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Band diagrams of immersed surfaces in 4-manifolds

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We study immersed surfaces in smooth 4-manifolds via singular banded unlink diagrams. Such a diagram consists of a singular link with bands inside a Kirby diagram of the ambient 4-manifold, representing a level set of the surface with respect to an associated Morse function. We show that every self-transverse immersed surface in a smooth, orientable, closed 4-manifold can be represented by a singular banded unlink diagram, and that such representations are uniquely determined by the ambient isotopy or equivalence class of the surface up to a set of singular band moves which we define explicitly. By introducing additional finger, Whitney and cusp diagrammatic moves, we can use these singular band moves to describe homotopies or regular homotopies as well.

Using these techniques, we introduce bridge trisections of immersed surfaces in arbitrary trisected 4-manifolds and prove that such bridge trisections exist and are unique up to simple perturbation moves. We additionally give some examples of how singular banded unlink diagrams may be used to perform computations or produce explicit homotopies of surfaces.

57K45; 57K40

1. Introduction 1731
2. Singular banded unlink diagrams 1735
3. Bridge trisections 1759
4. Some example applications 1782
References 1790

1 Introduction

Immersed surfaces are fundamental objects in low-dimensional topology, showing up frequently in the study of 4-manifolds. For example, immersed disks play a key role in Freedman's proof of the topological h-cobordism theorem and the homeomorphism classification of simply connected smooth 4-manifolds [6]. One reason for the prominent part they play lies in how abundant they are when compared to their embedded counterparts. In particular, maps of surfaces into smooth 4-manifolds can always be perturbed slightly to yield smooth immersions with transverse double points.

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Despite their importance, immersed surfaces and their isotopies are difficult to describe explicitly outside of a few concrete examples. While diagrammatic techniques have been developed to describe both smooth 4-manifolds and embedded surfaces (see eg Carter and Saito [3; 4], Hughes, Kim and Miller [14], Kamada [17; 19], Meier and Zupan [26; 27] and Roseman [29]), methods of studying immersed surfaces diagrammatically have not been established as fully in the literature, aside from a few examples (see eg Kamada, Kawauchi, Kim and Lee [20] for a diagrammatic framework for representing immersed surfaces in $\mathbb{R}^4$ via marked graph diagrams).

In this paper, we introduce a new diagrammatic system for describing immersed surfaces in smooth, oriented, closed 4-manifolds, called *singular banded unlink diagrams*. Such a diagram consists of a Kirby diagram for the ambient 4-manifold along with a decorated singular (4-valent) link with bands attached away from vertices (see Section 2.2 for details). As a Kirby diagram of X is uniquely determined by a Morse function h and its gradient ∇h, given two singular banded unlink diagrams in the same Kirby diagram (induced by the same Morse function on X), it makes sense to ask whether they determine isotopic surfaces. Even with singular banded unlink diagrams in two different Kirby diagrams of X, we can still ask whether they describe equivalent surfaces. With this in mind, we define a set of moves, called *singular band moves*, in Figures 3 and 4, which allow us to relate the diagrams of any two immersed surfaces which are ambiently isotopic. When combined with Kirby moves to the ambient diagram, these moves are also sufficient to relate equivalent surfaces. That is, we show the equivalence

$$\frac{\{\text{singular banded unlink diagrams}\}}{\text{singular band moves}} \leftrightarrow \frac{\{\text{self-transversely immersed surfaces in 4-manifolds}\}}{\text{ambient diffeomorphism}}.$$

We make this equivalence precise in Corollary 2.40 (and for isotopy rather than diffeomorphism in Theorem 2.39). This work generalizes earlier results in [14], where the authors define *banded unlink diagrams* of smoothly *embedded* surfaces in smooth 4-manifolds, and present a family of moves (called *band moves*) to describe isotopies between such surfaces. More precisely, given a smoothly embedded surface Σ in a smooth oriented closed 4-manifold X and a self-indexing Morse function $h\colon X \to \mathbb{R}$, we obtain a diagram $\mathcal{D}(\Sigma)$ which is well defined up to band moves and depends only on the ambient isotopy class of Σ inside X. Furthermore, given the diagram $\mathcal{D}(\Sigma)$, we may recover the pair (X, Σ) up to diffeomorphism. If we also specify the Morse function $h\colon X \to \mathbb{R}$, then the surface $\Sigma \subset X$ is determined up to isotopy. In the special case that $X^4 = S^4$ and h is standard (ie h has no index 1, 2 or 3 critical points), these results are originally due to Swenton [32] and Kearton and Kurlin [22].

Unless otherwise stated, we will assume that X is a closed, smooth, oriented 4-manifold. Our main theorems are as follows:

Theorem 2.39 *Let Σ be a smoothly immersed, self-transverse surface in a 4-manifold X. Then any choice of a self-indexing Morse function $h\colon X \to \mathbb{R}$ (with one index 0 point) and a gradientlike vector field ∇h on X induces a singular banded unlink diagram $\mathcal{D}(\Sigma)$ of (X, Σ) that is well defined up to singular band moves.*

Furthermore, let $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ be singular banded unlink diagrams of immersed surfaces Σ and Σ' in X.

(i) *The diagrams $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ are related by band moves and Kirby moves if and only if there is a diffeomorphism $(X, \Sigma) \cong (X, \Sigma')$.*

(ii) *If $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ are induced by the same self-indexing Morse function h and gradientlike vector field ∇h (which are suitably generic so as to ensure the underlying Kirby diagrams of $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ agree), then $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ are related by band moves if and only if Σ and Σ' are ambiently isotopic.*

In other words, if $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ are banded unlink diagrams whose underlying Kirby diagrams are identified, then Σ and Σ' are smoothly ambiently isotopic if and only if $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ are related by singular band moves.

In the opening paragraph of Theorem 2.39, we say that $\mathcal{D}(\Sigma)$ is well defined only up to singular band moves, even though ∇h is specified. This is because, in order to obtain $\mathcal{D}(\Sigma)$, we also need to choose a gradientlike vector field of $h|_\Sigma$, which is not canonically determined by $(h, \nabla h, \Sigma)$.

Note that part (ii) of Theorem 2.39 clearly implies part (i), so we will focus on proving part (ii). Furthermore, since Kirby diagrams of two 4-manifolds can be related by a sequence of Kirby moves if and only if they are diffeomorphic, we obtain the following corollary:

Corollary 2.40 *Let $\mathcal{D}$ and $\mathcal{D}'$ be singular banded unlink diagrams of surfaces Σ and Σ' self-transversely immersed in diffeomorphic 4-manifolds X and X'. There is a diffeomorphism taking (X, Σ) to (X', Σ') if and only if there is a sequence of singular band moves and Kirby moves taking $\mathcal{D}$ to $\mathcal{D}'$.*

Without much extra work, we may also extend Theorem 2.39 to consider homotopy instead of isotopy:

Corollary 2.41 *Let Σ and Σ' be self-transverse surfaces smoothly immersed in X, and let $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ be singular banded unlink diagrams in the same Kirby diagram of X.*

(i) *The surfaces Σ and Σ' are regularly homotopic if and only if $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ can be related by a sequence of singular band moves and the finger/Whitney moves illustrated in Figure 15.*

(ii) *The surfaces Σ and Σ' are homotopic (without specifying regularity) if and only if $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ are related by singular band moves, finger/Whitney moves and cusp moves as illustrated in Figure 15.*

One application of the authors' results in [14] was to prove the uniqueness of bridge trisections of surfaces in arbitrary trisected 4-manifolds up to perturbation. In Section 3.2, we define the notions of *bridge position* and *bridge trisections* for immersed surfaces in trisected 4-manifolds, and in Section 3.5 we prove an analogous uniqueness statement:

Theorem 3.36 *Let $(X^4, \mathcal{T})$ be a trisected 4-manifold. Let Σ be a self-transverse immersed surface in X^4. Then Σ can be isotoped into bridge position with respect to $\mathcal{T}$, yielding a bridge trisection of Σ*

with respect to $\mathcal{T}$. Moreover, any two bridge trisections of Σ with respect to $\mathcal{T}$ are related by $\mathcal{T}$-preserving isotopy, perturbations and vertex perturbations (and their inverses).

The moves referenced in Theorem 3.36 are defined in Section 3.1. For experts, we will say now that the perturbation move is the standard perturbation move that increases the number of disks of Σ in one section of the trisection, while vertex perturbation is supported in a neighborhood of the trisection surface and simply passes a self-intersection of Σ from one piece of the trisection to another.

Organization

In Section 2 we lay out the framework of singular banded unlink diagrams. We begin in Section 2.1 with a discussion on marked singular banded links. In Section 2.2, we describe how to use these decorated singular links to obtain immersed surfaces. In Section 2.3, we discuss two subclasses of immersed surfaces that will be needed to prove Theorem 2.39 and its corollaries in Section 2.4.

In Section 3 we turn our attention to bridge trisections. We review the theory of bridge trisections of embedded surfaces in Section 3.1. In Section 3.2, we adapt the notions of trivial tangles and bridge position to singular links, before defining bridge position for immersed surfaces in Section 3.3 and showing that every immersed surface in a smooth 4-manifold can be arranged in this position. It is here that we define the various moves on immersed bridge trisections referenced in Theorem 3.36. In Section 3.4, we then proceed to adapt the singular banded unlinks developed in Section 2 to bridge trisections, before using the uniqueness results for singular banded unlinks to prove Theorem 3.36 in Section 3.5.

In Section 4 we give some additional sample applications of the usefulness of singular banded unlink diagrams. In Section 4.1, we show how one may compute the Kirk invariant (see Schneiderman and Teichner [30]) of a spherical link using these diagrams. In Section 4.2, we prove that homologous immersed oriented surfaces with the same number of positive and negative self-intersections are stably isotopic (ie become isotopic after surgery along some collection of arcs). Finally, in Section 4.3, we show that certain 2-spheres embedded in S^4 can be trivialized by a single finger and Whitney move (recovering a fact originally proved by Joseph, Klug, Ruppik and Schwartz [16]).

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2 Singular banded unlink diagrams

2.1 Marked singular banded links

In this section we introduce marked singular banded links, which are the combinatorial objects we will use to describe self-transverse immersed surfaces in 4-manifolds. In what follows, all manifolds and maps between them should be assumed to be smooth. All isotopies of immersed (or embedded) submanifolds are assumed to be ambient isotopies unless otherwise specified. Note that we are isotoping the images of immersions rather than immersions themselves.

2.1.1 Marked singular links We begin by defining special singular links with additional data recorded at their double points.

Definition 2.1 Let M^3 be an orientable 3-manifold. A *singular link* L in M is the image of an immersion $\iota\colon S^1 \sqcup \cdots \sqcup S^1 \to M$ which is injective except at isolated double points that are not tangencies. At every double point p we include a small disk $v \cong D^2$ embedded in M such that $(v, v \cap L) \cong \big(D^2, \{(x,y) \in D^2 \mid xy = 0\}\big)$. We refer to these disks as the *vertices* of L.

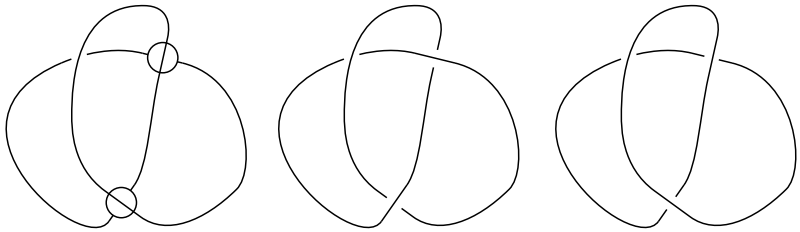
(Equivalently, a singular link is a 4-valent fat-vertex graph smoothly embedded in M.) For now, our motivating idea is that M will correspond to some level set of a 4-manifold X, and the double points of a singular link L in M will correspond to the isolated double points an immersed surface in X. Because these double points are isolated, we expect the singularities of L to be resolved away from the level set M. We must make a choice of how to resolve each double point.

Definition 2.2 A *marked singular link* (L, σ) in M is a singular link L along with decorations σ on the vertices of L, as follows: say that v is a vertex of L, with $\partial v \cap \overline{(L \setminus v)}$ consisting of the four points p_1, p_2, p_3, p_4 in cyclic order. Choose a coorientation of the disk v. On the positive side of v, add an arc connecting p_1 and p_3. On the negative side of v, add an arc connecting p_2 and p_4. See Figure 1, left. A choice of σ involves making a fixed choice of decoration on v for all vertices v of L.

Note that, if L has n vertices, there are 2^n choices of decorations σ such that (L, σ) is a marked singular link.

Definition 2.3 Let (L, σ) be a marked singular link in a 3-manifold M. Let v be a vertex of L; say that on the positive side of v there is an arc with endpoints p_1 and p_3, and on the negative side of v there is an arc with endpoints p_2 and p_4.

Let L^+ denote the link in M obtained from (L, σ) by pushing the arc of L between p_1 and p_3 off v in the positive direction, and repeating for each vertex in L. We call L^+ the *positive resolution* of (L, σ) (see Figure 1).



(L,σ) L^- L^+

Figure 1: Left: a marked singular link (L,σ). Middle and right: the negative and positive resolutions of (L,σ), respectively.

Similarly, let L^- denote the link in M obtained from (L,σ) by pushing the arc of L between p_1 and p_3 off v in the negative direction, and repeating for each vertex in L. We call L^- the *negative resolution* of (L,σ) (see Figure 1).

Informally, L^+ is obtained from (L,σ) by turning the decorations of σ into new overstrands, while L^- is obtained by turning the decorations of σ into new understrands.

To ease notation, from now on we will always take singular links to be marked. We will generally not specify the decorations σ, and will instead write "L is a marked singular link", with σ implicitly fixed.

2.1.2 Banded singular links Let L be a singular link, and let Δ_L denote the union of the vertices of L. A *band* b attached to L is the image of an embedding $\phi\colon I\times I \hookrightarrow M\setminus\Delta_L$, where $I=[-1,1]$, and $b\cap L=\phi(\{-1,1\}\times I)$. We call $\phi\big(I\times\{\frac{1}{2}\}\big)$ the *core* of the band b. Let L_b be the singular link defined by

$$L_b=\big(L\setminus\phi(\{-1,1\}\times I)\big)\cup\phi(I\times\{-1,1\}).$$

Then we say that L_b is the result of performing *band surgery* to L along b. If B is a finite family of pairwise disjoint bands for L, then we will let L_B denote the link we obtain by performing band surgery along each of the bands in B. We say that L_B is the result of *resolving* the bands in B. Note that the self-intersections of L_B naturally correspond to those of L, so a choice of markings for L yields markings for L_B. A triple (L,σ,B), where (L,σ) is a marked singular link and B is a family of disjoint bands for L, is called a *marked singular banded link*. To ease notation, we may refer to the pair (L,B) as a *singular banded link* and implicitly remember that L is actually a *marked* singular link.

2.2 Singular banded links describing surfaces

In this section, we use marked singular banded links to describe surfaces in 4-manifolds. Thinking of M as a level set of the 4-manifold X, we'll begin by defining what the surface looks like in a product tubular neighborhood of M.

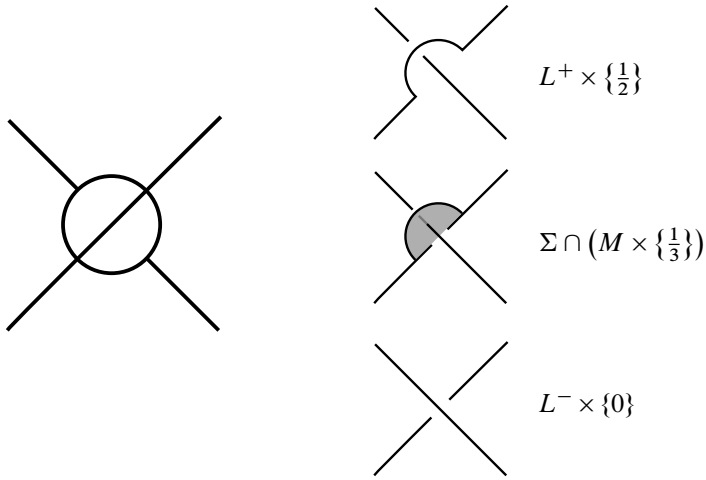


Figure 2: Left: a vertex v of a marked singular link (L, B). Right: part of the surface Σ built from (L, B) near v.

2.2.1 Realizing surfaces in $M^3 \times [0, 1]$ Let (L, B) be a marked singular banded link in the oriented 3-manifold M. We will describe how to construct a surface Σ in $M \times [0, 1]$ using (L, B).

Recall first that L^- is the (nonsingular) link obtained by negatively resolving each vertex of L. Also notice that L^- differs from L^+ only in a neighborhood of the vertices of L, where at each vertex a single strand of L is pushed in the positive direction to give L^+, and the negative direction to give L^-. For each vertex v of L, these two opposite pushoffs form a bigon in a neighborhood of v, which bounds an embedded disk D_v. This disk D_v can be chosen so that its interior intersects L transversely in a single point near v. For each vertex v, select such a disk D_v (ensuring that all of these disks are pairwise disjoint), and let D_L denote the union of all of these embedded disks.

We can then define the surface $\Sigma \subset M \times [0, 1]$ by

$$\begin{aligned}
\Sigma \cap \big(M \times \big[0, \tfrac{1}{3}\big)\big) &= L^- \times \big[0, \tfrac{1}{3}\big), \\
\Sigma \cap \big(M \times \{\tfrac{1}{3}\}\big) &= (L^- \cup D_L) \times \{\tfrac{1}{3}\}, \\
\Sigma \cap \big(M \times \big(\tfrac{1}{3}, \tfrac{2}{3}\big)\big) &= L^+ \times \big(\tfrac{1}{3}, \tfrac{2}{3}\big), \\
\Sigma \cap \big(M \times \{\tfrac{2}{3}\}\big) &= \big(L^+ \cup B\big) \times \{\tfrac{2}{3}\}, \\
\Sigma \cap \big(M \times \big(\tfrac{2}{3}, 1\big]\big) &= L^+_B \times \big(\tfrac{2}{3}, 1\big].
\end{aligned}$$

In total, Σ is a surface properly immersed in $M \times [0, 1]$ with boundary $(L^- \times \{0\}) \sqcup (L^+{}_B \times \{1\})$, and with isolated transverse self-intersections all contained in $M \times \{\frac{1}{3}\}$ corresponding to the vertices of L.

Definition 2.4 Let $\overline{\Sigma}(L, B)$ be a surface properly immersed in $M \times [0, 1]$ obtained from Σ by smoothing corners. We refer to $\overline{\Sigma}(L, B)$ as a *surface segment realizing* (L, B).

Proposition 2.5 *Up to ambient isotopy of $M \times [0,1]$, the surface segment $\overline{\Sigma}(L,B)$ depends only on the singular banded link (L,B).*

Proof There is a unique way (up to isotopy) to smooth the corners of Σ in a neighborhood of $M \times \{\frac{1}{3}, \frac{2}{3}\}$. The disks D_v in $M \times \{\frac{1}{3}\}$ are determined up to isotopy by the links L^- and L^+, which are well defined up to isotopy in M. No other choices were made in constructing $\overline{\Sigma}(L,B)$. □

Note that, by rescaling the interval parameter, we can similarly define a surface segment realizing (L,B) in any product of the form $M \times [t_1, t_2]$. As above, the ambient isotopy class of $\overline{\Sigma}(L,B)$ will depend only on (L,B).

2.2.2 Morse functions and projections between level sets Before describing how to construct a closed realizing surface in a 4-manifold from a singular banded unlink, it will be convenient to take a brief detour to set up some useful notation. Let X be a closed, oriented, 4-manifold equipped with a self-indexing Morse function $h\colon X \to [0,4]$, where h has exactly one index 0 critical point. In what follows, it will be helpful to have a way of identifying subsets of distinct level sets $h^{-1}(t_1)$ and $h^{-1}(t_2)$.

Suppose then that $t_1 \le t_2$, and let $x_1, \dots, x_p$ denote the critical points of h which satisfy $t_1 \le h(x_j) \le t_2$. Let X_{t_1,t_2} denote the complement in X of the ascending and descending manifolds of the critical points $x_1, \dots, x_p$. Then the gradient flow of h defines a diffeomorphism $\rho_{t_1,t_2}\colon h^{-1}(t_1) \cap X_{t_1,t_2} \to h^{-1}(t_2) \cap X_{t_1,t_2}$.

Definition 2.6 We call ρ_{t_1,t_2} the *projection of $h^{-1}(t_1)$ to $h^{-1}(t_2)$*. Similarly, we call ρ_{t_1,t_2}^{-1} the *projection of $h^{-1}(t_2)$ to $h^{-1}(t_1)$*, which we likewise denote by ρ_{t_2,t_1}.

Note that, despite calling ρ_{t_1,t_2} the projection from $h^{-1}(t_1)$ to $h^{-1}(t_2)$, it is only defined on the complement of the ascending and descending manifolds of the critical points that sit between t_1 and t_2. These projection maps allow us to define local product structures away from the ascending and descending manifolds of the critical points of h.

2.2.3 Singular banded unlinks and closed realizing surfaces We are now able to define a closed realizing surface associated to a given singular banded *unlink*, which we define below. As above, let X be a closed, oriented 4-manifold equipped with a self-indexing Morse function $h\colon X \to [0,4]$, with exactly one index 0 critical point.

Definition 2.7 Let (L,B) be a marked singular banded link in the 3-manifold $M := h^{-1}\left(\frac{3}{2}\right)$ such that $L, B \subset X_{1/2,5/2}$. Suppose furthermore that $\rho_{3/2,1/2}(L^-)$ bounds a collection of disjoint embedded disks D_- in $h^{-1}\left(\frac{1}{2}\right)$, and that $\rho_{3/2,5/2}(L_B^+)$ bounds a collection of disjoint embedded disks D_+ in $h^{-1}\left(\frac{5}{2}\right)$. Then we say that (L,B) is a *singular banded unlink* in the manifold X.

In plain English, (L, B) is a singular banded unlink when both

(1) L^- is an unlink when viewed as a link in $h^{-1}\left(\frac{3}{2}\right)$ ("below the 2-handles"),

(2) L_B^+ is an unlink when viewed as a link in $h^{-1}\left(\frac{5}{2}\right)$ ("above the 2-handles").

Fix $\varepsilon \in \left(0, \frac{1}{2}\right)$. Given a singular banded unlink (L, B) in $M = h^{-1}\left(\frac{3}{2}\right)$, and families of disks D_+ and D_- as in Definition 2.7, we can construct an immersed surface with corners $\Sigma \subset X$ as follows:

(i) $\Sigma \cap h^{-1}(t) = \varnothing$ for $t < \frac{1}{2}$ or $t > \frac{5}{2}$.

(ii) $\Sigma \cap h^{-1}\left(\frac{1}{2}\right) = D_-$.

(iii) $\Sigma \cap h^{-1}(t) = \rho_{1/2,t}(\partial D_-)$ for $t \in \left(\frac{1}{2}, \frac{3}{2} - \varepsilon\right)$.

(iv) $\Sigma \cap h^{-1}\left(\left(\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon\right)\right)$ is a realizing surface segment in the product $h^{-1}\left(\left(\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon\right)\right) \cong M \times \left(\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon\right)$ for the singular banded link (L, B) in M.

(v) $\Sigma \cap h^{-1}(t) = \rho_{5/2,t}(\partial D_+)$ for $t \in \left(\frac{3}{2} + \varepsilon, \frac{5}{2}\right)$.

(vi) $\Sigma \cap h^{-1}\left(\frac{5}{2}\right) = D_+$.

That is, Σ consists from bottom to top of minimum disks, a realizing surface segment (which we recall has self-intersections and index 1 critical points) and maximum disks.

Note that the identification of $h^{-1}\left(\left(\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon\right)\right)$ with $M \times \left(\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon\right)$ in part (iv) above is made using the projection maps $\rho_{3/2,t} : h^{-1}\left(\frac{3}{2}\right) \to h^{-1}(t)$, which is a diffeomorphism for $t \in \left(\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon\right)$ and small ε. Under this identification, the boundary of the realizing surface segment will be precisely $\rho_{5/2,3/2+\varepsilon}(\partial D_+) \sqcup \rho_{1/2,3/2-\varepsilon}(\partial D_-)$, and hence the surface Σ constructed above will be closed.

Definition 2.8 Let $\Sigma(L, B)$ be an immersed surface in X obtained from Σ by smoothing corners. We refer to $\Sigma(L, B)$ as a (*closed*) *realizing surface* for the singular banded unlink (L, B) in X.

The surface $\Sigma(L, B)$ is an immersed surface in X with isolated, transverse self-intersections. Note that $\Sigma(L, B)$ is obtained (up to isotopy) by smoothing the result of capping off the boundary components of $\overline{\Sigma}(L, B)$ by horizontal disks, which is possible exactly when (L, B) is a singular banded *unlink*.

Proposition 2.9 *Any two realizing surfaces for the singular banded unlink (L, B) are smoothly isotopic.*

Proof We first note that choosing a different value for ε changes Σ by an isotopy through realizing surfaces. Second, by Proposition 2.5 any two choices of surface segment $\overline{\Sigma}(L, B) \subset h^{-1}\left(\left[\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon\right]\right)$ are isotopic, and this isotopy can be extended to the rest of $\Sigma \cap h^{-1}\left(\left(\frac{1}{2}, \frac{5}{2}\right)\right)$ using the projection maps ρ_{t_1,t_2}. Finally, any choice of embedded disks $\Sigma \cap h^{-1}\left(\frac{1}{2}\right)$ and $\Sigma \cap h^{-1}\left(\frac{5}{2}\right)$ are isotopic rel boundary in $h^{-1}\left(\left[0, \frac{1}{2}\right]\right)$ and $h^{-1}\left(\left[\frac{5}{2}, 4\right]\right)$, respectively, which follows from the fact that $h^{-1}\left(\left[0, \frac{1}{2}\right]\right) \cong B^4$ and $h^{-1}\left(\left[\frac{5}{2}, 4\right]\right) \cong \natural_k (S^1 \times B^3)$. □

As the realizing surface $\Sigma(L, B)$ is determined by the singular banded unlink (L, B) up to isotopy, we will often think of $\Sigma(L, B)$ as representing an isotopy class of immersed surfaces, rather than a particular representative.

2.2.4 Singular banded unlink diagrams and Kirby diagrams We now make sense of how to describe a realizing surface as in Section 2.2.3 via a Kirby diagram. If one is comfortable with these diagrams, then the contents of this subsection are clear from Definition 2.7: simply draw L and B inside a diagram for X in a natural way. We now review some basic notions about Kirby diagrams.

Let $h\colon X \to \mathbb{R}$ be a self-indexing Morse function with a unique index 0 critical point, and let n be the number of index 1 critical points of h. Fix a gradientlike vector field ∇h for h. Let $M = h^{-1}\left(\frac{3}{2}\right)$, and let L_2 be the intersection of M with the descending manifolds of the index 2 critical points of h. Perturb ∇h slightly if necessary so that this intersection is transverse, so that L_2 is a link in the 3-manifold $M \cong \#_n S^1 \times S^2$. To each component of L_2, assign the framing induced by the descending manifold of the associated index 2 critical point, so that L_2 is actually a framed link in M.

Fix an n-component unlink L_1 in S^3. Let V denote the complement of the unique (up to isotopy rel boundary) boundary-parallel disks bounded by L_1 in B^4. Then V is diffeomorphic to $\natural_n S^1 \times B^3$, and we can therefore find a diffeomorphism $\phi\colon V \to h^{-1}\left(\left[0, \frac{3}{2}\right]\right)$. By Laudenbach and Poénaru [24] and Laudenbach [23], the choice of ϕ is natural up to isotopy and moves that correspond to slides of L_1 (as a 0-framed link) in S^3. Moreover, ∂V can be naturally identified with the result of performing 0-surgery on S^3 along L_1, which we denote by $S^3_0(L_1)$. By perturbing ∇h, we may assume that $\phi^{-1}(L_2) \subset \partial V \cong S^3_0(L_1)$ is disjoint from the surgery solid tori, and hence we can think of $\phi^{-1}(L_2)$ as a link in S^3. By abuse of notation, we will also refer to this link as L_2.

Definition 2.10 Let $\mathcal{K} := (L_1, L_2)$ be a pair of disjoint links in S^3 with L_1 an unlink and L_2 framed. Suppose there is a 4-manifold X, a Morse function $h\colon X \to \mathbb{R}$ and a gradientlike vector field ∇h for h such that $h^{-1}\left(\frac{3}{2}\right)$ may be identified with $S^3_0(L_1)$ and the descending manifolds of the index 2 critical points of h meet $h^{-1}\left(\frac{3}{2}\right)$ in the framed link L_2. Then we call $\mathcal{K}$ a *Kirby diagram of X corresponding to* $(h, \nabla h)$.

Remark 2.11 In [28], the third author and Naylor showed that a smooth, closed, *nonorientable* 4-manifold X^4 is also determined up to diffeomorphism by (framed) attaching regions of 0-, 1- and 2-handles. If desired, one could thus make sense of diagrams of closed (immersed) surfaces in Kirby diagrams of nonorientable 4-manifolds. We choose not to pursue this explicitly in this paper for sake of simplicity.

Remark 2.12 Given h and ∇h, a Kirby diagram $\mathcal{K}$ corresponding to $(h, \nabla h)$ is well defined up to isotopy and slides over L_1 as long as there is no flowline of ∇h between two index 2 critical points of h. That is, generically we expect h and ∇h to determine a Kirby diagram.

Conversely, given $\mathcal{K}$, the triple $(X, h, \nabla h)$ is determined up to diffeomorphism.

Let $E(\mathcal{K})$ denote the complement $S^3 \setminus \nu(\mathcal{K})$ of a small tubular neighborhood of the links L_1 and L_2 that form a Kirby diagram $\mathcal{K}$. Then we may think of a link $L \subset E(\mathcal{K})$ as describing a link in $h^{-1}(t)$ for any $t \in (0, 3)$ via the projection map $\rho_{3/2,t}$.

Definition 2.13 A *singular banded unlink diagram* in the Kirby diagram $\mathcal{K} = (L_1, L_2)$ is a triple $(\mathcal{K}, L, B)$, where $L \subset E(\mathcal{K})$ is a marked singular link and $B \subset E(\mathcal{K})$ is a finite family of disjoint bands for L, such that L^- bounds a family of pairwise disjoint embedded disks in $h^{-1}\left(\frac{1}{2}\right)$, and L_B^+ bounds a family of pairwise disjoint embedded disks in $h^{-1}\left(\frac{5}{2}\right)$.

By comparing Definition 2.13 to Definition 2.7, we see that a singular banded unlink diagram describes an immersed realizing surface, as follows. We first note that we can identify $E(\mathcal{K})$ with a subset of $h^{-1}\left(\frac{3}{2}\right)$ in a natural way (ie via ∇h). Since the banded link $L \cup B$ is disjoint from L_1, it can be identified with a subset of $h^{-1}\left(\frac{3}{2}\right)$, which we denote by $L' \cup B'$. This subset avoids the descending manifolds of the index 2 critical points of h.

Since L'^- is disjoint from L_1, we can isotope it vertically downwards via the projection map $\rho_{3/2,t}$ from $h^{-1}\left(\frac{3}{2}\right)$ to $h^{-1}\left(\frac{1}{2}\right)$, where it can be capped off by a family of disjoint embedded disks in $h^{-1}\left(\frac{1}{2}\right)$. Similarly, we can extend the surgered link $L'^+_{B'}$ vertically upwards from $h^{-1}\left(\frac{3}{2}\right)$ to $h^{-1}\left(\frac{5}{2}\right)$, where it can be capped off by disks. As these families of disks are unique up to isotopy rel boundary, the surface we obtain in this way from the banded unlink diagram $(\mathcal{K}, L, B)$ is well defined up to isotopy. (See also Proposition 2.9.) We denote this surface by $\Sigma(\mathcal{K}, L, B)$.

Definition 2.14 We say that $\Sigma(\mathcal{K}, L, B)$ is a *realizing surface for* $(\mathcal{K}, L, B)$, or that $(\mathcal{K}, L, B)$ *describes the surface* $\Sigma(\mathcal{K}, L, B)$.

Definition 2.15 If Σ is a realizing surface of a singular banded unlink diagram $(\mathcal{K}, L, B)$, then we say that $(\mathcal{K}, L, B)$ is a *singular banded unlink diagram* for Σ, and we write $\mathcal{D}(\Sigma) := (\mathcal{K}, L, B)$. (In practice, we might drop the word "singular", since this will be clear when Σ is immersed.) Note that Σ determines $\mathcal{D}(\Sigma)$ uniquely up to isotopy, assuming that Σ is a realizing surface for some diagram.

Definition 2.16 Let Σ be a subset of X. Then we say that $h|_\Sigma$ is *Morse* if there is a surface F and an immersion $f : F \to X$ such that $\Sigma = f(F)$, and such that $h \circ f$ is a Morse function on F. An index k critical point of $h|_\Sigma$ is a point of the form $f(p)$, where p is an index k critical point of $h \circ f$.

Lemma 2.17 *Let X be a closed 4-manifold, and $\mathcal{K}$ a Kirby diagram for X. Then any immersed surface Σ in X is ambient isotopic to a realizing surface $\Sigma(\mathcal{K}, L, B)$ for some singular banded unlink diagram $(\mathcal{K}, L, B)$.*

Proof After a small ambient isotopy we may assume that $h|_\Sigma$ is Morse. Isotope all of the maxima of Σ vertically upward into $h^{-1}\left(\left(\frac{5}{2}, 4\right)\right)$ (generically, maxima of Σ do not lie in the descending manifolds of index 1 or 2 critical points of h). Similarly isotope the minima of Σ vertically downward into $h^{-1}\left(\left(0, \frac{3}{2}\right)\right)$. Isotope all of the index 1 critical points of $h|_\Sigma$ vertically into $h^{-1}\left(\left(\frac{3}{2}, \frac{5}{2}\right)\right)$ (again, index 1 critical points of $h|_\Sigma$ generically do not lie in the ascending manifolds of index 3 critical points or the descending manifolds of index 1 critical points). Finally, isotope the self-intersections of Σ to lie in $h^{-1}\left(\left(\frac{3}{2}, \frac{5}{2}\right)\right)$ in such a way that they do not coincide with index 1 critical points of $h|_\Sigma$.

Now flatten Σ as in [21]. In words, notice that h and $-\nabla h$, when restricted to Σ, generically induce a CW decomposition of Σ in which 0-cells are the index 0 critical points of $h|_\Sigma$, one point in the interior of each 1-cell is an index 1 critical point of $h|_\Sigma$, and one point in the interior of each 2-cell is an index 2 critical point of $h|_\Sigma$. Perturb, if necessary, so that self-intersections of Σ all lie outside the descending and ascending manifolds in Σ of index 1 critical points of $h|_\Sigma$.

The family of gradient flowlines of ∇h in X which originate on the ascending manifolds of an index 1 critical point of $h|_\Sigma$ is 2-dimensional, as is the family of gradient flowlines of $-\nabla h$ in X which originate on the descending manifolds of an index 1 critical point of $h|_\Sigma$. Thus, we may generically take them all to be disjoint and also disjoint from ascending and descending manifolds of index 2 points of h. (We discuss this more in Section 2.3. While this condition is generic, it is not natural — this lack of generality precisely corresponding to the singular band moves of Theorem 2.39.)

Fix $\varepsilon > 0$, and let $L^- = \Sigma \cap h^{-1}\left(\frac{3}{2} - \varepsilon\right)$. Isotope Σ near height $\frac{3}{2}$ so that $\Sigma \cap h^{-1}\left(\left[\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon\right]\right)$ is of the form $L^- \times \left[\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon\right]$. A neighborhood of each 1-cell of Σ can be isotoped via $-\nabla h$ to a band in $h^{-1}\left(\frac{3}{2}\right)$ that is attached to a parallel copy of L^-. Let B be the collection of all such bands (one for each 1-cell in Σ).

Now isotope Σ near each self-intersection s of Σ as in Figure 2, right, ie make one of the sheets of Σ at s include a small region that is horizontal with respect to h and which contains s. Isotope this sheet via $-\nabla h$ to push this horizontal region to $h^{-1}\left(\frac{3}{2}\right)$, where it can be interpreted as a marked fat vertex as in Figure 2, left. Repeating for every self-intersection of Σ, we obtain a marked singular banded link L in $h^{-1}\left(\frac{3}{2}\right)$ whose negative resolution is L^-.

Now Σ intersects regions of X in the following ways:

$$\begin{aligned}
h^{-1}\left(\left[0, \tfrac{3}{2} - \varepsilon\right]\right) &\quad \text{in boundary parallel disks with boundary } L^-,\\
h^{-1}\left(\left[\tfrac{3}{2} - \varepsilon, \tfrac{3}{2} + \varepsilon\right]\right) &\quad \text{in the realizing surface segment for } (L, B),\\
h^{-1}\left(\left[\tfrac{3}{2} + \varepsilon, \tfrac{5}{2}\right]\right) &\quad \text{in an embedded surface on which } h \text{ has no critical points},\\
h^{-1}\left(\left[\tfrac{5}{2}, 4\right]\right) &\quad \text{in boundary parallel disks with boundary } L^+_B.
\end{aligned}$$

We conclude that Σ is isotopic to $\Sigma(\mathcal{H}, L, B)$. □

Remark 2.18 In the proof of Lemma 2.17, we made several references to genericity. That is, we made several choices of how to perturb Σ in order to obtain $(\mathcal{H}, L, B)$. It may be helpful to imagine the lower-dimensional analogue of knots in S^3: every knot in S^3 is isotopic to one that projects to a knot diagram. However, not every knot in S^3 actually projects to a knot diagram. An arbitrary knot may, for example, have a projection that includes a cusp, self-tangency or triple point. These conditions are not generic and can be corrected by a slight perturbation, but therein involves a choice that can yield diagrams differing by a Reidemeister move (RI, RII or RIII, respectively). There are, of course, even "worse" conditions, such as a knot whose projection involves a quadruple intersection. However, this condition is even "less" generic, by which we mean:

- A generic knot in S^3 admits a projection with no triple points.
- A generic 1-parameter family of smoothly varying knots in S^3 admits projections with finitely many triple points but no quadruple points.
- A generic 2-parameter family of smoothly varying knots in S^3 admits projections with 1-dimensional families of triple points and finitely many quadruple points.

Thus, in a 1-parameter family of knots (ie a knot isotopy), we expect to obtain diagrams that differ by an RIII move (and similarly for RI and RII), but need never consider moves involving quadruple intersections.

Moving back to the 4-dimensional world, in order to understand to what extent a singular banded unlink diagram is well defined up to isotopy of an immersed surface, we must understand which nongeneric behaviors of projections we expect to see a finite number of times in a 1-dimensional family of immersed surfaces. We discuss this more formally in Sections 2.3 and 2.4.

2.2.5 Singular band moves The Kirby diagram $\mathcal{K}$ only determines the described 4-manifold X up to diffeomorphism. Therefore, $(\mathcal{K}, L, B)$ only determines the pair $(X, \Sigma(\mathcal{K}, L, B))$ up to diffeomorphism; it does not make sense to say that $(\mathcal{K}, L, B)$ determines $\Sigma(\mathcal{K}, L, B)$ up to isotopy. However, if we have already identified X with the manifold described by $\mathcal{K}$, then we can consider $\Sigma(\mathcal{K}, L, B)$ up to isotopy. In particular, given another singular banded unlink diagram $(\mathcal{K}, L', B')$ in the same Kirby diagram $\mathcal{K}$, there is a natural (up to isotopy) diffeomorphism between the 4-manifolds containing $\Sigma(\mathcal{K}, L', B')$ and $\Sigma(\mathcal{K}, L, B)$. Therefore, it *does* make sense to ask whether $\Sigma(\mathcal{K}, L, B')$ and $\Sigma(\mathcal{K}, L, B)$ are ambiently isotopic, regularly homotopic or homotopic in X. In this section, we define moves of singular banded unlink diagrams that describe ambient isotopies of immersed surfaces; in Sections 2.3 and 2.4, we show that indeed these moves are sufficient.

Definition 2.19 Let $\mathcal{D} := (\mathcal{K}, L, B)$ and $\mathcal{D}' := (\mathcal{K}, L', B')$ be singular banded unlink diagrams. We say that $\mathcal{D}'$ *is related to* $\mathcal{D}$ *by singular band moves* if $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a finite sequence of the moves in Figures 3 and 4, which we call *singular band moves*. (This relationship is clearly symmetric.)

Specifically, the singular band moves (illustrated in Figures 3 and 4) are

(i) isotopy in $E(\mathcal{K})$,

(ii) cup/cap moves,

(iii) band slides,

(iv) band swims,

(v) slides of bands over components of L_2 (band/2-handle slide),

(vi) swims of bands about L_2 (band/2-handle swim),

(vii) slides of unlinks and bands over L_1,

(viii) sliding a vertex over a band (intersection/band slide),

(ix) passing a vertex past the edge of a band (intersection/band pass).

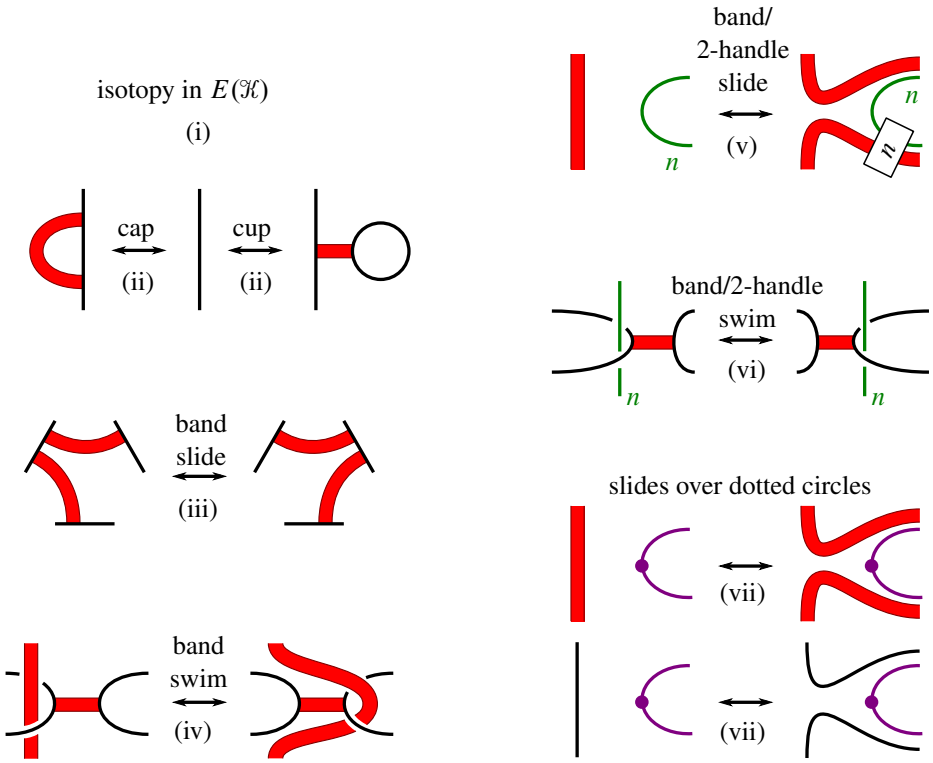


Figure 3: The band moves that do not involve the self-intersections of the described surface.

We may refer to moves (i)–(vii) (illustrated in Figure 3) as *band moves* (omitting the word "singular") since they do not involve the self-intersections of L. The remaining moves are illustrated in Figure 4.

Exercise 2.20 If $\mathcal{D}$ and $\mathcal{D}'$ are related by singular band moves, then $\Sigma(\mathcal{D})$ and $\Sigma(\mathcal{D}')$ are ambiently isotopic.

In the future, we will refer to moves by name rather than number to avoid confusion.

In Figures 5–10, we illustrate some other useful moves on singular banded unlink diagrams that are achievable by a combination of singular band moves. We call these moves $\star$ (Figure 5), the intersection/band swim (Figure 6), the upside-down intersection/band swim (Figure 7), the intersection pass (Figure 8), the intersection swim (Figures 9 and 10), the intersection/2-handle slide (Figure 11) and the intersection/2-handle swim (Figure 12).

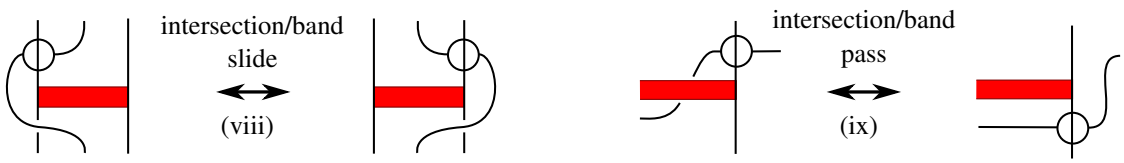


Figure 4: The singular band moves that involve self-intersections of the described surface.

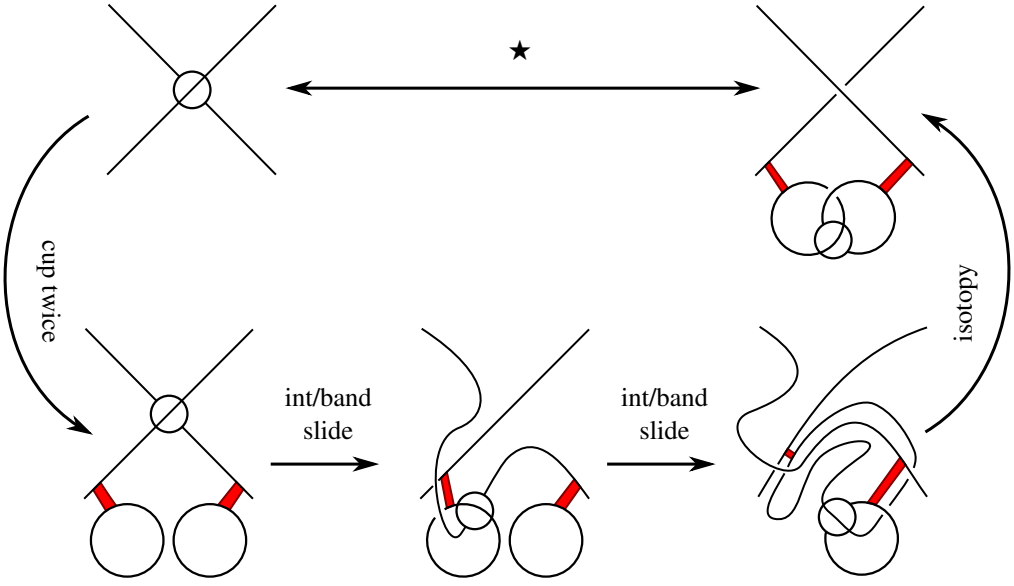


Figure 5: The $\star$ move moves a vertex onto two new unlink components (or the reverse). In Figures 7, 9 and 10, we see that the $\star$-move can be used (in conjunction with singular band moves) to achieve other seemingly natural moves.

In an earlier version of this paper, we included the intersection/band swim of Figure 6 as one of the singular band moves (as move (x)). Jablonowski [15] noticed that this move is redundant, so we have modified the list accordingly.

Remark 2.21 While the length of the list in Definition 2.19 may seem unwieldy, there is a general principle at play: singular band moves allow us to isotope a singular banded unlink (L, B) within $\mathcal{H}$; or to

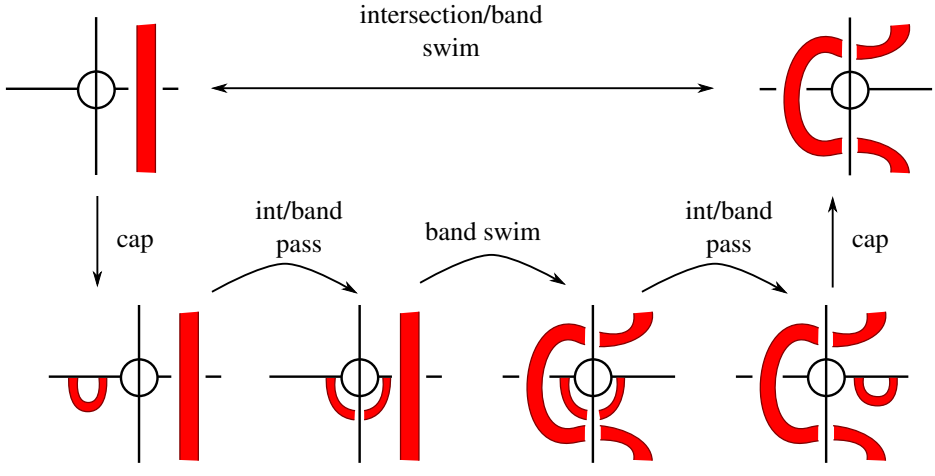


Figure 6: We can achieve an intersection/band swim by performing singular band moves. This sequence of moves was observed by Jablonowski [15].

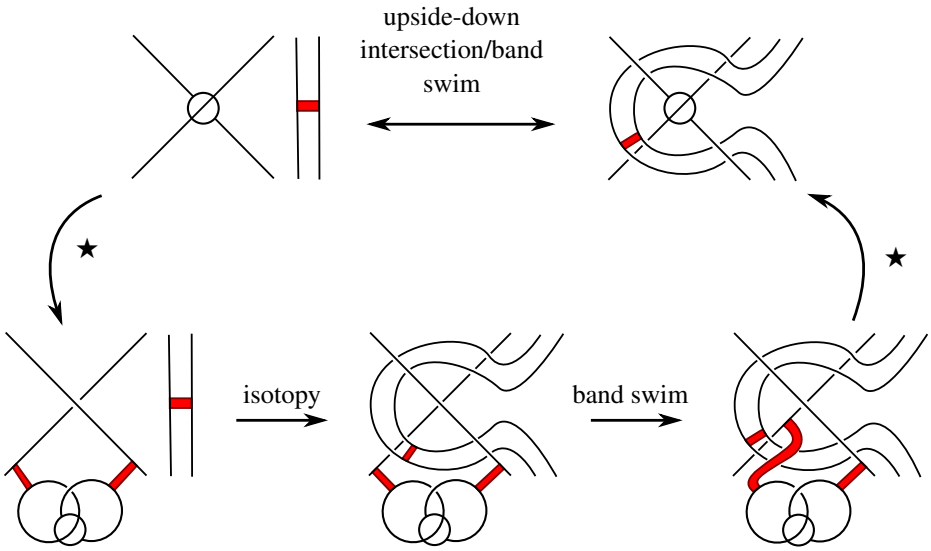


Figure 7: We can achieve the upside-down intersection/band swim by performing $\star$ and singular band moves.

push any vertex in L or band in B slightly into the past or future, do further isotopy there, and then push the vertex or band back into the present. In practice, when using these diagrams, we do not explicitly break a described isotopy into a sequence of the moves of Definition 2.19, just as how in practice one does not typically break an isotopy of a knot explicitly into a sequence of Reidemeister moves.

2.3 Ascending/descending manifolds and 0- and 1-standard surfaces

So far, we have only used singular banded unlink diagrams to describe realizing surfaces, which are incredibly nongeneric. One goal of this paper is to use singular banded unlink diagrams to describe any

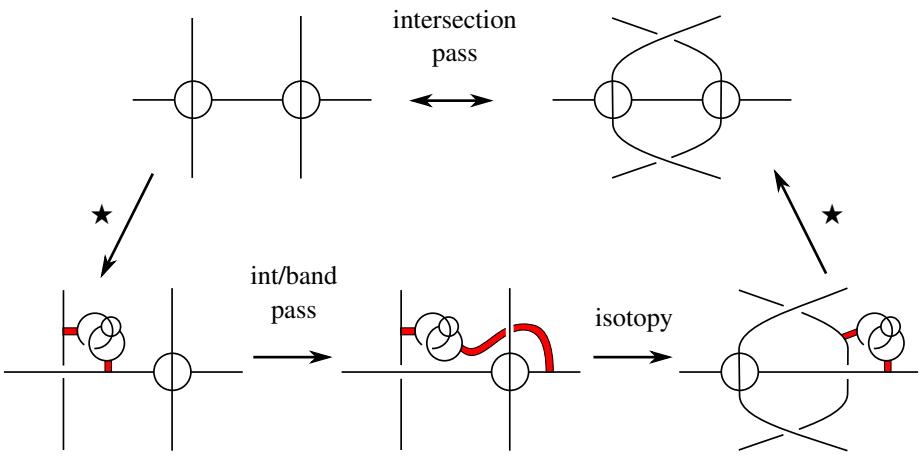


Figure 8: We can achieve an intersection pass by performing $\star$ and singular band moves.

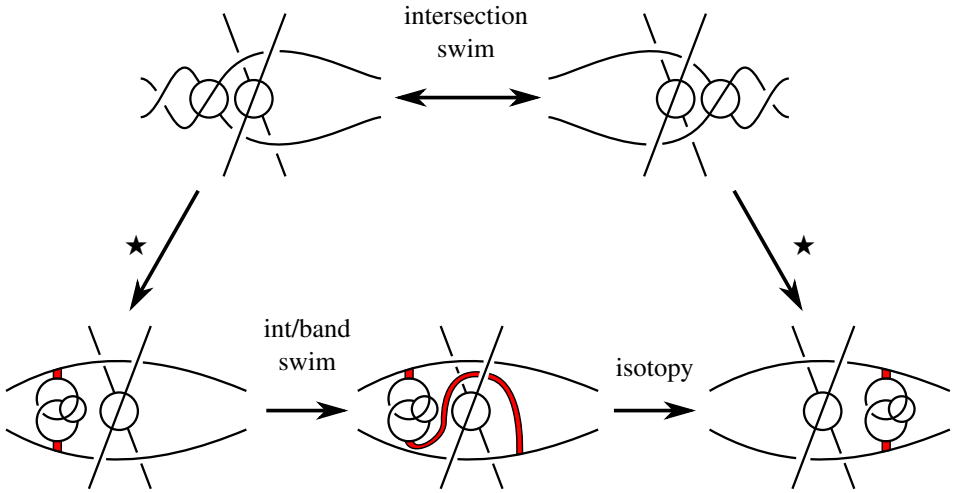


Figure 9: We can achieve an intersection swim by performing $\star$ and singular band moves.

self-transverse immersed surface Σ. In Lemma 2.17, we showed that any such Σ is isotopic to a realizing surface. However, it is not obvious that any two realizing surfaces isotopic to Σ have singular banded unlink diagrams that are related by singular band moves. In order to prove this, we must first restrict ourselves to understanding surfaces that intersect the ascending and descending manifolds of critical points of h in prescribed ways, but yet are still more generic than realizing surfaces.

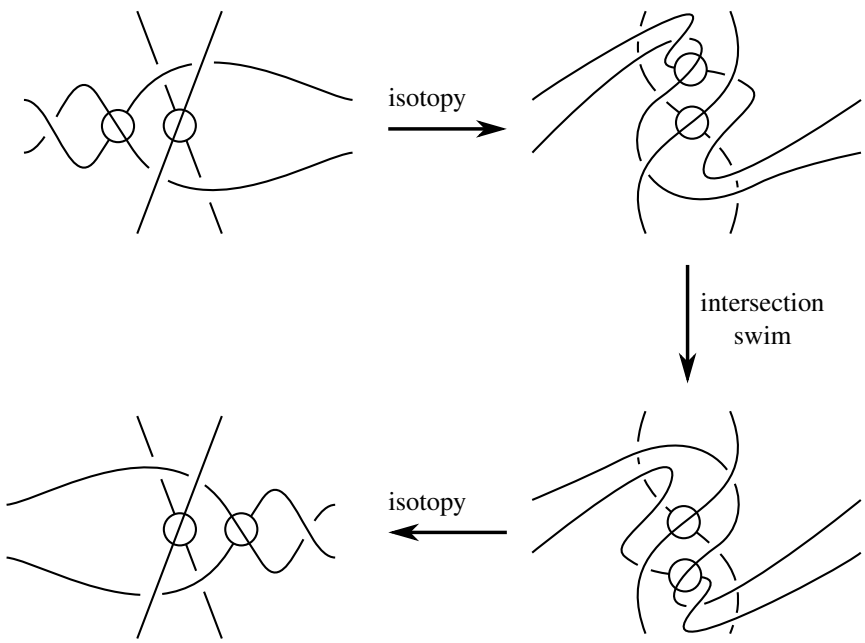


Figure 10: We achieve an alternative version of the intersection swim of Figure 9, in which one marking and one crossing are changed via isotopy and intersection swim.

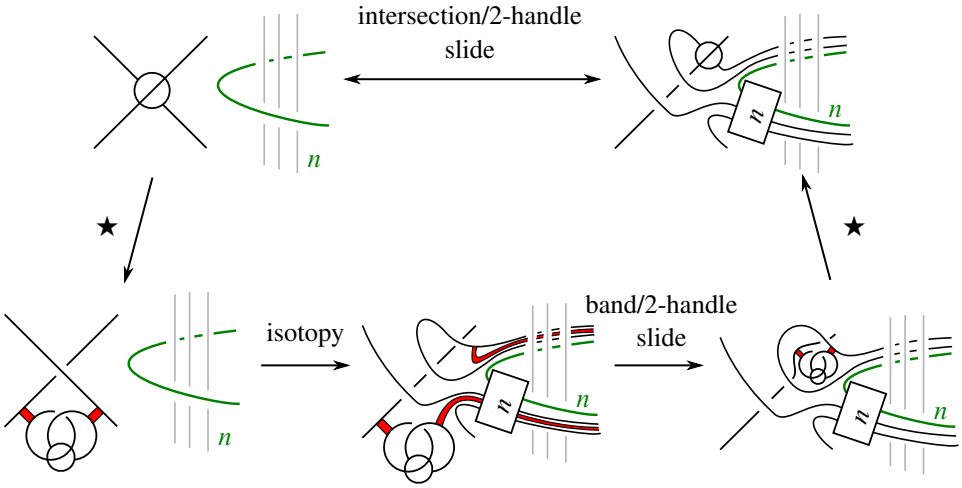


Figure 11: We achieve an intersection/2-handle slide by performing $\star$ and singular band moves.

We will now consider not only the ascending/descending manifolds of critical points of h, but also the ascending and descending manifolds of critical points of the restricted Morse function $h|_\Sigma$. From now on, fix a gradientlike vector field ∇h for the Morse function $h\colon X \to \mathbb{R}$, and let Z denote $X^4 \setminus \nu(\Sigma)$.

In order to obtain a gradientlike vector field on Σ itself, we choose a splitting $TX|_\Sigma = T\Sigma \oplus N$ and let $\mathrm{proj}_{T\Sigma}\colon TX|_\Sigma \to T\Sigma$ be the associated bundle projection. We can assume that the splitting is chosen so

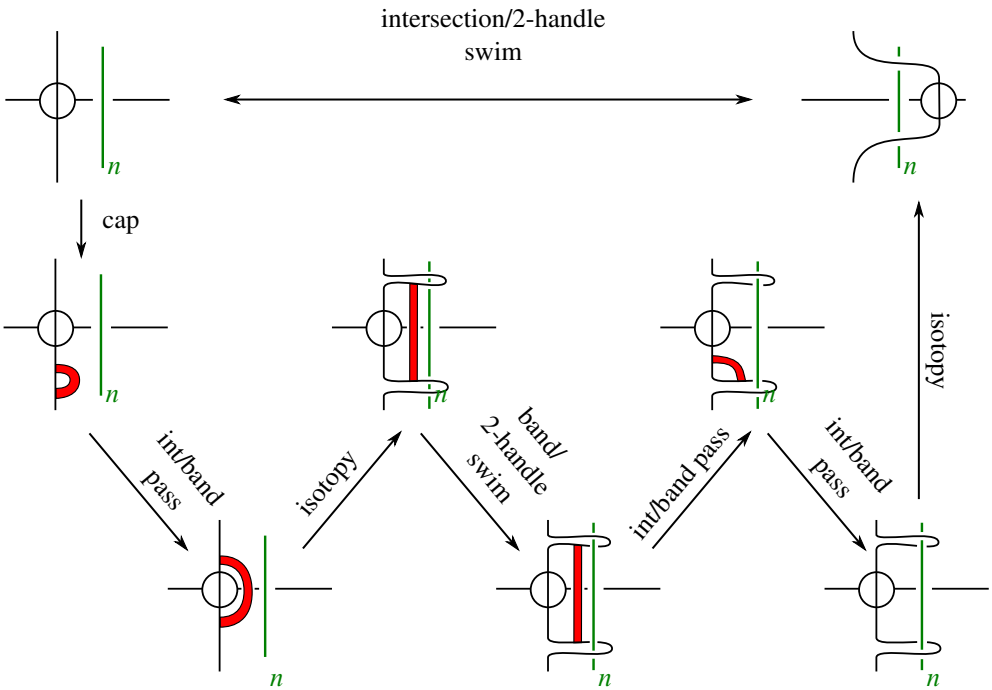


Figure 12: We achieve an intersection/2-handle swim by performing singular band moves.

that $\mathrm{proj}_{T\Sigma}(\nabla h)|_\Sigma$ is a gradientlike vector field for $h|_\Sigma$ on Σ, which we denote by $\nabla(h|_\Sigma)$. Note that this is actually *not* a vector field on the immersed surface Σ (although we could pull it back to a vector field on the abstract surface F), since there are two associated vectors at each point of self-intersection of Σ (the projections of ∇h onto the tangent planes of each local sheet); however, we think that the language "gradientlike vector field" is not confusing in this context. The vector field $\nabla(h|_\Sigma)$ is *not* canonically determined by h, ∇h and Σ, since to obtain it we have to choose a splitting of TX_Σ.

In what follows, we will often refer to the ascending or descending manifolds of critical points of $h|_\Sigma$ or of self-intersections of Σ. Unless we specify otherwise, assume that this always refers to the corresponding manifolds in X with respect to ∇h as defined above, rather than ascending or descending manifolds in Σ with respect to $\nabla(h|_\Sigma)$. These points are generally not critical points of h, but their ascending and descending manifolds can be studied as usual.

2.3.1 1-standard surfaces Suppose that Σ is a self-transverse immersed surface in X. The following definition will be important as we consider 1-parameter families of immersed surfaces:

Definition 2.22 We say that Σ is 1-*standard* if the following are true:

(1) The surface Σ is disjoint from the critical points of h.

(2) The restriction $h|_\Sigma$ is Morse except for possibly at most one birth/death degeneracy, ie a point of Σ about which $h|_\Sigma$ can be represented as $h|_\Sigma(x, y) = x^2 - y^3$ in some local coordinates on Σ.

(3) For $k \geq n+1$, the descending manifolds (with respect to ∇h) of index n critical points of h and index $n-1$ critical points of $h|_\Sigma$ are disjoint from the ascending manifolds of index k critical points of h and index $k-1$ critical points of $h|_\Sigma$. Moreover, self-intersections of Σ are disjoint from the ascending manifolds of index 3 critical points of h and descending manifolds of index 1 critical points of h. In other words, we ask for n-dimensional descending manifolds to be disjoint from $(4-n-1)$-dimensional ascending manifolds.

Remark 2.23 Definition 2.22 is essentially a list of all ascending/descending manifold pairs that we expect to be disjoint in a 1-parameter family of immersed surfaces by dimensional considerations, as explained in Proposition 2.24. This motivates the name "1-standard".

Proposition 2.24 *Let Σ_t be an isotopy between 1-standard surfaces Σ_0 and Σ_1. After an arbitrarily small perturbation of the isotopy Σ_t, we can assume that Σ_t is 1-standard for all t.*

Proof We prove that, after a small perturbation, Σ_t satisfies each property of Definition 2.22 for all t.

(1) The critical point set of h in $X \times I$ is 1-dimensional, while the isotopy Σ_t in $X \times I$ is 3-dimensional. Generically, we do not expect Σ_t to intersect a critical point of h for any t.

(2) This follows from Cerf's filtration on the space of surfaces (see eg [9, Chapter 1, Section 2]). This is a filtration on the space $C(F)$ of all smooth maps $F \to X^4$ for F a surface. The codimension-0

stratum consists of all maps $f : F \to X^4$ with $h|_{f(F)}$ Morse with critical points at distinct heights. The codimension-1 stratum includes f if either of the following is true:

- The restriction $h|_{f(F)}$ is Morse with exactly two critical points at the same height, but all other critical points sit at distinct heights.
- The restriction $h|_{f(F)}$ is Morse except for one birth or death degeneracy. This degeneracy and all critical points are at distinct heights.

Suppose Σ_0 has n points of self-intersection. Fix $2n$ points $x_1, y_2, \ldots, x_n, y_n$ in F and choose $f_t : F \to X$ so that $f_t(F) = \Sigma_t$ and $f_t(x_i) = f_t(y_i)$ for all i and t. Now a small perturbation of the path f_t from f_0 to f_1 in $C(F)$ yields a path g_t that is completely contained in the codimension-0 and codimension-1 strata of Cerf's filtration with $g_0 = f_0, g_1 = f_1$. Since g_t lies in these strata, $g_t(F)$ has property (2) of Definition 2.22 for all t. Moreover, if the perturbation is sufficiently small, we may assume that $g_t(F)$ is an immersed surface with n transverse double points for all t, all of which are contained in a fixed small tubular neighborhood of Σ_t. (Recall that smooth or PL self-transversely immersed surfaces in 4-manifolds have tubular neighborhoods; use local coordinates to choose a tubular neighborhood near each of the finitely many self-intersections and then extend over the whole surface using the tubular neighborhood theorem.)

While g_t is a homotopy from g_0 to g_1, we may view its image as an isotopy between the singular submanifolds Σ_0 and Σ_1 in X. We must now check that this isotopy extends to an ambient isotopy of X. That is, while we have argued that we may perturb f_t to achieve property (2), we must explain why this perturbation may be achieved by perturbing the ambient isotopy from Σ_0 to Σ_1, since there is a distinction between the immersions f_t and their images $f_t(F) = \Sigma_t$. This is relatively standard (and indeed stated without proof in eg [7]): choose small disjoint closed disks D_{x_i}, D_{y_i} $(i = 1, \ldots, n)$ in F, centered at x_i and y_i, respectively. We can fix a family of coordinates on a closed tubular neighborhood of $g_t(F)$ near the self-intersections such that, centered about $g_t(x_i) = g_t(y_i)$, we have a closed ball $B_i = g_t(D_{x_i}) \times g_t(D_{y_i})$ intersecting $g_t(F)$ in

$$\frac{(g_t(D_{x_i}) \times \{0\}) \cup (\{0\} \times g_t(D_{y_i}))}{(g_t(x_i) \times 0) \sim (0 \times g_t(y_i))}.$$

Now we may extend the isotopy $\Sigma_0 \to \Sigma_1$ that is the image of g_t to an isotopy ϕ_t of $\Sigma_0 \cup B_1 \cup \cdots \cup B_n$ by specifying that $\phi_t(g_0(a), g_0(b)) = (g_t(a), g_t(b))$ for all $a \in D_{x_i}, b \in D_{y_i}$, since $B_i = g_0(D_{x_i}) \times g_0(D_{y_i})$. Then $\phi_t(B_i) = g_t(D_{x_i}) \times g_t(D_{y_i})$. Since the B_i are balls, the isotopy $\phi_t|_{\bigcup_i B_i}$ extends to an ambient isotopy ψ_t of X. The composition $\psi_t^{-1}\phi_t$ then fixes B_i pointwise for each i.

Now, since $\Sigma_0 \cap (X \setminus \mathrm{int}(B_1 \sqcup \cdots \sqcup B_n))$ is an embedded submanifold (whose boundary is not tangent to the boundary of $X \setminus \mathrm{int}(B_1 \cup \cdots \cup B_n)$; ie $\Sigma_0 \cap (X \setminus \mathrm{int}(B_1 \sqcup \cdots \sqcup B_n))$ is neat in the sense of [12]) whose boundary is fixed by $\psi_t^{-1}\phi_t$, we may use usual isotopy extension to extend $\psi_t^{-1}\phi_t$ to an ambient isotopy. Then, since ψ_t^{-1} is an ambient isotopy (and hence a diffeotopy starting at the identity map), we conclude that ϕ_t extends to a diffeotopy starting at the identity map, ie an ambient isotopy.

We conclude that our original ambient isotopy from Σ_0 to Σ_1 may be perturbed to another ambient isotopy of Σ_0 to Σ_1 which satisfies property (2) of 1-standardness at all times.

(3) Note that both ascending and descending manifolds are parallel to ∇h, so, rather than counting transverse intersections, we count the dimension of the space of line intersections (parallel to ∇h) of these ascending and descending manifolds. (In other words, we count the dimension of the moduli space of unparametrized flowlines of $-\nabla h$ from one critical or intersection point to another.) An n-dimensional descending manifold and a $(4-k)$-dimensional ascending manifold thus have expected dimension

$$(n-1)+((4-k)-1)-(4-1)=n-k-1$$

as a space of lines. For $k \geq n+1$, this expected dimension is at most -2, so we conclude that we may perturb Σ_t (which by the previous item we see may be obtained by perturbing a path of immersions f_t in $C(F)$) to achieve property (3). □

2.3.2 0-standard surfaces In Remark 2.23, we explained that the definition of 1-standardness comes from studying generic 1-parameter families. That is, the conditions in Definition 2.22 are generically true for 1-parameter families of surfaces. We now define a slightly more restrictive condition on the surfaces we study, which we expect to be violated a finite number of times in a generic 1-parameter family.

Definition 2.25 We say that Σ is 0-*standard* if it is 1-standard and the following are true:

(1) The restriction $h|_\Sigma$ is Morse.

(2) Whenever p and q are either index 2 critical points of h, index 1 critical points of $h|_\Sigma$, or self-intersections of Σ (not necessarily of the same type), and $p \neq q$, the descending manifold of p is disjoint from the ascending manifold of q. In short: 2-dimensional descending manifolds are disjoint from 2-dimensional ascending manifolds.

Remark 2.26 Roughly speaking, a surface Σ is 0-standard if its index 1 critical points (viewed as bands) and self-intersections do not lie above each other, or above or below any index 2 critical points of h. This is all with respect to ∇h; we are *not* discussing $\nabla(h|_\Sigma)$. These forbidden conditions, allowed in a 1-standard surface, would cause a projection of Σ to a singular banded unlink diagram to not be well defined, motivating the cup/cap moves, band swims, band/2-handle slides and swims. Most of the other singular band moves are related to the choice of $\nabla(h|_\Sigma)$ (specifically the band slide, intersection/band slide and pass, and intersection pass). Isotopy in $E(\mathcal{H})$ and slides over L_1 correspond to horizontal isotopy.

Proposition 2.27 *Let Σ_t be an isotopy between 0-standard surfaces Σ_0 and Σ_1. After an arbitrarily small perturbation of the isotopy Σ_t, it is true that Σ_t is 1-standard for all t, and 0-standard for all but finitely many t.*

Proof It follows from Proposition 2.24 that 1-parameter families Σ_t of surfaces are generically 1-standard for all t. We now consider the conditions of Definition 2.25 separately.

(1) This is well known, due to Cerf (see eg [9, Chapter 1, Section 2]).

(2) A pair of complementary-dimension descending and ascending manifolds meet with expected dimension -1 (as a space of lines parallel to ∇h). Therefore, property (2) is generically true at all but finitely many times during a 1-parameter family of surfaces. □

Proposition 2.28 *Suppose Σ is 0-standard. Fix $\nabla(h|_\Sigma)$ with the property that, for p and q distinct index 1 points of $h|_\Sigma$ or self-intersections of Σ, the descending manifold of p with respect to $\nabla(h|_\Sigma)$ is disjoint from the ascending manifold of q with respect to $\nabla(h|_\Sigma)$. Then there is a singular banded unlink diagram $\mathcal{D}$ determined by Σ, ∇h and $\nabla(h|_\Sigma)$ up to isotopy and slides over the 1-handle circles L_1.*

Proof Since Σ is 0-standard (and hence 1-standard), we may vertically isotope Σ so that the minima of $h|_\Sigma$ lie below $h^{-1}\left(\frac{3}{2}\right)$, the maxima of $h|_\Sigma$ lie above $h^{-1}\left(\frac{5}{2}\right)$, and the self-intersections/bands of Σ lie in $h^{-1}\left(\left(\frac{3}{2}, \frac{5}{2}\right)\right)$.

By assumption, the descending manifolds (using $\nabla(h|_\Sigma)$) of index 1 critical points of $h|_\Sigma$ end at index 0 points of $h|_\Sigma$ without meeting any index 1 points or self-intersections of Σ. Similarly, flowlines of $-\nabla(h|_\Sigma)$ originating at self-intersections of Σ also end at index 0 points of $h|_\Sigma$ without meeting any other index 1 critical points or self-intersections of Σ.

Now let S be the 1-skeleton of Σ determined by $\nabla(h|_\Sigma)$, ie the 1-complex with

(1) 0-cells at index 0 points of $h|_\Sigma$,

(2) 1-cells along the descending manifolds of index 1 critical point of $h|_\Sigma$,

(3) additional 1-cells consisting of pairs of flowlines of $-\nabla(h|_\Sigma)$ glued together at self-intersections of Σ.

Isotope Σ vertically so that the index 1 critical points of $h|_\Sigma$ and self-intersections of Σ lie disjointly in $h^{-1}\left(\frac{3}{2}\right)$. (Here we are implicitly using the fact that, since Σ is 0-standard, these points do not lie directly above one another nor above index 2 critical points of h.) Flatten Σ near $h^{-1}\left(\frac{3}{2}\right)$ to turn index 1 points of $h|_\Sigma$ into bands whose cores are contained in 1-cells of S.

Since Σ is 0-standard, the bands and self-intersections of $\Sigma \cap h^{-1}\left(\frac{3}{2}\right)$ are disjoint from the descending manifolds of index 2 critical points of h, ie they are disjoint from the attaching circles L_2 of the 2-handles in $\mathcal{H}$.

Then $\Sigma \cap h^{-1}\left(\frac{3}{2}\right)$ is a singular banded link (L, B), where L^- is isotopic to $\Sigma \cap h^{-1}\left(\frac{3}{2} - \varepsilon\right)$, and L_B^+ is isotopic to $\Sigma \cap h^{-1}\left(\frac{3}{2} + \varepsilon\right)$. We conclude that (L, B) is well defined up to isotopy in $h^{-1}\left(\frac{3}{2}\right) \setminus$ (descending manifolds of index 2 critical points of h). Therefore, $(\mathcal{H}, L, B)$ is well defined up to slides of L and B over the dotted circles L_1 of $\mathcal{H}$. □

Corollary 2.29 *Let Σ_0 and Σ_1 be 0-standard surfaces. Suppose there is an isotopy Σ_t from Σ_0 to Σ_1 that is 0-standard for all t, with $\nabla(h|_{\Sigma_1})$ obtained from $\nabla(h|_{\Sigma_0})$ by the isotopy-induced map on $T\Sigma$. Then the singular banded unlink diagrams $\mathcal{D}_0$ and $\mathcal{D}_1$ for $\mathcal{H}_0$ and $\mathcal{H}_1$ produced by Proposition 2.28 are related by isotopy in $E(\mathcal{H})$ and slides over L_1.*

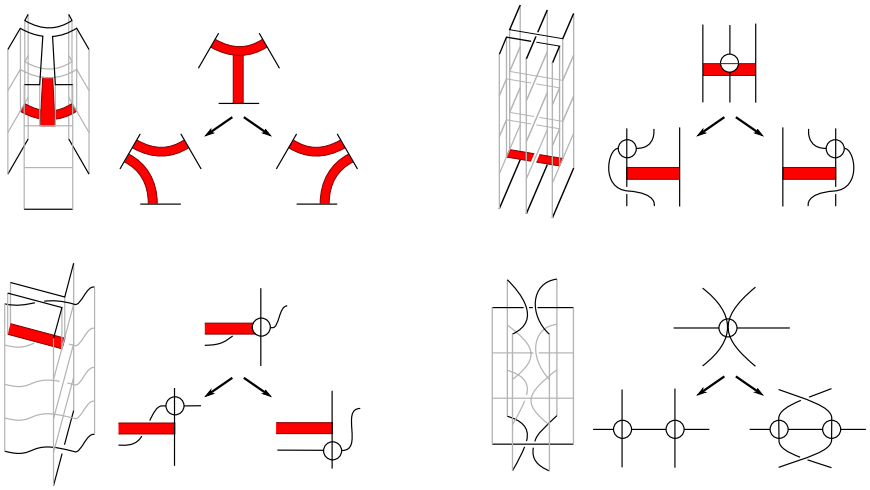


Figure 13: The cases of Proposition 2.30. At the left of each quadrant we draw a local model about the flowline of $-\nabla(h|_{\Sigma_{1/2}})$ that causes Proposition 2.28 to not apply. At the top right of each quadrant, we draw a schematic of the projection of $\Sigma_{1/2}$ to $E(\mathcal{H})$, where two bands, two self-intersections, or one of each coincide. We draw arrows to indicate the two diagrams that arise if we perturb $\Sigma_{1/2}$ to be 0-standard.

We can improve Proposition 2.28 by considering the difference between two choices for $\nabla(h|_\Sigma)$. First note that, if V_0 and V_1 are two such vector fields, then by considering the expected dimension of the space of flowlines between critical points of a Morse function on a surface, we find that V_0 and V_1 are isotopic through a sequence V_t of gradientlike vector fields for $\nabla(h|_\Sigma)$ with the property that, for all but finitely many t, V_t satisfies the conditions of Proposition 2.28. We can take the exceptional $V_{t_1}, \dots, V_{t_n}$ to each satisfy the conditions of Proposition 2.28 except for one disallowed flowline from an index 1 point or self-intersection to another (not necessarily the same type).

Proposition 2.30 *Suppose V_t satisfies the conditions of Proposition 2.28 except for $t \neq \frac{1}{2}$. Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be the singular banded unlink diagrams obtained from Σ as in Proposition 2.28 using V_0 and V_1, respectively. Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are related by isotopy in $E(K)$, slides over L_1, and possibly a band slide, intersection/band slide, intersection/band pass or intersection pass.*

Proof Let p and q be the index 1 or self-intersection points in Σ with a flowline of $-V_{1/2}$ from p to q. The proof of Proposition 2.28 fails for $\Sigma_{1/2}$ precisely because p lying above q in Σ causes indeterminacy in the 1-skeleton S. There are then two choices (up to small isotopy through 0-standard surfaces) in how to perturb Σ near p to obtain a 0-standard surface. See Figure 13. The resulting two singular banded unlink diagrams differ by one of the following moves:

$$\left.\begin{array}{r}\text{band slide}\\ \text{intersection/band slide}\\ \text{intersection/band pass}\\ \text{intersection pass}\end{array}\right\} \quad \text{if} \quad \left\{\begin{array}{l} p \text{ and } q \text{ are index 1 points}\\ p \text{ is a self-intersection and } q \text{ is an index 1 point}\\ p \text{ is an index 1 point and } q \text{ is a self-intersection}\\ p \text{ and } q \text{ are self-intersections}\end{array}\right.$$

Letting $\mathcal{D}_t$ denote the diagram obtained using V_t for $t \neq \frac{1}{2}$, we conclude that $\mathcal{D}_{1/2-\varepsilon}$ and $\mathcal{D}_{1/2+\varepsilon}$ are either isotopic or isotopic after one of the above moves. The same is then true of $\mathcal{D}_0$ and $\mathcal{D}_1$ by Corollary 2.29. □

The following proposition and corollary now follow immediately from Propositions 2.28 and 2.30:

Proposition 2.31 *Suppose Σ is 0-standard. Then there is a singular banded unlink diagram $\mathcal{D}$ determined by $\Sigma, \nabla h$ up to isotopy in $E(\mathcal{H})$, slides over L_1, band slides, intersection/band slides, intersection/band passes and intersection passes.*

Corollary 2.32 *Let Σ_0 and Σ_1 be 0-standard surfaces. Suppose there is an isotopy Σ_t from Σ_0 to Σ_1 that is 0-standard for all t, with $\nabla(h|_{\Sigma_1})$ obtained from $\nabla(h|_{\Sigma_0})$ by the isotopy-induced map on $T\Sigma$. Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are related by isotopy in $E(\mathcal{H})$, slides over L_1, band slides, intersection/band slides, intersection/band passes and intersection passes.*

2.4 Conclusion: uniqueness of singular banded unlink diagrams

2.4.1 Singular band moves and isotopy We are now in a position to prove our main results.

Theorem 2.33 *Let Σ_0 and Σ_1 be 0-standard self-transverse immersed surfaces. Suppose there exists an isotopy Σ_t such that Σ_t is 1-standard for all t, and 0-standard for all $t \neq \frac{1}{2}$.*

Set $\mathcal{D}_t := \mathcal{D}(\Sigma_t)$. Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are related by singular band moves.

We break Theorem 2.33 into Propositions 2.34–2.37, in which we separately consider different ways in which $\Sigma_{1/2}$ may fail to be 0-standard.

Proposition 2.34 *Suppose that $\Sigma_{1/2}$ would be 0-standard if not for a single birth or death degeneracy. Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are related by the singular band moves appearing in Proposition 2.30 and possibly a cup or cap move.*

Proof Combined with Corollary 2.32, this is a standard fact about the local model of a degenerate critical point appearing in a generic 1-parameter family of Morse functions. See eg [1]. □

Proposition 2.35 *Suppose that $\Sigma_{1/2}$ would be 0-standard if not for the descending manifold of p with respect to ∇h meeting the ascending manifold of q with respect to ∇h, where p and q are each index 1 critical points of $h|_\Sigma$ or self-intersections of Σ, and their ascending/descending manifolds intersect in their interiors (rather than in just Σ, as in Proposition 2.30). Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are related by the singular band moves appearing in Proposition 2.30 and possibly a band swim, intersection/band swim, upside-down intersection/band swim or intersection swim.*

Proof The proof of Proposition 2.31 fails for $\Sigma_{1/2}$ because, when we attempt to project the 1-skeleton of Σ to $h^{-1}\left(\frac{3}{2}\right)$, the edges corresponding to p and q will intersect. There are then two choices (up to small isotopy through 0-standard surfaces) in how to perturb Σ near p to obtain a 0-standard surface. See

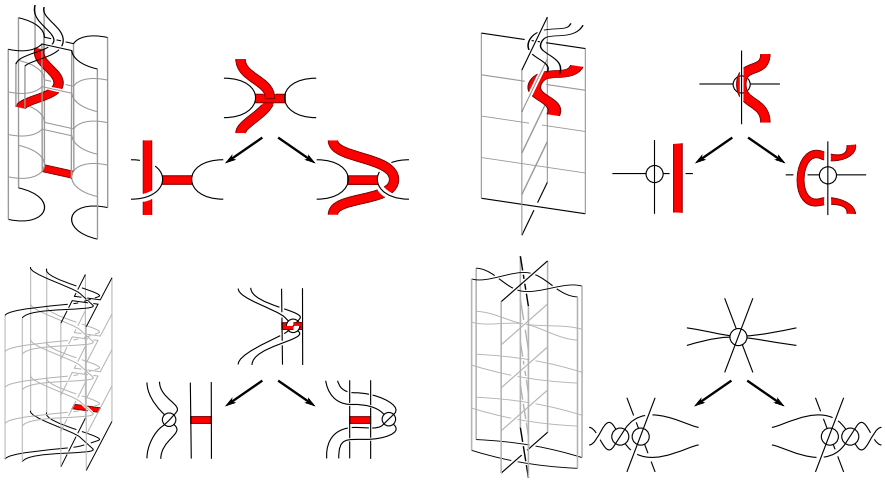


Figure 14: The cases of Proposition 2.35. At the left of each quadrant we draw a local model about the flowline that causes $\Sigma_{1/2}$ to not be 0-standard. At the top right of each quadrant, we draw a schematic of the projection of $\Sigma_{1/2}$ to $E(\mathcal{K})$, where two bands, two self-intersections, or one of each coincide. We draw arrows to indicate the two diagrams that arise if we perturb $\Sigma_{1/2}$ to be 0-standard.

Figure 14. The resulting two singular banded unlink diagrams differ by one of the following moves:

$$\left.\begin{array}{r}\text{band swim}\\ \text{intersection/band swim}\\ \text{upside-down intersection/band swim}\\ \text{intersection swim}\end{array}\right\} \quad \text{if} \quad \left\{\begin{array}{l}p \text{ and } q \text{ are index 1 points}\\ p \text{ is a self-intersection } q \text{ is an index 1 point}\\ p \text{ is an index 1 point } q \text{ is a self-intersection}\\ p \text{ and } q \text{ are self-intersections}\end{array}\right.$$

We conclude that $\mathcal{D}_{1/2-\varepsilon}$ and $\mathcal{D}_{1/2+\varepsilon}$ are either isotopic or isotopic after one of the above moves. The same is then true of $\mathcal{D}_0$ and $\mathcal{D}_1$ (up to the relevant moves) by Corollary 2.32. □

Proposition 2.36 *Suppose that $\Sigma_{1/2}$ would be 0-standard if not for the descending manifold of p intersecting the ascending manifold of q, where p is an index 1 critical point of $h|_\Sigma$ or a self-intersection of Σ and q is an index 2 critical point of h. Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are related by the singular band moves appearing in Proposition 2.30 and possibly a band/2-handle slide or intersection/2-handle slide.*

Proof The proof of Proposition 2.31 fails because we cannot project the edge of the 1-skeleton of Σ corresponding to p to the level $h^{-1}\left(\frac{3}{2}\right)$. There are then two choices (up to small isotopy through 0-standard surfaces) in how to perturb Σ near p to obtain a 0-standard surface, with resulting singular banded unlink diagrams differing by a slide over a 2-handle. That is, the resulting two singular banded unlink diagrams differ by one of the following moves:

$$\left.\begin{array}{r}\text{band/2-handle slide}\\ \text{intersection/2-handle slide}\end{array}\right\} \quad \text{if} \quad \left\{\begin{array}{l}p \text{ is an index 1 point}\\ p \text{ is a self-intersection}\end{array}\right.$$

We conclude that $\mathcal{D}_{1/2-\varepsilon}$ and $\mathcal{D}_{1/2+\varepsilon}$ are either isotopic or isotopic after one of the above moves. The same is then true of $\mathcal{D}_0$ and $\mathcal{D}_1$ (up to the relevant moves) by Corollary 2.32. □

Proposition 2.37 *Suppose that $\Sigma_{1/2}$ would be 0-standard if not for the descending manifold of p intersecting the ascending manifold of q, where p is an index 2 critical point of h and q is either an index 1 critical point of $h|_\Sigma$ or a self-intersection of Σ. Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are related by the singular band moves appearing in Proposition 2.30 and possibly a band/2-handle swim or intersection/2-handle swim.*

Proof The proof of Proposition 2.31 fails for $\Sigma_{1/2}$ because after we project the 1-skeleton of Σ to $h^{-1}\left(\frac{3}{2}\right)$, the edge corresponding to q will intersect the component of $L_2 \subset \mathcal{K}$ corresponding to p. There are then two choices (up to small isotopy through 0-standard surfaces) in how to perturb Σ near p to obtain a 0-standard surface, with resulting singular banded unlink diagrams differing by a swim through a 2-handle attaching circle. That is, the resulting two singular banded unlink diagrams differ by one of the following moves:

$$\left.\begin{array}{r}\text{band/2-handle swim}\\ \text{intersection/2-handle swim}\end{array}\right\} \quad \text{if} \quad \left\{\begin{array}{l}p \text{ is an index 1 point}\\ p \text{ is a self-intersection}\end{array}\right.$$

We conclude that $\mathcal{D}_{1/2-\varepsilon}$ and $\mathcal{D}_{1/2+\varepsilon}$ are either isotopic or isotopic after one of the above moves. The same is then true of $\mathcal{D}_0$ and $\mathcal{D}_1$ (up to the relevant moves) by Corollary 2.32. □

This completes the proof of Theorem 2.33, since Propositions 2.34–2.37 cover all of the cases in which $\Sigma_{1/2}$ is 1-standard and not 0-standard (of course, if $\Sigma_{1/2}$ is 0-standard then Theorem 2.33 follows from Corollary 2.32) except for the case that there are flowlines of $-\nabla h$ between index 2 critical points. However, h and ∇h are fixed during the isotopy, so this does not happen. □

Corollary 2.38 *Let Σ_0 and Σ_1 be 0-standard self-transverse immersed surfaces. Suppose there exists an isotopy Σ_t and values $t_1 < t_2 < \cdots < t_n \in (0,1)$ such that Σ_t is 0-standard for all $t \notin \{t_1, t_2, \dots, t_n\}$, and Σ_{t_i} is 1-standard for each $i = 1, 2, \dots, n$.*

Let $\mathcal{D}_t := \mathcal{D}(\Sigma_t)$. Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are related by a sequence of singular band moves.

Proof For each $i = 1, \dots, n-1$, let s_i be a value in (t_i, t_{i+1}). By Corollary 2.32:

- $\mathcal{D}_0$ is related to $\mathcal{D}_{s_1}$ by singular band moves.
- $\mathcal{D}_{s_i}$ is related to $\mathcal{D}_{s_{i+1}}$ by singular band moves for $i = 1, \dots, n-1$.
- $\mathcal{D}_{s_{n-1}}$ is related to $\mathcal{D}_1$ by singular band moves.

We conclude that $\mathcal{D}_0$ and $\mathcal{D}_1$ are related by singular band moves. □

2.4.2 Proof of uniqueness theorems We finally prove that singular banded unlink diagrams of isotopic (resp. regularly homotopic, homotopic) surfaces exist for arbitrary immersed self-transverse surfaces and are well defined up to singular band moves. At this point, not much is left to do — the material in Section 2.4 is essentially the whole proof that diagrams exist and are unique up to singular band moves.

Theorem 2.39 *Let Σ be a self-transverse smoothly immersed surface in X. Then there is a singular banded unlink diagram $\mathcal{D}(\Sigma)$, well defined up to singular band moves, such that Σ is isotopic to the closed realizing surface for $\mathcal{D}(\Sigma)$. Moreover, if Σ is isotopic to Σ', then $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ are related by singular band moves.*

We say that $\mathcal{D}(\Sigma)$ is *a singular banded unlink diagram for* Σ, or simply that $\mathcal{D}(\Sigma)$ is a diagram for Σ.

Proof Via a small perturbation, Σ is isotopic to a 0-standard surface Σ_0. Set $\mathcal{D}(\Sigma) := \mathcal{D}(\Sigma_0)$. To show that $\mathcal{D}(\Sigma)$ is well defined, suppose that Σ_1 is another 0-standard surface that is isotopic to Σ, and hence isotopic to Σ_0. By Proposition 2.27, there is an isotopy Σ_t from Σ_0 to Σ_1 such that Σ_t is 1-standard for all t and 0-standard for all but finitely many t. By Corollary 2.38, $\mathcal{D}(\Sigma_0)$ and $\mathcal{D}(\Sigma_1)$ are related by singular band moves.

Since this argument applies to any 0-standard surface Σ_1 isotopic to Σ, we conclude that, if Σ and Σ' are isotopic, then $\mathcal{D}(\Sigma)$ and $\mathcal{D}(\Sigma')$ are related by singular band moves. □

Corollary 2.40 *Let $\mathcal{D}$ and $\mathcal{D}'$ be singular banded unlink diagrams of surfaces Σ and Σ' immersed in diffeomorphic 4-manifolds X and X'. There is a diffeomorphism taking (X, Σ) to (X', Σ') if and only if there is a sequence of singular band moves and Kirby moves taking $\mathcal{D}$ to $\mathcal{D}'$.*

In addition, we can use these moves to describe homotopies of surfaces in terms of singular banded unlink diagrams.

Corollary 2.41 *Let $\mathcal{D}$ and $\mathcal{D}'$ be singular banded unlink diagrams for surfaces Σ and Σ' immersed in X. If Σ and Σ' are homotopic, then $\mathcal{D}$ and $\mathcal{D}'$ are related by a finite sequence of singular band moves and the following moves (illustrated in Figure 15):*

- *introducing or canceling two oppositely marked vertices (a "finger move" or "Whitney move"), as illustrated;*
- *replacing a nugatory crossing with a vertex or vice versa (a "cusp move"), as illustrated.*

In addition, if Σ and Σ' are **regularly** *homotopic, then $\mathcal{D}$ and $\mathcal{D}'$ are related by a finite sequence of singular band moves, finger moves and Whitney moves (ie a sequence of the given moves that does not include any cusp moves).*

Proof Say Σ and Σ' are homotopic and have self-intersection numbers s and s', respectively. By work of Hirsch [11] and Smale [31], Σ and Σ' are regularly homotopic if and only if $s = s'$.

After performing a cusp move on $\mathcal{D}$, a realizing surface for the resulting diagram has self-intersection $s \pm 1$, with sign depending on the choice of cusp move. Perform $|s' - s|$ cusp moves of the appropriate sign to $\mathcal{D}$ to obtain a diagram $\mathcal{D}_2$ whose realizing surface Σ_2 has self-intersection number s'. Now Σ_2 and Σ' are regularly homotopic.

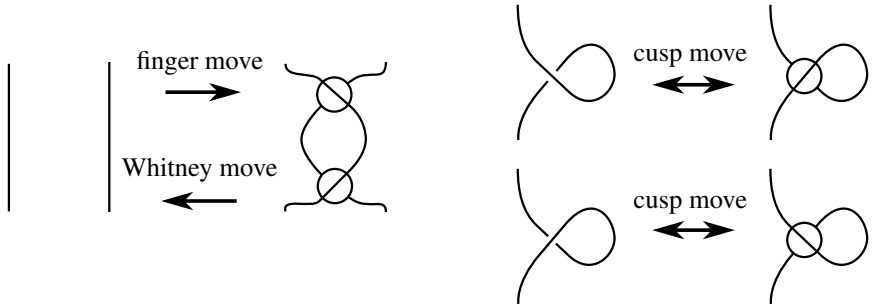


Figure 15: The new moves describing homotopy of a surface in a 4-manifold. There are two versions of the cusp move. One involves a positive self-intersection and one involves a negative self-intersection of the described immersed surface. To describe regular homotopy we only need finger and Whitney moves.

We recommend the reference [7] for exposition on regular homotopy of surfaces. In brief, there exists a sequence of finger moves on Σ_2 along framed arcs $\eta_1, \dots, \eta_n$ yielding a surface Σ_3, and a sequence of finger moves on Σ' along framed arcs $\eta'_1, \dots, \eta'_m$ yielding a surface Σ'', so that Σ_3 and Σ'' are ambiently isotopic.

We isotope η_1 to lie completely in $h^{-1}\left(\frac{3}{2}\right)$ (which may involve isotopy of Σ_2 inducing singular band moves on its singular banded unlink diagram according to Theorem 2.39) and then shrink η_1 to be short and contained in a neighborhood identical to the far left of Figure 15. Twist the diagram as necessary so

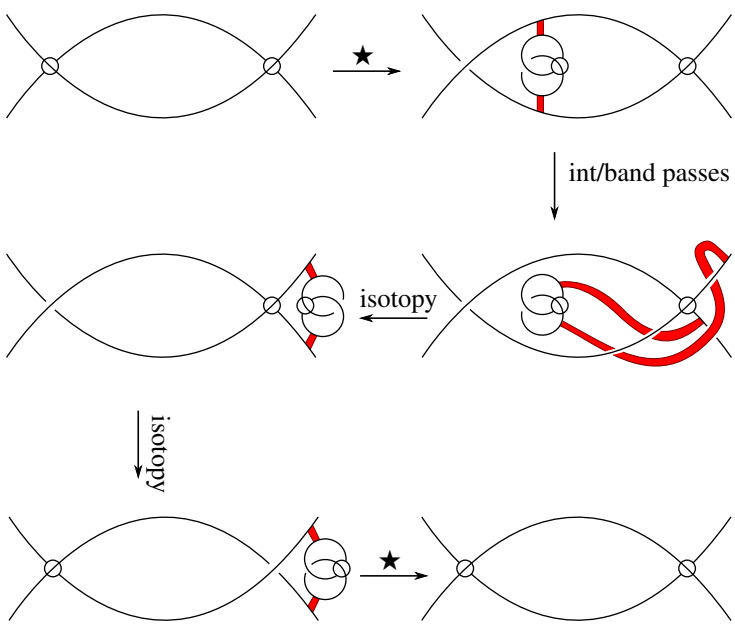


Figure 16: There are two seemingly different finger moves (differing in the decorations on the relevant vertices), but they yield singular banded unlink diagrams that differ by singular band moves.

that the framing of η_1 is untwisted. Then we perform a finger move to $\mathcal{D}_2$ in that neighborhood. Repeat for each $i = 2, \dots, n$, and call the resulting diagram $\mathcal{D}_3$. A realizing surface for $\mathcal{D}_3$ is isotopic to Σ_3.

Now repeat for Σ' by performing singular band moves and finger moves to its diagram $\mathcal{D}'$ until obtaining a diagram $\mathcal{D}''$ whose realizing surface is isotopic to Σ''. Since Σ'' and Σ_3 are isotopic, by Theorem 2.39 it follows that $\mathcal{D}_3$ and $\mathcal{D}''$ are related by singular band moves.

We thus conclude that $\mathcal{D}$ can be transformed into $\mathcal{D}'$ by a sequence of singular band moves, cusp moves, finger moves and Whitney moves (which are the inverses to finger moves). □

Remark 2.42 When performing a finger move to a singular banded unlink diagram, there are seemingly two choices (related by a local symmetry) of how to mark the new vertices. However, the choices yield diagrams related by singular band moves, as shown in Figure 16.

3 Bridge trisections

3.1 Bridge trisections of embedded surfaces

In Section 3.2, we prove that self-transverse immersed surfaces in 4-manifolds can be put into *bridge position*, a notion introduced for embedded surfaces by Meier and Zupan [26; 27]. Meier and Zupan showed that a bridge trisection of a surface in S^4 (with respect to a standard trisection of S^4) is unique up to perturbation [26], using the work of Swenton [32] and Kearton and Kurlin [22] on banded unlink diagrams in S^4. The authors of this paper then used a general version of this theorem in arbitrary 4-manifolds to show that bridge trisections of surfaces in any trisected manifold are unique up to perturbation. In what follows, we will apply Theorem 2.39 to prove an analogous uniqueness result for bridge trisections of immersed surfaces. In this section, we will review the situation where the surface is embedded.

First, we recall the definition of a trisection of a closed 4-manifold. Similar exposition can be found in [14]. We do not require much knowledge of trisections; for more detailed exposition, the interested reader may refer to [8].

Definition 3.1 [8] Let X^4 be a connected, closed, oriented 4-manifold. A (g,k)-*trisection* of X^4 is a triple (X_1, X_2, X_3) where

(i) $X_1 \cup X_2 \cup X_3 = X^4$,

(ii) $X_i \cong \natural_{k_i} S^1 \times B^3$,

(iii) $X_i \cap X_j = \partial X_i \cap \partial X_j \cong \natural_g S^1 \times B^2$,

(iv) $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$,

where Σ_g is a closed orientable surface of genus g. Here, g is an integer while $k = (k_1, k_2, k_3)$ is a triple of integers. If $k_1 = k_2 = k_3$, then the trisection is said to be *balanced*.

Briefly, a trisection is a decomposition of a 4-manifold into three elementary pieces, analogous to a Heegaard splitting of a 3-manifold into two elementary pieces.

Theorem 3.2 [8] *Any smooth, connected, closed, oriented 4-manifold X^4 admits a trisection. Moreover, any two trisections of X^4 are related by a stabilization operation.*

Note that, from the definition, Σ_g is a Heegaard surface for ∂X_i, inducing the Heegaard splitting $(X_i \cap X_j, X_i \cap X_k)$. By Laudenbach and Poénaru [24], X^4 is specified by its *spine*, $\Sigma_g \cup \bigcup_{i \neq j}(X_i \cap X_j)$. Therefore, we usually describe a trisection (X_1, X_2, X_3) by a *trisection diagram* $(\Sigma_g; \alpha, \beta, \gamma)$. Here each of α, β and γ consist of g independent curves on Σ_g (abusing notation to take Σ_g as both an abstract surface and the surface $X_1 \cap X_2 \cap X_3$ in X), which bound disks in the handlebodies $X_1 \cap X_2$, $X_2 \cap X_3$ and $X_1 \cap X_3$, respectively. Given (X_1, X_2, X_3), such a diagram is well defined up to slides of α, β and γ and automorphisms of Σ_g.

Definition 3.3 Let X^4 be a 4-manifold with trisection $\mathcal{T} = (X_1, X_2, X_3)$. We say that an isotopy f_t of X^4 is *$\mathcal{T}$-regular* if $f_t(X_i) = X_i$ for each $i = 1, 2, 3$ and for all t.

Definition 3.4 The *standard trisection of S^4* is the unique $(0,0)$-trisection (X_1^0, X_2^0, X_3^0). View $S^4 = \mathbb{R}^4 \cup \infty$, with coordinates (x, y, r, θ) on $\mathbb{R}^4$, where (x, y) are Cartesian planar coordinates and (r, θ) are polar planar coordinates. Up to isotopy, $X_i^0 = \{\theta \in [\frac{2}{3}\pi i, \frac{2}{3}\pi(i+1)]\} \cup \infty$. Then $X_i^0 \cong B^4$, $X_i^0 \cap X_{i+1}^0 = \{\theta = \frac{2}{3}\pi(i+1)\} \cup \infty \cong B^3$, and $X_1^0 \cap X_j^0 \cap X_k^0 = \{r = 0\} \cup \infty \cong S^2$.

From a trisection (X_1, X_2, X_3) of X^4, we can obtain a handle decomposition of X^4 in which X_1 contains the 0- and 1-handles, X_2 is built from $(X_1 \cap X_2) \times I$ by attaching the 2-handles, and X_3 contains the 3- and 4-handles. The following definition encapsulates this construction:

Definition 3.5 Let $\mathcal{T} = (X_1, X_2, X_3)$ be a trisection of a 4-manifold X^4. Let $h\colon X^4 \to [0,4]$ be a self-indexing Morse function. We say that h is *$\mathcal{T}$-compatible* if all of the following are true:

(i) $X_1 = h^{-1}([0, \frac{3}{2}])$.

(ii) $X_2 \subset h^{-1}([\frac{3}{2}, \frac{5}{2}))$ contains all of the index 2 critical points of h.

(iii) $X_1 \cup X_2$ contains the descending manifolds of all index 2 critical points of h.

Given any trisection $\mathcal{T}$, there always exists a Morse function compatible with $\mathcal{T}$ (see [8] or [25]).

Meier and Zupan used trisections to give a new way of describing a surface smoothly embedded in a 4-manifold.

Definition 3.6 [26; 27] Let $\mathcal{T} = (X_1, X_2, X_3)$ be a trisection of a closed 4-manifold X^4. Let S be a surface embedded in X^4. We say that S is in (b, c)-bridge position with respect to $\mathcal{T}$ if, for every $i \neq j \in \{1, 2, 3\}$:

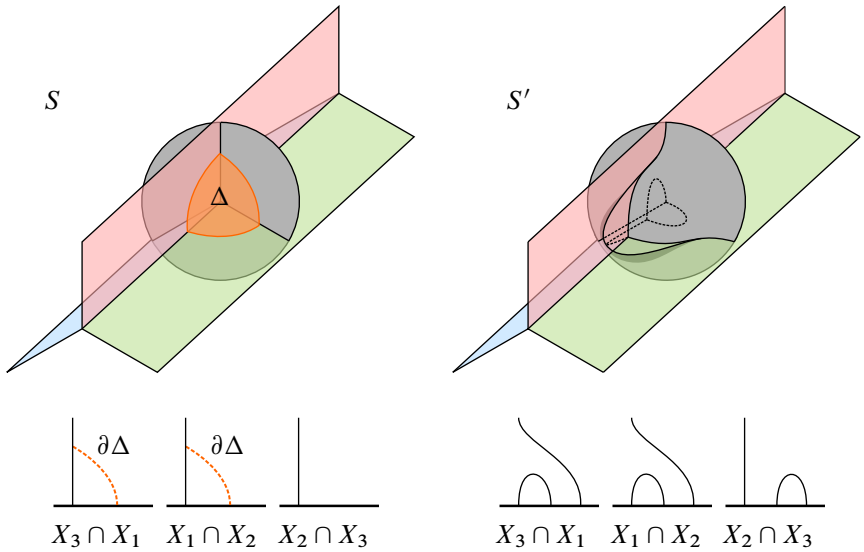


Figure 17: Left: a surface S in (b,c)-bridge position with respect to a trisection $\mathcal{T}$. We draw a neighborhood of an intersection of S with the central surface of $\mathcal{T}$. Right: we perturb S to obtain a surface S' in $(c', b+1)$-bridge position.

(i) $S \cap X_i$ is a disjoint union of c_i boundary parallel disks.

(ii) $S \cap X_i \cap X_j$ is a trivial tangle of b arcs.

Here b is an integer and $c = (c_1, c_2, c_3)$ is a triple of integers. Note that $\chi(S) = \sum c_i - b$.

Theorem 3.7 [26; 27] *Let S be a surface embedded in a 4-manifold X^4 with trisection $\mathcal{T} = (X_1, X_2, X_3)$. Then, for some c and b, S can be isotoped into (b,c)-bridge position with respect to $\mathcal{T}$. We may take $c_1 = c_2 = c_3$.*

Because a collection of boundary parallel disks in $\natural(S^1 \times B^3)$ is uniquely determined by its boundary (up to isotopy rel boundary), a surface S in bridge position is determined up to isotopy by $S \cap \left(\bigcup_{i \neq j} X_i \cap X_j\right)$.

There is a natural perturbation of a surface in bridge position, analogous to perturbation of a knot in bridge position within a 3-manifold. We define the simplest version of Meier and Zupan's original perturbation operation [26; 27].

Definition 3.8 Let $S \subset X^4$ be a surface in (b,c)-bridge position with respect to $\mathcal{T} = (X_1, X_2, X_3)$. Let S' be the surface obtained from S as in Figure 17. In words, we take a small disk D contained in $S \cap X_1$ whose boundary consists of an arc δ_1 in the interior of X_1, an arc δ_2 in $X_1 \cap X_2$, and an arc δ_3 in $X_3 \cap X_1$. We take a parallel copy Δ of D pushed off S away from δ_1, so Δ meets S in the arc $\delta_1 \subset \partial\Delta$ and the remaining boundary of Δ is an arc δ' in ∂X_1 that meets $X_1 \cap X_2 \cap X_3$ transversely in one point. Using the direction from which we obtained Δ from D, we frame Δ and isotope S along Δ to introduce two more intersection points between S and $X_1 \cap X_2 \cap X_3$. We call the resulting surface S' and say that S' is

obtained from S by *elementary perturbation*. We likewise say that S is obtained from S' by *elementary deperturbation*.

We may exchange the roles of X_1, X_2 and X_3 cyclically when performing this operation, ie alternatively obtain S' from this compression operation in either X_2 or X_3. We still say S' is obtained from S by elementary perturbation and that S is obtained from S' by elementary deperturbation.

Proposition 3.9 [27, Lemma 5.2] *Let S be a surface in (b,c)-bridge position with respect to a trisection $\mathcal{T}=(X_1,X_2,X_3)$, with $c=(c_1,c_2,c_3)$. Let S' be obtained from S by elementary perturbation, using a disk in X_i. Then S' is in $(c',b+1)$-bridge position with respect to $\mathcal{T}$, with $c'_j=c_j$ for $j\neq i$ and $c'_i=c_i+1$.*

In previous work, the authors of this paper showed that any two bridge trisections of a surface are related by elementary perturbations.

Theorem 3.10 [14] *Let S and S' be surfaces in bridge position with respect to a trisection $\mathcal{T}$ of a 4-manifold X^4. Suppose S is isotopic to S'. Then S can be taken to S' by a sequence of elementary perturbations and deperturbations, followed by a $\mathcal{T}$-regular isotopy.*

When $\mathcal{T}$ is the standard trisection of S^4, Theorem 3.10 is a result of Meier and Zupan [26].

3.2 Basic definitions for singular links and immersed surfaces

In Definition 3.6 of a bridge trisection of an embedded surface, we cut a 4-manifold into simple pieces so that an embedded surface is cut into systems of boundary-parallel disks. To describe immersed surfaces, we need to describe this notion with slightly different language.

Definition 3.11 Let $C_1,\dots,C_k$ be arcs properly immersed in a 3-manifold M^3. Assume that all intersections (including self-intersections) of $C_1,\dots,C_k$ are isolated points that are not tangencies. Let $V=(\partial M^3)\times I$ be a collar neighborhood of ∂M^3 and let $h\colon V\to I$ be projection onto the second factor.

We say that $(C_1,\dots,C_k)$ is a *trivial immersed tangle* if the following are satisfied:

(i) Each C_i is contained in V.

(ii) All self-intersections of C_i and intersections of C_i with C_j are contained in the interior of M.

(iii) There is an immersed tangle $(C'_1,\dots,C'_k)$ that is isotopic rel boundary to $(C_1,\dots,C_k)$ so that $h|C'_i$ is Morse with a single critical point for all i.

Definition 3.12 Let $D_1,\dots,D_k$ be 2-dimensional disks properly immersed in a 4-manifold X^4. Assume that all intersections (including self-intersections) of $D_1,\dots,D_k$ are isolated, transverse intersections contained in ∂X^4 (so $\partial\big(\bigcup D_i\big)$ is a singular link in ∂X). Let $V=\partial X\times I$ be a neighborhood of ∂X and let $h\colon V\to I$ be projection onto the second factor.

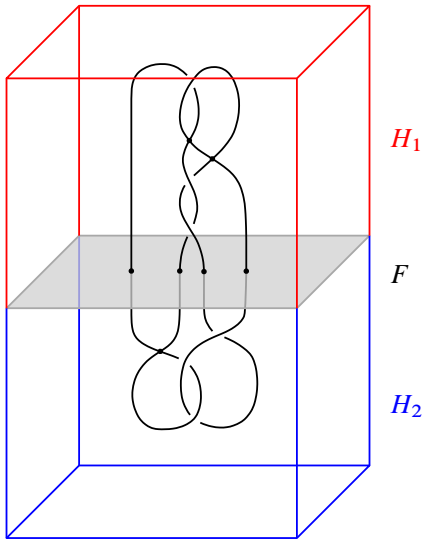


Figure 18: A singular link in bridge position.

We say that $(D_1, \dots, D_k)$ is a *trivial immersed disk system* if the following are satisfied (up to isotopy rel boundary):

(i) Each D_i is contained in V.

(ii) The restriction $h|D_i$ is Morse with a single critical point for all i.

Trivial immersed tangles and disk systems are the immersed analogue to systems of boundary parallel embedded tangles and disks. With immersed tangles we can easily define an analogue of bridge position for singular links.

Definition 3.13 Let L be a singular link in a 3-manifold M with a Heegaard splitting (H_1, H_2). Let $F := H_1 \cap H_2$.

We say that L is in *bridge position* with respect to F if $L \cap H_i$ is a trivial immersed tangle for $i = 1, 2$. See Figure 18. If (L, σ) is a marked singular link, then we say that (L, σ) is in *bridge position* if L is in bridge position.

We can perturb immersed tangles just as we perturb embedded tangles, but we must also account for vertices.

Definition 3.14 Let L be a marked singular link in a 3-manifold M with Heegaard splitting (H_1, H_2). Suppose L is in bridge position with respect to $\Sigma := H_1 \cup H_2$.

Let L' be a marked singular link obtained from L by perturbation near Σ, as in Figure 19. Note that we allow up to one vertex of L to be between the original intersection of L with Σ and the newly created pair of intersections. Then we say L' is obtained from L by *elementary perturbation*, and L is obtained from L' by *elementary deperturbation*.

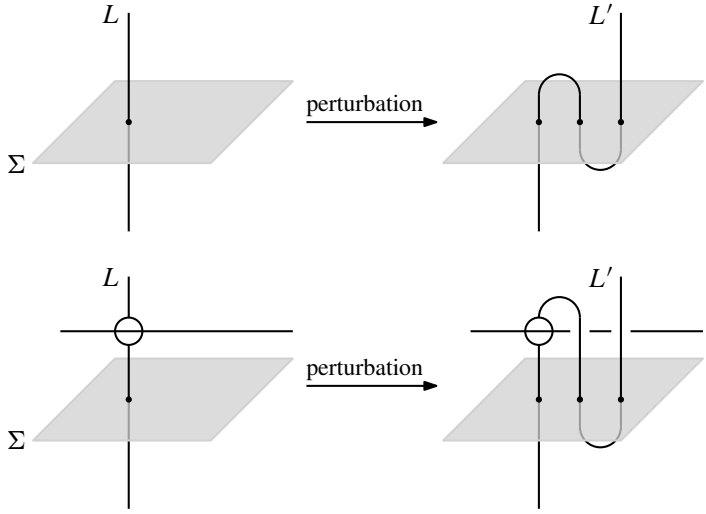


Figure 19: An elementary perturbation of a marked singular link in bridge position.

Let L'' be a marked singular link obtained from L by moving a vertex in L through Σ as in the local model shown in Figure 20. Then we say L'' is obtained from L (and vice versa) by *vertex perturbation*.

Theorem 3.15 *Let L and L' be isotopic marked singular links in a 3-manifold M with Heegaard splitting (H_1, H_2). Assume L and L' are in bridge position with respect to $\Sigma := H_1 \cap H_2$. Then there exists a marked singular link L'' that can be obtained from L and from L' by sequences of elementary perturbations, vertex perturbations and isotopies fixing Σ setwise.*

Proof When L and L' are nonsingular, this is a theorem of Hayashi and Shimokawa [10]. We will apply a version of this theorem for nonsingular banded links due to Meier and Zupan [26; 27] by using the following observation. First, recall from Section 2.1 that, if L is a marked singular link, then L^+ denotes the nonsingular link obtained by positively resolving the vertices of L.

Observation 3.16 *There exist disjoint framed arcs $a_1, \dots, a_n$ with endpoints on L^+ such that contracting L^+ along $a_1, \dots, a_n$ yields L.*

Similarly, let $a'_1, \dots, a'_n$ be framed arcs with endpoints on L'^+ such that contracting L'^+ along $a'_1, \dots, a'_n$ yields L'.

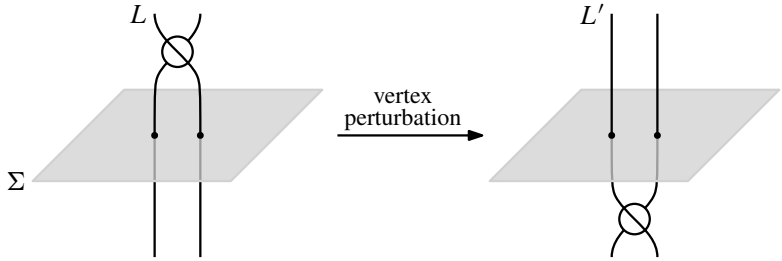


Figure 20: A vertex perturbation of a marked singular link in bridge position.

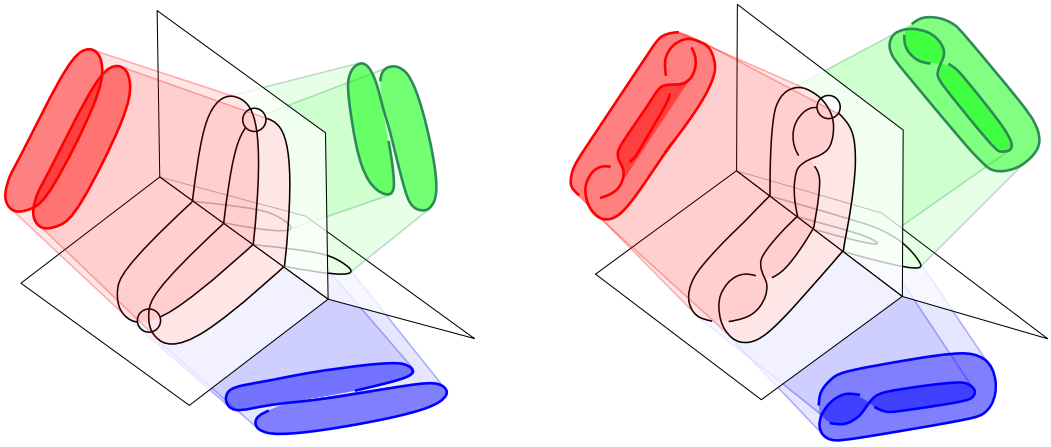


Figure 21: Two $((2,1,1),2)$-bridge trisections of immersed 2-spheres in S^4. Left: this 2-sphere has a pair of self-intersections of opposite sign. Right: this 2-sphere has a single self-intersection.

Now, by Meier and Zupan [26; 27], there exists a link J that can be obtained from L^+ and from L'^+ by elementary perturbations and isotopies fixing Σ setwise. Moreover, these isotopies and perturbations may be chosen to carry a_i and a_i' to framed arcs b_i and b_i', respectively, with endpoints on J, so that b_i and b_i' are parallel to Σ with surface framing, and are parallel to each other (though possibly on opposite sides of Σ). In Meier and Zupan's construction, during this sequence of perturbations and isotopies of L^+ (resp. L'^+), a_i (resp. a_i') never intersect Σ, so these perturbations and isotopies may be achieved by perturbations and isotopies of L (resp. L'). Let $\widehat{J}$ and $\widehat{J}'$ be the marked singular links obtained by contracting J along $\bigcup b_i$ and $\bigcup b_i'$, respectively, and with markings induced by those of L and L'. Then $\widehat{J}'$ can be transformed into $\widehat{J}$ by isotopy fixing Σ and a vertex perturbation for each pair a_i, a_i' separated in different components of $M \setminus \Sigma$. Therefore, the claim holds with $L'' = \widehat{J}$. □

3.3 Bridge trisections of immersed surfaces

We now use the definitions from Section 3.2 to define bridge trisections of self-transverse immersed surfaces.

Definition 3.17 Let $\mathcal{T} = (X_1, X_2, X_3)$ be a trisection of a closed 4-manifold X^4. Let S be a self-transverse immersed surface in X^4. We say that S is in (b,c)-bridge position with respect to $\mathcal{T}$ if, for each $i \neq j \in \{1,2,3\}$:

(i) $S \cap X_i$ is a trivial immersed disk system of c_i disks.

(ii) $S \cap X_i \cap X_j$ is a trivial immersed tangle of b strands.

Here, b is a positive integer and $c = (c_1, c_2, c_3)$ is a triple of positive integers.

In Figure 21, we give some small examples of bridge trisections of 2-spheres immersed in S^4.

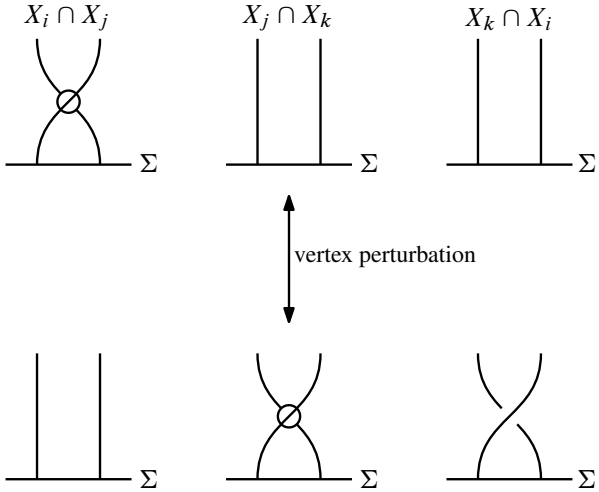


Figure 22: A vertex perturbation of a triplane diagram.

There is again a natural notion of perturbing an immersed surface in (b, c)-bridge position. More precisely, the notion of perturbing an embedded surface in bridge position works perfectly well for an immersed surface in bridge position. We write the definition below, believing that the value of transparency outweighs the cost of redundancy.

Definition 3.18 Let S be a self-transverse immersed surface in bridge position with respect to a trisection $\mathcal{T} = (X_1, X_2, X_3)$. In Figure 17, we depict a small neighborhood of a point in $S \cap \Sigma$ for $\Sigma := X_1 \cap X_2 \cap X_3$. Let S' be the surface obtained from S as in Figure 17. We say that S' is obtained from S by *elementary perturbation*, and that S is obtained from S' by *elementary deperturbation*.

If S is in bridge position with respect to a trisection $\mathcal{T} = (X_1, X_2, X_3)$, then elementary perturbation and $\mathcal{T}$-regular isotopy cannot move a self-intersection of S from X_i to X_j for $i \neq j$. Thus, we introduce one new kind of perturbation for immersed surfaces in bridge position, based on the most elementary way one might move a self-intersection of S from X_i to X_j.

Definition 3.19 Let v be a vertex of the singular link $S \cap X_i \cap X_{i+1}$ for some i (where the indices are understood to be taken mod 3), so that v is a self-intersection of S. Suppose v has a neighborhood as in Figure 22, so that v is near $\Sigma := X_1 \cap X_2 \cap X_3$. We may isotope S to move v into Σ and then into either $X_{i+1} \cap X_{i+2}$ or $X_{i-1} \cap X_i$, producing a new surface S' in (b, c)-bridge position. See Figures 22 and 23. We say that S' is obtained from S (and vice versa) by *vertex perturbation*.

Remark 3.20 Let S be an immersed surface in $(b; c_1, c_2, c_3)$-bridge position with respect to $\mathcal{T} = (X_1, X_2, X_3)$.

(1) If S' is obtained from S by elementary perturbation along a disk in X_i, then S' is in $(b+1; c'_1, c'_2, c'_3)$-bridge position with $c'_i = c_i + 1$ and $c'_j = c_j$ for $j \neq i$.

(2) If S' is obtained from S by vertex perturbation, then S' is in $(b; c_1, c_2, c_3)$-bridge position.

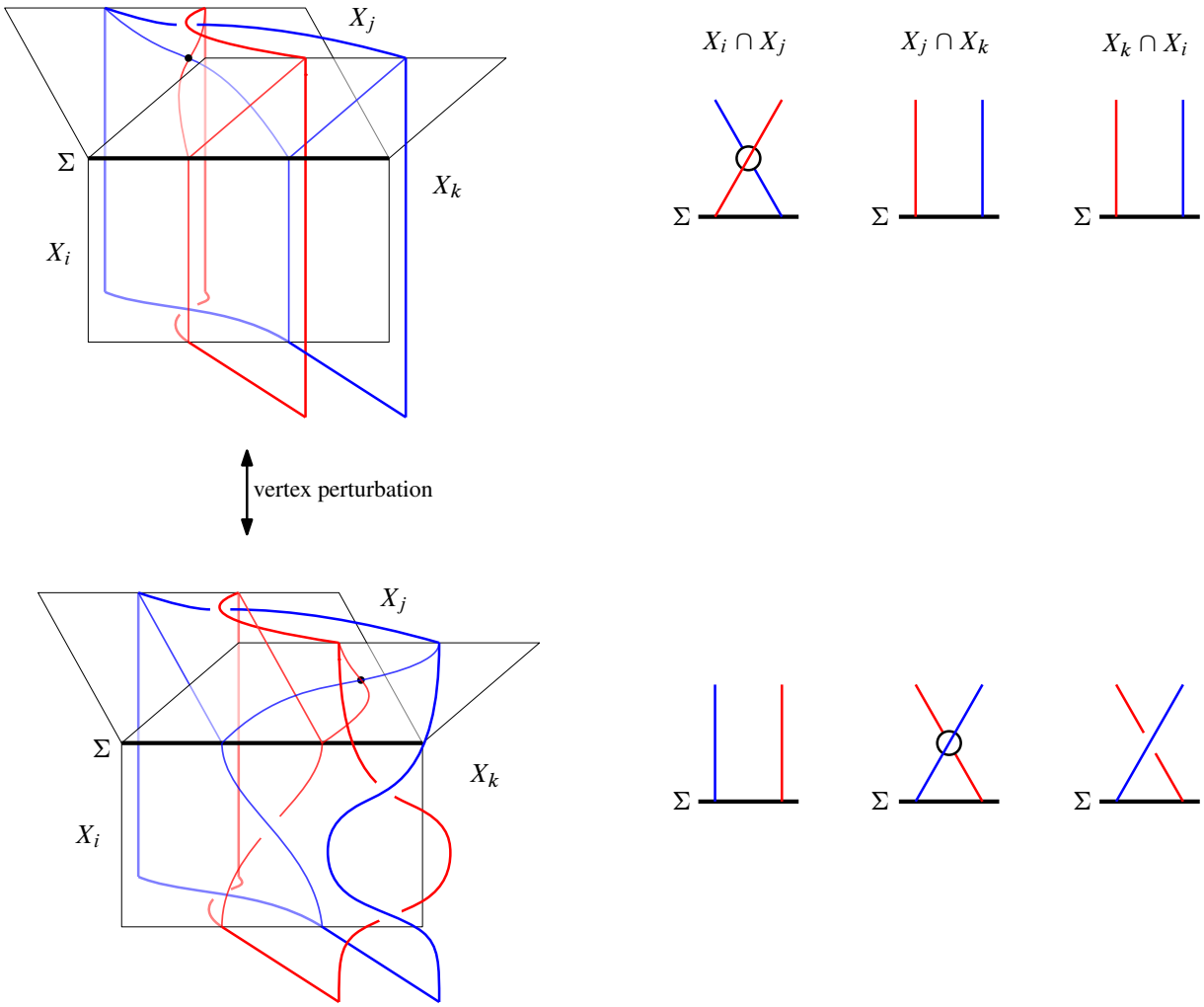


Figure 23: Pushing a self-intersection point from $X_i \cap X_j$ to $X_j \cap X_k$ during a vertex perturbation.

Definition 3.21 If a surface S' in bridge position with respect to a trisection $\mathcal{T}$ is obtained from a surface S in bridge position with respect to $\mathcal{T}$ by a sequence of elementary and vertex perturbations, then we simply say that S' is obtained from S by perturbation (with $\mathcal{T}$ implicit). If S' is obtained from S by a sequence of elementary perturbations and deperturbations and vertex perturbations, then we say that S' is obtained from S (or "related to S") by perturbation and deperturbation.

Theorem 3.22 *Let S be a self-transverse immersed surface in a 4-manifold X^4 with trisection $\mathcal{T} = (X_1, X_2, X_3)$. Then, for some c and b, S can be isotoped into (b, c)-bridge position with respect to $\mathcal{T}$.*

Proof Let h be a self-indexing Morse function of X^4 that is $\mathcal{T}$-compatible. Let (L, B) be a singular banded unlink diagram for S, so L is a singular link in $M := h^{-1}\left(\frac{3}{2}\right)$, and B is a set of bands for L in M. Let $H_1 := X_3 \cap X_1$ and $H_2 := X_1 \cap X_2$, so that $\Sigma := H_1 \cap H_2$ is a Heegaard surface for M.

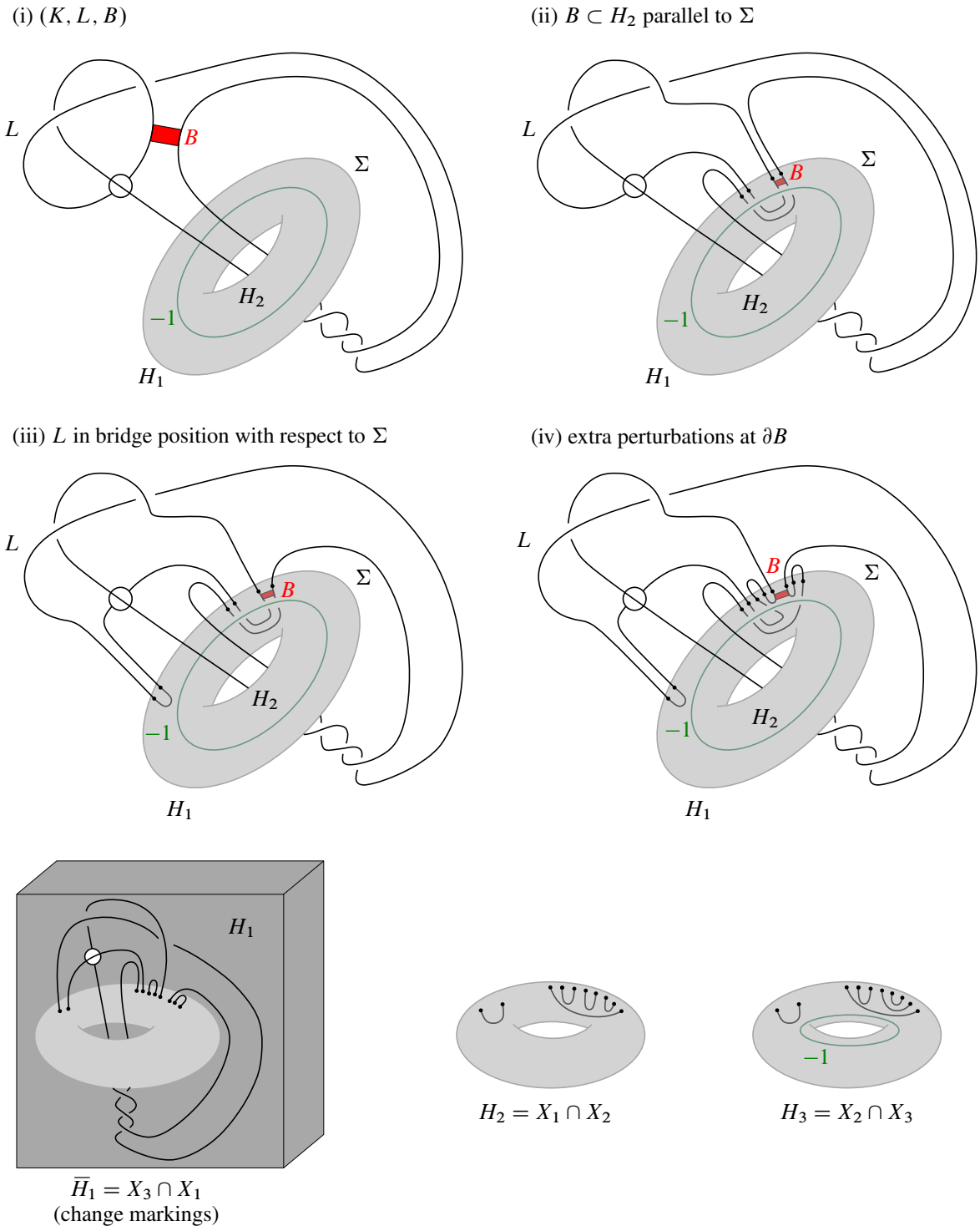


Figure 24: We illustrate how a surface that realizes a banded unlink diagram $(\mathcal{H}, L, B)$ may be isotoped to lie in bridge position. See the proof of Theorem 3.22.

By dimensionality, we may isotope L, B to be contained in $\Sigma \times [-1,1] \subset M$ (ie we isotope L and B to avoid a 1-skeleton of H_1 and H_2), with $\Sigma \times [-1,0] \subset H_1$, $\Sigma \times [0,1] \subset H_2$. Isotope L so that the vertices of L are disjoint from Σ, and so that B consists of short straight bands parallel to Σ in H_2 that are far from each other, as in Figure 24(ii). Let $\pi\colon \Sigma \times [0,1] \to [0,1]$ be the projection, and perform a small isotopy of L so that $\pi|_L$ is Morse. Isotope the index 0 critical points of $\pi|_L$ vertically with respect to π to be contained in H_1, and the index 1 critical points of $\pi|_L$ vertically with respect to π to be contained in H_2, isotoping horizontally first if necessary to avoid introducing self-intersections of L or intersections of L with B. Now L is in bridge position with respect to Σ. Perturb L again near ∂B as in Figure 24(iv), and isotope all bands in B to lie in H_2.

By Theorem 2.39, S is isotopic to $S' := \Sigma(L, B)$. We investigate the intersections of S' with the pieces of $\mathcal{T}$:

(i) $S' \cap X_1 = S' \cap h^{-1}\left(\frac{3}{2}\right)$ consists of the minimum disks of S'. All self-intersections of S' are contained in ∂X_1.

(ii) $S' \cap X_2$ contains the index 1 critical points of $h|_{S'}$. This surface is built from the singular tangle $L \cap H_2$ by extending vertically and then attaching bands according to B. By construction, these bandings are trivial and the components of $S' \cap X_2$ are boundary-parallel away from the intersections.

(iii) $S' \cap X_3$ contains the maximum disks of $h|_{S'}$. In particular, $(X_3, S' \cap X_3)$ can be strongly deformation retracted to $\left(h^{-1}\left(\left[\frac{5}{2}, 4\right]\right), S' \cap h^{-1}\left(\left[\frac{5}{2}, 4\right]\right)\right)$.

(iv) $S' \cap X_1 \cap X_2 = L \cap H_2$.

(v) $S' \cap X_2 \cap X_3$ is equivalent to the tangle obtained from $L^+ \cap H_2$ by surgery on B.

(vi) $S' \cap X_3 \cap X_1 = \overline{L \cap H_1}$. Note the reversed orientation; this is because H_1 is oriented as being in the boundary of X_1, but $X_3 \cap X_1$ is oriented as the boundary of X_3.

We conclude that S' is in (b, c)-bridge position with respect to $\mathcal{T}$ for some b and c. □

3.4 Bridge splittings of singular banded links

The proof of Theorem 3.22 motivates the following definition:

Definition 3.23 Let L be a singular link in a 3-manifold M, and let $B = b_1, \dots, b_n$ be a set of bands for L. Let F be a Heegaard surface for M. We say that the singular banded link (L, B) is in *bridge position* with respect to F if L is in bridge position with respect to F, and each band b_i is contained in a 3-ball U_i as in Figure 25, with $U_i \cap U_j = \varnothing$ for $i \neq j$.

The proof of Theorem 3.22 can be broken down into the following two lemmas, which are useful to state directly:

Lemma 3.24 *Let L be a singular link in a 3-manifold M, and let B be a set of bands for L. Fix a Heegaard surface F for M. Then (L, B) can be isotoped to lie in bridge position with respect to F.*

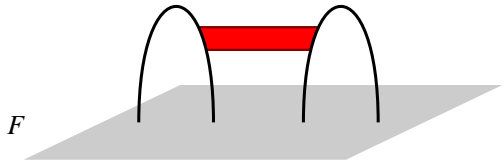


Figure 25: If a singular banded link (L, B) is in bridge position with respect to a Heegaard surface F, then every band in B has a neighborhood as pictured here. That is, every band in B has a neighborhood U containing two components C_1, C_2 of $L \setminus F$ (on which B has ends), meeting F in a disk, and not meeting any other bands in B or other components of $L \setminus F$. Moreover, $\overline{C}_1 \cup \overline{C}_2 \cup B$ may be isotoped rel $\partial(\overline{C}_1 \cup \overline{C}_2)$ in U to lie in F.

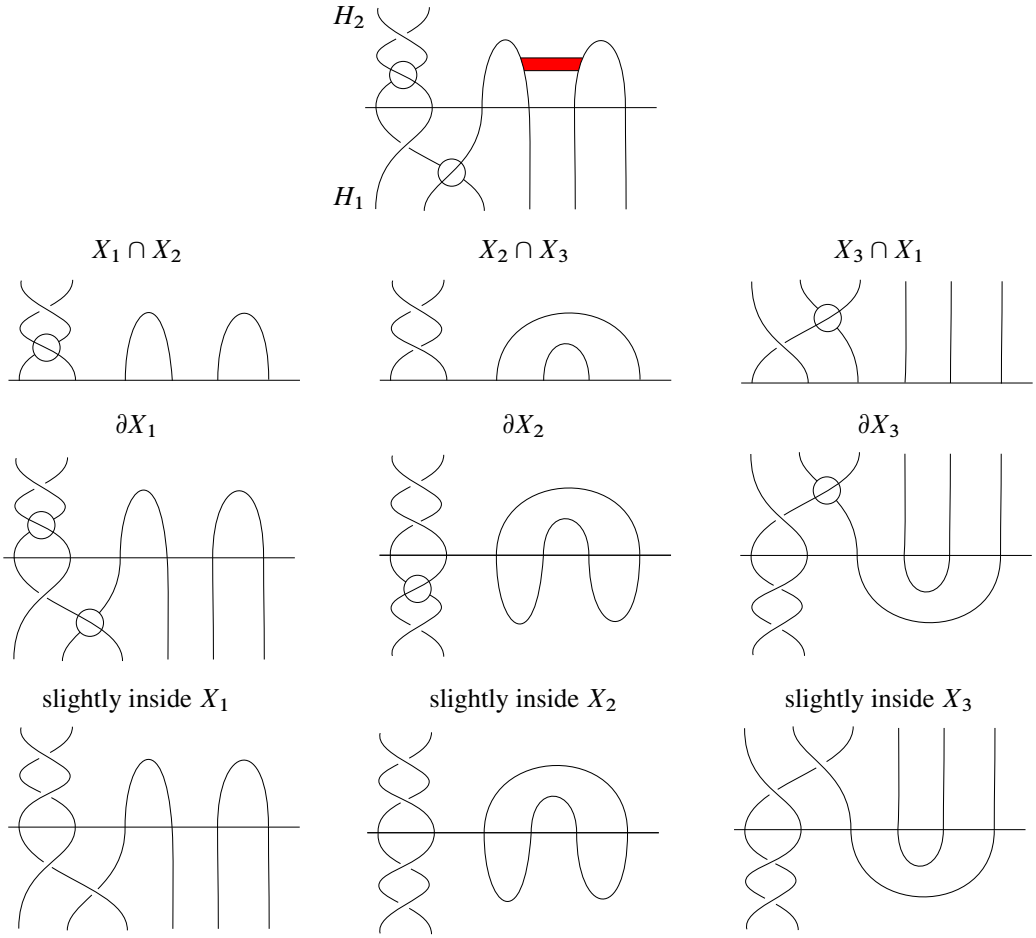


Figure 26: Top row: part of a singular banded unlink in bridge position. Second row: we obtain the singular tangles T_1, T_2 and T_3 as in Definition 3.26. Third row: the singular links that are the intersection of the associated bridge trisected surface with ∂X_1, ∂X_2 and ∂X_3. Bottom row: we draw the resolutions of these tangles in the interiors of X_1, X_2 and X_3. Note that vertices in $X_i \cap X_{i+1}$ are resolved negatively into X_i, while vertices in $X_{i-1} \cap X_i$ are resolved positively into X_i.

Lemma 3.25 *Let $\mathcal{T} = (X_1, X_2, X_3)$ be a trisection of a 4-manifold X^4. Let h be a $\mathcal{T}$-compatible Morse function on X^4, and $\mathcal{H}$ a Kirby diagram induced by h and a gradientlike vector field ∇h. Then $H_1 = X_3 \cap X_1$ and $H_2 = X_1 \cap X_2$ give a Heegaard splitting (H_1, H_2) for $h^{-1}(\frac{3}{2})$, in which $\Sigma := H_1 \cap H_2 \subset E(\mathcal{H})$ is a Heegaard surface.*

Suppose a banded unlink $(\mathcal{H}, L, B)$ is in bridge position with respect to Σ. Then a realizing surface $\Sigma(\mathcal{H}, L, B)$ is in bridge position with respect to $\mathcal{T}$.

Definition 3.26 Let S be a self-transverse immersed surface in a 4-manifold X^4 with trisection $\mathcal{T} = (X_1, X_2, X_3)$. Assume S is in (b, c)-bridge position. We call the triple of singular marked tangles $(T_1, T_2, T_3) = (S \cap X_1 \cap X_2, S \cap X_2 \cap X_3, S \cap X_3 \cap X_1)$ a *bridge trisection diagram* of S. The markings of each tangle should be chosen so that:

- In X_i, cross-sections of S are the negative resolution of $S \cap X_i \cap X_{i+1}$.
- In X_i, cross-sections of S are the positive resolution of $S \cap X_{i-1} \cap X_i$.

Note that we choose the marking convention to be symmetric with respect to the trisection, even though in the construction of Theorem 3.22, we used a Morse function h in which the pieces X_1, X_2 and X_3 were not symmetric. If (L, B) is a singular banded unlink diagram for S and we follow the construction of Theorem 3.22, then we obtain a bridge trisection diagram (T_1, T_2, T_3) of S with:

(i) $T_1 = L \cap H_2$ with markings agreeing with those of L.

(ii) $T_2 = (L \cap H_2)^+_B$.

(iii) $T_3 = L \cap \overline{H}_1$ with markings *opposite* those of L.

We include a local example in Figure 26.

From a bridge trisection diagram of S, we can reconstruct a surface that is ambiently isotopic to S as usual. For convenience (to mirror the construction in Theorem 3.22), we assume all self-intersections lie in H_1 and H_2 (ie in ∂X_1 and not in $X_2 \cap X_3$).

Lemma 3.27 *Let S be a self-transverse immersed surface in a 4-manifold X^4 that is in bridge position with respect to a trisection $\mathcal{T} = (X_1, X_2, X_3)$. Assume that S has no self-intersections in $X_2 \cap X_3$.*

Let h be a $\mathcal{T}$-compatible Morse function on X^4, and fix a gradientlike vector field ∇h inducing a Kirby diagram $\mathcal{H}$. Then there is a singular banded unlink diagram $(\mathcal{H}, L, B)$ such that (L, B) is in bridge position with respect to the Heegaard surface $\Sigma = X_1 \cap X_2 \cap X_3 \subset E(\mathcal{H})$, and S is $\mathcal{T}$-regularly isotopic to the surface $\Sigma(\mathcal{H}, L, B)$.

Proof Isotope S to be 0-standard (with respect to $h, \nabla h$). Since S is in bridge position, we may take this isotopy to be $\mathcal{T}$-regular.

Let $L := S \cap h^{-1}(\frac{3}{2})$. Recall $h^{-1}(\frac{3}{2}) = \partial X_1 = H_1 \cup H_2$, where $H_1 = X_3 \cap X_1$ and $H_2 = X_1 \cap X_2$. Then L is a singular link whose vertices are either in H_1 or H_2. Mark L so that the negative resolutions

of the vertices in H_1 and the positive resolutions of the vertices in H_2 correspond to the resolutions of the immersed disk system $S \cap X_1$. Then L is a marked singular link and L^- is an unlink.

Now $S \cap X_2$ is a trivial immersed disk system with all intersections in $X_1 \cap X_2$. Let $\widetilde{X}_2$ be obtained from X_2 by deleting a small neighborhood of each intersection, so that $\widetilde{X}_2$ is still a 4-dimensional 1-handlebody, but $S \cap \widetilde{X}_2$ is a trivial embedded disk system D. Let $\widetilde{H}_2$ denote the closure of $(\partial \widetilde{X}_2) \setminus (X_2 \cap X_3)$.

Now D is a collection of boundary parallel disks in $\widetilde{X}_2$, and $\partial \widetilde{X}_2$ has a Heegaard splittings $(\widetilde{H}_2, X_2 \cap X_3)$, which in respect to ∂D is in bridge position. We proceed as in [26, Lemma 3.3]: For each component D_i of D, let a_i be one component of $\overline{\partial D \setminus (X_2 \cap X_3)}$. Then let y_i be an arc in $\partial \widetilde{X}_2$ parallel to $\partial D_i \setminus a_i$ with endpoints on ∂D, with framing induced by D_i. Isotope y_i in ∂X_2 into the Heegaard surface for $\partial \widetilde{X}_2$, twisting y_i around ∂D as necessary so that the framing of y_i agrees with the framing induced by the Heegaard surface. Finally, project y_i to ∂X_2, push slightly into H_2, and thicken (according to the framing of y_i) to obtain a band attached to $S \cap H_2$ (ie a band b_i in $h^{-1}\left(\frac{3}{2}\right)$ attached to L, with b_i in H_2 parallel to $H_1 \cap H_2$).

Repeat this for every component of D to obtain a collection B of bands for L. By construction, L_B^+ is an unlink when projected to $h^{-1}\left(\frac{5}{2}\right)$. More specifically, in $\mathcal{H}$ the link L_B^+ (projected to $h^{-1}\left(\frac{5}{2}\right)$) can be made to agree with the link $S \cap h^{-1}\left(\frac{5}{2}\right)$ via an isotopy rel boundary in H_2 and slides in H_2 over curves in $\mathcal{H}$.

We conclude immediately that $(\mathcal{H}, L, B)$ is a singular banded unlink for some surface $S' := \Sigma(\mathcal{H}, L, B)$ in X. Moreover, S' is in bridge position with respect to $\mathcal{T}$, and by the above paragraph can be $\mathcal{T}$-regularly isotoped so that it agrees with S in $X_i \cap X_j$ for all $i \neq j$. Therefore, S and S' are $\mathcal{T}$-regularly isotopic. □

Remark 3.28 Fix a trisection $\mathcal{T} = (X_1, X_2, X_3)$ of X, a $\mathcal{T}$-compatible Morse function h and a gradientlike vector field ∇h, so that $(h, \nabla h)$ induce a Kirby diagram $\mathcal{H}$ of X in which $\Sigma := X_1 \cap X_2 \cap X_3$ is a Heegaard surface. Definition 3.26 and Lemma 3.27 can be combined to form the equivalence

$$\frac{\{\text{bridge trisections with respect to } \mathcal{T} \text{ with no self-intersections in } X_2 \cap X_3\}}{\mathcal{T}\text{-regular isotopy}} \leftrightarrow \frac{\{\text{SBUDs in } \mathcal{H} \text{ in bridge position with respect to } \Sigma\}}{\text{singular band moves preserving } \Sigma \text{ setwise}}.$$

The restriction of bridge position to not include self-intersections in $X_2 \cap X_3$ is merely a diagrammatic convenience from the viewpoint of singular banded unlinks diagrams (SBUDs).

Lemma 3.29 *Let S be in bridge position with respect to $\mathcal{T} = (X_1, X_2, X_3)$. There exists a sequence of perturbations of S yielding a surface S' in bridge position such that S' has no self-intersections in $X_2 \cap X_3$.*

To inductively prove Lemma 3.29, it is clearly sufficient to prove the following proposition:

Proposition 3.30 *Suppose there are $n > 0$ self-intersections of S in $X_2 \cap X_3$. Then, after $\mathcal{T}$-regular isotopy of S, there is a surface S' obtained from vertex perturbation on S such that S' has $n - 1$ self-intersections in $X_2 \cap X_3$.*

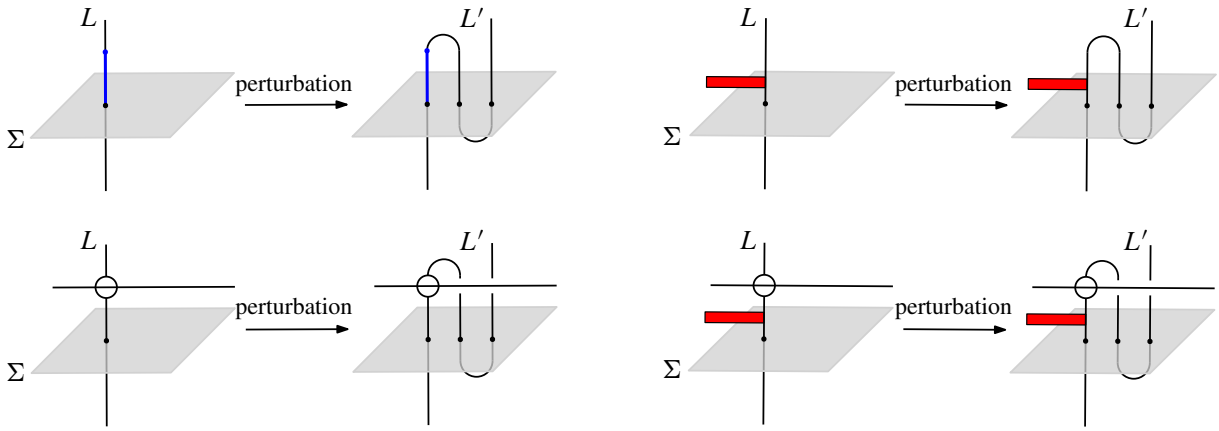


Figure 27: When performing a perturbation on the diagram in the top left, we allow the blue arc to intersect at most one band and one vertex, as shown in the other three diagrams.

Proof Following from Definition 3.11 of a trivial immersed tangle, some $\mathcal{T}$-regular isotopy of S can arrange for the tangle $T = S \cap X_2 \cap X_3$ to lie inside a collar neighborhood $\Sigma \times I \subset X_2 \cap X_3$ of $\partial(X_2 \cap X_3) = \Sigma$, so that projection to the I factor is Morse on T with one maximum on each arc component. Further isotope so that the self-intersections of S in $\Sigma \times I$ lie at different values of the I factor. In particular, one self-intersection c lies strictly closest to Σ. Then by $\mathcal{T}$-regular isotopy of S near Σ (sometimes called "mutual braid transposition" when performed diagrammatically), we can arrange for c to have a neighborhood as in Figure 22, and thus apply a vertex perturbation to S to obtain a surface S' with one less self-intersection in $X_2 \cap X_3$. □

3.5 Uniqueness of bridge trisections of immersed surfaces

Perturbation of bridge trisections is conveniently very similar to perturbation of a banded link in bridge position. When perturbing a banded link (L, B) with respect to a Heegaard surface Σ, we allow at most one band and one vertex to be between the intersection of L and Σ at which the perturbation is based and the two newly introduced intersections. See Figure 27.

Lemma 3.31 *Let $\mathcal{T} = (X_1, X_2, X_3)$ be a trisection of a 4-manifold X^4. Let h be a $\mathcal{T}$-compatible Morse function on X^4, and $\mathcal{K}$ a Kirby diagram induced by h. Let $H_1 := X_3 \cap X_1$ and $H_2 := X_1 \cap X_2$ give the usual Heegaard splitting (H_1, H_2) for $\mathcal{K}$, in which $\Sigma := H_1 \cap H_2$ is the Heegaard surface.*

Suppose a singular banded unlink diagram $(\mathcal{K}, L, B)$ is in bridge position with respect to Σ. Let $(\mathcal{K}, L', B')$ be obtained from $(\mathcal{K}, L, B)$ by perturbation near $L \cap \Sigma$. Then $\Sigma(\mathcal{K}, L', B')$ can be obtained from $\Sigma(\mathcal{K}, L, B)$ by perturbation followed by $\mathcal{T}$-regular isotopy.

Proof See Figure 28, top. □

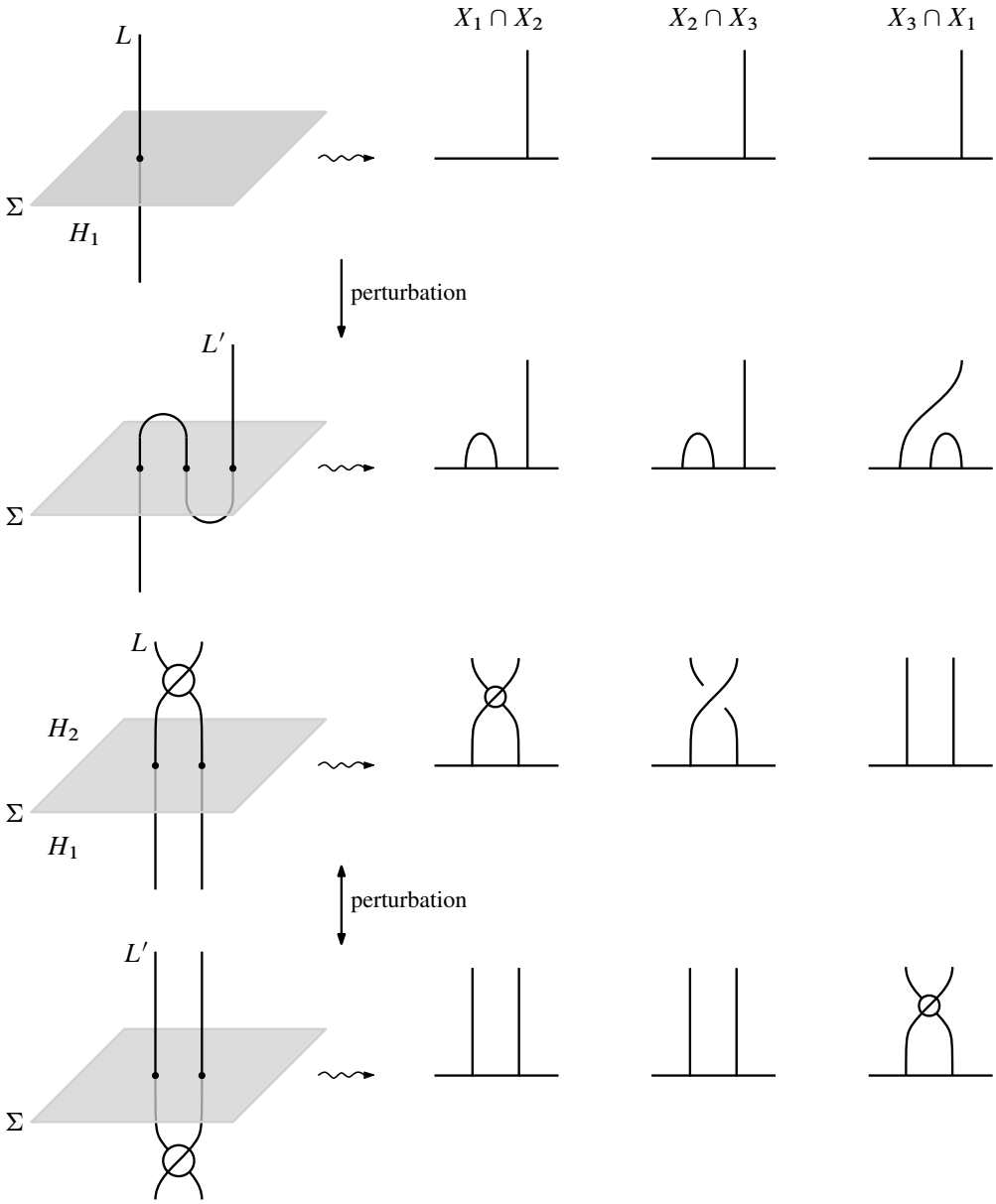


Figure 28: Perturbation of a singular banded unlink (L, B) in bridge position induces perturbation of $\Sigma(L, B)$. Top: elementary perturbation. Bottom: vertex perturbation.

Lemma 3.32 *Let $\mathcal{T} = (X_1, X_2, X_3)$ be a trisection of a 4-manifold X^4. Let h be a $\mathcal{T}$-compatible Morse function on X^4, and $\mathcal{K}$ a Kirby diagram induced by h. Let $H_1 := X_3 \cap X_1$ and $H_2 := X_1 \cap X_2$ give the usual Heegaard splitting (H_1, H_2) for $\mathcal{K}$, in which $\Sigma := H_1 \cap H_2$ is the Heegaard surface.*

Suppose a singular banded unlink diagram $(\mathcal{K}, L, B)$ is in bridge position with respect to Σ and that v is a vertex of L that is close to Σ as in Figure 20. Let $(\mathcal{K}, L', B')$ be obtained from $(\mathcal{K}, L, B)$ by isotoping

v through Σ. (We call this a *vertex perturbation of the banded link* (L, B).) *Then $\Sigma(\mathcal{K}, L', B')$ can be obtained from $\Sigma(\mathcal{K}, L, B)$ by one vertex perturbation followed by $\mathcal{T}$-regular isotopy.*

Proof See Figure 28, bottom. □

The following uniqueness of bridge splittings of banded links motivates the uniqueness of bridge trisections:

Theorem 3.33 *Let (L, B) and (L', B') be isotopic banded singular marked links in a 3-manifold M that has a Heegaard splitting (H_1, H_2). Assume that both (L, B) and (L', B') are in bridge position with respect to $\Sigma := H_1 \cap H_2$, and that B and B' are both contained in H_2. Then there exists a banded singular marked link (L'', B'') in bridge position with respect to Σ that can be obtained from both (L, B) and (L', B') by sequences of elementary perturbations, vertex perturbations and isotopies that fix Σ setwise.*

Theorem 3.33 is similar to a theorem for nonsingular banded links due to Meier and Zupan [26; 27].

Remark 3.34 Meier and Zupan study banded links by viewing each band as a framed arc with endpoints on a link. They give moves to perturb a link in order to make these framed arcs parallel to a bridge surface with correct framing. In the setting of singular banded links, we are able to use their proof by viewing both self-intersections and bands as framed arcs, applying the theorem and then contracting the self-intersection arcs to yield a singular link in bridge position.

Proof As in Theorem 3.15, there exist disjoint framed arcs $a_1, \dots, a_n$ with endpoints on L^+ such that contracting L^+ along $a_1, \dots, a_n$ yields L. Similarly, there exist framed arcs $a'_1, \dots, a'_n$ with endpoints on L'^+ such that contracting L'^+ along $a'_1, \dots, a'_n$ yields L'.

Now, by Meier and Zupan [26; 27], there exists a link J that can be obtained from L^+ and from L'^+ by elementary perturbations and isotopies fixing Σ setwise. Moreover, these isotopies and perturbations carry a_i and a'_i to framed arcs b_i and b'_i, respectively, with endpoints on J, so that b_i and b'_i are each parallel to Σ with surface framing, and either agree or could be isotoped to agree if the endpoints of b'_i were allowed to pass through Σ (ie b_i and b'_i are parallel and both lie close to Σ, but potentially on opposite sides). Moreover, during the perturbations and isotopies of L^+ (resp. L'^+), a_i (resp. a'_i) never intersect Σ, so these perturbations and isotopies may be achieved by perturbations and isotopies of L (resp. L').

Meier and Zupan's proof allows us to not only control the framed arcs a_i, a'_i, but also the framed arcs that are the cores of the bands B and B'. That is, by perhaps perturbing J even further, we may also assume that B and B' are taken to bands B_J, and B'_J whose i^{th} bands either agree or are parallel and close to Σ but on opposite sides, and that (J, B_J) and (J, B'_J) are both in bridge position. Let $\widehat{J}$ and $\widehat{J}'$ be the marked singular links obtained by contracting J along b_i and b'_i, respectively, and with markings induced by L and L'. Then $\widehat{J}'$ can be transformed into $\widehat{J}$ by isotopy fixing Σ and a vertex perturbation for each pair a_i and a'_i in different components of $M \setminus \Sigma$. Therefore, the claim holds with $L'' = \widehat{J}$ and $B'' = B_J$. □

Corollary 3.35 *If $\mathcal{D} = (L, B)$ and $\mathcal{D}' = (L', B')$ are isotopic banded unlink diagrams that are each in bridge position with respect to Σ, then $S := \Sigma(\mathcal{D})$ and $S' := \Sigma(\mathcal{D}')$ are related by elementary perturbation and deperturbation, vertex perturbation and $\mathcal{T}$-regular isotopy.*

Proof By Theorem 3.33, $\mathcal{D}$ and $\mathcal{D}'$ are related by a sequence of elementary perturbations and deperturbations, vertex perturbations and isotopies fixing Σ setwise. It is therefore sufficient to show that the claim is true if $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a single one of these moves. We have already shown the claim to be true when $\mathcal{D}'$ is obtained from $\mathcal{D}$ by either a perturbation/deperturbation (Lemma 3.31), or a vertex perturbation (Lemma 3.32). So suppose that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by an isotopy f_t of M that fixes Σ setwise.

The surface $\Sigma_{3/2} := \Sigma$ is a separating surface in $M = h^{-1}\left(\frac{3}{2}\right)$. For every $t \in [0, 4]$, there is a separating surface Σ_t in $h^{-1}(t)$ that is vertically above or below Σ. Then f_t can be extended to a horizontal isotopy of the whole 4-manifold X^4 that fixes every Σ_t horizontally. Since all index 2 critical points of h are contained in one component of $X^4 \setminus \bigcup_t \Sigma_t$, this isotopy can be chosen to take S to S'. Since this isotopy is horizontal, it fixes $X_1 = h^{-1}\left(\left[0, \frac{3}{2}\right]\right)$ and $X_2 \cup X_3 = h^{-1}\left(\left[\frac{3}{2}, 4\right]\right)$ setwise. Since this isotopy fixes $X_2 \cap X_3 = \bigcup_{[3/2,4]} \Sigma_t$ setwise, it also fixes X_2 and X_3 setwise. Therefore, this is a $\mathcal{T}$-regular isotopy. □

The main theorem of this section is that bridge position and hence bridge trisection diagrams are essentially unique. The proof uses Theorem 2.39.

Theorem 3.36 *Let S and S' be self-transverse immersed surfaces in bridge position with respect to a trisection $\mathcal{T} = (X_1, X_2, X_3)$ of a closed 4-manifold X^4. Suppose S is ambiently isotopic to S'. Then S can be taken to S' by a sequence of elementary perturbations and deperturbations, vertex perturbations and $\mathcal{T}$-regular isotopy.*

Proof Let $h\colon X \to [0, 4]$ be a $\mathcal{T}$-compatible Morse function on X^4. Let $\mathcal{K}$ be a Kirby diagram for X induced by h and a fixed choice of ∇h. As usual, we view $\Sigma := X_1 \cap X_2 \cap X_3$ as a Heegaard surface for the ambient space of $\mathcal{K}$, with the dotted circles of $\mathcal{K}$ contained in one handlebody H_1 of this splitting and the 2-handle circles of $\mathcal{K}$ contained in the other handlebody H_2.

By Lemma 3.29, we may $\mathcal{T}$-regularly isotope and perturb S and S' so that they do not include self-intersections in $X_2 \cap X_3$. Then, by Lemma 3.27, there are banded unlink diagrams $\mathcal{D} := (\mathcal{K}, L, B)$ and $\mathcal{D}' := (\mathcal{K}, L', B')$ such that (L, B) and (L', B') are in bridge position with respect to Σ and such that S and S' are $\mathcal{T}$-regular isotopic to $\Sigma(\mathcal{D})$ and $\Sigma(\mathcal{D}')$, respectively.

By Theorem 2.39, $\mathcal{D}$ and $\mathcal{D}'$ are related by a sequence of singular band moves. By Corollary 3.35, if $\mathcal{D}$ and $\mathcal{D}'$ are isotopic, then the theorem holds.

Assume that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by one singular band move (other than isotopy). We will show that S' and S become $\mathcal{T}$-regular isotopic after some sequence of perturbations and deperturbations. The theorem will then hold via induction on the length of a sequence of band moves relating $\mathcal{D}$ and $\mathcal{D}'$.

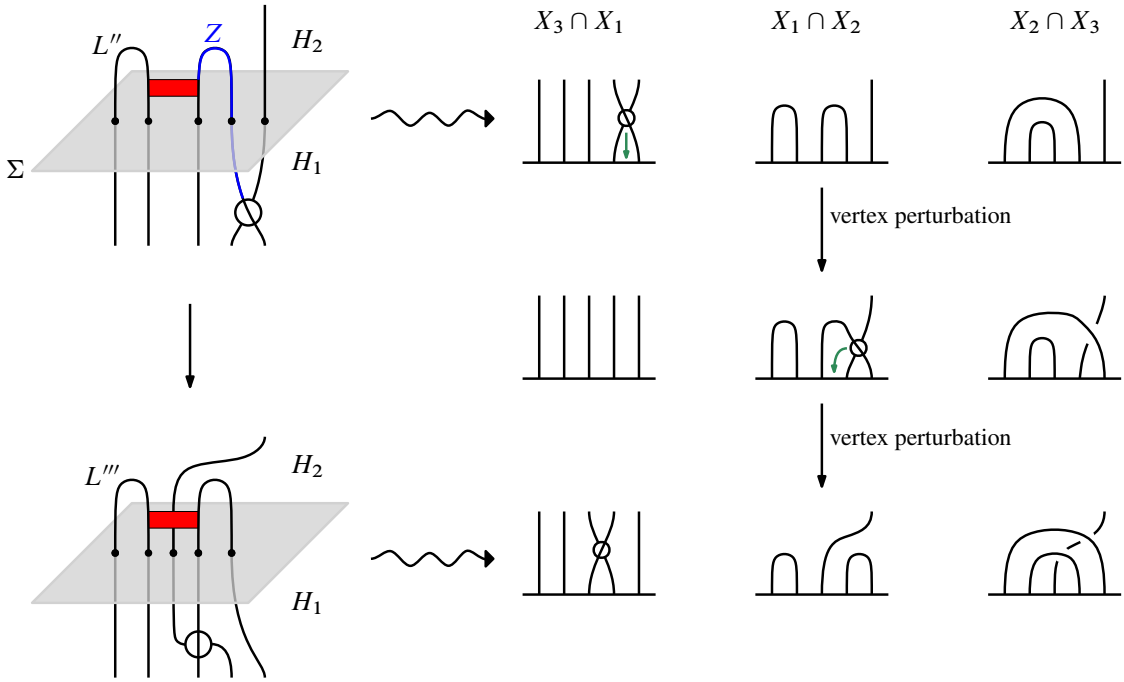


Figure 29: Left: the singular banded unlink (L''', B''') is obtained from (L'', B'') by an intersection/band pass. Right: we show that $\Sigma(L''', B''')$ (bottom) may be obtained from $\Sigma(L'', B'')$ (top) by two vertex perturbations and $\mathcal{T}$-regular isotopy.

Meier and Zupan [26] previously showed that the claim holds when the move turning $\mathcal{D}$ into $\mathcal{D}'$ is a cup, cap, band swim or band slide. The authors of this paper [14] showed the claim is true when the move is a 2-handle/band slide, 2-handle/band swim or dotted circle slides. These arguments were technically only made for nonsingular banded unlinks, so we repeat them in the singular setting for clarity, often repeating Meier and Zupan's arguments. In the following paragraphs, we consider every singular band move that might transform $\mathcal{D}$ into $\mathcal{D}'$.

Intersection/band pass Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by an intersection/band pass along a framed arc z in L between a vertex of L and a band in B. Isotope (L, B) so that z is as in Figure 29, top left. Then isotope the rest of L and B outside a neighborhood of z to obtain a banded link (L'', B'') in bridge position. This banded singular link is isotopic to (L, B), so, by Corollary 3.35, $S'' := \Sigma(L'', B'')$ is obtainable from S by (de)perturbations and $\mathcal{T}$-regular isotopy. Let (L''', B''') be obtained from (L'', B'') by performing the intersection/band pass along z, and let $S''' := \Sigma(L''', B''')$. Now the intersection of S''' with each $X_i \cap X_j$ is isotopic rel boundary to the intersection of S'' with $X_i \cap X_j$, so S''' is $\mathcal{T}$-regular isotopic to S''. Finally, by Corollary 3.35, we find that S''' can be transformed into S' by (de)perturbations and $\mathcal{T}$-regular isotopy.

Intersection/band slide Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by an intersection/band slide along a framed arc z in L between a vertex of L and a band in B. Isotope (L, B) so that z is short and contained

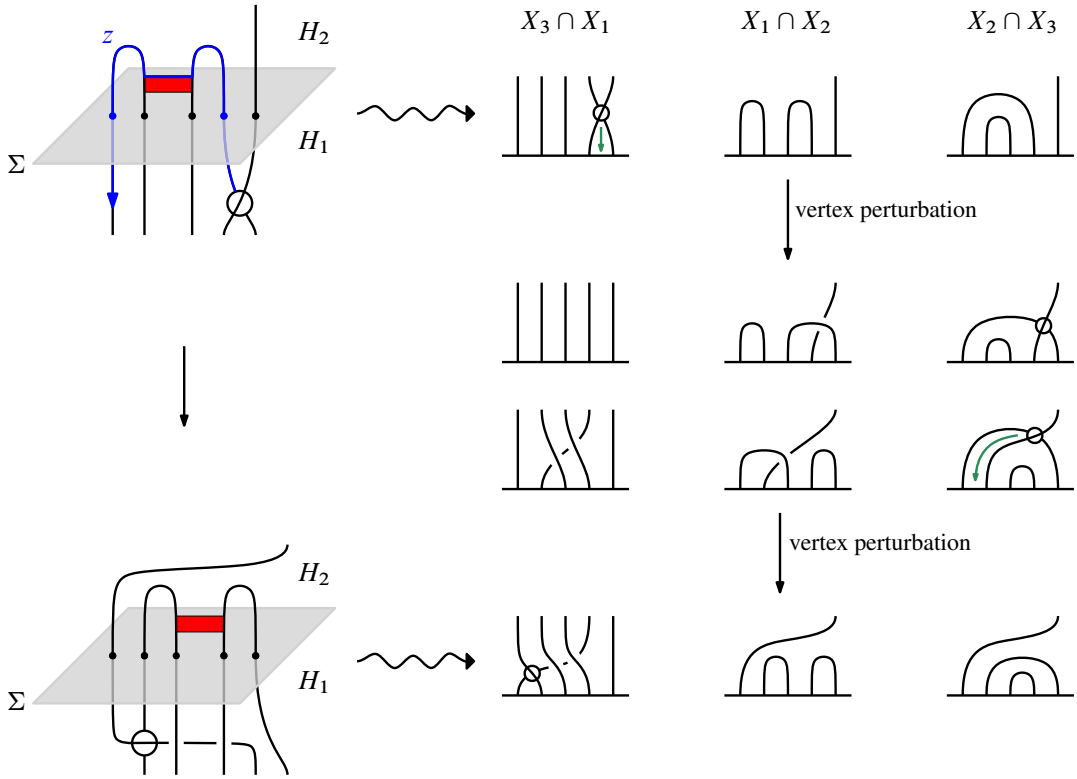


Figure 30: Left: the singular banded unlink (L''', B''') is obtained from (L'', B'') by an intersection/band slide. Right: we show that $\Sigma(L''', B''')$ (bottom) may be obtained from $\Sigma(L'', B'')$ (top) by two vertex perturbations and $\mathcal{T}$-regular isotopy.

in H_2 in a neighborhood as in Figure 30. Then isotope the rest of L and B outside this neighborhood to obtain a banded link (L'', B'') in bridge position. This banded singular link is isotopic to (L, B), so, by Corollary 3.35, $S'' := \Sigma(L'', B'')$ is obtainable from S by (de)perturbations and $\mathcal{T}$-regular isotopy. Let (L''', B''') be obtained from (L'', B'') by performing the intersection/band slide along z, and let $S''' := \Sigma(L''', B''')$. In Figure 30, we show that S''' can be obtained from S'' by perturbation and $\mathcal{T}$-regular isotopy. Finally, by Corollary 3.35, S''' can be transformed into S' by (de)perturbations and $\mathcal{T}$-regular isotopy.

Cup Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a cup move. It does not matter in which direction we take the move, so assume that L' is obtained from L by adding a new unlink component O contained in a ball not meeting L or B, and B' is obtained from B by adding a trivial band b_O from L to O. By isotopy and intersection/band passes, we may take O to be in 1-bridge position with respect to Σ, and b_O to be in H_2, contained in a neighborhood as in Figure 31. Performing the cup move yields a diagram $\mathcal{D}''$ that is related to $\mathcal{D}'$ by isotopy and intersection/band passes; by Corollary 3.35 and the already-considered intersection/band pass case, $\Sigma(\mathcal{D}'')$ can be transformed into S' by perturbation and $\mathcal{T}$-regular isotopy. Finally, we observe that $\Sigma(\mathcal{D}'')$ is obtained from the (perturbed) surface S by perturbation (see Figure 31).

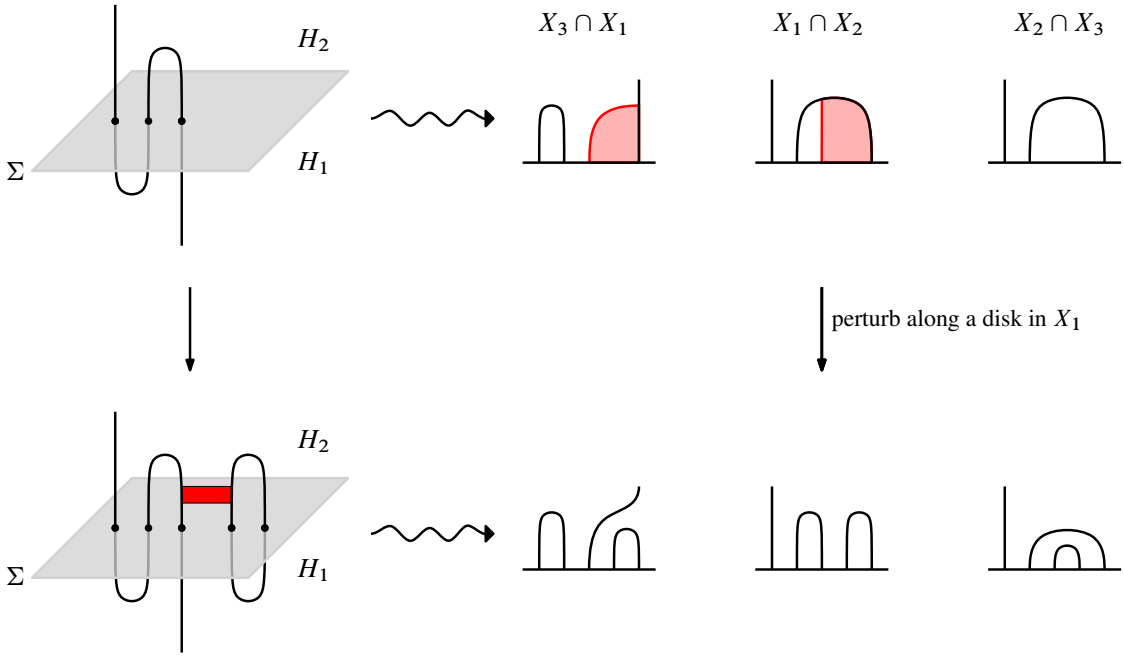


Figure 31: Left: the singular banded unlink (L''', B''') is obtained from (L'', B'') by a cup move. Right: we show that $\Sigma(L''', B''')$ (bottom) may be obtained from $\Sigma(L'', B'')$ (top) by an elementary perturbation and $\mathcal{T}$-regular isotopy.

Cap Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a cap move. Again, it does not matter in which direction we take the move, so assume that $L' = L$ and B' is obtained from B by adding a trivial band b. By isotopy and intersection/band passes, we may take b to have a neighborhood as in Figure 32. Performing the cap move yields a diagram $\mathcal{D}''$ that is related to $\mathcal{D}'$ by isotopy and intersection/band passes; by Corollary 3.35 and the case for intersection/band pass, $\Sigma(\mathcal{D}'')$ can be transformed into S' by perturbation and $\mathcal{T}$-regular isotopy. Finally, we observe that $\Sigma(\mathcal{D}'')$ is obtained from the (perturbed) surface S by perturbation and deperturbation (see Figure 32).

Band swim Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a band swim. Isotope $\mathcal{D}$ to obtain a diagram in which the band swim looks as in Figure 33. Perform the band swim to obtain a diagram $\mathcal{D}''$ that is related to $\mathcal{D}'$ by isotopy; by Corollary 3.35 and the intersection/band swim case, $\Sigma(\mathcal{D}'')$ can be transformed into S' by perturbation and $\mathcal{T}$-regular isotopy. Finally, we observe that $\Sigma(\mathcal{D}'')$ is obtained from the (perturbed) surface S by $\mathcal{T}$-regular isotopy (see Figure 33).

Band slide Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a band slide. Isotope $\mathcal{D}$ to obtain a diagram in bridge position in which the desired band slide looks like Figure 34. By Corollary 3.35, the effect on S can be achieved by (de)perturbation and $\mathcal{T}$-regular isotopy. Call the result of the band slide $\mathcal{D}''$; by Corollary 3.35, the surface $\Sigma(\mathcal{D}'')$ can be transformed into S' by (de)perturbation and $\mathcal{T}$-regular isotopy. In Figure 34, we observe that $\Sigma(\mathcal{D}'')$ is obtained from S by perturbation and deperturbation.

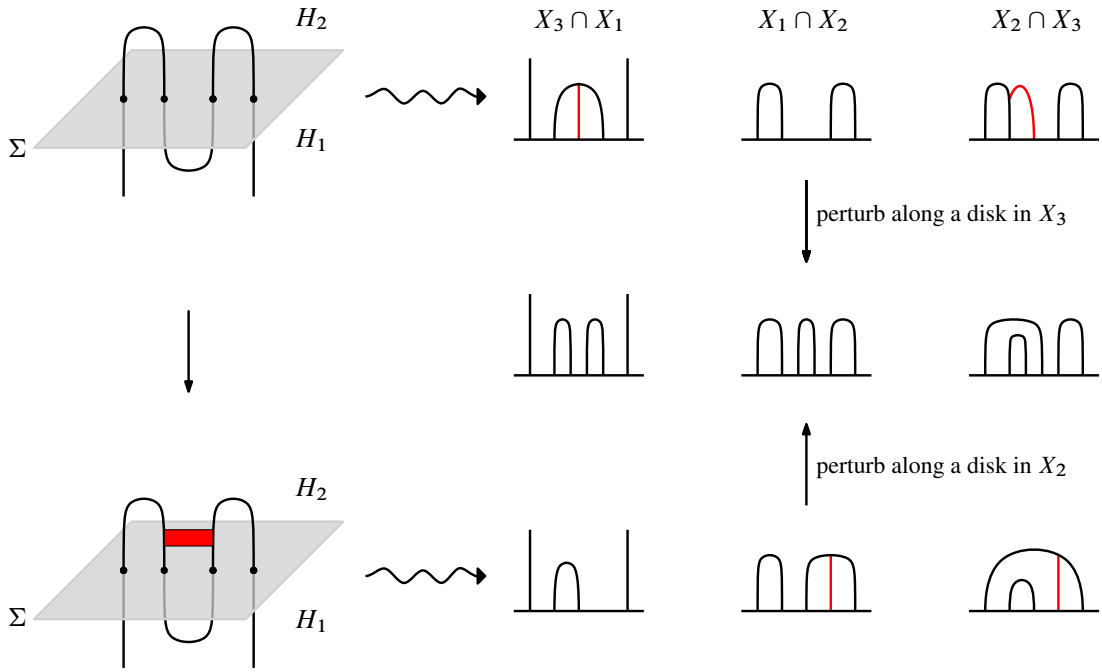


Figure 32: Left: the singular banded unlink $\mathcal{D}''$ is obtained from $\mathcal{D}$ by a cap move. Right: we show that $\Sigma(\mathcal{D}'')$ (bottom) may be obtained from $\Sigma(\mathcal{D})$ (top) by an elementary perturbation and deperturbation and $\mathcal{T}$-regular isotopy.

2-handle/band slide Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by sliding a band over a 2-handle via a framed arc z between a band in B and a 2-handle attaching circle in $\mathcal{K}$. As in the band slide case, we may perturb $\mathcal{D}$ so that z is contained in H_2 (see Figure 35). Now, performing the slide along z yields a diagram $\mathcal{D}''$ that is

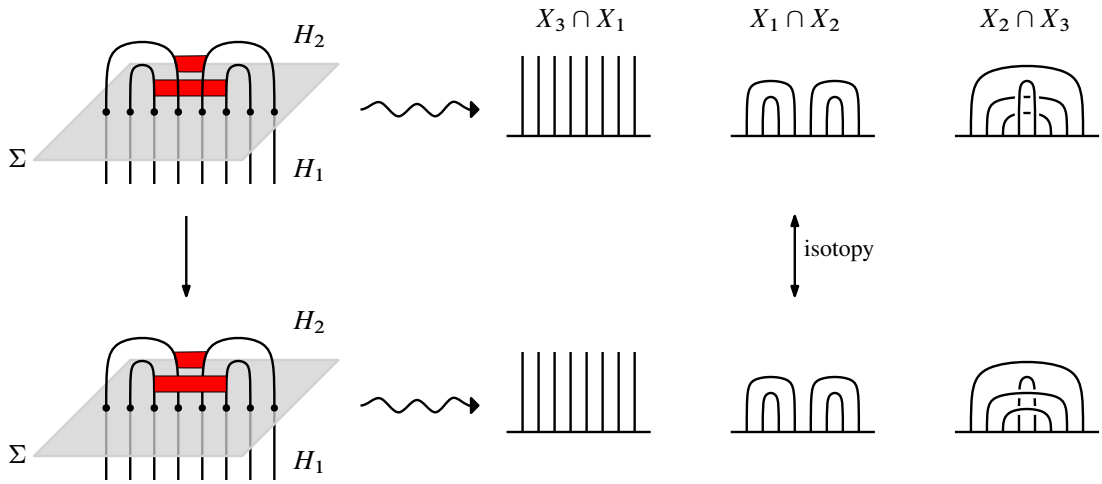


Figure 33: Left: the singular banded unlink $\mathcal{D}''$ is obtained from $\mathcal{D}$ by a band swim. Right: we show that $\Sigma(\mathcal{D}'')$ (bottom) may be obtained from $\Sigma(\mathcal{D})$ (top) by $\mathcal{T}$-regular isotopy.

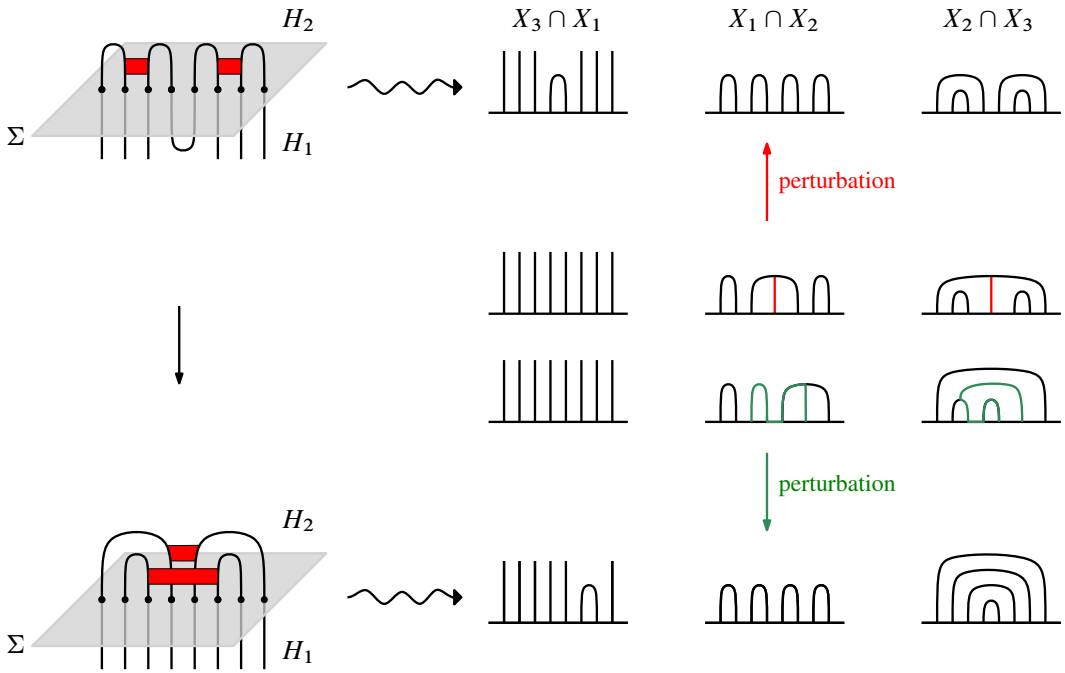


Figure 34: Left: the singular banded unlink $\mathcal{D}''$ is obtained from $\mathcal{D}$ by a band slide. Right: we show that $\Sigma(\mathcal{D}'')$ (bottom) may be obtained from $\Sigma(\mathcal{D})$ (top) by an elementary perturbation and deperturbation and $\mathcal{T}$-regular isotopy.

related to $\mathcal{D}'$ by isotopy; by Corollary 3.35, the surface $\Sigma(\mathcal{D}'')$ can be transformed into S' by perturbation and $\mathcal{T}$-regular isotopy. Finally, we observe that $\Sigma(\mathcal{D}'')$ is obtained from the (perturbed) surface S by $\mathcal{T}$-regular isotopy supported in X_2 and X_3.

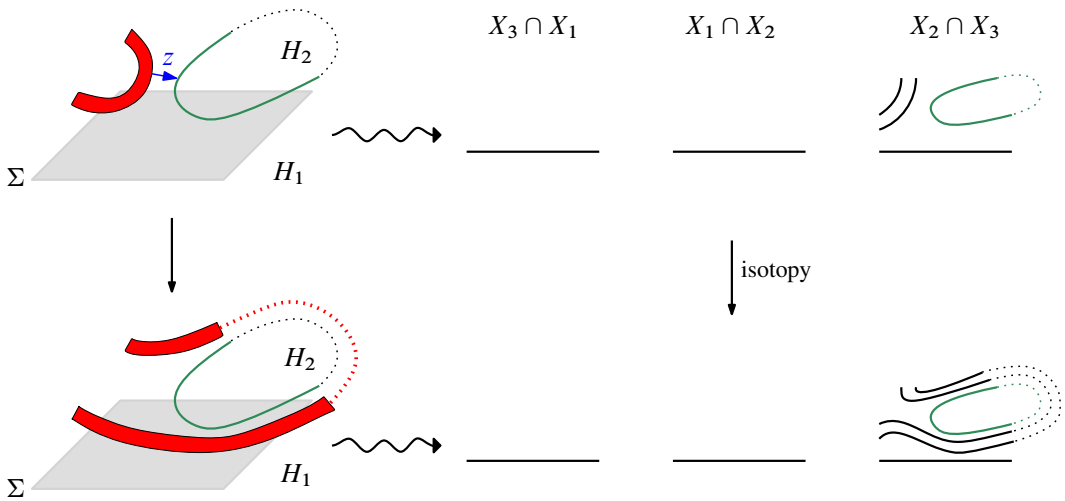


Figure 35: Left: the singular banded unlink $\mathcal{D}''$ is obtained from $\mathcal{D}$ by a 2-handle/band slide. Right: we show that $\Sigma(\mathcal{D}'')$ (bottom) may be obtained from $\Sigma(\mathcal{D})$ (top) by $\mathcal{T}$-regular isotopy.

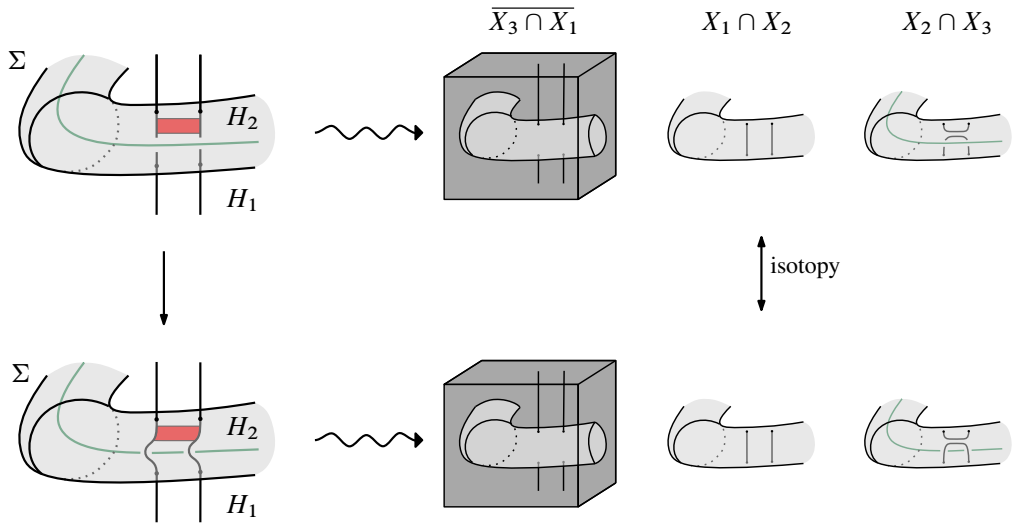


Figure 36: Left: the singular banded unlink $\mathcal{D}''$ is obtained from $\mathcal{D}$ by a 2-handle/band swim. Right: we show that $\Sigma(\mathcal{D}'')$ (bottom) may be obtained from $\Sigma(\mathcal{D})$ (top) by $\mathcal{T}$-regular isotopy.

2-handle/band swim Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by swimming a 2-handle through a band. Isotope $\mathcal{D}'$ so that the swim looks like the one in Figure 36. By Corollary 3.35, this can be achieved by (de)perturbations and $\mathcal{T}$-regular isotopy of S. Now, performing the swim along z yields a diagram $\mathcal{D}''$ that is related to $\mathcal{D}'$ by isotopy; by Corollary 3.35, the surface $\Sigma(\mathcal{D}'')$ can be transformed into S' by perturbation and $\mathcal{T}$-regular isotopy. Finally, we observe that $\Sigma(\mathcal{D}'')$ is obtained from the (perturbed) surface S by $\mathcal{T}$-regular isotopy supported in X_2 and X_3.

Slide of a band or L over a dotted circle This follows from Theorem 3.33, as slides over dotted circles are simply isotopies of the banded link (L, B) in $M_{3/2}$. □

4 Some example applications

In this (comparatively short) section, we give a few sample applications of the diagrammatic theory of singular banded unlink diagrams.

4.1 Calculating the Kirk invariant

Schneiderman and Teichner [30] classified all 2-component spherical links in S^4 up to link homotopy using the Kirk invariant $\sigma_i(F_1, F_2) := \lambda(F_i, F_i')$. Here $i \in \{1, 2\}$, F_i' is a parallel pushoff of F_i, and $\lambda(F_i, F_i')$ is Wall's intersection invariant. Furthermore, F_i denotes an oriented immersed 2-sphere in S^4, with F_1 and F_2 disjoint. The Kirk invariant takes values in $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[x^{\pm}]$.

Schneiderman and Teichner showed that the set of all 2-component spherical links in S^4 up to link homotopy is a free R-module, where $R = \mathbb{Z}[z_1, z_2]/(z_1 z_2)$ is freely generated by the Fenn–Rolfsen link FR depicted in Figure 37.

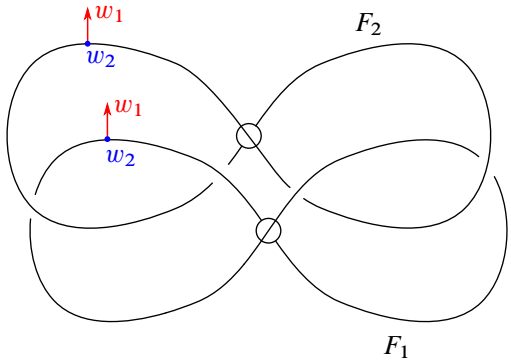


Figure 37: The Fenn–Rolfsen link. At the indicated points with arrows, a positive basis of the normal bundle is (w_1, w_2), where w_1 is the drawn arrow pointing upward and w_2 points out of the page toward the reader.

In this subsection, we show how to compute the Kirk invariant of FR. This computation can be adapted to compute Wall's self-intersection invariant for general 2-component spherical links in arbitrary closed orientable 4-manifolds. Since FR has a symmetry between its two components that reverses the orientation on one component, we have $\sigma_2 = -\sigma_1$ and thus only compute σ_1.

Consider the singular banded unlink diagram of $\mathrm{FR} = F_1 \sqcup F_2$ as in Figure 37. Choose a basepoint p far away from FR and an arc γ from p to a point q in F_1. Take a pushoff F_1' of F_1 that transversely intersects F_1; simultaneously push off γ to obtain an arc γ' from p to a point q' of F_1'.

We thus have two parallel arcs γ' and γ from p to F_1' and from p to F_1, respectively (as in Figure 38). Now delete a neighborhood of F_2 as in Figure 39.

Pick a vertex v between the diagrams of F_1 and F_1', and choose arcs η, η' contained in F_1 and F_1', respectively, from q and q', respectively, to v. Let C_v be the based loop obtained by concatenating γ, η,

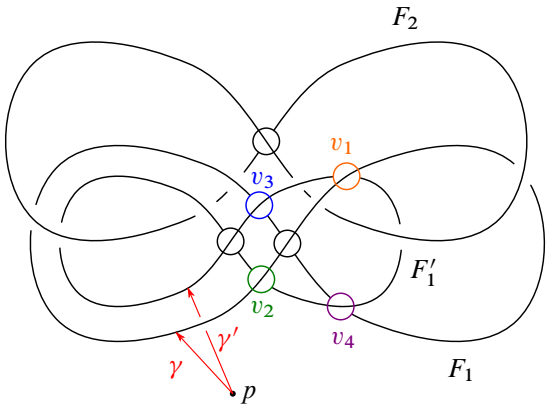


Figure 38: A parallel pushoff F_1' of F_1 that intersects F_1 transversely in four points yielding vertices v_1, v_2, v_3 and v_4 in the singular banded unlink diagram. The intersections have respective signs $s_{v_1} = 1$, $s_{v_2} = -1$, $s_{v_3} = -1$ and $s_{v_4} = 1$.

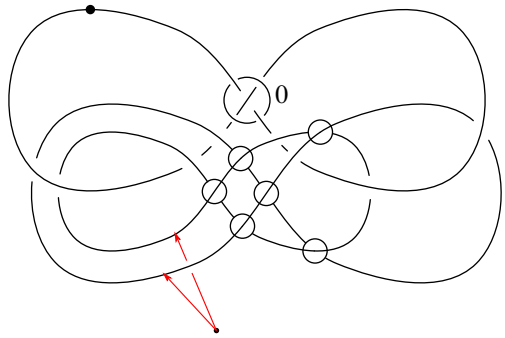


Figure 39: We delete a neighborhood of F_2. The resulting singular banded unlink diagram of $F_1 \cup F_1'$ is in a Kirby diagram with one 1-handle and one 2-handle.

$-\eta'$ and $-\gamma'$. There are four vertices v_1, v_2, v_3 and v_4 shared between the diagrams of F_1 and F_1'; see Figure 40 for potential loops C_{v_i} for all $i = 1, 2, 3, 4$. Note that each loop might pass through the other intersections in the singular banded unlink diagram, but we always can perturb each loop a little bit on the actual surface FR to miss the intersections.

Now each loop C_{v_i} represents some element of $H_1(S^4 - F_2) = \mathbb{Z}$. In addition, each vertex has a sign $s_{v_i} \in \{-1, +1\}$ given by the sign of the corresponding intersection of F_1 and F_1', which agrees with the sign of the crossing when the marking is resolved negatively. The values of $[C_{v_i}]$ and s_{v_i} are as follows:

i	s_{v_i}	$[C_{v_i}]$
1	1	0
2	-1	1
3	-1	-1
4	1	0

The Kirk invariant σ_1 is then given by

$$\sigma_1(\mathrm{FR}) = \sum_{i=1}^{4} s_{v_i} x^{[C_{v_i}]} = -x + 2 - x^{-1}.$$

The above computation generalizes for any singular banded unlink diagram of a 2-component spherical link (F_1, F_2) in S^4; use whiskers from a basepoint p to F_1 and a parallel pushoff F_1' intersecting F_1 in $v_1, \dots, v_n$ to form a loop C_{v_i} for each v_i representing $[C_{v_i}] \in H_1(S \setminus F_2) = \mathbb{Z}$. Then $\sigma_1(F_1, F_2) = \sum_{i=1}^{n} s_{v_i} x^{[C_{v_i}]}$.

4.2 Immersed surfaces and stabilization

Hosokawa and Kawauchi [13] showed that any pair of embedded oriented surfaces in S^4 become isotopic after some number of *stabilizations*.

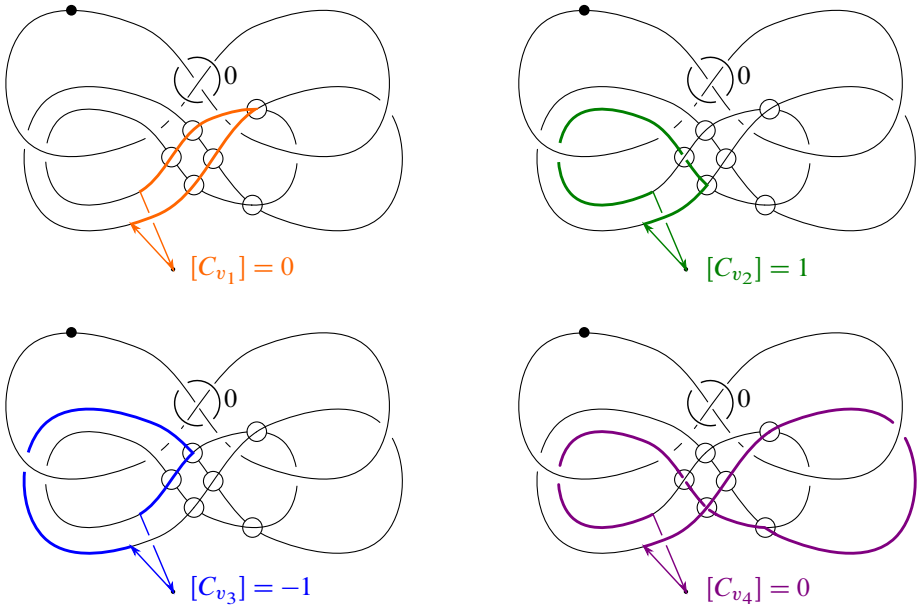


Figure 40: The loops C_{v_1}, C_{v_2}, C_{v_3} and C_{v_4}, respectively, represent the elements 0, 1, -1 and 0 in $H_1(S^4 \setminus F_2) = \mathbb{Z}$.

Definition 4.1 Let F be a connected, self-transversely immersed genus g oriented surface in S^4. Let γ be an arc with endpoints on F and which is normal to F near $\partial\gamma$, but with the interior of γ disjoint from F. Frame γ so that $\gamma \times D^2$ is a 3-dimensional 1-handle with ends on F, and so that surgering F along this 1-handle yields an oriented genus $g+1$ surface F'. Then we say F' is obtained from F by *stabilization*.

Remark 4.2 In Definition 4.1, there are two distinct ways to frame γ to obtain a 3-dimensional 1-handle with ends on F. However, one of these choices will yield a nonorientable surface after surgery, so in fact the framing of γ need not be specified.

More generally, Baykur and Sunukjian [2] extended this result for any pair of homologous embedded oriented surfaces in a closed orientable 4-manifold, and Kamada [18] extended it to immersed oriented surfaces in S^4 using singular braid charts. In this subsection, we extend these results in full generality, ie for any pair of homologous immersed surfaces in a closed orientable 4-manifold.

Theorem 4.3 *Let F and F' be oriented self-transversely immersed surfaces in a closed, orientable 4-manifold X which are homologous and have the same number of transverse double points of each sign. Then F and F' become isotopic after a sequence of stabilizations.*

To prove Theorem 4.3, we rely on the following diagrammatic lemma:

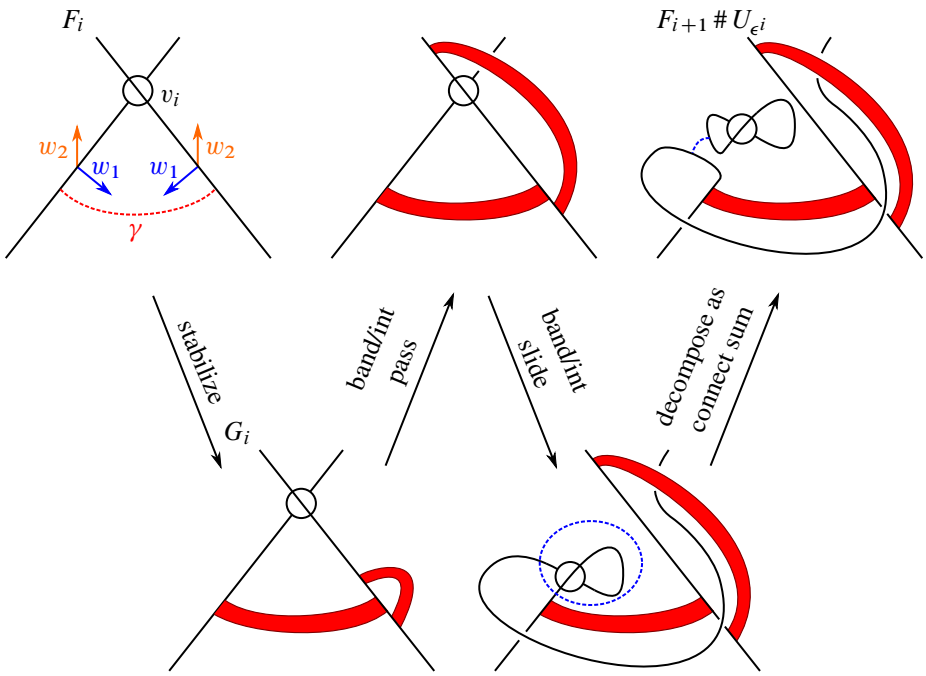


Figure 41: Top left: F_i is an oriented surface with $k-i>0$ transverse self-intersections. Here we draw part of a singular banded unlink diagram for F_i near a vertex v_i representing a self-intersection of F_i. (In this drawing, it is a negative self-intersection. Changing the marking at v_i yields a positive self-intersection.) We draw a positive normal basis (w_1, w_2) along each local sheet of F_i and indicate an arc γ along which we may stabilize F_i. From left to right following the arrows: we stabilize F_i to obtain a surface G_i, and then isotope G_i to realize a connect sum of a surface F_{i+1} with U_{ϵ^i}, where ϵ^i is the sign of the self-intersection represented by v_i.

Lemma 4.4 *Let F be an oriented self-transversely immersed surface in a closed, orientable 4-manifold. Suppose F has p positive and n negative self-intersections. After some number of stabilizations, F becomes isotopic to the connected-sum of an embedded surface with p copies of U_+ and n copies of U_-, where $U_\pm$ denotes the result of performing a cusp move to the embedded unknotted 2-sphere to create a $\pm$ self-intersection.*

Proof Let $(\mathcal{H}, L, B)$ be a singular banded unlink diagram of $F_0 := F$. Suppose that F has $k = p+n > 0$ self-intersections. Fix a vertex v_0 of L. Stabilize F_0 as in Figure 41, ie along an arc in $h^{-1}\left(\frac{3}{2}\right)$ that lies close to v_0. Call the resulting surface G_0. Now perform singular band moves as in Figure 41 to see that G_0 is isotopic to a connect sum $F_1 \# U_{\epsilon^0}$, where ϵ^0 is the sign of v_0 and F_1 is a self-transverse immersed surface with $k-1$ self-intersections.

If $k-1>0$, then repeat this argument on F_1 near another vertex v_1, stabilizing F_1 to obtain a surface G_1 that is isotopic to $F_2 \# U_{\epsilon^1}$, where F_2 has $k-2$ self-intersections. Note F is then stably isotopic to $F_2 \# U_{\epsilon^1} \# U_{\epsilon^0}$.

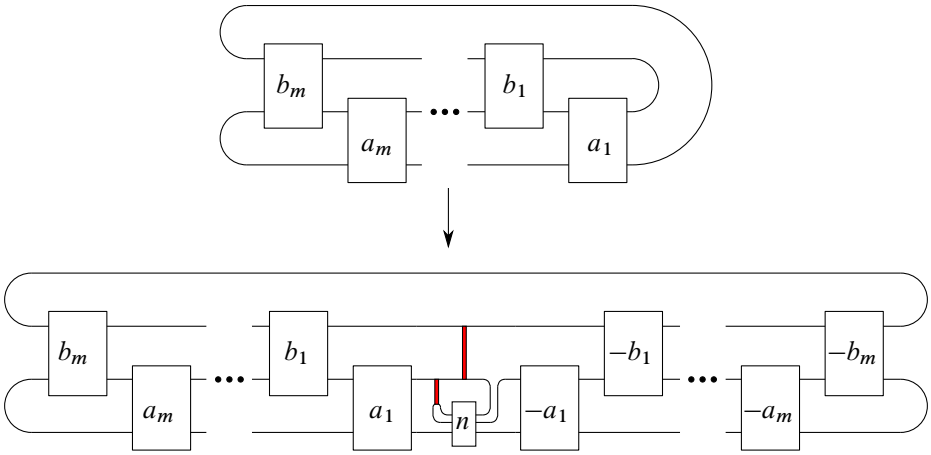


Figure 42: Top: a 2-bridge knot K in normal form. Here, a_i and b_i indicate signed numbers of whole twists (so each box has an even number of half-twists). Bottom: the n-twist spin $\tau^n K$ of K.

Repeat inductively to find that F is stably isotopic to $F_k \# (\#_p U_+) \# (\#_n U_-)$ for F_k an embedded surface, as desired. □

Proof of Theorem 4.3 By Lemma 4.4, F may be stabilized to a surface isotopic to $\widehat{F}\#(\#_p U_+)\#(\#_n U_-)$, where $\widehat{F}$ is an embedded surface and p and n are (respectively) the numbers of positive and negative

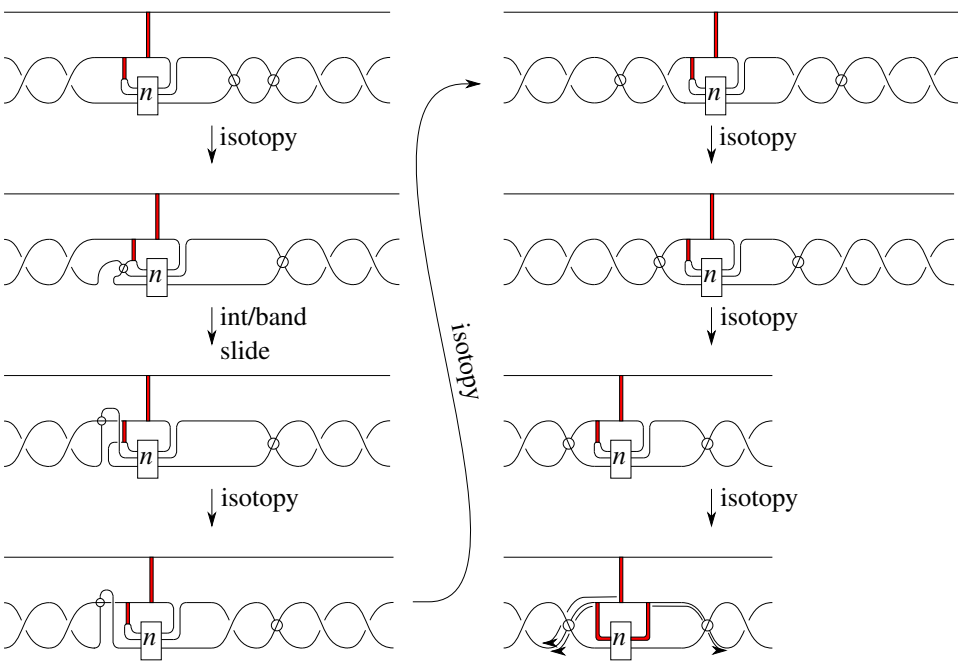


Figure 43: The first frame is (a portion of the diagram) obtained from Figure 42, bottom, by a finger move. We begin applying singular band moves with the goal of decreasing $|a_1|$ by 1. In the last frame we indicate three band/intersection passes that yield the first frame of Figure 44.

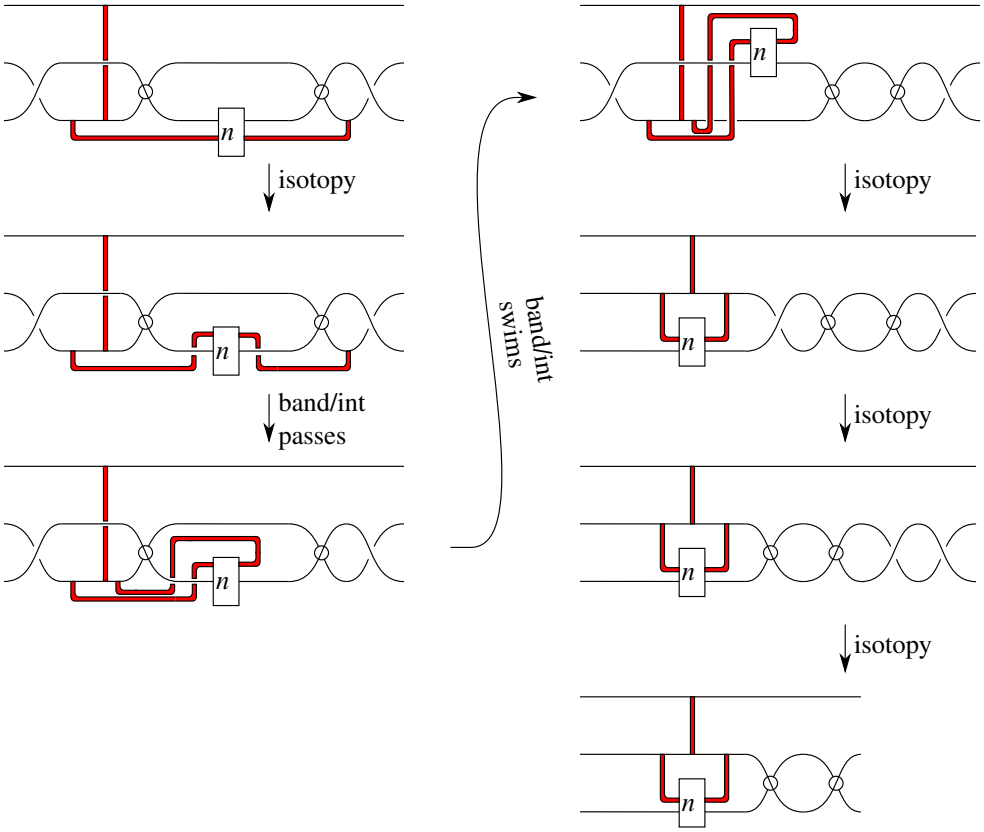


Figure 44: Continuing from Figure 43, we perform more singular band moves. In the last frame, the two vertices can be removed by a Whitney move, yielding the diagram from Figure 42, bottom, but with $|a_1|$ decreased by 1.

self-intersections of F. Applying the lemma also to F' (recalling that F' also has p positive and n negative self-intersections), we find that, after suitable stabilizations, F' becomes isotopic to

$$\widehat{F}' \# (\#_p U_+) \# (\#_n U_-)$$

for some embedded surface $\widehat{F}'$. Since $U_\pm$ is nullhomologous, $\widehat{F}$ and $\widehat{F}'$ are homologous to F and F' and hence to each other. Then, by [2], we know that $\widehat{F}$ and $\widehat{F}'$ (and hence F and F') are stably isotopic. □

4.3 Unknotting 2-knots with regular homotopies

Joseph, Klug, Ruppik and Schwartz [16] introduced the notion of the *Casson–Whitney number* of a 2-knot, which is half the minimal number of finger and Whitney moves needed to change a given 2-knot to an unknot. They showed that the Casson–Whitney number of any nontrivial twist spin of a 2-bridge knot is 1; ie that any nontrivial twist spin of a 2-bridge knot can be unknotted via one finger move followed by one Whitney move. In this subsection, we explicitly realize such a regular homotopy via singular banded unlink diagrams.

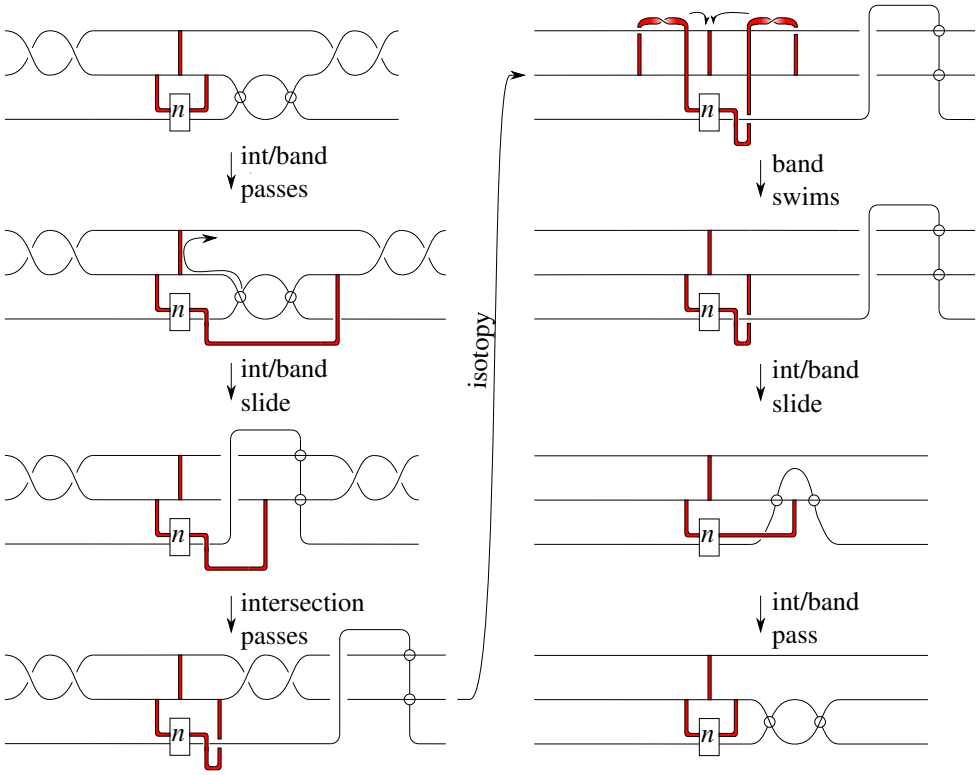


Figure 45: The first frame agrees with the last frame of Figure 44 after $|a_1|$ is decreased to zero. We can then perform singular band moves to the diagram to decrease $|b_1|$ by 1.

Theorem 4.5 [16] *The Casson–Whitney number of the n-twist spin ($|n| \neq 1$) $\tau^n K$ of a 2-bridge knot K is* 1.

Proof First, as in [16], we assume that the 2-bridge knot K is in normal form [5] with the number of half-twists in each twist region even, as in Figure 42. That is, using the standard correspondence between 2-bridge link diagrams and continued fraction expansion, we arrange for a diagram of K to correspond to a continued fraction $(a_1, b_1, \dots, a_m, b_m)$ of all even integers. We write $K = K(a_1, b_1, \dots, a_m, b_m)$.

Apply a finger move to the diagram of $\tau^n K$ in Figure 42 to obtain the first frame of Figure 43 (the visible twists are contained in the $\pm a_1$ twist boxes). In Figures 43 and 44, we show how to perform singular band moves with the result of decreasing $|a_1|$ by 1. Repeating this sequence, we eventually arrange for a_1 to become 0.

In Figure 45, we give another sequence of band moves (now assuming $a_1 = 0$) that decrease $|b_1|$ by 1. Repeating this sequence, we eventually arrange for $a_1 = b_1 = 0$.

We repeat these sequences of band moves to undo the twist boxes labeled $\pm a_2, \pm b_2, \dots, \pm a_m, \pm b_m$, and then finally apply a Whitney move to remove the two vertices and obtain a singular banded unlink diagram

for the n-twist spin of the unknot. This is an unknotted sphere, so we conclude that the Casson–Whitney number of $\tau^n K$ is 1. □

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A-polynomials, Ptolemy equations and Dehn filling 1265
JOSHUA A HOWIE, DANIEL V MATHEWS and JESSICA S PURCELL

The Alexandrov theorem for $2+1$ flat radiant spacetimes 1321
LÉO MAXIME BRUNSWIC

Real algebraic overtwisted contact structures on 3-spheres 1377
ŞEYMA KARADERELI and FERIT ÖZTÜRK

Fully augmented links in the thickened torus 1411
ALICE KWON

Unbounded $\mathfrak{sl}_3$-laminations and their shear coordinates 1433
TSUKASA ISHIBASHI and SHUNSUKE KANO

Bridge trisections and Seifert solids 1501
JASON JOSEPH, JEFFREY MEIER, MAGGIE MILLER and ALEXANDER ZUPAN

Random Artin groups 1523
ANTOINE GOLDSBOROUGH and NICOLAS VASKOU

A deformation of Asaeda–Przytycki–Sikora homology 1545
ZHENKUN LI, YI XIE and BOYU ZHANG

Cubulating a free-product-by-cyclic group 1561
FRANÇOIS DAHMANI and SURAJ KRISHNA MEDA SATISH

Virtual domination of 3-manifolds, III 1599
HONGBIN SUN

The Kakimizu complex for genus one hyperbolic knots in the 3-sphere 1667
LUIS G VALDEZ-SÁNCHEZ

Band diagrams of immersed surfaces in 4-manifolds 1731
MARK HUGHES, SEUNGWON KIM and MAGGIE MILLER

Anosov flows and Liouville pairs in dimension three 1793
THOMAS MASSONI

Hamiltonian classification of toric fibres and symmetric probes 1839
JOÉ BRENDEL

An example of higher-dimensional Heegaard Floer homology 1877
YIN TIAN and TIANYU YUAN

Fibered 3-manifolds and Veech groups 1897
CHRISTOPHER J LEININGER, KASRA RAFI, NICHOLAS ROUSE, EMILY SHINKLE and YVON VERBERNE