simplicial category (that is, all its hom-objects are Kan complexes), its homotopy coherent nerve is a quasicategory. Taking this fact as a starting point, our main result is the following.

Theorem *There is a right-adjoint functor*

$$N^{\mathrm{hc}}_{\mathcal{V}} : \mathcal{V}\mathrm{Cat}_\Delta \to S_\otimes \mathcal{V}$$

from the category of small $S\mathcal{V}$-enriched categories to the category of templicial objects in $\mathcal{V}$ with the following properties:

1. *If* $\mathcal{V} = \mathrm{Set}$, *then $N^{\mathrm{hc}}_{\mathcal{V}}$ recovers the classical homotopy coherent nerve.*
2. *If $\mathcal{C}$ is a small $S\mathcal{V}$-enriched category whose underlying simplicial category is locally Kan*, *then $N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})$ is a quasicategory in $\mathcal{V}$.*

We call this functor the *templicial homotopy coherent nerve*. It is constructed in Section 4.B and the theorem is proven in Corollary 5.12. Some other enriched versions of the homotopy coherent nerve exist in the cartesian context. See the work of Gindi [17] and Moser, Rasekh and Rovelli [34]. We will investigate the relation of the latter nerve with ours in subsequent work [33].

The model structure on $S\mathcal{V}$-enriched categories generalises the one of Bergner on simplicial categories. We expect that a generalisation of Joyal's model structure on SSet exists for $S_\otimes\mathcal{V}$ (under suitable conditions on $\mathcal{V}$), having quasicategories in $\mathcal{V}$ as fibrant objects and making the templicial homotopy coherent nerve into a Quillen equivalence. The weak equivalences are likely reflected by the left-adjoint, which we call the *categorification functor*, for instance in the case of left transfer. This is work in progress. Note that, by [19], this would establish quasicategories in $\mathcal{V}$ as a model for ∞-categories enriched over $S\mathcal{V}$ in the sense of [16].

1.B Templicial objects and necklace categories

In order to define quasicategories in a monoidal category $\mathcal{V}$ and prove the theorem from Section 1.A, two larger categories play an important role: the category $S_\otimes\mathcal{V}$ of templicial objects and the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ of necklace categories. In this section, we will explain and motivate their occurrence, starting with the former.

Given a small category $\mathcal{C}$, recall that its nerve $N(\mathcal{C})$ is the simplicial set whose set of n-simplices is given by

$$N(\mathcal{C})_n = \coprod_{A_0,\dots,A_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(A_0, A_1) \times \dots \times \mathcal{C}(A_{n-1}, A_n).$$

The inner face maps d_j for $0 < j < n$ are given by composing two consecutive morphisms in a sequence, and the degeneracy maps s_i for $0 \le i \le n$ are given by inserting an identity in the sequence. The outer face maps d_0 and d_n are defined by deleting respectively the first and last entry in a sequence. Now suppose $\mathcal{C}$ is enriched over the (possibly noncartesian) monoidal category $\mathcal{V}$. When defining the nerve of $\mathcal{C}$, a natural first attempt is to put

$$N(\mathcal{C})_n = \coprod_{A_0,\dots,A_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(A_0, A_1) \otimes \dots \otimes \mathcal{C}(A_{n-1}, A_n) \in \mathcal{V}$$

and try to make this into a simplicial object in $\mathcal{V}$. It is readily seen that we can define inner face morphisms and degeneracy morphisms in the same way. However, the same is not true for the outer face morphisms because in general there are no projections out of a tensor product, whence we cannot "project away" the factor $\mathcal{C}(A_0, A_1)$ or $\mathcal{C}(A_{n-1}, A_n)$ in the expression above. As a consequence, we do not obtain a simplicial object, but the above data can be organised into a colax monoidal functor

$$X : \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V},$$

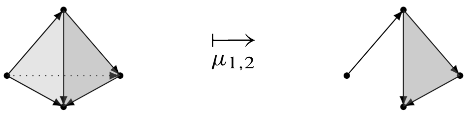
where $\mathbf{\Delta}_f$ is the monoidal category of finite intervals (see Section 1.D). Restricting from the usual simplex category $\mathbf{\Delta}$ to $\mathbf{\Delta}_f$, it follows that X no longer has any outer face maps, a loss which is compensated by the colax monoidal structure. It was shown by Leinster [26, Proposition 3.1.7] (also see Proposition 2.1 below) that, if $\mathcal{V}$ is cartesian, X may still be identified with a simplicial object in $\mathcal{V}$.

The philosophy of introducing coalgebraic structure in the noncartesian context is not uncommon. For example, Hopf algebras may be considered the group objects internal to $\mathrm{Mod}(k)$. Similarly, in their PhD thesis [1], Aguiar introduced graphs and categories internal to a monoidal category by means of bicomodules over comonoids. Such structure is invisible in a cartesian monoidal category because then every object has a unique comonoid structure. The same philosophy was applied by Bacard in their definition of $\mathcal{M}$-enriched Segal categories [3]. These are many-object versions of Leinster's homotopy monoids [26], based on the colax monoidal functors above.

Let us describe templicial objects in a little more detail. In a similar but nonequivalent way to [3], we define templicial objects as certain colax monoidal functors on $\mathbf{\Delta}_f$ with a discrete set of vertices. More precisely, a templicial object in $\mathcal{V}$ with vertex set S is a strongly unital, colax monoidal functor

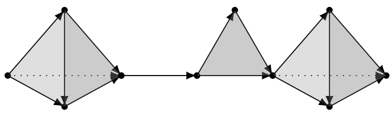
$$X : \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S,$$

where $\mathcal{V}\mathrm{Quiv}_S$ denotes the category of $\mathcal{V}$-enriched quivers with vertex set S. The colax monoidal structure equips X with quiver morphisms $\mu_{k,l} : X_{k+l} \to X_k \otimes_S X_l$ for all $k, l \ge 0$. For example, $\mu_{1,2}$ may be pictured as



Intuitively, $\mu_{k,l}$ involves pulling apart $(k+l)$-simplices into k-simplices attached to l-simplices at a vertex. We can thus no longer access outer faces of a simplex directly. The shift of focus to faces joint at a vertex naturally leads us to considering necklaces (see Section 3).

Necklaces were introduced by Baues [4] (under a different name) and popularised by Dugger and Spivak [14] in their description of the categorification functor. Roughly, a necklace is a sequence of simplices glued at the endpoints:



$$\Delta^3 \vee \Delta^1 \vee \Delta^2 \vee \Delta^3$$

Necklaces naturally occur in the interpretation of the comultiplications, with $\mu_{1,2}$ being parametrised by the necklace map

$$\nu_{1,2} \colon \Delta^1 \vee \Delta^2 \to \Delta^3$$

(see Notation 3.10). As such, they allow us to turn colax monoidal functors on $\mathbf{\Delta}_f$ into ordinary functors on the category $\mathcal{N}ec$ of necklaces, putting $X(\Delta^1 \vee \Delta^2) = X_1 \otimes_S X_2$ and $X(\nu_{1,2}) = \mu_{1,2}$ in the above example. Endowing the functor category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ with the Day convolution, we define a necklace category to be a category enriched in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. This allows us to realise $S_\otimes\mathcal{V}$ as a coreflective subcategory of the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ of necklace categories,

$$S_\otimes\mathcal{V} \hookrightarrow \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}. \tag{1}$$

This embedding will turn up as a crucial intermediate step in defining the templicial homotopy coherent nerve in Section 4. In the definition of quasicategories in $\mathcal{V}$ in Section 5, the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ also plays a fundamental role as the context in which we express the familiar lifting property with respect to inner horn inclusions.

1.C Overview of the paper

Next we give an overview of the contents of the paper. In Section 2, we formally introduce templicial objects and prove some basic properties. Starting in Section 2.A, we compare them to simplicial sets. In Section 2.B, we construct the templicial analogue of the classical nerve functor for small $\mathcal{V}$-enriched categories. For templicial objects which have nondegenerate simplices in an appropriate sense, in Section 2.C we prove a version of the Eilenberg–Zilber lemma. In general, the structure of a templicial object X is considerably richer than that of the underlying simplicial set $\widetilde{U}(X)$ (see Proposition 2.8 and

Remark 2.9). In particular, unlike in the classical case, simplices are no longer represented by morphisms from standard simplices (the "representation problem"; see Example 2.10).

This representation problem is solved in Section 3 with the introduction of necklace categories. In Section 3.A, we recall necklaces and give a combinatorial characterisation of their category $\mathcal{N}ec$. In Section 3.B, we define necklace categories and realise the category of templicial objects as a coreflective subcategory of the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ (Theorem 3.12). Finally, in Section 3.C, we observe that both the underlying simplicial set functor $\widetilde{U}$ and the templicial nerve $N_{\mathcal{V}}$ naturally factor through $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$.

In Section 4, we generalise the classical homotopy coherent nerve and the categorification functor (Definition 4.9). We follow the elegant approach from [14], which we recall in Section 4.A before presenting our enriched counterpart in Section 4.B. The key observation relating the two is a description of the categorification by means of a weighted colimit (Proposition 4.6).

Starting from the embedding (1), in order to construct the categorification, we construct a functor from necklace categories to $S\mathcal{V}$-enriched categories. Following [14], the categorification is simplified by using flanked flags in Section 4.C, in the presence of the nondegenerate simplices from Section 2.C. Finally, in Section 4.D, we show that the templicial homotopy coherent nerve reduces to the templicial nerve in the desired way. As usual, for a templicial object X, we naturally obtain a $\mathcal{V}$-enriched homotopy category $h_{\mathcal{V}}X$ as π_0 of the categorification. In general, the underlying simplicial set and underlying category functors do not commute with taking homotopy categories, as shown in Example 4.22.

In Section 5, we introduce the natural analogue of quasicategories in the templicial setting, which will remedy the aforementioned failure to commute. A quasicategory in $\mathcal{V}$ is defined as a templicial object satisfying a familiar lifting property with respect to inner horn inclusions (Definition 5.4). In contrast to the classical setup, this lifting property is considered in the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ rather than $S_{\otimes}\mathcal{V}$ because of the representation problem. The resulting notion is in general strictly stronger than requiring the underlying simplicial set $\widetilde{U}(X)$ to be a quasicategory. Nonetheless, when $\mathcal{V} = \mathrm{Set}$, we still recover ordinary quasicategories. We also show our main result (Corollary 5.12; see the theorem above). Finally, in Section 5.C, we show how the description of the homotopy category $h_{\mathcal{V}}X$ can be simplified when X is a quasicategory in $\mathcal{V}$. Moreover, the underlying category of the homotopy category corresponds to the homotopy category of the underlying ordinary quasicategory.

1.D Notation and conventions

(1) Throughout the text, we let $(\mathcal{V}, \otimes, I)$ be a fixed bicomplete, symmetric monoidal closed category (ie a Bénabou cosmos in the sense of Street [41]). Up to natural isomorphism, there is a unique colimit-preserving functor $F\colon \mathrm{Set} \to \mathcal{V}$ such that $F(\{*\}) = I$. This functor is left-adjoint to the forgetful functor $U = \mathcal{V}(I, -)\colon \mathcal{V} \to \mathrm{Set}$. Endowing Set with the cartesian monoidal structure, F is strong monoidal and U is lax monoidal. This notation will remain fixed as well.

(2) Let $(\mathcal{W}, \otimes, I)$ be an arbitrary monoidal category. Given a set S, we refer to a collection $Q = (Q(a,b))_{a,b \in S}$ with $Q(a,b) \in \mathcal{W}$ as a *$\mathcal{W}$-enriched quiver* with S its set of *vertices*. A *quiver morphism* $f : Q \to P$ is a collection $(f_{a,b})_{a,b \in S}$ of morphisms $f_{a,b} : Q(a,b) \to P(a,b)$ in $\mathcal{W}$. $\mathcal{W}$-enriched quivers with a fixed set of vertices S and morphisms between them form a category, which we denote by

$$\mathcal{W}\mathrm{Quiv}_S.$$

This category is monoidal with product $\otimes_S$ and unit I_S defined by

$$(Q \otimes_S P)(a,b) = \coprod_{c \in S} Q(a,c) \otimes P(c,b) \qquad \text{and} \qquad I_S(a,b) = \begin{cases} I & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

for all $Q, P \in \mathcal{W}\mathrm{Quiv}_S$ and $a, b \in S$.

(3) Let $f : S \to T$ be a map of sets. We have an induced lax monoidal functor $f^* : \mathcal{W}\mathrm{Quiv}_T \to \mathcal{W}\mathrm{Quiv}_S$ given by $f^*(Q)(a,b) = Q(f(a), f(b))$ for all $\mathcal{W}$-enriched quivers Q and $a, b \in S$. The functor f^* has a left-adjoint, which we denote by $f_! : \mathcal{W}\mathrm{Quiv}_S \to \mathcal{W}\mathrm{Quiv}_T$. It is given by

$$f_!(Q)(x,y) = \coprod_{\substack{a,b \in S \\ f(a)=x \\ f(b)=y}} Q(a,b)$$

for all $Q \in \mathcal{W}\mathrm{Quiv}_S$ and $x, y \in T$. As f^* is canonically lax monoidal, $f_!$ comes equipped with an induced colax monoidal structure.

(4) To relate $\mathcal{V}$-enriched and $S\mathcal{V}$-enriched categories to templicial objects (see Sections 2.B and 4.B), it will be more convenient for us to consider a $\mathcal{W}$-enriched category (or $\mathcal{W}$-category for short) as a pair $(\mathcal{C}, \mathrm{Ob}(\mathcal{C}))$ with $\mathrm{Ob}(\mathcal{C})$ its set of objects and $\mathcal{C}$ a monoid in $\mathcal{W}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$. Note that this convention implies that the composition in $\mathcal{C}$ is given by a collection of morphisms in $\mathcal{W}$, for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$,

$$m_{\mathcal{C}} : \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \to \mathcal{C}(A,C),$$

as opposed to the more conventional $\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \to \mathcal{C}(A,C)$. A $\mathcal{W}$-functor $\mathcal{C} \to \mathcal{D}$ is then a pair (H, f) with $f : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ a map of sets and $H : \mathcal{C} \to f^*(\mathcal{D})$ a morphism of monoids in $\mathcal{W}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$, where we used the lax structure of f^*. We denote the category of small $\mathcal{W}$-categories and $\mathcal{W}$-functors between them by

$$\mathcal{W}\mathrm{Cat}.$$

(5) We will make use of the simplex categories $\mathbf{\Delta}_{\mathrm{surj}} \subseteq \mathbf{\Delta}_f \subseteq \mathbf{\Delta}$, where:

- $\mathbf{\Delta}$ is the ordinary *simplex category*. Its objects are the posets $[n] = \{0, \dots, n\}$ with $n \geq 0$, and its morphisms are the order morphisms $[m] \to [n]$.
- $\mathbf{\Delta}_f$ is the category of *finite intervals*, which is the subcategory of $\mathbf{\Delta}$ consisting of all morphisms $f : [m] \to [n]$ that preserve the endpoints, that is, $f(0) = 0$ and $f(m) = n$.
- $\mathbf{\Delta}_{\mathrm{surj}}$ is the subcategory of $\mathbf{\Delta}$ of all surjective morphisms $[m] \twoheadrightarrow [n]$.

Unlike $\mathbf{\Delta}$, both $\mathbf{\Delta}_f$ and $\mathbf{\Delta}_{\mathrm{surj}}$ carry a monoidal structure $(+,[0])$, which is given by identifying their respective top and bottom endpoints, as follows. For all $m,n \geq 0$,

$$[m]+[n]=[m+n].$$

For morphisms $f\colon [m]\to[m']$ and $g\colon [n]\to[n']$ in $\mathbf{\Delta}_f$ or $\mathbf{\Delta}_{\mathrm{surj}}$,

$$(f+g)(i)=\begin{cases} f(i) & \text{if } i\leq m,\\ m'+g(i-m) & \text{if } i\geq m.\end{cases}$$

Note that, for any morphism $f\colon[m]\to[n]$ in $\mathbf{\Delta}_f$ and $k,l\geq 0$ such that $k+l=m$, there exist unique morphisms $f_1\colon[k]\to[p]$ and $f_2\colon[l]\to[q]$ in $\mathbf{\Delta}_f$ such that $f_1+f_2=f$.

There is a well-known monoidal equivalence between $\mathbf{\Delta}_f^{\mathrm{op}}$ and the augmented simplex category $\mathbf{\Delta}_+$ (equipped with the join as monoidal product). This is known as Joyal's duality; see [21].

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2 Templicial objects

Aguiar [1] defined a graph internal to a monoidal category $\mathcal{W}$ as a pair (G_1,G_0) where G_0 is a comonoid in $\mathcal{W}$ and G_1 is a bicomodule over G_0. When $\mathcal{W}$ is cartesian monoidal, this recovers the usual notion of a graph internal to a category, namely a pair of morphisms $s,t\colon G_1\rightrightarrows G_0$ expressing the source and target. Extending this philosophy to higher dimensions, we propose to define a *simplicial object internal to a monoidal category* $\mathcal{W}$ as a colax monoidal functor

$$X\colon \mathbf{\Delta}_f^{\mathrm{op}}\to\mathcal{W},$$

where $\mathbf{\Delta}_f$ is the monoidal category of finite intervals (see Section 1.D). The restriction to $\mathbf{\Delta}_f$ precisely gets rid of the outer face maps, which are replaced by the colax monoidal structure. To justify this change, let us remark that the colax structure of X provides $X_0\in\mathcal{W}$ with the structure of a comonoid in $\mathcal{W}$ and X_1 with that of bicomodule over X_0. In other words, (X_1,X_0) is a graph internal to $\mathcal{W}$ in the sense of [1]. Moreover, it was shown by Leinster [26] (reappearing here as Proposition 2.1) that, if $\mathcal{W}$ is cartesian, then X recovers a simplicial object in $\mathcal{W}$.

Because enriched categories have a set of objects, we also equip such colax monoidal functors with a discrete set of vertices. Formally, we achieve this by putting $\mathcal{W} = \mathcal{V}\mathrm{Quiv}_S$ for a set S (see Section 1.D) and requiring the functor to be strongly unital. This leads to the definition of our main objects of study, templicial objects (Definition 2.3). We then define the natural analogue of the classical nerve functor in this context (Definition 2.11), taking $\mathcal{V}$-categories to templicial objects in $\mathcal{V}$. Finally, we show a generalisation of the Eilenberg–Zilber lemma (Lemma 2.19).

2.A Simplicial versus templicial objects

Let us make explicit what data are contained in a colax monoidal functor $X \colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ for a monoidal category $\mathcal{W}$ and compare it to the data of a simplicial object. For general background on (co)lax monoidal functors, see eg [2]. Explicitly, such a functor X consists of a collection of objects $X_0, X_1, X_2, \dots$ of $\mathcal{W}$ along with

- inner face morphisms $d_j^X \colon X_n \to X_{n-1}$ for all $0 < j < n$,
- degeneracy morphisms $s_i^X \colon X_n \to X_{n+1}$ for all $0 \le i \le n$,
- comultiplication morphisms $\mu_{k,l}^X \colon X_{k+l} \to X_k \otimes X_l$ for all $k, l \ge 0$,
- a counit morphism $\epsilon^X \colon X_0 \to I$.

These data moreover must satisfy:

- **Simplicial identities** For all $i, j \ge 0$, whenever these equations are well defined,
$$d_i^X s_j^X = \begin{cases} s_{j-1}^X d_i^X & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1, \\ s_j^X d_{i-1}^X & \text{if } i > j+1, \end{cases}$$
$$d_i^X d_j^X = d_{j-1}^X d_i^X \quad \text{if } i < j, \qquad s_i^X s_j^X = s_j^X s_{i-1}^X \quad \text{if } i > j.$$
- **Naturality of $\boldsymbol{\mu}^X$** For all $k, l \ge 0$ and $0 < j < k+l+1$, $0 \le i \le k+l-1$,
$$\mu_{k,l}^X d_j^X \begin{cases} (d_j^X \otimes \mathrm{id}_{X_l})\mu_{k+1,l}^X & \text{if } j \le k, \\ (\mathrm{id}_{X_k} \otimes d_{j-k}^X)\mu_{k,l+1}^X & \text{if } j > k, \end{cases} \qquad \mu_{k,l}^X s_i^X = \begin{cases} (s_i^X \otimes \mathrm{id}_{X_l})\mu_{k-1,l}^X & \text{if } i < k, \\ (\mathrm{id}_{X_k} \otimes s_{i-k}^X)\mu_{k,l-1}^X & \text{if } i \ge k. \end{cases}$$
- **Coassociativity of $\boldsymbol{\mu}^X$** For all $r, s, t \ge 0$,
$$(\mathrm{id}_{X_r} \otimes \mu_{s,t}^X)\mu_{r,s+t}^X = (\mu_{r,s}^X \otimes \mathrm{id}_{X_t})\mu_{r+s,t}^X.$$
- **Counitality of $\boldsymbol{\mu}^X$ with $\boldsymbol{\epsilon}^X$** For all $n \ge 0$,
$$(\mathrm{id}_{X_n} \otimes \epsilon^X)\mu_{n,0}^X = \mathrm{id}_{X_n} = (\epsilon^X \otimes \mathrm{id}_{X_n})\mu_{0,n}^X.$$

Note that, by the coassociativity, we have a well-defined morphism
$$\mu_{k_1,\dots,k_n}^X \colon X_{k_1+\dots+k_n} \to X_{k_1} \otimes \dots \otimes X_{k_n}$$
for all $n \ge 2$ and $k_1, \dots, k_n \ge 0$. Further, we will set $\mu_{k_1,\dots,k_n}^X$ to be the identity on X_{k_1} if $n = 1$, and to be the counit ϵ^X if $n = 0$.

Moreover, under these identifications, a monoidal natural transformation $\alpha\colon X \to Y$ between colax monoidal functors $X, Y : \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ is equivalent to a collection of morphisms $(\alpha_n : X_n \to Y_n)_{n\geq 0}$ which satisfy:

- **Naturality of $\boldsymbol{\alpha}$** For all $0 < j < n$ and $0 \leq i \leq n$,

$$\alpha_{n-1} d_j^X = d_j^Y \alpha_n \quad \text{and} \quad \alpha_{n+1} s_i^X = s_i^Y \alpha_n.$$

- **Monoidality of $\boldsymbol{\alpha}$** For all $k, l \geq 0$,

$$\mu_{k,l}^Y \alpha_{k+l} = (\alpha_k \otimes \alpha_l)\mu_{k,l}^X \quad \text{and} \quad \epsilon^Y \alpha_0 = \epsilon^X.$$

Often we will drop the superscript X when it is clear from context.

We denote by $\mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}}, \mathcal{W})$ the category of colax monoidal functors $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ and monoidal natural transformations between them.

Proposition 2.1 [26, Proposition 3.1.7] *Let $\mathcal{W}$ be a cartesian monoidal category. There is an isomorphism of categories*

$$\mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}}, \mathcal{W}) \simeq S\mathcal{W}.$$

Remark 2.2 Suppose $\mathcal{W}$ is cartesian and let X be a simplicial object in $\mathcal{W}$. Its associated colax monoidal functor $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ has comultiplication morphisms given by

$$\mu_{k,l} : X_{k+l} \xrightarrow{(d_{k+1}\dots d_n, d_0 \dots d_0)} X_k \times X_l.$$

Conversely, suppose $X : \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ is a colax monoidal functor. The outer face morphisms of the associated simplicial object are obtained by using the projections of the product

$$d_0 : X_n \xrightarrow{\mu_{1,n-1}} X_1 \times X_{n-1} \xrightarrow{\pi_2} X_{n-1} \quad \text{and} \quad d_n : X_n \xrightarrow{\mu_{n-1,1}} X_{n-1} \times X_1 \xrightarrow{\pi_1} X_{n-1}.$$

If $\mathcal{W}$ is not cartesian, these projections are not available in general and the comultiplication μ of a colax monoidal functor can be considered as a replacement for the outer face morphisms in the monoidal context.

From now on, we will only consider such colax monoidal functors $X : \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$ with a discrete set of vertices. We formalise this by replacing $\mathcal{W}$ by a category $\mathcal{V}\mathrm{Quiv}_S$ of $\mathcal{V}$-enriched quivers (see Section 1.D) for some set S, and requiring that X be strongly unital.

An alternative but nonequivalent way to realise a set of vertices S consists in turning the monoidal category $\mathbf{\Delta}_f$ (which is a one-object bicategory) into a bicategory with object set S. This approach goes back to [28] and was used in [39; 3].

Definition 2.3 A *tensor-simplicial* or *templicial object* in $\mathcal{V}$ is a pair (X, S) with S a set and

$$X\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S$$

a colax monoidal functor which is strongly unital, ie its counit $\epsilon\colon X_0 \to I_S$ is an isomorphism. We call the elements of S the *vertices* of X. For $n > 0$, an *n-simplex* of X is an element of the set $U(X_n(a,b)) \in \mathcal{V}$ for some $a, b \in S$.

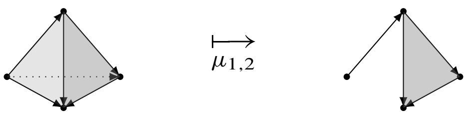
Let (X, S) and (Y, T) be templicial objects. A *templicial morphism* $(X, S) \to (Y, T)$ is a pair (α, f) with $f : S \to T$ a map of sets and $\alpha\colon f_! X \to Y$ a monoidal natural transformation between colax monoidal functors $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_T$. Here, we used the colax monoidal structure of $f_!$.

Sometimes we will denote a templicial object (X, S) or a templicial morphism (α, f) simply by X or α, respectively, assuming the underlying set or map of sets is clear.

Remark 2.4 Let (X, S) be a templicial object in $\mathcal{V}$ and consider $a, b \in S$. Then $X_n(a,b) \in \mathcal{V}$ should be interpreted as the *object of n-simplices of X with first vertex a and last vertex b*. Moreover, for all $k, l \geq 0$ and $a, b \in S$, the comultiplication morphism

$$(\mu_{k,l}^X)_{a,b}\colon X_{k+l}(a,b) \to \coprod_{c \in S} X_k(a,c) \otimes X_l(c,b)$$

should be interpreted as taking a $(k+l)$-simplex from a to b and sending it to a k-simplex from a to some $c \in S$, along with an l-simplex from c to b, which are outer faces of the original $(k+l)$-simplex:



Unlike for simplicial objects, we thus no longer have direct access to the outer faces of a simplex, only to outer faces which are glued at a vertex.

Definition 2.5 Given maps of sets $f : S \to T$ and $g\colon T \to U$, there is a canonical monoidal natural isomorphism $(gf)_! \simeq g_! f_!$ between colax monoidal functors $\mathcal{V}\mathrm{Quiv}_S \to \mathcal{V}\mathrm{Quiv}_U$. Consequently, we can define the *composition* of two templicial morphisms $(\alpha, f)\colon (X, S) \to (Y, T)$ and $(\beta, g)\colon (Y, T) \to (Z, U)$ as the templicial morphism (γ, gf) with

$$\gamma\colon (gf)_! X \simeq g_! f_! X \xrightarrow{g_! \alpha} g_! Y \xrightarrow{\beta} Z.$$

Further, we have a canonical monoidal natural isomorphism $\varphi\colon (\mathrm{id}_S)_! \xrightarrow{\sim} \mathrm{id}_{\mathcal{V}\mathrm{Quiv}_S}$ for any set S, and the *identity* at (X, S) is defined as the templicial morphism $(\varphi X, \mathrm{id}_S)$. It is then easy to see that templicial objects in $\mathcal{V}$ and templicial morphisms between them form a category, which we denote by

$$S_\otimes \mathcal{V}.$$

Remark 2.6 A more abstract construction of the category $S_\otimes\mathcal{V}$ of templicial objects is as follows. Given a set S, consider the category $\Phi(S) = \mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}}, \mathcal{V}\mathrm{Quiv}_S)$. For a map of sets $f: S \to T$, let $\Phi(f)\colon \Phi(S) \to \Phi(T)$ be the functor given by postcomposition with $f_!$. Then Φ is not a functor since it does not preserve composition on the nose. But one can show that the isomorphisms $(gf)_! \simeq g_! f_!$ above do define a pseudofunctor $\Phi\colon \mathrm{Set} \to \mathrm{Cat}$. Taking the Grothendieck construction $\int \Phi$, we find $S_\otimes\mathcal{V}$ as the full subcategory spanned by the strongly unital colax functors $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S$.

Proposition 2.7 *There is an equivalence of categories*

$$S_\times \mathrm{Set} \simeq \mathrm{SSet}.$$

Proof Let K be a simplicial set. By Proposition 2.1, we may consider K as a colax monoidal functor $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathrm{Set}$ with comultiplication μ and counit ϵ. Then define, for all $n \geq 0$ and $a, b \in K_0$,

$$K_n(a,b) = \{\sigma \in K_n \mid d_1 \dots d_n(\sigma) = a,\ d_0 \dots d_0(\sigma) = b\}.$$

Given $f: [m] \to [n]$ in $\mathbf{\Delta}_f$, it follows from the simplicial identities that $K(f)\colon K_n \to K_m$ restricts to a map $K(f)_{a,b}\colon K_n(a,b) \to K_m(a,b)$. Moreover, it is clear that, for all $k, l \geq 0$ and $a, b \in K_0$, $\mu_{k,l}$ restricts to

$$\mu_{k,l}|_{K_{k+l}(a,b)}\colon K_{k+l}(a,b) \to \coprod_{c \in K_0} K_k(a,c) \times K_l(c,b)$$

and $K_0(a,a) = \{a\}$ if $a = b$, while $K_0(a,b) = \varnothing$ if $a \neq b$. Consequently, the functor

$$\varphi(K)\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathrm{Quiv}_{K_0}, \quad [n] \mapsto (K_n(a,b))_{a,b \in K_0},$$

is strongly unital and colax monoidal. Hence $(\varphi(K), K_0)$ is a templicial object.

Conversely, if (X, S) is a templicial object in Set, then we can define a simplicial set $\mathfrak{c}(X)$ by setting, for all $n \geq 0$,

$$\mathfrak{c}(X)_n = \coprod_{a,b \in S} X_n(a,b).$$

It is readily verified that the assignments $K \mapsto \varphi(K)$ and $X \mapsto \mathfrak{c}(X)$ can be extended to mutually inverse equivalences between SSet and $S_\times$ Set. □

As $F: \mathrm{Set} \to \mathcal{V}$ preserves colimits and is strong monoidal, postcomposition with F induces a functor

$$\tilde{F}\colon \mathrm{SSet} \simeq S_\times \mathrm{Set} \to S_\otimes\mathcal{V}.$$

More precisely, given a simplicial set K, $\tilde{F}(K)$ has vertex set K_0 and, for all $a, b \in K_0$ and $n \geq 0$,

$$\tilde{F}(K)_n(a,b) = F(K_n(a,b)),$$

where $K_n(a,b) = \{\sigma \in K_n \mid d_1 \dots d_n(\sigma) = a,\ d_0 \dots d_0(\sigma) = b\}$ is the set of n-simplices of K with first vertex a and last vertex b, as above.

Proposition 2.8 *The category of templicial objects $S_\otimes\mathcal{V}$ is cocomplete and the functor $\widetilde{F}\colon \mathrm{SSet}\to S_\otimes\mathcal{V}$ has a right-adjoint*

$$\widetilde{U}\colon S_\otimes\mathcal{V}\to \mathrm{SSet}.$$

Proof As $\mathcal{V}$ is cocomplete, so is $\mathcal{V}\mathrm{Quiv}_S$. It is readily verified that then also $\mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}},\mathcal{V}\mathrm{Quiv}_S)$ is cocomplete with colimits given pointwise. Now consider a diagram $D\colon \mathcal{J}\to S_\otimes\mathcal{V}$, $j\mapsto (X^j,S^j)$, with $\mathcal{J}$ a small category. Let $S=\mathrm{colim}_{j\in\mathcal{J}} S^j$ in Set and write $\iota_j\colon S^j\to S$ for the canonical map. Then consider the colimit $X=\mathrm{colim}_{j\in\mathcal{J}}(\iota_j)_! X^j$ in $\mathrm{Colax}(\mathbf{\Delta}_f^{\mathrm{op}},\mathcal{V}\mathrm{Quiv}_S)$. The counit of X is

$$\operatorname*{colim}_{j\in\mathcal{J}}(\iota_j)_!(X_0^j)\xrightarrow{\mathrm{colim}_j(\iota_j)_!(\epsilon_{X^j})}\operatorname*{colim}_{j\in\mathcal{J}}(\iota_j)_!(I_{S^j})\xrightarrow{\sim} I_S,$$

which is an isomorphism since each ϵ_{X^j} is. Thus the pair (X,S) is a templicial object, which is easily seen to be the colimit of the diagram D in $S_\otimes\mathcal{V}$.

With the above description of the colimits in $S_\otimes\mathcal{V}$, it is clear that $\widetilde{F}$ preserves colimits and therefore has a right-adjoint $\widetilde{U}\colon S_\otimes\mathcal{V}\to\mathrm{SSet}$ given by $\widetilde{U}(X)_n=S_\otimes\mathcal{V}(\widetilde{F}(\Delta^n),X)$ for all templicial objects X and integers $n\geq 0$. □

Remark 2.9 Let us make the right-adjoint $\widetilde{U}$ a bit more explicit. Given a templicial object (X,S), an n-simplex of $\widetilde{U}(X)$ is a templicial morphism $\widetilde{F}(\Delta^n)\to X$, which is equivalent to a pair

$$((a_i)_{i=0}^n,(\alpha_{i,j})_{0\leq i<j\leq n})$$

with $a_i\in S$ and $\alpha_{i,j}\in U(X_{j-i}(a_i,a_j))$ such that, for all $0\leq i<k<j\leq n$,

$$\mu_{k-i,j-k}(\alpha_{i,j})=\alpha_{i,k}\otimes\alpha_{k,j}$$

in $U((X_{k-i}\otimes_S X_{j-k})(a_i,a_j))$. For example, $\widetilde{U}(X)_0$ recovers the set S and $\widetilde{U}(X)_1$ is given by the disjoint union of all sets $U(X_1(a,b))$ with $a,b\in S$.

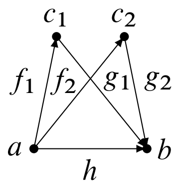
Unlike the case for simplicial objects $S\mathcal{V}$, not every n-simplex of a templicial object $(X,S)\in S_\otimes\mathcal{V}$ is uniquely represented by a morphism $\widetilde{F}(\Delta^n)\to X_n$. More precisely, the canonical map

$$\widetilde{U}(X)_n\to\coprod_{a,b\in S}U(X_n(a,b)),\quad (\alpha_{i,j})_{i,j}\mapsto\alpha_{0,n}, \tag{2}$$

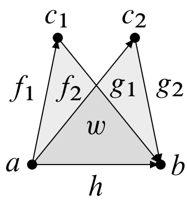
need not be injective or surjective for $n\geq 2$, as is shown in Example 2.10.

The lack of representation of simplices by morphisms makes templicial objects considerably harder to work with than ordinary simplicial sets. In an effort to resolve this issue, we extend $S_\otimes\mathcal{V}$ to a category of enriched categories $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ in Section 3.B.

Example 2.10 Let $\mathcal{V} = \mathrm{Ab}$ be the monoidal category of abelian groups with the tensor product as monoidal product and $\mathbb{Z}$ as monoidal unit. Consider the simplicial set $K = \partial\Delta^2 \amalg_{\Delta^1} \partial\Delta^2$:



We can extend $\widetilde{F}(K)$ to a templicial object X with a 2-simplex $w \in X_2(a,b)$ by setting $X_2(a,b) = \mathbb{Z}w \oplus F(K_2(a,b))$ and similarly adding the degeneracies of w. The inner face maps and comultiplication maps of X are uniquely determined by setting $d_1(w) = h$ and $\mu_{1,1}(w) = f_1 \otimes g_1 + f_2 \otimes g_2$:



Then w does not lie in the image of the map (2) for $n = 2$. Indeed, this would require a 2-simplex $(\alpha_{i,j})_{0\le i<j\le 2}$ of $\widetilde{U}(X)$ with $\alpha_{0,2} = w$. But $\mu_{1,1}(w)$ is not a pure tensor while $\mu_{1,1}(\alpha_{0,2}) = \alpha_{0,1} \otimes \alpha_{1,2}$. Moreover, since $\mu_{1,1}(2s_0(h)) = 2s_0(a) \otimes h = s_0(a) \otimes 2h$, we have two distinct 2-simplices of $\widetilde{U}(X)_2$ which map to $2s_0(h) \in X_2(a,b)$.

2.B The templicial nerve

Given a monoidal category $\mathcal{W}$, there is a well-known equivalence (this goes back to Mac Lane [31, Section V.II])

$$\mathrm{Mon}(\mathcal{W}) \simeq \mathrm{StrMon}(\mathbf{\Delta}_+, \mathcal{W}) \tag{3}$$

between the categories of monoids in $\mathcal{W}$ and strong monoidal functors $\mathbf{\Delta}_+ \to \mathcal{W}$. Due to the monoidal equivalence $\mathbf{\Delta}_+ \simeq \mathbf{\Delta}_f^{\mathrm{op}}$, we may as well consider strong monoidal functors $\mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{W}$.

Definition 2.11 Let $\mathcal{C}$ be a small $\mathcal{V}$-category, which we consider as a monoid $(\mathcal{C}, m_{\mathcal{C}}, u_{\mathcal{C}})$ in $\mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$. Applying (3) to the case $\mathcal{W} = \mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$, we obtain an associated strong monoidal functor, which we denote by

$$N_{\mathcal{V}}(\mathcal{C})\colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}.$$

In particular, the pair $(N_{\mathcal{V}}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$ forms a templicial object, which we call the *templicial nerve* of the $\mathcal{V}$-category $\mathcal{C}$.

Explicitly, $N_{\mathcal{V}}(\mathcal{C})$ is given by taking the n-fold monoidal product of the $\mathcal{V}$-quiver $\mathcal{C}$,

$$N_{\mathcal{V}}(\mathcal{C})_n = \mathcal{C}^{\otimes n},$$

for all integers $n \geq 0$. Further, the inner face and degeneracy morphisms are

$$d_j = \mathrm{id}_{\mathcal{C}}^{\otimes j-1} \otimes_S m_{\mathcal{C}} \otimes_S \mathrm{id}_{\mathcal{C}}^{\otimes n-j-1} \colon \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes n-1}, \qquad s_i = \mathrm{id}_{\mathcal{C}}^{\otimes i} \otimes_S u_{\mathcal{C}} \otimes_S \mathcal{C}^{\otimes n-i} \colon \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes n+1}$$

for all $0 \leq i \leq n$ and $0 < j < n$. Finally, the comultiplication morphisms and counit are given by the canonical isomorphisms

$$\mu_{k,l} \colon \mathcal{C}^{\otimes k+l} \xrightarrow{\sim} \mathcal{C}^{\otimes k} \otimes_S \mathcal{C}^{\otimes l} \quad \text{and} \quad \epsilon \colon \mathcal{C}^{\otimes 0} \xrightarrow{\sim} I_{\mathrm{Ob}(\mathcal{C})}$$

for any $k, l \geq 0$.

Recall the base change functor $f_! \colon \mathcal{V}\mathrm{Quiv}_S \to \mathcal{V}\mathrm{Quiv}_T$ and its right-adjoint $f^* \colon \mathcal{V}\mathrm{Quiv}_T \to \mathcal{V}\mathrm{Quiv}_S$ for a given map of sets $f : S \to T$ (see Section 1.D).

Lemma 2.12 *Let (X, S) be a templicial object, $\mathcal{C}$ a small $\mathcal{V}$-enriched category and $f : S \to \mathrm{Ob}(\mathcal{C})$ a map of sets. Then we have a bijection between monoidal natural transformations $f_! X \to N_{\mathcal{V}}(\mathcal{C})$ and quiver morphisms $H \colon X_1 \to f^*(\mathcal{C})$ such that the diagrams*

$$\begin{tikzcd} X_1^{\otimes 2} \arrow[r, "H^{\otimes 2}"] & f^*(\mathcal{C})^{\otimes 2} \arrow[r] & f^*(\mathcal{C}^{\otimes 2}) \arrow[d, "f^*(\tilde{m}_{\mathcal{C}})"] \\ X_2 \arrow[u, "\mu_{1,1}"] \arrow[r, "d_1"'] & X_1 \arrow[r, "H"'] & f^*(\mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} & I_S \arrow[r] & f^*(I_{\mathrm{Ob}(\mathcal{C})}) \arrow[d, "f^*(u_{\mathcal{C}})"] \\ X_0 \arrow[ur, "\epsilon", "\sim"'] \arrow[r, "s_0"'] & X_1 \arrow[r, "H"'] & f^*(\mathcal{C}) \end{tikzcd} \tag{4}$$

commute.

Proof Given a monoidal natural transformation $\alpha \colon f_! X \to N_{\mathcal{V}}(\mathcal{C})$, define $H_\alpha \colon X_1 \to f^*(\mathcal{C})$ to be adjoint to $\alpha_1 \colon f_!(X_1) \to \mathcal{C}$. It follows from the monoidality of α that, for all $n \geq 0$, α_n is the composite

$$f_!(X_n) \xrightarrow{f_!(\mu_{1,\dots,1})} f_!(X_1^{\otimes n}) \to f_!(X_1)^{\otimes n} \xrightarrow{\alpha_1^{\otimes n}} \mathcal{C}^{\otimes n},$$

where we used the colax monoidal structure of $f_!$. So the assignment $\alpha \mapsto H_\alpha$ is injective. Moreover, it then follows from the naturality of α that H_α satisfies (4). Conversely, if $H \colon X_1 \to f^*(\mathcal{C})$ satisfies (4), then, defining α_1 as adjoint to H and α_n as above, it follows that $\alpha \colon f_! X \to N_{\mathcal{V}}(\mathcal{C})$ is a natural transformation. It is immediate that α is also monoidal. □

Proposition 2.13 *The assignment $\mathcal{C} \mapsto N_{\mathcal{V}}(\mathcal{C})$ of Definition 2.11 extends to a fully faithful functor $N_{\mathcal{V}} \colon \mathcal{V}\mathrm{Cat} \to S_\otimes \mathcal{V}$. The essential image of $N_{\mathcal{V}}$ consists of all templicial objects (X, S) for which $X \colon \mathbf{\Delta}_f^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S$ is strong monoidal.*

Proof Let $\mathcal{C}$ and $\mathcal{D}$ be small $\mathcal{V}$-enriched categories, $f : \mathrm{Ob}(\mathcal{D}) \to \mathrm{Ob}(\mathcal{C})$ a map of sets and $H : \mathcal{D} \to f^*(\mathcal{C})$ a morphism in $\mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{D})}$. Then the diagrams (4) with $X = N_{\mathcal{V}}(\mathcal{D})$ precisely express that (H, f) is a $\mathcal{V}$-functor $\mathcal{D} \to \mathcal{C}$, and thus we have a bijection $\mathcal{V}\mathrm{Cat}(\mathcal{D}, \mathcal{C}) \simeq S_\otimes \mathcal{V}(N_{\mathcal{V}}(\mathcal{D}), N_{\mathcal{V}}(\mathcal{C}))$. More precisely, the templicial morphism $N_{\mathcal{V}}(H)$ corresponding to some $\mathcal{V}$-functor $H : \mathcal{D} \to \mathcal{C}$ is given by

$$N_{\mathcal{V}}(H)_n \colon f_!(\mathcal{D}^{\otimes n}) \to f_!(\mathcal{D})^{\otimes n} \xrightarrow{N_{\mathcal{V}}(H)_1^{\otimes n}} \mathcal{C}^{\otimes n}$$

for all $n \geq 0$, where $N_{\mathcal{V}}(H)_1 \colon f_!(\mathcal{D}) \to \mathcal{C}$ is adjoint to $H \colon \mathcal{D} \to f^*(\mathcal{D})$. Thus clearly this defines a functor which is necessarily fully faithful. The characterisation of the essential image follows immediately from (3). $\square$

Remark 2.14 When $(\mathcal{V}, \otimes, I) = (\mathrm{Set}, \times, \{*\})$, the templicial nerve functor $N_{\mathcal{V}} \colon \mathcal{V}\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ clearly recovers the classical nerve functor $N \colon \mathrm{Cat} \to \mathrm{SSet}$.

The adjunction $F \dashv U$ induces an adjunction between small categories and small $\mathcal{V}$-enriched categories, which we denote by

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\mathrm{Cat}. \arrow[l, shift left, "\mathcal{U}"] \end{tikzcd}$$

Proposition 2.15 *We have natural isomorphisms*

$$N_{\mathcal{V}} \circ \mathcal{F} \simeq \tilde{F} \circ N \quad \text{and} \quad \tilde{U} \circ N_{\mathcal{V}} \simeq N \circ \mathcal{U}.$$

Proof The first isomorphism follows from the construction of $N_{\mathcal{V}}$ and the fact that F is strong monoidal and preserves colimits. Further let $\mathcal{C}$ be a small $\mathcal{V}$-category and $n \geq 0$. Then we have isomorphisms, natural in $\mathcal{C}$ and n,

$$\begin{aligned} \tilde{U}(N_{\mathcal{V}}(\mathcal{C}))_n = S_{\otimes}\mathcal{V}(\tilde{F}(\Delta^n), N_{\mathcal{V}}(\mathcal{C})) &\simeq S_{\otimes}\mathcal{V}\big(N_{\mathcal{V}}(\mathcal{F}([n])), N_{\mathcal{V}}(\mathcal{C})\big) \\ &\simeq \mathcal{V}\mathrm{Cat}(\mathcal{F}([n]), \mathcal{C}) \simeq \mathrm{Cat}([n], \mathcal{U}(\mathcal{C})) \simeq N(\mathcal{U}(\mathcal{C}))_n, \end{aligned}$$

where we subsequently used the first isomorphism and Proposition 2.13. $\square$

2.C The Eilenberg–Zilber lemma

Recall the classical Eilenberg–Zilber lemma for simplicial sets [15, (8.3)]. It states that, for any n-simplex x of a simplicial set K, there is a unique nondegenerate k-simplex y of K and a unique surjective map $\sigma \colon [n] \twoheadrightarrow [k]$ in $\mathbf{\Delta}$ such that $x = K(\sigma)(y)$. Equivalently, there exists a bijection

$$K_n \simeq \coprod_{\sigma \colon [n] \twoheadrightarrow [k] \text{ in } \mathbf{\Delta}_{\mathrm{surj}}} K_k^{\mathrm{nd}},$$

where $K_k^{\mathrm{nd}} \subseteq K_k$ denotes the subset of nondegenerate k-simplices of K, and $\mathbf{\Delta}_{\mathrm{surj}} \subseteq \mathbf{\Delta}$ is the subcategory of surjective maps (see Section 1.D).

The analogous statement for templicial objects (Lemma 2.19) also holds, but this requires an extra condition to ensure that they have a well-behaved notion of nondegenerate simplices. We will make use of this lemma when we reformulate the left-adjoint of the templicial homotopy coherent nerve in Section 4.C.

Definition 2.16 Consider a functor $X \colon \mathbf{\Delta}_{\mathrm{surj}}^{\mathrm{op}} \to \mathcal{W}$ with $\mathcal{W}$ a cocomplete category. For every integer $n \geq 0$, we let

$$X_n^{\mathrm{deg}} = \operatorname*{colim}_{\substack{\sigma \colon [n] \twoheadrightarrow [k] \text{ in } \mathbf{\Delta}_{\mathrm{surj}} \\ 0 \leq k < n}} X_k.$$

Note that we have a canonical morphism $X_n^{\mathrm{deg}} \to X_n$ in $\mathcal{W}$.

Let (X,S) be a templicial object and consider the restricted functor $X|_{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}}}\colon \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}} \to \mathcal{V}\mathrm{Quiv}_S$. For $n \geq 0$, we call X^{deg}_n the *quiver of degenerate n-simplices* of X. We say X *has nondegenerate simplices* if for every $n \geq 0$, the quiver morphism $X^{\mathrm{deg}}_n \to X_n$ is isomorphic to a coprojection

$$X^{\mathrm{deg}}_n \to X^{\mathrm{deg}}_n \amalg N$$

for some $N \in \mathcal{V}\mathrm{Quiv}_S$. In this case, we'll often denote N by X^{nd}_n. When considering an abstract templicial object that has nondegenerate simplices, we implicitly assume a choice for X^{nd}_n in each dimension has been made. Note that $X^{\mathrm{deg}}_0 = 0$ and so we always have $X^{\mathrm{nd}}_0 \simeq X_0$.

Example 2.17 Let (X,S) be a templicial object and suppose the underlying functor $X|_{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}}}\colon \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}} \to \mathcal{V}\mathrm{Quiv}_S$ is isomorphic to FZ for some $Z\colon \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}} \to \mathrm{Quiv}_S$ with $Z^{\mathrm{deg}}_n(a,b) \to Z_n(a,b)$ injective for all $a,b \in S$ and $n \geq 0$. Then X has nondegenerate simplices. Indeed, simply set

$$X^{\mathrm{nd}}_n(a,b) = F(Z_n(a,b) \setminus Z^{\mathrm{deg}}_n(a,b)).$$

In particular, for any simplicial set K, the templicial object $\widetilde{F}(K)$ has nondegenerate simplices.

Certainly not every templicial object has nondegenerate simplices, as the following example shows.

Example 2.18 Consider the monoidal category $\mathcal{V} = \mathrm{Ab}$ of abelian groups with the tensor product. Let $S = \{*\}$ be a singleton and define a functor $X\colon \mathbf{\Delta}^{\mathrm{op}}_f \to \mathrm{Ab}$ by setting $X_n = \mathbb{Z}$ for all $n \geq 0$ with

$$s_0\colon X_0 = \mathbb{Z} \xrightarrow{2\cdot} X_1 = \mathbb{Z}$$

and all other face and degeneracy maps given by the identity on $\mathbb{Z}$. Then X is a strongly unital, colax monoidal functor with comultiplication map $\mu_{k,l}$ for $k,l \geq 0$ given by

$$\mu_{k,l}\colon X_{k+l} = \mathbb{Z} \to X_k \otimes X_l \simeq \mathbb{Z}, \qquad z \mapsto \begin{cases} 2z & \text{if } k,l > 0, \\ z & \text{if } k = 0 \text{ or } l = 0. \end{cases}$$

We thus find a templicial abelian group (X,S) for which $X^{\mathrm{deg}}_1 \to X_1$ is given by the inclusion $2\mathbb{Z} \subseteq \mathbb{Z}$, which doesn't have a direct complement.

Lemma 2.19 *Let X be a templicial object and assume it has nondegenerate simplices. For any integer $n \geq 0$, we have an isomorphism of quivers*

$$X_n \simeq \coprod_{\sigma\colon [n] \twoheadrightarrow [k] \text{ in } \mathbf{\Delta}_{\mathrm{surj}}} X^{\mathrm{nd}}_k.$$

Proof By definition, $X_0 = X^{\mathrm{nd}}_0$. Take $n > 0$; then it follows by induction that

$$\begin{aligned} X_n \simeq X^{\mathrm{nd}}_n \amalg X^{\mathrm{deg}}_n = X^{\mathrm{nd}}_n \amalg \operatorname*{colim}_{\substack{[n]\twoheadrightarrow[k]\\ 0\leq k<n}} X_k \simeq X^{\mathrm{nd}}_n \amalg \operatorname*{colim}_{\substack{[n]\twoheadrightarrow[k]\\ 0\leq k<n}} \coprod_{\sigma\colon [k]\twoheadrightarrow[l]} X^{\mathrm{nd}}_l \\ \simeq X^{\mathrm{nd}}_n \amalg \coprod_{\substack{\sigma\colon [n]\twoheadrightarrow[l]\\ 0\leq l<n}} \operatorname*{colim}_{\substack{[n]\overset{\sigma_1}{\twoheadrightarrow}[k]\overset{\sigma_2}{\twoheadrightarrow}[l]\\ \sigma=\sigma_2\sigma_1}} X^{\mathrm{nd}}_l \simeq X^{\mathrm{nd}}_n \amalg \coprod_{\substack{\sigma\colon [n]\twoheadrightarrow[l]\\ 0\leq l<n}} X^{\mathrm{nd}}_l. \end{aligned}$$

The last isomorphism is obtained by noting that the colimit on the left-hand side is taken over a category which has a terminal object given by the factorisation $[n] \xrightarrow{=} [n] \xrightarrow{\sigma} [l]$. □

3 Necklaces and necklace categories

Necklaces were first introduced by Baues [4] and popularised by Dugger and Spivak [14]. Their category $\mathcal{N}ec$ will play a crucial role in what follows. Morally a necklace is simply a sequence of simplices glued together at vertices. In view of Remark 2.4, necklaces appear naturally when applying the comultiplication morphism $\mu_{k,l}$ of a templicial object X. In this way, maps between necklaces parametrise the degeneracy and inner face morphisms of X, as well as its comultiplication morphisms. This change in perspective leads us to consider the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ of small categories enriched in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. We call these necklace categories and show in Section 3.B that we can recover $S_{\otimes}\mathcal{V}$ as a coreflective subcategory of $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ (see Theorem 3.12).

3.A Necklaces

We quickly recall the definition of a necklace and in Proposition 3.4 we give a combinatorial description of the category $\mathcal{N}ec$ of necklaces, which also appears in [18].

Definition 3.1 We denote by $\mathrm{SSet}_{*,*} = (\partial\Delta^1 \downarrow \mathrm{SSet})$ the category of *bipointed simplicial sets*. Its objects can be identified with tuples (K, a, b) where K is a simplicial set and $a, b \in K_0$ are called the *distinguished points* of K. We will also write $K_{a,b} = (K, a, b)$. A morphism $K_{a,b} \to L_{c,d}$ in $\mathrm{SSet}_{*,*}$ is a simplicial map $f : K \to L$ such that $f(a) = c$ and $f(b) = d$.

Let $K_{a,b}$ and $L_{c,d}$ be bipointed simplicial sets. The *wedge sum* $K \vee L$ of K and L is constructed by gluing K and L at the distinguished points b and c. More precisely, $K \vee L$ is given by the coequaliser

$$\Delta^0 \overset{b}{\underset{c}{\rightrightarrows}} K \amalg L \twoheadrightarrow K \vee L.$$

We consider $K \vee L$ again as bipointed with distinguished points (a, d).

Remark 3.2 It is not difficult to verify that the wedge $\vee$ is a monoidal product on the category of bipointed simplicial sets $\mathrm{SSet}_{*,*}$ whose unit is given by Δ^0.

Definition 3.3 For any $n \geq 0$, we consider the standard simplex Δ^n as bipointed with distinguished points 0 and n. A *necklace* T is an iterated wedge of standard simplices. That is,

$$T = \Delta^{n_1} \vee \cdots \vee \Delta^{n_k} \in \mathrm{SSet}_{*,*}$$

for some $k \geq 0$ and $n_1, \ldots, n_k > 0$ (if $k = 0$, then $T = \Delta^0$). We refer to the standard simplices $\Delta^{n_1}, \ldots, \Delta^{n_k}$ as the *beads* of T. The distinguished points in every bead are called the *joints* of T.

We let $\mathcal{N}ec$ denote the full subcategory of $\mathrm{SSet}_{*,*}$ spanned by all necklaces. By construction, $(\mathcal{N}ec, \vee, \Delta^0)$ is again a monoidal category.

Proposition 3.4 *The category of necklaces $\mathcal{N}ec$ is equivalent to the category defined as follows:*

The objects are pairs (T, p) with $p \geq 0$ and $\{0 < p\} \subseteq T \subseteq [p]$. The morphisms $(T, p) \to (U, q)$ are morphisms $f : [p] \to [q]$ in $\mathbf{\Delta}_f$ such that $U \subseteq f(T)$, with compositions and identities defined as in $\mathbf{\Delta}_f$.

Moreover, under this equivalence, the wedge $\vee$ corresponds to

$$(T, p) \vee (U, q) = (T \cup (p + U), p + q),$$

where $p + U = \{p + u \mid u \in U\}$.

Proof We may identify a necklace $T = \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$ with $T = \{0 < n_1 < n_1 + n_2 < \cdots < p\} \subseteq [p]$, where $p = n_1 + \cdots + n_k$, and we will do so for the rest of this proof. Note that, under this identification, $[p]$ is the set of vertices and T is the set of joints of the necklace. Further, a necklace map $T \to U$ is completely determined on vertices, which is a morphism $[p] \to [q]$ in $\mathbf{\Delta}_f$. It remains to show that a morphism $f : [p] \to [q]$ in $\mathbf{\Delta}_f$ is the vertex map of a necklace map $T \to U$ if and only if $U \subseteq f(T)$.

Suppose f is the vertex map of some necklace map $T \to U$. Assume that there exists a $u \in U \setminus f(T)$. Then we may choose subsequent joints $t < t'$ in T such that $f(t) < u < f(t')$. Now the unique edge in T between t and t' must be sent to an edge in U between $f(t)$ and $f(t')$. But there is no such edge. Conversely, assume $U \subseteq f(T)$. We may write $T = \{0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = p\}$ and $U = \{0 = u_0 < u_1 < \cdots < u_{l-1} < u_l = q\}$. Then $f = f_1 + \cdots + f_k$ for some unique maps $f_i : [t_i - t_{i-1}] \to [f(t_i) - f(t_{i-1})]$ in $\mathbf{\Delta}_f$. Fixing $i \in \{1, \ldots, k\}$, there is a unique $j \in \{1, \ldots, l\}$ such that $u_{j-1} \leq f(t_{i-1}) \leq f(t_i) \leq u_j$. So we can extend f_i to an order morphism $[t_i - t_{i-1}] \to [u_j - u_{j-1}]$, which induces a simplicial map $\Delta^{t_i - t_{i-1}} \to \Delta^{u_j - u_{j-1}} \to U$. These maps combine to give a map of necklaces $T \to U$ whose vertex map is f.

Clearly, this correspondence is functorial and preserves the wedge sum $\vee$. □

Henceforth, we will identify $\mathcal{N}ec$ with the category described in Proposition 3.4. So we will also use the notation

$$T = \{0 = t_0 < t_1 < t_2 < \cdots < t_k = p\}$$

to refer to the necklace $\Delta^{t_1} \vee \Delta^{t_2 - t_1} \vee \cdots \vee \Delta^{p - t_{k-1}}$. We will often refer to a necklace (T, p) just by its underlying set of joints T.

Definition 3.5 Let $f : (T, p) \to (U, q)$ be a map of necklaces. We say f is *inert* if $p = q$ and $f = \mathrm{id}_{[p]}$. We say f is *active* if $f(T) = U$.

Remark 3.6 Every necklace map $f : (T, p) \to (U, q)$ can be uniquely factored as an active necklace map $(T, p) \to (f(T), q)$ followed by an inert necklace map $(f(T), q) \to (U, q)$. In fact, it is easy to see that the active and inert necklace maps form an (orthogonal) factorisation system on $\mathcal{N}ec$ in the sense of [9].

Remark 3.7 A simplex Δ^n, considered as a necklace with a single bead, is represented in $\mathcal{N}ec$ by the pair $(\{0 < n\}, n)$. On the other hand, the necklace $([n], n)$ represents the spine of Δ^n, that is, the union of the edges $0 \to 1 \to \cdots \to n$ in Δ^n.

More generally, for any necklace (T, p), we can consider $([p], p)$, which is the spine passing through all the vertices of T. Note that there is a unique inert necklace map $([p], p) \to (T, p)$ which represents the inclusion of the spine into T. Further, there is a unique order isomorphism $[k] \simeq T$, where k is the number of beads of T. Thus there is a unique active map $([k], k) \to (T, p)$, which is the inclusion of the spine passing through all the joints of T.

3.B Necklace categories

Consider the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ of functors $\mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$. As $\mathcal{N}ec^{\mathrm{op}}$ and $\mathcal{V}$ are both monoidal categories, we can endow $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ with the (nonsymmetric) monoidal structure given by Day convolution (see [12]). We denote the resulting monoidal category by $(\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}, \otimes_{\mathrm{Day}}, \underline{I})$.

Given two functors $X, Y : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$, their Day convolution $X \otimes_{\mathrm{Day}} Y$ is obtained by the left Kan extension of the composite

$$\mathcal{N}ec^{\mathrm{op}} \times \mathcal{N}ec^{\mathrm{op}} \xrightarrow{X \times Y} \mathcal{V} \times \mathcal{V} \xrightarrow{-\otimes-} \mathcal{V}$$

along $\vee : \mathcal{N}ec^{\mathrm{op}} \times \mathcal{N}ec^{\mathrm{op}} \to \mathcal{N}ec^{\mathrm{op}}$,

$$X \otimes_{\mathrm{Day}} Y = \mathrm{Lan}_{\vee}(X(-) \otimes Y(-)).$$

Further, the monoidal unit of $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ is given by the representable functor on the monoidal unit $\{0\}$ of $\mathcal{N}ec$. As $\{0\}$ is also the terminal object of $\mathcal{N}ec$, we find that $F(\mathcal{N}ec(-, \{0\})) \simeq \underline{I}$ is the constant functor on I, the monoidal unit of $\mathcal{V}$.

Definition 3.8 Consider the category

$$\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} = \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}\text{-Cat}$$

of small categories enriched in the monoidal category $(\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}, \otimes_{\mathrm{Day}}, \underline{I})$. We call the objects of $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ *necklace categories* and its morphisms *necklace functors*.

If $\mathcal{V} = \mathrm{Set}$, we simply write $\mathrm{Cat}_{\mathcal{N}ec}$ for $\mathrm{Set}\,\mathrm{Cat}_{\mathcal{N}ec}$.

Construction 3.9 We construct a functor

$$(-)^{\mathrm{nec}} : S_{\otimes}\mathcal{V} \to \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$$

as follows. Let (X, S) be a templicial object. Define

$$X_T = X_{t_1} \otimes_S \cdots \otimes_S X_{n-t_{k-1}} \in \mathcal{V}\mathrm{Quiv}_S$$

for any necklace $T = \{0 = t_0 < t_1 < \cdots < t_k = p\}$. We will also write μ_T for the quiver morphism $\mu_{t_1, t_2-t_1, \ldots, p-t_{k-1}} : X_p \to X_T$.

This extends to a functor $X^{\mathrm{nec}}_\bullet\colon \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}\mathrm{Quiv}_S$ as follows. Take a necklace map $f\colon (T,p)\to(U,q)$ and write $U = \{0 = u_0 < u_1 < \cdots < u_l = q\}$.

- If f is inert, then $p = q$ and $U \subseteq T$. Then there exist unique necklaces $(T_i, u_i - u_{i-1})$ for $i \in \{1,\dots,l\}$ such that $T = T_1 \vee \cdots \vee T_l$. Now set
$$X(f)\colon X_U \xrightarrow{\mu_{T_1}\otimes\cdots\otimes\mu_{T_l}} X_T.$$
- If f is active, then there exist unique $f_i\colon [t_i - t_{i-1}] \to [f(t_i) - f(t_{i-1})]$ in $\mathbf{\Delta}_f$ for all $i \in \{1,\dots,k\}$ such that $f = f_1 + \cdots + f_k$. Now set
$$X(f)\colon X_U \simeq X_{f(t_1)} \otimes_S \cdots \otimes_S X_{q-f(t_{k-1})} \xrightarrow{X(f_1)\otimes\cdots\otimes X(f_k)} X_T,$$
where the isomorphism is induced by the strong unitality of X and the fact that $U = f(T)$.

It follows from the coassociativity of μ that $X_\bullet$ is functorial on inert morphisms, and from the functoriality of X that $X_\bullet$ is functorial on active morphisms. Then it follows from the naturality of μ that $X_\bullet$ is functorial on all morphisms.

If we fix vertices $a, b \in S$, then we obtain a functor
$$X_\bullet(a,b)\colon \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}, \quad T \mapsto X_T(a,b).$$

Now let X^{nec} denote the necklace category with S as its object set and $X_\bullet(a,b)$ as its hom-object for all $a, b \in S$. The composition $X_\bullet(a,b) \otimes_{\mathrm{Day}} X_\bullet(b,c) \to X_\bullet(a,c)$ for $a,b,c \in S$ is induced by the canonical morphism
$$X_T(a,b) \otimes X_U(b,c) \to X_{T\vee U}(a,c)$$
and the identities are given by the morphism $\underline{I} \to X_\bullet(a,a)$ for $a \in S$ induced by the isomorphism $I \simeq X_0(a,a) = X_{\{0\}}(a,a)$.

This clearly extends to a functor $(-)^{\mathrm{nec}}\colon S_\otimes\mathcal{V} \to \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$.

Notation 3.10 As in $\mathbf{\Delta}$, we distinguish some special maps in $\mathcal{N}ec$:

- For any $0 < j < n$, we write
$$\delta_j\colon \{0 < n-1\} \to \{0 < n\}$$
for the active necklace map whose underlying morphism in $\mathbf{\Delta}_f$ is the coface map $\delta_j\colon [n-1] \to [n]$, ie $\delta_j(i) = i$ if $i < j$ and $\delta_j(i) = j+1$ if $j \geq i$.
- For any $k, l > 0$, we write
$$\nu_{k,l}\colon \{0 < k < k+l\} \to \{0 < k+l\}$$
for the unique inert necklace map.

Construction 3.11 Let $\mathcal{C}$ be a necklace category with set of objects S. We construct a templicial object $(\mathcal{C}^{\text{temp}}, S)$ as follows. For every necklace T, we have a $\mathcal{V}$-enriched quiver $\mathcal{C}_T = (\mathcal{C}_T(a,b))_{a,b\in S}$. Then the composition and identities of $\mathcal{C}$ induce quiver morphisms

$$m_{U,V}\colon \mathcal{C}_U \otimes_S \mathcal{C}_V \to \mathcal{C}_{U\vee V} \quad \text{and} \quad u\colon I_S \to \mathcal{C}_{\{0\}}$$

for all necklaces U and V. Set $\mathcal{C}_0^{\text{temp}} = I_S$ and $p_0 = u\colon \mathcal{C}_0^{\text{temp}} \to \mathcal{C}_{\{0\}}$. Now let $n > 0$. We inductively define an object $\mathcal{C}_n^{\text{temp}} \in \mathcal{V}\mathrm{Quiv}_S$ along with morphisms p_n and $\mu_{k,l}$ as the limit of the diagram of solid arrows in $\mathcal{V}\mathrm{Quiv}_S$

$$\begin{tikzcd}
\mathcal{C}_n^{\text{temp}} \arrow[r, dashed, "(\mu_{k,l})_{k,l}"] \arrow[d, dashed, "p_n"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} \mathcal{C}_k^{\text{temp}} \otimes_S \mathcal{C}_l^{\text{temp}} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\prod_{k,l} p_k\otimes p_l"] & \prod\limits_{\substack{r,s,t>0\\r+s+t=n}} \mathcal{C}_r^{\text{temp}} \otimes_S \mathcal{C}_s^{\text{temp}} \otimes_S \mathcal{C}_t^{\text{temp}} \\
\mathcal{C}_{\{0<n\}} \arrow[dr, "(\mathcal{C}(\nu_{k,l}))_{k,l}"'] & \prod\limits_{\substack{k,l>0\\k+l=n}} \mathcal{C}_{\{0<k\}} \otimes_S \mathcal{C}_{\{0<l\}} \arrow[d, "\prod_{k,l} m_{\{0<k\},\{0<l\}}"] & \\
 & \prod\limits_{\substack{k,l>0\\k+l=n}} \mathcal{C}_{\{0<k<k+l\}} &
\end{tikzcd} \tag{5}$$

where α and β are defined by

$$\pi_{r,s,t}\alpha = (\mathrm{id}_r \otimes \mu_{s,t})\pi_{r,s+t} \quad \text{and} \quad \pi_{r,s,t}\beta = (\mu_{r,s} \otimes \mathrm{id}_t)\pi_{r+s,t}.$$

For example, $\mathcal{C}_1^{\text{temp}} = \mathcal{C}_{\{0<1\}}$ with $p_1 = \mathrm{id}_{\mathcal{C}_{\{0,1\}}}$, and $\mathcal{C}_2^{\text{temp}}$ is the pullback of $m_{\{0<1\},\{0<1\}}$ and $\mathcal{C}(\nu_{1,1})$. We further set $\mu_{0,n}$ and $\mu_{n,0}$ to be the left and right unit isomorphisms, respectively,

$$\mathcal{C}_n^{\text{temp}} \xrightarrow{\sim} \mathcal{C}_0^{\text{temp}} \otimes_S \mathcal{C}_n^{\text{temp}}, \quad \mathcal{C}_n^{\text{temp}} \xrightarrow{\sim} \mathcal{C}_n^{\text{temp}} \otimes_S \mathcal{C}_0^{\text{temp}}.$$

Further, let $f\colon [m] \to [n]$ be a morphism in $\mathbf{\Delta}_f$. We define a quiver morphism $\mathcal{C}^{\text{temp}}(f)\colon \mathcal{C}_n^{\text{temp}} \to \mathcal{C}_m^{\text{temp}}$ by induction on m. Set $\mathcal{C}^{\text{temp}}(\mathrm{id}_{[0]})$ to be the identity on I_S. If $m > 0$, we let $\mathcal{C}^{\text{temp}}(f)$ be the unique morphism satisfying, for all $k, l > 0$ with $k + l = m$,

$$\mu_{k,l}\mathcal{C}^{\text{temp}}(f) = (\mathcal{C}^{\text{temp}}(f_1) \otimes_S \mathcal{C}^{\text{temp}}(f_2))\mu_{p,q} \quad \text{and} \quad p_m\mathcal{C}^{\text{temp}}(f) = \mathcal{C}(f)p_n,$$

where $f_1\colon [k] \to [p]$ and $f_2\colon [l] \to [q]$ are unique in $\mathbf{\Delta}_f$ such that $f_1 + f_2 = f$. (Note that, when $m = 1$, the first condition is empty and $\mathcal{C}^{\text{temp}}(f)$ is just $\mathcal{C}(f)p_n$.)

We have thus constructed a well-defined functor

$$\mathcal{C}^{\text{temp}}\colon \mathbf{\Delta}_f^{\text{op}} \to \mathcal{V}\mathrm{Quiv}_S.$$

By construction, $\mathcal{C}^{\text{temp}}$ is strongly unital and colax monoidal with comultiplication given by the morphisms $(\mu_{k,l})_{k,l\geq 0}$.

Theorem 3.12 *The functor $(-)^{\text{nec}}\colon S_\otimes\mathcal{V} \to \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ is fully faithful and left-adjoint to a functor $(-)^{\text{temp}}\colon \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \to S_\otimes\mathcal{V}$ which is given on objects by the assignment $\mathcal{C} \mapsto \mathcal{C}^{\text{temp}}$ of Construction 3.11.*

Proof Let $\mathcal{C}$ be a necklace category and define a necklace functor $\varepsilon_{\mathcal{C}}\colon (\mathcal{C}^{\mathrm{temp}})^{\mathrm{nec}} \to \mathcal{C}$ by the quiver morphism

$$\varepsilon_{\mathcal{C}_T}\colon (\mathcal{C}^{\mathrm{temp}})_T \xrightarrow{p_{t_1}\otimes\cdots\otimes p_{p-t_{k-1}}} \mathcal{C}_{\{0<t_1\}}\otimes\cdots\otimes\mathcal{C}_{\{0<p-t_{k-1}\}} \xrightarrow{m_{\mathcal{C}}} \mathcal{C}_T$$

for any necklace $T=\{0=t_0<t_1<\cdots<t_k=p\}$.

Let (X,S) be a templicial object and $H\colon X^{\mathrm{nec}}\to\mathcal{C}$ a necklace functor with object map $f\colon S\to\mathrm{Ob}(\mathcal{C})$. We will construct a templicial morphism $(\alpha,f)\colon(X,S)\to(\mathcal{C}^{\mathrm{temp}},\mathrm{Ob}(\mathcal{C}))$ by induction. Let α_0 be the canonical quiver morphism $f_!(X_0)\simeq f_!(I_S)\to I_{\mathrm{Ob}(\mathcal{C})}=\mathcal{C}^{\mathrm{temp}}_0$. For $n>0$, let $\beta_n\colon f_!(X_n)=f_!(X_{\{0<n\}})\to\mathcal{C}_{\{0<n\}}$ be adjoint to $H_{\{0<n\}}\colon X_{\{0<n\}}\to f^*(\mathcal{C}_{\{0<n\}})$. By Construction 3.11, we have a unique morphism $\alpha_n\colon f_!(X_n)\to\mathcal{C}^{\mathrm{temp}}_n$ such that

$$p_n\alpha_n=\beta_n$$

and, for all $k,l>0$ with $k+l=n$, $\mu_{k,l}\circ\alpha_n$ is equal to the composite

$$f_!(X_n)\xrightarrow{f_!(\mu^X_{k,l})} f_!(X_k\otimes X_l)\to f_!(X_k)\otimes f_!(X_l)\xrightarrow{\alpha_k\otimes\alpha_l}\mathcal{C}^{\mathrm{temp}}_k\otimes\mathcal{C}^{\mathrm{temp}}_l,$$

where we used the colax monoidal structure of $f_!$. It follows that (α,f) is a well-defined templicial morphism which is unique such that $\varepsilon_{\mathcal{C}}\circ\alpha^{\mathrm{nec}}=H$. Hence, the assignment $\mathcal{C}\mapsto\mathcal{C}^{\mathrm{temp}}$ extends to a right-adjoint of $(-)^{\mathrm{nec}}$.

Now let X be a templicial object. Since the composition $X^{\mathrm{nec}}_T\otimes_S X^{\mathrm{nec}}_U\to X^{\mathrm{nec}}_{T\vee U}$ of X^{nec} is an isomorphism for all $T,U\in\mathcal{N}ec$, it follows from Construction 3.11 that $p_n\colon (X^{\mathrm{nec}})^{\mathrm{temp}}_n\to X_{\{0<n\}}=X_n$ is an isomorphism. We thus obtain a natural isomorphism $(-)^{\mathrm{temp}}\circ(-)^{\mathrm{nec}}\simeq\mathrm{id}_{S_\otimes\mathcal{V}}$, which shows that $(-)^{\mathrm{nec}}$ is fully faithful. $\square$

3.C Some constructions revisited

We show that the functor $\widetilde{U}\colon S_\otimes\mathcal{V}\to\mathrm{SSet}$ (Proposition 2.8) and the templicial nerve $N_{\mathcal{V}}\colon\mathcal{V}\mathrm{Cat}\to S_\otimes\mathcal{V}$ (Definition 2.11) factor through the category $\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$ of necklace categories.

Notation 3.13 By postcomposition, the adjunction $F\colon\mathrm{Set}\leftrightarrows\mathcal{V}:U$ induces an adjunction $F\colon\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}\leftrightarrows\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}:U$. Note that, as F is strong monoidal and preserves colimits, the induced functor $F\colon\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}\to\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ is strong monoidal as well. Therefore, we have an induced adjunction, which we denote by

$$\begin{tikzcd}\mathrm{Cat}_{\mathcal{N}ec} \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}. \arrow[l, shift left, "\mathcal{U}"]\end{tikzcd}$$

Proposition 3.14 *There is diagram of adjunctions*

$$\begin{tikzcd}\mathrm{SSet} \arrow[r, shift left, "\widetilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, hook, shift right, "(-)^{\mathrm{nec}}"'] \arrow[d, phantom, "\dashv"] & S_\otimes\mathcal{V} \arrow[l, shift left, "\widetilde{U}"] \arrow[d, hook, shift right, "(-)^{\mathrm{nec}}"'] \arrow[d, phantom, "\dashv"] \\ \mathrm{Cat}_{\mathcal{N}ec} \arrow[u, shift right, "(-)^{\mathrm{temp}}"'] \arrow[r, shift left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \arrow[l, shift left, "\mathcal{U}"] \arrow[u, shift right, "(-)^{\mathrm{temp}}"']\end{tikzcd}$$

which commutes in the sense that we have natural isomorphisms

$$(-)^{\mathrm{nec}} \circ \widetilde{F} \simeq \mathcal{F} \circ (-)^{\mathrm{nec}} \quad \text{and} \quad \widetilde{U} \circ (-)^{\mathrm{temp}} \simeq (-)^{\mathrm{temp}} \circ \mathcal{U}.$$

In particular, we have a natural isomorphism

$$\widetilde{U} \simeq (-)^{\mathrm{temp}} \circ \mathcal{U} \circ (-)^{\mathrm{nec}}.$$

Proof It suffices to show the commutativity of the left-adjoints. But this immediately follows from the fact that $F : \mathrm{Set} \to \mathcal{V}$ is strong monoidal and preserves colimits. The final isomorphism $\widetilde{U} \simeq (-)^{\mathrm{temp}} \circ \mathcal{U} \circ (-)^{\mathrm{nec}}$ follows from the fact that $(-)^{\mathrm{nec}}$ is fully faithful. □

Lemma 3.15 *Let $\mathcal{C}$ be a finitely complete category. Let $f : A \to B$ be a morphism in $\mathcal{C}$ and $n \geq 2$. Then A is the limit of the diagram of solid arrows*

$$\begin{tikzcd}
A \arrow[r, dashed, "\Delta"] \arrow[d, dashed, "f"'] & \prod\limits_{\substack{k,l>0 \\ k+l=n}} A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\prod_{k,l} f"] & \prod\limits_{\substack{r,s,t>0 \\ r+s+t=n}} A \\
B \arrow[r, "\Delta"'] & \prod\limits_{\substack{k,l>0 \\ k+l=n}} B &
\end{tikzcd}$$

where Δ is the diagonal morphism and α and β are defined by

$$\pi_{r,s,t}\alpha = \pi_{r+s,t} \quad \text{and} \quad \pi_{r,s,t}\beta = \pi_{r,s+t}$$

for all $r, s, t > 0$ with $r + s + t = n$.

Proof This is an easy verification. □

Let $\underline{(-)} : \mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ denote the diagonal functor associating to every object $V \in \mathcal{V}$ the constant functor on V. Then $\underline{(-)}$ is easily seen to be strong monoidal and thus it induces a functor

$$\underline{(-)} : \mathcal{V}\mathrm{Cat} \to \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}.$$

Proposition 3.16 *We have a natural isomorphism*

$$N_{\mathcal{V}} \simeq (-)^{\mathrm{temp}} \circ \underline{(-)}.$$

Proof Let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category and $n \geq 0$. Applying Lemma 3.15 to the n-fold composition $m_{\mathcal{C}}^{(n)} : \mathcal{C}^{\otimes n} \to \mathcal{C}$ as a morphism in $\mathcal{V}\mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}$, it follows from Construction 3.11 that $\underline{\mathcal{C}}_n^{\mathrm{temp}} \simeq \mathcal{C}^{\otimes n}$ by induction on n. It quickly follows that this identification induces an isomorphism of templicial objects $\underline{\mathcal{C}}^{\mathrm{temp}} \simeq N_{\mathcal{V}}(\mathcal{C})$, which is clearly natural in $\mathcal{C}$. □

4 Enriching the homotopy coherent nerve

Let Cat_Δ denote the category of small simplicial categories, that is categories enriched in the cartesian monoidal category of simplicial sets $(\mathrm{SSet}, \times, \Delta^0)$. In this section we generalise the classical adjunction between the categorification functor $\mathfrak{C}\colon \mathrm{SSet} \to \mathrm{Cat}_\Delta$ and the homotopy coherent nerve $N^{\mathrm{hc}}\colon \mathrm{Cat}_\Delta \to \mathrm{SSet}$ to the templicial level, yielding an adjunction $\mathfrak{C}_\mathcal{V} \dashv N^{\mathrm{hc}}_\mathcal{V}$ which depends on $\mathcal{V}$ (Definition 4.9). We first recall in Section 4.A the homotopy coherent nerve due to Cordier [10]. It is most easily constructed as the formal right-adjoint to the categorification functor $\mathfrak{C}$, which has historically gone through several different equivalent descriptions. This goes back to Cordier and Porter [11] and a different definition is given in [28], which we outline below. Later, Dugger and Spivak [14] gave a very elegant and simple description of $\mathfrak{C}$ by means of necklaces. We will closely follow their approach and adapt it to the templicial setting in Section 4.C.

4.A The classical homotopy coherent nerve

We recall Cordier's homotopy coherent nerve. Further, we give a new expression for its left-adjoint (Proposition 4.6) which will allow us to generalise it more easily to the templicial setting in Section 4.B.

Notation 4.1 Given a necklace (T, p), consider the poset

$$\mathcal{P}_T = \{U \subseteq [p] \mid T \subseteq U\}$$

ordered by inclusion. Equivalently, it is the poset of inert necklace maps $U \hookrightarrow T$.

If $T = \{0 < p\}$ is a simplex, we also write $\mathcal{P}_T = \mathcal{P}_p$.

Remark 4.2 It is easy to see that the assignment $T \mapsto \mathcal{P}_T$ extends to a strong monoidal functor

$$\mathcal{P}\colon \mathcal{N}ec \to \mathrm{Cat},$$

where, for every necklace map $f : T \to U$, we have $\mathcal{P}(f)(V) = f(V)$ for all $V \in \mathcal{P}_T$. For necklaces T and U, the monoidal structure is given by

$$\mathcal{P}_T \times \mathcal{P}_U \to \mathcal{P}_{T \vee U}, \quad (V, W) \mapsto (V \vee W),$$

which is clearly an order isomorphism.

In [28, Section 1.1.5], a simplicial category $\mathfrak{C}[\Delta^n]$ is constructed as follows. Its objects are given by the set $[n]$ and, for all $i, j \in [n]$, we have

$$\mathfrak{C}[\Delta^n](i,j) = \begin{cases} N(\mathcal{P}_{j-i}) & \text{if } i \leq j, \\ \varnothing & i > j. \end{cases}$$

Note that $N(\mathcal{P}_{j-i}) \simeq (\Delta^1)^{\times j-i-1}$ if $i < j$ and $N(\mathcal{P}_{j-i}) \simeq \Delta^0$ if $i = j$. Further, given $i \leq j \leq k$ in $[n]$, the composition

$$m_{i,j,k}\colon \mathfrak{C}[\Delta^n](i,j) \times \mathfrak{C}[\Delta^n](j,k) \to \mathfrak{C}[\Delta^n](i,k)$$

is given by applying N to the order morphism

$$\mathcal{P}_{j-i} \times \mathcal{P}_{k-j} \simeq \mathcal{P}_{\{0<j-i<k-i\}} \hookrightarrow \mathcal{P}_{k-i}, \quad (T, U) \mapsto T \vee U.$$

Finally, the identities are given by the unique vertex of $\mathfrak{C}[\Delta^n](i,i) \simeq \Delta^0$ for $i \in [n]$.

It is now easy to see that the above construction extends to a functor

$$\mathfrak{C}[\Delta^{(-)}] \colon \mathbf{\Delta} \to \mathrm{Cat}_\Delta.$$

Then, by left Kan extension along the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathrm{SSet}$, the cosimplicial object $\mathfrak{C}[\Delta^{(-)}]$ induces an adjunction

$$\begin{tikzcd} \mathrm{SSet} \arrow[r, shift left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_\Delta . \arrow[l, shift left, "N^{\mathrm{hc}}"] \end{tikzcd}$$

For all small simplicial categories $\mathcal{C}$ and $n \geq 0$,

$$N^{\mathrm{hc}}(\mathcal{C})_n \simeq \mathrm{Cat}_\Delta(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

Example 4.3 Given a small simplicial category $\mathcal{C}$, let us describe its homotopy coherent nerve in low dimensions:

- The vertices of $N^{\mathrm{hc}}(\mathcal{C})$ are given by the set of objects $\mathrm{Ob}(\mathcal{C})$.
- The edges of $N^{\mathrm{hc}}(\mathcal{C})$ are given by the morphisms of $\mathcal{C}$ (that is, vertices $f \in \mathcal{C}_0(A, B)$ for some $A, B \in \mathrm{Ob}(\mathcal{C})$).
- A 2-simplex of $N^{\mathrm{hc}}(\mathcal{C})$ is given by a (not necessarily commutative) diagram of morphisms in $\mathcal{C}$

$$\begin{tikzcd} & C \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\sigma"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & B \end{tikzcd}$$

along with an edge σ of $\mathcal{C}(A, B)$ from h to the composition $g \circ f$.

Proposition 4.4 [14, Proposition 3.7] *There is an isomorphism of simplicial sets*

$$\mathfrak{C}[T](0, p) \simeq N(\mathcal{P}_T)$$

that is natural in all necklaces $(T, p) \in \mathcal{N}ec$.

Proposition 4.5 [14, Proposition 4.3] *For every simplicial set K with vertices a and b, there is an isomorphism of simplicial sets*

$$\mathfrak{C}[K](a,b) \simeq \operatorname*{colim}_{\substack{T \to K_{a,b} \text{ in } \mathrm{SSet}_{*,*} \\ (T,p) \in \mathcal{N}ec}} \mathfrak{C}[T](0, p).$$

Recall that by Proposition 2.8 we may view a simplicial set K as a templicial set and thus, for any $a, b \in K_0$, we can apply Construction 3.9 to obtain a functor $K_\bullet(a,b) \colon \mathcal{N}ec^{\mathrm{op}} \to \mathrm{Set}$, $T \mapsto K_T(a,b)$. It follows from Yoneda's lemma that we have a canonical bijection, natural in $T \in \mathcal{N}ec$,

$$\mathrm{SSet}_{*,*}(T, K_{a,b}) \simeq K_T(a,b).$$

With this in mind, we introduce another description of $\mathfrak{C}$ by means of a weighted colimit. For background on weighted colimits, see [38, Definition 7.4.1], for example.

Proposition 4.6 *For any simplicial set K with vertices a and b, $\mathfrak{C}[K](a,b)$ is isomorphic to the weighted colimit in* SSet

$$\operatorname{colim}^{K_\bullet(a,b)} N\mathcal{P}_{(-)}$$

of $N\mathcal{P}_{(-)}\colon \mathcal{N}ec \to \mathrm{SSet}$ with weight $K_\bullet(a,b)\colon \mathcal{N}ec^{\mathrm{op}} \to \mathrm{Set}$.

Proof From Propositions 4.4 and 4.5, it is clear that $\mathfrak{C}[K](a,b)$ is given by the coequaliser in SSet

$$\coprod_{\substack{T\to U\to K_{a,b}\\ T,U\in\mathcal{N}ec}} N(\mathcal{P}_T) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{\substack{T\to K_{a,b}\\ T\in\mathcal{N}ec}} N(\mathcal{P}_T) \twoheadrightarrow \mathfrak{C}[K](a,b),$$

where α and β are given by respectively projecting onto $T\to K_{a,b}$ and applying $N\mathcal{P}_{(-)}$ to $T\to U$ for any $T\to U\to K_{a,b}$ in $\mathrm{SSet}_{*,*}$.

Since morphisms $T\to K_{a,b}$ in $\mathrm{SSet}_{*,*}$ with T a necklace correspond to elements of the set $K_T(a,b)$, we obtain a coequaliser diagram

$$\coprod_{T\to U \text{ in } \mathcal{N}ec} K_U(a,b)\times N(\mathcal{P}_T) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{T\in\mathcal{N}ec} K_T(a,b)\times N(\mathcal{P}_T) \twoheadrightarrow \mathfrak{C}[K](a,b),$$

where α and β are given by respectively applying $K_\bullet(a,b)$ and $N\mathcal{P}_{(-)}$ to $T\to U$ in $\mathcal{N}ec$. But this coequaliser is precisely the weighted colimit described in the statement. □

4.B The templicial homotopy coherent nerve

Consider the category $S\mathcal{V}$ of simplicial objects in $\mathcal{V}$, with the pointwise symmetric monoidal structure induced by that of $\mathcal{V}$. Note that its monoidal unit is the simplicial object $F(\Delta^0)=\underline{I}$ (the constant functor on I). Further, $S\mathcal{V}$ is canonically enriched and tensored over $\mathcal{V}$. We denote the enrichment over $\mathcal{V}$ by $[-,-]$. For every $V\in\mathcal{V}$ and $A\in S\mathcal{V}$, the tensoring $V\cdot A$ is given by the monoidal product $\underline{V}\otimes A$, where $\underline{V}$ denotes the constant functor on V.

We denote the category of small $S\mathcal{V}$-enriched categories by $\mathcal{V}\mathrm{Cat}_\Delta$. Note that the adjunction $F\colon \mathrm{Set}\leftrightarrows \mathcal{V}\colon U$ induces an adjunction $F\colon \mathrm{SSet}\leftrightarrows S\mathcal{V}\colon U$ by postcomposition, for which F is still strong monoidal. Hence, we have an induced adjunction between simplicial categories and $S\mathcal{V}$-categories, which we denote by

$$\mathrm{Cat}_\Delta \underset{\mathcal{U}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{V}\mathrm{Cat}_\Delta, \qquad \mathcal{F}\dashv\mathcal{U}.$$

From Proposition 4.6, it is easy to see that the adjunction $\mathfrak{C}\dashv N^{\mathrm{hc}}$ actually factors through the category $\mathrm{Cat}_{\mathcal{N}ec}$ of Definition 3.8. Moreover, it suggests that we can define the templicial categorification functor by means of a similar weighted colimit.

Construction 4.7 We construct an adjunction

$$\begin{tikzcd} \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}} \arrow[r, shift left, "\mathfrak{s}"] \arrow[r, phantom, "\perp"] & S\mathcal{V} \arrow[l, shift left, "\mathfrak{n}"] \end{tikzcd}$$

as follows. Given a functor $X : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$, consider the weighted colimit in $S\mathcal{V}$

$$\mathfrak{s}(X) = \mathrm{colim}^X \, FN\mathcal{P}_{(-)}$$

of the composite $\mathcal{N}ec \xrightarrow{\mathcal{P}_{(-)}} \mathrm{Cat} \xrightarrow{N} \mathrm{SSet} \xrightarrow{F} S\mathcal{V}$ with weight X. Explicitly, $\mathfrak{s}(X)$ may be realised as the coequaliser in $S\mathcal{V}$

$$\coprod_{f:T\to U \text{ in } \mathcal{N}ec} X_U \otimes FN(\mathcal{P}_T) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{T\in\mathcal{N}ec} X_T \otimes FN(\mathcal{P}_T) \twoheadrightarrow \mathfrak{s}(X), \tag{6}$$

where α and β are given by respectively applying X and $FN\mathcal{P}_{(-)}$ to a necklace morphism $f : T \to U$.

As a weighted colimit, $\mathfrak{s}(X)$ fits into a canonical bijection of sets

$$S\mathcal{V}(\mathfrak{s}(X), Y) \simeq \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}(X, [FN\mathcal{P}_{(-)}, Y])$$

which is natural in $Y \in S\mathcal{V}$. Hence, the assignment $X \mapsto \mathfrak{s}(X)$ extends to a functor $\mathfrak{s}\colon \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}} \to S\mathcal{V}$ which is left-adjoint to the functor

$$\mathfrak{n}\colon S\mathcal{V} \to \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}, \quad Y \mapsto [FN\mathcal{P}_{(-)}, Y].$$

Proposition 4.8 *The functor $\mathfrak{s}\colon \mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}} \to S\mathcal{V}$ of Construction 4.7 is strong monoidal.*

Proof For functors $X, Y : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$,

$$\begin{aligned} \mathfrak{s}(X \otimes_{\mathrm{Day}} Y) &= \mathrm{colim}_{T\in\mathcal{N}ec}^{(X\otimes_{\mathrm{Day}}Y)(T)} FN\mathcal{P}_T \simeq \mathrm{colim}_{U,V\in\mathcal{N}ec}^{X(U)\otimes Y(V)} FN\mathcal{P}_{U\vee V} \\ &\simeq \mathrm{colim}_{U\in\mathcal{N}ec}^{X(U)} FN\mathcal{P}_U \otimes \mathrm{colim}_{V\in\mathcal{N}ec}^{Y(V)} FN\mathcal{P}_V = \mathfrak{s}(X)\otimes\mathfrak{s}(Y), \end{aligned}$$

where we have used the presentation of $X \otimes_{\mathrm{Day}} Y$ as a left Kan extension and the strong monoidality of F, N and $\mathcal{P}_{(-)}$ (see Remark 4.2). Further, since $\underline{I} = F(\mathcal{N}ec(-, \{0\}))$,

$$\mathfrak{s}(\underline{I}) = \mathrm{colim}_{T\in\mathcal{N}ec}^{\underline{I}} FN\mathcal{P}_T \simeq FN\mathcal{P}_{\{0\}} \simeq F(\Delta^0). \qquad \square$$

Definition 4.9 By virtue of Proposition 4.8, the adjunction $\mathfrak{s} \dashv \mathfrak{n}$ between $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ and $S\mathcal{V}$ induces an adjunction

$$\begin{tikzcd} \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \arrow[r, shift left, "\mathfrak{s}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\mathrm{Cat}_{\Delta} \arrow[l, shift left, "\mathfrak{n}"] \end{tikzcd}$$

We call the composite

$$\mathfrak{C}_{\mathcal{V}}\colon S_{\otimes}\mathcal{V} \xrightarrow{(-)^{\mathrm{nec}}} \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \xrightarrow{\mathfrak{s}} \mathcal{V}\mathrm{Cat}_{\Delta}$$

the *categorification functor*. It is left-adjoint to the composite

$$N_{\mathcal{V}}^{\mathrm{hc}}\colon \mathcal{V}\mathrm{Cat}_{\Delta} \xrightarrow{\mathfrak{n}} \mathcal{V}\mathrm{Cat}_{\mathcal{N}ec} \xrightarrow{(-)^{\mathrm{temp}}} S_{\otimes}\mathcal{V},$$

which we call the *templicial homotopy coherent nerve*.

Remark 4.10 Suppose $\mathcal{V} = \mathrm{Set}$. Then the adjunction $\mathfrak{C}_{\mathcal{V}} \dashv N^{\mathrm{hc}}_{\mathcal{V}}$ reduces to the classical adjunction $\mathfrak{C} \dashv N^{\mathrm{hc}}$. Indeed, it suffices to note that $\mathfrak{C}_{\mathcal{V}}$ reduces to $\mathfrak{C}$, which follows from Proposition 4.6 and Construction 4.7.

Example 4.11 Let $\mathcal{C}$ be a small $S\mathcal{V}$-category. We describe the templicial object $N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})$ in low dimensions using Construction 3.11. Note the analogy with Example 4.3.

- The vertex set of $N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})$ is simply $\mathrm{Ob}(\mathcal{C})$.
- Further, for any $A, B \in \mathrm{Ob}(\mathcal{C})$, it follows from $N(\mathcal{P}_{\{0<1\}}) \simeq \Delta^0$ that

$$N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})_1(A,B) = \mathfrak{n}(\mathcal{C})_{\{0<1\}}(A,B) = [FN(\mathcal{P}_{\{0<1\}}), \mathcal{C}(A,B)] \simeq \mathcal{C}_0(A,B).$$

- In dimension 2, it follows from $N(\mathcal{P}_{\{0<2\}}) \simeq \Delta^1$ and $N(\mathcal{P}_{\{0<1<2\}}) \simeq \Delta^0$ that

$$\mathfrak{n}(\mathcal{C})_{\{0<2\}}(A,B) = [FN(\mathcal{P}_{\{0<2\}}), \mathcal{C}(A,B)] \simeq \mathcal{C}_1(A,B),$$
$$\mathfrak{n}(\mathcal{C})_{\{0<1<2\}}(A,B) = [FN(\mathcal{P}_{\{0<1<2\}}), \mathcal{C}(A,B)] \simeq \mathcal{C}_0(A,B).$$

Here, the morphism $\mathfrak{n}(\mathcal{C})_{\{0<2\}}(A,B) \to \mathfrak{n}(\mathcal{C})_{\{0<1<2\}}(A,B)$ is induced by the inert necklace map $\nu_{1,1}\colon \{0<1<2\} \hookrightarrow \{0<2\}$ and thus corresponds to the face map $d_0\colon \mathcal{C}_1(A,B) \to \mathcal{C}_0(A,B)$. It follows from (5) that we have a pullback diagram

$$\begin{tikzcd}
N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})_2(A,B) \arrow[r] \arrow[d] & \coprod_{C \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}_0(A,C) \otimes \mathcal{C}_0(C,B) \arrow[d, "m_{0,0}"] \\
\mathcal{C}_1(A,B) \arrow[r, "d_0"'] & \mathcal{C}_0(A,B)
\end{tikzcd}$$

In particular, we see that the underlying set of the object $N^{\mathrm{hc}}_{\mathcal{V}}(\mathcal{C})_2(A,B)$ consists of pairs (σ,α) with $\alpha \in U\big(\coprod_{C \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}_0(A,C) \otimes \mathcal{C}_0(C,B)\big)$ and $\sigma \in U(\mathcal{C}_1(A,B))$ an edge from $h = d_1(\sigma)$ to $m(\alpha)$.

Proposition 4.12 *There are canonical natural isomorphisms*

$$\mathfrak{C}_{\mathcal{V}} \circ \tilde{F} \simeq \mathcal{F} \circ \mathfrak{C} \quad \textit{and} \quad \tilde{U} \circ N^{\mathrm{hc}}_{\mathcal{V}} \simeq N^{\mathrm{hc}} \circ \mathcal{U}.$$

Proof As $\mathfrak{C}_{\mathcal{V}} \dashv N^{\mathrm{hc}}_{\mathcal{V}}$, $\mathfrak{C} \dashv N^{\mathrm{hc}}$, $\tilde{F} \dashv \tilde{U}$ and $\mathcal{F} \dashv \mathcal{U}$, it suffices to show the first natural isomorphism. Since $F\colon \mathrm{SSet} \to S\mathcal{V}$ preserves colimits and is strong monoidal, it is clear that

$$\operatorname*{colim}_{T \in \mathcal{N}ec}{}^{FX_T} FN\mathcal{P}_T \simeq F\big(\operatorname*{colim}_{T \in \mathcal{N}ec}{}^{X_T} N\mathcal{P}_T\big)$$

for any functor $X\colon \mathcal{N}ec^{\mathrm{op}} \to \mathrm{Set}$. It follows that we have a natural isomorphism $F \circ \mathfrak{s} \simeq \mathfrak{s} \circ F$ of functors $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}} \to S\mathcal{V}$, and thus also $\mathcal{F} \circ \mathfrak{s} \simeq \mathfrak{s} \circ \mathcal{F}$ of functors $\mathrm{Cat}_{\mathcal{N}ec} \to \mathcal{V}\mathrm{Cat}_{\Delta}$. Thus, by Proposition 3.14, $\mathcal{F} \circ \mathfrak{C} \simeq \mathfrak{C}_{\mathcal{V}} \circ \tilde{F}$ as well. □

4.C Simplification of the categorification functor

Following [14, Section 4], we give a simplified description of the categorification functor $\mathfrak{C}_{\mathcal{V}}\colon S_{\otimes}\mathcal{V} \to \mathcal{V}\mathrm{Cat}_{\Delta}$ of Definition 4.9. Let us first recall their approach.

Let (T, p) be a necklace and $n \geq 0$. A *flag of length* n on T is defined as an n-simplex of the nerve $N(\mathcal{P}_T)$. Explicitly, a flag of length n on T is a sequence of inclusions

$$\vec{T} = (T_0 \subseteq \cdots \subseteq T_n)$$

such that $T \subseteq T_0$ and $T_n \subseteq [p]$. We call a flag $\vec{T}$ on a necklace T *flanked* if $T = T_0$ and $T_n = [p]$.

Consider a simplicial set K with vertices a and b, and $T = \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$ a necklace. A map $T \to K_{a,b}$ in $\mathrm{SSet}_{*,*}$ is *totally nondegenerate* if, for every $i \in \{1, \ldots, k\}$, the composite map in SSet

$$\Delta^{n_i} \hookrightarrow T \to K_{a,b}$$

represents a nondegenerate n_i-simplex of K.

As an immediate consequence of Proposition 4.5, an n-simplex of $\mathfrak{C}[K](a,b)$ consists of an equivalence class

$$[T, T \to K_{a,b}, \vec{T}] \tag{7}$$

of triples $(T, T \to K_{a,b}, \vec{T})$ where

- T is a necklace,
- $T \to K_{a,b}$ is a map in $\mathrm{SSet}_{*,*}$ (or equivalently an element of $K_T(a,b)$),
- $\vec{T}$ is a flag of length n on T.

The equivalence relation is generated by setting two triples $(T, T \to K_{a,b}, \vec{T})$ and $(U, U \to K_{a,b}, \vec{U})$ to be equivalent if there exists a map of necklaces $f : T \to U$ making the obvious diagram commute and such that $f(T_i) = U_i$ for all $0 \leq i \leq n$.

Then one can make the following reductions:

(1) In every equivalence class (7), there exists a triple $(T, T \to K_{a,b}, \vec{T})$ such that $\vec{T}$ is flanked. Moreover, two such triples are equivalent if and only if they can be connected by a zigzag of morphisms of flagged necklaces in which every flag is flanked.

(2) In every equivalence class (7), there exists a *unique* triple $(T, T \to K_{a,b}, \vec{T})$ such that $\vec{T}$ is flanked and $T \to K_{a,b}$ is totally nondegenerate. In other words, there is a bijection

$$\mathfrak{C}[K]_n(a,b) \simeq \coprod_{\substack{T \in \mathcal{N}ec \\ \vec{T} \text{ flanked flag} \\ \text{of length } n}} K_T^{\mathrm{nd}}(a,b),$$

where $K_T^{\mathrm{nd}}(a,b) \subseteq K_T(a,b)$ is the subset of totally nondegenerate maps $T \to K_{a,b}$.

The main ingredient in the first reduction is flankification. Given a necklace T with a flag $\vec{T} = (T_0 \subseteq \cdots \subseteq T_n)$, there is a unique order isomorphism $T_n \simeq [k]$, where k is the number of beads of T_n. For all $i \in [n]$, write T'_i for the image of T_i under this isomorphism, so that $T'_0 \subseteq \cdots \subseteq T'_n = [k]$. Further set $T' = T'_0$, so that the flag $\vec{T}'$ is flanked on T'. Then $(T', \vec{T}')$ is the *flankification* of $(T, \vec{T})$.

We now proceed to adapting these steps to the templicial setting. Generalising the first of the above reductions to templicial objects is fairly straightforward. This is done in Proposition 4.15. For the second reduction, we have to restrict to templicial objects that have nondegenerate simplices (Definition 2.16). Our definition of $\mathfrak{C}_{\mathcal{V}}[X]$ has the advantage that there is no reference to X in the indexing category of the colimit involved, which allows for more categorical and shorter proofs of the reduction steps.

Notation 4.13 Given an integer $n \geq 0$, let us write

$$\mathcal{N}ec^{\urcorner}[n]$$

for the category of pairs $(T, \vec{T})$ where T is a necklace and $\vec{T} = (T_0, \ldots, T_n)$ is a flag of length n on T. A morphism $(T, \vec{T}) \to (U, \vec{U})$ in $\mathcal{N}ec^{\urcorner}[n]$ is a necklace map $f : T \to U$ such that $f(T_i) = U_i$ for all $i \in [n]$. Further, we let

$$\mathcal{N}ec^{\urcorner}_f[n]$$

denote the full subcategory of $\mathcal{N}ec^{\urcorner}[n]$ spanned by flagged necklaces whose flags are flanked. Note that a morphism in $\mathcal{N}ec^{\urcorner}_f[n]$ is necessarily active and surjective on vertices.

Lemma 4.14 *Let $n \geq 0$. Flankification extends to a functor $\mathcal{N}ec^{\urcorner}[n] \to \mathcal{N}ec^{\urcorner}_f[n]$ which is right-adjoint to the inclusion $\iota \colon \mathcal{N}ec^{\urcorner}_f[n] \hookrightarrow \mathcal{N}ec^{\urcorner}[n]$.*

Proof Write $\gamma(T, \vec{T})$ for the flankification of a flagged necklace $(T, \vec{T})$. If k is the number of beads of T, we obtain a morphism $\epsilon \colon \iota\gamma(T, \vec{T}) \to (T, \vec{T})$ in $\mathcal{N}ec^{\urcorner}[n]$ with underlying morphism $[k] \simeq T_n \hookrightarrow [p]$ in $\mathbf{\Delta}_f$. Given $(U, \vec{U}) \in \mathcal{N}ec^{\urcorner}_f[n]$ with (U, q) a necklace, and a morphism $f : \iota(U, \vec{U}) \to (T, \vec{T})$ in $\mathcal{N}ec^{\urcorner}[n]$, we have in particular that $T_n = f(U_n) = f([q])$. So the morphism $f : [q] \to [p]$ in $\mathbf{\Delta}_f$ factors uniquely as some $g \colon [q] \to [k]$ followed by $[k] \hookrightarrow [p]$. Moreover, g defines a morphism $(U, \vec{U}) \to \gamma(T, \vec{T})$ in $\mathcal{N}ec^{\urcorner}_f[n]$ such that $\epsilon \circ \iota(g)$. $\square$

Proposition 4.15 *Let (X, S) be a templicial object and $a, b \in S$. Then, for every $n \geq 0$, we have a canonical isomorphism*

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a, b) \simeq \operatorname*{colim}_{(T,\vec{T}) \in \mathcal{N}ec^{\urcorner}_f[n]} X_T(a, b).$$

Proof In view of (6), we have a coequaliser, for every integer $n \geq 0$,

$$\coprod_{\substack{f : T \to U \\ \vec{T} \text{ flag on } T \\ \text{of length } n}} X_U(a, b) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{\substack{T \in \mathcal{N}ec \\ \vec{T} \text{ flag on } T \\ \text{of length } n}} X_T(a, b) \twoheadrightarrow \mathfrak{C}_{\mathcal{V}}[X]_n(a, b),$$

where α is given by $X(f)$ and β is given by applying f to $\vec{T}$, for a necklace morphism $f: T \to U$. We thus have a canonical isomorphism

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) \simeq \operatorname*{colim}_{(T,\vec{T})\in \mathcal{N}ec^{\urcorner}[n]} X_T(a,b).$$

Now, as the inclusion $\mathcal{N}ec_f^{\urcorner}[n] \hookrightarrow \mathcal{N}ec^{\urcorner}[n]$ is a left-adjoint by Lemma 4.14, the corresponding functor between opposite categories is a right-adjoint and thus a final functor. Hence, the result follows. $\square$

Remark 4.16 The simplicial structure of $\mathfrak{C}_{\mathcal{V}}[X](a,b) = \operatorname{colim}^{X_\bullet(a,b)} N\mathcal{P}_{(-)}$ is given by that of $N\mathcal{P}_T$, ie by deleting and copying terms in a flag, but the simplicial structure on $\operatorname{colim}_{(T,\vec{T})\in\mathcal{N}ec_f^{\urcorner}} X_T(a,b)$ is slightly more difficult. The degeneracy maps and inner face maps are still given by respectively copying and deleting terms in the flags. The outer face maps however are given by first deleting the term T_0 or T_n from a flag $(T_0,\dots,T_n)$ and then applying the flankification functor.

Let (X,S) be a templicial object with nondegenerate simplices and T a necklace, which we write as $\{0 = t_0 < t_1 < t_2 < \cdots < t_k = p\}$. Then let

$$X_T^{\mathrm{nd}} = X_{t_1}^{\mathrm{nd}} \otimes_S X_{t_2-t_1}^{\mathrm{nd}} \otimes_S \cdots \otimes_S X_{p-t_{k-1}}^{\mathrm{nd}} \in \mathcal{V}\mathrm{Quiv}_S,$$

where X_n^{nd} denotes the quiver of nondegenerate simplices of Definition 2.16.

Proposition 4.17 *Let (X,S) be a templicial object with nondegenerate simplices. For all $n \geq 0$ and $a,b \in S$, we have an isomorphism in $\mathcal{V}$,*

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) \simeq \coprod_{\substack{T\in\mathcal{N}ec \\ \vec{T}\text{ flanked flag} \\ \text{of length } n}} X_T^{\mathrm{nd}}(a,b).$$

Proof By Proposition 4.15 and Lemma 2.19, we have an isomorphism

$$\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) \simeq \operatorname*{colim}_{(T,\vec{T})\in\mathcal{N}ec_f^{\urcorner}[n]} \coprod_{\substack{f_i:[t_i-t_{i-1}]\twoheadrightarrow[n_i] \\ i\in\{1,\dots,k\}}} (X_{n_1}^{\mathrm{nd}} \otimes_S \cdots \otimes_S X_{n_k}^{\mathrm{nd}})(a,b),$$

where we've written $T = \{0 = t_0 < t_1 < \cdots < t_k = p\}$ for any $(T,\vec{T}) \in \mathcal{N}ec_f^{\urcorner}[n]$. Now let $f:(T,p)\to(U,q)$ be an active necklace map whose underlying morphism $f:[p]\to[q]$ in $\mathbf{\Delta}_f$ is surjective. We can uniquely decompose $f = f_1 + \cdots + f_k$ with $f_i:[t_i - t_{i-1}] \twoheadrightarrow [n_i]$ in $\mathbf{\Delta}_{\mathrm{surj}}$ for all $i \in \{1,\dots,n\}$. Moreover, given a flag $\vec{T}$ of length n on T, there is a unique flanked flag $\vec{U} = (U_0,\dots,U_n)$ on U such that $f: T\to U$ lifts to a morphism $f:(T,\vec{T})\to(U,\vec{U})$ in $\mathcal{N}ec_f^{\urcorner}[n]$ (simply set $U_i = f(T_i)$). It follows that

$$\begin{aligned}\mathfrak{C}_{\mathcal{V}}[X]_n(a,b) &\simeq \operatorname*{colim}_{(T,\vec{T})\in\mathcal{N}ec_f^{\urcorner}[n]} \coprod_{(T,\vec{T})\to(U,\vec{U})\text{ in }\mathcal{N}ec_f^{\urcorner}[n]} X_U^{\mathrm{nd}}(a,b) \\ &\simeq \coprod_{(U,\vec{U})\in\mathcal{N}ec_f^{\urcorner}[n]} \operatorname*{colim}_{(T,\vec{T})\to(U,\vec{U})\text{ in }\mathcal{N}ec_f^{\urcorner}[n]} X_U^{\mathrm{nd}}(a,b) \simeq \coprod_{(U,\vec{U})\in\mathcal{N}ec_f^{\urcorner}[n]} X_U^{\mathrm{nd}}(a,b).\end{aligned}$$

The last isomorphism is obtained by noting that the colimit on the left-hand side is indexed over the category $((\mathcal{N}ec^{\urcorner}_f[n])_{/(U,\vec{U})})^{\mathrm{op}}$, which is connected, and the functor involved is constant on $X_U^{\mathrm{nd}}(a,b)$. □

4.D Comparison with the templicial nerve

Analogous to the classical homotopy coherent nerve, we show that the templicial homotopy coherent nerve $N_{\mathcal{V}}^{\mathrm{hc}}$ restricts to the templicial nerve $N_{\mathcal{V}}$ (see Section 2.B) when applied to ordinary $\mathcal{V}$-enriched categories.

Consider the $\mathcal{V}$-enriched left-adjoint $\pi_0\colon S\mathcal{V}\to\mathcal{V}$ to the functor $\underline{(-)}\colon\mathcal{V}\to S\mathcal{V}$ sending every object $V\in\mathcal{V}$ to the constant functor on V. Then, for any $Y\in S\mathcal{V}$, we have a reflexive coequaliser

$$Y_1 \overset{d_0}{\underset{d_1}{\underset{\xleftarrow{s_0}}{\rightrightarrows}}} Y_0 \twoheadrightarrow \pi_0(Y). \tag{8}$$

For example, if $\mathcal{V}=\mathrm{Set}$, then π_0 is the functor taking the set of connected components of a simplicial set.

As the monoidal product $\otimes$ of $\mathcal{V}$ preserves colimits in each variable, it follows that π_0 is strong monoidal and thus we have an induced adjunction

$$\mathcal{V}\mathrm{Cat}_{\Delta} \underset{\underline{(-)}}{\overset{\pi_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{V}\mathrm{Cat}.$$

Proposition 4.18 *We have a natural isomorphism*

$$N_{\mathcal{V}}^{\mathrm{hc}}\circ\underline{(-)}\simeq N_{\mathcal{V}}.$$

Proof By Proposition 3.16, $N_{\mathcal{V}}\simeq(-)^{\mathrm{temp}}\circ\underline{(-)}$ for $\underline{(-)}\colon\mathcal{V}\mathrm{Cat}\to\mathcal{V}\mathrm{Cat}_{\mathcal{N}ec}$. Thus it suffices to show that we have an isomorphism $\mathfrak{n}\circ\underline{(-)}\simeq\underline{(-)}$ of functors $\mathcal{V}\to\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. Take an object $A\in\mathcal{V}$ and write $[-,-]$ for the internal hom of $\mathcal{V}$. Since the simplicial set $N(\mathcal{P}_T)$ clearly only has one connected component, it follows from the fact that F preserves colimits that

$$[FN\mathcal{P}_T,\underline{A}]\simeq[\pi_0FN\mathcal{P}_T,A]\simeq[F(\pi_0N\mathcal{P}_T),A]\simeq[F(\{*\}),A]\simeq A$$

for all necklaces T. It follows that $\mathfrak{n}(\underline{A})$ is isomorphic to the constant functor on A. Clearly, this isomorphism is natural in A, as desired. □

Definition 4.19 It immediately follows from Proposition 4.18 that the templicial nerve $N_{\mathcal{V}}$ has a left-adjoint given by the composite

$$h_{\mathcal{V}}=\pi_0\circ\mathfrak{C}_{\mathcal{V}}\colon S_{\otimes}\mathcal{V}\to\mathcal{V}\mathrm{Cat},$$

which we call the *homotopy category functor*.

Remark 4.20 By Remark 2.14, $h_{\mathcal{V}}$ necessarily recovers the classical homotopy category functor $h\colon\mathrm{SSet}\to\mathrm{Cat}$ when $\mathcal{V}=\mathrm{Set}$.

Corollary 4.21 *Let (X, S) be a templicial object with $a, b \in S$. Then we have a reflexive coequaliser*

$$\coprod_{\substack{T\in\mathcal{N}ec\\ T\neq\{0\}}} X_T(a,b) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{p>0} X_1^{\otimes p}(a,b) \twoheadrightarrow h_{\mathcal{V}}X(a,b)$$

with α and β induced by the unique active and inert necklace maps $([k],k)\to(T,p)$ and $([p],p)\to(T,p)$ of Remark 3.7, respectively, for any necklace (T,p) with k beads.

Proof It directly follows from Proposition 4.15 and (8) that, for all $a, b \in S$, we have the (reflexive) coequaliser

$$\operatorname*{colim}_{\substack{T\in\mathcal{N}ec_-\\ T\neq\{0\}}} X_T(a,b) \underset{\beta'}{\overset{\alpha'}{\rightrightarrows}} \operatorname*{colim}_{\substack{[p]\in\mathbf{\Delta}_{\mathrm{surj}}\\ p>0}} X_1^{\otimes p}(a,b) \twoheadrightarrow h_{\mathcal{V}}X(a,b), \tag{9}$$

where $\mathcal{N}ec_-$ denotes the subcategory of $\mathcal{N}ec$ consisting of all active necklace maps that are surjective on vertices, and α' and β' are defined similarly to α and β. Via the epimorphism $\coprod_T X_T(a,b) \twoheadrightarrow \operatorname{colim}_T X_T(a,b)$, we may replace the left-hand colimit by $\coprod_T X_T(a,b)$.

To show that we may also replace the right-hand colimit, observe that any surjective necklace map $f\colon([p],p)\twoheadrightarrow([q],q)$ with $q>0$ can be factored as an inert map $([p],p)\to(T,p)$ followed by some $\sigma\colon(T,p)\to([q],q)$ such that T has q beads and the unique active map $([q],q)\hookrightarrow(T,p)$ is a section of σ. Indeed, let $\sigma\colon[p]\twoheadrightarrow[q]$ be the underlying morphism of f in $\mathbf{\Delta}_f$. Then σ has a section δ in $\mathbf{\Delta}_f$. Now simply set $T=\delta([p])$. □

Recall the commutativity results for the templicial nerve (Proposition 2.15). While it follows immediately from Proposition 4.12 that $h_{\mathcal{V}}\circ\widetilde{F}\simeq\mathcal{F}\circ h$, the homotopy category functors and forgetful functors do not commute in general, as the following example shows.

Example 4.22 Let $\mathcal{V}=\mathrm{Mod}(k)$ be the category of k-modules over with k an arbitrary unital commutative ring. Let $h_k=h_{\mathrm{Mod}(k)}$. Consider the templicial k-module $X=\widetilde{F}(\partial\Delta^2)$. Then the hom-object $(h_kX)(0,2)\in\mathrm{Mod}(k)$ is isomorphic to

$$F(h(\partial\Delta^2)(0,2)) = F\left(\left\{ \begin{tikzcd}[column sep=small, row sep=small] & \bullet^1 \arrow[dr] & \\ 0\bullet \arrow[ur] & & \bullet 2 \end{tikzcd},\ 0\bullet\longrightarrow\bullet 2 \right\}\right) \simeq k\oplus k.$$

On the other hand, note that each edge in $\widetilde{U}(X)$ between two given vertices is uniquely determined by an element $a_i\in k$. So the set $h\widetilde{U}(X)(0,2)$ consists of equivalence classes of sequences of edges $(a_1,\dots,a_n)$ from 0 to 2 in $\widetilde{U}(X)$. One can check that

$$h\widetilde{U}\widetilde{F}(\partial\Delta^2)(0,2)\simeq U(k)\amalg_{U(0)}U(k),$$

which identifies a sequence $(a_1,\dots,a_n)$ with its product $a_n\cdots a_1$ in k. The two terms $U(k)$ correspond to paths either passing through the vertex 1 or not. Now the induced map $h\widetilde{U}\widetilde{F}(\partial\Delta^2)(0,2)\to U((h_kX)(0,2))$ on hom-sets corresponds to the canonical map

$$U(k)\amalg_{U(0)}U(k)\to U(k\oplus k),$$

which is certainly not a bijection if k is not the zero ring. Hence, the canonical functor

$$h\widetilde{U}(X) \to \mathcal{U}(h_k X)$$

is not an equivalence of categories.

In the next section, we will restrict to a special class of templicial objects, which we call quasicategories in $\mathcal{V}$. It turns out that, for a quasicategory X in $\mathcal{V}$, the canonical functor $h\widetilde{U}(X) \to \mathcal{U}(h_{\mathcal{V}} X)$ is always an isomorphism (under suitable hypotheses on $\mathcal{V}$); see Corollary 5.23 below.

5 Quasicategories in a monoidal category

Quasicategories are models for $(\infty, 1)$-categories first introduced by Joyal [22] as simplicial sets satisfying the weak Kan condition in the sense of Boardman and Vogt [8]. That is, a simplicial set X is a quasicategory if every simplicial map $\Lambda^n_j \to X$ from an inner horn can be extended to a map $\Delta^n \to X$ from the standard simplex. In [23], Joyal equips SSet with a model structure in which the fibrant objects are precisely the quasicategories. In this section, we introduce the natural analogue of quasicategories in the templicial context (Definition 5.4). However, in view of Example 2.10, we express the lifting condition in the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$, rather than $S_{\otimes}\mathcal{V}$. Nonetheless, we still recover classical quasicategories when $\mathcal{V} = \mathrm{Set}$ (Proposition 5.8). We continue in Section 5.B by showing our main result: that the templicial nerve produces quasicategories in $\mathcal{V}$ from locally Kan $S\mathcal{V}$-categories (Corollary 5.12). In Section 5.C, we discuss the homotopy category of a quasicategory in $\mathcal{V}$.

5.A Horn filling in necklaces

For integers $0 \le j \le n$, we denote by Λ^n_j the j^{th} horn of the n-simplex. That is, Λ^n_j is the union of all the faces of Δ^n, except the j^{th} face. In order to define quasicategories in $\mathcal{V}$, we wish to consider the usual horn lifting property in the category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ via Construction 3.9. In this case, it is convenient to express the horn as a union of necklaces, rather than faces.

Proposition 5.1 *For all integers $0 < j < n$,*

$$(\Lambda^n_j)_\bullet(0,n) = \bigcup_{\substack{i=1 \\ i \neq j}}^{n-1} \delta_i(\Delta^{n-1})_\bullet(0,n) \cup \bigcup_{k=1}^{n-1} (\Delta^k \vee \Delta^{n-k})_\bullet(0,n)$$

as a subfunctor of $\Delta^n_\bullet(0,n)$ in $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}$.

Proof For all $0 < k, i < n$ with $i \neq j$, we have inclusions $\Delta^k \vee \Delta^{n-k} \subseteq \Lambda^n_j$ and $\delta_i(\Delta^{n-1}) \subseteq \Lambda^n_j$ in SSet. It follows that

$$\bigcup_{\substack{i=1 \\ i \neq j}}^{n-1} \delta_i(\Delta^{n-1})_\bullet(0,n) \cup \bigcup_{k=1}^{n-1} (\Delta^k \vee \Delta^{n-k})_\bullet(0,n) \subseteq (\Lambda^n_j)_\bullet(0,n).$$

Conversely, let $f: T \to (\Lambda^n_j)_{0,n}$ be a map in $\mathrm{SSet}_{*,*}$ with (T, p) a necklace. Suppose first that f is surjective on vertices. As the unique nondegenerate n-simplex of Δ^n is not contained in Λ^n_j, there must be some $k \in T$ such that $0 < f(k) < n$. Therefore, f factors through $\Delta^l \vee \Delta^{n-l}$ with $l = f(k)$. Now suppose that f is not surjective on vertices. Then f must factor through $\delta_i(\Delta^{n-1})$ for some $i \in [n] \setminus \{j\}$. As a map in $\mathrm{SSet}_{*,*}$, f always reaches the vertices 0 and n of Δ^n, and thus $0 < i < n$. □

Example 5.2 The outer horns aren't as well behaved in $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}$ as the inner horns. For example, Λ^2_0 is the pushout $\Delta^1 \amalg_{\{0\}} \Delta^1$ in SSet, but $(\Lambda^2_0)_\bullet(0,2)$ is isomorphic to just $\Delta^1_\bullet(0,1)$ as all maps $T \to (\Lambda^2_0)_{0,2}$ in $\mathrm{SSet}_{*,*}$ must factor through the edge $0 \to 2$ of Λ^2_0.

The following corollary expresses the advantage of working in the functor category $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. While not every simplex of a templicial object is represented by a templicial morphism (see Example 2.10), it is represented by a morphism in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$.

Corollary 5.3 *Let (X, S) be a templicial object with $a, b \in S$.*

(1) *Let (T, p) be a necklace. There is a bijective correspondence between morphisms $\widetilde{F}(T)_\bullet(0, p) \to X_\bullet(a, b)$ in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ and elements $\sigma \in U(X_T(a, b))$.*

(2) *Let $0 < j < n$ be integers. There is a bijective correspondence between morphisms $\widetilde{F}(\Lambda^n_j)_\bullet(0, n) \to X_\bullet(a, b)$ in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ and elements*

$$x_k \in U((X_k \otimes_S X_{n-k})(a, b)) \quad \text{and} \quad y_i \in U(X_{n-1}(a, b))$$

for all $0 < k, i < n$ with $i \neq j$, which satisfy:

- *For all $0 < i < i' < n$ with $i \neq j \neq i'$,*

$$d_{i'-1}(y_i) = d_i(y_{i'}).$$

- *For all $0 < k < l < n$,*

$$(\mathrm{id}_{X_k} \otimes \mu_{l-k,n-l})(x_k) = (\mu_{k,l-k} \otimes \mathrm{id}_{X_{n-l}})(x_l).$$

- *For all $0 < k < n-1$ and $0 < i < n$ with $i \neq j$,*

$$\mu_{k,n-k-1}(y_i) = \begin{cases} (d_i \otimes \mathrm{id}_{X_{n-k-1}})(x_{k+1}) & \text{if } i \le k, \\ (\mathrm{id}_{X_k} \otimes d_{i-k})(x_k) & \text{if } i > k. \end{cases}$$

Proof A morphism $F(T_\bullet(0, p)) \simeq \widetilde{F}(T)_\bullet(0, p) \to X_\bullet(a, b)$ is equivalent to a map $T_\bullet(0, p) \to U(X_\bullet(a, b))$ in $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}$, which corresponds to an element $\sigma \in U(X_T(a, b))$ by the Yoneda lemma. This shows (1). Statement (2) follows from Proposition 5.1. □

Definition 5.4 Let $Y : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$ be a functor. We say Y *lifts inner horns* if, for all $0 < j < n$, any lifting problem

$$\begin{tikzcd} \tilde{F}(\Lambda^n_j)_\bullet(0,n) \arrow[r] \arrow[d] & Y \\ \tilde{F}(\Delta^n)_\bullet(0,n) \arrow[ur, dashed] & \end{tikzcd}$$

has a solution in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. We say Y *lifts inner horns uniquely* if every such lifting problem has a unique solution in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$.

We call a templicial object (X, S) in $\mathcal{V}$ a *quasicategory in* $\mathcal{V}$ if the functor $X_\bullet(a,b)$ lifts inner horns for all $a, b \in S$. In this case, we will refer to the elements of S as the *objects* of X and to elements of $U(X_1(a,b))$ as the *morphisms* $a \to b$ in X.

Remark 5.5 Let $Y : \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$ be a functor. By the adjunction $F \dashv U$, Y lifts inner horns in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$ if and only if the composite $UY : \mathcal{N}ec^{\mathrm{op}} \to \mathrm{Set}$ lifts inner horns in $\mathrm{Set}^{\mathcal{N}ec^{\mathrm{op}}}$.

As for ordinary quasicategories, there is an elementwise characterisation of quasicategories in $\mathcal{V}$, although it is bit more cumbersome to describe.

Proposition 5.6 *Let (X, S) be a templicial object. The following statements are equivalent*:

(1) *X is a quasicategory in $\mathcal{V}$.*

(2) *Let $a, b \in S$ and $0 < j < n$. For all collections of elements $(x_k)_{k=1}^{n-1}$, $(y_i)_{i=1, i\neq j}^{n-1}$ satisfying the conditions of Corollary 5.3(2), there is an element $z \in U(X_n(a,b))$ such that*

$$\mu_{k,n-k}(z) = x_k \quad \text{and} \quad d_i(z) = y_i$$

for all $0 < k, i < n$ with $i \neq j$.

Proof This immediately follows from Corollary 5.3. □

Remark 5.7 Note the similarities with the classical elementwise characterisation of a quasicategory. The elements y_i with $0 < i < n$, $i \neq j$ represent all inner faces of the horn Λ^n_j. They still have to satisfy the same conditions as usual. However, the two outer faces of the horn are replaced by the elements x_k with $0 < k < n$. The two new conditions of Corollary 5.3(2) merely express that these outer faces are glued to each other and to the inner faces in the appropriate way.

Indeed, when $\mathcal{V} = \mathrm{Set}$, we recover the classical notion of a quasicategory.

Proposition 5.8 *A simplicial set is a quasicategory if and only if it is a quasicategory in* Set (*in the sense of Definition 5.4*).

Proof Let X be a simplicial set, considered as a templicial set with X_0 its set of vertices. Then the assignment $(x_k)_{k=1}^{n-1} \mapsto (x^1_{n-1}, x^2_1)$ defines a bijection between the set of all collections of elements

$$(x_k = (x^1_k, x^2_k) \in X_k \times X_{n-k})_{k=1}^{n-1}$$

satisfying $(x^1_k, \mu_{l-k,n-l}(x^2_k)) = (\mu_{k,l-k}(x^1_k), x^2_l)$ for all $0<k<l<n$, and the set of all pairs $(y_n, y_0) \in X_{n-1}\times X_{n-1}$ satisfying $d_{n-1}(y_0) = d_0(y_n)$. It follows that condition (2) of Proposition 5.6 is equivalent to:

(2′) Let $0<j<n$. Consider elements $y_i \in X_{n-1}$ for all $0 \le i \le n$ with $i \ne j$ which satisfy, for all $0 \le i < i' \le n$ with $i \ne j \ne i'$,
$$d_{i'-1}(y_i) = d_i(y_{i'}).$$
Then there is an element $z \in X_n$ such that $d_i(z) = y_i$ for all $0 \le i \le n$ with $i \ne j$.

But this precisely expresses that X is a quasicategory. □

5.B Nerves and quasicategories

We show that the earlier-defined templicial versions of classical nerves give examples of quasicategories in $\mathcal{V}$.

Lemma 5.9 *Let $\mathcal{C}$ be a necklace category with objects A and B. Consider the canonical morphism $\epsilon\colon \mathcal{C}^{\mathrm{temp}}_\bullet(A,B) \to \mathcal{C}_\bullet(A,B)$ induced by the counit of the adjunction $(-)^{\mathrm{nec}} \dashv (-)^{\mathrm{temp}}$. Given integers $0<j<n$, any lifting problem in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$*

$$\begin{tikzcd} \widetilde{F}(\Lambda^n_j)_\bullet(0,n) \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{temp}}_\bullet(A,B) \arrow[d, "\epsilon"] \\ \widetilde{F}(\Delta^n)_\bullet(0,n) \arrow[r] \arrow[ur, dashed] & \mathcal{C}_\bullet(A,B) \end{tikzcd}$$

has a unique solution.

Proof The top horizontal morphism corresponds to some collections of elements $(x_k)_{k=1}^{n-1}$ and $(y_i)_{i=1,i\ne j}^{n-1}$ with $x_k \in U((\mathcal{C}^{\mathrm{temp}}_k \otimes \mathcal{C}^{\mathrm{temp}}_{n-k})(a,b))$ and $y_i \in U(\mathcal{C}^{\mathrm{temp}}_{n-1}(a,b))$, satisfying the conditions of Corollary 5.3(2). Moreover, the bottom horizontal morphism corresponds to an element $z' \in U(\mathcal{C}_{\{0<n\}}(a,b))$ and the commutativity of the diagram comes down to the condition that $\mathcal{C}(\nu_{k,n-k})(z') = m_{\{0<k\},\{0<n-k\}}(p_k \otimes p_{n-k})(x_k)$ and $\mathcal{C}(\delta_i)(z') = p_{n-1}(y_i)$ for all $0<k,i<n$ with $i \ne j$.

Then, by the limit diagram (5), there exists a unique element $z \in U(\mathcal{C}^{\mathrm{temp}}_n(a,b))$ such that $\mu_{k,n-k}(z) = x_k$ for all $0<k<n$, and $p_n(z) = z'$. Again by (5), for all $0<k,i<n$ with $i\ne j$,

$$\begin{aligned} \mu_{k,n-1-k}(d_i(z)) &= \begin{cases} (d_i \otimes \mathrm{id}_{\mathcal{C}^{\mathrm{temp}}_{n-k-1}})(\mu_{k+1,n-k}(z)) & \text{if } i \le k,\\ (\mathrm{id}_{\mathcal{C}^{\mathrm{temp}}_k} \otimes d_{i-k})(\mu_{k,n-k}(z)) & \text{if } i > k,\end{cases} \\ &= \mu_{k,n-1-k}(y_i), \\ p_{n-1}(d_i(z)) &= \mathcal{C}(\delta_i)p_n(z) = \mathcal{C}(\delta_i)(z') = p_{n-1}(y_i), \end{aligned}$$

and thus $d_i(z) = y_i$. Hence, the element z determines the unique solution to the lifting problem. □

Proposition 5.10 *Let $\mathcal{C}$ be a necklace category with object set S. Suppose that, for all $A, B \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}_\bullet(A, B)$ lifts inner horns. Then $\mathcal{C}^{\mathrm{temp}}$ is a quasicategory in $\mathcal{V}$.*

Proof This is immediate from Lemma 5.9. ◻

Corollary 5.11 *For any small $\mathcal{V}$-category $\mathcal{C}$, the templicial object $N_{\mathcal{V}}(\mathcal{C})$ is a quasicategory in $\mathcal{V}$.*

Proof This follows from Propositions 3.16 and 5.10. ◻

Corollary 5.12 *Let $\mathcal{C}$ be small simplicial $\mathcal{V}$-category. Assume that, for all objects A and B of $\mathcal{C}$, the simplicial set $U(\mathcal{C}(A, B))$ is a Kan complex. Then the templicial object $N_{\mathcal{V}}^{\mathrm{hc}}(\mathcal{C})$ is a quasicategory in $\mathcal{V}$.*

Proof By Proposition 5.10, it suffices to check that, for all $A, B \in \mathrm{Ob}(\mathcal{C})$, the functor

$$\mathfrak{n}(\mathcal{C}(A, B))_\bullet = [FN\mathcal{P}_{(-)}, \mathcal{C}(A, B)] \colon \mathcal{N}ec^{\mathrm{op}} \to \mathcal{V}$$

lifts inner horns in $\mathcal{V}^{\mathcal{N}ec^{\mathrm{op}}}$. By the adjunction $\mathfrak{s} \dashv \mathfrak{n}$, this is equivalent to showing that, for all $0 < j < n$, every morphism $\mathfrak{s}(\widetilde{F}(\Lambda^n_j)_\bullet(0, n)) \to \mathcal{C}(A, B)$ in $S\mathcal{V}$ extends to $\mathfrak{s}(\widetilde{F}(\Delta^n)_\bullet(0, n))$. Now, by Proposition 4.12 and the adjunction $F \dashv U$, this is further equivalent to the lifting problem in SSet

$$\begin{tikzcd}
\mathfrak{C}[\Lambda^n_j](0, n) \arrow[r] \arrow[d] & U(\mathcal{C}(A, B)) \\
\mathfrak{C}[\Delta^n](0, n) \arrow[ur, dashed] &
\end{tikzcd}$$

This has a solution because $U(\mathcal{C}(A, B))$ is a Kan complex, as was shown in [28, Proposition 1.1.5.10] and is given in more detail in [30, Tag 00LH] (beware that, in the latter, the notation Path is used instead of $\mathfrak{C}$). ◻

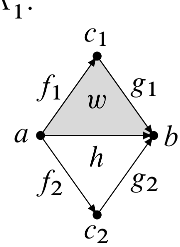
Corollary 5.13 *Let X be a quasicategory in $\mathcal{V}$. Then $\widetilde{U}(X)$ is a quasicategory.*

Proof This follows from Propositions 3.14, 5.8 and 5.10. ◻

The converse to Corollary 5.13 does not hold in general.

Example 5.14 Consider the over category $\mathcal{V} = \mathrm{Ab}/\mathbb{Z}$ of abelian groups A with a $\mathbb{Z}$-linear map $p \colon A \to \mathbb{Z}$. Then $\mathcal{V}$ is bicomplete and symmetric monoidal closed with monoidal unit given by $\mathrm{id}_{\mathbb{Z}} \colon \mathbb{Z} \to \mathbb{Z}$. The forgetful functor $U \colon \mathcal{V} \to \mathrm{Set}$ associates to every map $p \colon A \to \mathbb{Z}$ the set $\{a \in A \mid p(a) = 1\}$.

Now consider the simplicial set $\Delta^2 \amalg_{\{0,2\}} \Lambda^2_1$:



Set $X = \widetilde{F}(\Delta^2 \amalg_{\{0,2\}} \Lambda^2_1) \in S_\otimes \mathrm{Ab}$. We can promote X to a templicial object in $\mathcal{V}$ by equipping it with $\mathbb{Z}$-linear maps $p\colon X_n(x,y) \to \mathbb{Z}$ defined by

$$p(f_1) = p(f_2) = p(g_2) = p(h) = 1, \quad p(g_2) = 2 \quad \text{and} \quad p(w) = 1$$

Then, for example, $U(X_1(a,c_2)) = \{f_2\}$ but $U(X_1(c_2,b)) = \varnothing$. Consider the functor $\widetilde{U}\colon S_\otimes\mathcal{V} \to \mathrm{SSet}$ as induced by U above (not by $\mathrm{Ab} \to \mathrm{Set}$). Then $\widetilde{U}(X) \simeq \Delta^2 \amalg_{\{0\}} \Delta^1$, which is clearly a quasicategory. However, X is not a quasicategory in $\mathcal{V}$. To see this, consider the element

$$\alpha = f_2 \otimes g_2 - f_1 \otimes g_1 \in U((X_1 \otimes X_1)(a,b))$$

(note that, indeed, $(p \otimes p)(\alpha) = p(f_2)p(g_2) - p(f_1)p(g_1) = 1$). But there exists no element $\xi \in U(X_2(a,b))$ such that $\mu_{1,1}(\xi) = \alpha$.

We end this subsection by characterising the essential image of the templicial nerve functor $N_\mathcal{V}\colon \mathcal{V}\mathrm{Cat} \to S_\otimes\mathcal{V}$ in terms of horn fillings.

Proposition 5.15 *Let $(X,S) \in S_\otimes\mathcal{V}$. Consider the following statements*:

(1) *(X,S) is isomorphic to the templicial nerve of a small $\mathcal{V}$-category.*

(2) *For all $a,b \in S$, $X_\bullet(a,b)$ lifts inner horns uniquely.*

Then (1) *implies* (2). *Moreover*, *if the functor $U:\mathcal{V} \to \mathrm{Set}$ is conservative*, *then* (1) *and* (2) *are equivalent.*

Proof Let $\mathcal{C}$ be a small $\mathcal{V}$-category. We wish to show that $N_\mathcal{V}(\mathcal{C})_\bullet(A,B)$ lifts inner horns uniquely for all $A,B \in \mathrm{Ob}(\mathcal{C})$. Since $N_\mathcal{V} \simeq (-)^{\mathrm{temp}} \circ \underline{(-)}$ (Proposition 3.16), it suffices by Lemma 5.9 to note that the lifting problem

$$\begin{tikzcd} \widetilde{F}(\Lambda^n_j)_\bullet(0,n) \arrow[r] \arrow[d] & \underline{\mathcal{C}}(A,B) \\ \widetilde{F}(\Delta^n)_\bullet(0,n) \arrow[ur, dashed] & \end{tikzcd}$$

has a unique solution for all $0 < j < n$, which is clear.

Assume that (2) holds and that U is conservative. By (3), it suffices to show that each comultiplication morphism $\mu_{k,n-k}$ with $0 < k < n$ is an isomorphism. Take $x_k \in U(X_k \otimes_S X_{n-k})$. By induction on n, we can define, for any $0 < l < n$ with $l \neq k$,

$$x_l = \begin{cases} (\mathrm{id}_{X_l} \otimes \mu^{-1}_{k-l,n-k})(\mu_{l,k-l} \otimes \mathrm{id}_{X_{n-k}})(x_k) & \text{if } l < k, \\ (\mu^{-1}_{k,l-k} \otimes \mathrm{id}_{X_{n-l}})(\mathrm{id}_{X_k} \otimes \mu_{l-k,n-l})(x_k) & \text{if } l > k. \end{cases}$$

Further set, for all $0 < i < n$ with $i \neq k$,

$$y_i = \begin{cases} \mu^{-1}_{k-1,n-k}(d_i \otimes \mathrm{id}_{X_{n-k}})(x_k) & \text{if } i < k, \\ \mu^{-1}_{k,n-k-1}(\mathrm{id}_{X_k} \otimes d_{i-k})(x_k) & \text{if } i > k. \end{cases}$$

It follows that the elements $(x_l)_{l=1}^{n-1}$ and $(y_i)_{i=1,i\neq k}^{n-1}$ satisfy the conditions of Corollary 5.3(2) and thus there is a unique element $z \in U(X_n(a,b))$ such that $\mu_{l,n-l}(z) = x_l$ and $d_i(z) = y_i$ for all $0 < l,i < n$ with $i \neq k$. In particular, $\mu_{k,n-k}(z) = x_k$. For any other $z' \in U(X_n(a,b))$ with $\mu_{k,n-k}(z') = x_k$, it follows from the definitions of the x_l and y_i that also $\mu_{l,n-l}(z') = x_l$ and $d_i(z') = y_i$ for all $0 < l, i < n$ with $i \neq k$. Thus $z' = z$ and hence the map

$$U(\mu_{k,n-k})\colon U(X_n(a,b)) \to U((X_k \otimes X_{n-k})(a,b))$$

is a bijection. As U is conservative, $\mu_{k,n-k}\colon X_n \to X_k \otimes X_{n-k}$ is an isomorphism of $\mathcal{V}$-enriched quivers. □

5.C Simplification of the homotopy category

We now turn our attention to the homotopy category $h_{\mathcal{V}}X$ when X is a quasicategory in $\mathcal{V}$. As is the case in the classical situation, this allows for a simpler description of $h_{\mathcal{V}}X$.

Construction 5.16 Let (X, S) be a templicial object and $a, b \in S$. We define an object $\mathrm{Hom}^L_X(a,b)_1 \in \mathcal{V}$ by the pullback

$$\begin{tikzcd} \mathrm{Hom}^L_X(a,b)_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X_2(a,b) \arrow[d, "\mu^X_{1,1}"] \\ X_1(a,b) \arrow[r, "-\otimes s^X_0"'] & (X_1 \otimes_S X_1)(a,b) \end{tikzcd}$$

Further, we let $d_1 = \pi_1$, $d_0 = d^X_1 \pi_2$ and we let $s_0\colon X_1(a,b) \to \mathrm{Hom}^L_X(a,b)_1$ be the unique morphism such that $\pi_1 s_0 = \mathrm{id}_{X_1(a,b)}$ and $\pi_2 s_0 = s^X_1$. We obtain a reflexive pair

$$\mathrm{Hom}^L_X(a,b)_1 \underset{d_1}{\overset{d_0}{\underset{\xleftarrow{s_0}}{\rightrightarrows}}} X_1(a,b)$$

and we define an object $h'_{\mathcal{V}}X(a,b)$ as the coequaliser of this pair,

$$\mathrm{Hom}^L_X(a,b)_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_1(a,b) \xrightarrow{q} h'_{\mathcal{V}}X(a,b) \tag{10}$$

Remark 5.17 It is possible to extend Construction 5.16 to obtain a simplicial object $\mathrm{Hom}^L_X(a,b)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{V}$ which generalises the *left-pinched morphism space* of a simplicial set (as defined in [30, Tag 01KX]). In particular, $\mathrm{Hom}^L_X(a,b)_0 = X_1(a,b)$. Then the morphisms $d_0, d_1\colon \mathrm{Hom}^L_X(a,b)_1 \rightrightarrows X_1(a,b)$ and $s_0\colon X_1(a,b) \to \mathrm{Hom}^L_X(a,b)_1$ constitute the lowest-dimensional face and degeneracy morphisms of $\mathrm{Hom}^L_X(a,b)$. We will not go into them here however, and leave their investigation to later research.

Remark 5.18 As U preserves pullbacks, $U(\mathrm{Hom}^L_X(a,b)_1)$ is the set of all 2-simplices $\sigma \in \widetilde{U}(X)$ with $d_0(\sigma) = s_0(b)$ and $d_1 d_2(\sigma) = a$. In other words, it describes homotopies between two edges $a \to b$ in $\widetilde{U}(X)$.

Assuming that $\widetilde{U}(X)$ is a quasicategory and that U preserves reflexive coequalisers, it follows that we have an isomorphism

$$U(h'_{\mathcal{V}}X(a,b)) \simeq h\widetilde{U}(X)(a,b)$$

and the canonical morphism $X_1(a,b) \twoheadrightarrow h'_{\mathcal{V}}X(a,b)$ precisely takes the homotopy class $[f]$ in $h\widetilde{U}(X)$ of any $f \in U(X_1(a,b))$.

Lemma 5.19 *Assume that $U\colon \mathcal{V} \to \mathrm{Set}$ preserves reflexive coequalisers. Let X be a quasicategory in $\mathcal{V}$ with objects a and b. For any $w, w' \in U(X_2(a,b))$,*

$$(q\otimes q)\mu_{1,1}(w) = (q\otimes q)\mu_{1,1}(w') \implies q(d_1^X(w)) = q(d_1^X(w'))$$

in $h'_{\mathcal{V}}X(a,b)$.

Proof Let Q be the quiver given by $\mathrm{Hom}_X^L(a,b)_1$ for all objects a and b of X. Let $\sigma \in U((Q\otimes Q)(a,b))$ and $w, w' \in U(X_2(a,b))$ be such that $\mu_{1,1}(w) = (d_0\otimes d_0)(\sigma)$ and $\mu_{1,1}(w') = (d_1\otimes d_1)(\sigma)$. Then:

- Consider $x_1 = (d_1 \otimes s_0^X d_1)(\sigma) \in U((X_1\otimes X_2)(a,b))$, $x_2 = (\pi_2\otimes d_1)(\sigma) \in U((X_2\otimes X_1)(a,b))$ and $y_2 = w \in U(X_2(a,b))$. These define a morphism $\widetilde{F}(\Lambda_1^3)_\bullet(0,3) \to X_\bullet(a,b)$, which extends to an element $z \in U(X_3(a,b))$. Setting $w'' = d_1^X(z) \in U(X_2(a,b))$, we have $d_1^X(w'') = d_1^X(w)$.
- Similarly, consider $x_1 = (d_0\otimes \pi_2)(\sigma) \in U((X_1\otimes X_2)(a,b))$, $x_2 = w''\otimes s_0^X(b) \in U((X_2\otimes X_1)(a,b))$ and $y_2 = w'$. These define a morphism $\widetilde{F}(\Lambda_1^3)_\bullet(0,3) \to X_\bullet(a,b)$, which extends to an element $z \in U(X_3(a,b))$. Then set $\tau = d_1^X(z) \in U(X_2(a,b))$.

It follows that $\mu_{1,1}(\tau) = d_1^X(w)\otimes s_0^X(b)$ and $d_1^X(\tau) = d_1^X(w')$. Hence, $qd_1^X(w) = qd_1^X(w')$.

As the diagram (10) is a reflexive coequaliser, it is preserved by $-\otimes-$ in both variables simultaneously, so that we again have a reflexive coequaliser

$$(Q\otimes Q)(a,b) \overset{d_0\otimes d_0}{\underset{d_1\otimes d_1}{\rightrightarrows}} (X_1\otimes X_1)(a,b) \xrightarrow{q\otimes q} (h'_{\mathcal{V}}X\otimes h'_{\mathcal{V}}X)(a,b)$$

Now assume that $(q\otimes q)\mu_{1,1}(w) = (q\otimes q)\mu_{1,1}(w')$. As U preserves reflexive coequalisers, there exist $\alpha_0,\dots,\alpha_n \in U((X_1\otimes X_1)(a,b))$ such that $\mu_{1,1}(w) = \alpha_0$, $\alpha_n = \mu_{1,1}(w')$ and, for all $i\in\{1,\dots,n\}$, there exists a $\sigma \in U((Q\otimes Q)(a,b))$ such that

$$\alpha_{i-1} = (d_0\otimes d_0)(\sigma) \quad\text{and}\quad (d_1\otimes d_1)(\sigma) = \alpha_i \qquad\text{or}\qquad \alpha_{i-1} = (d_1\otimes d_1)(\sigma) \quad\text{and}\quad (d_0\otimes d_0)(\sigma) = \alpha_i.$$

For every $0 < i < n$, α_i defines a horn $\widetilde{F}(\Lambda_1^2)_\bullet(0,2) \to X_\bullet(a,b)$, which we can extend to an element $w_i \in U(X_2(a,b))$ so that $\mu_{1,1}(w_i) = \alpha_i$. Thus, it follows by the previous that

$$qd_1(w) = qd_1(w_1) = \cdots = qd_1(w_{n-1}) = qd_1(w'). \qquad \square$$

Lemma 5.20 *Assume that $U\colon \mathcal{V}\to \mathrm{Set}$ is faithful. Let $g\colon X\to Y$ and $f\colon X\to Z$ be morphisms in $\mathcal{V}$ such that g is a regular epimorphism. Suppose that, for all $x, y \in U(X)$,*

$$g(x) = g(y) \implies f(x) = f(y).$$

Then there exists a unique morphism $h\colon Y\to Z$ such that $hg = f$.

Proof Denote the kernel pair $X\times_Y X \rightrightarrows X$ of g by π_1 and π_2. Since g is the coequaliser of this pair, it suffices to show that $f\pi_1 = f\pi_2$. As U is faithful, this is equivalent to showing that, for all $(x,y) \in U(X)\times_{U(Y)} U(X)$, we have $f(x) = f(y)$. But this is equivalent to the hypothesis on f and g. $\square$

Construction 5.21 Assume that $U : \mathcal{V} \to \mathrm{Set}$ is faithful and preserves and reflects reflexive coequalisers. Let (X, S) be a quasicategory in $\mathcal{V}$. We construct a $\mathcal{V}$-enriched category $h'_{\mathcal{V}}X$ whose hom-objects are given by $h'_{\mathcal{V}}X(a,b)$ of Construction 5.16. Let $h'_{\mathcal{V}}X$ denote the quiver given by $h'_{\mathcal{V}}X(a,b)$ for all $a, b \in S$, and let $q : X_1 \to h'_{\mathcal{V}}X$ denote the canonical quiver morphism.

First define $u : I_S \xrightarrow{s_0} X_1 \xrightarrow{q} h'_{\mathcal{V}}X$. Note that U also reflects regular epimorphisms (as they are the coequaliser of their kernel pair). Thus, as X is a quasicategory in $\mathcal{V}$, the comultiplication $\mu_{1,1} : X_2 \to X_1 \otimes_S X_1$ is a regular epimorphism. Further, q is a regular epimorphism by definition. Now $- \otimes -$ preserves reflexive coequalisers in each variable and thus also regular epimorphisms. It follows that $q^{\otimes 2} \circ \mu_{1,1}$ is a regular epimorphism as well. Using Lemmas 5.19 and 5.20, we have a unique quiver morphism $m : h'_{\mathcal{V}}X \otimes_S h'_{\mathcal{V}}X \to h'_{\mathcal{V}}X$ such that the following diagram commutes:

$$\begin{tikzcd} X_2 \arrow[r, "\mu_{1,1}"] \arrow[dr, "d_1"'] & X_1^{\otimes 2} \arrow[r, "q^{\otimes 2}"] & (h'_{\mathcal{V}}X)^{\otimes 2} \arrow[d, "m"] \\ & X_1 \arrow[r, "q"'] & h'_{\mathcal{V}}X \end{tikzcd}$$

Given a 2-simplex $(\alpha_{i,j})_{1 \le i < j \le 2}$ (see Remark 2.9) of $\widetilde{U}(X)$ with vertices a, b and c, we have $\mu_{1,1}(\alpha_{02}) = \alpha_{01} \otimes \alpha_{12}$ and thus $m(q(\alpha_{01}) \otimes q(\alpha_{02})) = q(d_1(\alpha_{02}))$. Therefore, the induced map

$$U(h'_{\mathcal{V}}X(a,b)) \times U(h'_{\mathcal{V}}X(b,c)) \to U(h'_{\mathcal{V}}X(a,b) \otimes h'_{\mathcal{V}}X(b,c)) \xrightarrow{U(m_{a,b,c})} U(h'_{\mathcal{V}}X(a,c))$$

coincides with the composition law of $h\widetilde{U}(X)$ under the isomorphisms supplied by Remark 5.18. The element $u_a = q(s_0(a)) : I \to h'_{\mathcal{V}}X(a,a)$ is clearly the identity at a in $h\widetilde{U}(X)$. It then follows from the faithfulness of U that m is associative and unital with respect to u. So we obtain a $\mathcal{V}$-category $h'_{\mathcal{V}}X$.

Note that, by construction, we have an isomorphism of categories

$$\mathcal{U}(h'_{\mathcal{V}}X) \simeq h\widetilde{U}(X).$$

Proposition 5.22 *Assume that $U : \mathcal{V} \to \mathrm{Set}$ is faithful and preserves and reflects reflexive coequalisers. The assignment $X \mapsto h'_{\mathcal{V}}X$ of Construction 5.21 extends to a functor $h'_{\mathcal{V}}$ from the full subcategory of $S_{\otimes}\mathcal{V}$ spanned by all quasicategories in $\mathcal{V}$ to $\mathcal{V}\mathrm{Cat}$, which is left-adjoint to the templicial nerve functor $N_{\mathcal{V}}$.*

In particular, there exists a canonical isomorphism of $\mathcal{V}$-enriched categories

$$h_{\mathcal{V}}X \simeq h'_{\mathcal{V}}X$$

for every quasicategory X in $\mathcal{V}$.

Proof It follows from Construction 5.21 and Lemma 2.12 that we have a unique templicial morphism $\eta_X : X \to N_{\mathcal{V}}(h'_{\mathcal{V}}X)$ such that $\eta_{X_1} : X_1 \to h'_{\mathcal{V}}X$ is precisely q. We claim that η_X is the unit of an adjunction $h'_{\mathcal{V}} \dashv N_{\mathcal{V}}$.

Now let $\mathcal{C}$ be an arbitrary small $\mathcal{V}$-category and $(\zeta, f) : X \to N_{\mathcal{V}}(\mathcal{C})$ a templicial morphism. Then, by Lemma 2.12, $\zeta : f_!X \to N_{\mathcal{V}}(\mathcal{C})$ corresponds to a quiver morphism $H : X_1 \to f^*(\mathcal{C})$ such that the diagrams (4) commute. Letting Q denote the quiver given by $\mathrm{Hom}^{L}_{X}(a,b)_1$ for all objects a and b of X, we have a commutative diagram

$$\begin{tikzcd}
Q \arrow[r, "\pi_2"'] \arrow[rr, bend left, "d_0"] \arrow[d, "d_1=\pi_1"'] & X_2 \arrow[r, "d_1^X"'] \arrow[d, "\mu^X_{1,1}"] & X_1 \arrow[r, "H"] & f^*(\mathcal{C}) \\
X_1 \arrow[r, "-\otimes s_0^X"] \arrow[rr, bend right, "H\otimes u"'] & X_1^{\otimes 2} \arrow[r, "H^{\otimes 2}"] & f^*(\mathcal{C})^{\otimes 2} \arrow[r] & f^*(\mathcal{C}^{\otimes 2}) \arrow[u, "f^*(m_{\mathcal{C}})"']
\end{tikzcd}$$

It follows that $Hd_0 = Hd_1 \colon Q \to f^*(\mathcal{C})$ and thus there exists a unique quiver morphism $H' \colon h'_{\mathcal{V}}X \to f^*(\mathcal{C})$ such that $H'q = H$. By construction, H' defines a $\mathcal{V}$-functor $h'_{\mathcal{V}}X \to \mathcal{C}$, which is clearly unique, such that $N_{\mathcal{V}}(H) \circ \eta_X = (\zeta, f)$. □

Corollary 5.23 *Assume that* $U \colon \mathcal{V} \to \mathrm{Set}$ *is faithful and preserves and reflects reflexive coequalisers. Let* X *be a quasicategory in* $\mathcal{V}$*. The canonical functor*

$$h\widetilde{U}(X) \to \mathcal{U}(h_{\mathcal{V}}X)$$

is an isomorphism of categories.

Proof This is now an immediate consequence of Proposition 5.22 and the fact that $\mathcal{U}(h'_{\mathcal{V}}X) \simeq h\widetilde{U}(X)$. □

References

[1] **M Aguiar**, *Internal categories and quantum groups*, PhD thesis, Cornell University (1997) MR Available at https://www.proquest.com/docview/304347376

[2] **M Aguiar**, **S Mahajan**, *Monoidal functors, species and Hopf algebras*, CRM Monograph Series 29, Amer. Math. Soc., Providence, RI (2010) MR Zbl

[3] **H V Bacard**, *Segal enriched categories and applications*, Theory Appl. Categ. 35 (2020) 1227–1267 MR Zbl

[4] **H J Baues**, *Geometry of loop spaces and the cobar construction*, Mem. Amer. Math. Soc. 230, Amer. Math. Soc., Providence, RI (1980) MR Zbl

[5] **C Berger**, **I Moerdijk**, *On the homotopy theory of enriched categories*, Q. J. Math. 64 (2013) 805–846 MR Zbl

[6] **J E Bergner**, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. 359 (2007) 2043–2058 MR Zbl

[7] **J E Bergner**, *Three models for the homotopy theory of homotopy theories*, Topology 46 (2007) 397–436 MR Zbl

[8] **J M Boardman**, **R M Vogt**, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math. 347, Springer (1973) MR Zbl

[9] **A K Bousfield**, *Constructions of factorization systems in categories*, J. Pure Appl. Algebra 9 (1976/77) 207–220 MR Zbl

[10] **J-M Cordier**, *Sur la notion de diagramme homotopiquement cohérent*, Cahiers Topologie Géom. Différentielle 23 (1982) 93–112 MR Zbl

[11] **J-M Cordier**, **T Porter**, *Vogt's theorem on categories of homotopy coherent diagrams*, Math. Proc. Cambridge Philos. Soc. 100 (1986) 65–90 MR Zbl

[12] **B Day**, *On closed categories of functors*, from "Reports of the Midwest Category Seminar, IV" (S MacLane, editor), Lecture Notes in Math. 137, Springer (1970) 1–38 MR Zbl

[13] **E Dimitriadis Bermejo**, *A new model for dg-categories*, PhD thesis, Université de Toulouse (2022) Available at `https://theses.hal.science/tel-03966749`

[14] **D Dugger**, **D I Spivak**, *Rigidification of quasi-categories*, Algebr. Geom. Topol. 11 (2011) 225–261 MR Zbl

[15] **S Eilenberg**, **J A Zilber**, *Semi-simplicial complexes and singular homology*, Ann. of Math. 51 (1950) 499–513 MR Zbl

[16] **D Gepner**, **R Haugseng**, *Enriched ∞-categories via non-symmetric ∞-operads*, Adv. Math. 279 (2015) 575–716 MR Zbl

[17] **H Gindi**, *Coherent nerves for higher quasicategories*, Theory Appl. Categ. 37 (2021) 709–817 MR Zbl

[18] **D Grady**, **D Pavlov**, *Extended field theories are local and have classifying spaces*, preprint (2023) arXiv 2011.01208

[19] **R Haugseng**, *Rectification of enriched ∞-categories*, Algebr. Geom. Topol. 15 (2015) 1931–1982 MR Zbl

[20] **A Hirschowitz**, **C Simpson**, *Descente pour les n-champs*, preprint (2001) arXiv math/9807049

[21] **A Joyal**, *Disks*, *duality*, *and Theta-categories*, unpublished manuscript (1997) Available at `https://ncatlab.org/nlab/files/JoyalThetaCategories.pdf`

[22] **A Joyal**, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra 175 (2002) 207–222 MR Zbl

[23] **A Joyal**, *Notes on quasi-categories*, unpublished manuscript (2004) Available at `https://tinyurl.com/JoyalNotes`

[24] **A Joyal**, *Quasi-categories vs simplicial categories*, unpublished manuscript (2007) Available at `http://www.math.uchicago.edu/~may/IMA/Incoming/Joyal/QvsDJan9(2007).pdf`

[25] **A Joyal**, **M Tierney**, *Quasi-categories vs Segal spaces*, from "Categories in algebra, geometry and mathematical physics" (A Davydov, M Batanin, M Johnson, S Lack, A Neeman, editors), Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007) 277–326 MR Zbl

[26] **T Leinster**, *Homotopy algebras for operads*, preprint (2000) arXiv math/0002180

[27] **W Lowen**, **A Mertens**, *Frobenius templicial modules and the dg-nerve*, preprint (2023) arXiv 2005.04778

[28] **J Lurie**, *Higher topos theory*, Annals of Mathematics Studies 170, Princeton Univ. Press (2009) MR Zbl

[29] **J Lurie**, *Higher algebra*, preprint (2017) Available at `https://url.msp.org/Lurie-HA`

[30] **J Lurie**, *Kerodon*, online resource (2018) Available at `https://kerodon.net`

[31] **S MacLane**, *Categories for the working mathematician*, Graduate Texts in Math. 5, Springer (1971) MR Zbl

[32] **A Mertens**, *Templicial objects*: *simplicial objects in a monoidal category*, PhD thesis, University of Antwerp (2022) Available at `https://repository.uantwerpen.be/docman/irua/10a493/190384.pdf`

[33] **A Mertens**, *Nerves of enriched categories via necklaces*, preprint (2024) arXiv 2408.10049

[34] **L Moser**, **N Rasekh**, **M Rovelli**, *Homotopy coherent nerves of enriched categories*, preprint (2022) arXiv 2208.02745

[35] **F Muro**, *Dwyer–Kan homotopy theory of enriched categories*, J. Topol. 8 (2015) 377–413 MR Zbl

[36] **D G Quillen**, *Homotopical algebra*, Lecture Notes in Math. 43, Springer (1967) MR Zbl

[37] **C Rezk**, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. 353 (2001) 973–1007 MR Zbl

[38] **E Riehl**, *Categorical homotopy theory*, New Mathematical Monographs 24, Cambridge Univ. Press (2014) MR Zbl

[39] **C Simpson**, *Homotopy theory of higher categories*, New Mathematical Monographs 19, Cambridge Univ. Press (2012) MR Zbl

[40] **A E Stanculescu**, *Constructing model categories with prescribed fibrant objects*, Theory Appl. Categ. 29 (2014) 635–653 MR Zbl

[41] **R Street**, *Elementary cosmoi, I*, from "Category seminar" (G M Kelly, editor), Lecture Notes in Math. 420, Springer (1974) 134–180 MR Zbl

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Closures of T-homogeneous braids are real algebraic

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A link in S^3 is called real algebraic if it is the link of an isolated singularity of a polynomial map from $\mathbb{R}^4$ to $\mathbb{R}^2$. It is known that every real algebraic link is fibered and it is conjectured that the converse is also true. We prove this conjecture for a large family of fibered links, which includes closures of both T-homogeneous (and therefore also homogeneous) braids and braids that can be written as a product of the dual Garside element and a positive word in the Birman–Ko–Lee presentation. The proof offers a construction of the corresponding real polynomial maps, which can be written as semiholomorphic functions. We obtain information about their polynomial degrees.

57K10; 14P25, 32S55

1 Introduction

It is well known that the set of algebraic links, the links that arise as the links of isolated singularities of complex polynomials, consists of certain iterated cables of torus links; see Brauner [15; 16; 17]. In the last chapter of his seminal work [25], Milnor discusses the real analogue of this classification problem. Let $f:\mathbb{R}^4 \to \mathbb{R}^2$ be a polynomial map with $f(0)=0$ whose Jacobian Df vanishes at the origin, but has full rank at all other points of some neighborhood of the origin. In this case, we say that the origin is an *isolated singularity* of f. Exactly as in the complex case, the intersection $f^{-1}(0)\cap S^3_\rho$ of the vanishing set of f and a 3-sphere of sufficiently small radius $\rho>0$ is a link, whose link type does not depend on ρ. We call this link the *link of the singularity* and denote it by L_f.

Analogous to the terminology from the complex setting, we say that a link L is *real algebraic* if $L=L_f$ for some polynomial f. In contrast to the set of algebraic links, the set of real algebraic links has not been classified yet. Note that a generic real polynomial in these dimensions does not have an isolated singularity. The difficulty of constructing polynomials with isolated singularities and a given link type has already been noted by Milnor [25].

While there are many differences between the complex and the real setting, the importance of (Milnor) fibrations can be found in both. In particular, Milnor [25] showed that every real algebraic link is fibered. It is an open conjecture by Benedetti and Shiota [6] that the set of real algebraic links and the set of fibered links are identical. In addition to the complex algebraic links, which are obviously also real algebraic, several families of fibered links have been proved to be real algebraic. However, compared with the set of all fibered links, these families are still comparatively small. Looijenga [23] proved that every fibered link that satisfies a certain "odd" symmetry is real algebraic. This includes the connected sum $K \# K$ of any fibered knot K with itself. Answering a question of Milnor [25], Perron [28] constructed a real polynomial map that realizes the figure-eight knot as a real algebraic knot. Later, Rudolph [32] came up with a different polynomial for the figure-eight knot and gave the example of certain unions of complex algebraic links (up to orientation or mirror reflections). These were then studied in more generality by Pichon [29], who proved that $L_f \cup \overline{L}_g$ is real algebraic if and only if it is fibered. Here the overline denotes a reversal of orientation and L_f and L_g are links of singularities of holomorphic polynomials f and g. The corresponding real polynomial map can be written as $f\bar{g}\colon \mathbb{R}^4 \cong \mathbb{C}^2 \to \mathbb{C} \cong \mathbb{R}^2$, where $\bar{g}$ is the complex conjugate of g. In earlier work [9], we proved that closures of braids of the form B^2 for any homogeneous braid B are real algebraic. Examples of real polynomial maps with isolated singularities have also been constructed by A'Campo [1] and Seade [35; 34]. Another result that can be interpreted as evidence for Benedetti and Shiota's conjecture is due to Kauffman and Neumann [22]. It states that if we consider real analytic maps with so-called tame singularities instead of polynomial maps, then the conjecture is true.

The main result of this paper is a construction of certain polynomials with isolated singularities, which allows us to prove that a large family of fibered links is real algebraic.

All links in this family are closures of so-called *P-fibered braids* or, equivalently, bindings of *braided open book decompositions*, which can be described in terms of certain diagrams called *Rampichini diagrams*; see Bode [12], Morton and Rampichini [26] and Rampichini [30]. We introduce an operation on Rampichini diagrams detailed in the later sections. We say that one P-fibered braid B_2 is obtained from another B_1 by the insertion of *inner loops* if their Rampichini diagrams are related by such operations. The family of fibered links for which we can prove real algebraicity can be characterized in terms of properties of their Rampichini diagrams.

Theorem 1.1 *Let B_1 be a P-fibered braid on n strands with an odd, pure Rampichini diagram. Let B_2 be a P-fibered braid that is obtained from B_1 by the insertion of inner loops. Then the closure of B_2 is real algebraic.*

Furthermore, the corresponding real polynomial map with an isolated singularity can be taken to be semiholomorphic (ie it can be written as a polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ in complex variables u, v and the complex conjugate $\bar{v}$) and of degree n with respect to the complex variable u.

The properties of being odd and pure will be defined at a later point (Definitions 3.5 and 2.6). It is not straightforward to decide if a given link satisfies the condition from Theorem 1.1. However, for large families of fibered links we prove that this is the case.

Theorem 1.2 *Let B be a T-homogeneous braid. Then the closure of B is real algebraic and the corresponding polynomial can be taken to be semiholomorphic.*

The family of T-homogeneous braids (see Definition 2.11) was introduced by Rudolph [33] as a generalization of the family of homogeneous braids (whose closures are known to be fibered; see Stallings [36]), so we immediately obtain the following corollary.

Corollary 1.3 *Let B be a homogeneous braid. Then the closure of B is real algebraic and the corresponding polynomial can be taken to be semiholomorphic.*

While previous constructions resulted in links that are unions of complex algebraic links (up to orientation or mirror reflection) (see Rudolph [32] and Pichon [29]) or in links that satisfy, in addition to their fiberedness, certain symmetry properties (being "odd" in the case of Looijenga [23], or the closure of the square of a braid in the case of Perron's [28] and Rudolph's figure-eight knot [32] as well as in our work [9; 13]), Theorem 1.2 and Corollary 1.3 have no such constraints. To be precise, together [33; 12] imply that every T-homogeneous braid is P-fibered. Combining this with [9; 13] shows that the closure of B^2 is real algebraic for every T-homogeneous braid B. Theorem 1.2 goes beyond that by stating that it is not necessary to square the braid. Since in general closures of T-homogeneous braids are neither odd nor 2-periodic nor are all of their components (mirror images of) complex algebraic knots, this yields a much larger family than previous constructions.

In the context of braided open books and Rampichini diagrams it is useful to represent braids in terms of the BKL-generators $a_{i,j}$ of Birman, Ko and Lee [8], also called band generators, instead of Artin generators σ_i. A word in BKL-generators is called BKL-positive if it does not include inverses of any of the generators $a_{i,j}$. Let $\delta = a_{1,2}a_{2,3}\dots a_{n-1,n} = \sigma_1\sigma_2\dots\sigma_{n-1}$ denote the dual Garside element [8].

Theorem 1.4 *Let $B = \delta P$, where P is a BKL-positive word. Then the closure of B is real algebraic.*

Banfield [4] showed that closures of braids of the form $\delta' P$, where $\delta' = a_{n-1,n}a_{n-2,n-1}\dots a_{1,2} = \sigma_{n-1}\sigma_{n-2}\dots\sigma_1$ and P is a BKL-positive braid, are fibered. The same arguments apply to δP. Likewise, [8] defines the dual Garside normal form using δ' as the dual Garside element. But the same arguments can be used to define a dual Garside normal form in terms of δ, which is for example used by Ito and Wiest [21]. This means that in the BKL-presentation every braid on n strands can be written as $\delta^k P$ for some $k \in \mathbb{Z}$, where $P = A_1A_2\dots A_m$ is BKL-positive. This representation becomes unique for every given braid once appropriate conditions on the factors A_i are imposed [8]. We then call $\delta^k P$ the dual Garside normal form of the braid. Since $\delta^{k-1}P$ is BKL-positive if $k \geq 1$, Theorem 1.4 immediately implies the following corollary.

Corollary 1.5 *Let B be a braid whose dual Garside normal form contains a positive power of the dual Garside element δ. Then the closure of B is real algebraic and the corresponding polynomial can be taken to be semiholomorphic.*

The remainder of the paper is structured as follows. Section 2 reviews the relevant concepts in the study of P-fibered braids and Rampichini diagrams. We also introduce the definitions of inner loops and pure Rampichini diagrams, which feature in Theorem 1.1. Section 3 introduces odd Rampichini diagrams and proves a result about corresponding (singular) P-fibered braids with a certain "odd" symmetry. The proof of Theorem 1.1 then follows a similar strategy as the proof of our main result in [11], which constructs weakly isolated singularities (as opposed to isolated singularities) for any link type. In comparison to the work in [11], the individual steps require much more care as the fibration properties and isolation of the singularity are very strong conditions. We first use trigonometric interpolation and approximation to find a loop in the space of monic, complex polynomials h_t of fixed degree, whose coefficients are finite Fourier series and whose roots form a singular braid B_{sing} that is related to the braids B_1 and B_2 and that has to satisfy the "odd" symmetry requirement (see Section 4). From this we construct a radially weighed homogeneous polynomial that is degenerate in the sense of Oka [27] or Araújo dos Santos, Bode and Sanchez Quiceno [2] and whose zeros form the cone over the singular braid B_{sing}. In Section 5 we show that trigonometric interpolation techniques can also be used to find a deformation of this polynomial that has an isolated singularity and whose link is obtained from the singular braid by an appropriate resolution of its singular crossings. In Section 6 the resulting link is shown to be the closure of the desired braid B_2, which concludes the proof of Theorem 1.1. Then Theorems 1.2 and 1.4 along with their corollaries are shown in Section 7.

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2 Background

2.1 Braids

A geometric braid on n strands is a loop in the configuration space of n distinct unmarked points in the complex plane. Via the fundamental theorem of algebra this space is identified with the space X_n of monic polynomials in one complex variable of degree n and with distinct roots. This way a braid parametrized as $(z_1(t), z_2(t), \dots, z_n(t))$, where t is going from 0 to 2π, corresponds to the loop $g_t(u) := \prod_{j=1}^{n}(u - z_j(t))$. We call an equivalence class of geometric braids under braid isotopies a braid. Braid isotopies correspond to homotopies of loops (with fixed basepoint) in X_n, so the braid group $\mathbb{B}_n$ on n strands can be defined to be the fundamental group of X_n. It is common to visualize geometric braids as the set of parametric curves $\bigcup_{j=1}^{n}(z_j(t), t)$ in $\mathbb{C} \times [0, 2\pi]$.

Definition 2.1 We say that a geometric braid B is *P-fibered* if the corresponding loop of polynomials g_t defines a fibration via $\arg(g)\colon (\mathbb{C} \times S^1) \setminus B \to S^1$, $g(u, \mathrm{e}^{\mathrm{i}t}) := g_t(u)$. We say that a braid is P-fibered if it can be represented by a P-fibered geometric braid.

The condition of being P-fibered has the following geometric interpretation in terms of the critical values $v_j(t)$ for $j = 1, 2, \dots, n-1$ of $g_t\colon \mathbb{C} \to \mathbb{C}$. Since the roots of g_t are simple, all of its critical values $v_j(t)$ are nonzero. A geometric braid is P-fibered if and only if, for every $j = 1, 2, \dots, n-1$, the derivative $\partial \arg(v_j(t))/\partial t$ never vanishes. Thus, as t increases from 0 to 2π, every critical value twists around $0 \in \mathbb{C}$

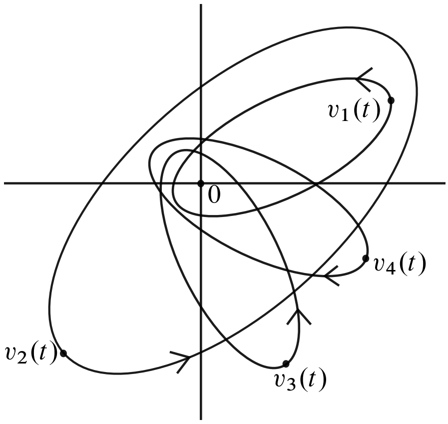

Figure 1: The motion in the complex plane of the critical values $v_j(t)$ of the loop of polynomials g_t corresponding to a P-fibered geometric braid.

in a fixed direction, either clockwise $(\partial\arg(v_j(t))/\partial t < 0)$ or counterclockwise $(\partial\arg(v_j(t))/\partial t > 0)$. This is illustrated in Figure 1.

The closure of a P-fibered braid B is a fibered link in the 3-sphere. Via this closing procedure (described in more detail in [12]), the braided surface $F_\varphi = \arg(g_t)^{-1}(\varphi)$ is identified with a fiber surface, whose boundary is the closure of B. A fibration of a link complement over a circle that comes from a P-fibered geometric braid and the corresponding loop of polynomials is also called a *braided open book decomposition* of S^3 [12].

Instead of the more common Artin presentation of the braid group $\mathbb{B}_n$, we will mostly work with the presentation of Birman, Ko and Lee [8], where the generators are $a_{i,j}$ for $1 \le i < j \le n$ and the relations are

$$a_{i,j}a_{k,m} = a_{k,m}a_{i,j} \qquad \text{if } (i-k)(i-m)(j-k)(j-m) > 0, \tag{1}$$

$$a_{i,j}a_{j,k} = a_{i,k}a_{i,j} = a_{j,k}a_{i,k} \qquad \text{for all } i,j,k \text{ with } 1 \le k < j < i \le n, \tag{2}$$

where we set $a_{j,i} = a_{i,j}$ for all $i,j \in \{1,2,\dots,n\}$. A geometric realization of a BKL-generator (also called *band generator*) $a_{i,j}$ is depicted in Figure 2. The better-known Artin generator σ_i is equal to $a_{i,i+1}$.

We can associate in a straightforward way a *braided surface* (in the sense of Rudolph [31; 33]) or *banded surface* to a given BKL-word. Starting with n parallel disks, the braided surface is obtained by inserting a half-twisted band between the i^{th} and j^{th} disk for each generator $a_{i,j}$ in the given word, where the sign of the twist corresponds to the sign of the generator. Note that the topology of this surface does not change under the BKL-relations (1) and (2). However, it does change under the trivial group relation $a_{i,j}a_{i,j}^{-1} = a_{i,j}^{-1}a_{i,j} = e$.

In the construction of polynomials described in the later sections we will also encounter singular braids. These differ from usual (geometric) braids in that we allow a finite number of simple intersections (double

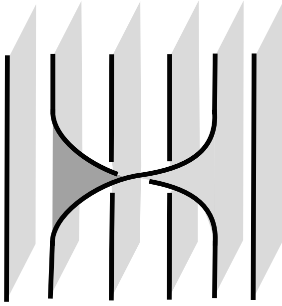

Figure 2: The band generator $a_{2,5}$ in $\mathbb{B}_6$.

points, or *singular crossings*) between the strands of a singular braid. There is now a rich theory of such singular braids [3; 7] and related variants, such as welded braids [20], but we will not really need any of these results apart from the definition itself.

2.2 Rampichini diagrams

In this subsection we review Rampichini diagrams and their connection to P-fibered braids. More details and proofs can be found in [12; 30].

Remark 2.2 Throughout this paper the expression $k \bmod n$ refers to the representative of k modulo n in $\{1, 2, \dots, n\}$. We use the cycle notation with arrows for permutations, ie $(1 \to 2)$ denotes the transposition that swaps 1 and 2. In diagrams, where there is little space and no chance of confusion with vectors, we also use $(1, 2)$ for $(1 \to 2)$.

Let $g\colon \mathbb{C} \to \mathbb{C}$ be an element of X_n, that is, a monic polynomial of degree n with distinct roots. Since g is monic, it satisfies $\lim_{\rho\to\infty} \arg(g(\rho e^{i\chi})) = n\chi$. Thus, viewing the closed disk D as a compactification of $\mathbb{C}$, we can interpret g as a simple branched cover from the disk D to itself with $g(e^{i\chi}) = e^{in\chi}$. The n^{th} roots of unity (labeled clockwise by 1 through n as in Figure 3(a)) are thus the n preimage points of $\arg(g) = 0$ on ∂D. They divide ∂D into n arcs A_i, where A_i connects i and $i+1 \bmod n$. The choice of which root of unity is labeled 1 is not important, but once we have made that choice, it should be the same for all polynomials.

Furthermore, we assume that the critical values v_j for $j = 1, 2, \dots, n-1$ of g have distinct arguments and are ordered so that $0 < \arg(v_1) < \arg(v_2) < \cdots < \arg(v_{n-1}) < 2\pi$. We denote the $n-1$ critical points of g by c_j for $j = 1, 2, \dots, n-1$, indexed so that $g(c_j) = v_j$. The map $\arg(g)$ induces a singular foliation on D, with the roots of g as elliptic singular points and the critical points as hyperbolic points (see Figure 3(a)).

The leaf of a critical point c_k has the shape of a cross, consisting of one line that connects two roots and one line that connects two points on ∂D, say on A_i and A_j, intersecting in c_k. Then we associate to a critical point c_k the transposition $\tau_k = (i \to j)$ (see Figure 3(b)). Since there is a one-to-one

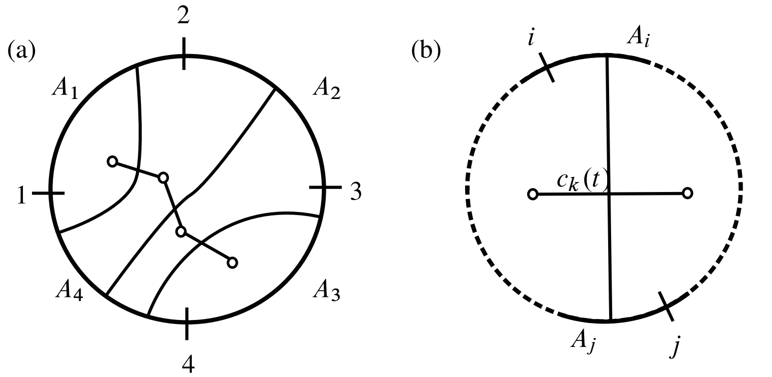

Figure 3: (a) The singular leaves of the singular foliation on D_t induced by $\arg(g_t)$ and the definition of the segments A_i. Small circles are roots of g_t. (b) The singular leaf containing $c_k(t)$. The transposition associated to $c_k(t)$ and $v_k(t) = g_t(c_k(t))$ is $(i \to j)$.

correspondence between critical points and critical values, we can also think of τ_k as a transposition associated with the critical value v_k. The ordered set of transpositions τ_k for $k = 1, 2, \dots, n-1$ is called the *cactus* of g and satisfies $\prod_{k=1}^{n-1} \tau_k = (1 \to 2 \to 3 \to \dots \to n-1 \to n)$.

Polynomials like g above, that is, monic, of degree n, with distinct roots and critical values that have distinct arguments, are generic in the following sense. Consider the complex vector space $\mathbb{C}^n$ with the surjective map π that sends an n-tuple of complex numbers $(z_1, z_2, \dots, z_n)$ to the polynomial $\prod_{j=1}^{n}(z - z_j)$. The map π is a branched covering map and the preimage of X_n under π is the complement of $\frac{1}{2}n(n-1)$ complex hyperplanes in $\mathbb{C}^n$. If we also want to account for distinct critical values, it is better to use a different projection map π, namely the one that sends an n-tuple $(c_1, c_2, \dots, c_{n-1}, a_0)$ to the unique monic polynomial of degree n with critical points $c_1, c_2, \dots, c_{n-1}$ and constant term a_0. The preimage points of polynomials with nondistinct roots or nondistinct critical values correspond to the solutions of a polynomial equation [5], so the space of polynomials with distinct roots and distinct critical values X_n' is the image of a Zariski-open subset of $\mathbb{C}^n$, the complement of a number of algebraic hypersurfaces of complex dimension $n-1$, under π. Within X_n' the polynomials where the k distinct critical values have the same argument form real submanifolds of real dimension $k + 2(n-1-k) + 1$, where we have k dimensions corresponding to motions of one of the k critical values along the fixed half-line of constant argument, $2(n-1-k)$ dimensions for the unrestricted motion of the remaining $n-1-k$ critical values and one dimension for rotating the half-line of constant argument with k critical values on it. The set of polynomials like g above is therefore the complement of such manifolds of dimension at most $2n-3$ (for $k = 2$) in X_n', which is itself the image of the complement of a number of complex hypersurfaces in $\mathbb{C}^n$ under the branched covering map π. In other words, the properties of g are generic.

Consider now a path g_t in X_n, with the parameter t going from 0 to 2π, whose endpoints satisfy the properties of g above. After a small homotopy of the path that does not change its endpoints, we may

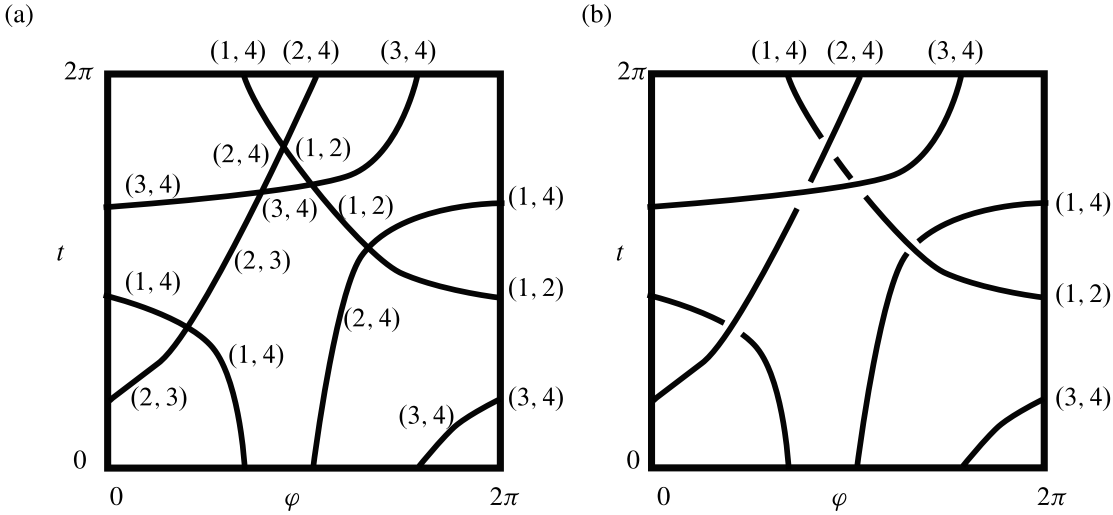

Figure 4: A Rampichini diagram. The transpositions at $t = 0$ are $\tau_1 = (1 \to 4)$, $\tau_2 = (2 \to 4)$ and $\tau_3 = (3 \to 4)$. A band word for the fiber $F_{\varphi=2\pi-\varepsilon}$ is given by $a_{3,4}a_{1,2}^{-1}a_{1,4}$.

assume that for all but finitely many values of $t \in [0, 2\pi]$ the arguments of the critical values $v_j(t)$ are distinct and at the remaining values of t there are only two critical values with the same argument (but different absolute values). For every value of t we order the critical values by their argument in $[0, 2\pi)$. Note that this means that the labeling of the critical values changes at the values of t where two critical values have the same argument or where one critical value crosses the line $\arg = 0$ (either from below or from above). We have a cactus $\tau_k(t)$ for $k = 1, 2, \ldots, n-1$, for every value of t for which the arguments of the critical values are distinct.

We store this information on the different cacti in a so-called *square diagram* as in Figure 4(a). It consists of a square containing curves that are labeled by transpositions in S_n, the symmetric group on n elements. The square represents a cylinder $S^1 \times [0, 2\pi]$ with coordinates (φ, t), both of which go from 0 to 2π, where φ is 2π-periodic. A point (φ, t) in the square lies on one of the curves in the diagram if and only if there is a critical value $v_j(t)$ of g_t with $\arg(v_j(t)) = \varphi$. Every point on a curve in the diagram thus corresponds to a critical value $v_j(t)$. Note that this means that every horizontal line ($t =$ constant) has exactly $n-1$ intersections with the curves in the square, when counted with multiplicities. The label of a point on a curve (away from intersection points) is the transposition $\tau_j(t)$ associated to the corresponding critical value $v_j(t)$. At intersection points of curves in the square there is no well-defined transposition.

The labels in a square diagram can only change at the right edge ($\varphi = 2\pi$) of the square, where all labels are shifted by $-1 \bmod n$ or at intersection points between the different curves. An intersection point between two curves in the square at (φ_*, t_*) occurs if and only if $\arg(v_j(t_*)) = \arg(v_{j+1}(t_*)) = \varphi_*$ for some j, where the indexing of the critical values at $t = t_*$ is taken to be that at $t_* - \varepsilon$ for sufficiently small $\varepsilon > 0$. The labels τ_j and τ_{j+1} at the intersection point change as follows. If $|v_j(t)(t_*)| < |v_{j+1}(t_*)|$, then

$$\tau_j(t_*+\varepsilon) = \tau_{j+1}(t_*-\varepsilon), \quad \tau_{j+1}(t_*+\varepsilon) = \tau_{j+1}(t_*-\varepsilon)\tau_j(t_*-\varepsilon)\tau_{j+1}(t_*-\varepsilon), \tag{3}$$

and if $|v_j(t)(t_*)| > |v_{j+1}(t_*)|$, then

$$\tau_j(t_*+\varepsilon) = \tau_j(t_*-\varepsilon)\tau_{j+1}(t_*-\varepsilon)\tau_j(t_*-\varepsilon), \quad \tau_{j+1}(t_*+\varepsilon) = \tau_j(t_*-\varepsilon). \tag{4}$$

In other words, the indices of the labels are swapped and the label of the critical value with smaller absolute value is conjugated by the label of the other critical value. In Figure 4(a) we only label arcs of the curves with transpositions (as opposed to every single point), since it is understood that the labels do not change along the arcs.

If g_t is a loop, then its square diagram can be interpreted as a torus as the critical values at $t=0$ along with their labels match those at $t=2\pi$. The roots of g_t form a P-fibered geometric braid if and only if the curves in the corresponding square diagram are strictly monotone increasing or strictly monotone decreasing, ie they can be locally interpreted as graphs of strictly monotone increasing functions $t(\varphi)$ (corresponding to $\partial\arg(v_j(t))/\partial t > 0$) or strictly monotone decreasing functions (corresponding to $\partial\arg(v_j(t))/\partial t < 0$). In this case we call the square diagram a *Rampichini diagram*.

Instead of using transpositions as labels we may use band generators, where $(i \to j)$ is replaced by $a_{i,j}$ if the corresponding line in the Rampichini diagram is strictly monotone increasing and by $a_{i,j}^{-1}$ if the line is strictly monotone decreasing. This way, the rules (3) and (4) become the band relations (1) and (2).

Rampichini diagrams have the following nice properties. At every fixed value of t apart from those with nondistinct arguments of critical values, the labels $\tau_j(t) \in S_n$ at that height t, indexed with increasing φ, satisfy $\prod_{j=1}^n \tau_j(t) = (1 \to 2 \to 3 \to \cdots \to n-1 \to n)$, since we have a cactus at all such values of t. At every fixed value of φ (away from intersection points of curves in the square) the labels (thought of as BKL-generators) spell a band word for the fiber surface $F_\varphi = \arg(g_t)^{-1}(\varphi)$ when read from the bottom to the top (ie with increasing t).

Every P-fibered braid can be represented by a Rampichini diagram (after a small homotopy of g_t in X_n to ensure that the critical values are distinct) and, conversely, every Rampichini diagram describes a P-fibered braid [12; 30]. We say that a BKL-word w is a *P-fibered braid word* if it appears in a Rampichini diagram R as the band word for a fiber surface F_φ. In this case we also say that R is a Rampichini diagram for the P-fibered braid word w.

Remark 2.3 In [26; 12], the roots of unity on ∂D are labeled counterclockwise instead of clockwise. As pointed out in [12], this different convention means that a cactus $\{\tau_j\}_{j=1,2,\dots,n-1}$ is defined by the property $\prod_{j=1}^n \tau_j = (1 \to n \to n-1 \to \cdots \to 3 \to 2)$ instead of $(1 \to 2 \to 3 \to \cdots \to n-1 \to n)$. Another consequence of this is that in [26; 12] the labels on the right edge of a Rampichini diagram increase by 1, while in this paper they decrease by 1 (modulo n). There are two reasons to prefer the convention that we use in this paper. Firstly, the definition of a cactus in this paper matches the one that is commonly used in the study of polynomials and branched coverings [19]. Secondly and more importantly, in contrast to what is claimed in [26; 12] the labels (i,j) in a Rampchini diagram coming from the counterclockwise convention do not directly correspond to the band generators $a_{i,j}$. Using the usual definition of the BKL-generators, where strands are labeled from left to right and bands are in front of all intermediate strands, the label (i,j) in a Rampichini diagram in [26] or [12] actually corresponds to $a_{n+1-i,n+1-j}$. This does not really affect any of the results in [26] or [12], but it is important to remember how extracting band words from a Rampichini diagram depends on the convention. For example, Morton and Rampichini [26] describe an algorithm based on their labeling convention that is meant to check if a given band word $B=\prod_{k=1}^{l}(a_{i_k,j_k})^{\varepsilon_k}$ represents a P-fibered braid. However, taking B as an input, the algorithm actually determines whether $\prod_{k=1}^{l}(a_{n+1-i_k,n+1-j_k})^{\varepsilon_k}$ is a P-fibered braid. Note that these are not only different braids, but have in general different closures; see Remark 2.10. Therefore, choosing the labels of the roots of unity on ∂D in order to define the transpositions in a Rampichini diagram is not simply a matter of taste, as was suggested in [12]. If we want the correspondence between labels in Rampichini diagrams and band generators, we need to label the roots of unity clockwise.

In order to keep Rampichini diagrams clear and less cluttered, we will from now on omit the labels apart from those at the edges of the square. Instead we will at each intersection point in the diagram mark one curve as the undercrossing curve by deleting the curve in a small neighborhood of the intersection (as is common in knot diagrams); see Figure 4(b). By choosing the curve whose critical value has the smaller absolute value to be the undercrossing strand, we can move between these two visual representations of Rampichini diagrams without losing any information, since at each crossing the label of the undercrossing curve is conjugated by the label of the overcrossing curve.

Remark 2.4 Originally, the decision to use crossings (as in knot diagrams) instead of labels in Rampichini diagrams was motivated by purely aesthetic aspects, as Rampichini diagrams with many intersection points needed many labels, which makes the visual representation rather confusing. However, it turns out that it is no coincidence that the resulting Figure 4(b) is a link diagram of a link in a thickened torus (ie a virtual link) with some labels on the edge of the square. We know that every P-fibered braid corresponds to a Rampichini diagram and vice versa. The critical values $v_j(t)$ of the corresponding polynomial g_t, which are known to be nonzero, form a closed (affine) braid

$$\bigcup_{t\in[0,2\pi]}\bigcup_{j=1}^{n}(v_j(t),\mathrm{e}^{\mathrm{i}t}) \tag{5}$$

in $(\mathbb{C}\setminus\{0\})\times S^1$, which is a thickened torus (see [10] and also recent work by Manturov and Nikonov [24]). It is this link that appears in a Rampichini diagram with our new crossing convention. The resulting virtual link is braided both in the t-direction and the φ-direction, ie transverse to all lines of constant t and all lines of constant φ. Any virtual Reidemeister move that preserves this braiding property corresponds to a homotopy of the loop in the space of critical values, which lifts to a homotopy of the loop in X_n [12]. In other words, the braided open book described by a Rampichini diagram does not change under isotopies of the corresponding virtual link as long as the link remains doubly braided throughout the isotopy. (In fact, it is sufficient if it remains braided in the t-direction throughout the isotopy and "doubly braided" at the end of the isotopy.) We thus have a 1-1-correspondence between Rampichini diagrams and links in a thickened torus that are "doubly braided" and labeled with certain transpositions. (Note that not every choice of labels on the boundary of the square diagram can be completed to a Rampichini diagram [26; 30].)

Definition 2.5 We call a Rampichini diagram R simple if the corresponding banded fiber surface consists of n disks connected by exactly $n-1$ bands. In other words, any horizontal line ($t=$ constant) has $n-1$ intersections with the curves in R (counted with multiplicity) and any vertical line ($\varphi=$ constant) also has $n-1$ intersections with the curves in R (counted with multiplicity).

Every fiber surface F_φ coming from a simple Rampichini diagram is a braided surface with n disks and $n-1$ bands. Thus they all have genus 0 and a connected boundary. Therefore, they are bounded by an unknot. The Rampichini diagram in Figure 4(a) is simple. It represents a braid on four strands, since every horizontal line has three intersection points with the curves in the square (counted with multiplicities), and every fiber surface F_φ has three bands, since each vertical line has three intersection points with the curves (again counted with multiplicities).

Let

$$\widetilde{V}_n=\{(v_1,v_2,\dots,v_{n-1})\in(\mathbb{C}\setminus\{0\})^{n-1}: v_i\neq v_j \text{ if } i\neq j\} \tag{6}$$

and

$$V_n=\widetilde{V}_n/S_{n-1}, \tag{7}$$

where the action of the symmetric group S_{n-1} permutes the different v_i. The fundamental group of V_n is the affine braid group $\mathbb{B}_{n-1}^{\mathrm{aff}}$, the subgroup of the braid group $\mathbb{B}_n$ where one of the strands is a vertical line, in this case $\{0\}\times[0,2\pi]$. This is discussed in more detail in [10]. The fundamental group of $\widetilde{V}_n$ is the intersection of the affine braid group and the pure braid group on n strands, both considered as subgroups of $\mathbb{B}_n$.

Let X_n' be the space of polynomials in X_n with distinct critical values. Then we may think of V_n as the space of sets of critical values of polynomials in X_n'. The map $\theta_n\colon X_n'\to V_n$ that sends a polynomial to its set of critical values maps a loop g_t in X_n' to a loop in V_n.

Definition 2.6 Let g_t be a loop in X_n' and R its Rampichini diagram. Then we say that R is pure if the loop $\theta_n(g_t)$ in V_n lifts to a loop in $\widetilde{V}_n$.

In other words, a Rampichini diagram is pure if the corresponding critical values form a pure braid in $(\mathbb{C}\setminus\{0\})\times S^1$. In this case we have two different ways to index the critical values. The first one is described above, where at every value of t we order the critical values by their arguments. This means that the indices are not constant along curves of critical values. They change at intersection points in the

Rampichini diagram and at the right edge of the square. The second one is given by the indices at $t=0$ and keeping the indices constant along each curve of critical values, ie the indexing comes from thinking of $\theta_n(g_t)$ as a loop in $\widetilde{V}_n$. In order to distinguish these two different orderings of critical values, we write $v(t)_j$ for the first ($0 \le \arg(v(t)_j) < \arg(v(t)_k) < 2\pi$ if and only if $j<k$) and $v_j(t)$ for the second ($j<k$ if and only if $0 \le \arg(v(0)_j) < \arg(v(0)_k) < 2\pi$). Note that the index of a transposition $\tau_j(t)$ always refers to the indexing of the first kind, ie $\tau_j(t)$ is the transposition associated with $v(t)_j$.

Critical values of loops of polynomials corresponding to P-fibered braids satisfy $\partial\arg(v_j(t))/\partial t \neq 0$, so every curve of critical values $v_j(t)$ has an associated sign $\varepsilon_j \in \{\pm 1\}$, which is equal to $\operatorname{sign}\bigl(\partial\arg(v_j(t))/\partial t\bigr)$ and which determines whether the corresponding curve in the Rampichini diagram is strictly monotone increasing or decreasing. We call ε_j the *sign of the critical value* $v_j(t)$ and say that a critical value v_j is positive/negative if ε_j is positive/negative.

The composition of two paths in V_n corresponds to a square diagram that is obtained by gluing the top edge of one square diagram along the bottom edge of the other square diagram. Since the paths have matching endpoints, the labels and endpoints in the respective square diagrams also match. Likewise, if the φ-coordinates and corresponding labels at the top edge of one square diagram match those at the bottom edge of another square diagram, we may glue the two diagrams along these edges to obtain a new diagram.

2.3 Inner loops

Let $(v_1, v_2, \dots, v_{n-1}) \in \widetilde{V}_n$ with $\arg(v_j) \neq \arg(v_k)$ if $j \neq k$. The v_i are not necessarily ordered by their arguments.

Definition 2.7 An inner loop is a loop γ_j (or γ_j^{-1}) in $\widetilde{V}_n$ based at $(v_1, v_2, \dots, v_{n-1})$, where every critical value except one is stationary and one critical value v_j moves in a counterclockwise (or clockwise) loop exactly once around the origin, such that whenever $\arg(v_j(t)) = \arg(v_i(t))$ with $j \neq i$ we have $|v_j(t)| < |v_i(t)|$. We call v_j the moving critical value of this inner loop.

Figure 5 shows the motion of the critical values during an inner loop. For a path γ in a topological space parametrized by t in $[0, 2\pi]$, we write $\gamma|_a^b$ for the segment of the path γ between $\gamma(a)$ and $\gamma(b)$.

Definition 2.8 Let g_t be a loop in X_n' whose roots form a P-fibered braid and let R be the corresponding Rampichini diagram, which we assume to be pure. Then $\theta_n(g_t)$ can be viewed as a loop in $\widetilde{V}_n$. Let $t_i \in [0, 2\pi]$ for $i = 1, 2, \dots, M$ be such that the arguments of the critical values of g_{t_i} are distinct for each fixed t_i and let $j(i) \in \{1, 2, \dots, n-1\}$ for $i = 1, 2, \dots, M$. Let $\varepsilon_{j(i)}$ be the sign of the critical value $v_{j(i)}(t)$. Let R' be the square diagram corresponding to the following loop in $\widetilde{V}_n$:

$$\theta_n(g_t)|_0^{t_1} \circ \gamma_{j(1)}^{\varepsilon_{j(1)}} \circ \theta_n(g_t)|_{t_1}^{t_2} \circ \gamma_{j(2)}^{\varepsilon_{j(2)}} \circ \cdots \circ \theta_n(g_t)|_{t_i}^{t_{i+1}} \circ \gamma_{j(i+1)}^{\varepsilon_{j(i+2)}} \circ \cdots \circ \gamma_{j(M)}^{\varepsilon_{j}(M)} \circ \theta_n(g_t)|_{t_M}^{2\pi}. \tag{8}$$

Then we say that R' is obtained from R by inserting inner loops.

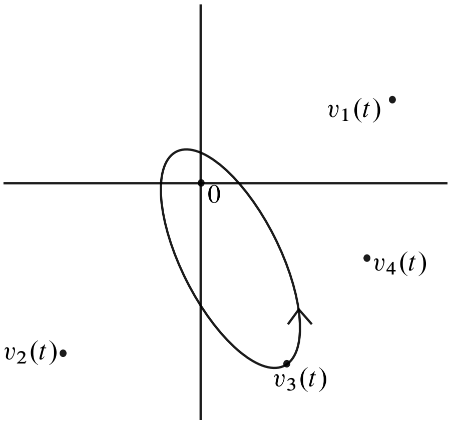

Figure 5: The inner loop γ_3.

First note that every loop in $\widetilde{V}_n$ corresponds to the critical values of some path in X_n [12], so R' is actually well defined as the square diagram of that path. (The path is not unique, but the choice of path does not matter.) For the construction of R' we may assume that the loop in (8) is rescaled so that it is parametrized by $t \in [0, 2\pi]$. The following lemma establishes that the corresponding path in X'_n is actually a loop and, moreover, R' is a Rampichini diagram.

Lemma 2.9 *Let R' be a square diagram that is obtained from a Rampichini diagram R by inserting inner loops. Then, after a small isotopy of the loop in (8), R' is a Rampichini diagram.*

Proof Figure 6 shows how the insertion of an inner loop affects a Rampichini diagram. It corresponds to the insertion of a line that is almost horizontal and that loops around the φ-coordinate exactly once. Since $|v_j(t)| < |v_i(t)|$ at all intersection points, the inserted line lies below all other curves in the diagram and thus its label must be conjugated at each crossing point in the diagram, while the other labels remain

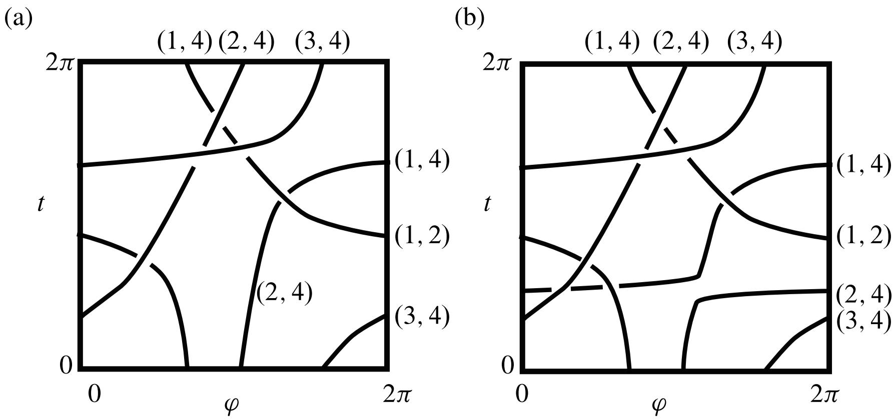

Figure 6: (a) A simple, pure Rampichini diagram. (b) The Rampichini diagram after the insertion of an inner loop.

unaffected. Since the sign of the inner loop $\gamma_j^{\varepsilon_j}$ matches the sign ε_j of the label of the moving critical value $v_j(t)$, the curves in R' are again strictly monotone after a small deformation of the curves (so that the critical values that are not moving during an inner loop become nonstationary). We now have to show that the labels $\tau_j(2\pi)$ at the top edge of the square diagram are identical to the labels $\tau_j(0)$ at the bottom edge. By the arguments in the proof of Theorem 5.6 in [12], this implies that the corresponding path in X'_n is a loop and hence R' is a Rampichini diagram.

We know that during an inner loop only the label of the moving critical value changes. We have to show that after the completion of the inner loop it is again the original label. Let $\tau_j(t_i)$ for $j = 1, 2, \dots, n-1$ be the transpositions in R associated with the critical values at the height $t = t_i$ at which an inner loop will be inserted, ordered by increasing value of φ. Let k be the index of the moving critical value for the inserted inner loop $\gamma_{j(i)}^{\varepsilon_{j(i)}}$, ie $v_{j(i)}(t_i) = v(t_i)_k$ in R. Let τ'_k denote its label after the inner loop.

By one of the defining properties of Rampichini diagrams, $\prod_{j=1}^{n-1} \tau_j(t_i) = (1 \to 2 \to 3 \to \dots \to n-1 \to n)$. In fact, this equality is true for any height t in a square diagram, not just at $t = t_i$. We thus have

$$\prod_{j=1}^{n-1} \tau_j(t_i) = (1 \to 2 \to 3 \to \dots \to n-1 \to n) = \biggl(\prod_{j=1}^{k-1} \tau_j(t_i)\biggr)\tau'_k\biggl(\prod_{j=k+1}^{n-1} \tau_j(t_i)\biggr), \tag{9}$$

where the left-hand side describes the cactus at $t = t_i - \varepsilon$ and the right-hand side the cactus at $t = t_i + \varepsilon$ for some small positive ε. This implies that $\tau_k(t_i) = \tau'_k$ after multiplying by $\bigl(\prod_{j=1}^{k-1} \tau_j(t_i)\bigr)^{-1}$ on the left and by $\bigl(\prod_{j=k+1}^{n-1} \tau_j(t_i)\bigr)^{-1}$ on the right. Thus the labels after an inserted inner loop are the same as the labels before the inner loop. Since R is a Rampichini diagram, the labels at its top edge are equal to the label at its bottom edge. It follows that the same holds for R', which proves the lemma. □

Remark 2.10 Figure 6(b) illustrates the arguments of Remark 2.3. It shows that the braid $a_{3,4}a_{2,4}a_{1,2}^{-1}a_{1,4}$, which closes to the Hopf link, is a P-fibered braid. If we had used the convention from [26] for labels on Rampichini diagrams, the labels on the right edge would read from the bottom to the top: $(1,2)$, $(1,3)$, $(3,4)$ and $(1,4)$. However, the closure of $a_{1,2}a_{1,3}a_{3,4}^{-1}a_{1,4}$ is a 2-component unlink and therefore not fibered. This is not a contradiction. We simply have to remember that labels (i,j) in the convention from [26] do not really correspond to band generators $a_{i,j}$, but to $a_{n+1-i,n+1-j}$. This means that some examples in [26] are not quite correct. Figure 10 in [26] for example displays a Rampichini diagram with their labeling convention, which is supposed to show that the braid $a_{3,4}a_{1,2}^{-1}a_{2,3}$ is P-fibered. Actually, it shows that $a_{1,2}a_{3,4}^{-1}a_{2,3}$ is P-fibered.

Instead of inserting inner loops into Rampichini diagrams, we can also insert them into a diagram as in Figure 7 corresponding to a trivial loop in X'_n, where all curves in the diagram are vertical. We call such a square diagram *trivial*. Note that there are also nontrivial paths, whose square diagrams are trivial, since the only requirement is that the argument of the critical values do not change. Since there are no intersections of the curves with each other or with the right edge of the square, the labels in a trivial square diagram never change.

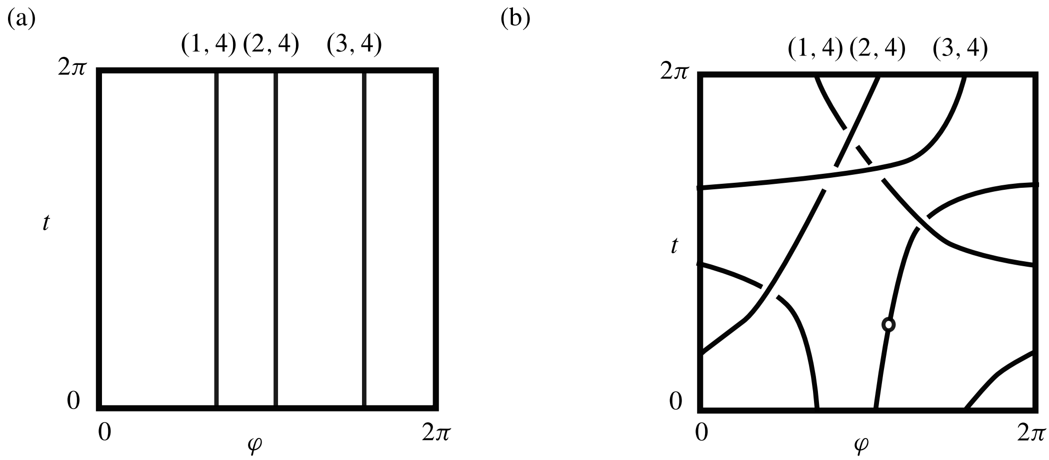

Figure 7: (a) A trivial square diagram corresponding to a constant loop in the space of polynomials. (b) A singular Rampichini diagram.

A trivial square diagram exists for any choice of cactus τ_j for $j = 1, 2, \ldots, n-1$, which by definition satisfies $\prod_{i=1}^{n} \tau_i = (1 \to 2 \to 3 \to \cdots \to n-1 \to n)$, and signs $\varepsilon_j \in \{\pm 1\}$ for $j = 1, 2, \ldots, n-1$. Inserting a finite number of inner loops $\gamma_{j(i)}^{\varepsilon_{j(i)}}$ for $i = 1, 2, \ldots, M$ into the diagram results in a Rampichini diagram R' by the same arguments as in Lemma 2.9.

We can read off a band word for the resulting fiber surface $F_{2\pi-\varepsilon}$ of the corresponding braid from the Rampichini diagram R'. Note that every critical value $v_j(t)$ always contributes the same letter, which we call $a_j^{\varepsilon_j}$. Its sign is always equal to the sign ε_j of the critical value, while its associated transposition is $\tau_j^{\prod_{i=j+1}^{n+1} \tau_i}$. Here a^b denotes conjugation of a group element a by another group element b, that is, $a^b = b^{-1}ab$. The definition of $a_j^{\varepsilon_j}$ clearly depends on the choice of cactus $\{\tau_j\}_{j=1,2,\ldots,n-1}$.

Definition 2.11 A braid B is called T-homogeneous if there is a choice of cactus $\{\tau_j\}_{j=1,2,\ldots,n-1}$ and signs $\varepsilon_j \in \{\pm 1\}$ such that B can be represented by a word that only contains the letters $a_j^{\varepsilon_j}$ and that contains $a_j^{\varepsilon_j}$ for every $j = 1, 2, \ldots, n-1$.

In some texts these braids are also called *strict*(*ly*) *T-homogeneous braids* to emphasize the second property, that every $a_j^{\varepsilon_j}$ appears in the word [33].

T-homogeneous braids are thus exactly those braids that can be obtained from a trivial square diagram as in Figure 7(a) by insertion of inner loops. Each inserted inner loop can be interpreted as a loop in $\widetilde{V}_n$ and a trivial square diagram corresponds to a constant loop in $\widetilde{V}_n$. Therefore every T-homogeneous braid has a pure Rampichini diagram.

Example 2.12 We may choose $\tau_j = (j \to n)$ for $j = 1, 2, \ldots, n-1$ and obtain $a_j^{\varepsilon_j} = a_{j,j+1}^{\varepsilon_j} = \sigma_j^{\varepsilon_j}$, so T-homogeneous braids for this choice of cactus are exactly the *homogeneous braids* in Artin generators σ_j.

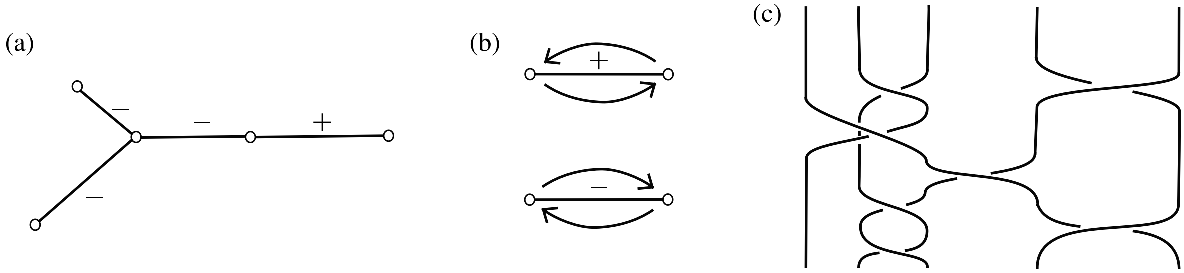

Figure 8: (a) An embedded tree T in the complex plane. The signs ε_e are drawn on each edge e. (b) The loop γ_e exchanges the roots on the endpoints of e as in the upper picture if $\varepsilon_e = +1$ and as in the lower picture if $\varepsilon_e = -1$. (c) A T-homogeneous braid with T as in (a).

Definition 2.11 describes the family of T-homogeneous braids in terms of their braid words. An alternative, equivalent definition explains the origin of the "T" in the name. Let T be an embedded tree in $\mathbb{C}$ with n vertices. We may assign to every edge e a sign $\varepsilon_e \in \{\pm 1\}$. We define a loop γ_e in X_n as follows. The basepoint of γ_e is the polynomial in X_n whose roots are the positions of the vertices of T. Throughout the loop all roots that do not correspond to endpoints of the edge e remain stationary, while the two endpoints of e exchange their position as in Figure 8(b), where the way in which they swap depends on the sign ε_e. A geometric braid B is T-homogeneous if its corresponding loop of polynomials g_t is the composition of loops of the form γ_e, where e runs through the set of edges of T, and for every edge e the corresponding loop γ_e appears in the factorization, ie $g_t = \prod_{i=1}^{l} \gamma_{e_i}$, where $\prod$ denotes composition of loops, and for every edge e there is an $i \in \{1, 2, \dots, l\}$ with $e_i = e$. The two definitions are equivalent with different choices of an embedded tree T (up to planar isotopy) corresponding to different choices of a cactus $\{\tau_j\}_{j=1,2,\dots,n-1}$. Homogeneous braids correspond to a path graph (or "linear graph").

Homogeneous braids are known to close to fibered links [36]. In fact, all T-homogeneous braids are known to close to bindings of totally braided open books [33], which is a property that is equivalent to being P-fibered [12].

Let B be a P-fibered geometric braid with corresponding loop of polynomials g_t and a pure Rampichini diagram R. Deforming its loop of critical values $(v_1(t), v_2(t), \dots, v_{n-1}(t))$ in V_n lifts to a deformation of g_t and, as long as the deformation only changes the absolute values of the $v_j(t)$, the Rampichini diagram and the fiberedness property remain unchanged. Now consider such a deformation, where we only change one of the curves of critical values $v_j(t)$ so that it becomes 0 at some $t = \tau \in [0, 2\pi]$ and call the lifted loop $\tilde{g}_t$. Then $\tilde{g}_\tau$ has a double root and therefore the roots of $\tilde{g}_t$ do not form a braid, but a singular braid. Since the argument of the critical value at $t = \tau$ is not defined, we cannot associate to it a label or a transposition like we usually do in Rampichini diagrams. We define a *singular Rampichini diagram* $R_{\{(t_i, j(i))\}_{i=1,2,\dots,M}}$ to be a Rampichini diagram R where, at a finite number of distinct values of $t = t_i$ for $i = 1, 2, \dots, M$, one of the curves, corresponding to the critical value $v_{j(i)}(t_i)$, in R at height t is replaced by a small circle. The rest of the diagram is unchanged. This is displayed in Figure 7b), which can be obtained from the Rampichini diagram in Figure 6(a). The circles represent values of t at which a critical value becomes zero. While Rampichini diagrams visualize certain loops in V_n that lift to loops in X_n, singular Rampichini diagrams visualize certain loops in the space of $n-1$ distinct complex numbers (a space that is homeomorphic to X_{n-1}), which lift to loops in the space of monic polynomials of degree n, but not to loops in X_n.

Note that not every loop h_t in the space of monic polynomials of fixed degree n gives rise to a singular Rampichini diagram. Apart from the usual condition that $\partial \arg(v_j)(t)/\partial t \neq 0$ for all $v_j(t) \neq 0$, it is necessary that $\lim_{t\to t_i} \arg(v_{j(i)}(t))$ and $\lim_{t\to t_i} \partial \arg(v_{j(i)}(t))/\partial t$ are well defined, with the latter being nonzero. Furthermore, it was an assumption that there is only a finite number of singular crossings and no higher multiplicity than 2 for any root of h_t.

Since the roots of a loop of polynomials h_t corresponding to a singular Rampichini diagram form a singular braid, the level sets of $\arg(h_t)$ are not braided surfaces anymore. Therefore, the property of Rampichini diagrams that we can read off band words for the fibers is not true anymore.

3 Odd P-fibered braids

P-fibered braids play a big role in constructions of real algebraic links. If B is a P-fibered braid, then the closure of B^2 is real algebraic [9]. The corresponding semiholomorphic polynomial can be constructed explicitly as

$$f(u, r\mathrm{e}^{\mathrm{i}t}) = r^{2kn} g\left(\frac{u}{r^{2k}}, \mathrm{e}^{2\mathrm{i}t}\right), \tag{10}$$

where k is a sufficiently large integer, n is the number of strands and $g_t(u) = g(u,t)$ is the loop corresponding to the P-fibered geometric braid B parametrized in terms of trigonometric polynomials, so that the coefficients of g (as a polynomial in u) are polynomials in $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$. Note that f is radially weighted homogeneous, ie $f(\lambda^{2k}u, \lambda v) = \lambda^{2kn} f(u,v)$ for all $\lambda \in \mathbb{R}$. This construction requires 2-periodicity of a braid (ie B^2 instead of B), since this guarantees that f, which is a priori a polynomial in u, v, $\bar{v}$ and $\sqrt{v\bar{v}}$, is actually a polynomial in u, v and $\bar{v}$, since all terms with square roots come with an even exponent.

In [9] we discuss another symmetry, which will be essential here. Instead of the even symmetry of B^2, where all frequencies in the trigonometric parametrization of the roots are even and thus $g_{t+\pi} = g_t$, we require the odd symmetry $g_{t+\pi}(u) = -g_t(-u)$ for all $t \in [0,\pi)$ and all $u \in \mathbb{C}$. In this case, we say that g_t is *odd*.

Polynomials in $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$ with complex coefficients are also called finite Fourier series or complex trigonometric polynomials. We reserve the term trigonometric polynomial for finite Fourier series that are real functions. They can thus be expressed as $\mathbb{R}$-linear combinations of 1, $\cos(kt)$ and $\sin(kt)$, where k goes through a finite range of natural numbers.

We write $f^{(i)}$ for the i^{th} derivative of a function f. The C^k-norm is $|f|_k = \sup_{i\leq k} \sup_{x\in\mathbb{R}} |f^{(i)}(x)|$.

A weaker version of the following result, where all components were required to have the same number of strands, was proved in [9, Lemma V.4].

Theorem 3.1 *Let B be a P-fibered geometric braid on n strands with corresponding loop of polynomials g_t. Assume that the coefficients of g_t are finite Fourier series. If n is odd and g_t is odd, then the closure of B is real algebraic, with the corresponding polynomial map given by*

$$f(u, re^{it}) = r^{kn} g\left(\frac{u}{r^k}, e^{it}\right), \tag{11}$$

where $g(u,t) = g_t(u)$ and k is a large odd integer.

Proof Equation (11) can be rewritten as

$$f(u,v) = \sqrt{v\bar{v}}^{kn} g\left(\frac{u}{\sqrt{v\bar{v}}^k}, \frac{v}{\sqrt{v\bar{v}}}\right). \tag{12}$$

As in the case of the even symmetry of B^2, choosing k sufficiently large clears the denominator and, since k, n and g_t are all odd, all monomials involving $\sqrt{v\bar{v}}$ come with an even exponent, so f is a polynomial in u, v and $\bar{v}$.

Note that f is radially weighted homogeneous, since $f(\lambda^k u, \lambda v) = \lambda^{kn} f(u,v)$. It then follows by the same arguments as in the proof of Theorem I.1 in [9] that the origin is an isolated singularity if and only if B is P-fibered. Furthermore, for each fixed r the zeros of $f|_{|v|=r}$ form the closed braid B in $\mathbb{C} \times rS^1$. By the same arguments as in [14] or [9], the link of the singularity is the closure of B. □

Note that the condition that coefficients of g_t should be finite Fourier series is not really a restriction. For an odd loop of polynomials g_t with $\deg_u(g_t)$ odd, the coefficient $a_j(t)$ of u^j is an odd function, ie $a_j(t+\pi) = -a_j(t)$, if j is even, and an even function, ie $a_j(t+\pi) = a_j(t)$, if j is odd. Odd/even trigonometric polynomials are dense in the space of odd/even 2π-periodic functions in the C^k-norm for any natural number k. Therefore, if B is a P-fibered geometric braid whose corresponding loop of polynomials g_t is odd, then an arbitrarily close approximation of B is an isotopic P-fibered braid whose corresponding loop of polynomials is odd and has coefficients that are finite Fourier series.

Definition 3.2 A cactus $\{\tau_j\}_{j=1,2,\dots,2n-2}$ with $\tau_j = (k(j) \to l(j))$ is called odd if it satisfies

$$\tau_{j+n-1} = (k(j) + n \pmod{2n-1} \to l(j) + n \pmod{2n-1}) \tag{13}$$

for all $j = 1, 2, \dots, n-1$.

Lemma 3.3 *Let $g\colon \mathbb{C} \to \mathbb{C}$, $g(u) = \prod_{j=1}^{2n-1}(u - z_j)$, be a monic polynomial of degree $2n-1$ with an odd cactus and critical values v_j for $j = 1, 2, \dots, 2n-2$ that satisfy*

$$0 < \arg(v_1) < \arg(v_2) < \cdots < \arg(v_{n-1}) < \pi < \arg(v_n) < \arg(v_{n+1}) < \cdots < \arg(v_{2n-2}) < 2\pi. \tag{14}$$

Then $\tilde{g}\colon \mathbb{C} \to \mathbb{C}$, $\tilde{g}(z) = \prod_{j=1}^{2n-1}(u + z_j)$, has the same cactus as g.

Proof By assumption, the critical values v_j for $j = 1, 2, \dots, n-1$ of g have an argument between 0 and π, while the critical values v_{j+n-1} for $j = 1, 2, \dots, n-1$ of g have an argument between π and 2π. Note that the critical values of $\tilde{g}$ are $\tilde{v}_{j+n-1} = -v_j$ for $j = 1, 2, \dots, 2n-2$, where the indices are taken mod $2n-2$. Hence the critical values $\tilde{v}_j$ for $j = 1, 2, \dots, n-1$ of $\tilde{g}$ have an argument between 0 and π and the critical values $\tilde{v}_{j+n-1}$ for $j = 1, 2, \dots, n-1$ have an argument between π and 2π.

The singular foliation of the disk induced by $\arg(\tilde{g})$ is exactly the singular foliation induced by $\arg(g)$ rotated by π. Thus, if $j \in \{1, 2, \dots, n-1\}$ and $\tau_j = (k(j) \to l(j))$, then $\arg(\tilde{v}_{j+n-1}) \in (\pi, 2\pi)$ with label $\tilde{\tau}_{j+n-1} = (k(j)+n \to l(j)+n) = \tau_{j+n-1}$, since g has an odd cactus.

Likewise, from $\tau_{j+n-1} = (k(j)+n \pmod{2n-1} \to l(j)+n \pmod{2n-1})$ for $j = 1, 2, \dots, n-1$, we get that $\tilde{v}_j$ for $j = 1, 2, \dots, n-1$ has the label $\tilde{\tau}_j = (k(j) \to l(j)) = \tau_j$. Thus $\tilde{g}$ has the same cactus as g. $\square$

Lemma 3.4 *Let g be a polynomial in X'_{2n-1} with an odd cactus and critical values v_j for $j = 1, 2, \dots, 2n-2$ that satisfy the following property. For every $j \in \{1, 2, \dots, 2n-2\}$ there is a $k \in \{1, 2, \dots, 2n-2\}$ such that $\arg(v_k) = \arg(v_j) + \pi$. Then there is a path in X'_{2n-1} from g to $\tilde{g}$ whose square diagram is trivial, where $\tilde{g}$ is as in Lemma 3.3.*

Proof Riemann's uniqueness theorem states that (once we have fixed a labeling of the arcs A_j for $j = 1, 2, \dots, 2n-1$ on ∂D as we have done at the beginning of Section 2.2), for every cactus and set of critical values v_j, there is a unique monic polynomial with the given cactus and set of critical values v_j, up to translations in the complex plane $z \mapsto z + b$ [12].

In particular, we can translate the roots of g in parallel so that one of them becomes 0 without changing the cactus or the set of critical values. In other words, fixing one $k \in \{1, 2, \dots, 2n-1\}$, the path in X'_{2n-1} that corresponds to the path of polynomials with roots $z_j(t) = z_j - \frac{t}{2\pi} z_k$, with t going from 0 to 2π, has a trivial square diagram.

By Lemma 3.3, g and $\tilde{g}$ have the same cactus and, by the given symmetry of the critical values v_j of g, the critical values of $\tilde{g}$ have the same arguments as v_j for $j = 1, 2, \dots, 2n-2$. Therefore there is a path in V_{2n-1} that does not change the arguments of the critical values and goes from the set of critical values of g to the set of critical values of $\tilde{g}$. Then this path lifts to a unique path in X'_{2n-1} if we require that throughout the path the constant term of every polynomial is 0 [5]. By Riemann's uniqueness theorem, this path thus goes from the translated g to a translation of $\tilde{g}$. Since the arguments of the critical values do not change along the corresponding path in V_{2n-1}, the path in X'_{2n-1} has a trivial square diagram. The composition of the path that translates g and this path, followed by a translation that ends at $\tilde{g}$, is then the desired path in X'_{2n-1} with a trivial square diagram. $\square$

We know from [12] that every Rampichini diagram arises from some loop g_t in X_n corresponding to a P-fibered braid.

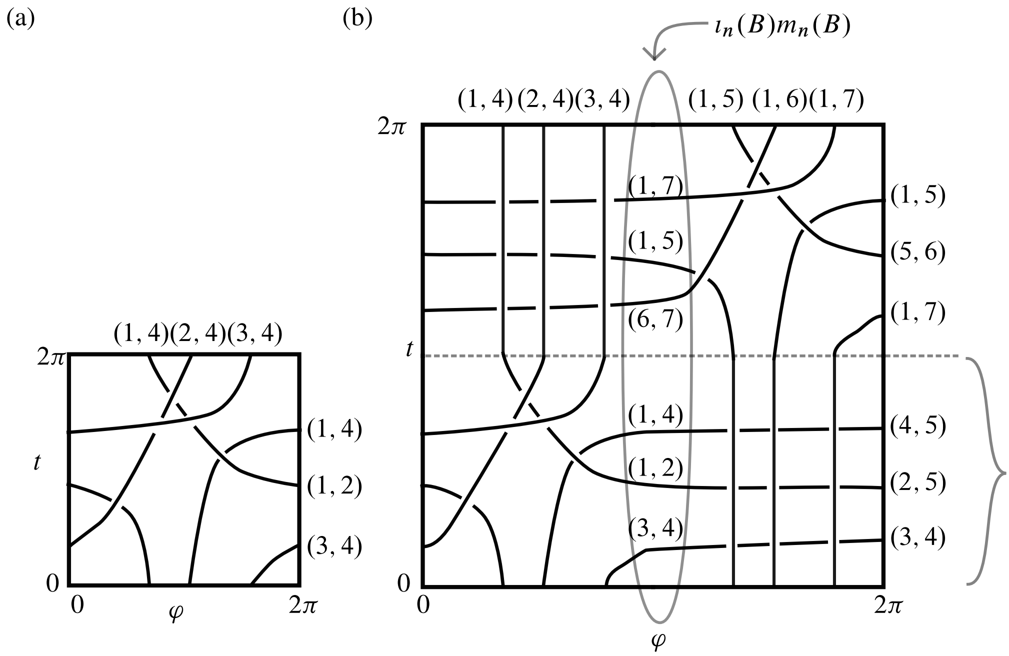

Figure 9: (a) A Rampichini diagram R. (b) The corresponding Rampichini diagram R'' with odd symmetry. Its lower half is the Rampichini diagram R'. The labels at $\varphi=\pi$ (interpreted as band generators) read from the bottom to the top spell $\iota_n(B)m_n(B)$.

Definition 3.5 We say that a Rampichini diagram is odd if there is a corresponding loop g_t in X_n that is odd.

We define $m_n\colon \mathbb{B}_n \to \mathbb{B}_{2n-1}$ to be the group homomorphism that sends a generator $a_{i,j}$ with $i,j \neq 1$ to $a_{i-n,j-n}$ and $a_{i,1}=a_{1,i}$ to $a_{1,i-n}$, where indices are understood modulo $2n-1$. Furthermore, we write $\iota_n\colon \mathbb{B}_n \to \mathbb{B}_{2n-1}$ for the inclusion which sends $a_{i,j}$ to $a_{i,j}$.

Lemma 3.6 *Let B be a P-fibered braid word on n strands. Then there is an odd Rampichini diagram for the P-fibered braid word $\iota_n(B)m_n(B)$.*

Proof Let R be the Rampichini diagram (see Figure 9(a)) whose labels at $\varphi=2\pi-\varepsilon$ spell the word B. We can define a square diagram R' (see the lower half of Figure 9(b)) by gluing a trivial square diagram with $n-1$ vertical curves as in Figure 7(a) on the right edge of R and continuing all curves on the right edge of R towards the right edge of the new diagram R', crossing below the curves of the trivial diagram. The new cactus at $t=0$ is given by $\tau_j(0)=(k(j)\to l(j))$, the original labels of R, for $j=1,2,\dots,n-1$, and $\tau_{j+n-1}=(k(j)+n \pmod{2n-1} \to l(j)+n \pmod{2n-1})$ for $j=1,2,\dots,n-1$. It is thus an odd cactus.

The square diagram R' corresponds to a loop in V_n and we claim that this loop lifts to a loop in X'_n. By the proof of Theorem 5.6 in [12], it is enough to check that the labels at the top edge of the square are the same as the labels at the bottom when we follow the rules of how labels change at crossings and at the right edge of the square. Since the vertical curves are the overcrossing strand in each crossing and since they never cross the right edge of the square, their labels do not change at all.

We call the curves in R and in R' that are strictly monotone increasing *positive curves* and the curves that are strictly monotone decreasing *negative curves*. We have to show that shifting the indices of every label of a positive curve on the right edge of R by $-1 \bmod n$ is equal to shifting the indices of the corresponding labels on the right edge of R' by $-1 \bmod 2n-1$. Likewise we have to show that the labels of the negative curves in R' at $\varphi = \pi$ are equal to the corresponding labels on the right edge of R. This implies that the labels in the left half of R' are equal to the corresponding labels in R.

If a label of a positive curve on the right edge of R is $(i \to j)$, then the corresponding label on the right edge of R' is given by the conjugate of $(i \to j)$ by $\prod_{k=1}^{n-1} \tau_{k+n-1} = (1 \to n+1 \to n+2 \to \cdots \to 2n-2 \to 2n-1)$. If both i and j are different from 1, the result of this conjugation is equal to $(i \to j)$, since $i, j \notin \{1, n+1, n+2, \ldots, 2n-1\}$. If one of them (say i) is equal to 1, then we obtain $(n+1 \to j)$. This is precisely what we wanted, since

$$(1-1 \to j-1) \bmod n = (n \to j-1) = (n+1-1 \to j-1) \bmod 2n-1. \tag{15}$$

If a label of a negative curve on the left edge of R is $(i \to j)$, then the corresponding label on the right edge of R is $(i+1 \to j+1)$ mod n, while it is $(i+1 \to j+1) \bmod 2n-1$ on the right edge of R'. The label of the same curve in R' at $\varphi = \pi$ after it has crossed all vertical curves is the conjugate of $(i+1 \to j+1) \bmod 2n-1$ by $\prod_{k=n-1}^{1} \tau_{k+n-1} = (1 \to 2n-1 \to 2n-2 \to \cdots \to n+1)$. If both i and j are different from n, this is simply $(i+1 \to j+1) \bmod 2n-1 = (i+1 \to j+1) \bmod n$, which is the desired label. If one of them (say j) is equal to n, then we obtain $(1 \to i+1) \bmod 2n-1 = (n+1 \to i+1) \bmod n$. Thus in all cases the labels in the left half of R' are the same as the corresponding labels in R. Since R is a Rampichini diagram, its labels at $t=0$ are equal to its labels at $t = 2\pi$ and thus R' corresponds to a loop g'_t in X'_n.

We may deform R' and the corresponding loop of polynomials g'_t in a neighborhood of $t=0$ such that the basepoint g'_0 satisfies the property from Lemma 3.4: for every critical value $v'_j(0)$ of g'_0 there is a critical value $v'_k(0)$ with $k \in \{1, 2, \ldots, 2n-2\}$ such that $\arg(v'_j(0)) = \arg(v'_k(0)) + \pi$.

Note that R' was constructed in such a way that g'_0 has an odd cactus. By Lemma 3.4 there is a path in X'_{2n-1} from g'_0 to $\tilde{g}'_0$ (the monic polynomial satisfying $\tilde{g}'_0(u) = -g'_0(-u)$) with trivial square diagram. The square diagram that is the composition of R' and this trivial square diagram can be interpreted as a square diagram of a path γ_t from g'_0 to $\tilde{g}'_0$.

For a path γ_t in X'_{2n-1} given by $\prod_{j=1}^{n}(u - z_j(t))$ we denote by $\tilde{\gamma}_t$ the path given by $\prod_{j=1}^{n}(u + z_j(t))$. If γ_t is the path from g'_0 to $\tilde{g}'_0$ described above, we can associate to $\tilde{\gamma}_t$ a square diagram (see the top half of Figure 9(b)). Its labels at the bottom edge match its labels at the top edge by the same arguments as above.

We now compose γ_t and $\tilde{\gamma}_t$, which is by construction an odd loop in X'_{2n-1}. Its corresponding square diagram R'' (see Figure 9(b)) is thus odd. It contains curves with vertical segments, but after a small isotopy that maintains oddness all curves are strictly monotone increasing or decreasing. Therefore, it is a Rampichini diagram.

We claim that the labels at $\varphi = \pi$ in this Rampichini diagram spell the word $\iota_n(B)m_n(B)$. We already know that the lower half of the diagram spells $\iota_n(B)$, since the lower left quadrant has the same labels as the original diagram R. By the same arguments, the upper right quadrant is the original diagram R except that all labels on the right edge have been shifted by n mod $2n-1$. It follows that the labels on the left edge in the upper half of R'' are the same as the labels on the left edge of R shifted by n mod $2n-1$. Therefore the labels in the upper half of R'' at $\varphi = \pi$ are the labels on the right edge of R shifted by $n-1 = -n$ mod $2n-1$, except if the corresponding label in R was of the form $a_{i,1}$, which is the only label that is affected by conjugation by $\prod_{j=1}^{n-1} \tau_j$. In this case the label on the right edge in the upper right quadrant is $a_{i+n,n+1}$. On the left edge of R'' it becomes $a_{n,i+n-1 \bmod 2n-1} = a_{n,i-n \bmod 2n-1}$ and after conjugation by $\prod_{j=1}^{n-1} \tau_j = (1 \to 2 \to 3 \to \cdots \to n-1 \to n)$ we have $a_{1,i-n \bmod 2n-1}$. Hence the labels in the top half at $\varphi = \pi$ spell $m_n(B)$. □

Note that if a knot K is a closure of a P-fibered braid B on n strands, then the closure of $\iota_n(B)m_n(B)$ is the connected sum $K \# K$. Following the construction above we obtain an odd loop of polynomials for the P-fibered braid $\iota_n(B)m_n(B)$ on $2n-1$ strands. By Theorem 3.1 we get a semiholomorphic polynomial with an isolated singularity at the origin and $K \# K$ as the link of that singularity. This proves the P-fibered/semiholomorphic version of a result by Looijenga [23].

Theorem 3.7 *Let K be a knot that is the closure of a P-fibered braid on n strands. Then $K \# K$ is real algebraic and the corresponding real polynomial can be taken to be semiholomorphic and of degree $2n-1$ with respect to the complex variable u.*

4 Trigonometric interpolation and approximation

The goal of this section is to prove that any singular Rampichini diagram that is obtained from a pure Rampichini diagram by inserting circles denoting singularities can be realized by a loop of polynomials h_t whose coefficients are polynomials in $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$.

The following result follows immediately from a theorem by Deutsch [18], which is a generalization of work by Walsh [37].

Lemma 4.1 *Let $f:\mathbb{R}\to\mathbb{R}$ be a 2π-periodic C^k-function. Then, for any chosen set of points z_j for $j = 1, 2, \ldots, M$ and any $\varepsilon > 0$, there is a trigonometric polynomial f_{trig} with $f_{\mathrm{trig}}^{(i)}(z_j) = f^{(i)}(z_j)$ for all $j = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, k$, and $|f - f_{\mathrm{trig}}|_k < \varepsilon$.*

Proof The proof is completely analogous to that of Corollary 3.2 in [18]. For every $j = 1, 2, \dots, M$ and $i = 1, 2, \dots, k$, we can define the operator $L_{i,j}(g) = g^{(i)}(z_j)$ on the space of 2π-periodic C^k-functions, which evaluates the i^{th} derivative of a function at the point z_j. Then every $L_{i,j}$ is a continuous linear operator, since

$$|L_{i,j}(g)| = |g^{(i)}(z_j)| \leq \sup_{x\in\mathbb{R}} |g^{(i)}(x)| \leq \sum_{i=1}^{k} \sup_{x\in\mathbb{R}} |g^{(i)}(x)| = |g|_k. \tag{16}$$

The lemma then follows from [18, Theorem 0], since trigonometric polynomials are dense in the space of 2π-periodic functions with the C^k-norm. □

Applying this lemma to the real and imaginary part of a 2π-periodic C^k-function $f : \mathbb{R} \to \mathbb{C}$ results in the analogous result for finite Fourier series.

For a power series $f = \sum_{j=0}^{\infty} a_j t^j$, we call the smallest index j with $a_j \neq 0$ the *lowest order* of f.

Lemma 4.2 *Let B be a P-fibered geometric braid on n strands with a pure Rampichini diagram R. Let $v_j(t)$ for $j = 1, 2, \dots, n-1$ denote the critical values of the corresponding loop of polynomials. Let $\{(t_i, j(i))\}_{i=1,2,\dots,M}$ with $t_i \in [0, 2\pi]$, $t_i \neq t_j$ if $i \neq j$, and $j(i) \in \{1, 2, \dots, n-1\}$ be such that for every $i \in \{1, 2, \dots, M\}$ we have $\arg(v_{j(i)}(t_i)) \neq \arg(v_k(t_i))$ for all $k \neq j(i)$. Then there exists a loop h_t in the space of monic polynomials with fixed degree n such that all of its coefficients are finite Fourier series and h_t has an associated singular Rampichini diagram given by $R_{\{(t_i, j(i))\}_{i=1,2,\dots,M}}$.*

Proof Since B is a P-fibered geometric braid, we know that the critical values $v_j(t)$ of the corresponding loop of polynomials satisfy $\partial \arg(v_j(t))/\partial t \neq 0$ for all $t \in [0, 2\pi]$. This property does not change under small C^1-deformations of $v_j(t)$, ie there is an ε_0 such that, for all positive $\varepsilon < \varepsilon_0$ and any smooth function $F : [0, 2\pi] \to \mathbb{C}$ with $|F|_1 < \varepsilon$, we have $v_j(t) + F(f) \neq 0$ and $\partial \arg(v_j(t) + F(t))/\partial t \neq 0$ for all t. This can be seen from $\arg(v_j(t) + F(t)) = \operatorname{Im}\bigl(\operatorname{Log}(v_j(t) + F(t))\bigr)$ and thus

$$\frac{\partial \arg(v_j + F)}{\partial t} = \frac{\operatorname{Re}(v_j + F)\, \partial \operatorname{Im}(v_j + F)/\partial t - \operatorname{Im}(v_j + F)\, \partial \operatorname{Re}(v_j + F)/\partial t}{(\operatorname{Re}(v_j + F))^2 + (\operatorname{Im}(v_j + F))^2}. \tag{17}$$

Let $\varepsilon < \varepsilon_0$ be such a small positive number. In particular, the above implies that $|v_j(t)| > \varepsilon$ for all j and t. Pick a small number $\delta > 0$ and define V_i as the open interval $(t_i - \delta, t_i + \delta)$. In particular, δ should be chosen so that in every V_i the argument of $v_{j(i)}(t)$ differs from the argument of any other critical value. Let $f_j : [0, 2\pi] \to \mathbb{R}$ be the smooth function defined to be $-\mathrm{e}^{-((t-t_i)/((t-t_i)^2-\delta^2))^2}$ on all V_i with $j(i) = j$ and everywhere else constant 0.

We may deform the critical values $v_j(t)$ without changing $\arg v_j(t)$ such that $|v_{j(i)}(t)|$ becomes small in V_i for all i. More precisely, we want that $|v_{j(i)}(t_i)|$ is a global minimum of $|v_{j(i)}(t)|$ (and in particular $|v_{j(i)}(t_i)| = |v_{j(i')}(t_{i'})|$ for all i and i' with $j(i) = j(i')$) and $|f_j v_{j(i)}(t)|_1 < \varepsilon/n$ for all t in any V_i with $j(i) = j$. Note that ε is the same as above and does not depend on i. This deformation of the critical values can be performed without changing $v_j(t)$ in any V_i with $j(i) \neq j$.

This deformation lifts to a homotopy of the loop of polynomials with distinct roots and fixed degree [12]. We can now approximate the critical points $c_j(t)$ for $j=1,2,\dots,n-1$ of the resulting polynomials g_t (the endpoint of the homotopy) and its constant term $a_0(t)$ by finite Fourier series using the C^2-norm, while interpolating the values of $c_j(t)$, $a_0(t)$ and their first and second derivatives at $t=t_i$ for $i=1,2,\dots,M$. (It is at this point that we use the fact that R is pure. Since $v_j(2\pi)=v_j(0)$, we have $c_j(2\pi)=c_j(0)$, which is necessary for a finite Fourier series.) This approximation and simultaneous interpolation exists by Lemma 4.1. By choosing the approximation close enough to the original $c_j(t)$ and $a_0(t)$, we can guarantee that there are still no argument-critical points of g_t. Since all critical values are smooth (in fact, analytic) functions of the critical points and the constant term, we can arrange that the critical values $w_j(t)$ for $j=1,2,\dots,n-1$ of g_t satisfy $|w_j(t)-v_j(t)|_1<\varepsilon/n$ on all V_i with $j(i)\neq j$. Furthermore, since all critical points and the constant term are parametrized by finite Fourier series, every coefficient of g_t is a finite Fourier series. It follows that the critical values $w_j(t)$ for $j=1,2,\dots,n-1$ of g_t are also parametrized by finite Fourier series and are nonzero. Moreover, the interpolation guarantees that we still have $|w_{j(i)}(t_i)|$ as a global minimum of $|w_{j(i)}(t)|$ and choosing the approximation sufficiently close gives $|f_j w_j(t)|_1<\varepsilon/n$. Note that this inequality is automatically satisfied outside of the V_i with $j(i)=j$, where $f_j\equiv 0$.

We claim that we can obtain the desired loop of polynomials h_t by only changing the constant term of g_t. More specifically, we are going to add $n-1$ finite Fourier series $A_1(t), A_2(t),\dots,A_{n-1}(t)$ to its constant term. These functions are defined successively as follows. We perform a simultaneous trigonometric interpolation and approximation of the function f_j and its first N derivatives, where N is some sufficiently large natural number. The resulting interpolating trigonometric polynomial F_j should agree with the values of f_j and its first N derivatives on all $t=t_i$. Then we define $A_j(t)=F_j(t)w_{j,j-1}(t)$, where $w_{k,j}(t)=w_k(t)+\sum_{i=1}^{j}A_i(t)$. Note that adding a term $A_j(t)$ to the constant term of a loop of polynomials shifts all critical values by $A_j(t)$, so $w_{k,j}(t)$ for $k=1,2,\dots,n-1$ are the critical values of $g_t+\sum_{i=1}^{j}A_i(t)$. We prove the following claim by induction on j.

Claim *If the approximation of f_j by F_j is sufficiently close for all $j=1,2,\dots,n-1$, then the following properties hold for all $j=1,2,\dots,n-1$:*

- $w_{k,j}(t)=0$ *if and only if* $k\leq j$ *and* $t=t_i$ *for some* i *with* $j(i)=k$.
- $\arg(w_{k,j}(t))$ *and* $\partial\arg(w_{k,j}(t))/\partial t$ *have well-defined limits as* t *goes to* t_i *for all* i *with* $j(i)=k$, *with the latter limit being nonzero.*
- *The critical values* $w_{k,j}(t)$ *for* $k=1,2,\dots,n-1$ *have no argument-critical points, wherever* $w_{k,j}(t)\neq 0$.
- *For all* k *we have* $|w_{k,j}-v_k|_1<(j+1)\varepsilon/n$ *on all* V_i *with* $j(i)\neq k$.
- *For all* $k>j$ *we have* $|f_k w_{k,j}|_1<\varepsilon/n$ *on all* V_i.
- $|w_{k,j}(t_i)|$ *is a global minimum of* $|w_{k,j}(t)|$ *for all* i *with* $j(i)=k$.

Before we go into the proof of the claim, note that each A_j is indeed a finite Fourier series, since F_j is a trigonometric polynomial and w_j is a finite Fourier series for all $j=1,2,\dots,n-1$.

Base case $j=1$ First we focus on $w_{1,1}$. The function $A_1(t)$ was constructed such that $w_{1,1}(t)=0$ if and only if $t=t_i$ and $j(i)=1$. This can be seen as follows. There is a neighborhood of $t=t_i$ with $j(i)=1$ such that in that neighborhood any sufficiently close approximation of f_1 in the C^2-norm has positive second derivative. It follows that $t=t_i$ is the unique point where F_1 takes the value -1 in that neighborhood and thus $t=t_i$ is the only zero of $w_{1,1}$ in that neighborhood. We can guarantee that there are no zeros of $w_{1,1}$ outside of these neighborhoods, since their complement in $[0,2\pi]$ is compact and we may choose the approximation arbitrarily close.

Furthermore, $\partial\arg(w_{1,1}(t))/\partial t\neq 0$ whenever $w_{1,1}\neq 0$, since in this case $\arg w_{1,1}(t)=\arg w_1(t)$. This equality also implies that $\arg(w_{1,1}(t))$ and $\partial\arg(w_{1,1}(t))/\partial t$ have well-defined limits as t goes to t_i with $j(i)=1$, which are equal to $\arg(w_1(t_i))$ and $\partial\arg(w_1)/\partial t(t_i)\neq 0$, respectively.

We know that $w_1(t)$ differs from $v_1(t)$ on V_i with $j(i)\neq 1$ by at most ε/n in the C^1-norm. Since $f_1\equiv 0$ on every such V_i, it follows that if F_1 is a sufficiently close approximation of f_1, it satisfies $|F_1(t)w_1(t)|_1<\varepsilon/n$ on all such V_i. We thus obtain

$$|w_{1,1}(t)-v_1(t)|_1\leq |F_1(t)w_1(t)|_1+|w_1(t)-v_1(t)|_1<\frac{2\varepsilon}{n} \tag{18}$$

on all V_i with $j(i)\neq 1$.

Now we turn our attention to $w_{k,1}$ with $k>1$. We want to show that for any $k\neq 1$ the new critical values $w_{k,1}(t)$ are nonzero and satisfy $\partial w_{k,1}(t)/\partial t\neq 0$. We know that $|f_1w_1|_1<\varepsilon/n$ on all V_i (on those with $j(i)=1$ by construction, on the others because $f_1\equiv 0$). It follows that if F_1 is a sufficiently close approximation of f_1, it satisfies $|F_1(t)w_1(t)|_1<\varepsilon/n$ on all V_i. Recall that in the V_i with $j(i)\neq k$, the critical value $w_k(t)$ with $k\neq 1$ differs from the original $v_k(t)$ by at most ε/n in the C^1-norm, so

$$|w_{k,1}(t)-v_k(t)|_1\leq |F_1(t)w_1(t)|_1+|w_k(t)-v_k(t)|_1<\frac{2\varepsilon}{n}. \tag{19}$$

Thus $w_{k,1}(t)$ is nonzero and has no argument-critical points in V_i for all i with $j(i)\neq k$.

Outside of the V_i with $j(i)\neq k$ the same is true if $|F_1w_1|_1$ is sufficiently small on $[0,2\pi]\setminus\big(\bigcup_{i\mid j(i)\neq k}V_i\big)$, which can be arranged by choosing F_1 sufficiently close to f_1 on the compact set $[0,2\pi]\setminus\big(\bigcup_{i\mid j(i)\neq k}V_i\big)$.

Obviously, $\arg(w_{k,1}(t))$ and $\partial\arg(w_{k,1}(t))/\partial t$ have well-defined limits as t approaches t_i with $j(i)=k>1$, since $w_{k,1}$ with $k>1$ has no zeros. Thus its argument and its derivative are well defined at $t=t_i$. Since there are no argument-critical points, this value of the derivative is nonzero. We have thus proved the first four items of the claim for $j=1$.

Since for all i with $j(i)\neq 1$ we have $f_1\equiv 0$ on all V_i, any sufficiently close approximation F_1 satisfies $|f_kw_{k,1}|_1=|f_k(w_k+F_1w_1)|_1<\varepsilon/n$ on V_i for $k>1$. Here we have used that $|f_kw_k|_1<\varepsilon/n$ on V_i with $j(i)=k$ and $f_k\equiv 0$ everywhere else.

The sixth item is obviously true for $k=1$, since $w_{1,1}(t)=0$ if and only if $t=t_i$ by the first item. For $k>1$, note that by the interpolation property of F_1 we have

$$F_1(t_i)=F_1^{(1)}(t_i)=F_1^{(2)}(t_i)=\cdots=F_1^{(N)}(t_i)=0 \tag{20}$$

for all i with $j(i)>1$. Since $w_k(t)$ is given by a nonvanishing finite Fourier series, it is real-analytic and $|w_k(t)|$ is real-analytic, too. We want the natural number N, which determines to which order F_1 interpolates the derivatives of f_1 at $t=t_i$ with $j(i)>1$ to be larger than the lowest order of the power series of $|w_k(t)|-|w_k(t_i)|$ centered at $t=t_i$. (We are omitting the fact that this lowest order could depend on i and k. We simply choose N so that it is sufficient for all i and k.) Since $|w_k(t_i)|$ is a local minimum, this lowest order has a positive coefficient and by the interpolation property $|w_{k,1}(t)|-|w_{k,1}(t_i)|$ has the same lowest-order term. It follows that $|w_{k,1}(t_i)|$ is a local minimum and the only local minimum in a neighborhood. On the compact complement of these neighborhoods in $[0,2\pi]$ we can guarantee that $|w_{k,1}(t)|>|w_{k,1}(t_i)|$ by choosing a sufficiently close approximation F_1 of f_1. On $[0,2\pi]\setminus\big(\bigcup_{i\mid j(i)=1}V_i\big)$, where $f_1\equiv 0$, this is obvious. On V_i with $j(i)=1$ we may use that $|w_k(t)|=|v_k(t)|>\varepsilon$, so

$$\begin{aligned}|w_{k,1}(t)|=|w_k(t)+A_1(t)|\geq\big||w_k(t)|-|A_1(t)|\big|>\varepsilon-\frac{\varepsilon}{n}>\frac{\varepsilon}{n}\\ >|f_kw_k|_1\geq|f_k(t_i)w_k(t_i)|=|w_{k,1}(t_i)|.\end{aligned} \tag{21}$$

Here we have used that $|A_1(t)|=|F_1(t)w_1(t)|<\varepsilon/n$. Thus $|w_{k,1}(t_i)|$ with $j(i)=k$ are global minima of $|w_{k,1}(t)|$ for all k. This concludes the proof of the six items in the claim for the base case of $j=1$.

Induction We assume that the claim holds for some $j-1\in\{1,2,\dots,n-2\}$ and want to show that it then also holds for j.

By the same arguments as in the base case, we have $w_{j,j}(t)=0$ if and only if $t=t_i$ for i with $j(i)=j$. Away from these points we have $\arg(w_{j,j}(t))=\arg(w_{j,j-1}(t))$, so there are no argument-critical points of $w_{j,j}$ and $\arg(w_{j,j}(t))$ and $\partial\arg(w_{j,j}(t))/\partial t$ have well-defined limits, equal to $\arg(w_{j,j-1}(t_i))$ and $\partial\arg(w_{j,j-1})/\partial t(t_i)$, as t goes to t_i with $j(i)=j$.

For $w_{k,j}$ consider that, by the induction hypothesis, $|w_{k,j-1}-v_k|_1<j\varepsilon/n$ on all V_i with $j(i)\neq k$. The induction hypothesis also states that $|f_jw_{j,j-1}|_1<\varepsilon/n$ on all V_i. This means that on all V_i with $j(i)\neq k$ we have

$$|w_{k,j}-v_k|_1\leq|F_jw_{j,j-1}|_1+|w_{k,j-1}-v_k|_1<\frac{(j+1)\varepsilon}{n} \tag{22}$$

if F_j is sufficiently close to f_j. This implies that $w_{k,j}(t)$ is nonzero and has no argument-critical points in V_i with $j(i)\neq k$. Since for $k>j$ the critical value $w_{k,j-1}(t)$ is nonzero and has no argument-critical points, the same is true for $w_{k,j}(t)$ on the compact set $[0,2\pi]\setminus\big(\bigcup_{i\mid j(i)\neq k}V_i\big)$ if the approximation of f_j by F_j is sufficiently close. Since $w_{k,j}(t)$ with $k>j$ is nonvanishing, its argument and the derivative of its argument are well defined everywhere and naturally have a well-defined, nonzero limit at $t=t_i$. This proves the first four items of the claim for $w_{k,j}$ with $k\geq j$ and the fourth item for $w_{k,j}$ with $k<j$.

For the fifth item, consider that, on V_i with $j(i) \neq k$, the function f_k is constant zero, while on V_i with $j(i) = k > j$ the function f_j is constant zero. In either case we have that

$$|f_k w_{k,j}|_1 = |f_k(w_{k,j-1} + f_j w_{j,j-1})|_1 < \frac{\varepsilon}{n} \tag{23}$$

on V_i by the induction hypothesis.

The sixth item in the claim is trivial for all k with $k \leq j$, since in these cases we already know that $t = t_i$ with $j(i) = k$ are the only zeros of $w_{k,j}(t)$. For $k > j$ it is the same argument as in the base case ($j = 1$). The fact that F_j interpolates f_j and its first N derivatives guarantees that $t = t_i$ is a local minimum and, by choosing the approximation sufficiently close, we can show that these are global minima of $|w_{k,j}(t)|$.

What is left to show are the first three items of the claim for $w_{k,j}(t)$ with $k < j$. This is particularly subtle in neighborhoods of $t = t_i$ with $j(i) = k$, where $w_{k,j}$ has a zero.

Since $w_{k,j-1}$ for $k < j$ is a finite Fourier series, both the numerator and the denominator of $\partial \arg w_{k,j-1}/\partial t$, given by

$$\mathrm{NUM} = \mathrm{Re}(w_{k,j-1}(t)) \frac{\partial \mathrm{Im}(w_{k,j-1})}{\partial t}(t) - \mathrm{Im}(w_{k,j-1}(t)) \frac{\partial \mathrm{Re}(w_{k,j-1})}{\partial t}(t) \tag{24}$$

and

$$\mathrm{DEN} = \big(\mathrm{Re}(w_{k,j-1}(t))\big)^2 + \big(\mathrm{Im}(w_{k,j-1}(t))\big)^2, \tag{25}$$

are real analytic. As both of these go to zero as t approaches t_i with $j(i) = 1$, the derivative $\partial \arg w_{k,j-1}/\partial t$ is not defined at these points. However, by assumption it has a well-defined limit as t approaches t_i and, by the interpolation property of each F'_j for $j' < j$, it is equal to $\partial w_k/\partial t(t_i) \neq 0$.

Since both NUM and DEN are real-analytic, we may consider their power series at $t = t_i$. Let $\sum_m b_m (t - t_i)^m$ be the power series of DEN and let $m(i)$ denote the lowest index with $b_{m(i)} \neq 0$. (Since the limit of their quotient is well defined and nonzero, using the power series of DEN to define $m(i)$ would have the same result.) From the interpolation properties of F_i for $i = 1, 2, \ldots, j$, we know that the lowest-order term of F_k is quadratic and the lowest order of F_i for $i \neq k$ is $N + 1 > 2$. Since $w_{k,k-1}(t)$ is nonzero for all t, we have that $m(i) = 4$. We know thus that for any $\delta' > 0$ there are neighborhoods $W_i \subset V_i$ of t_i on which $|\mathrm{NUM}|$ and $|\mathrm{DEN}|$ are both greater than $\delta' |(t - t_i)^m|$, where $m > m(i) = 4$.

Since we chose F_j so that $|F_j|_N < \varepsilon'$ outside of V_i with $j(i) = j$ for some arbitrarily small ε', we have $|F_j(t)| < |(t - t_i)^N| \varepsilon'$ and $|\partial F_j(t)/\partial t| < |(t - t_i)^{N-1}| \varepsilon'$ for all $t \in V_i$ with $j(i) \neq j$. It follows that, in $W_i \subset V_i$ with $j(i) \neq j$, we have $|w_{k,j}(t)| \geq \delta' |(t - t_i)^m| - |(t - t_i)^N| \varepsilon'' \max_{t \in W_i} |w_{j,j-1}(t)|$ with $N > m > m(i) = 4$, which has no zero in W_i apart from $t = t_i$ if ε'' is sufficiently small. Therefore, the t_i with $j(i) = k$ are the only zeros of $w_{k,j}$ in W_i when $j(i) \neq j$. Since these were also the only zeros of $w_{k,j-1}$ anywhere, a sufficiently close approximation F_j of f_j guarantees that there are no zeros of $w_{k,j}$ outside of the W_i with $j(i) \neq j$ either. As in the previous cases this follows from $f_j \equiv 0$ outside of V_i with $j(i) = j$, and $|w_{k,j}| \geq |v_k| - |w_{k,j} - v_k| > (n - j - 1)\varepsilon/n$ on V_i with $j(i) = j$.

Similarly, the modulus of the numerator $\mathrm{NUM}_{k,j}$ of $\partial \arg w_{k,,j}/\partial t$ satisfies

$$
\begin{aligned}
(26)\quad |\mathrm{NUM}_{k,j}| &= \left|\mathrm{Re}(w_{k,j-1}+A_j)\frac{\partial \mathrm{Im}(w_{k,j-1}+A_j)}{\partial t} - \mathrm{Im}(w_{k,j-1}+A_j)\frac{\partial \mathrm{Re}(w_{k,j-1}+A_j)}{\partial t}\right| \\
&= \left|\mathrm{NUM}_{k,j-1} + \mathrm{Re}(w_{k,j-1})\frac{\partial \mathrm{Im}(A_j)}{\partial t} - \mathrm{Im}(w_{k,j-1})\frac{\partial \mathrm{Re}(A_j)}{\partial t} + \mathrm{Re}(A_j)\frac{\partial \mathrm{Im}(w_{k,j-1})}{\partial t}\right. \\
&\qquad \left. - \mathrm{Im}(A_j)\frac{\partial \mathrm{Re}(w_{k,j-1})}{\partial t} + \mathrm{Re}(A_j)\frac{\partial \mathrm{Im}(A_j)}{\partial t} - \mathrm{Im}(A_j)\frac{\partial \mathrm{Re}(A_j)}{\partial t}\right| \\
&> \delta|(t-t_i)^m| - 2|(t-t_i)^N|\varepsilon' \max_{t\in W_i}\left|\frac{\partial w_{k,j-1}}{\partial t}(t)\right| \max_{t\in W_i}|w_{j,j-1}(t)| \\
&\qquad - 2|(t-t_i)^{N-1}|\varepsilon' \max_{t\in W_i}|w_{k,j-1}(t)| \max_{t\in W_i}|w_{j,j-1}(t)| \\
&\qquad - 2|(t-t_i)^N|\varepsilon' \max_{t\in W_i}|w_{k,j-1}(t)| \max_{t\in W_i}\left|\frac{\partial w_{j,j-1}}{\partial t}(t)\right| \\
&\qquad - 2|(t-t_i)^{2N-1}|\varepsilon'^2 \max_{t\in W_i}|w_{j,j-1}(t)|^2 \\
&\qquad - 4|(t-t_i)^{2N}|\varepsilon'\left|\frac{\partial w_{j,j-1}}{\partial t}(t)\right| \max_{t\in W_i}|w_{j,j-1}(t)| \\
&> 0
\end{aligned}
$$

in W_i with $j(i)\neq j$, where we use $A_j = F_j w_{j,j-1}$ and $|\mathrm{Re}(z)|\leq |z|$ for all $z\in\mathbb{C}$ (and likewise for $\mathrm{Im}(z)$). The last inequality holds because ε' can be chosen arbitrarily small and $N>m$.

Thus there are no argument-critical points of $w_{k,j}$ in W_i with $j(i)\neq j$. We already showed above that there are no argument-critical points of $w_{k,j}$ in V_i with $j(i)\neq k$. Since $w_{k,j-1}$ is nonzero and has no argument-critical points in the complement of these W_i and V_i, a sufficiently close approximation F_j of f_j guarantees that there are no argument-critical points of $w_{k,j}$ in the complement either.

Note that the interpolation property of F_j guarantees that $\arg(w_{k,j}(t))$ and $\partial\arg(w_{k,j}(t))/\partial t$ have well-defined limits as t goes to t_i with $j(i)=k$, which equal $\arg(w_{k,j-1}(t_i))$ and $\partial\arg(w_{k,j-1})/\partial t(t_i)\neq 0$, respectively. This concludes the proof of the claim by induction.

The desired loop of polynomials h_t is given by $g_t+\sum_{j=1}^{n-1}A_j(t)$ with critical values $w_{j,n-1}(t)$ for $j=1,2,\dots,n-1$. The first three items of the claim imply that the square diagram associated to h_t is a singular Rampichini diagram and, by construction, it is equal to $R_{\{(t_i,j(i))\}_{i=1,2,\dots,M}}$. □

Lemma 4.3 *Let B be a P-fibered braid on $n=2n'-1$ strands with an odd, pure Rampichini diagram R. Suppose that the set $\{(t_i,j(i))\}_{i=1,2,\dots,M}$ with M even from Lemma 4.2 satisfies the following symmetry. For every $i=1,2,\dots,\frac{1}{2}M$, we have $t_{i+M/2}=t_i+\pi \bmod 2\pi$ and $k\big(i+\frac{1}{2}M\big)=k(i)+n'-1 \bmod 2n'-2$, where $v_{j(i+M/2)}(t_{i+M/2})=v(t_{i+M/2})_{k(i+M/2)}$ and $v_{j(i)}(t_i)=v(t_i)_{k(i)}$. Then h_t in Lemma 4.2 can be taken to be odd.*

Proof This lemma only requires small modifications to the proof of Lemma 4.2. Since R is odd, we may take g_t to be odd. Since the set $\{(t_i,j(i))\}_{i=1,2,\dots,M}$ has the symmetry above, the data set for the interpolation can be taken to have an odd symmetry. In other words, all functions f_j that are approximated can be taken to satisfy $f_j(x+\pi) = -f_j(x)$. Since the set of odd trigonometric polynomials is dense in the space of odd periodic functions, Lemma 4.1 guarantees that the interpolating function F_j can also be taken to be odd. Note that the critical values $w_j(t)$ of g_t are odd functions, so A_j is also odd. We obtain $h_{t+\pi}(u) = g_{t+\pi}(u) + \sum_{j=1}^{n-1} A_j(t+\pi) = -g_t(-u) - \sum_{j=1}^{n-1} A_j(t) = -h_t(-u)$. □

Lemma 4.4 *The functions* $|w_{j,n-1}(t)|$ *are real-analytic.*

Proof Since $w_{j,n-1}(t)$ is a finite Fourier series, its absolute value is real-analytic, except possibly at values of t, where $w_{j,n-1}(t) = 0$. By construction,

$$w_{j,n-1}(t) = (1 + F_j(t))w_{j,j-1}(t) + \sum_{k>j} A_k(t) \tag{27}$$

and each $A_k(t)$ has a lowest order of at least 5, while the power series of $(1 + F_j(t))^2 |w_{j,j-1}(t)|^2$ has lowest order equal to 4. Thus the root of $|w_{j,n-1}(t)|^2$ at $t = t_i$ is exactly of order 4. We can thus write $|w_{j,n-1}(t)|^2 = (t-t_i)^4 W(t)$, where W is real-analytic with $W(t_i) > 0$. Then $|w_{j,n-1}(t)|$ is given by $(t-t_i)^2\sqrt{W(t)}$, which is a well-defined nonnegative function which squares to $|w_{j,n-1}(t)|^2$ and is real-analytic, since $W(t)$ is nonzero in a neighborhood of t_i. □

5 Isolated singularities

Let R be an odd, pure Rampichini diagram corresponding to a P-fibered braid B on n strands. By Lemma 4.2, we obtain a singular Rampichini diagram $R_{\{t_i, j(i)\}_{i=1,2,\dots,M}}$ from a loop h_t in the space of polynomials, whose coefficients are trigonometric polynomials, for any given set of t_i and $j(i)$. Furthermore, by Lemma 4.3 the new trigonometric loop h_t still is odd, satisfying $h_{t+\pi}(u) = -h_t(-u)$, if the data $(t_i, j(i))$ displays an odd symmetry. We write B_{sing} for the singular braid that is formed by the roots of h_t.

We can use the construction from Theorem 3.1 to obtain a semiholomorphic, radially weighted homogeneous polynomial

$$f(u, r\mathrm{e}^{\mathrm{i}t}) = r^{kn} h\left(\frac{u}{r^k}, \mathrm{e}^{\mathrm{i}t}\right), \tag{28}$$

where $h(u,t) = h_t(u)$, $n = \deg(h_t)$ and k is a sufficiently large odd integer. The fact that this is a polynomial (as opposed to a rational map or a function involving square roots) follows from the same reasoning as Theorem 3.1.

Note that the variety of f is the cone of the closed singular braid B_{sing}. Since the singular crossings are critical points, f does not have an isolated singularity (not even weakly isolated). It is a degenerate mixed function in the sense of Oka [27] and in the sense of [2]. In [27], Oka generalizes the definition of Newton polygons of complex polynomials to mixed functions and in [2] we investigate deformations of nondegenerate mixed functions and found that adding higher-order terms (above the boundary of the Newton polygon of the mixed function) does not change the link of a singularity. Since we are now dealing with a degenerate function, our situation is different. We claim that we can add a term of the form $r^m A(\mathrm{e}^{\mathrm{i}t})$, where m is a large even integer and A is an even polynomial in $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$, ie $A(t+\pi) = A(t)$, and construct in this way a polynomial with an isolated singularity. The function A is found via trigonometric interpolation.

Recall that the critical values of h_t are denoted by $w_{j,n-1}(t)$, whose zeros occur at $t = t_i$ for $i = 1, 2, \dots, M$, corresponding to singular crossings in B_{sing} and singularities in the singular Rampichini diagram. We assume that the conditions from Lemma 4.3 are satisfied, so M is even and the t_i are indexed so that $t_{i+M/2} = t_i + \pi \bmod 2\pi$ for all $i = 1, 2, \dots, \frac{1}{2}M$. A singular crossing of B_{sing} corresponds to a circle in the singular Rampichini diagram at t_i on the curve corresponding to the critical value $w_{j(i),n-1}(t_i)$. This means that $w_{j,n-1}(t) = 0$ if and only if $j = j(i)$ and $t = t_i$. By construction, $\arg(w_{j(i),n-1}(t))$ and $\partial\arg(w_{j(i),n-1})/\partial t$ have well-defined limits as t goes to t_i, which we denote by $\arg(w_{j(i),n-1}(t_i))$ and $\partial\arg(w_{j(i),n-1})/\partial t(t_i)$, respectively.

We will find A via a trigonometric interpolation that fixes the values of A and $\partial A/\partial t$ at each t_i for $i = 1, 2, \dots, M$. We can take a solution of this interpolation problem to be of degree M. Set $m = \max\{kn+1, M\}$, where k is the integer chosen in (28).

We define $A\colon \mathbb{R} \to \mathbb{C}$ to be the even finite Fourier series that solves

$$\arg(A(t_i)) = \arg(w_{j(i),n-1}(t_i)) + \pi, \qquad \frac{\partial |A|}{\partial t}(t_i) = 0, \qquad \frac{\partial \arg(A)}{\partial t} = \frac{kn+m}{2m-kn}\frac{\partial \arg(w_{j(i),n-1})}{\partial t}(t_i) \tag{29}$$

for all $i = 1, 2, \dots, \frac{1}{2}M$. By the even symmetry of A, this also fixes the value of A and its derivative at t_i for $i = \frac{1}{2}M+1, \frac{1}{2}M+2, \dots, M$. It is implicit in the interpolation data that $A(t_i) \neq 0$ for all i.

Lemma 5.1 *Let $A\colon \mathbb{R} \to \mathbb{C}$ be an even finite Fourier series as above. Then $p = f + r^m A$ is a semiholomorphic polynomial with an isolated singularity.*

Proof The function f is a semiholomorphic polynomial. Since m is at least the degree of A, we can cancel the denominator of $r^m A(\mathrm{e}^{\mathrm{i}t}) = (v\bar{v})^{m/2} A(v/\sqrt{v\bar{v}})$. Since A is even, all remaining factors of $\sqrt{v\bar{v}}$ come with an even exponent, so $f + r^m A$ is a polynomial in u, v and $\bar{v}$.

One advantage of working with semiholomorphic polynomials is that a critical point must satisfy $\partial p/\partial u = 0$. Since $\partial p/\partial u = \partial f/\partial u$, this can only happen at critical points of f, considered as a family of complex polynomials, parametrized by v. Note that $p|_{v=0} = u^n$, so the origin is the only critical point with $v = 0$. The u-coordinate of a point with $\partial f/\partial u = 0$ is thus $r^k c_j(t)$, where $c_j(t)$ is a critical point of h_t. Thus all points where the derivative of p with respect to u vanishes are of the form $(r^k c_j(t), r\mathrm{e}^{\mathrm{i}t})$ with $p(r^k c_j(t), r\mathrm{e}^{\mathrm{i}t}) = r^{kn} w_{j,n-1}(t) + r^m A(t)$. We now consider the real Jacobian matrix of p at such a point in coordinates $(\mathrm{Re}(u), \mathrm{Im}(u), r, t) = (\mathrm{Re}(u), \mathrm{Im}(u), |v|, \arg(v))$:

$$\begin{pmatrix} 0 & 0 & knr^{kn-1}\,\mathrm{Re}(w_{j,n-1}(t)) + mr^{m-1}\,\mathrm{Re}(A(t)) & r^{kn}\,\partial\mathrm{Re}(w_{j,n-1}(t))/\partial t + r^m\,\partial\mathrm{Re}(A(t))/\partial t \\ 0 & 0 & knr^{kn-1}\,\mathrm{Im}(w_{j,n-1}(t)) + mr^{m-1}\,\mathrm{Im}(A(t)) & r^{kn}\,\partial\mathrm{Im}(w_{j,n-1}(t))/\partial t + r^m\,\partial\mathrm{Im}(A(t))/\partial t \end{pmatrix}. \tag{30}$$

In order to show that the singularity at the origin is isolated, we show that for small positive values of r this matrix has full rank. For this we compute the determinant of the 2-by-2 matrix given by the nonzero entries above. The determinant is equal to

$$\begin{aligned}(31)\quad & knr^{2kn-1}\big(\operatorname{Re}(w_{j,n-1}(t))\operatorname{Im}(w'_{j,n-1}(t))-\operatorname{Im}(w_{j,n-1}(t))\operatorname{Re}(w'_{j,n-1}(t))\big)\\ &\quad +knr^{kn+m-1}\big(\operatorname{Re}(w_{j,n-1}(t))\operatorname{Im}(A'(t))-\operatorname{Im}(w_{j,n-1}(t))\operatorname{Re}(A'(t))\big)\\ &\qquad +mr^{kn+m-1}\big(\operatorname{Re}(A(t))\operatorname{Im}(w'_{j,n-1}(t))-\operatorname{Im}(A(t))\operatorname{Re}(w'_{j,n-1}(t))\big)\\ &\qquad\quad +mr^{2m-1}\big(\operatorname{Re}(A(t))\operatorname{Im}(A'(t))-\operatorname{Im}(A(t))\operatorname{Re}(A'(t))\big),\end{aligned}$$

where the dash denotes the derivative with respect to t. By assumption we have that $m > kn$.

For a set of open neighborhoods U_i (to be determined later) of the t_i, the resulting expression goes to

$$(32)\quad \operatorname{Re}(w_{j,n-1}(t))\operatorname{Im}(w'_{j,n-1}(t))-\operatorname{Im}(w_{j,n-1}(t))\operatorname{Re}(w'_{j,n-1}(t)) = |w_{j,n-1}(t)|^2\arg(w_{j,n-1})'(t)\neq 0$$

on $[0,2\pi]\setminus\big(\bigcup_{i\,|\,j(i)=j}U_i\big)$ as r goes to zero. Thus we obtain a nonzero determinant for all points $(r^kc_j(t), re^{it})$ with sufficiently small r except possibly for those with $j=j(i)$ and $t\in U_i$ for some i.

For every $i=1,2,\dots,M$ we define $R_1(t)=|A(t)|$ and $\delta_i(t)$ such that

$$(33)\qquad A(t)=R_1(t)e^{i\big(\arg(w_{j(i),n-1}(t))+\delta_i(t)\big)}$$

on U_i. Furthermore, we write $\alpha_i(t)$ for $\arg(w_{j(i),n-1}(t))$ and $R_2(t)=|w_{j(i),n-1}(t)|$. This means that

$$(34)\qquad \begin{aligned}\operatorname{Re}(A(t)) &= R_1(t)\big(\cos\arg(w_{j(i),n-1}(t))\cos\delta_i(t)-\sin\arg(w_{j(i),n-1}(t))\sin\delta_i(t)\big),\\ \operatorname{Im}(A(t)) &= R_1(t)\big(\cos\arg(w_{j(i),n-1}(t))\sin\delta_i(t)+\sin\arg(w_{j(i),n-1}(t))\cos\delta_i(t)\big).\end{aligned}$$

We now look at the individual terms in (31). First,

$$(35)\qquad \operatorname{Re}(w_{j(i),n-1}(t))\operatorname{Im}(w'_{j(i),n-1}(t))-\operatorname{Im}(w_{j(i),n-1}(t))\operatorname{Re}(w'_{j(i),n-1}(t)) = R_2(t)^2\alpha_i(t)',$$

which is 0 if and only if $t=t_i$.

Second,

$$\begin{aligned}(36)\quad & \operatorname{Re}(w_{j(i),n-1}(t))\operatorname{Im}(A'(t))-\operatorname{Im}(w_{j(i),n-1}(t))\operatorname{Re}(A(t))'\\ &= R_2(t)\big\{\cos\alpha_i(t)\big[R_1(t)\big(-\sin\alpha_i(t)\alpha_i(t)'\sin\delta_i(t)+\cos\alpha_i(t)\cos\delta_i(t)\,\delta_i(t)'\\ &\qquad +\cos\alpha_i(t)\,\alpha_i(t)'\cos\delta_i(t)-\sin\alpha_i(t)\sin\delta_i(t)\,\delta_i(t)'\big)\\ &\qquad +R_1'(t)(\cos\alpha_i(t)\sin\delta_i(t)+\sin\alpha_i(t)\cos\delta_i(t))\big]\\ &\quad -\sin\alpha_i(t)\big[R_1(t)\big(-\sin\alpha_i(t)\,\alpha_i(t)'\cos\delta_i(t)-\cos\alpha_i(t)\sin\delta_i(t)\,\delta_i(t)'\\ &\qquad -\cos\alpha_i(t)\,\alpha_i(t)'\sin\delta_i(t)-\sin\alpha_i(t)\cos\delta_i(t)\,\delta_i(t)'\big)\big]\\ &\qquad +R_1'(t)(\cos\alpha_i(t)\cos\delta_i(t)-\sin\alpha_i(t)\sin\delta_i(t))\big\}\\ &= R_2(t)(R_1(t)\cos\delta_i(t)\,(\delta_i(t)'+\alpha_i(t)')+R_1'\sin\delta_i(t)).\end{aligned}$$

Furthermore,

$$\begin{aligned}(37)\quad &\mathrm{Re}(A(t))\,\mathrm{Im}(w_{j(i),n-1}(t))' - \mathrm{Im}(A(t))\,\mathrm{Re}(w_{j(i),n-1}(t))' \\ &= R_1(t)\big[(\cos\alpha_i(t)\cos\delta_i(t) - \sin\alpha_i(t)\sin\delta_i(t))(R_2'(t)\sin\alpha_i(t) + R_2(t)\cos\alpha_i(t)\,\alpha_i(t)') \\ &\qquad - (\cos\alpha_i(t)\sin\delta_i(t) + \sin\alpha_i(t)\cos\delta_i(t))(R_2'(t)\cos\alpha_i(t) - R_2(t)\sin\alpha_i(t)\,\alpha_i(t)')\big] \\ &= R_1(t)R_2(t)\cos\delta_i(t)\alpha_i(t)' - R_1(t)R_2'(t)\sin\delta_i(t).\end{aligned}$$

Lastly,

$$(38)\qquad \mathrm{Re}(A(t))\,\mathrm{Im}(A(t))' - \mathrm{Im}(A(t))\,\mathrm{Re}(A(t))' = R_1(t)^2(\alpha_i(t)' + \delta_i(t)').$$

Thus (31) becomes

$$\begin{aligned}(39)\quad &\alpha_i(t)'(knr^{2kn-1}R_2(t)^2 + (kn+m)r^{kn+m-1}R_1(t)R_2(t)\cos\delta_i(t) + mr^{2m-1}R_1(t)^2) \\ &\quad + \delta_i(t)'(knr^{kn+m-1}R_1(t)R_2(t)\cos\delta_i(t) + mr^{2m-1}R_1(t)^2) \\ &\quad + \sin\delta_i(t)\,r^{kn+m-1}(knR_1'(t)R_2(t) - mR_1(t)R_2'(t)).\end{aligned}$$

Each term is a real analytic function of t (see Lemma 4.4) and thus can be locally written as a power series centered at t_i. Consider the lowest-order term of the series of $\sin\delta_i(t)\,(knR_1'(t)R_2(t) - mR_1(t)R_2'(t))$ and recall that 2 is the lowest order of R_2. Since 0 is the lowest order of R_1 and 1 is the lowest order of $\sin\delta_i$ (since $\delta_i(t_i)$ is either 0 or π, and $\delta_i'(t_i) \neq 0$ by the interpolation property), the lowest order of $\sin\delta_i\; mR_1R_2'$ is 2, while the lowest order of $\sin\delta_i\; knR_1'R_2$ is greater than 2. Likewise, the lowest orders of the series for $\alpha_i'(kn+m)R_1R_2\cos\delta_i$ and $\delta_i'knR_1R_2\cos\delta_i$ centered at t_i are also both 2.

Let r_2 denote the lowest-order coefficient of R_2. We compute the coefficient of $(t-t_i)^2$ for

$$\begin{aligned}(40)\quad &\alpha_i(t)'(kn+m)R_1(t)R_2(t)\cos\delta_i(t) + \delta_i(t)'knR_1(t)R_2(t)\cos\delta_i(t) \\ &\quad + \sin\delta_i(t)\,(knR_1'(t)R_2(t) - mR_1(t)R_2(t)')\end{aligned}$$

and obtain

$$\begin{aligned}(41)\quad &\alpha_i'(t_i)(kn+m)r_2R_1(t_i)\cos\delta_i(t) + \delta_i'(t_i)knr_2R_1(t_i)\cos\delta_i(t) - \cos\delta_i(t)\delta_i'(t_i)mR_1(t_i)2r_2 \\ &= r_2R_1(t_i)\cos\delta_i(t_i)\,(\alpha_i'(t_i)(kn+m) + \delta_i'(t_i)(kn-2m)),\end{aligned}$$

which is 0, since $\delta_i'(t_i) = \alpha_i(t_i)(kn+m)/(2m-kn)$ by the interpolation property.

Therefore, the lowest order of (40) is at least 3. It follows that there is a neighborhood U_i of t_i where the absolute value of (40) is less than or equal to the absolute value of

$$(42)\qquad 2R_1(t)R_2(t)\sqrt{knm}\,\alpha_i(t)',$$

whose lowest order is 2. This implies that the absolute value of the determinant in (39) is at least

$$\begin{aligned}(43)\quad &|\alpha_i(t)'r^{2kn-1}(\sqrt{kn}R_2(t) - \sqrt{m}R_1(t)r^{m-kn})^2 + \delta_i(t)'mr^{2m-1}R_1(t)^2| \\ &= |\alpha_i(t)'|r^{2kn-1}(\sqrt{kn}R_2(t) - \sqrt{m}R_1(t)r^{m-kn})^2 + |\delta_i(t)'|mr^{2m-1}R_1(t)^2 \\ &\geq |\delta_i(t)'|mr^{2m-1}R_1(t)^2 > 0,\end{aligned}$$

since $r > 0$ and U_i can be chosen such that $R_1(t) > 0$ and $|\delta_i(t)'| > 0$. The first equality holds because $\alpha_i(t)'$ and $\delta_i(t)'$ have the same sign.

Thus there are no critical points $(r^k c_j(t), r\mathrm{e}^{\mathrm{i}t})$ of p with $t \in U_i$ and $r > 0$. Note that the U_i do not depend on r. Since $S^1 \setminus \left(\bigcup_{i=1}^M U_i\right)$ is compact, sufficiently small choices of $r > 0$ guarantee that $(r^k c_j(t), r\mathrm{e}^{\mathrm{i}t})$ with $t \in [0, 2\pi] \setminus \left(\bigcup_{i=1}^M U_i\right)$ is not a critical point of p either (see (32)). Therefore, the origin is an isolated singularity of p. $\square$

6 The proof of Theorem 1.1

In order to finish the proof of Theorem 1.1 we have to determine the links of the constructed singularities via their Rampichini diagrams.

Lemma 6.1 *Let B be the vanishing set of $p|_{r=\varepsilon}\colon \mathbb{C} \times S^1 \to \mathbb{C}$, with $\varepsilon > 0$ sufficiently small. Then B is a closed geometric braid, whose braid isotopy class is P-fibered.*

A Rampichini diagram for B is obtained from $R_{\{(t_i, j(i))\}_{i=1,2,\dots,M}}$ by inserting inner loops $\gamma_{j(i)}^{\varepsilon_i}$ at $t = t_i$ for $i = 1, 2, \dots, \frac{1}{2}M$, where $\varepsilon_i = \operatorname{sign}(\partial \arg(w_{j(i),n-1})/\partial t(t_i))$, and removing the circles at $t = t_i$ for $i = \frac{1}{2}M + 1, \frac{1}{2}M + 2, \dots, M$.

Remark 6.2 We do not claim that B itself is a P-fibered *geometric* braid, as $p|_{r=\varepsilon}$ might have argument-critical points. However, B is braid isotopic to a P-fibered geometric braid.

Proof The function $p|_{r=\varepsilon}$ corresponds to a loop of monic complex polynomials of degree n with critical values $\varepsilon^{kn} w_{j,n-1}(t) + \varepsilon^m A(t)$.

We first have to show that no critical value is equal to 0, which implies that B is a closed braid. For this note that at $t = t_i$ we have $w_{j(i),n-1}(t_i) = 0$ and $A(t_i) \neq 0$, so $\varepsilon^{kn} w_{j(i),n-1}(t_i) + \varepsilon^m A(t_i) \neq 0$. Using the notation from the previous section we may write

$$\varepsilon^{kn} w_{j,n-1}(t) + \varepsilon^m A(t) = \mathrm{e}^{\mathrm{i}\alpha_i(t)} \varepsilon^{kn} (R_2(t) + R_1(t) \varepsilon^{m-kn} \mathrm{e}^{\mathrm{i}\delta_i(t)}). \tag{44}$$

Since $\delta_i(t_i) = 0$ and $\delta_i'(t_i) \neq 0$, there is a neighborhood U_i of t_i, where $\delta_i(t)$ is neither 0 nor π, such that (44) is nonzero for all t in U_i. The complement of these neighborhoods is a compact subset of S^1 on which R_2 is nonzero, so we can ensure that (44) is nonzero by choosing ε sufficiently small. This shows that no critical value is 0. Thus there are no double roots of $p|_{r=\varepsilon}$. It follows that B is a closed braid, since p is a semiholomorphic polynomial.

Now we consider the square diagram associated to the loop of polynomials $p|_{r=\varepsilon}$. Outside of the U_i the critical values approximate $w_{j,n-1}(t)$ arbitrarily well, so outside the U_i the square diagram looks exactly like the square diagram of h_t, which is a singular Rampichini diagram, all of whose singularities lie inside the U_i. Likewise, the critical values whose corresponding curves do not have a singularity in U_i are also arbitrarily well approximated by the critical values of $p|_{r=\varepsilon}$. Thus we only have to study what happens in a neighborhood of the singular points of the singular Rampichini diagram associated to h_t.

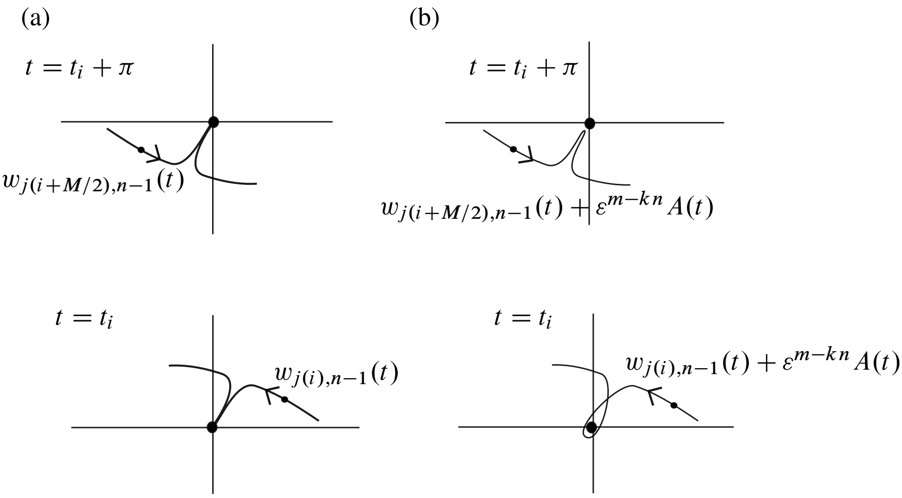

Figure 10: (a) The motion of the critical values $w_{j(i),n-1}(t)$ and $w_{j(i+M/2),n-1}(t)$ in neighborhoods of $t = t_i$ and $t = t_{i+M/2} = t_i + \pi$, respectively. (b) The motion (up to isotopy in $(\mathbb{C} \setminus \{0\}) \times S^1$) of the deformed critical values $w_{j(i),n-1}(t) + \varepsilon^{m-kn} A(t)$ and $w_{j(i+M/2),n-1}(t) + \varepsilon^{m-kn} A(t)$ in neighborhoods of $t = t_i$ and $t = t_{i+M/2} = t_i + \pi$, respectively.

There are two cases to consider. First, let $i \in \{1, 2, \dots, \frac{1}{2}M\}$. Equation (44) shows that $t = t_i$ is the unique value in U_i for which $\arg\bigl(\mathrm{e}^{\mathrm{i}\alpha_i(t)} \varepsilon^{kn} (R_2(t) + R_1(t) \varepsilon^{m-kn} \mathrm{e}^{\mathrm{i}\delta_i(t)})\bigr) = \alpha_i(t_i) + \pi$. At $t = t_i$ the derivative of $\arg\bigl(\mathrm{e}^{\mathrm{i}\alpha_i(t)} \varepsilon^{kn} (R_2(t) + R_1(t) \varepsilon^{m-kn} \mathrm{e}^{\mathrm{i}\delta_i(t)})\bigr)$ with respect to t is equal to $\delta_i'(t_i)$, which by construction has the same sign as $\alpha_i'(t_i)$.

Let $\tilde{\varepsilon}_i > 0$ and $U_i = (t_i - \tilde{\varepsilon}_i, t_i + \tilde{\varepsilon}_i)$. Note that we may choose $\tilde{\varepsilon}_i$ so small that both $\arg(w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i))$ and $\arg(w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i))$ are arbitrarily close to $\arg(w_{j(i),n-1}(t_i))$. Furthermore, we can assume that in U_i the critical value $\varepsilon^{kn} w_{j(i),n-1}(t) + \varepsilon^m A(t)$ has the smallest absolute value of all critical values. Then the argument of $\varepsilon^{kn} w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i) + \varepsilon^m A(t_i - \tilde{\varepsilon}_i)$ is arbitrarily close to the argument of $w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i)$ and the argument of $\varepsilon^{kn} w_{j(i),n-1}(t_i + \tilde{\varepsilon}_i) + \varepsilon^m A(t_i + \tilde{\varepsilon}_i)$ is arbitrarily close to the argument of $w_{j(i),n-1}(t_i + \tilde{\varepsilon}_i)$. Since the path $\varepsilon^{kn} w_{j(i),n-1}(t) + \varepsilon^m A(t)$ crosses the line $\varphi = \alpha_i(t_i) + \pi$ exactly once and since it has the smallest absolute value among all critical values, this path can be deformed into one without any argument-critical points and that winds around the origin exactly once (see Figure 10).

This deformation of the loop of critical values lifts to a deformation of the braid B [12]. By construction the direction of this path (clockwise (-1) or counterclockwise $(+1)$) is given by $\mathrm{sign}(\alpha_i'(t_i))$ and thus consistent with the monotonicity of the corresponding curve in the rest of the square diagram. Note that the deformation of the path taken by $\varepsilon^{kn} w_{j(i),n-1}(t) + r^m A(t)$ is an inner loop $\gamma_{j(i)}^{\mathrm{sign}(\alpha_i'(t_i))}$, since the moving critical value has the smallest absolute value throughout U_i.

Thus, after a deformation of the critical values, which lifts to a braid isotopy of B, there are no argument-critical points in U_i for $i = 1, 2, \dots, \frac{1}{2}M$, and by Lemma 2.9 the labels at $t = t_i - \tilde{\varepsilon}_i$ match the labels at $t = t_i + \tilde{\varepsilon}_i$.

Now we consider $i = \frac{1}{2}M + 1, \frac{1}{2}M + 2, \dots, M$. We know that at $t = t_i$ we have

$$\arg(\mathrm{e}^{\mathrm{i}\alpha_i(t)} \varepsilon^{kn} (R_2 + R_1 \varepsilon^{m-kn} \mathrm{e}^{\mathrm{i}\delta_i(t)})) = \alpha_i(t_i). \tag{45}$$

Equation (44) shows that $t = t_i$ is the unique point in U_i where this is true and there is no point in U_i with $\arg(\mathrm{e}^{\mathrm{i}\alpha_i(t)} \varepsilon^{kn} (R_2 + R_1 \varepsilon^{m-kn} \mathrm{e}^{\mathrm{i}\delta_i(t)})) = \alpha_i(t_i) + \pi$. It follows that the path taken by the critical value $\varepsilon^{kn} w_{j(i),n-1}(t) + \varepsilon^m A(t)$ in U_i does not twist around the origin. It has winding number zero and can thus be deformed to a curve without argument-critical points, connecting $\varepsilon^{kn} w_{j(i),n-1}(t_i - \tilde{\varepsilon}_i) + \varepsilon^m A(t_i - \tilde{\varepsilon}_i)$ and $\varepsilon^{kn} w_{j(i),n-1}(t_i + \tilde{\varepsilon}_i) + \varepsilon^m A(t_i + \tilde{\varepsilon}_i)$ (see Figure 10).

Again this deformation of the loop of critical values lifts to a deformation of the polynomial and thus to a braid isotopy of B. In the U_i with $i \in \left\{\frac{1}{2}M + 1, \frac{1}{2}M + 2, \dots, M\right\}$ the deformed square diagram of $p|_{r=\varepsilon}$ thus looks exactly like the singular Rampichini diagram of h_t, except that the singularities have been removed (see Figure 10).

Therefore, the complete deformed square diagram of $p|_{r=\varepsilon}$ is obtained from the singular Rampichini diagram $R_{\{(t_i, j(i))\}_{i=1,2,\dots,M}}$ of h_t by inserting inner loops of the appropriate signs at the singularities at $t = t_i$ for $i = 1, 2, \dots, \frac{1}{2}M$, and simply removing the circles representing the singularities at $t = t_i$ for $i = \frac{1}{2}M + 1, \frac{1}{2}M + 2, \dots, M$.

Since there are no argument-critical points, the deformed diagram is a Rampichini diagram and B is P-fibered. □

By the same arguments as in [14, Theorem I.1] the link of the singularity is equivalent to the closure of B.

Note that this completes the proof of Theorem 1.1.

Proof of Theorem 1.1 Let B_2 be a P-fibered braid, whose Rampichini diagram R' can be obtained from an odd, pure Rampichini diagram R by the insertion of inner loops. Then there is a corresponding odd loop of polynomials h_t, parametrized by trigonometric polynomials, whose singular Rampichini diagram is R, but with singularities at $t = t_i$ for $i = 1, 2, \dots, \frac{1}{2}M$, the positions where we want to insert inner loops, and at $t = t_{i+M/2} = t_i + \pi$. We construct the functions f, A and p and by the previous lemmas p has an isolated singularity at the origin and the closure of B_2 is the link of the singularity. The polynomial p is semiholomorphic and its degree with respect to u is equal to n, the number of strands of B_2. □

Theorem 1.1 deals with P-fibered braids whose Rampichini diagrams are obtained from odd, pure Rampichini diagrams by inserting inner loops. The result remains true if we replace "odd" by "simple".

Theorem 6.3 *Let B_1 be a P-fibered braid on n strands with a simple, pure Rampichini diagram. Let B_2 be a P-fibered braid whose Rampichini diagram is obtained from that of B_1 by the insertion of inner loops. Then the closure of B_2 is real algebraic.*

Furthermore, the corresponding real polynomial map with an isolated singularity can be taken to be semiholomorphic (ie it can be written as a polynomial in complex variables u, v and the complex conjugate $\bar{v}$) and of degree $2n-1$ with respect to the complex variable u.

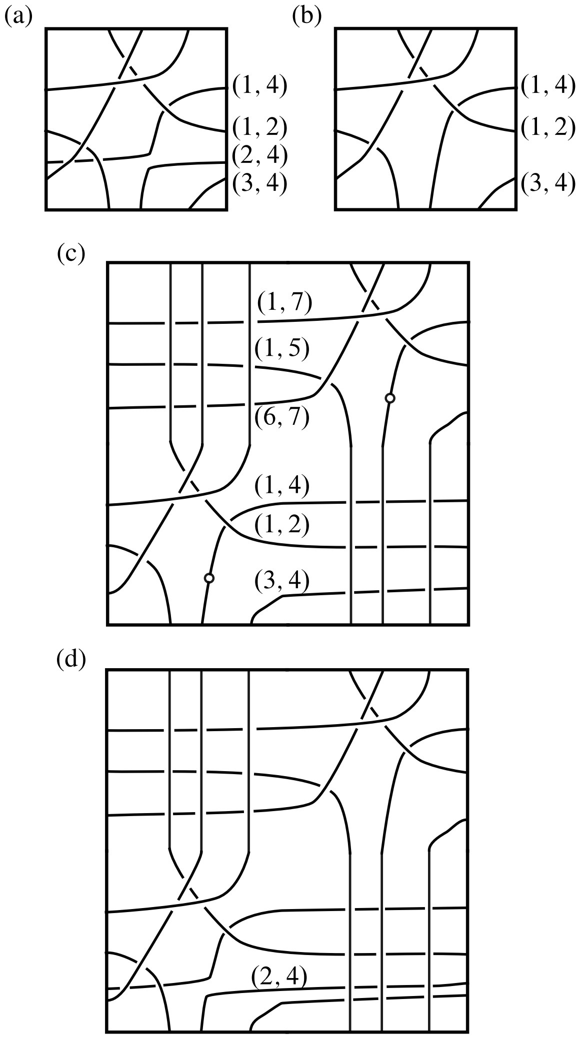

Figure 11: A sequence of Rampichini diagrams, where we have omitted several labels. (a) A Rampichini diagram that is obtained from a simple, pure Rampichini diagram by inserting an inner loop. (b) The corresponding simple, pure Rampichini diagram. (c) The singular Rampichini diagram with odd symmetry. (d) The final Rampichini diagram.

Proof Examples of Rampichini diagrams of B_2 and B_1 are shown in Figure 11(a) and (b), respectively. We want to show that B_2 can also be obtained from a P-fibered braid with odd, pure Rampichini diagram by the insertion of inner loops. As in Section 3 we can construct an odd Rampichini diagram R_{odd} for $\iota_n(B_1)m_n(B_1)$. Then the desired Rampichini diagram is obtained from R_{odd} by inserting inner loops into the lower left quadrant of R_{odd} (Figure 11(d)). We can thus construct an odd singular Rampichini diagram with singularities at $t = t_i \in (0, \pi)$ for $i = 1, 2, \dots, \frac{1}{2}M$ with $j(i) \in \{1, 2, \dots, n-1\}$ and $t = t_{i+M/2} = t_i + \pi$ for $i = 1, 2, \dots, \frac{1}{2}M$ with $j\left(i + \frac{1}{2}M\right) \in \{n, n+1, \dots, 2n-2\}$, corresponding to an odd loop of polynomials h_t as in the proof of Theorem 1.1 (see Figure 11(c)).

The braid that one obtains from this odd singular Rampichini diagram by inserting inner loops at the singularities with $t_i \in (0,\pi)$ and deleting the circles of the singularities with $t_i \in (\pi, 2\pi)$ is then $\iota_n(B_2)m_n(B_1)$. Since the Rampichini diagram of B_1 is simple, its closure is the unknot O, so the closure of $\iota_n(B_2)m_n(B_1)$ is a connected sum of the closure of B_2 and O and hence equal to the closure of B_2.

By Theorem 1.1, every P-fibered braid that can be obtained from an odd, pure Rampichini diagram closes to a real algebraic link. Thus the closure of B_2 is real algebraic. □

Example 6.4 Figure 11(b) shows a Rampichini diagram for the braid $B_1 = a_{3,4}a_{1,2}^{-1}a_{1,4}$, which closes to the unknot. We obtain a Rampichini diagram for the braid $B_2 = a_{3,4}a_{2,4}a_{1,2}^{-1}a_{1,4}$ in Figure 11(a) by inserting an inner loop. A braid word in BKL-generators for a braid with the same closure but with an odd, pure Rampichini diagram is then $\iota_n(B_2)m_n(B_1) = a_{3,4}a_{2,4}a_{1,2}^{-1}a_{1,4}a_{6,7}a_{1,5}^{-1}a_{1,7}$, which closes to the Hopf link. In fact, we could have inserted any number of inner loops into the Rampichini diagram of B_1, so that for example the closure of $a_{3,4}^{k_1}a_{2,4}^{k_2}a_{1,2}^{-k_3}a_{1,4}^{k_4}$ is real algebraic for every set of positive integers k_1, k_2, k_3, k_4. We obtain this braid word by inserting k_1 inner loops in the Rampichini diagram in Figure 11(b) into the curve in the bottom right corner, k_2 inner loops in the same position as in Figure 11(a), and k_3 and k_4 inner loops into the curves in Figure 11 next to the labels $(1,2)$ and $(1,4)$, respectively.

The proof of Theorem 1.1 is quite constructive. It describes explicitly the different approximations and interpolation steps that have to be performed in order to obtain the desired parametrization of the critical values. From the corresponding loop of polynomials h_t we obtain the function f as in (28) and the additional term A is again found via a very concrete interpolation process. However, it is still very challenging to produce examples of the resulting polynomials $p = f + A$.

The step that is difficult to perform in practice is the one that takes us from a parametrization of the critical values with labels (or, equivalently, from a Rampichini diagram) to the loop of polynomials h_t. For this it is necessary to solve a system of polynomial equations for every value of $t \in [0, 2\pi]$. This is very challenging even with the use of mathematical software. Recall that the different interpolation and approximation steps outlined in the previous sections were quite subtle. We can therefore not expect to obtain a loop of polynomials with the desired properties by simply solving the system of polynomials for a finite set of values of t in $[0, 2\pi]$ and then interpolating between these.

7 T-homogeneous braids and braids with positive Garside factor

We want to show that the family of T-homogeneous braid closures can be constructed as real algebraic links as in Theorems 1.1 and 6.3.

Definition 7.1 We say a braid word A in BKL-generators on n strands is a subword of a braid word B on n strands if deleting some letters (ie bands) from B results in A.

Proof of Theorem 1.2 We know that every T-homogeneous braid B on n strands is P-fibered and by definition B can be represented by a strictly homogeneous word w in a certain subset of BKL-generators, say $a_1, a_2, \dots, a_{n-1}$ with signs $\varepsilon_j \in \{\pm 1\}$ for $j = 1, 2, \dots, n$. We can choose a subword of w of length $n-1$ containing each $a_j^{\varepsilon_j}$. This subword is also T-homogeneous and hence P-fibered. Let R denote its Rampichini diagram, which is simple and pure.

It follows from the definition of T-homogeneity that the Rampichini diagram of B can be obtained from R by inserting inner loops. Theorem 6.3 then implies that the closure of B is real algebraic. □

Corollary 1.3 immediately follows, since homogeneous braids are T-homogeneous. By construction the corresponding polynomial is semiholomorphic. The degree of the polynomial with respect to the complex variable u is $2n-1$, where n is the number of strands of the homogeneous braid representative.

We proved in [13] that every link L is a sublink of a real algebraic link that consists of two nontrivially linked copies of L. Stallings showed that for every link L there is an unknot $O \in S^3 \setminus L$, nontrivially linked with L, such that $L \cup O$ is the closure of a homogeneous braid [36]. Combining this with Corollary 1.3 results in a proof of the following corollary.

Corollary 7.2 *For every link L in S^3 there is an unknot $O \in S^3 \setminus L$, nontrivially linked with L, such that $L \cup O$ is real algebraic.*

Now we prove Theorem 1.4. It states that any braid that is the product of a positive power of the dual Garside element δ and a BKL-positive word closes to a real algebraic link. As in the previous proof we show that this family of braids is obtained from a simple, pure Rampichini diagram by inserting inner loops.

Lemma 7.3 *For every positive generator $a_{i,j}$ there is a sequence of BKL-moves on the braid word $a_{1,n}a_{2,n}\dots a_{n-1,n}$ that results in a BKL-word whose last letter is $a_{i,j}$.*

Proof Assume $i < j$ and take the generator $a_{i,n}$ and move it towards $a_{j,n}$ conjugating the labels in between by $a_{i,n}$ at each step. After this the word looks like

$$a_{1,n}a_{2,n}\dots a_{i-1,n}a_{i,i+1}a_{i,i+2}\dots a_{i,j-1}a_{i,n}a_{j,n}a_{j+1,n}\dots a_{n-1,n}. \tag{46}$$

Then exchange $a_{i,n}$ and $a_{j,n}$, conjugating $a_{i,n}$ by $a_{j,n}$, to obtain

$$a_{1,n}a_{2,n}\dots a_{i-1,n}a_{i,i+1}a_{i,i+2}\dots a_{i,j-1}a_{j,n}a_{i,j}a_{j+1,n}\dots a_{n-1,n}. \tag{47}$$

Now move the generator $a_{i,j}$ all the way to the end of the word conjugating every generator in between by $a_{i,j}$. We thus obtain a word whose last letter is $a_{i,j}$. □

Note that such a sequence can be represented by a square diagram R, whose labels at the bottom edge spell $a_{1,n}a_{2,n}\dots a_{n-1,n}$ and whose labels at the top edge spell the final word of the sequence whose last letter is $a_{i,j}$. The lines are strictly monotone increasing and the words spelled by the labels at any horizontal slice of the diagram spell the intermediate BKL-word in the sequence.

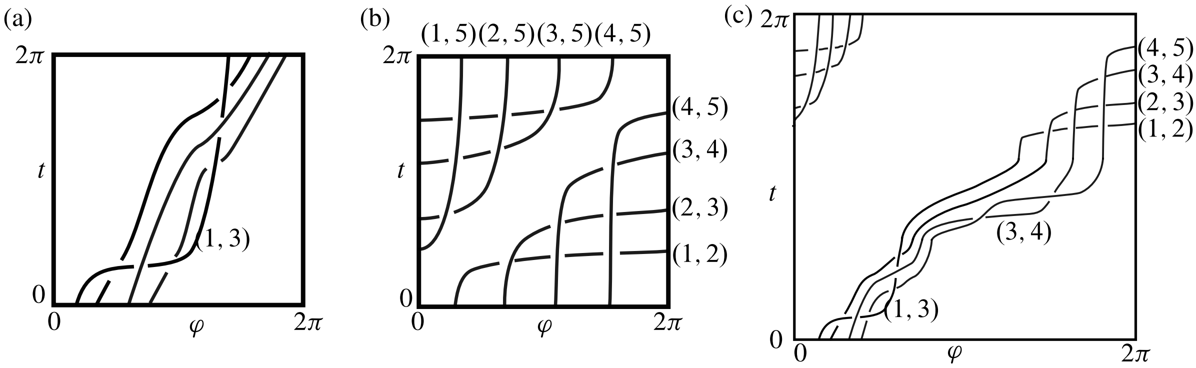

Figure 12: (a) $R'(a_{1,3})$ for $n = 5$. The transpositions at the top and bottom edge both read $(1,5)$, $(2,5)$, $(3,5)$, $(4,5)$ from left to right. (b) A Rampichini diagram for δ with $n = 5$. (c) A Rampichini diagram for δ built from $R'(a_{1,3})$, $R'(a_{3,4})$ and the diagram in (b). Inserting inner loops where the labels $(1,3)$ and $(3,4)$ are results in a Rampichini diagram for $a_{1,3}a_{3,4}\delta$.

Therefore, we can construct for every positive generator a square diagram $R'(a_{i,j})$, whose labels at the bottom and top edge both spell $a_{1,n}a_{2,n}\dots a_{n-1,n}$, whose curves do not intersect the right edge of the square ($\varphi = 2\pi$), and that for some value of t has the property that the label of the critical value with largest φ-coordinate is $a_{i,j}$. This diagram is obtained from R by composing it with the inverse of R, ie a square diagram that reverses the sequence of BKL-words from Lemma 7.3. Note that since all labels are positive, the inverse of R is still realized by strictly monotone increasing lines, see Figure 12(a). The reason why $R'(a_{i,j})$ is not a Rampichini diagram is that it is not 2π-periodic with respect to the t-coordinate. The endpoints at the top edge are strictly to the right of the starting points at the bottom edge.

Proof of Theorem 1.4 Let $B = \delta P$. Note that δ is a P-fibered braid with a simple, pure Rampichini diagram as in Figure 12(b). In particular, the labels at $t = 0$ read $a_{1,n}, a_{2,n}, \dots, a_{n-1,n}$ from left to right. We now have to show that every band in P can be obtained by inserting inner loops. Note that, since δ is a positive word, all curves in the Rampichini diagram are strictly monotone increasing.

Let $P = \prod_{k=1}^{M} a_{i_k,j_k}$. We glue the bottom edge of the square diagram $R'(a_{i_k,j_k})$ along the top edge of $R'(a_{i_{k-1},j_{k-1}})$ for all $k = 2, 3, \dots, M$. This results in a square diagram whose labels at the bottom edge and at the top edge are $a_{1,n}, a_{2,n}, \dots, a_{n-1,n}$. However the endpoints of the curves at the top edge of the square are strictly to the right of the endpoints of the curves at the bottom edge. We glue the top edge of this square diagram along the bottom edge of the Rampichini diagram of δ. In order to be able to perform these concatenations of square diagrams, we have to shift the curves in the upper square diagram to the right so that the endpoints of its curves on its bottom edge coincide with the endpoints of the lower square diagram on its top edge. Likewise we have to shift curves (in parallel) so that the endpoints at the top edge of the new diagram coincide with the endpoints on the bottom edge of $R'(a_{i_1,j_1})$. Note that none of these parallel shifts affect the crossing pattern or cactus at any value of t, nor do they change the strict monotonicity of the curves (see Figure 12).

The resulting square diagram is a simple, pure Rampichini diagram of δ and there are t_k for $k = 1, 2, \ldots, M$ with $t_k > t_{k-1}$ such that $\tau_{n-1}(t_k) = a_{i_k, j_k}$. Thus inserting inner loops whose moving critical value corresponds to $\tau_{n-1}(t_k)$ results in a Rampichini diagram of $P\delta$.

It follows from Theorem 6.3 that the closure of $P\delta$, which is also the closure of B, is real algebraic. □

References

[1] **N A'Campo**, *Le nombre de Lefschetz d'une monodromie*, Indag. Math. 76 (1973) 113–118 MR Zbl

[2] **R N Araújo dos Santos**, **B Bode**, **E L Sanchez Quiceno**, *Links of singularities of inner non-degenerate mixed functions*, Bull. Braz. Math. Soc. 55 (2024) art. id. 34 MR Zbl

[3] **J C Baez**, *Link invariants of finite type and perturbation theory*, Lett. Math. Phys. 26 (1992) 43–51 MR Zbl

[4] **I M Banfield**, *Almost all strongly quasipositive braid closures are fibered*, J. Knot Theory Ramifications 31 (2022) art. id. 2250073 MR Zbl

[5] **A F Beardon**, **T K Carne**, **T W Ng**, *The critical values of a polynomial*, Constr. Approx. 18 (2002) 343–354 MR Zbl

[6] **R Benedetti**, **M Shiota**, *On real algebraic links in S^3*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 1 (1998) 585–609 MR Zbl

[7] **J S Birman**, *New points of view in knot theory*, Bull. Amer. Math. Soc. 28 (1993) 253–287 MR Zbl

[8] **J Birman**, **K H Ko**, **S J Lee**, *A new approach to the word and conjugacy problems in the braid groups*, Adv. Math. 139 (1998) 322–353 MR Zbl

[9] **B Bode**, *Constructing links of isolated singularities of polynomials $\mathbb{R}^4 \to \mathbb{R}^2$*, J. Knot Theory Ramifications 28 (2019) art. id. 1950009 MR Zbl

[10] **B Bode**, *Real algebraic links in S^3 and braid group actions on the set of n–adic integers*, J. Knot Theory Ramifications 29 (2020) art. id. 2050039 MR Zbl

[11] **B Bode**, *All links are semiholomorphic*, Eur. J. Math. 9 (2023) art. id. 85 MR Zbl

[12] **B Bode**, *Braided open book decompositions in S^3*, Rev. Mat. Iberoam. 39 (2023) 2187–2232 MR Zbl

[13] **B Bode**, *Twisting and satellite operations on P-fibered braids*, Comm. Anal. Geom. 31 (2023) 2013–2038 MR Zbl

[14] **B Bode**, **M R Dennis**, *Constructing a polynomial whose nodal set is any prescribed knot or link*, J. Knot Theory Ramifications 28 (2019) art. id. 1850082 MR Zbl

[15] **K Brauner**, *Zur Geometrie der Funktionen zweier komplexen Veränderlichen, II: Das Verhalten der Funktionen in der Umgebung ihrer Verzweigungsstellen*, Abh. Math. Sem. Univ. Hamburg 6 (1928) 1–8 MR Zbl

[16] **K Brauner**, *Zur Geometrie der Funktionen zweier komplexen Veränderlichen, III: Klassifikation der Singularitäten algebroider Kurven*, Abh. Math. Sem. Univ. Hamburg 6 (1928) 8–34 MR Zbl

[17] **K Brauner**, *Zur Geometrie der Funktionen zweier komplexen Veränderlichen, IV: Die Verzweigungsgruppen*, Abh. Math. Sem. Univ. Hamburg 6 (1928) 34–55 MR Zbl

[18] **F Deutsch**, *Simultaneous interpolation and approximation in topological linear spaces*, SIAM J. Appl. Math. 14 (1966) 1180–1190 MR Zbl

[19] **M El Marraki**, **N Hanusse**, **J Zipperer**, **A Zvonkin**, *Cacti, braids and complex polynomials*, Sém. Lothar. Combin. 37 (1996) art. id. B37b MR Zbl

[20] **R Fenn**, **R Rimányi**, **C Rourke**, *The braid-permutation group*, Topology 36 (1997) 123–135 MR Zbl

[21] **T Ito**, **B Wiest**, *Lawrence–Krammer–Bigelow representations and dual Garside length of braids*, Geom. Topol. 19 (2015) 1361–1381 MR Zbl

[22] **L H Kauffman**, **W D Neumann**, *Products of knots, branched fibrations and sums of singularities*, Topology 16 (1977) 369–393 MR Zbl

[23] **E Looijenga**, *A note on polynomial isolated singularities*, Indag. Math. 74 (1971) 418–421 MR Zbl

[24] **V O Manturov**, **I M Nikonov**, *Maps from braids to virtual braids and braid representations*, preprint (2022) arXiv 2210.06862

[25] **J Milnor**, *Singular points of complex hypersurfaces*, Ann. of Math. Stud. 61, Princeton Univ. Press (1968) MR Zbl

[26] **H R Morton**, **M Rampichini**, *Mutual braiding and the band presentation of braid groups*, from "Knots in Hellas '98", Ser. Knots Everything 24, World Sci., River Edge, NJ (2000) 335–346 MR Zbl

[27] **M Oka**, *Non-degenerate mixed functions*, Kodai Math. J. 33 (2010) 1–62 MR Zbl

[28] **B Perron**, *Le nœud "huit" est algébrique réel*, Invent. Math. 65 (1982) 441–451 MR Zbl

[29] **A Pichon**, *Real analytic germs $f\bar{g}$ and open-book decompositions of the 3–sphere*, Int. J. Math. 16 (2005) 1–12 MR Zbl

[30] **M Rampichini**, *Mutually braided links, films and exchangeability*, J. Knot Theory Ramifications 10 (2001) 739–761 MR Zbl

[31] **L Rudolph**, *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helv. 58 (1983) 1–37 MR Zbl

[32] **L Rudolph**, *Isolated critical points of mappings from $\mathbb{R}^4$ to $\mathbb{R}^2$ and a natural splitting of the Milnor number of a classical fibered link, I: Basic theory, examples*, Comment. Math. Helv. 62 (1987) 630–645 MR Zbl

[33] **L Rudolph**, *Isolated critical points of maps from $\mathbb{R}^4$ to $\mathbb{R}^2$ and a natural splitting of the Milnor number of a classical fibred link, II*, from "Geometry and topology", Lecture Notes in Pure and Appl. Math. 105, Dekker, New York (1987) 251–263 MR Zbl

[34] **J Seade**, *Fibred links and a construction of real singularities via complex geometry*, Bol. Soc. Brasil. Mat. 27 (1996) 199–215 MR Zbl

[35] **J Seade**, *Open book decompositions associated to holomorphic vector fields*, Bol. Soc. Mat. Mex. 3 (1997) 323–335 MR Zbl

[36] **J R Stallings**, *Constructions of fibred knots and links*, from "Algebraic and geometric topology, II", Proc. Sympos. Pure Math. 32, Amer. Math. Soc., Providence, RI (1978) 55–60 MR Zbl

[37] **J L Walsh**, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloq. Publ. 20, Amer. Math. Soc., Providence, RI (1935) Zbl

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A new invariant of equivariant concordance and results on 2-bridge knots

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We study the equivariant concordance classes of 2-bridge knots and we prove that no 2-bridge knot is equivariantly slice. Finally, we introduce a new equivariant concordance invariant for strongly invertible knots. Using this invariant as an obstruction we strengthen the result on 2-bridge knots, proving that every 2-bridge knot has infinite order in the equivariant concordance group.

57K10; 57R85

1 Introduction

A strongly invertible knot is a pair (K, ρ), where $K \subseteq S^3$ is an oriented knot and $\rho \in \mathrm{Diffeo}^+(S^3)$ is an involution such that $\rho(K) = K$ and ρ reverses the orientation on K. By the resolution of the Smith conjecture [16] it is known that $\mathrm{Fix}(\rho)$ is an unknot which intersects K in two points. Sakuma [17] gave a well-defined notion of *equivariant connected sum* for strongly invertible knots by endowing them with a *direction*. Furthermore, Sakuma [17] studied strongly invertible knots up to *equivariant concordance* and introduced the *equivariant concordance group* $\widetilde{\mathcal{C}}$.

The equivariant concordance group is far from being understood. However, the first author proved in [7] that $\widetilde{\mathcal{C}}$ is not abelian, and several authors defined new invariants for equivariant concordance and obstructions for equivariant sliceness; see for example Boyle and Issa [2], Alfieri and Boyle [1], Dai, Hedden and Mallick [5], Dai, Hedden and Stoffregen [6] and Miller and Powell [15]. In particular, Boyle and Issa [2] defined the *butterfly link* associated with a directed strongly invertible knot and used it to define several equivariant concordance invariants.

We study the equivariant concordance of 2-bridge knots. In Proposition 3.2 we provide a formula to compute the *butterfly polynomial* [2] for a certain class of strongly invertible 2-bridge knots. Our initial goal was to prove Proposition 4.1 by combining the equivariant slice obstructions from the Kojima–Yamasaki *η-function* of Sakuma's link [17] and of the butterfly link [2]. This approach was inconclusive. However, using this formula we prove Corollary 3.3, which is a statement analogous to [17, Theorem II] in the case of the butterfly polynomial.

The main result of the paper is Theorem 5.10, which states the following:

Theorem *Let K be a directed strongly invertible knot and let $\widehat{L}_b(K)$ be its butterfly link endowed with the opposite semiorientation. If the Conway polynomial of $\widehat{L}_b(K)$ is nonzero then K is not equivariantly slice and has infinite order in $\widetilde{\mathcal{C}}$.*

Using this result we are able to prove Proposition 5.11, showing that every 2-bridge knot has infinite order in the equivariant concordance group.

Organisation of the paper

Section 2 contains a brief recap on the results on directed strongly invertible knots that we need later in the paper. In Section 3 we provide a formula for the butterfly polynomial [2, Definition 4.7] of 2-bridge knots. In Section 4 we prove that every 2-bridge knot is not equivariantly slice, using the nullity of the butterfly link as an obstruction to equivariant sliceness. Finally, in Section 5 we define a new equivariant concordance invariant for strongly invertible knots. We use this invariant to show that the equivariant concordance order of every 2-bridge knot is infinite.

2 Preliminaries

2.1 Directed strongly invertible knots

We briefly recall the notion of direction for a strongly invertible knot and of the equivariant concordance group. For the details see [17; 2].

Definition 2.1 A *direction* on a strongly invertible knot (K,ρ) is the choice of an oriented half-axis h, ie one of the two connected components of $\mathrm{Fix}(\rho)\setminus K$.

We call the triple (K,ρ,h) a *directed strongly invertible knot*. We write K instead of (K,ρ,h) when it is not strictly necessary to specify the choice of strong inversion and direction.

Definition 2.2 Let (K,ρ,h) be a directed strongly invertible knot. We define

- the *mirror* of (K,ρ,h) by $mK=(mK,\rho,h)$,
- the *axis-inverse* of (K,ρ,h) by $iK=(K,\rho,-h)$, where $-h$ is the direction given by the half-axis h with the opposite orientation,
- the *antipode* of (K,ρ,h) by $aK=(K,\rho,h')$, where h' is the direction given by the half-axis complementary to h. The orientation on h' is the one coherent with h.

Definition 2.3 We say that two directed strongly invertible knots (K_i,ρ_i,h_i), $i=0,1$, are *equivariantly concordant* if there exists a smooth properly embedded annulus $C\cong S^1\times I\subset S^3\times I$, invariant with respect to some involution ρ of $S^3\times I$ such that

- $\partial(S^3\times I,C)=(S^3,K_0)\sqcup -(S^3,K_1)$,

- ρ is in an extension of the strong inversion $\rho_0 \sqcup \rho_1$ on $S^3 \times 0 \sqcup S^3 \times 1$,
- the orientations of h_0 and $-h_1$ induce the same orientation on the annulus $\mathrm{Fix}(\rho)$, and h_0 and h_1 are contained in the same component of $\mathrm{Fix}(\rho) \setminus C$.

The *equivariant concordance group* is the set $\widetilde{\mathcal{C}}$ of equivalence classes of directed strongly invertible knots up to equivariant concordance, endowed with the operation induced by the *equivariant connected sum*, which we denote by $\widetilde{\#}$ (see [17; 2] for details).

Remark 2.4 It is easy to see that the mirror, axis-inverse and antipode induce involutive maps from the equivariant concordance group to itself. From the definition of equivariant connected sum we can easily deduce the following properties. Given two directed strongly invertible knots K and J, we have

- $m(K \mathbin{\widetilde{\#}} J) = mK \mathbin{\widetilde{\#}} mJ$,
- $i(K \mathbin{\widetilde{\#}} J) = iJ \mathbin{\widetilde{\#}} iK$,
- $a(K \mathbin{\widetilde{\#}} J) = aJ \mathbin{\widetilde{\#}} aK$.

Equivalently, we can say that m is an automorphism of $\widetilde{\mathcal{C}}$, while i and a are antiautomorphisms.

Remark 2.5 As a consequence, the equivariant concordance order of a directed strongly invertible knot (K, ρ, h) does not depend on the choice of a direction and it does not change when taking the mirror of the knot.

2.2 Butterfly links

Boyle and Issa [2, Definition 4.1] associated a directed strongly invertible knot (K, ρ, h) with a 2-component 2-periodic link (ie the involution ρ exchanges its components), called the *butterfly link* $L_b(K)$, defined as follows. Take an equivariant band B, parallel to the preferred half-axis h, which attaches to K at the two fixed points. Performing a band move on K along B produces a 2-component link. The link $L_b(K)$ is the one obtained from such a band move on K so that the linking number between its components is 0. Observe that $\partial B \setminus K$ consists of two arcs parallel to h, which we orient as h. The arcs lie in different components of $L_b(K)$ and the orientation on each component of $L_b(K)$ is the one induced from the orientation of the respective arc.

The following result can be proven easily by adapting the proof of [2, Proposition 7]. We report the proof because it will be useful for Proposition 5.4.

Proposition 2.6 *Let (K_i, ρ_i, h_i), $i = 0, 1$, be two equivariantly concordant directed strongly invertible knots. Then, $L_b(K_0)$ and $L_b(K_1)$ are equivariantly concordant (as 2-periodic links).*

Proof Let $C \subset S^3 \times I$ be an annulus between (K_0, ρ_0, h_0) and (K_1, ρ_1, h_1) that is invariant with respect to an extension $\rho\colon S^3 \times I \to S^3 \times I$ of $\rho_0 \sqcup \rho_1$. Let $A = \mathrm{Fix}(\rho)$ be the annulus of fixed points of ρ. Observe that $C \cap A = \alpha \cup \beta$, where α and β are two curves joining respectively the initial and final points of the half-axes of K_0 and K_1.

Now $A \setminus (\alpha \cup \beta)$ has two connected components: call D the component containing h_0 and h_1. Choose an equivariant tubular neighbourhood N of D and observe that $N \cap C$ is a D^1-subbundle of $N|_{\alpha\cup\beta}$. Consider two equivariant bands $B_i \subset S^3 \times \{i\}$, $i = 0, 1$, with B_i intersecting K_i and containing the half-axis h_i. We can choose B_i in such a way that $B_i \setminus K_i \subset N$; hence $B_0 \cup B_1 \cup C$ intersects N in a D^1-subbundle of $N|_{\partial D}$.

Choose B_0 so that the band move of K_0 along B_0 produces $L_b(K_0)$, and take B_1 so that the D^1-subbundle of $N|_{\partial D}$ given by $N \cap (B_0 \cup B_1 \cup C)$ extends to $E \subset N$, which is a D^1-bundle over D. Call L the 2-component link obtained from K_1 by the band move along B_1.

Now $E \cong D^1 \times D \cong D^1 \times D^1 \times D^1$, where $0 \times \partial D^1 \times D^1 = \alpha \cup \beta$. Then

$$C_b = (C \setminus D^1 \times \partial D^1 \times D^1) \cup \partial D^1 \times D^1 \times D^1$$

is an equivariant concordance between $L_b(K_0)$ and L. Since the linking number between components is a concordance invariant of 2-component links, we have $L = L_b(K_1)$. □

Corollary 2.7 *A directed strongly invertible knot is equivariantly slice if and only if its butterfly link is equivariantly slice* (*as a* 2*-periodic* 2*-component link*).

Proof Let K be a directed strongly invertible knot. Suppose that K is equivariantly slice, ie equivariantly concordant to the unknot. Then $L_b(K)$ is equivariantly concordant to the butterfly link of the unknot, which is easily seen to be a 2-component unlink, and hence $L_b(K)$ is equivariantly slice. Conversely, suppose that $L_b(K)$ bounds the disjoint union $D_1 \sqcup D_2$ of two disks in B^4 which is invariant under an extension of the involution of $L_b(K)$. Observe that the equivariant band move on K that gives $L_b(K)$ can be seen as an equivariant cobordism $C \subset S^3 \times [0,1]$ of genus 0 (ie a pair of pants) between K and $L_b(K)$. Then $C \cup D_1 \cup D_2$ is an equivariant slice disk with boundary K. □

2.3 Strong inversions on 2-bridge knots

Let $K = K(p,q) \subseteq S^3$ be a 2-bridge knot. From [19] we know that we can write p/q as a continued fraction

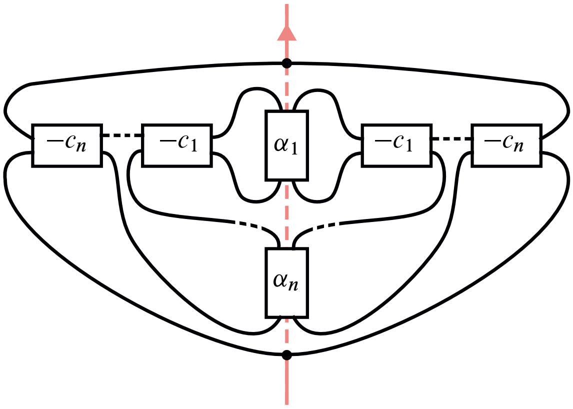
$$[a_1, \dots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

where $a_1, \dots, a_n$ and n are even nonzero integers.

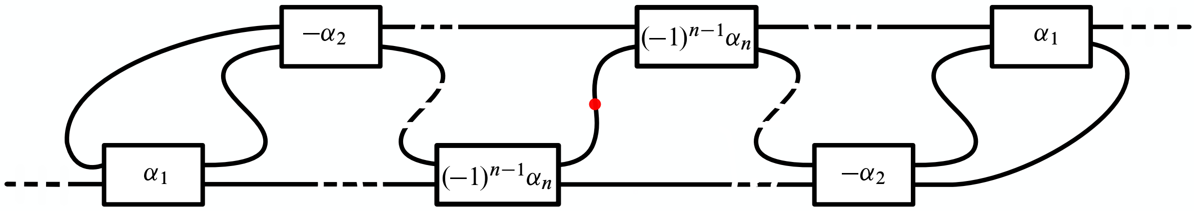
Recall that every 2-bridge knot is simple (see [8]). Sakuma [17] showed that a hyperbolic 2-bridge knot $K(p,q)$ admits exactly two inequivalent structures of strongly invertible knot. The first structure is given by the diagram

$$I_1\left(a_1, a_3, \dots, a_{n-1}; \tfrac{1}{2}a_2, \tfrac{1}{2}a_4, \dots, \tfrac{1}{2}a_n\right),$$

where the strong involution on a diagram $I_1(\alpha_1, \dots, \alpha_n; c_1 \dots, c_n)$ is given by the π-rotation around the vertical axis (see Figure 1).



$I_1(\alpha_1, \dots, \alpha_n; c_1 \dots, c_n)$



$I_2(\alpha_1, \alpha_2, \dots, \alpha_n)$

Figure 1: The two families of symmetric diagrams.

If $a_i = -a_{n-i+1}$ for all i, the second symmetry on $K(p,q)$ is represented by the diagram

$$I_2(a_1, a_2, \dots, a_{n/2}),$$

where the strong involution is the π-rotation around the central dot (see Figure 1, bottom). Otherwise it is given by

$$I_1\left(-a_n, -a_{n-2}, \dots, -a_2; -\tfrac{1}{2}a_{n-1}, \dots, -\tfrac{1}{2}a_3, -\tfrac{1}{2}a_1\right).$$

If $K(p,q)$ is a torus knot, it admits one only strong inversion, namely the one described by

$$I_1\left(a_1, a_3, \dots, a_{n-1}; \tfrac{1}{2}a_2, \tfrac{1}{2}a_4, \dots, \tfrac{1}{2}a_n\right).$$

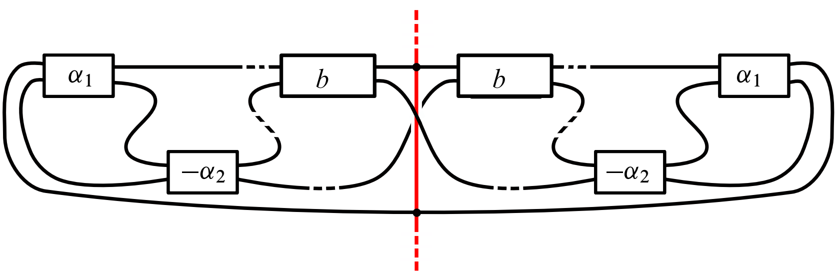


Figure 2: A diagram equivalent to the one in Figure 1, bottom.

Observe that the knot represented in Figure 1, bottom, can be equivariantly isotoped into the strongly invertible knot given in Figure 2, where $b = (-1)^{n-1}\alpha_n + 1$.

In the following, we will consider either $I_1(\alpha_1, \dots, \alpha_n; c_1 \dots, c_n)$ (Figure 1, top) or $I_2(\alpha_1, \dots, \alpha_n)$ (Figure 2) as a diagram for the directed strongly invertible knot $K = K(p, q)$, with the direction given by the oriented unbounded half-axis in the figures, unless the direction is otherwise specified. Here $n > 0$, $\alpha_1, \dots, \alpha_n \in 2\mathbb{Z} \setminus \{0\}$ and $c_1, \dots, c_n \in \mathbb{Z} \setminus \{0\}$.

3 η-function

Let $K \cup J$ be a 2-component link with linking number 0 between components. Kojima and Yamasaki [14] introduced the η-function, which is a topological concordance invariant for such links.

We briefly recall the construction of this invariant. Let X_K be the complement of K in S^3 and let $\widetilde{X}_K \to X_K$ be its infinite cyclic covering. Denote by t a generator of the deck transformation of $\widetilde{X}_K$. Recall that the Alexander module of K is $H_1(\widetilde{X}_K, \mathbb{Z})$ endowed with the $\mathbb{Z}[t, t^{-1}]$-module structure induced by the action of t. Now let l be the canonical longitude of J and let $\tilde{l}$ and $\tilde{J}$ be two nearby lifts of l and J to $\widetilde{X}_K$.

Since $\operatorname{lk}(K, J) = 0$, we know $\tilde{l}$ and $\tilde{J}$ are closed curves; hence they can be seen as classes in $H_1(\widetilde{X}_K, \mathbb{Z})$. Since the Alexander module is a torsion $\mathbb{Z}[t, t^{-1}]$-module, there exists a nonzero $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that $f(t) \cdot \tilde{l} = 0$, ie we can find a 2-chain Δ such that $\partial\Delta = f(t) \cdot \tilde{l}$. Then the η-function is defined as

$$\eta(K, J; t) = \frac{1}{f(t)} \sum_{n \in \mathbb{Z}} \#(\Delta \cap t^n \tilde{J}) \cdot t^n,$$

where $\#(\Delta \cap t^n \tilde{J})$ is the algebraic intersection.

One can check that η is well defined and has the following properties (see [14]):

(i) $\eta(K, J; t) = \eta(K, J; t^{-1})$.

(ii) $\eta(K, J; 1) = 0$.

(iii) η does not depend on the orientation of the link.

(iv) η is an invariant of topological concordance of links.

Observe that in general η does depend on the order of the components of the link, ie $\eta(K, J; t) \neq \eta(J, K; t)$.

In the following we will denote by $\mathbb{Z}\langle t \rangle \subseteq \mathbb{Z}[t, t^{-1}]$ the subgroup of Laurent polynomials satisfying properties (i) and (ii).

Boyle and Issa [2] defined the *butterfly polynomial* $\eta(L_b(K))$ of a directed strongly invertible knot K as the η-function of its butterfly link (since $L_b(K)$ is 2-periodic it does not depend on the order of its components) and showed that it induces a group homomorphism

$$\eta(L_b(-)) \colon \widetilde{\mathcal{C}} \to \mathbb{Q}(t).$$

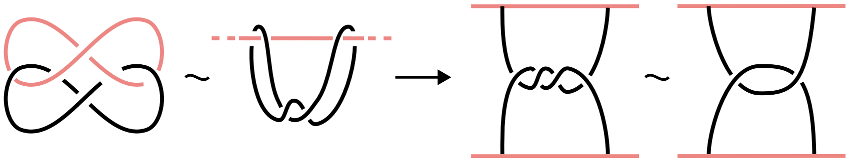


Figure 3: Fundamental domain of the Whitehead link.

Remark 3.1 Since the butterfly polynomial induces a homomorphism, every directed strongly invertible knot with nontrivial butterfly polynomial has infinite order in $\widetilde{\mathcal{C}}$.

We now describe a formula to compute the butterfly polynomial of 2-bridge knots. To do so, we report a convenient algorithm [17; 20] to compute $\eta(K,J;t)$ when K is unknotted.

In this special case the infinite cyclic cover of X_K is diffeomorphic to $\mathbb{R}\times D^2$, and the η-function of L is simply

$$\eta(K,J;t)=\sum_{i\in\mathbb{Z}}\operatorname{lk}(\tilde{l},t^i(\tilde{J}))t^i.$$

The algorithm consists of the following four steps.

(1) Start by noting that $\operatorname{lk}(\tilde{l},t^i(\tilde{J}))t^i=\operatorname{lk}(\tilde{J},t^i(\tilde{J}))t^i$ for $i\neq 0$, since $\tilde{l}$ is a nearby perturbation of $\tilde{J}$. Therefore, letting $r=\operatorname{lk}(\tilde{l},\tilde{J})$, we get

$$\sum_{i\in\mathbb{Z}}\operatorname{lk}(\tilde{l},t^i(\tilde{J}))t^i=\sum_{i\in\mathbb{Z}\setminus 0}\operatorname{lk}(\tilde{J},t^i(\tilde{J}))t^i+r.$$

In the following steps we compute $\bar{\eta}(t)=\sum_{i\in\mathbb{Z}\setminus 0}\operatorname{lk}(\tilde{J},t^i(\tilde{J}))t^i$. Since $\eta(1)=0$, it is easy to retrieve $r=-\bar{\eta}(1)$.

(2) Draw a fundamental domain of the infinite cyclic cover $\widetilde{X}_K$ (see Figure 3). Assign a label and an orientation to each arc as follows:

- (i) The arc starting from the top right point has index 0 and is oriented downwards.
- (ii) Suppose an arc α is already labelled and oriented. Let A be the end point of α and B be the point opposite to A. Call β the strand that starts from B (by saying this we are orienting it). Define $\operatorname{index}(\beta)$ (the label on β) to be $\operatorname{index}(\alpha)+1$ if B lies on the lower side of the domain or $\operatorname{index}(\alpha)-1$ if B is on the upper side.

The labels we put on the strands keep track of which translate of $\tilde{J}$ in $\widetilde{X}_K$ they correspond to. A strand labelled by i is a portion of $t^i(\tilde{J})$. Hence a crossing where an arc labelled by i overcrosses an arc labelled by j corresponds to a crossing between $t^i(\tilde{J})$ and $t^j(\tilde{J})$ or, equivalently, between $\tilde{J}$ and $t^{i-j}(\tilde{J})$. This motivates the following step.

(3) Assign to each double point P a sign $\epsilon_P\in\{+,-\}$ and an integer $d_P\in\mathbb{Z}$ as follows. The sign ϵ_P is the sign of the crossing and d_P is the difference between the label on the overcrossing arc and the label of the undercrossing arc.

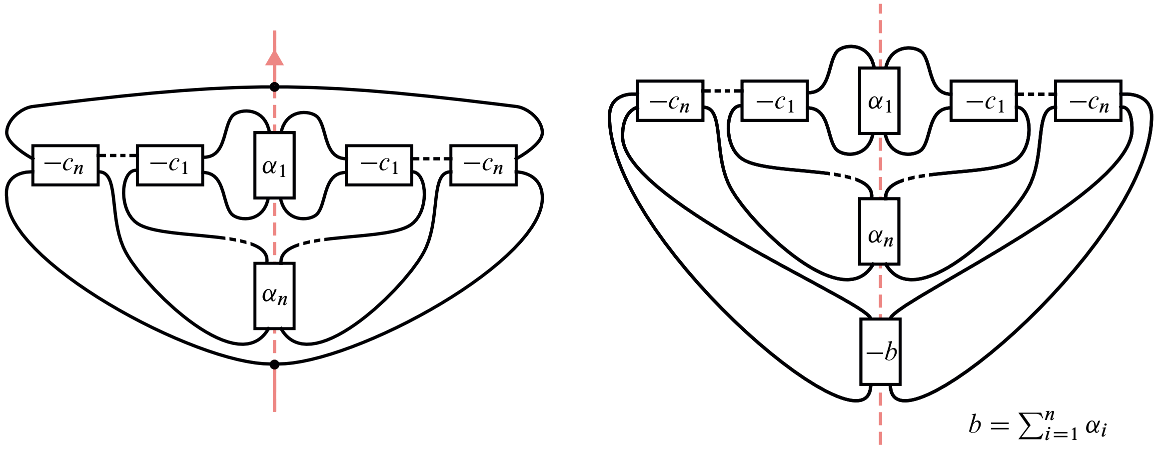


Figure 4: Strong inversion on a 2-bridge knot and construction of the butterfly link.

(4) Now let

$$\bar{\eta}(t) = \sum_P \epsilon_P t^{d_P} \quad \text{and} \quad r = -\bar{\eta}(1).$$

Then $\eta(t)$ is obtained as $\eta(t) = \bar{\eta}(t) + r$.

We will use this algorithm to prove Proposition 3.2.

Proposition 3.2 *Let $I_1(\alpha_1, \dots, \alpha_n; c_1, \dots, c_n)$ be a diagram for the directed strongly invertible knot $K = K(p, q)$. Then the butterfly polynomial of K is given by*

$$\eta_{L_b(K)}(t) = \sum_{i=1}^{n} c_i (t^{\sigma_i} + t^{-\sigma_i}) - 2 \sum_{i=1}^{n} c_i,$$

where $\sigma_i = \frac{1}{2} \sum_{j=1}^{i} \alpha_j$.

Proof Figure 4 shows the construction of the butterfly link $L_b(K) = K_0 \sqcup K_1$ that starts from $I_1(\alpha_1, \dots, \alpha_n; c_1, \dots, c_n)$. We only add a box of $-2\sigma_n$ crossings, so that $\mathrm{lk}(K_0, K_1) = 0$. Observe that $L_b(K)$ is the *denominator closure* of the rational tangle T in Figure 5 (for details see [10, Figure 5]).

Using the procedure described in [11, Section 2], we can compute the coefficients of the continued fraction associated with this tangle, finding that they are

$$[-b, 2c_n, \alpha_n, \dots, 2c_1, \alpha_1].$$

The coefficients are all nonzero even integers; then, by [13, Exercise 2.1.14], we find that $L_b(K)$ is equivalent to the link represented by the diagram in Figure 5.

The diagram in Figure 5 has unknotted components, so we can compute the η-function of the link $L_b(K)$ drawing a fundamental domain (Figure 6) and applying the algorithm previously described.

Observe that we can assume $b \geq 0$. In fact it follows from [2, Proposition 11] that $\eta_{L_b(mK)}(t) = -\eta_{L_b(K)}(t)$.

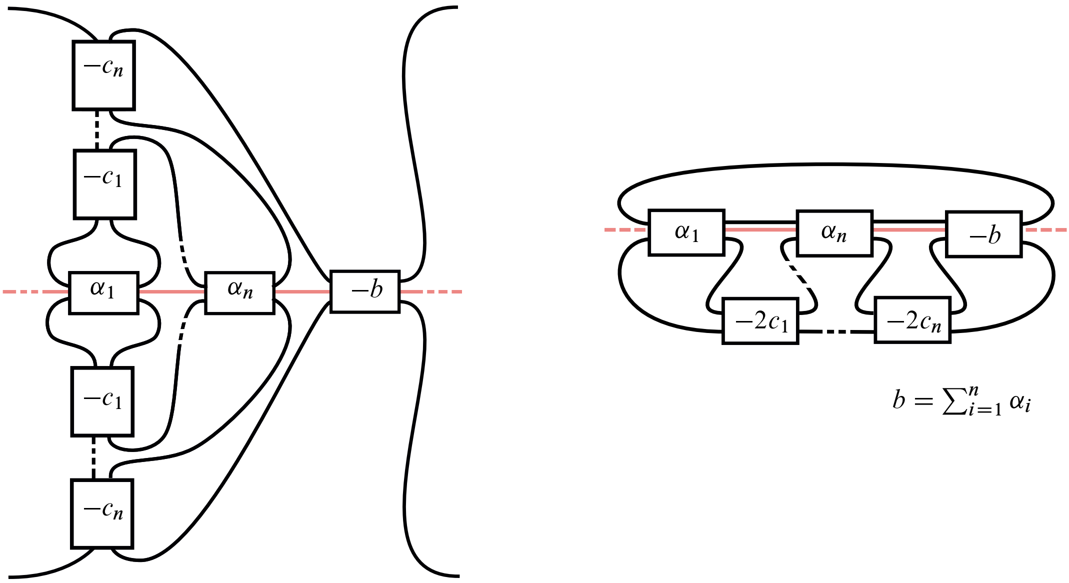


Figure 5: Left: rational tangle T. Right: canonical form of $L_b(K)$.

We start by labelling and orienting the arcs. Call γ the top-right arc, labelled 0. The arc γ runs across each $-2c_i$ box reaching the left side of the domain. If $\alpha_1 > 0$, then γ rises and ends in the upper horizontal bar. Then we will run across $\frac{1}{2}\alpha_1$ arcs riding from the lower bar to the upper bar, labelled with increasing indexes. If $\alpha_1 < 0$, then γ descends ending in the lower horizontal bar and the following $\frac{1}{2}\alpha_1$ arcs ride from the upper bar to the lower one with decreasing labels. This means that in both cases the last arc of this group is labelled by $\frac{1}{2}\alpha_1$.

This goes for all the groups of $\frac{1}{2}\alpha_i$ vertical arcs: in each group the labels increase or decrease by 1 and the arc entering the $-2c_i$ box, after the group of $\frac{1}{2}\alpha_i$ vertical arcs, is labelled by $\sum_{j=1}^{i} \frac{1}{2}\alpha_j = \sigma_i$.

The arc exiting the n^{th} box is labelled by $\sigma_n = \sum_{i=1}^{n} \frac{1}{2}\alpha_i$. The last group of vertical arcs consists of exactly $|\sigma_n|$ arcs oriented upwards or downwards depending on whether $\sigma_n < 0$ or $\sigma_n > 0$. It follows that the labels increase, or decrease, by 1 until they reach 0, at this point we meet the first arc, which is already oriented and labelled.

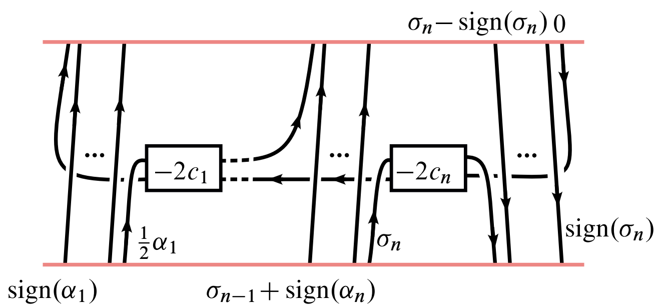


Figure 6: Fundamental domain with labelled oriented arcs.

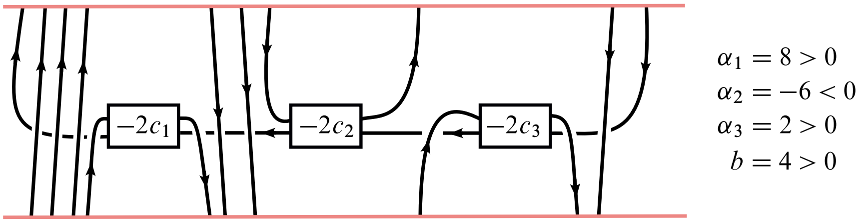


Figure 7: Example of fundamental domain.

At last, we count the crossings. We must count the crossings in the groups of vertical arcs and in the boxes for $i \in \{1, \dots, n\}$ and in the last group of $|\sigma_n|$ vertical arcs. For each $i \in \{1, \dots, n\}$ in the i^{th} box we find that the two strands run in opposite direction, one labelled 0 and one labelled σ_i. Hence we count

- $|c_i|$ crossings with $\epsilon = \operatorname{sign}(c_i)$ and $d = \sigma_i$,
- $|c_i|$ crossings with $\epsilon = \operatorname{sign}(c_i)$ and $d = -\sigma_i$.

Regarding the crossings in the vertical arcs, the idea is that they do not contribute inasmuch they simplify with each other. In fact the undercrossing arc is the same strand for each crossing and the labels of the overcrossing arcs start from one value and after increasing and decreasing they get back to the same value. This is due to the fact that the linking number between the components of the butterfly link is 0 and the two components only cross each other in the boxes of crossings that go from α_1 to α_n and in the box containing $-b$ crossings. Let us be more precise.

The $\frac{1}{2}|\alpha_i|$ crossings in the i^{th} group of vertical arcs contribute to the η-function with a polynomial that depends on the sign of α_i and of b in the following way:

$$f(\alpha_i, b) = \begin{cases} +t^{\sigma_{i-1}} \sum_{h=1}^{\alpha_i/2} t^h, & a_i > 0, \\ -t^{\sigma_{i-1}} \sum_{h=0}^{|\alpha_i|/2-1} t^{-h}, & a_i < 0. \end{cases}$$

Similarly, the crossings in the last group of vertical arcs contribute with

$$f(b) = -\sum_{h=1}^{\sigma_n} t^h.$$

See Figure 7 for an example. To prove that these two functions are correct we must examine the two possible cases.

(1) If $\alpha_i > 0$, then each crossing is positive, the undercrossing arc is labelled by 0 and the labels on the $\frac{1}{2}\alpha_i$ overcrossing arcs go from $\sigma_{i-1} + 1$ to $\sigma_{i-1} + \frac{1}{2}\alpha_i$.

(2) If $\alpha_i < 0$, then each crossing is negative, the undercrossing arc is labelled by 0 and the labels on the $\frac{1}{2}|\alpha_i|$ overcrossing arcs go from σ_{i-1} to $\sigma_{i-1} + 1 + \frac{1}{2}\alpha_i$.

The computation for the last group of vertical arcs works in the same way.

It follows that the count of the crossings on the vertical groups of arcs for $i \in \{1, \ldots, n\}$ simplifies with the count of the crossings on the last group of vertical arcs:

$$\left(\sum_{i=1}^{n} f(\alpha_i, b)\right) + f(b) = 0.$$

This means that

$$\bar{\eta}(t) = \sum_{i=1}^{n} c_i(t^{\sigma_i} + t^{-\sigma_i}) \quad \text{and} \quad \bar{\eta}(1) = 2\sum_{i=1}^{n} c_i;$$

hence $\eta(t) = \sum_{i=1}^{n} c_i(t^{\sigma_i} + t^{-\sigma_i}) - 2\sum_{i=1}^{n} c_i$. □

In analogy with Sakuma's result [17, Theorem II], we can observe the following corollary to Proposition 3.2.

Corollary 3.3 *Every Laurent polynomial in $\mathbb{Z}\langle t\rangle$ is realised as butterfly polynomial of some directed strongly invertible knot.*

Proof Notice that the η-function of the butterfly link of the 2-bridge knot $K_n = I_1(2n; 1)$ is

$$\eta(t) = t^n + t^{-n} - 2,$$

and these form a set of generators for $\mathbb{Z}\langle t\rangle$. □

Remark 3.4 Using only the formula of Proposition 3.2 we are not able to deduce that no 2-bridge knot is equivariantly slice. As an example, the butterfly polynomial vanishes on the family of directed strongly invertible knots given by $I_1(2a, -2a, 2a, -2a; b, c, -b, d)$, with $a, b, c, d \in \mathbb{Z} \setminus \{0\}$.

4 No 2-bridge knot is equivariantly slice

In this section we prove the following two propositions.

Proposition 4.1 *The strongly invertible knot $K = I_1(\alpha_1, \ldots, \alpha_n; c_1, \ldots, c_n)$ is not equivariantly slice.*

Proposition 4.2 *The strongly invertible knot $K = I_2(\alpha_1, \ldots, \alpha_n)$ is not equivariantly slice.*

To prove Proposition 4.1, we use the *nullity* of the butterfly link as an obstruction to equivariant sliceness. Let $L \subset S^3$ be a link, and denote by $\Sigma(L)$ the 2-fold cover of S^3 branched over L. Recall that the *nullity* of L is defined as

$$n(L) = 1 + \dim(H_1(\Sigma(L), \mathbb{Q}))$$

and that the nullity is an invariant for link concordance, as shown in [12].

Proof of Proposition 4.1 Consider on K the direction specified in Figure 4. Recall that the fraction associated with K is

$$p/q = [\alpha_1, 2c_1, \ldots, \alpha_n, 2c_n].$$

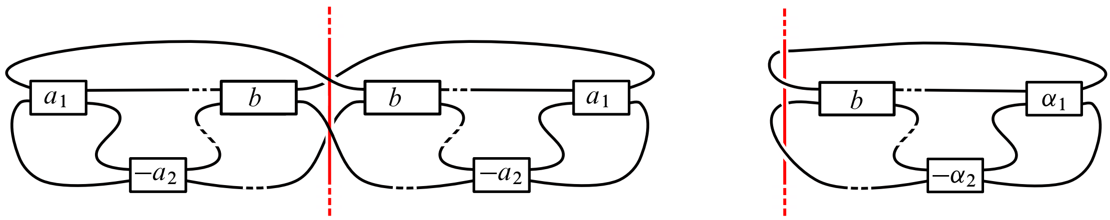


Figure 8: Left: butterfly link of $I_2(a_1, a_2, \dots, a_{n/2})$. Right: the quotient knot $\overline{K}$ and the quotient axis $\overline{A}$ (in red).

As shown in the proof of Proposition 3.2, observe that $L_b(K)$ is a 2-bridge link with continued fraction $\left[-\sum_{i=1}^n \alpha_i, 2c_n, \alpha_n, \dots, 2c_1, \alpha_1\right]$ (see Figure 5). It follows that the associated rational number is

$$p''/q'' = -\sum_{i=1}^{n} \alpha_i + q'/p',$$

where $p'/q' = [2c_n, \alpha_n, \dots, 2c_1, \alpha_1]$. It is a well-known fact (see [10, Theorem 4]) that $p = p'$ and q' is such that $q \cdot q' \equiv -1 \mod p$. It follows that $p''/q'' \in \mathbb{Q} \setminus \{0\}$. Since the 2-fold cover $\Sigma(L_b(K))$ of S^3 branched over $L_b(K)$ is a lens space and $p''/q'' \neq 0$, it is a rational homology S^3. In particular, the nullity of $L_b(K)$ is $n(L_b(K)) = 1$. Since the nullity of the 2-component unlink is easily seen to be 2, $L_b(K)$ is not concordant to the unlink. Therefore K is not equivariantly slice. □

Proof of Proposition 4.2 By [17, Proposition 2.3], if $\sum_{i\geq 0} \alpha_{2i+1} \neq 0$, then Sakuma's η-polynomial is nonvanishing for K, and hence K has infinite order in $\widetilde{\mathcal{C}}$.

Therefore, we can assume $\sum_{i\geq 0} \alpha_{2i+1} = 0$, and hence $n > 2$. In this case, we can see from Figure 8, left, that the butterfly link $L_b(K)$ of K is a 2-bridge link. In order to conclude the proof as in Proposition 4.1, we just have to show that $L_b(K)$ is not the unlink. Let $\overline{K}$ be the knot obtained by quotienting $(S^3, L_b(K))$ by the involution ρ, as depicted in Figure 8, right.

Suppose by contradiction that $L_b(K)$ is the unlink, and hence $\Sigma(L_b(K)) \cong S^1 \times S^2$. Observe that $\overline{K}$ is a 2-bridge knot with continued fraction $p/q = [\alpha_1, \dots, \alpha_n \pm 1]$.

We want to show that $\overline{K}$ is not an unknot. Observe that $\overline{K}$ is isotopic to the 2-bridge knot given by the continuous fraction $[-\alpha_n \mp 1, -\alpha_{n-1}, \dots, -\alpha_1]$. If $\overline{K}$ is unknotted, by [18] we know that $[-\alpha_n \mp 1, -\alpha_{n-1}, \dots, -\alpha_1] = 1/s$ for some $s \in \mathbb{Z}$. Then

$$\frac{\pm s + 1}{s} = [-\alpha_n, -\alpha_{n-1}, \dots, -\alpha_1].$$

Computing the continuous fraction term by term, one finds that

$$[\alpha_1, \dots, \alpha_n] = [\pm 2, \mp 2, \dots].$$

This proves that $\overline{K}$ cannot be an unknot, since these coefficients do not satisfy the constraint

$$\sum_{i\geq 0} \alpha_{2i+1} = 0.$$

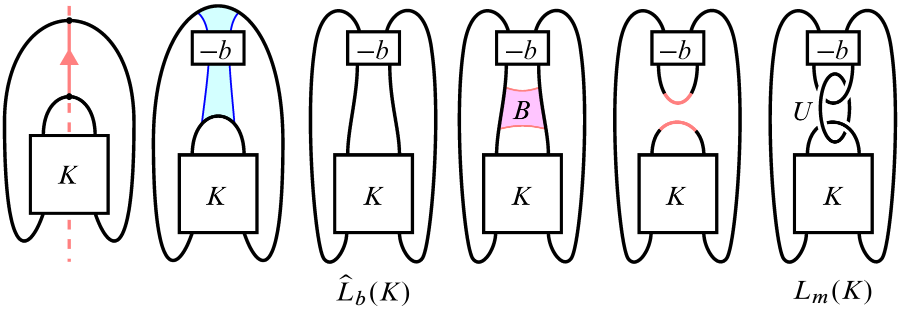


Figure 9: Construction of the moth link.

Now, since $L_b(K)$ is obtained from $\overline{K}$ by taking the double cover branched over the quotient of the axis $\overline{A}$, $\Sigma(L_b(K))$ is a double branched cover over $\Sigma(\overline{K})$. Since the transfer map (see [3, Chapter III, Section 2])

$$H_1\big(\Sigma(\overline{K}), \mathbb{Z}[\tfrac{1}{2}]\big) \cong \mathbb{Z}/p\mathbb{Z} \hookrightarrow H_1\big(\Sigma(L_b(K)), \mathbb{Z}[\tfrac{1}{2}]\big) \cong \mathbb{Z}[\tfrac{1}{2}]$$

is injective and $\mathbb{Z}[\frac{1}{2}]$ is torsion-free, we obtain a contradiction. It follows that $L_b(K)$ is a nontrivial 2-bridge link, $n(L_b(K)) = 1$ and hence that K is not equivariantly slice. □

5 A new invariant of equivariant concordance

Recall that a *semiorientation* on a link L is the choice of an orientation on each component of L, up to reversing the orientation on all components simultaneously.

Definition 5.1 Let K be a directed strongly invertible knot. We define $\widehat{L}_b(K)$ to be the 2-periodic, semioriented link obtained by endowing $L_b(K)$ with the opposite semiorientation.

Observe that $\widehat{L}_b(K)$ is obtained from K via a band move coherent with the (unique) semiorientation of K. Conversely, we can attach another equivariant band B to $\widehat{L}_b(K)$ obtaining again K (see Figure 9).

Definition 5.2 We define the *moth link of* K to be the link $L_m(K)$ given by the union of K and a meridian U of the core of the band B, as described in Figure 9. Observe that this meridian can be chosen so that $L_m(K)$ is a 2-component strongly invertible link.

Remark 5.3 Using the notation of [9], $L_m(K)$ is the *strong fusion* of the link $\widehat{L}_b(K)$ along the band B.

Proposition 5.4 *Let (K_0, ρ_0, h_0) and (K_1, ρ_1, h_1) be two equivariantly concordant directed strongly invertible knots. Then $L_m(K)$ is equivariantly concordant to $L_m(J)$.*

Proof Let $C \subset S^3 \times I$ be a concordance between (K_0, ρ_0, h_0) and (K_1, ρ_1, h_1) equivariant with respect to an extension $\rho\colon S^3 \times I \to S^3 \times I$ of $\rho_0 \sqcup \rho_1$. In the proof of Proposition 2.6 we found an equivariant embedding of $E \cong D^1 \times D^1 \times D^1$ in $S^3 \times I$ which intersects the concordance C in $D^1 \times \partial D^1 \times D^1$ and such that $C_b = (C \setminus E) \cup \partial D^1 \times D^1 \times D^1$ exhibits an equivariant concordance between $L_b(K_0)$ and $L_b(K_1)$.

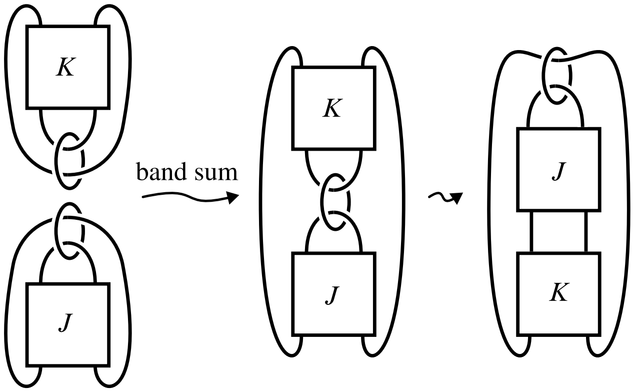


Figure 10: The band sum of $L_m(K)$ and $L_m(J)$ is $L_m(K \mathbin{\widetilde{\#}} J)$.

Now let N be a tubular neighbourhood of $D^1 \times 0 \times D^1$ in $S^3 \times I$ and let c be the arc $0 \times 0 \times D^1$. We can take $\epsilon > 0$ small such that $E_\epsilon = D^1 \times \epsilon D^1 \times D^1 \subset N$. Then $(C_b \setminus E_\epsilon) \cup D^1 \times \partial(\epsilon D^1) \times D^1 \cup \partial N|_c$ is an equivariant concordance between $L_m(K_0)$ and $L_m(K_1)$. □

Definition 5.5 Let K be a directed strongly invertible knot. We define the *moth polynomial* of K as the η-function of $L_m(K) = K \cup U$, taken with respect to the component K, ie

$$\eta(L_m(K))(t) = \eta(K, U; t).$$

Proposition 5.6 *The moth polynomial induces a group homomorphism*

$$\eta(L_m(-)) : \widetilde{\mathcal{C}} \to \mathbb{Q}(t).$$

Proof Let K and J be two directed strongly invertible knots. By Proposition 5.4 if K and J are equivariantly concordant then $L_m(K)$ and $L_m(J)$ are concordant. Since the η-function is a concordance invariant, $\eta(L_m(K)) = \eta(L_m(J))$; therefore $\eta(L_m(-))$ is well defined. Next we have to show that $\eta(L_m(K \mathbin{\widetilde{\#}} J)) = \eta(L_m(K)) + \eta(L_m(J))$. This follows by observing that $L_m(K \mathbin{\widetilde{\#}} J)$ is obtained from $L_m(K)$ and $L_m(J)$ by a band sum, as shown in Figure 10 and using [4, Theorem 7.1]. □

We provide now a formula to compute the moth polynomial of a directed strongly invertible knot K from the Conway polynomial of K and $\widehat{L}_b(K)$.

Proposition 5.7 *The moth polynomial of a directed strongly invertible knot K can be computed by the formula*

$$\eta(L_m(K))(t) = \frac{\nabla_{\widehat{L}_b(K)}(z)}{z \nabla_K(z)},$$

where $\nabla_L(z)$ is the Conway polynomial of an oriented (or semioriented) link L and $z = i(2 - t - t^{-1})^{1/2}$.

The proposition above is an immediate consequence of Propositions 5.8 and 5.9.

Proposition 5.8 [9, Proposition 1] *Let L be a 2-component link and let b be a band with ends on different components of L. Denote by $L(b)$ the knot obtained by performing a band move on L along b and by $\widehat{L}(b)$ the strong fusion of L given by b. Then the Cochran invariant of $\widehat{L}(b)$ is given by*

$$\beta(\widehat{L}(b))(x) = \frac{-ix^{-1/2}\nabla_L(ix^{1/2})}{\nabla_{L(b)}(ix^{1/2})}.$$

Proposition 5.9 [4, Theorem 7.1] *The Kojima–Yamasaki η-function can be expanded in powers of $x = (1-t)(1-t^{-1})$ so that*

$$\eta(L)(t) = \beta(L)(x).$$

Since $\eta(L_m(-))$ is a homomorphism, Proposition 5.7 implies the following result.

Theorem 5.10 *Let K be a directed strongly invertible knot such that $\nabla_{\widehat{L}_b(K)}(z) \neq 0$. Then K is not equivariantly slice and has infinite order in $\widetilde{\mathcal{C}}$.*

As an immediate application of Theorem 5.10 we have the following refinement of the results in Section 4 on 2-bridge knots.

Proposition 5.11 *Every 2-bridge knot has infinite order in $\widetilde{\mathcal{C}}$, independently of the choice of strong inversion and direction.*

Proof First of all, by Remark 2.5 it is sufficient to show that a directed strongly invertible knot K of type $I_1(\alpha_1, \dots, \alpha_n, c_1, \dots, c_n)$ or $I_2(a_1, \dots, a_n)$, with the direction specified in Figures 1, top, and 2, has infinite order in $\widetilde{\mathcal{C}}$.

As proven in Propositions 4.1 and 4.2, either Sakuma's η-polynomial of K is nonzero, and hence K has infinite order, or $\Sigma(L_b(K)) = \Sigma(\widehat{L}_b(K))$ is a rational homology S^3.

Recall now that, for a link $L \subset S^3$, we have $|H_1(\Sigma(L), \mathbb{Z})| = |\Delta_L(-1)|$, where 0 means that the group is infinite. Since $H_1(\Sigma(\widehat{L}_b(K)), \mathbb{Z})$ is finite, we deduce that the Alexander polynomial of $\widehat{L}_b(K)$ is nonzero, and hence by Theorem 5.10 that K has infinite order in $\widetilde{\mathcal{C}}$. □

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References

[1] **A Alfieri**, **K Boyle**, *Strongly invertible knots, invariant surfaces, and the Atiyah–Singer signature theorem*, Michigan Math. J. 74 (2024) 1–17 MR Zbl

[2] **K Boyle**, **A Issa**, *Equivariant 4-genera of strongly invertible and periodic knots*, J. Topol. 15 (2022) 1635–1674 MR Zbl

[3] **G E Bredon**, *Introduction to compact transformation groups*, Pure and Appl. Math. 46, Academic Press, New York (1972) MR Zbl

[4] **T D Cochran**, *Geometric invariants of link cobordism*, Comment. Math. Helv. 60 (1985) 291–311 MR Zbl

[5] **I Dai**, **M Hedden**, **A Mallick**, *Corks, involutions, and Heegaard Floer homology*, J. Eur. Math. Soc. 25 (2023) 2319–2389 MR Zbl

[6] **I Dai**, **A Mallick**, **M Stoffregen**, *Equivariant knots and knot Floer homology*, J. Topol. 16 (2023) 1167–1236 MR Zbl

[7] **A Di Prisa**, *The equivariant concordance group is not abelian*, Bull. Lond. Math. Soc. 55 (2023) 502–507 MR Zbl

[8] **A Hatcher**, **W Thurston**, *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. 79 (1985) 225–246 MR Zbl

[9] **U Kaiser**, *Strong band sum and dichromatic invariants*, Manuscripta Math. 74 (1992) 237–251 MR Zbl

[10] **L H Kauffman**, **S Lambropoulou**, *On the classification of rational knots*, Enseign. Math. 49 (2003) 357–410 MR Zbl

[11] **L H Kauffman**, **S Lambropoulou**, *On the classification of rational tangles*, Adv. Appl. Math. 33 (2004) 199–237 MR Zbl

[12] **L H Kauffman**, **L R Taylor**, *Signature of links*, Trans. Amer. Math. Soc. 216 (1976) 351–365 MR Zbl

[13] **A Kawauchi**, *A survey of knot theory*, Birkhäuser, Basel (1996) MR Zbl

[14] **S Kojima**, **M Yamasaki**, *Some new invariants of links*, Invent. Math. 54 (1979) 213–228 MR Zbl

[15] **A N Miller**, **M Powell**, *Strongly invertible knots, equivariant slice genera, and an equivariant algebraic concordance group*, J. Lond. Math. Soc. 107 (2023) 2025–2053 MR Zbl

[16] **J W Morgan**, **H Bass** (editors), *The Smith conjecture*, Pure Appl. Math. 112, Academic, Orlando, FL (1984) MR Zbl

[17] **M Sakuma**, *On strongly invertible knots*, from "Algebraic and topological theories", Kinokuniya, Tokyo (1986) 176–196 MR Zbl

[18] **H Schubert**, *Knoten mit zwei Brücken*, Math. Z. 65 (1956) 133–170 MR Zbl

[19] **L Siebenmann**, *Exercices sur les nœuds rationnels*, preprint (1975)

[20] **M Snape**, *Homological invariants of strongly invertible knots*, PhD thesis, University of Glasgow (2018) Available at `https://theses.gla.ac.uk/39015/`

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Recipes to compute the algebraic K-theory of Hecke algebras of reductive p-adic groups

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We compute the algebraic K-theory of the Hecke algebra of a reductive p-adic group G using the fact that the Farrell–Jones conjecture is known in this context. The main tools will be the properties of the associated Bruhat–Tits building and an equivariant Atiyah–Hirzebruch spectral sequence. In particular, the projective class group can be written as the colimit of the projective class groups of the compact open subgroups of G.

55P91; 19D50, 20C08

1 Introduction

We begin with stating the main theorem of this paper; explanation will follow:

Main Theorem 1.1 *Let G be a td-group which is modulo a normal compact subgroup a subgroup of a reductive p-adic group. Let R be a uniformly regular ring with $\mathbb{Q} \subseteq R$. Choose a model $E_{\mathcal{C}\mathrm{op}}(G)$ for the classifying space for proper smooth G-actions. Let $\mathcal{I} \subseteq \mathcal{C}\mathrm{op}$ be the set of isotropy groups of points in $E_{\mathcal{C}\mathrm{op}}(G)$.*

Then:

(i) *The map induced by the projection $E_{\mathcal{C}\mathrm{op}}(G) \to G/G$ induces for every $n \in \mathbb{Z}$ an isomorphism*
$$H_n^G(E_{\mathcal{C}\mathrm{op}}(G); \boldsymbol{K}_R) \to H_n^G(G/G; \boldsymbol{K}_R) = K_n(\mathcal{H}(G;R)).$$

(ii) *There is a (strongly convergent) spectral sequence*
$$E^2_{p,q} = SH_p^{G,\mathcal{I}}\big(E_{\mathcal{C}\mathrm{op}}(G); \overline{K_q(\mathcal{H}(?;R))}\big) \Rightarrow K_{p+q}(\mathcal{H}(G;R)),$$
whose E^2-term is concentrated in the first quadrant.

(iii) *The canonical map induced by the various inclusions $K \subseteq G$,*
$$\operatorname*{colim}_{K \in \mathrm{Sub}_{\mathcal{I}}(G)} K_0(\mathcal{H}(K;R)) \to K_0(\mathcal{H}(G;R)),$$
can be identified with the isomorphism appearing in assertion (i) in degree $n = 0$ and hence is bijective.

(iv) *We have $K_n(\mathcal{H}(G;R)) = 0$ for $n \le -1$.*

We proved assertion (i) of Theorem 1.1 in [2, Corollary 1.8]. So this paper deals with implications of it concerning computations of the algebraic K-groups $K_n(\mathcal{H}(G))$ of the Hecke algebra of G.

A *td-group* G is a locally compact, second countable, totally disconnected topological Hausdorff group. It is *modulo a normal compact subgroup a subgroup of a reductive* p*-adic group* if it contains a (not necessarily open) normal compact subgroup K such that G/K is isomorphic to a subgroup of some reductive p-adic group.

A ring is called *uniformly regular* if it is Noetherian and there exists a natural number l such that any finitely generated R-module admits a resolution by projective R-modules of length at most l. We write $\mathbb{Q} \subseteq R$ if, for any integer n, the element $n \cdot 1_R$ is a unit in R. Examples of uniformly regular rings R with $\mathbb{Q} \subseteq R$ are fields of characteristic zero.

We denote by $\mathcal{H}(G; R)$ the *Hecke algebra* consisting of locally constant functions $s\colon G \to R$ with compact support, where the additive structure comes from the additive structure of R and the multiplicative structure from the convolution product. Note that $\mathcal{H}(G; R)$ is a ring without unit.

We denote by $E_{\mathcal{C}\mathrm{op}}(G)$ a model for the *classifying space for proper smooth* G*-actions*, ie a G-CW-complex, whose isotropy groups are all compact open subgroups of G and for which $E_{\mathcal{C}\mathrm{op}}(G)^H$ is weakly contractible for any compact open subgroup $H \subseteq G$. Two such models are G-homotopy equivalent. Hence $H_n^G(E_{\mathcal{C}\mathrm{op}}(G); \boldsymbol{K}_R)$ is independent of the choice of a model. If G is a reductive p-adic group with compact center, then its Bruhat–Tits building is a model for $E_{\mathcal{C}\mathrm{op}}(G)$. If the center is not compact, one has to pass to the extended Bruhat–Tits building.

We will construct a *smooth* G*-homology theory* $H_*^G(-; \boldsymbol{K}_R)$ in Section 3. It assigns to a smooth G-CW-pair (X, A) a collection of abelian groups $\mathcal{H}_n^G(X, A; \boldsymbol{K}_R)$ for $n \in \mathbb{Z}$ that satisfies the expected axioms, ie long exact sequence of a pair, G-homotopy invariance, excision, and the disjoint union axiom. Moreover, for every open subgroup $U \subseteq G$ and $n \in \mathbb{Z}$, we have

$$H_n^G(G/U; \boldsymbol{K}_R) \cong K_n(\mathcal{H}(U; R)). \tag{1.2}$$

Let $\mathcal{F}$ be a collection of open subgroups of G which is closed under conjugation. Examples are the set $\mathcal{C}\mathrm{op}$ of compact open subgroups of G and the set $\mathcal{I}$ of isotropy groups of points of some model for $E_{\mathcal{C}\mathrm{op}}(G)$. The subgroup category $\mathsf{Sub}_{\mathcal{F}}(G)$ appearing in Theorem 1.1(iii) has $\mathcal{F}$ as set of objects and will be described in detail in Section 2.A.

The abelian groups $SH_p^{G,\mathcal{F}}\big(E_{\mathcal{F}}(G); \overline{K_q(\mathcal{H}(?; R))}\big)$ appearing in Theorem 1.1(ii) will be defined for the covariant functor $\overline{K_q(\mathcal{H}(?; R))}\colon \mathsf{Sub}_{\mathcal{F}}(G) \to \mathbb{Z}\text{-}\mathsf{Mod}$, whose value at $U \in \mathcal{F}$ is $K_n(\mathcal{H}(U; R))$, in Section 2.B. They are closely related to the *Bredon homology groups* $BH_p^{G,\mathcal{F}}\big(E_{\mathcal{F}}(G); K_q(\mathcal{H}(?; R))\big)$.

The proof of Theorem 1.1 will be given in Section 4.

The relevance of the Hecke algebra $\mathcal{H}(G; R)$ is that the category of nondegenerate modules over it is isomorphic to the category of smooth G-representations with coefficients in R; see for instance

Bernstein [5] or Garrett [13]. Hence, in particular, its projective class group $K_0(\mathcal{H}(G;R))$ is important. The various inclusions $K \to G$ for $K \in \mathcal{C}\mathrm{op}$ induce a map

(1.3)
$$\bigoplus_{K \in \mathcal{C}\mathrm{op}} K_0(\mathcal{H}(K;R)) \to K_0(\mathcal{H}(G;R)),$$

which factors over the canonical epimorphism $\bigoplus_{K \in \mathcal{C}\mathrm{op}} K_0(\mathcal{H}(K;R)) \to \operatorname{colim}_{K \in \mathrm{Sub}_{\mathcal{I}}(G)} K_0(\mathcal{H}(K;R))$ to the isomorphism appearing in Theorem 1.1(iii) and is hence surjective. Dat [10] has shown that the map (1.3) is rationally surjective for G a reductive p-adic group and $R = \mathbb{C}$. In particular, the cokernel of it is a torsion group. Dat [9, Conjecture 1.11] conjectured that this cokernel is $\widetilde{w}_G$-torsion. Here $\widetilde{w}_G$ is a certain multiple of the order of the Weyl group of G. Dat [9, Proposition 1.13] proved this conjecture for $G = \mathrm{GL}_n(F)$ for a p-adic field F of characteristic zero and asked about the integral version — see the comment following [9, Proposition 1.10] — which is now proven by Theorem 1.1(iii).

The computations simplify considerably in the case of a reductive p-adic group thanks to the associated (extended) Bruhat–Tits building; see Sections 5 and 7. As an illustration we analyze the projective class groups of the Hecke algebras of $\mathrm{SL}_n(F)$, $\mathrm{PGL}_n(F)$ and $\mathrm{GL}_n(F)$ in Section 6.

One of our main tools will be the *smooth equivariant Atiyah–Hirzebruch spectra sequence*, which we will establish and examine in Section 2.

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2 The smooth equivariant Hirzebruch spectral sequence

Throughout this section we fix a set $\mathcal{F}$ of open subgroups of G which is closed under conjugation. Our main examples for $\mathcal{F}$ are the family $\mathcal{O}\mathrm{p}$ of all open subgroups and the family $\mathcal{C}\mathrm{op}$ of all compact open subgroups. An $\mathcal{F}$-G-CW-*complex* X is a G-CW-complex X such that, for every $x \in X$, its isotropy group G_x belongs to $\mathcal{F}$. A smooth G-CW-complex is the same as an $\mathcal{O}\mathrm{p}$-CW-complex and a proper smooth G-CW-complex is the same as a $\mathcal{C}\mathrm{op}$-CW-complex. Let $\mathcal{H}_*^G$ be a smooth G-homology theory.

The main result of this section is:

Theorem 2.1 *Consider a pair (X,A) of $\mathcal{F}$-G-CW-complexes and a smooth G-homology theory $\mathcal{H}_*^G$. Then there is an equivariant Atiyah–Hirzebruch spectral sequence converging to $\mathcal{H}_{p+q}^G(X,A)$, whose E^2-term is given by*

$$E_{p,q}^2 = BH_p^{G,\mathcal{F}}(X,A;\mathcal{H}_q^G)$$

for the Bredon homology $BH_p^{G,\mathcal{F}}(X,A;\mathcal{H}_q^G)$ of (X,A) with coefficients in the covariant $\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)$-module $\mathcal{H}_q^G$ that sends G/H to $\mathcal{H}_q^G(G/H)$.

The remainder of this section is devoted to the definition of the Bredon homology, the construction of the equivariant Atiyah–Hirzebruch spectral sequence, and some general calculations concerning the E^2-term. Convergence means that there is an ascending filtration $F_{l,m-l}\mathcal{H}_m^G(X,A)$ for $l=0,1,2,\dots$ of $\mathcal{H}_m^G(X,A)$ such that $F_{p,q}\mathcal{H}_{p+q}^G(X,A)/F_{p-1,q+1}\mathcal{H}_{p+q}^G(X,A)\cong E_{p,q}^\infty$ for $E_{p,q}^\infty=\operatorname{colim}_{r\to\infty} E_{p,q}^r$.

2.A The smooth orbit category and the smooth subgroup category

The *$\mathcal{F}$-orbit category* $\mathrm{Or}_{\mathcal{F}}(G)$ has as objects homogeneous G-spaces G/H with $H\in\mathcal{F}$. Morphisms from G/H to G/K are G-maps $G/H\to G/K$. We will put no topology on $\mathrm{Or}_{\mathcal{F}}(G)$. For any G-map $f\colon G/H\to G/K$ of smooth homogeneous spaces, there is an element $g\in G$ such that $gHg^{-1}\subseteq K$ and f is the G-map $R_{g^{-1}}\colon G/H\to G/K$ sending $g'H$ to $g'g^{-1}K$. Given two elements $g_0,g_1\in G$ such that $g_iHg_i^{-1}\subseteq K$ for $i=0,1$, we have $R_{g_0^{-1}}=R_{g_1^{-1}}$ if and only if $g_1g_0^{-1}\in K$. We get a bijection

$$K\backslash\{g\in G\mid gHg^{-1}\subseteq K\}\xrightarrow{\cong}\operatorname{map}_G(G/H,G/K),\quad g\mapsto R_{g^{-1}}. \tag{2.2}$$

The *$\mathcal{F}$-subgroup category* $\mathrm{Sub}_{\mathcal{F}}(G)$ has $\mathcal{F}$ as the set of objects. For $H,K\in\mathcal{F}$, denote by $\operatorname{conhom}_G(H,K)$ the set of group homomorphisms $f\colon H\to K$ for which there exists an element $g\in G$ with $gHg^{-1}\subset K$ such that f is given by conjugation with g, ie $f=c(g)\colon H\to K$, $h\mapsto ghg^{-1}$. Note that $c(g)=c(g')$ holds for two elements $g,g'\in G$ with $gHg^{-1}\subset K$ and $g'Hg'^{-1}\subset K$ if and only if $g^{-1}g'$ lies in the centralizer $C_GH=\{g\in G\mid gh=hg \text{ for all } h\in H\}$ of H in G. The group of inner automorphisms $\mathrm{Inn}(K)$ of K acts on $\operatorname{conhom}_G(H,K)$ from the left by composition. Define the set of morphisms

$$\operatorname{mor}_{\mathrm{Sub}_{\mathcal{C}\mathrm{op}}(G)}(H,K):=\mathrm{Inn}(K)\backslash\operatorname{conhom}_G(H,K).$$

There is an obvious bijection

$$K\backslash\{g\in G\mid gHg^{-1}\subseteq K\}/C_GH\xrightarrow{\cong}\mathrm{Inn}(K)\backslash\operatorname{conhom}_G(H,K),\quad KgC_GH\mapsto[c(g)], \tag{2.3}$$

where $[c(g)]\in\mathrm{Inn}(K)\backslash\operatorname{conhom}_G(H,K)$ is the class represented by the element $c(g)\colon H\to K$, $h\mapsto ghg^{-1}$, in $\operatorname{conhom}_G(H,K)$ and K acts from the left and C_GH from the right on $\{g\in G\mid gHg^{-1}\subseteq K\}$ by the multiplication in G.

Let

$$P\colon\mathrm{Or}_{\mathcal{F}}(G)\to\mathrm{Sub}_{\mathcal{F}}(G) \tag{2.4}$$

be the canonical projection which sends an object G/H to H and is given on morphisms by the obvious projection under the identifications (2.2) and (2.3).

2.B Cellular chain complexes and Bredon homology

Given an $\mathcal{F}$-G-CW-complex X, we obtain a contravariant $\mathtt{Or}_{\mathcal{F}}(G)$-space $O_X\colon \mathtt{Or}_{\mathcal{F}}(G) \to \mathtt{Spaces}$ by sending G/H to $\mathrm{map}_G(G/H, X) = X^H$. We get a contravariant $\mathtt{Sub}_{\mathcal{F}}(G)$-space $S_X\colon \mathtt{Sub}_{\mathcal{F}}(G) \to \mathtt{Spaces}$ by sending H to $C_G H\backslash\mathrm{map}_G(G/H, X) = C_G H\backslash X^H$. A morphism $H \to K$ given by an element $g \in G$ satisfying $gHg^{-1} \subseteq K$ is sent to the map $C_G K\backslash X^K \to C_G H\backslash X^H$ induced by the map $X^K \to X^H$, $x \mapsto g^{-1}x$.

Given a pair (Y, A) with a filtration $A = Y_{-1} \subseteq Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y$ with $Y = \mathrm{colim}_{n\to\infty} Y_n$, we associate to it a $\mathbb{Z}$-chain complex $C_*^c(Y, A)$, whose n^{th} chain module is the singular homology $H_n^{\mathrm{sing}}(Y_n, Y_{n-1})$ of the pair (Y_n, Y_{n-1}) (with coefficients in $\mathbb{Z}$) and whose n^{th} differential is given by the composite

$$H_n^{\mathrm{sing}}(Y_n, Y_{n-1}) \xrightarrow{\partial_n} H_{n-1}^{\mathrm{sing}}(Y_{n-1}) \xrightarrow{H_{n-1}^{\mathrm{sing}}(i_{n-1})} H_{n-1}^{\mathrm{sing}}(Y_{n-1}, Y_{n-2})$$

for ∂_n the boundary operator of the pair (Y_n, Y_{n-1}) and $i_{n-1}\colon Y_{n-1} = (Y_{n-1}, \varnothing) \to (Y_{n-1}, Y_{n-2})$ the inclusion.

Given a pair of $\mathcal{F}$-G-CW-complexes (X, A), the filtration by its skeletons induces filtrations on X^H and $C_G H\backslash X^H$ for every $H \in \mathcal{F}$. We get a contravariant $\mathbb{Z}\mathtt{Or}_{\mathcal{F}}(G)$-chain complex $C_*^{\mathtt{Or}_{\mathcal{F}}(G)}(X, A)\colon \mathtt{Or}_{\mathcal{F}}(G) \to \mathbb{Z}$-$\mathtt{Ch}$ and a contravariant $\mathbb{Z}\mathtt{Sub}_{\mathcal{F}}(G)$-chain complex $C_*^{\mathtt{Sub}_{\mathcal{F}}(G)}(X, A)\colon \mathtt{Sub}_{\mathcal{F}}(G) \to \mathbb{Z}$-$\mathtt{Ch}$ by putting

$$\begin{aligned} C_*^{\mathtt{Or}_{\mathcal{F}}(G)}(X, A)(G/H) &:= C_*^c(O_X(G/H), O_A(G/H)) = C_*^c(X^H, A^H), \\ C_*^{\mathtt{Sub}_{\mathcal{F}}(G)}(X, A)(H) &:= C_*^c(S_X(X)(H), S_A(H)) = C_*^c(C_G H\backslash X^H, C_G H\backslash A^H). \end{aligned}$$

Choose a G-pushout

(2.5)
$$\begin{tikzcd} \coprod_{i\in I_n} G/H_i \times S^{n-1} \arrow[r, "\coprod_{i\in I_n} q_i^n"] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_{i\in I_n} G/H_i \times D^n \arrow[r, "\coprod_{i\in I_n} Q_i^n"'] & X_{n-1} \end{tikzcd}$$

It induces, for every closed subgroup $H \subseteq G$, pushouts

$$\begin{tikzcd} \coprod_{i\in I_n} (G/H_i)^H \times S^{n-1} \arrow[r, "\coprod_{i\in I_n} (q_i^n)^H"] \arrow[d] & X_{n-1}^H \arrow[d] \\ \coprod_{i\in I_n} (G/H_i)^H \times D^n \arrow[r, "\coprod_{i\in I_n} (Q_i^n)^H"'] & X_{n-1}^H \end{tikzcd}$$

and

$$\begin{tikzcd} \coprod_{i\in I_n} C_G H\backslash(G/H_i)^H \times S^{n-1} \arrow[r, "\coprod_{i\in I_n} C_G H\backslash(q_i^n)^H"] \arrow[d] & C_G H\backslash X_{n-1}^H \arrow[d] \\ \coprod_{i\in I_n} C_G H\backslash(G/H_i)^H \times D^n \arrow[r, "\coprod_{i\in I_n} C_G H\backslash(Q_i^n)^H"'] & C_G H\backslash X_{n-1}^H \end{tikzcd}$$

Note that $(G/H_i)^H$ agrees with $\mathrm{mor}_{\mathrm{Or}_{\mathcal{F}}(G)}(G/H, G/H_i) = \mathrm{map}_G(G/H, G/H_i)$. In the sequel, for a set S we denote by $\mathbb{Z}S$ the free $\mathbb{Z}$-module with basis S. Since singular homology satisfies the disjoint union axiom, homotopy invariance and excision, we obtain an isomorphism of contravariant $\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)$-modules

$$\bigoplus_{i \in I_n} \mathbb{Z}\mathrm{mor}_{\mathrm{Or}_{\mathcal{F}}(G)}(?, G/H_i) \xrightarrow{\cong} C_n^{\mathrm{Or}_{\mathcal{F}}(G)}(X, A), \tag{2.6}$$

where $\mathbb{Z}\mathrm{mor}_{\mathrm{Or}_{\mathcal{F}}(G)}(?, G/H_i)$ is the free $\mathbb{Z}\mathrm{Or}(G)$-module based at the object G/H_i [14, Example 9.8 on page 164], and analogously an isomorphism of contravariant $\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)$-modules

$$\bigoplus_{i \in I_n} \mathbb{Z}\mathrm{mor}_{\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)}(?, H_i) \xrightarrow{\cong} C_n^{\mathrm{Sub}_{\mathcal{F}}(G)}(X, A). \tag{2.7}$$

If $P_* C_*^{\mathrm{Or}_{\mathcal{F}}(G)}(X, A)$ is the $\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)$-chain complex obtained by induction with $P\colon \mathrm{Or}_{\mathcal{F}}(G) \to \mathrm{Sub}_{\mathcal{F}}(G)$ from $C_*^{\mathrm{Or}_{\mathcal{F}}(G)}(X, A)$ — see [14, Example 9.15 on page 166] — we conclude from (2.6) and (2.7) that the canonical map of $\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)$-chain complexes

$$P_* C_*^{\mathrm{Or}_{\mathcal{F}}(G)}(X, A) \xrightarrow{\cong} C_*^{\mathrm{Sub}_{\mathcal{F}}(G)}(X, A) \tag{2.8}$$

is an isomorphism.

For a covariant $\mathbb{Z}\mathrm{Or}(G)$-module M, we get from the tensor product over $\mathrm{Or}_{\mathcal{F}}(G)$ — see [14, 9.13 on page 166] — a $\mathbb{Z}$-chain complex $C_*^{\mathrm{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)} M$.

Definition 2.9 (Bredon homology) We define the n^{th} *Bredon homology* to be the $\mathbb{Z}$-module

$$BH_n^{G,\mathcal{F}}(X, A; M) = H_n(C_*^{\mathrm{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)} M).$$

Given a covariant $\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)$-module N, define analogously

$$SH_n^{G,\mathcal{F}}(X, A; N) = H_n(C_*^{\mathrm{Sub}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)} N).$$

Given a covariant $\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)$-module N, define the covariant $\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)$-module P^*N to be $N \circ P$. We get from the adjunction of [14, 9.22 on page 169] and (2.8) a natural isomorphism of $\mathbb{Z}$-chain complexes

$$C_*^{\mathrm{Sub}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)} N \xrightarrow{\cong} C_*^{\mathrm{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)} P^*N \tag{2.10}$$

and hence a natural isomorphism of $\mathbb{Z}$-modules

$$BH_n^{G,\mathcal{F}}(X, A; P^*N) \xrightarrow{\cong} SH_n^{G,\mathcal{F}}(X, A; N). \tag{2.11}$$

Let (X, A) be a pair of $\mathcal{F}$-CW-complexes. Denote by $\mathcal{I}$ the set of isotropy groups of points in X. Let M be a covariant $\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)$-module and N be a covariant $\mathrm{Sub}_{\mathcal{F}}(G)$-module. Denote by $M|_{\mathcal{I}}$ and $N|_{\mathcal{I}}$ their restrictions to $\mathrm{Or}_{\mathcal{I}}(G)$ and $\mathrm{Sub}_{\mathcal{I}}(G)$. Then one easily checks using [11, Lemma 1.9] that there are canonical isomorphisms

$$BH_n^{G,\mathcal{I}}(X, A; M|_{\mathcal{I}}) \cong BH_n^{G,\mathcal{F}}(X, A; M), \tag{2.12}$$

$$SH_n^{G,\mathcal{I}}(X, A; N|_{\mathcal{I}}) \cong BH_n^{G,\mathcal{F}}(X, A; N). \tag{2.13}$$

2.C The construction of the equivariant Atiyah–Hirzebruch spectral sequence

Proof of Theorem 2.1 Since (X, A) comes with the skeletal filtration, there is by a general construction a spectral sequence

$$E^r_{p,q}, \quad d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

converging to $\mathcal{H}^G_{p+q}(X, A)$, whose E_1-term is given by

$$E^1_{p,q} = \mathcal{H}^G_{p+q}(X_p, X_{p-1}),$$

and the first differential is the composite

$$d^1_{p,q}\colon E^1_{p,q} = \mathcal{H}^G_{p+q}(X_p, X_{p-1}) \to \mathcal{H}^G_{p+q-1}(X_{p-1}) \to \mathcal{H}^G_{p+q-1}(X_{p-1}, X_{p-2}) = E^1_{p-1,q},$$

where the first map is the boundary operator of the pair (X_p, X_{p-1}) and the second is induced by the inclusion. The elementary construction is explained for trivial G for instance in [18, 15.6 on page 339]. The construction carries directly over to the equivariant setting.

The straightforward proof of the identification of $E^2_{p,q}$ with $BH^{G,\mathcal{F}}_p(X, A; \mathcal{H}_q)$ is left to the reader. □

2.D Passing to the subgroup category

Condition 2.14 $(\mathrm{Sub}|_{\mathcal{F}})$ Let $\mathcal{H}^G_*(-)$ be a smooth G-homology theory. Then $\mathcal{H}^G_*(-)$ satisfies the condition $(\mathrm{Sub}|_{\mathcal{F}})$ if, for any $H \in \mathcal{F}$ and $g \in C_G H$, the G-map $R_{g^{-1}}\colon G/H \to G/H$ sending $g'H$ to $g'g^{-1}H$ induces the identity on $\mathcal{H}^G_q(G/H)$, ie $\mathcal{H}^G_q(R_{g^{-1}}) = \mathrm{id}_{\mathcal{H}^G_q(G/H)}$.

Remark 2.15 Suppose that the G-homology theory $\mathcal{H}^G_*$ satisfies the condition $(\mathrm{Sub}|_{\mathcal{F}})$. Then the covariant $\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)$-module $\mathcal{H}^G_q$ sending G/H with $H \in \mathcal{F}$ to $\mathcal{H}^G_q(G/H)$ defines a covariant $\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)$-module $\overline{\mathcal{H}^G_q}\colon \mathrm{Sub}_{\mathcal{F}}(G) \to \mathbb{Z}\text{-}\mathrm{Mod}$ uniquely determined by $\mathcal{H}^G_q = \overline{\mathcal{H}^G_q} \circ P$ for the projection $P\colon \mathrm{Or}_{\mathcal{F}}(G) \to \mathrm{Sub}_{\mathcal{F}}(G)$. Moreover, we obtain from (2.11), for every pair (X, A) of $\mathcal{F}$-G-CW-complexes, natural isomorphisms

$$BH^{G,\mathcal{F}}_n(X, A; \mathcal{H}^G_q(-)) \xrightarrow{\cong} SH^{G,\mathcal{F}}_n(X, A; \overline{\mathcal{H}^G_q(-)}).$$

Note that the right-hand side is often easier to compute than the left-hand side. One big advantage of $\mathrm{Sub}(G)$ in comparison with $\mathrm{Or}(G)$ is that, for a finite subgroup $H \subseteq G$, the set of automorphisms of H is the group $N_G H/H \cdot C_G H$, which is finite, whereas the set of automorphisms of G/H in $\mathrm{Or}(G)$ for a finite group H is the group $N_G H/H$, which is not necessarily finite. This is a key ingredient in the construction of an equivariant Chern character for discrete groups G and proper G-CW-complexes in [15; 16].

If G is abelian, $\mathrm{Sub}_{\mathcal{F}}(G)$ reduces to the poset of open subgroups of G ordered by inclusion.

2.E The connective case

Theorem 2.16 (i) *Suppose that $\mathcal{H}_q^G(G/H) = 0$ for every $H \in \mathcal{F}$ and $q \in \mathbb{Z}$ with $q < 0$. Then we get, for every pair (X, A) of $\mathcal{F}$-G-CW-complexes and every $q \in \mathbb{Z}$ with $q < 0$,*

$$\mathcal{H}_q^G(X, A) = 0.$$

(ii) *Choose a model $E_{\mathcal{C}\mathrm{op}}(G)$ for the classifying space of smooth proper G-actions. Let $\mathcal{I}$ be the set of isotropy groups of points in $E_{\mathcal{C}\mathrm{op}}(G)$. Suppose that $\mathcal{H}_q^G(G/H) = 0$ for every open $H \in \mathcal{I}$ and $q \in \mathbb{Z}$ with $q < 0$.*

(a) *Then, for every $q < 0$, we have $\mathcal{H}_q^G(E_{\mathcal{C}\mathrm{op}}(G)) = 0$, the edge homomorphism induces an isomorphism*

$$BH_0^G(E_{\mathcal{C}\mathrm{op}}(G); \mathcal{H}_q^G(-)) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{F}}(G))$$

and the canonical map

$$\operatorname*{colim}_{G/H \in \mathrm{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G(G/H) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{F}}(G))$$

is bijective.

(b) *Suppose additionally that $\mathcal{H}_*^G$ satisfies condition $(\mathrm{Sub}_{\mathcal{I}})$; see Condition 2.14. Then the edge homomorphism induces an isomorphism*

$$SH_0^G(E_{\mathcal{C}\mathrm{op}}(G); \overline{\mathcal{H}_q^G(-)}) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{F}}(G))$$

and the canonical map

$$\operatorname*{colim}_{H \in \mathrm{Sub}_{\mathcal{I}}(G)} \overline{\mathcal{H}_0^G}(H) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{C}\mathrm{op}}(G))$$

is bijective.

Proof (i) This follows directly from the smooth equivariant Atiyah–Hirzebruch spectral sequence of Theorem 2.1.

(a) We get $\mathcal{H}_q^G(E_{\mathcal{C}\mathrm{op}}(G)) = 0$ for $q < 0$ from assertion (i).

We get from the smooth equivariant Atiyah–Hirzebruch spectral sequence of Theorem 2.1 an isomorphism

$$BH_0^{G,\mathcal{I}}(E_{\mathcal{C}\mathrm{op}}(G); \mathcal{H}_0^G) = H_0(C_*^{\mathrm{Or}_{\mathcal{I}}(G)}(E_{\mathcal{C}\mathrm{op}}(G)) \otimes_{\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{C}\mathrm{op}}(G))$$

since $E_{p,q}^2 = BH_0^{G,\mathcal{I}}(E_{\mathcal{C}\mathrm{op}}(G); \mathcal{H}_q^G) = 0$ is valid for $p, q \in \mathbb{Z}$ if $p < 0$ or $q < 0$. Since the $\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)$-module $C_n^{\mathrm{Or}_{\mathcal{I}}(G)}(E_{\mathcal{C}\mathrm{op}}(G))$ is free in the sense of [14, 9.16 on page 167] for $n \geq 0$ by (2.6) and $E_{\mathcal{C}\mathrm{op}}(G)^H$ is weakly contractible for $H \in \mathcal{I}$, the $\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)$-chain complex $C_*^{\mathrm{Or}_{\mathcal{I}}(G)}(E_{\mathcal{C}\mathrm{op}}(G))$ is a projective $\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)$-resolution of the constant contravariant $\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)$-module $\underline{\mathbb{Z}}$, whose value is $\mathbb{Z}$ at each object and which assigns to any morphism $\mathrm{id}_{\mathbb{Z}}$. Since $- \otimes_{\mathbb{Z}\otimes_{\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)}} \mathcal{H}_q^G$ is right exact by [14, 9.23 on page 169], we get an isomorphism

$$H_0(C_*^{\mathrm{Or}_{\mathcal{I}}(G)}(E_{\mathcal{C}\mathrm{op}}(G)) \otimes_{\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G) \cong \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G.$$

We conclude from the adjunction appearing in [14, 9.21 on page 169] and the universal property of the colimit that there is a canonical isomorphism

$$\operatorname*{colim}_{G/H\in \mathrm{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G(G/H) \cong \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G.$$

This finishes the proof of assertion (a).

(b) This follows from assertion (a), since we get from ($\mathrm{Sub}_{\mathcal{I}}$) a canonical isomorphism

$$\operatorname*{colim}_{G/H\in \mathrm{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G(G/H) \xrightarrow{\cong} \operatorname*{colim}_{H\in \mathrm{Sub}_{\mathcal{I}}(G)} \overline{\mathcal{H}_q^G}(H)$$

for the covariant $\mathbb{Z}\mathrm{Sub}_{\mathcal{I}}(G)$-module $\overline{\mathcal{H}_q^G}$ determined by the covariant $\mathbb{Z}\mathrm{Or}_{\mathcal{I}}(G)$-module $\mathcal{H}_q^G$; see Remark 2.15. □

2.F The first differential

Let X be an $\mathcal{F}$-G-CW-complex. Suppose that $X_0 = \coprod_{j\in J} G/V_j$ and that X_1 is given by the G-pushout

(2.17)
$$\begin{tikzcd}
\coprod_{i\in I} G/U_i \times S^0 \arrow[r, "\coprod_{i\in I_n} q_i"] \arrow[d] & X_0 \arrow[d] \\
\coprod_{i\in I} G/U_i \times D^1 \arrow[r, "\coprod_{i\in I} Q_i"'] & X_1
\end{tikzcd}$$

We want to figure out the map of $\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)$-modules γ making the diagram

$$\begin{tikzcd}
\bigoplus_{i\in I} \mathbb{Z}\mathrm{mor}_{\mathrm{Or}_{\mathcal{F}}(G)}(?, G/U_i) \arrow[r, "\gamma"] \arrow[d, "\cong"'] & \bigoplus_{j\in J} \mathbb{Z}\mathrm{mor}_{\mathrm{Or}_{\mathcal{F}}(G)}(?, G/V_j) \arrow[d, "\cong"] \\
C_1^{\mathrm{Or}_{\mathcal{F}}(G)}(X) \arrow[r, "c_n"'] & C_0^{\mathrm{Or}_{\mathcal{F}}(G)}(X)
\end{tikzcd}$$

commute, where the vertical isomorphisms come from the isomorphisms (2.6). In order to describe γ, we have to define for each $i \in I$ and $j \in J$ a map of $\mathbb{Z}\mathrm{Or}(G)$-modules

$$\gamma_{i,j} : \mathbb{Z}\mathrm{mor}_{\mathrm{Or}_{\mathcal{F}}(G)}(?, G/H_i) \to \mathbb{Z}\mathrm{mor}_{\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)}(?, G/K_j)$$

such that $\{j \in I_{n-1} \mid \gamma_{i,j} \neq 0\}$ is finite for every $i \in I_n$. Note that $\gamma_{i,j}$ is determined by the image of id_{G/H_i}. Hence we need to specify for $i \in I$ and $j \in J$ an element

(2.18)
$$\bar{\gamma}_{i,j} \in \mathbb{Z}\mathrm{mor}_{\mathrm{Or}_{\mathcal{F}}(G)}(G/U_i, G/V_j) = \mathbb{Z}\mathrm{map}_G(G/U_i, G/V_j).$$

For each $i \in I$, there are two elements $j_-(i)$ and $j_+(i)$ in J such that the image of $G/H_i \times \{\pm 1\}$ under the map q_i appearing in (2.17) is the summand $G/K_{j_\pm}(i)$ belonging to $j_\pm(i)$ of $\coprod_{j\in I_0} G/K_j$ if we write $S^0 = \{-1, 1\}$. Denote by $(q_i^1)_{\pm 1} : G/H_i \to G/K_{j_\pm}$ the restriction of q_i^1 to $G/H_i \times \{\pm 1\}$. We leave the elementary proof of the next lemma to the reader.

Lemma 2.19 *In* $\mathbb{Z}\mathrm{map}_G(G/H_i, G/K_j)$,

$$\bar{\gamma}_{i,j} = \begin{cases} \pm[(q_i^1)_{\pm 1}] & \text{if } j = j_\pm(i) \text{ and } j_-(i) \neq j_+(i), \\ [(q_i^1)_{+1}] - [(q_i^1)_{-1}] & \text{if } j = j_-(i) = j_+(i), \\ 0 & \text{if } j \notin \{j_-(i), j_+(i)\}. \end{cases}$$

Remark 2.20 This implies, for the $\mathbb{Z}$-chain complex $C_*^{\mathrm{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathrm{Or}_{\mathcal{F}}(G)} M$ for a covariant $\mathbb{Z}\mathrm{Or}_{\mathcal{F}}(G)$-module M, that its first differential agrees with the $\mathbb{Z}$-homomorphism

$$\alpha = (\alpha_{i,j})_{i \in I, j \in J} \colon \bigoplus_{i \in I} M(G/U_i) \to \bigoplus_{j \in J} M(G/V_j),$$

where the $\mathbb{Z}$-homomorphisms $\alpha_{i,j} : M(G/U_i) \to M(G/V_j)$ are given, in the notation of Lemma 2.19, by

$$\alpha_{i,j} = \begin{cases} \pm M((q_i^1)_\pm) & \text{if } j = j_\pm(i) \text{ and } j_-(i) \neq j_+(i), \\ M((q_i^1)_{+1}) - M((q_i^1)_{-1}) & \text{if } j = j_-(i) = j_+(i), \\ 0 & \text{if } j \notin \{j_-(i), j_+(i)\}. \end{cases}$$

Note that the cokernel of α is $BH_0^{G,\mathcal{F}}(X; M)$.

We get a computation of the first differential of $C_*^{\mathrm{Sub}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)} N$ for a covariant $\mathbb{Z}\mathrm{Sub}_{\mathcal{F}}(G)$-module N from the isomorphism (2.10). Explicitly, the first differential is given by

$$\beta = (\beta_{i,j})_{i \in I, j \in J} \colon \bigoplus_{i \in I_n} N(U_i) \to \bigoplus_{j \in I_{n-1}} N(V_j),$$

where the $\mathbb{Z}$-homomorphisms $\beta_{i,j} : N(G/U_i) \to N(G/V_j)$ are given as follows. Choose for the map $(q_i)_\pm \colon G/U_i \to G/V_j$ an element $(g_i)_\pm$ with $(q_i)_\pm(eU_i) = (g_i)_\pm^{-1} V_j$. Let $[c(g_i)_\pm] \colon U_i \to V_j$ be the morphism in $\mathrm{Sub}_{\mathcal{F}}(G)$ represented by $c(g_i)_\pm \colon U_i \to V_j$ sending u to gug^{-1}. Then

$$\beta_{i,j} = \begin{cases} \pm N([c(g_i)_\pm]) & \text{if } j = j_\pm(i) \text{ and } j_-(i) \neq j_+(i), \\ N([c(g_i)_+]) - N([c(g_i)_-]) & \text{if } j = j_-(i) = j_+(i), \\ 0 & \text{if } j \notin \{j_-(i), j_+(i)\}. \end{cases}$$

Note that the cokernel of β is $SH_0^{G,\mathcal{F}}(X; N)$.

3 A brief review of the Farrell–Jones conjecture for the algebraic K-theory of Hecke algebras

In this section we give a review of the Farrell–Jones conjecture for the algebraic K-theory of Hecke algebras. Further information can be found in [4; 2].

Let R be a (not necessarily commutative) associative unital ring with $\mathbb{Q} \subseteq R$. Let G be a td-group. Let $\mathcal{H}(G; R)$ be the associated Hecke algebra.

One can construct a covariant functor

$$\boldsymbol{K}_R \colon \mathrm{Or}_{\mathcal{O}\mathrm{p}}(G) \to \mathsf{Spectra}$$

such that $\pi_n(\boldsymbol{K}_R(Q'/U')) \cong K_n(\mathcal{H}(U;R))$ for any $n \in \mathbb{Z}$ and open subgroup $U \subseteq Q$. Associated to it is a smooth G-homology theory $H_*^G(-;\boldsymbol{K}_R)$ such that

(3.1) $$H_n^G(G/U;\boldsymbol{K}_R) \cong K_n(\mathcal{H}(U;R))$$

for every $n \in \mathbb{Z}$ and every open subgroup $U \subseteq Q$.

The next result follows from [2, Corollary 1.8].

Theorem 3.2 *Let G be a td-group which is modulo a normal compact subgroup a subgroup of a reductive p-adic group. Let R be a uniformly regular ring with $\mathbb{Q} \subseteq R$.*

Then the map induced by the projection $E_{\mathcal{C}\mathrm{op}}(G) \to G/G$ induces for every $n \in \mathbb{Z}$ an isomorphism

$$H_n^G(E_{\mathcal{C}\mathrm{op}}(G);\boldsymbol{K}_R) \xrightarrow{\cong} H_n^G(G/G;\boldsymbol{K}_R) = K_n(\mathcal{H}(G;R)).$$

4 Proof of Theorem 1.1

Proof of Theorem 1.1 (i) This is exactly Theorem 3.2.

(ii) Since an open group homomorphism $U \to V$ between two td-groups induces a ring homomorphism $\mathcal{H}(U;R) \to \mathcal{H}(V;R)$ between the Hecke algebras and hence a homomorphism $K_n(\mathcal{H}(U;R)) \to K_n(\mathcal{H}(V;R))$, and inner automorphisms of a td-group U induce the identity on $K_n(\mathcal{H}(U;R))$, we get a covariant $\mathbb{Z}\mathsf{Sub}_{\mathcal{C}\mathrm{om}}(G)$-module $K_n(\mathcal{H}(?;R))$ whose value at U is $K_n(\mathcal{H}(U;R))$. Since the isomorphism (3.1) is natural, we get an isomorphism of covariant $\mathbb{Z}\mathsf{Or}_{\mathcal{O}\mathrm{p}}(G)$-modules

$$P^*K_n(\mathcal{H}(?;R)) \xrightarrow{\cong} \pi_n(\boldsymbol{K}_R)$$

for the projection $P\colon \mathsf{Or}_{\mathcal{O}\mathrm{p}}(G) \to \mathsf{Sub}_{\mathcal{O}\mathrm{p}}(G)$ of (2.4). So the smooth equivariant Atiyah–Hirzebruch spectral sequence applied to the smooth homology theory $H_*^G(-;\boldsymbol{K}_R)$ takes, for an $\mathcal{F}$-G-CW-complex X, the form

(4.1) $$E^2_{p,q} = SH_q^{G,\mathcal{F}}\big(X;K_q(\mathcal{H}(?;R))\big) \Rightarrow H^G_{p+q}(X;\boldsymbol{K}_R).$$

Now assertion (ii) follows from the special case $X = E_{\mathcal{C}\mathrm{op}}(G)$ and assertion (i).

(iii)–(iv) As $K_q(\mathcal{H}(K;R))$ vanishes for every compact td-group K and every $q \leq -1$ [4, Lemma 8.1], assertions (iii)–(iv) follow from Theorem 2.16 applied in the case $X = E_{\mathcal{C}\mathrm{op}}(G)$ and from assertion (i). □

5 The main recipe for the computation of the projective class group

Throughout this section, G will be a td-group and R a uniformly regular ring with $\mathbb{Q} \subseteq R$, eg a field of characteristic zero. We will assume that the assembly map $H_n^G(E_{\mathcal{C}\mathrm{op}}(G);\boldsymbol{K}_R) \to H_n^G(G/G;\boldsymbol{K}_R) = K_n(\mathcal{H}(G;R))$ is bijective for all $n \in \mathbb{Z}$. This is known to be true for subgroups of reductive p-adic groups by Theorem 3.2.

5.A The general case

Let X be an abstract simplicial complex with a simplicial G-action such that all isotropy groups are compact open, the G-action is cellular, and $|X|^K$ is nonempty and connected for every compact open subgroup K of G.

We can choose a subset V of the set of vertices of X such that the G-orbit through any vertex in X meets V in precisely one element. Fix a total ordering on V. Let E be the subset of $V \times V$ consisting of those pairs (v,w) such that $v \le w$ and there exists $g \in G$ for which v and gw satisfy $v \ne gw$ and span an edge $[v,gw]$ in X. For $(v,w) \in E$, define $\overline{F(v,w)}$ to be the subset of $G_v\backslash G/G_w$ consisting of elements x for which v and gw satisfy $v \ne gw$ and span an edge $[v,gw]$ in X for some (and hence all) representative g of x. Choose a subset $F(v,w)$ of G such that the projection $G \to G_v\backslash G/G_w$ induces a bijection $F(v,w) \to \overline{F(v,w)}$.

Then, for every edge of X, the G-orbit through it meets the set $\{[v,gv] \mid (v,w) \in E,\ g \in F(v,w)\}$ in precisely one element. Moreover, the 0-skeleton of $|X|$ is given by $|X|_0 = \coprod_{u\in V} G/G_u$ and $|X|_1$ is given by the G-pushout

$$\begin{tikzcd}
\coprod_{(v,w)\in E}\coprod_{g\in F(v,w)} G/(G_v\cap G_{gw})\times S^0 \arrow[r, "\coprod_{(v,w)\in E}\coprod_{g\in F(v,w)} q_{(v,w),g}"] \arrow[d] & {|X|_0} \arrow[d] \\
\coprod_{(v,w)\in E}\coprod_{g\in F(v,w)} G/(G_v\cap G_{gw})\times D^1 \arrow[r] & {|X|_1}
\end{tikzcd}$$

where $q_{(v,w),g}\colon G/(G_v\cap G_{gw})\times S^0 \to |X|_0 = \coprod_{u\in V} G/G_u$ is defined as follows. Write $S^0=\{-1,1\}$. The restriction of $q_{(v,w),g}$ to $G/(G_v\cap G_{gw})\times\{-1\}$ lands in the summand G/G_v and is given by the canonical projection. The restriction of $q_{(v,w),g}$ to $G/(G_v\cap G_{gw})\times\{1\}$ lands in the summand G/G_w and is given by the G-map $R_{g^{-1}}\colon G/(G_v\cap G_{gw}) \to G/G_w$ sending $z(G_v\cap G_{gw})$ to zgG_w.

Next we define a map

$$\beta=(\beta_{(v,w),g,u})\colon \bigoplus_{(v,w)\in E}\ \bigoplus_{g\in F(v,w)} K_0(\mathcal{H}(G_v\cap G_{gw};R)) \to \bigoplus_{u\in V} K_0(\mathcal{H}(G_u;R)).$$

If $u=v$, then $\beta_{(v,w),g,v}\colon K_0(\mathcal{H}(G_v\cap G_{gw};R)) \to K_0(\mathcal{H}(G_v;R))$ is the map induced by the inclusion $G_v\cap G_{gw}\to G_v$ multiplied with -1. If $u=w$, then $\beta_{(v,w),g,w}\ K_0(\mathcal{H}(G_v\cap G_{gw};R)) \to K_0(\mathcal{H}(G_w;R))$ is the map induced by the group homomorphism $G_v\cap G_{gw}\to G_w$ sending z to $g^{-1}zg$. If $u\notin\{v,w\}$, then $\beta_{(v,w),g,u}=0$.

Lemma 5.1 *The cokernel of β is isomorphic to $K_0(\mathcal{H}(G;R))$.*

Proof We conclude from Remark 2.20 that the cokernel of β is $SH_0^{G,\mathcal{C}\mathrm{op}}(X;\overline{K_0^G(-)})$. The up-to-$G$-homotopy unique G-map $f\colon X\to E_{\mathcal{C}\mathrm{op}}(G)$ induces for every compact open subgroup $K\subset G$ a 1-connected map $f^K\colon |X|^K \to E_{\mathcal{C}\mathrm{op}}(G)^K$. This implies that the map $SH_0^{G,\mathcal{C}\mathrm{op}}(X;\overline{K_0^G(-)}) \to SH_0^{G;\mathcal{C}\mathrm{op}}(E_{\mathcal{C}\mathrm{op}}(G);\overline{K_0^G(-)})$ induced by f is an isomorphism; see [14, Proposition 23(iii) on page 35].

Theorem 2.16(b) implies $SH_0^G(E_{\mathcal{C}\mathrm{op}}(G); \overline{K_0^G(-)}) \cong H_0^G(E_{\mathcal{C}\mathrm{op}}(G); \boldsymbol{K}_R)$. Since by assumption we have $H_0^G(E_{\mathcal{C}\mathrm{op}}(G); \boldsymbol{K}_R) \cong K_0(\mathcal{H}(G;R))$, Lemma 5.1 follows. □

Remark 5.2 Suppose additionally that X possesses a strict fundamental domain Δ, ie a simplicial subcomplex Δ that contains exactly one simplex from each orbit for the G-action on the set of simplices of X. Then one can take V to be the set of vertices of Δ and, for $(v,w) \in E$, the set $F(v,w)$ to be $\{e\}$. Moreover, β reduces to the map

$$\beta = (\beta_{(v,w,u)})\colon \bigoplus_{(v,w)\in E} K_0(\mathcal{H}(G_v \cap G_w; R)) \to \bigoplus_{u\in V} K_0(\mathcal{H}(G_u; R)),$$

where $\beta_{(v,w),u}$ is the map induced by the inclusion $G_v \cap G_w \to G_v$ multiplied with -1 for $u = v$, the map induced by the inclusion $G_v \cap G_w \to G_w$ for $u = w$, and zero for $u \notin \{v,w\}$. Note that E is the subset of $V \times V$ consisting of elements (v,w) for which $v < w$ holds and v and w span an edge $[v,w]$ in Δ.

5.B A variation

Consider a central extension $1 \to \widetilde{C} \to \widetilde{G} \xrightarrow{\mathrm{pr}} G \to 1$ of td-groups together with a group homomorphism $\mu\colon \widetilde{G} \to \mathbb{Z}$ such that $\widetilde{C} \cap \widetilde{M}$ is compact for $\widetilde{M} := \ker(\mu)$. We still consider the abstract simplicial complex X of Section 5.A coming with a simplicial G-action such that all isotropy groups are compact open, and $|X|^K$ is nonempty and connected for every compact open subgroup K of G. Furthermore, we will assume that the assembly map $H_n^{\widetilde{G}}(E_{\mathcal{C}\mathrm{op}}(\widetilde{G}); \boldsymbol{K}_R) \to H_n^{\widetilde{G}}(\widetilde{G}/\widetilde{G}; \boldsymbol{K}_R) = K_0(\mathcal{H}(\widetilde{G};R))$ is bijective for all $n \in \mathbb{Z}$.

If $\widetilde{C}$ is compact, then we can consider X as a $\widetilde{G}$-CW-complex by restricting the G-action with pr and Section 5.A applies. Hence we will assume that $\widetilde{C}$ is not compact, or, equivalently, that $\widetilde{C}$ is not contained in the kernel $\widetilde{M} := \ker(\mu)$. Then the index $m := [\mathbb{Z} : \mu(C)]$ is a natural number $m \geq 1$. We fix an element $\tilde{c} \in \widetilde{C}$ with $\mu(\tilde{c}) = m$. In the sequel, we choose for every $g \in G$ an element $\tilde{g}$ in $\widetilde{G}$ satisfying $\mathrm{pr}(\tilde{g}) = g$ and denote, for an open subgroup $U \subseteq G$, by $\widetilde{U} \subseteq \widetilde{G}$ its preimage under $\mathrm{pr}\colon \widetilde{G} \to G$. Let

$$\gamma\colon \bigoplus_{(v,w)\in E} \bigoplus_{g\in F(v,w)} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R)) \to \bigoplus_{u\in V} K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R))$$

be the map whose component for $(v,w) \in E$, $g \in F(v,w)$ and $u \in V$ is the map

$$\gamma_{(v,w),g,u}\colon K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R)) \to K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R)) \tag{5.3}$$

defined next. If $u = v$, it is the map coming from the inclusion $\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M} \to \widetilde{G}_v \cap \widetilde{M}$ multiplied with -1. If $u = w$, it is the map coming from the group homomorphism $\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M} \to \widetilde{G}_w \cap \widetilde{M}$ sending x to $\tilde{g}x\tilde{g}^{-1}$. If $u \notin \{v,w\}$, it is trivial. Note that this definition is independent of the choice of $\tilde{g} \in \widetilde{G}$ satisfying $\mathrm{pr}(\tilde{g}) = g$ for $g \in F(v,w)$.

Lemma 5.4 *The cokernel of γ is $K_0(\mathcal{H}(\widetilde{G};R))$.*

Proof Note that $|X| \times \mathbb{R}$ carries the $G\times\mathbb{Z}$-CW-complex structure coming from the product of the G-CW-complex structure on $|X|$ and the standard free $\mathbb{Z}$-CW-structure on $\mathbb{R}$. Since the $\mathbb{Z}$-CW-complex $\mathbb{R}$

has precisely one equivariant 1-cell and one equivariant 0-cell, the set of equivariant 0-cells of the $G\times\mathbb{Z}$-CW-complex $|X|\times\mathbb{R}$ can be identified with the set V and the set of equivariant 1-cells can be identified with the disjoint union of V and the set $\coprod_{(v,w)\in E} F(v,w)$. Now the 0-skeleton of $|X|\times\mathbb{R}$ is given by the disjoint union $\coprod_{u\in V} \widetilde{G}/\widetilde{G}_u\times\mathbb{Z}$ and the 1-skeleton of $|X|\times\mathbb{R}$ is given by the $G\times\mathbb{Z}$-pushout

(5.5)
$$\begin{array}{ccc}
\coprod_{v\in V} \widetilde{G}/\widetilde{G}_v\times\mathbb{Z}\times S^0 \amalg \coprod_{(v,w)\in E}\coprod_{g\in F(v,w)} \widetilde{G}/(\widetilde{G}_v\cap\widetilde{G}_{gw})\times\mathbb{Z}\times S^0 & \xrightarrow{\tilde{q}} & \coprod_{u\in V} \widetilde{G}/\widetilde{G}_u\times\mathbb{Z} \\
\downarrow & & \downarrow \\
\coprod_{v\in V} \widetilde{G}/\widetilde{G}_v\times\mathbb{Z}\times D^1 \amalg \coprod_{(v,w)\in E}\coprod_{g\in F(v,w)} \widetilde{G}/(\widetilde{G}_v\cap\widetilde{G}_{gw})\times\mathbb{Z}\times S^0 & \longrightarrow & (|X|\times\mathbb{R})_1
\end{array}$$

where $\tilde{q}$ is given as follows. Write $S^0=\{-1,1\}$. Fix $u\in V$. The restriction of $\tilde{q}$ to the summand $\widetilde{G}/\widetilde{G}_v\times\mathbb{Z}\times\{\epsilon\}$ lands in the summand $\widetilde{G}/\widetilde{G}_v\times\mathbb{Z}$ and is given by id for $\epsilon=-1$ and by $\mathrm{id}\times\mathrm{sh}_1$ for $\epsilon=1$, where $\mathrm{sh}_a\colon\mathbb{Z}\to\mathbb{Z}$ sends b to $a+b$ for $a,b\in\mathbb{Z}$. Fix $(v,w)\in E$ and $g\in F(v,w)$. The restriction of $\tilde{q}$ to the summand $\widetilde{G}/(\widetilde{G}_v\cap\widetilde{G}_{gw})\times\mathbb{Z}\times\{-1\}$ belonging to (v,w) and g lands in the summand for $u=v$ and is the canonical projection $\widetilde{G}/(\widetilde{G}_v\cap\widetilde{G}_{gw})\times\mathbb{Z}\to\widetilde{G}/\widetilde{G}_v\times\mathbb{Z}$. The restriction of $\tilde{q}$ to the summand $\widetilde{G}/(\widetilde{G}_v\cap\widetilde{G}_{gw})\times\mathbb{Z}\times\{1\}$ belonging to (v,w) and g lands in the summand for $u=w$ and is the map $R_{\tilde{g}^{-1}}\times\mathrm{id}_{\mathbb{Z}}\colon\widetilde{G}/(\widetilde{G}_v\cap\widetilde{G}_{gw})\times\mathbb{Z}\to\widetilde{G}/\widetilde{G}_w\times\mathbb{Z}$, where $R_{\tilde{g}^{-1}}$ sends $\tilde{z}(\widetilde{G}_v\cap\widetilde{G}_{gw})$ to $\tilde{z}\tilde{g}^{-1}\widetilde{G}_w$.

We have the group homomorphism

$$\iota:=\mathrm{pr}\times\mu\colon\widetilde{G}\to G\times\mathbb{Z}.$$

Its kernel is $\widetilde{C}\cap\widetilde{M}$. Its image has finite index in $G\times\mathbb{Z}$, which agrees with the index m of the image of μ in $\mathbb{Z}$.

We are interested in the $\widetilde{G}$-CW-complex $\iota^*(|X|\times\mathbb{R})$ obtained by restriction with ι from the $G\times\mathbb{Z}$-CW-complex $|X|\times\mathbb{R}$. So we have to analyze how the $G\times\mathbb{Z}$-cells in $\iota^*(|X|\times\mathbb{R})$ viewed as $\widetilde{G}$-spaces decompose as disjoint unions of $\widetilde{G}$-cells. Consider any open subgroup $U\subseteq G$. Then we obtain a $\widetilde{G}$-homeomorphism

$$\alpha(U)\colon\coprod_{p=0}^{m-1}\widetilde{G}/(\widetilde{U}\cap\widetilde{M})\xrightarrow{\cong}\iota^*(G/U\times\mathbb{Z})$$

by sending the element $\tilde{z}(\widetilde{U}\cap\widetilde{M})$ in the p^{th} summand to $(\mathrm{pr}(\tilde{z})U,\mu(\tilde{z})+p)$. Next we have to analyze the naturality properties of $\alpha(U)$. The diagram, for $a\in\mathbb{Z}$,

$$\begin{array}{ccc}
\coprod_{p=0}^{m-1}\widetilde{G}/(\widetilde{U}\cap\widetilde{M}) & \xrightarrow{\alpha(U)} & \iota^*(G/U\times\mathbb{Z}) \\
{\scriptstyle\hat{\pi}^a}\downarrow & & \downarrow{\scriptstyle\mathrm{id}\times\mathrm{sh}_a} \\
\coprod_{p=0}^{m-1}\widetilde{G}/(\widetilde{U}\cap\widetilde{M}) & \xrightarrow[\alpha(U)]{} & \iota^*(G/U\times\mathbb{Z})
\end{array}$$

commutes, where $\hat{\pi}$ sends the summand for $p=0,\dots,m-2$ by the identity to the summand for $p+1$ and sends the summand for $p=m-1$ to the summand for $p=0$ by the map $R_{\tilde{c}}\colon\widetilde{G}/(\widetilde{U}\cap\widetilde{M})\to\widetilde{G}/(\widetilde{U}\cap\widetilde{M})$ for $\tilde{c}\in\widetilde{C}$ satisfying $\mu(\tilde{c})=m$. Note for the sequel that the endomorphism $\pi_n(\boldsymbol{K}_R(R_{\tilde{c}}))$

of $\pi_n(\boldsymbol{K}_R(\widetilde{G}/\widetilde{U} \cap \widetilde{M})) = K_0(\mathcal{H}(\widetilde{U} \cap \widetilde{M}))$ is the identity, since conjugation with $\tilde{c}$ induces the identity on $\widetilde{U} \cap \widetilde{M}$.

Consider two open subgroups U and V of G and an element $g \in G$ with $gUg^{-1} \subseteq V$. Then we get well-defined $\widetilde{G}$-maps $R_{\tilde{g}^{-1}}\colon \widetilde{G}/(\widetilde{U} \cap \widetilde{M}) \to \widetilde{G}/(\widetilde{V} \cap \widetilde{M})$ sending $\tilde{z}(\widetilde{U} \cap \widetilde{M})$ to $\tilde{z}\tilde{g}^{-1}(\widetilde{V} \cap \widetilde{M})$ and $R_{g^{-1}} \times \mathrm{id}\colon \iota^*(G/U \times \mathbb{Z}) \to \iota^*(G/V \times \mathbb{Z})$ sending (zU, n) to $(zg^{-1}V, n)$ and the following diagram commutes:

$$\begin{tikzcd}
\coprod_{p=0}^{m-1} \widetilde{G}/(\widetilde{U} \cap \widetilde{M}) \arrow[rr, "\alpha(U)", "\cong"'] \arrow[d, "\coprod_{p=0}^{m-1} R_{\tilde{g}^{-1}}"'] & & \iota^*(G/U \times \mathbb{Z}) \arrow[d, "R_{g^{-1}} \times \mathrm{sh}_{\mu(\tilde{g}^{-1})}"] \\
\coprod_{p=0}^{m-1} \widetilde{G}/(\widetilde{V} \cap \widetilde{M}) \arrow[rr, "\cong", "\alpha(V)"'] & & \iota^*(G/V \times \mathbb{Z})
\end{tikzcd}$$

In particular the following diagram commutes:

$$\begin{tikzcd}
\coprod_{p=0}^{m-1} \widetilde{G}/(\widetilde{U} \cap \widetilde{M}) \arrow[rr, "\alpha(U)", "\cong"'] \arrow[d, "\pi^{\mu(\tilde{g})} \circ (\coprod_{p=0}^{m-1} R_{\tilde{g}^{-1}})"'] & & \iota^*(G/U \times \mathbb{Z}) \arrow[d, "R_{g^{-1}} \times \mathrm{id}"] \\
\coprod_{p=0}^{m-1} \widetilde{G}/(\widetilde{V} \cap \widetilde{M}) \arrow[rr, "\cong", "\alpha(V)"'] & & \iota^*(G/V \times \mathbb{Z})
\end{tikzcd}$$

Now we obtain from the $G \times \mathbb{Z}$-pushout (5.5) by applying restriction with ι and the maps α_U above a $\widetilde{G}$-pushout describing how the 1-skeleton of the $\widetilde{G}$-CW-complex $\iota^*(|X| \times \mathbb{R})$ is obtained from its 0-skeleton and explicit descriptions of the attaching maps.

In the sequel A^m stands for the m-fold direct sum of copies of A for an abelian group A and $\pi\colon A^m \to A^m$ denotes the permutation map sending $(a_1, a_2, \dots, a_m)$ to $(a_m, a_1, \dots, a_{m-1})$ and $\mathrm{aug}\colon A^m \to A$ denotes the augmentation map sending $(a_1, \dots, a_m)$ to $a_1 + \dots + a_m$.

Let δ be the map given by the direct sum

$$\delta = \bigoplus_{v \in V} \delta_v \colon \bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \to \bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m,$$

where $\delta_v\colon K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \to K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m$ is $\pi - \mathrm{id}$. Let

$$\epsilon\colon \bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R))^m \to \bigoplus_{u \in V} K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R))^m$$

be the map given by the components $\epsilon_{(v,w),g,u}$ defined as follows. For $u = v$, the map $\epsilon_{(v,w),g,v}$ is the m-fold direct sum $\gamma^m_{(v,w),g,v}$ of the maps $\gamma_{(v,w),g,v}$ defined in (5.3). For $u = w$, we put

$$\epsilon_{(v,w),g,w}\colon K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R))^m \xrightarrow{\gamma^m_{(v,w),g,u}} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \xrightarrow{\pi^{\mu(\tilde{g})}} K_0(\mathcal{H}(\widetilde{G}_w \cap \widetilde{M}; R))^m.$$

Since $\pi^m = \mathrm{id}$, the map $\pi^{\mu(\tilde{g})}$ depends only on $\bar{\mu}(g)$, where $\bar{\mu}\colon G \to \mathbb{Z}/m$ sends g to the image of $\tilde{g}$ under the projection $\mathbb{Z} \to \mathbb{Z}/m$ for any choice of an element $\tilde{g} \in \widetilde{G}$ with $\mathrm{pr}(\tilde{g}) = g$.

The cokernel of the map

$$\delta \oplus \epsilon \colon \Big(\bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \Big) \oplus \Big(\bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R))^m \Big) \\ \to \bigoplus_{u \in V} K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R))^m$$

is $K_0(\mathcal{H}(\widetilde{G}; R))$ because of Theorem 2.16(b) and Remark 2.20, by the same argument as appears in the proof of Lemma 5.1, since $(\iota^*(|X| \times \mathbb{R}))^K$ is connected for every compact open subgroup K of $\widetilde{G}$. It does not matter that $\iota^*(|X| \times \mathbb{R})$ is a $\widetilde{G}$-CW-complex but not a simplicial complex, since in the description of $\beta_{i,j}$ appearing in Remark 2.20 the case $j_i(+) = j_-(i)$ never occurs.

We can identify $\bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))$ and the cokernel of δ, since we have the exact sequence $A^m \xrightarrow{\pi - \mathrm{id}} A^m \xrightarrow{\alpha} A \to 0$ for every abelian group A. The cokernel of $\delta \oplus \epsilon$ is isomorphic to the cokernel of the composite of ϵ with the map

$$\bigoplus_{v \in V} \mathrm{aug} \colon \bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \to \bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R)) = \mathrm{cok}(\delta).$$

For every $(v,w) \in E$, $g \in F(v,w)$ and $u \in V$, the diagram

$$\begin{tikzcd} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R))^m \arrow[r, "\epsilon_{(v,w),u}"] \arrow[d, "\mathrm{aug}"'] & K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R))^m \arrow[d, "\mathrm{aug}"] \\ K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R)) \arrow[r, "\gamma_{(v,w),u}"'] & K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R)) \end{tikzcd}$$

commutes, since $\alpha \circ \pi = \alpha$. □

6 The projective class group of the Hecke algebras of $\mathrm{SL}_n(F)$, $\mathrm{PGL}_n(F)$ and $\mathrm{GL}_n(F)$

Next we apply the recipes of Section 5 to some prominent reductive p-adic groups G as an illustration. For the remainder of this section R is a uniformly regular ring with $\mathbb{Q} \subseteq R$.

Note that, for a reductive p-adic group G, the assembly map $H_n^G(E_{\mathcal{C}\mathrm{op}}(G); \boldsymbol{K}_R) \to H_n^G(G/G; \boldsymbol{K}_R) = K_n(\mathcal{H}(G; R))$ is bijective for all $n \in \mathbb{Z}$ by Theorem 3.2. Moreover, the Bruhat–Tits building X of G or of $G/\mathrm{cent}(G)$ can serve as the desired simplicial complex X appearing in Section 5. The original construction of the Bruhat–Tits building can be found in [8]. For more information about buildings, we refer to [1; 6; 7; 17]. The space X carries a CAT(0)-metric, which is invariant under the action of G or $G/\mathrm{cent}(G)$; see [6, Theorem 10A.4 on page 344]. Hence $|X|^H$ is contractible for any compact open subgroup H of G or $G/\mathrm{cent}(G)$, since X^H is a convex nonempty subset of X and hence contractible by [6, Corollary II.2.8 on page 179]. Therefore the geometric realization of the Bruhat–Tits building X is (after possibly subdividing to achieve a cellular action) a model for $E_{\mathcal{C}\mathrm{op}}(G)$ or $E_{\mathcal{C}\mathrm{op}}(G/\mathrm{cent}(G))$.

6.A $\mathrm{SL}_n(F)$

We begin with computing $K_0\bigl(\mathcal{H}(\mathrm{SL}_n(F);R)\bigr)$, where F is a non-Archimedean local field with valuation $v\colon F\to\mathbb{Z}\cup\{\infty\}$. The following claims about the Bruhat–Tits building X for $\mathrm{SL}_n(F)$ (and later about X') can all be verified from the description of X in [1, Section 6.9].

For $l=0,\dots,n-1$, let U_l^{S} be the compact open subgroup of $\mathrm{SL}_n(F)$ consisting of all matrices (a_{ij}) in $\mathrm{SL}_n(F)$ satisfying $v(a_{i,j})\geq -1$ for $1\leq i\leq n-l<j\leq n$, $v(a_{i,j})\geq 1$ for $1\leq j\leq n-l<i\leq n$ and $v(a_{i,j})\geq 0$ for all other i and j. In particular, $U_0^{\mathrm{S}}=\mathrm{SL}_n(\mathcal{O})$, where $\mathcal{O}=\{z\in F\mid v\geq 0\}$. The intersection of the U_l^{S} is the Iwahori subgroup I^{S} of $\mathrm{SL}_n(F)$. It is given by those matrices A in $\mathrm{SL}_n(F)$ for which $v(a_{i,j})\geq 1$ for $i>j$ and $v(a_{i,j})\geq 0$ for $i\leq j$.

The $(n-1)$-simplex Δ can be chosen with an ordering on its vertices such that the isotropy group of its l^{th} vertex v_l is U_l^{S}. The isotropy group of a face σ of Δ is the intersection of the isotropy groups of the vertices of σ. In particular, the isotropy group of Δ is the Iwahori subgroup I^S of $\mathrm{SL}_n(F)$. Consider the map

$$d^{\mathrm{SL}_n(F)}\colon \bigoplus_{0\leq i<j\leq n-1} K_0(\mathcal{H}(U_i^{\mathrm{S}}\cap U_j^{\mathrm{S}};R))\to\bigoplus_{0\leq l\leq n-1} K_0(\mathcal{H}(U_l^{\mathrm{S}};R))$$

for which the component $d_{i<j,l}^{\mathrm{SL}_n(F)}\colon K_0(\mathcal{H}(U_i^{\mathrm{S}}\cap U_j^{\mathrm{S}};R))\to K_0(\mathcal{H}(U_l^{\mathrm{S}};R))$ is given by $-K_0(\mathcal{H}(f_{i<j}^{i};R))$ if $l=i$, by $K_0(\mathcal{H}(f_{i<j}^{j};R))$ if $l=j$, and zero if $l\notin\{i,j\}$, where $f_{i<j}^{k}\colon U_i^{\mathrm{S}}\cap U_j^{\mathrm{S}}\to U_k^{\mathrm{S}}$ is the inclusion for $k=i,j$.

Then the cokernel of $d^{\mathrm{SL}_n(F)}$ is $K_0\bigl(\mathcal{H}(\mathrm{SL}_n(F);R)\bigr)$ by Lemma 5.1 and Remark 5.2.

6.B $\mathrm{PGL}_n(F)$

Next we compute $K_0\bigl(\mathcal{H}(\mathrm{PGL}_n(F);R)\bigr)$. The action of $\mathrm{SL}_n(F)$ on X extends to an action of $\mathrm{GL}_n(F)$. This action factors through the canonical projection $\mathrm{pr}\colon\mathrm{GL}_n(F)\to\mathrm{PGL}_n(F)$ to an action of $\mathrm{PGL}_n(F)$. These actions are still simplicial, but no longer cellular. Let

$$\hat{h}:=\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1\\ \zeta & & & \end{pmatrix}\in\mathrm{GL}_n(F),$$

where we choose a uniformizer $\zeta\in F$, ie an element in F satisfying $v(\zeta)=1$. Obviously $\hat{h}^n$ is the diagonal matrix $\zeta\cdot I_n$, all of whose diagonal entries are ζ, and hence is central in $\mathrm{GL}_n(F)$. Define $h\in\mathrm{PGL}_n(F)$ by $h=\mathrm{pr}(\hat{h})$. Then $hv_l=v_{l+1}$ for $l=0,\dots,n-2$ and $hv_{n-1}=v_0$ and h^n is the unit in $\mathrm{PGL}_n(F)$. In particular, the action of $\mathrm{PGL}_n(F)$ is transitive on the vertices of X. To obtain a cellular action, X can be subdivided to X' as follows. The $(n-2)$-skeleton of X is unchanged, while the $(n-1)$-simplices of X are replaced in X' with cones on their boundary. More formally, the vertices of X' are the vertices of X and the barycenters b_σ of $(n-1)$-simplices σ of X. A set S of vertices of X' is a simplex of X' if and only if either S is a k-simplex of X and $k<n-1$ or S contains exactly one barycenter b_σ and

all $v \in S \setminus \{b_\sigma\}$ are vertices of σ (in the simplicial structure of X). The action of $\mathrm{PGL}_n(F)$ on X' is then cellular and is transitive on $(n-1)$-simplices of X'. There are two orbits of vertices, represented by v_0 and b_Δ. Let $k := \lfloor \frac{1}{2} n \rfloor$. There are $k+1$ orbits of 1-simplices, represented by $\{v_0, v_1\}, \dots, \{v_0, v_k\}$ and $\{v_0, b_\Delta\}$. Next we describe some isotropy groups.

For an open subgroup $W \subseteq \mathrm{PGL}_n(F)$, let $\widetilde{W}$ be its preimage under the projection $\mathrm{pr}\colon \mathrm{GL}_n(F) \to \mathrm{PGL}_n(F)$. For $l = 0, \dots, n-1$, let U_l^{G} be the compact open subgroup of $\mathrm{GL}_n(F)$ given by $\hat{h}^l \, \mathrm{GL}_n(\mathcal{O}) \hat{h}^{-l} = \widetilde{\mathrm{PGL}_n(F)_{v_l}} = \widetilde{\mathrm{PGL}_n(F)_{h_l v_0}}$. In particular, $U_0^{\mathrm{G}} = \mathrm{GL}_n(\mathcal{O})$. Note that

$$U_l^{\mathrm{G}} \cap \mathrm{SL}_n(F) = (\hat{h}^l \, \mathrm{GL}_n(\mathcal{O}) \hat{h}^{-l}) \cap \mathrm{SL}_n(F) = \hat{h}^l \, \mathrm{SL}_n(\mathcal{O}) \hat{h}^{-l} = U_l^S.$$

The intersection of the U_l^{G} is the Iwahori subgroup I^{G} of $\mathrm{GL}_n(F)$. Let U_l^{P} be the image of U_l^{G} in $\mathrm{PGL}_n(F)$. This is the isotropy group of the vertex v_l for the action of $\mathrm{PGL}_n(F)$. The Iwahori subgroup I^{P} of $\mathrm{PGL}_n(F)$ is the image of I^{G} under pr. It is the pointwise isotropy subgroup for Δ. Let H be the subgroup generated by the image of h in $\mathrm{PGL}_n(F)$. It is a cyclic subgroup of order n that cyclically permutes the vertices of Δ. This subgroup normalizes I^{P} and the isotropy group of b_Δ is the product HI^{P}. Recall that $v_l = h^l v_0$ and hence $U_l^P = h^l U_0 P h^{-l}$.

Write $i_H\colon I^{\mathrm{P}} \to HI^{\mathrm{P}}$, $i_0\colon I^{\mathrm{P}} \to U_0^{\mathrm{P}}$, $c_0\colon U_0^{\mathrm{P}} \cap U_i^{\mathrm{P}} \to U_0^{\mathrm{P}}$ for the inclusions and define $c_l\colon U_0^{\mathrm{P}} \cap U_l^{\mathrm{P}} \to U_0^{\mathrm{P}}$ by $z \mapsto h^{-l} z h^l$. Let

$$d^{\mathrm{PGL}_n(F)}\colon K_0(\mathcal{H}(I^{\mathrm{P}}; R)) \oplus \bigoplus_{l=1}^{k} K_0(\mathcal{H}(U_0^{\mathrm{P}} \cap U_l^{\mathrm{P}}; R)) \to K_0(\mathcal{H}(HI^{\mathrm{P}}; R)) \oplus K_0(\mathcal{H}(U_0^{\mathrm{P}}; R))$$

be the map that is $K_0(i_H) \times -K_0(i_0)$ on $K_0(\mathcal{H}(I^{\mathrm{P}}; R))$ and $0 \times (K_0(c_l) - K_0(c_0))$ on $K_0(\mathcal{H}(U_0^{\mathrm{P}} \cap U_l^{\mathrm{P}}; R))$. The cokernel of the homomorphism $d^{\mathrm{PGL}_n(F)}$ agrees with $SH_0^{\mathrm{PGL}_n(F)}\big(X'; K_0(\mathcal{H}(?; R))\big)$ by Lemma 5.1 if, using the notation of Section 5.A, we put $E = \{v_0, b_\Delta\}$ with $v_0 < b_\Delta$, $F(v_0.v_0) = \{h, h^2, \dots, h^k\}$ and $F(v_0, b_\Delta) = \{e\}$.

6.C $\mathrm{GL}_n(F)$

Next we compute $K_0\big(\mathcal{H}(\mathrm{GL}_n(F); R)\big)$. Note that $\mathrm{GL}_n(F)$ has a noncompact center. Hence Section 5.A does not apply and we have to pass to the setting of Section 5.B using the short exact sequence $1 \to C = \mathrm{cent}(\mathrm{GL}_n(F)) \to \mathrm{GL}_n(F) \xrightarrow{\mathrm{pr}} \mathrm{PGL}_n(F) \to 1$, the discussion in Section 6.B and Lemma 5.4.

Let $\widetilde{M}$ be the kernel of the composite $\mu\colon \mathrm{GL}_n(F) \xrightarrow{\det} F^\times \xrightarrow{\nu} \mathbb{Z}$. Let $\hat{H} \subseteq \mathrm{GL}_n(F)$ be the infinite cyclic subgroup generated by the element $\hat{h}$. Note that $\widetilde{M} \cap C$ consists of those diagonal matrices whose entries on the diagonal are all the same and are sent to 0 under ν. We conclude $(\mathrm{GL}_n(\mathcal{O}) \cdot C) \cap \widetilde{M} = \mathrm{GL}_n(\mathcal{O})$ from $C \cap \widetilde{M} \subseteq \mathrm{GL}_n(\mathcal{O}) \subseteq \widetilde{M}$. Recall that, for $W \subseteq \mathrm{PGL}_n(F)$, we denote by $\widetilde{W}$ its preimage under $\mathrm{pr}\colon \mathrm{GL}_n(F) \to \mathrm{PGL}_n(F)$. Since $\mathrm{pr}(U_l^{\mathrm{G}}) = U_l^{\mathrm{P}}$, we get, for $l = 0, \dots, n-1$,

$$\widetilde{U_l^{\mathrm{P}}} \cap \widetilde{M} = (U_l^{\mathrm{G}} \cdot C) \cap \widetilde{M} = (\hat{h}^l \, \mathrm{GL}_n(\mathcal{O}) \hat{h}^{-l} \cdot C) \cap \widetilde{M} = \hat{h}^l \big((\mathrm{GL}_n(\mathcal{O}) \cdot C) \cap \widetilde{M}\big) \hat{h}^{-l} = \hat{h}^l \, \mathrm{GL}_n(\mathcal{O}) \hat{h}^{-l} = U_l^{\mathrm{G}}.$$

Now one easily checks $\widetilde{I^{\mathrm{P}}} \cap \widetilde{M} = I^{\mathrm{G}}$. Finally we show $\widetilde{HI^{\mathrm{P}}} \cap \widetilde{M} = I^{\mathrm{G}}$. We get $I^{\mathrm{G}} \subseteq \widetilde{HI^{\mathrm{P}}} \cap \widetilde{M}$ from $\widetilde{I^{\mathrm{P}}} \cap \widetilde{M} = I^{\mathrm{G}}$. Consider an element $A \in \widetilde{HI^{\mathrm{P}}} \cap \widetilde{M}$. We can find an integer b, an element $B \in I^{\mathrm{G}}$, and an element $D \in C$ such that $A = \hat{h}^b BD$ and $\nu(A) = 0$. From $I^{\mathrm{G}} \subseteq \widetilde{M}$, we conclude $\hat{h}^b D \in \widetilde{M}$. Since $\mu(D)$ is divisible by n and $\mu(\hat{h}) = 1$ holds, b is divisible by n. This implies $\hat{h}^b \in C$ and hence $\hat{h}^b D \in C \cap \widetilde{M}$. As $(C \cap \widetilde{M})I^{\mathrm{G}} = I^{\mathrm{G}}$, we conclude $A \in I^{\mathrm{G}}$. Hence $\widetilde{HI^{\mathrm{P}}} \cap \widetilde{M} = I^{\mathrm{G}}$.

Let $\tilde{\imath}_0 \colon I^{\mathrm{G}} \to U_0^{\mathrm{G}}$ and $\tilde{c}_0 \colon U_0^{\mathrm{G}} \cap U_i^{\mathrm{G}} \to U_0^{\mathrm{G}}$ be the inclusions and let $\tilde{c}_l \colon U_0^{\mathrm{G}} \cap U_l^{\mathrm{G}} \to U_0^{\mathrm{G}}$ be the map sending $\tilde{z}$ to $\hat{h}^{-l}\tilde{z}\hat{h}^l$. Let

$$\bar{d}^{\mathrm{GL}_n(F)} \colon K_0(\mathcal{H}(I^{\mathrm{G}}; R)) \oplus \bigoplus_{l=1}^{k} K_0(\mathcal{H}(U_0^{\mathrm{G}} \cap U_l^{\mathrm{G}}; R)) \to K_0(\mathcal{H}(I^{\mathrm{G}}; R)) \oplus K_0(\mathcal{H}(U_0^{\mathrm{G}}; R))$$

be the map that is $\mathrm{id}_{K_0(I^{\mathrm{G}})} \times - K_0(\tilde{\imath}_0)$ on $K_0(\mathcal{H}(I^{\mathrm{G}}; R))$ and $0 \times (K_0(\tilde{c}_l) - K_0(\tilde{c}_0))$ on $K_0(\mathcal{H}(U_0^{\mathrm{G}} \cap U_i^{\mathrm{G}}; R))$. The cokernel of the map $\bar{d}^{\mathrm{GL}_n(F)}$ is $K_0\big(\mathcal{H}(\mathrm{GL}_n(F); R)\big)$ by Lemma 5.4. Let

$$\tilde{d}^{\mathrm{GL}_n(F)} \colon \bigoplus_{l=1}^{k} K_0(\mathcal{H}(U_0^{\mathrm{G}} \cap U_l^{\mathrm{G}}; R)) \to K_0(\mathcal{H}(U_0^{\mathrm{G}}; R))$$

be the map which is given by $K_0(\tilde{c}_l) - K_0(\tilde{c}_0)$ on $K_0(\mathcal{H}(U_0^{\mathrm{G}} \cap U_l^{\mathrm{G}}; R))$. Since $\tilde{d}^{\mathrm{GL}_n(F)}$ has the same cokernel as $\bar{d}^{\mathrm{GL}_n(F)}$, the cokernel of $\tilde{d}^{\mathrm{GL}_n(F)}$ is $K_0\big(\mathcal{H}(\mathrm{GL}_n(F); R)\big)$.

7 Homotopy colimits

7.A The Farrell–Jones assembly map as a map of homotopy colimits

Next we want to extend the considerations of Section 6 to the higher K-groups. For this purpose and the proofs appearing in [2], it is worthwhile to write down the assembly map in terms of homotopy colimits. The projections $G/U \to G/G$ for U compact open in G induce a map

(7.1)
$$\operatorname*{hocolim}_{G/U \in \mathrm{Or}_{\mathcal{C}\mathrm{op}}(G)} \boldsymbol{K}_R(G/U) \to \boldsymbol{K}_R(G/G) \simeq \boldsymbol{K}(\mathcal{H}(G; R)).$$

This map can be identified after applying π_n with the assembly map appearing in Theorems 1.1(i) and 3.2. This follows from [11, Section 5].

7.B Simplifying the source of the Farrell–Jones assembly map

Let X be an abstract simplicial complex with simplicial G-action such that the isotropy group of each vertex is compact open and the G-action is cellular. Furthermore we assume that $|X|^K$ is weakly contractible for any compact open subgroup of G. Then $|X|$ is a model for $E_{\mathcal{C}\mathrm{op}}(G)$.

Let C be a collection of simplices of X that contains at least one simplex from each orbit of the action of G on the set of simplices of X. Define a category $\mathcal{C}(C)$ as follows. Its objects are the simplices from C. A morphism $gG_\sigma \colon \sigma \to \tau$ is an element $gG_\sigma \in G/G_\sigma$ satisfying $g\sigma \subseteq \tau$. The composite of $gG_\sigma \colon \sigma \to \tau$ with $hG_\tau \colon \tau \to \rho$ is $hgG_\sigma \colon \sigma \to \rho$. Define a functor

(7.2)
$$\iota_C \colon \mathcal{C}(C)^{\mathrm{op}} \to \mathrm{Or}_{\mathcal{C}\mathrm{op}}(G)$$

by sending an object σ to G/G_σ and a morphism $gG_\sigma \colon \sigma \to \tau$ to $R_g \colon G/G_\tau \to G/G_\sigma$, $g'G_\tau \mapsto g'gG_\sigma$.

Lemma 7.3 *Under the assumptions above, the map induced by the functor ι_C,*

$$\operatorname*{hocolim}_{\sigma \in \mathcal{C}(C)^{\mathrm{op}}} \boldsymbol{K}_R(G/G_\sigma) \xrightarrow{\simeq} \operatorname*{hocolim}_{G/U \in \mathrm{Or}_{\mathcal{C}\mathrm{op}}(G)} \boldsymbol{K}_R(G/U),$$

is a weak homotopy equivalence.

Proof We want to apply the criterion [12, 9.4]. So we have to show that the geometric realization of the nerve of the category $G/K \downarrow \iota_C$ is a contractible space for every object G/K in $\mathrm{Or}_{\mathcal{C}\mathrm{op}}(G)$. An object in $G/K \downarrow \iota_C$ is a pair (σ, u) consisting of an element $\sigma \in C$ and a G-map $u\colon G/K \to G/G_\sigma$. A morphism $(\sigma, u) \to (\tau, v)$ in $G/K \downarrow \iota_C$ is given by a morphism $gG_\tau\colon \tau \to \sigma$ in $\mathcal{C}(C)$ such that the G-map $R_g\colon G/G_\sigma \to G/G_\tau$ sending zG_σ to zgG_τ satisfies $v \circ R_g = u$.

Let $\mathcal{P}(X^K)$ be the poset given by the simplices of X^K ordered by inclusion. Then we get an equivalence of categories

$$F\colon \mathcal{P}(X^K)^{\mathrm{op}} \xrightarrow{\simeq} G/K \downarrow \iota_C$$

as follows. It sends a simplex σ to the object $(\sigma, \mathrm{pr}_\sigma\colon G/K \to G/G_\sigma)$ for the canonical projection pr_σ. A morphism $\sigma \to \tau$ in $\mathcal{P}(X^K)^{\mathrm{op}}$ is sent to the morphism $(\sigma, \mathrm{pr}_\sigma) \to (\tau, \mathrm{pr}_\tau)$ in $G/K \downarrow \iota_C$ which is given by the morphism $eG_\tau\colon \tau \to \sigma$ in $\mathcal{C}(C)$.

Consider an object (σ, u) in $G/K \downarrow \iota_C$. We want to show that it is isomorphic to an object in the image of F. Choose $g \in G$ such that $g^{-1}Kg \subseteq G_\sigma$ and let u be the G-map $R_g\colon G/K \to G/G_\sigma$ sending zK to zgG_σ. Then $K \subseteq G_{g\sigma}$ and we can consider the object $F(g\sigma) = (g\sigma, \mathrm{pr}_{g\sigma})$ for the projection $\mathrm{pr}_{g\sigma}\colon G/K \to G_{g\sigma}$. Now the isomorphism $gG_\sigma\colon \sigma \to g\sigma$ in $\mathcal{C}(C)$ induces an isomorphism $F(g\sigma) \xrightarrow{\cong} (\sigma, u)$ in $G/K \downarrow \iota_C$.

Obviously F is faithful. It remains to show that F is full. Fix two objects σ and τ in $\mathcal{P}(X^K)$. Consider a morphism $f\colon F(\sigma) = (\sigma, \mathrm{pr}_\sigma) \to F(\tau) = (\tau, \mathrm{pr}_\tau)$ in $G/K \downarrow \iota_C$. It is given by a morphism $gG_\tau\colon \tau \to \sigma$ in $\mathcal{C}(C)$ such that the composite of $R_g\colon G/G_\sigma \to G/G_\tau$ with pr_σ is pr_τ. This implies $gG_\tau = G_\tau$ and hence $g \in G_\tau$. Since $g\tau \subseteq \sigma$ by the definition of a morphism in $\mathcal{C}(C)$, we get $\tau \subseteq \sigma$. Hence f is the image of the morphism $\sigma \to \tau$ under F. This shows that F is full.

Hence it remains to show that geometric realization of the nerve of $\mathcal{P}(X^K)^{\mathrm{op}}$ is contractible. Since this is the barycentric subdivision of $|X|^K$, this follows from the assumptions. □

Suppose additionally that X admits a strict fundamental domain Δ, ie a simplicial subcomplex Δ that contains exactly one simplex from each orbit for the G-action on the set of simplices of X. Then we can take for C the simplices from Δ. In this case $\mathcal{C}(C)$ can be identified with the poset $\mathcal{P}(\Delta)$ of simplices of Δ. Recall that, for any open subgroup U of G, there is an explicit weak homotopy equivalence $\boldsymbol{K}(\mathcal{H}(U;R)) \xrightarrow{\simeq} \boldsymbol{K}_R(G/U)$, where the source is the K-theory spectrum $\boldsymbol{K}(\mathcal{H}(U;R))$ of the Hecke algebra $\mathcal{H}(U;R)$; see [3, (5.6) and Remark 6.7]. Lemma 7.3 implies:

Theorem 7.4 *Let X be an abstract simplicial complex with a simplicial G-action such that the isotropy group of each vertex is compact open, the G-action is cellular, and $|X|^K$ is weakly contractible for every compact open subgroup K of G. Let Δ be a strict fundamental domain.*

Then the assembly map

(7.5)
$$\operatorname*{hocolim}_{\sigma \in \mathcal{P}(\Delta)^{\mathrm{op}}} \boldsymbol{K}(\mathcal{H}(G_\sigma; R)) \to \operatorname*{hocolim}_{G/U \in \mathrm{Or}_{\mathcal{C}\mathrm{op}}(G)} \boldsymbol{K}_R(G/U)$$

induced by the functor $\mathcal{P}(\Delta)^{\mathrm{op}} \to \mathrm{Or}_{\mathcal{C}\mathrm{op}}(G)$ sending a simplex σ to G_σ is a weak homotopy equivalence.

Example 7.6 ($\mathrm{SL}_n(F)$) Let X be the Bruhat–Tits building for $\mathrm{SL}_n(F)$. Then the canonical $\mathrm{SL}_n(F)$ action on X is cellular. We will use again the notation introduced in Section 6. The $(n-1)$-simplex Δ, viewed as a subcomplex of X, is a strict fundamental domain. Applying this in the case $n = 2$ yields the homotopy pushout diagram

$$\begin{tikzcd}
\boldsymbol{K}(\mathcal{H}(I^{\mathrm{S}}; R)) \arrow[r] \arrow[d] & \boldsymbol{K}(\mathcal{H}(U_1^{\mathrm{S}}; R)) \arrow[d] \\
\boldsymbol{K}(\mathcal{H}(U_0^{\mathrm{S}}; R)) \arrow[r] & \boldsymbol{K}\big(\mathcal{H}(\mathrm{SL}_2(F); R)\big)
\end{tikzcd}$$

For the K-groups this yields a Mayer–Vietoris sequence, infinite to the left,

(7.7)
$$\begin{aligned}
\cdots \to K_n(\mathcal{H}(I^{\mathrm{S}}; R)) \to K_n(\mathcal{H}(U_1^{\mathrm{S}}; R)) \oplus K_n(\mathcal{H}(U_0^{\mathrm{S}}; R)) \to K_n\big(\mathcal{H}(\mathrm{SL}_2(F); R)\big) \\
\to K_{n-1}(\mathcal{H}(I^{\mathrm{S}}; R)) \to K_{n-1}(\mathcal{H}(U_1^{\mathrm{S}}; R)) \oplus K_{n-1}(\mathcal{H}(U_0^{\mathrm{S}}; R)) \to \cdots \\
\cdots \to K_0(\mathcal{H}(I^{\mathrm{S}}; R)) \to K_0(\mathcal{H}(U_1^{\mathrm{S}}; R)) \oplus K_0(\mathcal{H}(U_0^{\mathrm{S}}; R)) \to K_0\big(\mathcal{H}(\mathrm{SL}_2(F); R)\big) \to 0
\end{aligned}$$

and $K_n\big(\mathcal{H}(\mathrm{SL}_2(F); R)\big) = 0$ for $n \le -1$.

For $n = 3$ we obtain the homotopy pushout diagram

$$\begin{tikzcd}
 & \boldsymbol{K}(\mathcal{H}(U_{12}^{\mathrm{S}}; R)) \arrow[rr] \arrow[dd] & & \boldsymbol{K}(\mathcal{H}(U_2^{\mathrm{S}}; R)) \arrow[dd] \\
\boldsymbol{K}(\mathcal{H}(I^{\mathrm{S}}; R)) \arrow[ur] \arrow[rr] \arrow[dd] & & \boldsymbol{K}(\mathcal{H}(U_{02}^{\mathrm{S}}; R)) \arrow[ur] \arrow[dd] & \\
 & \boldsymbol{K}(\mathcal{H}(U_1^{\mathrm{S}}; R)) \arrow[rr] & & \boldsymbol{K}\big(\mathcal{H}(\mathrm{SL}_3(F); R)\big) \\
\boldsymbol{K}(\mathcal{H}(U_{01}^{\mathrm{S}}; R)) \arrow[ur] \arrow[rr] & & \boldsymbol{K}(\mathcal{H}(U_0^{\mathrm{S}}; R)) \arrow[ur] &
\end{tikzcd}$$

where we abbreviated $U_{ij}^{\mathrm{S}} := U_i^{\mathrm{S}} \cap U_j^{\mathrm{S}}$. In general, for $\mathrm{SL}_n(F)$ we obtain a homotopy pushout diagram whose shape is an n-cube.

To such an n-cube there is assigned a spectral sequence concentrated in the region for $p \ge 0$ and $0 \le q \le n-1$, which corresponds to the spectral sequence appearing in Theorem 1.1(ii).

8 Allowing central characters and actions on the coefficients

So far we have only considered the standard Hecke algebra $\mathcal{H}(G; R)$. There are more general Hecke algebras $\mathcal{H}(G; R, \rho, \omega)$ [4], and all the discussions of this paper carry over to them in the obvious way.

References

[1] **P Abramenko**, **K S Brown**, *Buildings: theory and applications*, Graduate Texts in Math. 248, Springer (2008) MR Zbl

[2] **A Bartels**, **W Lück**, *Algebraic K-theory of reductive p-adic groups*, preprint (2023) arXiv 2306.03452

[3] **A Bartels**, **W Lück**, *Inheritance properties of the Farrell–Jones conjecture for totally disconnected groups*, preprint (2023) arXiv 2306.01518

[4] **A Bartels**, **W Lück**, *On the algebraic K-theory of Hecke algebras*, from "Mathematics going forward: collected mathematical brushstrokes", Lecture Notes in Math. 2313, Springer (2023) 241–277 MR Zbl

[5] **J Bernstein**, *Representations of p-adic groups*, draft of lecture notes, Harvard University (1992) Available at `http://www.math.tau.ac.il/~bernstei/Unpublished_texts/Unpublished_list.html`

[6] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. 319, Springer (1999) MR Zbl

[7] **K S Brown**, *Buildings*, Springer (1989) MR Zbl

[8] **F Bruhat**, **J Tits**, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. 41 (1972) 5–251 MR Zbl

[9] **J-F Dat**, *Quelques propriétés des idempotents centraux des groupes p-adiques*, J. Reine Angew. Math. 554 (2003) 69–103 MR Zbl

[10] **J-F Dat**, *Théorie de Lubin–Tate non-abélienne et représentations elliptiques*, Invent. Math. 169 (2007) 75–152 MR Zbl

[11] **J F Davis**, **W Lück**, *Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory*, K-Theory 15 (1998) 201–252 MR Zbl

[12] **W G Dwyer**, **D M Kan**, *A classification theorem for diagrams of simplicial sets*, Topology 23 (1984) 139–155 MR Zbl

[13] **P Garrett**, *Smooth representations of totally disconnected groups*, unpublished notes (2012) Available at `https://www-users.cse.umn.edu/~garrett/m/v/smooth_of_td.pdf`

[14] **W Lück**, *Transformation groups and algebraic K-theory*, Lecture Notes in Math. 1408, Springer (1989) MR Zbl

[15] **W Lück**, *Chern characters for proper equivariant homology theories and applications to K- and L-theory*, J. Reine Angew. Math. 543 (2002) 193–234 MR Zbl

[16] **W Lück**, *The relation between the Baum–Connes conjecture and the trace conjecture*, Invent. Math. 149 (2002) 123–152 MR Zbl

[17] **M Ronan**, *Lectures on buildings*, Perspectives Math. 7, Academic, Boston, MA (1989) MR Zbl

[18] **R M Switzer**, *Algebraic topology: homotopy and homology*, Grundl. Math. Wissen. 212, Springer (1975) MR Zbl

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All known realizations of complete Lie algebras coincide

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We prove that for any reduced differential graded Lie algebra L, the classical Quillen geometrical realization $\langle L\rangle_Q$ is homotopy equivalent to the realization $\langle L\rangle = \mathrm{Hom}_{\mathrm{cdgl}}(\mathfrak{L}_\bullet, L)$ constructed via the cosimplicial free complete differential graded Lie algebra $\mathfrak{L}_\bullet$. As the latter is a deformation retract of the Deligne–Getzler–Hinich realization $\mathrm{MC}_\bullet(L)$ we deduce that, up to homotopy, all known topological realization functors of complete differential graded Lie algebras coincide. Immediate consequences of our main result include an elementary proof of the Baues–Lemaire conjecture and the description of the Quillen realization as a representable functor.

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Introduction

In [13], Quillen constructed a geometrical realization functor

$$\langle\,\cdot\,\rangle_Q : \mathbf{dgl_1} \to \mathbf{sset_1}$$

from the category of simply connected differential graded Lie algebras to the category of reduced simplicial sets. This was the starting point of rational homotopy theory, from the Lie approach, as this functor induces an equivalence between the corresponding homotopy categories when considering rational reduced simplicial sets. Later on [8; 9], the *Deligne–Getzler–Hinich groupoid* functor

$$\mathrm{MC}_\bullet : \mathbf{cdgl} \to \mathbf{sset}$$

was defined in the category of complete differential graded Lie algebras, for which we use the acronym cdgl henceforth. Given L a cdgl, $\mathrm{MC}_\bullet(L) = \mathrm{MC}(\mathscr{A}_\bullet \widehat{\otimes} L)$ is the simplicial set of Maurer–Cartan elements of the simplicial cdgl $\mathscr{A}_\bullet \widehat{\otimes} L$ in which $\mathscr{A}_\bullet$ denotes the simplicial commutative differential graded algebra of PL-differential forms on the standard simplices.

Finally — see Buijs, Félix, Murillo and Tanré [4] — there is a realization functor for cdgl's based on a quite geometrical cosimplicial cdgl $\mathfrak{L}_\bullet$,

$$\langle\,\cdot\,\rangle : \mathbf{cdgl} \to \mathbf{sset}, \quad \langle L\rangle = \mathrm{Hom}_{\mathrm{cdgl}}(\mathfrak{L}_\bullet, L). \tag{1}$$

Buijs, Félix, Murillo and Tanré [3, Theorem 0.2] and Robert-Nicoud [14, Theorem 5.2] showed that, for any cdgl L, $\langle L\rangle$ is simplicially isomorphic to $\gamma_\bullet(L)$, the *nerve of* L [8, Section 5], which is a deformation retract of $\mathrm{MC}_\bullet(L)$.

It is important to remark that the realization functor of (1), in fact the adjoint pair of functors in (2), has been widely and highly nontrivially extended to the L_∞-algebra setting by Robert-Nicoud and Vallette [15, Sections 2 and 3] by means of the so-called *integration functor* R, which in turn recovers again the Getzler nerve functor $\gamma_\bullet$.

Here, we close up the circle and prove:

Theorem 0.1 *For any simply connected differential graded Lie algebra L, the simplicial sets $\langle L\rangle$ and $\langle L\rangle_Q$ have the same weak homotopy type.*

A precise and slightly more general statement of this result is Theorem 2.1. The first immediate consequence is that the functors induced in the respective homotopy categories by the global model and realization functors, see [4],

$$\mathbf{sset} \underset{\langle\cdot\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl} \tag{2}$$

extend (in a unique way) the classical equivalence due to Quillen between the homotopy categories of rational reduced simplicial sets and that of simply connected differential graded Lie algebras. In particular, and under no restriction, the Quillen original realization functor is representable by the cosimplicial Lie algebra $\mathfrak{L}_\bullet$ and thus, it can be finally regarded as the Eckmann–Hilton dual of the realization functor of commutative differential graded algebras — see Sullivan [16] — which is corepresentable by $\mathscr{A}_\bullet$. That is a question which has puzzled rational homotopists since the birth of both the Sullivan and Quillen approaches to rational homotopy theory.

Theorem 0.1 was already known for simply connected differential graded Lie algebras of finite type [4, Corollary 11.17] but its proof heavily relied in the Baues–Lemaire conjecture [1, Conjecture 3.5] proved by Majewski [10]. The opposite procedure is now available. Namely, the self-contained proof of Theorem 0.1 under no restriction lets us trivially reprove this conjecture by which, given L the minimal Quillen model of a simply connected complex of finite type X, the commutative differential graded algebra $\mathscr{C}^*(L)$ of the Chevalley–Eilenberg cochain functor on L has the homotopy type of the Sullivan minimal model of X.

For completeness, we also collect extensions of this conjecture to the nonfinite type and/or not simply connected case and describe, in this extended scenario, the result of composing the Sullivan realization functor of commutative differential graded algebras [16],

$$\langle\cdot\rangle_S\colon \mathbf{cdga}\to\mathbf{sset},\quad \langle A\rangle_S=\mathrm{Hom}_{\mathrm{cdga}}(A,\mathscr{A}_\bullet),$$

and the appropriate extension of the Chevalley–Eilenberg cochain functor adapted to the specific class of considered differential graded Lie algebras.

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1 Preliminaries

Unless explicitly stated otherwise, any algebraic object is assumed to be $\mathbb{Z}$-graded and having $\mathbb{Q}$ as coefficient field. We denote in bold the category containing such object.

A *differential graded Lie algebra* (dgl henceforth) is a graded vector space L endowed with a *Lie bracket* $[\cdot,\cdot]$ satisfying graded antisymmetry and Jacobi identity, and a linear derivation d of degree -1 such that $d^2=0$. An element $a\in L_{-1}$ is *Maurer–Cartan* (MC element hereafter) if $da=-\frac{1}{2}[a,a]$. Given an MC element $a\in L_{-1}$ the map $d_a=d+\mathrm{ad}_a$ is a new differential on L. The *component* of L at a is the connected sub-dgl L^a of (L,d_a) given by

$$L^a_p=\begin{cases}\ker d_a & \text{if } p=0,\\ L_p & \text{if } p>0.\end{cases}$$

A dgl L is called *free* if it is free as a Lie algebra, that is, $L=\mathbb{L}(V)$ for some graded vector space V. A dgl L is *simply connected or reduced* if it is concentrated in positive degrees, $L=\bigoplus_{p\geq 1}L_p$.

A *complete differential graded Lie algebra*, cdgl henceforth, is a dgl L equipped with a decreasing filtration of differential ideals,

$$L=F^1\supset\cdots\supset F^n\supset F^{n+1}\supset\cdots,$$

with $[F^p,F^q]\subset F^{p+q}$ for $p,q\geq 1$ and such that the natural map

$$L\xrightarrow{\cong}\varprojlim_n L/F^n$$

is a dgl isomorphism. The lower central series of a given dgl,

$$L^1\supset\cdots\supset L^n\supset L^{n+1}\supset\cdots,\quad L^1=L,\quad L^n=[L,L^{n-1}],$$

is always a filtration. A *morphism* $f\colon L\to L'$ between cdgl's is a filtration preserving dgl morphism. Note that any simply connected dgl is always complete.

Given a free Lie algebra $\mathbb{L}(V)$ we write

$$\widehat{\mathbb{L}}(V)=\varprojlim_n\mathbb{L}(V)/\mathbb{L}(V)^n$$

where, as before, $\{\mathbb{L}(V)^n\}_{n\geq 0}$ denotes the lower central series of $\mathbb{L}(V)$. This Lie algebra is complete with respect to the filtration $F^n=\ker(\widehat{\mathbb{L}}(V)\to\mathbb{L}(V)/\mathbb{L}(V)^n)$ for $n\geq 1$.

The homotopy theory of cdgl's, for which we refer to [4], is based on the pair of adjoint functors, (*global*) *model* and *realization*,

(3) $$\mathbf{sset} \underset{\langle\cdot\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl},$$

which highly rely on the cosimplicial cdgl $\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n\geq 0}$; see [4, Chapter 6]. For each $n \geq 0$,

$$\mathfrak{L}_n = \big(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d\big)$$

where $s^{-1}\Delta^n$ denotes the desuspension of the nondegenerate simplicial chains on the simplicial set $\underline{\Delta}^n$. That is, for any $p \geq 0$, a generator of degree $p-1$ of $s^{-1}\Delta^n$ can be written as $a_{i_0\cdots i_p}$ with

$$0 \leq i_0 < \cdots < i_p \leq n.$$

The cofaces and codegeneracies in $\mathfrak{L}_\bullet$ are induced by those on the cosimplicial chain complex $s^{-1}\Delta^n$: For each $0 \leq i \leq n$, the map $\delta_i : \{0, \ldots, n-1\} \to \{0, \ldots, n\}$,

$$\delta_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i, \end{cases}$$

defines the cdgl morphism

$$\delta_i : \widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}) \to \widehat{\mathbb{L}}(s^{-1}\Delta^n), \quad \delta_i(a_{\ell_0\cdots\ell_p}) = a_{\delta_i(\ell_0)\cdots\delta_i(\ell_p)}.$$

On the other hand, the map $\sigma_i : \{0, \ldots, n+1\} \to \{0, \ldots, n\}$,

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i, \end{cases}$$

also defines the cdgl morphism $\sigma_i : \widehat{\mathbb{L}}(s^{-1}\Delta^{n+1}) \to \widehat{\mathbb{L}}(s^{-1}\Delta^n)$,

$$\sigma_i(a_{\ell_0\cdots\ell_q}) = \begin{cases} a_{\sigma_i(\ell_0)\cdots\sigma_i(\ell_q)} & \text{if } \sigma_i(\ell_0) < \cdots < \sigma_i(\ell_q), \\ 0 & \text{otherwise.} \end{cases}$$

The differential d on each $\mathfrak{L}_n$ satisfies:

(1) For each $i = 0, \ldots, n$, the generators $a_0, \ldots, a_n \in s^{-1}\Delta^n$, corresponding to vertices, are MC elements.
(2) The linear part of d is induced by the boundary operator of $s^{-1}\Delta^n$.
(3) The cofaces and codegeneracies are cdgl morphisms.

Moreover, two differentials satisfying these properties produce isomorphic cdgl's [4, Theorem 6.1].

The realization of any cdgl L is defined as the simplicial set,

$$\langle L\rangle = \mathrm{Hom}_{\mathrm{cdgl}}(\mathfrak{L}_\bullet, L), \quad d_i = \mathrm{Hom}_{\mathrm{cdgl}}(\delta_i, L), \quad s_i = \mathrm{Hom}_{\mathrm{cdgl}}(\sigma_i, L).$$

If L is connected, that is, nonnegatively graded, then $\langle L\rangle$ is a connected simplicial set and for any $n \geq 1$, the map

(4) $$\rho_n : \pi_n\langle L\rangle \xrightarrow{\cong} H_{n-1}(L), \quad \rho_n[\varphi] = [\varphi(a_{0\cdots n})]$$

is a group isomorphism [2, Theorem 4.6; 4, Theorem 7.18]. Here, the group law in $H_0(L)$ is given by the Baker–Campbell–Hausdorff product (BCH product henceforth).

On the other hand, in the seminal paper [13], for which we refer for details of what follows, D Quillen introduced a couple of functors,

$$\mathbf{sset_1} \underset{\langle\cdot\rangle_Q}{\overset{\lambda}{\rightleftarrows}} \mathbf{dgl_1},$$

which are the composition of the following pairs of adjoint functors (the upper arrow denotes left adjoint),

$$\lambda\colon \mathbf{sset_1} \underset{W}{\overset{G}{\rightleftarrows}} \mathbf{sgp_0} \underset{\mathcal{G}}{\overset{\widehat{\mathbb{Q}}}{\rightleftarrows}} \mathbf{sch_0} \underset{\mathcal{P}}{\overset{\widehat{U}}{\leftrightarrows}} \mathbf{sla_1} \underset{N}{\overset{N^*}{\leftrightarrows}} \mathbf{dgl_1} :\langle\cdot\rangle_Q.$$

Here, $\mathbf{sset_1}$, $\mathbf{sgp_0}$, $\mathbf{sch_0}$ and $\mathbf{sla_1}$ denote, respectively, the categories of *reduced simplicial sets* (those with only one simplex in dimensions 0 and 1), *connected simplicial groups* ($G_0 = \{1\}$), *connected complete Hopf algebras* ($A_{<0} = 0$ and $A_0 = \mathbb{Q}$), and *reduced simplicial Lie algebras*. Each of these pairs induces Quillen equivalences on the corresponding homotopy categories when localizing on rational weak homotopy equivalences in $\mathbf{sset_1}$ and $\mathbf{sgp_0}$, on weak homotopy equivalences in $\mathbf{sch_0}$ and $\mathbf{sla_1}$, and on quasi-isomorphisms in $\mathbf{dgl_1}$ [13, Theorem I].

Recall that, given X a reduced simplicial set, ΩX is simplicially modeled by the *Kan simplicial group* $G(X)$ where, for any $n \geq 0$, $(GX)_n$ is the free group generated by $X_{n+1} \setminus s_0 X_n$ or, equivalently, the quotient group $(GX)_n = \mathrm{Free}(X_{n+1})/(s_0 X_n)$. Faces and degeneracies are defined as

$$\begin{aligned}
&\bar{s}_i\colon (GX)_n \to (GX)_{n+1}, && \bar{s}_i x = s_{i+1} x, && x \in X_{n+1}, \quad i = 0, \dots, n,\\
&\bar{d}_i\colon (GX)_n \to (GX)_{n-1}, && \bar{d}_i x = d_{i+1} x, && x \in X_{n+1}, \quad i = 1, \dots, n,\\
&\bar{d}_0\colon (GX)_n \to (GX)_{n-1}, && \bar{d}_0 x = (d_0 x)^{-1}(d_1 x), && x \in X_{n+1}.
\end{aligned}$$

Next, see for instance [7, Proposition 8.4.1], the composition $\mathcal{P}\widehat{\mathbb{Q}}GX$ is the simplicial free complete Lie algebra

$$\mathbb{L}_n = \widehat{\mathbb{L}}(X_{n+1}/s_0 X_n), \quad n \geq 0,$$

in which X_n denotes here the vector space generated by the n-simplices of X. We denote by $[\![\cdot,\cdot]\!]$ the Lie bracket in this simplicial Lie algebra. The faces and degeneracies, $\bar{d}_i$'s and $\bar{s}_i$'s, act as above on generators and are extended as Lie morphisms. Note that, since the multiplication on GX is taken to the BCH product on $\mathbb{L}$, we have

$$\bar{d}_0 x = (-d_0 x) * (d_1 x), \quad x \in X_n,$$

where $*$ denotes the BCH product.

Finally, $\lambda(X) = N\mathbb{L}$ is the reduced dgl given by the *normalized chain complex* on $\mathbb{L}$,

$$(N\mathbb{L})_n = \bigcap_{i=1}^{n} \ker(\bar{d}_i\colon \mathbb{L}_n \to \mathbb{L}_{n-1}), \quad n \geq 1, \quad d = \bar{d}_0,$$

endowed with the bracket induced by the simplicial Eilenberg–Zilber formula: given $\omega \in (N\mathbb{L})_n$ and $\omega' \in (N\mathbb{L})_m$,

$$[\omega,\omega'] = \sum_{(\mu,\nu)\in S^0_{n,m}} \varepsilon_{\mu,\nu} [\![\bar{s}_{\nu_m}\bar{s}_{\nu_{m-1}}\cdots\bar{s}_{\nu_1}\omega, \bar{s}_{\mu_n}\bar{s}_{\mu_{n-1}}\cdots\bar{s}_{\mu_1}\omega']\!],$$

where $S^0_{n,m}$ denotes the (n,m)-shuffles of $\{0,1,\dots,n+m-1\}$ and $\varepsilon_{\mu,\nu}$ is the sign of such a shuffle.

In general, given a simplicial vector space V we denote by $C(V)$ its simplicial chain complex in which $C_n(V)=V_n$ and the boundary operator is given by $\sum_i(-1)^i d_i$. Given another simplicial vector space W, there are natural chain maps, Eilenberg–Zilber and Alexander–Whitney (see for instance [11, Section 29]),

$$C(V)\otimes C(W) \underset{\Delta}{\overset{\nabla}{\rightleftarrows}} C(V\otimes W),$$

in which $V\otimes W$ is the simplicial tensor product, given by

$$\nabla(\alpha\otimes\beta) = \sum_{(\mu,\nu)\in S^0_{n,m}} \varepsilon_{\mu,\nu} s_\nu\alpha\otimes s_\mu\beta, \quad \alpha\in V_n,\ \beta\in W_m,$$

where s_ν and s_μ stand for $s_{\nu_m}s_{\nu_{m-1}}\cdots s_{\nu_1}$ and $s_{\mu_n}s_{\mu_{m-1}}\cdots s_{\mu_1}$, and

$$\Delta(\alpha\otimes\beta) = \sum_{k=0}^n d^{n-k}_{k+1}\alpha\otimes d^k_0\beta = \sum_{k=0}^n d_{k+1}d_{k+2}\cdots d_n\alpha\otimes d_0d_1\cdots d_{k-1}\beta, \quad \alpha\otimes\beta\in V_n\otimes W_n.$$

These maps restrict to the corresponding normalized chain complexes

$$NV\otimes NW \underset{\Delta}{\overset{\nabla}{\rightleftarrows}} N(V\otimes W)$$

where they satisfy

$$\Delta\nabla = \mathrm{id}_{NV\otimes NW}, \quad \nabla\Delta \simeq \mathrm{id}_{N(V\otimes W)}\,. \tag{5}$$

With this notation, the Lie bracket on $\lambda(L) = N\mathbb{L}$ is given by

$$[\,\cdot\,,\cdot\,] = [\![\,\cdot\,,\cdot\,]\!]\circ\nabla\colon N\mathbb{L}\otimes N\mathbb{L}\to N\mathbb{L}. \tag{6}$$

2 The proof and consequences

In view of (4), if L is a simply connected dgl then $\langle L\rangle$ is a rational simplicial set. As the Quillen functors induce equivalences between the homotopy categories of reduced rational simplicial sets and that of simply connected dgl's, Theorem 0.1 is an immediate consequence of the following, more general, result:

Theorem 2.1 *Given a simply connected dgl L, there is a surjective quasi-isomorphism*

$$\Phi\colon \lambda\langle L\rangle \xrightarrow{\simeq} L.$$

Proof Write $X_\bullet = \langle L\rangle = \mathrm{Hom}_{\mathrm{cdgl}}(\mathfrak{L}_\bullet, L)$. We first define a linear map

$$\Phi\colon C(\mathbb{L}) \to L$$

recursively on each $\mathbb{L}_n^m = \widehat{\mathbb{L}}^m(X_{n+1}/s_0X_n)$, with $n \geq 1$ the simplicial degree and $m \geq 1$ the word length:

If $\varphi\colon \mathfrak{L}_2 \to L \in X_2$ is an indecomposable of $\mathbb{L}_1$ define $\Phi(\varphi) = -\varphi(a_{012}) \in L_1$. Observe that if $\varphi = s_0\eta$, then $\Phi(\varphi) = -\eta(\sigma_0(a_{012})) = -\eta(0) = 0$. Set $\Phi(\mathbb{L}_1^{\geq 2}) = 0$.

Assume Φ is built on $\mathbb{L}_{<n}$ and define

$$\Phi\colon \mathbb{L}_n \to L$$

as follows. Again, for an indecomposable $\varphi\colon \mathfrak{L}_{n+1} \to L \in X_{n+1}$ define

$$\Phi(\varphi) = (-1)^n\varphi(a_{0\cdots n+1}).$$

Once more, if $\varphi = s_0\eta$, then $\Phi(\varphi) = 0$ and thus Φ is well defined on X_{n+1}/s_0X_n. Now, if Φ has been defined for $\alpha, \beta \in \mathbb{L}_n$, set

$$\Phi[\![\alpha, \beta]\!] = [\,\cdot\,,\cdot\,] \circ (\Phi \otimes \Phi) \circ \Delta(\alpha \otimes \beta). \tag{7}$$

Note that Φ is well defined since, given its formula, it inductively maps $\mathbb{L}_n^m$ into L^m.

Finally, recall that $\lambda(X) = N\mathbb{L}$ and define

$$\Phi\colon \lambda\langle L\rangle \to L$$

as the restriction of Φ to $N\mathbb{L} \subset C(\mathbb{L})$.

(i) **$\Phi\colon N\mathbb{L} \to L$ is a morphism of Lie algebras** Indeed, by (6), (7) and (5) respectively, we have the identities

$$\Phi \circ [\,\cdot\,,\cdot\,] = \Phi \circ [\![\,\cdot\,,\cdot\,]\!] \circ \nabla = [\,\cdot\,,\cdot\,] \circ (\Phi \otimes \Phi) \circ \Delta \circ \nabla = [\,\cdot\,,\cdot\,] \circ (\Phi \otimes \Phi).$$

(ii) **$\Phi\colon N\mathbb{L} \to L$ is a surjective chain map** We first check that it is enough to show that $\Phi d = d\Phi$ for the indecomposable elements of $N\mathbb{L}$ (with respect to the Lie bracket $[\![\,\cdot\,,\cdot\,]\!]$). Assume that this is the case and suppose, inductively, that $\Phi(d\alpha) = d\Phi(\alpha)$ and $\Phi(d\beta) = d\Phi(\beta)$ for some elements $\alpha, \beta \in (N\mathbb{L})_n$. Then

$$\begin{aligned}
\Phi(d[\![\alpha, \beta]\!]) &= \Phi \circ [\![\,\cdot\,,\cdot\,]\!] \circ (d \otimes d)(\alpha \otimes \beta)\\
&= [\,\cdot\,,\cdot\,] \circ (\Phi \otimes \Phi) \circ \Delta \circ (d \otimes d)(\alpha \otimes \beta)\\
&= [\,\cdot\,,\cdot\,] \circ (\Phi \otimes \Phi) \circ (d \otimes \mathrm{id} + (-1)^n \,\mathrm{id} \otimes d) \circ \Delta(\alpha \otimes \beta)\\
&= [\,\cdot\,,\cdot\,] \circ (d \otimes \mathrm{id} + (-1)^n \,\mathrm{id} \otimes d) \circ (\Phi \otimes \Phi)\Delta(\alpha \otimes \beta)\\
&= d \circ [\,\cdot\,,\cdot\,] \circ (\Phi \otimes \Phi)\Delta(\alpha \otimes \beta) = d\Phi[\![\alpha, \beta]\!],
\end{aligned}$$

where, for the above identities we have used, respectively, the following facts: $d = \bar{d}_0$ in $N\mathbb{L}$ so it commutes with $[\![\,\cdot\,,\cdot\,]\!]$; formula (7); Δ is a chain map; induction hypothesis; d is a derivation on L; and again formula (7).

We next prove that Φ commutes with the differential for indecomposable elements of $N\mathbb{L}$. Note that, for a fixed degree $n\geq 1$, the indecomposable subspace of $(N\mathbb{L})_n$ is the vector subspace of X_{n+1} generated by the nondegenerate simplices.

Choose $\varphi\colon \mathfrak{L}_{n+1}\to L$ such a generator so that $\bar{d}_i\varphi=0$ for $i=1,\dots,n$, which is equivalent to $\varphi\circ\delta_i=0$ for $i=2,\dots,n+1$. This implies that the only possible nonzero images of generators are

$$\varphi(a_{01\cdots n+1})=x_{01},\quad \varphi(a_{02\cdots n+1})=x_0,\quad \varphi(a_{12\cdots n+1})=x_1,\quad \varphi(a_{23\cdots n+1})=x_2, \tag{8}$$

which are elements in L of degree $n,n-1,n-1$ and $n-2$ respectively. Recall that, in $\mathfrak{L}_{n+1}$,

$$da_{01\cdots n+1}=\sum_{i=0}^{n+1}(-1)^i a_{0\cdots\hat{i}\cdots n+1}+\Omega_n$$

where Ω_n is a decomposable element of degree $n-1$. By degree reasons, one easily sees that $\varphi(\Omega_n)=0$ except, at most, for $n=3$ and as long as Ω_3 contains $[a_{234},a_{234}]$ as a summand. The technical Lemma 2.2 below shows that this is not the case for an appropriate choice of the differential in $\mathfrak{L}_4$, which is not canonical as remarked in Section 1. Therefore,

$$\varphi(da_{01\cdots n+1})=\varphi(a_{12\cdots n+1})-\varphi(a_{02\cdots n+1}).$$

Another easy inspection lets us also write

$$\varphi(da_{02\cdots n+1})=\varphi(da_{12\cdots n+1})=\varphi(a_{2\cdots n+1}),$$

and we conclude that

$$dx_{01}=x_1-x_0\quad\text{and}\quad dx_0=dx_1=x_2. \tag{9}$$

On the one hand,

$$d\Phi(\varphi)=(-1)^n dx_{01}=(-1)^n(x_1-x_0).$$

On the other hand,

$$\begin{aligned}\Phi(d\varphi)=\Phi(\bar{d}_0\varphi)=\Phi\big((-d_0\varphi)*(d_1\varphi)\big)&=\Phi(d_1\varphi-d_0\varphi)+\Phi(\Psi)\\&=(-1)^{n-1}(x_0-x_1)+\Phi(\Psi)=(-1)^n(x_1-x_0)+\Phi(\Psi),\end{aligned}$$

where Ψ is an infinite sum of decomposable $[\![\cdot,\cdot]\!]$-brackets whose letters are either $d_0\varphi$ and $d_1\varphi$. Taking into account how Φ operates in $[\![\cdot,\cdot]\!]$-brackets, see (7), the only possible nonzero values of φ, see (8), and degree arguments, one checks that $\Phi(\Psi)=0$ and therefore $d\Phi(\varphi)=\Phi(d\varphi)$.

Finally, to see that Φ is surjective, given $x\in L_n$ consider the decomposable element φ in $(N\mathbb{L})_n$ defined by

$$\varphi(a_{01\cdots n+1})=(-1)^n x,\quad \varphi(a_{12\cdots n+1})=(-1)^n dx,\quad \varphi(a_{02\cdots n+1})=\varphi(a_{23\cdots n+1})=0,$$

In view of (8) and (9) φ is well defined and $\Phi(\varphi)=x$.

(iii) **$\Phi\colon N\mathbb{L} \to L$ is a quasi-isomorphism** Recall our notation $X = \mathrm{Hom}_{\mathrm{cdgl}}(\mathfrak{L}_\bullet, L)$ and consider, for any $n \geq 1$, the composition

$$\rho_{n+1}\colon \pi_{n+1}(X) \xrightarrow[\cong]{\gamma} H_n(N\mathbb{L}) \xrightarrow{H_n(\Phi)} H_n(L)$$

where $\gamma\colon \pi_{n+1}(X) \xrightarrow{\cong} H_n(\lambda X)$ is the isomorphism induced by λ in homotopy and homology groups respectively. Recall once again that $\lambda = N\mathcal{P}\widehat{\mathbb{Q}}G$ and through this composition the class of an $(n+1)$-simplex φ representing an element in $\pi_{n+1}(X)$ is taken precisely to the homology class of the indecomposable element φ whose degree has been shifted by 1.

Indeed, $[\varphi] \in \pi_{n+1}(X)$ is trivially sent to $[\varphi] \in \pi_n(GX)$. By [13, Section 3] this is taken to the homotopy class of the generator φ of the simplicial free Lie algebra $\mathbb{L}_n$. Finally, by (4.1) of [13, Section 4] this homotopy class is sent to the homology class $[\varphi] \in (N\mathbb{L})_n$

Hence $\rho_{n+1}[\varphi] = (-1)^n[\varphi(a_{01\cdots n+1})]$ which is, up to the sign, the isomorphism (4) and we conclude that $H_n(\Phi)$ is an isomorphism for any $n \geq 1$. □

Lemma 2.2 *The differential d in the term $\mathfrak{L}_4$ of the cosimplicial cdgl $\mathfrak{L}_\bullet$ can be chosen so that $[a_{234}, a_{234}]$ does not appear as a summand of da_{01234}.*

Recall that the differential on each $\mathfrak{L}_n$ is not canonical and might be modified to produce isomorphic copies of this cdgl.

Proof Write $V = s^{-1}\Lambda^4_4 \subset s^{-1}\Delta^4$ where Λ^4_4 denotes the 4[th] horn of Δ^4. Recall from the proof of [4, Theorem 6.7] that in $\mathfrak{L}_4$ the differential of the top generator is given by

$$d_{a_0} a_{01234} = a_{0123} - \Gamma$$

where $\Gamma \in \widehat{\mathbb{L}}(V)$ is any solution of the equation

$$d_{a_0} a_{0123} = \partial_{a_0}\Gamma, \tag{10}$$

which always exists as $H(\widehat{\mathbb{L}}(V), d_{a_0}) = 0$. Write $\Gamma = \sum_{j\geq 1}\Gamma_j$, $d_{a_0}a_{0123} = \sum_{k\geq 1}\omega_k$, where the subscripts denotes word lengths, and $d_{a_0} = \sum_{i\geq 1} d_i$ where each d_i (not to be confused with face maps!) increases the word length by $i-1$. Then (10) becomes the following family of equations in $\widehat{\mathbb{L}}(V)$:

$$\sum_{j+i=k+1} d_i\Gamma_j = \omega_k, \quad k \geq 1,$$

which can be recursively solved by choosing Γ_k so that

$$d_1\Gamma_k = \omega_k - \partial_2\Gamma_{k-1} - \cdots - \partial_{k+1}\Gamma_1. \tag{11}$$

Note that such an element exists since $H(\widehat{\mathbb{L}}(V), d_1) = 0$.

Next, we modify such a solution as follows. With the help of a computer we find the element in $\mathbb{L}^2(V)$,

$$\gamma = 2[a_{24} - a_{23} - a_{34}, a_{0234}] - [a_{023}, a_{023}] + 2[a_{023}, a_{024}] - 2[a_{023}, a_{034}] - [a_{024}, a_{024}] \\ + 2[a_{024}, a_{034}] - [a_{034}, a_{034}] + [a_{234}, a_{234}],$$

which contains the summand $[a_{234}, a_{234}]$ and satisfies $d_1\gamma = 0$, as can be easily checked. Hence, for a good choice of $\lambda \in \mathbb{Q}$, the term $[a_{234}, a_{234}]$ does not appear in the element $\Gamma_2 + \lambda\gamma$. Furthermore,

$$d_1(\Gamma_2 + \lambda\gamma) = d_1\Gamma_2 = \omega_2 - d_2\Gamma_1,$$

so (11) holds for $\Gamma_1' = \Gamma_1$ and $\Gamma_2' = \Gamma_2 + \lambda\gamma$. Solving, as stated, (11) recursively for $k \geq 3$, we get a sequence Γ_k' of elements in $\mathbb{L}^k(V)$ such that $\Gamma' = \sum_{k=1}^{\infty} \Gamma_k'$ is another solution of (10). Thus, in $\mathfrak{L}_4$, the differential of the top generator, which is now $d_{a_0}a_{01234} = a_{0123} - \Gamma'$, does not contain the term $[a_{234}, a_{234}]$. □

Recall from [4, Chapter 8] that (3) constitutes a Quillen pair with respect to the usual model category on **sset** and the one on **cdgl** given in [4, Section 8.1]. Then, the first consequence of Theorem 0.1 reads:

Corollary 2.3 *The adjoint functors*

$$\operatorname{Ho}\mathbf{sset} \underset{\langle\cdot\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \operatorname{Ho}\mathbf{cdgl}$$

extend the equivalences

$$\operatorname{Ho}\mathbf{sset}_{\mathbf{1}}^{\mathbb{Q}} \underset{\langle\cdot\rangle_Q}{\overset{\lambda}{\rightleftarrows}} \operatorname{Ho}\mathbf{dgl}_{\mathbf{1}}.$$

Here, $\operatorname{Ho}\mathbf{sset}_{\mathbf{1}}^{\mathbb{Q}}$ denotes the homotopy category of rational reduced simplicial sets.

Proof On the one hand, by Theorem 0.1, $\langle L\rangle \simeq \langle L\rangle_Q$ for any simply connected dgl L.

On the other hand, for any reduced simplicial set X, consider $\mathfrak{L}_X^a$ where a is the only 0-simplex of X (see [4, Chapters 6 and 8]). We finish by checking that $\mathfrak{L}_X^a \simeq \lambda(X)$. For this, write X as the homotopy colimit $\underleftarrow{\operatorname{hocolim}}_i X_i$ where X_i are finite type subsimplicial sets of X. As λ and $\mathfrak{L}$ are, respectively, equivalence and left adjoint functors between the corresponding homotopy categories, they both preserve homotopy colimits. Hence,

$$\lambda(X) = \lambda(\underleftarrow{\operatorname{hocolim}}_i X_i) \simeq \underleftarrow{\operatorname{hocolim}}_i \lambda(X_i) \simeq \underleftarrow{\operatorname{hocolim}}_i \mathfrak{L}_{X_i}^a \simeq \mathfrak{L}_X^a,$$

where the third identity is [4, Theorem 10.2]. □

Remark 2.4 In fact, the model and realization functors constitute the only Quillen pair extending, up to homotopy, the classical Quillen functors in the following sense: It is known that a Quillen model $(\mathbb{L}(V), d)$ of a given reduced simplicial set X can be chosen to be generated by the desuspension of nondegenerate simplices and where the differential reflects the simplicial structure of X. In particular, if we write $d = \sum_{i\geq 1} d_i$ where d_i increases the bracket length by $i-1$, it follows that $d_1\colon V \to V$ is the desuspension of the chain differential on the nondegenerate simplices and $d_2\colon V \to V \otimes V$ corresponds to an approximation of the diagonal. If we now allow simplices in degree 0 and 1, then vertices of X

must correspond to MC elements of V. As asserted in [4, Proposition 7.8] there is a unique differential d in $\widehat{\mathbb{L}}(V)$ fulfilling this properties and $\mathfrak{L}(X) = (\widehat{\mathbb{L}}(V), d)$ is the model functor whose (unique) adjoint is necessarily the realization functor $\langle \cdot \rangle$.

Corollary 2.5 (Baues–Lemaire conjecture) *Let L be the minimal Quillen model of a simply connected complex of finite type X. Then the commutative differential graded algebra $\mathscr{C}^*(L)$ given by the Chevalley–Eilenberg cochain functor on L has the homotopy type of the Sullivan minimal model of X.*

Proof It is enough (and in fact equivalent) to check that the Sullivan realization of $\mathscr{C}^*(L)$ has the homotopy type of that of the Sullivan minimal model of X which is its rationalization $X_{\mathbb{Q}}$. Indeed,

$$\langle \mathscr{C}^*(L) \rangle_S \simeq \langle L \rangle \simeq \langle L \rangle_Q \simeq X_{\mathbb{Q}}.$$

For the first equivalence one can use different arguments; a direct proof is in [5, Theorem 8.1]. Also one can consider the simplicial isomorphism $\langle \mathscr{C}^*(L) \rangle_S \cong \mathrm{MC}_\bullet(L)$, which is classical and easy to prove (see the bibliographical notes of [4, Chapter 11]), and then take into account that $\langle L \rangle$ is a deformation retract of $\mathrm{MC}_\bullet(L)$. The second equivalence is Theorem 0.1. $\square$

3 Extending the Baues–Lemaire conjecture

Recall from [4, Section 3.2] that the *minimal model* of a connected cdgl M is a connected cdgl of the form $(\widehat{\mathbb{L}}(V), d)$, in which d is decomposable, together with a quasi-isomorphism

$$(\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} M.$$

The *minimal model* of a connected simplicial set X is the minimal model of $\mathfrak{L}_X^a$ where a is any of the 0-simplices of X regarded as a Maurer–Cartan element of $\mathfrak{L}_X$; see [4, Section 8.4].

In this context, the Baues–Lemaire conjecture has already been extended to the connected case: if L is the minimal Lie model of a connected simplicial set of finite type X then [4, Theorem 10.8] guarantees that $\varinjlim_n \mathscr{C}^*(L/L^n)$ is a Sullivan model of X.

On the other hand, the generalization of rational homotopy theory to nonnilpotent spaces given by Pridham in [12], by means of the pro-category of nilpotent, finite type, differential graded Lie algebras, also admits a deep extension of the Baues–Lemaire conjecture [12, Corollary 4.41 and Remark 4.42].

Another way to interpret this conjecture in this extended scenario is via the category **edgl** of *complete enriched* dgl's (edgl henceforth); see [6, Section 12] for details. A connected dgl L is complete enriched if there is a family $\{I_\alpha\}_{\alpha \in \mathcal{J}}$ of ideals of L, where $\mathcal{J}$ is well ordered and directed, satisfying

(i) $\alpha \geq \beta$ if and only if $I_\beta \subset I_\alpha$;

(ii) L/I_α is a finite type nilpotent dgl for each $\alpha \in \mathcal{J}$;

(iii) $L = \varprojlim_{\alpha \in \mathcal{J}} L/I_\alpha$.

Given an edgl L one can define the *enriched cochains* by

$$\mathscr{C}_e^*(L) = \varinjlim_{\alpha \in \mathcal{J}} \mathscr{C}^*(L/I_\alpha).$$

By Theorem 4 of [6, Section 15] the enriched cochains functors establishes an equivalence between the homotopy categories of edgl's and connected commutative differential graded algebras. In particular, the Sullivan model of any connected simplicial set can be regarded as the enriched cochains of some edgl.

We remark that the categories **edgl** and **cdgl** are different in general, but if L is a dgl such that $L/[L,L]$ is of finite type, then L is a cdgl if and only if it is an edgl. In particular $\mathfrak{L}_\bullet$ is a cosimplicial edgl and for each edgl L we may define its realization as the simplicial set

$$\langle L \rangle_e = \operatorname{Hom}_{\mathrm{edgl}}(\mathfrak{L}_\bullet, L). \tag{12}$$

By the same observation, if L is a cdgl such that $L/[L,L]$ is of finite type then $\langle L \rangle = \langle L \rangle_e$. However, in general, $\langle \cdot \rangle_e$ coincides with the Sullivan realization of the enriched cochains:

Proposition 3.1 *Let L be any edgl. Then $\langle \mathscr{C}_e^*(L) \rangle_S \simeq \langle L \rangle_e$.*

Proof Since the Sullivan realization functor is a contravariant right adjoint,

$$\langle \mathscr{C}_e^*(L) \rangle_S \simeq \varprojlim_{\alpha \in \mathcal{J}} \langle \mathscr{C}^*(L/I_\alpha) \rangle_S.$$

But, as each $L/I\alpha$ is of finite type, the above observation tells us that $\langle \mathscr{C}^*(L/I_\alpha) \rangle_S \simeq \langle L/I_\alpha \rangle \simeq \langle L/I_\alpha \rangle_e$. Finally, by (12), $\langle \cdot \rangle_e$ commutes with limits and we conclude that $\langle \mathscr{C}_e^*(L) \rangle_S \simeq \langle L \rangle_e$. □

Example 3.2 In the nonfinite type setting, the realization of a Sullivan model of a space which, as remarked above, is the enriched cochain functor of some edgl, may not coincide with the rationalization of the space. Or equivalently, with the realization of the (unenriched) Quillen model of the space. For instance, let $X = \bigvee_{i \geq 1} S_i^2$ be a countably infinite wedge of 2-spheres. Its Quillen model is the simply connected (and hence complete) dgl $L = (\mathbb{L}(V), 0)$ where V is a countably infinite-dimensional vector space concentrated in degree 1. Its realization $\langle L \rangle \simeq \langle L \rangle_Q$ is then homotopy equivalent to $\bigvee_{i \geq 1} (S_i^2)_{\mathbb{Q}}$.

On the other hand, let $\{v_i\}_{i \geq 1}$ be a basis of V, and denote $V(n) \subset V$ the vector space generated by the v_i for $i \geq n$. Then the family of ideals $\{I_n\}_{n \geq 1}$ of L, where $I_n = \bigcup_{j=0}^{n-1} \mathbb{L}^{\geq n-j}(V(j+1))$, defines an edgl $L' = \varprojlim_n L/I_n$ whose realization $\langle L' \rangle_e$, as remarked above, is weakly homotopy equivalent to the realization of the Sullivan minimal model of X. That is, $\varprojlim_n \bigvee_{i \leq n} (S_i^2)_{\mathbb{Q}}$.

References

[1] **H J Baues**, **J-M Lemaire**, *Minimal models in homotopy theory*, Math. Ann. 225 (1977) 219–242 MR Zbl

[2] **A Berglund**, *Rational homotopy theory of mapping spaces via Lie theory for L_∞-algebras*, Homology Homotopy Appl. 17 (2015) 343–369 MR Zbl

[3] **U Buijs**, **Y Félix**, **A Murillo**, **D Tanré**, *The infinity Quillen functor*, *Maurer–Cartan elements and DGL realizations*, preprint (2017) arXiv 1702.04397

[4] **U Buijs**, **Y Félix**, **A Murillo**, **D Tanré**, *Lie models in topology*, Progr. Math. 335, Birkhäuser, Cham (2020) MR Zbl

[5] **U Buijs**, **Y Félix**, **A Murillo**, **D Tanré**, *Lie models of simplicial sets and representability of the Quillen functor*, Israel J. Math. 238 (2020) 313–358 MR Zbl

[6] **Y Félix**, **S Halperin**, *Enriched differential Lie algebras in topology*, preprint (2022) arXiv 2208.12171

[7] **B Fresse**, *Homotopy of operads and Grothendieck–Teichmüller groups, I: The algebraic theory and its topological background*, Math. Surv. Monogr. 217, Amer. Math. Soc., Providence, RI (2017) MR Zbl

[8] **E Getzler**, *Lie theory for nilpotent L_∞-algebras*, Ann. of Math. 170 (2009) 271–301 MR Zbl

[9] **V Hinich**, *Descent of Deligne groupoids*, Int. Math. Res. Not. 1997 (1997) 223–239 MR Zbl

[10] **M Majewski**, *Rational homotopical models and uniqueness*, Mem. Amer. Math. Soc. 682, Amer. Math. Soc., Providence, RI (2000) MR Zbl

[11] **J P May**, *Simplicial objects in algebraic topology*, Van Nostrand Math. Stud. 11, Van Nostrand, Princeton, NJ (1967) MR Zbl

[12] **J P Pridham**, *Pro-algebraic homotopy types*, Proc. Lond. Math. Soc. 97 (2008) 273–338 MR Zbl

[13] **D Quillen**, *Rational homotopy theory*, Ann. of Math. 90 (1969) 205–295 MR Zbl

[14] **D Robert-Nicoud**, *Representing the deformation ∞-groupoid*, Algebr. Geom. Topol. 19 (2019) 1453–1476 MR Zbl

[15] **D Robert-Nicoud**, **B Vallette**, *Higher Lie theory*, preprint (2020) arXiv 2010.10485

[16] **D Sullivan**, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. 47 (1977) 269–331 MR Zbl

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The space of nonextendable quasimorphisms

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For a pair (G, N) of a group G with normal subgroup N, we consider the space of quasimorphisms and quasicocycles on N nonextendable to G. To treat this space, we establish the five-term exact sequence of cohomology relative to the bounded subcomplex. As an application, we study the spaces associated with the kernel of the (volume) flux homomorphism, the IA-automorphism group of a free group, and certain normal subgroups of Gromov-hyperbolic groups.

Furthermore, we employ this space to prove that the stable commutator length is equivalent to the stable mixed commutator length for certain pairs of a group and normal subgroup.

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1 Introduction

1.1 Motivations

A *quasimorphism* on a group G is a real-valued function $f: G \to \mathbb{R}$ on G satisfying

$$D(f) := \sup\{|f(xy) - f(x) - f(y)| : x, y \in G\} < \infty.$$

We call $D(f)$ the *defect* of the quasimorphism f. A quasimorphism f on G is said to be *homogeneous* if $f(x^n) = n \cdot f(x)$ for every $x \in G$ and every integer n. Let $\mathrm{Q}(G)$ denote the real vector space consisting of homogeneous quasimorphisms on G. The (homogeneous) quasimorphisms are closely related to the second bounded cohomology group $\mathrm{H}_b^2(G) = \mathrm{H}_b^2(G;\mathbb{R})$, and have been extensively studied in geometric group theory and symplectic geometry (see Calegari [26], Frigerio [40] and Polterovich and Rosen [88]).

In this paper, we consider a pair (G, N) of a group G with normal subgroup N. Let $i : N \to G$ be the inclusion map. In this setting, we can construct the following two real vector spaces:

- the space $\mathrm{Q}(N)^G$ of all G*-invariant homogeneous quasimorphisms on* N, where $f : N \to \mathbb{R}$ is said to be G*-invariant* if $f(gxg^{-1}) = f(x)$ for every $g \in G$ and every $x \in N$;
- the space $\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, where $\mathrm{H}^1(N)^G$ is the space of all G-invariant homomorphisms from N to $\mathbb{R}$ and i^* is the linear map from $\mathrm{Q}(G)$ to $\mathrm{Q}(N)$ induced by $i : N \hookrightarrow G$.

An element $f \in \mathrm{Q}(N)$ belongs to $i^*\mathrm{Q}(G)$ if and only if there exists $\hat{f} \in \mathrm{Q}(G)$ such that $\hat{f}|_N \equiv f$; in this case, we say that f is *extendable* to G. Since a homogeneous quasimorphism is conjugation-invariant (see Lemma 3.1), the space $i^*\mathrm{Q}(G)$ is contained in $\mathrm{Q}(N)^G$. The *extendability problem* asks whether there exists $f \in \mathrm{Q}(N)^G$ that is not extendable to G or, equivalently, whether the quotient vector space

$$\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$$

is nonzero. A stronger version of this problem asks whether the quotient space

$$\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$$

is nonzero. We have some reasons to take the quotient vector space over $\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, instead of one over $i^*\mathrm{Q}(G)$. Elements in $\mathrm{H}^1(N)^G$ seem "trivial" as quasimorphisms in $\mathrm{Q}(N)^G$; also, when we apply the Bavard duality theorem for stable mixed commutator lengths (see Theorem 7.1), precisely the elements in $\mathrm{H}^1(N)^G$ behave trivially. An example of a pair (G, N) such that $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$ is nonzero is provided by Shtern [90], and an example of a pair such that $\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ is nonzero is provided by the first and second authors [58]. Some of the authors generalize the result of [58] and provide an extrinsic application in [61] (see Theorem 1.3).

Here we show that, under a certain condition on $\Gamma = G/N$ and a mild condition on G, the quotient real vector space $\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ is finite-dimensional. For example, amenability of Γ and finite presentability of G suffice. We exhibit here two such examples: one corresponds to a surface group (Theorem 1.1), and the other to the fundamental group of a hyperbolic mapping torus (Theorem 1.2). We discuss this point in more detail in the latter part of this subsection. We remark that in Theorem 1.2 the group quotient $\Gamma = G/N$ is nonabelian solvable in general. The main novel point of these theorems is that we obtain nonzero finite-dimensionality of vector spaces associated with quasimorphisms: the (quotient) spaces of homogeneous quasimorphisms modulo genuine homomorphisms tend to be either zero- or infinite-dimensional for groups naturally appearing in geometric group theory. To be more precise, if a group admits hyperbolicity in a certain weak sense, then the space is infinite-dimensional (see Bestvina and Fujiwara [12]); the space vanishes for higher-rank lattices (see Burger and Monod [22; 23]). We also mention that there are some exceptions in the world of one-dimensional dynamics to the aforementioned tendency, such as certain Thompson-type groups; see Fournier-Facio and Lodha [36] and Calegari [26, Chapter 5].

Theorem 1.1 (nonzero finite-dimensionality in surface groups) *Let l be an integer greater than* 1, *$G=\pi_1(\Sigma_l)$ the surface group with genus l, and N the commutator subgroup $[\pi_1(\Sigma_l),\pi_1(\Sigma_l)]$ of $\pi_1(\Sigma_l)$. Then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = l(2l-1) \quad \textit{and} \quad \dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\bigr)=1.$$

For $l\in\mathbb{N}$, let $\mathrm{Mod}(\Sigma_l)$ be the mapping class group of the surface Σ_l and $s_l\colon \mathrm{Mod}(\Sigma_l)\to \mathrm{Sp}(2l,\mathbb{Z})$ the symplectic representation. For a mapping class $\psi\in\mathrm{Mod}(\Sigma_l)$, we take a diffeomorphism f representing ψ and let T_f denote the mapping torus of f. The fundamental group of T_f is isomorphic to the semidirect product $\pi_1(\Sigma_l)\rtimes_{f_*}\mathbb{Z}$ and surjects onto $\mathbb{Z}^{2l}\rtimes_{s_l(\psi)}\mathbb{Z}$ via the abelianization map $\pi_1(\Sigma_l)\to \mathrm{H}_1(\Sigma_l;\mathbb{Z})$. Note that the kernel of the surjection is equal to the commutator subgroup of $\pi_1(\Sigma_l)$.

Theorem 1.2 (nonzero finite-dimensionality in hyperbolic mapping tori) *Let l be an integer greater than* 1, *$\psi\in\mathrm{Mod}(\Sigma_l)$ a pseudo-Anosov element and f a diffeomorphism representing ψ. Let G be the fundamental group of the mapping torus T_f and N the kernel of the surjection $G\to\mathbb{Z}^{2l}\rtimes_{s_l(\psi)}\mathbb{Z}$. Then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim\mathrm{Ker}(I_{2l}-s_l(\psi)) + \dim\mathrm{Ker}\bigl(I_{\binom{2l}{2}}-\textstyle\bigwedge^2 s_l(\psi)\bigr)$$

and

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\bigr) = \dim\mathrm{Ker}(I_{2l}-s_l(\psi))+1.$$

Here, for $n\in\mathbb{N}$, I_n denotes the identity matrix of size n, and $\bigwedge^2 s_l(\psi)\colon \bigwedge^2\mathbb{R}^{2l}\to\bigwedge^2\mathbb{R}^{2l}$ is the map induced by $s_l(\psi)$.

In particular, if ψ is in the Torelli group (that is, $s_l(\psi)=I_{2l}$), then

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = 2l+\binom{2l}{2} \quad \textit{and} \quad \dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\bigr)=2l+1.$$

In Theorem 1.2, the pseudo-Anosov property for ψ is assumed to ensure hyperbolicity of G; see Theorem 4.8. In Theorems 4.5 and 4.11, we also obtain results analogous to Theorems 1.1 and 1.2 in the free group setting.

In study of quasimorphisms, it is often quite hard to obtain nonzero finite-dimensionality. For instance, if a group G can act nonelementarily in a certain good manner on a Gromov-hyperbolic geodesic space, then the dimension of $\mathrm{Q}(G)$ is of the cardinal of the continuum [12]; contrastingly, a higher-rank lattice G has zero $\mathrm{Q}(G)$ [22]. For a group G such that the dimension of $\mathrm{Q}(G)$ is of the cardinal of the continuum, understanding *all* quasimorphisms on G might have been considered an impossible subject. Our study of the space of nonextendable quasimorphisms might have some possibility of shedding light on this problem modulo "trivial or extendable" quasimorphisms.

Theorems 1.1 and 1.2 treat the case where G is a nonelementary Gromov-hyperbolic group and N a subgroup with solvable quotient. In this case, a result of Epstein and Fujiwara [34] implies that the dimension of $\mathrm{Q}(G)$ is the cardinal of the continuum. The kernel of the restriction $i^*:\mathrm{Q}(G)\to\mathrm{Q}(N)^G$ can be identified with $\mathrm{Q}(\Gamma)$. Since Γ is now finitely generated solvable, the dimension of $\mathrm{Q}(\Gamma)=\mathrm{H}^1(\Gamma)$ is finite. This implies that the dimension of $\mathrm{Q}(N)^G$ is also the cardinal of the continuum. Nevertheless, our results (Theorems 1.9 and 1.10) imply that the spaces $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))$ are always both finite-dimensional. Theorems 1.1 and 1.2 provide nonvanishing examples, and it might be an interesting problem to understand *all* quasimorphism classes in these examples.

We outline how we deduce finite-dimensionality of $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))$ under certain conditions in our results. Our main theorem, Theorem 1.5 (stated in Section 1.2), establishes the five-term exact sequence of the cohomology $\mathrm{H}^\bullet_{/b}$ associated with a short exact sequence of groups

$$1\to N\to G\to\Gamma\to 1.$$

Here, $\mathrm{H}^\bullet_{/b}$ relates the bounded cohomology $\mathrm{H}^\bullet_b$ with the ordinary cohomology $\mathrm{H}^\bullet$; see Section 1.2 for the precise definition of $\mathrm{H}^\bullet_{/b}$. In Theorems 1.9 and 1.10, we assume that Γ is *boundedly* 3*-acyclic*, meaning that $\mathrm{H}^2_b(\Gamma;\mathbb{R})=0$ and $\mathrm{H}^3_b(\Gamma;\mathbb{R})=0$ (Definition 1.8). Then the five-term exact sequence enables us to relate $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))$, respectively, to the ordinary second cohomology $\mathrm{H}^2(\Gamma)=\mathrm{H}^2(\Gamma;\mathbb{R})$ and $\mathrm{H}^2(G)=\mathrm{H}^2(G;\mathbb{R})$. Since second ordinary cohomology is finite-dimensional under certain mild conditions, we obtain the desired finite-dimensionality results. In this point of view, our main theorem (Theorem 1.5) might be regarded as filling in a missing piece between the bounded cohomology theory and the ordinary cohomology theory.

We also note that the extendability and nonextendability of invariant quasimorphisms themselves have applications. See Section 2.1 for an application to the stable (mixed) commutator lengths, and Section 2.3 for one to symplectic geometry. As a notable extrinsic application, we recall our following theorem:

Theorem 1.3 [61, Theorem 1.1] *Let Σ_l be a closed orientable surface whose genus l is at least two and Ω an area form on S. Let $\mathrm{Diff}_0(\Sigma_l,\Omega)$ denote the identity component of the group of diffeomorphisms of Σ_l that preserve Ω. Assume that a pair $f,g\in\mathrm{Diff}_0(\Sigma_l,\Omega)$ satisfies $fg=gf$. Let $\mathrm{Flux}_\Omega:\mathrm{Diff}_0(\Sigma_l,\Omega)\to\mathrm{H}^1(\Sigma_l;\mathbb{R})$ be the volume flux homomorphism and $\smile:\mathrm{H}^1(\Sigma_l;\mathbb{R})\times\mathrm{H}^1(\Sigma_l;\mathbb{R})\to\mathrm{H}^2(\Sigma_l;\mathbb{R})\cong\mathbb{R}$ the cup product. Then*

$$\mathrm{Flux}_\Omega(f)\smile\mathrm{Flux}_\Omega(g)=0.$$

The statement of Theorem 1.3 might not seem to have any relation to quasimorphisms. Nevertheless, the key to the proof is comparison between vanishing and nonvanishing of $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$, where $G=\mathrm{Flux}_\Omega^{-1}(\langle\mathrm{Flux}_\Omega(f),\mathrm{Flux}_\Omega(g)/k\rangle)$ for a sufficiently large integer k and $N=\mathrm{Ker}(\mathrm{Flux}_\Omega)$; see Section 2.3 for basic concepts around volume flux homomorphisms. (We discuss a related example in Example 7.15.)

1.2 Main theorem

To treat the spaces of nonextendable quasimorphisms, we establish the five-term exact sequence of group cohomology $\mathrm{H}^{\bullet}_{/b}$ relative to bounded cochain complexes. Throughout the paper, the coefficient module of the cohomology groups is the field $\mathbb{R}$ of real numbers unless otherwise specified. The main reason we are interested in this cohomology $\mathrm{H}^{\bullet}_{/b}$ is that $\mathrm{H}^1_{/b}(G)$ and $\mathrm{Q}(G)$ are isomorphic (see Remark 1.6).

Now we start the definition of $\mathrm{H}^{\bullet}_{/b}$. Let V be a left normed G-module, and $C^n(G;V)$ the space of functions from the n-fold direct product $G^{\times n}$ of G to V. The group cohomology is defined by the cohomology group of $C^n(G;V)$ with a certain differential (see Section 3 for the precise definition). Recall that the spaces $C^n_b(G;V)$ of the bounded functions form a subcomplex of $C^{\bullet}(G;V)$, and its cohomology group is the bounded cohomology group of G. We write $C^{\bullet}_{/b}(G;V)$ to indicate the quotient complex $C^{\bullet}(G;V)/C^{\bullet}_b(G;V)$, and write $\mathrm{H}^{\bullet}_{/b}(G;V)$ to mean its cohomology group.

Our main result is the five-term exact sequence of the cohomology $\mathrm{H}^{\bullet}_{/b}$. Before stating our main theorem, we first recall the five-term exact sequence of ordinary group cohomology.

Theorem 1.4 (see for example Brown [19, Corollary VII.6.4]) *Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence of groups and V a left $\mathbb{R}[\Gamma]$-module. Then there exists an exact sequence*

$$0 \to \mathrm{H}^1(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^1(G;V) \xrightarrow{i^*} \mathrm{H}^1(N;V)^G \xrightarrow{\tau} \mathrm{H}^2(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^2(G;V).$$

The following theorem is the main result in this paper:

Theorem 1.5 (main theorem) *Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence of groups and V a left Banach $\mathbb{R}[\Gamma]$-module equipped with a Γ-invariant norm $\|\cdot\|$. Then there exists an exact sequence*

$$0 \to \mathrm{H}^1_{/b}(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^1_{/b}(G;V) \xrightarrow{i^*} \mathrm{H}^1_{/b}(N;V)^G \xrightarrow{\tau_{/b}} \mathrm{H}^2_{/b}(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^2_{/b}(G;V). \tag{1-1}$$

Moreover, the exact sequence above is compatible with the five-term exact sequence of group cohomology; that is, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(\Gamma;V) \arrow[r, "p^*"] \arrow[d, "\xi_1"] & \mathrm{H}^1(G;V) \arrow[r, "i^*"] \arrow[d, "\xi_2"] & \mathrm{H}^1(N;V)^G \arrow[r, "\tau"] \arrow[d, "\xi_3"] & \mathrm{H}^2(\Gamma;V) \arrow[r, "p^*"] \arrow[d, "\xi_4"] & \mathrm{H}^2(G;V) \arrow[d, "\xi_5"] \\
0 \arrow[r] & \mathrm{H}^1_{/b}(\Gamma;V) \arrow[r, "p^*"] & \mathrm{H}^1_{/b}(G;V) \arrow[r, "i^*"] & \mathrm{H}^1_{/b}(N;V)^G \arrow[r, "\tau_{/b}"] & \mathrm{H}^2_{/b}(\Gamma;V) \arrow[r, "p^*"] & \mathrm{H}^2_{/b}(G;V)
\end{tikzcd} \tag{1-2}$$

Here the ξ_i are the maps induced from the quotient map $C^{\bullet} \to C^{\bullet}_{/b}$.

Remark 1.6 By the definition of $\mathrm{H}^{\bullet}_{/b}$, there exists a natural map $\mathrm{Q}(G) \to \mathrm{H}^1_{/b}(G) = \mathrm{H}^1_{/b}(G;\mathbb{R})$, and it is known that this is an isomorphism (see the proof of [26, Theorem 2.50]). Then diagram (1-2) gives rise to

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(\Gamma) \arrow[r, "p^*"] \arrow[d, "\xi_1"] & \mathrm{H}^1(G) \arrow[r, "i^*"] \arrow[d, "\xi_2"] & \mathrm{H}^1(N)^G \arrow[r, "\tau"] \arrow[d, "\xi_3"] & \mathrm{H}^2(\Gamma) \arrow[r, "p^*"] \arrow[d, "\xi_4"] & \mathrm{H}^2(G) \arrow[d, "\xi_5"] \\
0 \arrow[r] & \mathrm{Q}(\Gamma) \arrow[r, "p^*"] & \mathrm{Q}(G) \arrow[r, "i^*"] & \mathrm{Q}(N)^G \arrow[r, "\tau_{/b}"] & \mathrm{H}^2_{/b}(\Gamma) \arrow[r, "p^*"] & \mathrm{H}^2_{/b}(G)
\end{tikzcd} \tag{1-3}$$

Note that the exactness of the sequence

$$0 \to \mathrm{Q}(\Gamma) \xrightarrow{p^*} \mathrm{Q}(G) \xrightarrow{i^*} \mathrm{Q}(N)^G$$

is well known (see [26, Remark 2.90]).

Remark 1.7 It is straightforward to show that the quotient space $\mathrm{H}^1_{/b}(N;V)^G/i^*\mathrm{H}^1_{/b}(G;V)$ is isomorphic to $\widehat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}/i^*\widehat{\mathrm{Q}}Z(G;V)$, where $\widehat{\mathrm{Q}}Z(G;V)$ and $\widehat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$ are the spaces of quasicocycles on G and G-quasiequivariant V-valued quasimorphisms on N, respectively (see Definition 6.1 and Section 8.2; see also Remark 6.4). In Section 8.2, we will apply Theorem 1.5 to the extension problem of G-quasiequivariant quasimorphisms on N to quasicocycles on G. We also mention that, for a group–subgroup pair (G,H) for H (*not* being normal in G, but) being hyperbolically embedded into G, certain extension theorems of quasicocycles have been obtained by Hull and Osin [49] and Frigerio, Pozzetti and Sisto [41].

This theorem provides several arguments to estimate the dimensions of the spaces $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ as follows. Here we recall the definition of *bounded k-acyclicity* of groups from Ivanov [53] and Moraschini and Raptis [82].

Definition 1.8 (bounded k-acyclicity) Let k be a positive integer. A group G is said to be *boundedly k-acyclic* if $\mathrm{H}^i_b(G) = 0$ for every positive integer i with $i \le k$.

We note that $\mathrm{H}^1_b(G) = 0$ for every group G. We recall properties and examples of boundedly k-acyclic groups in Theorem 3.6. In particular, we recall that amenable groups, such as abelian groups, are boundedly k-acyclic for all k (Theorem 3.5(5)).

Theorem 1.9 *If the quotient group $\Gamma = G/N$ is boundedly 3-acyclic, then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2(\Gamma).$$

Moreover, if G is Gromov-hyperbolic, then

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma).$$

In fact, the assumption of Gromov-hyperbolicity of G can be weakened to the surjectivity of the comparison map $c_G\colon \mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$; see Theorem 4.1.

On the space $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$, we also obtain the following:

Theorem 1.10 *If $\Gamma = G/N$ is boundedly 3-acyclic, then the map $p^* \circ (\xi_4)^{-1} \circ \tau_{/b}$ induces an isomorphism*

$$\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) \cong \mathrm{Im}(p^*) \cap \mathrm{Im}(c_G),$$

where $c_G\colon \mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is the comparison map. In particular, if Γ is boundedly 3-acyclic, then

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) \le \dim \mathrm{H}^2(G).$$

When $N = [G, G]$, we have a more precise calculation of $\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big)$ (see Corollary 4.3). As we mentioned in the previous subsection, there are many examples of a finitely presented group whose space of homogeneous quasimorphisms is infinite-dimensional, for instance any nonelementary Gromov-hyperbolic group [34]. Nevertheless, if we assume that $\Gamma = G/N$ is boundedly 3-acyclic, then we have the following two statements: the space $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ is finite-dimensional if G is finitely presented (following from Theorem 1.10); and the space $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ is finite-dimensional if Γ is finitely presented (following from Theorem 1.9).

There are several known conditions that guarantee $\mathrm{Q}(N)^G = i^*\mathrm{Q}(G)$, ie every G-invariant quasimorphism is extendable (see Malyutin [67], Ishida [50; 51] Shtern [90] and Kawasaki, Kimura, Matsushita and Mimura [60]). We say that a group homomorphism $p\colon G \to \Gamma$ *virtually splits* if there exist a subgroup Λ of finite index in Γ and a group homomorphism $s\colon \Lambda \to G$ such that $f \circ s(x) = x$ for every $x \in \Lambda$. The first, second, fourth and fifth authors showed that, if the group homomorphism $p\colon G \to \Gamma$ virtually splits, then $\mathrm{Q}(N)^G = i^*\mathrm{Q}(G)$ [60]. Thus the space $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$, which we consider in Theorem 1.9, can be seen as a space of obstructions to the existence of virtual splittings.

2 Other applications of the main theorem

In this section, we provide several other applications of our main theorem (Theorem 1.5); we also use its corollaries, Theorems 1.9 and 1.10. To conclude the section, we briefly describe the organization of the rest of the paper.

2.1 On equivalences of scl_G and $\mathrm{scl}_{G,N}$

As an application of the spaces of nonextendable quasimorphisms, we treat the equivalence problems of the stabilizations of usual and mixed commutator lengths. For two nonnegative-valued functions μ and ν on a group G, we say that μ and ν are *bi-Lipschitzly equivalent* (or *equivalent* in short) if there exist positive constants C_1 and C_2 such that $C_1\nu \le \mu \le C_2\nu$. By Theorem 1.10, $\mathrm{H}^2(G) = 0$ implies that $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) = 0$ if $\Gamma = G/N$ is boundedly 3-acyclic. We show that the condition $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) = 0$ implies that certain two stable word lengths related to commutators are bi-Lipschitzly equivalent.

Let G be a group and N a normal subgroup. A (G,N)*-commutator* is an element of G of the form $[g,x] = gxg^{-1}x^{-1}$ for some $g \in G$ and $x \in N$. Let $[G,N]$ be the group generated by the set of (G,N)-commutators. Then $[G,N]$ is a normal subgroup of G and is included in N. For an element x in $[G,N]$, the (G,N)*-commutator length* or the *mixed commutator length* of x is defined to be the minimum number n such that there exist n (G,N)-commutators $c_1,\dots,c_n$ such that $x = c_1\cdots c_n$, and is denoted by $\mathrm{cl}_{G,N}(x)$. Then there exists a limit

$$\mathrm{scl}_{G,N}(x) := \lim_{n\to\infty}\frac{\mathrm{cl}_{G,N}(x^n)}{n}$$

and we call $\mathrm{scl}_{G,N}(x)$ the *stable* (G,N)*-commutator length* of x.

When $N = G$, $\mathrm{cl}_{G,G}(x)$ and $\mathrm{scl}_{G,G}(x)$ equal the commutator length and stable commutator length of x, respectively. We write $\mathrm{cl}_G(x)$ and $\mathrm{scl}_G(x)$ instead of $\mathrm{cl}_{G,G}(x)$ and $\mathrm{scl}_{G,G}(x)$. The commutator lengths and stable commutator lengths have a long history (see [26]), for instance in the study of theory of mapping class groups (see [32; 27; 10]) and diffeomorphism groups (see [21; 95; 96; 97; 17]). The celebrated Bavard duality theorem [6] describes the relationship between homogeneous quasimorphisms and the stable commutator length. In particular, for an element $x \in [G, G]$, $\mathrm{scl}_G(x)$ is nonzero if and only if there exists a homogeneous quasimorphism f on G with $f(x) \neq 0$.

In [58; 60], we construct a pair (G, N) such that scl_N and $\mathrm{scl}_{G,N}$ are not bi-Lipschitzly equivalent on $[N, N]$. Contrastingly, in several cases it is known that scl_G and $\mathrm{scl}_{G,N}$ are bi-Lipschitzly equivalent on $[G, N]$. For example, if the map $p\colon G \to \Gamma = G/N$ virtually splits, then scl_G and $\mathrm{scl}_{G,N}$ are bi-Lipschitzly equivalent on $[G, N]$. In this paper, the vanishing of $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ implies the equivalence of scl_G and $\mathrm{scl}_{G,N}$ as follows. We note that $\mathrm{H}^2(G) = 0$ implies $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) = 0$ by Theorem 1.10 when $\Gamma = G/N$ is boundedly 3-acyclic.

Theorem 2.1 *Assume that* $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. *Then*:

(1) scl_G *and* $\mathrm{scl}_{G,N}$ *are bi-Lipschitzly equivalent on* $[G, N]$.

(2) *If* $\Gamma = G/N$ *is amenable, then* $\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x) \le 2 \cdot \mathrm{scl}_G(x)$ *for all* $x \in [G, N]$.

(3) *If* $\Gamma = G/N$ *is solvable, then* $\mathrm{scl}_G(x) = \mathrm{scl}_{G,N}(x)$ *for all* $x \in [G, N]$.

Remark 2.2 Recently, several examples of nonamenable boundedly acyclic groups have been constructed (see [38; 37; 79; 78]; we also recall some of them in Theorem 3.6). However, our proof of Theorem 2.1(2) does *not* work if the assumption of amenability of Γ in (2) is replaced with bounded 3-acyclicity. Indeed, in our proof, we use the fact that $\mathrm{H}_b^2(G) \to \mathrm{H}_b^2(N)^G$ is *isometric*, which is deduced from the amenability of Γ [74, Proposition 8.6.6].

By Theorem 1.10, when G/N is boundedly 3-acyclic, $\mathrm{H}^2(G) = 0$ implies $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, and hence $\mathrm{scl}_{G,N}$ and scl_G are equivalent on $[G, N]$. There are plenty of examples of groups whose second cohomology groups vanish, as follows:

- The free groups F_n.
- Let l be a positive integer. Let N_l be the nonorientable closed surface with genus l, and set $G = \pi_1(N_l)$. Then, $G = \langle a_1, \ldots, a_l \mid a_1^2 \cdots a_l^2 \rangle$ and $\mathrm{H}^2(G) = \mathrm{H}^2(N_l) = 0$.
- Let K be a knot in S^3. Then the knot group G of K is defined to be the fundamental group of the complement $S^3 \setminus K$. Since $S^3 \setminus K$ is an Eilenberg–Mac Lane space, we have $\mathrm{H}^2(G) = \mathrm{H}^2(S^3 \setminus K) = \widetilde{\mathrm{H}}_0(K) = 0$.
- The braid group B_n. Akita and Liu [1, Corollary 3.21] gave sufficient conditions on a labeled graph Γ for the real second cohomology group of the Artin group $A(\Gamma)$ to vanish.
- The Baumslag–Solitar groups $\mathrm{BS}(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$ with $m \neq n$ (see [13], for example).
- Free products of the above groups.

For other examples satisfying $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, see Example 4.13 and Corollaries 4.12, 4.14 and 4.16.

Finally, we discuss pairs (G, N) with nonequivalent scl_G and $\mathrm{scl}_{G,N}$. In [58], the first and second authors provided the first example of such (G, N) (Example 7.14); we obtain another example with smaller G in Example 7.15, which follows from [61]. These two examples may be seen as one example, coming from symplectic geometry. Unfortunately, in the present paper, we are unable to provide any new example from a different background. We remark that some of the authors [69] provided new examples after our work here; see the discussion below Problem 9.9. By Theorem 2.1, the vanishing of $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ implies the equivalence of scl_G and $\mathrm{scl}_{G,N}$. After this work, the authors proved that its converse holds if $N = [G, G]$ [59]. We discuss problems on this equivalence/nonequivalence in more detail in Section 9.2.

2.2 The case of IA-automorphism groups of free groups

Here we provide an example where $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ but Γ is not amenable. Our example comes from the automorphism group of a free group and the IA-automorphism group. The group of automorphisms of a group G is denoted by $\mathrm{Aut}(G)$. Let IA_n be the IA-automorphism group of the free group F_n, ie the kernel of the natural homomorphism $\mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$. Let $\mathrm{Aut}(F_n)_+$ denote the preimage of $\mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{Aut}(F_n)$. The following theorem will be proved in Section 8; see Theorems 8.9 and 8.17 for more general statements.

Theorem 2.3 (1) *For every $n \geq 2$, we have $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}(F_n)} = i^*\mathrm{Q}(\mathrm{Aut}(F_n))$ and $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}_+(F_n)} = i^*\mathrm{Q}(\mathrm{Aut}_+(F_n))$.*

(2) *For every $n \geq 6$ and every subgroup G of $\mathrm{Aut}(F_n)$ of finite index, $\mathrm{Q}(N)^G = i^*\mathrm{Q}(G)$. Here, $N = \mathrm{IA}_n \cap G$.*

Remark 2.4 (1) The bound $n \geq 6$ in Theorem 2.3(2) comes from Theorem 8.6(1), which treats an effective bound of the *Borel stable range* for second ordinary cohomology with the trivial real coefficients of SL_n.

In fact, by appealing to a recent result of Bader and Sauer [3], we are able to improve this bound to $n \geq 4$. We will see this in Theorem 8.17.

(2) Corollary 3.8 of [43] implies that $\mathrm{H}^2(\mathrm{Aut}(F_n)) = 0$ for $n \geq 5$. However, $\mathrm{H}^2(\Lambda)$ for a subgroup Λ of finite index in $\mathrm{Aut}(F_n)$ is mysterious in general. Even on H^1, only quite recently it has been proved that $\mathrm{H}^1(\Lambda) = 0$ if $n \geq 4$; the proof is based on Kazhdan's property (T) for $\mathrm{Aut}(F_n)$ for $n \geq 4$. See [55; 54; 86]. We refer to [8] for a comprehensive treatise on property (T). Contrastingly, by [71], there exists a subgroup Λ of finite index in $\mathrm{Aut}(F_3)$ such that $\mathrm{H}^1(\Lambda) \neq 0$.

(3) The same conclusions as in Theorem 2.3 hold if we replace $\mathrm{Aut}(F_n)$ and IA_n with $\mathrm{Out}(F_n)$ and $\overline{\mathrm{IA}}_n$, respectively. Here, $\overline{\mathrm{IA}}_n$ denotes the kernel of the natural map $\mathrm{Out}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$. Indeed, the proofs which will be presented in Section 8 work without any essential change.

(4) If $n \geq 3$ and G is a subgroup of $\mathrm{Aut}(F_n)$ of finite index, then the real vector space $i^*\mathrm{Q}(G)$ is infinite-dimensional. Indeed, we can apply [12] to the acylindrically hyperbolic group $\mathrm{Out}(F_n)$, whose amenable radical is trivial. Moreover, thanks to [39, Corollary 4.3], we may construct an infinite collection of homogeneous quasimorphisms on $\mathrm{Out}(F_n)$ which is linearly independent even when these quasimorphisms are restricted to $[\overline{\mathrm{IA}}_n \cap \overline{G}, \overline{\mathrm{IA}}_n \cap \overline{G}]$. Here $\overline{G}$ is the image of G under the natural projection $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$. Then consider the restriction of this collection on $\overline{G}$, and take the pullback of it under the projection $G \to \overline{G}$.

In fact, Corollary 1.2 of [9] treats quasicocycles into unitary representations. Then the following may be deduced in a similar manner to the above: Let G be a subgroup of $\mathrm{Aut}(F_n)$ of finite index with $n \geq 3$ and $\Gamma := G/(\mathrm{IA}_n \cap G)$. Let $(\pi, \mathcal{H})$ be a unitary Γ-representation, and $(\bar{\pi}, \mathcal{H})$ the pullback of it under the projection $G \to \Gamma$. Then the vector space $i^*\widehat{\mathrm{Q}}Z(G, \bar{\pi}, \mathcal{H})$ of the quasicocycles is infinite-dimensional. Furthermore, Corollary 1.2 of [9] and its proof can be employed to obtain the corresponding result in the setting where G is a subgroup of $\mathrm{Mod}(\Sigma_l)$ of finite index with $l \geq 3$ and $(\pi, \mathcal{H})$ is a unitary representation of $G/(\mathcal{I}(\Sigma_l) \cap G)$. Here $\mathcal{I}(\Sigma_l)$ denotes the Torelli group.

If (G, N) equals $(\mathrm{Mod}(\Sigma_l), \mathcal{I}(\Sigma_l))$ or its analogue for the setting of subgroups of finite index, then the situation is subtle. See Theorems 8.10 and 8.14 for our results. We remark that the question on the extendability of quasimorphisms might be open; see Problem 8.18.

2.3 Applications to volume flux homomorphisms

In Section 5, we will provide applications of Theorem 1.10 to diffeomorphism groups.

We study the problem to determine which cohomology class admits a bounded representative. Notably, the problem on (subgroups of) diffeomorphism groups is interesting and has been studied in view of characteristic classes of fiber bundles. However, the problem is often quite difficult and, in fact, there are only a few cohomology classes that are known to be bounded or not. Here we restrict our attention to the case of degree two cohomology classes. The best-known example is the Euler class of $\mathrm{Diff}_+(S^1)$, which has a bounded representative. The Godbillon–Vey class integrated along the fiber defines a cohomology class of $\mathrm{Diff}_+(S^1)$, which has no bounded representatives [91]. It was shown in [25] that the Euler class of $\mathrm{Diff}_0(\mathbb{R}^2)$ is unbounded. In the case of three-dimensional manifolds, the identity components of the diffeomorphism groups of several closed Seifert-fibered three-manifolds admit cohomology classes of degree two which do not have bounded representatives [68].

Let M be an m-dimensional manifold and Ω a volume form. Then we can define the flux homomorphism (on the universal covering) $\widetilde{\mathrm{Flux}}_\Omega \colon \widetilde{\mathrm{Diff}}_0(M, \Omega) \to \mathrm{H}^{m-1}(M)$, the flux group Γ_Ω, and the flux homomorphism $\mathrm{Flux}_\Omega \colon \mathrm{Diff}_0(M, \Omega) \to \mathrm{H}^{m-1}(M)/\,\Gamma_\Omega$; see Section 5 for the precise definitions.

As applications of Theorem 1.10, we have a few results related to the comparison maps $\mathrm{H}_b^2(\mathrm{Diff}_0(M, \Omega)) \to \mathrm{H}^2(\mathrm{Diff}_0(M, \Omega))$ and $\mathrm{H}_b^2(\widetilde{\mathrm{Diff}}_0(M, \Omega)) \to \mathrm{H}^2(\widetilde{\mathrm{Diff}}_0(M, \Omega))$.

It is known that the spaces $\mathrm{H}^2(\mathrm{Diff}_0(M,\Omega))$ and $\mathrm{H}^2(\widetilde{\mathrm{Diff}}_0(M,\Omega))$ can be very large (see for instance the following proposition in Kotschick and Morita [65]). Note that $\mathrm{H}^n(\mathbb{R}^m;\mathbb{R})$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}\big(\bigwedge^n_{\mathbb{Z}}(\mathbb{R}^m);\mathbb{R}\big)$.

Proposition 2.5 [65] *The homomorphisms*

$$\mathrm{Flux}_\Omega^*\colon \mathrm{H}^2(\mathrm{H}^{m-1}(M)/\Gamma_\Omega)\to \mathrm{H}^2(\mathrm{Diff}_0(M,\Omega)),\quad \widetilde{\mathrm{Flux}}_\Omega^*\colon \mathrm{H}^2(\mathrm{H}^{m-1}(M))\to \mathrm{H}^2(\widetilde{\mathrm{Diff}}_0(M,\Omega))$$

induced by the flux homomorphisms are injective.

As an application of Theorem 1.10, we have the following theorem:

Theorem 2.6 *Let (M,Ω) be an m-dimensional closed manifold with volume form Ω. Then the following hold*:

(1) *If $m=2$ and the genus of M is at least 2, then there exists at least one nontrivial element of $\mathrm{Im}(\mathrm{Flux}_\Omega^*)$ represented by a bounded 2-cochain.*
(2) *Otherwise, every nontrivial element of $\mathrm{Im}(\mathrm{Flux}_\Omega^*)$ and $\mathrm{Im}(\widetilde{\mathrm{Flux}}_\Omega^*)$ cannot be represented by a bounded 2-cochain.*

Note that in case (1) it is known that $\pi_1(\mathrm{Diff}_0(M,\Omega))=0$ (see for example [87, Section 7.2.B]); in particular, the flux group Γ_Ω is zero.

In the proof of Theorem 2.6(1), we essentially prove the nontriviality of the cohomology class $c_P\in \mathrm{Im}(\mathrm{Flux}_\Omega^*)$, called the *Py class*. In Section 9.1, we provide some observations on the Py class.

Organization of the paper

Section 3 collects preliminary facts. In Section 4, we first prove Theorems 1.9 and 1.10, assuming Theorem 1.5. Secondly, we show Theorems 1.1 and 1.2. In Section 5, we provide applications of Theorem 1.5 to the volume flux homomorphisms. Section 6 is devoted to the proof of Theorem 1.5. In Section 7, we prove Theorem 2.1. In Section 8, we prove Theorem 2.3, as well as Theorems 8.9 and 8.10 (and furthermore Theorems 8.17 and 8.14). In Section 9, we provide several open problems.

3 Preliminaries

Before proceeding to the main part of this section, we collect basic properties of quasimorphisms and state them as Lemmas 3.1 and 3.2; see [26, Sections 2.2 and 2.4] for more details. Lemma 3.1 follows from the equality $(ghg^{-1})^n=gh^ng^{-1}$ for every $g,h\in G$ and every $n\in\mathbb{Z}$.

Lemma 3.1 *A homogeneous quasimorphism is conjugation-invariant.*

In particular, the restriction of a homogeneous quasimorphism f of G to a normal subgroup N is G-invariant.

For a quasimorphism $f\colon G\to\mathbb{R}$, the Fekete lemma guarantees that the limit

$$\bar{f}(x)=\lim_{n\to\infty}\frac{f(x^n)}{n}$$

exists. The function $\bar{f}$ defined by the above equation is called the *homogenization of* f.

Lemma 3.2 (1) *$\bar{f}$ is a homogeneous quasimorphism.*

(2) *$|\bar{f}(x)-f(x)|\le D(f)$ for every $x\in G$.*

(3) [26, Lemma 2.58] *$D(\bar{f})\le 2D(f)$.*

In this section, we recall definitions and facts related to the ordinary and bounded cohomology of groups. For a more comprehensive introduction to this subject, we refer to [45; 26; 40].

Let V be a left $\mathbb{R}[G]$-module and $C^n(G;V)$ the vector space consisting of functions from the n-fold direct product $G^{\times n}$ to V. Let $\delta\colon C^n(G;V)\to C^{n+1}(G;V)$ be the $\mathbb{R}$-linear map defined by

$$(\delta f)(g_0,\dots,g_n)=g_0\cdot f(g_1,\dots,g_n)+\sum_{i=1}^{n}(-1)^i f(g_0,\dots,g_{i-1}g_i,\dots,g_n)+(-1)^{n+1}f(g_0,\dots,g_{n-1}).$$

Then $\delta^2=0$ and its n^{th} cohomology is the *ordinary group cohomology* $\mathrm{H}^n(G;V)$.

Next, suppose that V is equipped with a G-invariant norm $\|\cdot\|$, ie $\|g\cdot v\|=\|v\|$ for every $g\in G$ and $v\in V$. Define $C^n_b(G;V)$ to be the subspace

$$C^n_b(G;V)=\Bigl\{f\colon G^{\times n}\to V \mid \sup_{(g_1,\dots,g_n)\in G^{\times n}}\|f(g_1,\dots,g_n)\|<\infty\Bigr\}$$

of $C^n(G;V)$. Then $C^\bullet_b(G;V)$ is a subcomplex of $C^\bullet(G;V)$, and we call the n^{th} cohomology of $C^\bullet_b(G;V)$ the n^{th} *bounded cohomology of* G, and denote it by $\mathrm{H}^n_b(G;V)$. The inclusion $C^\bullet_b(G;V)\to C^\bullet(G;V)$ induces a map $c_G\colon \mathrm{H}^\bullet_b(G;V)\to\mathrm{H}^\bullet(G;V)$, called the *comparison map*.

Let $\mathrm{H}^\bullet_{/b}(G;V)$ denote their relative cohomology, that is, the cohomology of the quotient complex $C^\bullet_{/b}(G;V)=C^\bullet(G;V)/C^\bullet_b(G;V)$. Then the short exact sequence of cochain complexes

$$0\to C^\bullet_b(G;V)\to C^\bullet(G;V)\to C^\bullet_{/b}(G;V)\to 0$$

induces the cohomology long exact sequence

$$\cdots\to\mathrm{H}^n_b(G;V)\xrightarrow{c_G}\mathrm{H}^n(G;V)\to\mathrm{H}^n_{/b}(G;V)\to\mathrm{H}^{n+1}_b(G;V)\to\cdots. \tag{3-1}$$

If we need to specify the G-representation ρ, we may use the symbols $\mathrm{H}^\bullet(G;\rho,V)$, $\mathrm{H}^\bullet_b(G;\rho,V)$ and $\mathrm{H}^\bullet_{/b}(G;\rho,V)$ instead of $\mathrm{H}^\bullet(G;V)$, $\mathrm{H}^\bullet_b(G;V)$ and $\mathrm{H}^\bullet_{/b}(G;V)$, respectively. Let $\mathbb{R}$ denote the field of real numbers equipped with the trivial G-action. In this case, we write $\mathrm{H}^n(G)$, $\mathrm{H}^n_b(G)$ and $\mathrm{H}^n_{/b}(G)$ instead of $\mathrm{H}^n(G;\mathbb{R})$, $\mathrm{H}^n_b(G;\mathbb{R})$ and $\mathrm{H}^n_{/b}(G;\mathbb{R})$, respectively.

Let N be a normal subgroup of G. Then G acts on N by conjugation, and hence G acts on $C^n(N;V)$. This G-action is described by

$$({}^g f)(x_1,\dots,x_n)=g\cdot f(g^{-1}x_1g,\dots,g^{-1}x_ng).$$

The action induces G-actions on $\mathrm{H}^n(N;V)$, $\mathrm{H}^n_b(N;V)$ and $\mathrm{H}^n_{/b}(N;V)$. When $N=G$, these G-actions on $\mathrm{H}^n(G;V)$, $\mathrm{H}^n_b(G;V)$ and $\mathrm{H}^n_{/b}(G;V)$ are trivial. Indeed, for $h\in G$, the conjugation by h on $C^\bullet(G;V)$ is homotoped to the identity map by the chain homotopy $\Phi_h\colon C^n(G;V)\to C^{n-1}(G;V)$ defined by

$$(\Phi_h(c))(g_1,\dots,g_{n-1})=\sum_{i=0}^{n-1}(-1)^i c(g_1,\dots,g_i,h,h^{-1}g_{i+1}h,\dots,h^{-1}g_{n-1}h).$$

This chain homotopy induces ones between the conjugations by h and the identity maps on $C^\bullet_b(G;V)$ and $C^\bullet_{/b}(G;V)$. By definition, a cocycle $f\colon N\to V$ in $C^1_{/b}(N;V)$ defines a class of $\mathrm{H}^1_{/b}(N;V)^G$ if and only if the function ${}^gf-f\colon N\to V$ is bounded for every $g\in G$.

Until the end of Section 5, we consider the case of trivial real coefficients. Let $f\colon G\to\mathbb{R}$ be a homogeneous quasimorphism. Then f is considered as an element of $C^1(G)$, and its coboundary δf is

$$(\delta f)(x,y)=f(x)-f(xy)+f(y).$$

Since f is a quasimorphism, the coboundary δf is a bounded cocycle. Hence we obtain a map $\delta\colon\mathrm{Q}(G)\to\mathrm{H}^2_b(G)$ given by $f\mapsto[\delta f]$. Then the following lemma is well known:

Lemma 3.3 *The following sequence is exact*:

$$0\to\mathrm{H}^1(G)\to\mathrm{Q}(G)\xrightarrow{\delta}\mathrm{H}^2_b(G)\xrightarrow{c_G}\mathrm{H}^2(G).$$

Let $\varphi\colon G\to H$ be a group homomorphism. A *virtual section of* φ is a pair (Λ,x) consisting of a subgroup Λ of finite index in H and a group homomorphism $s\colon\Lambda\to G$ satisfying $\varphi(s(x))=x$ for every $x\in\Lambda$. The group homomorphism φ is said to *virtually split* if φ admits a virtual section. As mentioned at the end of the introduction, some of the authors showed the following proposition. For a further generalization of this result, see [61, Theorem 1.4].

Proposition 3.4 [60, Proposition 6.4] *If the projection $p\colon G\to\Gamma$ virtually splits, then the map $i^*\colon\mathrm{Q}(G)\to\mathrm{Q}(N)^G$ is surjective.*

In this paper, we often consider amenable groups and boundedly acyclic groups. Here we review basic properties related to them. First we collect those for amenable groups (see for example [40] for more details).

Theorem 3.5 (known results for amenable groups) (1) *Every finite group is amenable.*

(2) *Every abelian group is amenable.*

(3) *Every subgroup of an amenable group is amenable.*

(4) *Let $1\to N\to G\to\Gamma\to 1$ be an exact sequence of groups. Then G is amenable if and only if N and Γ are amenable.*

(5) *Every amenable group is boundedly k-acyclic for all $k\geq 1$.*

Secondly, we collect known results on bounded k-acyclicity for $k \geq 3$ due to various researchers; these results are not used in this paper, but it might be convenient to the reader to have some examples of boundedly 3-acyclic groups that are nonamenable. See also Remark 8.8 for one more example.

Theorem 3.6 (known results for boundedly acyclic groups) (1) [70] *Let $n \in \mathbb{N}$. Then the group $\mathrm{Homeo}_c(\mathbb{R}^n)$ of homeomorphisms on $\mathbb{R}^n$ with compact support is boundedly acyclic.*

(2) [75; 80] *For $n \geq 3$, every lattice in $\mathrm{SL}(n, \mathbb{R})$ is 3-boundedly acyclic.*

(3) [20] *Burger–Mozes groups* [24] *are 3-boundedly acyclic.*

(4) (see [82]) *Let $k \in \mathbb{N}$. Let $1 \to N \to G \to \Gamma \to 1$ be a short exact sequence of groups. Assume that N is boundedly k-acyclic. Then G is boundedly k-acyclic if and only if Γ is.*

(5) [38] *Every binate group* (see [38, Definition 3.1]) *is boundedly acyclic.*

(6) [37] *There exist continuum many nonisomorphic 5-generated nonamenable groups that are boundedly acyclic. There exists a finitely presented nonamenable group that is boundedly acyclic.*

(7) [78] *Thompson's group F is boundedly acyclic.*

(8) [78] *Let L be an arbitrary group. Let Γ be an infinite amenable group. Then the wreath product $L \wr \Gamma = \left(\bigoplus_\Gamma L\right) \rtimes \Gamma$ is boundedly acyclic.*

(9) [79] *For every integer n at least two, the identity component $\mathrm{Homeo}_0(S^n)$ of the group of orientation-preserving homeomorphisms of S^n is boundedly 3-acyclic. The group $\mathrm{Homeo}_0(S^3)$ is boundedly 4-acyclic.*

On (7), we remark that it is a major open problem whether Thompson's group F is amenable.

The seven-term exact sequence and the calculation of first cohomology mentioned below will be used in the proof of Theorem 4.11.

Theorem 3.7 (seven-term exact sequence; see [31] for example) *Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence. Then we have the exact sequence*

$$0 \to \mathrm{H}^1(\Gamma) \xrightarrow{p^*} \mathrm{H}^1(G) \xrightarrow{i^*} \mathrm{H}^1(N)^G \to \mathrm{H}^2(\Gamma) \\ \to \mathrm{Ker}(i^*\colon \mathrm{H}^2(G) \to \mathrm{H}^2(N)) \xrightarrow{\rho} \mathrm{H}^1(\Gamma; \mathrm{H}^1(N)) \to \mathrm{H}^3(\Gamma).$$

Here $\mathrm{H}^1(N)$ is regarded as a left $\mathbb{R}[\Gamma]$-module by the Γ-action induced from the conjugation G-action on N.

Lemma 3.8 *For a left $\mathbb{R}[\mathbb{Z}]$-module V, let $\rho\colon \mathbb{Z} \to \mathrm{Aut}(V)$ be the representation. Then the first cohomology group $\mathrm{H}^1(\mathbb{Z}; V)$ is isomorphic to $V/\mathrm{Im}(\mathrm{id}_V - \rho(1))$.*

Proof By definition, the set $Z^1(\mathbb{Z}; V)$ of cocycles on $\mathbb{Z}$ with coefficients in V is equal to the set of crossed homomorphisms, that is,

$$\{h\colon \mathbb{Z} \to V \mid h(n+m) = \rho(n)(h(m)) + h(n) \text{ for every } n, m \in \mathbb{Z}\}.$$

Since every crossed homomorphism on $\mathbb{Z}$ is determined by its value on $1 \in \mathbb{Z}$, we have $Z^1(\mathbb{Z}; V) \cong V$. The set $B^1(\mathbb{Z}; V)$ of coboundaries on $\mathbb{Z}$ with coefficients in V is equal to

$$\{h\colon \mathbb{Z} \to V \mid h(1) = v - \rho(1)(v) \text{ for some } v \in V\}.$$

Hence, $B^1(\mathbb{Z}; V) \cong \mathrm{Im}(\mathrm{id}_V - \rho(1))$, and the lemma follows. □

4 The spaces of nonextendable quasimorphisms

The purpose of this section is to provide several applications of our main theorem (Theorem 1.5) to the spaces $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ and $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$. In Section 4.1 we prove Theorems 1.9 and 1.10 modulo Theorem 1.5, and in Sections 4.2–4.4 we provide several examples of pairs (G, N) such that the space $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))$ does not vanish (Theorems 1.1, 1.2 and 4.18).

Here we restate Theorems 1.9 and 1.10 for the convenience of the reader.

Theorem 1.9 *If the quotient group $\Gamma = G/N$ is boundedly 3-acyclic, then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2(\Gamma).$$

Moreover, if G is Gromov-hyperbolic, then

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma).$$

Theorem 1.10 *If $\Gamma = G/N$ is boundedly 3-acyclic, then the map $p^* \circ (\xi_4)^{-1} \circ \tau_{/b}$ induces an isomorphism*

$$\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)) \cong \mathrm{Im}(p^*) \cap \mathrm{Im}(c_G),$$

where $c_G\colon \mathrm{H}_b^2(G) \to \mathrm{H}^2(G)$ is the comparison map. In particular, if Γ is boundedly 3-acyclic, then

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) \le \dim \mathrm{H}^2(G).$$

4.1 Proofs of Theorems 1.9 and 1.10

The goal of this section is to prove Theorems 1.9 and 1.10 modulo Theorem 1.5.

First we prove Theorem 1.9. Recall that, if G is Gromov-hyperbolic, then the comparison map $\mathrm{H}_b^2(G) \to \mathrm{H}^2(G)$ is surjective [73; 46; 84]. Hence, Theorem 1.9 follows from the following:

Theorem 4.1 *Let $1 \to N \to G \to \Gamma \to 1$ be an exact sequence of groups. Assume that Γ is boundedly 3-acyclic. Then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2(\Gamma).$$

Moreover, if the comparison map $c_G\colon \mathrm{H}_b^2(G) \to \mathrm{H}^2(G)$ is surjective, then

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma).$$

Proof By Theorem 1.5, we have the exact sequence

$$\mathrm{Q}(G) \xrightarrow{i^*} \mathrm{Q}(N)^G \xrightarrow{\tau_{/b}} \mathrm{H}_{/b}^2(\Gamma).$$

Hence,

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2_{/b}(\Gamma).$$

Since Γ is boundedly 3-acyclic, the map $\xi_4\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2_{/b}(\Gamma)$ is an isomorphism by (3-1), and therefore

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) \le \dim \mathrm{H}^2(\Gamma).$$

Next we show the latter assertion. Suppose that the comparison map $c_G\colon \mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is surjective. Then $\xi_5\colon \mathrm{H}^2(G) \to \mathrm{H}^2_{/b}(G)$ is the zero map. Since $\xi_4\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2_{/b}(\Gamma)$ is an isomorphism, the map $p^*\colon \mathrm{H}^2_{/b}(\Gamma) \to \mathrm{H}^2_{/b}(G)$ is also zero. Hence,

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2_{/b}(\Gamma) = \dim \mathrm{H}^2(\Gamma). \qquad \square$$

To prove Theorem 1.10, we use the following lemma in homological algebra:

Lemma 4.2 *For a commutative diagram of $\mathbb{R}$-vector spaces*

$$\begin{tikzcd}
 & & & C \arrow[d, "c"] \\
 & B_2 \arrow[r, "b_2"] \arrow[d, "c_2"] & B_3 \arrow[r, "b_3"] \arrow[d, "c_3", "\cong"'] & B_4 \arrow[d, "c_4"] \\
A_1 \arrow[r, "a_1"] & A_2 \arrow[r, "a_2"] & A_3 \arrow[r, "a_3"] & A_4
\end{tikzcd}$$

where the rows and the last column are exact and c_3 is an isomorphism, the map $b_3 \circ c_3^{-1} \circ a_2$ induces an isomorphism

$$A_2/(\mathrm{Im}(a_1) + \mathrm{Im}(c_2)) \cong \mathrm{Im}(b_3) \cap \mathrm{Im}(c).$$

Because the proof of Lemma 4.2 is done by a standard diagram chase, we omit it.

Proof of Theorem 1.10 If $\Gamma = G/N$ is boundedly 3-acyclic, $\xi_4\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2_{/b}(\Gamma)$ is an isomorphism. Therefore, Theorem 1.10 follows by applying Lemma 4.2 to the commutative diagram (1-3). $\square$

The following corollary of Theorem 1.10 will be used in the proof of Theorem 1.1:

Corollary 4.3 *Assume that N is contained in the commutator subgroup $[G,G]$ of G, and Γ is boundedly 3-acyclic. Then*

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) \le \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G.$$

Moreover, if the comparison map $\mathrm{H}^2_b(G) \to \mathrm{H}^2(G)$ is surjective, then

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) = \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G.$$

Proof Since N is contained in the commutator subgroup of G, the map $i^*\colon \mathrm{H}^1(G) \to \mathrm{H}^1(N)^G$ is zero and hence $\dim \mathrm{Im}(p^*) = \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G$. Therefore, Theorem 1.10 implies the corollary. $\square$

Remark 4.4 Let $1 \to N \to G \to \Gamma \to 1$ be an exact sequence and suppose that the group N is amenable. Then it is known that the map $\xi_3 \colon \mathrm{H}^1(N)^G \to \mathrm{Q}(N)^G$ in (1-3) is an isomorphism. Hence, Lemma 4.2 implies that the composite $\tau \circ \xi_3^{-1} \circ i^*$ induces an isomorphism

$$\mathrm{Q}(G)/(\mathrm{H}^1(G) + p^*\mathrm{Q}(\Gamma)) \cong \mathrm{Im}(\tau) \cap \mathrm{Im}(c_\Gamma).$$

This isomorphism was obtained in [62] in a different way and applied to study boundedness of characteristic classes of foliated bundles.

4.2 Proof of Theorem 1.1

The goal of this subsection is to prove Theorem 1.1 by using the results proved in the previous subsection. This theorem treats surface groups. Before proceeding to this case, we first prove the following theorem for free groups:

Theorem 4.5 (computations of dimensions for free groups) *For $n \geq 1$, set $G = F_n$ and $N = [F_n, F_n]$. Then*

$$\dim(\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)) = \tfrac{1}{2}n(n-1) \quad \text{and} \quad \dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) = 0.$$

Proof By Theorem 4.1,

$$\dim(\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(G/N) = \dim \mathrm{H}^2(\mathbb{Z}^n) = \tfrac{1}{2}n(n-1).$$

By Theorem 1.10, we obtain

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) \leq \dim \mathrm{H}^2(G) = 0. \qquad \square$$

Next we show Theorem 1.1. In the proof, we need the precise description of the space $\mathrm{H}^1([F_n, F_n])^{F_n}$ of F_n-invariant homomorphisms on the commutator subgroup $[F_n, F_n]$ of the free group F_n. Throughout this subsection, we write $a_1, \dots, a_n$ to mean the canonical basis of F_n.

Lemma 4.6 *Let i and j be integers such that $1 \leq i < j \leq n$. Then there exist F_n-invariant homomorphisms $\alpha_{i,j} \colon [F_n, F_n] \to \mathbb{R}$ such that, for $k, l \in \mathbb{Z}$ with $1 \leq k < l \leq n$,*

$$\alpha_{i,j}([a_k, a_l]) = \begin{cases} 1 & \text{if } (i,j) = (k,l), \\ 0 & \text{otherwise.} \end{cases} \tag{4-1}$$

Moreover, the $\alpha_{i,j}$ are a basis of $\mathrm{H}^1([F_n, F_n])^{F_n}$. In particular,

$$\dim \mathrm{H}^1([F_n, F_n])^{F_n} = \tfrac{1}{2}n(n-1).$$

Proof When $G = F_n$ and $N = [F_n, F_n]$, the five-term exact sequence (Theorem 1.4) implies that the dimension of $\mathrm{H}^1([F_n, F_n])^{F_n}$ is $\frac{1}{2}n(n-1)$. Hence it suffices to construct $\alpha_{i,j}$ satisfying (4-1).

We first consider the case $n = 2$. Since $\dim(\mathrm{H}^1([F_2, F_2])^{F_2}) = 1$, it suffices to show that there exists an F_2-invariant homomorphism $\alpha \colon [F_2, F_2] \to \mathbb{R}$ with $\alpha([a_1, a_2]) \neq 0$. Let $\varphi \colon [F_2, F_2] \to \mathbb{R}$ be a nontrivial F_2-invariant homomorphism. Then there exists a pair x and y of elements of F_2 such that $\varphi([x, y]) \neq 0$.

Let $f\colon F_2 \to F_2$ be the group homomorphism sending a_1 to x and a_2 to y. Since φ is F_2-invariant, we have

$$\varphi \circ f(gxg^{-1}) = \varphi(f(g)f(x)f(g)^{-1}) = \varphi \circ f(x)$$

for every $g \in F_2$ and every $x \in [F_2, F_2]$. Hence, $\varphi \circ (f|_{[F_2,F_2]})\colon [F_2, F_2] \to \mathbb{R}$ is an F_2-invariant homomorphism satisfying $\varphi \circ f([a_1, a_2]) \neq 0$. This completes the proof of the case $n = 2$.

Suppose that $n \geq 2$. Then, for $i, j \in \{1, \dots, n\}$ with $i < j$, define a homomorphism $q_{i,j}\colon F_n \to F_2$ which sends a_i to a_1, a_j to a_2, and a_k to the unit element of F_2 for $k \neq i, j$. Then $q_{i,j}$ induces a surjection $[F_n, F_n]$ to $[F_2, F_2]$, and induces a homomorphism $q_{i,j}^*\colon \mathrm{H}^1([F_2, F_2])^{F_2} \to \mathrm{H}^1([F_n, F_n])^{F_n}$. Set $\alpha_{i,j} = \alpha_{1,2} \circ q_{i,j}$. Then $\alpha_{i,j}$ clearly satisfies (4-1), and this completes the proof. □

Theorem 1.1 follows from Corollary 4.3 and the following proposition:

Proposition 4.7 *For $l \geq 1$,*

$$\dim \mathrm{H}^1\big([\pi_1(\Sigma_l), \pi_1(\Sigma_l)]\big)^{\pi_1(\Sigma_l)} = l(2l-1) - 1.$$

Proof Recall that $\pi_1(\Sigma_l)$ has the presentation

$$\langle a_1, \dots, a_{2l} \mid [a_1, a_2] \cdots [a_{2l-1}, a_{2l}] \rangle.$$

Let $f\colon F_{2l} \to \pi_1(\Sigma_l)$ be the natural epimorphism sending a_i to a_i, and K the kernel of f, ie K is the normal subgroup generated by $[a_1, a_2] \cdots [a_{2l-1}, a_{2l}]$ in F_{2l}. Then f induces an epimorphism $f|_{[F_{2l},F_{2l}]}\colon [F_{2l}, F_{2l}] \to [\pi_1(\Sigma_l), \pi_1(\Sigma_l)]$ between their commutator subgroups, and its kernel coincides with K since K is contained in $[F_{2l}, F_{2l}]$. This means that, for a homomorphism $\varphi\colon [F_{2l}, F_{2l}] \to \mathbb{R}$, φ induces a homomorphism $\bar{\varphi}\colon [\pi_1(\Sigma_l), \pi_1(\Sigma_l)] \to \mathbb{R}$ if and only if

$$\varphi([a_1, a_2] \cdots [a_{2l-1}, a_{2l}]) = 0.$$

It is straightforward to show that φ is F_{2l}-invariant if and only if $\bar{\varphi}$ is $\pi_1(\Sigma_l)$-invariant. Hence the image of the monomorphism $\mathrm{H}^1\big([\pi_1(\Sigma_l), \pi_1(\Sigma_l)]\big)^{\pi_1(\Sigma_l)} \to \mathrm{H}^1([F_{2l}, F_{2l}])^{F_{2l}}$ is the subspace consisting of elements

$$\sum_{i<j} k_{ij}\alpha_{ij}$$

such that

$$k_{1,2} + k_{3,4} + \cdots + k_{2l-1,2l} = 0.$$

Since the dimension of $\mathrm{H}^1([F_{2l}, F_{2l}])^{F_{2l}}$ is $l(2l-1)$ (see Lemma 4.6), this completes the proof. □

Proof of Theorem 1.1 Since the abelianization $\Gamma = \pi_1(\Sigma_l)/[\pi_1(\Sigma_l), \pi_1(\Sigma_l)]$ of the surface group is isomorphic to $\mathbb{Z}^{2l}$, we have $\dim \mathrm{H}^2(\Gamma) = l(2l-1)$. Thus the first assertion follows from Theorem 4.1. Since the comparison map $\mathrm{H}_b^2(\pi_1(\Sigma_l)) \to \mathrm{H}^2(\pi_1(\Sigma_l))$ is surjective, we obtain

$$\dim\big(\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) = 1$$

by Corollary 4.3 and Proposition 4.7. □

4.3 Proof of Theorem 1.2 and a related example

To prove Theorem 1.2, we now recall some terminology of mapping class groups.

Let l be an integer at least 2 and Σ_l the oriented closed surface with genus l. The *mapping class group* $\mathrm{Mod}(\Sigma_l)$ *of* Σ_l is the group of isotopy classes of orientation-preserving diffeomorphisms on Σ_l. By considering the action on the first homology group, $\mathrm{Mod}(\Sigma_l)$ has a natural epimorphism $s_l\colon \mathrm{Mod}(\Sigma_l) \to \mathrm{Sp}(2l;\mathbb{Z})$, called the *symplectic representation*.

For $\psi \in \mathrm{Mod}(\Sigma_l)$, we take a diffeomorphism f that represents ψ. The mapping torus T_f is an orientable closed 3-manifold equipped with a natural fibration structure $\Sigma_l \to T_f \to S^1$. The following is known:

Theorem 4.8 [93] *A mapping class ψ is a pseudo-Anosov element if and only if the mapping torus T_f is a hyperbolic manifold.*

Set $\Gamma = \mathbb{Z}^{2l} \rtimes_{s_l(\psi)} \mathbb{Z}$ and

$$G = \pi_1(T_f) = \pi_1(\Sigma_l) \rtimes_{f_*} \mathbb{Z} \tag{4-2}$$
$$= \langle a_1, \dots, a_{2l+1} \mid [a_1, a_2] \cdots [a_{2l-1}, a_{2l}] = 1_G,\ a_{2l+1} \cdot a_i = (f_* a_i) \cdot a_{2l+1} \text{ for every } 1 \le i \le 2l \rangle,$$

where $f_*\colon \pi_1(\Sigma_l) \to \pi_1(\Sigma_l)$ is the pushforward of f.

Lemma 4.9 (1) $\dim \mathrm{H}^2(\Gamma) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + \dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \bigwedge^2 s_l(\psi)\big)$.

(2) $\dim \mathrm{H}^2(G) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + 1$.

Proof Let T^{2l} be the $2l$-dimensional torus. By the natural inclusion $\mathrm{Sp}(2l;\mathbb{Z}) \to \mathrm{Homeo}(T^{2l})$, we regard the element $s_l(\psi)$ as a homeomorphism of T^{2l}. Let $M_{s_l(\psi)}$ be the mapping torus of $s_l(\psi) \in \mathrm{Homeo}(T^{2l})$. Since $M_{s_l(\psi)}$ is a $K(\Gamma, 1)$-manifold, we have $\dim \mathrm{H}^2(\Gamma) = \dim \mathrm{H}^2(M_{s_l(\psi)})$.

For the mapping torus $M_{s_l(\psi)}$, we have the long exact sequence

$$\cdots \to \mathrm{H}^1(T^{2l}) \xrightarrow{\delta_1} \mathrm{H}^1(T^{2l}) \to \mathrm{H}^2(M_{s_l(\psi)}) \to \mathrm{H}^2(T^{2l}) \xrightarrow{\delta_2} \mathrm{H}^2(T^{2l}) \to \cdots, \tag{4-3}$$

where the map δ_n is given by

$$\mathrm{id}_{\mathrm{H}^n(T^{2l})} - s_l(\psi)^* \colon \mathrm{H}^n(T^{2l}) \to \mathrm{H}^n(T^{2l})$$

(see [48, Example 2.48]). This, together with (4-3) and the fact that $H^2(T^{2l}) \cong \bigwedge^2 H^1(T^{2l})$, implies that

$$\dim \mathrm{H}^2(\Gamma) = \dim \mathrm{H}^2(M_{s_l(\psi)}) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + \dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \textstyle\bigwedge^2 s_l(\psi)\big).$$

The computation of $\dim \mathrm{H}^2(G)$ is done in a similar manner. □

Let N be the kernel of the natural epimorphism $G \to \Gamma$. Note that N is isomorphic to the commutator subgroup of $\pi_1(\Sigma_l)$.

Lemma 4.10 $$\dim \mathrm{H}^1(N)^G \le \dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \textstyle\bigwedge^2 s_l(\psi)\big) - 1.$$

Proof We set $\mathrm{H} = \mathrm{H}_1(\Sigma_l;\mathbb{Z})$. Let $\iota\colon \mathrm{H}^1(N)^G \to \mathrm{Hom}\big(\bigwedge^2 \mathrm{H}, \mathbb{R}\big)$ be the map defined by

$$\iota(h)(q(x)\wedge q(y)) = h([x,y]),$$

where $x, y \in \pi_1(\Sigma_l)$ and $q\colon \pi_1(\Sigma_l) \to \mathrm{H}$ is the abelianization map. We claim that this map ι is well defined. To verify this, let $h \in \mathrm{H}^1(N)^G$. By commutator calculus, $[x_1x_2, y] = x_1[x_2, y]x_1^{-1}\cdot[x_1, y]$ for every $x_1, x_2, y \in \pi_1(\Sigma_l)$. Since h is G-invariant, this implies that

$$h([x_1x_2, y]) = h([x_1, y]) + h([x_2, y]).$$

In a similar manner to the above, we can see that

$$h([xz, yw]) = h([x, y])$$

for every $x, y \in \pi_1(\Sigma_l)$ and every $z, w \in N = [\pi_1(\Sigma_l), \pi_1(\Sigma_l)]$. Now it is straightforward to confirm that ι is well defined. Moreover, since N is normally generated by $\{[a_i, a_j]\}_{1\le i<j\le 2l}$ in G, the map ι is injective.

We set

$$\mathrm{Hom}\big(\textstyle\bigwedge^2 \mathrm{H}, \mathbb{R}\big)^{\bigwedge^2 s_l(\psi)} = \big\{h \in \mathrm{Hom}\big(\textstyle\bigwedge^2 \mathrm{H}, \mathbb{R}\big) \mid h \circ \textstyle\bigwedge^2 s_l(\psi) = h\big\}.$$

Then the image of ι is contained in $\mathrm{Hom}\big(\bigwedge^2 \mathrm{H}, \mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$. Indeed, for $1 \le i < j \le 2l$ and $h \in \mathrm{H}^1(N)^G$,

$$\begin{aligned}\iota(h)\big(\textstyle\bigwedge^2 s_l(\psi)(q(a_i)\wedge q(a_j))\big) &= h([f_*a_i, f_*a_j])\\ &= h([a_{2l+1}\cdot a_i\cdot a_{2l+1}^{-1}, a_{2l+1}\cdot a_j\cdot a_{2l+1}^{-1}])\\ &= h([a_i, a_j]) = \iota(h)(q(a_i)\wedge q(a_j)),\end{aligned}$$

where the second equality comes from the relation in (4-2) and the third equality comes from the G-invariance of h.

Since $\dim \mathrm{Hom}\big(\bigwedge^2 \mathrm{H}, \mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$ is equal to $\dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \bigwedge^2 s_l(\psi)\big)$, it suffices to show that the map

$$\iota\colon \mathrm{H}^1(N)^G \to \mathrm{Hom}\big(\textstyle\bigwedge^2 \mathrm{H}, \mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$$

is not surjective. We set $v_1 = q(a_1)\wedge q(a_2) + \cdots + q(a_{2l-1})\wedge q(a_{2l}) \in \bigwedge^2 \mathrm{H}$. Then the map $\bigwedge^2 s_l(\psi)\colon \bigwedge^2 \mathrm{H} \to \bigwedge^2 \mathrm{H}$ preserves v_1. Hence, for a suitable basis containing v_1, the dual v_1^* is contained in $\mathrm{Hom}\big(\bigwedge^2 \mathrm{H}, \mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$. However, v_1^* is not contained in the image of ι. Indeed, for every $h \in \mathrm{H}^1(N)^G$,

$$\iota(h)(v_1) = h([a_1, a_2]\cdots[a_{2l-1}, a_{2l}]) = 0.$$

Hence the map $\iota\colon \mathrm{H}^1(N)^G \to \mathrm{Hom}\big(\bigwedge^2 \mathrm{H}, \mathbb{R}\big)^{\bigwedge^2 s_l(\psi)}$ is not surjective, and the lemma follows. □

Proof of Theorem 1.2 The group G is Gromov-hyperbolic by Theorem 4.8 and Γ is amenable by Theorem 3.5(4). Hence, Theorem 1.9, together with Lemma 4.9(1), asserts that

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + \dim \mathrm{Ker}\big(I_{\binom{2l}{2}} - \textstyle\bigwedge^2 s_l(\psi)\big).$$

By Theorem 1.10 and Lemma 4.9(2), we obtain

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) \le \dim \mathrm{H}^2(G) = \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + 1.$$

On the other hand,

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) = \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G \ge \dim \mathrm{Ker}(I_{2l} - s_l(\psi)) + 1$$

by Corollary 4.3 and Lemmas 4.9(1) and 4.10. □

As we mentioned in the introduction, we obtain an analogue (Theorem 4.11) of Theorem 1.2 in the free group setting. For $n \in \mathbb{N}$, let $\mathrm{Aut}(F_n)$ be the automorphism group of F_n. Let $t_n \colon \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$ be the representation induced by the action of $\mathrm{Aut}(F_n)$ on the abelianization of F_n. Then the group $F_n \rtimes_\psi \mathbb{Z}$ naturally surjects onto $\mathbb{Z}^n \rtimes_{t_n(\psi)} \mathbb{Z}$ via the abelianization of F_n. We say that an automorphism ψ of F_n is *atoroidal* if it has no periodic conjugacy classes, that is, there does not exist a pair $(a,k) \in F_n \times \mathbb{Z}$ with $a \ne 1_{F_n}$ and $k \ne 0$ such that $\psi^k(a)$ is conjugate to a. Bestvina and Feighn [11] showed that $\psi \in \mathrm{Aut}(F_n)$ is atoroidal if and only if $F_n \rtimes_\psi \mathbb{Z}$ is Gromov-hyperbolic.

Theorem 4.11 (computations of dimensions for free-by-cyclic groups) *Let n be an integer greater than 1 and $\psi \in \mathrm{Aut}(F_n)$ an atoroidal automorphism. Set $G = F_n \rtimes_\psi \mathbb{Z}$ and let N be the kernel of the surjection $G \to \mathbb{Z}^n \rtimes_{t_n(\psi)} \mathbb{Z}$ defined via the abelianization map $F_n \to \mathbb{Z}^n$. Then*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{Ker}(I_n - t_n(\psi)) + \dim \mathrm{Ker}\big(I_{\binom{n}{2}} - \textstyle\bigwedge^2 t_n(\psi)\big)$$

and

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) = \dim \mathrm{Ker}(I_n - t_n(\psi)),$$

where $\bigwedge^2 t_n(\psi)$ is the map induced by $t_n(\psi)$.

Proof Since G/N is isomorphic to $\Gamma = \mathbb{Z}^n \rtimes_{t_n(\psi)} \mathbb{Z}$, Theorem 1.9 implies that $\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)) = \dim \mathrm{H}^2(\Gamma)$. Moreover,

$$\dim \mathrm{H}^2(\Gamma) = \dim \mathrm{Ker}(I_n - t_n(\psi)) + \dim \mathrm{Ker}\big(I_{\binom{n}{2}} - \textstyle\bigwedge^2 t_n(\psi)\big)$$

by the same argument as in the proof of Lemma 4.9(1). Hence the former statement holds.

We set $\mathrm{H} = \mathrm{H}_1(F_n;\mathbb{Z}) = F_n/[F_n, F_n]$. As in Lemma 4.10, we can define a monomorphism $\iota\colon \mathrm{H}^1(N)^G \to \mathrm{Hom}\big(\bigwedge \mathrm{H}, \mathbb{R}\big)^{\bigwedge^2 t_n(\psi)}$. Hence, together with Corollary 4.3, we obtain

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) \ge \dim \mathrm{Ker}(I_n - t_n(\psi)).$$

On the other hand,

$$\dim\big(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\big) \le \dim \mathrm{H}^2(G)$$

by Theorem 1.10. By the seven-term exact sequence (Theorem 3.7) applied to the short exact sequence $1 \to F_n \to G \to \mathbb{Z} \to 1$,

$$\mathrm{H}^2(G) \cong \mathrm{H}^1(\mathbb{Z}; \mathrm{H}^1(F_n)).$$

By Lemma 3.8, $\mathrm{H}^1(\mathbb{Z};\mathrm{H}^1(F_n))$ is isomorphic to

$$\mathrm{H}^1(F_n)/\mathrm{Im}(\mathrm{id}_{\mathrm{H}^1(F_n)}-\psi^*),$$

where $\psi^*\colon \mathrm{H}^1(F_n)\to \mathrm{H}^1(F_n)$ is the pullback of ψ. Hence,

$$\dim \mathrm{H}^2(G)=\dim \mathrm{H}^1(\mathbb{Z};\mathrm{H}^1(F_n))=\dim \mathrm{Ker}(I_n-t_n(\psi))$$

and the theorem follows. □

4.4 Other examples

It follows from Theorem 1.10 that $\mathrm{H}^2(G)=0$ implies $\mathrm{Q}(N)^G=\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G)$, and we provide several examples of groups G with $\mathrm{H}^2(G)=0$ in Section 2.1.

As an application of [42, Theorem 2.4], we provide another example of a group G satisfying $\mathrm{Q}(N)^G=\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G)$.

Corollary 4.12 *Let L be a hyperbolic link in S^3 such that the number of the connected components of L is two. Let G be the link group of L (ie the fundamental group of the complement $S^3\setminus L$ of L) and N the commutator subgroup of G. Then $\mathrm{Q}(N)^G=\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G)$.*

Proof By Theorem 1.10, it suffices to show that the comparison map $c_G\colon \mathrm{H}^2_b(G)\to \mathrm{H}^2(G)$ is equal to zero. Theorem 2.4 of [42] gives $\mathrm{Im}(c_G)\neq \mathrm{H}^2(G)$. Since the number of the connected components of L is two, the second cohomology group $\mathrm{H}^2(G)$ is isomorphic to $\mathbb{R}$. Hence, $\mathrm{Im}(c_G)=0$. □

Here we provide other examples (G,N) such that $\mathrm{H}^2(G)\neq 0$ and $\mathrm{Q}(N)^G=\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G)$.

Example 4.13 Let $n\in\mathbb{N}$. For $i=1,2,\dots,n$, let H_i be a boundedly 2-acyclic group and assume that $\mathrm{H}^2(H_1)\neq 0$ (for example, we can take $H_1=\mathbb{Z}^2$). Set $G=H_1*H_2*\cdots*H_n$ and $N=[G,G]$. Then $\mathrm{H}^2(G)=\mathrm{H}^2(H_1)\oplus\mathrm{H}^2(H_2)\oplus\cdots\oplus\mathrm{H}^2(H_n)\neq 0$ but the comparison map $c_G\colon \mathrm{H}^2_b(G)\to\mathrm{H}^2(G)$ is the zero map. It follows from Theorem 1.10 that $\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))=0$.

We considered free products in Example 4.13, but the comparison map in degree 2 is also trivial for graph products of amenable groups (such as RAAGs) and graphs of groups with amenable vertex groups (see [66, Example 4.7]). See [28, Corollary 5.4] for more cases in which the comparison map in degree 2 is trivial.

Corollary 4.14 *Let $E\to\Sigma_l$ be a nontrivial orientable circle bundle over a closed oriented surface of genus $l>1$. For the fundamental group $G=\pi_1(E)$ and its normal subgroup $N=[G,G]$,*

$$\dim(\mathrm{Q}(N)^G/i^*\mathrm{Q}(G))=l(2l-1)\quad\textit{and}\quad\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\bigr)=0.$$

Remark 4.15 (1) The dimension of $\mathrm{Q}(G)$ (and hence the dimension of $\mathrm{Q}(N)^G$) is the cardinal of the continuum since G surjects onto the surface group $\pi_1(\Sigma_l)$.

(2) The dimension of $\mathrm{H}^2(G)$ is equal to $2l$. Indeed, $\mathrm{H}^2(G)$ is isomorphic to $\mathrm{H}^2(E)$ since E is a $K(G,1)$-space. Moreover, $\mathrm{H}^2(E)$ is isomorphic to $\mathrm{H}^1(\Sigma_l)$ by the Thom–Gysin sequence.

Proof of Corollary 4.14 Let n be the Euler number of the bundle $E \to \Sigma_l$. Note that n is nonzero since the bundle is nontrivial (see [40, Theorem 11.16]). Since the group G has a presentation

$$G = \pi_1(E) = \langle a_1, \dots, a_{2l+1} \mid [a_1, a_2] \cdots [a_{2l-1}, a_{2l}] = a_{2l+1}^{-n}, [a_i, a_{2l+1}] = 1_G \text{ for every } 1 \le i \le 2l \rangle,$$

the abelianization $\Gamma = G/N$ is isomorphic to $\mathbb{Z}^{2l} \times (\mathbb{Z}/n\mathbb{Z})$. Hence, $\dim \mathrm{H}^2(\Gamma) = l(2l-1)$. By the relation $[a_i, a_{2l+1}] = 1_G$ for each i and the fact that N is normally generated by $\{[a_i, a_j]\}_{1 \le i < j \le 2l+1}$ in G, we obtain $\dim \mathrm{H}^1(N)^G = l(2l+1) - 2l = l(2l-1)$ by an argument similar to the proof of Proposition 4.7. Hence Corollary 4.3 asserts that

$$\dim\bigl(\mathrm{Q}(N)^G / (\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) \le \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G = 0.$$

Since N is the commutator subgroup of G, the space $\mathrm{H}^1(N)^G$ injects into $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$. Hence,

$$\dim(\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)) \ge l(2l-1).$$

On the other hand, Theorem 4.1 asserts that

$$\dim(\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)) \le \mathrm{H}^2(\Gamma) = l(2l-1). \qquad \square$$

For elements $r_1, \dots, r_m \in G$, we write $\langle\!\langle r_1, \dots, r_m \rangle\!\rangle$ to mean the normal subgroup of G generated by $r_1, \dots, r_m$.

Corollary 4.16 *Let $r_1, \dots, r_m \in [F_n, [F_n, F_n]]$ and set*

$$G = F_n / \langle\!\langle r_1, \dots, r_m \rangle\!\rangle.$$

Then $\mathrm{Q}([G,G])^G = \mathrm{H}^1([G,G])^G + i^*\mathrm{Q}(G)$.

Proof Let q be the natural projection $F_n \to G$. Then the image of the monomorphism $q^* \colon \mathrm{H}^1([G,G])^G \to \mathrm{H}^1([F_n, F_n])^{F_n}$ is the space of F_n-invariant homomorphisms $f \colon [F_n, F_n] \to \mathbb{R}$ satisfying $f(r_1) = \cdots = f(r_m) = 0$. Since every F_n-invariant homomorphism of $[F_n, F_n]$ vanishes on $[F_n, [F_n, F_n]]$, we conclude that q^* is an isomorphism, and hence $\dim \mathrm{H}^1([G,G])^G = \frac{1}{2}n(n-1)$. Since $\Gamma = G/[G,G] = \mathbb{Z}^n$, we have $\dim \mathrm{H}^2(\Gamma) = \frac{1}{2}n(n-1)$. Hence Corollary 4.3 implies that $\mathrm{Q}([G,G])^G / (\mathrm{H}^1([G,G])^G + i^*\mathrm{Q}(G))$ is trivial. $\square$

Remark 4.17 Suppose that N is the commutator subgroup of G. As will be seen in Corollaries 6.20 and 7.11, the sum $\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ is actually a direct sum in this case, and the map $\mathrm{H}^1(N)^G \to \mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$ is an isomorphism. Hence, if G is a group as provided in Corollary 4.16 and N is the commutator subgroup of G, then the basis of $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$ is provided by the G-invariant homomorphism $\alpha'_{i,j} \colon N \to \mathbb{R}$ for $1 \le i < j \le n$, which is the homomorphism induced by $\alpha_{i,j} \colon [F_n, F_n] \to \mathbb{R}$ described in Lemma 4.6.

As an example of a pair (G, N) satisfying $\mathrm{Q}(N) \ne \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, we provide a certain family of one-relator groups. Recall that a *one-relator group* is a group isomorphic to $F_n / \langle\!\langle r \rangle\!\rangle$ for some positive integer n and an element r of F_n.

Theorem 4.18 *Let n and k be integers at least 2 and r an element of $[F_n, F_n] \setminus [F_n, [F_n, F_n]]$. Set $G = F_n/\langle\langle r^k \rangle\rangle$ and $N = [G, G]$. Then*

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) = 1.$$

Note that $r \in [F_n, F_n]\setminus[F_n,[F_n, F_n]]$ is equivalent to the existence of $f_0 \in \mathrm{H}^1([F_n, F_n])^{F_n}$ with $f_0(r) \neq 0$.

Proof of Theorem 4.18 By Newman's spelling theorem [85], every one-relator group with torsion is hyperbolic, and hence G is hyperbolic. Indeed, r does not belong to $\langle\langle r^k \rangle\rangle$ since $f_0(x)$ belongs to $k f_0(r)\mathbb{Z}$ for every element x of $\langle\langle r^k \rangle\rangle$. Since $\Gamma = G/N$ is abelian, Γ is boundedly 3-acyclic. By Corollary 4.3, it suffices to show

$$\dim\bigl(\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G + i^*\mathrm{Q}(G))\bigr) = \dim \mathrm{H}^2(\Gamma) - \dim \mathrm{H}^1(N)^G = 1.$$

Since $r^k \in [F_n, F_n]$, we have $\Gamma = \mathbb{Z}^n$, and $\dim \mathrm{H}^2(\Gamma) = \frac{1}{2}n(n-1)$. Hence, it only remains to show that

$$\dim \mathrm{H}^1(N)^G = \tfrac{1}{2}n(n-1) - 1. \tag{4-4}$$

Let $q\colon F_n \to G = F_n/\langle\langle r^k \rangle\rangle$ be the natural quotient. Then q induces a monomorphism $q^*\colon \mathrm{H}^1(N)^G \to \mathrm{H}^1([F_n, F_n])^{F_n}$. As in the proof of Proposition 4.7, it is straightforward to show that the image of $q^*\colon \mathrm{H}^1(N)^G \to \mathrm{H}^1([F_n, F_n])^{F_n}$ is the space of F_n-invariant homomorphisms $f\colon [F_n, F_n] \to \mathbb{R}$ such that $f(r) = 0$. Since there exists an element f_0 of $\mathrm{H}^1([F_n, F_n])^{F_n}$ with $f_0(r) \neq 0$, the codimension of the image of $q^*\colon \mathrm{H}^1(N)^G \to \mathrm{H}^1([F_n, F_n])^{F_n}$ is 1, which implies (4-4). □

After the authors completed this work, they obtained a generalization of Theorem 4.18; see [59, Theorem 11.15].

Remark 4.19 Let k be a positive integer. Here we construct a finitely presented group G satisfying

$$\dim\bigl(\mathrm{Q}([G,G])^G/(\mathrm{H}^1([G,G])^G + i^*\mathrm{Q}(G))\bigr) = k. \tag{4-5}$$

Let $F_{2k} = \langle a_1, \dots, a_{2k} \rangle$ be a free group and define G by

$$G = \langle a_1, \dots, a_{2k} \mid [a_1, a_2]^2, \dots, [a_{2k-1}, a_{2k}]^2 \rangle.$$

Set $H = \langle a_1, a_2 \mid [a_1, a_2]^2 \rangle$. Then G is the k-fold free product of H. Since H is a one-relator group with torsion, H is hyperbolic. Since a finite free product of hyperbolic groups is hyperbolic, G is hyperbolic. Hence the comparison map $\mathrm{H}_b^2(G) \to \mathrm{H}^2(G)$ is surjective.

Let $q\colon F_{2k} \to G$ be the natural quotient. Then $q^*\colon \mathrm{H}^1([G,G])^G \to \mathrm{H}^1([F_{2k}, F_{2k}])^{F_{2k}}$ is a monomorphism whose image comprises the F_{2k}-invariant homomorphisms $\varphi\colon [F_{2k}, F_{2k}] \to \mathbb{R}$ such that $\varphi([a_{2i-1}, a_{2i}]) = 0$ for $i = 1, \dots, k$. Therefore Corollary 4.3 implies (4-5).

5 Cohomology classes induced by the flux homomorphism

First we review the definition of the (volume) flux homomorphism (see for instance [5]).

Let $\mathrm{Diff}(M,\Omega)$ denote the group of diffeomorphisms on an m-dimensional smooth manifold M which preserve a volume form Ω on M, $\mathrm{Diff}_0(M,\Omega)$ the identity component of $\mathrm{Diff}(M,\Omega)$, and $\widetilde{\mathrm{Diff}}_0(M,\Omega)$ the universal cover of $\mathrm{Diff}_0(M,\Omega)$. Then the (*volume*) *flux homomorphism* $\widetilde{\mathrm{Flux}}_\Omega\colon \widetilde{\mathrm{Diff}}_0(M,\Omega)\to \mathrm{H}^{m-1}(M)$ is defined by

$$\widetilde{\mathrm{Flux}}_\Omega([\{\psi^t\}_{t\in[0,1]}]) = \int_0^1 [\iota_{X_t}\Omega]\,dt,$$

where ι is the inner product, $\{\psi^t\}_{t\in[0,1]}$ is a path representing an element of $\widetilde{\mathrm{Diff}}_0(M,\Omega)$, and X_t is the time-dependent vector field generating the isotopy $\{\psi^t\}_{t\in[0,1]}$. The value $\widetilde{\mathrm{Flux}}_\Omega([\{\psi^t\}_{t\in[0,1]}])$ does not depend on the choice of the isotopy $\{\psi^t\}_{t\in[0,1]}$ and thus the map $\widetilde{\mathrm{Flux}}_\Omega$ is well defined. Moreover, $\widetilde{\mathrm{Flux}}_\Omega$ is a homomorphism. The image of $\pi_1(\mathrm{Diff}_0(M,\Omega))$ under $\widetilde{\mathrm{Flux}}_\Omega$ is called the *flux group* of the pair (M,Ω) and denoted by Γ_Ω. The flux homomorphism $\widetilde{\mathrm{Flux}}_\Omega$ descends to a homomorphism

$$\mathrm{Flux}_\Omega\colon \mathrm{Diff}_0(M,\Omega)\to \mathrm{H}^{m-1}(M)/\Gamma_\Omega.$$

These homomorphisms are fundamental objects in theory of diffeomorphism groups and have been extensively studied (see for example [63; 52]).

As we wrote in Section 2.3, Proposition 2.5 is essentially in [65]; we prove it for the reader's convenience.

Proof of Proposition 2.5 Suppose that the pair (G,N) of groups is $(\mathrm{Diff}_0(M,\Omega),\mathrm{Ker}(\mathrm{Flux}_\Omega))$ or $(\widetilde{\mathrm{Diff}}_0(M,\Omega),\mathrm{Ker}(\widetilde{\mathrm{Flux}}_\Omega))$. Since the kernels of the homomorphisms Flux_Ω and $\widetilde{\mathrm{Flux}}_\Omega$ are perfect (see [92; 4] and also [5, Theorems 4.3.1 and 5.1.3]), we have $\mathrm{H}^1(N)=0$. Hence this proposition follows from the five-term exact sequence (Theorem 1.4). □

To prove Theorem 2.6(1), we use Py's Calabi quasimorphism $f_P\colon \mathrm{Ker}(\mathrm{Flux}_\Omega)\to\mathbb{R}$, which was introduced in [89]. For an oriented closed surface whose genus l is at least 2 and a volume form Ω on M, Py constructed a $\mathrm{Diff}_0(M,\Omega)$-invariant homogeneous quasimorphism $f_P\colon \mathrm{Ker}(\mathrm{Flux}_\Omega)\to\mathbb{R}$ on $\mathrm{Ker}(\mathrm{Flux}_\Omega)$.

Proof of Theorem 2.6 First we prove (1). Suppose that Σ_l is an oriented closed surface whose genus l is at least 2, and let Ω be its volume form. Since in this case Γ_Ω is trivial (as mentioned just after Theorem 2.6), the two flux homomorphisms Flux_Ω and $\widetilde{\mathrm{Flux}}_\Omega$ coincide.

Set $G=\mathrm{Diff}_0(\Sigma_l,\Omega)$ and $N=\mathrm{Ker}(\mathrm{Flux}_\Omega)$. Since N is perfect [4, théorème II.6.1], we have $\mathrm{H}^1(N)=\mathrm{H}^1(N)^G=0$. Since G/N is abelian, Theorem 1.10 implies that

$$\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)=\mathrm{Q}(N)^G/(\mathrm{H}^1(N)^G+i^*\mathrm{Q}(G))\cong \mathrm{Im}(\mathrm{Flux}_\Omega^*)\cap\mathrm{Im}(c_G).$$

Since Py's Calabi quasimorphism f_P is not extendable to $G=\mathrm{Diff}_0(\Sigma_l,\omega)$ [58, Theorem 1.11], $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$ is not trivial. Hence, $\mathrm{Flux}_\Omega^*\circ\xi_4^{-1}\circ\tau_{/b}([f_P])\in\mathrm{Im}(\mathrm{Flux}_\omega^*)\cap\mathrm{Im}(c_G)$ is nonzero.

Now we show (2). Suppose that $m=2$. The case that M is a 2-sphere is clear since $\mathrm{H}^1(M)=0$ and hence the flux homomorphisms are trivial. The case of M a torus follows from the fact that both Flux_Ω and $\widetilde{\mathrm{Flux}}_\Omega$ have section homomorphisms: hence, by Proposition 3.4, $\mathrm{Im}(\mathrm{Flux}_\Omega^*)\cap\mathrm{Im}(c_G)\cong\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)=0$.

Suppose that $m\ge 3$. Then Proposition 5.1 below implies that Flux_Ω has a section homomorphism. Hence, by Proposition 3.4, $\mathrm{Im}(\mathrm{Flux}_\Omega^*)\cap\mathrm{Im}(c_G)\cong\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)=0$. □

Proposition 5.1 [35, Proposition 6.1] *Let m be an integer at least 3, M an m-dimensional differential manifold, and Ω a volume form on M. Then there exists a section homomorphism of the reduced flux homomorphism $\mathrm{Flux}_\Omega\colon \mathrm{Diff}_0(M,\Omega) \to \mathrm{H}^{m-1}(M,\Omega)/\Gamma_\Omega$. In addition, there exists a section homomorphism of $\widetilde{\mathrm{Flux}}_\Omega\colon \widetilde{\mathrm{Diff}}_0(M,\Omega) \to \mathrm{H}^{m-1}(M,\Omega)$.*

The idea of Theorem 2.6 is also useful in (higher-dimensional) symplectic geometry. For notions in symplectic geometry, see for example [5; 88]. For a symplectic manifold (M,ω), let $\mathrm{Ham}(M,\omega)$ denote the group of Hamiltonian diffeomorphisms with compact support. For an exact symplectic manifold (M,ω), let $\mathrm{Cal}_\omega\colon \mathrm{Ham}(M,\omega) \to \mathbb{R}$ denote the Calabi homomorphism. We note that the map Cal_ω^* is injective, where $\mathrm{Cal}_\omega^*\colon \mathrm{H}^2(\mathbb{R};\mathbb{R}) \to \mathrm{H}^2(\mathrm{Ham}(M,\omega);\mathbb{R})$ is the homomorphism induced by Cal_ω. Here $\mathrm{H}^2(\mathbb{R};\mathbb{R})$ denotes the group cohomology of $\mathbb{R}$ (as a discrete group) with trivial real coefficients and is isomorphic to $\mathrm{Hom}_\mathbb{Z}\left(\bigwedge_\mathbb{Z}^2(\mathbb{R});\mathbb{R}\right)$ due to the discussion before Proposition 2.5. Indeed, because $\mathrm{Ker}(\mathrm{Cal}_\omega)$ is perfect [4], we can prove the injectivity of Cal_ω^* similarly to the proof of Proposition 2.5. Then we have the following theorem:

Theorem 5.2 *For an exact symplectic manifold (M,ω), every nontrivial element of $\mathrm{Im}(\mathrm{Cal}_\omega^*)$ cannot be represented by a bounded 2-cochain.*

Note that $\mathrm{Cal}_\omega\colon \mathrm{Ham}(M,\omega) \to \mathbb{R}$ has a section homomorphism. Indeed, for a (time-independent) Hamiltonian function whose integral over M is 1 and its Hamiltonian flow $\{\phi^t\}_{t\in\mathbb{R}}$, the homomorphism $t \mapsto \phi^t$ is a section of the Calabi homomorphism Cal_ω. Hence, the proof of Theorem 5.2 is similar to Theorem 2.6.

6 Proof of the main theorem

The goal in this section is to prove Theorem 1.5, which is the five-term exact sequence of the cohomology of groups relative to the bounded subcomplex; we restate it for the convenience of the reader:

Theorem 1.5 (main theorem) *Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence of groups and V a left Banach $\mathbb{R}[\Gamma]$-module equipped with a Γ-invariant norm $\|\cdot\|$. Then there exists an exact sequence*

$$0 \to \mathrm{H}^1_{/b}(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^1_{/b}(G;V) \xrightarrow{i^*} \mathrm{H}^1_{/b}(N;V)^G \xrightarrow{\tau_{/b}} \mathrm{H}^2_{/b}(\Gamma;V) \xrightarrow{p^*} \mathrm{H}^2_{/b}(G;V).$$

Moreover, the exact sequence above is compatible with the five-term exact sequence of group cohomology; that is, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(\Gamma;V) \arrow[r, "p^*"] \arrow[d, "\xi_1"] & \mathrm{H}^1(G;V) \arrow[r, "i^*"] \arrow[d, "\xi_2"] & \mathrm{H}^1(N;V)^G \arrow[r, "\tau"] \arrow[d, "\xi_3"] & \mathrm{H}^2(\Gamma;V) \arrow[r, "p^*"] \arrow[d, "\xi_4"] & \mathrm{H}^2(G;V) \arrow[d, "\xi_5"] \\
0 \arrow[r] & \mathrm{H}^1_{/b}(\Gamma;V) \arrow[r, "p^*"] & \mathrm{H}^1_{/b}(G;V) \arrow[r, "i^*"] & \mathrm{H}^1_{/b}(N;V)^G \arrow[r, "\tau_{/b}"] & \mathrm{H}^2_{/b}(\Gamma;V) \arrow[r, "p^*"] & \mathrm{H}^2_{/b}(G;V)
\end{tikzcd}$$

Here the ξ_i are the maps induced from the quotient map $C^\bullet \to C^\bullet_{/b}$.

Notation Throughout this section, V denotes a Banach space equipped with the norm $\|\cdot\|$ and an isometric G-action whose restriction to N is trivial. For a nonnegative real number $D \geq 0$ and $v, w \in V$, the symbol $v \approx_D w$ means that $\|v - w\| \leq D$. For functions $f, g \colon S \to V$ on a set S, the symbol $f \approx_D g$ means that $f(s) \approx_D g(s)$ for every $s \in S$.

6.1 N-quasicocycle

To define the map $\tau_{/b} \colon \mathrm{H}^1_{/b}(N; V)^G \to \mathrm{H}^2_{/b}(\Gamma; V)$ in Theorem 1.5, it is convenient to introduce the notion of an N-quasicocycle. First, we recall the definition of quasicocycles.

Definition 6.1 Let G be a group and V a left $\mathbb{R}[G]$-module with a G-invariant norm $\|\cdot\|$. A function $F \colon G \to V$ is called a *quasicocycle* if there exists a nonnegative number D such that

$$F(g_1 g_2) \approx_D F(g_1) + g_1 \cdot F(g_2)$$

for every $g_1, g_2 \in G$. The smallest such D is called the *defect* of F and denoted by $D(F)$. Let $\widehat{\mathrm{Q}}Z(G; V)$ denote the $\mathbb{R}$-vector space of all quasicocycles on G.

Remark 6.2 If we need to specify the G-representation ρ, we write $\widehat{\mathrm{Q}}Z(G; \rho, V)$ instead of $\widehat{\mathrm{Q}}Z(G; V)$.

We introduce the concept of N-quasicocycles, which is a generalization of the concept of partial quasimorphisms introduced in [33] (see also [81; 56; 64; 18; 60]).

Definition 6.3 Let N be a normal subgroup of G. A function $F \colon G \to V$ is called an *N-quasicocycle* if there exists a nonnegative number D'' such that

$$F(ng) \approx_{D''} F(n) + F(g) \quad \text{and} \quad F(gn) \approx_{D''} F(g) + g \cdot F(n) \tag{6-1}$$

for every $g \in G$ and $n \in N$. The smallest such D'' is called the *defect* of the N-quasicocycle F and denoted by $D''(F)$. Let $\widehat{\mathrm{Q}}Z_N(G; V)$ denote the $\mathbb{R}$-vector space of all N-quasicocycles on G.

If the G-action on V is trivial, then a quasicocycle is also called a *V-valued quasimorphism*. In this case, we use the symbol $\widehat{\mathrm{Q}}(G; V)$ instead of $\widehat{\mathrm{Q}}Z(G; V)$ to denote the space of V-valued quasimorphisms. A V-valued quasimorphism F is said to be *homogeneous* if $F(g^k) = k \cdot F(g)$ for every $g \in G$ and every $k \in \mathbb{Z}$. The homogenization of V-valued quasimorphisms is well defined, as in the case of ($\mathbb{R}$-valued) quasimorphisms. We write $\mathrm{Q}(G; V)$ for the space of V-valued homogeneous quasimorphisms.

Recall that in our setting the restriction of the G-action on V to N is always trivial. Then a left G-action on $\mathrm{Q}(N; V)$ is defined by

$$({}^g f)(n) = g \cdot f(g^{-1} n g)$$

for every $g \in G$ and every $n \in N$. We call an element of $\mathrm{Q}(N; V)^G$ a *G-equivariant V-valued homogeneous quasimorphism*.

Remark 6.4 An element $f \in \mathrm{Q}(N;V)$ belongs to $\mathrm{Q}(N;V)^G$ if and only if

$$g \cdot f(n) = f(gng^{-1})$$

for every $g \in G$ and every $n \in N$. This is why we call an element of $\mathrm{Q}(N;V)^G$ G-equivariant.

Remark 6.5 The isomorphism $\mathrm{H}^1_{/b}(N;V) \to \mathrm{Q}(N;V)$ given by the homogenization is compatible with the G-actions. In particular, this isomorphism induces an isomorphism $\mathrm{H}^1_{/b}(N;V)^G \to \mathrm{Q}(N;V)^G$.

The elements of $\mathrm{Q}(N;V)^G = \mathrm{H}^1_{/b}(N;V)^G$ are G-invariant (as cohomology classes). However, respecting the condition $g \cdot f(n) = f(gng^{-1})$ for $f \in \mathrm{Q}(N;V)^G$, we call the elements of $\mathrm{Q}(N;V)^G$ *G-equivariant V-valued homogeneous quasimorphisms*.

Lemma 6.6 *Let N be a normal subgroup of G and V a left $\mathbb{R}[G]$-module. Assume that the induced N-action on V is trivial. Then, for an N-quasicocycle $F \in \widehat{\mathrm{Q}}Z_N(G;V)$, there exists a bounded cochain $b \in C^1_b(G;V)$ such that the restriction $(F+b)|_N$ is in $\mathrm{Q}(N;V)^G$.*

Proof By the definition of N-quasicocycles, the restriction $F|_N\colon N \to V$ is a quasimorphism. Let $\bar{F}|_N$ be the homogenization of $F|_N$. Then the map

$$b' = \bar{F}|_N - F|_N\colon N \to V$$

is bounded. Define $b\colon G \to V$ by

$$b(g) = \begin{cases} b'(g) & \text{if } g \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Then the map b is also bounded. Set $\Phi = F + b$; then $\Phi|_N = (F+b)|_N = \bar{F}|_N$. Since Φ is an N-quasicocycle, we have

$$({}^g\Phi)(n) = g \cdot \Phi(g^{-1}ng) \approx_{D''(\Phi)} \Phi(g \cdot g^{-1}ng) - \Phi(g) = \Phi(ng) - \Phi(g) \approx_{D''(\Phi)} \Phi(n)$$

for $g \in G$ and $n \in N$. Hence the difference ${}^g\Phi - \Phi$ is in $C^1_b(N;V)$. Since $({}^g\Phi)|_N$ and $\Phi|_N$ are homogeneous quasimorphisms, we have ${}^g\Phi|_N - \Phi|_N = 0$, and this implies that the element $\Phi|_N = (F+b)|_N$ belongs to $\mathrm{Q}(N;V)^G$. □

If V is the trivial G-module $\mathbb{R}$, then N-quasicocycles are also called N-quasimorphisms (this word was first introduced in [57]). In this case, Lemma 6.6 reads as follows:

Corollary 6.7 *Let N be a normal subgroup of G. For an N-quasimorphism $F \in \widehat{\mathrm{Q}}_N(G)$, there exists a bounded cochain $b \in C^1_b(G)$ such that the restriction $(F+b)|_N$ is in $\mathrm{Q}(N)^G$.*

6.2 The map $\tau_{/b}$

Now we proceed to the proof of Theorem 1.5. The goal in this subsection is to construct the map $\tau_{/b}\colon \mathrm{H}^1_{/b}(N)^G \to \mathrm{H}^2_{/b}(G)$. Here we only present the proofs in the case where the coefficient module V

is the trivial module $\mathbb{R}$. When $V \neq \mathbb{R}$, the proofs work without any essential change (see Remarks 6.5, 6.8, 6.14 and 6.16).

First we define the map $\tau_{/b}\colon \mathrm{H}^1_{/b}(N)^G \to \mathrm{H}^2_{/b}(\Gamma)$. Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be a group extension. As a special case of Remark 6.5, we have isomorphisms $\mathrm{H}^1_{/b}(N) \to \mathrm{Q}(N)$ and $\mathrm{H}^1_{/b}(N)^G \to \mathrm{Q}(N)^G$. By using these, we identify $\mathrm{H}^1_{/b}(N)$ and $\mathrm{H}^1_{/b}(N)^G$ with $\mathrm{Q}(N)$ and $\mathrm{Q}(N)^G$, respectively.

Let $\overline{\mathrm{Q}}_N(G) = \overline{\mathrm{Q}}_N(G;\mathbb{R})$ be the $\mathbb{R}$-vector space of all N-quasimorphisms whose restrictions to N are homogeneous quasimorphisms on N; that is,

$$\overline{\mathrm{Q}}_N(G) = \{F\colon G \to \mathbb{R} \mid F \text{ is an } N\text{-quasimorphism such that } F|_N \in \mathrm{Q}(N)^G\} \subset \widehat{\mathrm{Q}}_N(G).$$

By definition, the restriction of the domain defines a map

$$i^*\colon \overline{\mathrm{Q}}_N(G) \to \mathrm{Q}(N)^G.$$

Remark 6.8 When the G-action on V is nontrivial, we need to replace the space $\overline{\mathrm{Q}}_N(G)$ by

$$\overline{\mathrm{Q}}Z^1_N(G;V) = \{F\colon G \to V \mid F \text{ is an } N\text{-quasicocycle such that } F|_N \in \mathrm{Q}(N;V)^G\}.$$

Lemma 6.9 *The map $i^*\colon \overline{\mathrm{Q}}_N(G) \to \mathrm{Q}(N)^G$ is surjective.*

Proof Let $s\colon \Gamma \to G$ be a set-theoretic section of p satisfying $s(1_\Gamma) = 1_G$. For $f \in \mathrm{Q}(N)^G$, define a map $F_{f,s}\colon G \to \mathbb{R}$ by

$$F_{f,s}(g) = f(g \cdot sp(g)^{-1})$$

for $g \in G$. Then $F_{f,s}|_N = f$ since $sp(n) = 1_G$ for every $n \in N$. Moreover, $F_{f,s}$ is an N-quasimorphism. Indeed,

$$F_{f,s}(ng) = f(ng \cdot sp(ng)^{-1}) = f(ng \cdot sp(g)^{-1}) \approx_{D(f)} f(n) + f(g \cdot sp(g)^{-1}) = F_{f,s}(n) + F_{f,s}(g)$$

and

$$\begin{aligned} F_{f,s}(gn) = F_{f,s}(gng^{-1}g) &\approx_{D(f)} F_{f,s}(gng^{-1}) + F_{f,s}(g) \\ &= f(gng^{-1}) + F_{f,s}(g) = f(n) + F_{f,s}(g) = F_{f,s}(n) + F_{f,s}(g) \end{aligned}$$

by the definition of quasimorphisms and the G-invariance of f. This means $i^*(F_{f,s}) = f$, and hence the map i^* is surjective. □

Lemma 6.10 *For $F \in \overline{\mathrm{Q}}_N(G)$ and $g_i, g_i' \in G$ satisfying $p(g_i) = p(g_i') \in \Gamma$,*

$$\delta F(g_1, g_2) \approx_{4D''(F)} \delta F(g_1', g_2').$$

Proof By the assumption, there exist $n_1, n_2 \in N$ satisfying $g_1' = n_1 g_1$ and $g_2' = g_2 n_2$. Therefore,

$$\begin{aligned} \delta F(g_1', g_2') &= F(g_2 n_2) - F(n_1 g_1 g_2 n_2) + F(n_1 g_1) \\ &\approx_{4D''(F)} F(g_2) + F(n_2) - (F(n_1) + F(g_1 g_2) + F(n_2)) + F(n_1) + F(g_1) \\ &= \delta F(g_1, g_2). \end{aligned}$$ □

For $F \in \overline{\mathrm{Q}}_N(G)$ and a section $s\colon \Gamma \to G$ of p, we set $\alpha_{F,s} = s^*\delta F \in C^2(\Gamma)$. By Lemma 6.10, the element $[\alpha_{F,s}] \in C^2_{/b}(\Gamma) = C^2(\Gamma)/C^2_b(\Gamma)$ is independent of the choice of the section s. Therefore we set $\alpha_F = [\alpha_{F,s}] \in C^2_{/b}(\Gamma)$.

Lemma 6.11 *The cochain α_F is a cocycle on $C^\bullet_{/b}(\Gamma)$.*

Proof It suffices to show that the coboundary $\delta\alpha_{F,s}$ belongs to $C^3_b(\Gamma)$. For $f, g, h \in \Gamma$,

$$\begin{aligned}\delta\alpha_{F,s}(f,g,h) &= \delta F(s(g),s(h)) - \delta F(s(fg),s(h)) + \delta F(s(f),s(gh)) - \delta F(s(f),s(g))\\ &\approx_{8D''(F)} \delta F(s(g),s(h)) - \delta F(s(f)s(g),s(h)) + \delta F(s(f),s(g)s(h)) - \delta F(s(f),s(g))\\ &= \delta(\delta F)(s(f),s(g),s(h)) = 0\end{aligned}$$

by Lemma 6.10. $\square$

By Lemmas 6.9 and 6.11, we obtain a map

$$\overline{\mathrm{Q}}_N(G) \to \mathrm{H}^2_{/b}(\Gamma);\ F \mapsto [\alpha_F]. \tag{6-2}$$

Lemma 6.12 *The cohomology class $[\alpha_F] \in \mathrm{H}^2_{/b}(\Gamma)$ depends only on the restriction $F|_N$.*

Proof Let $s\colon \Gamma \to G$ be a section of p and Φ an element of $\overline{\mathrm{Q}}_N(G)$ satisfying $\Phi|_N = F|_N$. Then, for every $g, h \in \Gamma$,

$$\begin{aligned}(\alpha_{F,s}-\alpha_{\Phi,s})(g,h) &= \delta F(s(g),s(h)) - \delta\Phi(s(g),s(h))\\ &= F(s(h)) - F(s(g)s(h)) + F(s(g)) - \big(\Phi(s(h)) - \Phi(s(g)s(h)) + \Phi(s(g))\big)\\ &= \delta(F\circ s)(g,h) - \delta(\Phi\circ s)(g,h) + F(s(gh)) - F(s(g)s(h)) - \big(\Phi(s(gh)) - \Phi(s(g)s(h))\big).\end{aligned}$$

Since F and Φ are N-quasimorphisms, we have

$$\begin{aligned}F(s(gh)) - F(s(g)s(h)) &\approx_{D''(F)} F(s(gh)s(h)^{-1}s(g)^{-1}),\\ \Phi(s(gh)) - \Phi(s(g)s(h)) &\approx_{D''(\Phi)} \Phi(s(gh)s(h)^{-1}s(g)^{-1}).\end{aligned}$$

Together with $F(s(gh)s(h)^{-1}s(g)^{-1}) = \Phi(s(gh)s(h)^{-1}s(g)^{-1})$, we have

$$\alpha_{F,s} - \alpha_{\Phi,s} \approx_{D''(F)+D''(\Phi)} \delta(F\circ s - \Phi\circ s),$$

and this implies $[\alpha_F] = [\alpha_\Phi] \in \mathrm{H}^2_{/b}(\Gamma)$. $\square$

By Lemma 6.12, the map defined in (6-2) descends to a map $\tau_{/b}\colon \mathrm{Q}(N)^G \to \mathrm{H}^2_{/b}(\Gamma)$; that is, $\tau_{/b}$ is defined by

$$\tau_{/b}(f) = [\alpha_F],$$

where F is an element of $\overline{\mathrm{Q}}_N(G)$ satisfying $F|_N = f$. Under the isomorphism $\mathrm{Q}(N)^G \cong \mathrm{H}^1_{/b}(N)^G$, we obtain the map

$$\tau_{/b}\colon \mathrm{H}^1_{/b}(N)^G \to \mathrm{H}^2_{/b}(\Gamma).$$

6.3 Proof of the exactness

Now we proceed to the proof of the exactness of the sequence

$$0 \to \mathrm{H}^1_{/b}(\Gamma) \xrightarrow{p^*} \mathrm{H}^1_{/b}(G) \xrightarrow{i^*} \mathrm{Q}(N)^G \xrightarrow{\tau_{/b}} \mathrm{H}^2_{/b}(\Gamma) \xrightarrow{p^*} \mathrm{H}^2_{/b}(G), \tag{6-3}$$

where we identify $\mathrm{Q}(N)^G$ with $\mathrm{H}^1_{/b}(N)^G$.

Lemma 6.13 *The sequence (6-3) is exact at $\mathrm{H}^1_{/b}(\Gamma)$ and $\mathrm{H}^1_{/b}(G)$.*

Remark 6.14 In the case of trivial real coefficients, this proposition is well known (see [26]). Indeed, the spaces $\mathrm{H}^1_{/b}(\Gamma)$ and $\mathrm{H}^1_{/b}(G)$ are isomorphic to $\mathrm{Q}(\Gamma)$ and $\mathrm{Q}(G)$, respectively, and exactness of the sequence

$$0 \to \mathrm{Q}(\Gamma) \to \mathrm{Q}(G) \to \mathrm{Q}(N)^G$$

follows from the homogeneity of the elements of $\mathrm{Q}(\Gamma)$. However, in general, the spaces $\mathrm{H}^1_{/b}(\Gamma; V)$ and $\mathrm{H}^1_{/b}(G; V)$ are not isomorphic to the spaces of V-valued homogeneous quasimorphisms $\mathrm{Q}(\Gamma; V)$ and $\mathrm{Q}(G; V)$, respectively. Therefore, we present a proof of Lemma 6.13 which can be modified to the case of nontrivial coefficients without any essential change.

Proof of Lemma 6.13 We first show the exactness at $\mathrm{H}^1_{/b}(\Gamma)$. Let $a \in \mathrm{H}^1_{/b}(\Gamma)$ and suppose $p^*a = 0$. Let $f \in C^1(\Gamma)$ be a representative of a. Since $p^*a = 0$ in $\mathrm{H}^1_{/b}(G)$, there exists $c \in \mathbb{R} \cong C^0(\Gamma)$ such that $p^*f - \delta c = p^*f$ is bounded. Since p is surjective, f is bounded, and hence $a = 0$. This implies the exactness at $\mathrm{H}^1_{/b}(\Gamma)$.

Next we prove the exactness at $\mathrm{H}^1_{/b}(G)$. Since the map $p \circ i$ is zero, the composite $i^* \circ p^*$ is also zero. For $a \in \mathrm{H}^1_{/b}(G)$ satisfying $i^*a = 0$, it follows from Lemma 6.6 that there exists a representative $f \in C^1(G)$ of a satisfying $f|_N = 0$. For a section $s\colon \Gamma \to G$ of p, set $f_s = s^*f\colon \Gamma \to \mathbb{R}$. Then f_s is a quasimorphism on Γ. Indeed, since f is a quasimorphism on G,

$$\begin{aligned} f_s(g_1g_2) = f(s(g_1g_2)) &= f(s(g_1g_2)s(g_2)^{-1}s(g_1)^{-1}s(g_1)s(g_2)) \\ &\approx_{D(f)} f(s(g_1g_2)s(g_2)^{-1}s(g_1)^{-1}) + f(s(g_1)s(g_2)) = f(s(g_1)s(g_2)) \\ &\approx_{D(f)} f(s(g_2)) + f(s(g_1)) = f_s(g_2) + f_s(g_1) \end{aligned}$$

by the triviality $f|_N = 0$. Hence the cochain f_s is a cocycle on $C^1_{/b}(\Gamma)$; let $a_s \in \mathrm{H}^1_{/b}(\Gamma)$ denote the relative cohomology class represented by f_s. For $g \in G$,

$$p^*f_s(g) = f(sp(g)) = f(sp(g)g^{-1}g) \approx_{D(f)} f(sp(g)g^{-1}) + f(g) = f(g).$$

Therefore, the cochain p^*f_s is equal to f as relative cochains on G, and this implies $p^*a_s = a$. □

Lemma 6.15 *The sequence (6-3) is exact at $\mathrm{Q}(N)^G$.*

Proof Note that every $\mu \in \mathrm{Q}(G)$ is an N-quasimorphism. Hence, by the definition of $\tau_{/b}\colon \mathrm{Q}(N)^G \to \mathrm{H}^2_{/b}(\Gamma)$,

$$\tau_{/b}(i^*(\mu)) = [\alpha_\mu] = [[\alpha_{\mu,s}]]$$

for a section $s\colon \Gamma \to G$. Since μ is a quasimorphism, $\alpha_{\mu,s} = s^*\delta\mu$ is an element of $C^2_b(\Gamma)$ and hence $\alpha_\mu = [\alpha_{\mu,s}] \in C^2_{/b}(\Gamma)$ is zero. This implies $\tau_{/b} \circ i^* = 0$.

Suppose that $f \in \mathrm{Q}(N)^G$ satisfies $\tau_{/b}(f) = 0$. By Lemma 6.9, we obtain $F \in \overline{\mathrm{Q}}_N(G)$ satisfying $F|_N = f$. Let $s\colon \Gamma \to G$ be a section of p. The triviality of $[\alpha_F] = \tau_{/b}(f) = 0$ implies that there exist $\beta \in C^1(\Gamma)$ and $b \in C^2_b(\Gamma)$ satisfying

$$\alpha_{F,s} - \delta\beta = b.$$

For $g_i \in G$,

$$\delta F(g_1, g_2) \approx_{4D''(F)} \delta F(sp(g_1), sp(g_2)) = \alpha_{F,s}(p(g_1), p(g_2))$$

by Lemma 6.10. Hence,

$$\delta(F - p^*\beta)(g_1, g_2) \approx_{4D''(F)} (\alpha_{F,s} - \delta\beta)(p(g_1), p(g_2)) = p^*b(g_1, g_2). \tag{6-4}$$

Since the cochain b is bounded, the function $F - p^*\beta$ is a quasimorphism. By the fact that the restriction $(F - p^*\beta)|_N$ is equal to the homogeneous quasimorphism f, the homogenization $\overline{F - p^*\beta}$ of $F - p^*\beta$ satisfies $i^*(\overline{F - p^*\beta}) = f$. This implies $\operatorname{Ker} \tau_{/b} \subset \operatorname{Im} i^*$. □

Remark 6.16 The last step of the proof of Lemma 6.15 uses the homogenization of a quasimorphism. In the case where the G-action on V is nontrivial (in particular, where we cannot use the homogenization), we replace the argument of the last step with the following: we have shown that the cochain $F - p^*\beta$ is a cocycle in $C^1_{/b}(G)$ by (6-4). Since $F|_N = f$, the restriction $(F - p^*\beta + \beta(1_\Gamma))|_N$ is equal to f. Therefore, $i^*\big([F - p^*\beta + \beta(1_\Gamma)]\big) = f$, and this implies $\operatorname{Ker} \tau_{/b} \subset \operatorname{Im} i^*$.

Lemma 6.17 *The sequence (6-3) is exact at* $\mathrm{H}^2_{/b}(\Gamma)$.

Proof For $f \in \mathrm{Q}(N)^G$, we have $F \in \overline{\mathrm{Q}}_N(G)$ satisfying $F|_N = f$ by Lemma 6.9. Then a representative of $p^*(\tau_{/b}(f)) \in \mathrm{H}^2_{/b}(G)$ is given by $p^*\alpha_{F,s} \in C^2(G)$ for some section $s\colon \Gamma \to G$ of p. For $g_i \in G$,

$$p^*\alpha_{F,s}(g_1, g_2) = s^*\delta F(p(g_1), p(g_2)) = \delta F(sp(g_1), sp(g_2)) \approx_{4D''(F)} \delta F(g_1, g_2)$$

by Lemma 6.10. This implies $p^*(\tau_{/b}(f)) = 0$.

For $a \in \mathrm{H}^2_{/b}(\Gamma)$ satisfying $p^*a = 0$, let $\alpha \in C^2(\Gamma)$ be a representative of a. We can assume that the cochain satisfies

$$\alpha(1_\Gamma, 1_\Gamma) = 0. \tag{6-5}$$

Indeed, if $\alpha(1_\Gamma, 1_\Gamma) = c \in \mathbb{R}$, then the cochain $\alpha - c$ satisfies (6-5) and is also a representative of a since the constant function c is bounded. Note that the cocycle condition of $C^\bullet_{/b}(\Gamma)$ implies that there exists a nonnegative constant D such that

$$\delta\alpha \approx_D 0.$$

Hence, for $\gamma_1, \gamma_2 \in \Gamma$,

$$0 \approx_D \delta\alpha(\gamma_1, 1_\Gamma, \gamma_2) = \alpha(1_\Gamma, \gamma_2) - \alpha(\gamma_1, 1_\Gamma).$$

In particular,

$$\alpha(1_\Gamma, \gamma) \approx_D \alpha(1_\Gamma, 1_\Gamma) = 0 \quad \text{and} \quad \alpha(\gamma, 1_\Gamma) \approx_D \alpha(1_\Gamma, 1_\Gamma) = 0 \tag{6-6}$$

for every $\gamma \in \Gamma$. The equality $p^* a = 0$ implies that there exists $\beta \in C^1(G)$ and a nonnegative constant D' satisfying

$$p^*\alpha - \delta\beta \approx_{D'} 0. \tag{6-7}$$

Define a cochain $\zeta \colon G \to \mathbb{R}$ by

$$\zeta(g) = \beta(g) - \alpha(p(g), 1_\Gamma); \tag{6-8}$$

then it is an N-quasimorphism. Indeed, by using $p(n) = 1_\Gamma$, we have

$$\delta\zeta(n, g) = \delta\beta(n, g) - \big(\alpha(p(g), 1_\Gamma) - \alpha(p(g), 1_\Gamma) + \alpha(1_\Gamma, 1_\Gamma)\big) \approx_D (\delta\beta - p^*\alpha)(g, n) \approx_{D'} 0$$

and

$$\delta\zeta(g, n) = \delta\beta(g, n) - \big(\alpha(1_\Gamma, 1_\Gamma) - \alpha(p(g), 1_\Gamma) + \alpha(p(g), 1_\Gamma)\big) \approx_D (\delta\beta - p^*\alpha)(g, n) \approx_{D'} 0$$

by (6-6) and (6-7). By Lemma 6.6, there exists a bounded cochain $b \in C^1_b(G)$ such that the restriction $(\zeta + b)|_N$ is in $\mathrm{Q}(N)^\Gamma$. Set $\Phi = \zeta + b \in \overline{\mathrm{Q}}_N(G)$; then a representative of $\tau_{/b}(\Phi|_N)$ is given by $\alpha_{\Phi,s}$ for some section $s \colon \Gamma \to G$ of p. For $g_1, g_2 \in \Gamma$,

$$(\alpha_{\Phi,s} - \alpha)(g_1, g_2) = (\delta\Phi - p^*\alpha)(s(g_1), s(g_2)) \approx_{D'} (\delta\Phi - \delta\beta)(s(g_1), s(g_2))$$

by (6-7). By (6-8),

$$(\Phi - \beta)(g) = (\zeta + b - \beta)(g) = b(g) - \alpha(p(g), 1_\Gamma).$$

Together with (6-6) and the boundedness of b, the cochain $\Phi - \beta \colon G \to \mathbb{R}$ is bounded. Hence the cochain $\alpha_{\Phi,s} - \alpha$ is also bounded, and this implies $a = [\alpha_\Phi] = \tau_{/b}(\Phi|_N)$. □

Proof of Theorem 1.5 The exactness is obtained from Lemmas 6.13, 6.15 and 6.17. Commutativity of the first, second and fourth squares is obtained from the cochain level calculations. The commutativity of the third square follows from the definition of the map $\tau_{/b}$ and Proposition 6.18 below. □

Proposition 6.18 [83, Proposition 1.6.6] *Let $1 \to N \to G \to \Gamma \to 1$ be an exact sequence and V an Γ-module. For a G-invariant homomorphism $f \in \mathrm{H}^1(N; V)^G$, there exists a map $F \colon G \to V$ such that the restriction $F|_N$ is equal to f and the coboundary δF descends to a group two cocycle $\alpha_F \in C^2(\Gamma; V)$; that is, $p^*\alpha_F = \delta F$. Then the map $\tau \colon \mathrm{H}^1(N; V)^G \to \mathrm{H}^2(\Gamma; V)$ in the five-term exact sequence of group cohomology is obtained by $\tau(f) = [\alpha_F]$.*

We conclude this section with the following applications of Theorem 1.5 to the extendability of G-invariant homomorphisms:

Proposition 6.19 *Let $\Gamma = G/N$. Assume that $\mathrm{H}_b^2(\Gamma) = 0$ and $f\colon N \to \mathbb{R}$ is a G-invariant homomorphism on N. If f is extended to a quasimorphism on G, then f is extended to a homomorphism on G.*

Proof Note that the assumption $\mathrm{H}_b^2(\Gamma) = 0$ implies that the map $\mathrm{H}^2(\Gamma) \to \mathrm{H}_{/b}^2(\Gamma)$ is injective. By diagram chasing on (1-3), the proposition holds. □

This proposition immediately implies the following corollary:

Corollary 6.20 *Let $\Gamma = G/N$. Assume that $\mathrm{H}_b^2(\Gamma) = 0$ and N is a subgroup of $[G, G]$. Then every nonzero G-invariant homomorphism $f\colon N \to \mathbb{R}$ cannot be extended to G as a quasimorphism. Namely, $\mathrm{H}^1(N)^G \cap i^*\mathrm{Q}(G) = 0$.*

Proof Assume that a homomorphism $f\colon N \to \mathbb{R}$ can be extended to G as a quasimorphism. Then Proposition 6.19 implies that there exists a homomorphism $f'\colon G \to \mathbb{R}$ with $f'|_N = f$. Since f' vanishes on $[G, G]$, we have $f = f'|_N = 0$. □

Without the assumption $\mathrm{H}_b^2(\Gamma) = 0$, there exists a G-invariant homomorphism which is extendable to G as a quasimorphism but is *not* extendable to G as a (genuine) homomorphism. To see this, let $G = \widetilde{\mathrm{Homeo}}_+(S^1)$ and $N = \pi_1(\mathrm{Homeo}_+(S^1))$. Then $\Gamma = \mathrm{Homeo}_+(S^1)$ and hence $\mathrm{H}_b^2(\Gamma) \cong \mathbb{R} \neq 0$. Poincaré's rotation number $\rho\colon \widetilde{\mathrm{Homeo}}_+(S^1) \to \mathbb{R}$ is an extension of the homomorphism $\pi_1(\mathrm{Homeo}_+(S^1)) \cong \mathbb{Z} \hookrightarrow \mathbb{R}$. However, this homomorphism cannot be extendable to $\widetilde{\mathrm{Homeo}}_+(S^1)$ as a homomorphism since $\widetilde{\mathrm{Homeo}}_+(S^1)$ is perfect.

7 Proof of equivalences of scl_G and $\mathrm{scl}_{G,N}$

The goal of this section is to prove Theorem 2.1. Here we recall its precise statement.

Theorem 2.1 *Assume that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. Then:*

(1) *scl_G and $\mathrm{scl}_{G,N}$ are bi-Lipschitzly equivalent on $[G, N]$.*

(2) *If $\Gamma = G/N$ is amenable, then $\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x) \le 2 \cdot \mathrm{scl}_G(x)$ for all $x \in [G, N]$.*

(3) *If $\Gamma = G/N$ is solvable, then $\mathrm{scl}_G(x) = \mathrm{scl}_{G,N}(x)$ for all $x \in [G, N]$.*

In this section, in order to specify the domain of a quasimorphism, we use the symbols D_G and D_N to denote the defect of a quasimorphism on G and N, respectively. The main tool in this section is the Bavard duality theorem for $\mathrm{scl}_{G,N}$, which was proved by the first, second, fourth and fifth authors:

Theorem 7.1 (Bavard duality theorem for stable mixed commutator lengths [60]) *Let N be a normal subgroup of a group G. Then, for every $x \in [G, N]$,*

$$\mathrm{scl}_{G,N}(x) = \frac{1}{2} \sup_{f \in \mathrm{Q}(N)^G - \mathrm{H}^1(N)^G} \frac{|f(x)|}{D_N(f)}.$$

Here we set the supremum in the right-hand side of the above equality to be zero if $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G$.

This theorem yields the following criterion to show the equivalence of $\mathrm{scl}_{G,N}$ and scl_G:

Proposition 7.2 *Let C be a real number such that, for every $f \in \mathrm{Q}(N)^G$, there exists $f' \in \mathrm{Q}(G)$ satisfying $f'|_N - f \in \mathrm{H}^1(N)^G$ and $D_G(f') \le C \cdot D_N(f)$. Then, for every $x \in [G, N]$,*

$$\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x) \le C \cdot \mathrm{scl}_G(x).$$

The existence of a C as in the assumption of Proposition 7.2 is equivalent to saying that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. See Section 7.1 for details.

Proof Let $x \in [G, N]$. It is clear that $\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x)$. Let $\varepsilon > 0$. Then Theorem 7.1 implies that there exists $f \in \mathrm{Q}(N)^G$ such that

$$\mathrm{scl}_{G,N}(x) - \varepsilon \le \frac{f(x)}{2D_N(f)}.$$

By assumption, there exists $f' \in \mathrm{Q}(G)$ such that $f'' = f'|_N - f \in \mathrm{H}^1(N)^G$ and $D_G(f') \le C \cdot D_N(f)$. Since f'' is a G-invariant homomorphism and $x \in [G, N]$, we have $f''(x) = 0$ and hence $f'(x) = f(x)$. Hence,

$$\mathrm{scl}_{G,N}(x) - \varepsilon \le \frac{f(x)}{2D_N(f)} \le C \cdot \frac{f'(x)}{2D_G(f')} \le C \cdot \mathrm{scl}_G(x).$$

Here we use the original Bavard duality theorem [6] to prove the last inequality. Since ε is an arbitrary number, we complete the proof. □

In the proofs of Theorem 2.1(2)–(3), we use the following corollary of Proposition 7.2:

Corollary 7.3 *Assume that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ and that there exists $C \ge 1$ such that $f' \in \mathrm{Q}(G)$ implies that $D_G(f') \le C \cdot D_N(f'|_N)$. Then, for every $x \in [G, N]$,*

$$\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x) \le C \cdot \mathrm{scl}_G(x).$$

Proof Let $f \in \mathrm{Q}(N)^G$. Then, by the assumption that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, there exists $f' \in \mathrm{Q}(G)$ such that $f'|_N - f$ is a G-invariant homomorphism. Note that $D_N(f'|_N) = D_N(f)$. Indeed, for every $a, b \in N$,

$$f(ab) - f(a) - f(b) = f'(ab) - f'(a) - f'(b)$$

since $f'|_N - f$ is a homomorphism. Hence, $C \cdot D_N(f) = C \cdot D_N(f'|_N) \ge D_G(f')$. Hence, Proposition 7.2 implies that

$$\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x) \le C \cdot \mathrm{scl}_G(x)$$

for every $x \in [G, N]$. □

concept	defect	definition	vector space
quasimorphism on G	D	$f(g_1g_2)\approx_D f(g_1)+f(g_2)$	$\widehat{\mathrm{Q}}(G)$
G-quasi-invariant quasimorphism on N	D, D'	$f(x_1x_2)\approx_D f(x_1)+f(x_2)$, $f(gxg^{-1})\approx_{D'} f(x)$	$\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}$
N-quasimorphism on G	D''	$f(gx)\approx_{D''} f(g)+f(x)$, $f(xg)\approx_{D''} f(x)+f(g)$	$\widehat{\mathrm{Q}}_N(G)$

Table 1: The concepts and symbols on quasimorphisms.

7.1 Proof of Theorem 2.1(1)

The main difficulty in the proof of Theorem 2.1 is distilled in Theorem 7.4, stated below. Note that the defect D_N defines a seminorm on $\mathrm{Q}(N)^G$, and its kernel is $\mathrm{H}^1(N)^G$.

Theorem 7.4 *The normed space* $(\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G, D_N)$ *is a Banach space.*

To show this theorem, we recall some concepts introduced in [60]. Let $\widehat{\mathrm{Q}}_N(G)=\widehat{\mathrm{Q}}_N(G;\mathbb{R})$ denote the $\mathbb{R}$-vector space of N-quasimorphisms (see Definition 6.3 and Table 1). We call $f\in\widehat{\mathrm{Q}}_N(G)$ an *N-homomorphism* if $D''(f)=0$, and let $\mathrm{H}^1_N(G)$ denote the space of N-homomorphisms on G. It is clear that the defect D'' is a seminorm on $\widehat{\mathrm{Q}}_N(G)$, and, in fact, the norm space $\widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$ is complete:

Proposition 7.5 [60, Corollary 3.6] *The normed space* $(\widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G), D'')$ *is a Banach space.*

A quasimorphism $f\colon N\to\mathbb{R}$ is said to be *G-quasi-invariant* if the number

$$D'(f)=\sup_{g\in G, x\in N}|f(gxg^{-1})-f(x)|$$

is finite. Let $\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}$ denote the space of G-quasi-invariant quasimorphisms on N. The function D_N+D', which assigns $D_N(f)+D'(f)$ to $f\in\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}$, defines a seminorm on $\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}$. For an N-quasimorphism f on G, the restriction $f|_N$ is a G-quasi-invariant quasimorphism [60, Lemma 2.3]. Conversely, for every G-quasi-invariant quasimorphism f on N, there exists an N-quasimorphism $f'\colon G\to\mathbb{R}$ satisfying $f'|_N=f$ [60, Proposition 2.4]. We summarize the concepts and symbols on quasimorphisms in Table 1.

Lemma 7.6 *The normed space* $(\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G, D_N+D')$ *is a Banach space.*

Proof In what follows, we will define bounded operators

$$A\colon \widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)\to\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G,\quad B\colon \widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G\to\widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$$

such that $A\circ B$ is the identity of $\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G$. First, we define A by the restriction, ie $A(f)=f|_N$. Then the operator norm of A is at most 3 since $D_N\le D''$ and $D'\le 2D''$. Indeed, $D_N\le D''$ follows by definition, and $D'\le 2D''$ follows from the estimate

$$f(gxg^{-1})+f(g)\approx_{D''(f)} f(gx)\approx_{D''(f)} f(g)+f(x)$$

for $g\in G$ and $x\in N$.

Let S be a subset of G such that $1_G \in S$ and the map

$$S \times N \to G, \quad (s, x) \mapsto sx,$$

is bijective. For an $f \in \widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}$, define a function $B(f)\colon G \to \mathbb{R}$ by $B(f)(sx) = f(x)$ for $s \in S$ and $x \in N$. Then $B(f)$ is an N-quasimorphism on G satisfying $D''(B(f)) \le D_N(f) + D'(f)$. Hence, the map B induces a bounded operator $\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G \to \widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$ whose operator norm is at most 1, and we conclude that $\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G$ is isomorphic to $B(\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G)$. Proposition 7.5 implies that $\widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$ is a Banach space. Therefore, it suffices to show that $B(\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G)$ is a closed subset of $\widehat{\mathrm{Q}}_N(G)/\mathrm{H}^1_N(G)$, which follows from the well-known Lemma 7.7, given below. □

Lemma 7.7 *Let X be a topological subspace of a Hausdorff space Y. If X is a retract of Y, then X is a closed subset of Y.*

Proof Let $r\colon Y \to X$ be a retraction of the inclusion map $i\colon X \to Y$. Since $X = \{y \in Y \mid i \circ r(y) = y\}$ and Y is a Hausdorff space, we conclude that X is a closed subset of Y. □

Proof of Theorem 7.4 For $n \in \mathbb{Z}$ and $x \in N$, define a function $\alpha_{n,x}\colon \widehat{\mathrm{Q}}(N)^{\mathrm{Q}G} \to \mathbb{R}$ by

$$\alpha_{n,x}(f) = f(x^n) - n \cdot f(x).$$

Since $|\alpha_{n,x}(f)| \le (n-1) D_N(f)$, we conclude that $\alpha_{n,x}$ is bounded with respect to the norm $D_N + D'$, and hence $\alpha_{n,x}$ induces a bounded operator $\overline{\alpha}_{n,x}\colon \widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G \to \mathbb{R}$. Since

$$\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G = \bigcap_{n \in \mathbb{Z}, x \in N} \mathrm{Ker}(\overline{\alpha}_{n,x}),$$

the space $\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G$ is a closed subspace of the Banach space $\widehat{\mathrm{Q}}(N)^{\mathrm{Q}G}/\mathrm{H}^1(N)^G$ (see Lemma 7.6). Since $D' = 0$ on $\mathrm{Q}(N)^G$ [60, Lemma 2.1], $(\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G, D_N)$ is a Banach space. □

Proof of Theorem 2.1(1) It is clear that $\mathrm{scl}_G(x) \le \mathrm{scl}_{G,N}(x)$ for every $x \in [G, N]$. Hence, it suffices to show that there exists $C > 1$ such that, for every $x \in [G, N]$, we have $\mathrm{scl}_{G,N}(x) \le C \cdot \mathrm{scl}_G(x)$.

It follows from Theorem 7.4 that $(\mathrm{Q}(G)/\mathrm{H}^1(G), D_G)$ and $(\mathrm{Q}(N)^G/\mathrm{H}^1(N)^G, D_N)$ are Banach spaces. Let $T\colon \mathrm{Q}(G)/\mathrm{H}^1(G) \to \mathrm{Q}(N)^G/\mathrm{H}^1(N)^G$ be the bounded operator induced by the restriction $\mathrm{Q}(G) \to \mathrm{Q}(N)^G$. Let X be the kernel of T. Then T induces a bounded operator

$$\overline{T}\colon (\mathrm{Q}(G)/\mathrm{H}^1(G))/X \to \mathrm{Q}(N)^G/\mathrm{H}^1(N)^G.$$

The assumption $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ implies that the map T is surjective, and hence $\overline{T}$ is a bijective bounded operator. By the open mapping theorem, the inverse $S = \overline{T}^{-1}$ is a bounded operator, and we set $C = \|S\| + 1$, where $\|S\|$ denotes the operator norm of S. Then, for every $[f] \in \mathrm{Q}(N)^G/\mathrm{H}^1(N)^G$, there exists $f' \in \mathrm{Q}(G)$ such that $D_G(f') \le C \cdot D_N(f)$ and $f'|_N - f \in \mathrm{H}^1(N)^G$. Hence, Proposition 7.2 implies that

$$\mathrm{scl}_G \le \mathrm{scl}_{G,N} \le C \cdot \mathrm{scl}_G$$

on $[G, N]$. □

7.2 Proof of Theorem 2.1(2)

Here we recall the definition of the seminorm on $\mathrm{H}_b^n(G)$. The space $C_b^n(G;\mathbb{R})$ of bounded n-cochains on G is a Banach space with respect to the ∞-norm

$$\|\varphi\|_\infty = \sup\{|\varphi(x_1,\ldots,x_n)| : x_1,\ldots,x_n \in G\}.$$

Since $\mathrm{H}_b^n(G) = \mathrm{H}_b^n(G;\mathbb{R})$ is a subquotient of $C_b^n(G;\mathbb{R})$, it has a seminorm induced by the norm on $C_b^n(G;\mathbb{R})$. Namely, for $\alpha \in \mathrm{H}_b^n(G)$, the seminorm $\|\alpha\|$ on $\mathrm{H}_b^n(G)$ is defined by

$$\|\alpha\| = \inf\{\|\varphi\|_\infty \mid \varphi \text{ is a bounded } n\text{-cocycle on } G \text{ satisfying } [\varphi] = \alpha\}.$$

Theorem 7.8 (see [74, Proposition 8.6.6]) *If $\Gamma = G/N$ is amenable, then the map $\mathrm{H}_b^2(G) \to \mathrm{H}_b^2(N)^G$ is an isometric isomorphism.*

We recall the following estimate of the defect of the homogenization:

Lemma 7.9 [26, Lemma 2.58] *Let $\delta\colon \mathrm{Q}(G) \to \mathrm{H}_b^2(G)$ be the natural map. Then*

$$\|[\delta f]\| \le D_G(f) \le 2\cdot\|[\delta f]\|.$$

Proof of Theorem 2.1(2) Suppose that $\Gamma = G/N$ is amenable and $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. Let $f \in \mathrm{Q}(G)$. By Corollary 7.3, it suffices to show that $2D_N(f|_N) \ge D_G(f)$ for every $f \in \mathrm{Q}(G)$. This follows from

$$2D_N(f|_N) \ge 2\|[\delta f|_N]\| = 2\|[\delta f]\| \ge D_G(f),$$

where the equality and the inequalities are deduced from Theorem 7.8 and Lemma 7.9, respectively. □

7.3 Proof of Theorem 2.1(3)

Lemma 7.10 *Let $f\colon N \to \mathbb{R}$ be an extendable homogeneous quasimorphism on N. Then, for each $a, b \in G$ satisfying $[a,b] \in N$,*

$$|f([a,b])| \le D_N(f).$$

Proof We first prove

$$[a^n, b] = a^{n-1}[a,b]a^{-(n-1)} \cdot a^{(n-2)}[a,b]a^{-(n-2)} \cdots [a,b]. \tag{7-1}$$

Indeed,

$$[a^n,b] = a^nba^{-n}b^{-1} = a^{n-1}\cdot aba^{-1}b^{-1}\cdot a^{-(n-1)}\cdot a^{n-1}ba^{-(n-1)}b^{-1} = a^{n-1}[a,b]a^{-(n-1)}\cdot[a^{n-1},b].$$

By induction on n, (7-1) follows. Since f is G-invariant, we have

$$f([a^n,b]) \approx_{(n-1)D_N(f)} f(a^{n-1}[a,b]a^{-(n-1)}) + \cdots + f([a,b]) = n\cdot f([a,b]).$$

Therefore,

$$|f([a^n, b])| \geq n \cdot \big(|f([a, b])| - D_N(f)\big).$$

Suppose that $|f([a,b])| > D_N(f)$. Then the right side of the above inequality can be unbounded with respect to n. However, since f is extendable, the left side of the above inequality is bounded. This is a contradiction. □

In Corollary 6.20, we provide a condition that ensures a G-invariant homomorphism $f\colon N \to \mathbb{R}$ cannot be extended to G as a quasimorphism. Here we present another condition:

Corollary 7.11 *Let $f\colon N \to \mathbb{R}$ be a G-invariant homomorphism and assume that N is generated by simple commutators of G. If f is nonzero, then f is not extendable.*

Proof If f is extendable, then Lemma 7.10 implies that $f(c) = 0$ for every simple commutator c of G contained in N. Since N is generated by simple commutators of G, this means that $f = 0$. □

Lemma 7.12 *Let f be a homogeneous quasimorphism on G, and assume that $\Gamma = G/N$ is solvable. Then $D_G(f) = D_N(f|_N)$.*

Proof We first assume that Γ is abelian. It is known that $D_G(f) = \sup_{a,b\in G} |f([a,b])|$ (see [26, Lemma 2.24]). Applying Lemma 7.10 to $f|_N$, we have

$$D_G(f) = \sup_{a,b\in G} |f([a,b])| \leq D_N(f|_N) \leq D_G(f),$$

and, in particular, $D_G(f) = D_N(f|_N)$.

Next we consider the general case. Let $G^{(n)}$ denote the n^{th} derived subgroup of G. Then there exists a positive integer n such that $G^{(n)} \subset N$ since Γ is solvable. By the previous paragraph,

$$D_G(f) = D_{G^{(1)}}(f|_{G^{(1)}}) = \cdots = D_{G^{(n)}}(f|_{G^{(n)}}) \leq D_N(f|_N) \leq D_G(f). \qquad \square$$

Proof of Theorem 2.1(3) Combine Lemma 7.12 and Corollary 7.3. □

Next we provide some applications of Theorem 2.1(3):

Corollary 7.13 *If one of the following conditions holds, then $\mathrm{scl}_G = \mathrm{scl}_{G,N}$ on $[G,N]$ (here $\Gamma = G/N$):*

1. *Γ is a finite solvable group.*
2. *Γ is a finitely generated abelian group whose rank is at most 1.*

Proof This clearly follows from Proposition 3.4 and Theorem 2.1(3). □

In Section 9.2, we propose several problems on the coincidence and equivalence of scl_G and $\mathrm{scl}_{G,N}$.

7.4 Examples with nonequivalent scl_G and $\mathrm{scl}_{G,N}$

To conclude this section, we provide some examples of group pairs (G, N) for which scl_G and $\mathrm{scl}_{G,N}$ *fail* to be bi-Lipschitzly equivalent on $[G, N]$.

Example 7.14 Let l be an integer at least 2, and Ω be an area form of Σ_l. In this case, the flux group Γ_Ω is known to be trivial; thus we have the volume flux homomorphism $\mathrm{Flux}_\Omega\colon \mathrm{Diff}_0(\Sigma, \Omega) \to \mathrm{H}^1(\Sigma_l)$. In [58], the authors proved that, for the pair

$$(G, N) = (\mathrm{Diff}_0(\Sigma_l, \Omega), \mathrm{Ker}(\mathrm{Flux}_\Omega)),$$

scl_G and $\mathrm{scl}_{G,N}$ are *not* bi-Lipschitzly equivalent on $[G, N]$. More precisely, we found an element γ in $[G, N]$ such that

$$\mathrm{scl}_G(\gamma) = 0 \quad \text{but} \quad \mathrm{scl}_{G,N}(\gamma) > 0.$$

Example 7.15 We can provide the following example, which is related to Example 7.14 with smaller G, from results in [61]. We stick to the setting of Example 7.14. Take an arbitrary pair (v, w) with $v, w \in \mathrm{H}^1(\Sigma_l)$ that satisfies

$$v \smile w \neq 0. \tag{7-2}$$

Here, recall from Theorem 1.3 that $\smile\colon \mathrm{H}^1(\Sigma_l) \times \mathrm{H}^1(\Sigma_l) \to \mathrm{H}^2(\Sigma_l) \cong \mathbb{R}$ denotes the cup product. Then, from results in [61], we can deduce the following: there exists a positive integer k_0, depending only on w and the area of Σ_l, such that, for every $k \geq k_0$, if we set

$$\Lambda_k = \langle v, w/k \rangle,$$

namely the subgroup of $\mathrm{H}^1(\Sigma_l)$ generated by v and w/k, and

$$(G, N) = (\mathrm{Flux}_\Omega^{-1}(\Lambda_k), \mathrm{Ker}(\mathrm{Flux}_\Omega)),$$

then scl_G and $\mathrm{scl}_{G,N}$ are *not* bi-Lipschitzly equivalent on $[G, N]$. To see this, by following arguments in [61, Section 4], we construct a sequence $(\gamma_m)_{m \in \mathbb{N}}$ in $[G, N]$. Then [61, Proposition 4.6], together with Bavard's duality theorem, implies that

$$\sup_{m \in \mathbb{N}} \mathrm{scl}_G(\gamma_m) \leq \tfrac{3}{2}.$$

Contrastingly, Proposition 4.7(3) in [61], together with Theorem 7.1, implies that

$$\liminf_{m \to \infty} \frac{\mathrm{scl}_{G,N}(\gamma_m)}{m} \geq \frac{1}{2k \cdot D_N(f_P)} |\mathfrak{b}_I(v, w)| > 0.$$

Here $\mathfrak{b}_I(v, w) = \langle v \smile w, [\Sigma_l] \rangle_{\Sigma_l} \in \mathbb{R}$ is the intersection number of v and w, where $[\Sigma_l]$ is the fundamental class of Σ_l and $\langle \cdot , \cdot \rangle_{\Sigma_l} : \mathrm{H}^2(\Sigma_l) \times \mathrm{H}_2(\Sigma_l) \to \mathbb{R}$ denotes the Kronecker pairing of Σ_l. The map $f_P : N \to \mathbb{R}$ is Py's Calabi quasimorphism (recall Section 5; see also [61, Sections 2.4 and 2.5]). We also note that v, w and f_P here correspond to $\bar{v}$, $\bar{w}$ and μ_P in [61], respectively.

8 Aut(F_n) and Mod(Σ_l)

8.1 Proof of Theorem 2.3

An *IA-automorphism* of a group G is an automorphism f on G which acts as the identity on the abelianization $\mathrm{H}_1(G;\mathbb{Z})$ of G. We write IA_n to indicate the group of IA-automorphisms on F_n. Then we have exact sequences

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n,\mathbb{Z}) \to 1, \quad 1 \to \mathrm{IA}_n \to \mathrm{Aut}_+(F_n) \to \mathrm{SL}(n,\mathbb{Z}) \to 1.$$

Theorem 2.3(1) claims that $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}(F_n)} = i^*\mathrm{Q}(\mathrm{Aut}(F_n))$ and $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}_+(F_n)} = i^*\mathrm{Q}(\mathrm{Aut}_+(F_n))$. To show it, we use the following facts, which can be derived from the computation of the second integral homology $\mathrm{H}_2(\mathrm{SL}(n,\mathbb{Z});\mathbb{Z})$:

Theorem 8.1 (see [72]) *For $n \geq 3$, $\mathrm{H}^2(\mathrm{SL}(n,\mathbb{Z})) = 0$ and $\mathrm{H}^2(\mathrm{GL}(n,\mathbb{Z})) = 0$.*

The following is obtained from [77, Corollary 1.4; 75, Theorem 1.2]:

Theorem 8.2 *Let n be an integer at least 3 and Γ_0 a subgroup of finite index in $\mathrm{SL}(n,\mathbb{Z})$. Then $\mathrm{H}^3_b(\Gamma_0) = 0$.*

The following theorem is a special case of [74, Proposition 8.6.2]:

Theorem 8.3 *Let N be a subgroup of finite index in G and V a Banach G-module; then the restriction $\mathrm{H}^n_b(G;V) \to \mathrm{H}^n_b(N;V)$ is injective for every $n \geq 0$.*

Now we proceed to the proof of Theorem 2.3(1). First we show the following lemma:

Lemma 8.4 *Let n be an integer at least 3 and Γ_0 a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$. Then $\mathrm{H}^3_b(\Gamma_0) = 0$.*

Proof Since $\Gamma_0 \cap \mathrm{SL}(n,\mathbb{Z})$ is a subgroup of finite index in $\mathrm{SL}(n,\mathbb{Z})$, we have $\mathrm{H}^3_b(\Gamma_0 \cap \mathrm{SL}(n,\mathbb{Z})) = 0$ by Theorem 8.2. Since $\Gamma_0 \cap \mathrm{SL}(n,\mathbb{Z})$ is a subgroup of finite index in Γ_0, we obtain $\mathrm{H}^3_b(\Gamma_0) = 0$ by Theorem 8.3. □

Proof of Theorem 2.3(1) Suppose that $n = 2$. Then $\mathrm{GL}(n,\mathbb{Z})$ and $\mathrm{SL}(n,\mathbb{Z})$ have a subgroup of finite index which is isomorphic to a free group. Therefore this case is proved by Proposition 3.4. So suppose that $n > 2$. Let Γ be either $\mathrm{GL}(n,\mathbb{Z})$ or $\mathrm{SL}(n,\mathbb{Z})$. By Theorem 8.1, Lemma 8.4 and the cohomology long exact sequence, we have $\mathrm{H}^2_{/b}(\Gamma) = 0$. Hence, Theorem 1.5 implies that $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}(F_n)}/i^*\mathrm{Q}(\mathrm{Aut}(F_n)) = 0$ and $\mathrm{Q}(\mathrm{IA}_n)^{\mathrm{Aut}_+(F_n)}/i^*\mathrm{Q}(\mathrm{Aut}_+(F_n)) = 0$. □

Next we prove Theorem 2.3(2). First we recall the following application of the transfer:

Lemma 8.5 (see [19, Proposition III.10.4]) *Let N be a normal subgroup of finite index in G, V a real G-module and q a positive integer. Then the restriction induces an isomorphism $\mathrm{H}^q(G;V) \xrightarrow{\cong} \mathrm{H}^q(N;V)^{\Gamma}$, where $\Gamma = G/N$.*

In the proof of Theorem 2.3(2), we furthermore use the following theorem, due to Borel [14; 15; 16] and Hain [47] (and Tshishiku [94]):

Theorem 8.6 (1) *For every $n \geq 6$ and every subgroup Γ_0 of finite index in $\mathrm{GL}(n,\mathbb{Z})$, $\mathrm{H}^2(\Gamma_0) = 0$.*

(2) *For every $l \geq 3$ and every subgroup Γ_0 of finite index in $\mathrm{Sp}(2l,\mathbb{Z})$, the inclusion map $\Gamma_0 \hookrightarrow \mathrm{Sp}(2l,\mathbb{Z})$ induces an isomorphism of cohomology $\mathrm{H}^2(\mathrm{Sp}(2l,\mathbb{Z})) \cong \mathrm{H}^2(\Gamma_0)$. In particular, the cohomology $\mathrm{H}^2(\Gamma_0)$ is isomorphic to $\mathbb{R}$.*

For the convenience of the reader, we describe the deduction of Theorem 8.6 from the work of Borel, Hain and Tshishiku.

Proof First we discuss (2). This is stated in [47, Theorem 3.2]; see [94] for the complete proof.

Next we treat (1). Let $\Lambda = \Gamma_0 \cap \mathrm{SL}(n,\mathbb{Z})$. Then Λ is a subgroup of finite index both in Γ_0 and in $\mathrm{SL}(n,\mathbb{Z})$. By Lemma 8.5, the restriction $\mathrm{H}^2(\Gamma_0) \to \mathrm{H}^2(\Lambda)$ is injective. Hence, to prove (1), it suffices to show that $\mathrm{H}^2(\Lambda) = 0$. In what follows, we sketch the deduction of $\mathrm{H}^2(\Lambda) = 0$ from the work of Borel; see also the discussion in the introduction of [94].

We appeal to Borel's theorem [15, Theorem 1], with $G = \mathrm{SL}_n$, $\Gamma = \Lambda$ and r the trivial complex representation. (See also [14, Theorem 11.1].) Then there exists a natural homomorphism $\mathrm{H}^q(\mathfrak{g},\mathfrak{k};\mathbb{C})^{\Lambda} \to \mathrm{H}^q(\Lambda;\mathbb{C})$ and, if $q \leq \min\{c(\mathrm{SL}_n), \mathrm{rank}_{\mathbb{R}}(\mathrm{SL}(n,\mathbb{R})) - 1\}$, then this map is an isomorphism. Here $\mathrm{H}^q(\mathfrak{g},\mathfrak{k};\mathbb{C})^{\Lambda}$ is a Lie algebraic cohomology ($\mathfrak{g}$ and $\mathfrak{k}$ stand for the Lie algebras of $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n)$, respectively); it is known that $\mathrm{H}^2(\mathfrak{g},\mathfrak{k};\mathbb{C})^{\Lambda} = 0$; see [14, 11.4]. For the definition of the constant $c(G) = c(G,0)$ for G being a connected semisimple group defined over $\mathbb{Q}$, see [14, 7.1]. We remark that, for the trivial complex representation r, $c(G,r) = (c(G,0) =)c(G)$. Since SL_n is of type A_{n-1}, the constant $c(\mathrm{SL}_n)$ equals $\lfloor \frac{1}{2}(n-2) \rfloor$; see [15] (and [14, 9.1]). Here, $\lfloor \cdot \rfloor$ denotes the floor function. The number $\mathrm{rank}_{\mathbb{R}}(\mathrm{SL}(n,\mathbb{R}))$ means the real rank of $\mathrm{SL}(n,\mathbb{R})$; it equals $n-1$. Since $n \geq 6$, we hence have $\min\{c(\mathrm{SL}_n), \mathrm{rank}_{\mathbb{R}}(\mathrm{SL}(n,\mathbb{R})) - 1\} \geq 2$. Therefore, $\mathrm{H}^2(\Lambda;\mathbb{C}) = 0$. This immediately implies that $\mathrm{H}^2(\Lambda) = 0$, as desired. (Borel [16] considered a better constant $C(G)$ than $c(G)$ in general, but $C(\mathrm{SL}_n) = \lfloor \frac{1}{2}(n-2) \rfloor$; see [16; 94].) □

Remark 8.7 In the proof of Theorem 2.3(2), we only use Theorem 8.6(1). We will use Theorem 8.6(2) in the proofs of claims in the next subsection.

concept	defect	definition	vector space
quasicocycle on G	D	$F(g_1g_2) \approx_D F(g_1) + g_1 \cdot F(g_2)$	$\hat{\mathrm{Q}}Z(G;V)$
G-quasiequivalent quasimorphism on N	D, D'	$f(x_1x_2) \approx_D f(x_1) + f(x_2)$, $f(gxg^{-1}) \approx_{D'} g \cdot f(x)$	$\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$
N-quasicocycle on G	D''	$F(gx) \approx_{D''} F(g) + g \cdot F(x)$, $F(xg) \approx_{D''} F(x) + F(g)$	$\hat{\mathrm{Q}}Z_N(G;V)$

Table 2: The concepts and symbols on quasicocycles.

Proof of Theorem 2.3(2) Let n be an integer at least 6. Let G be a group of finite index in $\mathrm{Aut}(F_n)$. Set $N = G \cap \mathrm{IA}_n$ and $\Gamma = G/N$. Then we have an exact sequence

$$1 \to N \to G \to \Gamma \to 1$$

and Γ is a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$. By Lemma 8.4 and Theorem 8.6(1), the second relative cohomology group $\mathrm{H}^2_{/b}(\Gamma)$ is trivial. Therefore, by Theorem 1.5, $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G) = 0$. □

Remark 8.8 By [80, Theorem 1.4] and Theorem 8.2, for every $n \geq 3$, every subgroup of finite index in $\mathrm{SL}(n,\mathbb{Z})$ is boundedly 3-acyclic.

8.2 Quasicocycle analogues of Theorem 2.3: finite-dimensional unitary coefficients

To state our next result, we need some notation. In Section 7.1, we introduced the notion of G-quasiequivariant quasimorphism. Let V be an $\mathbb{R}[G]$-module whose G-action on V is trivial at N. The G-quasi-invariance can be extended to the V-valued quasimorphisms as the G-quasiequivariance. Recall from Remark 6.4 that a V-valued quasimorphism $f\colon N \to V$ is G-equivariant if $f(gxg^{-1}) - g \cdot f(x) = 0$. A V-valued quasimorphism $f\colon N \to V$ is said to be G-*quasiequivariant* if the number

$$D'(f) = \sup_{g \in G, x \in N} \|f(gxg^{-1}) - g \cdot f(x)\|$$

is finite. Let $\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$ denote the $\mathbb{R}$-vector space of all G-quasiequivariant V-valued quasimorphisms. Let $F\colon G \to V$ be a quasicocycle; then the restriction $F|_N$ belongs to $\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$ by definition. It is straightforward to show that $\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}/i^*\hat{\mathrm{Q}}Z(G;V)$ is isomorphic to $\mathrm{Q}(N;V)^G/i^*\mathrm{H}^1_{/b}(G;V) = \mathrm{H}^1_{/b}(N;V)^G/i^*\mathrm{H}^1_{/b}(G;V)$. We summarize the concepts and symbols on quasicocycles in Table 2.

Our main results in this section are the following two theorems. In fact, in Section 8.3, we will deduce further generalizations (Theorems 8.17 and 8.14) of Theorems 8.9 and 8.10, respectively, from a recent result of Bader and Sauer [3].

Theorem 8.9 (result for $\mathrm{Aut}(F_n)$) *Let n be an integer at least 6 and G a subgroup of finite index in $\mathrm{Aut}(F_n)$. Then, for every finite-dimensional unitary representation π of Γ,*

$$\hat{\mathrm{Q}}(N;\mathcal{H})^{\mathrm{Q}G} = i^*\hat{\mathrm{Q}}Z(G;\bar{\pi},\mathcal{H}).$$

Here $(\bar{\pi},\mathcal{H})$ is the pullback to G of the representation $(\pi,\mathcal{H})$ of Γ.

Theorem 8.10 (result for $\mathrm{Mod}(\Sigma_l)$) *Let l be an integer at least 3 and G a subgroup of finite index in $\mathrm{Mod}(\Sigma_l)$. Set $N = G \cap \mathcal{I}(\Sigma_l)$ and $\Gamma = G/N$. Let $(\pi, \mathcal{H})$ be a finite-dimensional unitary Γ-representation such that $\pi \not\supset 1$, ie $\mathcal{H}^{\pi(\Gamma)} = 0$. Then*

$$\widehat{\mathrm{Q}}(N;\mathcal{H})^{\mathrm{Q}G} = i^*\widehat{\mathrm{Q}}Z(G;\bar{\pi},\mathcal{H}).$$

Here $\bar{\pi}$ is the pullback of π by the quotient homomorphism $G \to \Gamma$.

Before proceeding to the proofs of Theorems 8.9 and 8.10, we mention some known results we need in the proofs. The following theorem is well known (see [7, Corollaries 4.C.16 and 4.B.6]):

Theorem 8.11 *Let Γ_0 be a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$ for $n \geq 3$ or $\mathrm{Sp}(2l,\mathbb{Z})$ for $l \geq 3$, and $(\pi,\mathcal{H})$ a finite-dimensional unitary Γ_0-representation. Then $\Gamma_0(\pi) := \mathrm{Ker}(\pi : \Gamma_0 \to \mathcal{U}(\mathcal{H}))$ is a subgroup of finite index in Γ_0, where $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$.*

Theorem 8.12 [77, Corollary 1.6] *Let l be an integer at least 2 and Γ_0 a subgroup of finite index in $\mathrm{Sp}(2l,\mathbb{Z})$. Let $(\pi,\mathcal{H})$ be a unitary Γ_0-representation with $\mathcal{H}$ separable and $\pi \not\supset 1$. Then $\mathrm{H}^3_b(\Gamma_0;\pi,\mathcal{H}) = 0$.*

Corollary 8.13 (1) *Let n be an integer at least 6 and Γ_0 a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$. Let $(\pi,\mathcal{H})$ be a finite-dimensional unitary Γ_0-representation. Then $\mathrm{H}^2(\Gamma_0;\pi,\mathcal{H}) = 0$.*

(2) *Let l be an integer at least 3, Γ_0 a subgroup of finite index in $\mathrm{Sp}(2l,\mathbb{Z})$, and $(\pi,\mathcal{H})$ a finite-dimensional unitary Γ_0-representation such that $\pi \not\supset 1$. Then $\mathrm{H}^2(\Gamma_0;\pi,\mathcal{H}) = 0$.*

Proof We first prove (2). Set $\Gamma_0(\pi) = \mathrm{Ker}(\pi)$. Then Theorem 8.11 implies that $\Gamma_0(\pi)$ is of finite index in Γ_0. Hence, by Lemma 8.5, $\mathrm{H}^2(\Gamma_0;\pi,\mathcal{H}) \cong \mathrm{H}^2(\Gamma_0(\pi);\mathcal{H})^{\Gamma_0/\Gamma_0(\pi)}$.

We now show the following claims:

Claim *The conjugation action by Γ_0 on the cohomology $\mathrm{H}^2(\Gamma_0(\pi))$ is trivial.*

Proof By Theorems 8.11 and 8.6(2), the inclusion $i : \Gamma_0(\pi) \hookrightarrow \Gamma_0$ induces an isomorphism $i^* : \mathrm{H}^2(\Gamma_0) \cong \mathrm{H}^2(\Gamma_0(\pi))$. Lemma 8.5 implies that the map i in fact induces an isomorphism $\mathrm{H}^2(\Gamma_0) \cong \mathrm{H}^2(\Gamma_0(\pi))^{\Gamma_0}$. Therefore, the Γ_0-action on $\mathrm{H}^2(\Gamma_0(\pi))$ is trivial. ◁

Claim *There exists a canonical isomorphism $\mathrm{H}^2(\Gamma_0(\pi);\mathcal{H}) \cong \mathcal{H}$, and this induces an isomorphism $\mathrm{H}^2(\Gamma_0(\pi);\mathcal{H})^{\Gamma_0/\Gamma_0(\pi)} \cong \mathcal{H}^{\Gamma_0/\Gamma_0(\pi)}$.*

Proof By Theorem 8.6(2), the cohomology $\mathrm{H}^2(\Gamma_0(\pi))$ is isomorphic to $\mathbb{R}$; hence, by the universal coefficient theorem, the cohomology $\mathrm{H}^2(\Gamma_0(\pi);\mathcal{H})$ is isomorphic to $\mathcal{H}$ (here, note that $\mathcal{H}$ is a trivial $\mathbb{R}[\Gamma_0(\pi)]$-module of finite dimension). In what follows, we exhibit a concrete isomorphism. For $\alpha \in \mathcal{H}$, we define a cochain $c_\alpha \in C^2(\Gamma_0(\pi);\mathcal{H})$ by

$$c_\alpha(\gamma_1,\gamma_2) = c(\gamma_1,\gamma_2)\cdot\alpha \in \mathcal{H},$$

where $c \in C^2(\Gamma_0(\pi))$ is a cocycle whose cohomology class corresponds to $1 \in \mathbb{R}$ under the isomorphism $\mathrm{H}^2(\Gamma_0(\pi)) \cong \mathbb{R}$. This cochain c_α is a cocycle since the $\Gamma_0(\pi)$-action on $\mathcal{H}$ is trivial. Then the map sending α to $[c_\alpha]$ gives rise to an isomorphism $\mathcal{H} \xrightarrow{\cong} \mathrm{H}^2(\Gamma_0(\pi); \mathcal{H})$. For $\gamma \in \Gamma_0$ and $\gamma_1, \gamma_2 \in \Gamma_0(\pi)$,

$$({}^\gamma c_\alpha)(\gamma_1, \gamma_2) = \pi(\gamma) \cdot c_\alpha(\gamma^{-1}\gamma_1\gamma, \gamma^{-1}\gamma_2\gamma) = \pi(\gamma) \cdot (({}^\gamma c)(\gamma_1, \gamma_2) \cdot \alpha) = ({}^\gamma c)(\gamma_1, \gamma_2) \cdot (\pi(\gamma) \cdot \alpha).$$

Moreover, by the claim above, there exists a cochain $b \in C^1(\Gamma_0(\pi))$ satisfying ${}^\gamma c = c + \delta b$. Hence,

$$({}^\gamma c_\alpha)(\gamma_1, \gamma_2) = ({}^\gamma c)(\gamma_1, \gamma_2) \cdot (\pi(\gamma) \cdot \alpha) = (c + \delta b)(\gamma_1, \gamma_2) \cdot (\pi(\gamma) \cdot \alpha) = (c + \delta b)_{\pi(\gamma) \cdot \alpha}(\gamma_1, \gamma_2).$$

Therefore the cohomology class ${}^\gamma[c_\alpha]$ corresponds to the element $\pi(\gamma) \cdot \alpha$ under the isomorphism, and this implies the claim. ◁

By the claims above and the assumption that π does not contain the trivial representation, we have $\mathrm{H}^2(\Gamma_0; \pi, \mathcal{H}) = 0$. This completes the proof of (2).

We can deduce (1) by the same arguments as above with Lemma 8.5 and Theorems 8.6 and 8.11. □

Proof of Theorem 8.9 Let n be an integer at least 6. Let G be a group of finite index in $\mathrm{Aut}(F_n)$. Set $N = G \cap \mathrm{IA}_n$ and $\Gamma = G/N$. Then we have an exact sequence

$$1 \to N \to G \to \Gamma \to 1$$

and Γ is a subgroup of finite index in $\mathrm{GL}(n, \mathbb{Z})$. Let $(\pi, \mathcal{H})$ be a finite-dimensional unitary Γ-representation. Set $\Gamma(\pi) = \mathrm{Ker}(\pi)$. By Theorem 8.11, $\Gamma(\pi)$ is a normal subgroup of finite index in Γ. By using Lemma 8.4, we can show that $\mathrm{H}_b^3(\Gamma(\pi); \mathcal{H}) = 0$. Here, we use [74, Corollary 8.2.10]; see also the proof of Theorem 8.17 in the next subsection. Together with Theorem 8.3, we obtain $\mathrm{H}_b^3(\Gamma; \pi, \mathcal{H}) = 0$. Hence, by Corollary 8.13(1), $\mathrm{H}^2_{/b}(\Gamma; \pi, \mathcal{H}) = 0$. Therefore, the quotient $\mathrm{H}^1_{/b}(N; \mathcal{H})/i^*\mathrm{H}^1_{/b}(G; \pi, \mathcal{H})$ is trivial by Theorem 1.5. Since $\mathrm{H}^1_{/b}(N; \mathcal{H})/i^*\mathrm{H}^1_{/b}(G; \pi, \mathcal{H})$ is isomorphic to $\widehat{\mathrm{Q}}(N; \mathcal{H})^{\mathrm{Q}G}/i^*\widehat{\mathrm{Q}}Z(G; \pi, \mathcal{H})$, this completes the proof. □

Proof of Theorem 8.10 Let l be an integer at least 3. Let G be a subgroup of finite index in $\mathrm{Mod}(\Sigma_l)$. Set $N = G \cap \mathcal{I}(\Sigma_l)$ and $\Gamma = G/N$. Let $(\pi, \mathcal{H})$ be a finite-dimensional unitary Γ-representation not containing the trivial representation. Then Theorem 8.12 and Corollary 8.13(2) imply that the second relative cohomology group $\mathrm{H}^2_{/b}(\Gamma; \pi, \mathcal{H})$ is trivial. Hence, by arguments similar to those in the proof of Theorem 8.9, we obtain the theorem. □

8.3 Quasicocycle analogues of Theorem 2.3: including infinite-dimensional unitary coefficients

After the submitted version of this paper was completed, work of Bader and Sauer [3] on vanishing of higher group cohomology with unitary coefficients for higher-rank Lie groups and their lattices has come out. This work enables us to improve Theorems 8.10 and 8.9 to the following Theorems 8.14 and 8.17, respectively. For the reader's convenience, we add this subsection to state and deduce these two theorems. We are grateful to one of the referees for informing us of [3]. We start with the following strengthening of Theorem 8.10:

Theorem 8.14 (stronger result for $\mathrm{Mod}(\Sigma_l)$) *Fix an integer l at least 3. Then, for every subgroup G of finite index in $\mathrm{Mod}(\Sigma_l)$ and every unitary representation $(\pi, \mathcal{H})$ of Γ such that $\pi \not\supset 1$,*

$$\widehat{\mathrm{Q}}(N; \mathcal{H})^{\mathrm{Q}G} = i^* \widehat{\mathrm{Q}}Z(G; \bar{\pi}, \mathcal{H}).$$

Here we set $N = G \cap \mathcal{I}(\Sigma_l)$ and $\Gamma = G/N$; the representation $(\bar{\pi}, \mathcal{H})$ of G is the pullback of the representation $(\pi, \mathcal{H})$ of Γ.

For the proof of Theorem 8.14, we employ the following definitions and theorem:

Definition 8.15 Let H be a locally compact second countable group. Let k be a positive integer.

(1) The group H is said to be *strongly k-Kazhdan* [30] or, alternatively, to have *property* $[\mathrm{T}_k]$ [2; 3] if, for every (strongly continuous) unitary H-representation $(\pi, \mathcal{H})$ and every positive integer i with $i \le k$, the continuous cohomology $\mathrm{H}_c^i(H; \pi, \mathcal{H})$ vanishes.

(2) [2; 3] The group H is said to have *property* (T_k) if, for every (strongly continuous) unitary H-representation $(\pi, \mathcal{H})$ *with* $\pi \not\supset 1$ and every positive integer i with $i \le k$, the continuous cohomology $\mathrm{H}_c^i(H; \pi, \mathcal{H})$ vanishes.

The celebrated Delorme–Guichardet theorem states that property $[\mathrm{T}_1]$ (the strong 1-Kazhdan property) and property (T_1) (for a locally compact second countable group) are both equivalent to Kazhdan's property (T); see [8] for details. By definition, for positive integers k_1 and k_2 with $k_1 \ge k_2$, property (T_{k_1}) implies property (T_{k_2}), and property $[\mathrm{T}_{k_1}]$ implies property $[\mathrm{T}_{k_2}]$ (we also remark that, if Λ is a discrete group, then, for a unitary Λ-representation $(\pi, \mathcal{H})$, continuous cohomology $\mathrm{H}_c^\bullet(\Lambda; \pi, \mathcal{H})$ coincides with $\mathrm{H}^\bullet(\Lambda; \pi, \mathcal{H})$). The main motivation in [30] comes from their theorem [30, Theorem 1.2], which states that a finitely presented group with property $[\mathrm{T}_2]$ is Frobenius stable. For applications of property (T_2) to the Frobenius stability, see [2].

Theorem 8.16 [3, Theorem B and Appendix A] *Let H be a connected semisimple Lie group with a finite center. Then H has property $(\mathrm{T}_{r_0(H)-1})$ (as a topological group), where $r_0(H)$ is the invariant given in* [3, Appendix A]. *Let Λ be an irreducible lattice in H. Then Λ has property (T_{r-1}), where $r = \min\{r_0(H), \mathrm{rank}_{\mathbb{R}}(H)\}$.*

Here, both for $H = \mathrm{SL}(n+1, \mathbb{R})$ with n at least 2 and for $H = \mathrm{Sp}(2n, \mathbb{R})$ with n at least 3, we have $r_0(H) = n = \mathrm{rank}_{\mathbb{R}}(H)$.

We note that $\mathrm{H}^2(\mathrm{Sp}(2n, \mathbb{Z})) = \mathbb{R}$ for every integer n at least 2; see [14]. In particular, $\mathrm{Sp}(2(n+1), \mathbb{Z})$ with $n \ge 2$ *fails* to have property $[\mathrm{T}_n]$, whereas Theorem 8.16 ensures property (T_n) for this group.

Proof of Theorem 8.14 using Theorem 8.16 First we prove the theorem under the additional assumption that $\mathcal{H}$ is separable. We appeal to Theorem 8.16 by setting $H = \mathrm{Sp}(2l, \mathbb{R})$ and $\Lambda = \Gamma$ (recall that l is assumed to be at least 3). Then Γ has property (T_2). Therefore, $\mathrm{H}^2(\Gamma; \pi, \mathcal{H}) = 0$. Since $\pi \not\supset 1$, Theorem 8.12 (Monod's theorem) shows that $\mathrm{H}_b^3(\Gamma; \pi, \mathcal{H}) = 0$. Now Theorem 1.5, together with the exact sequence (3-1) and Remark 1.7, ends our proof for the case where $\mathcal{H}$ is separable.

Now we treat the general case; here, we use the following trick to reduce to the separable case. Suppose that

$$\widehat{\mathrm{Q}}(N;\mathcal{H})^{\mathrm{Q}G} \setminus i^*\widehat{\mathrm{Q}}Z(G;\bar{\pi},\mathcal{H}) \neq \varnothing.$$

Take some f from the left-hand side. Let $\mathcal{K}$ be the closure of the linear span of $\{\pi(\gamma)f(x) \mid \gamma \in \Gamma,\, x \in N\}$ in $\mathcal{H}$. Then $\mathcal{K}$ is a closed $\pi(\Gamma)$-invariant subspace in $\mathcal{H}$ that is *separable* (note that Γ and N are both countable). Hence, we can view $\mathcal{K}$ as a unitary Γ-subrepresentation space of $\mathcal{H}$; we write $(\sigma,\mathcal{K})$ for this representation. We define a unitary G-representation $(\bar{\sigma},\mathcal{K})$ by taking the pullback of σ. Thus, we can view f as an element in $\widehat{\mathrm{Q}}(N;\mathcal{K})^{\mathrm{Q}G}$, where we view $(\bar{\sigma},\mathcal{K})$ as an $\mathbb{R}[G]$-module. Since $\mathcal{K}$ is separable and $\sigma \not\supset 1$, the first paragraph of this proof implies that

$$\widehat{\mathrm{Q}}(N;\mathcal{K})^{\mathrm{Q}G} = i^*\widehat{\mathrm{Q}}Z(G;\bar{\sigma},\mathcal{K}).$$

Hence, we can find an element $F \in \widehat{\mathrm{Q}}Z(G;\bar{\sigma},\mathcal{K})$ such that $f = i^*F$. However, since F may be seen as an element in $\widehat{\mathrm{Q}}Z(G;\bar{\pi},\mathcal{H})$, we have $f = i^*F \in i^*\widehat{\mathrm{Q}}Z(G;\bar{\pi},\mathcal{H})$, a contradiction. □

Next we state the following strengthening of Theorem 8.9:

Theorem 8.17 (stronger result for $\mathrm{Aut}(F_n)$) *Fix an integer n at least 4. Then, for every subgroup G of finite index in $\mathrm{Aut}(F_n)$ and every unitary representation $(\pi,\mathcal{H})$ of Γ such that $\mathcal{H}^\Gamma$ is finite-dimensional,*

$$\widehat{\mathrm{Q}}(N;\mathcal{H})^{\mathrm{Q}G} = i^*\widehat{\mathrm{Q}}Z(G;\bar{\pi},\mathcal{H}).$$

Here we set $N = G \cap \mathrm{IA}_n$ and $\Gamma = G/N$; the representation $(\bar{\pi},\mathcal{H})$ of G is the pullback of the representation $(\pi,\mathcal{H})$ of Γ.

In particular, the assumptions of $n \geq 6$ in Theorems 2.3(2) and 8.9 can both be weakened to $n \geq 4$.

Proof of Theorem 8.17 using Theorem 8.16 By employing the same trick as in the final part of the proof of Theorem 8.14, we may assume that $\mathcal{H}$ is separable throughout this proof. By Theorem 1.5 and the exact sequence (3-1), it suffices to prove that $\mathrm{H}^2(\Gamma;\pi,\mathcal{H}) = 0$ and $\mathrm{H}_b^3(\Gamma;\pi,\mathcal{H}) = 0$. Note that Γ is a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$. First we prove that $\mathrm{H}_b^3(\Gamma;\pi,\mathcal{H}) = 0$. Decompose the Γ-representation space $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}^\Gamma \oplus (\mathcal{H}^\Gamma)^\perp$, where $(\mathcal{H}^\Gamma)^\perp$ is the orthogonal complement of $\mathcal{H}^\Gamma$ in $\mathcal{H}$. Then the restriction of π on $\mathcal{H}^\Gamma$ is trivial, and it is finite-dimensional by assumption; the restriction π^{orth} of π on $(\mathcal{H}^\Gamma)^\perp$ does not admit a nonzero Γ-invariant vector. Now we claim that $\mathrm{H}_b^3(\Gamma;\pi^{\mathrm{inv}},\mathcal{H}^\Gamma) = 0$. Indeed, by Lemma 8.4, $\mathrm{H}_b^3(\Gamma) = 0$. Since we assume that $\mathcal{H}^\Gamma$ is finite-dimensional, we can decompose $\mathcal{H}^\Gamma$ as a finite direct sum of the one-dimensional trivial module; then [74, Corollary 8.2.10] ends the proof of the claim above. We also claim that $\mathrm{H}_b^3(\Gamma;\pi^{\mathrm{orth}},(\mathcal{H}^\Gamma)^\perp) = 0$. Indeed, this follows from another theorem of Monod [76, Theorem 2]; see [77, Corollary 1.6] for a more general statement and Theorem 8.3. Again by [74, Corollary 8.2.10],

$$\mathrm{H}_b^3(\Gamma;\pi,\mathcal{H}) \cong \mathrm{H}_b^3(\Gamma;\pi^{\mathrm{inv}},\mathcal{H}^\Gamma) \oplus \mathrm{H}_b^3(\Gamma;\pi^{\mathrm{orth}},(\mathcal{H}^\Gamma)^\perp);$$

hence $\mathrm{H}_b^3(\Gamma;\pi,\mathcal{H}) = 0$, as desired.

Next we will show that $\mathrm{H}^2(\Gamma;\pi,\mathcal{H})=0$ by appealing to Theorem 8.16; the proof of this part works without the assumption that $\mathcal{H}^\Gamma$ is finite-dimensional. Let $\Gamma_0=\Gamma\cap\mathrm{SL}(n,\mathbb{Z})$, which is a subgroup of finite index both of Γ and of $\mathrm{SL}(n,\mathbb{Z})$. First, apply Theorem 8.16 with $H=\mathrm{SL}(n,\mathbb{R})$ and $\Lambda=\mathrm{SL}(n,\mathbb{Z})$ (recall that n is assumed to be at least 4). Then $\mathrm{SL}(n,\mathbb{Z})$ has property (T_2). Together with Theorem 8.1 and the universal coefficient theorem, we conclude that $\mathrm{SL}(n,\mathbb{Z})$ has property $[\mathrm{T}_2]$. By [30, Proposition 4.4], property $[\mathrm{T}_2]$ passes to a subgroup of finite index. (Strictly speaking, it follows that property $[\mathrm{T}_2]$ passes to a normal subgroup of finite index; then we employ [19, Proposition III.10.4] to have the full heredity. Or, alternatively, we may apply the Shapiro lemma.) Hence, Γ_0 also has property $[\mathrm{T}_2]$. By Lemma 8.5, Γ has property $[\mathrm{T}_2]$ as well, and thus $\mathrm{H}^2(\Gamma;\pi,\mathcal{H})=0$. □

As we explained in the proof above, the main result of [3] in particular improves the Borel stable range for second ordinary cohomology with the trivial coefficients of SL_n from $n\geq 6$ to $n\geq 4$. We also note that the assumption of the finite-dimensionality of $\mathcal{H}^\Gamma$ in Theorem 8.17 is used to deduce $\mathrm{H}^3_b(\Gamma;\pi^{\mathrm{inv}},\mathcal{H}^\Gamma)=0$ from $\mathrm{H}^3_b(\Gamma)=0$. As we mentioned in the proof, for ordinary cohomology we do not need this finite-dimensionality assumption thanks to the universal coefficient theorem. For a Banach space V, the following question might be of interest: (for a fixed q, say 2 or 3) when does the vanishing of $\mathrm{H}^q_b(G)$ imply that of $\mathrm{H}^q_b(G;V)$ for a group G, where V is viewed as an $\mathbb{R}[G]$-module with the trivial action? For results in this direction, we refer the reader to [44; 82, Proposition 2.39].

The counterpart of Theorem 8.14 in the case of the trivial real coefficients is an open problem.

Problem 8.18 *Let G be a subgroup of finite index in $\mathrm{Mod}(\Sigma_l)$. Set $N=G\cap\mathcal{I}(\Sigma_l)$ and $\Gamma=G/N$. Then does $\mathrm{Q}(N)^G=i^*\mathrm{Q}(G)$ hold?*

Cochran, Harvey and Horn [29] constructed $\mathrm{Mod}(\Sigma)$-invariant quasimorphisms on $\mathcal{I}(\Sigma)$ for a surface Σ with at least one boundary component. The problem asking whether their quasimorphisms are extendable may be of special interest.

8.4 Extension theorem of quasicocycles

As an appendix to this section, we present the following extension theorem of quasicocycles. In this subsection we treat a general theory, and the topic has no specific relation to $\mathrm{Aut}(F_n)$ or $\mathrm{Mod}(\Sigma_l)$. Recall that every G-quasi-invariant quasimorphism on N is extendable to G if the projection $G\to G/N$ virtually splits (Proposition 3.4). This can be generalized as follows:

Theorem 8.19 *Let $1\to N\to G\xrightarrow{p}\Gamma\to 1$ be an exact sequence and V an $\mathbb{R}[\Gamma]$-module with a Γ-invariant norm $\|\cdot\|$. Assume that the exact sequence virtually splits. Then, for every V-valued G-quasiequivariant quasimorphism $f\in\hat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$, there exists a quasicocycle $F\in\hat{\mathrm{Q}}Z(G;V)$ such that $F|_N=f$ and $D(F)\leq D(f)+3D'(f)$.*

The proof is parallel to that of [60, Proposition 6.4] (Proposition 3.4 above). For the sake of completeness, we include the proof; see [loc. cit.] for more details.

Proof Let (s,Λ) be a virtual section of $p\colon G\to\Gamma$ (see Section 2). Let B be a finite subset of Γ such that the map $\Lambda\times B\to\Gamma$, $(\lambda,b)\mapsto\lambda b$, is bijective. Let $s'\colon B\to G$ be a map satisfying $p\circ s'(b)=b$ for every $b\in B$. Define a map $t\colon\Gamma\to G$ by setting $t(\lambda b)=s(\lambda)s'(b)$. Note that t is a (set-theoretic) section of p. Given $f\in\widehat{\mathrm{Q}}(N;V)^{\mathrm{Q}G}$, define a function $F\colon G\to V$ by

$$F(g)=\frac{1}{\#B}\sum_{b\in B} f\bigl(g\cdot t(b\cdot p(g))^{-1}\cdot t(b)\bigr).$$

Then $F|_N=f$. Moreover, for $g_1,g_2\in G$, by using that $f(h_1h_2)\approx_{D'(f)} p(h_1)\cdot f(h_2h_1)$ and $f(h_1h_2)\approx_{D'(f)} p(h_2)^{-1}\cdot f(h_2h_1)$ for every $h_1,h_2\in G$ with $h_1h_2\in N$, we have

$$\begin{aligned}
F(g_1g_2)&=\frac{1}{\#B}\sum_{b\in B} f(g_1g_2\cdot t(b\cdot p(g_1g_2))^{-1}t(b))\\
&\approx_{D'(f)}\frac{1}{\#B}\sum_{b\in B} b^{-1}\cdot f\bigl(t(b)\cdot g_1g_2\cdot t(b\cdot p(g_1g_2))^{-1}\bigr)\\
&=\frac{1}{\#B}\sum_{b\in B} b^{-1}\cdot f\bigl(t(b)\cdot g_1\cdot t(b\cdot p(g_1))^{-1}\cdot t(b\cdot p(g_1))\cdot g_2\cdot t(b\cdot p(g_1g_2))^{-1}\bigr)\\
&\approx_{D(f)}\frac{1}{\#B}\sum_{b\in B} b^{-1}\cdot\bigl(f\bigl(t(b)\cdot g_1\cdot t(b\cdot p(g_1))^{-1}\bigr)+f\bigl(t(b\cdot p(g_1))\cdot g_2\cdot t(b\cdot p(g_1g_2))^{-1}\bigr)\bigr)\\
&\approx_{2D'(f)}\frac{1}{\#B}\sum_{b\in B} f(g_1\cdot t(b\cdot p(g_1))^{-1}\cdot t(b))+\frac{1}{\#B}\sum_{b\in B} p(g_1)\cdot f\bigl(g_2\cdot t(b\cdot p(g_1g_2))^{-1}\cdot t(b\cdot p(g_1))\bigr)\\
&=F(g_1)+g_1\cdot\Bigl(\frac{1}{\#B}\sum_{b\in B} f\bigl(g_2\cdot t\bigl((b\cdot p(g_1))\cdot p(g_2)\bigr)^{-1}\cdot t(b\cdot p(g_1))\bigr)\Bigr).
\end{aligned}$$

By the arguments in the proof of [60, Proposition 6.4],

$$\frac{1}{\#B}\sum_{b\in B} f\bigl(g_2\cdot t\bigl((b\cdot p(g_1))\cdot p(g_2)\bigr)^{-1}\cdot t(b\cdot p(g_1))\bigr)=F(g_2).$$

Therefore, $F(g_1g_2)\approx_{D(f)+3D'(f)} F(g_1)+g_1\cdot F(g_2)$. □

9 Open problems

9.1 Mystery of the Py class

Let Σ_l be a closed connected orientable surface whose genus l is at least 2 and Ω a volume form on Σ_l. Recall that Py [89] constructed a Calabi quasimorphism f_P on $\mathrm{Ker}(\mathrm{Flux}_\Omega)$ which is $\mathrm{Diff}_0(\Sigma_l,\Omega)$-invariant, and the first and second authors showed that f_P is not extendable to $\mathrm{Diff}_0(\Sigma_l,\Omega)$ (recall Section 5 and Example 7.15). We define $\bar{c}_P\in\mathrm{H}^2(\mathrm{H}^1(\Sigma_l))$ and $c_P\in\mathrm{H}^2(\mathrm{Diff}_0(\Sigma_l,\Omega))$ by $\bar{c}_P=\xi_4^{-1}\circ\tau_{/b}(f_P)$ and $c_P=\mathrm{Flux}_\Omega^*(\bar{c}_P)$, respectively. We call c_P the *Py class*. Note that we essentially proved the nontriviality of the Py class in the proof of Theorem 2.6(1).

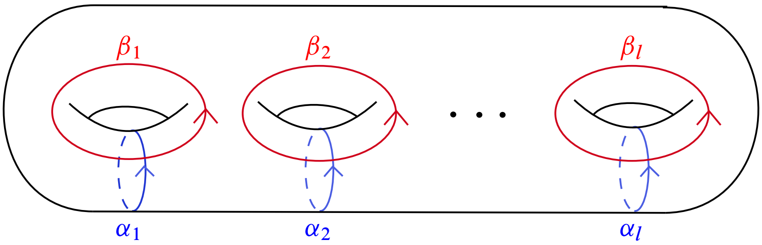

Figure 1: $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l : [0,1] \to \Sigma_l$.

When we constructed the class $\bar{c}_P \in \mathrm{H}^2(\mathrm{H}^1(\Sigma_l))$, we used the morphism $\xi_4 : \mathrm{H}^2(\mathrm{H}^1(\Sigma_l)) \to \mathrm{H}^2_{/b}(\mathrm{H}^1(\Sigma_l))$. Here we apply the exact sequence

$$1 \to \mathrm{Ker}(\mathrm{Flux}_\Omega) \to \mathrm{Diff}_0(\Sigma_l, \Omega) \xrightarrow{\mathrm{Flux}_\Omega} \mathrm{H}^1(\Sigma_l) \to 1$$

to diagram (1-3). Since the bounded cohomology groups of an amenable group are zero, the map ξ_4 is an isomorphism and we have the inverse $\xi_4^{-1} : \mathrm{H}^2_{/b}(\mathrm{H}^1(\Sigma_l)) \to \mathrm{H}^2(\mathrm{H}^1(\Sigma_l))$. Because the vanishing of the bounded cohomology of amenable groups is shown by a transcendental method, we do not have a precise description of the map ξ_4^{-1}.

Remark 9.1 If we fix a (right-)invariant mean m on the amenable group Γ, then we have the following description of the map ξ_4^{-1}. For a cocycle $[c] \in C^2_{/b}(\Gamma)$, a cocycle $f \in C^2(\Gamma)$ representing the class $\xi_4^{-1}([[c]])$ can be given by

$$f(\gamma_1, \gamma_2) = c(\gamma_1, \gamma_2) - m(\delta c(\cdot, \gamma_1, \gamma_2)).$$

However, we have the following observations on the Py class. Here we consider $\mathrm{H}^1(\Sigma_l)$ as a symplectic vector space by the intersection form.

Theorem 9.2 *Let Σ_l be a closed connected orientable surface whose genus l is at least 2 and Ω a volume form on Σ_l. For a subgroup Λ of $\mathrm{H}^1(\Sigma_l)$, let $\iota_\Lambda : \Lambda \to \mathrm{H}^1(\Sigma_l)$ be the inclusion map.*

(1) *Let v and w be elements in $\mathrm{H}^1(\Sigma_l)$ with $v \smile w \neq 0$. Then there exists a positive integer k_0 such that, for every integer k at least k_0, for the subgroup $\Lambda = \langle v, w/k \rangle$ of $\mathrm{H}^1(\Sigma_l)$, we have $\iota_\Lambda^* \bar{c}_P \neq 0$. Here $\smile$ denotes the cup product.*

(2) *If a subgroup Λ of $\mathrm{H}^1(\Sigma_l)$ is contained in a linear subspace $\langle [\alpha_1]^*, \dots, [\alpha_l]^* \rangle$ or $\langle [\beta_1]^*, \dots, [\beta_l]^* \rangle$, then $\iota_\Lambda^* \bar{c}_P = 0$, where $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l$ are the curves described in Figure 1.*

To prove Theorem 9.2, we use the following observation.

Let $1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1$ be an exact sequence of groups such that Γ is amenable. For a subgroup Γ^0 of Γ, $1 \to N \xrightarrow{i} p^{-1}(\Gamma^0) \xrightarrow{p} \Gamma^0 \to 1$ is also an exact sequence and Γ^0 is also amenable (Theorem 3.5(3)).

Then, by Theorem 1.5, we have the commuting diagrams

(9-1)
$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(\Gamma) \arrow[r, "p^*"] \arrow[d, "\xi_1"] & \mathrm{H}^1(G) \arrow[r, "i^*"] \arrow[d, "\xi_2"] & \mathrm{H}^1(N)^G \arrow[r, "\tau"] \arrow[d, "\xi_3"] & \mathrm{H}^2(\Gamma) \arrow[r, "p^*"] \arrow[d, "\xi_4"] & \mathrm{H}^2(G) \arrow[d, "\xi_5"] \\ 0 \arrow[r] & \mathrm{Q}(\Gamma) \arrow[r, "p^*"] & \mathrm{Q}(G) \arrow[r, "i^*"] & \mathrm{Q}(N)^G \arrow[r, "\tau_{/b}"] & \mathrm{H}^2_{/b}(\Gamma) \arrow[r, "p^*"] & \mathrm{H}^2_{/b}(G) \end{tikzcd}$$

(9-2)
$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(\Gamma^0) \arrow[r, "p^*"] \arrow[d, "\xi_1^0"] & \mathrm{H}^1(p^{-1}(\Gamma^0)) \arrow[r, "i^*"] \arrow[d, "\xi_2^0"] & \mathrm{H}^1(N)^{p^{-1}(\Gamma^0)} \arrow[r, "\tau^0"] \arrow[d, "\xi_3^0"] & \mathrm{H}^2(\Gamma^0) \arrow[r, "p^*"] \arrow[d, "\xi_4^0"] & \mathrm{H}^2(p^{-1}(\Gamma^0)) \arrow[d, "\xi_5^0"] \\ 0 \arrow[r] & \mathrm{Q}(\Gamma^0) \arrow[r, "p^*"] & \mathrm{Q}(p^{-1}(\Gamma^0)) \arrow[r, "i^*"] & \mathrm{Q}(N)^{p^{-1}(\Gamma^0)} \arrow[r, "\tau^0_{/b}"] & \mathrm{H}^2_{/b}(\Gamma^0) \arrow[r, "p^*"] & \mathrm{H}^2_{/b}(p^{-1}(\Gamma^0)) \end{tikzcd}$$

Since Γ and Γ^0 are boundedly 3-acyclic (Theorem 3.5(5)), $\xi_4\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2_{/b}(\Gamma)$ and $\xi_4^0\colon \mathrm{H}^2(\Gamma^0) \to \mathrm{H}^2_{/b}(\Gamma^0)$ are isomorphisms. The following lemma can be deduced from the definitions of $\tau_{/b}$ and $\tau^0_{/b}$:

Lemma 9.3 *We have*

$$(\xi_4^0)^{-1} \circ \tau^0_{/b} \circ I_1^* = I_2^* \circ (\xi_4)^{-1} \circ \tau_{/b},$$

where $I_1^*\colon \mathrm{Q}(N)^G \to \mathrm{Q}(N)^{p^{-1}(\Gamma^0)}$ and $I_2^*\colon \mathrm{H}^2(\Gamma) \to \mathrm{H}^2(\Gamma^0)$ are the homomorphisms induced from the inclusion $I\colon \Gamma^0 \to \Gamma$.

We employ the following theorem, which is related to Example 7.15, in order to prove Theorem 9.2:

Theorem 9.4 [61, Theorems 1.6 and 1.10] *Let Σ_l be a closed connected orientable surface whose genus l is at least 2 and Ω a volume form on Σ_l. Let Λ be a subgroup of $\mathrm{H}^1(\Sigma_l)$ and set $G = \mathrm{Flux}^{-1}(\Lambda)$ and $N = \mathrm{Ker}(\mathrm{Flux}_\Omega)$. Then:*

1. *Let v and w be elements in $\mathrm{H}^1(\Sigma_l)$ with $v \smile w \neq 0$. Then there exists a positive integer k_0 such that, for every integer k at least k_0, for $\Lambda = \langle v, w/k\rangle$, $[f_P]$ is a nontrivial element of $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$.*
2. *If Λ is contained in a linear subspace $\langle[\alpha_1]^*, \dots, [\alpha_l]^*\rangle$ or $\langle[\beta_1]^*, \dots, [\beta_l]^*\rangle$, then $[f_P]$ is the trivial element of $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$, where $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l$ are the curves described in Figure 1.*

On (2), see also [61, Remark 4.8].

Proof of Theorem 9.2 Set $\Gamma = \mathrm{H}^1(\Sigma_l)$, $\Gamma^0 = \Lambda$ and $G = \mathrm{Flux}_\Omega^{-1}(\Lambda)$. We use the notation in the diagrams (9-1) and (9-2).

First, to prove (1), suppose that the dimension of Λ is larger than l. Then, since $[f_P]$ is a nontrivial element of $\mathrm{Q}(N)^G/i^*\mathrm{Q}(G)$, by Theorem 1.10, $(\xi_4^0)^{-1} \circ \tau^0_{/b} \circ I_1^*(f_P)$ is also a nontrivial element of $\mathrm{H}^2(\Gamma^0) = \mathrm{H}^2(\Lambda)$. Hence, by Lemma 9.3, $\iota_\Lambda^* \bar{c}_P = I_2^* \circ (\xi_4)^{-1} \circ \tau_{/b}(f_P) = (\xi_4^0)^{-1} \circ \tau^0_{/b} \circ I_1^*(f_P)$ is also a nontrivial element of $\mathrm{H}^2(\Gamma^0) = \mathrm{H}^2(\Lambda)$.

Next, to prove (2), suppose that Λ is contained in linear subspaces $\langle [\alpha_1]^*, \dots, [\alpha_l]^* \rangle$ or $\langle [\beta_1]^*, \dots, [\beta_l]^* \rangle$. Then, since $[f_P]$ is the trivial element of $\mathrm{Q}(N)^G / i^*\mathrm{Q}(G)$, by Theorem 1.10 and Proposition 2.5, $(\xi_4^0)^{-1} \circ \tau^0_{/b} \circ I_1^*(f_P)$ is also the trivial element of $\mathrm{H}^2(\Gamma^0) = \mathrm{H}^2(\Lambda)$. Hence, by Lemma 9.3, $\iota_\Lambda^* \bar{c}_P = I_2^* \circ (\xi_4)^{-1} \circ \tau_{/b}(f_P) = (\xi_4^0)^{-1} \circ \tau^0_{/b} \circ I_1^*(f_P)$ is also the trivial element of $\mathrm{H}^2(\Gamma^0) = \mathrm{H}^2(\Lambda)$. □

Finally, we pose the following problems on the Py class:

Problem 9.5 *Give a precise description of a cochain representing* $\bar{c}_P \in \mathrm{H}^2(\mathrm{H}^1(\Sigma_l))$ *and a bounded cochain representing* $c_P \in \mathrm{H}^2(\mathrm{Diff}_0(M, \Omega))$.

Problem 9.6 *Let* Σ_l *be a closed connected orientable surface whose genus* l *is at least* 2 *and* Ω *a volume form on* Σ_l. *Is the vector space* $\mathrm{Im}(\mathrm{Flux}_\Omega^*) \cap \mathrm{Im}(c_{\mathrm{Diff}_0(\Sigma_l,\Omega)})$ *spanned by* c_P?

By Theorem 1.10, Problem 9.6 is rephrased as follows.

Problem 9.7 *Let* Σ_l *be a closed connected orientable surface whose genus* l *is at least* 2 *and* Ω *a volume form on* Σ_l. *Is the vector space* $\mathrm{Q}(\mathrm{Ker}(\mathrm{Flux}_\Omega))^{\mathrm{Diff}_0(\Sigma_l,\Omega)} / i^*\mathrm{Q}(\mathrm{Diff}_0(\Sigma_l, \Omega))$ *spanned by* $[f_P]$?

9.2 Problems on equivalences and coincidences of scl_G and $\mathrm{scl}_{G,N}$

By Theorem 2.1, $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ implies that scl_G and $\mathrm{scl}_{G,N}$ are equivalent on $[G, N]$. Moreover, if N is the commutator subgroup of G and $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$, then scl_G and $\mathrm{scl}_{G,N}$ coincide on $[G, N]$. Since $\mathrm{H}^2(G) = 0$ implies $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ (Theorem 1.10), there are several examples of pairs (G, N) such that $\mathrm{scl}_{G,N}$ and scl_G are equivalent (see Section 2.1). In Section 3, we provided several examples of groups G with $\mathrm{Q}(N)^G \neq \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ (see Theorems 1.1, 1.2 and 4.18), but we were unable to determine whether scl_G and $\mathrm{scl}_{G,N}$ are equivalent on $[G, N]$ in these examples. Hence, the example of $G = \mathrm{Diff}(\Sigma_l, \omega)$ with $l \geq 2$ and $N = [G, G]$ raised by [58] (see also [61]) has remained essentially the only known example where scl_G and $\mathrm{scl}_{G,N}$ are not equivalent on $[G, N]$. In fact, this is the only example where scl_G and $\mathrm{scl}_{G,N}$ do not coincide on $[G, N]$. Here we provide several problems on equivalences and coincidences of scl_G and $\mathrm{scl}_{G,N}$.

Problem 9.8 *Is it true that* $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ *implies that* $\mathrm{scl}_G = \mathrm{scl}_{G,N}$ *on* $[G, N]$?

Problem 9.9 *Find a pair* (G, N) *such that* G *is finitely generated and* scl_G *and* $\mathrm{scl}_{G,N}$ *are not equivalent. In particular, are* $\mathrm{scl}_{\pi_1(\Sigma_l)}$ *and* $\mathrm{scl}_{\pi_1(\Sigma_l),[\pi_1(\Sigma_l),\pi_1(\Sigma_l)]}$ *equivalent on* $[\pi_1(\Sigma_l), [\pi_1(\Sigma_l), \pi_1(\Sigma_l)]]$ *for* $l \geq 2$?

After the current work, Problem 9.9 was solved by some of the authors [69]: for $l \geq 2$, $\mathrm{scl}_{\pi_1(\Sigma_l)}$ and $\mathrm{scl}_{\pi_1(\Sigma_l),[\pi_1(\Sigma_l),\pi_1(\Sigma_l)]}$ are not equivalent. Moreover, the authors proved in [59] that scl_G and $\mathrm{scl}_{G,[G,G]}$ are not equivalent if $\mathrm{Q}([G,G])^G \neq \mathrm{H}^1([G,G])^G + i^*\mathrm{Q}(G)$.

We also pose the following problem. Let B_n be the n^{th} braid group and P_n the n^{th} pure braid group.

Problem 9.10 *For* $n \geq 3$, *does* $\mathrm{scl}_{B_n} = \mathrm{scl}_{B_n,[P_n,P_n]}$ *hold on* $[B_n, [P_n, P_n]]$?

In light of the following proposition, we can regard Problem 9.10 as a special case of Problem 9.8:

Proposition 9.11 *For $n \geq 2$, let $G = B_n$ and $N = [P_n, P_n]$. Then $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. In particular, $\mathrm{scl}_G(x) \leq \mathrm{scl}_{G,N}(x) \leq 2 \cdot \mathrm{scl}_G(x)$ for all $x \in [G, N]$.*

Proof Consider the exact sequence

$$1 \to P_n/[P_n, P_n] \to B_n/[P_n, P_n] \to \mathfrak{S}_n \to 1,$$

where $\mathfrak{S}_n$ is the symmetric group. By Theorem 3.5(1)–(2), $\mathfrak{S}_n$ and $P_n/[P_n, P_n]$ are amenable. Hence, Theorem 3.5(4) implies that $B_n/[P_n, P_n]$ is also amenable. As pointed out in Section 2.1, the second cohomology of the braid group B_n vanishes. Hence, Theorem 1.10 implies that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$. The equivalence between scl_{B_n} and $\mathrm{scl}_{B_n,[P_n,P_n]}$ follows from Theorem 2.1(2). □

As another special case of Problem 9.8, we provide the following problem:

Problem 9.12 *For $n \geq 2$, does $\mathrm{scl}_{\mathrm{Aut}(F_n)} = \mathrm{scl}_{\mathrm{Aut}(F_n),\mathrm{IA}_n}$ hold on $[\mathrm{Aut}(F_n), \mathrm{IA}_n]$?*

Even the following weaker variant of Problem 9.12 seems open. We note that Theorem 2.1(2) does *not* apply to the setting of Theorem 2.3.

Problem 9.13 *Let $n \geq 3$. Find an explicit real constant $C \geq 1$ such that $\mathrm{scl}_{\mathrm{Aut}(F_n),\mathrm{IA}_n} \leq C \cdot \mathrm{scl}_{\mathrm{Aut}(F_n)}$ on $[\mathrm{Aut}(F_n), \mathrm{IA}_n]$.*

In [60], the first, second, fourth and fifth authors considered the equivalence problem between cl_G and $\mathrm{cl}_{G,N}$. We provide the following problem:

Problem 9.14 *Is it true that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ implies the bi-Lipschitz equivalence of cl_G and $\mathrm{cl}_{G,N}$ on $[G, N]$?*

We note that Theorem 2.1(1) states that $\mathrm{Q}(N)^G = \mathrm{H}^1(N)^G + i^*\mathrm{Q}(G)$ implies the bi-Lipschitz equivalence of scl_G and $\mathrm{scl}_{G,N}$. To the best knowledge of the authors, Problem 9.14, even for the case where $1 \to N \to G \to \Gamma \to 1$ virtually splits, might be open in general.

In light of Proposition 9.11 and Theorem 2.3, we can regard the following problem as special cases of Problem 9.14:

Problem 9.15 *For (G, N) either $(B_n, [P_n, P_n])$ with $n \geq 3$ or $(\mathrm{Aut}(F_n), \mathrm{IA}_n)$ with $n \geq 2$, are cl_G and $\mathrm{cl}_{G,N}$ equivalent on $[G, N]$?*

Note that cl_G and $\mathrm{cl}_{G,N}$ are bi-Lipschitzly equivalent when $(G, N) = (B_n, P_n \cap [B_n, B_n] = [P_n, B_n])$ [60].

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References

[1] **T Akita**, **Y Liu**, *Second mod 2 homology of Artin groups*, Algebr. Geom. Topol. 18 (2018) 547–568 MR Zbl

[2] **U Bader**, **A Lubotzky**, **R Sauer**, **S Weinberger**, *Stability and instability of lattices in semisimple groups*, J. Anal. Math. 151 (2023) 1–23 MR Zbl

[3] **U Bader**, **R Sauer**, *Higher Kazhdan property and unitary cohomology of arithmetic groups*, preprint (2023) arXiv 2308.06517

[4] **A Banyaga**, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. 53 (1978) 174–227 MR Zbl

[5] **A Banyaga**, *The structure of classical diffeomorphism groups*, Math. Appl. 400, Kluwer, Dordrecht (1997) MR Zbl

[6] **C Bavard**, *Longueur stable des commutateurs*, Enseign. Math. 37 (1991) 109–150 MR Zbl

[7] **B Bekka**, **P de la Harpe**, *Unitary representations of groups, duals, and characters*, Math. Surv. Monogr. 250, Amer. Math. Soc., Providence, RI (2020) MR Zbl

[8] **B Bekka**, **P de la Harpe**, **A Valette**, *Kazhdan's property (T)*, New Math. Monogr. 11, Cambridge Univ. Press (2008) MR Zbl

[9] **M Bestvina**, **K Bromberg**, **K Fujiwara**, *Bounded cohomology with coefficients in uniformly convex Banach spaces*, Comment. Math. Helv. 91 (2016) 203–218 MR Zbl

[10] **M Bestvina**, **K Bromberg**, **K Fujiwara**, *Stable commutator length on mapping class groups*, Ann. Inst. Fourier (Grenoble) 66 (2016) 871–898 MR Zbl

[11] **M Bestvina**, **M Feighn**, *A combination theorem for negatively curved groups*, J. Differential Geom. 35 (1992) 85–101 MR Zbl

[12] **M Bestvina**, **K Fujiwara**, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. 6 (2002) 69–89 MR Zbl

[13] **R Bieri**, *Mayer–Vietoris sequences for HNN-groups and homological duality*, Math. Z. 143 (1975) 123–130 MR Zbl

[14] **A Borel**, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. 7 (1974) 235–272 MR Zbl

[15] **A Borel**, *Stable and L^2-cohomology of arithmetic groups*, Bull. Amer. Math. Soc. 3 (1980) 1025–1027 MR Zbl

[16] **A Borel**, *Stable real cohomology of arithmetic groups, II*, from "Manifolds and Lie groups", Progr. Math. 14, Birkhäuser, Boston, MA (1981) 21–55 MR Zbl

[17] **J Bowden**, **S W Hensel**, **R Webb**, *Quasi-morphisms on surface diffeomorphism groups*, J. Amer. Math. Soc. 35 (2022) 211–231 MR Zbl

[18] **M Brandenbursky**, **J Kędra**, *Fragmentation norm and relative quasimorphisms*, Proc. Amer. Math. Soc. 150 (2022) 4519–4531 MR Zbl

[19] **K S Brown**, *Cohomology of groups*, Graduate Texts in Math. 87, Springer (1982) MR Zbl

[20] **M Bucher**, **N Monod**, *The bounded cohomology of* SL_2 *over local fields and* S*-integers*, Int. Math. Res. Not. 2019 (2019) 1601–1611 MR Zbl

[21] **D Burago**, **S Ivanov**, **L Polterovich**, *Conjugation-invariant norms on groups of geometric origin*, from "Groups of diffeomorphisms", Adv. Stud. Pure Math. 52, Math. Soc. Japan, Tokyo (2008) 221–250 MR Zbl

[22] **M Burger**, **N Monod**, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. 1 (1999) 199–235 MR Zbl

[23] **M Burger**, **N Monod**, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. 12 (2002) 219–280 MR Zbl

[24] **M Burger**, **S Mozes**, *Finitely presented simple groups and products of trees*, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997) 747–752 MR Zbl

[25] **D Calegari**, *Circular groups, planar groups, and the Euler class*, from "Proceedings of the Casson Fest", Geom. Topol. Monogr. 7, Geom. Topol. Publ., Coventry (2004) 431–491 MR Zbl

[26] **D Calegari**, *scl*, MSJ Memoirs 20, Math. Soc. Japan, Tokyo (2009) MR

[27] **D Calegari**, **N Monden**, **M Sato**, *On stable commutator length in hyperelliptic mapping class groups*, Pacific J. Math. 272 (2014) 323–351 MR Zbl

[28] **P Capovilla**, **C Löh**, **M Moraschini**, *Amenable category and complexity*, Algebr. Geom. Topol. 22 (2022) 1417–1459 MR Zbl

[29] **T D Cochran**, **S Harvey**, **P D Horn**, *Higher-order signature cocycles for subgroups of mapping class groups and homology cylinders*, Int. Math. Res. Not. 2012 (2012) 3311–3373 MR Zbl

[30] **M De Chiffre**, **L Glebsky**, **A Lubotzky**, **A Thom**, *Stability, cohomology vanishing, and nonapproximable groups*, Forum Math. Sigma 8 (2020) art. id. e18 MR Zbl

[31] **K Dekimpe**, **M Hartl**, **S Wauters**, *A seven-term exact sequence for the cohomology of a group extension*, J. Algebra 369 (2012) 70–95 MR Zbl

[32] **H Endo**, **D Kotschick**, *Bounded cohomology and non-uniform perfection of mapping class groups*, Invent. Math. 144 (2001) 169–175 MR Zbl

[33] **M Entov**, **L Polterovich**, *Quasi-states and symplectic intersections*, Comment. Math. Helv. 81 (2006) 75–99 MR Zbl

[34] **D B A Epstein**, **K Fujiwara**, *The second bounded cohomology of word-hyperbolic groups*, Topology 36 (1997) 1275–1289 MR Zbl

[35] **A Fathi**, *Structure of the group of homeomorphisms preserving a good measure on a compact manifold*, Ann. Sci. École Norm. Sup. 13 (1980) 45–93 MR Zbl

[36] **F Fournier-Facio**, **Y Lodha**, *Second bounded cohomology of groups acting on* 1*-manifolds and applications to spectrum problems*, Adv. Math. 428 (2023) art. id. 109162 MR Zbl

[37] **F Fournier-Facio**, **C Löh**, **M Moraschini**, *Bounded cohomology and binate groups*, J. Aust. Math. Soc. 115 (2023) 204–239 MR Zbl

[38] **F Fournier-Facio**, **C Löh**, **M Moraschini**, *Bounded cohomology of finitely presented groups: vanishing, non-vanishing, and computability*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 25 (2024) 1169–1202 MR Zbl

[39] **F Fournier-Facio**, **R D Wade**, *Aut-invariant quasimorphisms on groups*, Trans. Amer. Math. Soc. 376 (2023) 7307–7327 MR Zbl

[40] **R Frigerio**, *Bounded cohomology of discrete groups*, Math. Surv. Monogr. 227, Amer. Math. Soc., Providence, RI (2017) MR Zbl

[41] **R Frigerio**, **M B Pozzetti**, **A Sisto**, *Extending higher-dimensional quasi-cocycles*, J. Topol. 8 (2015) 1123–1155 MR Zbl

[42] **K Fujiwara**, **T Soma**, *Bounded classes in the cohomology of manifolds*, Geom. Dedicata 92 (2002) 73–85 MR Zbl

[43] **S M Gersten**, *A presentation for the special automorphism group of a free group*, J. Pure Appl. Algebra 33 (1984) 269–279 MR Zbl

[44] **L Glebsky**, **A Lubotzky**, **N Monod**, **B Rangarajan**, *Asymptotic cohomology and uniform stability for lattices in semisimple groups*, preprint (2023) arXiv 2301.00476

[45] **M Gromov**, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. 56 (1982) 5–99 MR Zbl

[46] **M Gromov**, *Hyperbolic groups*, from "Essays in group theory", Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR Zbl

[47] **R Hain**, *Infinitesimal presentations of the Torelli groups*, J. Amer. Math. Soc. 10 (1997) 597–651 MR Zbl

[48] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[49] **M Hull**, **D Osin**, *Induced quasicocycles on groups with hyperbolically embedded subgroups*, Algebr. Geom. Topol. 13 (2013) 2635–2665 MR Zbl

[50] **T Ishida**, *Quasi-morphisms on the group of area-preserving diffeomorphisms of the* 2*-disk*, PhD thesis, University of Tokyo (2013) Available at `http://hdl.handle.net/2261/56009`

[51] **T Ishida**, *Quasi-morphisms on the group of area-preserving diffeomorphisms of the* 2*-disk via braid groups*, Proc. Amer. Math. Soc. Ser. B 1 (2014) 43–51 MR Zbl

[52] **T Ishida**, *Homomorphisms on groups of volume-preserving diffeomorphisms via fundamental groups*, from "Geometry, dynamics, and foliations", Adv. Stud. Pure Math. 72, Math. Soc. Japan, Tokyo (2017) 387–393 MR Zbl

[53] **N V Ivanov**, *Leray theorems in bounded cohomology theory*, preprint (2020) arXiv 2012.08038

[54] **M Kaluba**, **D Kielak**, **P W Nowak**, *On property (T) for* $\mathrm{Aut}(F_n)$ *and* $\mathrm{SL}_n(\mathbb{Z})$, Ann. of Math. 193 (2021) 539–562 MR Zbl

[55] **M Kaluba**, **P W Nowak**, **N Ozawa**, $\mathrm{Aut}(\mathbb{F}_5)$ *has property* (T), Math. Ann. 375 (2019) 1169–1191 MR

[56] **M Kawasaki**, *Relative quasimorphisms and stably unbounded norms on the group of symplectomorphisms of the Euclidean spaces*, J. Symplectic Geom. 14 (2016) 297–304 MR Zbl

[57] **M Kawasaki**, *Bavard's duality theorem on conjugation-invariant norms*, Pacific J. Math. 288 (2017) 157–170 MR Zbl

[58] **M Kawasaki**, **M Kimura**, $\hat{G}$*-invariant quasimorphisms and symplectic geometry of surfaces*, Israel J. Math. 247 (2022) 845–871 MR Zbl

[59] **M Kawasaki**, **M Kimura**, **S Maruyama**, **T Matsushita**, **M Mimura**, *Coarse group theoretic study on stable mixed commutator length*, preprint (2023) arXiv 2306.08618

[60] **M Kawasaki**, **M Kimura**, **T Matsushita**, **M Mimura**, *Bavard's duality theorem for mixed commutator length*, Enseign. Math. 68 (2022) 441–481 MR Zbl

[61] **M Kawasaki**, **M Kimura**, **T Matsushita**, **M Mimura**, *Commuting symplectomorphisms on a surface and the flux homomorphism*, Geom. Funct. Anal. 33 (2023) 1322–1353 MR Zbl

[62] **M Kawasaki**, **S Maruyama**, *On boundedness of characteristic class via quasi-morphism*, J. Topol. Anal. (online publication July 2024)

[63] **J Kędra**, **D Kotschick**, **S Morita**, *Crossed flux homomorphisms and vanishing theorems for flux groups*, Geom. Funct. Anal. 16 (2006) 1246–1273 MR Zbl

[64] **M Kimura**, *Conjugation-invariant norms on the commutator subgroup of the infinite braid group*, J. Topol. Anal. 10 (2018) 471–476 MR Zbl

[65] **D Kotschick**, **S Morita**, *Characteristic classes of foliated surface bundles with area-preserving holonomy*, J. Differential Geom. 75 (2007) 273–302 MR Zbl

[66] **K Li**, *Bounded cohomology of classifying spaces for families of subgroups*, Algebr. Geom. Topol. 23 (2023) 933–962 MR Zbl

[67] **A V Malyutin**, *Operators in the spaces of pseudocharacters of braid groups*, Algebra i Analiz 21 (2009) 136–165 MR Zbl In Russian; translated in St. Petersburg Math. J. 21 (2010) 261–280

[68] **K Mann**, *Unboundedness of some higher Euler classes*, Algebr. Geom. Topol. 20 (2020) 1221–1234 MR Zbl

[69] **S Maruyama**, **T Matsushita**, **M Mimura**, *Invariant quasimorphisms for groups acting on the circle and non-equivalence of SCL* (2022) arXiv 2203.09221 To appear in Israel J. Math.

[70] **S Matsumoto**, **S Morita**, *Bounded cohomology of certain groups of homeomorphisms*, Proc. Amer. Math. Soc. 94 (1985) 539–544 MR Zbl

[71] **J McCool**, *A faithful polynomial representation of* $\mathrm{Out}\,F_3$, Math. Proc. Cambridge Philos. Soc. 106 (1989) 207–213 MR Zbl

[72] **J Milnor**, *Introduction to algebraic K-theory*, Ann. of Math. Stud. 72, Princeton Univ. Press (1971) MR

[73] **I Mineyev**, *Straightening and bounded cohomology of hyperbolic groups*, Geom. Funct. Anal. 11 (2001) 807–839 MR Zbl

[74] **N Monod**, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Math. 1758, Springer (2001) MR Zbl

[75] **N Monod**, *Stabilization for* SL_n *in bounded cohomology*, from "Discrete geometric analysis", Contemp. Math. 347, Amer. Math. Soc., Providence, RI (2004) 191–202 MR

[76] **N Monod**, *Vanishing up to the rank in bounded cohomology*, Math. Res. Lett. 14 (2007) 681–687 MR Zbl

[77] **N Monod**, *On the bounded cohomology of semi-simple groups, S-arithmetic groups and products*, J. Reine Angew. Math. 640 (2010) 167–202 MR Zbl

[78] **N Monod**, *Lamplighters and the bounded cohomology of Thompson's group*, Geom. Funct. Anal. 32 (2022) 662–675 MR Zbl

[79] **N Monod**, **S Nariman**, *Bounded and unbounded cohomology of homeomorphism and diffeomorphism groups*, Invent. Math. 232 (2023) 1439–1475 MR Zbl

[80] **N Monod**, **Y Shalom**, *Cocycle superrigidity and bounded cohomology for negatively curved spaces*, J. Differential Geom. 67 (2004) 395–455 MR Zbl

[81] **A Monzner**, **N Vichery**, **F Zapolsky**, *Partial quasimorphisms and quasistates on cotangent bundles, and symplectic homogenization*, J. Mod. Dyn. 6 (2012) 205–249 MR Zbl

[82] **M Moraschini**, **G Raptis**, *Amenability and acyclicity in bounded cohomology*, Rev. Mat. Iberoam. 39 (2023) 2371–2404 MR Zbl

[83] **J Neukirch**, **A Schmidt**, **K Wingberg**, *Cohomology of number fields*, 2nd edition, Grundl. Math. Wissen. 323, Springer (2008) MR Zbl

[84] **W D Neumann**, **L Reeves**, *Central extensions of word hyperbolic groups*, Ann. of Math. 145 (1997) 183–192 MR Zbl

[85] **B B Newman**, *Some results on one-relator groups*, Bull. Amer. Math. Soc. 74 (1968) 568–571 MR Zbl

[86] **M Nitsche**, *Computer proofs for property* (T)*, and SDP duality*, preprint (2020) arXiv 2009.05134

[87] **L Polterovich**, *The geometry of the group of symplectic diffeomorphisms*, Birkhäuser, Basel (2001) MR Zbl

[88] **L Polterovich**, **D Rosen**, *Function theory on symplectic manifolds*, CRM Monogr. Ser. 34, Amer. Math. Soc., Providence, RI (2014) MR Zbl

[89] **P Py**, *Quasi-morphismes et invariant de Calabi*, Ann. Sci. École Norm. Sup. 39 (2006) 177–195 MR Zbl

[90] **A I Shtern**, *Extension of pseudocharacters from normal subgroups, III*, Proc. Jangjeon Math. Soc. 19 (2016) 609–614 MR Zbl

[91] **W Thurston**, *Noncobordant foliations of* S^3, Bull. Amer. Math. Soc. 78 (1972) 511–514 MR Zbl

[92] **W P Thurston**, *On the structure of the group of volume preserving diffeomorphisms*, preprint (1973)

[93] **W P Thurston**, *Hyperbolic structures on* 3*-manifolds, II: Surface groups and* 3*-manifolds which fiber over the circle*, preprint (1986) arXiv math/9801045 Reprinted as pages 79–110 in his *Collected works with commentary, II*, Amer. Math. Soc., Providence, RI (2022)

[94] **B Tshishiku**, *Borel's stable range for the cohomology of arithmetic groups*, J. Lie Theory 29 (2019) 1093–1102 MR Zbl

[95] **T Tsuboi**, *On the uniform perfectness of diffeomorphism groups*, from "Groups of diffeomorphisms", Adv. Stud. Pure Math. 52, Math. Soc. Japan, Tokyo (2008) 505–524 MR Zbl

[96] **T Tsuboi**, *On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds*, Comment. Math. Helv. 87 (2012) 141–185 MR Zbl

[97] **T Tsuboi**, *Several problems on groups of diffeomorphisms*, from "Geometry, dynamics, and foliations", Adv. Stud. Pure Math. 72, Math. Soc. Japan, Tokyo (2017) 239–248 MR Zbl

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Synthetic approach to the Quillen model structure on topological spaces

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We provide an axiomatic treatment of Quillen's construction of the model structure on topological spaces to make it applicable to a wider range of settings, including Δ-generated spaces and pseudotopological spaces. We use this axiomatization to construct a model structure on the category of locales.

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Introduction

The construction of Quillen's model structure on the category Top of topological spaces is a cornerstone result of modern homotopy theory [20, Section II.3]. The importance of many other model structures, like the Kan–Quillen model structure on simplicial sets or the Thomason model structure on the category of small categories [25], is justified by their equivalence with Quillen's structure on topological spaces. Numerous accounts of it have been given since, including by Hovey [11], Hirschhorn [9; 10] and May and Ponto [16, Section 17.2], further underscoring its significance.

Although the category of (all) topological spaces is perhaps most commonly studied, quite often one works with either subcategories of topological spaces (eg compactly generated, weakly Hausdorff spaces or Δ-generated spaces) or larger categories containing Top as a subcategory (eg pseudotopological spaces). The former are natural choices for a convenient category of spaces, ie a subcategory of Top with nice categorical properties, including cartesian closure. The latter, being a common generalization of both graphs and spaces, finds applications in topological data analysis when quantifying to what extent a space can be recovered by sampling only finitely many of its points; see Rieser [22; 21].

The purpose of this paper is to axiomatize the requirements on various categories of interest to make a more modern version of Quillen's argument work. Such a modern treatment differs from Quillen's argument in a variety of ways, including: isolating the pushout–product construction, utilizing characterization of the class of weak equivalences in terms of an "up to homotopy" lifting property, and identifying the key role of the subcategory of topological cubes. Our axiomatization first introduces the notion of a *category with intervals* (Definition 1.1), a framework in which one can speak of homotopies, defined using the topological interval, and their basic properties. We then introduce the framework of a *Q-structure*

(Definition 2.2), which is the structure on a category that permits Quillen's construction (Theorem 2.6). Moreover, in a *good* Q-structure, weak equivalences can be defined in terms of isomorphisms on the sets of connected components and all higher homotopy groups (Theorem 3.12).

Several examples of good Q-structures are identified and give rise to model structures on topological spaces (recovering the model structure Quillen originally constructed [20]), compactly generated, weakly Hausdorff spaces (recovering the model structure of Hovey [11]), Δ-generated spaces (recovering the model structure of Haraguchi [6]), sober spaces (Section 4), pseudotopological spaces (Section 5, recovering the model structure of Rieser [22]), and locales (Section 6). This in particular settles an open question of constructing a model structure on the category of locales.

We begin in Section 1 by introducing the notion of a category with intervals (Definition 1.1) and developing basic homotopy theory therein. We then proceed to introduce (good) Q-structures in Section 2 and define cofibrations and fibrations. We conclude our proof of the existence of a model structure in Section 3 before turning our attention to examples of Q-structures in Sections 4, 5 and 6.

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1 Basic notions in a nice category of spaces

In this section, we introduce the notion of a category with intervals (Definition 1.1) and show that it is sufficient to develop basics of the theory of homotopies.

Throughout the paper, we write Top for the category of (all) topological spaces and continuous maps. For notational convenience, the unit interval $[0,1]$ will be denoted by I.

Let $\mathcal{C}$ denote a bicomplete category, to be thought of as a "nice" category of spaces. As such, we will refer to objects in $\mathcal{C}$ as spaces, and morphisms in $\mathcal{C}$ as (continuous) maps. Let $\varnothing$ and $*$ denote the initial and terminal object of $\mathcal{C}$, respectively. Let $\square$ be the full subcategory of Top consisting of all n-cubes and their boundaries for $n \geq 0$. Let $\square_{\leq 2}$ denote the full subcategory of $\square$ whose objects are $*$, I, I^2, $\varnothing$, ∂I and ∂I^2.

Let $e_k : * \to I$ be the appropriate endpoint inclusion for $k \in \{0,1\}$. Denote the left, top, right and bottom edge inclusions $I \hookrightarrow I^2$ by $\partial_L = e_0 \times \mathrm{id}_I$, $\partial_T = \mathrm{id}_I \times e_1$, $\partial_R = e_1 \times \mathrm{id}_I$ and $\partial_B = \mathrm{id}_I \times e_0$, respectively. By embedding, we shall mean a faithful functor throughout this paper.

Definition 1.1 A *category with intervals*[1] is a bicomplete category $\mathcal{C}$ along with an embedding $\iota\colon \square_{\le 2} \hookrightarrow \mathcal{C}$ such that:

(S1) $\iota\varnothing$ is initial and $\iota *$ is terminal.

(S2) $\iota I^2 \cong \iota I \times \iota I$, where the product is taken in $\mathcal{C}$.

(S3) ι preserves a pushout of the form

$$\begin{tikzcd} * \arrow[r, "e_0"] \arrow[d, "e_1"'] & I \arrow[d, "b"] \\ I \arrow[r, "a"'] & I \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

such that $a(0) = 0$ and $b(1) = 1$, and $X \times -$ preserves this pushout for all $X \in \mathcal{C}$.

(S4) $\varnothing \times X \cong \varnothing$ for all $X \in \mathcal{C}$.

The axioms given above are fairly rigid in structure; as such, it is difficult to give a "noncanonical" example. We give several examples in Sections 4, 5 and 6, which will moreover be of good Q-structure (see Definition 2.2). All of these examples are given by restricting a functor $\mathrm{Top} \to \mathcal{C}$ to $\square$. We are unaware of any examples of categories which admit two distinct embeddings of $\square$ defining two distinct categories with intervals.

We require that the pushout in (S3) be of maps in $\square_{\le 2}$ to use relationships internal to $\square_{\le 2}$ to develop a basic theory of homotopies. Necessarily, a and b are injective, and hence homeomorphisms onto their image. Unless the distinction is necessary, we will identify elements in $\square_{\le 2}$ with their image in $\mathcal{C}$. For the rest of this section, fix an embedding ι making $(\mathcal{C}, \iota)$ into a category with intervals.

Homotopies

In this subsection, working in an arbitrary category with intervals (Definition 1.1), we develop a notion of homotopy, leading to the proof (Theorem 1.13) that any such category carries a natural structure of a 2-category.

Definition 1.2 Let X and Y be spaces and $f, g\colon X \to Y$ be maps. A *homotopy* from f to g is given by a map $H\colon X \times I \to Y$ such that the following commutes:

$$\begin{tikzcd} X \arrow[r, "e_0"] \arrow[dr, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, "e_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

If there exists a homotopy from f to g, then f *is homotopic to* g, denoted by $f \sim g$ or $H\colon f \sim g$.

In particular, we will be using $X \times I$ as a cylinder object for X, in which case our notion of homotopy is that of a *left homotopy*. In Section 2, we will assume further that $- \times I$ admits a right adjoint $(-)^I$, giving path spaces X^I; this gives an equivalent notion of *right homotopy*, which we will not explicitly use.

[1] Note that [17] introduces the notion of a *site with interval*. Our notion differs from theirs, as it is strongly based on the properties of the topological interval, whereas that of [17] is abstract and specific to the category of sheaves.

Let $f, g\colon X \to Y$, $u\colon W \to X$ and $v\colon Y \to X$ be maps, and H a homotopy from f to g. For a space A, let $\pi_A\colon A \times I \to A$ denote the projection. We will follow these notational conventions for homotopies:

(1) $Hu = H(u \times \mathrm{id}_I)$, which gives a homotopy from fu to gu.

(2) $\mathrm{const}_f = f\pi_X = \pi_Y(f \times \mathrm{id}_I)$.

Homotopies are subject to the following identities, which can easily be verified:

(1) $(u \times \mathrm{id}_I)e_k = e_k u$.

(2) $\pi_X e_k = \mathrm{id}_X$.

(3) $v \circ \mathrm{const}_f \circ (u \times \mathrm{id}_X) = \mathrm{const}_{vfu}$.

Definition 1.3 Let $f, g\colon X \to Y$ be maps, and suppose $i\colon A \to X$ is a map such that $fi = gi$. A *homotopy relative to A* is a homotopy $H\colon X \times I \to Y$ from f to g such that the following commutes:

$$\begin{tikzcd} A \times I \arrow[r, "\pi_A"] \arrow[d, hook, "i \times \mathrm{id}_I"'] & A \arrow[d, "fi=gi"] \\ X \times I \arrow[r, "H"'] & Y \end{tikzcd}$$

Explicitly, $Hi = \mathrm{const}_{fi}$. If such a homotopy exists, it is denoted by $f \sim g$ rel A.

Definition 1.4 Let $f, g, h\colon X \to Y$ be maps. Given a homotopy H from f to g and a homotopy K from g to h, their *track composite* $H \cdot K$ is the induced map in the pushout

$$\begin{tikzcd} X \arrow[r, "e_0"] \arrow[d, "e_1"'] & X \times I \arrow[d, "\mathrm{id}_X \times b"] \arrow[ddr, bend left, "K"] & \\ X \times I \arrow[r, "\mathrm{id}_X \times a"'] \arrow[drr, bend right, "H"'] & X \times I \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[dr, dashed, "H \cdot K"] & \\ & & Y \end{tikzcd}$$

Let $-\mathrm{id}_I\colon I \to I$ be the map $t \mapsto 1-t$. The *inverse homotopy* of H, denoted by $-H$, is $H(\mathrm{id}_X \times (-\mathrm{id}_I))$:

$$\begin{tikzcd} X \times I \arrow[r, "\mathrm{id}_X \times (-\mathrm{id}_I)"] \arrow[dr, "-H"'] & X \times I \arrow[d, "H"] \\ & Y \end{tikzcd}$$

Since $(\mathrm{id}_X \times a)e_0 = e_0$ and $(\mathrm{id}_X \times b)e_1 = e_1$ by (S3), $H \cdot K$ is a homotopy from f to h. Similarly, $-H$ gives a homotopy from g to f. Given maps $q\colon W \to X$ and $r\colon Y \to Z$, we have $q(H \cdot K)r = qHr \cdot qKr$. Thus, if $fi = gi = hi$ for $i\colon A \to X$, then $(H \cdot K)i = Hi \cdot Ki = \mathrm{const}_{fi}$, and similarly $-Hi = \mathrm{const}_{fi}$, showing that composition and inverse homotopies preserve relativity.

Definition 1.5 Let $H, K\colon X \times I \to Y$ be homotopies from f to g. Then H and K are *homotopic rel endpoints* if there is a map $\alpha\colon X \times I^2 \to Y$ such that $\alpha\partial_L = H$, $\alpha\partial_T = \mathrm{const}_g$, $\alpha\partial_R = K$ and $\alpha\partial_B = \mathrm{const}_f$.

Analogous comments to the above hold for homotopies rel endpoints. Thus, constant homotopies, inverse homotopies and track composites show respectively that (relative) homotopy is reflexive, symmetric and transitive. In particular, we obtain the following:

Corollary 1.6 *Homotopy defines an equivalence relation on $\mathcal{C}(X,Y)$ for every pair of objects $X,Y \in \mathcal{C}$. Similarly, for any fixed pair of maps $f,g\colon X \to Y$, homotopy rel endpoints defines an equivalence relation on the set of homotopies from f to g.* □

Lemma 1.7 *Let H and H' be homotopies from f to g, and K and K' be homotopies from g to h. If $H \sim H'$ rel endpoints and $K \sim K'$ rel endpoints, then $H \cdot K \sim H' \cdot K'$ rel endpoints. That is, track composition is well defined up to homotopy rel endpoints.*

Proof By the assumptions, we may choose a map $\alpha\colon X \times I^2 \to Y$ such that $\alpha\partial_L = H$, $\alpha\partial_T = \mathrm{const}_g$, $\alpha\partial_R = H'$ and $\alpha\partial_B = \mathrm{const}_f$; likewise, we may choose $\beta\colon X \times I^2 \to Y$ with $\beta\partial_L = K$, $\beta\partial_T = \mathrm{const}_h$, $\beta\partial_R = K'$ and $\beta\partial_B = \mathrm{const}_g$. Then the induced map in the pushout

$$\begin{tikzcd}
X\times I \arrow[r, "\mathrm{id}_X\times\partial_B"] \arrow[d, "\mathrm{id}_X\times\partial_T"'] & X\times I\times I \arrow[d] \arrow[ddr, bend left, "\beta"] & \\
X\times I\times I \arrow[r] \arrow[drr, bend right, "\alpha"'] & X\times I\times I \arrow[dr, dashed, "\zeta"] & \\
& & Y
\end{tikzcd}$$

gives a homotopy rel endpoints from $H \cdot K$ to $H' \cdot K'$. □

Lemma 1.8 *For any homotopy H from f to g, $H \cdot \mathrm{const}_g \sim H$ rel endpoints and $\mathrm{const}_f \cdot H \sim H$ rel endpoints.*

Proof Let $\Gamma\colon I^2 \to I$ be given by $\Gamma(s,t) = st + (1-s)a^{-1}\bigl(\min(t, a(1))\bigr)$. Then the map $\alpha = H\Gamma$ gives the required homotopy. To see this, note that $\Gamma(e_0 \times a) = \mathrm{id}_I$ and $\Gamma(e_0 \times b) = e_1$, so $\alpha\partial_L = H \cdot \mathrm{const}_g$, and clearly $\alpha\partial_R = H$, $\alpha\partial_B = \mathrm{const}_f$ and $\alpha\partial_T = \mathrm{const}_g$. □

Lemma 1.9 *For any homotopy H from f to g, $H \cdot (-H) \sim \mathrm{const}_f$ rel endpoints and $-H \cdot H \sim \mathrm{const}_g$ rel endpoints.*

Proof Let $\gamma\colon I \to I$ be given by $\gamma(t) = a^{-1}(t)$ for $t \le a(1)$ and $\gamma(t) = 1 - b^{-1}(t)$ for $t \ge b(0)$. Define $\Gamma\colon I^2 \to I$ by

$$\Gamma(s,t) = \begin{cases} \min(a^{-1}(t), 1-s) & \text{if } t \le u(1), \\ \min(1-b^{-1}(t), 1-s) & \text{if } t \ge v(0). \end{cases}$$

Then Γ is continuous by the pasting lemma, $\Gamma\partial_L = \gamma$ and $\Gamma\partial_T = \Gamma\partial_B = \Gamma\partial_R = 0$. Note that $\gamma a = \mathrm{id}_I$ and $\gamma b = -\mathrm{id}_I$, so $H(\mathrm{id}_X \times \gamma a) = H$ and $H(\mathrm{id}_X \times \gamma b) = -H$; hence, $H(\mathrm{id}_X \times \gamma) = H \cdot (-H)$. Thus, $H(\mathrm{id}_X \times \Gamma)\colon X \times I^2 \to Y$ gives a homotopy from $H \cdot (-H)$ to const_f rel endpoints. The other part is analogous. □

Lemma 1.10 *For any homotopies H from f_1 to f_2, J from f_2 to f_3, and K from f_3 to f_4, we have $(H \cdot J) \cdot K \sim H \cdot (J \cdot K)$ rel endpoints. That is, track composition of homotopies is associative up to homotopy rel endpoints.*

Proof Let $\Omega = a(1) = b(0)$. As before, note that a^{-1} is defined and continuous on $[0, \Omega]$, as is b^{-1} on $[\Omega, 1]$. Let $\gamma\colon I \to I$ be given by

$$\gamma(t) = \begin{cases} b^{-1}(t) & \text{if } t \in [b\Omega, 1], \\ aba^{-1}b^{-1}(t) & \text{if } t \in [\Omega, b\Omega], \\ a(t) & \text{if } t \in [0, \Omega]. \end{cases}$$

By the pasting lemma, γ is continuous, and $\gamma a = aa$, $\gamma ba = ab$ and $\gamma bb = b$. Let $\Gamma\colon I^2 \to I$ be defined by $\Gamma(s,t) = (1-s)t + s\gamma(t)$. Then $\Gamma\partial_B = e_0$, $\Gamma\partial_T = e_1$, $\Gamma\partial_L = \mathrm{id}_I$ and $\Gamma\partial_R = \gamma$. Let $\alpha = ((H \cdot J) \cdot K) \circ (\mathrm{id}_X \times \Gamma)$. Clearly $\alpha\partial_L = (H \cdot J) \cdot K$, $\alpha\partial_B = \mathrm{const}_{f_1}$ and $\alpha\partial_T = \mathrm{const}_{f_4}$. Moreover, $(\alpha\partial_R)a = ((H \cdot J) \cdot K)aa = H$, and similarly $(\alpha\partial_R)ba = J$ and $(\alpha\partial_R)bb = K$. Thus, both $\alpha\partial_R$ and $H \cdot (J \cdot K)$ fit in the following diagram, so by uniqueness they are equal, giving the required homotopy:

$$\begin{tikzcd} X \arrow[r, "e_0"] \arrow[d, "e_1"'] & X \times I \arrow[d, "b"] \arrow[ddr, bend left, "J \cdot K"] & \\ X \times I \arrow[r, "a"'] \arrow[drr, bend right, "H"'] & X \times I \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

∎

Lemma 1.11 *Let $f, g_0, g_1, h\colon X \to Y$ be maps and $A\colon f \sim g_0$, $B\colon g_0 \sim h$, $C\colon f \sim g_1$ and $D\colon g_1 \sim h$ be homotopies. If there is a map $\alpha\colon X \times I^2 \to Y$ with $\alpha\partial_B = A$, $\alpha\partial_R = B$, $\alpha\partial_L = C$ and $\alpha\partial_T = D$, then $A \cdot B \sim C \cdot D$ rel endpoints.*

Proof Let $\gamma_0, \gamma_1\colon I \to I^2$ be given by

$$\gamma_0(x) = \begin{cases} (a^{-1}(x), 0) & \text{if } x \le a(1), \\ (1, b^{-1}(x)) & \text{if } x \ge b(0), \end{cases} \qquad \text{and} \qquad \gamma_1(x) = \begin{cases} (0, a^{-1}(x)) & \text{if } x \le a(1), \\ (b^{-1}(x), 1) & \text{if } x \ge b(0). \end{cases}$$

Then $\gamma_0 a = \partial_B$, $\gamma_0 b = \partial_R$, $\gamma_1 a = \partial_L$ and $\gamma_1 b = \partial_T$. Let $\Gamma\colon I^2 \to I^2$ be given by $\Gamma(s,t) = s\gamma_1(t) + (1-s)\gamma_0(t)$. Since $\Gamma\partial_B = e_0 \times e_0$ and $\Gamma\partial_T = e_1 \times e_1$, $\alpha(\mathrm{id}_X \times \Gamma)$ gives a homotopy rel endpoints from $A \cdot B$ to $C \cdot D$. ∎

Lemma 1.12 [27, Lemma 1] *Let $u, v\colon X \to Y$ and $f, g\colon Y \to Z$ be maps. Given a homotopy H from u to v and a homotopy K from f to g, $fH \cdot Kv \sim Ku \cdot gH$ rel endpoints.*

Proof Let $\alpha\colon X \times I \times I \to Z$ be $K(H \times \mathrm{id}_I)$. Then $\alpha\partial_L = K(H \times \mathrm{id}_I)(e_0 \times \mathrm{id}_I) = K(He_0) = Ku$, and similarly $\alpha\partial_T = gH$, $\alpha\partial_R = Kv$ and $\alpha\partial_B = fH$. The result follows from Lemma 1.11. ∎

Theorem 1.13 *If $\mathcal{C}$ satisfies the assumptions in Definition 1.1, then $\mathcal{C}$ admits a 2-category structure, where each pair of spaces A and B in $\mathcal{C}$ are assigned the groupoid $\mathcal{C}(A, B)$ whose objects are maps $A \to B$ and morphisms are homotopy classes of homotopies rel endpoints.*

Proof The composition in $\mathcal{C}(A, B)$ is given by track composition as in Definition 1.2. Lemmas 1.7, 1.8, 1.9 and 1.10 assert that this is well defined and associative, that each map $f : A \to B$ has an identity homotopy const_f, and that each class of homotopies $[H]$ has an inverse $[-H]$. Thus, $\mathcal{C}(A, B)$ is a groupoid for every pair of spaces A and B. For each space A, the functor $I_A\colon [0] \to \mathcal{C}(A, A)$ maps the unique morphism in $[0]$ to $[\mathrm{const}_{\mathrm{id}_A} : \mathrm{id}_A \sim \mathrm{id}_A]$. The composition functors $c_{A,B,C} : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ act on objects by sending a pair of maps $(f : A \to B, g\colon B \to C)$ to $gf : A \to C$ and, by Lemma 1.12, there is a canonical, well-defined choice for a pair of homotopy classes $([H : u \sim v], [K : f \sim g])$, namely mapping them to $[fH \cdot Kv] = [Ku \cdot gH]$. □

We now turn our attention to defining homotopy equivalences.

Definition 1.14 A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists $g\colon Y \to X$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$. If $gf = \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$ rel X, then g is a *deformation retraction*.

Definition 1.15 Let $\mathrm{ho}\,\mathcal{C}$ denote the category whose objects are spaces in $\mathcal{C}$ and whose maps are homotopy classes of continuous maps. By Lemma 1.12, composition of homotopy classes is well defined. An isomorphism in this category is a class of homotopy equivalences. There is a canonical quotient functor $\Pi\colon \mathcal{C} \to \mathrm{ho}\,\mathcal{C}$, which maps $f : X \to Y$ to $[f]\colon X \to Y$.

We record two lemmas about homotopy equivalences for future use.

Lemma 1.16 *A map $f : X \to Y$ is a homotopy equivalence if and only if Πf is an isomorphism in* $\mathrm{ho}\,\mathcal{C}$.

Proof If f has a homotopy inverse $g : Y \to X$, then, since $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$, we have $[g][f] = [\mathrm{id}_X]$ and $[f][g] = [\mathrm{id}_Y]$, so $\Pi f = [f]$ is an isomorphism. Conversely, if $\Pi f = [f]$ is an isomorphism, choose a representative $g\colon Y \to X$ for $[f]^{-1}$. Then $[gf] = [g][f] = [\mathrm{id}_X]$, so $gf \sim \mathrm{id}_X$, and likewise $fg \sim \mathrm{id}_Y$, so f is a homotopy equivalence. □

Lemma 1.17 *The following properties hold for homotopy equivalences*:

1. *Homotopy equivalences are closed under* 2*-out-of-*6*.*
2. *A retract of a homotopy equivalence in the arrow category is again a homotopy equivalence.*
3. *If $f : X \to Y$ and $f'\colon X' \to Y'$ are homotopy equivalences, then so is $f \times f'$.*
4. *Assuming $- \times I$ preserves coproducts, then, if $\{f_k\}_{k \in K}$ is a family of homotopy equivalences, so is $\coprod_{k \in K} f_k$.*

Proof (1) and (2) are immediate from Lemma 1.16, since isomorphisms satisfy these in any category and homotopy equivalences are precisely the maps inverted by Π.

For (3), choose homotopy inverses $g\colon Y \to X$ and $g'\colon Y' \to X'$, as well as homotopies H from gf to id_X and H' from $g'f'$ for $\mathrm{id}_{X'}$. Then a homotopy from $gf \times g'f' = (g \times g')(f \times f')$ to $\mathrm{id}_{X \times X'}$ is given by (H, H'). Similar reasoning gives a homotopy from $(f \times f')(g \times g')$ to $\mathrm{id}_{Y \times Y'}$, so $f \times f'$ is a homotopy equivalence.

Lastly, for (4), given families $\{g_k\}_{k\in K}$ of homotopy inverses with $\{H_k\}_{k\in K}$ and $\{J_k\}_{k\in K}$ homotopies from $g_k f_k$ and $f_k g_k$ to the respective identities, $\coprod H_k$ and $\coprod J_k$ give homotopies from $\coprod g_k \circ \coprod f_k$ and $\coprod f_k \circ \coprod g_k$ to the respective identities since $\big(\coprod \mathrm{dom}(f_k)\big) \times I \cong \coprod(\mathrm{dom}(f_k) \times I)$. □

Pushout products

In this final subsection, we collect the requisite background on the Leibniz construction. Specifically, we recall the definitions of pushout product (Definition 1.18) and pullback power (Definition 1.20) and prove their basic properties in the category of topological spaces.

Definition 1.18 Let $f: X \to Y$ and $g: A \to B$ be maps in $\mathcal{C}$. Their *pushout product*, denoted by $f \mathbin{\hat{\times}} g$, is the factorization of $f \times g: X \times A \to Y \times B$ through the pushout in the following diagram:

$$\begin{tikzcd}
X \times A \arrow[r, "f \times \mathrm{id}_A"] \arrow[d, "\mathrm{id}_X \times g"'] & Y \times A \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_X \times g"] & \\
X \times B \arrow[r] \arrow[drr, bend right, "f \times \mathrm{id}_A"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f \hat{\times} g"] & \\
& & Y \times B
\end{tikzcd}$$

Note that, since $\times$ is symmetric, $f \mathbin{\hat{\times}} g \cong g \mathbin{\hat{\times}} f$, and that $\hat{\times}$ is associative up to isomorphism.

Lemma 1.19 *If $f: A \to B$ and $f': A' \to B'$ are isomorphic in the arrow category, as are $g: C \to D$ and $g': C' \to D'$, then $f \mathbin{\hat{\times}} g$ and $f' \mathbin{\hat{\times}} g'$ are isomorphic in the arrow category.*

Proof This is a standard diagram chase, given that naturally isomorphic diagrams induce an isomorphism between their colimits which commutes with the colimit legs. □

A functor $\otimes: \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is *divisible on the right* if, for every $B \in \mathcal{B}$, the functor $- \otimes B: \mathcal{A} \to \mathcal{C}$ admits a right adjoint. When the functor $- \times A: \mathcal{C} \to \mathcal{C}$ admits a right adjoint, A is an *exponentiable* space. The right adjoint will be denoted by $(-)^A: \mathcal{C} \to \mathcal{C}$, and its action on a morphism $f: X \to Y$ will be denoted by $f_*: X^A \to Y^A$. A map $g: A \to B$ between exponentiable spaces induces a natural transformation $g^*: (-)^B \Rightarrow (-)^A$.

Definition 1.20 Let $f: X \to Y$ and $g: A \to B$ be maps in $\mathcal{C}$ with A and B both exponentiable. Their *pullback power*, denoted by $f \triangleright g$, is the factorization of $f_* g^* = g^* f_*$ in the following diagram:

$$\begin{tikzcd}
X^B \arrow[dr, dashed, "f \triangleright g"] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "g^*"'] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^B \arrow[d, "g^*"] \\
& X^A \arrow[r, "f_*"'] & Y^A
\end{tikzcd}$$

Note that, unlike the pushout product, in general $f \triangleright g \neq g \triangleright f$.

The following statement holds in general for *closed* monoidal products in a given category. Since we will not be taking $\mathcal{C}$ to be cartesian closed, we need to ensure that we are only exponentiating spaces that we have assumed or shown to be exponentiable.

Lemma 1.21 *Let $i\colon A \to B$ be a map between exponentiable spaces. Then $f \mathbin{\hat{\times}} i$ has the left lifting property with respect to g if and only if f has the left lifting property with respect to $g \triangleright i$*

Proof This follows from [13, Proposition D.1.18], since the functor $-\times-\colon \mathcal{C}\times\mathcal{C}_{\mathrm{exp}} \to \mathcal{C}$ is divisible on the right, where $\mathcal{C}_{\mathrm{exp}}$ denotes the full subcategory of $\mathcal{C}$ consisting of exponentiable spaces. $\square$

Lemma 1.22 *Let A be a space and $!_A\colon \varnothing \to A$ the unique map. For any map $f\colon X \to Y$ and space A, we have $!_A \mathbin{\hat{\times}} f \cong \mathrm{id}_A \times f$.*

Proof This is immediate from (S4). $\square$

Lemma 1.23 *The following identities hold for the pushout product in the arrow category* $\mathsf{Top}^{\to}$, *where $!_X\colon \varnothing \to X$ and $i\colon \partial I \to I$ is the inclusion of the endpoints*:

(1) $e_0 \mathbin{\hat{\times}} e_0 \cong\, !_I \mathbin{\hat{\times}} e_0$.

(2) $i \mathbin{\hat{\times}} e_0 \cong\, !_I \mathbin{\hat{\times}} e_0$.

Proof For (1), it follows from Lemma 1.22 that we must find an automorphism $\phi\colon I\times I \to I \times I$ that restricts to an isomorphism $I\times\{0\} \cong I\times\{0\}\cup\{0\}\times I$, ie such that the following commutes:

$$\begin{tikzcd} I\times\{0\} \arrow[r, "\phi|_{I\times\{0\}}"] \arrow[d, hook] & I\times\{0\}\cup\{0\}\times I \arrow[d, hook] \\ I\times I \arrow[r, "\phi"'] & I\times I \end{tikzcd}$$

One such ϕ is given by the composite gf, where $g(x,y) = \bigl(\tfrac{1}{2}(1+x), y\bigr)$ and

$$f(x,y) = \begin{cases} (x,2y) & \text{if } y \le \tfrac{1}{2}x, \\ (2(x-y),x) & \text{if } \tfrac{1}{2}x \le y \le x, \\ (2(x-y),y) & \text{if } x \le y \le 2x, \\ (-y,2x) & \text{if } y \ge 2x. \end{cases}$$

The proof of (2) is similar. $\square$

Note that, since the pushout product is well defined up to isomorphism in the arrow category by Lemma 1.19, the isomorphism in (1) holds regardless of what pushout we take for $I \amalg_* I$, even though we chose a specific pushout for the calculation in (1). In particular, since we have taken I for the pushout $I \amalg_* I$ in $\mathcal{C}$, and $\square_{\le 2}$ to be a full subcategory, we may use (1) in any category with intervals.

2 Q-structures: fibrations and cofibrations

In this and the next section, we will show that the axioms of a Q-structure given in Definition 2.2 are sufficient for $\mathcal{C}$ to admit a model structure defined analogously to the standard model structure on Top. If

$\mathcal{C}$ satisfies the additional requirements for being a good Q-structure, given in (Q7) and (Q8), then the weak equivalences of the model structure are precisely the ones which induce isomorphisms on path-connected components and all homotopy groups, as defined in Definition 3.7.

We begin by introducing (relative) cell complexes.

Definition 2.1 Let $\mathcal{J}$ be a class of maps. A $\mathcal{J}$*-cell complex* is a transfinite composition of pushouts of coproducts of maps in $\mathcal{J}$, as in the following diagram:

$$\begin{tikzcd}
\coprod \mathrm{dom}(j_\alpha) \arrow[r, "\coprod j_\alpha"] \arrow[d] & \coprod \mathrm{cod}(j_\alpha) \arrow[d, "p_1"] & & & \\
X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \arrow[r] & \operatorname{colim} X_k \\
& \coprod \mathrm{dom}(j_\beta) \arrow[u] \arrow[r, "\coprod j_\beta"'] & \coprod \mathrm{cod}(j_\beta) \arrow[u, "p_2"'] & &
\end{tikzcd}$$

The collection of all $\mathcal{J}$-cell complexes will be denoted by $\mathrm{cell}(\mathcal{J})$, and $\mathrm{cof}(\mathcal{J})$ will denote the closure of $\mathrm{cell}(\mathcal{J})$ under retracts. Let $\mathrm{rlp}(\mathcal{J})$ denote the class of maps which have the right lifting property against $\mathcal{J}$. A standard proof shows that $\mathrm{rlp}(\mathcal{J}) = \mathrm{rlp}(\mathrm{cof}(\mathcal{J}))$ [9, Proposition 10.3.2].

Notation 2.1 Let $\mathcal{I} = \{i_n \colon \partial I^n \hookrightarrow I^n\}_{n \geq 0}$ and $\mathcal{J} = \{j_n \colon I^n \times \{0\} \hookrightarrow I^n \times I\}_{n \geq 0}$ be the subspace inclusions in Top.

As a slight abuse of notation, let $e_k \colon * \hookrightarrow \partial I$ for $k = 0, 1$ be the endpoint inclusions.

Definition 2.2 Let $\iota \colon \square \to \mathcal{C}$ be an embedding of $\square$ into a bicomplete category $\mathcal{C}$. The pair $(\mathcal{C}, \iota)$ is a *Q-structure* if the following hold:

(Q1) $\square_{\leq 2} \hookrightarrow \square \xrightarrow{\iota} \mathcal{C}$ makes $\mathcal{C}$ a category with intervals.

(Q2) $X \times \partial I$ is a coproduct $X \amalg X$ in $\mathcal{C}$, with inclusions $\mathrm{id}_X \times e_k$ for all $X \in \mathcal{C}$.

(Q3) $\mathcal{I} \mathbin{\hat{\times}} \mathcal{I} \subseteq \mathrm{cof}(\mathcal{I})$ and $\mathcal{I} \mathbin{\hat{\times}} \mathcal{J} \subseteq \mathrm{cof}(\mathcal{J})$.

(Q4) $\mathcal{J} \subseteq \mathrm{cof}(\mathcal{I})$ and $\{\varnothing \to \partial I^n\} \subseteq \mathrm{cof}(\mathcal{I})$.

(Q5) ∂I^n and I^n are exponentiable in $\mathcal{C}$ for all $n \geq 0$.

(Q6) ∂I^n and I^n are small relative to maps in $\mathrm{cell}(\mathcal{I})$ and $\mathrm{cell}(\mathcal{J})$, respectively, for all $n \geq 0$.

The pair $(\mathcal{C}, \iota)$ is a *good Q-structure* if, in addition:

(Q7) I^n is a product $\prod_{1 \leq k \leq n} I$ for all positive integers n.

(Q8) There are pushouts, for all $n \geq 1$, of the form

$$\begin{tikzcd}
\partial I^n \arrow[r, "e_1"] \arrow[d, "!"'] & \partial I^n \times I \arrow[d, "\rho"] \\
* \arrow[r, "c"'] & I^n
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial I^{n-1} \arrow[r, hook, "i_n"] \arrow[d, "!"'] & I^{n-1} \arrow[d, "\sigma"] \\
* \arrow[r, "(*)"'] & \partial I^n
\end{tikzcd}$$

in $\mathcal{C}$ such that $\rho e_0 = i_n$.

The axioms listed above are key properties used to construct the Quillen model structure on Top, restated for an arbitrary $\mathcal{C}$. We will use the axioms as follows: (Q1) allows us to make use of the homotopy theory developed in Section 1; (Q2) is used in lifting homotopies along certain maps, with prescribed endpoints; (Q3) and (Q5) allow us to obtain a weak version of the pushout product axiom; lastly, (Q4) and (Q6) are used in applying the small object argument. For good Q-structures, (Q7) will be used to define homotopy groups (see Definition 3.7) and (Q8) are some familiar quotient identities from Top, namely $S^{n-1} \times I / S^{n-1} \times \{0\} \cong D^n$ and $D^n / S^{n-1} \cong S^n$.

It follows from (Q5) that $- \times I^n$, as a left adjoint, preserves colimits for all $n \geq 0$. (Q6) is sufficient for $\mathcal{C}$ to admit a small object argument on $\mathcal{I}$ and $\mathcal{J}$, which generates two weak factorization systems in $\mathcal{C}$. For $\mathcal{K}$ equal to either $\mathcal{I}$ or $\mathcal{J}$, one has for the left class $\mathrm{cof}(\mathcal{K})$ and for the right class $\mathrm{rlp}(\mathcal{K})$ of the respective weak factorization system.

In Sections 4, 5 and 6, we discuss several examples of good Q-structures. We do not know of any examples of Q-structures that are not good; however, we do not expect the axioms (Q1)–(Q6) to imply (Q7)–(Q8), since the latter require the existence of certain pushouts squares which are otherwise not assumed to exist. We chose to isolate the axioms (Q1)–(Q6) to underscore the fact that they are the only ones required to establish a model structure, while (Q7)–(Q8) are used to characterize the class of weak equivalences.

We can now define the classes of maps that will form a model structure on $\mathcal{C}$.

Definition 2.3 (1) A *cofibration* is a map in $\mathrm{cof}(\mathcal{I})$.

(2) A *fibration* is a map in $\mathrm{rlp}(\mathcal{J})$.

(3) A *trivial cofibration* is a map in $\mathrm{cof}(\mathcal{J})$.

(4) A *trivial fibration* is a map in $\mathrm{rlp}(\mathcal{I})$.

Using the small object argument, we obtain the expected factorizations:

Lemma 2.4 *Every map in $\mathcal{C}$ can be factored as*

(1) *a cofibration followed by a trivial fibration, and*

(2) *a trivial cofibration followed by a fibration.* □

Definition 2.5 Let $f : X \to Y \in \mathcal{C}$ and n a nonnegative integer. Then:

(1) f is *n-compressible* if any square of the form

$$\begin{tikzcd} \partial I^n \arrow[r, "u"] \arrow[d, hook, "i_n"'] & X \arrow[d, "f"] \\ I^n \arrow[r, "v"] & Y \end{tikzcd}$$

admits a diagonal map $h\colon I^n \to X$ such that $hi_n = u$ and $fh \sim v$ rel ∂I^n.

(2) f is *n-connected* if it is k-compressible for all $k \leq n$.

(3) f is a *weak equivalence* if it is n-connected for all nonnegative integers.

A map f has the *weak right lifting property* against $i : A \to B$ if it has the lifting property as in (1) against i: given maps such that $fu = vi$, there is a filler h satisfying $hi = u$ and $fh \sim v$ rel A.

Theorem 2.6 *Any Q-structure $(\mathcal{C}, \iota)$ induces a model structure on $\mathcal{C}$ whose cofibrations and fibrations are as defined in Definition 2.3 and weak equivalences are as defined in Definition 2.5.*

We will defer the proof of this to Section 3.

Definition 2.7 A space X is *fibrant* if the map $X \to *$ is a fibration. Dually, X is *cofibrant* if the map $\varnothing \to X$ is a cofibration.

Lemma 2.8 *Every space is fibrant.*

Proof Given the square

$$\begin{tikzcd} I^n \arrow[r, "f"] \arrow[d, "j_n"'] & X \arrow[d, "!"] \\ I^n \times I \arrow[r, "!"] & * \end{tikzcd}$$

the map $f\pi_{I^n}$ gives a lift. □

Lemma 2.9 *Every trivial fibration is both a fibration and a weak equivalence.*

Proof Clearly, if f has the right lifting property with respect to $\mathcal{I}$, it is a weak equivalence, since every square of the form

$$\begin{tikzcd} \partial I^n \arrow[r, "u"] \arrow[d, hook, "i_n"'] & X \arrow[d, "f"] \\ I^n \arrow[r, "v"] & Y \end{tikzcd}$$

admits a strict lift, not just one up to homotopy on the lower triangle. Moreover, by (Q4), since f has the right lifting property with respect to all maps in $\mathrm{cof}(\mathcal{I})$, it has the right lifting property with respect to $\mathcal{J}$, and thus is a fibration. □

Proposition 2.10 *The following identities hold in $\mathcal{C}$:*

(1) $\mathrm{cof}(\mathcal{I}) \mathbin{\hat{\times}} \mathcal{I} \subseteq \mathrm{cof}(\mathcal{I})$.

(2) $\mathrm{cof}(\mathcal{I}) \mathbin{\hat{\times}} \mathcal{J} \subseteq \mathrm{cof}(\mathcal{J})$.

(3) $\mathcal{I} \mathbin{\hat{\times}} \mathrm{cof}(\mathcal{J}) \subseteq \mathrm{cof}(\mathcal{J})$.

Proof Since $\mathcal{I} \mathbin{\hat{\times}} \mathcal{I} \subseteq \mathrm{cof}(\mathcal{I})$ by (Q3), $\mathcal{I} \mathbin{\hat{\times}} \mathcal{I}$ has the left lifting property with respect to $\mathrm{rlp}(\mathcal{I})$, so $\mathcal{I}$ has the left lifting property with respect to $\mathrm{rlp}(\mathcal{I}) \triangleright \mathcal{I}$ by Lemma 1.21. Thus, $\mathrm{cof}(\mathcal{I})$ has the left lifting property with respect to $\mathrm{rlp}(\mathcal{I}) \triangleright \mathcal{I}$, and hence $\mathrm{cof}(\mathcal{I}) \mathbin{\hat{\times}} \mathcal{I} \subseteq \mathrm{cof}(\mathcal{I})$ by Lemma 1.21 again. The other parts are analogous. □

For closed monoidal products $\otimes$, we can obtain the stronger result that $\mathrm{cof}(\mathcal{I}) \mathbin{\hat{\otimes}} \mathrm{cof}(\mathcal{J}) \subseteq \mathrm{cof}(\mathcal{K})$ whenever $\mathcal{I} \mathbin{\hat{\otimes}} \mathcal{J} \subseteq \mathcal{K}$. In this case, maps in $\mathrm{cof}(\mathcal{I})$ might not be between exponentiable spaces, so we cannot apply the same reasoning again. Thus, some care will be needed when applying Proposition 2.10 to ensure that at least one map is in either $\mathcal{I}$ or $\mathcal{J}$.

Definition 2.11 Let $[-,-]$ denote the induced map from the pushout

$$\begin{tikzcd}
* \arrow[r, "e_0"] \arrow[d, "e_1"'] & I \arrow[d] \arrow[ddr, bend left, "\partial_T"] & \\
I \arrow[r] \arrow[drr, bend right, "\partial_L"'] & I \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "{[\partial_L,\partial_T]}"] & \\
& & I^2
\end{tikzcd}$$

The *open box inclusion* is the induced map in the pushout

$$\begin{tikzcd}
* \arrow[r, "e_0"] \arrow[d, "e_1"'] & I \arrow[d] \arrow[ddr, bend left, "-\partial_R"] & \\
I \arrow[r] \arrow[drr, bend right, "{[\partial_L,\partial_T]}"'] & I \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "{[[\partial_L,\partial_T],-\partial_R]}"] & \\
& & I^2
\end{tikzcd}$$

By Lemmas 1.19 and 1.23, we recognize the top pushout as $[\partial_L,\partial_T] \cong e_0 \mathbin{\hat{\times}} e_0 \cong {!_I} \mathbin{\hat{\times}} e_0 \cong \mathrm{id}_I \times e_0 = j_1$, so, by the same reasoning, $[[\partial_L,\partial_T],-\partial_R] \cong j_1$ in the arrow category. When we wish to make the distinction, we will use $\sqcap$ for I in the latter pushout and $\mathrm{inc}\colon \sqcap \hookrightarrow I^2$ for $[[\partial_L,\partial_T],-\partial_R]$. Geometrically, it should be thought of as the left, top, and right edges of the square, which we will use to "fill out" homotopies. By Proposition 2.10, if $i: A \rightarrowtail X$ is a cofibration, $i \mathbin{\hat{\times}} \mathrm{inc}$ is a trivial cofibration. To verify the equality of two maps from $\sqcap$ in this context, one of which is a composition including inc, it suffices to check that they agree after precomposing with b, aa and ab (see Definition 1.1).

Lemma 2.12 *Let $f: X \to Y$ be a homotopy equivalence. Then there is a map $g\colon Y \to X$ along with homotopies H from id_X to gf and K from id_Y to fg and a map $\alpha\colon X \times I^2 \to Y$ such that $\alpha\partial_L = fH$, $\alpha\partial_T = \mathrm{const}_{fgf}$, $\alpha\partial_R = Kf$ and $\alpha\partial_B = \mathrm{const}_f$.*

Proof This is the statement that any equivalence can be promoted to an adjoint equivalence, applied to the 2-category $\mathcal{C}$ as given in Theorem 1.13. □

Theorem 2.13 *Homotopy equivalences are weak equivalences.*

Proof Let $f: X \to Y$ be a homotopy equivalence and g, H, K and α be as in the statement of Lemma 2.12. Suppose that, for a nonnegative integer n, the following diagram commutes:

$$\begin{tikzcd}
\partial I^n \arrow[r, "u"] \arrow[d, hook, "i_n"'] & X \arrow[d, "f"] \\
I^n \arrow[r, "v"'] & Y
\end{tikzcd}$$

By Proposition 2.10, $i_n \mathbin{\hat{\times}} e_1 \in \mathrm{cof}(\mathcal{J})$, so there is a lift $J\colon I^n \times I \to X$ since X is fibrant:

$$\begin{tikzcd}
\partial I^n \times I \amalg_{\partial I^n \times \{1\}} I^n \times \{1\} \arrow[r, "{[Hu,gv]}"] \arrow[d, hook, "i_n \hat{\times} e_1"'] & X \\
I^n \times I \arrow[ur, dashed, "J"'] &
\end{tikzcd}$$

Let $w = Je_0\colon I^n \to X$. Since $He_0 = \mathrm{id}_X$, we have $wi_n = Je_0 i_n = J(i_n \times \mathrm{id}_I)e_0 = Hue_0 = u$. By the lifting triangle above, $fJi_n = fHu$, and, by the initial square, $Kvi_n = Kfu$. Taking the convention for $\mathrm{inc}\colon \sqcap \hookrightarrow I^2$ as in Definition 2.11,

$$\begin{aligned}
\alpha u \circ \mathrm{inc} \circ b &= \alpha u \circ -\partial_R = -Kfu = -Kvi_n,\\
\alpha u \circ \mathrm{inc} \circ aa &= \alpha u \partial_L = fHu = fJi_n,\\
\alpha u \circ \mathrm{inc} \circ ab &= \alpha u \partial_T = \mathrm{const}_{fgfu} = \mathrm{const}_{fgv} i_n;
\end{aligned}$$

hence, the top map in the diagram

$$\begin{tikzcd}
\partial I^n \times I^2 \amalg_{\partial I^n \times \sqcap} I^n \times \sqcap \arrow[r, "{[\alpha u,[[fJ,\mathrm{const}_{fgv}],-Kv]]}"] \arrow[d, "i_n \hat{\times} \mathrm{inc}"'] & Y \\
I^n \times I^2 \arrow[ur, dashed, "\widetilde{\alpha}"'] &
\end{tikzcd}$$

is defined, so it admits a lift $\widetilde{\alpha}\colon I^n \times I^2 \to Y$ by Proposition 2.10. The composite $\widetilde{\alpha}\partial_B$ gives a homotopy from fw to v rel ∂I^n, since $\widetilde{\alpha}\partial_B i_n = \widetilde{\alpha}(i_n \times \mathrm{id}_{I^2})\partial_B = \alpha u \partial_B = \mathrm{const}_{fu} = \mathrm{const}_{vi_n}$. □

3 Quillen model structure

In this section, we put all the pieces together and prove Theorem 2.6, while also characterizing weak equivalences in terms of homotopy groups (Theorem 3.12), which are introduced in Definition 3.7.

Lemma 3.1 *For a weak equivalence $f : X \to Y$ and a cofibration $i : A \rightarrowtail B$, if the square*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, tail, "i"'] & X \arrow[d, "f"] \\
B \arrow[r, "v"'] & Y
\end{tikzcd}$$

commutes, then there is a filler $h\colon B \to X$ such that $hi = u$ and $fh \sim v$ rel A.

Proof This is analogous to the proof that the left class of a weak factorization system is closed under coproducts, pushouts, transfinite composition and retracts, except special care is needed to check that relativity is preserved in each part. To get the induced homotopy on the colimits, one needs to use relativity and cocontinuity of $- \times I$. We will only give the proof for pushouts. Suppose that the square

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\
B \arrow[r, "\beta"'] & B' \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout and that f has the weak right lifting property (see Definition 2.5) against i. We can combine this square with

$$\begin{tikzcd} A' \arrow[r, "u"] \arrow[d, "i'"'] & X \arrow[d, "f"] \\ B' \arrow[r, "v"'] & Y \end{tikzcd}$$

to get a diagonal filler $h\colon B \to X$ with $hi = u\alpha$ and a homotopy $H\colon B \times I \to Y$ from fh to $v\beta$ rel A. This induces the map $h'\colon B' \to X$ and homotopy $H'\colon B' \times I \to Y$, by cocontinuity of $- \times I$, in

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \arrow[ddr, bend left, "u"] & \\ B \arrow[r, "\beta"'] \arrow[drr, bend right, "h"'] & B' \arrow[dr, dashed, "h'"] & \\ & & X \end{tikzcd} \qquad \begin{tikzcd} A \times I \arrow[r, "\alpha\times\mathrm{id}_I"] \arrow[d, "i\times\mathrm{id}_I"'] & A' \times I \arrow[d, "i'\times\mathrm{id}_I"] \arrow[ddr, bend left, "\mathrm{const}_{fu}"] & \\ B \times I \arrow[r, "\beta\times\mathrm{id}_I"'] \arrow[drr, bend right, "H"'] & B' \times I \arrow[dr, dashed, "H'"] & \\ & & Y \end{tikzcd}$$

since $Hi = \mathrm{const}_{fhi} = \mathrm{const}_{fu\alpha} = \mathrm{const}_{fu}\alpha$ by relativity of H. By commutativity, $H'i' = \mathrm{const}_{fu}$, so H' is relative to A', and one can verify through the pushout on the right that $H'e_0 = fh'$ and $H'e_1 = v$. □

Definition 3.2 Let $\mathrm{ho}\,\mathcal{C}_{\mathrm{cof}}$ denote the full subcategory of $\mathrm{ho}\,\mathcal{C}$ (Definition 1.15) consisting of cofibrant spaces.

Theorem 3.3 *A map $f\colon X \to Y$ is a weak equivalence if and only if*

$$f_*\colon [A, X] \to [A, Y] \tag{$*$}$$

is a bijection for all cofibrant spaces A.

Proof Suppose first that f is a weak equivalence, and let A be a cofibrant space. By the above lemma, for any $[\beta] \in [A, Y]$, there is a filler $\alpha\colon A \to X$ in the diagram

$$\begin{tikzcd} \varnothing \arrow[r, "!"] \arrow[d, "!"'] & X \arrow[d, "f"] \\ A \arrow[r, "\beta"'] & Y \end{tikzcd}$$

such that $f\alpha \sim \beta$; hence, f_* is surjective. If $f_*[\alpha] = f_*[\beta]$ for $[\alpha], [\beta] \in [A, X]$, then there is a homotopy $H\colon A \times I \to Y$ from $f\alpha$ to $f\beta$. Since A is cofibrant, $!_A \mathbin{\widehat{\times}} i_1 = \mathrm{id}_A \times i_1\colon A \times \partial I \to A \times I$ is a cofibration by Proposition 2.10 and Lemma 1.22. By (Q2), $A \times \partial I$ is a coproduct $A \amalg A$, so the square

$$\begin{tikzcd} A \times \partial I \arrow[r, "\alpha+\beta"] \arrow[d, "!_A \widehat{\times} i_1"'] & X \arrow[d, "f"] \\ A \times I \arrow[r, "H"'] & Y \end{tikzcd}$$

admits a filler $K\colon A \times I \to X$ such that the upper triangle commutes, so K is a homotopy from α to β; hence, f_* is injective.

Suppose instead that $f_*\colon [A,X]\to[A,Y]$ is a bijection for all cofibrant spaces A. Using Lemma 2.4, factor $\varnothing\to X$ as a cofibration followed by a trivial fibration, which is a weak equivalence and a fibration by Lemma 2.9: $\varnothing\rightarrowtail\widetilde{X}\xrightarrow{\sim}\!\!\!\!\rightarrow X$. Similarly, factor $\varnothing\to Y$ into $\varnothing\rightarrowtail\widetilde{Y}\xrightarrow{\sim}\!\!\!\!\rightarrow Y$. Then there is a lift $\tilde f\colon\widetilde{X}\to\widetilde{Y}$ in the diagram

$$\begin{tikzcd}
\varnothing \arrow[r, tail] \arrow[d, tail] & \widetilde{Y} \arrow[d, two heads, "\wr"] \\
\widetilde{X} \arrow[r] \arrow[ur, dashed, "\tilde f"] & Y
\end{tikzcd}$$

where the bottom arrow is the composite $\widetilde{X}\xrightarrow{\sim}\!\!\!\!\rightarrow X\xrightarrow{f}Y$. Since we have already shown that weak equivalences satisfy $(*)$, $\tilde f$ does as well, by the 2-out-of-3 property for isomorphisms, since the following diagram commutes:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, two heads, "\sim"] \arrow[d, "\tilde f"'] & X \arrow[d, "f"] \\
\widetilde{Y} \arrow[r, two heads, "\sim"] & Y
\end{tikzcd}$$

Then $[\tilde f]_*\colon \operatorname{ho}\mathcal{C}_{\mathrm{cof}}(-,\widetilde{X})\Rightarrow\operatorname{ho}\mathcal{C}_{\mathrm{cof}}(-,\widetilde{Y})$ is a natural isomorphism, so it follows from the Yoneda lemma that $[\tilde f]$ is an isomorphism in $\operatorname{ho}\mathcal{C}_{\mathrm{cof}}$, and hence a homotopy equivalence and a weak equivalence by Theorem 2.13. Suppose now that the following diagram commutes:

$$\begin{tikzcd}
\partial I^n \arrow[r, "u"] \arrow[d, "i_n"'] & X \arrow[d, "f"] \\
I^n \arrow[r, "v"'] & Y
\end{tikzcd}$$

Since ∂I^n is cofibrant by (Q4), we may choose lifts for the following squares:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, tail] & \widetilde{X} \arrow[d, two heads, "\wr"] \\
\partial I^n \arrow[r, "u"'] \arrow[ur, dashed, "\tilde u"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial I^n \arrow[r, "\tilde f\tilde u"] \arrow[d, tail, "i_n"'] & \widetilde{Y} \arrow[d, two heads, "\wr"] \\
I^n \arrow[r, "v"'] \arrow[ur, dashed, "\tilde v"] & Y
\end{tikzcd}$$

Then we obtain a map $\tilde h\colon I^n\to\widetilde{X}$ such that $\tilde h i_n=\tilde u$ and $\tilde f\circ\tilde h\sim\tilde v$ rel ∂I^n in the following diagram:

$$\begin{tikzcd}
\partial I^n \arrow[r, "\tilde u"] \arrow[d, "i_n"'] & \widetilde{X} \arrow[r, two heads] \arrow[d, "\tilde f"] & X \arrow[d, "f"] \\
I^n \arrow[r, "\tilde v"'] & \widetilde{Y} \arrow[r, two heads] & Y
\end{tikzcd}$$

Composing $\tilde h$ with $\widetilde{X}\xrightarrow{\sim}\!\!\!\!\rightarrow X$ gives the required map. □

Proposition 3.4 *Weak equivalences satisfy the* 2*-out-of-*3 *property.*

Proof This is immediate from Theorem 3.3 and the 2-out-of-3 property for isomorphisms. □

Theorem 3.5 *A map $f\colon X\to Y$ is a trivial fibration if and only if f is a weak equivalence and a fibration.*

Proof The forward direction is given in Lemma 2.9. Suppose instead that f is a weak equivalence and a fibration. Given a square of the form

$$\begin{tikzcd} \partial I^n \arrow[r, "u"] \arrow[d, hook, "i_n"'] & X \arrow[d, "f"] \\ I^n \arrow[r, "v"] & Y \end{tikzcd}$$

choose diagonal filler $w\colon I^n \to X$ such that $wi_n = u$ and a homotopy $H\colon I^n \times I \to Y$ from fw to v relative to ∂I^n. Since $i_n \hat{\times} j_0 \in \mathrm{cof}(\mathcal{J})$ by Proposition 2.10, we may choose a lift $\widetilde{H}\colon I^n \times I \to X$ for the following square:

$$\begin{tikzcd} \partial I^n \times I \amalg_{\partial I^n \times \{0\}} I^n \times \{0\} \arrow[r, "{[\mathrm{const}_u, w]}"] \arrow[d, hook, "i_n \hat{\times} j_0"'] & X \arrow[d, "f"] \\ I^n \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & Y \end{tikzcd}$$

Commutativity of this square comes from $He_0 = fw$ and H being relative to ∂I^n. Let $\widetilde{w} = \widetilde{H}e_1$. Then $\widetilde{w}i_n = \widetilde{H}i_n e_1 = \mathrm{const}_u e_1 = u$ and $f\widetilde{w} = He_1 = v$, so $\widetilde{w}$ is the required lift. □

Theorem 3.6 *A map $f\colon X \to Y$ is a trivial cofibration if and only if f is a weak equivalence and a cofibration.*

Proof It follows from (Q4) that $\mathrm{cof}(\mathcal{J}) \subseteq \mathrm{cof}(\mathcal{I})$, so every trivial cofibration is a cofibration. Suppose f is a trivial cofibration. Since X is fibrant, we obtain a lift $p\colon Y \to X$ in the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d, "!"] \\ Y \arrow[r, "!"'] \arrow[ur, dashed, "p"] & * \end{tikzcd}$$

such that $pf = \mathrm{id}_X$. We show that $fp \sim \mathrm{id}_Y$, from which it will follow from Theorem 2.13 that f is a weak equivalence. Since $f \hat{\times} i_1 \in \mathrm{cof}(\mathcal{J})$ by Proposition 2.10 and $Y \times \partial I$ is a coproduct $Y \amalg Y$ by (Q2), we obtain a lift $H\colon Y \times I \to Y$ in

$$\begin{tikzcd} X \times I \amalg_{X \times \partial I} (Y \times \partial I) \arrow[r, "{[\mathrm{const}_f, \mathrm{id}_Y + fp]}"] \arrow[d, hook, "f \hat{\times} i_1"'] & Y \arrow[d, "!"] \\ Y \times I \arrow[r, "!"'] \arrow[ur, dashed, "H"] & * \end{tikzcd}$$

By commutativity, H is a homotopy from fp to id_Y.

The converse follows immediately by the retract argument. □

We are now ready to show that the three classes given in Definitions 2.3 and 2.5 define a model structure on $\mathcal{C}$.

Proof of Theorem 2.6 By (Q6), $\mathcal{C}$ admits two weak factorization systems. The first, cofibrantly generated by $\mathcal{I}$, has cofibrations for its left class by definition and, by Theorem 3.5, the intersection of fibrations and weak equivalences for its right class. The second, cofibrantly generated by $\mathcal{J}$, has, by Theorem 3.6, the intersection of cofibrations and weak equivalences for its left class and fibrations for its right class by definition. Moreover, by Proposition 3.4, weak equivalences satisfy the 2-out-of-3 property. □

Homotopy groups in a good Q-structure

For the remainder of this section, we assume that $(\mathcal{C}, \iota)$ is a good Q-structure. For a space $X \in \mathcal{C}$, a *point* x_0 in X is a map $x_0\colon * \to X$; if there is a map $* \to X$, then X *admits points*. We will denote the composite $X \to * \xrightarrow{y_0} Y$ by const_{y_0} or just y_0. Let $\mathcal{C}_*$ be the slice category under $*$: its objects are pairs $(X, x_0\colon * \to X)$ and a morphism $f\colon (X, x_0) \to (Y, y_0)$ is a based map $f\colon X \to Y$ such that $f(x_0) = y_0$. Given two maps $f, g\colon (X, x_0) \to (Y, y_0)$, a homotopy between them in this category is a homotopy H from f to g relative to x_0. As in Section 1, this defines an equivalence relation on $\mathcal{C}_*((X, x_0), (Y, y_0))$.

Recall that, in (Q7), there is a point $(*)\colon * \to \partial I^n$ in the second pushout, which we will treat as a distinguished basepoint of ∂I^n to define homotopy groups. Such a point is given for all $n \geq 1$, though we will not make the distinction between $(*)$ for different n.

Definition 3.7 Let $n \geq 0$ and $x_0\colon * \to X$ a point in X. Let $\pi_n(X, x_0) = \mathcal{C}_*\big((\partial I^{n+1}, (*)), (X, x_0)\big)/{\sim}$ denote the set of based maps $(\partial I^{n+1}, (*)) \to (X, x_0)$ quotiented by homotopy rel $(*)$. By the pushout in (Q8), this is equivalently the set of all maps $f\colon I^n \to X$ such that $f i_n = \mathrm{const}_{x_0}$, quotiented by homotopy relative to ∂I^n. For $n \geq 1$, this set admits a group operation, with which it is the n^{th} *homotopy group* of X at x_0. The operation on $\pi_n(X, x_0)$ is defined by $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$, where $\alpha \cdot \beta$ is the induced map in

$$\begin{tikzcd}
I^{n-1} \arrow[r, "e_0"] \arrow[d, "e_1"'] & I^n \arrow[d, "\mathrm{id}_{I^{n-1}} \times v"'] \arrow[ddr, bend left, "\beta"] & \\
I^n \arrow[r, "\mathrm{id}_{I^{n-1}} \times u"'] \arrow[drr, bend right, "\alpha"'] & I^n \arrow[dr, dashed, "\alpha \cdot \beta"] & \\
& & X
\end{tikzcd}$$

since $\alpha e_1 = \beta e_0 = \mathrm{const}_{x_0}$.

Note that we are implicitly using (Q7) and (S3) in defining the group operation above to recognize $I^n \cong I^{n-1} \times I$. We will continue to make this identification without further reference. As in the topological case, $\pi_0(X, x_0)$ will not form a group, but rather acts as the set of path-connected components.

Theorem 3.8 *The operation on $\pi_n(X, x_0)$ defined above makes it a group for $n \geq 1$.*

Proof The identity is const_{x_0} by Lemma 1.8, associativity comes from Lemma 1.10, and the inverse of $[\alpha]$ is $[-\alpha]$ by Lemma 1.9. □

Lemma 3.9 *Let X be a space and $i\colon A \rightarrowtail X$ a cofibration. Suppose $f, g, h\colon X \to Y$ are continuous maps such that $f i = g i$. Given homotopies H from f to h and K from g to h, if $Hi \sim Ki$ rel endpoints then there is a homotopy from f to g that is relative to A.*

Proof Choose a map $\alpha\colon A\times I^2\to Y$ such that $\alpha\partial_L = Hi$, $\alpha\partial_T = \mathrm{const}_{hi}$, $\alpha\partial_R = Ki$ and $\alpha\partial_B = \mathrm{const}_{fi}$. By Proposition 2.10, $i \mathbin{\hat{\times}} \mathrm{inc}$ is in $\mathrm{cof}(\mathcal{J})$, where $\mathrm{inc}\colon \sqcap \to I^2$ is as in Definition 2.11. Since

$$\alpha\circ\mathrm{inc}\circ b = \alpha\circ -\partial_R = -Ki,\quad \alpha\circ\mathrm{inc}\circ aa = \alpha\partial_L = Hi,\quad \alpha\circ\mathrm{inc}\circ ab = \alpha\partial_T = \mathrm{const}_{hi},$$

the top map is defined, so there is a lift $\beta\colon X\times I^2\to Y$ in

$$\begin{tikzcd} A\times I^2 \amalg_{A\times\sqcap} X\times\sqcap \arrow[rr, "{[\alpha,[[Hi,\mathrm{const}_{hi}],-Ki]]}"] \arrow[d, "i\hat{\times}\mathrm{inc}"'] & & Y \\ X\times I^2 \arrow[urr, dashed, "\beta"'] & & \end{tikzcd}$$

The composite $\beta\partial_B$ gives a homotopy from f to g that is relative to A, since $\beta\partial_B i = \beta(i\times\mathrm{id}_{I^2})\partial_B = \alpha\partial_B = \mathrm{const}_{fi}$. □

Lemma 3.10 *Suppose $(\mathcal{C},\iota)$ is a good Q-structure and let $n\geq 1$. Then a map $u\colon\partial I^n\to X$ is homotopic to x rel $(*)$ if and only if there is a map $\tilde{u}\colon I^n\to X$ such that $\tilde{u}i_n = u$.*

Proof Suppose u is homotopic to the point $x = u(*)$ rel $(*)$. Then there is a homotopy $H\colon\partial I^n\times I\to X$ such that $He_0 = u$ and $He_1 = \mathrm{const}_x$. Thus, by the left pushout in (Q8), H factors through I^n to a map $\tilde{u}\colon I^n\to X$ such that $\tilde{u}i_n = He_0 = u$. Conversely, suppose that u admits an extension $\tilde{u}\colon I^n\to X$. Let $H\colon I^n\times I\to I^n$ be $H(s,t) = (1-t)s + t\cdot(*)$, the straight line homotopy from id_{I^n} to $(*)$, which satisfies $H\circ(*) = \mathrm{const}_{(*)}$. Then $\widetilde{H} = \tilde{u}Hi_n$ satisfies $\widetilde{H}e_0 = u$, $\widetilde{H}e_1 = u(*)$ and $\widetilde{H}(*) = \mathrm{const}_{u(*)} = \mathrm{const}_x$, so $\widetilde{H}$ is the required homotopy relative to $(*)$. □

Lemma 3.11 *Suppose $(\mathcal{C},\iota)$ is a good Q-structure and that X admits points. Let $f\colon X\to Y$ be continuous and n a positive integer. Then:*

(1) *f is 0-compressible if and only if $\pi_0 f$ is surjective.*

(2) *f is n-compressible if and only if, for all points x in X, $\pi_k f\colon\pi_k(X,x)\to\pi_k(Y,f(x))$ is injective for $k = n-1$ and surjective for $k = n$.*

Proof For (1), since $\partial I^0 = \varnothing$ and $I^0\cong *$ by (S1), the diagram in Definition 2.5 always admits a diagonal filler if and only if, for any point y in Y, we may choose a point x in X such that fx is homotopic to y, that is, $\pi_0 f$ is surjective. For (2), suppose first that f is n-compressible. Let $x\in X$ and fix $[\beta]\in\pi_n(Y,f(x))$. Regarding β as a map $I^n\to Y$ such that $\beta i_n = f(x)$, we get a diagonal filler $\alpha\colon I^n\to X$ for the square

$$\begin{tikzcd} \partial I^n \arrow[r, "x"] \arrow[d, hook, "i_n"'] & X \arrow[d, "f"] \\ I^n \arrow[r, "\beta"'] & Y \end{tikzcd}$$

where $\alpha i_n = x$, so $[\alpha]\in\pi_n(X,x)$, and $f\alpha\sim\beta$ rel ∂I^n, so $\pi_n f([\alpha]) = [\beta]$. Thus, $\pi_n f$ is surjective. For injectivity, when $n-1 = 0$, if $\pi_0 f[x] = [f(x)] = [f(x)'] = \pi_0 f[x']$ in $\pi_0 Y$ for $[x],[x']\in\pi_0 X$, then choosing a homotopy p from $f(x)$ to $f(x)'$ induces a diagonal filler in

$$\begin{tikzcd} \partial I \arrow[r, "{[x,x']}"] \arrow[d, hook, "i_1"'] & X \arrow[d, "f"] \\ I \arrow[r, "p"'] & Y \end{tikzcd}$$

since ∂I is a coproduct $* \amalg *$ by (Q2), which gives a homotopy from x to x'. When $n-1>0$, if $[f\alpha]=[\mathrm{const}_{f(x)}]$ for $[\alpha]\in\pi_{n-1}(X,x)$, choose an extension $\widetilde{f\alpha}\colon I^{n-1}\to Y$ given by Lemma 3.10, so there is a diagonal filler in

$$\begin{tikzcd} \partial I^{n-1} \arrow[r, "\alpha"] \arrow[d, hook, "i_{n-1}"'] & X \arrow[d, "f"] \\ I^{n-1} \arrow[r, "\widetilde{f\alpha}"'] & Y \end{tikzcd}$$

from which, by Lemma 3.10, $[\alpha]=0$; hence, $\pi_k f$ is injective. Conversely, suppose that π_k is injective for $k=n-1$ and surjective for $k=n$. Given the commutative square

$$\begin{tikzcd} \partial I^{n} \arrow[r, "u"] \arrow[d, hook, "i_{n}"'] & X \arrow[d, "f"] \\ I^{n} \arrow[r, "v"'] & Y \end{tikzcd}$$

let $x=u(*)$. By commutativity and Lemma 3.10, fu is homotopic to $\mathrm{const}_{f(x)}$ rel $(*)$, which, by injectivity of $\pi_{n-1}f$, gives that u is homotopic to const_x, say through a homotopy $H\colon \partial I^n\times I\to X$ rel $(*)$. Since $fu=vi_n$ and Y is fibrant, by Proposition 2.10 there is a lift in the diagram

$$\begin{tikzcd} \partial I^n\times I\cup I^n\times\{0\} \arrow[r, "{[fH,v]}"] \arrow[d, hook, "i_n\hat{\times} j_0"'] & Y \\ I^n\times I \arrow[ur, dashed, "J"'] & \end{tikzcd}$$

where the left map is in $\mathrm{cof}(\mathcal{J})$ by Proposition 2.10. Let $\tilde v=Je_1$. Since $\tilde v i_n=Ji_ne_1=fHe_1=\mathrm{const}_{f(x)}$, we can regard $\tilde v$ as a map on ∂I^{n+1} by the right pushout in (Q8), so $[\tilde v]\in\pi_n(Y,f(x))$. Since $\pi_n f$ is surjective, there is a $[\widetilde w]\in\pi_n(X,x)$ such that there is a homotopy $K\colon I^n\times I\to Y$ from $f\widetilde w$ to $\tilde v$ rel ∂I^n. Regarding $\widetilde w$ as a map on I^n, we obtain a lift in the diagram

$$\begin{tikzcd} \partial I^n\times I\cup I^n\times\{1\} \arrow[r, "{[H,\widetilde w]}"] \arrow[d, hook] & Y \\ I^n\times I \arrow[ur, dashed, "G"'] & \end{tikzcd}$$

since $He_1=\mathrm{const}_x$. Let $w=Ge_0$, so $wi_n=Gi_ne_0=He_0=u$, and $vi_n=fu=fwi_n$. Since $J\cdot{-K}$ is a homotopy from v to $f\widetilde w$, fG is a homotopy from fw to $f\widetilde w$ and

$$(J\cdot{-K})i_n=Ji_n\cdot{-Ki_n}=fH\cdot\mathrm{const}_{f(x)}\sim fH=fGi_n \text{ rel endpoints,}$$

we may apply Lemma 3.9 to get that $fw\sim v$ rel ∂I^n; hence, f is n-connected. □

Thus, we may conclude that, in a good Q-structure $(\mathcal{C},\iota)$, the definition of weak equivalences given in Definition 2.5 is equivalent to the classical definition given in terms of homotopy groups.

Theorem 3.12 *In a good Q-structure, if a space X admits points, then a map $f: X \to Y$ is n-connected if and only if, for all points x in X, $\pi_k f : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism for $k < n$ and a surjection for $k = n$. In particular, if X admits points, then a map $f: X \to Y$ is a weak equivalence if and only if $\pi_n f : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for every nonnegative integer n and every point x in X.* □

4 Example: subcategories of topological spaces

We begin by recovering the usual Quillen model structure on the category Top of topological spaces. From this, we will also be able to verify that the result holds for several subcategories of Top, as well as for pseudotopological spaces in PsTop and locales in Loc.

Theorem 4.1 *The category* Top *of topological spaces carries a model structure whose weak equivalences are the weak homotopy equivalences, fibrations are the Serre fibrations, and cofibrations are the Serre cofibrations.*

To prove the theorem, we will appeal to Theorem 2.6 with the obvious inclusion $\iota: \square \hookrightarrow \mathsf{Top}$. We begin however with a technical lemma, which verifies the only tricky axiom, (Q3):

Lemma 4.2 *For n and m nonnegative integers, $i_n \hat{\times} i_m \cong i_{n+m}$ and $i_n \hat{\times} j_m \cong j_{n+m}$ in the arrow category $\mathsf{Top}^{\to}$. Thus, in Notation 2.1, $\mathcal{I} \hat{\times} \mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I} \hat{\times} \mathcal{J} \subseteq \mathcal{J}$.*

Proof In Top, we may take the pushout product of two inclusions $A \hookrightarrow B$ and $C \hookrightarrow D$ to be the inclusion $A \times D \cup B \times C \hookrightarrow B \times D$. Thus, $i_n \hat{\times} i_m$ is the inclusion $\partial I^n \times I^m \cup I^n \times \partial I^m \hookrightarrow I^{n+m}$. Since $\partial I^n \times I^m$ is the subspace of I^{n+m} where at least one of the first n coordinates is either 0 or 1, and likewise for $I^n \times \partial I^m$ with the last m coordinates, we have $\partial I^n \times I^m \cup I^n \times \partial I^m \cong \partial I^{n+m}$. Thus, $i_n \hat{\times} i_m \cong i_{n+m}$. Similarly, $i_n \hat{\times} j_m$ is the inclusion $\partial I^n \times I^m \times I \cup I^n \times I^m \times \{0\} \hookrightarrow I^{n+m} \times I$. But $\partial I^n \times I \cup I^n \times \{0\} \cong I^n$, so $i_n \hat{\times} j_0 \cong j_0$ and thus, since $- \times I^m$ preserves colimits, $i_n \hat{\times} j_m \cong (I^n \times I^m \hookrightarrow I^{n+m} \times I) \cong j_{n+m}$. □

Proof of Theorem 4.1 Having fixed the inclusion $\iota: \square \hookrightarrow \mathsf{Top}$, (S1), (S2) and (S4) hold, and (S3) is given by letting $a(t) = \frac{1}{2}t$ and $b(t) = \frac{1}{2}(1+t)$. Moreover, (Q2) and (Q7) are obvious, (Q5) holds since ∂I^n and I^n are locally compact Hausdorff, and (Q6) is well known (see for instance [11, Proposition 2.4.2]). For (Q3), we then use Lemma 4.2.

(Q4) follows from the given pushouts

$$\begin{tikzcd} \partial I^n \arrow[r, hook, "i_n"] \arrow[d, hook, "i_n"'] & I^n \arrow[d] & \\ I^n \arrow[r, "\in \mathrm{cell}(\mathcal{I})"'] \arrow[ur, phantom, "\ulcorner"] & \partial I^{n+1} \arrow[r, hook, "i_{n+1}"'] & I^{n+1} \end{tikzcd} \qquad \begin{tikzcd} & \partial I^{n-1} \arrow[r, hook, "i_{n-1}"] \arrow[d] & I^{n-1} \arrow[d] \\ \varnothing \arrow[r, hook, "i_0"'] & * \arrow[r, "\in \mathrm{cell}(\mathcal{I})"'] \arrow[ur, phantom, "\ulcorner"] & \partial I^n \end{tikzcd}$$

Lastly, (Q8) is precisely the statement that, under the identifications $\partial I^n \times I / \partial I^n \times \{1\} \cong I^n$ and $I^{n-1}/\partial I^{n-1} \cong \partial I^n$, if we take ρ and σ to be the quotient maps, then any continuous map that is constant on fibers of the quotient factors through the quotient. By the description of the classes of maps, it is clear this is the classic Quillen model structure. $\square$

In a (co)reflective subcategory of Top, much of the work has already been done. In particular, if $\mathcal{C}$ is a (co)reflective subcategory of Top containing the CW complexes, then we may take $\iota\colon \square \hookrightarrow \mathcal{C}$ to be the canonical inclusion. In this case, the image of ι is naturally isomorphic to the image of $\square$ under the (co)reflection; hence, we may take each of the above pushouts to be the same as in Top. All that remains is to check that $X \times -$ preserves the pushout in (S3), as well as the conditions of (Q5) and (Q6). We introduce four (co)reflective subcategories which admit good Q-structures.

Let X be a topological space. Recall that a closed subset $F \subseteq X$ is *irreducible* if it cannot be written as the union of two proper closed subsets of F. That is, if $F = F_1 \cup F_2$ and F_1 and F_2 are closed, then either $F_1 = F$ or $F_2 = F$.

Definition 4.3 We take the following subcategories to all be full:

(1) Let cgHaus be the subcategory of compactly generated Hausdorff spaces.

(2) Let CGWH be the subcategory of compactly generated, weakly Hausdorff spaces.

(3) Let DTop be the subcategory of Δ-generated spaces. A Δ-generated space X is a topological space which has the final topology with respect to all maps from simplices $\Delta^n \to X$. There is an obvious functor Δ-ify: Top $\to$ DTop, which simply refines the topology on a space X to be the finest topology for which all maps $\Delta^n \to X$ are continuous. This forms an adjunction $\iota \dashv \Delta$-ify, making DTop a coreflective subcategory of Top [26, Corollary 1.4].

(4) Let Sob be the subcategory of sober spaces. A sober space X is a topological space such that every irreducible closed subset of X is the closure of exactly one point. In particular, every Hausdorff space is sober. There is a soberification functor sober: Top $\to$ Sob, which is a left adjoint to the inclusion $\iota\colon$ Sob $\hookrightarrow$ Top, making Sob a reflective subcategory of Top.

There is a chain of (co)reflections relating cgHaus and CGWH to Top, so the comments above apply to them. The categories cgHaus, CGWH and DTop are cartesian closed (see for instance [23; 24; 26], respectively). In Sob, the exponentiable spaces are precisely the locally compact spaces [12, Theorem VII.4.12]. Thus, in each of these categories, (Q5) is satisfied. A standard proof (such as the one in [11]) shows that (Q6) is also satisfied in cgHaus and CGWH, and DTop is locally presentable [5, Theorem 3.7] and hence admits the small object argument.

Most of the model structures on the above categories are well known (see for instance the proof in [10]). The only model structure that we are unaware of an existing reference for is on Sob, so we will carefully check the remaining axiom (Q6) there.

Lemma 4.4 *Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$. Then every map in* cell$(\mathcal{K})$ *is a closed T_1 inclusion; that is, if $f\colon X \to Y \in$* cell$(\mathcal{K})$*, then f is a closed map such that every point in $Y - fX$ is closed. Moreover, $Y - fX$ is Hausdorff in the subspace topology.*

Proof The first part is given in [11, Lemma 2.4.5]. For the second part, notice that Y/fX is a cell complex and hence Hausdorff, so the result follows since $Y - fX \cong Y/fX - [fX]$ as subspaces. $\square$

Lemma 4.5 *Coproducts of sober spaces, as formed in* Top, *are sober.*

Proof Let $\coprod X_\alpha$ be a coproduct in Top, where each X_α is sober. Clearly, any irreducible closed subset must be contained within exactly one component of $\coprod X_\alpha$, say X_β; hence, it is the closure of a single point since X_β is sober. $\square$

Lemma 4.6 *Suppose X is a space and $A, B \subseteq X$ are subsets such that X is the disjoint union of A and B as sets, A is closed in X, and every point of B is closed in X. If A is sober and B is Hausdorff under the subspace topology, then X is sober.*

Proof Let $F \subseteq X$ be an irreducible closed subset of X. Suppose F has at least two distinct points of B, say x and y. Then, since B is Hausdorff, there exist disjoint sets $U \subseteq B$ and $V \subseteq B$ that are open neighborhoods of x and y, respectively, in B. Since A is closed, B is open, so U and V are open in X. Then $F = (F - U) \cup (F - V)$ gives F as a union of two nonempty closed subsets of X, contradicting irreducibility. Thus, F has at most one point of B. If F intersected both A and B, then it would contain exactly one point of B, say x, and, since $\{x\}$ is closed in X, we would have $F = (F \cap A) \cup \{x\}$, contradicting irreducibility. Thus, F is either a single point of B or an irreducible closed subset of A; in either case, F is the closure of exactly one point since A is sober. $\square$

Corollary 4.7 *Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$, and let $\{k_\alpha\colon S_\alpha \to K_\alpha\}$ be a collection of maps in $\mathcal{K}$. If X is sober and the following is a pushout in* Top, *then Y is sober*:

$$\begin{tikzcd}
\coprod_\alpha S_\alpha \arrow[r, hook, "\coprod k_\alpha"] \arrow[d] & \coprod_\alpha K_\alpha \arrow[d] \\
X \arrow[r, hook, "f"'] & Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Proof This follows from Lemmas 4.4 and 4.6, given that Y is the disjoint union of fX and $Y - fX$ as sets. $\square$

In particular, this means that any time cells are attached to a sober space in Sob in a single step, we may take the resulting relative cell complex, as formed in Top, to be the pushout. In fact, this then holds for transfinite composites of such maps as well:

Lemma 4.8 *Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X = \operatorname{colim} X_\alpha$ be a transfinite composite of pushouts of coproducts of maps in $\mathcal{K}$, formed in* Top. *If X_0 is sober, then so is X.*

Proof Again, this holds by Lemmas 4.4 and 4.6. $\square$

Corollary 4.9 *In* Sob, *compact Hausdorff spaces are small relative to maps in* $\mathrm{cell}(\mathcal{I})$ *and* $\mathrm{cell}(\mathcal{J})$.

Proof By the above lemmas, transfinite composites of pushouts of maps in $\mathcal{I}$ and $\mathcal{J}$ are formed the same in Sob as in Top. Thus, the maps in $\mathrm{cell}(\mathcal{I}_{\mathsf{Sob}})$ and $\mathrm{cell}(\mathcal{J}_{\mathsf{Sob}})$ are precisely those in $\mathrm{cell}(\mathcal{I}_{\mathsf{Top}})$ and $\mathrm{cell}(\mathcal{J}_{\mathsf{Top}})$ whose domain is sober. The result follows from the usual statement in Top. $\square$

It is also easily verified that (S3), (Q7) and (Q8) hold in each of the categories in Definition 4.3, making each a good Q-structure with the canonical inclusions. Thus, we may apply Theorems 2.6 and 3.12 for each of the categories to get the following:

Theorem 4.10 *Each of the categories of*

- *compactly generated Hausdorff spaces* cgHaus,
- *compactly generated*, *weakly Hausdorff spaces* CGWH,
- Δ*-generated spaces* DTop,
- *sober spaces* Sob

admits a model structure by taking the weak homotopy equivalences, *Serre fibrations and relative cell complexes to be the weak equivalences*, *fibrations and cofibrations.* $\square$

Many of the model structures above are known to be Quillen equivalent to the standard model structure on Top. The only remaining one to our knowledge is the model structure on Sob, which we easily verify is Quillen equivalent as well:

Lemma 4.11 *A map $f: A \to B$ in* Sob *is a weak equivalence if and only if ιf is in* Top.

Proof We have characterized weak equivalences in both categories by maps from sober spaces. The result follows from Sob being full. $\square$

Theorem 4.12 *The adjunction* $\mathrm{sober} \dashv \iota$ *is a Quillen equivalence.*

Proof That it is a Quillen adjunction holds by construction, since $\mathrm{sober}(\mathcal{I}_{\mathsf{Top}}) \cong \mathcal{I}_{\mathsf{Sob}}$, and likewise with $\mathcal{J}$; hence, sober preserves (trivial) cofibrations since it preserves colimits. Thus, let $X \in \mathsf{Top}$ be cofibrant, which is sober as a Hausdorff space, and $Y \in \mathsf{Sob}$ be any sober space. By the following diagram and Lemma 4.11, $\mathrm{sober}\, X \to Y$ is a weak equivalence if and only if $X \to \iota Y$ is:

$$\begin{tikzcd} \iota \circ \mathrm{sober}\, X \arrow[r] & \iota Y \\ X \arrow[u, "\cong"] \arrow[ur] & \end{tikzcd}$$

$\square$

5 Example: pseudotopological spaces

Pseudotopological spaces are a generalization of topological spaces, and a more specialized type of convergence spaces. Rather than describing the structure in terms of open sets, one takes convergence of ultrafilters to be foundational. The category of pseudotopological spaces PsTop has several nice properties which Top lacks, most notably the fact that PsTop is quasitopos [7]. In fact, [8, Theorem 3.12] showed that PsTop is the cartesian closed topological hull of the category of pretopological spaces PreTop (also known as Čech closure spaces) in the category of convergence spaces Conv. In particular, since DiGraph and Graph both fully embed into PreTop [2, Proposition 1.56], the existence of a model structure on PsTop could be applicable to the homotopy theory of graphs and Čech closure spaces. We now properly introduce pseudotopological spaces and give a brief recollection of the necessary lemmas.

Definition 5.1 Let X be a set, $\mathbb{F}X$ be the set of filters on X, and βX be the set of ultrafilters on X. For each $x \in X$, let $x_\bullet = \{A \subseteq X \mid x \in A\}$ be the principal ultrafilter at x. A *pseudotopology* on X is a function $\lim_\xi \colon \beta X \to \mathcal{P}X$ such that $x \in \lim_\xi x_\bullet$ for all $x \in X$. A *pseudotopological space* is a pair $(X, \lim_\xi)$ such that $\lim_\xi$ is a pseudotopology on X.

Given a pseudotopology, we can define the limit of any filter $\mathcal{F}$ on X,

$$\lim_\xi \mathcal{F} = \bigcap_{\mathcal{U} \in \beta\mathcal{F}} \lim_\xi \mathcal{U},$$

where $\beta\mathcal{F} = \{\mathcal{U} \in \beta X \mid \mathcal{F} \subseteq \mathcal{U}\}$. By the ultrafilter lemma, $\beta\mathcal{F}$ is nonempty unless $\mathcal{F} = \mathcal{P}X$, in which case the empty intersection is understood to be X (that is, $\mathcal{P}X$ converges to every element of X).

Definition 5.2 Given pseudotopologies $\lim_\xi$ on X and $\lim_\zeta$ on Y, a function $f \colon X \to Y$ is *continuous* if $f(\lim_\xi \mathcal{F}) \subseteq \lim_\zeta f_*\mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}X$, where $f_*\mathcal{F} = \{B \subseteq Y \mid \text{there is } A \in \mathcal{F} \text{ such that } fA \subseteq B\}$ is the pushforward of $\mathcal{F}$. The category of pseudotopological spaces and continuous functions will be denoted by PsTop.

Definition 5.3 Given two pseudotopologies $\lim_\xi$ and $\lim_\zeta$ on X, $\lim_\xi$ is *finer* than $\lim_\zeta$ (and $\lim_\zeta$ is *coarser* than $\lim_\xi$), written $\zeta \leq \xi$, if $\lim_\xi \mathcal{U} \subseteq \lim_\zeta \mathcal{U}$ for all $\mathcal{U} \in \beta X$. Equivalently, $\lim_\xi$ is finer than $\lim_\zeta$ if and only if $\mathrm{id}_X \colon (X, \lim_\xi) \to (X, \lim_\zeta)$ is continuous.

Lemma 5.4 [4, Section III.6] *For any set X, the collection of pseudotopologies Ξ on X is a complete lattice.*

Proof Let $\mathcal{U}$ be an ultrafilter on X, and $\Gamma \subseteq \Xi$. Then an infimum is given by $\lim_{\bigwedge \Gamma} \mathcal{U} = \bigcup_{\xi \in \Gamma} \lim_\xi \mathcal{U}$ and a supremum is given by $\lim_{\bigvee \Gamma} \mathcal{U} = \bigcap_{\xi \in \Gamma} \lim_\xi \mathcal{U}$. □

The existence of a finest and a coarsest pseudotopology implies the existence of discrete and indiscrete functors from Set to PsTop. Indeed, these form the usual "discrete ⊣ forgetful ⊣ indiscrete" adjunctions, so any (co)limit in PsTop has, as its underlying set, the (co)limit set. Specifically, for the discrete pseudotopology, only the principal ultrafilter $x_\bullet$ converges to x, and, for the indiscrete pseudotopology, every ultrafilter converges to every point. Moreover, for any set of maps $\{f_k \colon X_k \to Y\}_{k \in K}$, we may take the *final pseudotopology* on Y, which is the finest pseudotopology for which each f_k is continuous [4, Section IV.3]. Dually, we may take the *initial pseudotopology* with respect to a set of maps with common domain, which is the coarsest pseudotopology for which each map is continuous.

Corollary 5.5 PsTop *is (co)complete.*

Proof Take the (co)limit of the underlying sets, and then take the initial (final) pseudotopology under the (co)limit cone. □

For any pair of pseudotopological spaces X and Y, there is a natural pseudotopology on $\mathsf{PsTop}(X,Y)$. We do not describe it here, instead referring the reader to [8, Theorem 3.7]:

Proposition 5.6 [8, Theorem 3.7] *The category* PsTop *is cartesian closed.* □

Every topology τ on X induces a pseudotopology, by taking $\lim_\tau \mathcal{U} = \{x \in X \mid \mathcal{N}(x) \subseteq \mathcal{U}\}$; that is, one takes the usual limits of the filter. Given a topology σ on Y, a function $f : X \to Y$ is continuous with respect to the topologies if and only if it is continuous with respect to the induced pseudotopologies. Thus, we get a full embedding $\iota\colon \mathsf{Top} \hookrightarrow \mathsf{PsTop}$. In the other direction, given any pseudotopology, we can take its topological modification.

Definition 5.7 Let $(X, \lim_\xi)$ be a pseudotopological space. A set $O \subseteq X$ is *open* if $\lim_\xi \mathcal{U} \cap O \neq \varnothing$ implies $O \in \mathcal{U}$. A set is *closed* if its complement is open.

One can verify that the collection of open sets $\tau\xi$ forms a topology, which is the *topological modification* of $\lim_\xi$; the topological modification of a space $(X, \lim_\xi)$ is $(X, \tau\xi)$. One can verify that this process preserves continuity, so this defines a functor $\tau\colon \mathsf{PsTop} \to \mathsf{Top}$ which is left adjoint to ι. This makes Top a reflective subcategory of PsTop. In particular, since the underlying sets and set functions have not changed, the unit is the identity map $\mathrm{id}_X : (X, \lim_\xi) \to (X, \tau\xi)$. Through this, we are able to verify most of the required axioms from the required identities holding in Top through a few lemmas.

Definition 5.8 [4, Definition III.3.1, Proposition IV.2.14] Let $(X, \lim_\xi)$ be a pseudotopological space and $i : A \hookrightarrow X$ a set inclusion. The *subspace pseudotopology*, denoted by $\lim_{\xi|_A}$, is the initial pseudotopology under i. Explicitly, this is given by $x \in \lim_{\xi|_A} \mathcal{F}$ if and only if $x \in \lim_\xi i_*\mathcal{F}$.

Lemma 5.9 [4, Proposition IV.2.15] *Let $f : (X, \lim_\xi) \to (Y, \lim_\zeta)$ be continuous and $A \subset X$ and $B \subset Y$ such that $fA \subseteq B$. Then $f|_A : (A, \lim_{\xi|_A}) \to (B, \lim_{\zeta|_B})$ is continuous.* □

Lemma 5.10 [4, Proposition V.4.24] *For any psuedotopological space $(X, \lim_\xi)$ and any set inclusion $i : A \hookrightarrow X$, the topological modification of the subspace pseudotopology, $\tau(\xi|_A)$, is finer than the subspace topology under the topological modification, $(\tau\xi)|_A$. That is, $\mathrm{id}_A : (A, \lim_{\tau(\xi|_A)}) \to (A, \lim_{(\tau\xi)|_A})$ is continuous.* □

Definition 5.11 Two collections of subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}X$ *mesh*, written $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \varnothing$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The *grill* of $\mathcal{A}$ is

$$\mathcal{A}^{\#} = \{B \subseteq X \mid A \cap B \neq \varnothing \text{ for all } A \in \mathcal{A}\}.$$

Definition 5.12 Let $(X, \lim_\xi)$ be a pseudotopological space. For $\mathcal{A} \subseteq \mathcal{P}X$, the *adherence* of $\mathcal{A}$ is

$$\operatorname{adh}_\xi \mathcal{A} = \bigcup_{\mathcal{F}\#\mathcal{A}} \lim_\xi \mathcal{F},$$

where the union is taken over all filters $\mathcal{F}$ that mesh with $\mathcal{A}$. Given two subsets $A, B \subseteq X$, A is *ξ-compact at B* if, for every filter $\mathcal{F}$, $A \in \mathcal{F}^{\#}$ implies $\operatorname{adh}_\xi \mathcal{F} \cap B \neq \varnothing$. A is *compact* if A is ξ-compact at A, and *relatively compact* if A is ξ-compact at X.

Note that, if A is ξ-compact at B and $B \subseteq C$, then A is ξ-compact at C. Moreover, if A and B are ξ-compact at C, then $A \cup B$ is ξ-compact at C. A topological space is compact in the above sense if and only if it is compact in the usual sense.

Lemma 5.13 [4, Proposition IX.1.26] *Let $f : (X, \lim_\xi) \to (Y, \lim_\zeta)$ be continuous. For $A, B \subseteq X$, if A is ξ-compact at B, then fA is ζ-compact at fB.* □

Proposition 5.14 [4, Corollary IX.1.18] *Let X be a compact pseudotopological space and Y a Hausdorff topological space. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.* □

Corollary 5.15 *Suppose*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i"] \\ C \arrow[r, "j"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout diagram in Top, *where B and C are compact and P is Hausdorff. Then P is the pushout in* PsTop.

Proof Let $\lim_\xi$ be the colimit pseudotopology on P, and τ the colimit topology. By Lemma 5.13, iB and jC are ξ-compact at P, so $P = iB \cup jC$ is compact. The result follows from Proposition 5.14, since, by the adjunction $\tau \dashv \iota$, the map $\mathrm{id}_P : (P, \lim_\xi) \to (P, \lim_\tau)$ is continuous. □

We are now ready to show that PsTop carries a model structure. Note that this is identical to the model structure defined in [22], but constructed in a different manner.

Theorem 5.16 *There is a model structure on* PsTop *whose cofibrations are maps in* $\operatorname{cof}(\mathcal{I})$, *fibrations are maps in* $\operatorname{rlp}(\mathcal{J})$, *and weak equivalences are maps which have the weak right lifting property against $\mathcal{I}$ or, equivalently by Theorem 3.12, maps which induce isomorphisms on all homotopy groups.*

Proof We are now ready to verify the axioms of Definition 2.2. That (S1) and (S4) hold is obvious, (S2) follows from ι being a right adjoint, and (S3) by Corollary 5.15 and Proposition 5.6. Thus, $(\mathcal{C}, \iota)$ is a category with intervals. For (Q2), note that $* \amalg *$ is given the finest pseudotopology for which the inclusions $* \hookrightarrow * \amalg *$ are continuous. Since these are always continuous, $* \amalg *$ must have the discrete topology; hence, $* \amalg * \cong \partial I$. The general case follows from Proposition 5.6. Next, (Q3) and (Q4) follow from Lemma 4.2 and Corollary 5.15, and (Q5) by Proposition 5.6. The verification of (Q6) is done in [22, Theorem 5.16]. Lastly, (Q7) follows from ι preserving limits, and (Q8) by Corollary 5.15. $\square$

Of course, since only limits of topological spaces are again topological in PsTop, not every cofibration in Top is necessarily a cofibration in PsTop. In fact, this need not hold for cofibrant cell complexes even.

Next we will verify that this model structure is Quillen equivalent to the standard model structure on Top, via the adjunction $\tau \dashv \iota$. We will first check that it is a Quillen adjunction as well as some intermediate lemmas.

Proposition 5.17 *The adjunction $\tau \dashv \iota$ is a Quillen adjunction.*

Proof Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$, and $\mathcal{K}_{\mathsf{Top}}$ and $\mathcal{K}_{\mathsf{PsTop}}$ be the respective generating (acyclic) cofibrations. Clearly, $\tau\mathcal{K}_{\mathsf{PsTop}} = \mathcal{K}_{\mathsf{Top}}$. Since τ is a left adjoint, it preserves colimits, so $\tau(\mathrm{cell}(\mathcal{K}_{\mathsf{PsTop}})) \subseteq \mathrm{cell}(\mathcal{K}_{\mathsf{Top}})$. Since functors preserve retracts in the arrow category, τ preserves (trivial) cofibrations. $\square$

Lemma 5.18 *A map $f : X \to Y$ between topological spaces is a weak equivalence in PsTop if and only if it is a weak equivalence in Top.*

Proof Since we have characterized weak equivalences by maps from topological spaces via lifting properties (and, for homotopies, ι preserves products), this follows from Top being a full subcategory of PsTop. $\square$

Let $\varnothing \rightarrowtail A$ be a cofibrant cell complex in PsTop, and fix a presentation of A as a transfinite composite, $A = \mathrm{colim}\, A_\alpha$. For each cell attached to A, there is a *characteristic map*, denoted by $\Phi_k : I^{n_k} \to A$, which is the composite $I^{n_k} \hookrightarrow \coprod_k I^{n_k} \to A_\alpha \to A$. A subset $S \subseteq A$ is a (finite) subcomplex if it can be written as a (finite) union, $S = \bigcup_k \Phi_k I^{n_k}$.

Lemma 5.19 *Let $(X, \lim_\xi)$ be a cofibrant cell complex in PsTop, and S a finite subcomplex of X. Then the subspace pseudotopology on S agrees with the subspace topology on τX.*

Proof Let $S = \bigcup_{k=1}^{N} \Phi_k I^{n_k}$ and let $\lim_{\xi|_S}$ be the subspace pseudotopology under $\lim_\xi$ and $\lim_{(\tau\xi)|_S}$ be the subspace topology under $\tau\xi$. By adjunction, $\mathrm{id}_S : (S, \lim_{\xi|_S}) \to (S, \lim_{\tau(\xi|_S)})$ is continuous, and, by Lemma 5.10, $\mathrm{id}_S : (S, \lim_{\tau(\xi|_S)}) \to (S, \lim_{(\tau\xi)|_S})$ is as well; thus their composite is continuous. Since τX is a cell complex in Top, it is Hausdorff, and since the restricted maps $\Phi_k : I^{n_k} \to (S, \lim_{\xi|_S})$ are continuous by Lemma 5.9, their image is compact in S. The result follows from Proposition 5.14: since S is a finite union of sets compact in $\lim_{\xi|_S}$, S itself is compact in $\lim_{\xi|_S}$, and $\lim_{(\tau\xi)|_S}$ is Hausdorff as a subspace of τX. $\square$

Theorem 5.20 *The adjunction $\tau \dashv \iota$ is a Quillen equivalence between the standard model structure on Top and the model structure on PsTop given in Theorem 5.16.*

Proof By [11, Corollary 1.3.16], since ι preserves weak equivalences by Lemma 5.18, it suffices to check that $\mathrm{id}_X \colon X \to \iota\tau X = \tau X$ is a weak equivalence for cofibrant X. The general case will follow from considering just $\mathcal{I}$-cell complexes. Suppose X is a cofibrant $\mathcal{I}$-cell complex and that the following diagram commutes:

$$\begin{tikzcd} \partial I^n \arrow[r, "u"] \arrow[d, "i_n"'] & X \arrow[d, "\mathrm{id}_X"] \\ I^n \arrow[r, "v"'] & \tau X \end{tikzcd}$$

Since τX is a cell complex in Top and vI^n is a compact subset, it is contained in a finite subcomplex, say S. By Lemma 5.19, the subspace topology under τX and subspace pseudotopology under X agree, so the inclusion maps $S \hookrightarrow X$ and $S \hookrightarrow \tau X$ are both continuous. Factoring v through S, we get the following diagram, which gives a strict lift:

$$\begin{tikzcd} \partial I^n \arrow[rr, "u"] \arrow[d, hook, "i_n"'] & & X \arrow[d, "\mathrm{id}_X"] \\ & S \arrow[ur, hook] \arrow[dr, hook] & \\ I^n \arrow[ur, dashed, "v"] \arrow[rr, "v"'] & & \tau X \end{tikzcd}$$

□

Of course, the approach in this paper is not suitable for every topological category. For instance, in the category PreTop of pretopological spaces [2; 21] (also referred to as Čech closure spaces), which is reflective in PsTop and contains Top as a reflective subcategory, most of the axioms assumed are suitable, except for the exponentiability requirement. In particular, [15, Theorem 3.4] showed that a pretopological space X is exponentiable if and only if every point in X has a smallest neighborhood, which of course means I is not exponentiable. Similarly, the only exponentiable T_0 uniform spaces are the discrete spaces [18, Corollary 3.4]; hence, Theorem 2.6 is not applicable to the category of uniform spaces either.

6 Example: locales

As the main application of Theorem 2.6, we will show that Loc admits a model structure. As with PsTop, we will first recall some basic theory about locales before verifying the assumptions of Definition 2.2.

Definition 6.1 A *frame* is a complete lattice $(L, \leq)$ which satisfies the infinitary distributive law

$$a \wedge \left(\bigvee_{k\in K} x_k\right) = \bigvee_{k\in K} (a \wedge x_k).$$

Given two frames L and M, a *frame homomorphism* is a monotone function $f \colon L \to M$ such that $f\left(\bigvee K\right) = \bigvee fK$ for all $K \subseteq L$ and $f\left(\bigwedge_{k=1}^n x_k\right) = \bigwedge_{k=1}^n f(x_k)$ for all $\{x_k\}_{k=1}^n \subseteq L$.

The category of frames and frame homomorphisms is denoted by Frm.

Definition 6.2 The category of locales, denoted by Loc, is the opposite of the category of frames, $\mathsf{Frm}^{\mathrm{op}}$.

Therefore, locales are frames, but a locale morphism $L \to M$ is a frame homomorphism $f: M \to L$. Since frames are complete lattices, there is a right adjoint $g: L \to M$ going in the localic direction; such a map is a *localic map*, which preserves all meets. When necessary, we distinguish between the left and right adjoint by writing f^L and f^R. Since a Galois connection $f: L \rightleftarrows M : g$ satisfies $fgf = f$ and $gfg = g$, f is injective if and only if g is surjective, and f is surjective if and only if g is injective.

Let $\Omega\colon \mathsf{Top} \to \mathsf{Loc}$ denote the functor taking a space X to its topology regarded as a frame, and a continuous map $f: X \to Y$ to a frame homomorphism $f^{-1}: \Omega Y \to \Omega X$. Restricting Ω to Sob, this becomes a full embedding which admits a right adjoint.

Proposition 6.3 [12, Corollary II.1.7] *There is an adjunction* $\Omega\colon \mathsf{Sob} \rightleftarrows \mathsf{Loc} : \mathrm{pt}$. *In this*, Ω *is a full embedding of* Sob *into* Loc, *making* Sob *a coreflective subcategory of* Loc. $\square$

We will take the restriction of Ω to $\square$ to be the inclusion ι.

Notation 6.1 We write $\mathcal{I}_{\mathsf{Loc}}$ and $\mathcal{J}_{\mathsf{Loc}}$ for the images of $\mathcal{I}$ and $\mathcal{J}$ under Ω, respectively.

A locale L is *spatial* if it is in the essential image of Ω, that is, if $L \cong \Omega X$ for some sober space X. As a left adjoint, Ω preserves colimits and, in particular, the pushouts described in the section on the model structure on Top.

For a frame L, a *prenucleus* is an order-preserving map $k_0: L \to L$ such that $x \le k_0(x)$ and $k_0(x) \wedge y \le k_0(x \wedge y)$ for all $x, y \in L$. A *nucleus* is a closure operator $k: L \to L$ that preserves binary meets. Any prenucleus k_0 generates a nucleus with the same fixed points as k_0 by letting $\mathrm{Fix}(k_0) \subseteq L$ be the set of fixed points of k_0 and defining $k(x) = \bigwedge\{y \in \mathrm{Fix}(k_0) \mid x \le y\}$. We will now fix a construction of the necessary limits and colimits in Frm and Loc.

Limits in Frm are formed as in Set, with the natural pointwise order on products: $(x_k) \le (y_k)$ if and only if $x_k \le y_k$ for all k. We use a description of binary coproducts in Frm as given by Banaschewski [1] and Chen [3]. For a frame L and $S \subseteq L$, let ${\downarrow}S = \{x \in L \mid x \le s \text{ for some } s \in S\}$. Define the downset functor $\mathfrak{D}\colon \mathsf{Frm} \to \mathsf{Frm}$ by $\mathfrak{D}X = \{U \subseteq \mathcal{P}X \mid U \text{ downward closed}\}$ as the set of downsets (ordered by inclusion, with intersection for meets and unions for joins) and $\mathfrak{D}f(U) = {\downarrow}f(U)$. To distinguish downsets from other sets, we will denote them by ${\downarrow}S$ from here on. Given two frames L_1 and L_2, let $L_1 \times L_2$ be their product. Define the following prenuclei on $\mathfrak{D}(L_1 \times L_2)$:

$$\sigma_0(U) = \left\{\bigvee D \mid D \subseteq U \text{ up-directed}\right\},$$
$$\pi_1(U) = \left\{\left(\bigvee X, y\right) \mid X \subseteq L_1,\ X \times \{y\} \subseteq U\right\},$$
$$\hat{\pi}_2(U) = \left\{\left(x, \bigvee Y\right) \mid Y \subseteq L_2 \text{ finite}, \{x\} \times Y \subseteq U\right\}.$$

Let σ be the associated nucleus of σ_0 and let $\pi = \sigma \circ \pi_1 \circ \hat{\pi}_2$. Then the set $\mathrm{Fix}(\pi)$ of fixed elements, also called π-saturated elements, constitutes the coproduct of L_1 and L_2 in Frm, denoted by $L_1 \otimes L_2$. For a downset ${\downarrow}S$ to be π-saturated means that, if $x \in L_1$ is such that $\{(x, y_\alpha)\} \subset {\downarrow}S$, then $\left(x, \bigvee y_\alpha\right) \in {\downarrow}S$, and likewise for joins in L_1. The coproduct inclusions $\iota_i: L_i \to L_1 \otimes L_2$ are given by $\iota_1(x) = \{(a, b) \mid a \le x\} \cup \bar{n}$, where $\bar{n} = \{(a,b) \mid a = 0 \text{ or } b = 0\}$, and likewise for ι_2. The elements of the form $\iota_1(x) \cap \iota_2(y) = {\downarrow}(x,y) \cup \bar{n}$ are denoted by $x \otimes y$. In particular, due to the possibility of an empty index set for π-saturation, $\bigl(x, \bigvee \varnothing\bigr) = (x,0) \in {\downarrow}S$, and similarly $(0,y) \in {\downarrow}S$ for all ${\downarrow}S \in L_1 \otimes L_2$, so $\bar{n}$ is the least element of $L_1 \otimes L_2$.

For arbitrary ${\downarrow}S \in L_1 \otimes L_2$, we have the identity ${\downarrow}S = \bigvee_{a\otimes b \le {\downarrow}S} a \otimes b = \bigcup_{a \otimes b \le {\downarrow}S} a \otimes b$; moreover, for $\{x_\alpha\} \subseteq L_1$ and $y \in L_2$, we have $\bigvee (x_\alpha \otimes y) = \bigl(\bigvee x_\alpha\bigr) \otimes y$ [19, Proposition IV.5.2]. Given a frame homomorphism $f : L_2 \to M$, the map $\mathrm{id}_{L_1} \otimes f : L_1 \otimes L_2 \to L_1 \otimes M$ satisfies

$$(\mathrm{id}_{L_1} \otimes f)(1 \otimes x) = (\mathrm{id}_{L_1} \otimes f)\iota_2(x) = \iota_2 \circ f(x) = (1 \otimes f(x)),$$

and likewise for ι_1; thus, $(\mathrm{id}_{L_1} \otimes f)(a \otimes b) = a \otimes f(b)$ and so $(\mathrm{id}_{L_1} \otimes f)({\downarrow}S) = \bigvee_{a \otimes b \le {\downarrow}S} a \otimes f(b)$.

Lastly, we may form the pushout of a span $B \xleftarrow{f} A \xrightarrow{g} C$ in Loc by taking the underlying set P to be the pullback of the left adjoints in Frm, $P = \{(b,c) \in B \times C \mid f^L(b) = g^L(c)\}$. In Frm, the projection maps are the obvious projections, so the pushout legs in Loc are the right adjoints, namely $i_B : B \to P$ given by $i_B(x) = \bigvee \{(b,c) \in P \mid b = i_B^L(b,c) \le x\}$, and analogously for i_C. Note that, since pullbacks of injections (surjections) are again injections (surjections) in Frm since they are in Set, the analogous statement holds for pushouts in Loc.

Proposition 6.4 [19, Corollary IV.4.3.5] *The categories* Frm *and* Loc *are bicomplete.* □

The coproduct in Frm distributes over arbitrary products or, equivalently, the product in Loc distributes over arbitrary coproducts.

Proposition 6.5 [14, Proposition I.5.2] *Let L be a frame and $\{M_k\}_{k\in K}$ be a collection of frames. Then $L \otimes \prod_{k\in K} M_k \cong \prod_{k \in K}(L \otimes M_k)$ is a product with projections $\mathrm{id}_L \otimes p_k$, where p_k are the projections from $\prod_{k\in K} M_k$.* □

Proposition 6.6 [12, Proposition II.2.13] *If X and Y are sober spaces with X locally compact, then $\Omega(X \times Y) \cong \Omega X \otimes \Omega Y$, where $\otimes$ is the product in* Loc. □

The initial object in Frm is the poset $T = \{0 < 1\}$: since every frame homomorphism preserves 0 and 1, this uniquely determines a morphism out of T. Terminal objects in Frm are trivial posets $*$: the constant map from any frame is clearly a frame homomorphism. Note that, since $0 = 1$ in $*$, there are no maps out of $*$, except to singletons. Considering the dual then, any singleton is an initial object in Loc, and T is the terminal object in Loc. From this, it is clear that $\Omega\varnothing$ is initial and $\Omega *$ is terminal. In particular, since there are no maps into $\Omega\varnothing$ except the identity, any product with $\Omega\varnothing$ is again $\Omega\varnothing$; hence, $\Omega\varnothing$ satisfies (S4).

In a frame, a is *way below* b, written $a \ll b$, if, whenever $b = \bigvee K$, there is a finite subset $K' \subseteq K$ such that $a \le \bigvee K'$. A frame is *continuous* or *locally compact* if $a = \bigvee \{x \in L \mid x \ll a\}$ for all $a \in L$; a locale is locally compact if it is locally compact as a frame. Note that, if X is a locally compact Hausdorff space, then ΩX is locally compact. Johnstone characterizes the exponentiable locales as the locally compact locales, as described in the following proposition:

Proposition 6.7 [12, Theorem VII.4.11] *A locale is exponentiable in* Loc *if and only if it is locally compact.* □

Hence, the images of ∂I^n and I^n under Ω are exponentiable in Loc. Let us now verify that $\mathcal{I}_{\mathsf{Loc}}$ and $\mathcal{J}_{\mathsf{Loc}}$ admit the small object argument.

Lemma 6.8 *Let $X_0 \to X = \operatorname{colim} X_n$ be an ω-transfinite composite of injective localic maps. Denote the colimit inclusions by $i_n \colon X_n \hookrightarrow X$. A map $f : A \to X$ factors through X_N if and only if $fA \subseteq i_N X_N$.*

Proof The forward direction is clear. Suppose conversely that $fA \subseteq i_N X_N$ for some positive integer N, so for each $a \in A$ there is $x_a \in X_N$ with $f(a) = i_N(x_a)$. Define $\bar{f} : A \to X_N$ by $\bar{f}(a) = x_a$. Clearly, $i_N \bar{f}(a) = f(a)$, so, since i_N is injective and both f and i_N are localic maps, $\bar{f}$ is a localic map as well by [19, IV.1.1(1)]. □

In the following, we write an element of $\coprod \Omega X_\alpha$ as a union $\bigcup U_\alpha$, where U_α is an open subset of X_α.

Proposition 6.9 *Let K be a compact Hausdorff space. Then ΩK is small relative to transfinite composites of pushouts of coproducts of maps in $\mathcal{I}_{\mathsf{Loc}}$ and $\mathcal{J}_{\mathsf{Loc}}$ in* Loc.

Proof Let $X_0 \to X = \operatorname{colim} X_n$ be an ω-transfinite composite of pushouts of coproducts of maps in $\mathcal{K}_{\mathsf{Loc}}$, where $\mathcal{K}_{\mathsf{Loc}} = \{\Omega i_n \colon \Omega S_n \to \Omega C_n\}_{n \geq 0}$ is either $\mathcal{I}_{\mathsf{Loc}}$ or $\mathcal{J}_{\mathsf{Loc}}$. Note that, when $X_0 \cong \Omega *$, any map $f : \Omega K \to X$ factors through some X_n since it does in Sob and Ω preserves colimits. We can reduce the general case to this as follows. Let $Y_0 = \Omega *$, and inductively define Y_{n+1} as the pushout

$$\begin{tikzcd}
\coprod \Omega S_{n_\alpha} \arrow[r, "\sigma"] \arrow[d, hook, "\coprod \Omega i_{n_\alpha}"'] & X_n \arrow[r, "p_n"] \arrow[d, hook, "i_X"] & Y_n \arrow[d, hook, "i_Y"] \\
\coprod \Omega C_{n_\alpha} \arrow[r] & X_{n+1} \arrow[r, dashed, "p_{n+1}"'] & Y_{n+1}
\end{tikzcd}$$

where the left square is the defining pushout for X_{n+1} and both the outer and right squares are pushouts. Explicitly, since $\left(\coprod \Omega i_{n_\alpha}\right)^L\left(\bigcup U_\alpha\right) = \left(\coprod i_{n_\alpha}\right)^{-1}\left(\bigcup U_\alpha\right) = \bigcup (U_\alpha \cap S_{n_\alpha})$ for $\bigcup U_\alpha \subseteq \bigcup C_{n_\alpha}$, we obtain

$$X_{n+1} = \left\{\left(x, \bigcup U_\alpha\right) \in X_n \times \coprod \Omega C_{n_\alpha} \mid \sigma^L(x) = \bigcup (U_\alpha \cap S_{n_\alpha})\right\},$$
$$Y_{n+1} = \left\{\left(y, x, \bigcup U_\alpha\right) \in Y_n \times X_{n+1} \mid p_n^L(y) = i_X^L\left(x, \bigcup U_\alpha\right) = x\right\}.$$

For $x \in X_n$, let $\mathfrak{m}(x) = \sigma^L(x) \cup \bigcup (C_{n_\alpha} - S_{n_\alpha})$, which is open since each i_{n_α} is a closed map. Then $i_X(x) = \bigvee \left\{\left(\tilde{x}, \bigcup V_\alpha\right) \in X_{n+1} \mid \tilde{x} \leq x\right\}$; clearly $(x, \mathfrak{m}(x))$ is the maximum within this set, so $i_X(x) = (x, \mathfrak{m}(x))$. Similarly, $i_Y(y) = \left(y, p_n^L(y), \mathfrak{m}(p_n^L(y))\right)$. Repeating this process for all n, we get an induced map $\hat{p} \colon X \to Y = \operatorname{colim} Y_n$ in the diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d, "p_0"'] & X_1 \arrow[r, hook] \arrow[d, "p_1"'] & X_2 \arrow[r, hook] \arrow[d, "p_2"'] & \cdots \arrow[r, hook] & X \arrow[d, "\hat{p}"] \\
Y_0 \arrow[r, hook] & Y_1 \arrow[r, hook] & Y_2 \arrow[r, hook] & \cdots \arrow[r, hook] & Y
\end{tikzcd}$$

By Lemma 6.8, a map $f\colon \Omega K \to X$ factors through X_N if and only if $f(\Omega K)$ is contained in X_N, identified with its inclusion into X. Suppose there was a map $f\colon \Omega K \to X$ that did not factor through any X_n. Necessarily, then, $f(\Omega K)$ intersects the image of $X_{n+1} - i_X X_n$ for arbitrarily large n; we will show that this holds for Y_n as well.

If $\bigl(x, \bigcup U_\alpha\bigr) \in f(\Omega K) \cap (X_{n+1} - i_X X_n)$, then necessarily $\bigcup U_\alpha \neq \mathfrak{m}(x)$. Since $\sigma^L(x) = \bigcup (U_\alpha \cap S_{n_\alpha})$, we have $\bigcup (U_\alpha \cap S_{n_\alpha}) = \mathfrak{m}(x) \cap \bigcup S_{n_\alpha}$; hence, there must be a point $t \in (\mathfrak{m}(x) - U_\alpha) \cap \bigl(\bigcup (C_{n_\alpha} - S_{n_\alpha})\bigr)$. In particular, $t \in \mathfrak{m}(x')$ for any $x' \in X_n$ since $\bigcup (C_{n_\alpha} - S_{n_\alpha}) \subseteq \mathfrak{m}(x')$. Let

$$\Bigl(\tilde{y}, \tilde{x}, \bigcup V_\alpha\Bigr) = p_{n+1}\Bigl(x, \bigcup U_\alpha\Bigr) = \bigvee \Bigl\{\Bigl(y', x', \bigcup V'_\alpha\Bigr) \in Y_{n+1} \mid \Bigl(x', \bigcup V'_\alpha\Bigr) \leq \Bigl(x, \bigcup U_\alpha\Bigr)\Bigr\}.$$

Then $\bigcup V_\alpha \subseteq \bigcup U_\alpha$, so $t \notin V_\alpha$. But, by the description of i_Y, if $(\tilde{y}, \tilde{x}, \bigcup V_\alpha) = i_Y(y)$ for some $y \in Y_n$, then $\bigcup V_\alpha = \mathfrak{m}(p_n^L(y))$, which cannot be true since $t \in \mathfrak{m}(p_n^L(y))$. Thus, $p_{n+1}\bigl(x, \bigcup U_\alpha\bigr) \notin i_Y Y_n$.

Hence, $\hat{p} f(\Omega K)$ intersects $Y_{n+1} - i_Y Y_n$ for arbitrarily large n too, which is a contradiction since $\hat{p} f$ must factor through some Y_N. Thus, no such f exists, so all maps $f\colon \Omega K \to X$ factor through some X_n. □

The only remaining condition to check is that products with L preserve the pushout in (S3) for any locale L, which we will verify the dual statement for in Frm. This however requires a few lemmas. Recall that, as in the beginning of Section 4, $a(t) = \frac{1}{2}t$ and $b(t) = \frac{1}{2}(t+1)$ are pushout legs $I \to I$ such that a restricts to a homeomorphism $[0,1) \to \bigl[0, \frac{1}{2}\bigr)$ and b restricts to a homeomorphism $(0,1] \to \bigl(\frac{1}{2}, 1\bigr]$.

Lemma 6.10 [3, Proposition 2.2] *If N is a continuous frame and $a \otimes b \leq \pi(U)$ in $L \otimes N$, then, for any $c \ll b$, we have $(a, c) \in \pi_1 \hat{\pi}_2(U)$.* □

Lemma 6.11 *Let L be a frame and ${\downarrow} S \in L \otimes \Omega I$. Then the following hold:*

1. *$\mathfrak{D}(\mathrm{id}_L \times a^{-1})({\downarrow} S) = (\mathrm{id}_L \times a^{-1})_*({\downarrow} S) = \{(x, a^{-1}U) \mid (x, U) \in {\downarrow} S\}$. That is, the image of ${\downarrow} S$ under $\mathrm{id}_L \times a^{-1}$ is a downset.*
2. *$(\mathrm{id}_L \times a^{-1})_*({\downarrow} S) = \hat{\pi}_2((\mathrm{id}_L \times a^{-1})_*({\downarrow} S)) = \pi_1((\mathrm{id}_L \times a^{-1})_*({\downarrow} S))$. In particular, $(\mathrm{id}_L \times a^{-1})_*({\downarrow} S)$ is fixed under $\pi_1 \hat{\pi}_2$.*
3. *$\mathrm{id}_L \otimes a^{-1}({\downarrow} S) \leq \pi \circ (\mathrm{id}_L \times a^{-1})_*({\downarrow} S)$.*

Analogous statements hold for b^{-1} in place of a^{-1}.

Proof Since ${\downarrow} S \in L \otimes \Omega I$, it is fixed under π; hence, if $\{(x, U_\alpha)\} \subseteq {\downarrow} S$, then $(x, \bigcup U_\alpha) \in {\downarrow} S$. For (1) then, suppose $(x, U) \in \mathfrak{D}(\mathrm{id}_L \times a^{-1})({\downarrow} S) = {\downarrow}[(\mathrm{id}_L \times a^{-1})({\downarrow} S)]$. Then there is $(y, V) \in {\downarrow} S$ such that $(x, U) \leq (y, a^{-1}V)$, so $x \leq y$ and $U \subseteq a^{-1}V$. If $1 \notin U$, then aU is open, $aU \subseteq aa^{-1}V \subseteq V$ and $a^{-1}aU = U$, so $(x, aU) \in {\downarrow} S$ satisfies $(\mathrm{id}_L \times a^{-1})(x, aU) = (x, U)$. Otherwise, if $1 \in U$ and hence $\frac{1}{2} \in V$, then $W = aU \cup \bigl(\bigl[\frac{1}{2}, 1\bigr] \cap V\bigr)$ is open and satisfies $a^{-1}W = U$ and $W \subseteq V$, so $(x, W) \in {\downarrow} S$ satisfies $(\mathrm{id}_L \times a^{-1})(x, W) = (x, U)$. Thus, the direction $\subseteq$ holds, and the other inclusion is clear, so they are equal.

For the remainder of the proof, let $\mathcal{S} = (\mathrm{id}_L \times a^{-1})_*({\downarrow} S)$. For (2), suppose first that $(x, V) \in \hat{\pi}_2 \mathcal{S}$. Then there must be a finite subset $\{a^{-1}U_n\}_{n=1}^N \subseteq \Omega I$ such that $\{x\} \times \{a^{-1}U_n\}_{n=1}^N \subseteq \mathcal{S}$ and $\bigcup_{n=1}^N a^{-1}U_n = V$.

Since each $(x, a^{-1}U_n)$ is in $\mathcal{S}$, $(x, U_n) \in {\downarrow}S$, so, since ${\downarrow}S$ is π-saturated, $\left(x, \bigcup_{n=1}^N U_n\right) \in {\downarrow}S$ and $(x, V) = \left(x, a^{-1}\left(\bigcup_{n=1}^N U_n\right)\right) \in \mathcal{S}$. Thus, $\hat{\pi}_2\mathcal{S} \subseteq \mathcal{S}$. Similarly, if $\left(\bigvee x_\alpha, a^{-1}U\right) \in \pi_1\mathcal{S}$ for some $\{(x_\alpha, a^{-1}U)\} \subseteq \mathcal{S}$, then $(x_\alpha, U) \in {\downarrow}S$ for each α; hence, by π-saturation, $\left(\bigvee x_\alpha, U\right) \in {\downarrow}S$ and $\left(\bigvee x_\alpha, a^{-1}U\right) \in \mathcal{S}$. Thus, $\pi_1\mathcal{S} \subseteq \mathcal{S}$. Since the reverse inclusions are clear, the equalities hold.

Lastly, for (3), for each $(x, U) \in {\downarrow}S$, we have $(x, a^{-1}U) \in \mathcal{S}$, so also in $\pi\mathcal{S}$, so $\mathrm{id}_L \otimes a^{-1}(x, U) = x \otimes a^{-1}U \le \pi\mathcal{S}$. Since this holds for all such (x, U),

$$\mathrm{id}_L \otimes a^{-1}({\downarrow}S) = \bigvee_{x\otimes U \le {\downarrow}S} x \otimes a^{-1}U \le \pi\mathcal{S}. \qquad \square$$

Lemma 6.12 *Let $f\colon M \to N$ be a frame homomorphism, and suppose $\{n_k\}_{k\in K} \subseteq N$ is such that every $n \in N$ is of the form $n = \bigvee_{k\in K'} n_k$ for some $K' \subseteq K$. If $\{n_k\}_{k\in K}$ is in the image of f, then f is surjective.*

Proof Suppose that $f(m_k) = n_k$ for each $k \in K$. Fix $n \in N$, and let $K' \subseteq K$ be such that $n = \bigvee_{k\in K'} n_k$. Then $\bigvee_{k\in K'} m_k$ satisfies $f\left(\bigvee_{k\in K'} m_k\right) = \bigvee_{k\in K'} f(m_k) = \bigvee_{k\in K'} n_k = n$. $\square$

We will identify $L \otimes \{0 < 1\}$ with L and write χ_k for $\mathrm{id}_L \otimes i_k\colon L \otimes \Omega I \to L$ for $k = 0, 1$. For $x \in L$ and $U \in \Omega I$, we have $\chi_k(x \otimes U) = x$ if $k \in U$ and $\chi_k(x \otimes U) = 0$ if $k \notin U$; for a general ${\downarrow}S \in L \otimes \Omega I$, $\chi_k({\downarrow}S) = \bigvee\{x \in L \mid \text{there is } U \in \Omega I \text{ with } k \in U \text{ and } x \otimes U \le {\downarrow}S\}$.

Proposition 6.13 *For any frame L, the following is a pullback square in* Frm:

$$\begin{tikzcd}
L \otimes \Omega I \arrow[r, "\mathrm{id}_L \otimes a^{-1}"] \arrow[d, "\mathrm{id}_L \otimes b^{-1}"'] & L \otimes \Omega I \arrow[d, "\chi_1"] \\
L \otimes \Omega I \arrow[r, "\chi_0"'] & L
\end{tikzcd}$$

Proof Let P be the canonical pullback of the square (constructed as a subset of the product); since limits in Frm have the corresponding limit in Set for their underlying set, we may take

$$P = \{({\downarrow}S, {\downarrow}T) \subseteq (L \otimes \Omega I)^2 \mid \chi_1({\downarrow}S) = \chi_0({\downarrow}T)\}$$

along with the usual projections pr_1 and pr_2. Let $\phi\colon L \otimes \Omega I \to P$ be the induced map, which is given by $\phi({\downarrow}S) = (\mathrm{id}_L \otimes a^{-1}({\downarrow}S), \mathrm{id}_L \otimes b^{-1}({\downarrow}S))$. We will show that ϕ is a bijection, and hence an isomorphism.

By Lemma 6.12, it suffices to show every $(x \otimes U, y \otimes V)$ is in the image of ϕ for surjectivity, since every element of P can be written as a join of such elements. Indeed, let $({\downarrow}S, {\downarrow}T) \in P$. For each $x \otimes U \le {\downarrow}S$, if $1 \notin U$, then let $t_{x\otimes U} = 0$, so $(x \otimes U, t_{x\otimes U}) \in P$. Otherwise, if $1 \in U$, then let $\mathcal{T} = \{y \otimes V \le {\downarrow}T \mid 0 \in V\}$, so $((x \wedge y) \otimes U, (x \wedge y) \otimes V) \in P$ for each $y \otimes V \in \mathcal{T}$. Since $\chi_0({\downarrow}T) = \chi_1({\downarrow}S)$ and this holds for all such $y \otimes V$,

$$\begin{aligned}
\bigvee_{y\otimes V\in\mathcal{T}} ((x \wedge y) \otimes U, (x \wedge y) \otimes V) &= \left(\left(\bigvee_{y\otimes V\in\mathcal{T}} (x \wedge y)\right) \otimes U, t_{x\otimes U}\right) \\
&= \left(\left(x \wedge \bigvee_{y\otimes V\in\mathcal{T}} y\right) \otimes U, t_{x\otimes U}\right) \\
&= \left((x \wedge \chi_1({\downarrow}S)) \otimes U, t_{x\otimes U}\right) = (x \otimes U, t_{x\otimes U}) \le ({\downarrow}S, {\downarrow}T),
\end{aligned}$$

where $t_{x\otimes U} = \bigvee_{y\otimes V\in\mathfrak{T}}((x\wedge y)\otimes V)$; the first equality holds since $\bigvee(x_\alpha\otimes y) = \big(\bigvee x_\alpha\big)\otimes y$, the second by distributivity, and the last since $x \le \chi_1(\downarrow S)$. Thus, whether $1\in U$ or not, $(x\otimes U, t_{x\otimes U})$ can be written as a join of pairs of tensor elements, and hence so can

$$\bigvee_{x\otimes U\le\downarrow S}(x\otimes U, t_{x\otimes U}) = \big(\bigvee_{x\otimes U\le\downarrow S} x\otimes U, t_{\downarrow S}\big) = (\downarrow S, t_{\downarrow S})\in P,$$

where $t_{\downarrow S} = \bigvee_{x\otimes U\le\downarrow S} t_{x\otimes U} \le \downarrow T$. Analogous reasoning for $\downarrow T$ shows $(\downarrow S,\downarrow T) = (\downarrow S, t_{\downarrow S})\vee(s_{\downarrow T},\downarrow T)$ can be written as a join of elements of the form $(x\otimes U, y\otimes V)$.

For surjectivity then, fix an element $(x\otimes U, y\otimes V)\in P$. Suppose first that $1\in U$. If $x=0$, then $\chi_1(x\otimes U) = 0 = \chi_0(y\otimes V)$; thus, either $0\notin V$ or $y=0$. If $y=0$, then $\phi(0) = (0,0) = (x\otimes U, y\otimes V)$, so suppose not. Then, since $0\notin V$, bV is open in I and $a^{-1}bV$ is empty, so $y\otimes bV$ satisfies

$$\phi(y\otimes bV) = (y\otimes a^{-1}bV, y\otimes b^{-1}bV) = (0, y\otimes V) = (x\otimes U, y\otimes V).$$

Suppose now that $x\ne 0$. Then $x = \chi_1(x\otimes U) = \chi_0(y\otimes V)$; necessarily then $0\in V$ and $y = x$. Then $C = aU\cup bV$ is open in I and satisfies $a^{-1}C = U$ and $b^{-1}C = V$, so $\phi(x\otimes C) = (x\otimes U, y\otimes V)$. Suppose now that $1\notin U$, so aU is open in I and $b^{-1}aU$ is empty. If $y=0$, then $\phi(x\otimes aU) = (x\otimes a^{-1}aU, 0) = (x\otimes U, y\otimes V)$. If $y\ne 0$, then, since $\chi_0(y\otimes V) = \chi_1(x\otimes U) = 0$, it must be that $0\notin V$. Thus, aU and bV are both open in I and $a^{-1}bV$ is empty, so

$$\phi((x\otimes aU)\vee(y\otimes bV)) = \phi(x\otimes aU)\vee\phi(y\otimes bV) = (x\otimes U,0)\vee(0,y\otimes V) = (x\otimes U, y\otimes V).$$

Thus, in any case we see that $(x\otimes U, y\otimes V)$ is in the image of ϕ; hence, ϕ is surjective.

For injectivity, let $\downarrow S\ne\downarrow T$ in $L\otimes\Omega I$. Without loss of generality, assume there is $(x,U)\in\downarrow S-\downarrow T$. Suppose that $(x\otimes a^{-1}U, x\otimes b^{-1}U)\le\phi(\downarrow T)$. Then $x\otimes a^{-1}U\le(\mathrm{id}_L\otimes a^{-1})(\downarrow T)\le\pi\circ(\mathrm{id}_L\times a^{-1})_*(\downarrow T)$. For each $t_a\in a^{-1}U$, by local compactness we may choose an open interval I_{t_a} (or half-open interval $[0,\alpha)$ or $(\beta,1]$ if $t_a = 0,1$ is in $a^{-1}U$), not containing 1 unless $t_a=1$, such that $I_{t_a}\ll a^{-1}U$. Thus, by Lemmas 6.10 and 6.11, $(x, I_{t_a})\in(\mathrm{id}_L\times a^{-1})_*(\downarrow T)$. Similarly, for each $t_b\in b^{-1}U$, there is an interval J_{t_b}, not containing 0 unless $t_b=0$, such that $(x, J_{t_b})\in(\mathrm{id}_L\times a^{-1})_*(\downarrow T)$. If $1\in a^{-1}U$, and hence $0\in b^{-1}U$, then we may choose some neighborhood $J_{1/2}\subseteq U$ of $\frac{1}{2}$ such that $(x, J_{1/2})\in\downarrow T$, $a^{-1}J_{1/2} = I_1$ and $b^{-1}J_{1/2} = J_0$; otherwise let $J_{1/2}=\varnothing$. For all $t_a\ne 1$ in $a^{-1}U$, $aI_{t_a}\subseteq U$ is open since $1\notin I_{t_a}$, and $(x, aI_{t_a})\in\downarrow T$ since aI_{t_a} is the least open set V such that $a^{-1}V = I_{t_a}$; likewise, for each $t_b\ne 0$, $bJ_{t_b}\subseteq U$ is open and $(x, bJ_{t_b})\in\downarrow T$. Thus, since a and b are jointly surjective, each $t\in U$ is in some aI_{t_a}, bJ_{t_b} or $J_{1/2}$, so their union is U. Since $\downarrow T$ is π-saturated,

$$\big(x, \bigvee_{t_a\in a^{-1}U} aI_{t_a}\bigvee_{t_b\in b^{-1}U} bJ_{t_b}\vee J_{1/2}\big) = (x,U)\in\downarrow T,$$

which contradicts our assumption. Thus, $(x\otimes a^{-1}U, x\otimes b^{-1}U)\not\le\phi(\downarrow T)$, but $(x\otimes a^{-1}U, x\otimes b^{-1}U)\le\phi(\downarrow S)$ and hence $\phi(\downarrow T)\ne\phi(\downarrow S)$. □

We are now ready to verify that (Loc,Ω) carries a model structure.

Theorem 6.14 *There is a model structure on* Loc *whose cofibrations are maps in* $\mathrm{cof}(\mathcal{I}_{\mathsf{Loc}})$ *(see Notation 6.1), fibrations are maps in* $\mathrm{rlp}(\mathcal{J}_{\mathsf{Loc}})$ *(see idem), and weak equivalences are maps which have the weak right lifting property against* $\mathcal{I}_{\mathsf{Loc}}$ *or, equivalently by Theorem 3.12, maps which induce isomorphisms on all homotopy groups.*

Proof We verify that (Loc, Ω) is a good Q-structure. By the description of Ω, (S1) and (S4) are clear, (S2) and (Q7) hold by Proposition 6.6, and (S3) holds by Proposition 6.13; thus, (Loc, Ω) is a category with intervals. By Proposition 6.5, (Q2) holds, (Q5) follows from Proposition 6.7, and (Q6) by Proposition 6.9. Lastly, (Q3), (Q4) and (Q8) hold since Ω preserves colimits. □

Since Ω preserves colimits and the generating (acyclic) cofibrations, we have an immediate corollary:

Proposition 6.15 *The adjunction* $\Omega \dashv \mathrm{pt}$ *is a Quillen adjunction between the Quillen model structures on* Sob *and* Loc.

We will show that $\Omega \dashv \mathrm{pt}$ is in fact a Quillen equivalence. Note that every object in Loc is fibrant and, for any $X \in \mathsf{Sob}$, $X \to \mathrm{pt}\,\Omega X$ is an isomorphism as Ω is fully faithful. Thus, by [11, Corollary 1.3.16(c)], it is necessary and sufficient to show that pt reflects weak equivalences. The following lemma will be useful:

Lemma 6.16 *Let* $i : A \to X$ *be a map between sober spaces and* $f, g : \Omega X \to L$ *be maps in* Loc. *Then* $f \sim g$ *rel* ΩA *in* Loc *if and only if* $\hat{f} \sim \hat{g}$ *rel* A *in* Sob, *where* $\hat{f}, \hat{g} : X \to \mathrm{pt}\,L$ *are the adjunct maps.*

Proof Recall from Proposition 6.6 that the canonical map $\Omega(X \times I) \to \Omega X \otimes \Omega I$ is an isomorphism as I is locally compact, where $\otimes$ is again the product in Loc. By this and adjointness, a map $K : X \times I \to \mathrm{pt}\,L$ makes the two diagrams

$$\begin{tikzcd} X \arrow[r, "e_0"] \arrow[dr, "\hat{f}"'] & X \times I \arrow[d, dashed, "K"] & X \arrow[l, "e_1"'] \arrow[dl, "\hat{g}"] \\ & \mathrm{pt}\,L & \end{tikzcd} \qquad \begin{tikzcd} A \times I \arrow[r, "\pi_A"] \arrow[d, "i \times \mathrm{id}_I"'] & A \arrow[d, "\hat{f}i = \hat{g}i"] \\ X \times I \arrow[r, "K"'] & \mathrm{pt}\,L \end{tikzcd}$$

commute in Sob if and only if $\Omega X \otimes \Omega I \cong \Omega(X \times I) \xrightarrow{\hat{K}} L$ makes the following two diagrams commute in Loc:

$$\begin{tikzcd} \Omega X \arrow[r, "e_0"] \arrow[dr, "f"'] & \Omega X \otimes \Omega I \arrow[d, dashed, "\hat{K}"] & \Omega X \arrow[l, "e_1"'] \arrow[dl, "g"] \\ & L & \end{tikzcd} \qquad \begin{tikzcd} \Omega A \otimes \Omega I \arrow[r, "\pi_{\Omega A}"] \arrow[d, "\Omega i \otimes \mathrm{id}_{\Omega I}"'] & \Omega A \arrow[d, "f\Omega i = g\Omega i"] \\ \Omega X \otimes \Omega I \arrow[r, "\hat{K}"'] & L \end{tikzcd}$$

□

Theorem 6.17 *The adjunction* $\Omega \dashv \mathrm{pt}$ *is a Quillen equivalence between the Quillen model structures on* Sob *and* Loc.

Proof Suppose that $f : X \to Y \in \mathsf{Loc}$ is a map such that $\mathrm{pt}\,f$ is a weak equivalence. Given the square on the left, we obtain the square on the right by adjointness:

$$
\begin{tikzcd}
\Omega\partial I^n \arrow[r, "u"] \arrow[d, hook, "\Omega i_n"'] & X \arrow[d, "f"] \\
\Omega I^n \arrow[r, "v"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial I^n \arrow[r, "\hat{u}"] \arrow[d, hook, "i_n"'] & \mathrm{pt}\, X \arrow[d, "\mathrm{pt}\, f"] \\
I^n \arrow[r, "\hat{v}"'] & \mathrm{pt}\, Y
\end{tikzcd}
$$

As pt f is a weak equivalence, we obtain a map $h\colon I^n \to \mathrm{pt}\, X$ with $h i_n = \hat{u}$ and a homotopy $K\colon \mathrm{pt}\, f \circ h \sim \hat{v}$ rel ∂I^n. Then, by adjointness, $\hat{h} \circ \Omega i_n = u$, and, by Lemma 6.16, $\hat{K}$ is a homotopy $f\hat{h} \sim v$ rel $\Omega(\partial I^n)$. Thus, $\hat{h}$ is the required filler in the original diagram; hence, f is a weak equivalence. □

In fact, recognizing that $\mathrm{rlp}(\mathcal{I}_{\mathsf{Loc}}) = \mathrm{rlp}(\Omega(\mathcal{I}_{\mathsf{Sob}})) = \mathrm{pt}^{-1}(\mathrm{rlp}(\mathcal{I}_{\mathsf{Sob}}))$ by adjunction, the weak equivalences and fibrations in Loc are preserved and reflected by pt. Thus, the model structure on Loc is precisely the transferred model structure from Sob.

Corollary 6.18 *The Quillen model structure on* Loc *is the right transferred model structure from* Sob *along* $\Omega \dashv \mathrm{pt}$. □

With this Quillen equivalence, we can also compare localic homotopy groups to their homotopy groups under pt. Since every locale is fibrant, [11, Corollary 1.3.16(b)] implies that the counit $\Omega\, \mathrm{pt}\, L \to L$ is a weak equivalence for every $L \in \mathsf{Loc}$. Thus, we get the following:

Proposition 6.19 *If (X, x_0) is a pointed sober space, then $\pi_n^{\mathsf{Sob}}(X, x_0) \cong \pi_n^{\mathsf{Loc}}(\Omega X, \Omega x_0)$ for any $n \geq 0$. In particular, for any pointed locale (L, l_0) and $n \geq 0$, $\pi_n^{\mathsf{Sob}}(\mathrm{pt}\, L, \mathrm{pt}\, l_0) \cong \pi_n^{\mathsf{Loc}}(L, l_0)$ since $\Omega\, \mathrm{pt}\, L \to L$ is a weak equivalence.*

Proof This follows from Lemma 6.16, as two maps $f, g\colon I^n \to X \cong \mathrm{pt}\, \Omega X$ are homotopic rel ∂I^n if and only if $\hat{f}, \hat{g}\colon \Omega I^n \to \Omega X$ are homotopic rel $\Omega(\partial I^n)$. □

References

[1] **B Banaschewski**, *Bourbaki's fixpoint lemma reconsidered*, Comment. Math. Univ. Carolin. 33 (1992) 303–309 MR Zbl

[2] **P Bubenik**, **N Milićević**, *Homotopy, homology, and persistent homology using closure spaces*, J. Appl. Comput. Topol. 8 (2024) 579–641 MR Zbl

[3] **X D Chen**, *On binary coproducts of frames*, Comment. Math. Univ. Carolin. 33 (1992) 699–712 MR Zbl

[4] **S Dolecki**, **F Mynard**, *Convergence foundations of topology*, World Scientific, Hackensack, NJ (2016) MR Zbl

[5] **L Fajstrup**, **J Rosický**, *A convenient category for directed homotopy*, Theory Appl. Categ. 21 (2008) 7–20 MR Zbl

[6] **T Haraguchi**, *On model structure for coreflective subcategories of a model category*, Math. J. Okayama Univ. 57 (2015) 79–84 MR Zbl

[7] **H Herrlich**, *Topological improvements of categories of structured sets*, Topology Appl. 27 (1987) 145–155 MR Zbl

[8] **H Herrlich**, **E Lowen-Colebunders**, **F Schwarz**, *Improving Top: PrTop and PsTop*, from "Category theory at work" (H Herrlich, H-E Porst, editors), Res. Exp. Math. 18, Heldermann, Berlin (1991) 21–34 MR Zbl

[9] **P S Hirschhorn**, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, Amer. Math. Soc., Providence, RI (2003) MR Zbl

[10] **P S Hirschhorn**, *The Quillen model category of topological spaces*, Expo. Math. 37 (2019) 2–24 MR Zbl

[11] **M Hovey**, *Model categories*, Mathematical Surveys and Monographs 63, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[12] **P T Johnstone**, *Stone spaces*, Cambridge Studies in Advanced Mathematics 3, Cambridge Univ. Press (1982) MR Zbl

[13] **A Joyal**, *The theory of quasi-categories and its applications*, lecture notes (2008) Available at `https://ncatlab.org/nlab/files/JoyalTheoryOfQuasiCategories.pdf`

[14] **A Joyal**, **M Tierney**, *An extension of the Galois theory of Grothendieck*, Mem. Amer. Math. Soc. 309, Amer. Math. Soc., Providence, RI (1984) MR Zbl

[15] **E Lowen-Colebunders**, **G Sonck**, *Exponential objects and Cartesian closedness in the construct Prtop*, Appl. Categ. Structures 1 (1993) 345–360 MR Zbl

[16] **J P May**, **K Ponto**, *More concise algebraic topology: localization, completion, and model categories*, Univ. of Chicago Press (2012) MR Zbl

[17] **F Morel**, **V Voevodsky**, *$\mathbb{A}^1$-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999) 45–143 MR Zbl

[18] **S B Niefield**, *Cartesianness: topological spaces, uniform spaces, and affine schemes*, J. Pure Appl. Algebra 23 (1982) 147–167 MR Zbl

[19] **J Picado**, **A Pultr**, *Frames and locales: topology without points*, Springer (2012) MR Zbl

[20] **D G Quillen**, *Homotopical algebra*, Lecture Notes in Math. 43, Springer (1967) MR Zbl

[21] **A Rieser**, *Čech closure spaces: a unified framework for discrete and continuous homotopy*, Topology Appl. 296 (2021) art. id. 107613 MR Zbl

[22] **A Rieser**, *Cofibration and model category structures for discrete and continuous homotopy*, preprint (2022) arXiv 2209.13510

[23] **N E Steenrod**, *A convenient category of topological spaces*, Michigan Math. J. 14 (1967) 133–152 MR Zbl

[24] **N P Strickland**, *The category of cgwh spaces*, preprint (2009) Available at `https://ncatlab.org/nlab/files/StricklandCGHWSpaces.pdf`

[25] **R W Thomason**, *Cat as a closed model category*, Cahiers Topologie Géom. Différentielle 21 (1980) 305–324 MR Zbl

[26] **R M Vogt**, *Convenient categories of topological spaces for homotopy theory*, Arch. Math. (Basel) 22 (1971) 545–555 MR Zbl

[27] **R M Vogt**, *A note on homotopy equivalences*, Proc. Amer. Math. Soc. 32 (1972) 627–629 MR Zbl

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