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A deformation of Asaeda–Przytycki–Sikora homology

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We define a 1-parameter family of homology invariants for links in thickened oriented surfaces. It recovers the homology invariant of Asaeda, Przytycki and Sikora (Algebr. Geom. Topol. 4 (2004) 1177–1210) and the invariant defined by Winkeler (Michigan Math. J. 74 (2024) 1–31). The new invariant can be regarded as a deformation of Asaeda–Przytycki–Sikora homology; it is not a Lee-type deformation as the deformation is only nontrivial when the surface is not simply connected. Our construction is motivated by computations in singular instanton Floer homology. We also prove a detection property for the new invariant, which is a stronger result than our previous work (Selecta Math. 29 (2023) art. id. 84).

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1 Introduction

Khovanov homology [9] is a link invariant that assigns a bigraded homology group to every oriented link in $\mathbb{R}^3$. Asaeda, Przytycki and Sikora [1] introduced a generalization of Khovanov homology for links in $(-1,1)$-bundles over surfaces, where the bundles are required to be oriented as 3-manifolds. Such $(-1,1)$-bundles are called *thickened surfaces*. When the surface is an annulus, Asaeda–Przytycki–Sikora homology is also called *annular Khovanov homology*. Khovanov homology and Asaeda–Przytycki–Sikora homology have been essential tools for the study of knots and links for decades. More recently, Winkeler [16] introduced another variation of Khovanov homology for links in thickened multipunctured disks, which is different from the invariant of Asaeda, Przytycki and Sikora.

Suppose Σ is an oriented surface. We define a 1-parameter family of homology invariants for oriented links in $(-1,1)\times\Sigma$. As bigraded modules, the new invariant recovers both Asaeda–Przytycki–Sikora homology and the invariant of Winkeler, and it can be interpreted as a 1-parameter deformation of Asaeda–Przytycki–Sikora homology. The deformation is not a Lee-type deformation as it is only nontrivial when the surface has a nontrivial fundamental group. The construction is motivated by computations from singular instanton Floer homology. We also use instanton Floer theory to prove a detection result for the deformed Asaeda–Przytycki–Sikora homology, which gives a stronger rank estimate than the main theorem of Li, Xie and Zhang [12].

The paper is organized as follows. Section 2 introduces some notation and conventions. Sections 3 and 4 define the differential map and prove that $d^2 = 0$. Section 5 defines the homology invariant and proves the invariance under Reidemeister moves. Section 6 explains the motivation from instanton Floer homology and proves the aforementioned detection result in Theorem 6.1.

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2 Notation

Throughout this paper we use R to denote a fixed commutative ring with unit. We use Σ to denote an oriented surface, possibly with boundary and possibly noncompact.

For every embedded closed 1-manifold $c \subset \Sigma$, we assign an R-module $V(c)$ to c as follows:

(1) If γ is a contractible simple closed curve on Σ, define $V(\gamma)$ to be the free R-module generated by $\boldsymbol{v}(\gamma)_+$ and $\boldsymbol{v}(\gamma)_-$, where $\boldsymbol{v}(\gamma)_+$ and $\boldsymbol{v}(\gamma)_-$ are formal generators associated with γ.

(2) If γ is a noncontractible simple closed curve, let $\mathfrak{o}$ and $\mathfrak{o}'$ be the two orientations of γ. Define $V(\gamma)$ to be the free module generated by $\boldsymbol{v}(\gamma)_{\mathfrak{o}}$ and $\boldsymbol{v}(\gamma)_{\mathfrak{o}'}$, where $\boldsymbol{v}(\gamma)_{\mathfrak{o}}$ and $\boldsymbol{v}(\gamma)_{\mathfrak{o}'}$ are formal generators.

(3) In general, suppose the connected components of c are $\gamma_1, \dots, \gamma_k$. Define $V(c)$ to be $\bigotimes_{i=1}^k V(\gamma_i)$.

When the choice of Σ needs to be emphasized, we will write $V(c)$ as $V^\Sigma(c)$, and write $\boldsymbol{v}(\gamma)_{\mathfrak{o}}$ and $\boldsymbol{v}(\gamma)_\pm$ as $\boldsymbol{v}^\Sigma(\gamma)_{\mathfrak{o}}$ and $\boldsymbol{v}^\Sigma(\gamma)_\pm$, respectively.

If $\mathfrak{o}$ is an orientation of a curve γ, we use $\gamma_{\mathfrak{o}}$ to denote the corresponding oriented curve.

3 Band surgery homomorphisms

Suppose c is an embedded closed 1-manifold on Σ, suppose b is an embedded disk on Σ such that the interior of b is disjoint from c and the boundary of b intersects c at two arcs (see Figure 1). The surgery of c along b yields another embedded closed 1-manifold on Σ, which we denote by c_b. We will call the disk b a *band* that is *attached* to c.

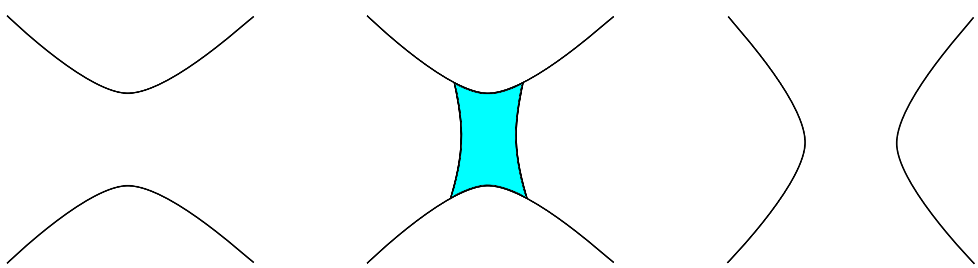

Figure 1: Band surgery.

For later reference, we record the following two elementary lemmas:

Lemma 3.1 *The change from c to c_b has three possibilities*:

(1) *two circle components of c are merged to one circle*,

(2) *one circle component of c is split to two circles*,

(3) *one circle component of c is modified by the surgery to another circle.*

Proof Since $\partial b \cap c$ contains two arcs, at most two components of c are affected by the surgery. If the arcs of $\partial b \cap c$ are on two different components of c, then the surgery merges these two components into one circle. If the arcs of $\partial b \cap c$ are on one component of c, then the boundary orientation of b defines an orientation on both components of $\partial b \cap c$, so we have two oriented arcs embedded in one component γ of c. If these two arcs induce the same orientation on γ, then the surgery splits one component of c to two circles. If these two arcs induce opposite orientations on γ, then the surgery changes this component to another circle. □

Recall that if $\mathfrak{o}$ is an orientation of a curve γ, we use $\gamma_{\mathfrak{o}}$ to denote the corresponding oriented curve.

Lemma 3.2 *Suppose γ is a simple closed curve on a connected surface Σ, and assume Σ is not diffeomorphic to S^2. Suppose $\mathfrak{o}$ and $\mathfrak{o}'$ are the two orientations of γ. Then $\gamma_{\mathfrak{o}}$ and $\gamma_{\mathfrak{o}'}$ are not isotopic on Σ.*

Proof If γ is nonseparating, there exists an oriented simple closed curve β such that the algebraic intersection number of β and γ is nonzero. Since isotopies preserve the sign of algebraic intersection numbers, the desired result follows.

If γ is separating and $\partial\Sigma \neq \varnothing$, then every orientation of γ defines an ordering of the two components of $\Sigma\backslash\gamma$, which defines an ordered partition of the components of $\partial\Sigma$. Since every isotopy of γ on Σ can be extended to an isotopy of Σ fixing the boundary, the desired result is proved.

If γ is separating and Σ is closed, then every orientation of γ defines an ordering of the two components of $\Sigma\backslash\gamma$. Suppose Σ_1 and Σ_2 are the two components of $\Sigma\backslash\gamma$ ordered by an orientation $\mathfrak{o}$ of γ. Since Σ is not a sphere, the images of $H_1(\Sigma_1;\mathbb{Z})$ and $H_1(\Sigma_2;\mathbb{Z})$ are distinct in $H_1(\Sigma;\mathbb{Z})$. The images of $H_1(\Sigma_1;\mathbb{Z})$ and $H_1(\Sigma_2;\mathbb{Z})$ are invariant under isotopies of $\gamma_{\mathfrak{o}}$, so the desired result is proved. □

Taking an arbitrary element $\lambda \in R$, we define a homomorphism

$$T_\lambda(b)\colon V(c) \to V(c_b)$$

associated with the band surgery along b. When the choice of Σ needs to be emphasized, we will write $T_\lambda(b)$ as $T_\lambda^\Sigma(b)$.

We first assume that the intersection of ∂b with every component of c is nonempty. The general case will be discussed later. By Lemma 3.1, if the intersection of ∂b with every component of c is nonempty, then there are three cases:

Case 1 (c has two components γ_1 and γ_2 and they are merged into one circle $\gamma = c_b$ after the surgery) In this case, we define $T_\lambda(b)\colon V(\gamma_1)\otimes V(\gamma_2)\to V(\gamma)$ as follows:

(1) If both γ_1 and γ_2 are contractible circles, then γ is also contractible, and we define $T_\lambda(b)$ by

$$\boldsymbol{v}(\gamma_1)_+\otimes \boldsymbol{v}(\gamma_2)_+\mapsto \boldsymbol{v}(\gamma)_+,\quad \boldsymbol{v}(\gamma_1)_+\otimes \boldsymbol{v}(\gamma_2)_-\mapsto \boldsymbol{v}(\gamma)_-,$$
$$\boldsymbol{v}(\gamma_1)_-\otimes \boldsymbol{v}(\gamma_2)_+\mapsto \boldsymbol{v}(\gamma)_-,\quad \boldsymbol{v}(\gamma_1)_-\otimes \boldsymbol{v}(\gamma_2)_-\mapsto 0.$$

(2) If γ_1 is contractible and γ_2 is noncontractible, then γ_2 is isotopic to γ. The existence of noncontractible curves on Σ implies that Σ is not diffeomorphic to S^2. By Lemma 3.2, the orientations of γ_2 are canonically identified with the orientations of γ via an isotopy. This identification defines a canonical isomorphism from $V(\gamma_2)$ to $V(\gamma)$, which we denote by ι. In this case, the homomorphism $T_\lambda(b)$ is defined by

$$\boldsymbol{v}(\gamma_1)_+\otimes x\mapsto \iota(x),\quad \boldsymbol{v}(\gamma_1)_-\otimes x\mapsto 0$$

for all $x\in V(\gamma_2)$.

(3) If γ_1 is noncontractible and γ_2 is contractible, define $T_\lambda(b)$ by requiring the map to be symmetric with respect to γ_1 and γ_2 and reducing to (2) above.

(4) If γ_1 and γ_2 are both noncontractible and γ_3 is contractible, then γ_1 and γ_2 must be isotopic. By Lemma 3.2, the orientations of γ_1 and γ_2 are canonically identified by the isotopy. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be the two orientations of γ_1, and use the same notation to denote the corresponding orientations of γ_2. The map $T_\lambda(b)$ is then defined by

$$\boldsymbol{v}(\gamma_1)_{\mathfrak{o}}\otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}}\mapsto 0,\qquad \boldsymbol{v}(\gamma_1)_{\mathfrak{o}'}\otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}'}\mapsto 0,$$
$$\boldsymbol{v}(\gamma_1)_{\mathfrak{o}}\otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}'}\mapsto \boldsymbol{v}(\gamma)_-,\quad \boldsymbol{v}(\gamma_1)_{\mathfrak{o}'}\otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}}\mapsto \boldsymbol{v}(\gamma)_-.$$

(5) If all of γ_1, γ_2, and γ are noncontractible, let N be the regular neighborhood of $b\cup\gamma_1\cup\gamma_2$. Then N is a sphere with three disks removed, and the three boundary components of N are parallel to γ_1, γ_2 and γ. Since $N\subset\Sigma$ is oriented, the boundary orientation of N defines an orientation on each of γ_1, γ_2 and γ, and we denote these orientations by $\mathfrak{o}_1$, $\mathfrak{o}_2$ and $\mathfrak{o}$, respectively. Denote their opposite orientations by $\mathfrak{o}_1'$, $\mathfrak{o}_2'$ and $\mathfrak{o}'$. Then $T_\lambda(b)$ is defined by

$$\boldsymbol{v}(\gamma_1)_{\mathfrak{o}_1'}\otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}_2'}\mapsto \lambda\cdot \boldsymbol{v}(\gamma)_{\mathfrak{o}},\quad \boldsymbol{v}(\gamma_1)_{\mathfrak{o}_1'}\otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}_2}\mapsto 0,$$
$$\boldsymbol{v}(\gamma_1)_{\mathfrak{o}_1}\otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}_2'}\mapsto 0,\qquad \boldsymbol{v}(\gamma_1)_{\mathfrak{o}_1}\otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}_2}\mapsto 0.$$

Case 2 (c contains one component γ and c_b has two components γ_1 and γ_2) In this case, we define $T_\lambda(b)\colon V(\gamma)\to V(\gamma_1)\otimes V(\gamma_2)$ as follows:

(1) If γ_1 and γ_2 are both contractible circles, then γ is also contractible, and we define $T_\lambda(b)$ by

$$\boldsymbol{v}(\gamma)_+\mapsto \boldsymbol{v}(\gamma_1)_+\otimes \boldsymbol{v}(\gamma_2)_- + \boldsymbol{v}(\gamma_1)_-\otimes \boldsymbol{v}(\gamma_2)_+,\quad \boldsymbol{v}(\gamma)_-\mapsto \boldsymbol{v}(\gamma_1)_-\otimes \boldsymbol{v}(\gamma_2)_-.$$

(2) If one of $\{\gamma_1,\gamma_2\}$ is contractible and the other is noncontractible, assume without loss of generality that γ_1 is contractible and γ_2 is noncontractible. Then γ is isotopic to γ_2, and the orientations of γ and

γ_2 are canonically identified. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be the two orientations of γ, and use the same notation to denote the corresponding orientations of γ_2. Define the map $T_\lambda(b)$ by

$$\boldsymbol{v}(\gamma)_{\mathfrak{o}} \mapsto \boldsymbol{v}(\gamma_1)_- \otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}}, \quad \boldsymbol{v}(\gamma)_{\mathfrak{o}'} \mapsto \boldsymbol{v}(\gamma_1)_- \otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}'}.$$

(3) If both γ_1 and γ_2 are noncontractible and γ is contractible, then γ_1 and γ_2 are isotopic to each other, and the orientations of γ_1 are γ_2 are canonically identified. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be the orientations of γ_1 and use the same notation for the orientations of γ_2. Define the map $T_\lambda(b)$ by

$$\boldsymbol{v}(\gamma)_+ \mapsto \boldsymbol{v}(\gamma_1)_{\mathfrak{o}} \otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}'} + \boldsymbol{v}(\gamma_1)_{\mathfrak{o}'} \otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}}, \quad \boldsymbol{v}(\gamma)_- \mapsto 0.$$

(4) If all of γ, γ_1 and γ_2 are noncontractible, let N be the regular neighborhood of $b \cup \gamma$. Then N is a sphere with three disks removed, and the three boundary components of N are parallel to γ_1, γ_2 and γ. The boundary orientation of N defines an orientation on each of γ_1, γ_2 and γ, and we denote them by $\mathfrak{o}_1$, $\mathfrak{o}_2$ and $\mathfrak{o}$, respectively. Denote their opposite orientations by $\mathfrak{o}_1'$, $\mathfrak{o}_2'$ and $\mathfrak{o}'$. Define the map $T_\lambda(b)$ by

$$\boldsymbol{v}(\gamma)_{\mathfrak{o}'} \mapsto \lambda \cdot \boldsymbol{v}(\gamma_1)_{\mathfrak{o}_1} \otimes \boldsymbol{v}(\gamma_2)_{\mathfrak{o}_2}, \quad \boldsymbol{v}(\gamma)_{\mathfrak{o}} \mapsto 0.$$

Case 3 (both c and c_b have exactly one component) In this case, define $T_\lambda(b)$ to be zero.

In general, suppose $c = c^{(1)} \sqcup c^{(2)}$ such that ∂b is disjoint from $c^{(2)}$ and intersects every component of $c^{(1)}$. We define the band surgery homomorphism $T_\lambda(b)\colon V_\lambda(c) \to V_\lambda(c_b)$ to be

$$T_\lambda(b) = T_\lambda(b)|_{V(c^{(1)})} \otimes \mathrm{id}|_{V(c^{(2)})}. \tag{3-1}$$

Remark 3.3 In the above definition, the coefficient λ only appeared in Cases 1(5) and 2(4).

4 Commutativity of band surgery homomorphisms

The main result of this section is the following proposition:

Proposition 4.1 *Suppose c is an embedded closed 1-manifold on Σ, and suppose b_1 and b_2 are two disjoint bands attached to c. Then for all $\lambda \in R$,*

$$T_\lambda(b_1) \circ T_\lambda(b_2) = T_\lambda(b_2) \circ T_\lambda(b_1). \tag{4-1}$$

The key idea is to use the following two lemmas to reduce Proposition 4.1 to the case when Σ has genus 0 or 1.

Lemma 4.2 *Suppose Σ is an oriented surface, and $\Sigma' \subset \Sigma$ is an embedded surface whose orientation is induced by Σ. Suppose the embedding of Σ' in Σ is π_1-injective. Suppose c is an embedded closed 1-manifold in Σ', and b_1 and b_2 are two disjoint bands in Σ' attached to c. Then*

$$T_\lambda^{\Sigma'}(b_1) \circ T_\lambda^{\Sigma'}(b_2) = T_\lambda^{\Sigma'}(b_2) \circ T_\lambda^{\Sigma'}(b_1)$$

on $V_{\Sigma'}(c)$ if and only if

$$T_\Sigma(b_1) \circ T_\Sigma(b_2) = T_\Sigma(b_2) \circ T_\Sigma(b_1)$$

on $V_\Sigma(c)$.

Proof Since the embedding of Σ' in Σ is π_1-injective, there is a canonical isomorphism from $V_{\Sigma'}(c)$ to $V_\Sigma(c)$ for every embedded 1-manifold $c \subset \Sigma'$ which takes the generators of $V_{\Sigma'}(c)$ to the corresponding generators of $V_\Sigma(c)$, and this isomorphism intertwines with $T_\lambda^{\Sigma'}$ and T_λ^{Σ}, so the lemma is proved. □

Lemma 4.3 *Assume Proposition 4.1 holds whenever Σ is a sphere, finitely punctured sphere, torus or finitely punctured torus. Then Proposition 4.1 holds for all cases.*

Proof Without loss of generality, we may assume that every component of c intersects b_1 and b_2 nontrivially, and that $c \cup b_1 \cup b_2$ is connected.

In this case, $c \cup b_1 \cup b_2$ is homotopy equivalent to the wedge sum of three circles. Therefore its Euler characteristic is -2.

Let N be a closed regular neighborhood of $c \cup b_1 \cup b_2$ in Σ. Let Σ' be obtained from N as follows: For each component γ of ∂N, if γ is contractible in Σ but not contractible in N, then γ bounds a disk D_γ in Σ such that $D_\gamma \cap N = \gamma$. Define Σ' to be the union of N and all disks D_γ as above. Then the embedding of Σ' in Σ is π_1-injective. Note that $\chi(\Sigma') \geq \chi(N) = -2$. If $\chi(\Sigma') = \chi(N) = -2$, then no disk D_γ appears in the construction of Σ', so $\partial\Sigma' \neq \varnothing$. Therefore the genus of Σ' is 0 or 1. By assumption, (4-1) holds on Σ'. Hence by Lemma 4.2, the desired equation also holds on Σ. □

The rest of this section proves Proposition 4.1 when Σ is a sphere, finitely punctured sphere, torus or finitely punctured torus.

4.1 The genus-zero case

We first establish (4-1) when Σ is a sphere or a finitely punctured sphere. Our argument here is inspired by the work of Winkeler [16].

Lemma 4.4 *Equation* (4-1) *holds if Σ is a sphere or a finitely punctured sphere.*

Proof If Σ is a sphere or a disk, then every curve is contractible, and Lemma 3.1(3) is not possible. In this case, our definition of $T_\lambda(b)$ does not depend on λ and it coincides with the definition of the merge and split maps in standard Khovanov theory. Therefore (4-1) holds.

When Σ has $n \geq 2$ boundary components, we view Σ as a disk B with $n-1$ interior disks $B_1, \dots, B_{n-1}$ removed. Assume the orientation of Σ is defined so that the boundary orientation on ∂B is given by the counterclockwise orientation, and the boundary orientation on ∂B_i is the clockwise orientation.

Recall that when the surface Σ needs to be emphasized, we write $V(c)$, $\boldsymbol{v}(\gamma)_\circ$, $\boldsymbol{v}(\gamma)_\pm$ and $T_\lambda(b)$ as $V^\Sigma(c)$, $\boldsymbol{v}^\Sigma(\gamma)_\circ$, $\boldsymbol{v}^\Sigma(\gamma)_\pm$ and $T_\lambda^\Sigma(b)$, respectively.

For each embedded closed 1-manifold $c \subset \Sigma$, define an isomorphism $\Phi\colon V^B(c) \to V^\Sigma(c)$ as follows. For each component γ of c, if γ is contractible in Σ, define

$$\Phi(\boldsymbol{v}^B(\gamma)_\pm) = \boldsymbol{v}^\Sigma(\gamma)_\pm.$$

If γ is noncontractible in Σ, let $\mathfrak{o}$ denote the counterclockwise orientation of γ, let $\mathfrak{o}'$ denote the clockwise orientation of γ, and define

$$\Phi(\boldsymbol{v}^B(\gamma)_+) = \boldsymbol{v}^\Sigma(\gamma)_{\mathfrak{o}}, \quad \Phi(\boldsymbol{v}^B(\gamma)_-) = \boldsymbol{v}^\Sigma(\gamma)_{\mathfrak{o}'}.$$

Since $T^B_\lambda(b)$ does not depend on λ, we denote it by $T^B(b)$. Then

$$\Phi \circ T^B(b) \circ \Phi^{-1}$$

is a homomorphism from $V^\Sigma(c)$ to $V^\Sigma(c_b)$.

For each $i \in \{1, \dots, n-1\}$, define a grading on $V^\Sigma(c)$ as follows. If a circle γ is a contractible curve on Σ, define the degree of $\boldsymbol{v}^\Sigma(\gamma)_\pm$ to be zero. If γ is noncontractible, for each orientation $\mathfrak{o}$ of γ, define the degree of $\boldsymbol{v}^\Sigma(\gamma)_{\mathfrak{o}}$ to be the rotation number of $\gamma_{\mathfrak{o}}$ around B_i. Here our convention on the rotation number is defined so that counterclockwise orientations always have nonnegative rotation numbers. Define the grading of the tensor product of a set of generators to be the sum of the grading of each generator.

By checking all the cases in the definition of $T_\lambda(b)$, it is straightforward to verify that the map $T^\Sigma(b)$ preserves all the $n-1$ gradings defined above. Moreover, for each $i \in \{1, \dots, n-1\}$, the map $\Phi \circ T^B(b) \circ \Phi^{-1}$ does not increase the i^{th} grading. The components of $\Phi \circ T^B(b) \circ \Phi^{-1}$ that preserve all the $n-1$ gradings is equal to the map $T^\Sigma_1(b)$, which is the map T^Σ_λ when $\lambda = 1$. Since $T^B(b_1) \circ T^B(b_2) = T^B(b_2) \circ T^B(b_1)$ on B, we conclude that (4-1) holds for T^Σ_1.

To show that (4-1) holds for general λ, define $T^\Sigma_\delta = T^\Sigma_1 - T^\Sigma_0$. Then

$$T^\Sigma_\lambda = T^\Sigma_0 + \lambda \cdot T^\Sigma_\delta.$$

We define another grading on $V^\Sigma(-)$ as follows. If a circle γ is a contractible curve on Σ, define the degree of $\boldsymbol{v}_\Sigma(\gamma)_\pm$ to be zero. If γ is noncontractible, for each orientation $\mathfrak{o}$ of γ, define the degree of $\boldsymbol{v}^\Sigma(\gamma)_{\mathfrak{o}}$ to be 1 if $\mathfrak{o}$ is the counterclockwise orientation, and define the degree of $\boldsymbol{v}^\Sigma(\gamma)_{\mathfrak{o}}$ to be -1 if $\mathfrak{o}$ is the clockwise orientation. Define the grading of the tensor product of a set of generators to be the sum of the grading of each generator.

By checking all the cases in the definition of T^Σ_λ, it is straightforward to verify that under the above grading, the map T^Σ_0 is homogeneous with degree 0, and T^Σ_δ is homogeneous with degree -1. Since (4-1) holds for $\lambda = 1$, we have

$$\begin{gathered}
T^\Sigma_0(b_1) \circ T^\Sigma_0(b_2) = T^\Sigma_0(b_2) \circ T^\Sigma_0(b_1), \\
T^\Sigma_\delta(b_1) \circ T^\Sigma_0(b_2) + T^\Sigma_0(b_1) \circ T^\Sigma_\delta(b_2) = T^\Sigma_\delta(b_2) \circ T^\Sigma_0(b_1) + T^\Sigma_0(b_2) \circ T^\Sigma_\delta(b_1), \\
T^\Sigma_\delta(b_1) \circ T^\Sigma_\delta(b_2) = T^\Sigma_\delta(b_2) \circ T^\Sigma_\delta(b_1).
\end{gathered}$$

Therefore (4-1) holds for all $\lambda \in R$. □

4.2 The genus-one case

Now we prove Proposition 4.1 when Σ is a torus or a finitely punctured torus. Let Σ_0 be a torus and suppose $\Sigma = \Sigma_0 \backslash \{p_1, \dots, p_n\}$ with $n \geq 0$. Let c, b_1 and b_2 be as in Proposition 4.1. By the definition of T_λ, we may assume without loss of generality that every component of c intersects $\partial(b_1 \cup b_2)$ nontrivially.

Lemma 4.5 *Assume every simple closed curve $\gamma_0 \subset \Sigma_0$ that is disjoint from $c \cup b_1 \cup b_2$ is contractible in Σ_0. Then up to orientation-preserving diffeomorphisms of Σ_0, there are only eight possible configurations of c, b_1 and b_2 as subsets of Σ_0, which are shown in Figure 2.*

In each case of Figure 2, the torus Σ_0 is the quotient space obtained by gluing the two boundary components of the annulus. The blue curves denote the 1-manifold c, and the disks b_1 and b_2 are defined to be the thickening of the red arcs.

Proof We discuss the following cases:

If c contains two circles γ_1 and γ_2, and both of them are contractible, let $D_1, D_2 \subset \Sigma$ denote the disks bounded by γ_1 and γ_2. Then $D_1 \cup D_2 \cup b_1 \cup b_2$ is a disk or an annulus, and hence there exists a circle γ_0 in the complement of $c \cup b_1 \cup b_2$ that is contractible, contradicting the assumptions.

If c contains two circles γ_1 and γ_2 such that both γ_1 and γ_2 are noncontractible, then γ_1 and γ_2 must be parallel to each other. The complement $\Sigma_0 \backslash (\gamma_1 \cup \gamma_2)$ contains two components. If every simple closed curve in $\Sigma_0 \backslash (c \cup b_1 \cup b_2)$ is contractible in Σ_0, then the interior of b_1 and b_2 must be contained in different components of $\Sigma_0 \backslash (\gamma_1 \cup \gamma_2)$, and ∂b_i must intersect both components of c for each i. Therefore, up to orientation-preserving diffeomorphisms of Σ_0, the configuration is given by Figure 2(1).

If c contains two circles γ_1 and γ_2, where γ_1 is contractible and γ_2 is not contractible, let D_1 be the disk bounded by γ_1. If either b_1 or b_2 is contained in D_1, then $D_1 \cup c \cup b_1 \cup b_2$ deformation retracts onto γ_2, so there exists a noncontractible simple closed curve in Σ_0 that is disjoint from $D_1 \cup c \cup b_1 \cup b_2$, which

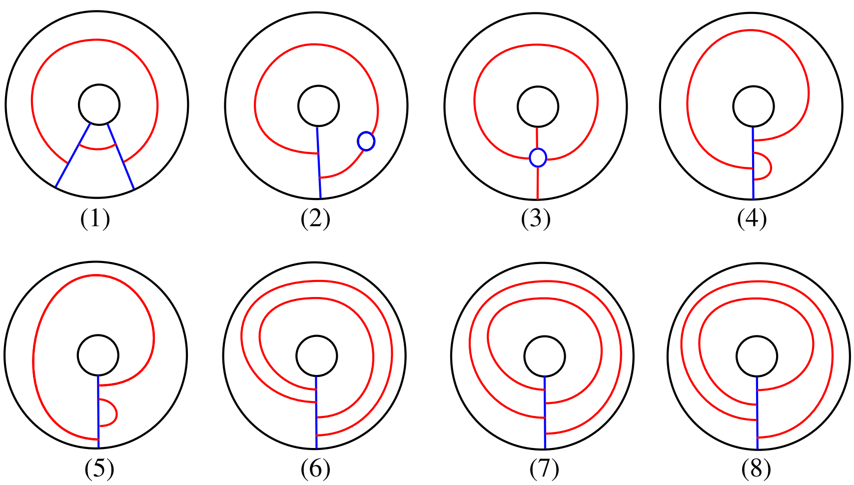

Figure 2: All possible configurations.

contradicts the assumptions. Therefore both b_1 and b_2 must be on the outside of D_1, so $b_1 \cup D_1 \cup b_2$ deformation retracts onto an arc with both endpoints on γ_2. The assumptions then imply that $c \cup b_1 \cup b_2$ is given by Figure 2(2) up to orientation-preserving diffeomorphisms of Σ_0.

If c consists of one simple closed curve γ that is contractible in Σ_0, let D be the disk bounded by γ. Then b_1 and b_2 must be the thickening of two disjoint arcs r_1 and r_2 in $\Sigma_0 \backslash D$. For $i = 1, 2$, let $\bar{r}_i$ be the circle obtained by the union of r_i with an arc in D. Since r_1 and r_2 are disjoint arcs, we may choose the arcs in D so that $\bar{r}_1$ and $\bar{r}_2$ are either disjoint or intersect transversely at one point. The assumptions then imply that $\bar{r}_1$ and $\bar{r}_2$ must intersect transversely at one point. Hence the configuration is given by Figure 2(3) up to orientation-preserving diffeomorphisms of Σ_0.

If c consists of one noncontractible simple closed curve, then the possible configurations are given by Figure 2(4)–(8). □

Lemma 4.6 *Equation* (4-1) *holds if* Σ *is a torus or a finitely punctured torus.*

Proof If there exists a noncontractible simple closed curve $\gamma_0 \subset \Sigma_0$ that is disjoint from $c \cup b_1 \cup b_2$, we may cut open Σ_0 along γ_0, and the desired result follows from Lemmas 4.4 and 4.2. Therefore, by Lemma 4.5, we only need to consider the eight cases given by Figure 2.

In (2) and (4)–(8), both sides of (4-1) are zero because Lemma 3.1(3) appears on both sides of the equations.

For (1) and (3), the complement $\Sigma \backslash (c \cup b_1 \cup b_2)$ has two connected components. Therefore, by Lemma 4.2 again, we only need to consider the cases when there is at most one puncture on each component.

Recall that n denotes the number of punctures on Σ_0. For (1) with $n = 0$ or 2, and for (3), there is an orientation-preserving diffeomorphism of Σ_0 that preserves c and Σ, is orientation-preserving on c, and switches b_1 and b_2. Therefore (4-1) holds.

For (1) with $n = 1$, it is straightforward to verify that both sides of (4-1) are zero. □

5 Khovanov homology

Suppose $L \subset (-1, 1) \times \Sigma$ is a link. For each λ, we define a homology invariant for L using the maps T_λ.

Suppose a link L is given by a diagram D on Σ with k crossings, and fix an ordering of the crossings. For $v = (v_1, v_2, \dots, v_k) \in \{0, 1\}^k$, resolving the crossings of D by a sequence of 0-smoothings and 1-smoothings (see Figure 3) by v turns D into an embedded closed 1-manifold in Σ. Denote the resolved diagram by D_v.

Whenever u is obtained from v by changing one coordinate from 0 to 1, there is a band b near the crossing such that v is obtained from u by a band surgery along b. Define $d^\lambda_{vu}\colon V(D_v) \to V(D_u)$ to be $T_\lambda(b)$. Let e_i be the i^{th} standard basis vector of $\mathbb{Z}^k$. Define

$$\mathrm{CKh}_{\Sigma,\lambda}(L) = \bigoplus_{v \in \{0,1\}^k} V(D_v),$$

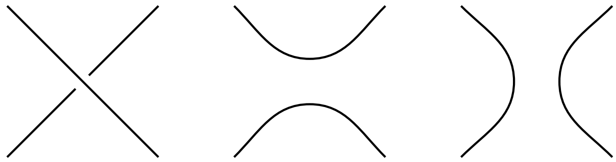

Figure 3: Two types of smoothings.

and define an endomorphisms on $\mathrm{CKh}_\Sigma(L)$ by

$$\mathcal{D}_{\Sigma,\lambda} = \sum_i \sum_{u-v=e_i} (-1)^{\sum_{i<j\le c} v_j} d_{vu}.$$

By (4-1), we have $\mathcal{D}^2_{\Sigma,\lambda} = 0$.

We define a quantum grading and a homological grading on $\mathrm{CKh}_{\lambda,\Sigma}(L)$ as follows. For each circle γ, if γ is noncontractible, define the quantum grading on $V(\gamma)$ to be zero. If γ is contractible, define the quantum grading of $\boldsymbol{v}(\gamma)_+$ to be 1 and the quantum grading of $\boldsymbol{v}(\gamma)_-$ to be -1. This grading then extends to a grading on $\mathrm{CKh}_{\lambda,\Sigma}(L)$. Define the homology grading of $V(D_v) \subset \mathrm{CKh}_{\lambda,\Sigma}(L)$ to be the sum of coordinates in v.

There is also a grading on $\mathrm{CKh}_{\lambda,\Sigma}(L)$ over $H_1(\Sigma;\mathbb{Z})$ defined as follows. For each circle γ, if γ is contractible, define the grading on $V(\gamma)$ to be zero. If γ is noncontractible, for each orientation $\mathfrak{o}$ of γ, define the grading of $\boldsymbol{v}(\gamma)_{\mathfrak{o}}$ to be the fundamental class of $\gamma_{\mathfrak{o}}$.

Following the standard convention, we use curly brackets $\{l\}$ to denote the shifting in quantum gradings by l (namely, adding the quantum grading to each homogeneous element by l); we use the square brackets $[l]$ to denote the shifting in homology gradings by l.

Theorem 5.1 *The homology of*

$$(\mathrm{CKh}_{\lambda,\Sigma}(L)[-n_-]\{n_+ - 2n_-\}, \mathcal{D}_{\Sigma,\lambda})$$

as a $\mathbb{Z}\oplus\mathbb{Z}\oplus H_1(\Sigma;\mathbb{Z})$-graded module is independent of the diagram or the ordering of the crossings, where n_+ and n_- denote the number of positive and negative crossings of the diagram.

Proof The proof is identical to the proof of the invariance of the standard Khovanov homology under Reidemeister moves in [6]. Besides (3-1) and (4-1), the only properties about the band homomorphisms $T_\lambda(b)$ needed in the proof are the following:

(1) If γ is a contractible circle, then $V(\gamma)$ is rank 2 with two generators $\boldsymbol{v}(\gamma)_\pm$.

(2) Suppose the band surgery along b merges two circles γ_1 and γ_2 to γ, where γ_1 is contractible. Then γ_2 and γ are isotopic, and this isotopy defines a canonical isomorphism $\iota\colon V(\gamma_2) \to V(\gamma)$. Then $T_\lambda(b)(\boldsymbol{v}(\gamma_1)_+ \otimes x) = \iota(x)$ for all $x \in V(\gamma_2)$.

(3) Suppose the band surgery along b splits one circle γ to circles γ_1 and γ_2, where γ_1 is contractible. Then γ_2 and γ are isotopic, and this isotopy defines a canonical isomorphism $\iota\colon V(\gamma) \to V(\gamma_2)$. Then the composition map

$$V(\gamma) \xrightarrow{T_\lambda(b)} V(\gamma_1) \otimes V(\gamma_2) \xrightarrow{/\boldsymbol{v}(\gamma_1)_+=0} \operatorname{span}\{\boldsymbol{v}(\gamma_1)_-\} \otimes V(\gamma_2)$$

is given by the tensor product with $\boldsymbol{v}(\gamma_1)_-$, where the second map above is a quotient map.

The only remark worth making is that there is a typo in the definition of the "transpose" map in [6, Section 3.5.5]. The map Υ on the top layer should map the *quotient image* of the pair (β_1, γ_1) to the *quotient image* of the pair (β_2, γ_2) *such that* $\gamma_1 + \tau_1\beta_1 = \gamma_2 + \tau_2\beta_2$. The italicized phrases and the last equation in the previous sentence were missing in [6]. □

Definition 5.2 We define the homology of

$$(\mathrm{CKh}_{\lambda,\Sigma}(L)[-n_-]\{n_+ - 2n_-\}, \mathcal{D}_{\Sigma,\lambda})$$

as a $\mathbb{Z}\oplus\mathbb{Z}\oplus H_1(\Sigma;\mathbb{Z})$-module to be the Khovanov invariant of $L \subset (-1,1)\times\Sigma$, and denote it by $\Sigma\mathrm{Kh}_{\lambda,\Sigma}(L;R)$.

When there is no risk of confusion on the surface Σ and the coefficient ring R, we will also denote $\Sigma\mathrm{Kh}_{\lambda,\Sigma}(L;R)$ by $\Sigma\mathrm{Kh}_\lambda(L)$.

Remark 5.3 When $\lambda = 0$, the differential map $\mathcal{D}_{\Sigma,\lambda}$ is identical to the differential map of Asaeda–Przytycki–Sikora homology defined in [1]. When $R = \mathbb{Z}$, $\lambda = 1$ and Σ is a punctured disk, the homology $\Sigma\mathrm{Kh}_\lambda$ recovers the invariant defined by Winkeler [16].

6 Relations with instanton Floer homology

This section explains the motivation of the definition of $T_\lambda(b)$ from instanton homology. We will also prove the following detection result:

Theorem 6.1 *Suppose that Σ is a surface with genus zero, and $L \subset (-1,1)\times\Sigma$ is a link. Then* $\operatorname{rank}_{\mathbb{Z}/2} \Sigma\mathrm{Kh}_1(L;\mathbb{Z}/2) \ge 2$, *and equality holds if and only if L is isotopic to an embedded knot in Σ.*

The detection problems of Khovanov homology and other quantum invariants of knots and links have attracted considerable attention since the introduction of the invariants. Kronheimer and Mrowka [10] proved that the standard Khovanov homology detects the unknot; see also [7; 8]. Since then, a large number of detection results on Khovanov homology were obtained using different versions of Floer theory, for example, by [2; 3; 4; 5; 13; 14; 19; 20]. The main theorem in [12] gave the first detection result on Khovanov homology that is valid on an infinite family of manifolds. Theorem 6.1 above is an improvement of the main theorem of [12].

In fact, by a spectral sequence of Winkeler [16, Theorem 1.3], we have

$$\operatorname{rank}_{\mathbb{Z}/2} \Sigma\mathrm{Kh}_0(L;\mathbb{Z}/2) \geq \operatorname{rank}_{\mathbb{Z}/2} \Sigma\mathrm{Kh}_1(L;\mathbb{Z}/2). \tag{6-1}$$

The main result in [12] states a classification of all links L such that $\Sigma\mathrm{Kh}_0(L;\mathbb{Z}/2)$ has the minimum possible rank. Theorem 6.1 immediately implies the result in [12] because of (6-1).

6.1 Motivation from instanton Floer homology

We start by discussing the motivation of the definition of $T_\lambda(b)$ from computations in instanton Floer homology. All instanton homology groups here will be defined with $\mathbb{C}$ coefficients. We refer the reader to [12, Section 2] for the general notation and properties of singular instanton Floer homology. In particular, we will use $\mathrm{I}(Y, L, \omega)$ to denote the instanton homology of a nonintegral triple (Y, L, ω), where Y is a closed 3-manifold, $L \subset Y$ is a link and $(\omega, \partial\omega) \subset (Y, L)$ is an embedded 1-manifold. The nonintegral condition is a technical condition to ensure that Floer homology is well-defined, and the statement of the condition can be found in [12, Section 2.3]. If $\Sigma \subset Y$ is an oriented embedded surface, then $\mathrm{I}(Y, L, \omega \,|\, \Sigma)$ denotes a subspace of $\mathrm{I}(Y, L, \omega)$ introduced by [18]; the complete definition can be found in [12, Definition 2.10]. If Σ is connected, one may regard $\mathrm{I}(Y, L, \omega \,|\, \Sigma)$ as the component of $\mathrm{I}(Y, L, \omega)$ at the maximum possible grading with respect to a grading induced by Σ.

Suppose Q is a *closed* oriented surface, and let L be a link in $(-1, 1) \times Q$. Let p be a point on Q that is disjoint from the projection of L to Q. In [12], the authors studied the instanton homology group

$$\Sigma\mathrm{HI}_{Q,p}(L) := \mathrm{I}(S^1 \times Q, L, S^1 \times \{p\} \,|\, \{t_*\} \times Q), \tag{6-2}$$

where S^1 is viewed as the quotient space of $[-1, 1]$ with -1 identified with 1, and $t_* \in S^1$ is a fixed basepoint.

Remark 6.2 The closed surface in (6-2) was denoted by R instead of Q in [12]. We use the notation Q here to avoid collision of notation with the coefficient ring.

Suppose c is an embedded 1-manifold in Q, and b is a band attached to c that is disjoint from p. Then the band surgery along b defines a link cobordism from c to c_b as links in $(-1, 1) \times Q$. Therefore it induces a cobordism map for Floer homology groups (up to sign)

$$\Sigma\mathrm{HI}_{Q,p}(b)\colon \Sigma\mathrm{HI}_{Q,p}(c) \to \Sigma\mathrm{HI}_{Q,p}(c_b).$$

It was proved in [12, Proposition 6.12] that the maps $\Sigma\mathrm{HI}_{Q,p}(b)$ are components of the second page of a variant of Kronheimer and Mrowka's spectral sequence which abuts to $\Sigma\mathrm{HI}_{Q,p}(L)$. In [12, Proposition 6.11], the cobordism maps $\Sigma\mathrm{HI}_{Q,p}(b)$ were computed for multiple special cases; in all the computed cases, $\Sigma\mathrm{HI}_{Q,p}(b)$ is equal to $T_\lambda(b)$ for some $\lambda \in \mathbb{C}$ in a suitable sense. This motivated our definition of the map $T_\lambda(b)$. It is natural to conjecture that the second page of Kronheimer and Mrowka's spectral sequence is isomorphic to $\Sigma\mathrm{Kh}_{\lambda,Q}(L;\mathbb{C})$ for some $\lambda \in \mathbb{C}$.

Conjecture 6.3 *Suppose Q is a closed oriented surface, and let L be a link in $(-1,1)\times Q$ given by a diagram D on Q. Let p be a fixed point on Q that is disjoint from D. Then there exist $\lambda\in\mathbb{C}$ and a spectral sequence that abuts to $\Sigma\mathrm{HI}_{Q,p}(L)$ whose second page is isomorphic to $(\mathrm{CKh}_{\lambda,Q}(L),\mathcal{D}_{\Sigma,\lambda})$ (with $\mathbb{C}$-coefficients) as a chain complex.*

6.2 Proof of Theorem 6.1

We may assume without loss of generality that Σ is connected and compact. If $\Sigma=S^2$, the desired result follows from the unknot detection theorem for the standard Khovanov homology [10]. We assume in the following that $\partial\Sigma\neq\varnothing$.

Assume F is a connected oriented surface such that ∂F equals $\partial\Sigma$ with the reversed orientation. Let $Q=\Sigma\cup_\partial F$. By [12, Proposition 6.12], there is a spectral sequence that abuts to $\Sigma\mathrm{HI}_{Q,p}(L)$ whose second page is given by maps of the form $\Sigma\mathrm{HI}_{Q,p}(b)$, where b is a band corresponding to a crossing change between different smoothings of the diagram D. By [12, Lemma 5.2], the second page of Kronheimer and Mrowka's spectral sequence, as a linear space, is isomorphic to $\mathrm{CKh}_{\lambda,Q}(L)$ (with $\mathbb{C}$-coefficients).

By [12, Proposition 6.11], after conjugating by an isomorphism from $V(-)$ to $\Sigma\mathrm{HI}_{Q,p}(-)$ defined in [12, Section 5.2], each component of the differential map on the second page has the form $i^kT_\lambda(b)$ for some $\lambda\in\mathbb{C}$ and $k\in\mathbb{Z}$. We show that it is possible to choose an isomorphism from $V(-)$ to $\Sigma\mathrm{HI}_{Q,p}(-)$ such that after conjugation, the coefficients λ are the same (up to sign) on all the components of the differential map. Moreover, we show that the coefficient λ must be nonzero.

In the following, we will denote $\Sigma\mathrm{HI}_{Q,p}(-)$ by $\Sigma\mathrm{HI}(-)$ to simplify notation.

Let $\lambda_1,\dots,\lambda_4$ be the constants from [12, Section 6]. By [12, Lemma 6.9], one can rescale the isomorphisms in [12, Section 5.2] so that $\lambda_1=\pm1$ and $\lambda_3=\pm1$.

Lemma 6.4 *Assume the generator w_0 defined in [12, Section 5.2.1] is chosen so that $\lambda_1=\pm1$ and $\lambda_3=\pm1$. Then $\lambda_2=\pm\lambda_4$.*

Proof Consider the two bands in Figure 4 and apply the TQFT property of $\Sigma\mathrm{HI}(b)$. □

Lemma 6.5 *The coefficients λ_2 and λ_4 are both nonzero.*

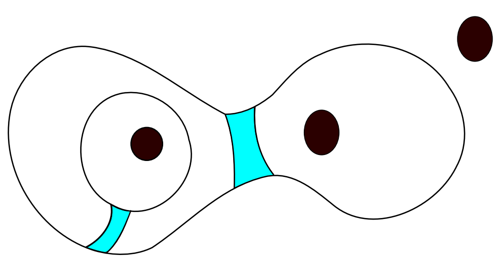

Figure 4: Two bands.

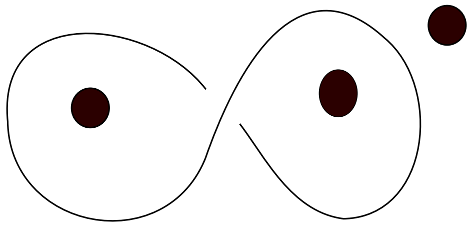

Figure 5: The diagram D on Σ.

Proof Suppose Σ is a sphere with three open disks removed, and let D be a diagram on Σ as shown in Figure 5. Let $D_0 = \gamma$ be the resolution of D into one circle, let $D_1 = \gamma_1 \cup \gamma_2$ be the resolution of D into two circles and let b be the band relating D_0 and D_1. Let K be the knot represented by D. Then by [10, Theorem 6.8], there is an exact triangle

$$\cdots \to \Sigma\mathrm{HI}(\gamma) \xrightarrow{\Sigma\mathrm{HI}(b)} \Sigma\mathrm{HI}(\gamma_1 \cup \gamma_2) \to \Sigma\mathrm{HI}(K) \to \Sigma\mathrm{HI}(\gamma) \xrightarrow{\Sigma\mathrm{HI}(b)} \Sigma\mathrm{HI}(\gamma_1 \cup \gamma_2) \to \cdots.$$

By [12, Lemma 6.4], $\lambda_2 = \lambda_4 = 0$ if and only if $\Sigma\mathrm{HI}(b) = 0$ in the above exact sequence. By [12, Lemma 5.2], we have $\dim \Sigma\mathrm{HI}(\gamma_1 \cup \gamma_2) = 4$ and $\dim \Sigma\mathrm{HI}(\gamma) = 2$. Therefore we only need to show

$$\dim \Sigma\mathrm{HI}(K) < 6. \tag{6-3}$$

Let L be the link in the thickened annulus as shown in Figure 6. Pick a meridional disk in the thickened annulus which intersects L at two points. We decompose the thickened annulus along this disk and obtain a product sutured thickened disk with a tangle T in it. The sutured instanton Floer homology of this sutured manifold with tangle T is isomorphic to $\mathrm{AHI}(L, 2)$ according to [11, Theorem 2.14], where $\mathrm{AHI}(L, 2)$ denotes the component of the annular instanton Floer homology with Alexander grading 2. (The Alexander grading is also called the annular grading, the f-grading or the k-grading in the literature. We follow the terminology of [15, Definition 2.2] here, which agrees with the notation in [12].)

The tangle T has two product vertical components. We remove the tubular neighborhoods of the two vertical components and add a meridian suture to the boundary of each neighborhood to obtain a sutured

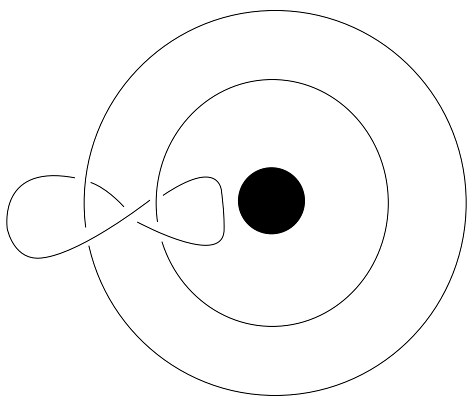

Figure 6: The annular link L.

manifold M' with a knot K' in it. Moreover, this process does not change the sutured instanton Floer homology according to [18, Lemma 7.10] and its proof. Therefore

$$\mathrm{SHI}(M', \gamma_{M'}, K') \cong \mathrm{AHI}(K, 2).$$

Notice that in the definition of sutured instanton Floer homology, the pairs (M', K') and (M, K) can be given the same closure; therefore their sutured instanton homologies are isomorphic. As a result, we have

$$\mathrm{SHI}([-1,1] \times \Sigma, \{0\} \times \Sigma, K) \cong \mathrm{AHI}(L, 2). \tag{6-4}$$

A straightforward calculation shows that

$$\mathrm{AKh}(L, 2; \mathbb{C}) \cong \mathbb{C}^4,$$

where $\mathrm{AKh}(L, 2; \mathbb{C})$ denotes the component of the annular Khovanov homology of L with Alexander grading 2 and with coefficient ring $\mathbb{C}$. According to [17, Theorem 5.16], we have

$$\dim \mathrm{AHI}(L, 2; \mathbb{C}) \le \dim \mathrm{AKh}(L, 2; \mathbb{C}) = 4.$$

Therefore, (6-4) implies

$$\dim \Sigma\mathrm{HI}(K) = \dim \mathrm{SHI}([-1,1] \times \Sigma, \{0\} \times \Sigma, K) \le 4.$$

This verifies (6-3), and hence the desired result is proved. □

Theorem 6.1 can now be proved using an argument from [12].

Proof of Theorem 6.1 When Σ is a compact surface with genus zero, there is a grading on $V(-)$ such that $T_0(b)$ is homogeneous with degree zero and $T_1(b)$ is homogeneous with degree -1. Since $\lambda_2 \neq 0$, we can rescale the map $\Theta_{w_0,\sigma}$ in [12] by a factor of λ_2^k at degree k. By the discussion in [12, Section 6], there is a spectral sequence of chain complexes in $\mathbb{C}$-coefficients that converges to $\mathrm{I}([-1,1] \times \Sigma, \{0\} \times \partial\Sigma, L)$, whose second page (E_2, d_2) is isomorphic to the chain complex $(\mathrm{CKh}_{\Sigma,1}(L), \mathcal{D}_{\Sigma,1})$ up to multiplications by integer powers of i on the components of the differential map. In other words, there exists a chain complex (C, d) defined with $\mathbb{Z}[i]$ coefficients, such that when reducing to $\mathbb{C}$ coefficients, it is isomorphic to (E_2, d_2); when reducing to $\mathbb{Z}[i]/(i-1) \cong \mathbb{Z}/2$ coefficients, it is isomorphic to the chain complex $(\mathrm{CKh}_{\Sigma,1}(L), \mathcal{D}_{\Sigma,1})$. By the universal coefficient theorem,

$$\begin{aligned}\mathrm{rank}_{\mathbb{Z}/2}\, \Sigma\mathrm{Kh}_{\Sigma,1}(L; \mathbb{Z}/2) &\ge \mathrm{rank}_{\mathbb{Z}[i]}\, H(C, d) = \dim_{\mathbb{C}} H(E_2, d_2) \\ &\ge \dim_{\mathbb{C}} \mathrm{I}([-1,1] \times \Sigma, \{0\} \times \partial\Sigma, L),\end{aligned}$$

and the desired result follows from [12, Theorem 1.3]. □

References

[1] **M M Asaeda**, **J H Przytycki**, **A S Sikora**, *Categorification of the Kauffman bracket skein module of I-bundles over surfaces*, Algebr. Geom. Topol. 4 (2004) 1177–1210 MR Zbl

[2] **J A Baldwin**, **N Dowlin**, **A S Levine**, **T Lidman**, **R Sazdanovic**, *Khovanov homology detects the figure-eight knot*, Bull. Lond. Math. Soc. 53 (2021) 871–876 MR Zbl

[3] **J A Baldwin**, **Y Hu**, **S Sivek**, *Khovanov homology and the cinquefoil*, J. Eur. Math. Soc. (online publication January 2024)

[4] **J A Baldwin**, **S Sivek**, *Khovanov homology detects the trefoils*, Duke Math. J. 171 (2022) 885–956 MR Zbl

[5] **J A Baldwin**, **S Sivek**, **Y Xie**, *Khovanov homology detects the Hopf links*, Math. Res. Lett. 26 (2019) 1281–1290 MR Zbl

[6] **D Bar-Natan**, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. 2 (2002) 337–370 MR Zbl

[7] **J E Grigsby**, **S M Wehrli**, *On the colored Jones polynomial, sutured Floer homology, and knot Floer homology*, Adv. Math. 223 (2010) 2114–2165 MR Zbl

[8] **M Hedden**, *Khovanov homology of the 2-cable detects the unknot*, Math. Res. Lett. 16 (2009) 991–994 MR Zbl

[9] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426 MR Zbl

[10] **P B Kronheimer**, **T S Mrowka**, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. 113 (2011) 97–208 MR Zbl

[11] **Z Li**, **Y Xie**, **B Zhang**, *On Floer minimal knots in sutured manifolds*, Trans. Amer. Math. Soc. Ser. B 9 (2022) 499–516 MR Zbl

[12] **Z Li**, **Y Xie**, **B Zhang**, *Instanton homology and knot detection on thickened surfaces*, Selecta Math. 29 (2023) art. id. 84 MR Zbl

[13] **R Lipshitz**, **S Sarkar**, *Khovanov homology detects split links*, Amer. J. Math. 144 (2022) 1745–1781 MR Zbl

[14] **G Martin**, *Khovanov homology detects $T(2,6)$*, Math. Res. Lett. 29 (2022) 835–849 MR Zbl

[15] **L P Roberts**, *On knot Floer homology in double branched covers*, Geom. Topol. 17 (2013) 413–467 MR Zbl

[16] **Z Winkeler**, *Khovanov homology for links in thickened multipunctured disks*, Michigan Math. J. 74 (2024) 1–31 MR Zbl

[17] **Y Xie**, *Instantons and annular Khovanov homology*, Adv. Math. 388 (2021) art. id. 107864 MR Zbl

[18] **Y Xie**, **B Zhang**, *Instanton Floer homology for sutured manifolds with tangles* (2019) arXiv 1907.00547 To appear in J. Differential Geom.

[19] **Y Xie**, **B Zhang**, *Classification of links with Khovanov homology of minimal rank*, J. Eur. Math. Soc. 27 (2025) 333–394 MR Zbl

[20] **Y Xie**, **B Zhang**, *Instantons and Khovanov skein homology on $I \times T^2$*, Quantum Topol. 16 (2025) 75–115 MR

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