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Léo Maxime Brunswic

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The Alexandrov theorem for $2+1$ flat radiant spacetimes

LÉO MAXIME BRUNSWIC

Fillastre showed that one can realize the universal covering of any locally Euclidean surface Σ with conical singularities of angle bigger than 2π as the boundary of a convex Fuchsian polyhedron in 3-dimensional Minkowski space in a unique manner, up to the action of $\mathrm{SO}(1,2) \ltimes \mathbb{R}^3$, the affine isometry group of Minkowski space. The proof used a so-called deformation method, which is nonconstructive. We adapt a variational method previously used by Volkov, Bobenko, Izmestiev, and Fillastre on similar problems to provide an effective proof of Fillastre's theorem. In passing, we extend Fillastre's theorem as follows. Without assumptions on the conical angles θ_i of Σ and for any choice of nonnegative $(\kappa_i)_{i\in[\![1,s]\!]}$ such that $\kappa_i < \theta_i$ and $\kappa_i \leq 2\pi$, there exists a unique couple (M,P) where M belongs to a class of singular locally Minkowski manifolds we define with s singular lines of respective conical angle κ_i, and P is a convex polyhedron in M whose boundary ∂P is a Cauchy surface isometric to Σ, the i^{th} conical singularity of ∂P lying on the i^{th} singular line of M. Our result unifies Fillastre's theorem and instances of Penner–Epstein convex hull constructions, corresponding respectively to $\kappa_i = 2\pi$ and $\kappa_i = 0$ for all i.

51M05, 52B10, 52B70, 53C50, 57K35; 53C42, 57M60

1 Introduction

1.1 The Alexandrov theorem

Let C be a cube in the 3-dimensional Euclidean space $\mathbb{E}^3$ and consider $\Sigma := \partial C$ its boundary, as represented in Figure 1. On the one hand, Σ is a surface homeomorphic to the 2-dimensional sphere $\mathbb{S}^2$; on the other hand, Σ is naturally endowed with a locally Euclidean metric with six conical singularities, each of angle $\frac{3}{2}\pi$.

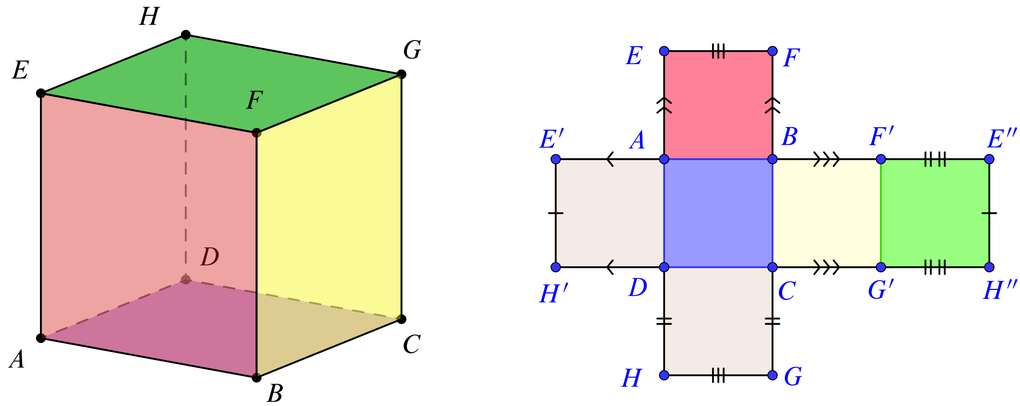

Figure 1

More generally, the boundary of any compact convex polyhedron in $\mathbb{E}^3$ is homeomorphic to the 2-dimensional sphere. It is naturally endowed with a locally Euclidean metric with conical singularities of angles less than 2π.

A classical theorem of Alexandrov [2] shows that this construction is actually bijective:

Theorem [2] *Let Σ be a locally Euclidean surface with conical singularities of angles less than 2π and homeomorphic to the sphere $\mathbb{S}^2$. There exists a compact convex polyhedron P in $\mathbb{E}^3$ such that ∂P is isometric to Σ. Furthermore, two such polyhedra are congruent.*

Using a so-called deformation method, Alexandrov proved generalizations to convex polyhedrons in $\mathbb{H}^3$ and $\mathbb{S}^3$; this method is, however, not effective since it does not provide an efficient way to construct the convex polyhedra these theorems predict.

1.2 Generalizations to space forms and main result

In the 2000s, Izmestiev and Bobenko gave a new proof of the Alexandrov theorem by a variational, therefore effective, method. See Kane, Price, and Demaine [23] for a complexity analysis of the resulting algorithm. Rivin, Hodgson, Schlenker, and Fillastre proved generalizations to Lorentzian space forms (Minkowski, de Sitter, and anti-de Sitter), in which case conical singularities of the locally Euclidean surface have angles greater than 2π. The Alexandrov problem can then be stated in a more general context that has been recently studied systematically by Fillastre and Izmestiev.

Problem *Let Σ be a closed surface of genus g endowed with a singular metric of constant curvature $K \in \{-1, 0, 1\}$ and cone angles all bigger that 2π (case $\varepsilon = -$) or all less than 2π (case $\varepsilon = +$). Denote by X_K^ε the model space of constant curvature K. It is Riemannian if $\varepsilon = +$ and Lorentzian if $\varepsilon = -$.*

Is there a convex polyhedral Fuchsian realization of Σ in X_K^ε? Furthermore, is this polyhedron unique up to congruence?

g	K	ε	DM	VM
0	0	$+$	[2]	[6]
0	-1	$+$	[3]	
0	1	$+$	[3]	
0	1	$-$	[21]	
1	-1	$+$		[18]
1	1	$-$		[19]
≥ 2	-1	$+$	[15]	
≥ 2	-1	$-$	[17]	
≥ 2	0	$-$	[17]	[B]
≥ 2	1	$-$	[31]	

Table 1: See Alexandrov [2; 3], Bobenko and Izmestiev [6], Fillastre [15; 17], Fillastre and Izmestiev [18; 19], Hodgson and Rivin [21], and Schlenker [31].

The signature of the X_K^ε and the Gauss–Bonnet formula impose constraints on (g, K, ε). Table 1 is based upon work of Fillastre [16] and sums up all possible situations, together with references to proofs by deformation (DM) and/or variational (VM) methods; [B] refers to the present work.

Proving Fillastre's theorem — the case where $(g, K, \varepsilon) = (\geq 2, 0, -)$ and X_K^ε is Minkowski space $\mathbb{E}^{1,2}$ — by a variational method is the primary motivation of the present work. Here "convex polyhedral Fuchsian realization" means that we build a triple (ρ, ι, P), where ρ is a representation of $\pi_1(\Sigma) \to \mathrm{Isom}(\mathbb{E}^{1,2})$, ι is a ρ-equivariant embedding $\iota\colon \widetilde{\Sigma} \to \mathbb{E}^{1,2}$ of the universal covering of Σ, and P is a convex globally ρ-invariant polyhedron, with the additional hypothesis that ρ fixes a point and acts cocompactly on the hyperboloid model of the hyperbolic plane $\mathbb{H}^2 = \{(t, x, y) \mid t^2 - x^2 - y^2 = 1, t > 0\} \subset \mathbb{E}^{1,2}$.

To this end, we adapt the variational method successfully used by Bobenko, Fillastre, and Izmestiev [6; 18; 19]; we derive Alexandrov–Fillastre and obtain a generalization to a class of singular locally Minkowski 3-manifolds: radiant singular flat spacetimes, which we shall describe thereafter.

Theorem *Let Σ be a closed locally Euclidean surface of genus g with s marked conical singularities*[1] *of angles $(\theta_i)_{i \in [\![1,s]\!]}$. For all*

$$\kappa \in \left(\prod_{i=1}^{s} [0, \min(\theta_i, 2\pi)]\right) \setminus \{(\theta_i)_{i \in [\![1,s]\!]}\},$$

there exists a radiant singular flat spacetime M homeomorphic to $\Sigma \times \mathbb{R}$ with exactly s singular lines of angles $\kappa_1, \dots, \kappa_s$ and a convex polyhedron $P \subset M$ whose boundary is isometric to Σ. The boundary of P is a Cauchy surface of M.

Furthermore, if for all $i \in [\![1, s]\!]$, $\kappa_i < \theta_i$, then (M, P) is unique up to equivalence.

Finally, if for some $i \in [\![1, s]\!]$, $\theta_i \leq \pi$ and $\kappa \in \mathbb{R}_+^s$ is such that $\kappa_i > \theta_i$, then there is no such couple (M, P).

[1] We allow marked conical singularities with angle 2π, which are hence not singular but marked nonetheless.

Remark By taking all the $\theta_i > 2\pi$ and $\kappa_i = 2\pi$ we obtain a manifold M whose universal covering is isomorphic to a subdomain of Minkowski space X_0^- (via a theorem of Mess [25]). Fillastre's theorem thus follows.

1.3 Radiant spacetimes

Before giving the outline of the variational method, we quickly describe radiant spacetimes. A more thorough description is given in the appendix, together with technical results. We denote by $\mathbb{E}^{1,2}$ the 3-dimensional Minkowski space (the oriented affine space $\mathbb{R}^3$ together with the quadratic form[2] $\boldsymbol{g} := \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2$ written in some fixed choice of Cartesian coordinates t, x, y) and by $\mathrm{Isom}_0(\mathbb{E}^{1,2})$ the identity component of the Lie group of affine isometries of $\mathbb{E}^{1,2}$, namely $\mathrm{SO}_0(1,2) \ltimes \mathbb{R}^3$. We denote by $O := (0,0,0) \in \mathbb{E}^{1,2}$ the origin of $\mathbb{E}^{1,2}$. A vector $u \neq 0$ is spacelike (resp. timelike, lightlike, causal) if $\boldsymbol{g}(u) < 0$ (resp. $\boldsymbol{g}(u) > 0$, $\boldsymbol{g}(u) = 0$, $\boldsymbol{g}(u) \geq 0$). A causal vector is future (resp. past) if its t coordinate is positive (resp. negative). Minkowski space is naturally endowed with two order relations: the causal order $\leq$ and the chronological order $\lesssim$ (the associated strict relation is denoted by $\ll$). Given $p, q \in \mathbb{E}^{1,2}$ then $p < q$ (resp. $p \ll q$) if $q - p$ is future causal (resp. future timelike). The group $\mathrm{Isom}_0(\mathbb{E}^{1,2})$ preserves the orientation of $\mathbb{E}^{1,2}$ as well as the causal and the chronological orders. We define the causal future of p, denoted by $J^+(p) := \{q \in M \mid p \leq q\}$, as well as the chronological future of p, denoted by $I^+(p) := \{q \in M \mid p \ll q\}$. The causal past, as well as the chronological past, are defined accordingly. A plane in $\mathbb{E}^{1,2}$ is spacelike (resp. timelike, lightlike) if the induced quadratic form is positive definite (resp. definite, degenerated), and a normal to such a plane is a timelike vector (resp. spacelike vector, lightlike vector). By convention, all spacelike and lightlike planes are oriented by a future normal vector.

Radiant spacetimes are obtained via gluings of cones in $J^+(O)$ of triangular basis, ie

$$C = \{ru \mid r \in \mathbb{R}_+^*, u \in T\},$$

with T some affine spacelike triangle in $J^+(O)$. We will not consider any such gluing with boundary.

Such gluings have a natural $(\mathrm{SO}_0(1,2), I^+(O))$-structure in the sense of Ehresmann [12], Thurston [33], or Goldman [20] on the complement of the edges of the cones (the 1-facet of the simplicial complex). These "singular" edges are one of two types:

- Timelike edges are locally modeled on so-called massive particles (the plane orthogonal to the given edge is a Euclidean conical singularity of some angle $\kappa > 0$).
- Lightlike edges are locally modeled on so-called extreme BTZ-like singularities (see the appendix and Barbot, Bonsante and, Schlenker [4] for more details). The convention is that such an edge bears a cone angle $\kappa = 0$.

[2]Beware we chose a sign convention for $\boldsymbol{g}$ different from most of the literature to favor positive values of $\boldsymbol{g}$ on the relevant domains and avoid defining two different quadratic forms.

For brevity sake, we will write $\mathcal{F}$ instead of $I^+(O)$ and $\mathcal{F}$-manifold instead of $(\mathrm{SO}_0(1,2), \mathcal{F})$-manifold. Cones in $J^+(O)$ have a natural $\mathrm{SO}_0(1,2)$-invariant 1-dimensional foliation formed by the rays from the origin of the form $\mathcal{R}_u := \{ru \mid r > 0\}$ with u in $J^+(O)$; therefore each radiant spacetime comes with such a foliation. The statement "the surface Σ is a Cauchy surface of the radiant spacetime M" is understood in our context as "the surface Σ is spacelike and intersects all rays of the natural foliation".

Equivalence in our context has to be understood in the following way: two couples (M, P) and (M', P') are equivalent if there exists an isomorphism $M \to M'$ of singular $\mathbb{E}^{1,2}$-manifolds (a homeomorphism sending regular domain to regular domain and which is an $\mathbb{E}^{1,2}$-morphism on the regular domain) which induces a bijection $P \to P'$.

1.4 The variational method

Now that the terminology is clarified, the variational method proceeds as follows:

(1) Consider a closed locally Euclidean surface Σ of genus g with $s \in \mathbb{N}^*$ marked conical singularities $\theta_1, \dots, \theta_s \in \mathbb{R}_+^*$ and define S the set of marked points.

(2) Choose an arbitrary couple $(\tau, \mathcal{T})$ with $\tau\colon S \to \mathbb{R}_+$ and $\mathcal{T}$ a triangulation of Σ whose set of vertices is S.

(3) For each triangle T of $\mathcal{T}$, choose a direct affine isometric embedding

$$\iota\colon T \to J^+(O) := \{t > 0, \boldsymbol{g} \geq 0\} \subset \mathbb{E}^{1,2}$$

in such a way that for each vertex s of T we have $\boldsymbol{g} \circ \iota(s) = \tau(s)$.

(4) To each triangle T is then associated the cone of rays from $O := (0,0,0)$ through T in $\mathbb{E}^{1,2}$; glue these cones together following the same combinatorics as $\mathcal{T}$. The gluing is a 3-manifold M endowed with a flat Lorentzian metric on the complement of the rays through the vertices of $\mathcal{T}$. Furthermore we have a natural embedding $\iota\colon \Sigma \to M$ in such a way that $\iota(\Sigma)$ is the boundary of the polyhedron $P := J^+(\iota(\Sigma))$ of M.

(5) Study the domain of $\tau \in (\mathbb{R}_+)^S$ such that the polyhedron P is convex; ι is then called convex, and show that for a given τ there is at most one triangulation $\mathcal{T}$ (up to equivalence) for which the embedding ι is convex; a τ is then admissible if it has such a triangulation.

(6) Choose some target Lorentzian angles $\bar{\kappa}$ and define an Einstein–Hilbert functional on the space of admissible $\tau \in (\mathbb{R}_+)^S$ in such a way that each of its critical points induces a manifold M with Lorentzian cone angle $\bar{\kappa}$ around the rays through the vertices of $\mathcal{T}$.

(7) Finally, study this functional and show it admits a unique critical point.

1.5 The special case $\kappa = 0$

Penner gives another viewpoint on our result [27; 28], constructing a cellulation of the decorated Teichmüller space of a closed surface Σ with s marked points $S = \{\sigma_1, \dots, \sigma_s\}$ viewed as the space of marked finite-volume complete hyperbolic surfaces with s cusps homeomorphic to $\Sigma \setminus S$ together with a choice of a positive number on each cusp. Consider such a surface Σ^*. The universal covering of Σ^* naturally identifies with the usual hyperbolic plane $\mathbb{H}^2 := \{(t,x,y) \in \mathbb{E}^{1,2} \mid \boldsymbol{g}(t,x,y) = 1,\, t > 0\}$ in $\mathbb{E}^{1,2}$, and the positive number λ_σ on each cusp σ corresponds to a point on the lightlike rays corresponding to the cusp:

- There exists a unique horocycle $\mathcal{H}_{\sigma,\lambda_\sigma}$ of length λ_σ around σ.
- Consider a ray $\mathcal{R}$ fixed by a parabolic holonomy of Σ^* and a point $p \in \mathcal{R}$. The intersection of the future light cone of p (the set $\{q \in \mathbb{E}^{1,2} \mid \boldsymbol{g}(q-p) = 0, t(q-p) > 0\}$) with $\mathbb{H}^2$ is a horocycle around $\mathcal{R}$, and every horocycle is obtained in this manner.

Penner then considers the surface obtained as the boundary of the convex hull of these points.[3] He shows the surface obtained is locally Euclidean, its quotient by the holonomy of Σ^* is a locally Euclidean surface $\Sigma_{\mathbb{E}^2}$ with s conical singularities. Furthermore, the convex hull is a polyhedron, the faces of which induce a cellulation on $\Sigma_{\mathbb{E}^2}$ with marked points S. He notes that this cellulation is simply the Delaunay cellulation of $(\Sigma_{\mathbb{E}^2}, S)$. It is not hard to see that

(1) this construction actually defines a natural bijection from the decorated Teichmüller space of (Σ, S) to the deformation space of locally Euclidean metrics on Σ with arbitrary conical singularities on S,

(2) the quotient by the holonomy of Σ^* of the union of $I^+(O)$ with the rays fixed by parabolic holonomy of Σ^* is a radiant spacetime with s conical singularities of angle 0.

Penner construction can thus be seen as the special case of our theorem where $\kappa = 0$ and (Σ, S) runs through all locally Euclidean surfaces with s conical singularities at S of arbitrary angles.

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[3] A joint work of Penner and Epstein [13] generalizes this construction.

2 Convex τ-suspension and polyhedral embedding

In the present section, we shall define and study τ-suspension of a singular locally Euclidean surface (Σ, S). A cellulation of Σ is a homeomorphism between Σ and a gluing of affine convex dimension-n polyhedra along $(n-1)$-facets. We identify k-facets with their image in Σ. All cellulations considered in this section have totally geodesic facets.

Definition 2.1 Let (Σ, S) be a compact Euclidean surface with conical singularities with a finite subset S of marked points such that $\mathrm{Sing}(\Sigma) \subset S$, and let $\mathcal{C}$ be a cellulation of (Σ, S). $\mathcal{C}$ is adapted if the set of vertices of $\mathcal{C}$ is exactly S.

Definition 2.2 Let (Σ, S) be a compact Euclidean surface with conical singularities with a finite subset S of marked points such that $\mathrm{Sing}(\Sigma) \subset S$. Let M be a singular $\mathbb{E}^{1,2}$-manifold. An embedding $\iota\colon \Sigma \to M$ is polyhedral if there exists a geodesic adapted cellulation $\mathcal{C}$ of (Σ, S) such that on each cell C, the restriction of ι to $\mathrm{Int}(C)$ is an isometric affine map into the regular locus of L.

The notion of an isometric affine map is well defined in this context. Indeed, both $\mathbb{E}^2$ and $\mathbb{E}^{1,2}$ are affine spaces endowed with a semi-Riemannian metric; the regular loci of Σ and M are endowed with an $\mathbb{E}^2$-structure and an $\mathbb{E}^{1,2}$-structure, respectively.

The quadratic form on $\mathbb{E}^{1,2}$ is a $\mathrm{SO}_0(1,2)$-invariant function defined on the underlying vector space $\overrightarrow{\mathbb{E}^{1,2}}$:

$$\boldsymbol{g}\colon \overrightarrow{\mathbb{E}^{1,2}} \to \mathbb{R}, \quad (t,x,y) \mapsto t^2 - x^2 - y^2.$$

We extend the definition of $\boldsymbol{g}$ to $\mathbb{E}^{1,2}$ via the identification $\mathbb{E}^{1,2} \to \overrightarrow{\mathbb{E}^{1,2}}$, $x \mapsto x - O$. The map $\boldsymbol{g}$ is positive on the future of the origin in $\mathbb{E}^{1,2}$, namely $J^+(O) := \{(t,x,y) \in \mathbb{R}^3 \mid t^2 - x^2 - y^2 \geq 0 \text{ and } t > 0\}$; furthermore, it induces a Cauchy time function on $I^+(O)$, ie an increasing map $(I^+(O), \leq) \to (\mathbb{R}_+^*, \leq)$ whose restriction to any nonextendible future causal curve of $I^+(O)$ is surjective (see the appendix for more details on the structure of singular $\mathcal{F}$-manifolds). Since $\boldsymbol{g}$ is $\mathrm{SO}_0(1,2)$-invariant, it induces a well-defined nondecreasing function on every radiant singular flat spacetime.

In a radiant singular flat spacetime, the surface $\boldsymbol{g} = 1$ is a hyperbolic surface with conical singularities and cusps, which is complete and has finite volume. One can prove that the association $M \mapsto \{\boldsymbol{g} = 1\}$ induces a bijection from the deformation space of marked radiant singular flat spacetimes to the deformation space of marked finite-volume complete hyperbolic surfaces with conical singularities and cusps; see Theorem 6 in the appendix.

2.1 Affine embedding of triangles into $\mathbb{E}^{1,2}$

The goal of this section is mainly to introduce terminology that will be used throughout the paper and to prove a parametrization of polyhedral embeddings into radiant singular flat spacetimes of a singular locally Euclidean surface by the class of *distance-like* function we introduce. This last point is the object of Theorem 1.

Lemma 2.3 *Let $T = [ABC]$ be a nondegenerated Euclidean triangle and let $\tau\colon \{A, B, C\} \to \mathbb{R}$. There exists a unique couple $(\tau_0, \omega) \in \mathbb{R} \times \mathbb{E}^2$ such that the map*

$$\tilde{\tau}\colon \mathbb{E}^2 \to \mathbb{R}, \quad x \mapsto \tau_0 - d(x,\omega)^2$$

extends τ.

Furthermore, if $\tau \geq 0$ then $\tau_0 > 0$ and $\tilde{\tau} > 0$ on the triangle $[ABC]$, except possibly at A, B, or C.

Proof Identify $\mathbb{E}^2$ to $\mathbb{R}^2$ via Cartesian coordinates (x, y); without loss of generality, we can assume $A = (0,0)$, and we write $B = (x_B, y_B)$ and $C = (x_C, y_C)$. Finding $\tilde{\tau}$ is equivalent to solving the following system in $\omega = (x_\omega, y_\omega)$ and τ_0:

$$\begin{cases} \tau_A = \tau_0 - x_\omega^2 - y_\omega^2, \\ \tau_B = \tau_0 - (x_\omega - x_B)^2 - (y_\omega - y_B)^2, \\ \tau_C = \tau_0 - (x_\omega - x_C)^2 - (y_\omega - y_C)^2, \end{cases} \iff \begin{cases} x_\omega^2 + y_\omega^2 + \tau_A = \tau_0, \\ \tau_B - \tau_A + x_B^2 + y_B^2 = 2x_\omega x_B + 2y_\omega y_B, \\ \tau_C - \tau_A + x_C^2 + y_C^2 = 2x_\omega x_C + 2y_\omega y_C. \end{cases}$$

Since A, B and C are in general position, the second and third lines form a nonsingular linear system of unknown (x_ω, y_ω). The first line is already solved. Existence and uniqueness of $\tilde{\tau}$ follows.

Assume $\tau \geq 0$, since A, B and C are distinct, ω is distinct from one of them, say $P \in \{A, B, C\}$. Then $0 \leq \tau_P = \tau_0 - d(P,\omega)^2 < \tau_0$. Furthermore, $\tilde{\tau}$ is strictly concave, so its minimum on $[ABC]$ is reached in the set of extremal points, eg $\{A, B, C\}$ and nowhere else. □

Lemma 2.4 *Let $A, B, A', B' \in J^+(O)$, $A \neq B$ and $A' \neq B'$ be such that $\boldsymbol{g}(A) = \boldsymbol{g}(A')$, $\boldsymbol{g}(B) = \boldsymbol{g}(B')$ and $\boldsymbol{g}(B - A) = \boldsymbol{g}(B' - A')$. Then there exists a unique isometry $\gamma \in \mathrm{SO}_0(1,2)$ such that $\gamma A = A'$ and $\gamma B = B'$. Furthermore, if C is on a given side of the oriented plane (OAB), then γC is on the same side of $(OA'B')$.*

Proof The group $\mathrm{SO}_0(1,2)$ acts transitively on each of the sets $(\boldsymbol{g}|_{J^+(O)\setminus\{O\}})^{-1}(\tau_0)$ for $\tau_0 \geq 0$. There thus exists some $\gamma_0 \in \mathrm{SO}_0(1,2)$ such that $\gamma_0 A = A'$. The stabilizer of A' under the action of $\mathrm{SO}_0(1,2)$ is a 1-parameter subgroup (either parabolic or elliptic depending on whether (OA') is lightlike or timelike); under its action, the orbit of $\gamma_0 B$ is

$$\{x \in J^+(O) \mid \boldsymbol{g}(x - A') = \boldsymbol{g}(\gamma_0 B - A') \text{ and } \boldsymbol{g}(x) = \boldsymbol{g}(\gamma_0 B)\}.$$

The stabilizer of A' acts freely on this set, so there exists a unique γ with the wanted properties. Finally, $\mathrm{SO}_0(1,2)$ preserves orientation, and the result follows. □

Proposition 2.5 *Suppose that $T = [ABC]$ is an oriented nondegenerated Euclidean triangle and let $\tau\colon \{A, B, C\} \to \mathbb{R}_+$. There exists a direct isometric affine embedding $\iota\colon T \to J^+(O)$ such that $\tau = \boldsymbol{g} \circ \iota|_{\{A,B,C\}}$, where $\iota(T)$ is endowed with the orientation induced by a future-pointing normal vector.*

Furthermore,

- *such an embedding is unique up to the action of $\mathrm{SO}_0(1,2)$,*
- *$\boldsymbol{g} \circ \iota = \tilde{\tau}$, where $\tilde{\tau}$ is given by Lemma 2.3.*

Proof Endow $\mathbb{E}^{1,2}$ with Cartesian coordinates (t, x, y), write $O = (0, 0, 0)$ the origin, and identify $\mathbb{E}^2$ with $\{t = 0\} \subset \mathbb{E}^{1,2}$. Take $(\tau_0, \omega) \in \mathbb{R} \times \mathbb{E}^2$ and $\tilde{\tau}$ given by Lemma 2.3, and define

$$\iota\colon T \to \mathbb{E}^{1,2}, \qquad x \mapsto x + \vec{u} \quad \text{with } \vec{u} = \begin{pmatrix} \sqrt{\tau_0} \\ -\overrightarrow{O\omega} \end{pmatrix}.$$

Write $\omega = (x_\omega, y_\omega)$. For $(x, y) \in T$, we have

$$\boldsymbol{g} \circ \iota(x, y) = \sqrt{\tau_0}^2 - (x - x_\omega)^2 - (y - y_\omega)^2 = \tilde{\tau}(x, y).$$

Since $\tau \geq 0$, by Lemma 2.3 $\tilde{\tau} \geq 0$; hence $\boldsymbol{g} \circ \iota|_T \geq 0$. Moreover, $\sqrt{\tau_0} > 0$, thus $\iota(T) \subset J^+(O)$. The existence statement follows, as well as the second additional point.

If ι and ι' are two such embeddings, by Lemma 2.4 there exists a unique isometry sending $\iota(A)$ on $\iota'(A)$ and $\iota(B)$ on $\iota'(B)$. There thus exist exactly two points $P_1, P_2 \in J^+(O)$ such that $\boldsymbol{g}(P_i) = \tau(C)$, $\mathrm{d}(A, C)^2 = \boldsymbol{g}(\iota(P_i) - \iota(A))$ and $\mathrm{d}(B, C)^2 = \boldsymbol{g}(\iota(P_i) - \iota(B))$ for $i \in \{1, 2\}$. Since these two points are each other's images by the reflection across the plane $(O\iota(A)\iota(B))$ which is orientation-reversing and preserves $\leq$, exactly one induces the right orientation. □

Definition 2.6 (f-triangulation) Let (Σ, S) be a singular locally Euclidean surface and let $f : \Sigma \to \mathbb{R}$. A triangulation $\mathcal{T}$ is an f-triangulation if $\mathcal{T}$ is a geodesic triangulation of Σ whose set of vertices contains S and such that for all triangles $T \in \mathcal{T}$, there exists $\omega \in \mathbb{E}^2$ and $\tau_0 \in \mathbb{R}$ such that

$$\text{for all } x \in T, \quad f(x) = \tau_0 - \mathrm{d}(\mathcal{D}(x), \omega)^2,$$

where $\mathcal{D}\colon T \to \mathbb{E}^2$ is a developing map of T.

Definition 2.7 (distance-like function) Let (Σ, S) be a singular locally Euclidean surface. A function $f : \Sigma \to \mathbb{R}$ is distance-like if it admits an f-triangulation.

Remark Let (Σ, S) be a singular locally Euclidean surface, and let M be a radiant spacetime. For any polyhedral embedding $\iota\colon \Sigma \to M$, the map $\boldsymbol{g} \circ \iota\colon \Sigma \to \mathbb{R}_+$ is distance-like.

Proposition 2.8 *Let (Σ, S) be singular locally Euclidean surface. Let $\mathcal{T}$ be an adapted triangulation of (Σ, S).*

For all $\tau : S \to \mathbb{R}$, there exists a unique distance-like extension $\tilde{\tau}$ such that $\mathcal{T}$ is a $\tilde{\tau}$-triangulation.

Proof Apply Lemma 2.3 to each triangle of $\mathcal{T}$. □

Definition 2.9 Let (Σ, S) be a singular locally Euclidean surface. Let $\mathcal{T}$ be an adapted triangulation of (Σ, S) and let $\tau : S \to \mathbb{R}_+$. We denote by $\tilde{\tau}_{\tau,\mathcal{T}}$ the extension of τ given by Proposition 2.8.

Definition 2.10 (equivalent triangulations) Let (Σ, S) be singular locally Euclidean surface. Let $\tau : S \to \mathbb{R}_+$. Two adapted triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of (Σ, S) are τ-equivalent if

$$\tilde{\tau}_{\tau,\mathcal{T}_1} = \tilde{\tau}_{\tau,\mathcal{T}_2}.$$

Definition 2.11 (τ-suspension) Let (Σ, S) be a singular locally Euclidean surface and $f:\Sigma \to \mathbb{R}_+$ be distance-like.

Choose an f-triangulation $\mathcal{T}$ not necessarily adapted to (Σ, S). For each $T \in \mathcal{T}$, denote by $\iota_T : T \to J^+(O)$ the affine embedding of T given by Proposition 2.5 and define $C_T := \{t \cdot \iota_T(x) \mid t \in \mathbb{R}_+^*, x \in T\}$. For each edge e of $\mathcal{T}$ bounding $T_1, T_2 \in \mathcal{T}$, let γ_e be the isometry given by Lemma 2.4 sending the face of C_{T_2} associated to e to the face of C_{T_1} associated to e.

Define $M(f)$ as the radiant spacetime obtained by gluing the family $(C_T)_{T\in\mathcal{T}}$ via the isometries $(\gamma_e)_{e\in \mathrm{edges}(\mathcal{T})}$

Proposition 2.12 *Let (Σ, S) be a singular locally Euclidean surface and $f:\Sigma\to\mathbb{R}_+$ be distance-like. The spacetime $M(f)$ does not depend on the choice of the f-triangulation $\mathcal{T}$.*

Proof Consider two geodesic f-triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$. There exists a geodesic f-triangulation of (Σ, S) such that any 2-facet of $\mathcal{T}_1$ or $\mathcal{T}_2$ is a union of adjacent 2-facets of $\mathcal{T}$. It thus suffices to show that on a given triangle $T \subset \Sigma$ on which $\tilde{\tau}$ is $\mathscr{C}^1$, any decomposition of T into smaller triangles $(T_i)_{i\in[\![1,n]\!]}$ induces a gluing isomorphic to C_T. We may assume T is obtained by inductively gluing T_{k+1} to $\bigcup_{i=1}^k T_i$ for $k \in [\![1, n-1]\!]$. We give ourselves an embedding $\iota_0 : T \to J^+(O)$ given by Proposition 2.5. Start from T_1 with an embedding $\iota: T_1 \to J^+(O)$, using Lemma 2.4. Without loss of generality, we may assume that $\iota|_{0T_1} = \iota$, then glue the C_{T_k} for $k \in [\![2, n]\!]$ naturally extending $\iota: \bigcup_{i=1}^k T_i \to J^+(O)$. By Lemma 2.4, at each step, there is only one way to glue a cone $C_{T_{k+1}}$ to $\bigcup_{i=1}^k C_{T_i}$ so that $\tilde{\tau} = \boldsymbol{g}\circ\iota$. Hence at each step there is at most one extension of ι to $\bigcup_{i=1}^k T_i$; the embedding ι thus coincides with the restriction of ι_0 at each step, and thus on the whole T. Finally, C_T is isomorphic to the gluing of the $(C_{T_i})_{i\in[\![1,n]\!]}$. □

Definition 2.13 (equivalent polyhedral embedding) Let (Σ, S) be a singular locally Euclidean surface and let (M_1, ι_1) and (M_2, ι_2) be two radiant spacetimes together with a polyhedral embedding of (Σ, S).

We say that (M_1, ι_1) is equivalent to (M_2, ι_2) if there exists an isomorphism $\varphi: M_1 \to M_2$ such that $\iota_2 = \varphi \circ \iota_1$.

Theorem 1 *Denoting by $\sim$ the equivalence relation among polyhedral embeddings, the function*

$$\{(M,\iota) \mid M \text{ radiant, } \iota \text{ polyhedral embedding}\}/\sim \;\to \{\tilde{\tau} \mid \tilde{\tau}:\Sigma\to\mathbb{R}_+ \text{ distance-like}\}, \quad (\iota, M) \mapsto \boldsymbol{g}\circ\iota$$

is bijective with inverse $\tilde{\tau} \mapsto M(\tilde{\tau})$.

Remark The proof depends on a description of radiant spacetimes as suspensions of singular hyperbolic surfaces; we give it in the appendix.

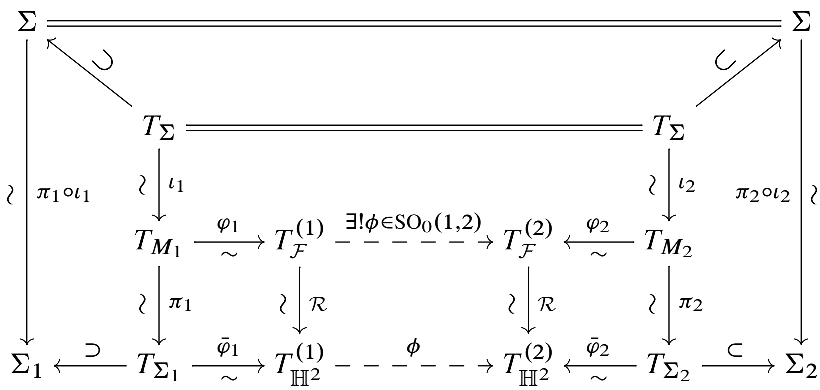
Proof Denote by Φ the function above. For any $\tilde{\tau}$ distance-like on (Σ, S), by Proposition 2.5 the construction of $M(\tilde{\tau})$ ensures $\Phi(M(\tilde{\tau})) = \tilde{\tau}$. Hence Φ is surjective. Let (M_1, ι_1) be the polyhedral embedding of (Σ, S), let $\tilde{\tau} := \Phi(M_1, \iota_1)$, and let $M_2 = M(\tilde{\tau})$ with its polyhedral embedding $\iota_2 : \Sigma \to M_2$.

By Theorem 6, for $i \in \{1,2\}$, M_i is isomorphic to $\mathrm{susp}(\Sigma_i)$ with Σ_i the space of rays of the natural causal foliation of M_i endowed with its $\mathbb{H}^2_{\geq 0}$-structure. Define the natural projections $\pi_i\colon M_i \to \Sigma_i$. Denote by $\mathcal{R}\colon \mathcal{F} \to \mathbb{H}^2$ the map that associates to any $x \in \mathcal{F}$ the intersection point of the ray through x with $\mathbb{H}^2 \subset \mathcal{F}$.

For $i \in \{1,2\}$, the map $\pi_i \circ \iota_i\colon \Sigma \to \Sigma_i$ is a homeomorphism. The map $h := \pi_2 \circ \iota_2 \circ (\pi_1 \circ \iota_1)^{-1}$ is then a homeomorphism. We shall prove g is an a.e. $\mathbb{H}^2$-morphism from Σ_1 to Σ_2 and hence that $\mathrm{susp}(h)\colon M_1 \to M_2$ is an isomorphism.

Choose a geodesic triangulation $\mathcal{T}$ of Σ adapted to $\tilde{\tau}$. Its image by $\pi_i \circ \iota_i$ is a geodesic triangulation of Σ_i. Note that h sends a cell of Σ_1 to a cell of Σ_2. Thus in order to prove that h is an $\mathbb{H}^2$-morphism, it suffices to prove that its restrictions to each cell of Σ_1 are isometries.

Let $T \in \mathcal{T}$, $x \in T \setminus S$, and, for $i \in \{1,2\}$, choose a chart $(\mathcal{U}_i, \mathcal{V}_i, \varphi_i)$ of M_i around $\iota_i(x)$ such that $\mathcal{V}_i$ is a cone of $\mathcal{F}$. Let $T_\Sigma \subset T \setminus S$ be a triangle of Σ containing x. For $i \in \{1,2\}$, write $T_{M_i} := \iota_i(T_\Sigma)$, $T_{\Sigma_i} := \pi_i \circ \iota_i(T_\Sigma)$, $T^{(i)}_{\mathcal{F}} := \varphi_i(T_{M_i})$, and $T^{(i)}_{\mathbb{H}^2} := \mathcal{R} \circ \varphi_i(T_{M_i})$. By construction of the $\mathbb{H}^2$-structure on Σ_i, φ_i induces a chart $\bar{\varphi}_i\colon T_{\Sigma_i} \to T^{(i)}_{\mathbb{H}^2}$. By Lemma 2.4 there exists a unique $\phi \in \mathrm{SO}_0(1,2)$ such that $\varphi_2 \circ \iota_2 = \phi \circ \varphi_1 \circ \iota_1$. Since $\mathcal{R}$ commutes with the action of $\mathrm{SO}_0(1,2)$, we then have $\mathcal{R} \circ \varphi_2 \circ \iota_2 = \phi \circ \mathcal{R} \circ \varphi_1 \circ \iota_1$. The following commutative diagram sums up the situation:



Therefore the (co)restriction of h from T_{Σ_1} to T_{Σ_2} is an isometry. It follows that h is an isometry from a triangle of $\pi_1 \circ \iota_1(\mathcal{T})$ to a triangle of $\pi_2 \circ \iota_2(\mathcal{T})$. □

2.2 Convex embeddings

We start by clarifying the notion of a convex embedding in Definition 2.14, and translate the notion in terms of a *Q-convex* distance-like function. Proposition 2.23 is the main result of this subsection. It provides a parametrization of convex polyhedral embeddings by a domain of $\mathbb{R}^S_+$. Throughout the section, (Σ, S) is a marked locally Euclidean surface with conical singularities included in the set of marked points S.

Definition 2.14 (convex polyhedral embedding) Let M be a radiant spacetime with $\iota\colon \Sigma \to M$ a polyhedral embedding.

The embedding ι is convex if $J^+(\iota(\Sigma))$ is convex in the sense that for any spacelike geodesic $c\colon[a,b]\to M$, if $\{c(a),c(b)\}\subset J^+(\iota(\Sigma))$ then $c([a,b])\subset J^+(\iota(\Sigma))$.

Definition 2.15 (Q-convexity on $\mathbb{R}$) Let $I\subset\mathbb{R}$ be an interval. A function $f: I\to\mathbb{R}$ is Q-convex (resp. Q-concave) if f is continuous, piecewise $\mathscr{C}^1$ and if for all $t_0\in I$,

$$\lim_{t_0^-} f' \leq \lim_{t_0^+} f' \quad \Big(\text{resp. } \lim_{t_0^-} f' \geq \lim_{t_0^+} f'\Big).$$

Definition 2.16 (Q-convexity on an $\mathbb{E}^2_{>0}$-surface) A function $\tilde\tau\colon\Sigma\to\mathbb{R}$ is Q-convex (resp. Q-concave) if for all geodesics $c\colon I\to\Sigma\setminus S$, the restriction of $\tilde\tau$ to c is Q-convex (resp. Q-concave).

Lemma 2.17 *Let $f,g\colon[a,b]\to\mathbb{R}$ be two continuous functions piecewise of the form $x\mapsto -x^2+\alpha x+\beta$ with f of class $\mathscr{C}^1$.*

- *If g is Q-convex with $f(a)\geq g(a)$ and $f(b)\geq g(b)$ then $g\leq f$.*
- *If g is Q-concave with $f(a)\leq g(a)$ and $f(b)\leq g(b)$ then $f\leq g$.*

Furthermore, if the Q-convexity (resp. Q-concavity) is strict, the inequalities are strict on $]a,b[$.

Proof First, $g-f$ is piecewise affine; since f is $\mathscr{C}^1$, the Q-convexity of $g-f$ (and hence its convexity) is the same as the Q-convexity of f. In the first (resp. second) case, since $g-f$ is nonpositive (resp. nonnegative) at a and b, it is thus nonpositive (resp. nonnegative) on $[a,b]$. The strict case is obtained the same way. □

Lemma 2.18 *Let M be a radiant singular flat spacetime and let $\Sigma\subset\mathcal{M}$ be a Cauchy surface. Denote by $\mathcal{R}\colon M\to\Sigma$ the function that associates to $x\in M$ the unique intersection point with Σ of the ray through x of the natural foliation of M; denote by $M_{>0}$ the complement in M of the singular lightlike lines.*

Then

$$J^+_M(\Sigma)=\overline{\{x\in M_{>0}\mid \boldsymbol{g}(x)\geq\boldsymbol{g}(\mathcal{R}(x))\}}.$$

Proof Since Σ is a Cauchy surface of M, $J^+_M(\Sigma)\cap J^-_M(\Sigma)=\Sigma$ and $J^+_M(\Sigma)\cup J^-_M(\Sigma)=M$. Since $\boldsymbol{g}$ is increasing toward the future along the timelike rays of the natural foliation of M,

$$\{x\in M_{>0}\mid \pm\boldsymbol{g}(x)\geq\pm\boldsymbol{g}(\mathcal{R}(x))\}\subset J^\pm_M(\Sigma).$$

Furthermore, since M is globally hyperbolic and Σ compact, $J^\pm_M(\Sigma)$ are closed. Hence

$$\overline{\{x\in M_{>0}\mid \pm\boldsymbol{g}(x)\geq\pm\boldsymbol{g}(\mathcal{R}(x))\}}\subset J^\pm_M(\Sigma).$$

Since $M_{>0}$ is dense in M, we have

$$\bigcup_{\epsilon\in\{+,-\}}\overline{\{x\in M_{>0}\mid \epsilon\boldsymbol{g}(x)\geq\epsilon\boldsymbol{g}(\mathcal{R}(x))\}}=M.$$

Furthermore

$$\Sigma \subset \bigcap_{\epsilon \in \{+,-\}} \overline{\{x \in M_{>0} \mid \epsilon \boldsymbol{g}(x) \geq \epsilon \boldsymbol{g}(\mathcal{R}(x))\}} \subset J_M^+(\Sigma) \cap J_M^-(\Sigma) = \Sigma,$$

and it follows that

$$\overline{\{x \in M_{>0} \mid \pm \boldsymbol{g}(x) \geq \pm \boldsymbol{g}(\mathcal{R}(x))\}} = J_M^{\pm}(\Sigma). \qquad \square$$

Proposition 2.19 *Let $\tilde{\tau}\colon \Sigma \to \mathbb{R}_+$ be distance-like, and $M := M(\tilde{\tau})$ with its associated polyhedral embedding $\iota\colon \Sigma \to M$.*

The embedding ι is convex if and only if $\tilde{\tau}$ is Q-convex.

Proof We identify Σ with $\iota(\Sigma)$ and denote by $\mathcal{R}\colon M \to \Sigma$ the map that associates to any $x \in M$ the intersection point of the ray (of the natural foliation) through x with Σ. Consider a spacelike geodesic $c\colon [a,b] \to M$ such that $c(a), c(b) \in J^+(\Sigma)$. A direct computation in a chart gives that both $\boldsymbol{g} \circ c$ and $\boldsymbol{g} \circ \mathcal{R} \circ c$ are continuous piecewise of the form $s \mapsto -s^2 + \alpha s + \beta$ and that $\boldsymbol{g} \circ c$ is $\mathscr{C}^1$. Furthermore, the derivatives of $\boldsymbol{g} \circ \mathcal{R} \circ c$ and $\tilde{\tau} \circ \mathcal{R} \circ c$ may be discontinuous at $s \in [a,b]$ only when the ray through $c(s)$ encounters an edge of Σ. At such an s, these two functions $\boldsymbol{g} \circ \mathcal{R} \circ c$ have the same Q-convexity.

- Assume that $\tilde{\tau}$ is Q-convex and consider a spacelike geodesic $c\colon [a,b] \to M$ such that $c(a), c(b) \in J^+(\Sigma)$. By Lemma 2.17, $\boldsymbol{g} \circ c - \boldsymbol{g} \circ \mathcal{R} \circ c$ is nonnegative and by Lemma 2.18 we thus have $c([a,b]) \subset J^+(\Sigma)$. Finally, $J^+(\Sigma)$ is convex, and hence ι is convex.
- Assume that $\tilde{\tau}$ is not Q-convex. There thus exists an edge e in Σ around which $\tilde{\tau}$ is strictly Q-concave. Consider two points x and y in Σ, each on a different side of said edge. We can choose x and y close enough so that they lie in a chart of M around $\iota(e)$. Then consider the geodesic $c\colon [a,b] \to M$ in this chart from x to y. It follows from Lemma 2.17 that $\boldsymbol{g} \circ c < \boldsymbol{g} \circ \mathcal{R} \circ c$ on $]a,b[$. Thus by Lemma 2.18 $c(]a,b[)$ is not in $J^+(\Sigma)$ and hence $J^+(\Sigma)$ is not convex; neither is ι. $\square$

Proposition 2.20 *Let $\tau \in \mathbb{R}_+^S$. Up to equivalence there is at most one adapted triangulation $\mathcal{T}$ such that the distance-like extension $\tilde{\tau}_{\tau,\mathcal{T}}\colon \Sigma \to \mathbb{R}_+$ is Q-convex.*

Proof Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two adapted triangulations (Σ, S) such that both $f_1 := \tilde{\tau}_{\tau,\mathcal{T}_1}$ and $f_2 := \tilde{\tau}_{\tau,\mathcal{T}_2}$ are Q-convex. For all edges e of $\mathcal{T}_1$, the function $f|_{1e}$ is continuous quadratic while the function $f|_{2e}$ is piecewise quadratic and Q-convex; also, they are equal on the vertices of e. By Lemma 2.17 it thus follows that $f_2 \leq f_1$ on e. For any triangle T of $\mathcal{T}_1$, $f_1 \geq f_2$ on ∂T, and applying again Lemma 2.17 along any segment $[a,b]$ of T with $a, b \in \partial T$, we deduce that $f_1 \geq f_2$ on T. Therefore $f_1 \geq f_2$ on the whole Σ. We show in the same way that $f_1 \leq f_2$, and hence $f_1 = f_2$. The triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are then equivalent. $\square$

Corollary 2.21 *Let $\tau \in \mathbb{R}_+^S$. There is at most one Q-convex distance-like extension $\tilde{\tau}$ of τ to the whole Σ.*

Definition 2.22 (admissible times) Define $\mathcal{P}$ to be the set of $\tau \in \mathbb{R}_+^S$ such that there exists an adapted triangulation $\mathcal{T}$ of (Σ, S) inducing a Q-convex distance-like extension $\tilde{\tau}_{\tau,\mathcal{T}}$. Elements of $\mathcal{P}$ are called admissible times.

For $\tau \in \mathcal{P}$, we denote by $\mathcal{T}_\tau$ the unique adapted triangulation of Σ (up to equivalence) such that $\tilde{\tau}_{\tau,\mathcal{T}_\tau}$ is Q-convex. We define as well $\tilde{\tau}_\tau := \tilde{\tau}_{\tau,\mathcal{T}_\tau}$ and $M(\tau) := M(\tilde{\tau}_\tau)$.

As a corollary of Proposition 2.20 and Theorem 1, we obtain the following:

Proposition 2.23 *With $\sim$ the equivalence relation between polyhedral embeddings, the function*

$$\{(M,\iota) \mid M \text{ radiant}, \iota\colon \Sigma \to M \text{ polyhedral convex embedding}\}/\sim \;\to \mathcal{P}, \quad (\iota, M) \mapsto (\boldsymbol{g} \circ \iota)|_S$$

is bijective.

3 The domain of admissible times

For this whole section, we give ourselves a marked locally Euclidean surface with conical singularities (Σ, S). While Proposition 2.23 parametrizes polyhedral embeddings by the domain $\mathcal{P} \subset \mathbb{R}^S$, for now, little is known about it, and before studying the image of $\tau \mapsto M(\tau)$ we shall provide a thorough description. More precisely, we prove the following:

Theorem 2 *Let $\mathbf{1}_S$ the indicator function of S, H the linear hyperplane of $\mathbb{R}^S$ orthogonal to $\mathbf{1}_S$, and π the orthogonal projection onto H. Define $\overline{\mathcal{P}} = \pi(\mathcal{P}) \subset H$. Then we have the following properties:*

- (a) *$\overline{\mathcal{P}}$ is a convex compact polyhedron.*
- (b) *$\mathcal{P} = (\overline{\mathcal{P}} + \mathbb{R} \cdot \mathbf{1}_S) \cap \mathbb{R}_+^S$.*
- (c) *The interior of $\overline{\mathcal{P}}$ contains $0 \in \mathbb{R}^S$.*
- (d) *With $\mathscr{T} := \{\mathcal{T}_\tau \mid \tau \in \mathcal{P}\}$, each $\overline{\mathcal{P}}_\mathcal{T} := \{\pi(\tau) \mid \mathcal{T}_\tau = \mathcal{T}\} \subset \overline{\mathcal{P}}$ is a convex polyhedron of H for $\mathcal{T} \in E$. Furthermore, the family $(\overline{\mathcal{P}}_\mathcal{T})_{\mathcal{T} \in \mathscr{T}}$ is a finite cellulation of $\overline{\mathcal{P}}$.*
- (e) *The support planes Π of $\mathcal{P}$ whose intersection with $\mathcal{P}$ has nonempty interior relative to Π are either of the form "$\tau_\sigma = 0$" for some $\sigma \in S$ or "$Q^*(\tau) = 0$" for some unflippable immersed hinge Q around an edge of a triangulation $\mathcal{T}_\tau$ for some $\tau \in \mathcal{P}$ (see Definitions 3.1, 3.5, 3.8 and 3.15).*

The starting point is to study "local" criteria for Q-convexity. By local, we mean at each edge of a given triangulation; the following definitions make this notion precise:

Definition 3.1 (hinge) A hinge is a quadrilateral $[ABCD] \subset \mathbb{E}^2$ together with a diagonal $[AC]$ such that $[AC] \subset [ABCD]$.

Beware that the quadrilateral of a hinge need not be convex. If convex with vertices in general positions, a quadrilateral may define two hinges: one for each interior diagonal. Otherwise only one hinge may be defined.

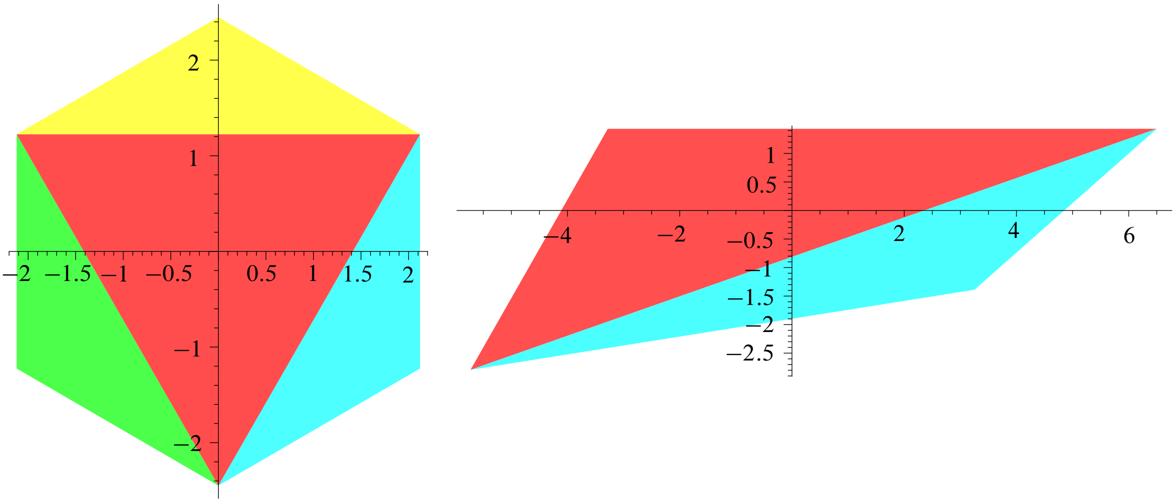


Figure 2: Projection of the domain of admissible τ. On the left, the domain $\overline{\mathcal{P}}$ of the surface is obtained by gluing two copies of an equilateral triangle's edges to edges. The central cell (red) corresponds to the Delaunay triangulation of the surface. In contrast, each of the other cells corresponds to the triangulations obtained after flipping an edge of the Delaunay triangulation. On the right, the domain $\overline{\mathcal{P}}$ of the surface is obtained by two copies of the triangle of vertices $(0,0)$, $(1,1)$ and $(0,3)$. The upper triangle corresponds to the Delaunay triangulation, while the lower one corresponds to the triangulation obtained after the only flip possible from the Delaunay triangulation. The domains are represented in an orthonormal basis of the plane H. The pictures were generated using SageMath [29].

Definition 3.2 (flippable hinge and hinge flipping) Let $Q=([ABCD],[AC])$ be a hinge. If $[ABCD]$ is convex and the four points A, B, C and D are in general position, then Q is flippable, and its flipping is the hinge $Q'=([ABCD],[DB])$. If $[ABCD]$ is not convex or A, B, C and D are not in general position, then Q is unflippable.

Definition 3.3 (weighted hinge) A weighted hinge is the datum of a hinge, $Q=([ABCD],[AC])$, and a function $\tau\colon\{A,B,C,D\}\to\mathbb{R}$.

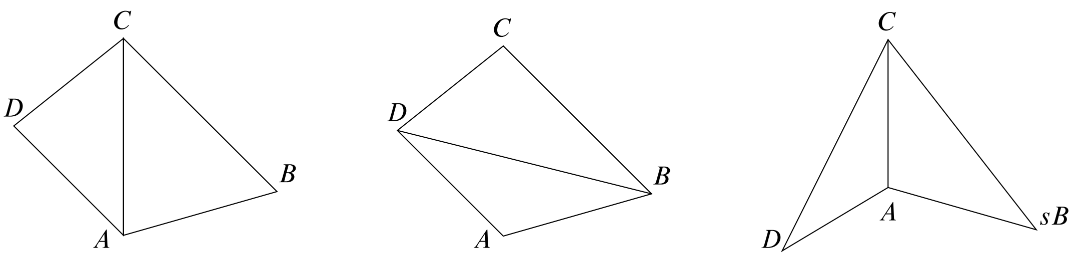


Figure 3: Different hinges. Left: a hinge $([ABCD],[AC])$. Center: its flipping $([ABCD],[DB])$. Right: a nonconvex hinge.

Definition 3.4 (τ-legal/τ-critical hinge) Let (Q,τ) be a weighted hinge. Denote by $\tilde{\tau}_{\tau,Q}\colon Q \to \mathbb{R}$ the distance-like function induced by the triangulation $\mathcal{T} = ([ABC],[ADC])$. A hinge Q is τ-legal (resp. τ-critical, τ-illegal) if $\tilde{\tau}_{\tau,Q}$ is Q-convex (resp. $\mathscr{C}^1$, strictly Q-concave).

Each edge e of a given triangulation $\mathcal{T}$ provides a hinge; indeed e bounds two triangles $T_1, T_2 \in \mathcal{T}$, and the gluing of these two triangles along e is a hinge. Beware that two such triangles might actually be the same in $\mathcal{T}$ (a triangle glued to itself), but we take two copies to construct the hinge. More generally, we will need to consider immersed hinges.

Definition 3.5 An immersed hinge is a couple (Q,η) with Q a hinge in $\mathbb{E}^2$ and $\eta\colon Q \to \Sigma$ an isometric immersion. An immersed hinge (Q,η) is embedded if the restriction $\eta|_{\mathrm{Int}(Q)}$ to the interior of Q is an embedding.

The hinge associated with an edge is embedded if and only if the triangles bounded by e are different in $\mathcal{T}$.

After an analysis of criteria ensuring τ-legality of a given hinge, we notice the set of τ for which a given hinge is τ-legal is the set of solutions of an affine inequality, and hence a convex set. Then, we turn to the whole surface and try to construct triangulations for which all hinges are τ-legal for a given τ.

Definition 3.6 (τ-Delaunay triangulation) Let $\mathcal{T}$ be an adapted triangulation of Σ.

The triangulation $\mathcal{T}$ is τ-Delaunay if the following equivalent properties are satisfied:

(i) $\tilde{\tau}_{\tau,\mathcal{T}}$ is Q-convex.

(ii) Every hinge of $\mathcal{T}$ is τ-legal.

For a given triangulation $\mathcal{T}$, the set of $\tau \in \mathbb{R}_+^S$ such that $\mathcal{T}$ is τ-Delaunay is the set solutions of a system of affine inequalities, and hence a convex set; hence the first part of Theorem 2(d). However, $\mathcal{P}$ is a possibly infinite union of such domains; therefore Theorem 2(a) and the second part of (d) are not direct corollaries. We thus reverse the problem and construct a τ-Delaunay triangulation with τ given a priori.

The definition of τ-Delaunay triangulation is coherent with the usual definition of Delaunay triangulation. Indeed, an adapted triangulation of (Σ, S) is a subtriangulation of the Delaunay cellulation if and only if it is 0-Delaunay. The Delaunay cellulation can either be constructed as the dual of the Voronoi cellulation (see [24] for a thorough exposition) or via a flipping algorithm starting from a given adapted triangulation. The flipping algorithm is based upon the following remark (Lemma 3.9): for a given τ, if a hinge is τ-illegal, then its flipping (if it exists) is τ-legal. The algorithm then proceeds by flipping τ-illegal hinges one by one in the hope that after finitely many iterations there will not be any τ-illegal hinges left. Proposition 3.17 ensures the algorithm behaves mostly as expected: it stops after finitely many iterations on a triangulation without any flippable τ-illegal hinges. To complete the analysis of the flipping algorithm, we show the resulting triangulation is τ-Delaunay if and only if there exists such a triangulation.

We end the section applying the results obtained on the flipping algorithm to prove Theorem 2.

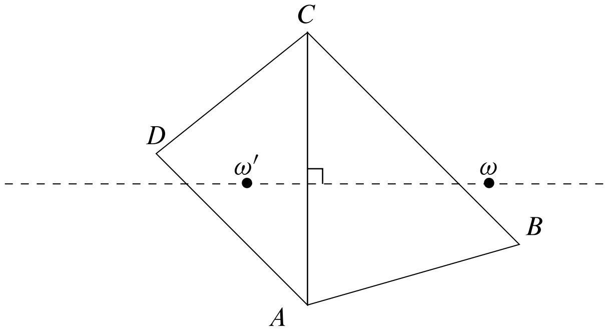


Figure 4

3.1 Q-convexity on hinges

Before going any further, we notice that the group $\mathrm{Isom}(\mathbb{E}^2)$ acts naturally on weighted hinges and preserves legality.

In this subsection, we give ourselves a hinge $Q=([ABCD],[AC])$ and some weights τ. For simplicity's sake, we choose a Cartesian coordinate system (x,y) of $\mathbb{E}^2$, set $A=O$ as the origin of this coordinate system, and put C on the vertical axis above A. Denote by ω and τ_0 (resp. ω' and τ_0') the parameters given by Lemma 2.3 on $[ABC]$ (resp. $[ADC]$) for the weights τ; define

$$\tau_{ABC}\colon \mathbb{E}^2\to\mathbb{R},\quad x\mapsto \tau_0-\mathrm{d}(x,\omega)^2,\qquad \tau_{ADC}\colon \mathbb{E}^2\to\mathbb{R},\quad x\mapsto \tau_0'-\mathrm{d}(x,\omega')^2.$$

Figure 4 sums up the situation.

Remark Note that $\mathrm{d}(\omega,C)^2-\mathrm{d}(\omega,A)^2=\mathrm{d}(\omega',C)^2-\mathrm{d}(\omega',A)^2$ and hence $\overrightarrow{\omega\omega'}\perp\overrightarrow{AC}$. More generally, from the proof of Lemma 2.3, one sees that the orthogonal projection of ω on the line (AC) only depends on A, C, τ_A, and τ_C.

Proposition 3.7 (Q-convexity criteria) *Under this subsection's hypotheses, the following are equivalent*:

(i) $\tilde\tau_{\tau,Q}$ *is Q-convex.*

(ii) $\tilde\tau_{\tau,Q}$ *is Q-convex along some segment crossing* $[AC]$.

(iii) $\tau_{ABC}\le\tau_{ACD}$ *on* $[ACD]$ *and* $\tau_{ABC}\ge\tau_{ACD}$ *on* $[ABC]$.

(iv) $\tau_{ABC}(D)\le\tau_{ACD}(D)$ *or* $\tau_{ABC}(B)\ge\tau_{ACD}(B)$.

(v) $x_\omega\ge x_{\omega'}$.

(vi) $$\left(\frac{y_B}{|x_B|}+\frac{y_D}{|x_D|}\right)\tau_C+\left(\frac{AC-y_B}{|x_B|}+\frac{AC-y_D}{|x_D|}\right)\tau_A\le\frac{AC}{|x_D|}\tau_D+\frac{AC}{|x_B|}\tau_B+K$$
with
$$K=\frac{AC}{|x_B|}(AB^2-AC\,y_B)+\frac{AC}{|x_D|}(AD^2-AC\,y_D).$$

(vii) *Denoting by $\vec{u}\wedge\vec{v}$ the determinant $|\vec{u}\vec{v}|$,*

$$(\overrightarrow{AB}\wedge\overrightarrow{AD})\tau_C+(\overrightarrow{CD}\wedge\overrightarrow{CB})\tau_A-(\overrightarrow{CA}\wedge\overrightarrow{CB})\tau_D-(\overrightarrow{AC}\wedge\overrightarrow{AD})\tau_B-K\le 0$$

with

$$K=\overrightarrow{AC}\wedge\overrightarrow{AD}(\overrightarrow{AB}\cdot\overrightarrow{CB})+\overrightarrow{CA}\wedge\overrightarrow{CB}(\overrightarrow{AD}\cdot\overrightarrow{CD}).$$

Proof • **(i) $\Longrightarrow$ (ii)** This follows by definition.

• **(ii) $\Longrightarrow$ (i)** Since the line $(\omega\omega')$ is perpendicular to (AC) it follows that $\partial\tau_{ABC}/\partial y=\partial\tau_{ACD}/\partial y$. Then $\overrightarrow{\text{grad}}\,\tau_{[ABC]}-\overrightarrow{\text{grad}}\,\tau_{[ACD]}$ is horizontal and the sign of $\langle\overrightarrow{\text{grad}}\,\tau_{[ABC]}-\overrightarrow{\text{grad}}\,\tau_{[ACD]}\mid\vec{u}\rangle$ does not depend on $\vec{u}$ as long as $\vec{u}$ is directed toward increasing x.

• **(i) $\Longrightarrow$ (v) and (v) $\Longrightarrow$ (ii)** We have that (v) is equivalent to $\partial\tau_{ABC}/\partial x\ge\partial\tau_{ACD}/\partial x$, which is equivalent to Q-convexity along the direction perpendicular to $[AC]$.

• **(i) $\Longrightarrow$ (iii)** Let $P\in[ABC]$ and choose some $P'\in[ACD]$ such that $[P'P]$ crosses $[AC]$. The function $\tau_{[ACD]}$ is $\mathscr{C}^1$ while $\tilde{\tau}_{\tau,Q}$ is Q-convex along $[P'P]$. The same argument as in the proof of Lemma 2.17 gives the first inequality. The second is proven the same way.

• **(iii) $\Longrightarrow$ (iv)** This is trivial.

• **(iv) $\Longrightarrow$ (ii)** Consider any segment $[PB]$ with $P\in[ACD]$. Along such a segment, $\tilde{\tau}_{\tau,Q}$ is either Q-convex or strictly Q-concave. The inequality $\tau_{ABC}(B)\ge\tau_{ACD}(B)$ implies it is the former. The same argument shows $\tau_{ABC}(D)\le\tau_{ACD}(D)\Longrightarrow$ (ii).

• **(v) $\Longleftrightarrow$ (vi)** Solve explicitly the system in the proof of Lemma 2.3 for both sides in (v).

• **(vii) $\Longleftrightarrow$ (vi)** These are geometric rewritings of each other, which can be checked by rewriting terms in coordinates. □

The previous proposition shows that Q-convexity is an affine constraint on τ for a given hinge. Since we will have to consider multiple hinges for multiple triangulations, we introduce the following:

Definition 3.8 (affine form of a hinge) Letting $Q=([ABCD],[AC])$ be a hinge, define the affine form associated to Q by

$$Q^*:\mathbb{R}_+^{\{A,B,C,D\}}\to\mathbb{R},\quad \tau\mapsto\lambda_C\tau_C+\lambda_A\tau_A-\lambda_D\tau_D-\lambda_B\tau_B-K,$$

where

$$\lambda_C=\overrightarrow{AB}\wedge\overrightarrow{AD},\quad \lambda_A=\overrightarrow{CD}\wedge\overrightarrow{CB},\quad \lambda_D=\overrightarrow{CA}\wedge\overrightarrow{CB},\quad \lambda_B=\overrightarrow{AC}\wedge\overrightarrow{AD},$$
$$K=\overrightarrow{AC}\wedge\overrightarrow{AD}(\overrightarrow{AB}\cdot\overrightarrow{CB})+\overrightarrow{CA}\wedge\overrightarrow{CB}(\overrightarrow{AD}\cdot\overrightarrow{CD}).$$

Remark The affine form Q^* is defined in such a way that $\tilde{\tau}_{\tau,Q}$ is Q-convex if and only if $Q^*(\tau)\le 0$.

Remark If (Q,η) is an immersed hinge of (Σ, S) with η sending vertices into S and with $Q = ([ABCD],[AC])$, we can then define a corresponding affine form $\mathbb{R}_+^S \to \mathbb{R}$

$$\mathbb{R}_+^S \to \mathbb{R}, \quad \tau \mapsto Q^*(\tau \circ \eta|_{\{A,B,C,D\}}).$$

If there is no ambiguity, we shall also denote it by Q^*.

Remark A hinge Q is τ-critical if and only if $Q^*(\tau) = 0$.

Lemma 3.9 *Let* $Q = ([ABCD],[AC])$ *be a flippable hinge and let* Q' *be its flipped hinge. As functions* $\mathbb{R}^{\{A,B,C,D\}} \to \mathbb{R}$ *we have*

$$Q'^* = -Q^*.$$

Proof This can, of course, be checked directly in coordinates, but we provide a more geometric proof. Following the notation of Definition 3.8 we write

$$Q^*\colon \mathbb{R}_+^{\{A,B,C,D\}} \to \mathbb{R}, \quad \tau \mapsto \lambda_C\tau_C + \lambda_A\tau_A - \lambda_D\tau_D - \lambda_B\tau_B - K,$$
$$Q'^*\colon \mathbb{R}_+^{\{A,B,C,D\}} \to \mathbb{R}, \quad \tau \mapsto \lambda'_C\tau_C + \lambda'_A\tau_A - \lambda'_D\tau_D - \lambda'_B\tau_B - K',$$

where

$$\lambda_C = \overrightarrow{AB} \wedge \overrightarrow{AD}, \quad \lambda_A = \overrightarrow{CD} \wedge \overrightarrow{CB}, \quad \lambda_D = \overrightarrow{CA} \wedge \overrightarrow{CB}, \quad \lambda_B = \overrightarrow{AC} \wedge \overrightarrow{AD},$$
$$\lambda'_D = -\overrightarrow{BC} \wedge \overrightarrow{BA}, \quad \lambda'_B = -\overrightarrow{DA} \wedge \overrightarrow{DC}, \quad \lambda'_A = -\overrightarrow{DB} \wedge \overrightarrow{DC}, \quad \lambda'_C = -\overrightarrow{BD} \wedge \overrightarrow{BA}.$$

We check that

$$\lambda'_A = -\overrightarrow{DB} \wedge \overrightarrow{DC} = -(\overrightarrow{DC} + \overrightarrow{CB}) \wedge \overrightarrow{DC} = -\overrightarrow{CB} \wedge \overrightarrow{DC} = -\overrightarrow{CD} \wedge \overrightarrow{CB} = -\lambda_A,$$

and we check the same way that $\lambda'_B = -\lambda_B$, $\lambda'_C = -\lambda_C$, and $\lambda'_D = -\lambda_D$.

A quick way to prove that $K' = -K$ is to notice that

$$K = (AB \cdot CB \cdot CD \cdot DA)\sin(\widehat{BAD} + \widehat{DCB}), \quad K' = (AB \cdot CB \cdot CD \cdot DA)\sin(\widehat{CBA} + \widehat{ADC}),$$

and that $\widehat{BAD} + \widehat{DCB} + \widehat{CBA} + \widehat{ADC} = 0 \bmod 2\pi$. □

Corollary 3.10 *Let* (Q,τ) *be a weighted flippable hinge. Then* Q *is* τ*-critical if and only if its flipping* Q' *is* τ*-critical.*

Corollary 3.11 *Let* (Q,τ) *be a weighted flippable hinge and* Q' *the flipping of* Q*. If* Q *is not* τ*-critical, then the following are equivalent:*

(i) Q *is* τ*-legal.*

(ii) Q' *is* τ*-illegal.*

Lemma 3.12 *For any hinge* Q*, the indicator function* $\mathbf{1}_S$ *is in the kernel of the linear part of* Q^**, eg*

$$\text{for all } \tau \in \mathbb{R}^S \text{ and } \lambda \in \mathbb{R}, \quad Q^*(\tau + \lambda\mathbf{1}_S) = Q^*(\tau).$$

Proof Using the notation of Definition 3.8, we have

$$\lambda_A+\lambda_C-\lambda_B-\lambda_D=\overrightarrow{CD}\wedge\overrightarrow{CB}+\overrightarrow{AB}\wedge\overrightarrow{AD}-\overrightarrow{AC}\wedge\overrightarrow{AD}-\overrightarrow{CA}\wedge\overrightarrow{CB}=\overrightarrow{AD}\wedge\overrightarrow{CB}+\overrightarrow{CB}\wedge\overrightarrow{AD}=0. \qquad \square$$

Corollary 3.13 *For all $\tau \in \mathcal{P}$ and all $\lambda \in \mathbb{R}$,*

$$\tau+\lambda\mathbf{1}_S \in \mathcal{P} \iff \tau+\lambda\mathbf{1}_S \geq 0.$$

Corollary 3.14 *With the notation of Theorem 2,*

$$\mathcal{P}=(\overline{\mathcal{P}}+\mathbb{R}\cdot\mathbf{1}_S)\cap\mathbb{R}_+^S.$$

3.2 The flipping algorithm

Let $\mathcal{T}$ be an adapted triangulation of (Σ, S). Consider (Q,η) an immersed hinge given by an edge of $\mathcal{T}$. We would like to flip (Q,η), ie construct a new triangulation of (Σ, S) with $\eta(Q)$ replaced by $\eta(Q')$ with Q' the flip of Q. There are three cases:

- η is not an embedding. Then the diagonal one wants to replace is also a side of the hinge. Hence one cannot simply replace it without modifying the triangulation $\mathcal{T}$ elsewhere.
- η is embedded but Q is not flippable.
- η is embedded and Q is flippable. Then the flipped hinge Q' is well defined, $\eta\colon Q' \to \Sigma$ is well defined, $\eta(Q')=\eta(Q)$ so that we only modify $\mathcal{T}$ locally, and the new triangulation $\mathcal{T}'$ is composed of nondegenerated triangles.

This remark motivates the following definitions:

Definition 3.15 (flippable immersed hinge) An immersed hinge (Q,η) is flippable if it is embedded and Q is flippable; it is unflippable otherwise.

Definition 3.16 (flipping algorithm) Let $\mathcal{T}_0$ be any adapted triangulation of (Σ, S) and let $\tau\colon S \to \mathbb{R}_+$. The flipping algorithm proceeds as follows:

(1) Set $i=0$.

(2) Let L_i be the set of τ-illegal flippable embedded hinges (Q,η) induced by the edges of the current triangulation $\mathcal{T}_i$.

(3) If L_i is nonempty,

(a) choose some immersed hinge (Q,η) in L_i,

(b) replace the hinge (Q,η) by its flipping (Q',η) in $\mathcal{T}_i$ to obtain a new triangulation $\mathcal{T}_{i+1}$,

(c) increment i and go to step (2).

(4) If L_i is empty, the algorithm stops and returns $\mathcal{T}_i$.

The goal of the section is to prove the following:

Proposition 3.17 *Let $\tau\colon S \to \mathbb{R}_+$. For any starting triangulation $\mathcal{T}_0$, the flipping algorithm for τ starting at $\mathcal{T}_0$ stops on some triangulation $\mathcal{T}_\tau$ after finitely many iterations and every flippable immersed hinge in $\mathcal{T}_\tau$ is τ-legal. Furthermore,*

- *$\tau \in \mathcal{P}$ if and only if $\mathcal{T}_\tau$ is τ-Delaunay,*
- $\max_\Sigma \tilde{\tau}_{\tau,\mathcal{T}_\tau} \leq \max_S \tau + \max_\Sigma \tilde{\tau}_{0,\mathcal{T}_0}$.

Remark The notation $\mathcal{T}_\tau$ of this last proposition is consistent with the one introduced in Definition 2.22.

Two lemmas are key to the proof; the first is Lemma 3.18, which states that $\tilde{\tau}_{\tau,\mathcal{T}_i}$ is decreasing along the iterations of the algorithm; the second is Lemma 3.22, which implies that immersed unflippable hinges are always τ-legal for $\tau \in \mathcal{P}$, even those that are not associated to an edge. Lemma 3.22 will again be useful in the following section.

Lemma 3.18 *Let $\tau\colon S \to \mathbb{R}_+$ and let $\mathcal{T}_0$ be an adapted triangulation. Let $(\mathcal{T}_i)_{i\in I}$ be the sequence of triangulation given by the flipping algorithm with weights τ and starting at $\mathcal{T}_0$, where $I = [\![0,n]\!]$ or $\mathbb{N}$.*

Then the associated sequence of distance-like functions $(\tilde{\tau}_{\tau,\mathcal{T}_i})_{i\in I}$ is decreasing:

- *for all $i,j \in I$ with $i \leq j$ we have $\tilde{\tau}_{\tau,\mathcal{T}_i} \geq \tilde{\tau}_{\tau,\mathcal{T}_j}$,*
- *for all $i,j \in I$ with $i < j$ there exists $x \in \Sigma$ such that*

$$\tilde{\tau}_{\tau,\mathcal{T}_i}(x) > \tilde{\tau}_{\tau,\mathcal{T}_j}(x).$$

Proof Let $i \in I$ be such that $i+1 \in I$. The triangulation $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by flipping an embedded hinge, say (Q,η), of $\mathcal{T}_i$ with $Q = ([ABCD],[AC])$. Then:

- For all $x \in \Sigma \setminus \eta(\mathrm{Int}(Q))$, $\tilde{\tau}_{\tau,\mathcal{T}_i}(x) = \tilde{\tau}_{\tau,\mathcal{T}_{i+1}}(x)$. Indeed, for $x \notin \eta(Q)$, the triangle containing x is the same in $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$.
- For all $x \in \eta(\mathrm{Int}(Q))$, $\tilde{\tau}_{\tau,\mathcal{T}_i}(x) > \tilde{\tau}_{\tau,\mathcal{T}_{i+1}}(x)$. Indeed, $\tilde{\tau}_{\tau,Q}$ and $\tilde{\tau}_{\tau,Q'}$ are equal on $[AB]$, $[BC]$, $[CD]$, and $[DA]$; by hypothesis $\tilde{\tau}_{\tau,Q}$ is strictly Q-concave and, from Corollary 3.11, $\tilde{\tau}_{\tau,Q'}$ is strictly Q-convex. Applying Lemma 2.17 on segments going from side to side of $[ABCD]$ we obtain

$$\text{for all } x \in \mathrm{Int}(Q), \quad \tilde{\tau}_{\tau,Q} > \tilde{\tau}_{\tau,Q'}. \qquad \square$$

Corollary 3.19 *No triangulation appears twice in the sequence $(\mathcal{T}_i)_{i\in I}$ given by the flipping algorithm.*

Lemma 3.20 *Let $\tilde{\tau}$ be a nonnegative distance-like function on (Σ, S). If $\tilde{\tau}$ is $\mathscr{C}^1$ on some geodesic of length ℓ then*

$$\max \tilde{\tau} \geq \tfrac{1}{4}\ell^2.$$

Proof Let $c\colon [a,b] \to \Sigma$ be an arc length parametrization of such a geodesic and let $f := \tilde{\tau} \circ c$. We have

$$f\colon [a,b] \to \mathbb{R}, \quad x \mapsto -x^2 + \alpha x + \beta,$$

for some $\alpha, \beta \in \mathbb{R}$. Furthermore $\tilde{\tau} \geq 0$, and so $f(a) \geq 0$ and $g(b) \geq 0$.

Define $u\colon [a,b] \to \mathbb{R}$ to be the unique affine function such that $u(a) = f(a)$ and $u(b) = b$. We thus have for all $x \in [a,b]$, $f(x) = u(x) - (x-a)(x-b)$. On the one hand, $f(a)$ and $f(b)$ are nonnegative, so u is nonnegative. On the other hand,

$$\max_{x \in [a,b]} (-(x-a)(x-b)) = \tfrac{1}{4}(b-a)^2 = \tfrac{1}{4}\ell^2. \qquad \square$$

Lemma 3.21 *For $C \in \mathbb{R}_+^*$, let E_C be the set of adapted triangulations $\mathcal{T}$ of (Σ, S) such that*

$$\textit{there exists } \tau \in \mathbb{R}_+^S \textit{ with } \max \tilde{\tau}_{\tau,\mathcal{T}} \leq C.$$

Then E_C is finite.

Proof Let $\mathcal{T}$ be an adapted triangulation such that there exists $\tau \in \mathbb{R}_+^S$ with $\max \tilde{\tau}_{\tau,\mathcal{T}} \leq C$. Choose such a τ. Let e be the longest edge of $\mathcal{T}$. From Lemma 3.20 with $L = \text{length}(e)$

$$\tfrac{1}{4}L^2 \leq \max_e \tilde{\tau}_{\tau,\mathcal{T}} \leq C,$$

and thus $L \leq 2\sqrt{C}$. Therefore the triangulation $\mathcal{T}$ only has edges of length less than $2\sqrt{C}$.

Consider a finite covering $\hat{\Sigma}$ of Σ branched above S such that all cone angles of $\hat{\Sigma}$ are bigger than 2π. Note that $\hat{\Sigma}$ is locally CAT(0), so its universal (unbranched) covering $\tilde{\hat{\Sigma}}$ is CAT(0) by [1, Theorem 3.3.1], and hence for any two points in $\tilde{\hat{\Sigma}}$ above S there exists at most one geodesic; see [1, Section 2.2]. Furthermore, any geodesic of length at most $2\sqrt{C}$ in Σ from a point A of S to a point B of S lifts to a geodesic in $\tilde{\hat{\Sigma}}$ of the same length starting from a fixed $\hat{A}$ to some unfixed lift $\hat{B}$ of B in the ball of radius $2\sqrt{L}$ around $\hat{A}$. There are finitely many such $\hat{B} \in \tilde{\hat{\Sigma}}$, thus finitely such geodesics in $\tilde{\hat{\Sigma}}$. There are thus only finitely many geodesics of Σ from S to S of length bounded by $2\sqrt{C}$; hence there are only finitely many triangulations with edges of length at most $2\sqrt{C}$. $\square$

Lemma 3.22 *Let Q be an unflippable hinge with $Q = ([ABCD],[AC])$. If there exists some distance-like Q-convex function f on $[ABCD]$ extending $\tau\colon \{A,B,C,D\} \to \mathbb{R}$, then Q is τ-legal.*

Remark Beware that f-triangulations of $[ABCD]$ may be very different from the one induced by the hinge, ie $([ABC],[ACD])$.

Proof Without loss of generality, we may assume that C is in the convex hull of $[ABD]$. Define $g := \tilde{\tau}_{\tau,Q}$ and h the distance-like extension of $\tau|_{\{A,B,D\}}$ on $[ABD]$ given by Lemma 2.3. Both functions f and g are defined on $[ABCD] \subset [ABD]$ and h is defined on $[ABD]$. Furthermore, g is either Q-convex or Q-concave.

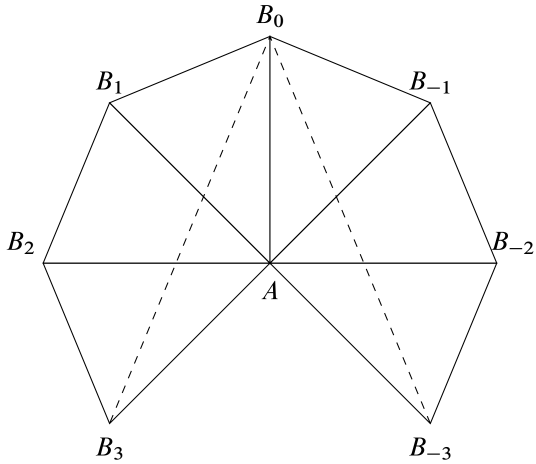


Figure 5

Applying Lemma 2.17 on sides of the hinge Q and then on any edge within Q and with extremities on the sides, we see that $f \le h$ and that either $h \ge g$ or $h \le g$, depending on whether g is Q-convex or Q-concave.

Since $g - h$ is affine on each triangle $[ACB]$ and $[ACD]$ and null at A, B, and D, we see that $g - h$ is nonpositive if and only if $g(C) - h(C) \le 0$. However, $g(C) = f(C) \le h(C)$, so $g \le h$, and hence $g = \tilde{\tau}_{\tau,Q}$ is Q-convex. □

Lemma 3.23 *Let $\tau\colon S \to \mathbb{R}_+$ and let (Q,η) be an immersed hinge with $Q = ([AB_{-1}B_0B_1],[AB_0])$ such that $[AB_0B_1]$ is obtained from $[AB_{-1}B_0]$ via a rotation and $\eta([AB_0]) = \eta([AB_1]) = \eta([AB_{-1}])$.*

Then there exists an immersed hinge $(\hat{Q},\hat{\eta})$ with $\hat{Q}$ unflippable such that (Q,η) is τ-legal if and only if $(\hat{Q},\hat{\eta})$ is τ-legal.

Proof Let $\theta = \widehat{B_0AB_1}$ and $n = \lceil \pi/\theta \rceil - 1$. If $\theta \ge \frac{1}{2}\pi$ then take $\hat{Q} = Q$ and $\hat{\eta} = \eta$.

Otherwise, construct the polygon $[AB_{-n}\cdots B_0 \cdots B_n]$ such that for each $k \in [\![1-n,n]\!]$, the triangle $[AB_kB_{k+1}]$ is obtained from $[AB_0B_1]$ via the rotation of center A and angle $\alpha_k = k\theta$. Define $\hat{Q} := [AB_nB_0B_{-n}]$ and

$$\hat{\eta}\colon \hat{Q} \to \Sigma, \quad x \in [AB_{2k}B_{2k+1}] \mapsto \eta(\rho_{-2\alpha_k}(x)), \quad x \in [AB_{2k-1}B_{2k}] \mapsto \eta(\rho_{-2\alpha_k}(x)),$$

where ρ_β denotes the rotation of center A and angle β; see Figure 5.

We have for all $k \in [\![-n,n]\!]$, $\hat{\eta}(B_k) = \eta(B_0)$. The weights $\tau \in \mathbb{R}^S$ thus induce weights $\hat{\tau} := \tau \circ \hat{\eta}$ such that $\hat{\tau}(B_k) = \tau(B_0) = \hat{\tau}(B_1) = \hat{\tau}(B_{-1})$. For $I, J, K \in \{A, B_{-n}, \dots, B_n\}$, denote by $\omega_{[IJK]}$ the center of $\tilde{\tau}_{\tau,[IJK]}$ and by $\omega_{[IJ]}$ the orthogonal projection of $\omega_{[IJK]}$ on the line (IJ). From the remark before Proposition 3.7, the orthogonal projection of $\omega_{[IJK]}$ on the line (IJ) only depends on $\hat{\tau}(I)$, $\hat{\tau}(J)$, and $[IJ]$; in other words $\omega_{[IJ]}$ does not depend on K.

Since $\hat\tau(B_n)=\hat\tau(B_0)=\hat\tau(B_{-1})$, we have that $\omega_{[B_0B_n]}$ (resp. $\omega_{[B_0B_{-n}]}$) is the middle of $[B_0B_n]$ (resp. of $[B_0B_{-n}]$). Since the lengths $(AB_k)_{k\in[\![-n,n]\!]}$ are equal, the perpendicular bisectors of $[B_0B_{-n}]$ and $[B_0B_n]$ intersect at A. Therefore $\omega_{AB_{-n}B_0}$ is on the right of $\omega_{AB_nB_0}$ on the perpendicular to (AB_0) at $\omega_{[AB_0]}$ if and only if $\omega_{[AB_0]}$ is on the ray $[AB_0)$. Hence, by Proposition 3.7(v), the hinge $\widehat{Q}$ is $\hat\tau$-legal if and only if $\omega_{[AB_0]}$ is on the ray $[AB_0)$.

The same argument shows Q is $\hat\tau$-legal if and only if $\omega_{[AB_0]}$ is on the ray $[AB_0)$. Finally, (Q,η) is τ-legal if and only if $(\widehat{Q},\hat\eta)$ is τ-legal. □

Proof of Proposition 3.17 By Lemma 3.18, the sequence of distance-like functions given by the flipping algorithm is bounded above by the first of the sequence $\tilde\tau_{\tau,\mathcal{T}_0}$. Since $\tilde\tau_{\tau,\mathcal{T}_0}-\tilde\tau_{0,\mathcal{T}_0}$ is affine on each triangle of $\mathcal{T}_0$ it is bounded by its value on S, and thus by $\max_S\tau$. Hence, for all i, $\tilde\tau_{\tau,\mathcal{T}_i}\le\max_S\tau+\max\tilde\tau_{0,\mathcal{T}_0}$. By Lemma 3.21, the flipping algorithm runs through a finite set of triangulations. Finally, by Corollary 3.19, the algorithm reaches a given triangulation at most once and thus stops after finitely many steps, say $n\in\mathbb{N}^*$. The algorithm stops when the set of flippable τ-illegal hinges is empty, so $\mathcal{T}_n$ has no flippable τ-illegal hinges.

If the final triangulation $\mathcal{T}_n$ is τ-Delaunay then by definition $\tau\in\mathcal{P}$. Assume $\tau\in\mathcal{P}$ and consider (Q,η) some unflippable hinge of $\mathcal{T}_n$. Either η is an embedding, in which case the weighted hinge $(Q,\tau\circ\eta)$ satisfies the hypotheses of Lemma 3.22 and the immersed hinge (Q,η) is then τ-legal, or η is not an embedding, in which case (Q,η) satisfies the hypotheses of Lemma 3.23, so the immersed hinge $(\widehat{Q},\hat\eta)$ provided by Lemma 3.23 satisfies the hypotheses of Lemma 3.22, thus being τ-legal, and so (Q,η) is τ-legal as well. Finally, $\mathcal{T}_n$ is τ-Delaunay. □

3.3 Description of the domain of admissible times

We may interpret Lemma 3.22 together with Lemma 3.23 in the following way: if $\tau\in\mathcal{P}$, then all unflippable immersed hinges of (Σ,S) with vertices in S are τ-legal. Furthermore, Proposition 3.17 shows the converse: the flipping algorithm stops on a triangulation $\mathcal{T}$, whose flippable hinges are all τ-legal if all unflippable hinges of (Σ,S) are τ-legal, in particular those of $\mathcal{T}$ are τ-legal, and hence $\mathcal{T}$ is τ-Delaunay. We thus proved the following:

Proposition 3.24 *Let* UFlip *be the set of the unflippable immersed hinges of* (Σ,S) *with vertices in* S*. Then*

$$\mathcal{P}=\bigcap_{(Q,\eta)\in\mathrm{UFlip}}(Q^*)^{-1}(\mathbb{R}_-).$$

In particular $\mathcal{P}$ *is a convex domain of* $\mathbb{R}_+^S$*.*

Remark Lemma 3.29 implies that UFlip is nonempty. We take the convention that the intersection is $\mathbb{R}_+^S$ if $\mathrm{UFlip}=\varnothing$.

Proposition 3.25 *For $\tau \in \mathcal{P}$, if $\tilde{\tau}$ is the unique Q-convex distance-like extension of τ to (Σ, S) then*

$$\tilde{\tau} = \min_{\mathcal{T}'} \tilde{\tau}_{\tau,\mathcal{T}'},$$

where $\mathcal{T}'$ runs through all adapted triangulations of (Σ, S).

Proof Take any adapted triangulation $\mathcal{T}$ of (Σ, S) and consider T a triangle of $\mathcal{T}$. On T, $\tilde{\tau}_{\tau,\mathcal{T}}$ is $\mathscr{C}^1$ while $\tilde{\tau}$ is Q-convex. By Lemma 2.17, $\tilde{\tau} \le \tilde{\tau}_{\tau,\mathcal{T}}$ on T. The triangle T is arbitrary; thus $\tilde{\tau} \le \tilde{\tau}_{\tau,\mathcal{T}}$ on Σ. □

Proposition 3.26 *The indicator function $\mathbf{1}_S$ of S is in the interior of $\mathcal{P}$.*

Proof To begin with, by [24, Theorem 4.4], each cell of the Delaunay cellulation $\mathcal{C}$ of (Σ, S) is isometric to a polygon inscribed into a circle of $\mathbb{E}^2$ whose center is a vertex of the Voronoi cellulation. For any given cell C of the Delaunay cellulation, with R_C the radius and $\omega \in \mathbb{E}^2$ the center of the circumscribed circle of the image of a development $\mathcal{D}\colon C \to \mathbb{E}^2$, the function

$$f : C \to \mathbb{R}_+, \quad x \mapsto R_C^2 - \mathrm{d}(\mathcal{D}(x), \omega)^2,$$

is distance-like $\mathscr{C}^1$ on C and $f(p) = 0$ for any vertex p of C; hence, for any adapted subtriangulation $\mathcal{T}$ of $\mathcal{C}$, for all $x \in C$, $\tilde{\tau}_{0,\mathcal{T}}(x) = f(\mathcal{D}(x))$.

Let e be an edge of the Delaunay cellulation, let C and C' be the two cells on each side of e, and denote by $\widetilde{C}$ and $\widetilde{C}'$ lifts in a covering branched above S such that $\widetilde{C} \neq \widetilde{C}'$ and such that $\widetilde{C} \cap \widetilde{C}' = \tilde{e}$ with $\tilde{e}$ a lift of e. Choose a development $\mathcal{D}$ of $\widetilde{C} \cup \widetilde{C}'$. By abuse of notation let ω and ω' denote the centers of the images of $\widetilde{C}$ and $\widetilde{C}'$, respectively.

Denote by $Q^*_{e,\mathcal{T}}$ the affine form associated with the hinge of axis e for any subtriangulation $\mathcal{T}$ of $\mathcal{C}$.

Claim $$Q^*_{e,\mathcal{T}}(0) \neq 0.$$

This is equivalent to $\omega \neq \omega'$. Assume for the sake of contradiction that $\omega = \omega'$. Then vertices of $\mathcal{D}(\widetilde{C}) \cup \mathcal{D}(\widetilde{C}')$ are cocyclic; hence C and C' are in the same Delaunay cell, ie C is glued to itself via e.

Without loss of generality, we may assume that ω is on the side (inclusively) of $\mathcal{D}(\widetilde{C}')$; hence e is strictly longer than every other edge of C. We deduce that C cannot be glued to itself via e, a contradiction.

Claim $$Q^*_{e,\mathcal{T}}(0) \le 0.$$

We may assume that the hinge at e is developed as in Figure 4. We take the notation of the proposition. Notice that assuming condition (v) is not satisfied, either B is in the interior of the circumscribed circle of ACD or D is in the interior of the circumscribed circle of ABC. This violates a characterization of the Delaunay cellulation.

Define

$$\mathcal{U} := \mathbb{R}^S_+ \cap \bigcap_{\mathcal{T} \in \boldsymbol{D}} \bigcap_{e} {Q^*_{e,\mathcal{T}}}^{-1}(\mathbb{R}^*_-),$$

where $\boldsymbol{D}$ is the set of adapted subtriangulations of the Delaunay cellulation, and e runs through the edges of the Delaunay cellulation. The intersection is finite since there are only finitely many such subtriangulations and edges. $\mathcal{U}$ is thus an open subset of $\mathbb{R}_+^S$ which contains $\mathbb{R}_+\mathbf{1}_S$.

We now show $\mathcal{U} \subset \mathcal{P}$. Apply the flipping algorithm for some $\tau \in \mathcal{U}$ and start from some $\mathcal{T}_0 \in \boldsymbol{D}$ of the Delaunay cellulation. Let $\mathcal{T}_0, \dots, \mathcal{T}_n$ be the sequence of triangulations given by the flipping algorithm. By induction we have $\mathcal{T}_0 \in \boldsymbol{D}$, and assuming $\mathcal{T}_k \in \boldsymbol{D}$ for some $k < n$, the conditions $Q^*_{e,\mathcal{T}_k}(\tau) < 0$ ensure that the edges e are τ-legal and thus not flipped. Hence $\mathcal{T}_{k+1} \in \boldsymbol{D}$. From Proposition 3.17, the triangulation $\mathcal{T}_n$ is such that all flippable hinges are τ-legal. On the one hand, the edges of $\mathcal{C}$ are τ-legal since $\tau \in \mathcal{U}$. On the other hand, all hinges inside a cell of the Delaunay cellulation are flippable. Finally, all the edges of $\mathcal{T}_n$ are τ-legal and $\mathcal{U}$ is a subset of $\mathcal{P}$. □

In order to obtain a finite cellulation of $\mathcal{P}$ as well as characterize its boundary, we prove its transverse compactness. By transverse compactness of $\mathcal{P}$ we mean that the projection of $\mathcal{P}$ into the hyperplane $\left\{\tau \in \mathbb{R}^S \mid \sum_{s\in S} \tau(s) = 0\right\}$ is compact. Note that, for instance, if $\mathcal{P}$ were equal to the whole $\mathbb{R}_+^S$ then it wouldn't be transversely compact in this sense. The proof that $\mathcal{P}$ is transversely compact relies upon the construction of affine constraints of the form $\tau_A - \tau_C \le \varepsilon(\tau_A + \tau_B + \tau_C + \tau_D) + K$ with $\varepsilon > 0$ arbitrarily small, and A and C arbitrary in S. Such constraints are provided by type-(x, L) hinges; see Definition 3.27, via Lemma 3.28. Lemma 3.29 focuses on the construction of such immersed hinges.

Definition 3.27 (type-(x, L) hinge) Let $x, L > 0$. A hinge $([ABCD], [AC])$ of $\mathbb{E}^2$ is of type (x, L) if it is nonconvex with $C \in [ABD]$ and

$$\mathrm{d}(B, \Delta) \le x, \quad \mathrm{d}(D, \Delta) \le x, \quad AB > L, \quad AD > L,$$

where Δ is the line (AC).

Lemma 3.28 *Let $l > 0$ and $x > 0$. For a hinge Q, write*

$$Q^*\colon \tau \mapsto \alpha(Q)\tau_A + \beta(Q)\tau_B + \gamma(Q)\tau_C + \delta(Q)\tau_D + K(Q)$$

for the affine form associated to Q.

Then, for all sequences $(Q_n)_{n\in\mathbb{N}}$ of hinges such that for all $n \in \mathbb{N}$, Q_n is of type (x, n) and axis length l, we have

$$\lim_{n\to+\infty} \frac{\alpha(Q_n)}{\gamma(Q_n)} = -1, \quad \lim_{n\to+\infty} \frac{\beta(Q_n)}{\gamma(Q_n)} = 0, \quad \lim_{n\to+\infty} \frac{\delta(Q_n)}{\gamma(Q_n)} = 0 \quad \textit{for all } n \in \mathbb{N},\ \gamma(Q_n) > 0.$$

Proof Let $L > 0$, and let $Q = ([ABCD], [AC])$ be a hinge of type (x, L) such that $AC = l$. Without loss of generality, we may choose Cartesian coordinates of $\mathbb{E}^2$ such that $A : (0, 0)$ is the origin, $C : (0, l)$, $x_B > 0$, and $x_D < 0$.

There exists some $\lambda > 0$ such that

$$\beta(Q) = \lambda \frac{l}{|x_B|}, \quad \alpha(Q) = \lambda\left(\frac{l - y_B}{|x_B|} + \frac{l - y_D}{|x_D|}\right), \quad \delta(Q) = \lambda \frac{l}{|x_D|}, \quad \gamma(Q) = \lambda\left(\frac{y_B}{|x_B|} + \frac{y_D}{|x_D|}\right).$$

We have $|x_B| \le x$, $|x_D| \le x$, $y_B \ge \sqrt{L^2 - x^2}$, and $y_D \ge \sqrt{L^2 - x^2}$; thus $\gamma(Q) > 0$ and

$$-1 \le \frac{\alpha(Q)}{f(Q)} \le -1 + \frac{l}{\sqrt{L^2 - x^2}}, \quad 0 \le \frac{\beta(Q)}{\gamma(Q)} \le \frac{l}{\sqrt{L^2 - x^2}}, \quad 0 \le \frac{\delta(Q)}{\gamma(Q)} \le \frac{l}{\sqrt{L^2 - x^2}}. \qquad \square$$

Lemma 3.29 *Let e be nontrivial geodesic segment of (Σ, S) going from some $\sigma_1 \in S$ to some $\sigma_2 \in S$ whose relative interior is in Σ^*.*

There exists $x_0 > 0$ such that for all $L > 0$, there is an immersed hinge $Q = ([ABCD], [AC], \eta)$ of type (x_0, L) such that $\eta([AC]) = e$.

Proof Let $M := \max_{x \in \Sigma} \mathrm{d}(x, S)$ and $m := \min_{s \in S} \min_{s' \in S \setminus \{s\}} \mathrm{d}(s, s')$.

Define $\Phi\colon \mathcal{U} \to \Sigma$ as the exponential map at σ_1 defined on some maximal star-shaped open neighborhood $\mathcal{U}$ of 0 in the tangent plane $T_{\sigma_1}\Sigma$ above σ_1 such that $\Phi(\mathcal{U} \setminus \{0\}) \subset \Sigma \setminus S$. We identify $T_{\sigma_1}\Sigma$ with $\mathbb{E}^2_\alpha$, where α is the cone angle at σ_1, so that Φ is an isometric immersion from an open set of $\mathbb{E}^2_\alpha$ to Σ. We choose polar coordinates (r, θ) of $\mathbb{E}^2_\alpha$ so that the direction $\theta = 0$ is the initial derivative of the segment e.

With $\beta = \min\left(\frac{1}{2}\alpha, \frac{1}{6}\pi\right)$ define

$$r_{\max}\colon]-\beta, \beta[\to \mathbb{R}_+^* \cup \{+\infty\}, \quad \theta \mapsto \max\{r \in \mathbb{R}_+ \mid (r, \theta) \in \mathcal{U}\},$$
$$R_\pm\colon]0, \beta[\to \mathbb{R}_+^* \cup \{+\infty\}, \quad \theta \mapsto \min_{\theta' \in]0, \theta]} r_{\max}(\pm\theta').$$

For any given $\theta \in]-\beta, \beta[$, if $r_{\max}(\theta) < +\infty$ we extend Φ continuously to $(r_{\max}(\theta), \theta)$; note that in this case $\Phi(r_{\max}(\theta), \theta) \in S$.

Claim
$$\limsup_{\theta \to 0^+} \theta R_\pm(\theta) \le 2M.$$

Let $\theta \in]0, \beta[$. Φ is defined on the interior of the triangle $[OAB] \subset \mathbb{E}^2_\alpha$ with $A = (R_+(\theta), 0)$ and $B = (R_+(\theta), \theta)$ in polar coordinates. The inscribed circle of $[OAB]$ bounds an open disc whose image by Φ does not contain any element of S, and hence the radius $\frac{1}{2}R_+(\theta)(\cos(\theta) + \sin(\theta) - 1)$ of this inscribed circle is less than M. One easily checks that $\cos(\theta) + \sin(\theta) - 1 \sim_{\theta \to 0^+} \theta$. The result follows for R_+, and one may proceed the same way for R_-; see Figure 6.

Claim
$$\lim_{\theta \to 0^+} R_\pm(\theta) = +\infty.$$

The function R_+ is nondecreasing by definition, so the limit is well defined. Define a sequence $(\theta_n)_{n \in \mathbb{N}}$ as follows: choose some $\theta_0 \in]0, \beta[$ such that $r_{\max}(\theta_0) = R_+(\theta_0)$ and $\sin(\theta_0) \le \frac{1}{2}m$; then for all $n \in \mathbb{N}$ take $\theta_{n+1} \in \left]0, \frac{1}{2}\theta_n\right[$ such that $r_{\max}(\theta_{n+1}) = R_+(\theta_{n+1})$. The map Φ can be continuously extended to the domain

$$D := \bigcup_{n \in \mathbb{N}} \{(r, \theta) \mid \theta \in [0, \theta_n], r \le R(\theta_n)\}.$$

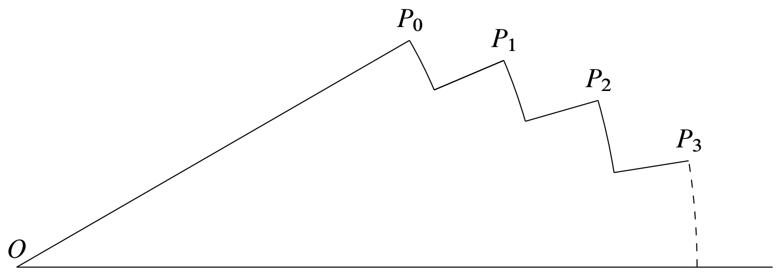


Figure 6

Write $P_n := (R_+(\theta_n), \theta_n)$; since for all $n \in \mathbb{N}$, $\Phi(P_n) \in S$, for all $n \in \mathbb{N}$,

$$R_+(\theta_{n+1}) - R_+(\theta_n) + \theta_n - \theta_{n+1} = \mathrm{d}_D(P_n, P_{n+1}) \geq \mathrm{d}_\Sigma(\Phi(P_n), \Phi(P_{n+1})) \geq m.$$

Thus

$$\text{for all } n \in \mathbb{N}, \quad R_+(\theta_n) \geq nm + R_+(\theta_0) + \theta_0 - \theta_n \xrightarrow{n \to +\infty} +\infty.$$

One may proceed the same way for R_-.

We now come back to the proof of the lemma. Take some $x_0 > M$, for any $L \in \mathbb{R}_+$. From the claims above, there exists some $\theta_+ \in]0, \beta[$ and $\theta_- \in]-\beta, 0[$ such that $|\sin(\theta_\pm) r_{\max}(\theta_\pm)| \leq x_0$ and $r_{\max}(\theta_\pm) \geq L$. Choose such a $\theta_\pm \in]-\beta, \beta[$ and notice Φ is well defined on the hinge $Q = [ABCD]$ with $A = O$, $B := (r_{\max}(\theta_-), \theta_-)$, $C := (\mathrm{length}(e), 0)$, and $D := (r_{\max}(\theta_+), \theta_+)$. The hinge Q is of type (x_0, L) and $\eta := \Phi|_Q$ is an isometric immersion. The immersed hinge (Q, η) is then of type (x_0, L) with vertices in S and such that $\Phi([AC]) = e$. □

Lemma 3.30 *There exists $C > 0$ such that for all $A, B \in S$ and all $\tau \in \mathcal{P}$,*

$$|\tau(A) - \tau(B)| \leq C.$$

Proof From Corollary 3.13, it is enough to find a $C > 0$ such that

$$\text{for all } \tau \in \mathcal{P}, \quad \min \tau = 0 \implies \max \tau \leq C.$$

From Lemmas 3.28 and 3.29 and from Proposition 3.24, for all $\varepsilon > 0$, and $A, B \in S$, if there exists a geodesic from A to B whose relative interior is in $\Sigma \setminus S$, then there exists $K > 0$ such that for all $\tau \in \mathcal{P}$, $|\tau_A - \tau_B| \leq \varepsilon \max \tau + K$. For all $A, B \in S$ there exists a geodesic from A to B possibly intersecting S in his relative interior. Hence

$$\forall \varepsilon > 0,\ \forall A, B \in S,\ \exists K > 0 \text{ such that } \forall \tau \in \mathcal{P},\ |\tau_A - \tau_B| \leq \varepsilon \max \tau + K.$$

Since S is finite,

$$\exists K > 0,\ \forall A, B \in S,\ \forall \tau \in \mathcal{P},\ |\tau_A - \tau_B| \leq \tfrac{1}{2} \max \tau + K.$$

Choose such a $K > 0$ and define $C = 2K$. Then for all $\tau \in \mathcal{P}$ such that $\min \tau = 0$,

$$\max \tau = |\max \tau - \min \tau| \leq \tfrac{1}{2} \max \tau + K.$$

Thus for such a τ

$$\max \tau \leq 2K = C. \qquad \square$$

Proof of Theorem 2 Let π be the orthogonal projection of $\mathbb{R}^S$ onto $H := \{\tau \in \mathbb{R}^S \mid \sum_{s\in S} \tau(s) = 0\}$. Note that the kernel of π is $\mathbb{R}\cdot \mathbf{1}_S$. For each triangulation $\mathcal{T}$, the set of $\tau \in \mathbb{R}_+^S$ such that $\tilde{\tau}_{\tau,\mathcal{T}}$ is Q-convex is the domain

$$\mathcal{P}_\mathcal{T} := \mathbb{R}_+^S \cap \bigcap_{e\in \mathrm{Edge}(\mathcal{T})} (Q_e^*)^{-1}(\mathbb{R}_-),$$

since $\mathbf{1}_S$ is in the kernel of the linear part of all the affine forms Q_e^*. Since the number of edges of $\mathcal{T}$ is finite, $\overline{\mathcal{P}_\mathcal{T}} := \pi(\mathcal{P}_\mathcal{T})$ is a convex polyhedron and $\mathcal{P}_\mathcal{T} = (\overline{\mathcal{P}_\mathcal{T}} + \mathbb{R}\cdot \mathbf{1}_S) \cap \mathbb{R}_+^S$.

On the one hand,

$$\mathcal{P} = \bigcup_{\mathcal{T}} P_\mathcal{T},$$

where $\mathcal{T}$ runs through all adapted triangulations of (Σ, S). Then defining $\overline{\mathcal{P}} := \bigcup_\mathcal{T} \overline{\mathcal{P}}_\mathcal{T}$, we have $\mathcal{P} = (\overline{\mathcal{P}} + \mathbb{R}\cdot \mathbf{1}_S) \cap \mathbb{R}_+^S$.

On the other hand, by Lemma 3.30, $\overline{\mathcal{P}} = \pi(\mathcal{P})$ is compact. Furthermore, by Proposition 3.24, $\mathcal{P}$ is convex. Hence $\overline{\mathcal{P}}$ is convex.

Then consider the set $\boldsymbol{T}$ of triangulations that are τ-Delaunay for some $\tau \in \mathcal{P}$. For any admissible $\tau \in \mathcal{P}$, it follows from Lemma 3.12 that $\tau' := \tau - \min \tau \in \mathcal{P}$ and that the set of τ-Delaunay triangulations is equal to the set of τ'-Delaunay triangulations. Therefore $\boldsymbol{T}$ is the set of triangulations that are τ-Delaunay for some $\tau \in \mathcal{P}_0 := \{\tau \in \mathcal{P} \mid \min \tau = 0\}$. By Lemma 3.30, there exists a constant C that only depends on Σ such that for all $\tau \in \mathcal{P}_0$, $\tau \leq C$. By Proposition 3.17, there thus exists a constant A that only depends on Σ such that $\tilde{\tau}_{\tau,\mathcal{T}} \leq A$ for all τ-Delaunay triangulation $\mathcal{T}$ and all $\tau \in \mathcal{P}_0$. Using notation of Lemma 3.21, we deduce that $\boldsymbol{T} \subset E_A$ is finite; hence $\boldsymbol{T}$ is finite. The domain $\overline{\mathcal{P}}$ is thus a polyhedron.

Choose any triangulation $\mathcal{T}_0$ and define $A := \sup_{\tau\in\mathcal{P}_0} \max_{x\in\Sigma} \tilde{\tau}_{\tau,\mathcal{T}_0}(x)$; by compactness of $\overline{\mathcal{P}}$, the set $\mathcal{P}_0$ is bounded. Hence $A < +\infty$. Consider the finite family $(Q_i)_{i\in[\![1,q]\!]}$ of unflippable immersed hinges around edges of triangulations in E_A and define $\mathcal{P}_A := \bigcap_{i=1}^q {Q_i^*}^{-1}(\mathbb{R}_-)$. By Proposition 3.24 $\mathcal{P}_A \supset \mathcal{P}$. In addition, for any $\tau \in \mathcal{P}_A$ the flipping algorithm starting at $\mathcal{T}_0 \in E_A$ stops after finitely many iterations on some $\mathcal{T}_n \in E_A$; Proposition 3.17 ensures that flippable hinges of $\mathcal{T}_n$ are τ-legal and the definition of $\mathcal{P}_A$ ensures that unflippable hinges of $\mathcal{T}_n$ are also τ-legal. We deduce that $\mathcal{T}_n$ is τ-Delaunay, and hence $\tau \in \mathcal{P}$. We conclude that $\mathcal{P} = \mathcal{P}_A$, so that essential support planes of $\mathcal{P}_A$ are either

- essential support planes of $\mathbb{R}_+^S$ and thus of the form $\tau_\sigma = 0$ for some $\sigma \in S$, or
- given by "$Q_i^* = 0$" for some $i \in [\![1,q]\!]$.

Finally, since $\mathcal{P}$ is a finite union of cells, essential support planes of the second kind correspond to a facet of some cell $\mathcal{P}_\mathcal{T}$. Theorem 2(e) follows. $\square$

4 The Volkov lemma for Lorentzian convex cones

In effective methods used to prove Alexandrov-like theorems, at some point a Volkov lemma bounding the cone angle Θ around a singular line of angle κ in a Riemannian manifold is needed. This is used to exclude some positions of critical points of the Einstein–Hilbert functional introduced in the following section.

We consider spacelike convex cones in $\mathbb{E}^{1,2}_\kappa$ for $\kappa > 0$, eg the model space of the timelike singular lines of angle κ as $\mathbb{R}^3$ endowed with the metric $\mathrm{d}t^2 - \mathrm{d}r^2 - (\kappa/(2\pi))^2\,\mathrm{d}\theta^2$. There are many ways to rigorously define a spacelike cone in $\mathbb{E}^{1,2}_\kappa$. In our context, we define a cone $\mathcal{D}$ as the graph of some Lipschitz 1-homogeneous function $t\colon \mathbb{R}^2 \mapsto \mathbb{R}$. The cone is spacelike if the graph in $\mathbb{R}^3$ identified to $\mathbb{E}^{1,2}_\kappa$ is spacelike. The cone $\mathcal{D}$ is then convex if the future $J^+(\mathcal{D})$ is convex in the sense that any spacelike geodesic with extremities in $J^+(\mathcal{D})$ is in $J^+(\mathcal{D})$. The Lorentzian structure of $\mathbb{E}^{1,2}_\kappa$ induces complete metric space structure on the cone, which is locally Euclidean except possibly at $\{r = 0\}$. In other words, $\mathcal{D}$ is isometric to $\mathbb{E}^2_\Theta$ for some $\Theta > 0$; this Θ is its so-called cone angle.

Let $\mathcal{D}$ be a cone defined as the graph of $t\colon \mathbb{R}^2 \to \mathbb{R}$. A *wedge* is the graph of t on some domain $\{\theta \in I,\, r \geq 0\}$ with I an interval; Such a wedge is *coplanar* if it is totally geodesic. A wedge is isometric to some domain $\{(r,\theta) \mid r \geq 0,\, 0 \leq \theta \leq \pi\}$ in $(\mathbb{R}^2, \mathrm{d}r^2 + (\alpha/\pi)^2 r^2\,\mathrm{d}\theta^2)$; the value of α is unique and we refer to it as the *Euclidean angle* of the wedge.

Theorem 3 *Let $\Theta > 0$ and $\kappa > 0$. Let $\mathcal{D}$ be a convex spacelike cone in $\mathbb{E}^{1,2}_\kappa$ of cone angle Θ whose vertex is on the singular line of $\mathbb{E}^{1,2}_\kappa$.*

Assuming $\mathcal{D}$ has a coplanar wedge of Euclidean angle at least $\min(\pi,\Theta)$,

- *if $\Theta > 2\pi$ then $\kappa > 2\pi$,*
- *if $\Theta = 2\pi$ then $\kappa = 2\pi$,*
- *if $\Theta \in\,]\pi, 2\pi[$ then $\kappa \geq \Theta$,*
- *if $\Theta = \pi$ then $\kappa = \pi$,*
- *if $\Theta < \pi$ then $\kappa \in\,]0,\Theta]$ with $\kappa = \Theta$ if and only if $\mathcal{D}$ is the horizontal plane,*

and all the bounds above are sharp.

Remark Though results such as stated above are used one way or another in [3; 6; 18; 19; 22; 26], to our knowledge, a complete proof of the bounds we use is not available in English (one may appear in the original thesis of Volkov which is in Russian, and only a summary is available in English [34]). We thus provide a complete proof.

Remark In Minkowski, a convex cone always has a cone angle bigger than 2π. One may expect this to be carried out in $\mathbb{E}^{1,2}_\kappa$ for arbitrary $\kappa \geq 0$. Theorem 3 shows this intuition is valid for $\kappa \in [0,\pi] \cup \{2\pi\}$ but not for $\kappa \in\,]\pi, 2\pi[$.

When considering a cone $\mathcal{D}$ in $\mathbb{E}^3$, an elementary remark is that the angle of the conical singularity is, in fact, the length of its *stalk*: the curve given by the intersection $\mathcal{D} \cap \mathbb{S}^2$. By extension "stalk" refers to curves in $\mathbb{S}^2$ or $\mathbb{S}^{1,1}$ that are graphs over the "equator". As in the Euclidean case, we may notice that the angle Θ of the conical singularity of a spacelike cone in $\mathbb{E}^{1,2}_\kappa$ is given by the length of the spacelike curve induced on $\mathbb{S}^{1,1}_\kappa := \{(t,r,\theta) \in \mathbb{E}^{1,2}_\kappa \mid r^2 - t^2 = 1\}$. However, the relation between κ and Θ is far from trivial, and the Lorentzian nature of $\mathbb{S}^{1,1}$ does not help. One may devise an analytical proof of the needed Volkov lemma [9], but a more geometrical one is provided based on a suggestion of Graham Smith.

The key idea developed in Section 4.1 is that to each cone stalk $\rho\colon \mathbb{R}/\kappa\mathbb{Z} \to \mathbb{S}^{1,1}_\kappa$ corresponds a dual stalk $\gamma\colon \mathbb{R}/\Theta\mathbb{Z} \to \mathbb{S}^2_\Theta$. The length of ρ is the Euclidean cone angle Θ while the length of γ is the Lorentzian cone angle κ.

4.1 Stalks of Lorentzian cones

Although we defined cones with Lipschitz regularity, we will focus our argumentation on polyhedral cones that are simpler to describe. A density argument allows us to generalize to lower regularity.

Definition 4.1 (stalk of a spacelike cone) Let $\kappa > 0$, $\Theta > 0$ and $\mathcal{D}$ be a spacelike cone of $\mathbb{E}^{1,2}_\kappa$ of cone angle Θ. In cylindrical coordinates (r,θ,t), the set $\mathbb{S}^{1,1}_\kappa \cap \mathcal{D}$ can be parametrized by arc length with increasing θ coordinate:

$$\mathcal{D} \cap \mathbb{S}^{1,1}_\kappa = \left\{ \begin{pmatrix} t(s) \\ r(s) \\ \theta(s) \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}.$$

The stalk $\rho_{\mathcal{D}}$ of $\mathcal{D}$ is the function $t\colon \mathbb{R} \to \mathbb{R}$ of this parametrization.

Remark The stalk ρ of a cone is unique up to precomposition by an affine transformation of slope ± 1.

Proposition 4.2 *Let $\kappa > 0$, $\Theta > 0$ and $\mathcal{D}$ be a cone of $\mathbb{E}^{1,2}_\kappa$ of cone angle Θ whose vertex is on the origin and of stalk $\rho := \rho_{\mathcal{D}}$. We have the following*:

(1) *$\rho\colon \mathbb{R} \to \mathbb{R}$ is Θ-periodic and Lipschitz continuous.*

(2) *$\mathcal{D}$ is polyhedral if and only if ρ is piecewise trigonometric* (*piecewise of the form $\theta \mapsto A\cos(\theta+\varphi)$*).

(3) *If $\mathcal{D}$ is polyhedral then*

$$\mathcal{D} \text{ is convex} \iff \rho \text{ is } Q\text{-convex}.$$

(4) $\kappa = \int_0^\Theta \sqrt{1+\rho(\theta)^2+\rho'(\theta)^2}/(1+\rho(\theta)^2)\,\mathrm{d}\theta.$

Proof The first three points are simple enough. To obtain the last item, we first choose a parametrization by arc length $s \mapsto (t,r,\theta)$ of $\mathcal{D} \cap \mathbb{S}^{1,1}$ with θ increasing and notice

$$2\pi = \int_0^\Theta \theta'(s)\,\mathrm{d}s, \quad r^2 - \rho^2 = 1, \quad -(\rho')^2 + (r')^2 + \left(\frac{\kappa}{2\pi}\right)^2 r^2 (\theta')^2 = 1.$$

Therefore $rr' = \rho\rho'$ and

$$(\theta')^2 = \left(\frac{2\pi}{\kappa}\right)^2 \frac{1+(\rho')^2-(r')^2}{r^2} = \left(\frac{2\pi}{\kappa}\right)^2 \frac{1+(\rho')^2-(\rho\rho'/r)^2}{1+\rho^2} = \left(\frac{2\pi}{\kappa}\right)^2 \frac{(1+(\rho')^2)(1+\rho^2)-\rho^2(\rho')^2}{(1+\rho^2)^2}$$
$$= \left(\frac{2\pi}{\kappa}\right)^2 \frac{1+(\rho')^2+\rho^2}{(1+\rho^2)^2},$$

and so

$$\theta' = \frac{2\pi}{\kappa} \frac{\sqrt{1+(\rho')^2+\rho^2}}{1+\rho^2}.$$

Insert the last line in $2\pi = \int_0^\Theta \theta'$ to get the result. □

Remark For $\rho\colon I \to \mathbb{R}$ continuous piecewise trigonometric, ρ is Q-convex if and only if $s \mapsto \rho(-s)$ is Q-convex.

Definition 4.3 (mass of a stalk) For $\rho\colon [a,b] \to \mathbb{R}$ (resp. $\rho\colon \mathbb{R}/\Theta\mathbb{Z} \to \mathbb{R}$), define

$$\kappa(\rho) := \int_a^b \frac{\sqrt{1+\rho^2+\rho'^2}}{1+\rho^2} \quad \left(\text{resp. } \int_0^\Theta \frac{\sqrt{1+\rho^2+\rho'^2}}{1+\rho^2}\right).$$

Remark Every $\rho\colon \mathbb{R} \to \mathbb{R}$ piecewise trigonometric Q-convex and Θ-periodic induces a convex polyhedral embedding of $\mathbb{E}^2_\Theta$ into $\mathbb{E}^{1,2}_{\kappa(\rho)}$. Furthermore, this embedding is essentially unique: from Proposition 4.2, the mass κ is given by ρ; there is thus no choice for the space $\mathbb{E}^{1,2}_\kappa$ and two embeddings of the same germ only differ by a rotation or a symmetry.

Corollary 4.4 *Let $\kappa > 0$, $\Theta > 0$ and $\mathcal{D}$ be a spacelike polyhedral cone in $\mathbb{E}^{1,2}_\kappa$ of cone angle Θ. If its stalk ρ is $\mathscr{C}^1$ then $\kappa(\rho) = \Theta$. Furthermore, if κ is not a multiple of 2π then $\rho = 0$.*

Proof To begin with, since ρ is piecewise trigonometric and continuously differentiable, ρ is in fact trigonometric. Then either 2π is the minimal period of ρ or $\rho = 0$. If $\rho = 0$, the result follows from Proposition 4.2(4). Otherwise $\Theta \in 2\pi\mathbb{N}$ and we notice that for any A and φ we have $\kappa(s \mapsto A\cos(s+\varphi)) = 2k\pi$ if $\Theta = 2k\pi$. □

Lemma 4.5 *Let I be an interval, $\rho\colon I \to \mathbb{R}$ be piecewise trigonometric Q-convex, and let $\theta_0 \in I$. Let $\bar\rho$ be the unique trigonometric function such that $\rho(\theta_0) = \bar\rho(\theta_0)$ and $\rho'(\theta_0^+) = \bar\rho'(\theta_0)$. Then for all $\theta \in I \cap [\theta_0, \theta_0+\pi]$,*

$$\rho(\theta) \geq \bar\rho(\theta).$$

Furthermore:

$$\text{there exists } \theta \in\,]\theta_0, \theta_0+\pi[\text{ such that } \bar\rho(\theta) = \rho(\theta) \iff \text{for all } \theta \in [\theta_0, \theta_0+\pi],\ \rho(\theta) = \bar\rho(\theta).$$

Proof Let $(\theta_0, \theta_1, \dots, \theta_n = \theta_0+\pi)$ be subdivision adapted to ρ. For $k \in [\![1,n]\!]$, denote by $\rho_k\colon \mathbb{R} \to \mathbb{R}$ the unique trigonometric function such that $\rho|_{k[\theta_{k-1},\theta_k]} = \rho|_{[\theta_{k-1},\theta_k]}$ and define $\rho_0 = \bar\rho$.

For $k \in [\![0, n-1]\!]$, we have $\rho_k(\theta_k) = \rho_{k+1}(\theta_k)$. If $\rho'_k(\theta_k) = \rho'_{k+1}(\theta_k)$ then $\rho_k = \rho_{k+1}$. Otherwise $\rho'_k(\theta_k) < \rho'_{k+1}(\theta_k)$; thus $\rho_k < \rho_{k+1}$ on a nontrivial interval $[\theta_k, \theta_k + \varepsilon]$. These two trigonometric functions are in particular distinct and intersect each other on the set $\theta_k + \pi\mathbb{Z}$. Hence $\rho_k - \rho_{k+1}$ has constant sign on the interval $[\theta_k, \theta_k + \pi]$ and $\rho_k \le \rho_{k+1}$ on $[\theta_k, \theta_k + \pi]$. By induction, the result follows. □

Definition 4.6 Let $\mathbb{S}^2_\infty$ be the universal covering of the round sphere branched over its north and south poles, eg $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right] \times \mathbb{R}/\sim$ endowed with the metric

$$ds^2 = d\phi^2 + \cos(\phi)^2\, d\theta^2,$$

where $\sim$ identifies all points such that $\phi = \frac{1}{2}\pi$ together as the north pole N and all points such that $\phi = -\frac{1}{2}\pi$ as the south pole S.

Definition 4.7 A piecewise geodesic curve $\gamma\colon I \to \mathbb{S}^2_\infty$ is Q-convex if $\theta \circ \gamma$ is injective and $\phi \circ \gamma$ is Q-convex.

Lemma 4.8 *Let $\rho\colon [a,b] \to \mathbb{R}$ be Lipschitz continuous and define $\gamma\colon [a,b] \to \mathbb{S}^2_\infty$, $\theta \mapsto (\arctan\rho(\theta), \theta)$. Then*

$$\kappa(\rho) = \operatorname{length}(\gamma).$$

Furthermore, γ is a piecewise geodesic Q-convex curve if and only if ρ is piecewise trigonometric Q-convex.

Proof By direct computation:

$$\begin{aligned}\operatorname{length}(\gamma) &= \int_a^b \sqrt{\frac{(\rho')^2(\theta)}{(1+\rho^2(\theta))^2} + \cos^2(\arctan\circ\rho(\theta))}\, d\theta = \int_a^b \sqrt{\frac{(\rho')^2(\theta)}{(1+\rho^2(\theta))^2} + \frac{1}{1+\rho^2(\theta)}}\, d\theta \\ &= \int_a^b \sqrt{\frac{(\rho')^2(\theta) + 1 + \rho^2(\theta)}{(1+\rho^2(\theta))^2}}\, d\theta = \kappa(\rho).\end{aligned}$$

Then it suffices to note that curves of the form $t \mapsto (\phi(t), \theta(t))$ with $\phi(t) = \arctan(\alpha\cos(\theta(t) + \phi_0) \notin \left\{-\frac{1}{2}\pi, \frac{1}{2}\pi\right\}$ and $\theta(t) = t$ are exactly the nonmeridional geodesic segment of $\mathbb{S}^2_\infty$. □

Proposition 4.9 *Let S_Θ be the set of Lipschitz stalks of convex spacelike cones of cone angle Θ admitting a coplanar wedge of Euclidean angle $\min(\Theta, \pi)$ endowed with the Lipschitz norm*

$$\|\rho\|_{\mathrm{Lip}} := \sup_{s \in I} |\rho(s)| + \sup_{s_1 \neq s_2} \left|\frac{\rho(s_1) - \rho(s_2)}{s_1 - s_2}\right|.$$

Then the subspace of piecewise trigonometric Q-convex functions is dense in S_Θ.

Sketch of proof Consider the stalk $\rho_{\mathcal{D}}\colon \mathbb{R} \to \mathbb{R}$ of a (Lipschitz) spacelike convex cone $\mathcal{D}$ and its associated geodesics γ in $\mathbb{S}^2_\infty$. Consider $[a, a+\alpha] + \Theta\mathbb{Z}$ with $\alpha \ge \min(\Theta, \pi)$ and $\rho|_{[a+k\Theta, a+k\Theta+\alpha)}$ trigonometric for all $k \in \mathbb{Z}$.

The curve γ divides $\mathbb{S}^2_\infty$ into two parts (north and south); the epigraph of γ is the northern domain. Convexity of the spacelike cone translates in $\mathbb{S}^2_\infty$ into the local convexity of the epigraph of γ. We may construct an approximating sequence $(\gamma_n)_{n\in\mathbb{N}}$ of γ interpolating by geodesics, say between points of the form $(s_k, \gamma(s_k))$ with $(s_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ increasing such that $|s_{k+1}-s_k| \le 1/(1+n)$, $\lim_{\pm\infty} s_k = \pm\infty$, and $\{a+k\Theta, a+\alpha+k\Theta \mid k\in\mathbb{Z}\} \subset \{s_k \mid k\in\mathbb{Z}\}$. Then notice that for n big enough, the geodesics are not meridional and thus correspond to a piecewise trigonometric Θ-periodic stalk ρ_n. By convexity of γ, each γ_n is Q-convex for n big enough, and so are the ρ_n. We note that γ is Lipschitz and that the sequence γ_n converges in Lipschitz norm to γ. □

4.2 Lower bounds

Lemma 4.8 provides a neat geometrical translation from Lorentzian to Riemannian. Indeed, an issue with the geometry of Lorentzian manifolds is that spacelike geodesics are not characterized as minimizers of the usual energy Lagrangian $\int g(\dot\gamma,\dot\gamma)$. The description of convex polyhedral cones as Q-convex piecewise geodesics in $\mathbb{S}^2_\infty$ allows us to leverage the usual Riemannian theory of geodesics.

Proposition 4.10 *Let $\Theta \ge \pi$. Then*

$$\inf_{\rho\in S_\Theta} \kappa(\rho) = \min(2\pi,\Theta),$$

the infimum being taken over the set S_Θ of stalks $\rho\colon \mathbb{R}/\Theta\mathbb{Z} \to \mathbb{R}$ of convex spacelike cones admitting a coplanar wedge of Euclidean angle at least π. Furthermore, the infimum is achieved if and only if $\Theta \le 2\pi$.

Proof Note that by Proposition 4.9, piecewise trigonometric elements of S_Θ form a dense subspace for a norm for which κ is continuous. By abuse of language, we say that "ρ is a Q-convex stalk", meaning that ρ is a Lipschitz limit of piecewise trigonometric Q-convex stalks.

- Assume $\Theta \ge 2\pi$. Consider for $\alpha > 0$ the stalk
$$\rho\colon \mathbb{R}/\Theta\mathbb{Z} \to \mathbb{R}, \quad \theta \mapsto \begin{cases} \sinh(\alpha)\sin(\theta) & \text{for } \theta \in \left[-\tfrac{3}{2}\pi, \tfrac{1}{2}\pi\right], \\ \sinh(\alpha) & \text{otherwise,}\end{cases}$$
so that ρ is Q-convex and $\kappa(\rho) = 2\pi + (\Theta - 2\pi)/\cosh(\alpha)$. As a result, $\inf_{\rho\in S_\Theta} \kappa(\rho) \le 2\pi$.

- Assume $\Theta \le 2\pi$. Then the stalk $\rho = 0$ is such that $\kappa(\rho) = \Theta$.

- Let $\gamma\colon [0, \Theta-\pi] \to \mathbb{S}^2_\infty$ be a Lipschitz curve from $(\phi_0, 0)$ to $(-\phi_0, \Theta-\pi)$ minimizing the length with $\gamma(0), \gamma(\Theta-\pi) \notin \{N, S\}$. The curve γ is a geodesic with possibly intermediate points in $\{N, S\}$.
  - Assume γ does not intersect $\{N, S\}$. Then up to reparametrization $\phi\circ\gamma$ is of the form $\theta \mapsto \arctan(\alpha\cos(\theta+\theta_0))$. If $\Theta \ge 2\pi$, then the length of such a curve is at least π. Otherwise, since $\phi\circ\gamma(0) = -\phi\circ\gamma(\Theta-\pi)$ up to reparametrization, $\rho(\theta) = \sinh(\alpha)\sin(\theta)$ with $\theta \in \left[m-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi - m\right]$ and $m = \pi - \tfrac{1}{2}\Theta$. Then
  $$\kappa(\rho) = \pi - 2\arctan\left(\frac{\tan(m)}{\cosh(\alpha)}\right),$$
  which is minimal if and only if $\alpha = 0$, in which case the length of γ is $\Theta - \pi$.

- Assume γ intersects $\{N, S\}$ exactly once. Then γ is formed of a geodesic from $\gamma(0)$ to N (resp. S) followed by a geodesic from N (resp. S) to $\gamma(\Theta)$. Such geodesics are meridional; hence the length of γ is exactly π.
- Assume γ intersects $\{N, S\}$ at least twice. Then it contains a meridional geodesic from N to S and its length is strictly bigger than π.

In any case, the length of the curve associated by Lemma 4.8 to a stalk ρ in S_Θ is bounded from below by π plus the length of such a minimizing curve γ. Hence $\inf_{\rho\in S_\Theta}\kappa(\rho)=\min(\Theta,2\pi)$. Furthermore, the infimum is achieved if and only if the minimizing geodesic can be associated with a stalk, which is only possible if the geodesic γ considered above reaches neither N nor S; this is possible only if $\Theta \le 2\pi$. Reciprocally, if $\Theta \le 2\pi$, then the stalk $\rho = 0$ achieves the infimum. □

Proposition 4.11 *Let $\Theta < \pi$. Then*

$$\inf_{S_\Theta}\kappa(\rho)=0,$$

the infimum being taken over the set S_Θ of stalks $\rho\colon \mathbb{R}/\Theta\mathbb{Z}\to\mathbb{R}$ of convex spacelike cones admitting a coplanar wedge of Euclidean angle at least Θ.

Furthermore, among such stalks, $\kappa(\rho)\le\Theta$ with equality if and only if $\rho=0$.

Proof Any element of S_Θ is of the form

$$\rho_\alpha\colon \mathbb{R}/\Theta\mathbb{Z}\to\mathbb{R},\quad \theta+\Theta\mathbb{Z},\theta\in[-\Theta/2,\Theta/2]\mapsto \sinh(\alpha)\cos(\theta+\theta_0),$$

for some $\alpha\in\mathbb{R}$, $\theta_0\in\,]-\pi,\pi[$. If $\alpha=0$ we may choose $\theta_0=0$; otherwise, up to translation, we may assume that $\rho\big(-\frac{1}{2}\Theta\big)=\rho\big(\frac{1}{2}\Theta\big)$, which implies that $\big(\frac{1}{2}\Theta+\theta_0\big)=\pm\big(-\frac{1}{2}\Theta+\theta_0\big)+2k\pi$ for some $k\in\mathbb{Z}$. Therefore either $\Theta=2k\pi$ or $\theta_0=k\pi$; since $0<\Theta<\pi$ and $|\theta_0|<\pi$, it follows that $\theta_0=0$.

In particular, all elements of S_Θ are piecewise trigonometric. On the one hand, since ρ_α is Q-convex only for $\alpha\ge 0$,

$$\text{for all } \alpha\in\mathbb{R},\quad \rho_\alpha\in S_\Theta \iff \alpha\ge 0.$$

On the other hand

$$\text{for all } \alpha\ge 0,\quad \kappa(\rho_\alpha)=2\arctan\left(\frac{\tan\big(\frac{1}{2}\Theta\big)}{\cosh(\alpha)}\right)\xrightarrow{\alpha\to+\infty}0.$$

It follows that $\inf_{\rho\in S_\Theta}\kappa(\rho)=0$. Note that $\alpha\mapsto\kappa(\rho_\alpha)$ is decreasing; the maximum is thus reached for $\alpha=0$; hence $\rho=0$. The formula above gives $\kappa(0)=\kappa(\rho_0)=2\arctan\big(\tan\big(\frac{1}{2}\Theta\big)\big)=\Theta$. □

4.3 Proof of the Volkov lemma

We now compile and complete the elements proven in the previous section.

Proof of Theorem 3 Proposition 4.10 implies the first and third claims, and partially the second. The fifth claim is a consequence of Proposition 4.11

To complete the second consider the stalk ρ of a convex spacelike cone of Euclidean angle 2π having a coplanar wedge of angle π. By Proposition 4.9 we may assume ρ is piecewise trigonometric. Using the remark just before Definition 4.3, assume without loss of generality that $\rho(0) = -\rho(\pi) \geq 0$ and $\rho|_{[\pi,2\pi]}$ is trigonometric. Using Lemma 4.5 we see that if ρ is not trigonometric on $[0,\pi]$ then $-\rho(0) = \rho(\pi) > \bar{\rho}(\pi) = -\bar{\rho}(0) = -\rho(0)$ for some trigonometric function $\bar{\rho}$, a contradiction. Therefore ρ is trigonometric on $[0,\pi]$ and on $[\pi,2\pi]$ so that $\kappa(\rho) = 2\pi$.

The same argument allows us to prove the fourth claim. □

5 The Einstein–Hilbert functional

We give ourselves a Euclidean surface Σ with conical singularities and marked points $S \supset \mathrm{Sing}(\Sigma)$; we will keep this surface fixed in the whole section.

To sum up the results of the preceding sections, we have a construction that associates to any $\tau \in \mathcal{P}$ a radiant spacetime $M(\tau)$ and a convex polyhedral embedding ι_τ of (Σ, S) into $M(\tau)$. We know from Proposition 2.23 this construction reaches every equivalence classes of such a couple (M,ι) and is injective. By Theorem 2, $\mathcal{P}$ is a convex domain of $\mathbb{R}_+^S$ and is the union of finitely many convex cells, each corresponding to a triangulation of (Σ, S).

The objective is now to construct polyhedral embeddings (M,ι) such that the singularities of M have cone angles we gave ourselves a priori.

Definition 5.1 (mass function) Let $\tau \in \mathcal{P}$ and $(M(\tau), \iota_\tau)$ be its associated polyhedral embedding of (Σ, S). For $\sigma \in S$ define $\kappa_\sigma(\tau)$ the (Lorentzian) cone angle of $M(\tau)$ at $\iota_\tau(\sigma) \in M(\tau)$.

We define $\kappa : \mathcal{P} \to \mathbb{R}_+^S$ the map that associates to τ the vector $(\kappa_\sigma(\tau))_{\sigma\in S}$.

Remark On each cell $\mathcal{P}_\mathcal{T} := \{\tau \in \mathcal{P} \mid \mathcal{T}_\tau = \mathcal{T}\}$, the function $\tau \mapsto \kappa(\tau)$ is continuous and furthermore $\mathscr{C}^1$. Since we will actually compute the derivative later on, we do not prove it now.

Furthermore, if $\tau \in \mathcal{P}_\mathcal{T} \cap \mathcal{P}_{\mathcal{T}'}$ the triangulations $\mathcal{T}$ and $\mathcal{T}'$ are τ-equivalent; κ computed with either triangulation yields the same result since $M(\tau)$ may be constructed using $\mathcal{T}$ or $\mathcal{T}'$. The map $\tau \mapsto \kappa$ is thus continuous on $\mathcal{P}$.

Remark As a complement to the previous remark, we do not neglect the limit case $\tau_\sigma = 0$ for which $\kappa_\sigma = 0$ by convention. One may check directly that $\lim_{\tau_\sigma \to 0^+} \kappa_\sigma(\tau) = 0$.

Reformulating with this notation, we thus aim to solve the following:

Problem *Let $\bar{\kappa} \in \mathbb{R}_+^S$. Is there some $\tau \in \mathcal{P}$ such that $\kappa(\tau) = \bar{\kappa}$, and if so, is it unique?*

There is a restriction on the possible $\bar{\kappa}$. Indeed, for any $\tau \in \mathcal{P}$, the spacetime $M(\tau)$ is the suspension of some marked closed hyperbolic surface with conical singularities and cusp[4] $(\Sigma_{\mathbb{H}^2}, S')$ marked by (Σ, S), and the

[4] See Definitions A.6 and A.7 in the appendix.

cone angles at S' are $\kappa(\tau)$. Therefore, by the Gauss–Bonnet formula, $\sum_{\sigma\in S}(2\pi-\kappa(\tau)_\sigma)-\mathrm{Area}(\Sigma_{\mathbb{H}^2}) = 2\pi\chi(\Sigma) = \sum_{\sigma\in S}(2\pi-\theta_\sigma)$. Hence

$$\text{for all } \tau\in\mathcal{P},\quad \sum_{\sigma\in S}\theta_\sigma > \sum_{\sigma\in S}\kappa(\tau)_\sigma.$$

In addition to this global constraint, there are local constraints due to upper bounds in the Volkov lemma. We do not systematically explore the local upper bounds and only provide the one that is consistent with the boundary condition (ie the last item of Theorem 3). We settle for an incomplete statement.

Theorem 4 *Let (Σ, S) be a closed locally Euclidean surface of genus g with conical singularities of angles $(\theta_\sigma)_{\sigma\in S}$. Using notation of the previous sections,*

$$\textit{for all } \bar\kappa \in \left(\prod_{\sigma\in S}[0,\min(\theta_\sigma,2\pi)]\right)\setminus\{(\theta_\sigma)_{\sigma\in S}\} \textit{ there exists } \tau\in\mathcal{P} \textit{ such that } \kappa(\tau)=\bar\kappa.$$

Furthermore, if for all $\sigma\in S, \bar\kappa_\sigma<\theta_\sigma$, then such a τ is unique. Finally, if $\theta_\sigma\le\pi$ for some $\sigma\in S$ then for all $\tau\in\mathcal{P}, \kappa_\sigma(\tau)\le\theta_\sigma$.

The proof relies on the analysis of a so-called Einstein–Hilbert functional; the first step is to define a functional $\mathcal{H}_{\bar\kappa}$ on $\mathcal{P}$ for a given $\bar\kappa$ whose critical points are solution to the problem before Theorem 4. In fact, one could check that such a functional exists by checking $\partial\kappa_{\sigma_1}/\partial h_{\sigma_2} = \partial\kappa_{\sigma_2}/\partial h_{\sigma_1}$.

For technical reasons which will shortly make themselves clear, it will be more appropriate to define such a functional on the domain $\mathcal{P}^{1/2} := \{h\in\mathbb{R}_+^S \mid h^2\in\mathcal{P}\}$. Elements of $\mathcal{P}^{1/2}$ will be denoted systematically by h, while elements of $\mathcal{P}$ will be denoted by τ. Going from the one to the other being simple, we extend all definitions to $\mathcal{P}^{1/2}$: $M(h) := M(h^2)$, etc.

A standard analysis of the critical points of $\mathcal{H}_{\bar\kappa}$ as well as its gradient on the boundary of $\mathcal{P}^{1/2}$ follows. Under the assumption that for all $\sigma\in S$, $\bar\kappa_\sigma$ is no greater than 2π and less than the cone angle of Σ at σ, we show that critical points of $\mathcal{H}_{\bar\kappa}$ are positive definite and that the gradient of $\mathcal{H}$ on the boundary of $\mathcal{P}$ is homotopic to an outward vector field.

5.1 Reminders on Lorentzian angles and Schläffli's Formula

The following is an adaptation of the exposition of Rabah Souam [32].

To begin with, the modulus $|u|$ of a vector u of $\mathbb{E}^{1,2}$ is

$$|u| = \sqrt{\langle u\mid u\rangle},$$

with the convention that when $\langle u\mid u\rangle<0$ we have that $|u|=\lambda i$ with $\lambda>0$ and $i^2=-1$. Let u and v be two vectors of $\mathbb{E}^{1,2}$. Then the angle $\angle uv$ is defined so that it satisfies the following properties:

(1) For all vectors u and v, $\angle uv \in \mathbb{R}+i\mathbb{R}/(2\pi\mathbb{Z})$.

(2) For all vectors u and v, $\langle u\mid v\rangle = |u||v|\cosh(\angle uv)$.

(3) For all vectors u, v and w coplanar, $\angle uv + \angle vw = \angle uv$.

Beware that if u and v are spacelike, $\angle uv$ is not the usual angle $\widehat{uv}$ but actually $\widehat{uv}\cdot i$. Angles are well defined only if neither u nor v are lightlike.

Definition 5.2 (type of a vector of $\mathbb{E}^{1,1}$) Choose a direct Cartesian coordinate system (t,x) of the vector space underlying $\mathbb{E}^{1,1}$. Let u be a nonlightlike vector of $\mathbb{E}^{1,1}$. The type $k_u \in \mathbb{Z}/4\mathbb{Z}$ of u is defined as follows:

- $k_u = 0$ if u is future timelike.
- $k_u = 1$ if u is spacelike with negative spacelike coordinate.
- $k_u = 2$ if u is past timelike.
- $k_u = 3$ if u is spacelike with positive spacelike coordinate.

Definition 5.3 Define $\mathbb{H}^1_+$ as the Riemannian submanifold of unit future timelike vectors in $\mathbb{E}^{1,1}$. We choose the orientation $\vec{x}$ of $\mathbb{H}^1_+$ so that $(\vec{x},\vec{n})$ induces the same orientation as $\mathbb{E}^{1,1}$ for any future timelike vector $\vec{n}$.

Definition 5.4 Let u and v be two linearly independent nonlightlike unit vectors in $\mathbb{E}^{1,2}$ and let Π the vectorial plane generated by u and v.

- If Π is spacelike,
$$\angle uv = i\theta,$$
with θ the angle from u to v in Π oriented by the future timelike normal.

- If Π is timelike and u and v of types k_u and k_v in Π are identified with $\mathbb{E}^{1,1}$ and oriented by the basis (u,v), then
$$\angle uv = \alpha + i(k_v - k_u)\tfrac{1}{2}\pi,$$
with α the (oriented) length of the geodesics from u' to v' in $\mathbb{H}^1_+$, where u' (resp. v') is the unique future unit timelike vector of Π orthogonal or colinear to u (resp. v).

Definition 5.5 (dihedral angle) Let Π_1 and Π_2 be two vectorial half-planes that intersect along their common boundary Δ. Assume none of Π_1, Π_2, and Δ are lightlike and write $\nu_i = \Delta^\perp \cap \Pi_i$ for $i \in \{1,2\}$. We choose some $u \in \Delta$ and for $i \in \{1,2\}$ define n_i, the unique unit vector normal to Π_i such that (u,ν_i,n_i) is a direct basis. The dihedral angle $\angle \Pi_1\Pi_2$ between the planes Π_1 and Π_2 is then defined as
$$\angle \Pi_1\Pi_2 := \begin{cases} \mathrm{Real}(\angle n_1 n_2) & \text{if } \Delta^\perp \text{ is Lorentzian,} \\ \mathrm{Im}(\angle n_1 n_2) \in \,]-\pi,\pi] & \text{if } \Delta^\perp \text{ is Riemannian.} \end{cases}$$

Remark In the definition above, the dihedral angle does not depend on the choice of u.

Definition 5.6 (1-parameter family of oriented polyhedra) A 1-parameter family of oriented locally Minkowski polyhedra is the data of an oriented simplicial complex $\mathcal{K}$ and a map $\psi : [0,1] \times \mathcal{K} \to \mathbb{E}^{1,2}$ such that

(1) for all simplices P of $\mathcal{K}$ and all $t \in [0,1]$, $\psi|_{\{t\}\times P}$ is an orientation-preserving smooth embedding and $\psi(t,P)$ a polyhedron of $\mathbb{E}^{1,2}$,

(2) for all simplices P the restriction of ψ to $[0,1]\times P$ is smooth.

Let $(\mathcal{K},\psi)$ be a 1-parameter family of locally Minkowski polyhedra. If e is an edge of $\mathcal{K}$, then for all $t \in [0,1]$, we write $l_{e,t} \geq 0$ for the length of the edge $\psi(e,t) \subset \mathbb{E}^{1,2}$ and $\theta_{e,t}$ for the sum of the dihedral angles between the faces of the simplices of $\mathcal{K}$ around the edge e.

We will also have to assume that adjacent 2-facets never change convexity. This can be made rigorous by saying that the family $\{u, v_1, v_2\}$ used in the definition of the dihedral angle above always is such that $\det(u v_1 v_2)$ has constant sign (but can be 0).

Theorem (Schläffli's formula [32]) *Let* $(\mathcal{K},\psi)$ *be a* 1*-parameter family of oriented locally Minkowski polyhedra such that none of its faces or edges are lightlike and such that adjacent* 2*-facets never change convexity. Denoting by* $\mathcal{E}$ *the set of edges of* $\mathcal{K}$, *we have*

$$\sum_{e\in\mathcal{E}} l_{e,t} \frac{\mathrm{d}\theta_{e,t}}{\mathrm{d}t} = 0.$$

Remark The convexity condition is always satisfied by construction for the polyhedra we consider.

5.2 Kites and angles

Consider an adapted triangulation $\mathcal{T}$ of (Σ, S) and consider a cell $\mathcal{P}^{1/2}_{\mathcal{T}}$ of $h \in \mathcal{P}^{1/2}$ of nonempty interior.

For $h \in \mathcal{P}^{1/2}_{\mathcal{T}}$, the past of Σ in $M(h)$ is a locally Minkowski polyhedron with each simplex being a pyramid of $\mathbb{E}^{1,2}$ as represented in Figure 7, the notation of which we give a more precise meaning. If T is a triangle of $\mathcal{T}$ of vertices σ_1, σ_2, and σ_3, while $e = \overrightarrow{\sigma_1\sigma_2}$ and $e' = \overrightarrow{\sigma_1\sigma_3}$ are two edges on the boundary of T, define ρ_e the real part of the angle from $\overrightarrow{\sigma_1 O}$ to $\overrightarrow{\sigma_1\sigma_2}$, $\theta_{ee'}$ the real part of the angle from $\overrightarrow{\sigma_1\sigma_2}$ to $\overrightarrow{\sigma_1\sigma_3}$ and α_e the real part of the dihedral angle from the plane $(O\sigma_1\sigma_1)$ to the plane $(\sigma_1\sigma_2\sigma_3)$. In this section, edges are oriented so that we distinguish α_e and α_{-e}: the angle α_e is on the left of e, and thus α_{-e} is the angle on the right of e.

We aim at proving κ is continuous and computing the partial derivatives

$$\frac{\partial \kappa_{\sigma_1}}{\partial h_{\sigma_2}} \quad \text{for } \sigma_1, \sigma_2 \in S.$$

If there is no edge from σ_1 to σ_2, then this derivative is null. If there is an edge e from σ_1 to σ_2, then in both pyramids P_+ and P_- on both sides of e, we need to study the variations of the dihedral angle

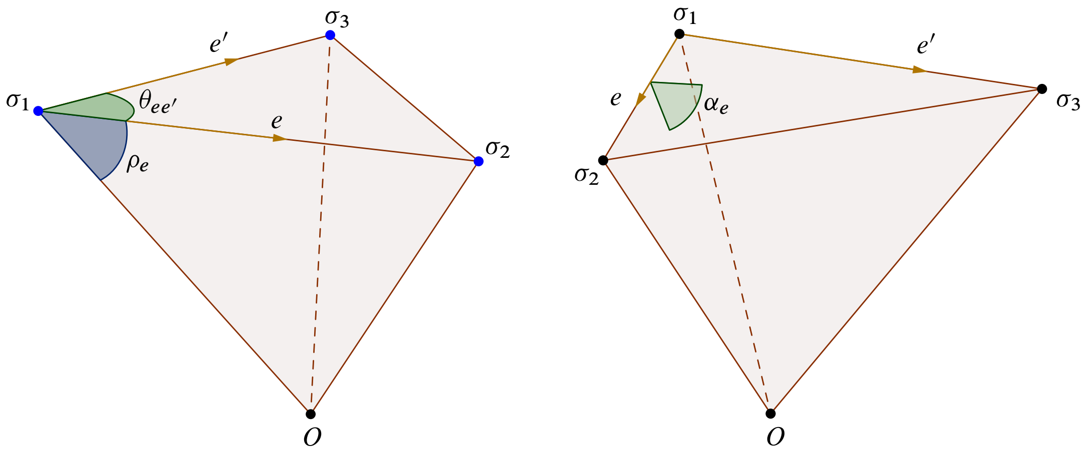


Figure 7: The simplex cell of the past of Σ in $M(h)$. The following angles are represented: ρ_e the angle from $\overrightarrow{\sigma_1 O}$ to $\overrightarrow{\sigma_1\sigma_2}$, $\theta_{ee'}$ the angle from $\overrightarrow{\sigma_1\sigma_2}$ to $\overrightarrow{\sigma_1\sigma_3}$ and α_e the angle from the plane $(0\sigma_1\sigma_1)$ to the plane $(\sigma_1\sigma_2\sigma_3)$.

on the edge $[O\sigma_1]$ with respect to h_{σ_2} and h_{σ_1}. Since the algebraic relationship between κ_σ and h_σ is complicated, a key to obtaining meaningful relations is to draw the kite associated with each embedded triangle $\mathbb{E}^{1,2}$.

Definition–Proposition 5.7 (kite, [14, pages 90–91]) *A hyperbolic kite* (*resp. Euclidean kite*) *is a quadrangle $ABCD$ in $X = \mathbb{H}^2$ (resp. in $X = \mathbb{E}^2$) with two opposite right angles and possibly with self-intersections. We parametrize kites by fixing a convex quadrangle decorated as in Figure 8 and constructing it as follows*:

1. *choose some point A in X and some direction $\vec{u} \in T_A X$*,
2. *move ρ_1 along the oriented line $(A\vec{u})$ to reach at $B = \exp_A(\rho_1 \vec{u})$*,
3. *turn $\frac{1}{2}\pi$ (counterclockwise) to obtain the new direction $\vec{v} \in T_B X$*,

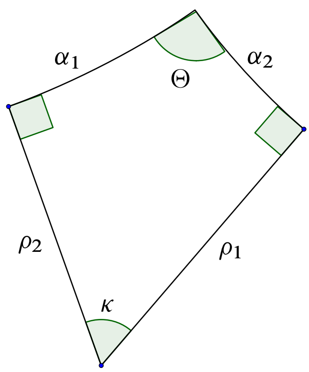


Figure 8

(4) *move* α_2 *on the oriented line* $(B\vec{v})$ *to reach* $C = \exp_B(\alpha_2\vec{v})$,

(5) *turn* $\pi - \Theta$ *to obtain* $\vec{w}$,

(6) *move a distance* α_1 *on the oriented line* $(C\vec{w})$ *to reach* D,

(7) *turn* $\frac{1}{2}\pi$ *to obtain* $\vec{k}$,

(8) *move a distance* ρ_2 *on the oriented line* $(D\vec{k})$ *to reach* A',

(9) *turn* $\pi - \kappa$ *to obtain* $\vec{u}'$.

For any choices of three out of the six parameters $\alpha_1, \alpha_2, \rho_1, \rho_2, \kappa$, *and* Θ, *there exists a unique choice for the three others so that the construction above yields a hyperbolic kite, ie* $X = \mathbb{H}^2$, $A' = A$, *and* $\vec{u}' = \vec{u}$. *Furthermore, for such six parameters,*

$$\cos(\kappa) = \frac{\sinh(\rho_1)\sinh(\rho_2) - \cos(\Theta)}{\cosh(\rho_1)\cosh(\rho_2)}, \quad \sinh(\rho_2) = \frac{\cos(\kappa)\sinh(\alpha_1) + \sinh(\alpha_2)}{\sin(\kappa)\cosh(\alpha_1)},$$
$$\frac{\sin(\kappa)}{\sin(\Theta)} = \frac{\cosh(\alpha_2)}{\cosh(\rho_2)} = \frac{\cosh(\alpha_1)}{\cosh(\rho_1)}.$$

Consider P_+ and use the notation of Figure 7. Then consider the quadrilateral of $\mathbb{H}^2$ given by the sequence of geodesics in the set of future unit timelike vectors identified with the hyperbolic plane $\mathbb{H}^2$:

$$(O\sigma_1) \to (O\sigma_1\sigma_2) \cap (\sigma_1\sigma_2)^\perp \to (\sigma_1\sigma_2)^\perp \cap (\sigma_1\sigma_3)^\perp \to (\sigma_1\sigma_3)^\perp \cap (O\sigma_1\sigma_3) \to (O\sigma_1).$$

To identify the parameters $(\rho_1, \alpha_2, \alpha_1, \rho_2, \kappa, \Theta)$ as in Figure 9, we note the following.

- $\rho_e := \mathrm{Real}(\angle \overrightarrow{\sigma_1 O}\,\overrightarrow{\sigma_1\sigma_2}) := d_{\mathbb{H}}((O\sigma_1), (O\sigma_1\sigma_2) \cap (\sigma_1\sigma_2)^\perp)$ since $(O\sigma_1\sigma_2)$ is the (timelike) vectorial plane containing both vectors and $\overrightarrow{\sigma_1\sigma_2}$ is spacelike, and $\overrightarrow{\sigma_1 O}$ is past timelike. So the (oriented) length of the geodesic $(O\sigma_1) \to (O\sigma_1\sigma_2) \cap (\sigma_1\sigma_2)^\perp$ is ρ_e, and thus $\rho_1 = \rho_e$.

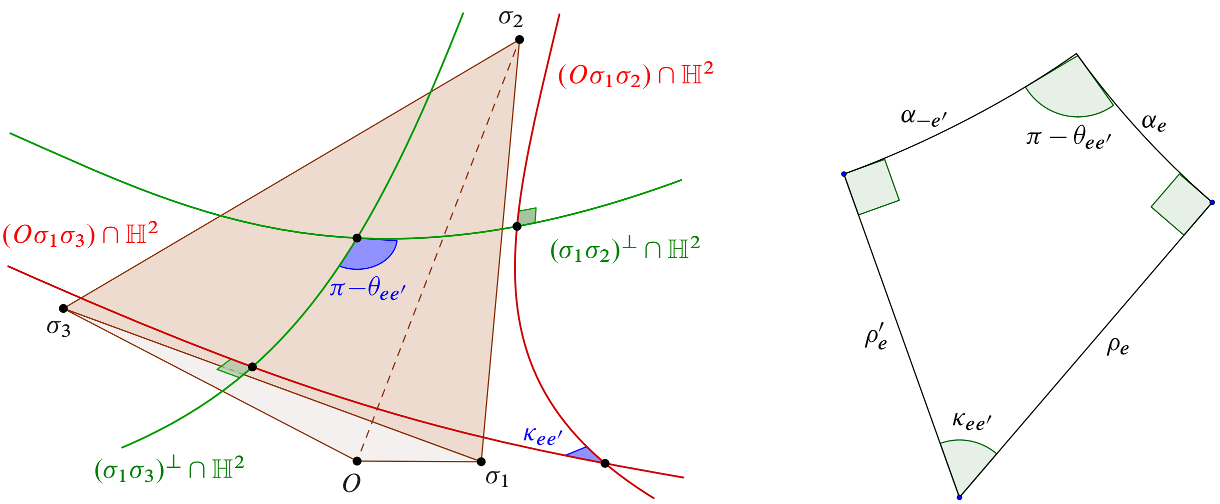


Figure 9: The kite associated to an edge, with e the edge $\overrightarrow{\sigma_1\sigma_2}$ and e' the edge $\overrightarrow{\sigma_1\sigma_3}$.

Mutatis mutandis, we show the same way $-\rho_2 = -\rho_{e'}$; thus $\rho_2 = \rho_{e'}$.

- $\alpha_e := \mathrm{Real}(\angle(O\sigma_1\sigma_2)(\sigma_1\sigma_2\sigma_3)) = \mathrm{Real}(\angle n_1 n_2)$, where n_1 and n_2 are respectively the normals to the planes $(O\sigma_1\sigma_2)$ and $(\sigma_1\sigma_2\sigma_3)$ such that $(\overrightarrow{\sigma_1\sigma_2}, \overrightarrow{\sigma_1 O}, n_1)$ and $(\overrightarrow{\sigma_1\sigma_2}, \overrightarrow{\sigma_1\sigma_3}, n_2)$ are direct bases. We thus have $n_2 \in (\sigma_1\sigma_2\sigma_3)^\perp = (\sigma_1\sigma_2)^\perp \cap (\sigma_1\sigma_3)^\perp$ future timelike and $n_1 \in (O\sigma_1\sigma_2)^\perp$ spacelike, so $\angle n_1 n_2 = \angle n_3 n_2$ with $n_3 \in ((O\sigma_1\sigma_2)^\perp)^\perp \cap (n_1 n_2) = (O\sigma_1\sigma_2) \cap (\sigma_1\sigma_2)^\perp$. Therefore α_e is the (oriented) length of the geodesic
$$(O\sigma_1\sigma_2) \cap (\sigma_1\sigma_2)^\perp \to (\sigma_1\sigma_2)^\perp \cap (\sigma_1\sigma_3)^\perp.$$

We thus have shown that $\alpha_2 = \alpha_e$.

Mutatis mutandis, we show the same way that $-\alpha_{-e'} = -\alpha_1$; thus $\alpha_{e'} = \alpha_1$.

- The parameter κ is given by the dihedral angle $\angle(O\sigma_1\sigma_2)(O\sigma_1\sigma_3)$; thus $\kappa = \kappa_{ee'}$.

- Finally, to compute Θ, notice that the radial projection of the hyperbolic kite on the spacelike plane $(\sigma_1\sigma_2\sigma_3)$ yields a Euclidean kite with the same signs of oriented lengths of sides and whose κ parameter is $\theta_{ee'}$. Furthermore, the plane $(\sigma_1\sigma_2\sigma_3)$ is orthogonal to the timelike line from O to $(\sigma_1\sigma_2)^\perp \cap (\sigma_1\sigma_3)^\perp$, so the angle in $\mathbb{H}^2$ at $\mathbb{H}^2 \cap (\sigma_1\sigma_2)^\perp \cap (\sigma_1\sigma_3)^\perp$ is the same as the Euclidean angle in $(\sigma_1\sigma_2\sigma_3)$ at $(\sigma_1\sigma_2\sigma_3) \cap (\sigma_1\sigma_2)^\perp \cap (\sigma_1\sigma_3)^\perp$. We deduce that $\Theta = \pi - \theta_{ee'}$.

Corollary 5.8 *Using the same notation as in Definition–Proposition 5.7 and choosing Θ, ρ_1, and ρ_2 as parameters*,
$$\frac{\partial\kappa}{\partial\rho_1} = -\frac{\tanh(\alpha_2)}{\cosh(\rho_1)}.$$

We thus need to compute the derivative of ρ_e with respect to the heights $(h_\sigma)_{\sigma\in S}$ for each edge e.

Lemma 5.9 *Using the notation of Figure 7*,
$$\mathrm{d}\rho_e = -\frac{(h_{\sigma_1}^2 + h_{\sigma_2}^2 + l_e^2)\,\mathrm{d}h_{\sigma_1} - 2h_{\sigma_1}h_{\sigma_2}\,\mathrm{d}h_{\sigma_2}}{2l_e h_{\sigma_1}^2 \cosh(\rho_e)}.$$

Proof From the cosine law in $\mathbb{E}^{1,2}$:
$$-h_{\sigma_2}^2 = -h_{\sigma_1}^2 + l_e^2 - 2l_e h_{\sigma_1}\sinh(\rho_e),$$
$$\cosh(\rho_e)\,\mathrm{d}\rho_e = \frac{h_{\sigma_1}(-2h_{\sigma_1}) - (h_{\sigma_2}^2 + l_e^2 - h_{\sigma_1}^2)}{2l_e h_{\sigma_1}^2}\,\mathrm{d}h_{\sigma_1} + \frac{2h_{\sigma_2}h_{\sigma_1}}{2l_e h_{\sigma_1}^2}\,\mathrm{d}h_{\sigma_2},$$
$$\mathrm{d}\rho_e = -\frac{(h_{\sigma_2}^2 + l_e^2 + h_{\sigma_1}^2)\,\mathrm{d}h_{\sigma_1} - 2h_{\sigma_1}h_{\sigma_2}\,\mathrm{d}h_{\sigma_2}}{2l_e h_{\sigma_1}^2\cosh(\rho_e)}.$$ □

5.3 The Einstein–Hilbert functional

We give ourselves some $Z \subset S$, and define $z := |Z|$ and $s := |S|$. Define $\mathcal{P}_Z := \{\tau \in \mathbb{R}^S \mid \forall\sigma \in Z,\ \tau_\sigma = 0\}$ as well as $\mathcal{P}_Z^{1/2} := \{\bar\kappa \in \mathbb{R}_+^S \mid \forall\sigma \in Z,\ \bar\kappa_\sigma = 0\}$. Recall that we set $\kappa_\sigma(\tau) = 0$ if $\tau_\sigma = 0$.

Definition 5.10 (Einstein–Hilbert functional) Let $\bar{\kappa} \in K_Z$. For $h \in \mathcal{P}_Z^{1/2}$ and for an edge e of $\mathcal{T}_h$, we denote by l_e the length of e and by θ_e the dihedral angle of the embedding ι_h at the edge e.

The Einstein–Hilbert functional is defined as follows:

$$\mathcal{H}_{\bar{\kappa}} : \mathcal{P}_Z^{1/2} \to \mathbb{R}, \quad h \mapsto \sum_{\sigma \in S} h_\sigma (\kappa_\sigma - \bar{\kappa}_\sigma) + \sum_{e \in \mathrm{Edge}(\mathcal{T}_h)} l_e \theta_e.$$

Lemma 5.11 *Letting* $\sigma \in S$, *the map* $h \mapsto \kappa_\sigma(h)$ *is continuous on* $\mathcal{P}^{1/2}$ *and* $\mathscr{C}^1$ *on each cell* $\mathcal{P}_{\mathcal{T}}^{1/2}$ *of* $\mathcal{P}^{1/2}$.

Proof From $-h_{\sigma_2}^2 = -h_{\sigma_1}^2 + l_e^2 - 2 l_e h_{\sigma_1} \sinh(\rho_e)$ — the cosine law in $\mathbb{E}^{1,2}$ — together with the first equality of Definition–Proposition 5.7, the restriction of κ_σ to $H_{\mathcal{T}}^{\sigma+} := \{h \in \mathcal{P}_{\mathcal{T}}^{1/2} \mid h(\sigma) > 0\}$ for each triangulation $\mathcal{T}$ is $\mathscr{C}^1$. Let $h \in \partial \mathcal{P}_{\mathcal{T}}^{1/2}$. Then for all cells $\mathcal{P}_{\mathcal{T}'}^{1/2}$ containing h, the triangulations $\mathcal{T}$ and $\mathcal{T}'$ are equivalent by Proposition 2.20, so $M_{h,\mathcal{T}} \simeq M_{h,\mathcal{T}'}$. Hence $\kappa_{\sigma,\mathcal{T}}(h) = \kappa_{\sigma,\mathcal{T}'}(h)$, and we deduce that κ_σ is continuous on $H^{\sigma+} := \{h \in \mathcal{P}^{1/2} \mid h(\sigma) > 0\}$. Again using the cosine law, for any edge $e = [\sigma\sigma']$ in some triangulation $\mathcal{T}$ and any $\bar{h} \in \mathcal{P}_{\mathcal{T}}^{1/2}$ such that $\bar{h}(\sigma) = 0$, we have

$$\lim_{\substack{h \to \bar{h} \\ h \in H_{\mathcal{T}}^{\sigma+}}} \rho_e = \lim_{\substack{h \to \bar{h} \\ h \in H_{\mathcal{T}}^{\sigma+}}} \sinh^{-1} \frac{l_e^2 - h_\sigma^2 + h_{\sigma'}^2}{l_e h_\sigma} = +\infty.$$

The first equality of Definition–Proposition 5.7 yields

$$\lim_{\substack{h \to \bar{\kappa} \\ h \in H_{\mathcal{T}}^{\sigma+}}} \cos \kappa_\sigma(h) = 1,$$

and since $0 \le \kappa \le \pi$ we deduce that $\lim_{h \to \bar{h}, h \in H_{\mathcal{T}}^{\sigma+}} \kappa_\sigma(h) = 0$. Hence κ_σ is continuous on $\mathcal{P}_{\mathcal{T}}^{1/2}$. From Corollary 5.8 and $\lim_{h \to \bar{h}, h \in H_{\mathcal{T}}^{\sigma+}} \rho_e = +\infty$, we also obtain that

$$\lim_{\substack{h \to \bar{h} \\ h \in H_{\mathcal{T}}^{\sigma+}}} \frac{\partial \kappa_\sigma}{\partial h_{\sigma''}}(h) = 0$$

for any $\sigma'' \in S$, and obviously $(\partial \kappa / \partial h_{\sigma''})(h) = 0$ if $h(\sigma) = 0$ and $\sigma'' \neq \sigma$. We deduce that κ is continuously differentiable on $\mathcal{P}_{\mathcal{T}}^{1/2}$.

Finally, since $\mathcal{P}^{1/2}$ is the union of finitely many such cells $\mathcal{P}_{\mathcal{T}}^{1/2}$, we get continuity on $\mathcal{P}^{1/2}$. □

Lemma 5.12 *Let* $\mathcal{T}$ *be a triangulation associated with a cell* $\mathcal{P}_{\mathcal{T}}^{1/2}$ *of* $\mathcal{P}^{1/2}$. *For any edge* e *of* $\mathcal{T}$, *the map* $h \mapsto \theta_e(h)$ *is* $\mathscr{C}^1$ *on* $\mathcal{P}_{\mathcal{T}}^{1/2}$.

Proof By construction of the polyhedral embedding, it suffices to show the "half" dihedral angle α_e is $\mathscr{C}^1$, effectively reducing the problem to the embedding of a fixed triangle $T = [\sigma_1\sigma_2\sigma_3] \in \mathcal{T}$. Note that although the edge $[O\sigma_i]$ may become lightlike when $h_{\sigma_i} \to 0$, the planes $(O\sigma_i\sigma_j)$ (resp. $(\sigma_1\sigma_2\sigma_3)$) are never degenerated and stay timelike (resp. spacelike) for $i, j \in \{1,2,3\}$. Therefore the angle α_e is well defined and depends in a $\mathscr{C}^1$ manner in the coordinates of the embeddings of the σ_i for $i \in \{1,2,3\}$.

Using notation of Lemma 2.3, the center ω is the orthogonal projection of O on $(\sigma_1\sigma_2\sigma_3)$. We may choose the embedding of T in such a way that $(\sigma_1\sigma_2\sigma_3)$ is the plane $\{t = \sqrt{\tau_0}\}$, ie $\omega = (t,0,0)$. Recall that $\tau_0(h)$ is positive and depends polynomially on h. We may in addition fix the embedding ι so that $\iota(\sigma_1) = (h,x,0)$ with $x > 0$. Then elementary trigonometry in the spacelike plane $(\sigma_1\sigma_2\sigma_3)$ yields that the coordinates of $\iota(\sigma_i)$ are $\mathscr{C}^1$ functions in h. □

Proposition 5.13 *Let $\bar\kappa \in \mathbb{R}_+^S$. The functional $\mathcal{H}_{\bar\kappa}$ is well defined, $\mathscr{C}^1$ on $\mathcal{P}_Z^{1/2}$, and*

$$\mathrm{d}\mathcal{H}_{\bar\kappa} = \sum_{\sigma\in S\setminus Z} (\kappa_\sigma - \bar\kappa_\sigma)\,\mathrm{d}h_\sigma.$$

Proof We prove the proposition for $Z = \varnothing$; the other cases are corollaries.

Consider the family of compact locally Minkowski polyhedra $(Q_h)_{h\in\mathcal{P}^{1/2}}$ given by the past of the polyhedral Cauchy surface $\iota_h(\Sigma) \subset M(h)$. For any triangulation $\mathcal{T}$ defining a cell $\mathcal{P}_\mathcal{T}$ of $\mathcal{P}$, the underlying simplicial complex $\mathcal{K}_h$ of Q_h is constant on $\mathcal{P}_\mathcal{T}$. The edges of $\mathcal{T}$ are always spacelike, κ is well defined and continuous on $\mathcal{P}_\mathcal{T}^{1/2}$, and $\mathcal{K}_h$ is a continuous family of polyhedra. All the angles in the definition of $\mathcal{H}_{\bar\kappa}$ are $\mathscr{C}^1$ on each cell $\mathcal{P}_\mathcal{T}^{1/2}$ by Lemmas 5.11 and 5.12. In addition, Lemma 5.11 gives continuity of $h \mapsto \sum_{\sigma\in S} h_\sigma(\kappa_\sigma - \bar\kappa_\sigma)$ on the whole $\mathcal{P}^{1/2}$. Continuity of $\sum_{e\in\mathrm{Edge}(\mathcal{T}_h)} l_e\theta_e$ follows from the remark that at some h on the interface of adjacent cells $\mathcal{P}_\mathcal{T}^{1/2}$ and $\mathcal{P}_{\mathcal{T}'}^{1/2}$, one obtains $\mathcal{T}'$ from $\mathcal{T}$ by flipping h-critical edges. On such edges e one has $\theta_e = 0$, so the sums $\sum_{e\in\mathrm{Edge}(\mathcal{T}_h)} l_e\theta_e$ and $\sum_{e\in\mathrm{Edge}(\mathcal{T}'_h)} l_e\theta_e$ only differ by null terms. We conclude that $\mathcal{H}_{\bar\kappa}$ is continuous on $\mathcal{P}^{1/2}$ and its restriction to any cell is $\mathscr{C}^1$.

Schläffli's formula thus applies to the interior of any cell $\mathcal{P}_\mathcal{T}^{1/2}$ where $h > 0$ and gives

$$\sum_{\sigma\in S} h_\sigma\,\mathrm{d}\kappa_\sigma + \sum_{e\in\mathcal{A}_h} l_e\,\mathrm{d}\theta_e = 0.$$

Hence

$$\mathrm{d}\mathcal{H}_{\bar\kappa} = \sum_{\sigma\in S}(\kappa_\sigma - \bar\kappa_\sigma)\,\mathrm{d}h_\sigma + \sum_{\sigma\in S} h_\sigma\,\mathrm{d}\kappa_\sigma + \sum_{e\in\mathcal{A}_h} l_e\,\mathrm{d}\theta_e = \sum_{\sigma\in S}(\kappa_\sigma - \bar\kappa_\sigma)\,\mathrm{d}h_\sigma.$$

We have thus proved the result for the restriction to the interior of any cell $\mathcal{P}_\mathcal{T}^{1/2}$, and hence on a dense subset of $\mathcal{P}^{1/2}$. Finally, by continuity of $h \mapsto \sum_{\sigma\in S}(\kappa_\sigma - \bar\kappa_\sigma)\,\mathrm{d}h_\sigma$ and $d\mathcal{H}_{\bar\kappa}$ on $\mathcal{P}^{1/2}$, the result follows. □

We now study the Hessian of the Einstein–Hilbert functional on the interior of the domain of admissible times $\mathcal{P}^{1/2}$.

Lemma 5.14 *The map κ is $\mathscr{C}^1$ on $\mathcal{P}_Z^{1/2}$, and for all h in $\mathcal{P}_Z^{1/2}$ and all $\sigma \in S \setminus Z$, we have*

$$\mathrm{d}_h\kappa_\sigma = \sum_{\substack{e \in \mathcal{E}_h,\, e\colon \sigma \rightsquigarrow \sigma' \\ \sigma' \in S\setminus Z}} (\tanh(\alpha_e) + \tanh(\alpha_{-e})) \frac{(h_{\sigma'}^2 + l_e^2 + h_\sigma^2)\,\mathrm{d}h_\sigma - 2h_\sigma h_{\sigma'}\,\mathrm{d}h_{\sigma'}}{2l_e c_e^2},$$

where $\mathcal{E}_h$ is the set of edges of any h-Delaunay triangulation and where

$$c_e = \begin{cases} \cosh(\rho_e) h_\sigma & \text{if } h_\sigma \neq 0, \\ (l_e^2 + h_{\sigma'}^2)/l_e & \text{if } h_\sigma = 0. \end{cases}$$

Proof By Lemma 5.11, the restriction of κ is $\mathscr{C}^1$ on each cell $\mathcal{P}_{Z,\mathcal{T}}^{1/2}$. To prove κ is $\mathscr{C}^1$ on the whole $\mathcal{P}_Z^{1/2}$, it suffices to show that the equality holds on the relative interiors of cells and that the right-hand side is well defined and continuous on the whole $\mathcal{P}_Z^{1/2}$.

As argued in the proof of Lemma 5.12, for any edge $e\colon \sigma \rightsquigarrow \sigma'$, the angles α_e and α_{-e} are well defined and continuous even at h with null coordinates. In addition, by the cosine law, when $h \to \bar{h}$ for some $\bar{h}$ such that $\bar{h}_\sigma = 0$, we have $\rho_e \to +\infty$ and

$$\cosh(\rho_e) h_\sigma \sim \sinh(\rho_e) h_\sigma \sim \frac{l_e^2 + h_{\sigma'}^2}{l_e}.$$

The right-hand side is then well defined and continuous when restricted to a given cell $\mathcal{P}_{Z,\mathcal{T}}^{1/2}$.

As before, critical edges e in the sum yield zero terms as $0 = \theta_e = \alpha_e + \alpha_{-e}$, ie $\alpha_e = -\alpha_{-e}$, so that $\tanh \alpha_e = -\tanh \alpha_{-e}$. We conclude that the right-hand side does not depend on the h-Delaunay triangulation and is thus well defined and continuous on the whole $\mathcal{P}_Z^{1/2}$.

For h in the relative interior of a cell $\mathcal{P}_{Z,\mathcal{T}}^{1/2}$ associated to a triangulation $\mathcal{T}$ and for $\sigma \in S \setminus Z$, denote by $(e_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ the family of outgoing edges from σ enumerated coherently with the orientation of Σ. Define $\sigma_i \in S$ the other end of e_i so that

$$\begin{aligned}
\mathrm{d}_h\kappa_\sigma &= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \mathrm{d}_h\kappa_{e_i e_{i+1}} = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \left(-\frac{\tanh(\alpha_{e_i})}{\cosh(\rho_{e_i})}\,\mathrm{d}\rho_{e_i} - \frac{\tanh(\alpha_{-e_{i+1}})}{\cosh(\rho_{e_{i+1}})}\,\mathrm{d}\rho_{e_{i+1}} \right) \\
&= -\sum_{i \in \mathbb{Z}/n\mathbb{Z}} \left(\frac{\tanh(\alpha_{e_i})}{\cosh(\rho_{e_i})} + \frac{\tanh(\alpha_{-e_i})}{\cosh(\rho_{e_i})} \right) \mathrm{d}\rho_{e_i} = -\sum_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{\tanh(\alpha_{e_i}) + \tanh(\alpha_{-e_i})}{\cosh(\rho_{e_i})}\,\mathrm{d}\rho_{e_i} \\
&= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{\tanh(\alpha_{e_i}) + \tanh(\alpha_{-e_i})}{\cosh(\rho_{e_i})} \frac{(h_\sigma^2 + h_{\sigma_i}^2 + l_e^2)\,\mathrm{d}h_\sigma - 2h_\sigma h_{\sigma_i}\,\mathrm{d}h_{\sigma_i}}{2l_e h_\sigma^2 \cosh(\rho_e)} \\
&= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{\tanh(\alpha_{e_i}) + \tanh(\alpha_{-e_i})}{\cosh^2(\rho_{e_i})} \frac{(h_\sigma^2 + h_{\sigma_i}^2 + l_e^2)\,\mathrm{d}h_\sigma - 2h_\sigma h_{\sigma_i}\,\mathrm{d}h_{\sigma_i}}{2l_e h_\sigma^2}.
\end{aligned}$$

□

Proposition 5.15 *For $\bar{\kappa} \in \mathbb{R}_+^S$, the functional $\mathcal{H}_{\bar{\kappa}}$ is convex on $\mathcal{P}_Z^{1/2}$ and strictly convex on the relative interior of $\mathcal{P}_Z^{1/2}$.*

Proof From Proposition 5.13 and Lemma 5.14, $\mathcal{H}_{\bar{\kappa}}$ is $\mathscr{C}^2$ on $\mathcal{P}_Z^{1/2}$ and its Hessian matrix H has the following coefficients for $h \in \mathcal{P}_Z^{1/2}$, an h-Delaunay triangulation being chosen, for all $\sigma, \sigma' \in S \setminus Z$ with $\sigma \neq \sigma'$:

$$H_{\sigma,\sigma'} = -\sum_{e\colon \sigma \rightsquigarrow \sigma'} (\tanh(\alpha_e) + \tanh(\alpha_{-e})) \frac{2h_\sigma h_{\sigma'}}{2l_e c_e^2} \leq 0$$

$$H_{\sigma,\sigma} = \sum_{\sigma' \in S} \sum_{e\colon \sigma \rightsquigarrow \sigma'} (\tanh(\alpha_e) + \tanh(\alpha_{-e})) \frac{h_\sigma^2 + h_{\sigma'}^2 + l_e^2}{2l_e c_e^2} - \sum_{e\colon \sigma \rightsquigarrow \sigma} (\tanh(\alpha_e) + \tanh(\alpha_{-e})) \frac{2h_\sigma h_\sigma}{2l_e c_e^2}.$$

Since the embedding of Σ into $M(h)$ is convex, $\tanh(\alpha_e) + \tanh(\alpha_{-e}) \geq 0$ with equality if and only if the edge is h-critical. Therefore, for all $\sigma \in S$,

$$\begin{aligned} H_{\sigma,\sigma} + \sum_{\sigma' \neq \sigma} H_{\sigma,\sigma'} &= \sum_{\sigma' \in S} \sum_{e\colon \sigma \rightsquigarrow \sigma'} (\tanh(\alpha_e) + \tanh(\alpha_{-e})) \frac{h_{\sigma'}^2 + l_e^2 + h_\sigma^2 - 2h_\sigma h_{\sigma'}}{2l_e c_e^2} \\ &= \sum_{\sigma' \in S} \sum_{e\colon \sigma \rightsquigarrow \sigma'} (\tanh(\alpha_e) + \tanh(\alpha_{-e})) \frac{(h_{\sigma'} - h_\sigma)^2 + l_e^2}{2l_e c_e^2} \geq 0. \end{aligned}$$

The Hessian matrix of $\mathcal{H}_{\bar{\kappa}}$ is thus diagonally dominant on $\mathcal{P}_Z^{1/2}$.

Consider some h in $\mathcal{P}_Z^{1/2}$ and $\sigma \in S \setminus Z$ such that $H_{\sigma,\sigma} - \sum_{\sigma' \neq \sigma} |H_{\sigma,\sigma'}| = 0$. Then all outgoing edges from σ are h-critical. We build a hinge as follows.

(1) Take any h-Delaunay triangulation of Σ and enumerate counterclockwise the p vertices $(\sigma_k)_{k \in \mathbb{Z}/p\mathbb{Z}}$ of the neighborhood of σ.

(2) Consider the hinge $Q = ([\sigma\sigma_{-2}\sigma_{-1}\sigma_0], [\sigma\sigma_{-1}])$.

(3) If Q is unflippable return Q.

(4) Otherwise, flip Q; the neighborhood vertices of σ are now $(\sigma_k)_{k \in \mathbb{Z}/(p-1)\mathbb{Z}}$. Then return to step (2).

Since at each step, the number of neighbors of σ decreases, the algorithm eventually stops after finitely many iterations and thus returns an unflippable immersed hinge in the neighborhood of σ. Such a hinge is h-critical and unflippable; hence h^2 is in a boundary facet of $\mathcal{P}$ not of the type $h_{\sigma'} = 0$. We conclude that h is not in the relative interior of $\mathcal{P}_Z^{1/2}$. Finally, the Hessian matrix H is strictly diagonally dominant on the relative interior of $\mathcal{P}_Z^{1/2}$. □

5.4 Proof of the main theorem

Theorem 5 *Let Σ be a closed locally Euclidean surface of genus g with s marked conical singularities of angles $(\theta_i)_{i \in [\![1,s]\!]}$. For all*

$$\bar{\kappa} \in \left(\prod_{i=1}^{s} [0, \min(\theta_i, 2\pi)] \right) \setminus \{(\theta_1, \dots, \theta_s)\},$$

there is a radiant singular flat spacetime M homeomorphic to $\Sigma \times \mathbb{R}$ with exactly s marked lines $\Delta_1, \dots, \Delta_s$ of respective cone angles $\bar{\kappa}_1, \dots, \bar{\kappa}_s$ and a convex polyhedral embedding $\iota\colon (\Sigma, S) \to (M, (\Delta_i)_{i \in [\![1,s]\!]})$.

Furthermore, if for all $i \in [\![1,s]\!]$, $\bar\kappa_i < \theta_i$, *then such a couple* (M,ι) *is unique up to equivalence.*

Finally, if for some $i \in [\![1,s]\!]$, $\theta_i \le \pi$, *there is no such convex polyhedral embedding such that* $\kappa_i > \theta_i$.

Denoting by $\kappa(x)$ the cone angle at x if x is a point in an $\mathbb{H}^2_{\geq 0}$-manifold, in view of Theorem 6 the main case of the theorem can also be stated as follows:

Corollary 5.16 *Let* Σ *be a closed locally Euclidean surface of genus* g *with* s *marked conical singularities of angles* $(\theta_\sigma)_{\sigma\in S}$. *For all* $\bar\kappa \in \prod_{\sigma\in S}[0,2\pi]\cap[0,\theta_\sigma[$, *there exists a closed* $\mathbb{H}^2_{\geq 0}$*-manifold* $\Sigma_{\bar\kappa}$ *together with a homeomorphism* $h\colon \Sigma \to \Sigma_{\bar\kappa}$ *and a convex polyhedral embedding* $\iota\colon (\Sigma, S) \to \mathrm{susp}(\Sigma_{\bar\kappa})_S$ *such that*

- *for all* $\sigma \in S$, $\bar\kappa_\sigma = \kappa(h(\sigma))$,
- *with* $\mathrm{susp}(\Sigma_{\bar\kappa}) \xrightarrow{\pi} \Sigma_{\bar\kappa}$ *the natural projection, we have* $\pi\circ\iota = h$.

Furthermore, such a triple $(\Sigma_{\bar\kappa}, h, \iota)$ *is unique up to equivalence.*

Remark Equivalence between triples $(\Sigma^{(i)}_{\bar\kappa}, h^{(i)}, \iota^{(i)})$ for $i\in\{1,2\}$ is understood as an isomorphism $\varphi\colon \Sigma^{(1)}_{\bar\kappa} \to \Sigma^{(2)}_{\bar\kappa}$ such that $\iota^{(2)} = \hat\varphi\circ\iota^{(1)}$ with $\hat\varphi\colon \mathrm{susp}(\Sigma^{(1)}_{\bar\kappa}) \xrightarrow{\sim} \mathrm{sup}(\Sigma^{(2)}_{\bar\kappa})$ the isomorphism induced by φ.

Let us prove a last lemma:

Lemma 5.17 *With* $\theta = (\theta_\sigma)_{\sigma\in S}$ *the cone angles of* Σ, *we have*

$$\lim_{\substack{\tau\in\mathcal{P}\\ \tau\to+\infty}} \kappa(\tau) = \theta.$$

Proof We use the same notation as in the preceding section. In a given cell $\mathcal{P}_{\mathcal{T}}$ of $\mathcal{P}$, for each vertex $\sigma\in S$ and for all edges e of $\mathcal{T}$ outgoing from σ to some σ_2, by the cosine law

$$-\tau_{\sigma_2} = -\tau_\sigma + l_e^2 - 2l_e\sqrt{\tau_\sigma}\sinh(\rho_e).$$

Since $|\tau_{\sigma_1}-\tau_{\sigma_2}|$ is uniformly bounded on $\mathcal{P}$ and l_e is constant, $\rho_e \xrightarrow{\tau\to+\infty} 0$. Then from Definition–Proposition 5.7, with e' the subsequent edge around σ, we have $\kappa_{ee'} \xrightarrow{\tau\to+\infty} \theta_{ee'}$. Hence,

$$\kappa_\sigma(\tau) \xrightarrow{\tau\in\mathcal{P}_{\mathcal{T}},\,\tau\to+\infty} \theta_\sigma.$$

Finally, there are only finitely many cells $\mathcal{P}_{\mathcal{T}}$, and S is finite. □

Proof of Theorem 5 Let $Z\subset S$. We prove the theorem for $\bar\kappa$ such that $\{\sigma\in S \mid \bar\kappa_\sigma = 0\} = Z$. It suffices to show that for such $\bar\kappa$ the Einstein–Hilbert functional $\mathcal{H}_{\bar\kappa}$ has exactly one critical point in $\mathcal{P}_Z^{1/2}$. Define $K_Z := \{\bar\kappa\in\prod_{\sigma\in S}[0,2\pi]\cap[0,\theta_\sigma[\mid \forall\sigma\in Z, \bar\kappa_\sigma = 0\}$. We need to prove the existence and uniqueness of critical points of $\mathcal{H}_{\bar\kappa}$ for any $\bar\kappa\in K_Z$.

If $z = s$ then $K_Z = \{0\}$ and $\mathcal{P}_Z = \{0\}$ by Theorem 2(c), and there is nothing else to prove. Otherwise, we proceed as follows.

By Proposition 5.15 the functional $\mathcal{H}_{\bar\kappa}$ is strictly convex in the relative interior of $\mathcal{P}_Z^{1/2}$; thus the critical points are of index 1 when considered as a function on the relative interior of $\mathcal{P}_Z$.

Let $\tau \in \partial\mathcal{P}_Z$, the relative boundary of $\mathcal{P}_Z$, and let $\bar\kappa \in K_Z$. By Theorem 2(e), on $\partial\mathcal{P}_Z$ there exists $\sigma \in S \setminus Z$ such that either $\tau_\sigma = 0$ or τ is in the kernel of the affine form of an unflippable immersed hinge. In the former situation, $0 = \kappa_\sigma < \bar\kappa_\sigma$. In the latter situation, consider such a hinge (Q,η) with $Q = ([ABCD],[AC])$.

- If (Q,η) is embedded, then Q is unflippable. Without loss of generality, we may assume $C \in [ABD]$, the cone around $\sigma = \eta(C)$ is then convex and contains a coplanar wedge of Euclidean angle at least π; in particular $\theta_\sigma \geq \pi$. By the Lorentzian Volkov's lemma (Theorem 3),
  - if $\theta_\sigma > 2\pi$ we have $\kappa_\sigma > 2\pi \geq \bar\kappa_\sigma$,
  - if $\pi \leq \theta_\sigma \leq 2\pi$ we have $\kappa_\sigma \geq \theta_\sigma > \bar\kappa_\sigma$.
- If η is not an embedding, then without loss of generality we may assume $\eta(A) = \eta(B) = \eta(D)$; being h-critical, all edges have null dihedral angles so that the stalk of the cone around $\sigma := \eta(C)$ is trigonometric (without breaking point). In particular, $\theta_\sigma = \kappa_\sigma > \bar\kappa$.

Either way, $\kappa_\sigma > \bar\kappa_\sigma$. Together with Proposition 5.13 this implies that $\mathcal{H}_{\bar\kappa}$ has no critical points on $\partial\mathcal{P}_Z^{1/2}$.

If $z = s-1$, then κ is a function defined on an interval, and is continuous and increasing from 0 to some $\kappa_{\max} > \bar\kappa$. The result follows.

We now assume $z \leq s-2$. Define $\overline{\mathcal{P}}_Z^{1/2} := \mathcal{P}_Z^{1/2}$ if $Z \neq \varnothing$ and $\overline{\mathcal{P}}_Z^{1/2} := \mathcal{P}_Z^{1/2} \cup \{\infty\}$ if $Z = \varnothing$. This way $\overline{\mathcal{P}}_Z^{1/2}$ is homeomorphic to an $s-z$ dimensional closed ball and its boundary $\partial\overline{\mathcal{P}}_Z^{1/2}$ is homeomorphic to an $(s-z-1)$-dimensional sphere. The homeomorphism may be made explicit by the radial map from some $\tau_0 \in \mathrm{Int}(\mathcal{P}_Z^{1/2})$, the relative interior of $\mathcal{P}_Z^{1/2}$. Consider the family of vector fields indexed on K_Z,

$$X_{\bar\kappa} : \overline{\mathcal{P}}_Z^{1/2} \to \mathbb{R}^{s-z}, \quad h \neq \infty \mapsto (\kappa_\sigma(h) - \bar\kappa_\sigma)_{\sigma\in S\setminus Z}, \quad \infty \mapsto (\theta_\sigma - \bar\kappa_\sigma)_{\sigma \in S\setminus Z},$$

and notice that $X|_{\bar\kappa\, \mathrm{Int}(\mathcal{P}_Z^{1/2})}$ is the gradient of $\mathcal{H}|_{\bar\kappa\, \mathrm{Int}(\mathcal{P}_Z^{1/2})}$ for $\bar\kappa \in K_Z$ by Proposition 5.13. By Lemma 5.17, X is continuous at ∞ if $Z = \varnothing$; thus $\bar\kappa, h \mapsto X_{\bar\kappa}(h)$ are continuous on $K_Z \times \overline{\mathcal{P}}_Z^{1/2}$ and, from the discussion above, nonsingular on the boundary of $\overline{\mathcal{P}}_Z^{1/2}$. By Proposition 5.15 and the Poincaré-Hopf theorem [8, Theorem 12.13], the number of singular points of the vector field $X_{\bar\kappa}$ in the interior of $\mathcal{P}_Z^{1/2}$ is equal to the index of $X_{\bar\kappa}/\|X_{\bar\kappa}\|$ on $\partial\overline{\mathcal{P}}_Z^{1/2}$. Since $\bar\kappa \mapsto X(\bar\kappa,\cdot)$ is continuous and K_Z is connected, the index of $X_{\bar\kappa}/\|X_{\bar\kappa}\|$ is independent from $\bar\kappa$.

Finally, take some $\bar\kappa \in K_Z$ and $\bar h$ in the interior of $\mathcal{P}_Z^{1/2}$ close enough to 0 that $\prod_{\sigma\in S\setminus Z}[0, 2\bar h_\sigma] \subset \mathcal{P}_Z^{1/2}$ and consider the vector field $Y : h \to (h-\bar h)/\|h-\bar h\|$ on $\partial\mathcal{P}_Z^{1/2}$, which can be continuously extended to the whole $\partial\overline{\mathcal{P}}_Z^{1/2}$ since $\lim_{h\to+\infty} Y(h) = \mathbf{1}_S$. On the one hand, for h on an "$h_\sigma = 0$" boundary component, $Y(h)_\sigma < 0$ while $\kappa_\sigma(h) = 0$; on the other hand, for h on a "$Q^*(h) = 0$" boundary component, there is a $\sigma \in S \setminus Z$ such that $\kappa_\sigma - \bar\kappa_\sigma > 0$, and on such a component, for all $\sigma' \in S \setminus Z$, $(h-\bar h)_{\sigma'} > 0$. At infinity, both X and Y have positive coordinates. In any case for all $h \in \partial\overline{\mathcal{P}}_Z^{1/2}$, $Y \neq -X_{\bar\kappa}/\|X_{\bar\kappa}\|$; thus $X_{\bar\kappa}/\|X_{\bar\kappa}\|$ is homotopic to Y among nonsingular vector fields on $\partial\overline{\mathcal{P}}_Z^{1/2}$. The latter has index 1; thus so does the former. Finally, for all $\bar\kappa \in K_Z$, $\mathcal{H}_{\bar\kappa}$ has exactly one critical point on $\mathcal{P}_Z^{1/2}$. Existence and uniqueness follow for $\bar\kappa \in K_Z$.

By continuity of κ and compactness of $\overline{\mathcal{P}}_Z^{1/2}$, any $\bar\kappa \in \prod_{\sigma\in S}[0,2\pi]\cap[0,\theta_\sigma]$ is in the image of $h \mapsto (\kappa_\sigma(h))_{\sigma\in S}$, except possibly $(\bar\kappa_\sigma)_{\sigma\in S} = (\theta_\sigma)_{\sigma\in S}$, which is the limit at ∞.

Finally, the last point follows from the case $\Theta \le \pi$ of Theorem 3. □

Appendix Radiant 2 + 1 singular spacetimes

Before providing a more thorough description of our singularities, allow us to stress that there is a subtle point one needs to be aware of. We construct 3-manifolds with a geometric structure locally modeled on the Minkowski space $\mathbb{E}^{1,2}$ except on a discrete family of lines we deem reasonable to call "singular". The geometric $\mathbb{E}^{1,2}$-structure (in a sense described below) on the complement of the singular lines is easily defined, but our manifolds are not naturally metric spaces; they are spacetimes and come with a natural local order relation: the causal order. As a consequence, characterizing the isomorphism classes of the singular lines requires some care in general, especially for lightlike lines. We refer to [4] for the zoology of Lorentzian singular lines obtained via finite polyhedra gluings in dimension $2+1$, which should convince the reader that one should be slightly careful.

The causal structure is a tool to characterize lightlike singularities; furthermore, the boundary of the polyhedron we will construct has a special role with respect to this structure: it is a *Cauchy surface*, as defined below.

In this section, we discuss the isomorphisms classes of singularities in our manifolds: their local description as well as constructions with the addition of some more general background.

A.1 Singular (G, X)-manifolds

Let (G, X) be an analytical structure, ie a group G acting on a locally connected Hausdorff space X by homeomorphisms so that any element $g \in G$ is completely determined by its action on a nontrivial open subset. Following [11], we define a singular (G, X)-manifold as a Hausdorff second countable topological M space endowed with a (G, X)-structure on an open and dense subset $\mathcal{U}$ locally connected in M. There exists a unique maximal extension of this (G, X)-structure to a maximal open and dense subset $\mathrm{Reg}(M)$ locally connected in M called the regular locus of M. An a.e. (G, X)-morphism is a continuous map sending regular locus to regular locus and which is a (G, X)-morphism on the regular locus.

A singular (G, X)-manifold is locally modeled on a family $(X_\alpha)_{\alpha\in A}$ if for all $\alpha \in A$, X_α is a singular (G, X)-manifold and for all $x \in M$, there exists a neighborhood $\mathcal{U}$ of x and an open $\mathcal{V}$ of some X_α such that $\mathcal{U}$ is isomorphic to $\mathcal{V}$.

In our situation, the singular locus is a union of 1-dimensional submanifolds of a 3-manifold. The hypotheses of [11] are then satisfied, and the isomorphism class of a singular point is thus well defined.

A.2 Local models of singular lines

We now introduce the local models of the singular $\mathcal{F}$-manifolds we will consider.

Definition A.1 (massive particles model space) Let $\alpha \in \mathbb{R}_+^*$. The manifold $\mathbb{E}_\alpha^{1,2}$ is $\mathbb{R}^3$ endowed with the flat Lorentzian metric

$$\mathrm{d}s_\alpha^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \left(\frac{\alpha}{2\pi} r\right)^2 \mathrm{d}\theta^2$$

on $\mathrm{Reg}(\mathbb{E}_\alpha^{1,2}) := \{r > 0\}$, the complement of the line $\mathrm{Sing}(\mathbb{E}_\alpha^{1,2}) := \{r = 0\}$, where (t, r, θ) are cylindrical coordinates of $\mathbb{R}^3$.

For $\alpha > 0$, the metric on $\mathbb{E}_\alpha^{1,2}$ induces a unique $(\mathrm{Isom}_0(\mathbb{E}^{1,2}), \mathbb{E}^{1,2})$-structure on $\mathrm{Reg}(\mathbb{E}_\alpha^{1,2})$ such that the curves $t \mapsto c(t) = (t, r_0, \theta_0)$ are future causal for $r_0 > 0$ and all $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$.

Definition A.2 (BTZ line model space) The manifold $\mathbb{E}_0^{1,2}$ is $\mathbb{R}^3$ endowed with the flat Lorentzian metric

$$\mathrm{d}s_0^2 = -2\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{r} + \mathrm{d}\mathfrak{r}^2 + \mathfrak{r}^2\,\mathrm{d}\theta^2$$

on $\mathrm{Reg}(\mathbb{E}_0^{1,2}) := \{\mathfrak{r} > 0\}$, the complement of the line $\mathrm{Sing}(\mathbb{E}_0^{1,2}) := \{\mathfrak{r} = 0\}$, where $(\tau, \mathfrak{r}, \theta)$ are cylindrical coordinates of $\mathbb{R}^3$.

The metric on $\mathbb{E}_0^{1,2}$ induces a unique $(\mathrm{Isom}_0(\mathbb{E}^{1,2}), \mathbb{E}^{1,2})$-structure on $\mathrm{Reg}(\mathbb{E}_0^{1,2})$ such that the curves $\tau \mapsto c(\tau) = (\tau, \mathfrak{r}_0, \theta_0)$ are future causal for $\mathfrak{r}_0 > 0$ and all $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$. The model spaces $\mathbb{E}_{\geq 0}^{1,2}$ are singular $\mathbb{E}^{1,2}$-manifolds but not singular $\mathcal{F}$-manifolds. We thus introduce the following:

Definition A.3 For $\alpha \geq 0$ define $\mathcal{F}_\alpha := \mathrm{Int}(J^+(O))$ with $O = (0,0,0) \in \mathbb{E}_\alpha^{1,2}$.

By [10, Proposition 1.3], if $\varphi\colon \mathcal{U}_\alpha \to \mathcal{U}_\beta$ is an a.e. $\mathrm{SO}_0(1,2)$-isomorphism between neighborhoods of singular points in $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$, then $\alpha = \beta$ and φ is induced by an element of $\mathrm{SO}_0(1,2)$. The local models are thus nonisomorphic as singular $\mathcal{F}$-manifolds. Note that the singular line of a massive particle is timelike while the singular line of $\mathbb{E}_0^{1,2}$ is lightlike.

A.3 Causal structure

An $\mathcal{F}$-manifold M comes with a causal structure, eg a family $(\leq_\mathcal{U}, \ll_\mathcal{U})_\mathcal{U}$ of transitive relations, each defined on an open subset $\mathcal{U}$ of M which is inherited from the causal and chronological relation of $\mathcal{F}$. The causal structure on $\mathrm{Reg}(\mathcal{F}_\alpha)$ can be extended to $\mathcal{F}_\alpha$ so that any $\mathcal{F}_{\geq 0}$-manifold M comes with a causal structure. A future causal curve is then a curve in M, which is locally increasing for $\leq$. The causal past/future of a point p can then be defined accordingly, and we denote them by $J^-(p)$ and $J^+(p)$, respectively.

Note that $\leq_{\mathcal{U}}$ is an order relation for $\mathcal{U}$ small enough, but this is not necessarily the case for $\leq_M$. We say that an $\mathcal{F}_{\geq 0}$-manifold M is *causal* if $\leq_M$ is an order relation; we say furthermore that M is *globally hyperbolic* if it is causal and for any $p, q \in M$, $J^+(p) \cap J^-(q)$ is compact. A *Cauchy surface* of M is a topological 2-dimensional submanifold Σ in M which intersects every future causal curve exactly once. One can prove a version of the Geroch theorem valid for $\mathcal{F}_{\geq 0}$-manifolds [5] which states that an $\mathcal{F}_{\geq 0}$-manifold M admits a Cauchy surface if and only if it is globally hyperbolic. An $\mathcal{F}_{\geq 0}$-manifold is *Cauchy-compact* if it admits a compact Cauchy surface.

A morphism $M_1 \to M_2$ between globally hyperbolic $\mathcal{F}_{\geq 0}$-manifolds is a Cauchy-embedding if it is injective and sends a Cauchy surface of M_1 to a Cauchy surface of M_2; the latter is then called a Cauchy-extension of M_1. A manifold M_1 is *Cauchy-maximal* if, for any Cauchy-embedding $M_1 \xrightarrow{\varphi} M_2$, the map φ is an isomorphism. One can prove [9; 10] a version of the Choquet–Bruhat–Geroch theorem for $\mathcal{F}_{\geq 0}$-manifolds following the lines of [30], which states that any $\mathcal{F}_{\geq 0}$-manifold admits a unique Cauchy-maximal Cauchy-extension.

A.4 Rays, suspensions, and the structure theorem

Letting M be an $\mathcal{F}_{\geq 0}$-manifold, $\mathrm{Reg}(M)$ admits a natural causal geodesic foliation, the leaves of which we call *rays*. We notice that in the model spaces, $\mathcal{F}_\alpha$ the foliation can be extended to the whole $\mathcal{F}_\alpha$; furthermore, the extended foliation to the whole $\mathcal{F}_\alpha$ induces a causal foliation on M.

Definition A.4 For $\alpha \in \mathbb{R}_+$, define $\mathbb{H}^2_\alpha$ as the space of ray of $\mathcal{F}_\alpha$ and define the natural projection $\pi_\alpha : \mathcal{F}_\alpha \to \mathbb{H}^2_\alpha$.

Proposition A.5 *For* $\alpha \geq 0$, $\mathbb{H}^2_\alpha$ *is homeomorphic to* $\mathbb{R}^2$ *and comes with a natural singular* $\mathbb{H}^2$*-structure whose singular locus contains at most one point. Furthermore,*

- *if* $\alpha = 2\pi$, $\mathbb{H}^2_\alpha$ *is regular and isomorphic to* $\mathbb{H}^2$,
- *if* $2\pi \neq \alpha > 0$, *the singular point is a conical singularity of angle* α,
- *if* $\alpha = 0$, *the singular point is a cusp.*

Proof • To begin with, in $\mathcal{F}_\alpha$, define the plane $\Pi := \{t = 1\}$ if $\alpha > 0$ and $\Pi := \{\tau = 1\}$ if $\alpha = 0$. The plane Π intersects each ray exactly once and $\pi|_\Pi$ is a homeomorphism.

• Define the surface $\mathcal{H}^* := \{\tau = (1+\mathfrak{r}^2)/(2\mathfrak{r}),\ \mathfrak{r} > 0\}$ if $\alpha = 0$ and $\mathcal{H}^* := \{t^2 - r^2 = 1,\ r > 0\}$ if $\alpha > 0$. The Lorentzian metric of $\mathcal{F}_\alpha$ induces a hyperbolic metric on $\mathcal{H}^*$ which intersects each ray of $\mathrm{Reg}(\mathcal{F}_\alpha)$ exactly once, and the projection $\mathcal{F}_\alpha \to \mathbb{H}^2_\alpha$ induces a homeomorphism $\mathcal{H}^* \simeq (\mathbb{H}^2_\alpha \setminus \mathrm{Sing}(\mathcal{F}_\alpha))$. Hence $\mathbb{H}^2_\alpha$ has an $\mathbb{H}^2$-structure defined on the complement of $\mathrm{Sing}(\mathcal{F}_\alpha)$, eg on the complement of a subset containing at most one point.

• If $\alpha = 2\pi$ then $\mathcal{F}_\alpha \simeq \mathcal{F}$ and the result follows.

• If $\alpha = 0$, one can check that $\mathcal{H}^*$ is complete and that the singular point of $\mathbb{H}^2_\alpha$ has a neighborhood of finite volume. The singular point is thus a cusp.

- If $2\pi \neq \alpha > 0$, then one can check that the length of the circle of radius $r > 0$ in $\mathbb{H}^2_\alpha$ around the singular point is αr. The singular point is a conical singularity of angle α. □

Definition A.6 An $\mathbb{H}^2_{\geq 0}$-manifold is a singular $\mathbb{H}^2$-manifold whose singular locus is locally modeled on $\mathbb{H}^2_\alpha$ for some $\alpha \geq 0$.

Definition A.7 Let Σ be an $\mathbb{H}^2_{\geq 0}$-manifold, let $(\mathcal{U}_i, \mathcal{V}_i, \varphi_i, \alpha_i)_{i\in I}$ be an $\mathbb{H}^2_{\geq 0}$-atlas of Σ with $\mathcal{V}_i \subset \mathbb{H}^2_{\alpha_i}$, and let $\mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j$ and $\mathcal{V}_{ij} := \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j)$ for $i, j \in I$ such that $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$. We add the convention that $\alpha_i \neq 2\pi$ if and only if $\mathcal{V}_i$ contains a neighborhood of the singular point of $\mathbb{H}^2_{\alpha_i}$ such that for any $i, j \in I$ where $\mathcal{U}_{ij} \neq \varnothing$ and $\mathcal{U}_i$ contains a singular point, $\alpha_i = \alpha_j$ and the change of charts $\mathcal{V}_{ij} \xrightarrow{\varphi_{ij}} \mathcal{V}_{ji}$ comes from some $\phi_{ij} \in \mathrm{SO}_0(1,2)$ acting both on $\mathbb{H}^2_{\alpha_i}$ and $\mathcal{F}_{\alpha_i}$.

Define the suspension $\mathrm{susp}(\Sigma)$ of Σ as the gluing of $(\pi_{\alpha_i}^{-1}(\mathcal{V}_i))_{i\in I}$ via $(\pi_{\alpha_i}^{-1}(\mathcal{V}_{ij}) \xrightarrow{\phi_{ij}} \pi_{\alpha_j}^{-1}(\mathcal{V}_{ji}))_{i,j\in I}$.

Remark The suspension susp is a functor from the category of $\mathbb{H}^2_{\geq 0}$-manifolds to the category of $\mathcal{F}_{\geq 0}$-manifolds.

Remark By construction, $\mathrm{susp}(\Sigma)$ is an $\mathcal{F}_{\geq 0}$-manifold with a natural projection $\mathrm{susp}(\Sigma) \to \Sigma$. One can check that diamonds $J^+(p) \cap J^-(q)$ are compact and that $\mathrm{susp}(\Sigma)$ is causal, and hence globally hyperbolic. Furthermore, the natural projection induces a homeomorphism $\pi : \Sigma_0 \to \Sigma$ for any Cauchy surface Σ_0.

Remark Be wary that the following simpler construction might be deceptively wrong. Start from $(\mathbb{H}^2_\alpha, \boldsymbol{h}_\alpha)$ as a hyperbolic conical singularity (or a cusp) with $\boldsymbol{h}_\alpha$ its Riemannian metric; then define the suspension as

$$\mathcal{F}'_\alpha := (\mathbb{R}^*_+ \times \mathbb{H}^2_\alpha, \boldsymbol{g}_\alpha), \quad \boldsymbol{g}_\alpha := -\mathrm{d}t^2 + t^2 \boldsymbol{h}_\alpha.$$

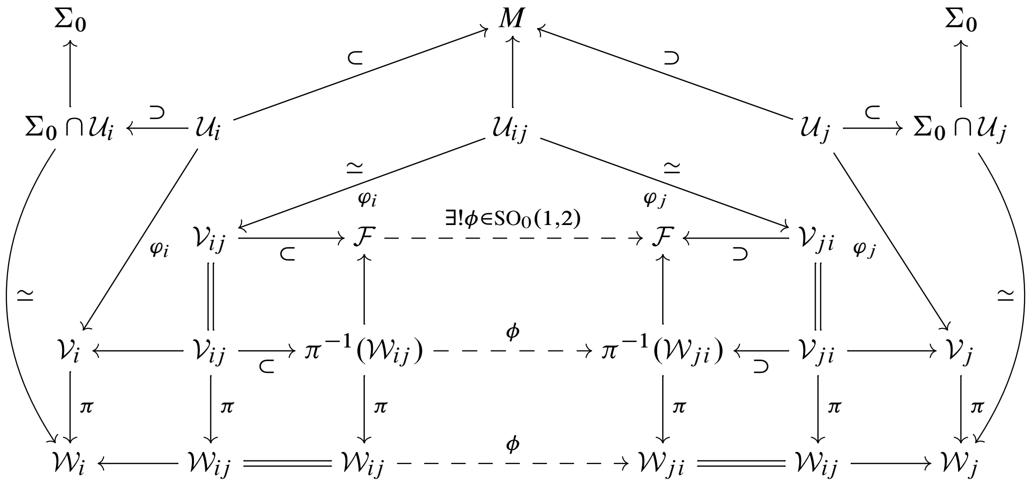
Though one indeed obtains $\mathcal{F}'_\alpha \simeq \mathcal{F}_\alpha$ for $\alpha > 0$ as well as $\mathrm{Reg}(\mathcal{F}_0) \simeq \mathrm{Reg}(\mathcal{F}'_0)$, note that $\mathcal{F}'_0$ is not isomorphic to $\mathcal{F}_0$ and not isomorphic to a neighborhood of a singular point of $\mathbb{E}^{1,2}_0$. To see this, notice that the past causal geodesics in $\mathrm{Reg}(\mathcal{F}'_\alpha)$ that "should" hit the singular line all converge to the same ideal point in the past (the origin) but never actually hit the singular line.

Definition A.8 A radiant spacetime is a Cauchy-compact Cauchy-maximal globally hyperbolic $\mathcal{F}_{\geq 0}$-manifold M.

We have a structure theorem for radiant spacetimes. This result is in the line of Mess's theorem [25] and is akin to previous results by Bonsante and Seppi [7], or the author [10] though in a much simpler context. To the author's knowledge, while this result is expected and "folkloric", there is no existing reference to point to. We therefore provide a proof.

Theorem 6 *Let M be a radiant spacetime. There exists a compact singular $\mathbb{H}^2_{\geq 0}$-manifold Σ such that $M \simeq \mathrm{susp}(\Sigma)$.*

Proof Let Σ_0 be a Cauchy surface of M and consider the natural projections $\pi_\alpha\colon \mathcal{F}_\alpha \to \mathbb{H}^2_\alpha$. Consider an $\mathcal{F}$-atlas $(\varphi_i, \mathcal{U}_i, \mathcal{V}_i)_{i\in I}$ of $\mathrm{Reg}(M)$ such that each $\mathcal{V}_i$ is causally convex in $\mathcal{F}$. Write $\mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j$ for $i \in I$, and for $i, j \in I$ such that $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$ write $\mathcal{V}_{ij} := \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j)$ as well as $\mathcal{W}_{ij} := \pi(\mathcal{V}_{ij}) \subset \mathbb{H}^2$. We then have a unique $\phi \in \mathrm{SO}_0(1,2)$ such that for all $x \in \mathcal{V}_{ij}$, $\varphi_j \circ \varphi_i(x) = \phi \cdot x$. Hence, for any $i, j \in I$ such that $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$, we have the following commutative diagram:



Since Σ_0 is acausal, the projection the maps $\Sigma_0 \cap \mathcal{U}_i \to \mathcal{W}_i$ are injective and by definition surjective; Σ_0 as well as all the $\mathcal{W}_i$ are 2-dimensional manifolds; by invariance of domain, the maps $\Sigma_0 \cap \mathcal{U}_i \to \mathcal{W}_i$ are then homeomorphisms. The $\mathcal{F}$-structure on M thus induces on Σ_0 a singular $\mathbb{H}^2$-structure; we call this singular $\mathbb{H}^2$-manifold Σ. Proceeding the same way around singular points of M, the local models $\mathcal{F}_\alpha$ of M induce a local model $\mathbb{H}^2_\alpha$ for each singular point of Σ. The suspension $\mathrm{susp}(\Sigma)$ of Σ is then given by the induced gluing of the cones $\pi_{\alpha_i}^{-1}(\mathcal{W}_i)$ along the $\pi_\alpha^{-1}(\mathcal{W}_{ij})$.

One can then define a natural map $M \xrightarrow{\iota} \mathrm{susp}(\Sigma)$ on each chart $(\mathcal{U}, \mathcal{V}, \varphi)$ of the $(\mathcal{F}_\alpha)_{\alpha\geq 0}$-atlas of M with $\mathcal{V} \subset \mathcal{F}_\alpha$ as $\iota\colon \mathcal{U} \to \pi_\alpha^{-1}(\pi_\alpha(\mathcal{V}))$, $x \mapsto \varphi(x)$. By construction, the map ι is an injective a.e. $\mathcal{F}$-morphism. Since M is Cauchy-maximal and Cauchy-compact, it follows from [10, Proposition 2.20] that the map ι is surjective, and thus an isomorphism. □

Corollary A.9 *Any radiant spacetime admits an embedded natural $\mathbb{H}^2_{>0}$-surface which is a Cauchy surface of its $\mathcal{F}_{>0}$ part.*

Another way to construct the suspension of an $\mathbb{H}^2_{\geq 0}$-surface Σ (and hence a radiant spacetime) is to choose a geodesic cellulation of Σ such that each cell is a polygon of $\overline{\mathbb{H}}^2$. The surface Σ can thus be seen as a gluing of a family of cells $\mathcal{P} = (P_i)_{i\in I}$ along their edges $\mathcal{E} = (e_i^{(j)})_{i\in I, j\in J_i}$ (where J_i parametrizes the edges of P_i) via isometries $\phi_{e,e'} \in \mathrm{SO}_0(1,2)$ sending the edge e to the edge e'. We denote by $\mathcal{G}$ the set of couples $(e,e') \in \mathcal{E}$ such that e is glued to e'. We can then construct $\mathrm{susp}(\Sigma)$ by gluing the cones $C_i := \pi^{-1}(P_i)$ for $i \in I$ along their faces $(\pi^{-1}(e))_{e\in\mathcal{E}}$ via the isometries $(\phi_{e,e'})_{(e,e')\in\mathcal{G}}$. We thus have the following:

Proposition A.10 *Any gluing of cones of $\overline{\mathcal{F}} = J^+(O)$ with polygonal basis, gluing couples of distinct 2-facets together via elements of $\mathrm{SO}_0(1,2)$ and without leaving unglued 2-facets, is a radiant spacetime.*

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Centre de recherche astrophysique de Lyon
Lyon, France

Current address: *Huawei, Noah Ark Laboratories*
Montreal QC, Canada

`leo@brunswic.fr`

`https://leo.brunswic.fr`

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